

**Generalization of the three-term recurrence formula and its
applications**

by

Yoon Seok Choun

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Professor Sultan Catto

Date

Chair of Examining Committee

Professor Igor L. Kuskovsky

Date

Executive Officer

Professor Brian Schwartz

Professor Ramzi Khuri

Professor David Tepper

Professor Namjong Moh

Professor Adrian Dumitru

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Advisor: Professor Sultan Catto

In an earlier paper we showed development of a bilocal baryon-meson field from two quark-antiquark fields. In the local approximation the hadron field was shown to exhibit supersymmetry which was then extended to hadronic mother trajectories and to inclusion of multiquark states. The Hamiltonian in the case of vanishing quark masses was shown to have a very good agreement with experiments. The theory for vanishing mass was solved using confluent hypergeometric functions. In order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalize all existing theories of confluent hypergeometric types. We call it the 'Grand Confluent Hypergeometric Function.' Our new solution produces previously unknown extra "hidden" quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

Furthermore, we show for the first time how to solve mathematical equations having three term recursion relations and go on producing the exact

solutions of some of the well-known special function theories that include Mathieu, Heun, Lamé and the Grand Confluent Hypergeometric Function. We hope these new functions and their solutions will produce remarkable new range of applications not only in supersymmetric field theories as is shown here, but in the areas of all different classes of mathematical physics, applied mathematics and in engineering applications.

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Contents

1	Introduction	1
1.1	Introduction	2
2	Grand Confluent Hypergeometric Equation (GCH)	8
2.1	Finite series	46
2.1.1	$\nu = \text{non-integer}$	46
2.1.2	$\nu = \text{integer}$	154
2.2	Infinite series	228
2.2.1	$\nu = \text{non-integer}$	228
2.2.2	$\nu = \text{integer}$	259
3	Three-term Recurrence Formula	271

3.1	Infinite series	274
3.2	Polynomial which makes B_n term terminated	291
3.3	Polynomial which makes A_n term terminated	312
3.4	Polynomial which makes A_n and B_n terms terminated	317
3.5	Summary	329
4	Analytic solution for GCH equation	337
4.0.1	Power series of polynomial in which B_n term is terminated: . .	339
4.0.2	infinite series	350
4.0.3	Integral formalism of polynomial which makes B_n term terminated:	353
4.0.4	Integral formalism of infinite series	365
4.0.5	Power series of polynomial in which makes A_n term terminated	369
4.0.6	Integral formalism of polynomial in which makes A_n term terminated	381
4.0.7	Power series of polynomial in which makes A_n and B_n term terminated	389

4.0.8	Integral formalism of polynomial in which makes A_n and B_n term terminated	397
4.0.9	Generating function of polynomial in which makes B_n term terminated	404
4.0.10	Generating function of polynomial in which makes A_n and B_n terms terminated	417
4.0.11	Asymptotic behavior of the function $y(x)$ and the boundary condition for x in the case of B_n term terminated	429
4.0.12	Asymptotic behavior of the function $y(x)$ and the boundary condition for x in the case of A_n term terminated	431
4.0.13	Asymptotic behavior of the function $y(x)$ and the boundary condition for x in the case of infinite series	433
4.1	Summary	434
5	Conclusion	455
	Bibliography	462

Chapter 1

Introduction

1.1 Introduction

The subject of special functions has a long and varied history. At the core of special functions lie the Gaussian hypergeometric function and its confluent forms, the confluent hypergeometric functions. The slight modification of the confluent hypergeometric functions are also known as Whittaker functions.

The Gaussian hypergeometric functions includes as special cases Legendre functions, the incomplete beta function, the complete elliptic functions of the first and second kinds, and most of the classical orthogonal polynomials.

The confluent hypergeometric functions include as special cases Bessel functions, parabolic cylinder functions, Coulomb wave functions, and incomplete gamma functions. Numerous properties of confluent hypergeometric functions flow directly from a knowledge of the Gaussian hypergeometric function, and a basic understanding of the Gaussian and the confluent hypergeometric functions is sufficient for the derivation of many characteristics of all the above named functions. A natural generalization of the Gaussian hypergeometric function is the generalized hypergeometric function, which in turn is generalized by Meijer's G-function. The theory of the generalized and the G-function is fundamental in the applications, since they contain as special cases all

commonly used functions of mathematical analysis, Further, these functions are the building blocks for many other functions which do not belong to the hypergeometric family.

Recently, taking our lead from the exact solution of a supersymmetric differential equation [1] describing quark-antiquark potential models and their relation to the multiquark systems, namely quark-diquark system and associated potential models (diquark is a pair of quarks formed as a point-like particle at one end of a baryon bag) in the description of relating the baryon masses to the meson masses, we are led into a discovery of a new kind of special function theory that generalizes the confluent hypergeometric series that we call Grand Confluent Hypergeometric functions, and show generation of a class of exact solutions that may be used in many areas of physics and mathematics and most importantly for our own purposes, the supersymmetric particle theories.

The supersymmetric theory we developed earlier [2], [3] based on supergroups of the type $SU(m/n)$ relates baryon masses and meson masses to each other with noteworthy accuracy, but also predicts multiquark states that have been observed experimentally, (such as a_0 and f_0 mesons made up of diquark-antidiquark pairs) with significant accuracy. In the process we have also developed a algebra of color[4], [5]

based on the units of highest division algebras, that of octonions that we call "split octonion algebra" that produces the correct color degrees of freedom for quark systems. Split octonion algebra based on the complexification of the octonionic units is a closed Heisenberg algebra whose properties have been described in some of our earlier papers[6], [7].

The notion of the solution having to give rise to an extra degree of a quantum number we designate as "*i*th kind of hidden radial quantum number" that will be expressed below, makes the theory especially applicable to all supersymmetric theories in nature. Indeed, it has an infinite number of eigenvalues, and each one has infinite eigennumbers. For example a hydrogen like atom wave function, only has one eigenvalue and has infinite eigennumbers. As we see in Regge trajectory plot of angular momentum vs. square of mass (J vs. m^2), there are many linearly increasing lines with same slopes including bunch of eigenvalues corresponding to fermions and bosons. It is not clear what the meaning of many eigenvalues are. With our analysis, it will be explained below. The formalism we develop in this paper is very general in scope, as it unifies all the special functions mentioned above in mathematical physics. It has a remarkable range of applications in mathematics and other sciences, as well as large range of applicability in all sorts of engineering problems.

We also generalized three term recurrence formula in linear differential equation. It is well known that all known special functions have only two term recursion relations. Linear differential equations has very long history over 500 years. During this period, mathematicians developed analytic solutions of only two term recursion relations. They do not know how to solve the case of three term recurrence formula. From our paper, we get exact solution of the three term one, and we can express it by integral formalism and generating function of it. For example of three recursion relations, We show exact solution of four special function: Mathieu, Heun, Lamé and Grand Confluent hypergeometric functions. Mathieu functions appear in physical problems involving elliptical shapes or periodic potentials, and were first introduced by Mathieu (1868)[8] when analyzing the motion of elliptical membranes. Unfortunately, the analytic determination of Mathieu functions "presents great difficulties" (Whittaker 1914[9], Frenkel and Portugal 2001[10]), and they are difficult to employ, "mainly because of the impossibility of analytically representing them in a simple and handy way" (Sips 1949[11], Frenkel and Portugal 2001). But it is not hard, yet extremely easy with our analysis.

For example, if one studies certain problems in astronomy or in general relativity, an encounter with Heun equation is inevitable. This is a general equation whose

special forms take names as Mathieu, Lamé and Coulomb spheroidal equations. Here the coefficients in a power series expansions do not have two term recursion relations. We have a relation at least between three or four different coefficients. In this thesis we will show how to get exact solution in power series, the integral formalism and generating functions of it. For the past 500 years, we have been only using in two term recurrence formulas. More than three term case we have been neglected because of its complexity. However, since 1930, we do not have simple problems to solve in theoretical particle physics and scientists and mathematicians doing research on this field have to tackle more difficult problems, either with more difficult metrics or in higher dimensions. Most of the difficult problems must include three term or more. With analysis of three term recurrence formulas, we can get exact solutions of higher term recurrence formula; four, five, \dots , m^{th} term and indeed, an infinite term recurrence formulas. It means that we can generalize all homogeneous and inhomogeneous linear differential equations. Most problems in nature turns out to be nonlinear. We usually linearize those system for simplification purposes. In linearizing the systems by certain methods of simplification we can approach the future with a good approximation. All physics theories ($E\&M$, Newtonian mechanics, quantum mechanic, QCD, supersymmetric field theories, string theories, general relativity, etc),

always involves solutions of linear differential equations, but unfortunately, there are no analytic solution of them in some important physical cases. We hope with the theory we developed here, we can get analytic solution for many linear systems, and point towards a future we can put under our control.

Chapter 2

Grand Confluent Hypergeometric

Equation (GCH)

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)9

There is the spin free hamiltonian involving only scalar potential for the $q - \bar{q}$ system:

$$H^2 = 4\left[\left(m + \frac{1}{2}br\right)^2 + P_r^2 + \frac{l(l+1)}{r^2}\right]; \left\{ \begin{array}{l} P_r^2 = -\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} \\ m = \text{mass} \\ b = \text{real positive constant} \\ l = \text{angular momentum quantum number} \end{array} \right. \quad (2.1)$$

The mass m is negligible because of its extremely small value. Eq(2.1) turns to be

$$H^2 = 4\left[\frac{1}{4}b^2r^2 + P_r^2 + \frac{l(l+1)}{r^2}\right] \quad (2.2)$$

When wave function $\psi(r)$ acts on both sides of eq(2.2), we have second order differential equation with respect to r

$$\frac{\partial^2\psi(r)}{\partial r^2} + \frac{2}{r} \frac{\partial\psi(r)}{\partial r} + \left[\frac{E^2}{4} - \frac{1}{4}b^2r^2 - \frac{l(l+1)}{r^2}\right]\psi(r) = 0 \quad (2.3)$$

Using the transformation $\psi = e^{-\frac{b}{4}r^2} r^l F(r)$, eq(2.3) becomes

$$\frac{\partial^2 F}{\partial r^2} + \left(\frac{2(l+1)}{r} - br \right) \frac{\partial F}{\partial r} + \left(\frac{E^2}{4} - \left(l + \frac{2}{3} \right) b \right) F = 0 \quad (2.4)$$

Let $z = \frac{1}{2}br^2$ then, Eq(2.4) becomes

$$z \frac{\partial^2 F}{\partial z^2} + (\gamma - z) \frac{\partial F}{\partial z} - \alpha_0 F(z) = 0 \quad (2.5)$$

where

$$\gamma = l + \frac{3}{2} \quad (2.6)$$

$$\alpha_0 = -\frac{1}{2b} \left(\frac{E^2}{4} - \left(l + \frac{3}{2} \right) b \right) \quad (2.7)$$

We see, eq(2.5) is equivalent to confluent hypergeometric equation. Also, α_0 must be zero or a negative integer. If it doesn't, polynomial $F(z)$ will blows up as z and n goes to infinity. Physically, we are not interested in this kind of situation. The function $F(z)$ must to be finite. Let's set $\alpha_0 = 1 - n_0$. Then,

$$\frac{1}{2b} \left(\frac{E^2}{4} - \left(l + \frac{3}{2} \right) b \right) = n_0 - 1 \quad (2.8)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)11

$$n_0 = 1, 2, 3, \dots ; \text{primary radial quantum number} \quad (2.9)$$

Then, eigenvalue is

$$E^2 = 4b \left[2|\alpha_0| + \left(l + \frac{3}{2} \right) \right] \quad (2.10)$$

Also wave function is equal to

$$\psi(r, \theta, \phi) = N r^l e^{-\frac{1}{4}br^2} F_{|\alpha_0|} \left(\gamma = l + \frac{3}{2}; z = \frac{1}{2}br^2 \right) Y_l^{m^*}(\theta, \phi) \quad (2.11)$$

where

$$\left\{ \begin{array}{l} |\alpha_0| = n_0 - 1 \\ l = \text{angular momentum quantum number} \\ m^* = \text{azimuthal quantum number} \\ N = \text{normalized constant} \\ F_{|\alpha_0|}(\gamma; z) = \text{the 1}^{\text{st}} \text{ kind of confluent hypergeometric function} \end{array} \right. \quad (2.12)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)12

The wave function is normalized so that

$$\int |\psi(r, \theta, \phi)|^2 d^3\mathbf{r} = 1 \quad (2.13)$$

Plug eq(2.11) in eq(2.13)

$$\begin{aligned} \int |\psi(r, \theta, \phi)|^2 d^3\mathbf{r} &= N^2 \int_0^\infty r^{2l+2} e^{-\frac{1}{2}br^2} |F_{|\alpha_0|}(\gamma; z)|^2 dr \underbrace{\int |Y_l^{m*}(\theta, \phi)|^2 d\Omega}_{=1} \\ &= 1 \end{aligned} \quad (2.14)$$

So,

$$N = \frac{1}{\sqrt{\int_0^\infty r^{2l+2} e^{-\frac{1}{2}br^2} |F_{|\alpha_0|}(\frac{1}{2}br^2)|^2 dr}} \quad (2.15)$$

By using orthogonal relation, eq(2.15) is equal to

$$N = \sqrt{\frac{(2b)^{l+\frac{3}{2}} (l+1)! (l+\frac{1}{2})!}{(2l+2)! |\alpha_0|! (l+|\alpha_0|+\frac{1}{2})! \sqrt{\pi}}} \quad (2.16)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)13

Putting eq(2.16) into eq(2.11), the normalized wave function is

$$\psi(r, \theta, \phi) = \sqrt{\frac{(2b)^{l+\frac{3}{2}}(l+1)!(l+\frac{1}{2})!}{(2l+2)!|\alpha_0|!(l+|\alpha_0|+\frac{1}{2})!\sqrt{\pi}}} r^l e^{-\frac{1}{4}br^2} F_{|\alpha_0|}(\gamma; z) Y_l^{m^*}(\theta, \phi) \quad (2.17)$$

where

$$\begin{cases} |\alpha_0| = n_0 - 1 \\ \gamma = l + \frac{2}{3} \\ z = \frac{1}{2}br^2 \end{cases} \quad (2.18)$$

Now let's define normalized wave function including the small mass m. We obtain schrödinger equation from eq(2.1) which is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \left(\frac{E^2}{4} - \frac{1}{4}b^2 \left(r + \frac{2m}{b} \right)^2 - \frac{l(l+1)}{r^2} \right) \psi = 0 \quad (2.19)$$

By the transformation $\psi = e^{-\frac{b}{4}\left(r+\frac{2m}{b}\right)^2} r^l y(r) Y_l^{m^*}(\theta, \phi)$, eq(2.19) becomes

$$r \frac{\partial^2 y}{\partial r^2} + (-br^2 - 2mr + 2(l+1)) \frac{\partial y}{\partial r} + (Qr - 2m(l+1)) y = 0 \quad (2.20)$$

where

$$Q = \frac{E^2}{4} - b \left(l + \frac{2}{3} \right) \quad (2.21)$$

Now, let's try to describe the function $y(r)$ as Frobinous series.

$$y(r) = \sum_{n=0}^{\infty} C_n x^{n+\lambda} \quad (2.22)$$

Plug eq(2.22) into eq(2.20), then we have

$$\begin{aligned} & \lambda(\lambda + 2l + 1)C_0 r^{\lambda-1} + (\lambda + 1)(\lambda + 2(l + 1))C_1 r^\lambda - 2m(\lambda + l + 1)C_0 r^\lambda \\ & + \sum_{n=1}^{\infty} \{ (n + \lambda + 1)(n + \lambda + 2(l + 1))C_{n+1} - 2m(n + \lambda + l + 1)C_n \\ & + (Q - b(n + \lambda - 1))C_{n-1} \} x^{n+\lambda} = 0 \end{aligned} \quad (2.23)$$

Indicial equation is

$$\lambda(\lambda + 2l + 1)C_0 r^{\lambda-1} + (\lambda + 1)(\lambda + 2(l + 1))C_1 r^\lambda - 2m(\lambda + l + 1)C_0 r^\lambda = 0 \quad (2.24)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)15

In eq(2.24), assume that $C_0 \neq 0$. Then we have two indicial roots

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -2l - 1 \quad (2.25)$$

and, recurrence formula for all n is

$$K_n = A_n + \frac{B_n}{K_{n-1}} \quad ; n \geq 1 \quad \left\{ \begin{array}{l} K_n = \frac{C_{n+1}}{C_n} \\ K_{n-1} = \frac{C_n}{C_{n-1}} \\ A_n = \frac{2m(n+\lambda+l+1)}{(n+\lambda+1)(n+\lambda+2(l+1))} \\ B_n = \frac{-Q+b(n+\lambda-1)}{(n+\lambda+1)(n+\lambda+2(l+1))} \end{array} \right. \quad (2.26)$$

Let's investigate function $y(r)$ as n and r go to infinity. As $n \gg 1$

$$\lim_{n \gg 1} A_n = \frac{2m}{n} \quad \lim_{n \gg 1} B_n = \frac{b}{n} \quad (2.27)$$

Plug eq(2.27) into eq(2.26)

$$\lim_{n \gg 1} K_n = \frac{2m}{n} + \frac{b/n}{\lim_{n \gg 1} K_{n-1}} \quad (2.28)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)16

We see the first term of r.h.s. in eq(2.28) is negligible, since mass m is extremely small and n is too large, respectively. Then, eq(2.28) is approximately equal to

$$\lim_{n \gg 1} K_n \simeq \frac{b/n}{\lim_{n \gg 1} K_{n-1}} \quad (2.29)$$

Let's now simplify this relation by giving K_n in terms of $\frac{C_1}{C_0} = m$ instead of K_{n-1} .

From eq(2.29)

$$\begin{aligned} K_0 &= \frac{C_1}{C_0} = m & K_1 &= \frac{b}{m} \\ K_2 &= \frac{m}{2} & K_3 &= \frac{2b}{3m} \\ K_4 &= \frac{3m}{2^2 2!} & K_5 &= \frac{8b}{1 \cdot 3 \cdot 5m} \\ K_6 &= \frac{1 \cdot 3 \cdot 5m}{2^3 3!} & K_7 &= \frac{2^3 3! b}{1 \cdot 3 \cdot 5 \cdot 7m} \\ K_8 &= \frac{1 \cdot 3 \cdot 5 \cdot 7m}{2^4 4!} & K_9 &= \frac{2^4 4! b}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9m} \\ K_{10} &= \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9m}{2^5 5!} & K_{11} &= \frac{2^5 5! b}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11m} \\ &\vdots & &\vdots \end{aligned} \quad (2.30)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)17

We can classify C_n to its even and odd parts from eq(2.30)

$$\begin{aligned}
 C_0 & & C_1 &= mC_0 \\
 C_2 &= bC_0 & C_3 &= \frac{b}{2}mC_0 \\
 C_4 &= \frac{1}{1 \cdot 3}b^2C_0 & C_5 &= \frac{b^2}{2^2 2!}mC_0 \\
 C_6 &= \frac{1}{1 \cdot 3 \cdot 5}b^3C_0 & C_7 &= \frac{b^3}{2^3 3!}mC_0 & (2.31) \\
 C_8 &= \frac{1}{1 \cdot 3 \cdot 5 \cdot 7}b^4C_0 & C_9 &= \frac{b^4}{2^4 4!}mC_0 \\
 C_{10} &= \frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}b^5C_0 & C_{11} &= \frac{b^5}{2^5 5!}mC_0 \\
 \vdots & & \vdots &
 \end{aligned}$$

We can describe $y(r)$ as power series by using eq(2.31). For simplicity, We suggest

$$C_0 = 1$$

$$\begin{aligned}
 \lim_{n \gg 1} y(r) &\simeq \sum_n C_{2n} r^{2n} + \sum_n C_{2n+1} r^{2n+1} \\
 &= \sum_n \frac{2^{2n} n!}{(2n)!} \left(\frac{1}{2} br^2\right)^n + mr \sum_n \frac{1}{n!} \left(\frac{1}{2} br^2\right)^n \\
 &= \sum_n \frac{(-\frac{1}{2})!}{(n - \frac{1}{2})!} \left(\frac{1}{2} br^2\right)^n + me^{\frac{1}{2} br^2} \\
 &> \sum_n \frac{1}{n!} \left(\frac{1}{2} br^2\right)^n + me^{\frac{1}{2} br^2} = (1 + mr)e^{\frac{1}{2} br^2} \quad (2.32)
 \end{aligned}$$

There is an important boundary condition for quantum wave function $\psi(r, \theta, \phi)$ which must go to zero as r goes to infinity. It is unacceptable that wave function $\psi(r, \theta, \phi)$ is divergent as r goes to infinity from the quantum mechanical point of view. As r is extremely large value, the big polynomial of degree n will take a dominant position. Substitute eq(2.32) into the wave function which gives $\psi(r, \theta, \phi) = N e^{-\frac{b}{4}(r + \frac{2m}{b})^2} r^l y(r) Y_l^{m*}(\theta, \phi)$; N is normalized constant.

$$\begin{aligned}
 \lim_{n \gg 1} \psi(r, \theta, \phi) &> N r^l e^{-\frac{b}{4}(r + \frac{2m}{b})^2} Y_l^{m*}(\theta, \phi) \lim_{n \gg 1} y(r) \\
 &= N r^l e^{-\frac{b}{4}(r + \frac{2m}{b})^2} (1 + mr) e^{\frac{1}{2} br^2} Y_l^{m*}(\theta, \phi) \\
 &\approx N (1 + mr) r^l e^{\frac{1}{4} br^2} Y_l^{m*}(\theta, \phi) \quad (2.33)
 \end{aligned}$$

Even if the mass m is extremely small, the wave function $\psi(r, \theta, \phi)$ will blow up as $r \rightarrow \infty$. Then it yields

$$\lim_{\substack{n \gg 1 \\ r \rightarrow \infty}} \psi(r, \theta, \phi) > \lim_{r \rightarrow \infty} N(1 + mr)r^l e^{\frac{1}{4}br^2} Y_l^{m*}(\theta, \phi) \rightarrow \infty \quad (2.34)$$

The wave function $\psi(r, \theta, \phi)$ blows up as r goes to infinity. All wave functions must go to zero as r goes to infinity from a quantum mechanical perspective. The first and second term of $y(r)$ must also be terminated to become a polynomial of degree n in this case. As we see in eq(2.32), the first term indicates even term of C_n , and the second term of it has odd term of C_n ; if $y(r)$ is not terminated at certain value of n , it will blow up as r goes to infinity. This is the reason why we argue that $y(r)$ must be terminated to become a polynomial of degree n . Now, let's try to define the first kind of independent solution as $\lambda_1 = 0$:

As $\lambda_1 = 0$,

$$A_n|_{\lambda=0} = \frac{2(n+l+1)m}{(n+1)(n+2(l+1))} \quad (2.35a)$$

$$B_n|_{\lambda=0} = \frac{-Q + b(n-1)}{(n+1)(n+2(l+1))} \quad (2.35b)$$

Put $n=0$ in eq(2.35a). We get $A_0 = m$. From eq(2.30), we have $K_0 = m$ which is

equal to A_0 . Plug $n = 1$ into recurrence formula.

$$K_1 = A_1 + \frac{B_1}{K_0} = A_1 + \frac{B_1}{A_0} = A_1 + \frac{B_1}{m} \quad (2.36)$$

We see from eq(2.36), A_n includes the first order of m which has extremely small value.. Then, we argue that $A_1 \ll \frac{B_1}{m}$. Eq(2.36) is approximately the same as

$$K_1 \simeq \frac{B_1}{m} \quad (2.37)$$

By using this process, we can simplify eq(2.26) by giving K_n in terms of A_n and B_n

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)21

instead of K_{n-1} . We define $B_{i,j,k,l}$ referring to $B_i B_j B_k B_l$. From eq(2.26),

$$\begin{aligned}
 K_0 &= A_0 \\
 K_1 &\simeq \frac{B_1}{A_0} \\
 K_2 &\simeq \frac{1}{B_1} (A_0 B_2 + A_2 B_1) \\
 K_3 &\simeq \frac{B_{1,3}}{A_0 B_2 + A_2 B_1} \\
 K_4 &\simeq \frac{1}{B_{1,3}} (A_0 B_{2,4} + A_2 B_{1,4} + A_4 B_{1,3}) \\
 K_5 &\simeq \frac{B_{1,3,5}}{A_0 B_{2,4} + A_2 B_{1,4} + A_4 B_{1,3}} \tag{2.38} \\
 K_6 &\simeq \frac{1}{B_{1,3,5}} (A_0 B_{2,4,6} + A_2 B_{1,4,6} + A_4 B_{1,3,6} + A_6 B_{1,3,5}) \\
 K_7 &\simeq \frac{B_{1,3,5,7}}{A_0 B_{2,4,6} + A_2 B_{1,4,6} + A_4 B_{1,3,6} + A_6 B_{1,3,5}} \\
 K_8 &\simeq \frac{1}{B_{1,3,5,7}} (A_0 B_{2,4,6,8} + A_2 B_{1,4,6,8} + A_4 B_{1,3,6,8} + A_6 B_{1,3,5,8} + A_8 B_{1,3,5,7}) \\
 K_9 &\simeq \frac{B_{1,3,5,7,9}}{A_0 B_{2,4,6,8} + A_2 B_{1,4,6,8} + A_4 B_{1,3,6,8} + A_6 B_{1,3,5,8} + A_8 B_{1,3,5,7}} \\
 \vdots & \qquad \qquad \qquad \vdots
 \end{aligned}$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)22

and we can classify C_n to have even and odd terms from eq(2.38)

$$\begin{aligned}
 C_0 & & C_1 &= C_0 A_0 \\
 C_2 &= C_0 B_1 & C_3 &= C_0 (A_0 B_2 + A_2 B_1) \\
 C_4 &= C_0 B_{1,3} & C_5 &= C_0 (A_0 B_{2,4} + A_2 B_{1,4} + A_4 B_{1,3}) \\
 C_6 &= C_0 B_{1,3,5} & C_7 &= C_0 (A_0 B_{2,4,6} + A_2 B_{1,4,6} + A_4 B_{1,3,6} + A_6 B_{1,3,5}) \\
 C_8 &= C_0 B_{1,3,5,7} & C_9 &= C_0 (A_0 B_{2,4,6,8} + A_2 B_{1,4,6,8} + A_4 B_{1,3,6,8} + A_6 B_{1,3,5,8} + A_8 B_{1,3,5,7}) \\
 C_{10} &= C_0 B_{1,3,5,7,9} & C_{11} &= C_0 (A_0 B_{2,4,6,8,10} + A_2 B_{1,4,6,8,10} + A_4 B_{1,3,6,8,10} + A_6 B_{1,3,5,8,10} \\
 & & & + A_8 B_{1,3,5,7,10} + A_{10} B_{1,3,5,7,9}) \\
 \vdots & & \vdots &
 \end{aligned} \tag{2.39}$$

Now $y(r)$ can be described as a power series, by using eq(2.39).

$$\begin{aligned}
 y(r) &= y(r)_{domin.} + y(r)_{small} = \sum_{n=0}^{\infty} C_{2n} r^{2n} + \sum_{n=0}^{\infty} C_{2n+1} r^{2n+1} \\
 &= C_0 \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} B_{2k+1} r^{2n} \right\} + C_0 \left\{ A_0 r + (A_0 B_2 + A_2 B_1) r^3 \right. \\
 &\quad \left. + \sum_{n=2}^{\infty} \left[A_0 \prod_{p=0}^{n-1} B_{2p+2} + \sum_{j=1}^{n-1} \left(A_{2j} \prod_{p=j}^{n-1} B_{2p+2} \prod_{k=0}^{j-1} B_{2k+1} \right) + A_{2n} \prod_{p=0}^{n-1} B_{2p+1} \right] r^{2n+1} \right\} \\
 & \tag{2.40}
 \end{aligned}$$

We choose one of B_{2k+1} to be zero, where $k = 0, 1, 2, 3, \dots$ in order to make a polynomial of degree n in $y(r)_{domin.}$ of eq(2.39). In other words, we might choose $Q = 2b(n_0 - 1)$ where $n_0 = 1, 2, 3, \dots$ in eq(2.35b). Then B_{2k+1} will be zero at certain value of k . However, since $y(r)_{small}$ in eq(2.40); look at the odd term of C_n in eq(2.39), it has combinations of B_{2k+2} and B_{2k+1} for every each of the odd term of C_n . It means that one of each of B_{2k+2} and B_{2k+1} terms must be zero at same time. In other words, $Q = 2b(n_0 - 1)$ and $Q = b(2n_1 - 1)$ where $n_1 = 0, 1, 2, 3, \dots$ must be satisfied in this series spontaneously. We call n_0 as primary radial quantum number and n_1 as the first kind of hidden radial quantum number. Then $y(r)$ will be a polynomial degree of n . $y(r)$ consists of two terms which are $y(r)_{domin.}$ and $y(r)_{small}$ in eq(2.40). The dominant wave function $y(r)_{domin.}$ which does not include small mass m must be terminated to become a polynomial of degree n in this case. But the $y(r)_{small}$ is extremely small wave function because it includes $A_n|_{\lambda=0} = \frac{2(n+l+1)m}{(n+1)(n+2(l+1))}$ which has small mass m . Also eq(2.21) is equivalent to $2b(n_0 - 1)$ and $b(2n_1 - 1)$ at same time.

For simplicity, plugging $C_0 = \frac{(l+n_0-\frac{1}{2})!}{(l+\frac{1}{2})!}$ in eq(2.40), we obtain

$$\begin{aligned}
 y(r) &= \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = l + \frac{3}{2}; z = \frac{1}{2}br^2 \right) \\
 &= F_{|\alpha_0|}(\gamma; z) + mr \prod_{|\alpha_0|}^{|\alpha_1|} (\gamma; z) \quad \text{only if } |\alpha_0| \leq |\alpha_1| \quad (2.41)
 \end{aligned}$$

where,

$$F_{|\alpha_0|}(\gamma; z) = \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} z^n = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \quad (2.42a)$$

$$\begin{aligned} \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \\ &\times \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{1}{2}(\gamma - \frac{1}{2})) \Gamma(n + \frac{1}{2}) \Gamma(\gamma + n - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \\ &= \frac{1}{2\pi i B(|\alpha_1| + 1, \frac{1}{2})} \mathcal{T}(s, t, p, u) \left(w_1 \partial_{w_1} + \frac{1}{2}(\gamma - \frac{1}{2}) \right) F_{|\alpha_0|}(\gamma; w_1) \end{aligned} \quad (2.42b)$$

And,

$$B(|\alpha_1| + 1, \frac{1}{2}) = \frac{\Gamma(|\alpha_1| + 1) \Gamma(1/2)}{\Gamma(|\alpha_1| + 3/2)} \quad (2.43a)$$

$$w_1 = \frac{z u s t}{(1 - u)} (1 - p^2) \quad (2.43b)$$

in the above, $\mathcal{T}(s, t, p, u)$ is the operator which acts on

$$\mathcal{T}(s, t, p, u) = \int_0^\infty ds s^{-1/2} (1 + s)^{-(|\alpha_1| + 3/2)} \int_0^1 dt t^{\gamma - 3/2} \int_{-1}^1 dp \oint du \frac{e^{-\frac{zu(1-t)}{(1-u)}(1-p^2)}}{u^{|\alpha_1| + 1} (1 - u)} \quad (2.44)$$

We see in eq(2.42a), it is the first kind of confluent hypergeometric polynomial of degree $|\alpha_0|$ surprisingly. Eq(2.41) denoted as $\mathcal{QW}_{|\alpha_0|,|\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = l + \frac{3}{2}; z = \frac{1}{2}br^2 \right)$ is called as the first kind of Grand confluent hypergeometric polynomial of degree $|\alpha_0|$ and $|\alpha_1|$ with the first order m .

Also, we obtain two eigenvalues which are

$$E_0^2 = 4b \left\{ l + 2n_0 - \frac{1}{2} \right\} \quad (2.45a)$$

$$E_1^2 = 4b \left\{ l + 2n_1 + \frac{1}{2} \right\} \quad (2.45b)$$

Eq(2.45a) is called as primary radial eigenvalue. And eq(2.45b) is called as the first kind of hidden radial eigenvalue.

As we let the small mass m goes to zero in eq(2.41), we have

$$\lim_{m \rightarrow 0} y(r) = F_{|\alpha_0|}(\gamma; z) \quad (2.46)$$

Eq(2.46) denoted as the 1st kind of confluent hypergeometric polynomial is exactly

satisfied with eq(2.5). From eq(2.41), the wave function of it is

$$\psi(r, \theta, \phi)_{n_0, n_1, l, m^*} = \bar{N} e^{-\frac{b}{4}(r + \frac{2m}{b})^2} r^l \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = l + \frac{3}{2}; z = \frac{1}{2} br^2 \right) Y_l^{m^*}(\theta, \phi) \quad (2.47)$$

By using orthogonal relation, normalized constant \bar{N} is

$$\begin{aligned} \bar{N} &= \left[\frac{2^{\gamma-1}}{b^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) - m \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}}}{b^{\gamma+\frac{1}{2}}} \left\{ \frac{2\Gamma(|\alpha_0| + \gamma - \frac{1}{2}) \Gamma(|\alpha_0| + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} \right. \right. \\ &\quad \left. \left. - \sum_{n=0}^{|\alpha_0|} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{1}{2}(\gamma - \frac{1}{2})) \Gamma(|\alpha_0| + \gamma) \Gamma(\gamma + n - \frac{1}{2}) (-|\alpha_0|)_n (n - |\alpha_1|)_k}{\Gamma(\gamma) \Gamma(k + n - |\alpha_0| + \frac{1}{2}) (\gamma)_n (n!)} \right\} \right]^{-\frac{1}{2}} \quad (2.48) \end{aligned}$$

Plug eq(2.48) in eq(2.47)

$$\begin{aligned} \psi(r, \theta, \phi)_{n_0, n_1}^{l, m^*} &= \left[\frac{2^{\gamma-1}}{b^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) \right. \\ &\quad \left. - m \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}}}{b^{\gamma+\frac{1}{2}}} \left\{ \frac{2\Gamma(|\alpha_0| + \gamma - \frac{1}{2}) \Gamma(|\alpha_0| + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} \right. \right. \\ &\quad \left. \left. - \sum_{n=0}^{|\alpha_0|} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{1}{2}(\gamma - \frac{1}{2})) \Gamma(|\alpha_0| + \gamma) \Gamma(\gamma + n - \frac{1}{2}) (-|\alpha_0|)_n (n - |\alpha_1|)_k}{\Gamma(\gamma) \Gamma(k + n - |\alpha_0| + \frac{1}{2}) (\gamma)_n (n!)} \right\} \right]^{-\frac{1}{2}} \\ &\quad \times r^l e^{-\frac{b}{4}(r + \frac{2m}{b})^2} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = l + \frac{3}{2}; z = \frac{1}{2} br^2 \right) Y_l^{m^*}(\theta, \phi) \quad (2.49) \end{aligned}$$

where,

$$\left. \begin{array}{l} |\alpha_0| = n_0 - 1 \\ |\alpha_1| = n_1 - 1 \end{array} \right\} \text{only if } |\alpha_0| \leq |\alpha_1| \quad (2.50)$$

Eq(2.49) is called the 1st kind of the normalized Grand confluent hypergeometric wave function of degree $|\alpha_0|$ and $|\alpha_1|$.

As the small mass m goes to zero in eq(2.49), it turns out

$$\lim_{m \rightarrow 0} \psi(r, \theta, \phi)_{n_0, n_1, l, m^*} = \sqrt{\frac{b^\gamma}{2^{\gamma-1} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma)}} r^l e^{-\frac{1}{4}br^2} F_{|\alpha_0|}(\gamma; z) Y_l^{m^*}(\theta, \phi) \quad (2.51)$$

Eq(2.51) is exactly equivalent to eq(2.17). We see in the new special function, there are two eigennumbers n_0 and n_1 . The first eigennumber called primary radial quantum number appears in zeroth order of m term, $y(r)_{\text{domin.}}$ and first order of m term $y(r)_{\text{small}}$. But the second eigennumber called the first kind of hidden radial quantum number only appears in the first order of the m term. As we neglect small mass m , the primary radial quantum number only appears in wave function. However, when we include the small mass m , the second eigennumber, 1st kind of hidden radial quantum number is created. When one sees any other special function, for example,

Laguerre and Associated laguerre function, Legendre and associated Legendre function, hypergeometric function, Kummer function, etc, those functions only have one eigennumber. This is quiet a strange function. Also, surprisingly, the story does not end here! In the new special function, there are two terms which are 0^{th} order of m term, $y(r)_{dominan.}$ and 1^{st} order of m term, $y(r)_{small.}$ Actually, higher order of m terms do exist. But the mass m is extremely small. So We neglect the m terms that are more than 2^{nd} order. We only count it up to a 1^{st} order of m term. The full description of function $y(r)$ include all higher order of mass m in the following way.

$$y(r) = \sum_{i=0}^{\infty} m^i \sum_{n=0}^{\infty} C_{n,i} r^n = \sum_{n=0}^{\infty} C_{n,0} r^n + m \sum_{n=0}^{\infty} C_{n,1} r^n + m^2 \sum_{n=0}^{\infty} C_{n,2} r^n + m^3 \sum_{n=0}^{\infty} C_{n,3} r^n + \dots \quad (2.52)$$

If the function $y(r)$ of eq(2.52) is infinite series, then the wave function $\psi(r, \theta, \phi)$ will blow up as we see in eq(2.34). So all of each of summation in eq(2.52) must be a polynomial. Then, eq(2.52) become

$$y(r) = \sum_{i=0}^{\infty} m^i \sum_{n=0}^{N_i} C_{n,i} r^n = \sum_{n=0}^{N_0} C_{n,0} r^n + m \sum_{n=0}^{N_1} C_{n,1} r^n + m^2 \sum_{n=0}^{N_2} C_{n,2} r^n + m^3 \sum_{n=0}^{N_3} C_{n,3} r^n + \dots \quad (2.53)$$

Then, all possible eigenvalues and eigennumbers of m^{th} order term are

$$\text{m}^0 \text{ order term; } \left\{ E_0^2 = 4b\left(l + 2n_0 - \frac{1}{2}\right) \quad n_0 = 1, 2, 3, 4, \dots \right\} \quad (2.54a)$$

$$\text{m}^1 \text{ order term; } \left\{ \begin{array}{l} E_0^2 = 4b\left(l + 2n_0 - \frac{1}{2}\right) \quad n_0 = 1, 2, 3, 4, \dots \\ E_1^2 = 4b\left(l + 2n_1 + \frac{1}{2}\right) \quad n_1 = 1, 2, 3, 4, \dots \end{array} \right\} \text{ only if } n_0 \leq n_1 \quad (2.54b)$$

$$\text{m}^2 \text{ order term; } \left\{ \begin{array}{l} E_0^2 = 4b\left(l + 2n_0 - \frac{1}{2}\right) \quad n_0 = 1, 2, 3, 4, \dots \\ E_1^2 = 4b\left(l + 2n_1 + \frac{1}{2}\right) \quad n_1 = 1, 2, 3, 4, \dots \\ E_2^2 = 4b\left(l + 2n_2 + \frac{3}{2}\right) \quad n_2 = 1, 2, 3, 4, \dots \end{array} \right\} \text{ only if } n_0 \leq n_1 \leq n_2 \quad (2.54c)$$

$$\text{m}^3 \text{ order term; } \left\{ \begin{array}{l} E_0^2 = 4b\left(l + 2n_0 - \frac{1}{2}\right) \quad n_0 = 1, 2, 3, 4, \dots \\ E_1^2 = 4b\left(l + 2n_1 + \frac{1}{2}\right) \quad n_1 = 1, 2, 3, 4, \dots \\ E_2^2 = 4b\left(l + 2n_2 + \frac{3}{2}\right) \quad n_2 = 1, 2, 3, 4, \dots \\ E_3^2 = 4b\left(l + 2n_3 + \frac{5}{2}\right) \quad n_3 = 1, 2, 3, 4, \dots \end{array} \right\} \quad (2.54d)$$

$$m^4 \text{ order term; } \left\{ \begin{array}{l} E_0^2 = 4b\left(l + 2n_0 - \frac{1}{2}\right) \quad n_0 = 1, 2, 3, 4, \dots \\ E_1^2 = 4b\left(l + 2n_1 + \frac{1}{2}\right) \quad n_1 = 1, 2, 3, 4, \dots \\ E_2^2 = 4b\left(l + 2n_2 + \frac{3}{2}\right) \quad n_2 = 1, 2, 3, 4, \dots \\ E_3^2 = 4b\left(l + 2n_3 + \frac{5}{2}\right) \quad n_3 = 1, 2, 3, 4, \dots \\ E_4^2 = 4b\left(l + 2n_4 + \frac{7}{2}\right) \quad n_4 = 1, 2, 3, 4, \dots \end{array} \right\} \quad (2.54e)$$

where

$$\left\{ \begin{array}{l} n_i \leq n_j \text{ where } i \leq j \text{ and } i, j = 0, 1, 2, \dots \\ E_0^2 = \text{primary radial eigenvalue} \\ E_i^2 = i^{\text{th}} \text{ type hidden radial eigenvalue} \\ n_0 = \text{primary radial quantum number} \\ n_i = i^{\text{th}} \text{ type hidden radial quantum number} \end{array} \right\} \quad (2.55)$$

We see from eq(2.54a) through eq(2.54e), i th term of m has $(i+1)$ different eigenvalues.

Actually, we don't have to think about functions of all higher order of m th term, since these functions create extremely small vibrations and variation. So zeroth order of m and 1st order of m terms are sufficient.

Now let's generalize this new special function. Suppose that there is a second

order differential equation which is

$$x^2 y''(x) + a_0 x y'(x) + (a_1 x^4 + b_1 x^3 + c_1 x^2 + d_1) y(x) = 0 \quad \text{where, } \begin{cases} a_0, a_1, b_1, c_1, d_1 = \Re \\ 0 \leq x \leq \infty \end{cases} \quad (2.56)$$

Eq(2.56) is equivalent to eq(2.19). All coefficients in the above exactly correspond in the following way.

$$\begin{aligned} a_0 &\longleftrightarrow 2 \\ a_1 &\longleftrightarrow -\frac{b^2}{4} \\ b_1 &\longleftrightarrow -mb \\ c_1 &\longleftrightarrow \left(\frac{E^2}{4} - m^2\right) \\ d_1 &\longleftrightarrow -l(l+1) \\ x &\longrightarrow r \end{aligned} \quad (2.57)$$

Let's say that

$$y(x) = e^{\frac{i}{2}\sqrt{a_1}\left(x+\frac{b_1}{2a_1}\right)^2} x^{\frac{1}{2}\left(-(a_0-1)\pm\sqrt{(a_0-1)^2-4d_1}\right)} g(x) \quad (2.58)$$

Plug eq(2.58) into eq(2.56).

$$xg''(x) + (\mu x^2 + \varepsilon x + \nu)g'(x) + (\Omega x + \varepsilon\omega)g(x) = 0 \quad (2.59)$$

where,

$$\begin{aligned} \mu &= 2i\sqrt{a_1} \\ \varepsilon &= \frac{ib_1}{\sqrt{a_1}} \\ \nu &= 1 \pm \sqrt{(a_0 - 1)^2 - 4d_1} \\ \Omega &= c_1 - \frac{b_1^2}{4a_1} + i\sqrt{a_1}(2 \pm \sqrt{(a_0 - 1)^2 - 4d_1}) \\ \omega &= \frac{1}{2}(1 \pm \sqrt{(a_0 - 1)^2 - 4d_1}) \end{aligned} \quad (2.60)$$

Eq(2.59) is equivalent to eq(2.20). All coefficients in the above exactly correspond to

the following:

$$\begin{aligned}
 \mu &\longleftrightarrow -b \\
 \varepsilon &\longleftrightarrow -2m \\
 \nu &\longleftrightarrow 2(l+1) \\
 \Omega &\longleftrightarrow Q = \frac{E^2}{4} - b\left(l + \frac{3}{2}\right) \\
 \omega &\longleftrightarrow (l+1) \\
 x &\longrightarrow r
 \end{aligned} \tag{2.61}$$

We first suggest that $|\frac{\varepsilon}{2}| = |\frac{ib_1}{2\sqrt{a_1}}| \ll 1$. Assume that

$$g(x) = \sum_{n=0}^{\infty} C_n x^{n+\lambda} \tag{2.62}$$

Plug eq(2.62) into eq(2.59). Then we get two indicial roots.

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 1 - \nu \tag{2.63}$$

And, recurrence formula for all n is

$$K_n = A_n + \frac{B_n}{K_{n-1}} \quad ; n \geq 1 \quad \left\{ \begin{array}{l} K_n = \frac{C_{n+1}}{C_n} \\ K_{n-1} = \frac{C_n}{C_{n-1}} \\ A_n = -\frac{\varepsilon(n+\lambda+\omega)}{(n+\lambda+1)(n+\lambda+\nu)} \\ B_n = -\frac{\Omega+\mu(n+\lambda-1)}{(n+\lambda+1)(n+\lambda+\nu)} \\ \frac{C_1}{C_0} = -\frac{\varepsilon}{2} \end{array} \right. \quad (2.64)$$

Let's test for the convergence of the function $g(x)$. As n goes to infinity, recurrence formula is approximately equal to

$$\lim_{n \gg 1} K_n \simeq \frac{-\mu/n}{\lim_{n \gg 1} K_{n-1}} \quad (2.65)$$

Plug eq(2.65) into eq(2.64)

$$C_{n+1} \simeq -\frac{\mu}{n} C_{n-1} \quad (2.66)$$

Now we can describe $g(x)$ as power series by substituting eq(2.66) into eq(2.62). For

simplicity, we suggest $C_0 = 1$

$$\begin{aligned}
 \lim_{n \gg 1} g(x) &\simeq \sum_n C_{2n} x^{2n} + \sum_n C_{2n+1} x^{2n+1} \\
 &= \sum_n \frac{2^{2n} n!}{(2n)!} \left(-\frac{1}{2} \mu x^2\right)^n - \frac{1}{2} \varepsilon x \sum_n \frac{1}{n!} \left(\frac{1}{2} b r^2\right)^n \\
 &= \sum_n \frac{(-\frac{1}{2})!}{(n - \frac{1}{2})!} \left(-\frac{1}{2} \mu x^2\right)^n - \frac{1}{2} \varepsilon x e^{-\frac{1}{2} \mu x^2} \\
 &= 1 + \sqrt{\pi} e^{(-\frac{1}{2} \mu x^2)} \sqrt{-\frac{1}{2} \mu x^2} \operatorname{Erf} \left[\sqrt{-\frac{1}{2} \mu x^2} \right] - \frac{1}{2} \varepsilon x e^{-\frac{1}{2} \mu x^2} \\
 &> \sum_n \frac{1}{n!} \left(-\frac{1}{2} \mu x^2\right)^n - \frac{1}{2} \varepsilon x e^{-\frac{1}{2} \mu x^2} = \left(1 - \frac{1}{2} \varepsilon x\right) e^{-\frac{1}{2} \mu x^2} \quad (2.67)
 \end{aligned}$$

Substitute eq(2.67) into eq(2.58)

$$\begin{aligned}
 \lim_{n \gg 1} y(x) &> e^{\frac{i}{2} \sqrt{a_1} \left(x + \frac{b_1}{2a_1}\right)^2} x^{\frac{1}{2} \left(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}\right)} \lim_{n \gg 1} g(x) \\
 &= e^{\frac{i}{2} \sqrt{a_1} \left(x + \frac{b_1}{2a_1}\right)^2} x^{\frac{1}{2} \left(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}\right)} \left(1 - \frac{1}{2} \varepsilon x\right) e^{-\frac{1}{2} \mu x^2} \\
 &\approx e^{-\frac{i}{2} \sqrt{a_1} x^2} x^{\frac{1}{2} \left(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}\right)} \left(1 - \frac{i b_1}{2 \sqrt{a_1}} x\right) \quad (2.68)
 \end{aligned}$$

(I) As $a_1 =$ real positive

(a) If $\frac{1}{2} \left(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}\right) < -1$,

Eq(2.68) turns to be

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) > \left(\cos\left(\frac{\sqrt{a_1}}{2}x^2\right) - i \sin\left(\frac{\sqrt{a_1}}{2}x^2\right) \right) x^{\frac{1}{2}(-a_0-1) \pm \sqrt{(a_0-1)^2-4d_1}} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x \right) \quad (2.69)$$

As $x \rightarrow 0$, it yields

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) > x^{\frac{1}{2}(-a_0-1) \pm \sqrt{(a_0-1)^2-4d_1}} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x \right) \rightarrow \infty \quad (2.70)$$

Clearly the function $y(x)$ is divergent. Also, if $x \rightarrow \infty$, then

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) > \left(\cos\left(\frac{\sqrt{a_1}}{2}x^2\right) - i \sin\left(\frac{\sqrt{a_1}}{2}x^2\right) \right) x^{\frac{1}{2}(-a_0-1) \pm \sqrt{(a_0-1)^2-4d_1}} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x \right) \rightarrow 0 \quad (2.71)$$

Then, the function $y(x)$ is undetermined. We have to prove this in a different way.

Plug the fourth term in eq(2.67) into eq(2.58).

$$\begin{aligned}
 \lim_{n \gg 1} y(x) &= e^{\frac{i}{2}\sqrt{a_1}\left(x+\frac{b_1}{2a_1}\right)^2} x^{\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right)} \\
 &\quad \times \left(1 + \sqrt{\pi} e^{-\frac{1}{2}\mu x^2} \sqrt{-\frac{1}{2}\mu x^2} \operatorname{Erf} \left[\sqrt{-\frac{1}{2}\mu x^2}\right] - \frac{1}{2}\varepsilon x e^{-\frac{1}{2}\mu x^2}\right) \\
 &\simeq x^{\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right)} \left(e^{\frac{i}{2}\sqrt{a_1}x^2} - \frac{ib_1}{2\sqrt{a_1}} x e^{-\frac{i}{2}\sqrt{a_1}x^2}\right. \\
 &\quad \left.+ (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} x e^{-\frac{i}{2}\sqrt{a_1}x^2} \operatorname{Erf} \left[(i)^{3/2}(a_1)^{1/4}x\right]\right) \quad (2.72)
 \end{aligned}$$

As x of $\operatorname{Erf} \left[(i)^{3/2}(a_1)^{1/4}x\right]$ function goes to ∞ , it's imaginary part oscillates around zero. As a_1 increases, its oscillation are really close to zero, and it's real part oscillates at around -1 . Also, as a_1 increases, its oscillation is really close to -1 . As x goes to ∞ in eq(2.72)

$$\begin{aligned}
 \lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) &\simeq x^{\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right)} \left(e^{\frac{i}{2}\sqrt{a_1}x^2} - \frac{ib_1}{2\sqrt{a_1}} x e^{-\frac{i}{2}\sqrt{a_1}x^2}\right. \\
 &\quad \left.+ (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} x e^{-\frac{i}{2}\sqrt{a_1}x^2} \operatorname{Erf} \left[(i)^{3/2}(a_1)^{1/4}x\right]\right) \rightarrow 0 \quad (2.73)
 \end{aligned}$$

The solution of real and imaginary parts in eq(2.73) both goes to zero as x goes to ∞ .

(b) If $\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right) = -1$,

Eq(2.68) and eq(2.72) turn out to be

$$\lim_{n \gg 1} y(x) > e^{-\frac{i}{2}\sqrt{a_1}x^2} \left(\frac{1}{x} - \frac{ib_1}{2\sqrt{a_1}} \right) \quad (2.74)$$

$$\lim_{n \gg 1} y(x) \simeq \left(\frac{1}{x} e^{\frac{i}{2}\sqrt{a_1}x^2} - \frac{ib_1}{2\sqrt{a_1}} e^{-\frac{i}{2}\sqrt{a_1}x^2} + (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} e^{-\frac{i}{2}\sqrt{a_1}x^2} \text{Erf} \left[(i)^{3/2} (a_1)^{1/4} x \right] \right) \quad (2.75)$$

As $x \rightarrow 0$ in eq(2.74), it then yields

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) > e^{-\frac{i}{2}\sqrt{a_1}x^2} \left(\frac{1}{x} - \frac{ib_1}{2\sqrt{a_1}} \right) \rightarrow \infty \quad (2.76)$$

As x goes to zero, the function $y(x)$ become divergent. Also, as $x \rightarrow \infty$ in eq(2.75),

we get

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) \simeq e^{-\frac{i}{2}\sqrt{a_1}x^2} \left(-\frac{ib_1}{2\sqrt{a_1}} + (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} \text{Erf} \left[(i)^{3/2} (a_1)^{1/4} x \right] \right) \quad (2.77)$$

As suggested by $|\frac{\varepsilon}{2}| = |\frac{ib_1}{2\sqrt{a_1}}| \ll 1$, Eq(2.77) is simply approximately

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) \simeq (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} e^{-\frac{i}{2}\sqrt{a_1}x^2} \text{Erf} \left[(i)^{3/2} (a_1)^{1/4} x \right] \quad (2.78)$$

As x goes to ∞ in eq(2.78), the function $y(x)$ is convergent.

$$(c) \text{ If } -1 < \frac{1}{2} \left(-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right) < 0,$$

As x goes to zero and ∞ in eq(2.68),

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0 \\ x \rightarrow \infty}} y(x) > e^{-\frac{i}{2}\sqrt{a_1}x^2} x^{\frac{1}{2} \left(-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right)} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x \right) \rightarrow \infty \quad (2.79)$$

We see eq(2.79), the function $y(x)$ is divergent as $x \rightarrow 0$ and $x \rightarrow \infty$.

$$(d) \text{ If } \frac{1}{2} \left(-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right) = 0,$$

Eq(2.68) and eq(2.72) turns out to be

$$\lim_{n \gg 1} y(x) > e^{-\frac{i}{2}\sqrt{a_1}x^2} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x \right) \quad (2.80)$$

$$\lim_{n \gg 1} y(x) \simeq \left(e^{\frac{i}{2}\sqrt{a_1}x^2} - \frac{ib_1}{2\sqrt{a_1}}x e^{-\frac{i}{2}\sqrt{a_1}x^2} + (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} x e^{-\frac{i}{2}\sqrt{a_1}x^2} \text{Erf} \left[(i)^{3/2} (a_1)^{1/4} x \right] \right) \quad (2.81)$$

As x goes in eq(2.81),

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) \simeq 1 \quad (2.82)$$

The function $y(x)$ become convergent, and as x goes to ∞ in eq(2.80),

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) > \infty \quad (2.83)$$

The function $y(x)$ is divergent.

(e) If $\frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) > 0$,

As x goes to ∞ in eq(2.68). the function $y(x)$ is divergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) > e^{-\frac{i}{2}\sqrt{a_1}x^2} x^{\frac{1}{2}(-(a_0-1) \pm \sqrt{(a_0-1)^2 - 4d_1})} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x\right) \rightarrow \infty \quad (2.84)$$

and as x goes to 0 in eq(2.72). the function $y(x)$ is convergent.

$$\begin{aligned} \lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) &\simeq x^{\frac{1}{2}(-(a_0-1) \pm \sqrt{(a_0-1)^2 - 4d_1})} \left(e^{\frac{i}{2}\sqrt{a_1}x^2} - \frac{ib_1}{2\sqrt{a_1}}xe^{-\frac{i}{2}\sqrt{a_1}x^2} \right. \\ &\quad \left. + (i)^{3/2}\sqrt{\pi}(a_1)^{1/4}xe^{-\frac{i}{2}\sqrt{a_1}x^2} \text{Erf} \left[(i)^{3/2}(a_1)^{1/4}x \right] \right) \rightarrow 0 \end{aligned} \quad (2.85)$$

(II) As $a_1 = 0$

Eq(2.72) simply turns to be

$$\lim_{n \gg 1} y(x) \simeq x^{\frac{1}{2}(-(a_0-1) \pm \sqrt{(a_0-1)^2 - 4d_1})} \left(1 - \frac{ib_1}{2\sqrt{a_1}}x\right) \quad (2.86)$$

We suggest that $|\frac{\varepsilon}{2}| = |\frac{ib_1}{2\sqrt{a_1}}| \ll 1$. Then as $a_1 = 0$ in the second term of the bracket in eq(2.86), $|\frac{\varepsilon}{2}| = |\frac{ib_1}{2\sqrt{a_1}}| \rightarrow \infty$. The function $y(x)$ will be divergent no matter what the value of x is. Therefore, there are no any independent solutions at all in the case of $a_1 = 0$.

(III) As $a_1 =$ real negative

Plug $a_1 = -|a_1|$ into eq(2.68) and eq(2.72).

$$\lim_{n \gg 1} y(x) > e^{\frac{1}{2}\sqrt{|a_1|x^2}} x^{\frac{1}{2}(- (a_0-1) \pm \sqrt{(a_0-1)^2 - 4d_1})} \left(1 - \frac{b_1}{2\sqrt{|a_1|}}x\right) \quad (2.87)$$

$$\begin{aligned} \lim_{n \gg 1} y(x) \simeq & x^{\frac{1}{2}(- (a_0-1) \pm \sqrt{(a_0-1)^2 - 4d_1})} \left(e^{-\frac{1}{2}\sqrt{|a_1|x^2}} \right. \\ & \left. + \sqrt{\pi}(|a_1|)^{1/4} x e^{\frac{1}{2}\sqrt{|a_1|x^2}} \operatorname{Erf} \left[(|a_1|)^{1/4} x \right] - \frac{b_1}{2\sqrt{|a_1|}} x e^{\frac{1}{2}\sqrt{|a_1|x^2}} \right) \end{aligned} \quad (2.88)$$

(a) If $\frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) < 0$,

As x goes to 0 and ∞ in eq(2.87), the function $y(x)$ is divergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0 \\ x \rightarrow \infty}} y(x) > e^{\frac{1}{2}\sqrt{|a_1|x^2}} x^{\frac{1}{2}(- (a_0-1) \pm \sqrt{(a_0-1)^2 - 4d_1})} \left(1 - \frac{b_1}{2\sqrt{|a_1|}}x\right) \rightarrow \infty \quad (2.89)$$

(b) If $\frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) = 0$,

Eq(2.88) simply turns to be

$$\lim_{n \gg 1} y(x) \simeq \left(e^{-\frac{1}{2}\sqrt{|a_1|x^2}} + \sqrt{\pi}(|a_1|)^{1/4} x e^{\frac{1}{2}\sqrt{|a_1|x^2}} \operatorname{Erf} \left[(|a_1|)^{1/4} x \right] - \frac{b_1}{2\sqrt{|a_1|}} x e^{\frac{1}{2}\sqrt{|a_1|x^2}} \right) \quad (2.90)$$

As x goes to 0 in eq(2.90), the function $y(x)$ become convergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) \rightarrow 1 \quad (2.91)$$

As x goes to ∞ in eq(2.90), the function $y(x)$ become divergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) \rightarrow \infty \quad (2.92)$$

(c) If $\frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) > 0$,

As x goes to 0 in eq(2.88), the function $y(x)$ is convergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) \rightarrow 0 \quad (2.93)$$

As x goes to ∞ in eq(2.88), the function $y(x)$ is divergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) \rightarrow \infty \quad (2.94)$$

We check all possible tests to determine if an infinite series of function $y(x)$ converges or diverges on the above. Now let's test for convergence as the function $y(x)$ as $g(x)$ is polynomial for the case of $a_1 = \text{real negative}$. Substitute $g(x) = \sum_{n=0}^N C_n x^{n+\lambda}$ into eq(2.58).

$$\begin{aligned} y(x) &= e^{-\frac{1}{2}\sqrt{|a_1|}\left(x - \frac{b_1}{2|a_1|}\right)^2} x^{\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right)} \sum_{n=0}^N C_n x^{n+\lambda} \\ &\simeq e^{-\frac{1}{2}\sqrt{|a_1|x^2} x^{\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right)}} \sum_{n=0}^N C_n x^{n+\lambda} \end{aligned} \quad (2.95)$$

(a) If $\frac{1}{2}\left(-\left(a_0-1\right) \pm \sqrt{\left(a_0-1\right)^2-4d_1}\right) < 0$,

As x goes to 0 in eq(2.95), the function $y(x)$ is divergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) \rightarrow \infty \quad (2.96)$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)44

As x goes to ∞ in eq(2.95), the function $y(x)$ is convergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) \rightarrow 0 \quad (2.97)$$

(b) If $\frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) = 0$,

As x goes to 0 in eq(2.95), the function $y(x)$ is convergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0}} y(x) \rightarrow 1 \quad (2.98)$$

As x goes to ∞ in eq(2.95), the function $y(x)$ is convergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow \infty}} y(x) \rightarrow 0 \quad (2.99)$$

c) If $\frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) > 0$,

As x goes to 0 and ∞ in eq(2.95), the function $y(x)$ is convergent.

$$\lim_{\substack{n \gg 1 \\ x \rightarrow 0 \\ x \rightarrow \infty}} y(x) \rightarrow 0 \quad (2.100)$$

Now let's try to investigate independent solutions of a function $g(x)$ for two cases

which are polynomial and infinite series. First of all, if we choose boundary conditions that a function $g(x)$ goes to zero as x goes to ∞ , and it is convergent as x goes to zero. Then, the function $g(x)$ must be polynomial. The boundary condition for the polynomial is the following:

$$\begin{cases} \lim_{x \rightarrow 0} g(x) \rightarrow \text{convergent} \\ \lim_{x \rightarrow \infty} g(x) \rightarrow 0 \end{cases} \quad (2.101)$$

and the necessary conditions of it is

$$\begin{cases} a_1 = \text{real negative} \\ \frac{1}{2}(- (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) \geq 0 \\ \left| \frac{\varepsilon}{2} \right| = \left| \frac{ib_1}{2\sqrt{a_1}} \right| \ll 1 \end{cases} \quad (2.102)$$

After we develop independent solutions for the polynomial, we can expand them as infinite series in simple ways. But even if this process is simple, it does not mean that polynomial series is just the special case of the infinite series. They are independent each other. When you try to find the solution of any differential equation, first of all, you must consider what physical circumstance and mathematical condition make

solutions as polynomial series and or infinite series. As we know, there are two indicial roots which are $\lambda_1 = 0$ and $\lambda_2 = 1 - \nu$.

2.1 Finite series

2.1.1 $\nu = \text{non-integer}$

(A) As $\lambda_1 = 0$

From eq(2.64)

$$A_n|_{\lambda=0} = -\frac{\varepsilon(n + \omega)}{(n + 1)(n + \nu)} \quad (2.103a)$$

$$B_n|_{\lambda=0} = -\frac{\Omega + \mu(n - 1)}{(n + 1)(n + \nu)} \quad (2.103b)$$

Put $n=0$ in eq(2.103a). We get $A_0 = -\frac{\omega}{\nu}\varepsilon = -\frac{\varepsilon}{2} = \frac{C_1}{C_0}$. From eq(2.64), we have $K_0 = \frac{C_1}{C_0}$ which is equal to A_0 . Plug $n = 1$ into recurrence formula.

$$K_1 = A_1 + \frac{B_1}{K_0} = A_1 + \frac{B_1}{A_0} = A_1 - \frac{B_1}{(\varepsilon/2)} \quad (2.104)$$

As we see in eq(2.104), A_n includes the first order of $\frac{\varepsilon}{2}$ in which is has an extremely small value. Then, we argue that $|A_1| \ll \frac{|B_1|}{A_0} = \frac{B_1}{(\varepsilon/2)}$. Eq(2.104) is approximatively

same as

$$K_1 \simeq \frac{B_1}{A_0} = -\frac{B_1}{(\varepsilon/2)} \quad (2.105)$$

By using this process, we can simplify eq(2.64) by giving K_n in terms of A_n and B_n instead of K_{n-1} . It is equal to eq(2.38). Also, even and odd terms of C_n one the same as in eq(2.39). We can describe $g(x)$ as power series by using eq(2.39).

$$\begin{aligned} g(x) &= g(x)_{domin.} + g(x)_{small} = \sum_{n=0}^{\infty} C_{2n} x^{2n} + \sum_{n=0}^{\infty} C_{2n+1} x^{2n+1} \\ &= C_0 \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} B_{2k+1} x^{2n} \right\} + C_0 \left\{ A_0 x + (A_0 B_2 + A_2 B_1) x^3 \right. \\ &\quad \left. + \sum_{n=2}^{\infty} \left[A_0 \prod_{p=0}^{n-1} B_{2p+2} + \sum_{j=1}^{n-1} \left(A_{2j} \prod_{p=j}^{n-1} B_{2p+2} \prod_{k=0}^{j-1} B_{2k+1} \right) + A_{2n} \prod_{p=0}^{n-1} B_{2p+1} \right] x^{2n+1} \right\} \end{aligned} \quad (2.106)$$

By using similar process from the previous case, there are two eigennumbers which are

$$-\frac{\Omega}{2\mu} = n_0 - 1 = |\alpha_0| \quad \text{where } n_0 = 1, 2, 3, \dots \quad (2.107a)$$

$$-\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right) = n_1 - 1 = |\alpha_1| \quad \text{where } n_1 = 1, 2, 3, \dots \quad (2.107b)$$

Eq(2.107a) makes B_{2k+1} term go to zero at certain value of k where $k = 0, 1, 2, \dots$.

And eq(2.107b) makes B_{2k+2} term go to zero at certain value of k . Eq(2.107a) and eq(2.107b) make $g(x)$ function a polynomial series. The $g(x)_{small}$ term is extremely small value relatively compared to $g(x)_{domin.}$ because it includes A_n term having $\frac{\epsilon}{2}$. First of all, let's investigate $g(x)_{domin.}$ how it can be described.

The first term of $g(x)$ in eq(2.106) is

$$g(x)_{domin.} = C_0 \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} B_{2k+1} x^{2n} \right\} \quad (2.108)$$

Replaced n by $2k + 1$ in eq(2.103b). Then plug it into eq(2.108).

$$g(x)_{domin.} = C_0 \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left(- \frac{\mu \left(\frac{\Omega}{2\mu} + k \right)}{2(k+1) \left(k + \frac{1}{2}(1+\nu) \right)} \right) x^{2n} \right\} \quad (2.109)$$

Plug eq(2.107a) into eq(2.109).

$$g(x)_{domin.} = C_0 \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left(\frac{\mu (|\alpha_0| - k)}{2(k+1) \left(k + \frac{1}{2}(1+\nu) \right)} \right) x^{2n} \right\} \quad (2.110)$$

Eq(2.110) simply turns to be

$$g(x)_{domin.} = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n (|\alpha_0|)! (\gamma - 1)!}{(n!) (|\alpha_0| - n)! (\gamma + n - 1)!} \left(-\frac{1}{2} \mu x^2 \right)^n \quad \text{where } \gamma = \frac{1}{2}(1 + \nu) \quad (2.111)$$

index n terminates at $|\alpha_0|$ in eq(2.111). We suggest to let $C_0 = \frac{(\gamma + |\alpha_0| - 1)!}{(\gamma - 1)!}$.

$$g(x)_{domin.} = \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (\gamma + |\alpha_0| - 1)!}{(n!) (|\alpha_0| - n)! (\gamma + n - 1)!} \left(-\frac{1}{2} \mu x^2 \right)^n \quad (2.112)$$

Eq(2.112) is exactly equivalent eq(2.42a).

$$g(x)_{domin.} = F_{|\alpha_0|}(\gamma; z) = \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (\gamma + |\alpha_0| - 1)!}{(n!) (|\alpha_0| - n)! (\gamma + n - 1)!} \left(-\frac{1}{2} \mu x^2 \right)^n \quad (2.113)$$

The second term of $g(x)$ in eq(2.106) plugging $C_0 = \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)}$ is

$$g(x)_{small} = \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} \left\{ A_0 x + (A_0 B_2 + A_2 B_1) x^3 \right. \quad (2.114) \\ \left. + \sum_{n=2}^{\infty} \left[A_0 \prod_{p=0}^{n-1} B_{2p+2} + \sum_{j=1}^{n-1} \left(A_{2j} \prod_{p=j}^{n-1} B_{2p+2} \prod_{k=0}^{j-1} B_{2k+1} \right) + A_{2n} \prod_{p=0}^{n-1} B_{2p+1} \right] x^{2n+1} \right\}$$

where,

$$A_{2j} = -\frac{\varepsilon(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \quad (2.115a)$$

$$A_{2n} = -\frac{\varepsilon(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \quad (2.115b)$$

$$A_0 = -\frac{\varepsilon(\frac{\omega}{2})}{2(\frac{1}{2})(\gamma - \frac{1}{2})} \quad (2.115c)$$

$$A_2 = -\frac{\varepsilon(1 + \frac{\omega}{2})}{2(\frac{3}{2})(\gamma + \frac{1}{2})} \quad (2.115d)$$

$$B_{2p+1} = \frac{2\mu(-\frac{\Omega}{2\mu} - p)}{(2p+2)(2p+1+\nu)} = \frac{\mu(|\alpha_0| - p)}{2(p+1)(p+\gamma)} \quad (2.115e)$$

$$B_{2k+1} = \frac{2\mu(-\frac{\Omega}{2\mu} - k)}{(2k+2)(2k+1+\nu)} = \frac{\mu(|\alpha_0| - k)}{2(k+1)(k+\gamma)} \quad (2.115f)$$

$$B_{2p+2} = \frac{\mu(-\frac{\Omega}{2\mu} - p)}{2(p+\frac{3}{2})(p+1+\frac{\nu}{2})} = \frac{\mu(|\alpha_1| - p)}{2(p+\frac{3}{2})(p+\gamma+\frac{1}{2})} \quad (2.115g)$$

$$B_1 = \frac{\mu|\alpha_0|}{2\gamma} \quad (2.115h)$$

$$B_2 = \frac{\mu|\alpha_1|}{2(\frac{3}{2})(\gamma + \frac{1}{2})} \quad (2.115i)$$

Substitute eq(2.115a) through eq(2.115i) into eq(2.114)

$$\begin{aligned}
 g(x)_{small} = & -\frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} x \left\{ \frac{\varepsilon(\frac{\omega}{2})}{2(\frac{1}{2})(\gamma - \frac{1}{2})} \right. \\
 & + \left(\frac{\varepsilon(\frac{\omega}{2})}{2(\frac{1}{2})(\gamma - \frac{1}{2})} \frac{\mu|\alpha_1|}{2(\frac{3}{2})(\gamma + \frac{1}{2})} + \frac{\varepsilon(1 + \frac{\omega}{2})}{2(\frac{3}{2})(\gamma + \frac{1}{2})} \frac{\mu|\alpha_0|}{2\gamma} \right) x^2 \\
 & + \sum_{n=2}^{\infty} \left[\frac{\varepsilon(\frac{\omega}{2})}{2(\frac{1}{2})(\gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{\mu(|\alpha_1| - p)}{2(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \right. \\
 & + \sum_{j=1}^{n-1} \left(\frac{\varepsilon(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \prod_{p=j}^{n-1} \frac{\mu(|\alpha_1| - p)}{2(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \prod_{k=0}^{j-1} \frac{\mu(|\alpha_0| - k)}{2(k + 1)(k + \gamma)} \right) \\
 & \left. + \frac{\varepsilon(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{\mu(|\alpha_0| - p)}{2(p + 1)(p + \gamma)} \right] x^{2n} \Big\} \quad (2.116)
 \end{aligned}$$

Eq(2.116) can be described in the following way.

$$\begin{aligned}
 g(x)_{small} = & -\frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} x \left\{ \sum_{n=0}^{\infty} \frac{\varepsilon(\frac{\omega}{2})}{2(\frac{1}{2})(\gamma - \frac{1}{2})} \frac{(-1)^n (\frac{1}{2})! (\gamma - \frac{1}{2})! (|\alpha_1|)!}{(n + \frac{1}{2})! (\gamma + n - \frac{1}{2})! (|\alpha_1| - n)!} \right. \\
 & + \sum_{n=1}^{\infty} \frac{\varepsilon(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(-1)^n (\gamma - 1)! (|\alpha_0|)!}{(n)! (\gamma + n - 1)! (|\alpha_0| - n)!} \\
 & + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{\varepsilon(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \\
 & \left. \times \frac{(-1)^n (\gamma - 1)! (|\alpha_0|)! (j + \frac{1}{2})! (j + \gamma - \frac{1}{2})! (|\alpha_1| - j)!}{(j)! (\gamma + j - 1)! (|\alpha_0| - j)! (n + \frac{1}{2})! (n + \gamma - \frac{1}{2})! (|\alpha_1| - n)!} \right\} x^{2n} \quad (2.117)
 \end{aligned}$$

where,

$$\gamma = \frac{1}{2}(1 + \nu) \quad (2.118a)$$

$$z = -\frac{1}{2}\mu x^2 \quad (2.118b)$$

First of all,

(a) $|\alpha_0| = 0$ in eq(2.117)

$$g(x)_{small} \Big|_{|\alpha_0|=0} = -x\varepsilon \frac{\left(\frac{\omega}{2}\right)}{2\left(\frac{1}{2}\right)(\gamma - \frac{1}{2})} \sum_{n=0}^{|\alpha_1|} \frac{(-1)^n \left(\frac{1}{2}\right)! (\gamma - \frac{1}{2})! (|\alpha_1|)!}{(n + \frac{1}{2})! (\gamma + n - \frac{1}{2})! (|\alpha_1| - n)!} z^n \quad (2.119)$$

(b) $|\alpha_0| = 1$ in eq(2.117)

$$\begin{aligned} g(x)_{small} \Big|_{|\alpha_0|=1} &= -x\varepsilon \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} \left\{ \frac{\left(\frac{\omega}{2}\right)}{2\left(\frac{1}{2}\right)(\gamma - \frac{1}{2})} \right. \\ &\times \sum_{n=0}^{|\alpha_1|} \frac{(-1)^n \left(\frac{1}{2}\right)! (\gamma - \frac{1}{2})! (|\alpha_1|)!}{(n + \frac{1}{2})! (\gamma + n - \frac{1}{2})! (|\alpha_1| - n)!} z^n + \frac{(1 + \frac{\omega}{2})}{2\left(\frac{3}{2}\right)(\gamma + \frac{1}{2})} \frac{1}{2\gamma} \\ &\times \left. \sum_{n=1}^{|\alpha_1|} \frac{(-1)^n (3!) (|\alpha_1| - 1)! \left(\frac{1}{2}\right)! (\gamma + \frac{1}{2})!}{2^1 (1!) (n + \frac{1}{2})! (|\alpha_1| - n)! (\gamma + n - \frac{1}{2})!} z^n \right\} \quad (2.120) \end{aligned}$$

(c) $|\alpha_0| = 2$ in eq(2.117)

$$\begin{aligned}
 g(x)_{small} \Big|_{|\alpha_0|=1} &= -x\varepsilon \frac{\Gamma(\gamma+2)}{\Gamma(\gamma)} \left\{ \frac{\left(\frac{\omega}{2}\right)}{2\left(\frac{1}{2}\right)\left(\gamma-\frac{1}{2}\right)} \sum_{n=0}^{|\alpha_1|} \frac{(-1)^n \left(\frac{1}{2}\right)! \left(\gamma-\frac{1}{2}\right)! (|\alpha_1|)!}{\left(n+\frac{1}{2}\right)! \left(\gamma+n-\frac{1}{2}\right)! (|\alpha_1|-n)!} z^n \right. \\
 &+ \frac{\left(1+\frac{\omega}{2}\right)}{2\left(\frac{3}{2}\right)\left(\gamma+\frac{1}{2}\right)} \frac{(2)!}{2(1)!\gamma} \sum_{n=1}^{|\alpha_1|} \frac{(-1)^n (3!) (|\alpha_1|-1)! \left(\frac{1}{2}\right)! \left(\gamma+\frac{1}{2}\right)!}{2^1 (1!) \left(n+\frac{1}{2}\right)! (|\alpha_1|-n)! \left(\gamma+n-\frac{1}{2}\right)!} z^n \\
 &+ \frac{\left(3+\frac{\omega}{2}\right)}{2\left(\frac{5}{2}\right)\left(\gamma+\frac{3}{2}\right)} \frac{(2)!}{2^2 (2)! \gamma (\gamma+1)} \\
 &\times \left. \sum_{n=2}^{|\alpha_1|} \frac{(-1)^n (5!) (|\alpha_1|-2)! \left(\frac{1}{2}\right)! \left(\gamma+\frac{3}{2}\right)!}{2^2 (2!) \left(n+\frac{1}{2}\right)! (|\alpha_1|-n)! \left(\gamma+n-\frac{1}{2}\right)!} z^n \right\}
 \end{aligned}
 \tag{2.121}$$

(d) $|\alpha_0| = 3$ in eq(2.117)

$$\begin{aligned}
 g(x)_{small} \Big|_{|\alpha_0|=1} &= -x\varepsilon \frac{\Gamma(\gamma+3)}{\Gamma(\gamma)} \left\{ \frac{\left(\frac{\omega}{2}\right)}{2\left(\frac{1}{2}\right)(\gamma-\frac{1}{2})} \sum_{n=0}^{|\alpha_1|} \frac{(-1)^n \left(\frac{1}{2}\right)! (\gamma-\frac{1}{2})! (|\alpha_1|)!}{(n+\frac{1}{2})! (\gamma+n-\frac{1}{2})! (|\alpha_1-n|)!} z^n \right. \\
 &+ \frac{\left(1+\frac{\omega}{2}\right)}{2\left(\frac{3}{2}\right)(\gamma+\frac{1}{2})} \frac{(3)!}{2(2)!(1)!\gamma} \sum_{n=1}^{|\alpha_1|} \frac{(-1)^n (3)! (|\alpha_1-1)! \left(\frac{1}{2}\right)! (\gamma+\frac{1}{2})!}{2^1 (1!) (n+\frac{1}{2})! (|\alpha_1-n)! (\gamma+n-\frac{1}{2})!} z^n \\
 &+ \frac{\left(3+\frac{\omega}{2}\right)}{2\left(\frac{5}{2}\right)(\gamma+\frac{3}{2})} \frac{(3)!}{2^2(1)!(2)!\gamma(\gamma+1)} \\
 &\times \sum_{n=2}^{|\alpha_1|} \frac{(-1)^n (5)! (|\alpha_1-2)! \left(\frac{1}{2}\right)! (\gamma+\frac{3}{2})!}{2^2(2!) (n+\frac{1}{2})! (|\alpha_1-n)! (\gamma+n-\frac{1}{2})!} z^n \\
 &+ \frac{\left(3+\frac{\omega}{2}\right)}{2\left(\frac{7}{2}\right)(\gamma+\frac{5}{2})} \frac{(3)!}{2^3(0)!(3)!\gamma(\gamma+1)(\gamma+2)} \\
 &\times \left. \sum_{n=3}^{|\alpha_1|} \frac{(-1)^n (7)! (|\alpha_1-3)! \left(\frac{1}{2}\right)! (\gamma+\frac{5}{2})!}{2^3(3!) (n+\frac{1}{2})! (|\alpha_1-n)! (\gamma+n-\frac{1}{2})!} z^n \right\}
 \end{aligned} \tag{2.122}$$

We can describe a function $g(x)_{small}$ of degree $|\alpha_0|$ and $|\alpha_1|$ according to eq(2.119) through eq(2.122).

$$\begin{aligned}
 g(x)_{small} &= -x\varepsilon \sum_{n=0}^{|\alpha_0|} \frac{(n+\frac{\omega}{2})(n-\frac{1}{2})! (|\alpha_0|)! (|\alpha_0+\gamma-\frac{1}{2})!}{(n+\gamma-\frac{1}{2})(n)! \left(-\frac{1}{2}\right)! (|\alpha_0-n)! (n+\gamma-1)!} \\
 &\times \sum_{k=n}^{|\alpha_1|} \frac{(-1)^k \left(\frac{1}{2}\right)! (n+\gamma-\frac{1}{2})! (|\alpha_1-n)!}{\left(k+\frac{1}{2}\right)! (|\alpha_1-k)! (k+\gamma-\frac{1}{2})!} z^k
 \end{aligned} \tag{2.123}$$

As we see in eq(2.123), maximum value of index n is $|\alpha_0|$. The range of index k is $n \leq k \leq |\alpha_1|$. In other words, $0 \leq n \leq |\alpha_0| \leq k \leq |\alpha_1|$. Then we obtain $|\alpha_0| \leq |\alpha_1|$. If $|\alpha_0| \geq |\alpha_1|$, the function $g(x)$ will be infinite series. Then, the function $y(x)$ will blow up as we plug $g(x)$ into it no matter what the value of x is. Such solution does not exist. When we see the second summation of eq(123), we can shift index k to zero at the beginning of summation. Then we can replace the interval of index k by $0 \leq k \leq |\alpha_1| - n$. Eq(2.123) which is simply described as

$$\begin{aligned}
 g(x)_{small} &= -x \frac{\varepsilon}{2} \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} z^n \\
 &\quad \times \sum_{k=0}^{|\alpha_1| - n} \frac{(-1)^n (n + \frac{\omega}{2}) (n - \frac{1}{2})! (n + \gamma - \frac{3}{2})! (|\alpha_1| - n)!}{(k + n + \frac{1}{2})! (|\alpha_1| - n - k)! (k + n + \gamma - \frac{1}{2})!} z^k \\
 &= -x \frac{\varepsilon}{2} \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \\
 &\quad \times \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \quad (2.124)
 \end{aligned}$$

We see the first summation of eq(2.124) which is $\frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n$, it is the first kind of confluent hypergeometric polynomial of degree $|\alpha_0|$ which is denoted

as $F_{|\alpha_0|}(\gamma; z)$ surprisingly. Substitute eq(2.113) and eq(2.124) into eq(2.106).

$$\begin{aligned} g(x) &= \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \\ &= F_{|\alpha_0|}(\gamma; z) - \frac{\varepsilon}{2} x \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) \quad \text{only if } |\alpha_0| \leq |\alpha_1| \end{aligned} \quad (2.125)$$

where,

$$F_{|\alpha_0|}(\gamma; z) = \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} z^n = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \quad (2.126a)$$

$$\begin{aligned} \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \\ &\times \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \end{aligned} \quad (2.126b)$$

Eq(2.125) denoted as $\mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ is called as the first kind of Grand confluent hypergeometric polynomial of degree $|\alpha_0|$ and $|\alpha_1|$ with the first order of $\frac{\varepsilon}{2}$.

We also we obtain two eigenvalues from eq(2.107a) and eq(2.107b), which are

$$\Omega_0^1 = -2\mu|\alpha_0| \quad \text{where } |\alpha_0| = 0, 1, 2, \dots \quad (2.127a)$$

$$\Omega_1^1 = -2\mu\left(|\alpha_1| + \frac{1}{2}\right) \quad \text{where } |\alpha_1| = 0, 1, 2, \dots \quad (2.127b)$$

As we know the first kind of confluent hypergeometric function is

$$F_{|\alpha_0|}(\gamma; z) = \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} z^n = \frac{e^z}{z^{\gamma-1}} \frac{d^{|\alpha_0|}}{dz^{|\alpha_0|}} (z^{|\alpha_0| + \gamma - 1} e^{-z}) \quad (2.128)$$

And the generating function of it is

$$\frac{e^{\frac{-zt}{(1-t)}}}{(1-t)^\gamma} = \sum_{|\alpha_0|=0}^{\infty} \frac{t^{|\alpha_0|}}{|\alpha_0|!} F_{|\alpha_0|}(z) \quad (2.129)$$

The Laguerre differential equation is

$$zL_n(z)'' + (1-z)L_n(z)' + nL_n(z) = 0 \quad (2.130)$$

And the solution of it is

$$L_n(z) = \sum_{k=0}^n \frac{(-1)^k (n)!}{(n-k)! (k!)^2} z^k = \sum_{k=0}^n \frac{(-1)^k \binom{n}{k}}{k!} z^k = \frac{e^z}{n!} \frac{d^n}{dz^n} (z^n e^{-z}) \quad (2.131)$$

The generating function of it is

$$\frac{e^{\frac{-zt}{1-t}}}{(1-t)} = \sum_{n=0}^{\infty} t^n L_n(z) \quad (2.132)$$

The associated Laguerre differential equation is

$$zL_n^k(z)'' + (k+1-z)L_n^k(z)' + nL_n^k(z) = 0 \quad (2.133)$$

And the solution of it is

$$L_n^k(z) = \sum_{j=0}^n \frac{(-1)^j (n+k)!}{(n-j)! (k+j)! (j)!} z^j = \frac{e^z z^{-k}}{n!} \frac{d^n}{dz^n} (z^{n+k} e^{-z}) \quad (2.134)$$

The generating function of it is

$$\frac{e^{\frac{-zt}{1-t}}}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} t^n L_n^k(z) \quad (2.135)$$

Eq(2.126b) might be described in the following way:

$$\prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) = \sum_{n=0}^{|\alpha_0|} F_n^{|\alpha_0|}(\gamma; z) \prod_n^{|\alpha_1|}(\gamma; z) \quad (2.136a)$$

where,

$$F_n^{|\alpha_0|}(\gamma; z) = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \quad (2.136b)$$

$$\prod_n^{|\alpha_1|}(\gamma; z) = \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \quad (2.136c)$$

(a) Plug $|\alpha_0| = 0$ into eq(2.125), eq(2.126a), eq(2.136a), eq(2.136b) and eq(2.136c)

$$\begin{aligned} \mathcal{QW}_{0,|\alpha_1|}(\gamma; z) &= F_0(\gamma; z) - \frac{\varepsilon}{2} x \prod_0^{|\alpha_1|}(\gamma; z) \\ &= 1 - \frac{\varepsilon}{2} x \prod_0^{|\alpha_1|}(\gamma; z) \\ &= 1 - \frac{\varepsilon}{2} x \sum_{k=0}^{|\alpha_1|} \frac{(\frac{\omega}{2}) \Gamma(\frac{1}{2}) \Gamma(\gamma - \frac{1}{2}) (-|\alpha_1|)_k}{\Gamma(k + \frac{3}{2}) \Gamma(k + \gamma + \frac{1}{2})} z^k \end{aligned} \quad (2.137)$$

The beta function is

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 dt t^{p-1}(1-t)^{q-1} = \int_0^\infty dt \frac{t^{p-1}}{(1+t)^{p+q}} = B(q, p) \quad (2.138)$$

And,

$$\frac{1}{\Gamma(k+1)} \int_{-1}^1 dp (1-p^2)^k = \frac{\Gamma(\frac{1}{2})}{\Gamma(k+\frac{3}{2})} \quad (2.139)$$

By using eq(2.139) into the second term of eq(2.137) on r.h.s

$$\prod_0^{|\alpha_1|}(\gamma; z) = \frac{\omega}{2} \int_{-1}^1 dp \sum_{k=0}^{|\alpha_1|} \frac{(-1)^k \Gamma(\gamma - \frac{1}{2}) \binom{|\alpha_1|}{k}}{\Gamma(k + \gamma + \frac{1}{2})} [z(1-p^2)]^k \quad (2.140)$$

Replace p and q by $\gamma - \frac{1}{2}$ and $k + 1$ in eq(2.138). Substitute it into eq(2.140).

$$\prod_0^{|\alpha_1|}(\gamma; z) = \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} \sum_{k=0}^{|\alpha_1|} \frac{(-1)^k \binom{|\alpha_1|}{k}}{\Gamma(k+1)} [z(1-t)(1-p^2)]^k \quad (2.141)$$

Replace n and z by $|\alpha_1|$ and $\lambda = z(1-t)(1-p^2)$ in eq(2.131).

$$L_{|\alpha_1|}(\lambda) = \sum_{k=0}^{|\alpha_1|} \frac{(-1)^k \binom{|\alpha_1|}{k}}{k!} \lambda^k \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.142)$$

Plug eq(2.142) into eq(2.141)

$$\prod_0^{|\alpha_1|}(\gamma; z) = \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} L_{|\alpha_1|}(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.143)$$

Plug eq(2.143) into eq(2.137)

$$\mathcal{QW}_{0,|\alpha_1|}(\gamma; z) = F_0(\gamma; z) - x \frac{\varepsilon}{2} \left(\frac{\omega}{2} \right) \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} L_{|\alpha_1|}(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.144)$$

(b) Plug $|\alpha_0| = 1$ into eq(2.125), eq(2.126a), eq(2.136a), eq(2.136b) and eq(2.136c)

$$\mathcal{QW}_{1,|\alpha_1|}(\gamma; z) = F_1(\gamma; z) - \frac{\varepsilon}{2} x \left\{ F_0^1(\gamma, z) \prod_0^{|\alpha_1|}(\gamma; z) + F_1^1(\gamma, z) \prod_1^{|\alpha_1|}(\gamma; z) \right\} \quad (2.145)$$

The second term inside of bracket in eq(2.145) is equal to

$$\prod_1^{|\alpha_1|}(\gamma; z) = \frac{1}{2} \left(1 + \frac{\omega}{2} \right) \sum_{k=0}^{|\alpha_1|-1} \frac{(-1)^k \Gamma(\frac{1}{2}) \Gamma(|\alpha_1|) \Gamma(\gamma + \frac{1}{2})}{\Gamma(k + \frac{5}{2}) \Gamma(|\alpha_1| - k) \Gamma(k + \gamma + \frac{3}{2})} z^k \quad (2.146)$$

Replace k by $k + 1$ in eq(2.139), and plug it into eq(2.146)

$$\begin{aligned} \prod_1^{|\alpha_1|}(\gamma; z) &= \frac{1}{2} \left(1 + \frac{\omega}{2}\right) \int_{-1}^1 dp (1 - p^2) \\ &\times \sum_{k=0}^{|\alpha_1|-1} \frac{(-1)^k \Gamma(|\alpha_1|) \Gamma(\gamma + \frac{1}{2})}{\Gamma(|\alpha_1| - k) \Gamma(k + \gamma + \frac{3}{2}) \Gamma(k + 2)} [z(1 - p^2)]^k \end{aligned} \quad (2.147)$$

Replace p and q by $\gamma + \frac{1}{2}$ and $k + 1$ in eq(2.138). Substitute it into eq(2.147).

$$\begin{aligned} \prod_1^{|\alpha_1|}(\gamma; z) &= \frac{1}{2} \left(1 + \frac{\omega}{2}\right) \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\gamma - \frac{1}{2}} \\ &\times \sum_{k=0}^{|\alpha_1|-1} \frac{(-1)^k \Gamma(|\alpha_1|)}{\Gamma(|\alpha_1| - k) \Gamma(k + 1) \Gamma(k + 2)} [z(1 - t)(1 - p^2)]^k \end{aligned} \quad (2.148)$$

Replace n, k, j and z by $|\alpha_1| - 1, 1, k$ and $\lambda = z(1 - t)(1 - p^2)$ in eq(2.134).

$$L_{|\alpha_1|-1}^1(\lambda) = \sum_{k=0}^{|\alpha_1|-1} \frac{(-1)^k \Gamma(|\alpha_1| + 1)}{\Gamma(|\alpha_1| - k) \Gamma(k + 1) \Gamma(k + 2)} \lambda^k \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.149)$$

Plug eq(2.149) into eq(2.148).

$$\prod_1^{|\alpha_1|}(\gamma; z) = \frac{1}{2|\alpha_1|} \left(1 + \frac{\omega}{2}\right) \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\gamma - \frac{1}{2}} L_{|\alpha_1|-1}^1(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.150)$$

Substitute eq(2.143) and eq(2.150) into eq(2.145).

$$\begin{aligned}
 \mathcal{QW}_{1,|\alpha_1|}(\gamma; z) &= F_1(\gamma; z) - \frac{\varepsilon}{2}x \left\{ F_0^1(\gamma, z) \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} L_{|\alpha_1|}(\lambda) \right. \\
 &\quad + F_1^1(\gamma, z) \frac{1}{2|\alpha_1|} \left(1 + \frac{\omega}{2} \right) \\
 &\quad \left. \times \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\gamma-\frac{1}{2}} L_{|\alpha_1|-1}^1(\lambda) \right\} \Big|_{\lambda=z(1-t)(1-p^2)}
 \end{aligned} \tag{2.151}$$

(c) Plug $|\alpha_0| = 2$ into eq(2.125),eq(2.126a), eq(2.136a), eq(2.136b) and eq(2.136c)

$$\begin{aligned}
 \mathcal{QW}_{2,|\alpha_1|}(\gamma; z) &= F_2(\gamma; z) - \frac{\varepsilon}{2}x \left\{ F_0^2(\gamma, z) \prod_0^{|\alpha_1|}(\gamma; z) + F_1^2(\gamma, z) \prod_1^{|\alpha_1|}(\gamma; z) \right. \\
 &\quad \left. + F_2^2(\gamma, z) \prod_2^{|\alpha_1|}(\gamma; z) \right\}
 \end{aligned} \tag{2.152}$$

The third term inside of bracket in eq(2.152) is equal to

$$\prod_2^{|\alpha_1|}(\gamma; z) = \frac{1}{2} \frac{3}{2} \left(2 + \frac{\omega}{2} \right) \sum_{k=0}^{|\alpha_1|-2} \frac{(-1)^k \Gamma(\frac{1}{2}) \Gamma(|\alpha_1| - 1) \Gamma(\gamma + \frac{3}{2})}{\Gamma(k + \frac{7}{2}) \Gamma(|\alpha_1| - 1 - k) \Gamma(k + \gamma + \frac{5}{2})} z^k \tag{2.153}$$

Replace k by $k + 2$ in eq(2.139).and plug it into eq(2.153)

$$\begin{aligned} \prod_2^{|\alpha_1|}(\gamma; z) &= \frac{1}{2} \frac{3}{2} \left(2 + \frac{\omega}{2}\right) \int_{-1}^1 dp (1-p^2)^2 \\ &\times \sum_{k=0}^{|\alpha_1|-2} \frac{(-1)^k \Gamma(|\alpha_1| - 1) \Gamma(\gamma + \frac{3}{2})}{\Gamma(|\alpha_1| - 1 - k) \Gamma(k + \gamma + \frac{5}{2}) \Gamma(k + 3)} [z(1-p^2)]^k \end{aligned} \quad (2.154)$$

Replace p and q by $\gamma + \frac{3}{2}$ and $k + 1$ in eq(2.138). Substitute it into eq(2.147).

$$\begin{aligned} \prod_2^{|\alpha_1|}(\gamma; z) &= \frac{1}{2} \frac{3}{2} \left(2 + \frac{\omega}{2}\right) \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\gamma+\frac{1}{2}} \\ &\times \sum_{k=0}^{|\alpha_1|-2} \frac{(-1)^k \Gamma(|\alpha_1| - 1) [z(1-t)(1-p^2)]^k}{\Gamma(|\alpha_1| - 1 - k) \Gamma(k + 1) \Gamma(k + 3)} \end{aligned} \quad (2.155)$$

Replace n, k, j and z by $|\alpha_1| - 2, 2, k$ and $\lambda = z(1-t)(1-p^2)$ in eq(2.134).

$$L_{|\alpha_1|-2}^2(\lambda) = \sum_{k=0}^{|\alpha_1|-2} \frac{(-1)^k \Gamma(|\alpha_1| + 1)}{\Gamma(|\alpha_1| - 1 - k) \Gamma(k + 1) \Gamma(k + 3)} \lambda^k \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.156)$$

Plug eq(2.156) into eq(2.155).

$$\prod_2^{|\alpha_1|}(\gamma; z) = \frac{1}{2} \frac{3}{2} \frac{1}{|\alpha_1| (|\alpha_1| - 1)} \left(2 + \frac{\omega}{2}\right) \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\gamma+\frac{1}{2}} L_{|\alpha_1|-2}^2(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.157)$$

Substitute eq(2.143), eq(2.150) and eq(2.157) into eq(2.152).

$$\begin{aligned} \mathcal{QW}_{2,|\alpha_1|}(\gamma; z) &= F_2(\gamma; z) - \frac{\varepsilon}{2} x \sum_{n=0}^2 \left\{ F_n^2(\gamma, z) \left(n + \frac{\omega}{2} \right) \frac{\Gamma(n + \frac{1}{2})\Gamma(|\alpha_1| + 1 - n)}{\Gamma(\frac{1}{2})\Gamma(|\alpha_1| + 1)} \right. \\ &\quad \left. \times \int_0^1 dt t^{\gamma - \frac{3}{2} + n} \int_{-1}^1 dp (1 - p^2)^n L_{|\alpha_1| - n}^n(\lambda) \right\} \Big|_{\lambda = z(1-t)(1-p^2)} \end{aligned} \quad (2.158)$$

We can describe a function $\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(\gamma; z)$ of degree $|\alpha_0|$ and $|\alpha_1|$ according to eq(2.144), eq(2.151) and eq(2.158).

$$\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(\gamma; z) = F_{|\alpha_0|}(\gamma; z) - \frac{\varepsilon}{2} x \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) \quad (2.159)$$

where,

$$\begin{aligned} \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \sum_{n=0}^{|\alpha_0|} \left\{ F_n^{|\alpha_0|}(\gamma, z) \left(n + \frac{\omega}{2} \right) \frac{\Gamma(n + \frac{1}{2})\Gamma(|\alpha_1| + 1 - n)}{\Gamma(\frac{1}{2})\Gamma(|\alpha_1| + 1)} \right. \\ &\quad \left. \times \int_0^1 dt t^{\gamma - \frac{3}{2} + n} \int_{-1}^1 dp (1 - p^2)^n L_{|\alpha_1| - n}^n(\lambda) \right\} \Big|_{\lambda = z(1-t)(1-p^2)} \end{aligned} \quad (2.160)$$

We know

$$\frac{\Gamma(n + \frac{1}{2})\Gamma(|\alpha_1| + 1 - n)}{\Gamma(\frac{1}{2})\Gamma(|\alpha_1| + 1)} = \frac{B(n + \frac{1}{2}, |\alpha_1| + 1 - n)}{B(\frac{1}{2}, |\alpha_1| + 1)} \quad (2.161)$$

Substitute eq(2.136b) and eq(2.161) into eq(2.160).

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \left\{ \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{(n-1)! (|\alpha_0| - n)! (n + \gamma - 1)!} \frac{B(n + \frac{1}{2}, |\alpha_1| + 1 - n)}{B(\frac{1}{2}, |\alpha_1| + 1)} \right. \\
 &\times \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp [zt(1 - p^2)]^n L_{|\alpha_1| - n}^n(\lambda) \\
 &+ \frac{\omega}{2} \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} \frac{B(n + \frac{1}{2}, |\alpha_1| + 1 - n)}{B(\frac{1}{2}, |\alpha_1| + 1)} \\
 &\left. \times \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp [zt(1 - p^2)]^n L_{|\alpha_1| - n}^n(\lambda) \right\} \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.162)
 \end{aligned}$$

Replace p and q by $n + \frac{1}{2}$ and $|\alpha_1| + 1 - n$ into eq(2.138).

$$B(n + \frac{1}{2}, |\alpha_1| + 1 - n) = \int_0^\infty ds \frac{s^{n - \frac{1}{2}}}{(1 + s)^{|\alpha_1| + \frac{3}{2}}} = \int_0^1 ds s^{n - \frac{1}{2}} (1 - s)^{|\alpha_1| - n} \quad (2.163)$$

Plug eq(2.163) into eq(2.162).

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{1}{B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1 + s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \\
 &\times \left\{ \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{(n-1)! (|\alpha_0| - n)! (n + \gamma - 1)!} [zts(1 - p^2)]^n L_{|\alpha_1| - n}^n(\lambda) \right. \\
 &+ \frac{\omega}{2} \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} [zts(1 - p^2)]^n L_{|\alpha_1| - n}^n(\lambda) \left. \right\} \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.164)
 \end{aligned}$$

Integral form of Associated Laguerre polynomial is

$$L_n^m(z) = \frac{1}{2\pi i} \oint du \frac{e^{-\frac{zu}{1-u}}}{u^{n+1}(1-u)^{m+1}} \quad (2.165)$$

Replace n , m and z by $|\alpha_1| - n$, n and λ in eq(2.165). Then plug it into eq(2.164).

$$\begin{aligned} \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{1}{2\pi i B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \\ &\times \int_{-1}^1 dp \oint du \frac{e^{-\frac{\lambda u}{1-u}}}{u^{|\alpha_1|+1}(1-u)} \left\{ zst(1-p^2) \frac{u}{1-u} \right. \\ &\times \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{(n-1)! (|\alpha_0| - n)! (n + \gamma - 1)!} \left[zts(1-p^2) \frac{u}{(1-u)} \right]^{n-1} \\ &\left. + \frac{\omega}{2} \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} \left[zts(1-p^2) \frac{u}{(1-u)} \right]^n \right\} \Bigg|_{\lambda=z(1-t)(1-p^2)} \quad (2.166) \end{aligned}$$

Replace z and λ by $w_1 = zts(1-p^2) \frac{u}{(1-u)}$ and $z(1-t)(1-p^2)$ into eq(2.113). And plug it into eq(2.166).

$$\begin{aligned} \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{1}{2\pi i B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \\ &\times \oint du \frac{e^{-\frac{zu}{(1-u)(1-t)(1-p^2)}}}{u^{|\alpha_1|+1}(1-u)} \left\{ w_1 \partial_{w_1} + \frac{\omega}{2} \right\} \\ &\times F_{|\alpha_0|} \left(\gamma; w_1 = zts(1-p^2) \frac{u}{(1-u)} \right) \quad (2.167) \end{aligned}$$

Eq(2.167) can be described as various integral forms of several different special function in the following way:

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{(|\alpha_0|)!}{2\pi i B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \\
 &\times \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \oint \frac{dv}{v^{|\alpha_0|+1} (1-v)^\gamma} \\
 &\times \left\{ \frac{zstv(1-p^2)}{(1-v)} \partial_{w_2} + \frac{\omega}{2} \right\} L_{|\alpha_1|} \left(w_2 = z(1-p^2) \left\{ (1-t) + \frac{tsv}{(1-v)} \right\} \right)
 \end{aligned} \tag{2.168}$$

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{(2|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \\
 &\times \int_0^1 dt t^{\gamma - \frac{3}{2}} \oint \frac{dv}{v^{|\alpha_0|+1} (1-v)^\gamma} \\
 &\times \oint \frac{du}{u^{|\alpha_1|+1} (1-u)} \left\{ \frac{zstvu}{(1-v)(1-u)} \left(\frac{1}{3} M\left(1, \frac{5}{2}, w_3\right) - M\left(1, \frac{3}{2}, w_3\right) \right) \right. \\
 &\left. + \frac{\omega}{2} M\left(1, \frac{3}{2}, w_3\right) \right\} \\
 \text{where, } w_3 &= \frac{zu}{(1-u)} \left\{ -(1-t) - \frac{stv}{(1-v)} \right\}
 \end{aligned} \tag{2.169}$$

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{(|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \\
 &\times \int_0^1 dp (1-p^2)^{-\frac{1}{2}} \oint \frac{dv}{v^{|\alpha_0|+1} (1-v)^\gamma} \\
 &\times \oint \frac{due^{-\frac{zup}{(1-u)}}}{u^{|\alpha_1|+1} (1-u)} \left\{ -\frac{zspvu}{(\gamma + \frac{1}{2})(1-v)(1-u)} M(\gamma + \frac{1}{2}, \gamma + \frac{3}{2}, w_4) \right. \\
 &+ \left. \frac{\omega}{2(\gamma - \frac{1}{2})} M(\gamma - \frac{1}{2}, \gamma + \frac{1}{2}, w_4) \right\} \\
 \text{where, } w_4 &= \frac{zup}{(1-u)} \left\{ 1 - \frac{sv}{(1-v)} \right\} \tag{2.170}
 \end{aligned}$$

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{\sqrt{\pi}(|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \oint \frac{dv}{v^{|\alpha_0|+1} (1-v)^\gamma} \\
 &\times \oint du \frac{e^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{|\alpha_1|+1} (1-u)} \\
 &\times \left\{ -\frac{ztvu(1-p^2)}{2(1-v)(1-u)} U(\frac{3}{2}, -|\alpha_1| + 1, w_5) + \frac{\omega}{2} U(\frac{1}{2}, -|\alpha_1|, w_5) \right\} \\
 \text{where, } w_5 &= \frac{ztuv}{(1-v)(1-u)} (1-p^2) \tag{2.171}
 \end{aligned}$$

In the above, $M(a, b, z)$ is the first kind of Kummer function which is

$$\begin{aligned}
 M(a, b, z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n = e^z M(b-a, b, -z) \\
 &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 du e^{zu} u^{a-1} (1-u)^{b-a-1} \\
 \text{where, } \operatorname{Re}(b) &> \operatorname{Re}(a) > 0 \tag{2.172}
 \end{aligned}$$

And $U(a, b, z)$ is the second kind of Kummer function which is

$$\begin{aligned}
 U(a, b, z) &= \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z) \\
 &= z^{1-b} U(1+a-b, 2-b, z) \\
 &= \frac{1}{\Gamma(a)} \int_0^\infty dt e^{-zt} t^{a-1} (1+t)^{b-a-1} \quad \text{where, } \operatorname{Re}(a) > 0 \quad (2.173)
 \end{aligned}$$

Integral form of the first kind of confluent hypergeometric function is

$$F_{|\alpha_0|}(z) = \frac{(|\alpha_0|)!}{2\pi i} \oint dv \frac{e^{-\frac{zv}{(1-v)}}}{v^{|\alpha_0|+1} (1-v)^\gamma} \quad (2.174)$$

Substitute eq(2.174) into eq(2.167).

$$\begin{aligned}
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{(|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1|+1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1|+\frac{3}{2})} \\
 &\times \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_{-1}^1 dp \oint \frac{dv}{v^{|\alpha_0|+1} (1-v)^\gamma} \\
 &\times \oint \frac{du}{u^{|\alpha_1|+1} (1-u)} \left\{ -\frac{zstuv(1-p^2)}{(1-u)(1-v)} + \frac{\omega}{2} \right\} e^{-\frac{zu}{(1-u)}(1-p^2)} \left\{ (1-t) + \frac{tsv}{1-v} \right\}
 \end{aligned} \quad (2.175)$$

Description we gave in eq(2.168), eq(2.169), eq(2.170), eq(2.171) as integral formalism is exactly equivalent eq(2.175).

$$\begin{aligned}
 \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) &= F_{|\alpha_0|}(\gamma; z) - \frac{\varepsilon}{2} x \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) \\
 \text{where,} \\
 F_{|\alpha_0|}(\gamma; z) &= \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|)! (|\alpha_0| + \gamma - 1)!}{n! (|\alpha_0| - n)! (n + \gamma - 1)!} z^n = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \\
 \prod_{|\alpha_0|}^{|\alpha_1|}(\gamma; z) &= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \\
 &= \frac{1}{2\pi i B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \oint du \frac{e^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{|\alpha_1|+1}(1-u)} \\
 &\quad \times \left\{ w_1 \partial_{w_1} + \frac{\omega}{2} \right\} F_{|\alpha_0|} \left(\gamma; w_1 = zts(1-p^2) \frac{u}{(1-u)} \right) \\
 &= \frac{(|\alpha_0|)!}{2\pi i B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \oint \frac{dv}{v^{|\alpha_0|+1}(1-v)^\gamma} \\
 &\quad \times \left\{ \frac{zstv(1-p^2)}{(1-v)} \partial_{w_2} + \frac{\omega}{2} \right\} L_{|\alpha_1|} \left(w_2 = z(1-p^2) \left\{ (1-t) + \frac{tstv}{(1-v)} \right\} \right) \\
 &= \frac{(2|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \oint \frac{dv}{v^{|\alpha_0|+1}(1-v)^\gamma} \\
 &\quad \times \oint \frac{du}{u^{|\alpha_1|+1}(1-u)} \left\{ \frac{zstvu}{(1-v)(1-u)} \left(\frac{1}{3} M\left(1, \frac{5}{2}, w_3\right) - M\left(1, \frac{3}{2}, w_3\right) \right) + \frac{\omega}{2} M\left(1, \frac{3}{2}, w_3\right) \right\} \\
 \text{where, } w_3 &= \frac{zu}{(1-u)} \left\{ -(1-t) - \frac{stv}{(1-v)} \right\} \\
 &= \frac{(|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dp (1-p^2)^{-\frac{1}{2}} \oint \frac{dv}{v^{|\alpha_0|+1}(1-v)^\gamma} \\
 &\quad \times \oint \frac{due^{-\frac{zup}{(1-u)}}}{u^{|\alpha_1|+1}(1-u)} \left\{ -\frac{zspvu}{(\gamma + \frac{1}{2})(1-v)(1-u)} M\left(\gamma + \frac{1}{2}, \gamma + \frac{3}{2}, w_4\right) \right. \\
 &\quad \left. + \frac{\omega}{2(\gamma - \frac{1}{2})} M\left(\gamma - \frac{1}{2}, \gamma + \frac{1}{2}, w_4\right) \right\} \quad \text{where, } w_4 = \frac{zup}{(1-u)} \left\{ 1 - \frac{sv}{(1-v)} \right\} \\
 &= \frac{\sqrt{\pi} (|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \oint \frac{dv}{v^{|\alpha_0|+1}(1-v)^\gamma} \oint \frac{due^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{|\alpha_1|+1}(1-u)} \\
 &\quad \times \left\{ -\frac{ztvu(1-p^2)}{2(1-v)(1-u)} U\left(\frac{3}{2}, -|\alpha_1| + 1, w_5\right) + \frac{\omega}{2} U\left(\frac{1}{2}, -|\alpha_1|, w_5\right) \right\} \\
 \text{where, } w_5 &= \frac{ztuv}{(1-v)(1-u)} (1-p^2)
 \end{aligned}$$

Table 2.1: Summary of the first kind of grand confluent hypergeometric function

We obtain the integral representation of the first kind of grand confluent hypergeometric function according to eq(2.174) and eq(2.175).

$$\mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) = \frac{(|\alpha_0|)!}{2\pi i} \oint dv \frac{e^{-\frac{zv}{(1-v)}}}{v^{|\alpha_0|+1}(1-v)^\gamma} - \frac{\varepsilon}{2} x \prod_{|\alpha_0|}^{|\alpha_1|} (\gamma; z)$$

where,

$$\prod_{|\alpha_0|}^{|\alpha_1|} (\gamma; z) = \frac{(|\alpha_0|)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha_1| + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(|\alpha_1| + \frac{3}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \oint \frac{dv}{v^{|\alpha_0|+1}(1-v)^\gamma}$$

$$\times \oint \frac{du}{u^{|\alpha_1|+1}(1-u)} \left\{ -\frac{zstuv(1-p^2)}{(1-u)(1-v)} + \frac{\omega}{2} \right\} e^{-\frac{zu}{(1-u)}(1-p^2)} \left\{ (1-t) + \frac{t\frac{sv}{1-v}}{1-v} \right\}$$

Table 2.2: Integral representation of the first kind of grand confluent hypergeometric function.

Now, let's try to get the generating function of the 1st kind of grand confluent hypergeometric function. First of all, we multiply $\sum_{|\alpha_1|=|\alpha_0|}^\infty B(|\alpha_1| + 1, \frac{1}{2}) v_0^{|\alpha_1|}$ on both sides of integral form of 1st kind of grand confluent hypergeometric function in Table 2.2 where $|v_0| < 1$.

$$\sum_{|\alpha_1|=|\alpha_0|}^\infty B(|\alpha_1| + 1, \frac{1}{2}) v_0^{|\alpha_1|} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) = F_{|\alpha_0|}(z) \text{I} - \frac{\varepsilon}{2} x \text{II} \quad (2.176)$$

where,

$$I = \sum_{|\alpha_1|=|\alpha_0|}^{\infty} B(|\alpha_1| + 1, \frac{1}{2}) v_0^{|\alpha_1|} \quad (2.177a)$$

$$II = \sum_{|\alpha_1|=|\alpha_0|}^{\infty} B(|\alpha_1| + 1, \frac{1}{2}) v_0^{|\alpha_1|} \prod_{|\alpha_0|}^{|\alpha_1|} (\gamma; z) \quad (2.177b)$$

The 1st kind of hypergeometric function is

$${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a}$$

only if $|z| < 1$, $\text{Re}(c) > \text{Re}(b) > 0$ (2.178)

Substitute eq(2.178) into eq(2.177a).

$$I = \frac{\Gamma(|\alpha_0| + 1)\Gamma(\frac{1}{2})}{\Gamma(|\alpha_0| + \frac{3}{2})} v_0^{|\alpha_0|} {}_2F_1(1, |\alpha_0| + 1, |\alpha_0| + \frac{3}{2}, v_0) = \int_0^1 dt (v_0 t)^{|\alpha_0|} (1-t)^{-\frac{1}{2}} (1-v_0 t)^{-1}$$

(2.179)

Substitute eq(2.175) into eq(2.177b).

$$II = \frac{(|\alpha_0|)!}{2\pi i} \int_0^{\infty} ds s^{-\frac{1}{2}} (1+s)^{-\frac{3}{2}} \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_0^1 dp (1-p)^{-\frac{1}{2}} \oint \frac{dv}{v^{|\alpha_0|+1} (1-v)^{\gamma}}$$

$$\times \left\{ \frac{1+s}{1+s-v_0} \left(-\frac{z v s t p v_0}{(1+s-v_0)(1-v)} + \frac{\omega}{2} \right) \right\} e^{-\frac{z p v_0}{1+s-v_0} \left\{ (1-t) + \frac{t s v}{1-v} \right\}} \quad (2.180)$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{v_1^{|\alpha_0|}}{(|\alpha_0|)!}$ with both sides of eq(2.176) where $|v_1| < 1$.

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} \frac{B(|\alpha_1| + 1, \frac{1}{2})}{(|\alpha_0|)!} v_0^{|\alpha_1|} v_1^{|\alpha_0|} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \\ & = A - \frac{\varepsilon}{2} x B \end{aligned} \quad (2.181)$$

where,

$$A = \sum_{|\alpha_0|=0}^{\infty} \frac{v_1^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(z) I \quad (2.182a)$$

$$B = \sum_{|\alpha_0|=0}^{\infty} \frac{v_1^{|\alpha_0|}}{(|\alpha_0|)!} II \quad (2.182b)$$

Plug eq(2.179) into eq(2.182a).

$$A = \int_0^1 dt (1-t)^{-\frac{1}{2}} (1-v_0 t)^{-1} \sum_{|\alpha_0|=0}^{\infty} \frac{(v_0 v_1 t)^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(z) \quad (2.183)$$

Replace t by tv_0v_1 in eq(2.129), and plug it into eq(2.183).

$$\begin{aligned}
 A &= \int_0^1 dt (1-t)^{-\frac{1}{2}}(1-v_0t)^{-1}(1-v_0v_1t)^{-\gamma} e^{-\frac{zv_0v_1t}{1-v_0v_1t}} \\
 &= \sum_{n=0}^{\infty} \frac{(-zv_0v_1)^n}{n!} \int_0^1 dt t^n (1-t)^{-\frac{1}{2}}(1-v_0t)^{-1}(1-v_0v_1t)^{-(\gamma+n)} \\
 &= \sum_{n,m=0}^{\infty} \frac{(\gamma+n+m-1)!(-zv_0v_1)^n(v_0v_1)^m}{n! m! (\gamma+n-1)!} \\
 &\quad \times \int_0^1 dt t^{n+m} (1-t)^{-\frac{1}{2}}(1-v_0t)^{-1} \tag{2.184}
 \end{aligned}$$

Replace a, b, c and z by $1, n+m+1, n+m+\frac{3}{2}$ and v_0 into eq(2.178).

$${}_2F_1\left(1, n+m+1, n+m+\frac{3}{2}, v_0\right) = \frac{\Gamma(n+m+\frac{3}{2})}{\Gamma(n+m+1)\Gamma(\frac{1}{2})} \int_0^1 dt t^{n+m} (1-t)^{-\frac{1}{2}}(1-v_0t)^{-1} \tag{2.185}$$

Substitute eq(2.185) into eq(2.184).

$$\begin{aligned}
 A &= \sum_{n,m=0}^{\infty} \frac{B(n+m+1, \frac{1}{2})(\gamma+n)_m (-zv_0v_1)^n (v_0v_1)^m}{n! m!} \\
 &\quad \times {}_2F_1\left(1, n+m+1, n+m+\frac{3}{2}, v_0\right) \\
 &= \sum_{n,m,j=0}^{\infty} \frac{B(n+m+j+1, \frac{1}{2})(\gamma+n)_m (-z)^n}{n! m!} v_0^{n+m+j} v_1^{n+m} \tag{2.186}
 \end{aligned}$$

CHAPTER 2. GRAND CONFLUENT HYPERGEOMETRIC EQUATION (GCH)76

We can describe eq(2.186) in a different way. First of all, there is the 1st kind of Appell hypergeometric function which is

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{j,k=0}^{\infty} \frac{(\alpha)_{j+k} (\beta)_j (\beta')_k}{j! k! (\gamma)_{j+k}} x^j y^k \quad \text{for } |x| < 1, |y| < 1 \quad (2.187)$$

The function F_1 can be expressed by simple integral

$$\frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha; \beta, \beta'; \gamma; x, y) = \int_0^1 du u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} (1-uy)^{-\beta'}$$

where $\text{Re}(\alpha) > 0, \text{Re}(\gamma - \alpha) > 0 \quad (2.188)$

Put $\alpha = n + 1, \gamma = n + \frac{3}{2}, \beta = 1, \beta' = \gamma + n, x = v_0$ and $y = v_0 v_1$ in eq(188).

$$\begin{aligned} & \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} F_1(n+1; 1, \gamma+n; n+\frac{3}{2}; v_0, v_0 v_1) \\ &= \int_0^1 du u^n (1-u)^{-\frac{1}{2}} (1-uv_0)^{-1} (1-uv_0 v_1)^{-(\gamma+n)} \end{aligned} \quad (2.189)$$

Substitute eq(2.189) into eq(2.184).

$$\begin{aligned}
 A &= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})(-zv_0v_1)^n}{\Gamma(n + \frac{3}{2})} F_1(n + 1; 1, \gamma + n; n + \frac{3}{2}; v_0, v_0v_1) \\
 &= \sum_{n,j,k=0}^{\infty} \frac{B(n + k + j + 1, \frac{1}{2})(\gamma + n)_k (-z)^n}{n! k!} v_0^{n+k+j} v_1^{n+k} \quad (2.190)
 \end{aligned}$$

The equation in the second line in eq(2.190) is exactly equivalent to the second line of eq(2.186). Substitute eq(2.180) into eq(2.182b).

$$B = \Gamma_1(s, p, t) + \Gamma_2(s, p, t) \quad (2.191)$$

where

$$\begin{aligned}
 \Gamma_1(s, p, t) &= \frac{-zv_0v_1}{(1-v_1)^{\gamma+1}} \int_0^{\infty} ds s^{\frac{1}{2}} (1+s)^{-\frac{1}{2}} (1+s-v_0)^{-2} \int_0^1 dt t^{\gamma-\frac{1}{2}} \\
 &\times \int_0^1 dp p(1-p)^{-\frac{1}{2}} e^{-\frac{zpv_0}{1+s-v_0}} \left\{ (1-t) + \frac{tsv_1}{1-v_1} \right\} \quad (2.192)
 \end{aligned}$$

$$\begin{aligned}
 \Gamma_2(s, p, t) &= \frac{\frac{\omega}{2}}{(1-v_1)^{\gamma}} \int_0^{\infty} ds s^{-\frac{1}{2}} (1+s)^{-\frac{1}{2}} (1+s-v_0)^{-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \\
 &\times \int_0^1 dp (1-p)^{-\frac{1}{2}} e^{-\frac{zpv_0}{1+s-v_0}} \left\{ (1-t) + \frac{tsv_1}{1-v_1} \right\} \quad (2.193)
 \end{aligned}$$

By using the first kind of Kummer function,

$$\int_0^1 dp p(1-p)^{-\frac{1}{2}} e^{-\alpha p} = e^{-\alpha} \left\{ 2M\left(\frac{1}{2}, \frac{3}{2}, \alpha\right) - \frac{2}{3}M\left(\frac{3}{2}, \frac{5}{2}, \alpha\right) \right\}$$

$$\text{where } \alpha = \frac{zv_0}{1+s-v_0} \left\{ (1-t) + \frac{tsv_1}{1-v_1} \right\} \quad (2.194)$$

Plug eq(2.194) into eq(2.192).

$$\Gamma_1(s, p, t) = \gamma_1(s, t) + \gamma_2(s, t) \quad (2.195)$$

$$\gamma_1(s, t) = \frac{-2zv_0v_1}{(1-v_1)^{\gamma+1}} \int_0^\infty ds s^{\frac{1}{2}} (1+s)^{-\frac{1}{2}} (1+s-v_0)^{-2} e^{-\frac{zv_0}{1+s-v_0}}$$

$$\times \int_0^1 dt t^{\gamma-\frac{1}{2}} e^{\frac{zv_0}{1+s-v_0} \left(1 - \frac{sv_1}{1-v_1}\right) t} M\left(\frac{1}{2}, \frac{3}{2}, \frac{zv_0}{1+s-v_0} \left\{ (1-t) + \frac{tsv_1}{1-v_1} \right\}\right) \quad (2.196)$$

$$\gamma_2(s, t) = \frac{2zv_0v_1}{3(1-v_1)^{\gamma+1}} \int_0^\infty ds s^{\frac{1}{2}} (1+s)^{-\frac{1}{2}} (1+s-v_0)^{-2} e^{-\frac{zv_0}{1+s-v_0}}$$

$$\times \int_0^1 dt t^{\gamma-\frac{1}{2}} e^{\frac{zv_0}{1+s-v_0} \left(1 - \frac{sv_1}{1-v_1}\right) t} M\left(\frac{3}{2}, \frac{5}{2}, \frac{zv_0}{1+s-v_0} \left\{ (1-t) + \frac{tsv_1}{1-v_1} \right\}\right) \quad (2.197)$$

There are addition theorem for $M(a, b, x + y)$.

$$M(a, b, x + y) = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{(b)_n n!} M(a + n, b + n, x) \quad (2.198a)$$

$$M(a, b, x + y) = e^y \sum_{n=0}^{\infty} \frac{(b - a)_n (-y)^n}{(b)_n n!} M(a, b + n, x) \quad (2.198b)$$

Put $a = \frac{1}{2}$, $b = \frac{3}{2}$, $x = \frac{zv_0}{1+s-v_0}$ and $y = -\frac{zv_0}{1+s-v_0} \left(1 - \frac{sv_1}{1-v_1}\right)t$ into eq(2.198b), and plug it into eq(2.196).

$$\begin{aligned} \gamma_1(s, t) &= -2v_1 \sum_{n=0}^{\infty} \frac{(1 - v_1)^{-(n+\gamma+1)} (zv_0)^{n+1}}{(\gamma + n + \frac{1}{2}) (\frac{3}{2})_n} \int_0^{\infty} ds s^{\frac{1}{2}} (1 + s)^{-\frac{1}{2}} (1 + s - v_0)^{-(n+2)} \\ &\quad \times (1 - v_1(1 + s))^n M\left(n + 1, n + \frac{3}{2}, -\frac{zv_0}{1 + s - v_0}\right) \end{aligned} \quad (2.199)$$

Put $a = \frac{3}{2}$, $b = \frac{5}{2}$, $x = \frac{zv_0}{1+s-v_0}$ and $y = -\frac{zv_0}{1+s-v_0} \left(1 - \frac{sv_1}{1-v_1}\right)t$ into eq(198.b), and plug it into eq(197).

$$\begin{aligned} \gamma_2(s, t) &= \frac{2v_1}{3} \sum_{n=0}^{\infty} \frac{(1 - v_1)^{-(n+\gamma+1)} (zv_0)^{n+1}}{(\gamma + n + \frac{1}{2}) (\frac{5}{2})_n} \int_0^{\infty} ds s^{\frac{1}{2}} (1 + s)^{-\frac{1}{2}} (1 + s - v_0)^{-(n+2)} \\ &\quad \times (1 - v_1(1 + s))^n M\left(n + 1, n + \frac{5}{2}, -\frac{zv_0}{1 + s - v_0}\right) \end{aligned} \quad (2.200)$$

Replace s by $\frac{1}{t} - 1$ in eq(2.199) and eq(2.200).

$$\begin{aligned} \gamma_1(s, t) &= 2 \sum_{n=0}^{\infty} \frac{(1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+1} (-v_1)^{n+1}}{(\gamma+n+\frac{1}{2})(\frac{3}{2})_n} \\ &\quad \times \int_0^1 dt (1-t)^{\frac{1}{2}} (1-v_0t)^{-(n+2)} \left(1 - \frac{1}{v_1}t\right)^n M\left(n+1, n+\frac{3}{2}, -\frac{ztv_0}{1-v_0t}\right) \end{aligned} \quad (2.201)$$

$$\begin{aligned} \gamma_2(s, t) &= -\frac{2}{3} \sum_{n=0}^{\infty} \frac{(1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+1} (-v_1)^{n+1}}{(\gamma+n+\frac{1}{2})(\frac{5}{2})_n} \\ &\quad \times \int_0^1 dt (1-t)^{\frac{1}{2}} (1-v_0t)^{-(n+2)} \left(1 - \frac{1}{v_1}t\right)^n M\left(n+1, n+\frac{5}{2}, -\frac{ztv_0}{1-v_0t}\right) \end{aligned} \quad (2.202)$$

We know that

$$M\left(n+1, n+\frac{3}{2}, \frac{-zv_0t}{1-v_0t}\right) = \sum_{m=0}^{\infty} \frac{(n+1)_m (-zv_0)^m}{(n+\frac{3}{2})_m m!} t^m (1-v_0t)^{-m} \quad (2.203a)$$

$$M\left(n+1, n+\frac{5}{2}, \frac{-zv_0t}{1-v_0t}\right) = \sum_{m=0}^{\infty} \frac{(n+1)_m (-zv_0)^m}{(n+\frac{5}{2})_m m!} t^m (1-v_0t)^{-m} \quad (2.203b)$$

Plug eq(2.203a) into eq(2.201).

$$\begin{aligned} \gamma_1(s, t) &= 2 \sum_{n,m=0}^{\infty} \frac{(-1)^m (1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+m+1} (-v_1)^{n+1}}{(\gamma+n+\frac{1}{2})(\frac{3}{2})_n (n+\frac{3}{2})_m m!} \\ &\quad \times \int_0^1 dt t^m (1-t)^{\frac{1}{2}} (1-v_0 t)^{-(n+m+2)} \left(1 - \frac{1}{v_1} t\right)^n \end{aligned} \quad (2.204)$$

Plug eq(2.203b) into eq(2.202).

$$\begin{aligned} \gamma_2(s, t) &= -\frac{2}{3} \sum_{n,m=0}^{\infty} \frac{(-1)^m (1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+m+1} (-v_1)^{n+1}}{(\gamma+n+\frac{1}{2})(\frac{5}{2})_n (n+\frac{5}{2})_m m!} \\ &\quad \times \int_0^1 dt t^m (1-t)^{\frac{1}{2}} (1-v_0 t)^{-(n+m+2)} \left(1 - \frac{1}{v_1} t\right)^n \end{aligned} \quad (2.205)$$

Put $\alpha = m + 1$, $\beta = n + m + 2$, $\beta' = -n$, $\gamma = m + \frac{5}{2}$, $x = v_0$ and $y = \frac{1}{v_1}$ in eq(188).

Then plug it into eq(2.204) and eq(2.205).

$$\begin{aligned} \gamma_1(s, t) &= 2 \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1} (1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+m+1} (v_1)^{n+1} \Gamma(\frac{3}{2})}{(\gamma+n+\frac{1}{2})(\frac{3}{2})_n (n+\frac{3}{2})_m \Gamma(m+\frac{5}{2})} \\ &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \end{aligned} \quad (2.206)$$

$$\begin{aligned} \gamma_2(s, t) &= -\frac{2}{3} \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1} (1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+m+1} (v_1)^{n+1} \Gamma(\frac{3}{2})}{(\gamma+n+\frac{1}{2})(\frac{5}{2})_n (n+\frac{5}{2})_m \Gamma(m+\frac{5}{2})} \\ &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \end{aligned} \quad (2.207)$$

Substitute eq(2.206) and eq(2.207) into eq(2.195).

$$\begin{aligned} \Gamma_1(s, p, t) &= \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1} (n+m+1) \Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) (1-v_1)^{-(n+\gamma+1)} (zv_0)^{n+m+1} (v_1)^{n+1}}{(\gamma+n+\frac{1}{2}) \Gamma(m+\frac{5}{2}) \Gamma(n+m+\frac{5}{2})} \\ &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \end{aligned} \quad (2.208)$$

Replace s and p by $\frac{1}{v} - 1$ and $1 - p^2$ in eq(2.193).

$$\begin{aligned} \Gamma_2(s, p, t) &= \frac{\frac{\omega}{2}}{(1-v_1)^\gamma} \int_0^1 dv (1-v)^{-\frac{1}{2}} (1-v_0v)^{-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} e^{-\frac{zv_0v}{1-v_0v} \left\{ (1-t) + \frac{tv_1(1-v)}{v(1-v_1)} \right\}} \\ &\quad \times \int_{-1}^1 dp e^{\frac{zv_0v}{1-v_0v} \left\{ (1-t) + \frac{tv_1(1-v)}{v(1-v_1)} \right\}} p^2 \\ &= \frac{\omega}{(1-v_1)^\gamma} \int_0^1 dv (1-v)^{-\frac{1}{2}} (1-v_0v)^{-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \\ &\quad \times M\left(1, \frac{3}{2}, \frac{-zv_0v}{1-v_0v} \left\{ 1-t + \frac{v_1(1-v)t}{v(1-v_1)} \right\}\right) \end{aligned} \quad (2.209)$$

Put $a = 1$, $b = \frac{3}{2}$, $x = -\frac{zv_0}{1-v_0v}$ and $y = \frac{zv_0v}{1-v_0v} \left\{ 1 - \frac{v_1(1-v)}{v(1-v_1)} \right\} t$ into eq(2.198a), and plug it into eq(2.209).

$$\begin{aligned}
 \Gamma_2(s, p, t) &= \frac{\omega}{(1-v_1)^\gamma} \sum_{n=0}^{\infty} \frac{(1-v_1)^{-n} (zv_0)^n}{(\gamma+n-\frac{1}{2})(\frac{3}{2})_n} \int_0^1 dv (1-v)^{-\frac{1}{2}} (1-v_0v)^{-(n+1)} (v-v_1)^n \\
 &\quad \times M\left(n+1, n+\frac{3}{2}, -\frac{zv_0v}{1-v_0v}\right) \\
 &= \frac{\omega}{(1-v_1)^\gamma} \sum_{n,m=0}^{\infty} \frac{v_1^n (1-v_1)^{-n} (-zv_0)^{n+m} (n+1)_m}{(\gamma+n-\frac{1}{2})(\frac{3}{2})_n (n+\frac{3}{2})_m m!} \\
 &\quad \times \int_0^1 dv v^m (1-v)^{-\frac{1}{2}} (1-v_0v)^{-(n+m+1)} \left(1-\frac{v}{v_1}\right)^n \tag{2.210}
 \end{aligned}$$

Put $\alpha = m+1$, $\gamma = m+\frac{3}{2}$, $\beta' = -n$, $\beta = n+m+1$, $x = v_0$ and $y = \frac{1}{v_1}$ in eq(2.188).

Then plug it into eq(2.210).

$$\begin{aligned}
 \Gamma_2(s, p, t) &= \omega \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m} \Gamma(\frac{1}{2}) (n+1)_m (1-v_1)^{-(n+\gamma)} (zv_0)^{n+m} (v_1)^n}{(\gamma+n-\frac{1}{2})(\frac{3}{2})_n (n+\frac{3}{2})_m \Gamma(m+\frac{3}{2})} \\
 &\quad \times F_1\left(m+1; n+m+1, -n; m+\frac{3}{2}; v_0, \frac{1}{v_1}\right) \tag{2.211}
 \end{aligned}$$

Substitute eq(2.208) and eq(2.211) into eq(2.191).

$$\begin{aligned}
 B &= \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1}(n+m+1)\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(1-v_1)^{-(n+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n+1}}{(\gamma+n+\frac{1}{2})\Gamma(m+\frac{5}{2})\Gamma(n+m+\frac{5}{2})} \\
 &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \\
 &+ \omega \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}\Gamma(\frac{1}{2})(n+1)_m(1-v_1)^{-(n+\gamma)}(zv_0)^{n+m}(v_1)^n}{(\gamma+n-\frac{1}{2})(\frac{3}{2})_n(n+\frac{3}{2})_m\Gamma(m+\frac{3}{2})} \\
 &\quad \times F_1\left(m+1; n+m+1, -n; m+\frac{3}{2}; v_0, \frac{1}{v_1}\right) \tag{2.212}
 \end{aligned}$$

Plug eq(2.190) and eq(2.212) into eq(2.181). Then we obtain the generating function for 1st kind of grand confluent hypergeometric function:

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} \frac{B(|\alpha_1|+1, \frac{1}{2})}{(|\alpha_0|)!} v_0^{|\alpha_1|} v_1^{|\alpha_0|} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1+\nu); z = -\frac{1}{2}\mu x^2 \right) \\
 &= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})(-zv_0v_1)^n}{\Gamma(n+\frac{3}{2})} F_1\left(n+1; 1, \gamma+n; n+\frac{3}{2}; v_0, v_0v_1\right) \\
 & - \frac{\varepsilon}{2} x \left\{ \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1}(n+m+1)\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(1-v_1)^{-(n+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n+1}}{(\gamma+n+\frac{1}{2})\Gamma(m+\frac{5}{2})\Gamma(n+m+\frac{5}{2})} \right. \\
 & \quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \\
 & \quad + \omega \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}\Gamma(\frac{1}{2})(n+1)_m(1-v_1)^{-(n+\gamma)}(zv_0)^{n+m}(v_1)^n}{(\gamma+n-\frac{1}{2})(\frac{3}{2})_n(n+\frac{3}{2})_m\Gamma(m+\frac{3}{2})} \\
 & \quad \left. \times F_1\left(m+1; n+m+1, -n; m+\frac{3}{2}; v_0, \frac{1}{v_1}\right) \right\} \\
 &= \sum_{n,j,k=0}^{\infty} \frac{B(n+k+j+1, \frac{1}{2})(\gamma+n)_k(-z)^n}{n! k!} v_0^{n+k+j} v_1^{n+k} - \frac{\varepsilon}{2} x \sum_{n,m,j,k,s=0}^{\infty} \left\{ \right. \\
 & \frac{(-1)^{n+m+1}(n+m+1)\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(n+\gamma+1)_s(m+1)_{j+k}(n+m+2)_j(-n)_k z^{n+m+1}}{s! j! k! (\gamma+n+\frac{1}{2})(m+\frac{3}{2})!(n+m+\frac{3}{2})!(m+\frac{5}{2})_{j+k}} \\
 & \times v_0^{n+m+j+1} v_1^{n+s-k+1} \\
 & \left. + \omega \frac{(-1)^{n+m}\Gamma(\frac{1}{2})(n+\gamma)_s(n+1)_m(m+1)_{j+k}(n+m+1)_j(-n)_k z^{n+m}}{s! j! k! (\gamma+n-\frac{1}{2})(m+\frac{1}{2})!(\frac{3}{2})_n(n+\frac{3}{2})_m(m+\frac{3}{2})_{j+k}} v_0^{n+m+j} v_1^{n+s-k} \right\}
 \end{aligned}$$

Table 2.3: The generating function for the first kind of grand confluent hypergeometric function

We can describe the generating function of it as the integral formalism by using eq(2.184), eq(2.192) and eq(2.193).

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} \frac{B(|\alpha_1|+1, \frac{1}{2})}{(|\alpha_0|)!} v_0^{|\alpha_1|} v_1^{|\alpha_0|} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1+\nu); z = -\frac{1}{2}\mu x^2 \right) \\
 &= \int_0^1 dt (1-t)^{-\frac{1}{2}} (1-v_0 t)^{-1} (1-v_0 v_1 t)^{-\gamma} e^{-\frac{z v_0 v_1 t}{1-v_0 v_1 t}} - \frac{\varepsilon}{2} \frac{x}{(1-v_1)^\gamma} \\
 &\times \int_0^1 du (1-u)^{-\frac{1}{2}} (1-v_0 u)^{-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_0^1 dp (1-p)^{-\frac{1}{2}} \\
 &\times \infty \left\{ \frac{\omega}{2} - \frac{z p t (1-u) v_0 v_1}{(1-v_1)(1-uv_1)} \right\} e^{-\frac{z p u v_0}{1-uv_0}} \left\{ 1 - \frac{t(u-v_1)}{u(1-v_1)} \right\}
 \end{aligned}$$

Table 2.4: Integral formalism of the generating function for the 1st kind of grand confluent hypergeometric function

From the differential equations satisfied by $\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(z)$ and $\mathcal{QW}_{|\beta_0|, |\beta_1|}(z)$, namely,

$$x \mathcal{QW}_{|\alpha_0|, |\alpha_1|}''(z) + (\mu x^2 + \varepsilon x + \nu) \mathcal{QW}_{|\alpha_0|, |\alpha_1|}'(z) + (\Omega_{|\alpha_0|, |\alpha_1|} x + \varepsilon \omega) \mathcal{QW}_{|\alpha_0|, |\alpha_1|}(z) = 0 \tag{2.213a}$$

$$x \mathcal{QW}_{|\beta_0|, |\beta_1|}''(z) + (\mu x^2 + \varepsilon x + \nu) \mathcal{QW}_{|\beta_0|, |\beta_1|}'(z) + (\Omega_{|\beta_0|, |\beta_1|} x + \varepsilon \omega) \mathcal{QW}_{|\beta_0|, |\beta_1|}(z) = 0 \tag{2.213b}$$

where,

$$\Omega_{|\alpha_0|,|\alpha_1|} = \begin{cases} -2\mu|\alpha_0| \\ -2\mu(|\alpha_1| + \frac{1}{2}) \end{cases} \quad \Omega_{|\beta_0|,|\beta_1|} = \begin{cases} -2\mu|\beta_0| \\ -2\mu(|\beta_1| + \frac{1}{2}) \end{cases} \quad (2.214)$$

multiplying eq(2.213a) and eq(2.213b) by $\mathcal{QW}_{|\beta_0|,|\beta_1|}(z)$ and $\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)$ respectively and subtracting, we have

$$x \frac{dM(z)}{dx} + (\mu x^2 + \varepsilon x + \nu)M(z) = (\Omega_{|\beta_0|,|\beta_1|} - \Omega_{|\alpha_0|,|\alpha_1|})x \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) \quad (2.215)$$

where

$$M(z) = \mathcal{QW}'_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) - \mathcal{QW}'_{|\beta_0|,|\beta_1|}(z) \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \quad (2.216)$$

Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x}$ on both side of eq(2.215).

$$\frac{d}{dx} \left\{ x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} M(z) \right\} = (\Omega_{|\beta_0|,|\beta_1|} - \Omega_{|\alpha_0|,|\alpha_1|}) x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) \quad (2.217)$$

Integrate eq(2.217) with respect to x from 0 to ∞ .

$$(\Omega_{|\beta_0|,|\beta_1|} - \Omega_{|\alpha_0|,|\alpha_1|}) \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) = \left[x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} M(z) \right] \Big|_0^\infty \quad (2.218)$$

Substitute eq(2.216) into eq(2.218).

$$\begin{aligned} & (\Omega_{|\beta_0|,|\beta_1|} - \Omega_{|\alpha_0|,|\alpha_1|}) \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) \quad (2.219) \\ &= \left[x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} (\mathcal{QW}'_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) - \mathcal{QW}'_{|\beta_0|,|\beta_1|}(z) \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)) \right] \Big|_0^\infty \end{aligned}$$

Therefore, only if $|\alpha_0| \neq |\beta_0|$ and $|\alpha_1| \neq |\beta_1|$, then

$$\int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \mathcal{QW}_{|\beta_0|,|\beta_1|}(z) = 0 \quad (2.220)$$

Let's think about the case in which are $|\alpha_0| = |\beta_0|$ and $|\alpha_1| = |\beta_1|$. First of all, from eq(2.125)

$$\begin{aligned} \left[\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \right]^2 &= \left[F_{|\alpha_0|}(z) \right]^2 - \varepsilon x F_{|\alpha_0|}(z) \prod_{|\alpha_0|}^{|\alpha_1|}(z) + \frac{\varepsilon^2}{4} x^2 \left[\prod_{|\alpha_0|}^{|\alpha_1|}(z) \right]^2 \\ &\simeq \left[F_{|\alpha_0|}(z) \right]^2 - \varepsilon x F_{|\alpha_0|}(z) \prod_{|\alpha_0|}^{|\alpha_1|}(z) \quad (2.221) \end{aligned}$$

ε is extremely small. So anything beyond the second order of ε is negligible in eq(2.221). Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x}$ in eq(2.221) and integrate them with respect to $x = [0, \infty]$.

$$\int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(z) \right]^2 = E_1 - \varepsilon E_2 \quad (2.222)$$

where

$$E_1 = \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[F_{|\alpha_0|}(z) \right]^2 \quad (2.223a)$$

$$E_2 = \int_0^\infty dx x^{\nu+1} e^{\frac{1}{2}\mu x^2 + \varepsilon x} F_{|\alpha_0|}(z) \prod_{|\alpha_0|}^{|\alpha_1|}(z) \quad (2.223b)$$

Replace t by v_0 in eq(2.129). Then square of both side of it

$$\frac{e^{-\frac{2zv_0}{(1-v_0)}}}{(1-v_0)^{2\gamma}} = \sum_{|\alpha_0|=0}^{\infty} \frac{\left[F_{|\alpha_0|}(z) \right]^2}{(|\alpha_0|!)^2} v_0^{2|\alpha_0|} \quad (2.224)$$

Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x}$ on both side of eq(2.224), and integrate them with respect to $x = [0, \infty]$.

$$I = (1-v_0)^{-2\gamma} \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} e^{-\frac{2zv_0}{(1-v_0)}} = \sum_{|\alpha_0|=0}^{\infty} \frac{v_0^{2|\alpha_0|}}{(|\alpha_0|!)^2} \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[F_{|\alpha_0|}(z) \right]^2 \quad (2.225)$$

We know $\mu = \text{real negative}$. Plug $z = \frac{1}{2}|\mu|x^2$ into the l.h.s of eq(2.225).

$$I = (1 - v_0)^{-2\gamma} e^{\frac{\varepsilon^2}{2|\mu|\sigma}} \int_0^\infty dx x^\nu e^{-\frac{1}{2}|\mu|\sigma \left(x - \frac{\varepsilon}{|\mu|\sigma}\right)^2} \quad \text{where } \sigma = \frac{(1 + v_0)}{(1 - v_0)} \quad (2.226)$$

We know that ε is very small. So we can neglect any terms which include more than second order of ε . We can say that $e^{\frac{\varepsilon^2}{2|\mu|\sigma}} \approx 1$, and lets say $K = \frac{\varepsilon}{|\mu|\sigma}$. Plug it into eq(2.226).

$$I \approx (1 - v_0)^{-2\gamma} \int_0^\infty dx x^\nu e^{-\frac{1}{2}|\mu|\sigma(x-K)^2} \quad (2.227)$$

Replaced $x - K$ by r in eq(2.227).

$$I \approx (1 - v_0)^{-2\gamma} \left\{ \int_0^\infty dr (r + K)^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \int_{-K}^0 dr (r + K)^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} \right\} \quad (2.228)$$

By using binomial expansion,

$$(r + K)^\nu = \sum_{k=0}^{\infty} \frac{\nu!}{(\nu - k)! k!} r^{\nu-k} K^k \quad \text{only if } \nu \neq \text{positive integer} \quad (2.229)$$

K includes the first order of ε . Neglecting any terms which include more than the

second order of ε (extremely small value). So eq(2.229) is approximately equal to

$$(r + K)^\nu \approx r^\nu + \nu K r^{\nu-1} \quad (2.230)$$

Plug eq(2.230) into eq(2.228).

$$\begin{aligned} I \approx & (1 - v_0)^{-2\gamma} \left\{ \int_0^\infty dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_0^\infty dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \right. \\ & \left. + \int_{-K}^0 dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_{-K}^0 dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \right\} \end{aligned} \quad (2.231)$$

Replace r by $\sqrt{\frac{2}{|\mu|\sigma}} t$ in the first and second terms of r.h.s. in eq(2.231).

$$\begin{aligned} & \int_0^\infty dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_0^\infty dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \\ & = \left(\frac{2}{|\mu|\sigma} \right)^{\frac{1}{2}(1+\nu)} \int_0^\infty dt t^{2(\frac{1}{2}(1+\nu))-1} e^{-t^2} \\ & + \nu K \left(\frac{2}{|\mu|\sigma} \right)^{\frac{\nu}{2}} \int_0^\infty dt t^{2(\frac{\nu}{2})-1} e^{-t^2} \end{aligned} \quad (2.232)$$

By using integral form of gamma function, eq(2.232) is equal to

$$\begin{aligned} & \int_0^\infty dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_0^\infty dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \\ &= \frac{1}{2} \left(\frac{2}{|\mu|\sigma} \right)^\gamma \Gamma(\gamma) + K \left(\frac{2}{|\mu|\sigma} \right)^{\gamma-\frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right) \\ & \text{where } \gamma = \frac{1}{2}(1 + \nu), \operatorname{Re}(\nu) > 0 \end{aligned} \quad (2.233)$$

By using binomial expansion, the third and fourth terms of rhs in eq(2.231) are

$$\begin{aligned} & \int_{-K}^0 dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_{-K}^0 dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^{j+\nu+1} (|\mu|)^j \sigma^j}{2^j (2j + \nu + 1) j!} K^{2j+\nu+1} + \nu \sum_{j=0}^{\infty} \frac{(-1)^{j+\nu} (|\mu|)^j \sigma^j}{2^j (2j + \nu) j!} K^{2j+\nu+1} \end{aligned} \quad (2.234)$$

The lowest order of K in eq(2.234) is

$$\int_{-K}^0 dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_{-K}^0 dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \approx \frac{-\nu}{1 + \nu} (-K)^{\nu+1} \quad (2.235)$$

We know that $\operatorname{Re}(\nu) > 0$. The first order of ε is only countable because of too small

value of ε . So eq(2.234) is negligible.

$$\int_{-K}^0 dr r^\nu e^{-\frac{1}{2}|\mu|\sigma r^2} + \nu K \int_{-K}^0 dr r^{\nu-1} e^{-\frac{1}{2}|\mu|\sigma r^2} \approx 0 \quad (2.236)$$

Substitute eq(2.233) and eq(2.236) into eq(2.231).

$$I \approx (1 - v_0)^{-2\gamma} \left\{ \frac{1}{2} \left(\frac{2}{|\mu|\sigma} \right)^\gamma \Gamma(\gamma) + K \left(\frac{2}{|\mu|\sigma} \right)^{\gamma-\frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right) \right\} \quad (2.237)$$

Plug $K = \frac{\varepsilon}{|\mu|\sigma}$ and $\sigma = \frac{(1+v_0)}{(1-v_0)}$ in eq(2.237).

$$I \approx \frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(\gamma) (1 - v_0^2)^{-\gamma} + \varepsilon \frac{2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \Gamma\left(\gamma + \frac{1}{2}\right) (1 - v_0)^{-\gamma+\frac{1}{2}} (1 + v_0)^{-\gamma-\frac{1}{2}} \quad (2.238)$$

By using binomial expansion, eq(2.238) is equivalent to

$$I \approx \sum_{|\alpha_0|=0}^{\infty} \frac{v_0^{2|\alpha_0|}}{(|\alpha_0|!)^2} \left\{ \frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) \right. \\ \left. + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}} \Gamma\left(|\alpha_0| + \gamma - \frac{1}{2}\right) \Gamma\left(|\alpha_0| + \gamma + \frac{1}{2}\right)}{|\mu|^{\gamma+\frac{1}{2}} \Gamma\left(\gamma - \frac{1}{2}\right)} \right\} \quad (2.239)$$

Eq(2.239) is equivalent to eq(2.225). Then, we obtain

$$\begin{aligned}
 E_1 &= \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[F_{|\alpha_0|}(z) \right]^2 \\
 &= \frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) \\
 &\quad + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}} \Gamma\left(|\alpha_0| + \gamma - \frac{1}{2}\right) \Gamma\left(|\alpha_0| + \gamma + \frac{1}{2}\right)}{|\mu|^{\gamma+\frac{1}{2}} \Gamma\left(\gamma - \frac{1}{2}\right)}
 \end{aligned} \tag{2.240}$$

Replace z by $\frac{1}{2}|\mu|x^2$ in eq(2.223b).

$$E_2 = \frac{2^{\gamma-\frac{3}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \int_0^\infty dz z^{\gamma-\frac{1}{2}} e^{-z+\varepsilon\sqrt{\frac{2z}{|\mu|}}} F_{|\alpha_0|}(z) \prod_{|\alpha_0|}^{|\alpha_1|}(z) \tag{2.241}$$

Substitute eq(2.126b) into eq(2.241).

$$\begin{aligned}
 E_2 &= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} \\
 &\quad \times \frac{2^{\gamma-\frac{3}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \int_0^\infty dz z^{\gamma+n+k-\frac{1}{2}} e^{-z+\varepsilon\sqrt{\frac{2z}{|\mu|}}} F_{|\alpha_0|}(z)
 \end{aligned} \tag{2.242}$$

Multiply $z^{\gamma+n+k-\frac{1}{2}} e^{-z+\varepsilon\sqrt{\frac{2z}{|\mu|}}}$ on both side of eq(2.224). And integrate them with

respect to $z = [0, \infty]$.

$$\begin{aligned}
 \text{II} &= \int_0^\infty dz z^{n+k+\gamma-\frac{1}{2}} e^{-z+\varepsilon\sqrt{\frac{2z}{|\mu|}}} \frac{e^{-\frac{zv_0}{(1-v_0)}}}{(1-v_0)^\gamma} \\
 &= \sum_{|\alpha_0|=0}^\infty \frac{v_0^{|\alpha_0|}}{|\alpha_0|!} \int_0^\infty dz z^{n+k+\gamma-\frac{1}{2}} e^{-z+\varepsilon\sqrt{\frac{2z}{|\mu|}}} F_{|\alpha_0|}(z)
 \end{aligned} \tag{2.243}$$

The lhs of eq(2.243) is approximately equal to

$$\begin{aligned}
 \text{II} &= (1-v_0)^{-\gamma} e^{\varepsilon^2 \frac{(1-v_0)}{2|\mu|}} \int_0^\infty dz z^{n+k+\gamma-\frac{1}{2}} e^{\frac{-1}{(1-v_0)} \left(\sqrt{z-\varepsilon\frac{(1-v_0)}{\sqrt{2|\mu|}}} \right)^2} \\
 &\approx (1-v_0)^{-\gamma} \int_0^\infty dz z^{n+k+\gamma-\frac{1}{2}} e^{\frac{-1}{(1-v_0)} \left(\sqrt{z-\varepsilon\frac{(1-v_0)}{\sqrt{2|\mu|}}} \right)^2}
 \end{aligned} \tag{2.244}$$

Let say $M = \varepsilon \frac{(1-v_0)}{\sqrt{2|\mu|}}$, and replace z by $(r+M)^2$ in eq(2.244).

$$\text{II} \approx 2(1-v_0)^{-\gamma} \left\{ \int_0^\infty dr (r+M)^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} + \int_{-M}^0 dr (r+M)^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} \right\} \tag{2.245}$$

By using binomial expansion,

$$(r+M)^{2(n+k+\gamma)} = \sum_{j=0}^\infty \frac{(2(n+k+\gamma))!}{(2(n+k+\gamma)-j)! j!} r^{2(n+k+\gamma)-j} M^j \tag{2.246}$$

M includes the first order of ε . More than the second order of ε is negligible because

of too small value of it. Then eq(246) is approximately equal to

$$(r + M)^{2(n+k+\gamma)} \approx r^{2(n+k+\gamma)} + 2(n + k + \gamma)Mr^{2(n+k+\gamma)-1} \quad (2.247)$$

Plug eq(2.247) into eq(2.245).

$$\begin{aligned} \text{II} = & 2(1 - v_0)^{-\gamma} \left\{ \int_0^\infty dr r^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} \right. \\ & + 2(n + k + \gamma)M \int_0^\infty dr r^{2(n+k+\gamma)-1} e^{-\frac{r^2}{(1-v_0)}} + \int_{-M}^0 dr r^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} \\ & \left. + 2(n + k + \gamma)M \int_{-M}^0 dr r^{2(n+k+\gamma)-1} e^{-\frac{r^2}{(1-v_0)}} \right\} \quad (2.248) \end{aligned}$$

By using gamma function, the first and second terms in eq(2.248) where $\text{Re}(n + k + \gamma) > 0$ are equal to

$$\begin{aligned} & 2(1 - v_0)^{-\gamma} \left\{ \int_0^\infty dr r^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} + 2(n + k + \gamma)M \int_0^\infty dr r^{2(n+k+\gamma)-1} e^{-\frac{r^2}{(1-v_0)}} \right\} \\ & = (1 - v_0)^{n+k+\frac{1}{2}} \Gamma\left(n + k + \gamma + \frac{1}{2}\right) + 2\Gamma(n + k + \gamma + 1)M(1 - v_0)^{n+k} \quad (2.249) \end{aligned}$$

By using binomial expansion, the third and fourth terms of rhs in eq(2.248) are

$$\begin{aligned} & \int_{-M}^0 dr r^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} + 2(n+k+\gamma)M \int_{-M}^0 dr r^{2(n+k+\gamma)-1} e^{-\frac{r^2}{(1-v_0)}} \\ &= 2 \sum_{j=0}^{\infty} \frac{(-1)^{j+1} (1-v_0)^{-j}}{j!} M^{2(n+k+j+\gamma)+1} \end{aligned} \quad (2.250)$$

We know that $\text{Re}(n+k+\gamma) > 0$. The first order of ε is only countable because of too small value of ε . So eq(2.250) is negligible.

$$\int_{-M}^0 dr r^{2(n+k+\gamma)} e^{-\frac{r^2}{(1-v_0)}} + 2(n+k+\gamma)M \int_{-M}^0 dr r^{2(n+k+\gamma)-1} e^{-\frac{r^2}{(1-v_0)}} \approx 0 \quad (2.251)$$

Substitute eq(2.249) and eq(2.251) into eq(2.248), and replace M by $\varepsilon \frac{(1-v_0)}{\sqrt{2|\mu|}}$.

$$\Pi \approx (1-v_0)^{n+k+\frac{1}{2}} \Gamma\left(n+k+\gamma+\frac{1}{2}\right) + 2 \frac{\varepsilon}{\sqrt{2|\mu|}} \Gamma(n+k+\gamma+1) (1-v_0)^{n+k+1} \quad (2.252)$$

By using binomial expansion, eq(2.252) is equivalent to

$$\begin{aligned} \text{II} \approx & \sum_{|\alpha_0|=0}^{\infty} \frac{v_0^{|\alpha_0|}}{|\alpha_0|!} \left\{ \frac{(-1)^{|\alpha_0|} \Gamma(n+k+\frac{3}{2}) \Gamma(n+k+\gamma+\frac{1}{2})}{\Gamma(n+k-|\alpha_0|+\frac{3}{2})} \right. \\ & \left. + \varepsilon \frac{2}{\sqrt{2|\mu|}} \frac{(-1)^{|\alpha_0|} \Gamma(n+k+\gamma+1) \Gamma(n+k+2)}{\Gamma(n+k-|\alpha_0|+2)} \right\} \end{aligned} \quad (2.253)$$

Eq(2.253) is equivalent to eq(2.243). Then we obtain

$$\begin{aligned} & \int_0^{\infty} dz z^{n+k+\gamma-\frac{1}{2}} e^{-z+\varepsilon\sqrt{\frac{2z}{|\mu|}}} F_{|\alpha_0|}(z) \\ &= \frac{(-1)^{|\alpha_0|} \Gamma(n+k+\frac{3}{2}) \Gamma(n+k+\gamma+\frac{1}{2})}{\Gamma(n+k-|\alpha_0|+\frac{3}{2})} \\ &+ \varepsilon \frac{2}{\sqrt{2|\mu|}} \frac{(-1)^{|\alpha_0|} \Gamma(n+k+\gamma+1) \Gamma(n+k+2)}{\Gamma(n+k-|\alpha_0|+2)} \end{aligned} \quad (2.254)$$

Substitute eq(2.254) into eq(2.242).

$$\begin{aligned} E_2 = & \frac{2^{\gamma-\frac{3}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \frac{\Gamma(|\alpha_0|+\gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n!(\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n+\frac{\omega}{2}) \Gamma(n+\frac{1}{2}) \Gamma(n+\gamma-\frac{1}{2}) (n-|\alpha_1|)_k}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\gamma+\frac{1}{2})} \\ & \times \left\{ \frac{(-1)^{|\alpha_0|} \Gamma(n+k+\frac{3}{2}) \Gamma(n+k+\gamma+\frac{1}{2})}{\Gamma(n+k-|\alpha_0|+\frac{3}{2})} \right. \\ & \left. + \varepsilon \frac{2}{\sqrt{2|\mu|}} \frac{(-1)^{|\alpha_0|} \Gamma(n+k+\gamma+1) \Gamma(n+k+2)}{\Gamma(n+k-|\alpha_0|+2)} \right\} \end{aligned} \quad (2.255)$$

Substitute eq(2.240) and eq(2.255) into eq(2.222).

$$\begin{aligned}
 & \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(z) \right]^2 \\
 &= \frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}} \Gamma\left(|\alpha_0| + \gamma - \frac{1}{2}\right) \Gamma\left(|\alpha_0| + \gamma + \frac{1}{2}\right)}{|\mu|^{\gamma+\frac{1}{2}} \Gamma\left(\gamma - \frac{1}{2}\right)} \\
 & - \frac{2^{\gamma-\frac{3}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} \\
 & \times \left\{ \varepsilon \frac{(-1)^{|\alpha_0|} \Gamma(n + k + \frac{3}{2}) \Gamma(n + k + \gamma + \frac{1}{2})}{\Gamma(n + k - |\alpha_0| + \frac{3}{2})} \right. \\
 & \left. + \varepsilon^2 \frac{2}{\sqrt{2|\mu|}} \frac{(-1)^{|\alpha_0|} \Gamma(n + k + \gamma + 1) \Gamma(n + k + 2)}{\Gamma(n + k - |\alpha_0| + 2)} \right\} \tag{2.256}
 \end{aligned}$$

Larger than second order of ε is negligible because of extremely small value of it.

Then, eq(2.256) is simply approximately equal to

$$\begin{aligned}
 & \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(z) \right]^2 \\
 &= \frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma\left(|\alpha_0| + \gamma - \frac{1}{2}\right) \Gamma\left(|\alpha_0| + \gamma + \frac{1}{2}\right)}{\Gamma\left(\gamma - \frac{1}{2}\right)} \right. \\
 & \left. - \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{2\Gamma(k + n + \frac{3}{2} - |\alpha_0|)} \right\} \tag{2.257}
 \end{aligned}$$

Therefore, we obtain the orthogonal relation according to eq(2.220) and eq(2.257):

$$\begin{aligned}
 & \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|, |\alpha_1|}(z) \mathcal{QW}_{|\beta_0|, |\beta_1|}(z) \\
 &= \left\{ \frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma\left(|\alpha_0| + \gamma - \frac{1}{2}\right) \Gamma\left(|\alpha_0| + \gamma + \frac{1}{2}\right)}{\Gamma\left(\gamma - \frac{1}{2}\right)} \right. \right. \\
 & \left. \left. - \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{2\Gamma(k + n + \frac{3}{2} - |\alpha_0|)} \right\} \right\} \delta_{|\alpha_0|, |\beta_0|} \delta_{|\alpha_1|, |\beta_1|}
 \end{aligned}$$

Table 2.5: The orthogonal relation for the 1st kind of grand confluent hypergeometric function

If there is an analytic function $\Psi(x)$ with interval $x = [0, \infty]$ which is

$$\Psi(x) = \sum_{|\alpha_0|=0}^{\infty} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} A_{|\alpha_0|, |\alpha_1|} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \quad (2.258)$$

Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ on both sides of eq(2.258) and integrate it with respect to $x=[0, \infty]$.

$$\begin{aligned}
 & \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \Psi(x) \\
 &= \sum_{|\alpha_0|=0}^{\infty} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} A_{|\alpha_0|, |\alpha_1|} \int_0^\infty x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{QW}_{|\alpha_0|, |\alpha_1|}(\gamma; z) \right]^2 \quad (2.259)
 \end{aligned}$$

Substitute eq(2.257) into eq(2.259).

$$\begin{aligned}
 A_{|\alpha_0|,|\alpha_1|} &= \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(\gamma; z) \Psi(x) \left[\frac{2^{\gamma-1}}{|\mu|^\gamma} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) \right. \\
 &\quad + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma(|\alpha_0| + \gamma - \frac{1}{2}) \Gamma(|\alpha_0| + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} \right. \\
 &\quad - \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \\
 &\quad \left. \left. \times \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{2\Gamma(k + n + \frac{3}{2} - |\alpha_0|)} \right\} \right]^{-1} \quad (2.260)
 \end{aligned}$$

If there is an analytic function $\Psi(x)$ having first and second continuous derivatives in $[0, \infty]$ and approaching zero when $x \rightarrow \infty$, they can be expanded in terms of

$\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(\gamma; z)$:

$$\Psi(x) = \sum_{|\alpha_0|=0}^{\infty} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} A_{|\alpha_0|,|\alpha_1|} \mathcal{QW}_{|\alpha_0|,|\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$$

where,

$$A_{|\alpha_0|,|\alpha_1|} = \int_0^{\infty} dx x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(\gamma; z) \Psi(x) \left[\frac{2^{\gamma-1}}{|\mu|^{\gamma}} \Gamma(|\alpha_0| + 1) \Gamma(|\alpha_0| + \gamma) + \varepsilon \frac{(-1)^{|\alpha_0|} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \right. \\ \left. \times \left\{ \frac{\Gamma(|\alpha_0| + \gamma - \frac{1}{2}) \Gamma(|\alpha_0| + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} - \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{2 \Gamma(k + n + \frac{3}{2} - |\alpha_0|)} \right\} \right]^{-1}$$

Table 2.6: Summary of the zero, one, two and three term of A_n 's that only makes B_n term terminated for polynomial case.

(B) As $\lambda_2 = 1 - \nu$

From eq(2.64)

$$A_n|_{\lambda=1-\nu} = -\frac{\varepsilon(n + \omega - \nu + 1)}{(n + 1)(n + 2 - \nu)} \quad (2.261a)$$

$$B_n|_{\lambda=1-\nu} = -\frac{\Omega + \mu(n - \nu)}{(n + 1)(n + 2 - \nu)} \quad (2.261b)$$

We can describe $g(x)$ as power series by plugging eq(2.39) into eq(2.62).

$$\begin{aligned}
 g(x) &= g(x)_{domin.} + g(x)_{small} = \sum_{n=0}^{\infty} C_{2n} x^{2n+1-\nu} + \sum_{n=0}^{\infty} C_{2n+1} x^{2n+2-\nu} \\
 &= C_0 x^{1-\nu} \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} B_{2k+1} x^{2n} \right\} + C_0 x^{1-\nu} \left\{ A_0 x + (A_0 B_2 + A_2 B_1) x^3 \right. \\
 &\quad \left. + \sum_{n=2}^{\infty} \left[A_0 \prod_{p=0}^{n-1} B_{2p+2} + \sum_{j=1}^{n-1} \left(A_{2j} \prod_{p=j}^{n-1} B_{2p+2} \prod_{k=0}^{j-1} B_{2k+1} \right) + A_{2n} \prod_{p=0}^{n-1} B_{2p+1} \right] x^{2n+1} \right\}
 \end{aligned} \tag{2.262}$$

And there are eigennumbers where $\gamma = \frac{1}{2}(1 + \nu)$ which are

$$-\frac{\Omega}{2\mu} + \gamma - 1 = \psi_0 \quad \text{where } \psi_0 = 1, 2, 3, \dots \tag{2.263a}$$

$$-\frac{\Omega}{2\mu} + \gamma - \frac{3}{2} = \psi_1 \quad \text{where } \psi_1 = 1, 2, 3, \dots \tag{2.263b}$$

Eq(2.263a) makes B_{2k+1} term go to zero at certain value of k where $k = 0, 1, 2, \dots$.

Eq(2.263b) makes B_{2k+2} term go to zero at certain value of k . Eq(2.263a) and

eq(2.263b) make $g(x)$ function as polynomial series. And $\psi_1 \geq \psi_0$ must also be

satisfied. If it doesn't, the function $g(x)$ will be divergent for every x . The $g(x)_{small}$

term has extremely small value, relatively compared to $g(x)_{domin.}$ because it includes

A_n term having $\frac{\epsilon}{2}$.

The first term of $g(x)$ in eq(2.262) is

$$g(x)_{domin.} = C_0 x^{1-\nu} \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} B_{2k+1} x^{2n} \right\} \quad (2.264)$$

Replace n by $2k + 1$ in eq(2.261b). Then plug it into eq(2.264).

$$g(x)_{domin.} = C_0 x^{1-\nu} \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left(-\frac{\mu(\frac{\Omega}{2\mu} + k + \frac{1}{2} - \frac{\nu}{2})}{2(k+1)(k + \frac{3}{2} - \frac{\nu}{2})} \right) x^{2n} \right\} \quad (2.265)$$

Plug eq(2.107a) into eq(2.109).

$$g(x)_{domin.} = C_0 x^{1-\nu} \left\{ 1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left(\frac{\mu(\psi_0 - k)}{2(k+1)(k+2-\gamma)} \right) x^{2n} \right\} \quad (2.266)$$

Eq(2.268) turns out to be

$$g(x)_{domin.} = C_0 x^{2(1-\gamma)} \sum_{n=0}^{\infty} \frac{(-1)^n \psi_0! (1-\gamma)!}{(\psi_0 - n)! (n+1-\gamma)! n!} \left(-\frac{1}{2} \mu x^2 \right)^n \quad (2.267)$$

index n terminates at ψ_0 in eq(2.267). We suggest to put $C_0 = \frac{(-1)^{1-\gamma} \mu^{1-\gamma} (\psi_0+1-\gamma)!}{2^{1-\gamma} (1-\gamma)!}$.

$$\begin{aligned} g(x)_{domin.} &= z^{1-\gamma} A_{\psi_0} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \\ &= \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! n!} z^{n+1-\gamma} \end{aligned} \quad (2.268)$$

The polynomial in eq(2.268) is denoted as $A_{\psi_0} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ is called as the second kind of confluent hypergeometric polynomial surprisingly. The second term of $g(x)$ in eq(2.262) plugging $C_0 = \frac{(-1)^{1-\gamma} \mu^{1-\gamma} (\psi_0+1-\gamma)!}{2^{1-\gamma} (1-\gamma)!}$ is

$$\begin{aligned} g(x)_{small} &= \frac{(-1)^{1-\gamma} \mu^{1-\gamma} (\psi_0 + 1 - \gamma)!}{2^{1-\gamma} (1 - \gamma)!} x^{2(1-\gamma)} \left\{ A_0 x + (A_0 B_2 + A_2 B_1) x^3 \right. \\ &+ \sum_{n=2}^{\infty} \left[A_0 \prod_{p=0}^{n-1} B_{2p+2} + \sum_{j=1}^{n-1} \left(A_{2j} \prod_{p=j}^{n-1} B_{2p+2} \prod_{k=0}^{j-1} B_{2k+1} \right) \right. \\ &\left. \left. + A_{2n} \prod_{p=0}^{n-1} B_{2p+1} \right] x^{2n+1} \right\} \end{aligned} \quad (2.269)$$

where,

$$A_{2j} = -\frac{\varepsilon(\omega + 2j + 1 - \nu)}{(2j + 1)(2j + 2 - \nu)} = -\frac{\varepsilon(\frac{\omega}{2} + j + 1 - \gamma)}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \quad (2.270a)$$

$$A_{2n} = -\frac{\varepsilon(\omega + 2n + 1 - \nu)}{(2n + 1)(2n + 2 - \nu)} = -\frac{\varepsilon(\frac{\omega}{2} + n + 1 - \gamma)}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \quad (2.270b)$$

$$A_0 = -\frac{\varepsilon(\frac{\omega}{2} + 1 - \gamma)}{2(\frac{1}{2})(\frac{3}{2} - \gamma)} \quad (2.270c)$$

$$A_2 = -\frac{\varepsilon(\frac{\omega}{2} + 2 - \gamma)}{2(\frac{3}{2})(\frac{5}{2} - \gamma)} \quad (2.270d)$$

$$B_{2p+1} = -\frac{\mu(\frac{\Omega}{\mu} + 2p + 1 - \nu)}{(2p + 2)(2p + 3 - \nu)} = \frac{\mu(\psi_0 - p)}{2(p + 1)(p + 2 - \gamma)} \quad (2.270e)$$

$$B_{2k+1} = -\frac{\mu(\frac{\Omega}{\mu} + 2k + 1 - \nu)}{(2k + 2)(2k + 3 - \nu)} = \frac{\mu(\psi_0 - k)}{2(k + 1)(k + 2 - \gamma)} \quad (2.270f)$$

$$B_{2p+2} = \frac{-\mu(\frac{\Omega}{\mu} + 2p + 2 - \nu)}{(2p + 3)(2p + 4 - \nu)} = \frac{\mu(\psi_1 - p)}{2(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \quad (2.270g)$$

$$B_1 = \frac{\mu\psi_0}{2(2 - \gamma)} \quad (2.270h)$$

$$B_2 = \frac{\mu\psi_1}{2(\frac{3}{2})(\frac{5}{2} - \gamma)} \quad (2.270i)$$

Substitute eq(2.270a) through eq(2.270i) into eq(2.269)

$$\begin{aligned}
 g(x)_{small} &= -\frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \varepsilon x \left(-\frac{1}{2} \mu x^2 \right)^{1-\gamma} \left\{ \frac{(\frac{\omega}{2} + 1 - \gamma)}{2(\frac{1}{2})(\frac{3}{2} - \gamma)} \right. \\
 &+ \left(\frac{(\frac{\omega}{2} + 1 - \gamma)}{2(\frac{1}{2})(\frac{3}{2} - \gamma)} \frac{\mu \psi_1}{2(\frac{3}{2})(\frac{5}{2} - \gamma)} + \frac{(\frac{\omega}{2} + 2 - \gamma)}{2(\frac{3}{2})(\frac{5}{2} - \gamma)} \frac{\mu \psi_0}{2(2 - \gamma)} \right) x^2 \\
 &+ \sum_{n=2}^{\infty} \left[\frac{(\frac{\omega}{2} + 1 - \gamma)}{2(\frac{1}{2})(\frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{\mu(\psi_1 - p)}{2(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \right. \\
 &+ \sum_{j=1}^{n-1} \left(\frac{(\frac{\omega}{2} + 1 + j - \gamma)}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \prod_{p=j}^{n-1} \frac{\mu(\psi_1 - p)}{2(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \right. \\
 &\times \left. \left. \prod_{k=0}^{j-1} \frac{\mu(\psi_0 - k)}{2(k + 1)(k + 2 - \gamma)} \right) \right. \\
 &+ \left. \frac{(\frac{\omega}{2} + n + 1 - \gamma)}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{\mu(\psi_0 - p)}{2(p + 1)(p + 2 - \gamma)} \right] x^{2n} \Big\} \quad (2.271)
 \end{aligned}$$

Eq(2.271) can be described in the following way.

$$\begin{aligned}
 g(x)_{small} &= -\frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \varepsilon x z^{1-\gamma} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{\omega}{2} + 1 - \gamma)}{2(\frac{1}{2})(\frac{3}{2} - \gamma)} \frac{(-1)^n (\frac{1}{2})! (\frac{3}{2} - \gamma)! (\psi_1)!}{(n + \frac{1}{2})! (n + \frac{3}{2} - \gamma)! (\psi_1 - n)!} \right. \\
 &+ \sum_{n=1}^{\infty} \frac{(\frac{\omega}{2} + n + 1 - \gamma)}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \frac{(-1)^n (1 - \gamma)! (\psi_0)!}{(n)! (n + 1 - \gamma)! (\psi_0 - n)!} \\
 &+ \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(\frac{\omega}{2} + 1 + j - \gamma)}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \\
 &\times \left. \frac{(-1)^n (1 - \gamma)! (\psi_0)! (j + \frac{1}{2})! (j + \frac{3}{2} - \gamma)! (\psi_1 - j)!}{(j)! (j + 1 - \gamma)! (\psi_0 - j)! (n + \frac{1}{2})! (n + \frac{3}{2} - \gamma)! (|\psi_1 - n|)!} \right\} z^n \quad (2.272)
 \end{aligned}$$

where,

$$\gamma = \frac{1}{2}(1 + \nu) \quad (2.273a)$$

$$z = -\frac{1}{2}\mu x^2 \quad (2.273b)$$

(a) $\psi_0 = 0$ in eq(2.272)

$$g(x)_{small} \Big|_{\psi_0=0} = -\varepsilon x z^{1-\gamma} \frac{\Gamma(2-\gamma)}{\Gamma(2-\gamma)} \frac{(\frac{\omega}{2} + 1 - \gamma)}{(\frac{3}{2} - \gamma)} \sum_{n=0}^{\psi_1} \frac{(-1)^n (\frac{1}{2})! (\frac{3}{2} - \gamma)! (\psi_1)!}{(n + \frac{1}{2})! (n + \frac{3}{2} - \gamma)! (\psi_1 - n)!} z^n \quad (2.274)$$

(b) $\psi_0 = 1$ in eq(2.272)

$$g(x)_{small} \Big|_{\psi_0=1} = -\varepsilon x z^{1-\gamma} \frac{\Gamma(3-\gamma)}{\Gamma(2-\gamma)} \left\{ \frac{(\frac{\omega}{2} + 1 - \gamma)}{(\frac{3}{2} - \gamma)} \sum_{n=0}^{\psi_1} \frac{(-1)^n (\frac{1}{2})! (\frac{3}{2} - \gamma)! (\psi_1)!}{(n + \frac{1}{2})! (n + \frac{3}{2} - \gamma)! (\psi_1 - n)!} z^n \right. \\ \left. + \frac{(\frac{\omega}{2} + 2 - \gamma)}{3(\frac{5}{2} - \gamma)} \frac{1}{2(2-\gamma)} \right. \\ \left. \times \sum_{n=1}^{\psi_1} \frac{(-1)^n (3!) (\psi_1 - 1)! (\frac{1}{2})! (\frac{5}{2} - \gamma)!}{2^1 (1!) (n + \frac{1}{2})! (\psi_1 - n)! (n + \frac{3}{2} - \gamma)!} z^n \right\} \quad (2.275)$$

(c) $\psi_0 = 2$ in eq(2.272)

$$\begin{aligned}
 g(x)_{small} \Big|_{\psi_0=2} &= -\varepsilon x z^{1-\gamma} \frac{\Gamma(4-\gamma)}{\Gamma(2-\gamma)} \left\{ \frac{(\frac{\omega}{2} + 1 - \gamma)}{(\frac{3}{2} - \gamma)} \sum_{n=0}^{\psi_1} \frac{(-1)^n (\frac{1}{2})! (\frac{3}{2} - \gamma)! (\psi_1)!}{(n + \frac{1}{2})! (n + \frac{3}{2} - \gamma)! (\psi_1 - n)!} z^n \right. \\
 &+ \frac{(\frac{\omega}{2} + 2 - \gamma) 2(1-\gamma)!}{3(\frac{5}{2} - \gamma) 2(2-\gamma)!} \sum_{n=1}^{\psi_1} \frac{(-1)^n (3!) (\psi_1 - 1)! (\frac{1}{2})! (\frac{5}{2} - \gamma)!}{2^1 (1!) (n + \frac{1}{2})! (\psi_1 - n)! (n + \frac{3}{2} - \gamma)!} z^n \\
 &+ \frac{(\frac{\omega}{2} + 3 - \gamma) 2!(1-\gamma)!}{5(\frac{7}{2} - \gamma) 2^2 2! (3-\gamma)!} \\
 &\times \left. \sum_{n=2}^{\psi_1} \frac{(-1)^n (5!) (\psi_1 - 2)! (\frac{1}{2})! (\frac{7}{2} - \gamma)!}{2^2 (2!) (n + \frac{1}{2})! (\psi_1 - n)! (n + \frac{3}{2} - \gamma)!} z^n \right\}
 \end{aligned}$$

(2.276)

(d) $\psi_0 = 3$ in eq(272)

$$\begin{aligned}
 g(x)_{small} \Big|_{\psi_0=3} &= -\varepsilon x z^{1-\gamma} \frac{\Gamma(5-\gamma)}{\Gamma(2-\gamma)} \left\{ \frac{(\frac{\omega}{2} + 1 - \gamma)}{(\frac{3}{2} - \gamma)} \sum_{n=0}^{\psi_1} \frac{(-1)^n (\frac{1}{2})! (\frac{3}{2} - \gamma)! (\psi_1)!}{(n + \frac{1}{2})! (n + \frac{3}{2} - \gamma)! (\psi_1 - n)!} z^n \right. \\
 &+ \frac{(\frac{\omega}{2} + 2 - \gamma)}{3(\frac{5}{2} - \gamma)} \frac{3!(1-\gamma)!}{2! (2-\gamma)!} \sum_{n=1}^{\psi_1} \frac{(-1)^n (3!) (\psi_1 - 1)! (\frac{1}{2})! (\frac{5}{2} - \gamma)!}{2^1 (1!) (n + \frac{1}{2})! (\psi_1 - n)! (n + \frac{3}{2} - \gamma)!} z^n \\
 &+ \frac{(\frac{\omega}{2} + 3 - \gamma)}{5(\frac{7}{2} - \gamma)} \frac{3!(1-\gamma)!}{2^2 2! 1! (3-\gamma)!} \\
 &\times \sum_{n=2}^{\psi_1} \frac{(-1)^n (5!) (\psi_1 - 2)! (\frac{1}{2})! (\frac{7}{2} - \gamma)!}{2^2 (2!) (n + \frac{1}{2})! (\psi_1 - n)! (n + \frac{3}{2} - \gamma)!} z^n \\
 &+ \frac{(\frac{\omega}{2} + 4 - \gamma)}{7(\frac{9}{2} - \gamma)} \frac{3!(1-\gamma)!}{2^3 3! 0! (4-\gamma)!} \\
 &\left. \times \sum_{n=3}^{\psi_1} \frac{(-1)^n (7!) (\psi_1 - 3)! (\frac{1}{2})! (\frac{9}{2} - \gamma)!}{2^3 (3!) (n + \frac{1}{2})! (\psi_1 - n)! (n + \frac{3}{2} - \gamma)!} z^n \right\} \quad (2.277)
 \end{aligned}$$

We can describe a function $g(x)_{small}$ of degree ψ_0 and ψ_1 according to eq(2.274) through eq(2.277).

$$\begin{aligned}
 g(x)_{small} &= -\varepsilon x \sum_{n=0}^{\psi_0} \frac{(\frac{\omega}{2} + n + 1 - \gamma) (n - \frac{1}{2})! (\psi_0)! (\psi_0 + 1 - \gamma)!}{(n + \frac{3}{2} - \gamma) (n)! (\psi_0 - n)! (n + 1 - \gamma)!} \\
 &\times \sum_{k=n}^{\psi_1} \frac{(-1)^k (n + \frac{3}{2} - \gamma)! (\psi_1 - n)!}{2(k + \frac{1}{2})! (\psi_1 - k)! (k + \frac{3}{2} - \gamma)!} z^{k+1-\gamma} \quad (2.278)
 \end{aligned}$$

As we see from eq(2.278), maximum value of index n is ψ_0 . And the range of index

k is $n \leq k \leq \psi_1$. In other words, $0 \leq n \leq \psi_0 \leq k \leq \psi_1$. Then we obtain $\psi_0 \leq \psi_1$. If $\psi_0 \geq \psi_1$, the function $g(x)$ will be infinite series. Then, the function $y(x)$ will blow up as we plug $g(x)$ into it no matter what the value of x is. Such solution does not exist. When we see the second summation of eq(278), we can shift index k to zero at the beginning of summation. Then replace the interval of index k by $0 \leq k \leq \psi_1 - n$. Eq(2.278) might be described as

$$\begin{aligned}
 g(x)_{small} &= -\frac{\varepsilon}{2}x \sum_{n=0}^{\psi_0} \frac{(-1)^n (\psi_0)! (\psi_0 + 1 - \gamma)!}{n! (\psi_0 - n)! (n + 1 - \gamma)!} z^n \\
 &\times \sum_{k=0}^{\psi_1 - n} \frac{(-1)^k (\frac{\omega}{2} + n + 1 - \gamma) (n - \frac{1}{2})! (n + \frac{1}{2} - \gamma)! (\psi_1 - n)!}{(k + n + \frac{1}{2})! (\psi_1 - k - n)! (k + n + \frac{3}{2} - \gamma)!} z^{k+1-\gamma}
 \end{aligned} \tag{2.279a}$$

$$\begin{aligned}
 &= -\frac{\varepsilon}{2}x \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n \\
 &\times \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} z^{k+1-\gamma}
 \end{aligned} \tag{2.279b}$$

We see in the first summation of eq(2.279b) which is $\frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2-\gamma)_n} z^n$, there is the second kind of confluent hypergeometric polynomial of degree ψ_0 which is

denoted as $A_{\psi_0}(\gamma; z)$ surprisingly. Substitute eq(2.268) and eq(2.279b) into eq(2.262).

$$\begin{aligned} g(x) &= \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \\ &= z^{1-\gamma} \left\{ A_{\psi_0}(\gamma; z) - \frac{\varepsilon}{2} x \mathcal{A}_{\psi_0}^{\psi_1}(\gamma; z) \right\} \quad \text{only if } \psi_0 \leq \psi_1 \end{aligned} \quad (2.280)$$

where,

$$A_{\psi_0}(\gamma; z) = \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! n!} z^n = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n \quad (2.281a)$$

$$\begin{aligned} \mathcal{A}_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n \\ &\times \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} z^k \end{aligned} \quad (2.281b)$$

Eq(2.280) denoted as $\mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ is called as the second kind of Grand confluent hypergeometric polynomial of degree ψ_0 and ψ_1 with the first order of $\frac{\varepsilon}{2}$.

Also, we obtain two eigenvalues from eq(2.263a) and eq(2.263b) in which are

$$\Omega_0^2 = -2\mu(\psi_0 + 1 - \gamma) \quad \text{where } \psi_0 = 0, 1, 2, \dots \quad (2.282a)$$

$$\Omega_1^2 = -2\mu\left(\psi_1 + \frac{3}{2} - \gamma\right) \quad \text{where } \psi_1 = 0, 1, 2, \dots \quad (2.282b)$$

We know the second kind of confluent hypergeometric function is

$$A_{\psi_0}(\gamma; z) = \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)!(n + 1 - \gamma)! n!} z^n = \frac{e^z}{z^{1-\gamma}} \frac{d^{\psi_0}}{dz^{\psi_0}} (z^{\psi_0+1-\gamma} e^{-z}) \quad (2.283)$$

and the generating function of it is

$$\frac{e^{\frac{-zt}{1-t}}}{(1-t)^{2-\gamma}} = \sum_{\psi_0=0}^{\infty} \frac{t^{\psi_0}}{\psi_0!} A_{\psi_0}(\gamma; z) \quad (2.284)$$

Eq(2.281b) might be described in the following way:

$$\Lambda_{\psi_0}^{\psi_1}(\gamma; z) = \sum_{n=0}^{\psi_0} A_n^{\psi_0}(\gamma; z) \Lambda_n^{\psi_1}(\gamma; z) \quad (2.285a)$$

where,

$$A_n^{\psi_0}(\gamma; z) = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \frac{(-\psi_0)_n}{n!(2 - \gamma)_n} z^n \quad (2.285b)$$

$$\bigwedge_n^{\psi_1}(\gamma; z) = \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma)\Gamma(n + \frac{1}{2})\Gamma(n + \frac{3}{2} - \gamma)(n - \psi_1)_k}{\Gamma(k + n + \frac{3}{2})\Gamma(k + n + \frac{5}{2} - \gamma)} z^k \quad (2.285c)$$

(a) Plug $\psi_0 = 0$ into eq(2.280),eq(2.281a), eq(2.285a), eq(2.285b) and eq(2.285c)

$$\begin{aligned} \mathcal{RW}_{0,\psi_1}(\gamma; z) &= z^{1-\gamma} \left\{ A_0(\gamma; z) - \frac{\varepsilon}{2} x \bigwedge_0^{\psi_1}(\gamma; z) \right\} \\ &= z^{1-\gamma} \left\{ 1 - \frac{\varepsilon}{2} x \bigwedge_0^{\psi_1}(\gamma; z) \right\} \\ &= z^{1-\gamma} \left\{ 1 - \frac{\varepsilon}{2} x \sum_{k=0}^{\psi_1} \frac{(\frac{\omega}{2} + 1 - \gamma)\Gamma(\frac{1}{2})\Gamma(\frac{3}{2} - \gamma)(-\psi_1)_k}{\Gamma(k + \frac{3}{2})\Gamma(k + \frac{5}{2} - \gamma)} z^k \right\} \end{aligned} \quad (2.286)$$

By using eq(2.139) into the second term in bracket of eq(2.286) on r.h.s

$$\bigwedge_0^{\psi_1}(\gamma; z) = \left(\frac{\omega}{2} + 1 - \gamma\right) \int_{-1}^1 dp \sum_{k=0}^{\psi_1} \frac{(-1)^k \Gamma(\frac{3}{2} - \gamma) \binom{\psi_1}{k}}{\Gamma(k + \frac{5}{2} - \gamma)} [z(1 - p^2)]^k \quad (2.287)$$

Replace p and q by $\frac{3}{2} - \gamma$ and $k + 1$ in eq(2.138). Substitute it into eq(2.287).

$$\Lambda_0^{\psi_1}(\gamma; z) = \left(\frac{\omega}{2} + 1 - \gamma\right) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2} - \gamma} \sum_{k=0}^{\psi_1} \frac{(-1)^k \binom{\psi_1}{k}}{\Gamma(k+1)} [z(1-t)(1-p^2)]^k \quad (2.288)$$

Replace n and z by ψ_1 and $\lambda = z(1-t)(1-p^2)$ in eq(2.131).

$$L_{\psi_1}(\lambda) = \sum_{k=0}^{\psi_1} \frac{(-1)^k \binom{\psi_1}{k}}{k!} \lambda^k \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.289)$$

Plug eq(2.289) into eq(2.288)

$$\Lambda_0^{\psi_1}(\gamma; z) = \left(\frac{\omega}{2} + 1 - \gamma\right) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2} - \gamma} L_{\psi_1}(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.290)$$

Plug eq(2.290) into eq(2.286)

$$\mathcal{RW}_{0,\psi_1}(\gamma; z) = z^{1-\gamma} \left\{ A_0(\gamma; z) - x \frac{\varepsilon}{2} \left(\frac{\omega}{2} + 1 - \gamma\right) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2} - \gamma} L_{\psi_1}(\lambda) \right\} \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.291)$$

(b) Plug $\psi_0 = 1$ into eq(2.280),eq(2.281a), eq(2.285a), eq(2.285b) and eq(2.285c)

$$\mathcal{RW}_{1,\psi_1}(\gamma; z) = z^{1-\gamma} \left\{ A_1(\gamma; z) - \frac{\varepsilon}{2} x \left\{ A_0^1(\gamma; z) \Lambda_0^{\psi_1}(\gamma; z) + A_1^1(\gamma; z) \Lambda_1^{\psi_1}(\gamma; z) \right\} \right\} \quad (2.292)$$

The third term inside of bracket in eq(2.292) is equal to

$$\Lambda_1^{\psi_1}(\gamma; z) = \frac{1}{2} \left(\frac{\omega}{2} + 2 - \gamma \right) \sum_{k=0}^{\psi_1-1} \frac{(-1)^k \Gamma(\frac{1}{2}) \Gamma(\frac{5}{2} - \gamma) \Gamma(\psi_1)}{\Gamma(k + \frac{5}{2}) \Gamma(k + \frac{7}{2} - \gamma) \Gamma(\psi_1 - k)} z^k \quad (2.293)$$

Replace k by $k + 1$ in eq(2.139).and plug it into eq(2.298)

$$\begin{aligned} \Lambda_1^{\psi_1}(\gamma; z) &= \frac{1}{2} \left(\frac{\omega}{2} + 2 - \gamma \right) \int_{-1}^1 dp (1 - p^2) \\ &\times \sum_{k=0}^{\psi_1-1} \frac{(-1)^k \Gamma(\frac{5}{2} - \gamma) \Gamma(\psi_1)}{\Gamma(k + \frac{7}{2} - \gamma) \Gamma(k + 2) \Gamma(\psi_1 - k)} [z(1 - p^2)]^k \end{aligned} \quad (2.294)$$

Replace p and q by $\frac{5}{2} - \gamma$ and $k + 1$ in eq(2.138). Substitute it into eq(2.294).

$$\begin{aligned} \Lambda_1^{\psi_1}(\gamma; z) &= \frac{1}{2} \left(\frac{\omega}{2} + 2 - \gamma \right) \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\frac{3}{2}-\gamma} \\ &\times \sum_{k=0}^{\psi_1-1} \frac{(-1)^k \Gamma(\psi_1)}{\Gamma(k + 1) \Gamma(k + 2) \Gamma(\psi_1 - k)} [z(1 - p^2)(1 - t)]^k \end{aligned} \quad (2.295)$$

Replace n, k, j and z by $\psi_1 - 1, 1, k$ and $\lambda = z(1-t)(1-p^2)$ in eq(2.134).

$$L_{\psi_1-1}^1(\lambda) = \sum_{k=0}^{\psi_1-1} \frac{(-1)^k \Gamma(\psi_1 + 1)}{\Gamma(\psi_1 - k) \Gamma(k + 1) \Gamma(k + 2)} \lambda^k \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.296)$$

Plug eq(2.296) into eq(2.295).

$$\Lambda_1^{\psi_1}(\gamma; z) = \frac{1}{2\psi_1} \left(\frac{\omega}{2} + 2 - \gamma \right) \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\frac{3}{2}-\gamma} L_{\psi_1-1}^1(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.297)$$

Substitute eq(2.290) and eq(2.297) into eq(2.292).

$$\begin{aligned} \mathcal{RW}_{0,\psi_1}(\gamma; z) &= z^{1-\gamma} \left\{ A_1(\gamma; z) - \frac{\varepsilon}{2} x \left\{ A_0^1(\gamma; z) \left(\frac{\omega}{2} + 1 - \gamma \right) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} L_{\psi_1}(\lambda) \right. \right. \\ &+ A_1^1(\gamma; z) \frac{1}{2\psi_1} \left(\frac{\omega}{2} + 2 - \gamma \right) \\ &\times \left. \left. \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\frac{3}{2}-\gamma} L_{\psi_1-1}^1(\lambda) \right\} \right\} \Big|_{\lambda=z(1-t)(1-p^2)} \end{aligned} \quad (2.298)$$

(c) Plug $\psi_0 = 2$ into eq(2.280),eq(2.281a), eq(2.285a), eq(2.285b) and eq(2.285c)

$$\begin{aligned} \mathcal{RW}_{2,\psi_1}(\gamma; z) &= z^{1-\gamma} \left\{ A_2(\gamma; z) - \frac{\varepsilon}{2} x \left\{ A_0^2(\gamma; z) \mathcal{A}_0^{\psi_1}(\gamma; z) + A_1^2(\gamma; z) \mathcal{A}_1^{\psi_1}(\gamma; z) \right. \right. \\ &\quad \left. \left. + A_2^2(\gamma; z) \mathcal{A}_2^{\psi_1}(\gamma; z) \right\} \right\} \end{aligned} \quad (2.299)$$

The fourth term inside of bracket in eq(2.299) is equal to

$$\mathcal{A}_2^{\psi_1}(\gamma; z) = \frac{1}{2} \frac{3}{2} \left(\frac{\omega}{2} + 3 - \gamma \right) \sum_{k=0}^{\psi_1-2} \frac{(-1)^k \Gamma(\frac{1}{2}) \Gamma(\frac{7}{2} - \gamma) \Gamma(\psi_1 - 1)}{\Gamma(k + \frac{7}{2}) \Gamma(k + \frac{9}{2} - \gamma) \Gamma(\psi_1 - 1 - k)} z^k \quad (2.300)$$

Replace k by $k + 2$ in eq(2.139).and plug it into eq(2.300)

$$\begin{aligned} \mathcal{A}_2^{\psi_1}(\gamma; z) &= \frac{1}{2} \frac{3}{2} \left(\frac{\omega}{2} + 3 - \gamma \right) \int_{-1}^1 dp (1 - p^2)^2 \\ &\quad \times \sum_{k=0}^{\psi_1-2} \frac{(-1)^k \Gamma(\psi_1 - 1) \Gamma(\frac{7}{2} - \gamma)}{\Gamma(k + 3) \Gamma(\psi_1 - 1 - k) \Gamma(k + \frac{9}{2} - \gamma)} [z(1 - p^2)]^k \end{aligned} \quad (2.301)$$

Replace p and q by $\frac{7}{2} - \gamma$ and $k + 1$ in eq(2.138). Substitute it into eq(2.301).

$$\begin{aligned} \mathcal{A}_2^{\psi_1}(\gamma; z) &= \frac{1}{2} \frac{3}{2} \left(\frac{\omega}{2} + 3 - \gamma \right) \int_{-1}^1 dp (1 - p^2)^2 \int_0^1 dt t^{\frac{5}{2}-\gamma} \\ &\quad \times \sum_{k=0}^{\psi_1-2} \frac{(-1)^k \Gamma(\psi_1 - 1)}{\Gamma(k + 1) \Gamma(k + 3) \Gamma(\psi_1 - 1 - k)} [z(1 - p^2)(1 - t)]^k \end{aligned} \quad (2.302)$$

Replace n, k, j and z by $|\alpha_1| - 2, 2, k$ and $\lambda = z(1-t)(1-p^2)$ in eq(2.134).

$$L_{\psi_1-2}^2(\lambda) = \sum_{k=0}^{\psi_1-2} \frac{(-1)^k \Gamma(\psi_1 + 1)}{\Gamma(\psi_1 - 1 - k) \Gamma(k + 1) \Gamma(k + 3)} \lambda^k \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.303)$$

Plug eq(2.303) into eq(2.302).

$$\bigwedge_2^{\psi_1}(\gamma; z) = \frac{1}{2} \frac{3}{2} \frac{1}{\psi_1(\psi_1 - 1)} \left(\frac{\omega}{2} + 3 - \gamma \right) \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\frac{5}{2}-\gamma} L_{\psi_1-2}^2(\lambda) \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.304)$$

Substitute eq(2.290), eq(2.297) and eq(2.304) into eq(2.299).

$$\begin{aligned} \mathcal{RW}_{2,\psi_1}(\gamma; z) &= z^{1-\gamma} \left\{ A_2(\gamma; z) \right. \\ &\quad - \frac{\varepsilon}{2} x \sum_{n=0}^2 \left\{ A_n^2(\gamma; z) \left(\frac{\omega}{2} + 1 + n - \gamma \right) \frac{\Gamma(n + \frac{1}{2}) \Gamma(\psi_1 + 1 - n)}{\Gamma(\frac{1}{2}) \Gamma(\psi_1 + 1)} \right. \\ &\quad \left. \left. \times \int_0^1 dt t^{\frac{1}{2}+n-\gamma} \int_{-1}^1 dp (1-p^2)^n L_{\psi_1-n}^n(\lambda) \right\} \right\} \Big|_{\lambda=z(1-t)(1-p^2)} \quad (2.305) \end{aligned}$$

We can describe a function $\mathcal{RW}_{\psi_0,\psi_1}(\gamma; z)$ of degree ψ_0 and ψ_1 according to eq(2.291), eq(2.298) and eq(2.305).

$$\mathcal{RW}_{\psi_0,\psi_1}(\gamma; z) = z^{1-\gamma} \left\{ A_{\psi_0}(\gamma; z) - \frac{\varepsilon}{2} x \bigwedge_{\psi_0}^{\psi_1}(\gamma; z) \right\} \quad (2.306)$$

where,

$$\begin{aligned} \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \sum_{n=0}^{\psi_0} \left\{ A_n^{\psi_0}(\gamma; z) \left(\frac{\omega}{2} + 1 + n - \gamma \right) \frac{\Gamma(n + \frac{1}{2})\Gamma(\psi_1 + 1 - n)}{\Gamma(\frac{1}{2})\Gamma(\psi_1 + 1)} \right. \\ &\quad \left. \times \int_0^1 dt t^{\frac{1}{2} + n - \gamma} \int_{-1}^1 dp (1 - p^2)^n L_{\psi_1 - n}^n(\lambda) \right\} \Bigg|_{\lambda = z(1-t)(1-p^2)} \end{aligned} \quad (2.307)$$

We know

$$\frac{\Gamma(n + \frac{1}{2})\Gamma(\psi_1 + 1 - n)}{\Gamma(\frac{1}{2})\Gamma(\psi_1 + 1)} = \frac{B(n + \frac{1}{2}, \psi_1 + 1 - n)}{B(\frac{1}{2}, \psi_1 + 1)} \quad (2.308)$$

Substitute eq(2.285b) and eq(2.308) into eq(2.307).

$$\begin{aligned} \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \left\{ \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)!(n + 1 - \gamma)! (n - 1)!} \frac{B(n + \frac{1}{2}, \psi_1 + 1 - n)}{B(\frac{1}{2}, \psi_1 + 1)} \right. \\ &\quad \times \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp [zt(1 - p^2)]^n L_{\psi_1 - n}^n(\lambda) \\ &\quad + \left(\frac{\omega}{2} + 1 - \gamma \right) \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)!(n + 1 - \gamma)! n!} \frac{B(n + \frac{1}{2}, \psi_1 + 1 - n)}{B(\frac{1}{2}, \psi_1 + 1)} \\ &\quad \left. \times \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp [zt(1 - p^2)]^n L_{\psi_1 - n}^n(\lambda) \right\} \Bigg|_{\lambda = z(1-t)(1-p^2)} \end{aligned} \quad (2.309)$$

Replace p and q by $n + \frac{1}{2}$ and $\psi_1 + 1 - n$ into eq(2.138).

$$B(n + \frac{1}{2}, \psi_1 + 1 - n) = \int_0^\infty ds \frac{s^{n - \frac{1}{2}}}{(1 + s)^{\psi_1 + \frac{3}{2}}} = \int_0^1 ds s^{n - \frac{1}{2}} (1 - s)^{\psi_1 - n} \quad (2.310)$$

Plug eq(2.310) into eq(2.309).

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{1}{B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \\
 &\times \left\{ \sum_{n=0}^{\psi_1} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! (n - 1)!} [zts(1 - p^2)]^n L_{\psi_1 - n}^n(\lambda) \right. \\
 &+ \left. \left(\frac{\omega}{2} + 1 - \gamma \right) \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! n!} \right. \\
 &\times \left. [zts(1 - p^2)]^n L_{\psi_1 - n}^n(\lambda) \right\} \Bigg|_{\lambda=z(1-t)(1-p^2)} \quad (2.311)
 \end{aligned}$$

Replace n, m and z by $\psi_1 - n$, n and λ in eq(2.165). Then plug it into eq(2.311).

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{1}{2\pi i B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \\
 &\times \oint du \frac{e^{-\frac{\lambda u}{(1-u)}}}{u^{\psi_1+1} (1-u)} \left\{ zst(1 - p^2) \frac{u}{1-u} \right. \\
 &\times \sum_{n=0}^{\psi_1} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! (n - 1)!} \left[zts(1 - p^2) \frac{u}{(1-u)} \right]^{n-1} \\
 &+ \left. \left(\frac{\omega}{2} + 1 - \gamma \right) \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! n!} \right. \\
 &\times \left. \left[zts(1 - p^2) \frac{u}{(1-u)} \right]^n \right\} \Bigg|_{\lambda=z(1-t)(1-p^2)} \quad (2.312)
 \end{aligned}$$

Replace z and λ by $w_1 = zts(1 - p^2) \frac{u}{(1-u)}$ and $z(1 - t)(1 - p^2)$ into eq(2.281a), and

plug it into eq(2.312).

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{1}{2\pi i B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2} - \gamma} \\
 &\times \int_{-1}^1 dp \oint du \frac{e^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{\psi_1+1}(1-u)} \left\{ w_1 \partial_{w_1} + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} \\
 &\times A_{\psi_0} \left(\gamma; w_1 = zts(1-p^2) \frac{u}{(1-u)} \right) \tag{2.313}
 \end{aligned}$$

Eq(2.313) can be described as various integral forms of several different special function in the following way:

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{\psi_0!}{2\pi i B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2} - \gamma} \\
 &\times \int_{-1}^1 dp \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}} \left\{ \frac{zstv(1-p^2)}{(1-v)} \partial_{w_2} + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} \\
 &\times L_{\psi_1} \left(w_2 = z(1-p^2) \left\{ (1-t) + \frac{tsv}{(1-v)} \right\} \right) \tag{2.314}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{2\psi_0!}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2}-\gamma} \\
 &\times \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}} \oint \frac{du}{u^{\psi_1+1}(1-u)} \left\{ \frac{zstvu}{(1-v)(1-u)} \right. \\
 &\times \left. \left(\frac{1}{3} M\left(1, \frac{5}{2}, w_3\right) - M\left(1, \frac{3}{2}, w_3\right) \right) + \left(\frac{\omega}{2} + 1 - \gamma \right) M\left(1, \frac{3}{2}, w_3\right) \right\} \\
 &\text{where, } w_3 = \frac{zu}{(1-u)} \left\{ - (1-t) - \frac{stv}{(1-v)} \right\} \tag{2.315}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{\psi_0!}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \\
 &\times \int_0^1 dp (1-p^2)^{-\frac{1}{2}} \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}} \\
 &\times \oint \frac{due^{-\frac{zup}{(1-u)}}}{u^{\psi_1+1}(1-u)} \left\{ - \frac{zspvu}{(1-v)(1-u)} \frac{1}{(\frac{5}{2} - \gamma)} M\left(\frac{5}{2} - \gamma, \frac{7}{2} - \gamma, w_4\right) \right. \\
 &+ \left. \frac{\left(\frac{\omega}{2} + 1 - \gamma\right)}{\left(\frac{3}{2} - \gamma\right)} M\left(\frac{3}{2} - \gamma, \frac{5}{2} - \gamma, w_4\right) \right\} \\
 &\text{where, } w_4 = \frac{zup}{(1-u)} \left\{ 1 - \frac{sv}{(1-v)} \right\} \tag{2.316}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{\psi_0! \sqrt{\pi}}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}} \\
 &\times \oint \frac{du e^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{\psi_1+1}(1-u)} \left\{ -\frac{ztvu(1-p^2)}{2(1-v)(1-u)} U\left(\frac{3}{2}, -\psi_1 + 1, w_5\right) \right. \\
 &+ \left. \left(\frac{\omega}{2} + 1 - \gamma\right) U\left(\frac{1}{2}, -\psi_1, w_5\right) \right\} \\
 \text{where, } w_5 &= \frac{ztuv}{(1-v)(1-u)}(1-p^2) \tag{2.317}
 \end{aligned}$$

Integral form of the second kind of confluent hypergeometric function is

$$A_{\psi_0}(z) = \frac{\psi_0!}{2\pi i} \oint dv \frac{e^{-\frac{zv}{(1-v)}}}{v^{\psi_0+1}(1-v)^{2-\gamma}} \tag{2.318}$$

Substitute eq(2.318) into eq(2.313).

$$\begin{aligned}
 \Lambda_{\psi_0}^{\psi_1}(\gamma; z) &= \frac{\psi_0!}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}}(1+s)^{-(\psi_1+\frac{3}{2})} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \\
 &\times \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}} \oint \frac{du}{u^{\psi_1+1}(1-u)} \left\{ -\frac{zstuv(1-p^2)}{(1-u)(1-v)} \right. \\
 &+ \left. \left(\frac{\omega}{2} + 1 - \gamma\right) \right\} e^{-\frac{zu}{(1-u)}(1-p^2)} \left\{ (1-t) + \frac{tsv}{1-v} \right\} \tag{2.319}
 \end{aligned}$$

As described by eq(2.314), eq(2.315), eq(2.316), eq(2.317) as integral formalism, it is exactly equivalent eq(2.319).

$$\mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) = z^{1-\gamma} \left\{ A_{\psi_0}(\gamma; z) - \frac{\varepsilon}{2} x \Lambda_{\psi_0}^{\psi_1}(\gamma; z) \right\}$$

where,

$$A_{\psi_0}(\gamma; z) = \sum_{n=0}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{(\psi_0 - n)! (n + 1 - \gamma)! n!} z^n = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n$$

$$\Lambda_{\psi_0}^{\psi_1}(\gamma; z) = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} z^k$$

$$= \frac{1}{2\pi i B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1 + s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \oint du \frac{e^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{\psi_1 + 1} (1 - u)}$$

$$\times \left\{ w_1 \partial_{w_1} + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} A_{\psi_0} \left(\gamma; w_1 = zts(1 - p^2) \frac{u}{(1 - u)} \right)$$

$$= \frac{\psi_0!}{2\pi i B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1 + s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \oint \frac{dv}{v^{\psi_0 + 1} (1 - v)^{2 - \gamma}}$$

$$\times \left\{ \frac{zstv(1 - p^2)}{(1 - v)} \partial_{w_2} + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} L_{\psi_1} \left(w_2 = z(1 - p^2) \left\{ (1 - t) + \frac{tsv}{(1 - v)} \right\} \right)$$

$$= \frac{2 \psi_0!}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1 + s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2} - \gamma} \oint \frac{dv}{v^{\psi_0 + 1} (1 - v)^{2 - \gamma}}$$

$$\times \oint \frac{du}{u^{\psi_1 + 1} (1 - u)} \left\{ \frac{zstvu}{(1 - v)(1 - u)} \left(\frac{1}{3} M\left(1, \frac{5}{2}, w_3\right) - M\left(1, \frac{3}{2}, w_3\right) \right) \right.$$

$$\left. + \left(\frac{\omega}{2} + 1 - \gamma \right) M\left(1, \frac{3}{2}, w_3\right) \right\} \quad \text{where, } w_3 = \frac{zu}{(1 - u)} \left\{ - (1 - t) - \frac{stv}{(1 - v)} \right\}$$

$$= \frac{\psi_0!}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1 + s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dp (1 - p^2)^{-\frac{1}{2}} \oint \frac{dv}{v^{\psi_0 + 1} (1 - v)^{2 - \gamma}}$$

$$\times \oint \frac{due^{-\frac{zup}{(1-u)}}}{u^{\psi_1 + 1} (1 - u)} \left\{ - \frac{zspvu}{(1 - v)(1 - u)} \frac{1}{(\frac{5}{2} - \gamma)} M\left(\frac{5}{2} - \gamma, \frac{7}{2} - \gamma, w_4\right) \right.$$

$$\left. + \frac{(\frac{\omega}{2} + 1 - \gamma)}{(\frac{3}{2} - \gamma)} M\left(\frac{3}{2} - \gamma, \frac{5}{2} - \gamma, w_4\right) \right\} \quad \text{where, } w_4 = \frac{zup}{(1 - u)} \left\{ 1 - \frac{sv}{(1 - v)} \right\}$$

$$= \frac{\psi_0! \sqrt{\pi}}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \oint \frac{dv}{v^{\psi_0 + 1} (1 - v)^{2 - \gamma}} \oint \frac{due^{-\frac{zu}{(1-u)}(1-t)(1-p^2)}}{u^{\psi_1 + 1} (1 - u)}$$

$$\times \left\{ - \frac{ztku(1 - p^2)}{2(1 - v)(1 - u)} U\left(\frac{3}{2}, -\psi_1 + 1, w_5\right) + \left(\frac{\omega}{2} + 1 - \gamma \right) U\left(\frac{1}{2}, -\psi_1, w_5\right) \right\}$$

$$\text{where, } w_5 = \frac{ztuv}{(1 - v)(1 - u)} (1 - p^2)$$

 Table 2.7: Summarization of the 2nd of grand confluent hypergeometric function

We obtain the integral representation of the second kind of grand confluent hypergeometric function according to eq(2.318) and eq(2.319).

$$\mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) = z^{1-\gamma} \left\{ \frac{\psi_0!}{2\pi i} \oint dv \frac{e^{-\frac{zv}{(1-v)}}}{v^{\psi_0+1}(1-v)^{2-\gamma}} - \frac{\varepsilon}{2} x \Lambda_{\psi_0}^{\psi_1}(\gamma; z) \right\}$$

where,

$$\Lambda_{\psi_0}^{\psi_1}(\gamma; z) = \frac{\psi_0!}{(2\pi i)^2 B(\frac{1}{2}, \psi_1 + 1)} \int_0^\infty ds s^{-\frac{1}{2}} (1+s)^{-(\psi_1 + \frac{3}{2})} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}}$$

$$\times \oint \frac{du}{u^{\psi_1+1}(1-u)} \left\{ -\frac{zstuv(1-p^2)}{(1-u)(1-v)} + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} e^{-\frac{zv}{(1-u)}(1-p^2)} \left\{ (1-t) + \frac{t\omega v}{1-v} \right\}$$

Table 2.8: Integral representation of the second kind of grand confluent hypergeometric function

Let's try to get the generating function of the 2nd kind of grand confluent hypergeometric function. First, multiply $\sum_{\psi_1=\psi_0}^\infty B(\psi_1 + 1, \frac{1}{2}) v_0^{\psi_1}$ on both sides of integral form of 2nd kind of grand confluent hypergeometric function in Table 2.8 where $|v_0| < 1$.

$$\sum_{\psi_1=\psi_0}^\infty B(\psi_1 + 1, \frac{1}{2}) v_0^{\psi_1} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) = z^{1-\gamma} \left\{ A_{\psi_0}(\gamma; z) \text{I} - \frac{\varepsilon}{2} x \text{II} \right\} \quad (2.320)$$

where,

$$I = \sum_{\psi_1=\psi_0}^{\infty} B(\psi_1 + 1, \frac{1}{2}) v_0^{\psi_1} \quad (2.321a)$$

$$II = \sum_{\psi_1=\psi_0}^{\infty} B(\psi_1 + 1, \frac{1}{2}) v_0^{\psi_1} \mathcal{L}_{\psi_0}^{\psi_1}(\gamma; z) \quad (2.321b)$$

Substitute eq(2.178) into eq(2.321a).

$$I = \frac{\Gamma(\psi_0 + 1)\Gamma(\frac{1}{2})}{\Gamma(\psi_0 + \frac{3}{2})} v_0^{\psi_0} {}_2F_1(1, \psi_0 + 1, \psi_0 + \frac{3}{2}, v_0) = \int_0^1 dt (v_0 t)^{\psi_0} (1-t)^{-\frac{1}{2}} (1-v_0 t)^{-1} \quad (2.322)$$

Replace s by $\frac{1}{y} - 1$ in eq(2.319) and substitute it into eq(2.321b).

$$II = \frac{\psi_0!}{2\pi i} \int_0^1 dy (1-y)^{-\frac{1}{2}} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \oint \frac{dv}{v^{\psi_0+1}(1-v)^{2-\gamma}} \quad (2.323)$$

$$\times \left\{ \frac{-zv_0 y v t (1-y)(1-p^2)}{y(1-v_0 y)^2(1-v)} + \frac{(\frac{\psi}{2} + 1 - \gamma)}{(1-v_0 y)} \right\} e^{-\frac{zv_0 y}{1-v_0 y}(1-p^2)} \left\{ 1 - \frac{t(y-v)}{y(1-v)} \right\}$$

Multiply $\sum_{\psi_0=0}^{\infty} \frac{v_1^{\psi_0}}{\psi_0!}$ on both sides of eq(2.320) where $|v_1| < 1$.

$$\sum_{\psi_0=0}^{\infty} \sum_{\psi_1=\psi_0}^{\infty} \frac{B(\psi_1 + 1, \frac{1}{2})}{\psi_0!} v_0^{\psi_1} v_1^{\psi_0} \mathcal{R}\mathcal{W}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$$

$$= z^{1-\gamma} \left\{ A - \frac{\varepsilon}{2} x B \right\} \quad (2.324)$$

where,

$$A = \sum_{\psi_0=0}^{\infty} \frac{v_1^{\psi_0}}{\psi_0!} A_{\psi_0}(z) \text{I} \quad (2.325a)$$

$$B = \sum_{\psi_0=0}^{\infty} \frac{v_1^{\psi_0}}{\psi_0!} \text{II} \quad (2.325b)$$

Plug eq(2.322) into eq(2.325a).

$$A = \int_0^1 dt (1-t)^{-\frac{1}{2}} (1-v_0t)^{-1} \sum_{\psi_0=0}^{\infty} \frac{(v_0v_1t)^{\psi_0}}{\psi_0!} A_{\psi_0}(z) \quad (2.326)$$

Replace t by tv_0v_1 in eq(2.284), and plug it into eq(2.326).

$$\begin{aligned} A &= \int_0^1 dt (1-t)^{-\frac{1}{2}} (1-v_0t)^{-1} (1-v_0v_1t)^{\gamma-2} e^{-\frac{zv_0v_1t}{1-v_0v_1t}} \\ &= \sum_{m=0}^{\infty} \frac{(-zv_0v_1)^m}{m!} \int_0^1 dt t^m (1-t)^{-\frac{1}{2}} (1-v_0t)^{-1} (1-v_0v_1t)^{-(m+2-\gamma)} \\ &= \sum_{n,m=0}^{\infty} \frac{\Gamma(n+m+2-\gamma) (-zv_0v_1)^m (v_0v_1)^n}{n! m! \Gamma(m+2-\gamma)} \\ &\quad \times \int_0^1 dt t^{n+m} (1-t)^{-\frac{1}{2}} (1-v_0t)^{-1} \end{aligned} \quad (2.327)$$

Substitute eq(2.185) into eq(2.327).

$$\begin{aligned}
 A &= \sum_{n,m=0}^{\infty} \frac{B(n+m+1, \frac{1}{2})(m+2-\gamma)_n (-zv_0v_1)^m (v_0v_1)^n}{n! m!} \\
 &\quad \times {}_2F_1\left(1, n+m+1, n+m+\frac{3}{2}, v_0\right) \\
 &= \sum_{n,m,k=0}^{\infty} \frac{B(n+m+k+1, \frac{1}{2})(m+2-\gamma)_n (-z)^m}{n! m!} v_0^{n+m+k} v_1^{n+m} \quad (2.328)
 \end{aligned}$$

Put $\alpha = m+1$, $\gamma = m+\frac{3}{2}$, $\beta = 1$, $\beta' = 2-\gamma+m$, $u = t$, $x = v_0$ and $y = v_0v_1$ in eq(2.188).

$$\begin{aligned}
 &\frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{3}{2})} F_1\left(m+1; 1, 2-\gamma+m; m+\frac{3}{2}; v_0, v_0v_1\right) \\
 &= \int_0^1 dt t^m (1-t)^{-\frac{1}{2}} (1-tv_0)^{-1} (1-tv_0v_1)^{-(2-\gamma+m)} \quad (2.329)
 \end{aligned}$$

Substitute eq(2.329) into eq(2.327).

$$\begin{aligned}
 A &= \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2})(-zv_0v_1)^m}{\Gamma(m+\frac{3}{2})} F_1\left(m+1; 1, 2-\gamma+m; m+\frac{3}{2}; v_0, v_0v_1\right) \\
 &= \sum_{m,j,k=0}^{\infty} \frac{B(m+k+j+1, \frac{1}{2})(m+2-\gamma)_k (-z)^m}{m! k!} v_0^{m+k+j} v_1^{m+k} \quad (2.330)
 \end{aligned}$$

The equation of second line in eq(2.330) is exactly equivalent to the second line of

eq(2.328). Substitute eq(2.323) into eq(2.325b).

$$B = \Gamma_1(y, t, p) + \Gamma_2(y, t, p) \quad (2.331)$$

where

$$\begin{aligned} \Gamma_1(y, t, p) &= \frac{-zv_0v_1}{(1-v_1)^{3-\gamma}} \int_0^1 dy (1-y)^{\frac{1}{2}} (1-v_0y)^{-2} \int_0^1 dt t^{\frac{3}{2}-\gamma} \\ &\times \int_{-1}^1 dp (1-p^2) e^{-\frac{zv_0y}{1-v_0y} \left\{ 1 - \frac{t(y-v_1)}{y(1-v_1)} \right\} (1-p^2)} \end{aligned} \quad (2.332)$$

$$\begin{aligned} \Gamma_2(y, t, p) &= \frac{\left(\frac{\omega}{2} + 1 - \gamma\right)}{(1-v_1)^{2-\gamma}} \int_0^1 dy (1-y)^{-\frac{1}{2}} (1-v_0y)^{-1} \int_0^1 dt t^{\frac{1}{2}-\gamma} \\ &\times \int_{-1}^1 dp e^{-\frac{zv_0y}{1-v_0y} \left\{ 1 - \frac{t(y-v_1)}{y(1-v_1)} \right\} (1-p^2)} \end{aligned} \quad (2.333)$$

By using the first kind of Kummer function,

$$\begin{aligned} \int_{-1}^1 dp (1-p^2) e^{-\eta(1-p^2)} &= 2M\left(1, \frac{3}{2}, -\eta\right) - \frac{2}{3}M\left(1, \frac{5}{2}, -\eta\right) \\ \text{where } \eta &= \frac{zv_0y}{1-v_0y} \left\{ 1 - \frac{t(y-v_1)}{y(1-v_1)} \right\} \end{aligned} \quad (2.334)$$

Plug eq(2.334) into eq(2.332).

$$\Gamma_1(y, t, p) = \gamma_1(y, t) + \gamma_2(y, t) \quad (2.335)$$

$$\begin{aligned} \gamma_1(y, t) &= \frac{-2zv_0v_1}{(1-v_1)^{3-\gamma}} \int_0^1 dy \frac{(1-y)^{\frac{1}{2}}}{(1-v_0y)^2} \int_0^1 dt t^{\frac{3}{2}-\gamma} \\ &\quad \times M\left(1, \frac{3}{2}, -\frac{zv_0y}{1-v_0y} \left\{1 - \frac{t(y-v_1)}{y(1-v_1)}\right\}\right) \end{aligned} \quad (2.336)$$

$$\begin{aligned} \gamma_2(y, t) &= \frac{2zv_0v_1}{3(1-v_1)^{3-\gamma}} \int_0^1 dy \frac{(1-y)^{\frac{1}{2}}}{(1-v_0y)^2} \int_0^1 dt t^{\frac{3}{2}-\gamma} \\ &\quad \times M\left(1, \frac{5}{2}, -\frac{zv_0y}{1-v_0y} \left\{1 - \frac{t(y-v_1)}{y(1-v_1)}\right\}\right) \end{aligned} \quad (2.337)$$

Put $a = 1$, $b = \frac{3}{2}$, $x = -\frac{zv_0y}{1-v_0y}$ and $y = \frac{zv_0}{1-v_0y} \frac{y-v_1}{1-v_1} t$ into eq(2.198a). And plug it into eq(2.336).

$$\begin{aligned} \gamma_1(y, t) &= 2 \sum_{n=0}^{\infty} \frac{(-zv_0v_1)^{n+1}}{(n + \frac{5}{2} - \gamma)(\frac{3}{2})_n (1-v_1)^{n+3-\gamma}} \int_0^1 dy (1-y)^{\frac{1}{2}} (1-v_0y)^{-(n+2)} \\ &\quad \times \left(1 - \frac{y}{v_1}\right)^n M\left(n+1, n + \frac{3}{2}, -\frac{zv_0y}{1-v_0y}\right) \end{aligned} \quad (2.338)$$

Put $a = 1$, $b = \frac{5}{2}$, $x = -\frac{zv_0y}{1-v_0y}$ and $y = \frac{zv_0}{1-v_0y} \frac{y-v_1}{1-v_1} t$ into eq(2.198a). And plug it into

eq(2.337).

$$\begin{aligned} \gamma_2(y, t) &= -\frac{2}{3} \sum_{n=0}^{\infty} \frac{(-zv_0v_1)^{n+1}}{(n + \frac{5}{2} - \gamma)(\frac{5}{2})_n(1 - v_1)^{n+3-\gamma}} \int_0^1 dy (1 - y)^{\frac{1}{2}}(1 - v_0y)^{-(n+2)} \\ &\quad \times \left(1 - \frac{y}{v_1}\right)^n M\left(n + 1, n + \frac{5}{2}, -\frac{zv_0y}{1 - v_0y}\right) \end{aligned} \quad (2.339)$$

We know that

$$M\left(n + 1, n + \frac{3}{2}, \frac{-zv_0y}{1 - v_0y}\right) = \sum_{m=0}^{\infty} \frac{(n + 1)_m(-zv_0)^m}{(n + \frac{3}{2})_m m!} y^m (1 - v_0y)^{-m} \quad (2.340a)$$

$$M\left(n + 1, n + \frac{5}{2}, \frac{-zv_0y}{1 - v_0y}\right) = \sum_{m=0}^{\infty} \frac{(n + 1)_m(-zv_0)^m}{(n + \frac{5}{2})_m m!} y^m (1 - v_0y)^{-m} \quad (2.340b)$$

Plug eq(2.340a) into eq(2.338).

$$\begin{aligned} \gamma_1(y, t) &= 2 \sum_{n,m=0}^{\infty} \frac{(n + 1)_m(-zv_0v_1)^{n+1}(-zv_0)^m}{(n + \frac{5}{2} - \gamma)(\frac{3}{2})_n(n + \frac{3}{2})_m(1 - v_1)^{n+3-\gamma} m!} \\ &\quad \times \int_0^1 dy y^m (1 - y)^{\frac{1}{2}}(1 - v_0y)^{-(n+m+2)} \left(1 - \frac{y}{v_1}\right)^n \end{aligned} \quad (2.341)$$

Plug eq(2.340b) into eq(2.339).

$$\begin{aligned} \gamma_2(y, t) &= -\frac{2}{3} \sum_{n,m=0}^{\infty} \frac{(n+1)_m (-zv_0v_1)^{n+1} (-zv_0)^m}{(n+\frac{5}{2}-\gamma)(\frac{5}{2})_n (n+\frac{5}{2})_m (1-v_1)^{n+3-\gamma} m!} \\ &\quad \times \int_0^1 dy y^m (1-y)^{\frac{1}{2}} (1-v_0y)^{-(n+m+2)} \left(1 - \frac{y}{v_1}\right)^n \end{aligned} \quad (2.342)$$

Put $\alpha = m + 1$, $\beta = n + m + 2$, $\beta' = -n$, $\gamma = m + \frac{5}{2}$, $u = y$, $x = v_0$ and $y = \frac{1}{v_1}$ in eq(2.188). Then plug it into eq(2.341) and eq(2.342).

$$\begin{aligned} \gamma_1(y, t) &= 2 \sum_{n,m=0}^{\infty} \frac{\Gamma(\frac{3}{2})(n+1)_m (-zv_0)^{n+m+1} v_1^{n+1} (1-v_1)^{-(n+3-\gamma)}}{(n+\frac{5}{2}-\gamma)\Gamma(m+\frac{5}{2})(\frac{3}{2})_n (n+\frac{3}{2})_m} \\ &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \end{aligned} \quad (2.343)$$

$$\begin{aligned} \gamma_2(y, t) &= -\frac{2}{3} \sum_{n,m=0}^{\infty} \frac{\Gamma(\frac{3}{2})(n+1)_m (-zv_0)^{n+m+1} v_1^{n+1} (1-v_1)^{-(n+3-\gamma)}}{(n+\frac{5}{2}-\gamma)\Gamma(m+\frac{5}{2})(\frac{5}{2})_n (n+\frac{5}{2})_m} \\ &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \end{aligned} \quad (2.344)$$

Substitute eq(2.343) and eq(2.344) into eq(2.335).

$$\begin{aligned} \Gamma_1(y, t, p) &= \sum_{n,m=0}^{\infty} \frac{(n+m+1)(n+1)_m \Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) (1-v_1)^{-(n+3-\gamma)} (v_1)^{n+1}}{(n+\frac{5}{2}-\gamma) \Gamma(m+\frac{5}{2}) \Gamma(n+m+\frac{5}{2})} \\ &\quad \times (-zv_0)^{n+m+1} F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \end{aligned} \quad (2.345)$$

We have

$$\int_{-1}^1 dp e^{-\eta(1-p^2)} = 2M\left(1, \frac{3}{2}, -\eta\right) \quad \text{where } \eta = \frac{zv_0y}{1-v_0y} \left\{1 - \frac{t(y-v_1)}{y(1-v_1)}\right\} \quad (2.346)$$

Plug eq(2.346) into eq(2.333).

$$\begin{aligned} \Gamma_2(y, t, p) &= \frac{2(\frac{\omega}{2} + 1 - \gamma)}{(1-v_1)^{2-\gamma}} \int_0^1 dy (1-y)^{-\frac{1}{2}} (1-v_0y)^{-1} \int_0^1 dt t^{\frac{1}{2}-\gamma} \\ &\quad \times M\left(1, \frac{3}{2}, -\frac{zv_0y}{1-v_0y} \left\{1 - \frac{t(y-v_1)}{y(1-v_1)}\right\}\right) \end{aligned} \quad (2.347)$$

Put $a = 1$, $b = \frac{3}{2}$, $x = -\frac{zv_0y}{1-v_0y}$ and $y = \frac{zv_0}{1-v_0y} \frac{(y-v_1)}{(1-v_1)} t$ into eq(2.198a). And plug it into

eq(2.347).

$$\begin{aligned}
 \Gamma_2(y, p, t) &= 2\left(\frac{\omega}{2} + 1 - \gamma\right) \sum_{n=0}^{\infty} \frac{(-zv_0v_1)^n}{(n + \frac{3}{2} - \gamma)(\frac{3}{2})_n(1 - v_1)^{n+2-\gamma}} \int_0^1 dy (1 - y)^{-\frac{1}{2}} \\
 &\quad \times (1 - v_0y)^{-(n+1)} \left(1 - \frac{y}{v_1}\right)^n M\left(n + 1, n + \frac{3}{2}, -\frac{zv_0y}{1 - v_0y}\right) \\
 &= 2\left(\frac{\omega}{2} + 1 - \gamma\right) \sum_{n,m=0}^{\infty} \frac{(n + 1)_m (-zv_0v_1)^n (-zv_0)^m}{(n + \frac{3}{2} - \gamma)(\frac{3}{2})_n (n + \frac{3}{2})_m (1 - v_1)^{n+2-\gamma} m!} \\
 &\quad \times \int_0^1 dy y^m (1 - y)^{-\frac{1}{2}} (1 - v_0y)^{-(n+m+1)} \left(1 - \frac{y}{v_1}\right)^n \quad (2.348)
 \end{aligned}$$

Put $\alpha = m + 1$, $\gamma = m + \frac{3}{2}$, $\beta' = -n$, $\beta = n + m + 1$, $u = y$, $x = v_0$ and $y = \frac{1}{v_1}$ in eq(2.188). Then plug it into eq(2.348).

$$\begin{aligned}
 \Gamma_2(y, p, t) &= 2\left(\frac{\omega}{2} + 1 - \gamma\right) \sum_{n,m=0}^{\infty} \frac{\Gamma(\frac{1}{2})(n + 1)_m (-zv_0)^{n+m} (v_1)^n (1 - v_1)^{-(n+2-\gamma)}}{(n + \frac{3}{2} - \gamma)(\frac{3}{2})_n (n + \frac{3}{2})_m \Gamma(m + \frac{3}{2})} \\
 &\quad \times F_1\left(m + 1; n + m + 1, -n; m + \frac{3}{2}; v_0, \frac{1}{v_1}\right) \quad (2.349)
 \end{aligned}$$

Substitute eq(2.345) and eq(2.349) into eq(2.331).

$$\begin{aligned}
 B &= \sum_{n,m=0}^{\infty} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(n+m+1)(n+1)_m(1-v_1)^{-(n+3-\gamma)}(-zv_0)^{n+m+1}(v_1)^{n+1}}{(n+\frac{5}{2}-\gamma)\Gamma(m+\frac{5}{2})\Gamma(n+m+\frac{5}{2})} \\
 &\quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \\
 &+ \left(\frac{\omega}{2} + 1 - \gamma\right) \sum_{n,m=0}^{\infty} \frac{(\Gamma(\frac{1}{2}))^2(n+1)_m(1-v_1)^{-(n+2-\gamma)}(-zv_0)^{n+m}(v_1)^n}{(n+\frac{3}{2}-\gamma)\Gamma(m+\frac{3}{2})\Gamma(n+m+\frac{3}{2})} \\
 &\quad \times F_1\left(m+1; n+m+1, -n; m+\frac{3}{2}; v_0, \frac{1}{v_1}\right) \tag{2.350}
 \end{aligned}$$

Plug eq(2.330) and eq(2.350) into eq(2.324). Then we obtain the generating function for 2nd kind of grand confluent hypergeometric function:

$$\begin{aligned}
 & \sum_{\psi_0=0}^{\infty} \sum_{\psi_1=\psi_0}^{\infty} \frac{B(\psi_1+1, \frac{1}{2})}{\psi_0!} v_0^{\psi_1} v_1^{\psi_0} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1+\nu); z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2})(-zv_0v_1)^m}{\Gamma(m+\frac{3}{2})} F_1\left(m+1; 1, 2-\gamma+m; m+\frac{3}{2}; v_0, v_0v_1\right) \right. \\
 & - \frac{\varepsilon}{2} x \left\{ \sum_{n,m=0}^{\infty} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})(n+m+1)(n+1)_m (1-v_1)^{-(n+3-\gamma)} (-zv_0)^{n+m+1} (v_1)^{n+1}}{(n+\frac{5}{2}-\gamma)\Gamma(m+\frac{5}{2})\Gamma(n+m+\frac{5}{2})} \right. \\
 & \quad \times F_1\left(m+1; n+m+2, -n; m+\frac{5}{2}; v_0, \frac{1}{v_1}\right) \\
 & \quad + \left(\frac{\omega}{2}+1-\gamma\right) \sum_{n,m=0}^{\infty} \frac{(\Gamma(\frac{1}{2}))^2(n+1)_m (1-v_1)^{-(n+2-\gamma)} (-zv_0)^{n+m} (v_1)^n}{(n+\frac{3}{2}-\gamma)\Gamma(m+\frac{3}{2})\Gamma(n+m+\frac{3}{2})} \\
 & \quad \left. \left. \times F_1\left(m+1; n+m+1, -n; m+\frac{3}{2}; v_0, \frac{1}{v_1}\right) \right\} \right\} \\
 &= z^{1-\gamma} \left\{ \sum_{m,j,k=0}^{\infty} \frac{B(m+k+j+1, \frac{1}{2})(m+2-\gamma)_k (-z)^m}{m! k!} v_0^{m+k+j} v_1^{m+k} - \frac{\varepsilon}{2} x \sum_{n,m,j,k,s=0}^{\infty} \left\{ \right. \\
 & \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})\Gamma(n+s+3-\gamma)\Gamma(n+m+j+2)(n+1)_m (m+1)_{j+k} (-n)_k (-z)^{n+m+1}}{s! j! k! (n+m)! (n+\frac{5}{2}-\gamma)\Gamma(m+\frac{5}{2})\Gamma(n+3-\gamma)\Gamma(n+m+\frac{5}{2})(m+\frac{5}{2})_{j+k}} v_0^{n+m+j+1} v_1^{n+s-k+1} \\
 & + \frac{(\Gamma(\frac{1}{2}))^2 (\frac{\omega}{2}+1-\gamma)\Gamma(n+s+2-\gamma)\Gamma(n+m+j+1)(n+1)_m (m+1)_{j+k} (-n)_k (-z)^{n+m}}{s! j! k! (n+m)! (n+\frac{3}{2}-\gamma)\Gamma(m+\frac{3}{2})\Gamma(n+2-\gamma)\Gamma(n+m+\frac{3}{2})(m+\frac{3}{2})_{j+k}} \\
 & \left. \left. \times v_0^{n+m+j} v_1^{n+s-k} \right\} \right\}
 \end{aligned}$$

Table 2.9: The generating function for the second kind of grand confluent hypergeometric function

We can describe the generating function of it as the integral formalism by using eq(2.327), eq(2.332) and eq(2.333):

$$\begin{aligned}
 & \sum_{\psi_0=0}^{\infty} \sum_{\psi_1=\psi_0}^{\infty} \frac{B(\psi_1+1, \frac{1}{2})}{\psi_0!} v_0^{\psi_1} v_1^{\psi_0} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1+\nu); z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \left\{ \int_0^1 dt (1-t)^{-\frac{1}{2}} (1-v_0 t)^{-1} (1-v_0 v_1 t)^{\gamma-2} e^{-\frac{z v_0 v_1 t}{1-v_0 v_1 t}} \right. \\
 & \quad - \frac{\varepsilon}{2} \frac{x}{(1-v_1)^{2-\gamma}} \int_0^1 dy (1-y)^{-\frac{1}{2}} (1-v_0 y)^{-1} \\
 & \quad \left. \times \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \left\{ \left(\frac{\omega}{2} + 1 - \gamma \right) - \frac{z v_0 v_1 t (1-y)}{(1-v_1)(1-v_1 y)} \right\} e^{-\frac{z v_0 y}{1-v_0 y}} \left\{ 1 - \frac{t(y-v_1)}{y(1-v_1)} \right\}^{(1-p^2)} \right\}
 \end{aligned}$$

Table 2.10: Integral formalism of the generating function for the second kind of grand confluent hypergeometric function

From the differential equations satisfied by $\mathcal{RW}_{\psi_0, \psi_1}(z)$ and $\mathcal{RW}_{\phi_0, \phi_1}(z)$, namely,

$$x \mathcal{RW}_{\psi_0, \psi_1}''(z) + (\mu x^2 + \varepsilon x + \nu) \mathcal{RW}_{\psi_0, \psi_1}'(z) + (\Omega_{\psi_0, \psi_1} x + \varepsilon \omega) \mathcal{RW}_{\psi_0, \psi_1}(z) = 0 \quad (2.351a)$$

$$x \mathcal{RW}_{\phi_0, \phi_1}''(z) + (\mu x^2 + \varepsilon x + \nu) \mathcal{RW}_{\phi_0, \phi_1}'(z) + (\Omega_{\phi_0, \phi_1} x + \varepsilon \omega) \mathcal{RW}_{\phi_0, \phi_1}(z) = 0 \quad (2.351b)$$

where,

$$\Omega_{\psi_0, \psi_1} = \begin{cases} -2\mu(\psi_0 + 1 - \gamma) \\ -2\mu(\psi_1 + \frac{3}{2} - \gamma) \end{cases} \quad \Omega_{\phi_0, \phi_1} = \begin{cases} -2\mu(\phi_0 + 1 - \gamma) \\ -2\mu(\phi_1 + \frac{3}{2} - \gamma) \end{cases} \quad (2.352)$$

multiplying eq(2.351a) and eq(2.351b) by $\mathcal{RW}_{\phi_0, \phi_1}(z)$ and $\mathcal{RW}_{\psi_0, \psi_1}(z)$ respectively and subtracting, we have

$$x \frac{dN(z)}{dx} + (\mu x^2 + \varepsilon x + \nu)N(z) = (\Omega_{\phi_0, \phi_1} - \Omega_{\psi_0, \psi_1})x \mathcal{RW}_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) \quad (2.353)$$

where

$$N(z) = \mathcal{RW}'_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) - \mathcal{RW}'_{\phi_0, \phi_1}(z) \mathcal{RW}_{\psi_0, \psi_1}(z) \quad (2.354)$$

Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x}$ on both side of eq(2.353).

$$\frac{d}{dx} \left\{ x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} N(z) \right\} = (\Omega_{\phi_0, \phi_1} - \Omega_{\psi_0, \psi_1}) x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) \quad (2.355)$$

Integrate eq(2.355) with respect to x from 0 to ∞ .

$$(\Omega_{\phi_0, \phi_1} - \Omega_{\psi_0, \psi_1}) \int_0^{\infty} dx x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) = \left[x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} N(z) \right] \Big|_0^{\infty} \quad (2.356)$$

Substitute eq(2.354) into eq(2.356).

$$\begin{aligned} & (\Omega_{\phi_0, \phi_1} - \Omega_{\psi_0, \psi_1}) \int_0^{\infty} dx x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) \\ &= \left[x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} (\mathcal{RW}'_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) - \mathcal{RW}'_{\phi_0, \phi_1}(z) \mathcal{RW}_{\psi_0, \psi_1}(z)) \right] \Big|_0^{\infty} \end{aligned} \quad (2.357)$$

Therefore, only if $\psi_0 \neq \phi_0$ and $\psi_1 \neq \phi_1$, then

$$\int_0^{\infty} dx x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) = 0 \quad (2.358)$$

Let's think about the case in which are $\psi_0 = \phi_0$ and $\psi_1 = \phi_1$. First of all, from

eq(2.280)

$$\begin{aligned}
 \left[\mathcal{RW}_{\psi_0, \psi_1}(z) \right]^2 &= z^{2(1-\gamma)} \left[A_{\psi_0}(z) \right]^2 - \varepsilon x z^{2(1-\gamma)} A_{\psi_0}(z) \bigwedge_{\psi_0}^{\psi_1}(z) \\
 &\quad + \frac{\varepsilon^2}{4} x^2 z^{2(1-\gamma)} \left[\bigwedge_{\psi_0}^{\psi_1}(z) \right]^2 \\
 &\simeq z^{2(1-\gamma)} \left[A_{\psi_0}(z) \right]^2 - \varepsilon x z^{2(1-\gamma)} A_{\psi_0}(z) \bigwedge_{\psi_0}^{\psi_1}(z) \quad (2.359)
 \end{aligned}$$

ε is extremely small. So more than second order of ε is negligible in eq(2.359).

Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x}$ in eq(2.359) and integrate them with respect to $x = [0, \infty]$.

$$\int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{RW}_{\psi_0, \psi_1}(z) \right]^2 = B_1 - \varepsilon B_2 \quad (2.360)$$

where

$$\begin{aligned}
 B_1 &= \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} z^{2(1-\gamma)} \left[A_{\psi_0}(z) \right]^2 \\
 &= \left(-\frac{\mu}{2} \right)^{2(1-\gamma)} \int_0^\infty dx x^{-2(\gamma - \frac{3}{2})} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[A_{\psi_0}(z) \right]^2 \quad (2.361a)
 \end{aligned}$$

$$B_2 = \left(-\frac{\mu}{2} \right)^{2(1-\gamma)} \int_0^\infty dx x^{2(2-\gamma)} e^{\frac{1}{2}\mu x^2 + \varepsilon x} A_{\psi_0}(z) \bigwedge_{\psi_0}^{\psi_1}(z) \quad (2.361b)$$

Replaced t by v_0 in eq(2.284). Then square of both side of it:

$$\frac{e^{-\frac{2zv_0}{(1-v_0)}}}{(1-v_0)^{2(2-\gamma)}} = \sum_{\psi_0=0}^{\infty} \frac{v_0^{2\psi_0}}{(\psi_0!)^2} [A_{\psi_0}(z)]^2 \quad (2.362)$$

Multiply $x^{-2(\gamma-\frac{3}{2})}e^{\frac{1}{2}\mu x^2+\varepsilon x}$ on both side of eq(2.362), and integrate them with respect to $x = [0, \infty]$.

$$\begin{aligned} \text{I} &= (1-v_0)^{2(\gamma-2)} \int_0^{\infty} dx x^{-2(\gamma-\frac{3}{2})} e^{\frac{1}{2}\mu x^2+\varepsilon x} e^{-\frac{2zv_0}{(1-v_0)}} \\ &= \sum_{\psi_0=0}^{\infty} \frac{v_0^{2\psi_0}}{(\psi_0!)^2} \int_0^{\infty} dx x^{-2(\gamma-\frac{3}{2})} e^{\frac{1}{2}\mu x^2+\varepsilon x} [A_{\psi_0}(z)]^2 \end{aligned} \quad (2.363)$$

We know $\mu = \text{real negative}$. Plug $z = \frac{1}{2}|\mu|x^2$ into the l.h.s of eq(2.363).

$$\text{I} = (1-v_0)^{2(\gamma-2)} e^{\frac{\varepsilon^2}{2|\mu|\sigma}} \int_0^{\infty} dx x^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|(x-\frac{\varepsilon}{|\mu|\sigma})^2} \quad \text{where } \sigma = \frac{(1+v_0)}{(1-v_0)} \quad (2.364)$$

We know that ε is so smalll, so we can neglect any terms which include more than second order of ε . We can say that $e^{\frac{\varepsilon^2}{2|\mu|\sigma}} \approx 1.$, and lets say $K = \frac{\varepsilon}{|\mu|\sigma}$. Plug it into eq(2.364).

$$\text{I} \approx (1-v_0)^{2(\gamma-2)} \int_0^{\infty} dx x^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma(x-K)^2} \quad (2.365)$$

Replace $x - K$ by r in eq(2.365).

$$I \approx (1 - v_0)^{2(\gamma-2)} \left\{ \int_0^\infty dr (r + K)^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} + \int_{-K}^0 dr (r + K)^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} \right\} \quad (2.366)$$

By using binomial expansion,

$$(r + K)^{-2(\gamma-\frac{3}{2})} = \sum_{k=0}^{\infty} \frac{(-1)^k (2(\gamma-2) + k)!}{k! (2(\gamma-2))!} r^{-(k+2(\gamma-\frac{3}{2}))} K^k \quad \text{where } |r| < K \quad (2.367)$$

K includes the first order of ε . Neglecting any terms which include more than the second order of ε (extremely small value), then eq(2.367) is approximately equal to

$$(r + K)^{-2(\gamma-\frac{3}{2})} \approx r^{-2(\gamma-\frac{3}{2})} - (2\gamma - 3)r^{-2(\gamma-1)}K \quad (2.368)$$

Plug eq(2.368) into eq(2.366).

$$I \approx (1 - v_0)^{2(\gamma-2)} \left\{ \int_0^\infty dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma - 3)K \int_0^\infty dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} + \int_{-K}^0 dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma - 3)K \int_{-K}^0 dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} \right\} \quad (2.369)$$

Replace r by $\sqrt{\frac{2}{|\mu|\sigma}} t$ in the first and second terms of rhs in eq(2.369).

$$\begin{aligned}
 & \int_0^\infty dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma-3)K \int_0^\infty dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} \\
 = & \left(\frac{2}{|\mu|\sigma}\right)^{2-\gamma} \int_0^\infty dt t^{2(2-\gamma)-1} e^{-t^2} \\
 & -2\left(\gamma-\frac{3}{2}\right) \left(\frac{2}{|\mu|\sigma}\right)^{-(\gamma-\frac{3}{2})} K \int_0^\infty dt t^{2(\frac{3}{2}-\gamma)-1} e^{-t^2} \tag{2.370}
 \end{aligned}$$

By using integral form of gamma function, eq(2.370) is equal to

$$\begin{aligned}
 & \int_0^\infty dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma-3)K \int_0^\infty dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} \\
 = & 2^{1-\gamma} |\mu|^{2-\gamma} \Gamma(2-\gamma) \sigma^{2-\gamma} + K \left(\frac{2}{|\mu|\sigma}\right)^{\frac{3}{2}-\gamma} \Gamma\left(\frac{5}{2}-\gamma\right) \sigma^{-(\frac{3}{2}-\gamma)} \\
 & \text{where } \operatorname{Re}(\gamma) < \frac{3}{2} \tag{2.371}
 \end{aligned}$$

By using binomial expansion, the third and fourth terms of rhs in eq(2.369) are

$$\begin{aligned}
 & \int_{-K}^0 dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma-3)K \int_{-K}^0 dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} \tag{2.372} \\
 = & \left\{ \sum_{j=0}^\infty \frac{(-1)^{j+1} (|\mu|)^j \sigma^j}{2^{j+1} (j+2-\gamma) j!} + \left(\frac{3}{2}-\gamma\right) \sum_{j=0}^\infty \frac{(-1)^j (|\mu|)^j \sigma^j}{2^{j+2} (j+\frac{3}{2}-\gamma) j!} \right\} K^{2(j+2-\gamma)}
 \end{aligned}$$

The lowest order of K in eq(2.372) is

$$\int_{-K}^0 dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma - 3)K \int_{-K}^0 dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} \approx -\frac{\gamma}{2^2(2-\gamma)} (K)^{2(2-\gamma)} \quad (2.373)$$

We know that $\text{Re}(\gamma) < \frac{3}{2}$. The first order of ε is only countable because of extremely small value of ε . So eq(2.373) is negligible.

$$\int_{-K}^0 dr r^{-2(\gamma-\frac{3}{2})} e^{-\frac{1}{2}|\mu|\sigma r^2} - (2\gamma - 3)K \int_{-K}^0 dr r^{-2(\gamma-1)} e^{-\frac{1}{2}|\mu|\sigma r^2} \approx 0 \quad (2.374)$$

Substitute eq(2.371) and eq(2.374) into eq(2.369).

$$I \approx (1 - v_0)^{2(\gamma-2)} \left\{ 2^{1-\gamma} |\mu|^{2-\gamma} \Gamma(2-\gamma) \sigma^{2-\gamma} + K \left(\frac{2}{|\mu|} \right)^{\frac{3}{2}-\gamma} \Gamma\left(\frac{5}{2}-\gamma\right) \sigma^{-(\frac{3}{2}-\gamma)} \right\} \quad (2.375)$$

Plug $K = \frac{\varepsilon}{|\mu|\sigma}$ and $\sigma = \frac{(1+v_0)}{(1-v_0)}$ in eq(2.375).

$$\begin{aligned} I &\approx 2^{1-\gamma} |\mu|^{2-\gamma} \Gamma(2-\gamma) (1+v_0)^{2-\gamma} (1-v_0)^{-3(2-\gamma)} \\ &+ \varepsilon 2^{\frac{3}{2}-\gamma} |\mu|^{-(\frac{5}{2}-\gamma)} \Gamma\left(\frac{5}{2}-\gamma\right) (1+v_0)^{-(\frac{5}{2}-\gamma)} (1-v_0)^{-(\frac{3}{2}-\gamma)} \end{aligned} \quad (2.376)$$

By using binomial expansion, eq(2.376) is equivalent to

$$\begin{aligned}
 I &\approx \sum_{\psi_0=0}^{\infty} \frac{v_0^{2\psi_0}}{(\psi_0!)^2} \left\{ 2^{1-\gamma} |\mu|^{2-\gamma} \frac{\Gamma(2-\gamma)B(3-\gamma, \psi_0)}{B(6-3\gamma, \psi_0)} \right. \\
 &\quad \left. + \varepsilon 2^{\frac{3}{2}-\gamma} |\mu|^{-(\frac{5}{2}-\gamma)} \frac{(-1)^{\psi_0} \Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \right\} \quad (2.377)
 \end{aligned}$$

Eq(2.377) is equivalent to eq(2.363). Then, we obtain

$$\begin{aligned}
 B_1 &= \left(\frac{|\mu|}{2}\right)^{2(1-\gamma)} \int_0^{\infty} dx x^{2(\frac{3}{2}-\gamma)} e^{-\frac{1}{2}|\mu|x^2 + \varepsilon x} \left[A_{\psi_0}(z)\right]^2 \\
 &= \frac{2^{\gamma-1}}{|\mu|^{3\gamma-4}} \frac{\Gamma(2-\gamma)B(3-\gamma, \psi_0)}{B(6-3\gamma, \psi_0)} \\
 &\quad + \varepsilon \frac{2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \frac{(-1)^{\psi_0} \Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \quad (2.378)
 \end{aligned}$$

Replace z by $\frac{1}{2}|\mu|x^2$ in eq(2.361b).

$$B_2 = \left(\frac{|\mu|}{2}\right)^{2(1-\gamma)} \int_0^{\infty} dx x^{2(2-\gamma)} e^{-\frac{1}{2}|\mu|x^2 + \varepsilon x} A_{\psi_0}(z) \Lambda_{\psi_0}^{\psi_1}(z) \quad (2.379)$$

Substitute eq(2.281b) into eq(2.379).

$$\begin{aligned}
 B_2 &= \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{|\mu|^n (-\psi_0)_n}{2^n n! (2 - \gamma)_n} \\
 &\times \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) |\mu|^k \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{2^k \Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} \\
 &\times \left(\frac{|\mu|}{2}\right)^{2(1-\gamma)} \int_0^\infty dx x^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|x^2 + \varepsilon x} A_{\psi_0}(z) \quad (2.380)
 \end{aligned}$$

Replace t and z by v_0 and $\frac{1}{2}|\mu|x^2$ in eq(2.284). Multiply $x^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|x^2 + \varepsilon x}$ on both side of it, and integrate them with respect to $x = [0, \infty]$.

$$\begin{aligned}
 \text{II} &= (1 - v_0)^{\gamma-2} \int_0^\infty dx x^{2(n+k+2-\gamma)} e^{-\frac{1}{2}\frac{|\mu|}{(1-v_0)}x^2 + \varepsilon x} \\
 &= \sum_{\psi_0=0}^{\infty} \frac{v_0^{\psi_0}}{\psi_0!} \int_0^\infty dx x^{n+k+2-\gamma} e^{-\frac{1}{2}|\mu|x^2 + \varepsilon x} A_{\psi_0}(z) \quad (2.381)
 \end{aligned}$$

The lhs of eq(2.381) is approximately equal to

$$\begin{aligned}
 \text{II} &= (1 - v_0)^{\gamma-2} e^{\varepsilon^2 \frac{(1-v_0)}{2|\mu|}} \int_0^\infty dx x^{n+k+2-\gamma} e^{-\frac{1}{2}\frac{|\mu|}{(1-v_0)}\left(x - \varepsilon \frac{(1-v_0)}{|\mu|}\right)^2} \\
 &\approx (1 - v_0)^{\gamma-2} \int_0^\infty dx x^{n+k+2-\gamma} e^{-\frac{1}{2}\frac{|\mu|}{(1-v_0)}\left(x - \varepsilon \frac{(1-v_0)}{|\mu|}\right)^2} \quad (2.382)
 \end{aligned}$$

Lets set $J = \varepsilon \frac{1}{|\mu|\rho}$ where $\rho = \frac{1}{(1-v_0)}$. And replace x by $s + J$ in eq(2.382).

$$\text{II} \approx (1-v_0)^{\gamma-2} \left\{ \int_0^\infty ds (s+J)^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} + \int_{-J}^0 ds (s+J)^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \right\} \quad (2.383)$$

By using binomial expansion,

$$(s+J)^{2(n+k+2-\gamma)} = \sum_{m=0}^{\infty} \frac{(2(n+k+2-\gamma))!}{(2(n+k+2-\gamma)-m)! m!} s^{2(n+k+2-\gamma)-m} J^m \quad (2.384)$$

J includes the first order of ε . Larger than the second order of ε is negligible because of too small value of it. Then eq(2.384) is approximately equal to

$$(s+J)^{2(n+k+2-\gamma)} \approx s^{2(n+k+2-\gamma)} + 2(n+k+2-\gamma) s^{2(n+k+\frac{3}{2}-\gamma)} J \quad (2.385)$$

Plug eq(2.385) into eq(2.383).

$$\begin{aligned} \text{II} = & (1-v_0)^{\gamma-2} \left\{ \int_0^\infty ds s^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} + 2(n+k+2-\gamma) J \right. \\ & \times \int_0^\infty ds s^{2(n+k+\frac{3}{2}-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} + \int_{-J}^0 ds s^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \\ & \left. + 2(n+k+2-\gamma) J \int_{-J}^0 ds s^{2(n+k+\frac{3}{2}-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \right\} \quad (2.386) \end{aligned}$$

By using gamma function, the first and second terms in eq(2.386) where $\text{Re}(\gamma) < \frac{5}{2}$ are equal to

$$\begin{aligned}
 & (1 - v_0)^{\gamma-2} \left\{ \int_0^\infty ds s^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \right. \\
 & \left. + 2(n+k+2-\gamma)J \int_0^\infty ds s^{2(n+k+\frac{3}{2}-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \right\} \quad (2.387) \\
 & = \frac{1}{2} \left(\frac{2}{|\mu|} \right)^{n+k+\frac{5}{2}-\gamma} \frac{\Gamma\left(n+k-\gamma+\frac{5}{2}\right)}{(1-v_0)^{-(n+k+\frac{1}{2})}} + \frac{\varepsilon}{|\mu|} \left(\frac{2}{|\mu|} \right)^{n+k+2-\gamma} \frac{\Gamma(n+k+3-\gamma)}{(1-v_0)^{-(n+k+1)}}
 \end{aligned}$$

By using binomial expansion, the third and fourth terms of rhs in eq(2.386) are

$$\begin{aligned}
 & \int_{-J}^0 ds s^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} + 2(n+k+2-\gamma)J \int_{-J}^0 ds s^{2(n+k+\frac{3}{2}-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \\
 & = \sum_{j=0}^{\infty} \frac{(-1)^{j-2\gamma} (|\mu|\rho)^j J^{2(n+k+j+\frac{5}{2}-\gamma)}}{2^j j!} \\
 & \times \left\{ \frac{1}{2(n+k+j+\frac{5}{2}-\gamma)} - \frac{(n+k+2-\gamma)}{(n+k+j+2-\gamma)} \right\} \quad (2.388)
 \end{aligned}$$

The first order of ε is only countable because of very small value of ε . Then eq(2.388) is negligible because of $\text{Re}(n+k+\frac{5}{2}-\gamma) > 0$.

$$\int_{-J}^0 ds s^{2(n+k+2-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} + 2(n+k+2-\gamma)J \int_{-J}^0 ds s^{2(n+k+\frac{3}{2}-\gamma)} e^{-\frac{1}{2}|\mu|\rho s^2} \approx 0 \quad (2.389)$$

Substitute eq(2.387) and eq(2.389) into eq(2.386).

$$\begin{aligned} \text{II} &\approx \frac{1}{2} \left(\frac{2}{|\mu|} \right)^{n+k+\frac{5}{2}-\gamma} \Gamma\left(n+k-\gamma+\frac{5}{2}\right) (1-v_0)^{(n+k+\frac{1}{2})} \\ &\quad + \frac{\varepsilon}{|\mu|} \left(\frac{2}{|\mu|} \right)^{n+k+2-\gamma} \Gamma(n+k+3-\gamma) (1-v_0)^{(n+k+1)} \end{aligned} \quad (2.390)$$

By using binomial expansion, eq(2.390) is equivalent to

$$\begin{aligned} \text{II} &\approx \sum_{\psi_0=0}^{\infty} \frac{v_0^{\psi_0}}{\psi_0!} \left\{ \frac{1}{2} \left(\frac{2}{|\mu|} \right)^{n+k+\frac{5}{2}-\gamma} \frac{(-1)^{\psi_0} \Gamma\left(n+k+\frac{3}{2}\right) \Gamma\left(n+k-\gamma+\frac{5}{2}\right)}{\Gamma\left(n+k-\psi_0+\frac{3}{2}\right)} \right. \\ &\quad \left. + \frac{\varepsilon}{|\mu|} \left(\frac{2}{|\mu|} \right)^{n+k+2-\gamma} \frac{(-1)^{\psi_0} \Gamma(n+k+3-\gamma) \Gamma(n+k+2)}{\Gamma(n+k+2-\psi_0)} \right\} \end{aligned} \quad (2.391)$$

Eq(2.391) is equivalent to eq(2.381). Then we obtain

$$\begin{aligned} &\int_0^{\infty} dx x^{n+k+2-\gamma} e^{-\frac{1}{2}|\mu|x^2+\varepsilon x} A_{\psi_0}(z) \\ &= \frac{1}{2} \left(\frac{2}{|\mu|} \right)^{n+k+\frac{5}{2}-\gamma} \frac{(-1)^{\psi_0} \Gamma\left(n+k+\frac{3}{2}\right) \Gamma\left(n+k-\gamma+\frac{5}{2}\right)}{\Gamma\left(n+k-\psi_0+\frac{3}{2}\right)} \\ &\quad + \frac{\varepsilon}{|\mu|} \left(\frac{2}{|\mu|} \right)^{n+k+2-\gamma} \frac{(-1)^{\psi_0} \Gamma(n+k+3-\gamma) \Gamma(n+k+2)}{\Gamma(n+k+2-\psi_0)} \end{aligned} \quad (2.392)$$

Substitute eq(2.392) into eq(2.380).

$$\begin{aligned}
 B_2 &= \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2 - \gamma)_n} \\
 &\times \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma)\Gamma(n + \frac{1}{2})\Gamma(n + \frac{3}{2} - \gamma)(n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2})\Gamma(n + k + \frac{5}{2} - \gamma)} \\
 &\times \left\{ \frac{1}{2} \left(\frac{2}{|\mu|} \right)^{\gamma + \frac{1}{2}} \frac{(-1)^{\psi_0} \Gamma(n + k + \frac{3}{2}) \Gamma(n + k - \gamma + \frac{5}{2})}{\Gamma(n + k - \psi_0 + \frac{3}{2})} \right. \\
 &\left. + \frac{\varepsilon}{|\mu|} \left(\frac{2}{|\mu|} \right)^{\gamma} \frac{(-1)^{\psi_0} \Gamma(n + k + 3 - \gamma) \Gamma(n + k + 2)}{\Gamma(n + k + 2 - \psi_0)} \right\} \quad (2.393)
 \end{aligned}$$

Substitute eq(2.378) and eq(2.393) into eq(2.360).

$$\begin{aligned}
 &\int_0^{\infty} dx x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{RW}_{\psi_0, \psi_1}(z) \right]^2 \\
 &= \frac{2^{\gamma-1}}{|\mu|^{3\gamma-4}} \frac{\Gamma(2 - \gamma) B(3 - \gamma, \psi_0)}{B(6 - 3\gamma, \psi_0)} + \varepsilon \frac{2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \frac{(-1)^{\psi_0} \Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \\
 &- \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2 - \gamma)_n} \\
 &\times \sum_{k=0}^{\psi_1 - n} \frac{(\frac{\omega}{2} + n + 1 - \gamma)\Gamma(n + \frac{1}{2})\Gamma(n + \frac{3}{2} - \gamma)(n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2})\Gamma(n + k + \frac{5}{2} - \gamma)} \\
 &\times \left\{ \frac{\varepsilon}{2} \left(\frac{2}{|\mu|} \right)^{\gamma + \frac{1}{2}} \frac{(-1)^{\psi_0} \Gamma(n + k + \frac{3}{2}) \Gamma(n + k - \gamma + \frac{5}{2})}{\Gamma(n + k - \psi_0 + \frac{3}{2})} \right. \\
 &\left. + \frac{\varepsilon^2}{|\mu|} \left(\frac{2}{|\mu|} \right)^{\gamma} \frac{(-1)^{\psi_0} \Gamma(n + k + 3 - \gamma) \Gamma(n + k + 2)}{\Gamma(n + k + 2 - \psi_0)} \right\} \quad (2.394)
 \end{aligned}$$

More than second order in ε is negligible because of extremely small value of it. Then, eq(2.394) is simply approximately equal to

$$\begin{aligned}
 & \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{RW}_{\psi_0, \psi_1}(z) \right]^2 \\
 &= \frac{|\mu|^{4-3\gamma} \Gamma(2-\gamma) B(3-\gamma, \psi_0)}{2^{1-\gamma} B(6-3\gamma, \psi_0)} + \varepsilon \frac{(-1)^{\psi_0} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \right. \\
 & \quad - \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2-\gamma)_n} \\
 & \quad \left. \times \sum_{k=0}^{\psi_1-n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2} - \psi_0)} \right\} \quad (2.395)
 \end{aligned}$$

Therefore, we obtain orthogonal relation according to eq(2.358) and eq(2.395):

$$\int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(z) \mathcal{RW}_{\phi_0, \phi_1}(z) = \left\{ \frac{|\mu|^{4-3\gamma} \Gamma(2-\gamma) B(3-\gamma, \psi_0)}{2^{1-\gamma} B(6-3\gamma, \psi_0)} + \varepsilon \frac{(-1)^{\psi_0} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \right. \right. \\
 \left. \left. - \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2-\gamma)_n} \sum_{k=0}^{\psi_1-n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2} - \psi_0)} \right\} \right\} \delta_{\psi_0, \phi_0} \delta_{\psi_1, \phi_1}$$

Table 2.11: The orthogonal relation for the 2nd kind of grand confluent hypergeometric function

If there is an analytic function $\Phi(x)$ with interval $x = [0, \infty]$ which is

$$\Phi(x) = \sum_{\psi_0=0}^{\infty} \sum_{\psi_1=\psi_0}^{\infty} B_{\psi_0, \psi_1} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \quad (2.396)$$

Multiply $x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ on both sides of eq(2.396) and integrate it with respect to $x=[0, \infty]$.

$$\begin{aligned} & \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \Phi(x) \\ &= \sum_{\psi_0=0}^{\infty} \sum_{\psi_1=\psi_0}^{\infty} B_{\psi_0, \psi_1} \int_0^\infty x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \left[\mathcal{RW}_{\psi_0, \psi_1}(\gamma; z) \right]^2 \end{aligned} \quad (2.397)$$

Substitute eq(2.395) into eq(2.397).

$$\begin{aligned} B_{\psi_0, \psi_1} &= \int_0^\infty dx x^\nu e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(\gamma; z) \Phi(x) \left[\frac{|\mu|^{4-3\gamma} \Gamma(2-\gamma) B(3-\gamma, \psi_0)}{2^{1-\gamma} B(6-3\gamma, \psi_0)} \right. \\ &+ \varepsilon \frac{(-1)^{\psi_0} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \right. \\ &- \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2-\gamma)_n} \\ &\left. \left. \times \sum_{k=0}^{\psi_1-n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2} - \psi_0)} \right\} \right]^{-1} \end{aligned} \quad (2.398)$$

If there is an analytic function $\Phi(x)$ having first and second continuous derivatives in $[0, \infty]$ and approaching zero when $x \rightarrow \infty$, it can be expanded in terms of $\mathcal{RW}_{\psi_0, \psi_1}(\gamma; z)$:

$$\Phi(x) = \sum_{\psi_0=0}^{\infty} \sum_{\psi_1=\psi_0}^{\infty} B_{\psi_0, \psi_1} \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$$

where,

$$B_{\psi_0, \psi_1} = \int_0^{\infty} dx x^{\nu} e^{\frac{1}{2}\mu x^2 + \varepsilon x} \mathcal{RW}_{\psi_0, \psi_1}(\gamma; z) \Phi(x) \left[\frac{|\mu|^{4-3\gamma}}{2^{1-\gamma}} \frac{\Gamma(2-\gamma)B(3-\gamma, \psi_0)}{B(6-3\gamma, \psi_0)} \right. \\ \left. + \varepsilon \frac{(-1)^{\psi_0} 2^{\gamma-\frac{1}{2}}}{|\mu|^{\gamma+\frac{1}{2}}} \left\{ \frac{\Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{5}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \right. \right. \\ \left. \left. - \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n!(2-\gamma)_n} \sum_{k=0}^{\psi_1-n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2} - \psi_0)} \right\} \right]^{-1}$$

Table 2.12: A complete set of orthonormal function for the 2nd kind of grand confluent hypergeometric function

2.1.2 $\nu = \text{integer}$

When ν is integer, one of two solution of the grand confluent hypergeometric equation does not have any meaning, because $y_2(x) = \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ can be described as $\mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right)$ as long as $|\lambda_1 - \lambda_2| = |\nu - 1| =$

integer. If $\nu = 1$, the two solutions are identical. Then two solutions can be solved by Case B) of theorem 1 in the appendix, and if ν is any integer except 1, then it can be calculated by Case C) and Case D) of theorem 1 in the appendix. Now let's investigate the second independent solution as ν being an integer.

(i) As $\nu =$ negative integer, 0, and 1

Theorem 1. $P(x)\frac{\partial^2 G}{\partial x^2} + Q(x)\frac{\partial G}{\partial x} + R(x)G = 0$

Suppose 0 is a regular singular point ; $G(x, \lambda) = \sum_{n=0}^{\infty} C_n(\lambda)x^{n+\lambda}$

(1)As $\lambda_1 \leq \lambda_2$: distinct roots of the indicial equation.

Case A) If $\lambda_1 - \lambda_2$ is not an integer, there are 2 independent Frobenius solutions

$$g_1(x) = G(x, \lambda) \Big|_{\lambda=\lambda_1} \qquad g_2(x) = G(x, \lambda) \Big|_{\lambda=\lambda_2}$$

Case B) If $\lambda_1 - \lambda_2 = 0$, there is a Frobenius solution

$$g_1(x) = G(x, \lambda) \Big|_{\lambda=\lambda_1} \qquad g_2(x) = \frac{\partial G(x, \lambda)}{\partial \lambda} \Big|_{\lambda=\lambda_1}$$

Case C) If $\lambda_1 - \lambda_2$ is an integer,

$$g_1(x) = G(x, \lambda) \Big|_{\lambda=\lambda_1} \qquad g_2(x) = \frac{\partial((\lambda-\lambda_2)G(x, \lambda))}{\partial \lambda} \Big|_{\lambda=\lambda_2}$$

(2)As $\lambda_2 < \lambda_1$: distinct roots of the indicial equation.

Case D) If $\lambda_2 - \lambda_1$ is not an integer,

$$g_1(x) = G(x, \lambda) \Big|_{\lambda=\lambda_2} \qquad g_2(x) = G(x, \lambda) \Big|_{\lambda=\lambda_1}$$

Case E) If $\lambda_2 - \lambda_1$ is an integer,

$$g_1(x) = G(x, \lambda) \Big|_{\lambda=\lambda_2} \qquad g_2(x) = \frac{\partial((\lambda-\lambda_1)G(x,\lambda))}{\partial\lambda} \Big|_{\lambda=\lambda_1}$$

Since ν is smaller than 1, $\lambda_2 = 1 - \nu$ is positive integer. So $\lambda_1 = 0$ is smaller than λ_2 . By using the Case C) in Theorem 1, $g_1(x) = G(x)|_{\lambda=\lambda_1=0}$ is equal to $\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2)$. Also, since ν is equal to 1, λ_1 is equal to λ_2 . So by using the Case B) in Theorem 1, $g_1(x) = G(x)|_{\lambda=\lambda_1=0}$ is equal to $\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2)$. It is same as before. Only difference is that in the previous case ν was equal to 0 or negative integer, and in this case ν is equal to 1. Now, let assume that the second independent solution is

$$g_2(x) = u(x)\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2) \tag{2.399}$$

Substitute eq(2.399) into the grand Confluent hypergeometric differential equation

$$\begin{aligned} & x (u(x)\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z))'' + (\mu x^2 + \varepsilon x + \nu) (u(x)\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z))' \\ & + (\Omega x + \varepsilon\omega) (u(x)\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)) = 0 \end{aligned} \tag{2.400}$$

When we calculate and simplify eq(2.400), it becomes

$$\frac{u''(x)}{u'(x)} + 2 \frac{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)'}{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)} + \left(\mu x + \varepsilon + \frac{\nu}{x}\right) = 0 \quad (2.401)$$

By the transformation $R(x) = u'(x) = \frac{du(x)}{dx}$ & $R'(x) = \frac{dR(x)}{dx} = u''(x)$, eq(2.401) turns to be

$$\frac{1}{R(x)} dR(x) + \frac{2}{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)} d\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) + \left(\mu x + \varepsilon + \frac{\nu}{x}\right) dx = 0 \quad (2.402)$$

Integrate the l.h.s. of eq(2.402) with respect to $R(x)$ and also integrate the r.h.s. on it with respect to x

$$R(x) = \frac{du(x)}{dx} = C_1 \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}^2(z)} \quad ; C_1 = \text{arbitrary constant} \quad (2.403)$$

Again, integrate eq(2.403) with respect x

$$u(x) = C_1 \int \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}^2(z)} dx + C_0 \quad ; C_0 = \text{arbitrary constant} \quad (2.404)$$

Substitute eq(2.404) into eq(2.399)

$$g_2(x) = C_1 \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \int \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}^2(z)} dx + C_0 \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \quad (2.405)$$

For simplicity, let say $C_0 = 0$ & $C_1 = 1$. Then,

$$\boxed{g_2(x) = \mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z) \int \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}_{|\alpha_0|,|\alpha_1|}^2(z)} dx} \quad (2.406)$$

The eq(2.406) is called "The first kind integral formalism of the second independent solution of the grand confluent hypergeometric function when $\nu =$ negative integer, 0 and 1."

(ii) Case of $\nu =$ positive integer except 1

When ν is greater than 1, $\lambda_2 = 1 - \nu$ is negative integer. So λ_2 is smaller than λ_1 , by using the Case E) in theorem 1, $g_1(x) = G(x)|_{\lambda=\lambda_2=1-\nu}$ is equal to $\mathcal{RW}_{\psi_0,\psi_1}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right)$. Now, let assume that the second independent solution is

$$g_2(x) = u(x)g_1(x) = u(x)\mathcal{RW}_{\psi_0,\psi_1}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \quad (2.407)$$

Substitute eq(2.407) into the grand confluent hypergeometric differential equation

$$\begin{aligned} & x \left(u(x) \mathcal{RW}_{\psi_0, \psi_1}(z) \right)'' + (\mu x^2 + \varepsilon x + \nu) \left(u(x) \mathcal{RW}_{\psi_0, \psi_1}(z) \right)' \\ & + (\Omega x + \varepsilon \omega) \left(u(x) \mathcal{RW}_{\psi_0, \psi_1}(z) \right) = 0 \end{aligned} \quad (2.408)$$

When we calculate eq(2.408), it turns to be

$$\begin{aligned} & u(x) \left\{ x \mathcal{RW}_{\psi_0, \psi_1}''(z) + (\mu x^2 + \varepsilon x + \nu) \mathcal{RW}_{\psi_0, \psi_1}'(z) + (\Omega x + \varepsilon \omega) \mathcal{RW}_{\psi_0, \psi_1}(z) \right\} \\ & + x \left\{ u''(x) \mathcal{RW}_{\psi_0, \psi_1}(z) + 2u'(x) \mathcal{RW}_{\psi_0, \psi_1}'(z) \right\} \\ & + (\mu x^2 + \varepsilon x + \nu) u'(x) \mathcal{RW}_{\psi_0, \psi_1}(z) = 0 \end{aligned} \quad (2.409)$$

From eq(2.351a), terms in the first bracket in eq(2.409) are equal to zero. Hence it is simply

$$x \left\{ u''(x) \mathcal{RW}_{\psi_0, \psi_1}(z) + 2u'(x) \mathcal{RW}_{\psi_0, \psi_1}'(z) \right\} + (\mu x^2 + \varepsilon x + \nu) u'(x) \mathcal{RW}_{\psi_0, \psi_1}(z) = 0 \quad (2.410)$$

Divide eq(2.410) by $x u(x)' \mathcal{RW}_{\psi_0, \psi_1}(z)$

$$\frac{u''(x)}{u'(x)} + 2 \frac{\mathcal{RW}_{\psi_0, \psi_1}(z)'}{\mathcal{RW}_{\psi_0, \psi_1}(z)} + \left(\mu x + \varepsilon + \frac{\nu}{x} \right) = 0 \quad (2.411)$$

By using simple algebra in eq(2.411), we obtain

$$u(x) = Q_1 \int \frac{x^{-\nu} e^{-(\frac{1}{2}\mu x^2 + \varepsilon x)}}{\mathcal{RW}_{\psi_0, \psi_1}^2(z)} dx + Q_0 \quad ; Q_0, Q_1 = \text{arbitrary constant} \quad (2.412)$$

Substitute eq(2.412) into eq(2.407)

$$g_2(x) = Q_1 \mathcal{RW}_{\psi_0, \psi_1}(z) \int \frac{x^{-\nu} e^{-(\frac{1}{2}\mu x^2 + \varepsilon x)}}{\mathcal{RW}_{\psi_0, \psi_1}^2(z)} dx + Q_0 \mathcal{RW}_{\psi_0, \psi_1}(z) \quad (2.413)$$

For simplicity, let say $Q_0 = 0$ & $Q_1 = 1$. Then,

$$\boxed{g_2(x) = \mathcal{RW}_{\psi_0, \psi_1}(z) \int \frac{x^{-\nu} e^{-(\frac{1}{2}\mu x^2 + \varepsilon x)}}{\mathcal{RW}_{\psi_0, \psi_1}^2(z)} dx} \quad (2.414)$$

The eq(2.414) is called "The second kind integral formalism of the second independent solution of the grand confluent hypergeometric function as $\nu =$ positive integer except 1." Eq(2.406) and eq(2.414) are inconvenient for direct calculation, as the integrand

contains $[\mathcal{QW}_{|\alpha_0|,|\alpha_1|}(z)]^{-2}$ and $[\mathcal{RW}_{\psi_0,\psi_1}(z)]^{-2}$. Now let's investigate eq(2.406) and eq(2.414) more precisely and see how these two analytic functions have power series. These two functions can be described by the Frobenius method.

(A) the case of $\nu = 0$ or negative integer

As ν is 0 or negative integer, $\lambda_2 = 1 - \nu$ is a positive integer. So $\lambda_1 = 0$ is smaller than λ_2 . By using Case C) in theorem 1, the first independent solution is

$$g_1(x) = G(x)|_{\lambda=\lambda_1=0} = \mathcal{QW}_{|\alpha_0|,|\alpha_1|}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \quad (2.415)$$

The second independent solution is

$$g_2(x) = g_1(x)\ln x + x^{\lambda_2} \sum_{n=0}^{\infty} b_n(\lambda_2)x^n \quad (2.416)$$

To generate this solution, keep the recurrence formula in term of λ and use it to find to coefficients $C_n(n \geq 1)$ in terms of both λ and C_0 , where coefficient C_0 remains arbitrary. Substitute these C_n into $G(x) = x^\lambda \sum_{n=0}^{\infty} C_n x^n$ to obtain a function $G(\lambda, x)$ which depends on the variables λ & x . From the Case C) in theorem 1, the second

independent solution is

$$g_2(x) = \left. \frac{\partial G(\lambda, x)}{\partial \lambda} \right|_{\lambda=\lambda_2} = \ln x \sum_{n=0}^{\infty} C_n x^{n+\lambda} \Big|_{\lambda=\lambda_2} + \sum_{n=0}^{\infty} \frac{\partial C_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=\lambda_2} \quad (2.417)$$

From eq(2.262), $G(x) = g(x)_{domin.} + g(x)_{small}$. Plug it into eq(2.417).

$$g_2(x) = \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=\lambda_2} + \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=\lambda_2} \quad (2.418)$$

The first term of r.h.s in eq(2.418) is

$$\begin{aligned} \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=1-\nu} &= \left. \frac{\partial}{\partial \lambda} \left\{ \sum_{n=0}^{\infty} C_{2n}(\lambda, n) x^{2n+\lambda} \right\} \right|_{\lambda=1-\nu} \\ &= \ln x \sum_{n=0}^{\infty} C_{2n}(\lambda, n) x^{2n+\lambda} \Big|_{\lambda=1-\nu} + \sum_{n=0}^{\infty} \frac{\partial C_{2n}(\lambda, n)}{\partial \lambda} x^{2n+\lambda} \Big|_{\lambda=1-\nu} \end{aligned} \quad (2.419a)$$

The second term of r.h.s in eq(2.418) is

$$\begin{aligned} \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=1-\nu} &= \left. \frac{\partial}{\partial \lambda} \left\{ \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \right\} \right|_{\lambda=1-\nu} \\ &= \ln x \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=1-\nu} \\ &\quad + \sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \Big|_{\lambda=1-\nu} \end{aligned} \quad (2.419b)$$

We know that $g(x)_{domin.}$ as $\lambda = 1 - \nu$ is equal to the second kind of confluent hypergeometric polynomial. So a solution for eq(2.419a) is similar to Remark 2 on appendix. Replace α , λ and λ_2 by $-\psi_0 - 1 + \gamma$, $\frac{\lambda}{2}$ and $\frac{\lambda_2}{2}$ in eq(2.182) on appendix. Then differentiate it with respect to λ . After that, plug $\lambda = 2(1 - \gamma)$ instead of $(1 - \gamma)$, and substitute it to see what we get in eq(2.181) on the appendix. Then we obtain

$$\begin{aligned} \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=1-\nu} &= (1 + \ln(x))z^{1-\gamma}A_{\psi_0}(\gamma; z) \\ &+ \sum_{n=1}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{2 n!(n + 1 - \gamma)! (\psi_0 - n)!} \\ &\times \sum_{k=0}^{n-1} \left\{ \frac{1}{(k - \psi_0)} - \frac{1}{(k + 2 - \gamma)} - \frac{1}{(k + 1)} \right\} z^{n+1-\gamma} \end{aligned} \quad (2.420)$$

The first summation of eq(2.419b) which is $\sum_{n=0}^{\infty} C_{2n+1}(\lambda, n)x^{2n+1+\lambda} \Big|_{\lambda=1-\nu}$ is equivalent

to eq(2.279b). Then we have

$$\begin{aligned}
 & \ln x \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=1-\nu} = -\frac{\varepsilon}{2} x \ln x z^{1-\gamma} \bigwedge_{\psi_0}^{\psi_1}(\gamma; z) \\
 &= -\frac{\varepsilon}{2} x \ln x \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n!(2 - \gamma)_n} \\
 & \quad \times \sum_{k=0}^{\psi_1-n} \frac{(\frac{\omega}{2} + n + 1 - \gamma) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} z^{k+1-\gamma}
 \end{aligned} \tag{2.421}$$

According to eq(2.262), we have

$$\left\{ \begin{array}{l} C_1 = C_0 A_0 \\ C_3 = C_0 (A_0 B_2 + A_2 B_1) \\ C_{2n+1} = C_0 \left\{ A_0 \prod_{p=0}^{n-1} B_{2p+2} + \sum_{j=1}^{n-1} \left(A_{2j} \prod_{p=j}^{n-1} B_{2p+2} \prod_{k=0}^{j-1} B_{2k+1} \right) + A_{2n} \prod_{p=0}^{n-1} B_{2p+1} \right\}; n \geq 2 \end{array} \right. \tag{2.422}$$

From eq(2.64), we get

$$A_n = -\frac{\varepsilon(n + \lambda + \omega)}{(n + 1 + \lambda)(n + \lambda + \nu)} = -\frac{\varepsilon(n + \lambda + \omega)}{(n + 1 + \lambda)(n - 1 + 2\gamma + \lambda)} \tag{2.423a}$$

$$B_n = -\frac{\mu(n-1+\lambda) + \Omega}{(n+1+\lambda)(n+\lambda+\nu)} = -\frac{\mu(n-1+\lambda) + \Omega}{(n+1+\lambda)(n-1+2\gamma+\lambda)} \quad (2.423b)$$

According to eq(2.263a) and eq(2.263b), $\Omega = -2\mu\left(\psi_0 + \frac{1}{2} - \frac{\nu}{2}\right) = -2\mu(\psi_0 + 1 - \gamma)$

as n is odd in eq(2.423b). When n is even, $\Omega = -2\mu\left(\psi_1 + 1 - \frac{\nu}{2}\right) = -2\mu\left(\psi_1 + \frac{3}{2} - \gamma\right)$.

where,

$$A_{2j} = -\frac{\varepsilon\left(j + \frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(j + \frac{1}{2} + \frac{\lambda}{2}\right)\left(j + \gamma - \frac{1}{2} + \frac{\lambda}{2}\right)} \quad (2.424a)$$

$$A_{2n} = -\frac{\varepsilon\left(n + \frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(n + \frac{1}{2} + \frac{\lambda}{2}\right)\left(n + \gamma - \frac{1}{2} + \frac{\lambda}{2}\right)} \quad (2.424b)$$

$$A_0 = -\frac{\varepsilon\left(\frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(\frac{1}{2} + \frac{\lambda}{2}\right)\left(\gamma - \frac{1}{2} + \frac{\lambda}{2}\right)} \quad (2.424c)$$

$$A_2 = -\frac{\varepsilon\left(1 + \frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(\frac{3}{2} + \frac{\lambda}{2}\right)\left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)} \quad (2.424d)$$

$$B_{2p+1} = -\frac{\mu(2p+\lambda) + \Omega}{(2(p+1)+\lambda)(2p+1+\lambda+\nu)} = \mu \frac{\psi_0 - p + 1 - \gamma - \frac{\lambda}{2}}{2(p+1+\frac{\lambda}{2})(p+\gamma+\frac{\lambda}{2})} \quad (2.424e)$$

$$B_{2k+1} = -\frac{\mu(2k+\lambda) + \Omega}{(2(k+1)+\lambda)(2k+1+\lambda+\nu)} = \mu \frac{\psi_0 - k + 1 - \gamma - \frac{\lambda}{2}}{2(k+1+\frac{\lambda}{2})(k+\gamma+\frac{\lambda}{2})} \quad (2.424f)$$

$$B_{2p+2} = -\frac{\mu(2p+1+\lambda) + \Omega}{(2p+3+\lambda)(2p+2+\nu+\lambda)} = \mu \frac{\psi_1 - p + 1 - \gamma - \frac{\lambda}{2}}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \quad (2.424g)$$

$$B_1 = \mu \frac{\psi_0 + 1 - \gamma - \frac{\lambda}{2}}{2\left(1 + \frac{\lambda}{2}\right)\left(\gamma + \frac{\lambda}{2}\right)} \quad (2.424h)$$

$$B_2 = \mu \frac{\psi_1 + 1 - \gamma - \frac{\lambda}{2}}{2(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \quad (2.424i)$$

Substitute eq(2.424a) through eq(2.424i) into eq(2.422).

$$C_1 = -\frac{\varepsilon(\frac{\omega}{2} + \frac{\lambda}{2})}{2(\frac{1}{2} + \frac{\lambda}{2})(\gamma - \frac{1}{2} + \frac{\lambda}{2})} C_0 \quad (2.425a)$$

$$\begin{aligned} C_3 = & -\frac{\varepsilon}{2} \mu \left(\frac{(\frac{\omega}{2} + \frac{\lambda}{2})}{(\frac{1}{2} + \frac{\lambda}{2})(\gamma - \frac{1}{2} + \frac{\lambda}{2})} \frac{\psi_1 + 1 - \gamma - \frac{\lambda}{2}}{2(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \right. \\ & \left. + \frac{(1 + \frac{\omega}{2} + \frac{\lambda}{2})}{(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \frac{\psi_0 + 1 - \gamma - \frac{\lambda}{2}}{2(1 + \frac{\lambda}{2})(\gamma + \frac{\lambda}{2})} \right) C_0 \end{aligned} \quad (2.425b)$$

$$\begin{aligned} C_{2n+1} = & -\varepsilon \left\{ \frac{(\frac{\omega}{2} + \frac{\lambda}{2})}{2(\frac{1}{2} + \frac{\lambda}{2})(\gamma - \frac{1}{2} + \frac{\lambda}{2})} \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \right. \\ & + \sum_{j=1}^{n-1} \left(\frac{(j+\frac{\omega}{2}+\frac{\lambda}{2})}{2(j+\frac{1}{2}+\frac{\lambda}{2})(j+\gamma-\frac{1}{2}+\frac{\lambda}{2})} \prod_{p=j}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \right. \\ & \times \left. \prod_{k=0}^{j-1} \frac{-\mu(k-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(k+1+\frac{\lambda}{2})(k+\gamma+\frac{\lambda}{2})} \right) \\ & \left. + \frac{(n+\frac{\omega}{2}+\frac{\lambda}{2})}{2(n+\frac{1}{2}+\frac{\lambda}{2})(n+\gamma-\frac{1}{2}+\frac{\lambda}{2})} \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(p+1+\frac{\lambda}{2})(p+\gamma+\frac{\lambda}{2})} \right\} C_0 \end{aligned} \quad (2.425c)$$

Differentiate eq(2.425a), eq(2.425b) and eq(2.425c) with respect to λ and plugging

$$\lambda = 1 - \nu = 2(1 - \gamma).$$

$$\left. \frac{\partial C_1}{\partial \lambda} \right|_{\lambda=1-\nu} = -\varepsilon C_0 \frac{(1 - \omega) - (3 - 2\gamma)(2 - 2\gamma + \omega)}{(3 - 2\gamma)^2} \quad (2.426a)$$

$$\begin{aligned} \left. \frac{\partial C_3}{\partial \lambda} \right|_{\lambda=1-\nu} &= -\varepsilon C_0 \left\{ \left(1 - \frac{2(2 - \gamma)(2 - 2\gamma + \omega)}{(3 - 2\gamma)} \right) \frac{2\mu\psi_1}{3(3 - 2\gamma)(5 - 2\gamma)} \right. \\ &\quad - \left(1 + \frac{2^2(4 - \gamma)\psi_1}{3(5 - 2\gamma)} \right) \frac{\mu(2 - 2\gamma + \omega)}{3(3 - 2\gamma)(5 - 2\gamma)} \\ &\quad + \left(1 - \frac{2(4 - \gamma)(4 - 2\gamma + \omega)}{3(5 - 2\gamma)} \right) \frac{\mu\psi_0}{2 \cdot 3(2 - \gamma)(5 - 2\gamma)} \\ &\quad \left. - \left(1 + \frac{(3 - \gamma)\psi_0}{(2 - \gamma)} \right) \frac{\mu(4 - 2\gamma + \omega)}{2^2 \cdot 3(2 - \gamma)(5 - 2\gamma)} \right\} \quad (2.426b) \end{aligned}$$

$$\begin{aligned}
 \left. \frac{\partial C_{2n+1}}{\partial \lambda} \right|_{\lambda=1-\nu} &= -\varepsilon C_0 \left\{ \frac{1}{(3-2\gamma)} \left(1 - \frac{2(2-\gamma)(2-2\gamma+\omega)}{(3-2\gamma)} \right) \prod_{p=0}^{n-1} \frac{-\mu(p-\psi_1)}{2(p+\frac{5}{2}-\gamma)(p+\frac{3}{2})} \right. \\
 &+ \sum_{j=1}^{n-1} \left[\frac{1}{(2j+3-2\gamma)(2j+1)} \left(1 - \frac{2(2j+2-\gamma)(2j+2-2\gamma+\omega)}{(2j+3-2\gamma)(2j+1)} \right) \right. \\
 &\times \left. \prod_{p=j}^{n-1} \frac{-\mu(p-\psi_1)}{2(p+\frac{5}{2}-\gamma)(p+\frac{3}{2})} \prod_{k=0}^{j-1} \frac{-\mu(k-\psi_0)}{2(k+2-\gamma)(k+1)} \right] \\
 &+ \frac{1}{(2n+3-2\gamma)(2n+1)} \left(1 - \frac{2(2n+2-\gamma)(2n+2-2\gamma+\omega)}{(2n+3-2\gamma)(2n+1)} \right) \\
 &\times \prod_{p=0}^{n-1} \frac{-\mu(p-\psi_0)}{2(p+2-\gamma)(p+1)} \\
 &+ \left[\frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \right\} \right. \\
 &+ \sum_{j=1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{3}{2}-\gamma)(j+\frac{1}{2})} \frac{\partial}{\partial \lambda} \left\{ \prod_{p=j}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \right. \\
 &\times \left. \prod_{k=0}^{j-1} \frac{-\mu(k-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(k+1+\frac{\lambda}{2})(k+\gamma+\frac{\lambda}{2})} \right\} \\
 &+ \left. \left. \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{3}{2}-\gamma)(n+\frac{1}{2})} \frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(p+1+\frac{\lambda}{2})(p+\gamma+\frac{\lambda}{2})} \right\} \right] \right|_{\lambda=1-\nu} \left. \right\} \\
 &\hspace{15em} (2.426c)
 \end{aligned}$$

By using simple algebra,

$$\begin{aligned} & \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \\ &= \frac{\mu^n(\psi_1+1-\gamma-\frac{\lambda}{2})! (\frac{1}{2}+\gamma)! (\gamma-\frac{1}{2}+\frac{\lambda}{2})!}{2^n(\psi_1+1-n-\gamma-\frac{\lambda}{2})! (n+\frac{1}{2}+\frac{\lambda}{2})! (n-\frac{1}{2}+\gamma+\frac{\lambda}{2})!} \end{aligned} \quad (2.427a)$$

$$\begin{aligned} & \prod_{p=j}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \prod_{k=0}^{j-1} \frac{-\mu(k-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(k+1+\frac{\lambda}{2})(k+\gamma+\frac{\lambda}{2})} \\ &= \left\{ \frac{\mu^{n-j}(\psi_1+1-j-\gamma-\frac{\lambda}{2})! (j+\frac{1}{2}+\frac{\lambda}{2})! (j-\frac{1}{2}+\gamma+\frac{\lambda}{2})!}{2^{n-j}(\psi_1+1-n-\gamma-\frac{\lambda}{2})! (n+\frac{1}{2}+\frac{\lambda}{2})! (n-\frac{1}{2}+\gamma+\frac{\lambda}{2})!} \right\} \\ & \times \left\{ \frac{\mu^j(\psi_0+1-\gamma-\frac{\lambda}{2})! (\frac{\lambda}{2})! (\gamma-1+\frac{\lambda}{2})!}{2^j(\psi_0+1-j-\gamma-\frac{\lambda}{2})! (j+\frac{\lambda}{2})! (j-1+\gamma+\frac{\lambda}{2})!} \right\} \end{aligned} \quad (2.427b)$$

$$\begin{aligned} & \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(p+1+\frac{\lambda}{2})(p+\gamma+\frac{\lambda}{2})} \\ &= \frac{\mu^n(\psi_0+1-\gamma-\frac{\lambda}{2})! (\frac{\lambda}{2})! (\gamma-1+\frac{\lambda}{2})!}{2^n (\psi_0+1-n-\gamma-\frac{\lambda}{2})! (n+\frac{\lambda}{2})! (n-1+\gamma+\frac{\lambda}{2})!} \end{aligned} \quad (2.427c)$$

Differentiate eq(2.427a), eq(2.427b), eq(2.427c) with respect to λ plugging $\lambda = 1-\nu =$

$2(1 - \gamma)$ and note that $\frac{\partial c_k}{\partial \lambda} = c_k \frac{\partial \ln c_k}{\partial \lambda}$, we obtain

$$\frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \right\} \Big|_{\lambda=1-\nu} \quad (2.428a)$$

$$= -\frac{1}{2} \prod_{p=0}^{n-1} \frac{\mu(\psi_1-p)}{2(p-\gamma+\frac{5}{2})(p+\frac{3}{2})} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1-k)} + \frac{1}{(k+\frac{5}{2}-\gamma)} + \frac{1}{(k+\frac{3}{2})} \right)$$

$$\frac{\partial}{\partial \lambda} \left\{ \prod_{p=j}^{n-1} \frac{-\mu(p-1-\psi_1+\gamma+\frac{\lambda}{2})}{2(p+\frac{3}{2}+\frac{\lambda}{2})(p+\gamma+\frac{1}{2}+\frac{\lambda}{2})} \prod_{k=0}^{j-1} \frac{-\mu(k-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(k+1+\frac{\lambda}{2})(k+\gamma+\frac{\lambda}{2})} \right\} \Big|_{\lambda=1-\nu}$$

$$= -\frac{1}{2} \prod_{p=j}^{n-1} \frac{\mu(\psi_1-p)}{2(p+\frac{3}{2})(p+\frac{5}{2}-\gamma)} \prod_{k=0}^{j-1} \frac{\mu(\psi_0-k)}{2(k+1)(k+2-\gamma)}$$

$$\times \left\{ \sum_{l=0}^{j-1} \left(\frac{1}{(\psi_0-l)} + \frac{1}{(l+2-\gamma)} + \frac{1}{(l+1)} \right) \right.$$

$$\left. + \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1-l)} + \frac{1}{(l+\frac{3}{2})} + \frac{1}{(l+\frac{5}{2}-\gamma)} \right) \right\}$$

(2.428b)

$$\frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{-\mu(p-1-\psi_0+\gamma+\frac{\lambda}{2})}{2(p+1+\frac{\lambda}{2})(p+\gamma+\frac{\lambda}{2})} \right\} \Big|_{\lambda=1-\nu} \quad (2.428c)$$

$$= -\frac{1}{2} \prod_{p=0}^{n-1} \frac{\mu(\psi_0-p)}{2(p+1)(p+2-\gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_0-k)} + \frac{1}{(k+2-\gamma)} + \frac{1}{(k+1)} \right)$$

Substitute eq(2.428a), eq(2.428b) and eq(2.428c) into eq(2.426c).

$$\left. \frac{\partial C_{2n+1}}{\partial \lambda} \right|_{\lambda=1-\nu} = -\varepsilon C_0 \left\{ A(\lambda) + B(\lambda) \right\} \Big|_{\lambda=1-\nu} \quad (2.429a)$$

where,

$$\begin{aligned} A(\lambda) \Big|_{\lambda=1-\nu} &= \frac{1}{(3-2\gamma)} \left(1 - \frac{2(2-\gamma)(2-2\gamma+\omega)}{(3-2\gamma)} \right) \prod_{p=0}^{n-1} \frac{-\mu(p-\psi_1)}{2(p+\frac{5}{2}-\gamma)(p+\frac{3}{2})} \\ &+ \sum_{j=1}^{n-1} \left[\frac{1}{(2j+3-2\gamma)(2j+1)} \left(1 - \frac{2(2j+2-\gamma)(2j+2-2\gamma+\omega)}{(2j+3-2\gamma)(2j+1)} \right) \right. \\ &\times \left. \prod_{p=j}^{n-1} \frac{-\mu(p-\psi_1)}{2(p+\frac{5}{2}-\gamma)(p+\frac{3}{2})} \prod_{k=0}^{j-1} \frac{-\mu(k-\psi_0)}{2(k+2-\gamma)(k+1)} \right] \\ &+ \frac{1}{(2n+3-2\gamma)(2n+1)} \left(1 - \frac{2(2n+2-\gamma)(2n+2-2\gamma+\omega)}{(2n+3-2\gamma)(2n+1)} \right) \\ &\times \prod_{p=0}^{n-1} \frac{-\mu(p-\psi_0)}{2(p+2-\gamma)(p+1)} \end{aligned} \quad (2.429b)$$

$$\begin{aligned}
 B(\lambda) \Big|_{\lambda=1-\nu} &= -\frac{1}{2} \frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \prod_{p=0}^{n-1} \frac{\mu(\psi_1-p)}{2(p-\gamma+\frac{5}{2})(p+\frac{3}{2})} \\
 &\times \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1-k)} + \frac{1}{(k+\frac{5}{2}-\gamma)} + \frac{1}{(k+\frac{3}{2})} \right) \\
 &- \frac{1}{2} \sum_{j=1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{3}{2}-\gamma)(j+\frac{1}{2})} \prod_{p=j}^{n-1} \frac{\mu(\psi_1-p)}{2(p+\frac{3}{2})(p+\frac{5}{2}-\gamma)} \\
 &\times \prod_{k=0}^{j-1} \frac{\mu(\psi_0-k)}{2(k+1)(k+2-\gamma)} \left\{ \sum_{l=0}^{j-1} \left(\frac{1}{(\psi_0-l)} + \frac{1}{(l+2-\gamma)} + \frac{1}{(l+1)} \right) \right. \\
 &\left. + \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1-l)} + \frac{1}{(l+\frac{3}{2})} + \frac{1}{(l+\frac{5}{2}-\gamma)} \right) \right\} \\
 &- \frac{1}{2} \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{3}{2}-\gamma)(n+\frac{1}{2})} \prod_{p=0}^{n-1} \frac{\mu(\psi_0-p)}{2(p+1)(p+2-\gamma)} \\
 &\times \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_0-k)} + \frac{1}{(k+2-\gamma)} + \frac{1}{(k+1)} \right)
 \end{aligned} \tag{2.429c}$$

Substitute eq(2.426a), eq(2.426b), eq(2.429a), eq(2.429b) and eq(2.429c) into the r.h.s of eq(2.419b) and plugging $C_0 = \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)} (-\frac{1}{2}\mu)^{1-\gamma}$.

$$\sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \Big|_{\lambda=1-\nu} = I_{\psi_0, \psi_1} + K_{\psi_0, \psi_1} \tag{2.430}$$

where,

$$\begin{aligned}
 I_{\psi_0, \psi_1} &= -\varepsilon \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left(-\frac{1}{2}\mu x^2\right)^{1-\gamma} \left\{ \frac{(1 - \omega) - (3 - 2\gamma)(2 - 2\gamma + \omega)}{(3 - 2\gamma)^2} x \right. \\
 &+ \left[\left(1 - \frac{2(2 - \gamma)(2 - 2\gamma + \omega)}{(3 - 2\gamma)}\right) \frac{2\mu\psi_1}{3(3 - 2\gamma)(5 - 2\gamma)} \right. \\
 &+ \left. \left. \left(1 - \frac{2(4 - \gamma)(4 - 2\gamma + \omega)}{3(5 - 2\gamma)}\right) \frac{\mu\psi_0}{2 \cdot 3(2 - \gamma)(5 - 2\gamma)} \right] x^3 \right. \\
 &+ \sum_{n=2}^{\infty} \left\{ \frac{1}{(3 - 2\gamma)} \left(1 - \frac{2(2 - \gamma)(2 - 2\gamma + \omega)}{(3 - 2\gamma)}\right) \prod_{p=0}^{n-1} \frac{-\mu(p - \psi_1)}{2(p + \frac{5}{2} - \gamma)(p + \frac{3}{2})} \right. \\
 &+ \sum_{j=1}^{n-1} \left[\frac{1}{(2j + 3 - 2\gamma)(2j + 1)} \left(1 - \frac{2(2j + 2 - \gamma)(2j + 2 - 2\gamma + \omega)}{(2j + 3 - 2\gamma)(2j + 1)}\right) \right. \\
 &\times \left. \prod_{p=j}^{n-1} \frac{-\mu(p - \psi_1)}{2(p + \frac{5}{2} - \gamma)(p + \frac{3}{2})} \prod_{k=0}^{j-1} \frac{-\mu(k - \psi_0)}{2(k + 2 - \gamma)(k + 1)} \right] \\
 &+ \frac{1}{(2n + 3 - 2\gamma)(2n + 1)} \left(1 - \frac{2(2n + 2 - \gamma)(2n + 2 - 2\gamma + \omega)}{(2n + 3 - 2\gamma)(2n + 1)}\right) \\
 &\times \left. \prod_{p=0}^{n-1} \frac{-\mu(p - \psi_0)}{2(p + 2 - \gamma)(p + 1)} \right\} x^{2n+1} \Big\}
 \end{aligned} \tag{2.431a}$$

$$\begin{aligned}
 K_{\psi_0, \psi_1} &= \frac{\varepsilon \Gamma(\psi_0 + 2 - \gamma)}{2 \Gamma(2 - \gamma)} \left\{ \sum_{n=1}^{\infty} \left\{ \frac{(1 - \gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{(p - \psi_1)}{(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \right. \right. \\
 &\times \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1 - k)} + \frac{2(k + 2 - \frac{\gamma}{2})}{(k + \frac{3}{2})(k + \frac{5}{2} - \gamma)} \right) \\
 &+ \frac{(n + 1 - \gamma + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{(p - \psi_0)}{(p + 1)(p + 2 - \gamma)} \\
 &\times \left. \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_0 - k)} + \frac{2(k + \frac{3}{2} - \frac{\gamma}{2})}{(k + 1)(k + 2 - \gamma)} \right) \right\} z^n \\
 &+ \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \prod_{p=j}^{n-1} \frac{(p - \psi_1)}{(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \prod_{k=0}^{j-1} \frac{(k - \psi_0)}{(k + 1)(k + 2 - \gamma)} \\
 &\times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(\psi_0 - l)} + \frac{2(l + \frac{3}{2} - \frac{\gamma}{2})}{(l + 1)(l + 2 - \gamma)} \right) \right. \\
 &\left. + \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1 - l)} + \frac{2(l + 2 - \frac{\gamma}{2})}{(l + \frac{3}{2})(l + \frac{5}{2} - \gamma)} \right) \right] z^n \Big\} z^{1-\gamma} \tag{2.431b}
 \end{aligned}$$

In the above of eq(2.431b), $z = -\frac{1}{2}\mu x^2$. First of all,

(a) $\psi_0 = 0$ in eq(2.431a)

$$\begin{aligned}
 I_{0, \psi_1} &= -\varepsilon x \frac{\Gamma(2 - \gamma)}{\Gamma(2 - \gamma)} \left(-\frac{1}{2}\mu x^2 \right)^{1-\gamma} \left(1 - \frac{2(2 - \gamma)(1 - \gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \right) \\
 &\times \sum_{k=0}^{\psi_1} \frac{(2)^{k-1} \mu^k k! (\psi_1)! (\frac{1}{2} - \gamma)!}{(2k + 1)! (k + \frac{3}{2} - \gamma)! (\psi_1 - k)!} x^{2k} \tag{2.432}
 \end{aligned}$$

(b) $\psi_0 = 1$ in eq(2.431a)

$$\begin{aligned}
 I_{1,\psi_1} &= -\varepsilon x \frac{\Gamma(3-\gamma)}{\Gamma(2-\gamma)} \left(-\frac{1}{2}\mu x^2 \right)^{1-\gamma} \left\{ \left(1 - \frac{2(2-\gamma)(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \right) \right. \\
 &\times \sum_{k=0}^{\psi_1} \frac{(2)^{k-1} \mu^k k! (\psi_1)! (\frac{1}{2}-\gamma)!}{(2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \\
 &+ \frac{\mu(1-\gamma)!}{2(2-\gamma)!} \left(1 - \frac{2(4-\gamma)(2-\gamma+\frac{\omega}{2})}{3(\frac{5}{2}-\gamma)} \right) \\
 &\times \left. \sum_{k=1}^{\psi_1} \frac{(2)^{k-1} \mu^{k-1} 2! k! (\psi_1-1)! (\frac{3}{2}-\gamma)!}{2^1 1! (2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \right\} \tag{2.433}
 \end{aligned}$$

(c) $\psi_0 = 2$ in eq(2.431a)

$$\begin{aligned}
 I_{2,\psi_1} &= -\varepsilon x \frac{\Gamma(4-\gamma)}{\Gamma(2-\gamma)} \left(-\frac{1}{2}\mu x^2\right)^{1-\gamma} \left\{ \left(1 - \frac{2(2-\gamma)(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)}\right) \right. \\
 &\times \sum_{k=0}^{\psi_1} \frac{(2)^{k-1} \mu^k k! (\psi_1)! (\frac{1}{2}-\gamma)!}{(2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \\
 &+ \frac{2! \mu(1-\gamma)!}{2 (1!)(1!) (2-\gamma)!} \left(1 - \frac{2(4-\gamma)(2-\gamma+\frac{\omega}{2})}{3(\frac{5}{2}-\gamma)}\right) \\
 &\times \sum_{k=1}^{\psi_1} \frac{(2)^{k-1} \mu^{k-1} 2! k! (\psi_1-1)! (\frac{3}{2}-\gamma)!}{2^1 1! (2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \\
 &+ \frac{2! \mu^2(1-\gamma)!}{2^2 (0!) (2!)(3-\gamma)!} \left(1 - \frac{2(6-\gamma)(3-\gamma+\frac{\omega}{2})}{5(\frac{7}{2}-\gamma)}\right) \\
 &\times \left. \sum_{k=2}^{\psi_1} \frac{(2)^{k-1} \mu^{k-2} 4! k! (\psi_1-2)! (\frac{5}{2}-\gamma)!}{2^2 2! (2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \right\}
 \end{aligned}$$

(2.434)

(d) $\psi_0 = 3$ in eq(2.431a)

$$\begin{aligned}
 I_{3,\psi_1} &= -\varepsilon x z^{1-\gamma} \frac{\Gamma(5-\gamma)}{\Gamma(2-\gamma)} \left(-\frac{1}{2}\mu x^2\right)^{1-\gamma} \left\{ \left(1 - \frac{2(2-\gamma)(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)}\right) \right. \\
 &\times \sum_{k=0}^{\psi_1} \frac{(2)^{k-1} \mu^k k! (\psi_1)! (\frac{1}{2}-\gamma)!}{(2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \\
 &+ \frac{3! \mu(1-\gamma)!}{2 (2!)(1!) (2-\gamma)!} \left(1 - \frac{2(4-\gamma)(2-\gamma+\frac{\omega}{2})}{3(\frac{5}{2}-\gamma)}\right) \\
 &\times \sum_{k=1}^{\psi_1} \frac{(2)^{k-1} \mu^{k-1} 2! k! (\psi_1-1)! (\frac{3}{2}-\gamma)!}{2^1 1! (2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \\
 &+ \frac{3! \mu^2(1-\gamma)!}{2^2 (1!) (2!)(3-\gamma)!} \left(1 - \frac{2(6-\gamma)(3-\gamma+\frac{\omega}{2})}{5(\frac{7}{2}-\gamma)}\right) \\
 &\times \sum_{k=2}^{\psi_1} \frac{(2)^{k-1} \mu^{k-2} 4! k! (\psi_1-2)! (\frac{5}{2}-\gamma)!}{2^2 2! (2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \\
 &+ \frac{3! \mu^3(1-\gamma)!}{2^3 (0!) (3!)(4-\gamma)!} \left(1 - \frac{2(8-\gamma)(4-\gamma+\frac{\omega}{2})}{7(\frac{9}{2}-\gamma)}\right) \\
 &\times \left. \sum_{k=3}^{\psi_1} \frac{(2)^{k-1} \mu^{k-3} 6! k! (\psi_1-3)! (\frac{7}{2}-\gamma)!}{2^3 3! (2k+1)! (k+\frac{3}{2}-\gamma)! (\psi_1-k)!} x^{2k} \right\}
 \end{aligned} \tag{2.435}$$

We can describe a function I_{ψ_0,ψ_1} of degree ψ_0 and ψ_1 according to eq(2.432) through

eq(2.435).

$$\begin{aligned}
 I_{\psi_0, \psi_1} &= -\varepsilon x \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left(-\frac{1}{2} \mu x^2 \right)^{1-\gamma} \sum_{n=0}^{\psi_0} \frac{(\psi_0)!(1-\gamma)!}{2^n n! (\psi_0 - n)!(n+1-\gamma)!} \\
 &\quad \times \left(1 - \frac{2(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 &\quad \times \sum_{k=n}^{\psi_1} \frac{2^{k-1} \mu^k k! (2n)! (\psi_1 - n)! (n + \frac{1}{2} - \gamma)!}{2^n n! (2k+1)! (\psi_1 - k)! (k + \frac{3}{2} - \gamma)!} x^{2k} \tag{2.436}
 \end{aligned}$$

By using simple algebra,

$$(2n)! = \frac{2^{2n} n! (n - \frac{1}{2})!}{(-\frac{1}{2})!} \quad (2k+1)! = \frac{2^{2k+1} k! (k + \frac{1}{2})!}{(-\frac{1}{2})!} \tag{2.437}$$

Plug eq(2.437) into eq(2.436).

$$\begin{aligned}
 I_{\psi_0, \psi_1} &= -\varepsilon x \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left(-\frac{1}{2} \mu x^2 \right)^{1-\gamma} \sum_{n=0}^{\psi_0} \frac{(\psi_0)!(1-\gamma)!}{2^n n! (\psi_0 - n)!(n+1-\gamma)!} \\
 &\quad \times \left(1 - \frac{2(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 &\quad \times \sum_{k=n}^{\psi_1} \frac{2^{n-k-2} \mu^k (n - \frac{1}{2})! (\psi_1 - n)! (n + \frac{1}{2} - \gamma)!}{(k + \frac{1}{2})! (\psi_1 - k)! (k + \frac{3}{2} - \gamma)!} x^{2k} \tag{2.438}
 \end{aligned}$$

We see in the second summation of eq(2.438), we can make index k start from 0.

Change the interval of index k as $0 \leq k \leq \psi_1 - n$. Eq(2.438) is then equal to

$$\begin{aligned}
 I_{\psi_0, \psi_1} &= -\frac{\varepsilon}{2}x \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} \left(-\frac{1}{2}\mu x^2\right)^n \\
 &\times \left(\frac{1}{2} - \frac{(n+1 - \frac{\gamma}{2})(n+1 - \gamma + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)}\right) \\
 &\times \sum_{k=0}^{\psi_1 - n} \frac{\Gamma(n + \frac{1}{2})\Gamma(n + \frac{3}{2} - \gamma)(n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2})\Gamma(n + k + \frac{5}{2} - \gamma)} \left(-\frac{1}{2}\mu x^2\right)^{k+1-\gamma} \quad (2.439)
 \end{aligned}$$

Plug eq(2.431b) and eq(2.439) into eq(2.430), and substitute it and eq(2.421) into eq(2.419b) after that, add eq(2.420) into eq(2.418):

$$\begin{aligned}
 g_2(x) &= \mathcal{RW}_{\psi_0, \psi_1}^F \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) A_{\psi_0}(\gamma; z) \right. \\
 &\quad + \sum_{n=1}^{\psi_0} \frac{(-1)^n \psi_0! (\psi_0 + 1 - \gamma)!}{2 n! (n+1 - \gamma)! (\psi_0 - n)!} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k - \psi_0)} - \frac{1}{(k + 2 - \gamma)} - \frac{1}{(k + 1)} \right\} z^n \\
 &\quad - \frac{\varepsilon}{2} x \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \ln x \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2 - \gamma)_n} \sum_{k=0}^{\psi_1 - n} \frac{(n + 1 - \gamma + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} z^k \right. \\
 &\quad + \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2 - \gamma)_n} \left(\frac{1}{2} - \frac{(n + 1 - \frac{\gamma}{2})(n + 1 - \gamma + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \right) \sum_{k=0}^{\psi_1 - n} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (n - \psi_1)_k}{\Gamma(n + k + \frac{3}{2}) \Gamma(n + k + \frac{5}{2} - \gamma)} z^k \\
 &\quad - \sum_{n=1}^{\infty} \left\{ \frac{(1 - \gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{(p - \psi_1)}{(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1 - k)} + \frac{1}{(k + \frac{5}{2} - \gamma)} + \frac{1}{(k + \frac{3}{2})} \right) \right. \\
 &\quad + \frac{(n + 1 - \gamma + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{(p - \psi_0)}{(p + 1)(p + 2 - \gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_0 - k)} + \frac{1}{(k + 2 - \gamma)} + \frac{1}{(k + 1)} \right) \left. \right\} z^n \\
 &\quad - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \prod_{p=j}^{n-1} \frac{(p - \psi_1)}{(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \prod_{k=0}^{j-1} \frac{(k - \psi_0)}{(k + 1)(k + 2 - \gamma)} \\
 &\quad \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(\psi_0 - l)} + \frac{1}{(l + 2 - \gamma)} + \frac{1}{(l + 1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1 - l)} + \frac{1}{(l + \frac{3}{2})} + \frac{1}{(l + \frac{5}{2} - \gamma)} \right) \right] z^n \left. \right\}
 \end{aligned}$$

Table 2.13: The second independent solution of grand confluent hypergeometric function as $\nu = 0$ or negative integer.

We can describe $\mathcal{RW}_{\psi_0, \psi_1}^F(\gamma; z)$ more precisely in the following way.

(1) As $\psi_0 = 0$ & $\psi_1 = 0$

$$\begin{aligned}
 & \mathcal{RW}_{0,0}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) - \frac{\varepsilon}{2} x \left\{ \left[\frac{(1-\gamma + \frac{\omega}{2})}{\frac{1}{2}(\frac{3}{2} - \gamma)} \ln(x) + \left(\frac{1}{(\frac{3}{2} - \gamma)} - \frac{2(2-\gamma)(1-\gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)^2} \right) \right] \right. \right. \\
 &+ \sum_{n=1}^{\infty} \left[\frac{(1-\gamma + \frac{\omega}{2})}{2} \frac{\Gamma(n)\Gamma(\frac{1}{2})\Gamma(\frac{3}{2} - \gamma)}{\Gamma(n + \frac{3}{2})\Gamma(n + \frac{5}{2} - \gamma)} + \frac{(n+1-\gamma + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \frac{1}{n(2-\gamma)_n} \right] z^n \\
 &+ \left. \left. \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j+1-\gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{\Gamma(n)(\frac{3}{2})_j(\frac{5}{2} - \gamma)_j}{j! \Gamma(n + \frac{3}{2})(2-\gamma)_j(\frac{3}{2})_n(\frac{5}{2} - \gamma)_n} z^n \right\} \right\} \quad (2.440a)
 \end{aligned}$$

(2) As $\psi_0 = 0$ & $\psi_1 = 1$

$$\begin{aligned}
 & \mathcal{RW}_{0,1}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) - \frac{\varepsilon}{2} x \left\{ \ln(x) \frac{(1-\gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \sum_{n=0}^1 \frac{(-1)^n \Gamma(\frac{1}{2})}{\Gamma(n + \frac{3}{2})(\frac{5}{2} - \gamma)_n} z^n \right. \right. \\
 &+ \left(1 - \frac{2(2-\gamma)(1-\gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \right) \sum_{n=0}^1 \frac{(-1)^n \Gamma(\frac{3}{2})\Gamma(\frac{3}{2} - \gamma)}{\Gamma(n + \frac{3}{2})\Gamma(n + \frac{5}{2} - \gamma)} z^n \\
 &+ \left(1 + \frac{(4-\gamma)}{\frac{3}{2}(\frac{5}{2} - \gamma)} \right) \frac{(1-\gamma + \frac{\omega}{2})}{\frac{3}{2}(\frac{3}{2} - \gamma)(\frac{5}{2} - \gamma)} z - \frac{(1-\gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{(\frac{3}{2})_n(\frac{5}{2} - \gamma)_n} z^n \\
 &+ \sum_{n=1}^{\infty} \frac{(n+1-\gamma + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \frac{1}{n(2-\gamma)_n} z^n \\
 &+ \left. \left. \sum_{n=3}^{\infty} \sum_{j=2}^{n-1} \frac{(j+1-\gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{\Gamma(n-1)(\frac{3}{2})_j(\frac{5}{2} - \gamma)_j}{j \Gamma(j-1)(2-\gamma)_j(\frac{3}{2})_n(\frac{5}{2} - \gamma)_n} z^n \right\} \right\} \quad (2.440b)
 \end{aligned}$$

(3) As $\psi_0 = 0$ & $\psi_1 \geq 2$

$$\begin{aligned}
 & \mathcal{RW}_{0,\psi_1}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) - \frac{\varepsilon}{2} x \left\{ \ln(x) \frac{(1-\gamma + \frac{\omega}{2})}{\frac{1}{2}(\frac{3}{2} - \gamma)} \sum_{n=0}^{\psi_1} \frac{(-1)^n}{(\psi_1)! (\frac{3}{2})_n (\frac{5}{2} - \gamma)_n} z^n \right. \right. \\
 &+ \left(1 - \frac{2(2-\gamma)(1-\gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \right) \sum_{n=0}^{\psi_1} \frac{(-1)^n (-\psi_1)_n \Gamma(\frac{3}{2} - \gamma)}{(\frac{3}{2})_n \Gamma(n + \frac{5}{2} - \gamma)} z^n \\
 &- \frac{(1-\gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \left[\sum_{n=1}^{\psi_1} \frac{(-\psi_1)_n}{(\frac{3}{2})_n (\frac{2}{5} - \gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1 - k)} + \frac{1}{(k + \frac{5}{2} - \gamma)} + \frac{1}{(k + \frac{3}{2})} \right) z^n \right. \\
 &- \left. \sum_{n=\psi_1+1}^{\infty} \frac{(-1)^{-\psi_1} \Gamma(\psi_1 + 1) \Gamma(n - \psi_1)}{(\frac{3}{2})_n (\frac{5}{2} - \gamma)_n} z^n \right] + \sum_{n=1}^{\infty} \frac{(n+1 - \gamma + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \frac{1}{n(2-\gamma)_n} z^n \\
 &+ \sum_{n=2}^{\psi_1} \sum_{j=1}^{n-1} \frac{(j+1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{(-\psi_1)_n (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j (-\psi_1)_j (2-\gamma)_j (\frac{3}{2})_n (\frac{5}{2} - \gamma)_n} z^n \quad (2.440c) \\
 &+ \left. \left. \sum_{n=\psi_1+2}^{\infty} \sum_{j=\psi_1+1}^{n-1} \frac{(j+1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{\Gamma(n - \psi_1) (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j \Gamma(j - \psi_1) (\frac{3}{2})_n (2-\gamma)_j (\frac{5}{2} - \gamma)_n} z^n \right\} \right\}
 \end{aligned}$$

(4) As $\psi_0 = 1$ & $\psi_1 = 1$

$$\begin{aligned}
 & \mathcal{RW}_{1,1}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) \sum_{n=0}^1 \frac{(-1)^n \Gamma(3-\gamma)}{\Gamma(n+2-\gamma)} z^n + \frac{(\frac{5}{2}-\gamma)}{(2-\gamma)} z \right. \\
 & \quad - \frac{\varepsilon}{2} (2-\gamma)x \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{(2-\gamma)_n} \sum_{k=0}^{1-n} \frac{(-1)^k (n+1-\gamma+\frac{\omega}{2}) \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma)}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\frac{5}{2}-\gamma)} z^{n+k} \right. \\
 & \quad + \sum_{n=0}^1 \frac{(-1)^n}{(2-\gamma)_n} \left(\frac{1}{2} - \frac{(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 & \quad \times \sum_{k=0}^{1-n} \frac{(-1)^k \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma)}{\Gamma(n+k+\frac{3}{2}) \Gamma(n+k+\frac{5}{2}-\gamma)} z^{n+k} \\
 & \quad + \left\{ \left(1 - \frac{(4-\gamma)}{\frac{3}{2}(\frac{5}{2}-\gamma)} \right) \frac{(1-\gamma+\frac{\omega}{2})}{\frac{3}{2}(\frac{3}{2}-\gamma)(\frac{5}{2}-\gamma)} + \left(1 + \frac{(3-\gamma)}{(2-\gamma)} \right) \frac{(2-\gamma+\frac{\omega}{2})}{3(2-\gamma)(\frac{5}{2}-\gamma)} \right\} z \\
 & \quad - \frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} z^n \\
 & \quad - \sum_{n=2}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \frac{\Gamma(n-1)}{\Gamma(n+1)(2-\gamma)_n} z^n \\
 & \quad \left. - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{\Gamma(n-1)(\frac{3}{2})_j (\frac{5}{2}-\gamma)_j}{j! (\frac{3}{2})_n (2-\gamma)_j (\frac{5}{2}-\gamma)_n} z^n \right\} \quad (2.440d)
 \end{aligned}$$

(5) As $\psi_0 = 1$ & $\psi_1 = 2$

$$\begin{aligned}
 & \mathcal{RW}_{1,2}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) \sum_{n=0}^1 \frac{(-1)^n \Gamma(3-\gamma)}{\Gamma(n+2-\gamma)} z^n + \frac{(\frac{5}{2}-\gamma)}{(2-\gamma)} z - \frac{\varepsilon}{2} (2-\gamma)x \right. \\
 & \times \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{(2-\gamma)_n} \sum_{k=0}^{2-n} \frac{(n+1-\gamma+\frac{\omega}{2}) \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma) (n-2)_k}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\frac{5}{2}-\gamma)} z^{n+k} \right. \\
 & + \sum_{n=0}^1 \frac{(-1)^n}{(2-\gamma)_n} \left(\frac{1}{2} - \frac{(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 & \times \sum_{k=0}^{2-n} \frac{\Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma) (n-2)_k}{\Gamma(n+k+\frac{3}{2}) \Gamma(n+k+\frac{5}{2}-\gamma)} z^{n+k} \\
 & + \left\{ \left(1 - \frac{2(4-\gamma)}{\frac{3}{2}(\frac{5}{2}-\gamma)} \right) \frac{(1-\gamma+\frac{\omega}{2})}{\frac{3}{2}(\frac{3}{2}-\gamma)(\frac{5}{2}-\gamma)} + \left(1 + \frac{(3-\gamma)}{(2-\gamma)} \right) \frac{(2-\gamma+\frac{\omega}{2})}{3(2-\gamma)(\frac{5}{2}-\gamma)} \right\} z \\
 & - \frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \left[\frac{\Gamma(\frac{1}{2}) \Gamma(\frac{2}{5}-\gamma)}{\Gamma(\frac{7}{2}) \Gamma(\frac{9}{2}-\gamma)} \sum_{k=0}^1 \left(\frac{1}{2-k} + \frac{1}{(k+\frac{5}{2}-\gamma)} + \frac{1}{(k+\frac{3}{2})} \right) z^2 \right. \\
 & + \sum_{n=3}^{\infty} \frac{(-1)^3 \Gamma(n-2)}{\frac{1}{2}(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} z^n \left. - \sum_{n=2}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \frac{\Gamma(n-1)}{\Gamma(n+1)(2-\gamma)_n} z^n \right. \\
 & - \frac{(2-\gamma+\frac{\omega}{2})}{3(\frac{5}{2}-\gamma)} \frac{1}{\frac{5}{2}(2-\gamma)(\frac{7}{2}-\gamma)} \left(\frac{17}{5} + \frac{1}{(2-\gamma)} + \frac{1}{(\frac{7}{2}-\gamma)} \right) z^2 \\
 & + \sum_{n=3}^{\infty} \frac{(2-\gamma+\frac{\omega}{2})}{2(2-\gamma)} \frac{\Gamma(n-2)}{(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} z^n \\
 & \left. - \sum_{n=4}^{\infty} \sum_{j=3}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{\Gamma(j-1) \Gamma(n-2) (\frac{3}{2})_j (\frac{5}{2}-\gamma)_j}{\Gamma(j+1) \Gamma(j-2) (\frac{3}{2})_n (2-\gamma)_j (\frac{5}{2}-\gamma)_n} z^n \right\} \left. \right\} \quad (2.440e)
 \end{aligned}$$

(6) As $\psi_0 = 1$ & $\psi_1 \geq 3$

$$\begin{aligned}
 & \mathcal{RW}_{1,\psi_1}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) \sum_{n=0}^1 \frac{(-1)^n \Gamma(3-\gamma)}{\Gamma(n+2-\gamma)} z^n + \frac{(\frac{5}{2}-\gamma)}{(2-\gamma)} z - \frac{\varepsilon}{2} (2-\gamma)x \left\{ \ln(x) \right. \right. \\
 & \times \sum_{n=0}^1 \frac{(-1)^n}{(2-\gamma)_n} \sum_{k=0}^{\psi_1-n} \frac{(n+1-\gamma+\frac{\omega}{2})\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2}-\gamma)(n-\psi_1)_k}{\Gamma(k+n+\frac{3}{2})\Gamma(k+n+\frac{5}{2}-\gamma)} z^{n+k} \\
 & + \sum_{n=0}^1 \frac{(-1)^n}{(2-\gamma)_n} \left(\frac{1}{2} - \frac{(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 & \times \sum_{k=0}^{\psi_1-n} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2}-\gamma)(\psi_1-2)_k}{\Gamma(n+k+\frac{3}{2})\Gamma(n+k+\frac{5}{2}-\gamma)} z^{n+k} \\
 & + \left\{ \left(1 - \frac{(4-\gamma)\psi_1}{\frac{3}{2}(\frac{5}{2}-\gamma)} \right) \frac{(1-\gamma+\frac{\omega}{2})}{\frac{3}{2}(\frac{3}{2}-\gamma)(\frac{5}{2}-\gamma)} + \left(1 + \frac{(3-\gamma)}{(2-\gamma)} \right) \frac{(2-\gamma+\frac{\omega}{2})}{3(2-\gamma)(\frac{5}{2}-\gamma)} \right\} z \\
 & - \frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \left[\sum_{n=2}^{\psi_1} \frac{(-\psi_1)_n}{(\frac{3}{2})_n(\frac{5}{2}-\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{\psi_1-k} + \frac{1}{(k+\frac{5}{2}-\gamma)} + \frac{1}{(k+\frac{3}{2})} \right) z^n \right. \\
 & + \left. \sum_{n=\psi_1+1}^{\infty} \frac{(-1)^{\psi_1+1} \Gamma(\psi_1+1) \Gamma(n-\psi_1)}{(\frac{3}{2})_n(\frac{5}{2}-\gamma)_n} z^n \right] \\
 & - \sum_{n=2}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \frac{\Gamma(n-1)}{\Gamma(n+1)(2-\gamma)_n} z^n
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{n=2}^{\psi_1} \sum_{j=1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{(-1)^j (-\psi_1)_n (\frac{3}{2})_j (\frac{5}{2}-\gamma)_j}{j! (-\psi_1)_j (\frac{3}{2})_j (\frac{5}{2}-\gamma)_n (2-\gamma)_j \Gamma(2-j)} \\
 & \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(1-l)} + \frac{1}{(l+2-\gamma)} + \frac{1}{(l+1)} \right) \right. \\
 & \left. + \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1-l)} + \frac{1}{(l+\frac{3}{2})} + \frac{1}{(l+\frac{5}{2}-\gamma)} \right) \right] z^n \\
 & - \sum_{n=3}^{\psi_1} \sum_{j=2}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{(-1)^n \Gamma(j-1) (\psi_1)! (\frac{3}{2})_j (\frac{5}{2}-\gamma)_j}{\Gamma(j+1) (-\psi_1)_j (\frac{3}{2})_n (2-\gamma)_j (\frac{5}{2}-\gamma)_n} z^n \\
 & + \sum_{n=\psi_1+1}^{\infty} \frac{(2-\gamma+\frac{\omega}{2})}{2} \frac{(-1)^{\psi_1} \Gamma(\psi_1) \Gamma(n-\psi_1)}{(\frac{3}{2})_n (3-\gamma)_n (\frac{5}{2}-\gamma)_n} z^n \tag{2.440f} \\
 & - \left. \sum_{n=\psi_1+2}^{\infty} \sum_{j=\psi_1+1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{\Gamma(j-1) \Gamma(n-\psi_1) (\frac{3}{2})_j (\frac{5}{2}-\gamma)_j}{\Gamma(j+1) \Gamma(j-\psi_1) (\frac{3}{2})_n (2-\gamma)_j (\frac{5}{2}-\gamma)_n} z^n \right\}
 \end{aligned}$$

(7) As $\psi_0 \geq 2$ & $\psi_0 \leq \psi_1 \leq \psi_0 + 1$

$$\begin{aligned}
 & \mathcal{RW}_{\psi_0, \psi_1}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) A_{\psi_0}(\gamma; z) + \sum_{n=1}^{\psi_0} \frac{(-\psi_0)_n (2-\gamma)_{\psi_0}}{2 n! (2-\gamma)_n} \right. \\
 & \times \sum_{k=0}^{n-1} \left\{ \frac{1}{(k-\psi_0)} - \frac{1}{(k+2-\gamma)} - \frac{1}{(k+1)} \right\} z^n - \frac{\varepsilon}{2} x \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)} \\
 & \times \left\{ \ln x \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2-\gamma)_n} \sum_{k=0}^{\psi_1-n} \frac{(n+1-\gamma+\frac{\omega}{2}) \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma) (n-\psi_1)_k}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\frac{5}{2}-\gamma)} z^k \right. \\
 & + \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2-\gamma)_n} \left(\frac{1}{2} - \frac{(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 & \times \sum_{k=0}^{\psi_1-n} \frac{\Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma) (n-\psi_1)_k}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\frac{5}{2}-\gamma)} z^k - \frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \left\{ \sum_{n=1}^{\psi_1} \frac{(-\psi_1)_n}{(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} \right. \\
 & \times \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1-k)} + \frac{1}{(k+\frac{5}{2}-\gamma)} + \frac{1}{(k+\frac{3}{2})} \right) z^n \\
 & \left. + \sum_{n=\psi_1+1}^{\infty} \frac{(-1)^{\psi_1+1} \Gamma(\psi_1+1) \Gamma(n-\psi_1)}{(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} z^n \right\} \\
 & - \sum_{n=1}^{\psi_1} \frac{(-\psi_0)_n}{n! (2-\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_0-k)} + \frac{1}{(k+2-\gamma)} + \frac{1}{(k+1)} \right) z^n \\
 & + \sum_{n=\psi_1+1}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \frac{(-1)^{\psi_0} \Gamma(\psi_0+1) \Gamma(n-\psi_0)}{n! (2-\gamma)_n} z^n \\
 & - \sum_{n=2}^{\psi_1} \sum_{j=1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{(-\psi_0)_j (-\psi_1)_n (\frac{5}{2}-\gamma)_j (\frac{3}{2})_j}{j! (-\psi_1)_j (\frac{5}{2}-\gamma)_n (2-\gamma)_n (\frac{3}{2})_n}
 \end{aligned}$$

$$\begin{aligned}
 & \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(\psi_0 - l)} + \frac{1}{(l + 2 - \gamma)} + \frac{1}{(l + 1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1 - l)} + \frac{1}{(l + \frac{3}{2})} + \frac{1}{(l + \frac{5}{2} - \gamma)} \right) \right] z^n \\
 & + \sum_{n=\psi_1+1}^{\infty} \sum_{j=1}^{\psi_0} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{(-1)^{\psi_1} (\psi_1)! \Gamma(n - \psi_1) (-\psi_0)_j (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j! (-\psi_1)_j (\frac{3}{2})_n (2 - \gamma)_j (\frac{5}{2} - \gamma)_n} z^n \\
 & + \sum_{n=\psi_1+2}^{\infty} \sum_{j=\psi_1+1}^{n-1} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \\
 & \times \left. \frac{(-1)^{\psi_1} \Gamma(n - \psi_1) \Gamma(j - \psi_0) (\psi_0)! (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j! \Gamma(j - \psi_1) (\frac{3}{2})_n (2 - \gamma)_j (\frac{5}{2} - \gamma)_n} z^n \right\} \quad (2.440g)
 \end{aligned}$$

(8) As $\psi_0 \geq 2$ & $\psi_1 \geq \psi_0 + 2$

$$\begin{aligned}
 & \mathcal{RW}_{\psi_0, \psi_1}^F(\gamma; z) \\
 &= z^{1-\gamma} \left\{ (1 + \ln(x)) A_{\psi_0}(\gamma; z) + \sum_{n=1}^{\psi_0} \frac{(-\psi_0)_n (2-\gamma)_{\psi_0}}{2 n! (2-\gamma)_n} \right. \\
 & \times \sum_{k=0}^{n-1} \left\{ \frac{1}{(k-\psi_0)} - \frac{1}{(k+2-\gamma)} - \frac{1}{(k+1)} \right\} z^n - \frac{\varepsilon}{2} x \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)} \\
 & \times \left\{ \ln x \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2-\gamma)_n} \sum_{k=0}^{\psi_1-n} \frac{(n+1-\gamma+\frac{\omega}{2}) \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma) (n-\psi_1)_k}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\frac{5}{2}-\gamma)} z^k \right. \\
 & + \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2-\gamma)_n} \left(\frac{1}{2} - \frac{(n+1-\frac{\gamma}{2})(n+1-\gamma+\frac{\omega}{2})}{(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \right) \\
 & \times \sum_{k=0}^{\psi_1-n} \frac{\Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2}-\gamma) (n-\psi_1)_k}{\Gamma(k+n+\frac{3}{2}) \Gamma(k+n+\frac{5}{2}-\gamma)} z^k - \frac{(1-\gamma+\frac{\omega}{2})}{(\frac{3}{2}-\gamma)} \left\{ \sum_{n=1}^{\psi_1} \frac{(-\psi_1)_n}{(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} \right. \\
 & \times \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_1-k)} + \frac{1}{(k+\frac{5}{2}-\gamma)} + \frac{1}{(k+\frac{3}{2})} \right) z^n \\
 & \left. + \sum_{n=\psi_1+1}^{\infty} \frac{(-1)^{\psi_1+1} \Gamma(\psi_1+1) \Gamma(n-\psi_1)}{(\frac{3}{2})_n (\frac{5}{2}-\gamma)_n} z^n \right\} \\
 & - \sum_{n=1}^{\psi_1} \frac{(-\psi_0)_n}{n! (2-\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(\psi_0-k)} + \frac{1}{(k+2-\gamma)} + \frac{1}{(k+1)} \right) z^n \\
 & + \sum_{n=\psi_1+1}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2})}{2(n+\frac{1}{2})(n+\frac{3}{2}-\gamma)} \frac{(-1)^{\psi_0} \Gamma(\psi_0+1) \Gamma(n-\psi_0)}{n! (2-\gamma)_n} z^n \\
 & - \sum_{n=2}^{\psi_1} \sum_{j=1}^{n-1} \frac{(j+1-\gamma+\frac{\omega}{2})}{2(j+\frac{1}{2})(j+\frac{3}{2}-\gamma)} \frac{(-\psi_0)_j (-\psi_1)_n (\frac{5}{2}-\gamma)_j (\frac{3}{2})_j}{j! (-\psi_1)_j (\frac{5}{2}-\gamma)_n (2-\gamma)_n (\frac{3}{2})_n}
 \end{aligned}$$

$$\begin{aligned}
 & \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(\psi_0 - l)} + \frac{1}{(l + 2 - \gamma)} + \frac{1}{(l + 1)} \right) \right. \\
 & + \left. \sum_{l=j}^{n-1} \left(\frac{1}{(\psi_1 - l)} + \frac{1}{(l + \frac{3}{2})} + \frac{1}{(l + \frac{5}{2} - \gamma)} \right) \right] z^n \\
 & + \sum_{n=\psi_0+2}^{\psi_1} \sum_{j=\psi_0+1}^{n-1} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{(-1)^{\psi_0} (-\psi_1)_n (\psi_0)! \Gamma(j - \psi_0) (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j! (-\psi_1)_j (\frac{3}{2})_n (2 - \gamma)_j (\frac{5}{2} - \gamma)_n} z^n \\
 & + \sum_{n=\psi_1+1}^{\infty} \sum_{j=1}^{\psi_0} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \frac{(-1)^{\psi_1} (\psi_1)! \Gamma(n - \psi_1) (-\psi_0)_j (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j! (-\psi_1)_j (\frac{3}{2})_n (2 - \gamma)_j (\frac{5}{2} - \gamma)_n} z^n \\
 & + \sum_{n=\psi_1+2}^{\infty} \sum_{j=\psi_1+1}^{n-1} \frac{(j + 1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \\
 & \times \left. \frac{(-1)^{\psi_1} \Gamma(n - \psi_1) \Gamma(j - \psi_0) (\psi_0)! (\frac{3}{2})_j (\frac{5}{2} - \gamma)_j}{j! \Gamma(j - \psi_1) (\frac{3}{2})_n (2 - \gamma)_j (\frac{5}{2} - \gamma)_n} z^n \right\} \quad (2.440h)
 \end{aligned}$$

(B) the case of $\nu =$ positive integer except 1

As ν is positive integer except 1, $\lambda_2 = 1 - \nu$ is negative integer. So $\lambda_2 = 1 - \nu$ is smaller than $\lambda_1 = 0$. By using Case E) in theorem 1, the first independent solution is

$$g_1(x) = G(x) \Big|_{\lambda=\lambda_2=1-\nu} = \mathcal{RW}_{\psi_0, \psi_1} \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \quad (2.441)$$

The second independent solution is

$$g_2(x) = g_1(x) \ln x + x^{\lambda_1} \sum_{n=0}^{\infty} b_n(\lambda_1) x^n \quad (2.442)$$

To generate this solution, keep the recurrence formula in term of λ and use it to find to coefficients $C_n(n \geq 1)$ in terms of both λ and C_0 , where coefficient C_0 remains arbitrary. Substitute these C_n into $G(x) = x^\lambda \sum_{n=0}^{\infty} C_n x^n$ to obtain a function $G(\lambda, x)$ which depends on the variables λ & x . From the Case E) in theorem 1, the second independent solution is

$$g_2(x) = \left. \frac{\partial G(\lambda, x)}{\partial \lambda} \right|_{\lambda=\lambda_1} = \ln x \sum_{n=0}^{\infty} C_n x^{n+\lambda} \Big|_{\lambda=\lambda_1} + \sum_{n=0}^{\infty} \frac{\partial C_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=\lambda_1} \quad (2.443)$$

From eq(2.262), $G(x) = g(x)_{domin.} + g(x)_{small}$. Plug it into eq(2.443).

$$g_2(x) = \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=\lambda_1} + \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=\lambda_1} \quad (2.444)$$

The first term of r.h.s in eq(2.444) is

$$\begin{aligned} \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=0} &= \left. \frac{\partial}{\partial \lambda} \left\{ \sum_{n=0}^{\infty} C_{2n}(\lambda, n) x^{2n+\lambda} \right\} \right|_{\lambda=0} \\ &= \ln x \sum_{n=0}^{\infty} C_{2n}(\lambda, n) x^{2n+\lambda} \Big|_{\lambda=0} + \sum_{n=0}^{\infty} \frac{\partial C_{2n}(\lambda, n)}{\partial \lambda} x^{2n+\lambda} \Big|_{\lambda=0} \end{aligned} \quad (2.445a)$$

The second term of r.h.s in eq(2.444) is

$$\begin{aligned}
 \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=0} &= \left. \frac{\partial}{\partial \lambda} \left\{ \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \right\} \right|_{\lambda=0} \\
 &= \ln x \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=0} \\
 &\quad + \sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \Big|_{\lambda=0} \tag{2.445b}
 \end{aligned}$$

We know that $g(x)_{domin.}$ as $\lambda = 0$ is equal to the first kind of confluent hypergeometric polynomial. So a solution for eq(2.445a) is similar to Remark 3 on appendix. Replace α , λ and λ_1 by $-|\alpha_0|$, $\frac{\lambda}{2}$, $\frac{\lambda_1}{2}$ in eq(2.198) on appendix, then differentiate it with respect to λ . After that, plug $\lambda = 0$, and substitute what we get into eq(2.197) on appendix.

Then we obtain

$$\begin{aligned}
 \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=0} &= (1 + \ln(x)) F_{|\alpha_0|}(\gamma; z) \\
 &\quad + \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=1}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{2 n! (\gamma)_n} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k - |\alpha_0|)} - \frac{1}{(k + 1)} - \frac{1}{(k + \gamma)} \right\} z^n \tag{2.446}
 \end{aligned}$$

The first summation of eq(2.445b) which is $\sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=0}$ is equivalent

to eq(2.124). Then, we obtain

$$\begin{aligned}
 & \ln x \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=0} = -\frac{\varepsilon}{2} x \ln x \prod_{|\alpha_0|}^{|\alpha_1|} (\gamma; z) \\
 = & -x \ln x \frac{\varepsilon}{2} \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \\
 & \times \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k
 \end{aligned} \tag{2.447}$$

According to eq(2.107a) and eq(2.107b), $\Omega = -2\mu|\alpha_0|$ as n is odd in eq(2.423b). When n is even, $\Omega = -2\mu\left(|\alpha_1| + \frac{1}{2}\right)$.

where,

$$A_{2j} = -\frac{\varepsilon(j + \frac{\omega}{2} + \frac{\lambda}{2})}{2(j + \frac{1}{2} + \frac{\lambda}{2})(j + \gamma - \frac{1}{2} + \frac{\lambda}{2})} \tag{2.448a}$$

$$A_{2n} = -\frac{\varepsilon(n + \frac{\omega}{2} + \frac{\lambda}{2})}{2(n + \frac{1}{2} + \frac{\lambda}{2})(n + \gamma - \frac{1}{2} + \frac{\lambda}{2})} \tag{2.448b}$$

$$A_0 = -\frac{\varepsilon(\frac{\omega}{2} + \frac{\lambda}{2})}{2(\frac{1}{2} + \frac{\lambda}{2})(\gamma - \frac{1}{2} + \frac{\lambda}{2})} \tag{2.448c}$$

$$A_2 = -\frac{\varepsilon(1 + \frac{\omega}{2} + \frac{\lambda}{2})}{2(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \tag{2.448d}$$

$$B_{2p+1} = -\frac{\mu(2p + \lambda) + \Omega}{(2(p + 1) + \lambda)(2p + 1 + \lambda + \nu)} = \mu \frac{|\alpha_0| - p - \frac{\lambda}{2}}{2(p + 1 + \frac{\lambda}{2})(p + \gamma + \frac{\lambda}{2})} \quad (2.448e)$$

$$B_{2k+1} = -\frac{\mu(2k + \lambda) + \Omega}{(2(k + 1) + \lambda)(2k + 1 + \lambda + \nu)} = \mu \frac{|\alpha_0| - k - \frac{\lambda}{2}}{2(k + 1 + \frac{\lambda}{2})(k + \gamma + \frac{\lambda}{2})} \quad (2.448f)$$

$$B_{2p+2} = -\frac{\mu(2p + 1 + \lambda) + \Omega}{(2p + 3 + \lambda)(2p + 2 + \nu + \lambda)} = \mu \frac{|\alpha_1| - p - \frac{\lambda}{2}}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \quad (2.448g)$$

$$B_1 = \mu \frac{(|\alpha_0| - \frac{\lambda}{2})}{2(1 + \frac{\lambda}{2})(\gamma + \frac{\lambda}{2})} \quad (2.448h)$$

$$B_2 = \mu \frac{(|\alpha_1| - \frac{\lambda}{2})}{2(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \quad (2.448i)$$

Substitute eq(2.448a) through eq(2.448i) into eq(2.422).

$$C_1 = -\frac{\varepsilon(\frac{\omega}{2} + \frac{\lambda}{2})}{2(\frac{1}{2} + \frac{\lambda}{2})(\gamma - \frac{1}{2} + \frac{\lambda}{2})} C_0 \quad (2.449a)$$

$$\begin{aligned} C_3 = & -\frac{\varepsilon}{2} \mu \left(\frac{(\frac{\omega}{2} + \frac{\lambda}{2})}{(\frac{1}{2} + \frac{\lambda}{2})(\gamma - \frac{1}{2} + \frac{\lambda}{2})} \frac{(|\alpha_1| - \frac{\lambda}{2})}{2(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \right. \\ & \left. + \frac{(1 + \frac{\omega}{2} + \frac{\lambda}{2})}{(\frac{3}{2} + \frac{\lambda}{2})(\gamma + \frac{1}{2} + \frac{\lambda}{2})} \frac{(|\alpha_0| - \frac{\lambda}{2})}{2(1 + \frac{\lambda}{2})(\gamma + \frac{\lambda}{2})} \right) C_0 \end{aligned} \quad (2.449b)$$

$$\begin{aligned}
 C_{2n+1} &= -\varepsilon \left\{ \frac{\left(\frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(\frac{1}{2} + \frac{\lambda}{2}\right)\left(\gamma - \frac{1}{2} + \frac{\lambda}{2}\right)} \prod_{p=0}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2\left(p + \frac{3}{2} + \frac{\lambda}{2}\right)\left(p + \gamma + \frac{1}{2} + \frac{\lambda}{2}\right)} \right. \\
 &+ \sum_{j=1}^{n-1} \left(\frac{\left(j + \frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(j + \frac{1}{2} + \frac{\lambda}{2}\right)\left(j + \gamma - \frac{1}{2} + \frac{\lambda}{2}\right)} \prod_{p=j}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2\left(p + \frac{3}{2} + \frac{\lambda}{2}\right)\left(p + \gamma + \frac{1}{2} + \frac{\lambda}{2}\right)} \right. \\
 &\times \left. \prod_{k=0}^{j-1} \frac{\mu(|\alpha_0| - k - \frac{\lambda}{2})}{2\left(k + 1 + \frac{\lambda}{2}\right)\left(k + \gamma + \frac{\lambda}{2}\right)} \right) \\
 &+ \left. \frac{\left(n + \frac{\omega}{2} + \frac{\lambda}{2}\right)}{2\left(n + \frac{1}{2} + \frac{\lambda}{2}\right)\left(n + \gamma - \frac{1}{2} + \frac{\lambda}{2}\right)} \prod_{p=0}^{n-1} \frac{\mu(|\alpha_0| - p - \frac{\lambda}{2})}{2\left(p + 1 + \frac{\lambda}{2}\right)\left(p + \gamma + \frac{\lambda}{2}\right)} \right\} C_0 \quad (2.449c)
 \end{aligned}$$

Differentiate eq(2.449a), eq(2.449b) and eq(2.449c) with respect to λ and plugging

$\lambda = 0$.

$$\left. \frac{\partial C_1}{\partial \lambda} \right|_{\lambda=0} = -\frac{\varepsilon}{2} \frac{\gamma(1-\omega) - \frac{1}{2}}{\left(\gamma - \frac{1}{2}\right)^2} C_0 \quad (2.450a)$$

$$\begin{aligned}
 \left. \frac{\partial C_3}{\partial \lambda} \right|_{\lambda=0} &= -\frac{\varepsilon}{2} \left\{ \frac{\frac{1}{2} - \gamma(1-\omega)}{\left(\gamma - \frac{1}{2}\right)^2} \frac{|\alpha_1|}{\frac{3}{2}\left(\gamma + \frac{1}{2}\right)} + \frac{\omega}{\left(\gamma - \frac{1}{2}\right)} \frac{\frac{3}{2}\left(\gamma + \frac{1}{2}\right) + (2+\gamma)|\alpha_1|}{3 \cdot \frac{3}{2}\left(\gamma + \frac{1}{2}\right)^2} \right. \\
 &+ \left. \frac{\frac{1}{2}(2+\gamma)(2+\omega) - \frac{3}{2}\left(\gamma + \frac{1}{2}\right)|\alpha_0|}{3 \cdot \frac{3}{2}\left(\gamma + \frac{1}{2}\right)^2} \frac{|\alpha_0|}{\gamma} + \frac{\gamma + (1+\gamma)|\alpha_0|}{2\gamma^2} \frac{(2+\omega)}{3\left(\gamma + \frac{1}{2}\right)} \right\} \left(-\frac{1}{2}\mu \right) C_0 \\
 &\quad (2.450b)
 \end{aligned}$$

$$\begin{aligned}
 \left. \frac{\partial C_{2n+1}}{\partial \lambda} \right|_{\lambda=0} &= -\varepsilon C_0 \left\{ \frac{(\gamma - \frac{1}{2}) - \gamma \omega}{2(\gamma - \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{\mu(|\alpha_1| - p)}{2(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \right. \\
 &+ \sum_{j=1}^{n-1} \left(\frac{(j + \frac{1}{2})(j + \gamma - \frac{1}{2}) - (2j + \gamma)(j + \frac{\omega}{2})}{2^2(j + \frac{1}{2})^2(j + \gamma - \frac{1}{2})^2} \prod_{p=j}^{n-1} \frac{\mu(|\alpha_1| - p)}{2(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \right. \\
 &\times \left. \prod_{k=0}^{j-1} \frac{\mu(|\alpha_0| - k)}{2(k + 1)(k + \gamma)} \right) \\
 &+ \frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2^2(n + \frac{1}{2})^2(n + \gamma - \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{\mu(|\alpha_0| - p)}{2(p + 1)(p + \gamma)} \\
 &+ \left[\frac{\frac{\omega}{2}}{(\gamma - \frac{1}{2})} \frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \right\} \right. \\
 &+ \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\partial}{\partial \lambda} \left\{ \prod_{p=j}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \right. \\
 &\times \left. \prod_{k=0}^{j-1} \frac{\mu(|\alpha_0| - k - \frac{\lambda}{2})}{2(k + 1 + \frac{\lambda}{2})(k + \gamma + \frac{\lambda}{2})} \right\} \\
 &+ \left. \left. \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{\mu(|\alpha_0| - p - \frac{\lambda}{2})}{2(p + 1 + \frac{\lambda}{2})(p + \gamma + \frac{\lambda}{2})} \right\} \right] \right|_{\lambda=0} \Bigg\} \\
 &\hspace{15em} (2.450c)
 \end{aligned}$$

By using simple algebra,

$$\prod_{p=0}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} = \frac{\mu^n(|\alpha_1| - \frac{\lambda}{2})! (\frac{1}{2} + \frac{\lambda}{2})! (\gamma - \frac{1}{2} + \frac{\lambda}{2})!}{2^n(|\alpha_1| - n - \frac{\lambda}{2})! (n + \frac{1}{2} + \frac{\lambda}{2})! (n - \frac{1}{2} + \gamma + \frac{\lambda}{2})!} \quad (2.451a)$$

$$\begin{aligned}
 & \prod_{p=j}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \prod_{k=0}^{j-1} \frac{\mu(|\alpha_0| - k - \frac{\lambda}{2})}{2(k + 1 + \frac{\lambda}{2})(k + \gamma + \frac{\lambda}{2})} \\
 &= \left\{ \frac{\mu^{n-j}(|\alpha_1| - j - \frac{\lambda}{2})! (j + \frac{1}{2} + \frac{\lambda}{2})! (j - \frac{1}{2} + \gamma + \frac{\lambda}{2})!}{2^{n-j}(|\alpha_1| - n - \frac{\lambda}{2})! (n + \frac{1}{2} + \frac{\lambda}{2})! (n - \frac{1}{2} + \gamma + \frac{\lambda}{2})!} \right\} \\
 &\times \left\{ \frac{\mu^j(|\alpha_0| - \frac{\lambda}{2})! (\frac{\lambda}{2})! (\gamma - 1 + \frac{\lambda}{2})!}{2^j(|\alpha_0| - j - \frac{\lambda}{2})! (j + \frac{\lambda}{2})! (j - 1 + \gamma + \frac{\lambda}{2})!} \right\} \tag{2.451b}
 \end{aligned}$$

$$\begin{aligned}
 & \prod_{p=0}^{n-1} \frac{\mu(|\alpha_0| - p - \frac{\lambda}{2})}{2(p + 1 + \frac{\lambda}{2})(p + \gamma + \frac{\lambda}{2})} \\
 &= \frac{\mu^n(|\alpha_0| - \frac{\lambda}{2})! (\frac{\lambda}{2})! (\gamma - 1 + \frac{\lambda}{2})!}{2^n (|\alpha_0| - n - \frac{\lambda}{2})! (n + \frac{\lambda}{2})! (n - 1 + \gamma + \frac{\lambda}{2})!} \tag{2.451c}
 \end{aligned}$$

Differentiate eq(2.451a), eq(2.451b), eq(2.451c) with respect to λ plugging $\lambda = 0$ and

note that $\frac{\partial c_k}{\partial \lambda} = c_k \frac{\partial \ln c_k}{\partial \lambda}$, we obtain

$$\begin{aligned}
 & \frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \right\} \Big|_{\lambda=0} \tag{2.452a} \\
 &= -\frac{1}{2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{1}{(k + \frac{3}{2})} + \frac{1}{(k + \gamma + \frac{1}{2})} \right) \left(-\frac{1}{2} \mu \right)^n
 \end{aligned}$$

$$\begin{aligned}
 & \left. \frac{\partial}{\partial \lambda} \left\{ \prod_{p=j}^{n-1} \frac{\mu(|\alpha_1| - p - \frac{\lambda}{2})}{2(p + \frac{3}{2} + \frac{\lambda}{2})(p + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \prod_{k=0}^{j-1} \frac{\mu(|\alpha_0| - k - \frac{\lambda}{2})}{2(k + 1 + \frac{\lambda}{2})(k + \gamma + \frac{\lambda}{2})} \right\} \right|_{\lambda=0} \\
 &= -\frac{1}{2} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \frac{1}{2} + \gamma)} \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)(k + \gamma)} \left(-\frac{1}{2} \mu \right)^n \\
 & \times \left\{ \sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{1}{(l + 1)} + \frac{1}{(l + \gamma)} \right) \right. \\
 & \left. + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{1}{(l + \frac{3}{2})} + \frac{1}{(l + \frac{1}{2} + \gamma)} \right) \right\}
 \end{aligned} \tag{2.452b}$$

$$\begin{aligned}
 & \left. \frac{\partial}{\partial \lambda} \left\{ \prod_{p=0}^{n-1} \frac{\mu(|\alpha_0| - p - \frac{\lambda}{2})}{2(p + 1 + \frac{\lambda}{2})(p + \gamma + \frac{\lambda}{2})} \right\} \right|_{\lambda=0} \tag{2.452c} \\
 &= -\frac{1}{2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)(p + \gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{1}{(k + 1)} + \frac{1}{(k + \gamma)} \right) \left(-\frac{1}{2} \mu \right)^n
 \end{aligned}$$

Substitute eq(2.452a), eq(2.452b) and eq(2.452c) into eq(2.450c).

$$\left. \frac{\partial C_{2n+1}}{\partial \lambda} \right|_{\lambda=0} = -\frac{\varepsilon}{2} C_0 \left\{ A(\lambda) + B(\lambda) \right\} \Big|_{\lambda=0} \tag{2.453a}$$

where,

$$\begin{aligned}
 A(\lambda) \Big|_{\lambda=0} &= \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \left(-\frac{1}{2}\mu\right)^n \\
 &+ \sum_{j=1}^{n-1} \left(\frac{(j + \frac{1}{2})(j + \gamma - \frac{1}{2}) - (2j + \gamma)(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2(j + \gamma - \frac{1}{2})^2} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \right. \\
 &\times \left. \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)(k + \gamma)} \right) \left(-\frac{1}{2}\mu\right)^n \\
 &+ \frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2(n + \gamma - \frac{1}{2})^2} \\
 &\times \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)(p + \gamma)} \left(-\frac{1}{2}\mu\right)^n \tag{2.453b}
 \end{aligned}$$

$$\begin{aligned}
 B(\lambda) \Big|_{\lambda=0} &= -\frac{\frac{\omega}{2}}{(\gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \\
 &\times \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{1}{(k + \frac{3}{2})} + \frac{1}{(k + \gamma + \frac{1}{2})} \right) \left(-\frac{1}{2}\mu \right)^n \\
 &- \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \frac{1}{2} + \gamma)} \\
 &\times \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)(k + \gamma)} \left(-\frac{1}{2}\mu \right)^n \\
 &\times \left\{ \sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{1}{(l + 1)} + \frac{1}{(l + \gamma)} \right) \right. \\
 &\left. + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{1}{(l + \frac{3}{2})} + \frac{1}{(l + \frac{1}{2} + \gamma)} \right) \right\} \\
 &- \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)(p + \gamma)} \\
 &\times \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{1}{(k + 1)} + \frac{1}{(k + \gamma)} \right) \left(-\frac{1}{2}\mu \right)^n
 \end{aligned} \tag{2.453c}$$

Substitute eq(2.450a), eq(2.450b), eq(2.453a), eq(2.453b) and eq(2.453c) into the r.h.s of eq(2.445b) and plugging $C_0 = \frac{\Gamma(\psi_0 + \gamma)}{\Gamma(\gamma)}$:

$$\sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \Big|_{\lambda=0} = I_{|\alpha_0|, |\alpha_1|} + K_{|\alpha_0|, |\alpha_1|} \tag{2.454}$$

where,

$$\begin{aligned}
 I_{|\alpha_0|,|\alpha_1|} &= -\frac{\varepsilon}{2}x \frac{\Gamma(\psi_0 + \gamma)}{\Gamma(\gamma)} \left\{ \frac{\gamma(1 - \omega) - \frac{1}{2}}{(\gamma - \frac{1}{2})^2} \right. \\
 &+ \left[\frac{\frac{1}{2} - \gamma(1 - \omega)}{(\gamma - \frac{1}{2})^2} \frac{|\alpha_1|}{\frac{3}{2}(\gamma + \frac{1}{2})} + \frac{\frac{1}{2}(2 + \gamma)(2 + \omega) - \frac{3}{2}(\gamma + \frac{1}{2})}{3 \cdot \frac{3}{2}(\gamma + \frac{1}{2})^2} \frac{|\alpha_0|}{\gamma} \right] z \\
 &+ \sum_{n=2}^{\infty} \left\{ \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \right. \\
 &+ \sum_{j=1}^{n-1} \left(\frac{(j + \frac{1}{2})(j + \gamma - \frac{1}{2}) - (2j + \gamma)(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2(j + \gamma - \frac{1}{2})^2} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \right. \\
 &\times \left. \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)(k + \gamma)} \right) \\
 &+ \left. \frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2(n + \gamma - \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)(p + \gamma)} \right\} z^n \Bigg\}
 \end{aligned}
 \tag{2.455a}$$

$$\begin{aligned}
 K_{|\alpha_0|,|\alpha_1|} &= \frac{\varepsilon}{2} x \frac{\Gamma(\psi_0 + \gamma)}{\Gamma(\gamma)} \left\{ \sum_{n=1}^{\infty} \left\{ \frac{\frac{\omega}{2}}{\left(\gamma - \frac{1}{2}\right)} \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{\left(p + \frac{3}{2}\right)\left(p + \gamma + \frac{1}{2}\right)} \right. \right. \\
 &\times \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{(2(k+1) + \gamma)}{\left(k + \frac{3}{2}\right)\left(k + \gamma + \frac{1}{2}\right)} \right) \\
 &+ \frac{\left(n + \frac{\omega}{2}\right)}{2\left(n + \frac{1}{2}\right)\left(n + \gamma - \frac{1}{2}\right)} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p+1)(p+\gamma)} \\
 &\times \left. \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{(2k+1 + \gamma)}{(k+1)(k+\gamma)} \right) \right\} z^n \\
 &+ \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{\left(j + \frac{\omega}{2}\right)}{2\left(j + \frac{1}{2}\right)\left(j + \gamma - \frac{1}{2}\right)} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{\left(p + \frac{3}{2}\right)\left(p + \frac{1}{2} + \gamma\right)} \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k+1)(k+\gamma)} \\
 &\times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{(2l+1 + \gamma)}{(l+1)(l+\gamma)} \right) \right. \\
 &\left. + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{(2(k+1) + \gamma)}{\left(l + \frac{3}{2}\right)\left(l + \frac{1}{2} + \gamma\right)} \right) \right] z^n \Big\} \tag{2.455b}
 \end{aligned}$$

In the above of eq(2.455a) and eq(2.455b), $z = -\frac{1}{2}\mu x^2$. First of all,

(a) $|\alpha_0| = 0$ in eq(2.455a)

$$I_{0,|\alpha_1|} = -\frac{\varepsilon}{2} x \frac{\Gamma(\gamma)}{\Gamma(\gamma)} \frac{\left(\gamma - \frac{1}{2}\right) - \gamma\omega}{\left(\gamma - \frac{1}{2}\right)^2} \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{\left(\frac{3}{2}\right)_n \left(\gamma + \frac{1}{2}\right)_n} z^n \tag{2.456}$$

(b) $|\alpha_0| = 1$ in eq(2.455a)

$$\begin{aligned}
 I_{1,|\alpha_1|} = & -\frac{\varepsilon}{2}x \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)} \left\{ \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right. \\
 & \left. + \frac{3(\gamma + \frac{1}{2}) - (\gamma + 2)(2 + \omega)}{3^2(\gamma + \frac{1}{2})^2} \frac{\Gamma(\gamma)}{2\Gamma(\gamma+1)} \frac{(\gamma + \frac{1}{2})}{|\alpha_1|} \sum_{n=1}^{|\alpha_1|} \frac{3! (-|\alpha_1|)_n}{2^1 1! (\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right\} \quad (2.457)
 \end{aligned}$$

(c) $|\alpha_0| = 2$ in eq(2.455a)

$$\begin{aligned}
 I_{2,|\alpha_1|} = & -\frac{\varepsilon}{2}x \frac{\Gamma(\gamma+2)}{\Gamma(\gamma)} \left\{ \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right. \\
 & + \frac{3(\gamma + \frac{1}{2}) - (\gamma + 2)(2 + \omega)}{3^2(\gamma + \frac{1}{2})^2} \frac{2\Gamma(\gamma)}{2\Gamma(\gamma+1)} \frac{\Gamma(|\alpha_1|)\Gamma(\gamma + \frac{3}{2})}{\Gamma(|\alpha_1| + 1)\Gamma(\gamma + \frac{1}{2})} \\
 & \times \sum_{n=1}^{|\alpha_1|} \frac{3! (-|\alpha_1|)_n}{2^1 1! (\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \\
 & + \frac{5(\gamma + \frac{3}{2}) - (\gamma + 4)(4 + \omega)}{5^2(\gamma + \frac{3}{2})^2} \frac{2! \Gamma(\gamma)}{2^2 2! \Gamma(\gamma+2)} \frac{\Gamma(|\alpha_1| - 1)\Gamma(\gamma + \frac{5}{2})}{\Gamma(|\alpha_1| + 1)\Gamma(\gamma + \frac{1}{2})} \\
 & \left. \times \sum_{n=2}^{|\alpha_1|} \frac{5! (-|\alpha_1|)_n}{2^2 2! (\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right\} \quad (2.458)
 \end{aligned}$$

(d) $|\alpha_0| = 3$ in eq(2.455a)

$$\begin{aligned}
 I_{3,|\alpha_1|} = & -\frac{\varepsilon}{2}x \frac{\Gamma(\gamma+3)}{\Gamma(\gamma)} \left\{ \frac{(\gamma-\frac{1}{2})-\gamma\omega}{(\gamma-\frac{1}{2})^2} \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n(\gamma+\frac{1}{2})_n} z^n \right. \\
 & + \frac{3(\gamma+\frac{1}{2})-(\gamma+2)(2+\omega)}{3^2(\gamma+\frac{1}{2})^2} \frac{3! \Gamma(\gamma)}{2 \ 2! \ 1! \Gamma(\gamma+1)} \frac{\Gamma(|\alpha_1|)\Gamma(\gamma+\frac{3}{2})}{\Gamma(|\alpha_1|+1)\Gamma(\gamma+\frac{1}{2})} \\
 & \times \sum_{n=1}^{|\alpha_1|} \frac{3! (-|\alpha_1|)_n}{2^1 \ 1! (\frac{3}{2})_n(\gamma+\frac{1}{2})_n} z^n \\
 & + \frac{5(\gamma+\frac{3}{2})-(\gamma+4)(4+\omega)}{5^2(\gamma+\frac{3}{2})^2} \frac{3! \Gamma(\gamma)}{2^2 \ 1! \ 2! \Gamma(\gamma+2)} \frac{\Gamma(|\alpha_1|-1)\Gamma(\gamma+\frac{5}{2})}{\Gamma(|\alpha_1|+1)\Gamma(\gamma+\frac{1}{2})} \\
 & \times \sum_{n=2}^{|\alpha_1|} \frac{5! (-|\alpha_1|)_n}{2^2 \ 2! (\frac{3}{2})_n(\gamma+\frac{1}{2})_n} z^n \\
 & + \frac{7(\gamma+\frac{5}{2})-(\gamma+6)(6+\omega)}{7^2(\gamma+\frac{5}{2})^2} \frac{3! \Gamma(\gamma)}{2^3 \ 0! \ 3! \Gamma(\gamma+3)} \frac{\Gamma(|\alpha_1|-2)\Gamma(\gamma+\frac{7}{2})}{\Gamma(|\alpha_1|+1)\Gamma(\gamma+\frac{1}{2})} \\
 & \left. \times \sum_{n=3}^{|\alpha_1|} \frac{7! (-|\alpha_1|)_n}{2^3 \ 3! (\frac{3}{2})_n(\gamma+\frac{1}{2})_n} z^n \right\}
 \end{aligned} \tag{2.459}$$

We can describe a function $I_{|\alpha_0|,|\alpha_1|}$ of degree $|\alpha_0|$ and $|\alpha_1|$ according to eq(2.456)

through eq(2.459).

$$\begin{aligned}
 I_{|\alpha_0|,|\alpha_1|} &= -\frac{\varepsilon}{2}x \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(|\alpha_0|)!(2n)!(\gamma - 1)!}{2^{2n} n! (|\alpha_0| - n)!(n + \gamma - 1)!} \\
 &\quad \times \frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})^2} \\
 &\quad \times \sum_{k=n}^{|\alpha_1|} \frac{(-1)^k (|\alpha_1| - n)! (\frac{1}{2})! (n + \gamma - \frac{1}{2})!}{n! (k + \frac{1}{2})! (|\alpha_1| - k)! (k + \gamma - \frac{1}{2})!} z^k \tag{2.460}
 \end{aligned}$$

Plug eq(2.437) into eq(2.460).

$$\begin{aligned}
 I_{|\alpha_0|,|\alpha_1|} &= -\frac{\varepsilon}{2}x \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-1)^n (-|\alpha_0|)_n (n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{n! (\gamma)_n (n + \frac{1}{2})(n + \gamma - \frac{1}{2})^2} \\
 &\quad \times \sum_{k=n}^{|\alpha_1|} \frac{(-1)^k (|\alpha_1| - n)! (n - \frac{1}{2})! (n + \gamma - \frac{1}{2})!}{2 (k + \frac{1}{2})! (|\alpha_1| - k)! (k + \gamma - \frac{1}{2})!} z^k \tag{2.461}
 \end{aligned}$$

We see the second summation of eq(2.461), where we can make index k start from 0.

Change the interval of index k as $0 \leq k \leq |\alpha_1| - n$. Eq(2.461) is equal to

$$\begin{aligned}
 I_{|\alpha_0|,|\alpha_1|} &= -\frac{\varepsilon}{2}x \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \left(\frac{1}{2} - \frac{(n + \frac{\gamma}{2})(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \right) \\
 &\quad \times \sum_{k=0}^{|\alpha_1| - n} \frac{\Gamma(n + \frac{1}{2})\Gamma(n + \gamma - \frac{1}{2})(n - |\alpha_1|)_k}{\Gamma(n + k + \frac{3}{2})\Gamma(n + k + \gamma + \frac{1}{2})} z^k \tag{2.462}
 \end{aligned}$$

Plug eq(2.455b) and eq(2.462) into eq(2.454), and substitute it and eq(2.447) into eq(2.445b) after that, add it and put eq(2.446) into eq(2.444):

$$\begin{aligned}
 g_2(x) &= \mathcal{QW}_{|\alpha_0|, |\alpha_1|}^F \left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2 \right) \\
 &= (1 + \ln(x))F_{|\alpha_0|}(\gamma; z) \\
 &+ \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=1}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{2 n! (\gamma)_n} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k - |\alpha_0|)} - \frac{1}{(k + 1)} - \frac{1}{(k + \gamma)} \right\} z^n \\
 &- x \frac{\varepsilon \Gamma(|\alpha_0| + \gamma)}{2 \Gamma(\gamma)} \left\{ \ln x \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} z^n \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \right. \\
 &+ \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \left(\frac{1}{2} - \frac{(n + \frac{\gamma}{2})(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \right) \sum_{k=0}^{|\alpha_1| - n} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (n - |\alpha_1|)_k}{\Gamma(n + k + \frac{3}{2}) \Gamma(n + k + \gamma + \frac{1}{2})} z^k \\
 &- \sum_{n=1}^{\infty} \left\{ \frac{\frac{\omega}{2}}{(\gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{(2(k + 1) + \gamma)}{(k + \frac{3}{2})(k + \gamma + \frac{1}{2})} \right) \right. \\
 &+ \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)(p + \gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{(2k + 1 + \gamma)}{(k + 1)(k + \gamma)} \right) \left. \right\} z^n \\
 &- \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})(p + \frac{1}{2} + \gamma)} \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)(k + \gamma)} \\
 &\times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{(2l + 1 + \gamma)}{(l + 1)(l + \gamma)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{(2(k + 1) + \gamma)}{(l + \frac{3}{2})(l + \frac{1}{2} + \gamma)} \right) \right] z^n \left. \right\}
 \end{aligned}$$

Table 2.14: The second independent solution of grand confluent hypergeometric function as $\nu =$ positive integer except 1. We can describe $\mathcal{QW}_{|\alpha_0|, |\alpha_1|}^F(\gamma; z)$ more precisely in the following way.

(1) As $|\alpha_0| = 0$ & $|\alpha_1| = 0$

$$\begin{aligned}
 \mathcal{QW}_{0,0}^F(\gamma; z) &= (1 + \ln(x)) - \frac{\varepsilon}{2}x \left\{ \left[\frac{\omega}{(\gamma - \frac{1}{2})} \ln(x) + \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \right] \right. \\
 &+ \sum_{n=1}^{\infty} \left[\frac{\omega}{2(\gamma - \frac{1}{2})} \frac{\Gamma(n)}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} + \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{1}{n(\gamma)_n} \right] z^n \\
 &\left. + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\Gamma(n)(\frac{3}{2})_j(\gamma + \frac{1}{2})_j}{j! (\frac{3}{2})_n(\gamma + \frac{1}{2})_n(\gamma)_j} z^n \right\} \quad (2.463a)
 \end{aligned}$$

(2) As $|\alpha_0| = 0$ & $|\alpha_1| = 1$

$$\begin{aligned}
 \mathcal{QW}_{0,1}^F(\gamma; z) &= (1 + \ln(x)) - \frac{\varepsilon}{2}x \left\{ \ln(x) \frac{\omega}{(\gamma - \frac{1}{2})} \sum_{n=0}^1 \frac{(-1)^n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right. \\
 &+ \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \sum_{n=0}^1 \frac{(-1)^n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n + \frac{\omega}{(\gamma - \frac{1}{2})} \frac{5(\gamma + \frac{11}{10})}{3^2(\gamma + \frac{1}{2})^2} z \\
 &- \frac{\omega}{2(\gamma - \frac{1}{2})} \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n + \sum_{n=1}^{\infty} \frac{(n + \omega)}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{1}{n(\gamma)_n} z^n \\
 &\left. + \sum_{n=3}^{\infty} \sum_{j=2}^{n-1} \frac{(j + \omega)}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\Gamma(n-1)(\frac{3}{2})_j(\gamma + \frac{1}{2})_j}{j (\frac{3}{2})_n(\gamma + \frac{1}{2})_n(\gamma)_j} z^n \right\} \quad (2.463b)
 \end{aligned}$$

(3) As $|\alpha_0| = 0$ & $|\alpha_1| \geq 2$

$$\begin{aligned}
 & \mathcal{QW}_{0,|\alpha_1|}^F(\gamma; z) \\
 &= (1 + \ln(x)) - \frac{\varepsilon}{2}x \left\{ \ln(x) \frac{\omega}{(\gamma - \frac{1}{2})} \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right. \\
 &+ \frac{(\gamma - \frac{1}{2}) - \gamma\omega}{(\gamma - \frac{1}{2})^2} \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \\
 &- \frac{\omega}{2(\gamma - \frac{1}{2})} \left[\sum_{n=1}^{|\alpha_1|} \frac{(-\psi_1)_n}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{1}{(k + \frac{3}{2})} + \frac{1}{(k + \gamma + \frac{1}{2})} \right) z^n \right. \\
 &+ \left. \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{-|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{(\frac{3}{2})_n(\gamma + \frac{1}{2})_n} z^n \right] + \sum_{n=1}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{1}{n (\gamma)_n} z^n \\
 &+ \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{(-|\alpha_1|)_n (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j (-|\alpha_1|)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \\
 &+ \left. \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\Gamma(n - |\alpha_1|) (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j \Gamma(j - |\alpha_1|) (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \right\} \quad (2.463c)
 \end{aligned}$$

(4) As $|\alpha_0| = 1$ & $|\alpha_1| = 1$

$$\begin{aligned}
 & \mathcal{QW}_{1,1}^F(\gamma; z) \\
 &= (1 + \ln(x))\gamma \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} z^n + \frac{(\gamma + \frac{1}{2})}{\gamma} z \\
 & - x \frac{\varepsilon}{2} \gamma \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} \sum_{k=0}^{1-n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(n-1)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^{n+k} \right. \\
 & + \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} \left(\frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2 (n + \gamma - \frac{1}{2})^2} \right) \\
 & \times \sum_{k=0}^{1-n} \frac{(n-1)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^{n+k} \\
 & + \left\{ \frac{\omega}{(\gamma - \frac{1}{2})} \frac{5(\gamma + \frac{11}{10})}{3^2 (\gamma + \frac{1}{2})^2} + \frac{(2 + \omega)}{3 \gamma^2} \right\} z - \frac{\omega}{2(\gamma - \frac{1}{2})} \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \\
 & - \sum_{n=2}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{\Gamma(n-1)}{\Gamma(n+1)(\gamma)_n} z^n \\
 & \left. - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\Gamma(n-1)(\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_n} z^n \right\} \tag{2.463d}
 \end{aligned}$$

(5) As $|\alpha_0| = 1$ & $|\alpha_1| = 2$

$$\begin{aligned}
 & \mathcal{QW}_{1,2}^F(\gamma; z) \\
 &= (1 + \ln(x))\gamma \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} z^n + \frac{(\gamma + \frac{1}{2})}{\gamma} z \\
 & - x \frac{\varepsilon}{2} \gamma \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} \sum_{k=0}^{2-n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(n-2)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^{n+k} \right. \\
 & + \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} \left(\frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2 (n + \gamma - \frac{1}{2})^2} \right) \\
 & \times \sum_{k=0}^{2-n} \frac{(n-2)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^{n+k} + \left. \left\{ \frac{\omega}{(\gamma - \frac{1}{2})} \frac{7(\gamma + \frac{19}{14})}{3^2 (\gamma + \frac{1}{2})^2} + \frac{(2 + \omega)}{3 \gamma^2} \right\} z \right. \\
 & - \frac{\omega}{(\gamma - \frac{1}{2})} \left[\frac{\Gamma(\frac{1}{2})\Gamma(\gamma + \frac{1}{2})}{\Gamma(\frac{7}{2})\Gamma(\gamma + \frac{5}{2})} \sum_{k=0}^1 \left(\frac{1}{(2-k)} + \frac{(2(k+1) + \gamma)}{(k + \frac{3}{2})(k + \gamma + \frac{1}{2})} \right) z^2 \right. \\
 & - \left. \sum_{n=3}^{\infty} \frac{\Gamma(n-2)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \right] - \sum_{n=2}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{\Gamma(n-1)}{\Gamma(n+1)(\gamma)_n} z^n \\
 & - \frac{(1 + \frac{\omega}{2})}{\frac{3}{2}(\gamma + \frac{1}{2})} \frac{1}{\frac{5}{2}\gamma(\gamma + \frac{3}{2})} \left(\frac{17}{5} + \frac{1}{\gamma} + \frac{1}{(\gamma + \frac{3}{2})} \right) z^2 + \sum_{n=3}^{\infty} \frac{(1 + \frac{\omega}{2})}{2^2 \gamma} \frac{\Gamma(n-2)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \\
 & - \left. \sum_{n=4}^{\infty} \sum_{j=3}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\Gamma(j-1)\Gamma(n-2)(\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{\Gamma(j+1)\Gamma(j-2)(\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \right\} \quad (2.463e)
 \end{aligned}$$

(6) As $|\alpha_0| = 1$ & $|\alpha_1| \geq 3$

$$\begin{aligned}
 & \mathcal{QW}_{1,|\alpha_1|}^F(\gamma; z) \\
 &= (1 + \ln(x))\gamma \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} z^n + \frac{(\gamma + \frac{1}{2})}{\gamma} z \\
 & - x \frac{\varepsilon}{2} \gamma \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(n - |\alpha_1|)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^{n+k} \right. \\
 & + \sum_{n=0}^1 \frac{(-1)^n}{(\gamma)_n} \left(\frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2 (n + \gamma - \frac{1}{2})^2} \right) \\
 & \times \sum_{k=0}^{|\alpha_1|-n} \frac{(n - |\alpha_1|)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^{n+k} + \left\{ \frac{\omega}{(\gamma - \frac{1}{2})} \frac{\frac{3}{2}(\gamma + \frac{1}{2}) + |\alpha_1|(\gamma + 2)}{3 \frac{3}{2}(\gamma + \frac{1}{2})^2} + \frac{(2 + \omega)}{3 \gamma^2} \right\} z \\
 & - \frac{\omega}{2(\gamma - \frac{1}{2})} \left[\sum_{n=2}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} \sum_{k=0}^1 \left(\frac{1}{(|\alpha_1| - k)} + \frac{(2(k+1) + \gamma)}{(k + \frac{3}{2})(k + \gamma + \frac{1}{2})} \right) z^2 \right. \\
 & - \left. \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \right] \\
 & - \sum_{n=2}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{\Gamma(n - 1)}{\Gamma(n + 1)(\gamma)_n} z^n \\
 & - \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{(-|\alpha_1|)_n \Gamma(j - 1) (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (-|\alpha_1|)_j (\gamma)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \\
 & - \sum_{n=|\alpha_1|+1}^{\infty} \frac{(1 + \frac{\omega}{2})}{2\gamma} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1|) \Gamma(n - |\alpha_1|)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \tag{2.463f} \\
 & - \left. \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{\Gamma(j - 1) \Gamma(n - |\alpha_1|) (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{\Gamma(j + 1) \Gamma(j - |\alpha_1|) (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \right\}
 \end{aligned}$$

(7) As $|\alpha_0| \geq 2$ & $|\alpha_0| \leq |\alpha_1| \leq |\alpha_0| + 1$

$$\begin{aligned}
 & \mathcal{QW}_{|\alpha_0|, |\alpha_1|}^F(\gamma; z) \\
 &= (1 + \ln(x))F_{|\alpha_0|}(\gamma; z) + \sum_{n=1}^{|\alpha_0|} \frac{(-|\alpha_0|)_n (\gamma)_{|\alpha_0|}}{2 n! (\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(k - |\alpha_0|)} - \frac{(2k + \gamma + 1)}{(k + 1)(k + \gamma)} \right) z^n \\
 & - \frac{\varepsilon}{2} x \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \left\{ \ln x \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{n! (\gamma)_n} \right. \\
 & \times \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(n - |\alpha_1|)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^k \\
 & + \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{n! (\gamma)_n} \frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2 (n + \gamma - \frac{1}{2})^2} \\
 & \times \sum_{k=0}^{|\alpha_1| - n} \frac{(n - |\alpha_1|)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})} z^k \\
 & + \left(\frac{\omega}{(\gamma - \frac{1}{2})} \frac{3(\gamma + \frac{1}{2}) + 2|\alpha_1|(2 + \gamma)}{3^2 (\gamma + \frac{1}{2})^2} + \frac{(2 + \omega) \gamma + |\alpha_0|(\gamma + 1)}{(\gamma + \frac{1}{2}) 2 \cdot \gamma^2} \right) z \\
 & - \frac{\omega}{2(\gamma - \frac{1}{2})} \left\{ \sum_{n=2}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{1}{(k + \frac{3}{2})} + \frac{1}{(k + \gamma + \frac{1}{2})} \right) z^n \right. \\
 & \left. - \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \right\} \\
 & - \sum_{n=2}^{|\alpha_0|} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{1}{(k + 1)} + \frac{1}{(k + \gamma)} \right) z^n
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=|\alpha_0|+1}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(-1)^{|\alpha_1|}(\psi_1)! \Gamma(n - |\alpha_1|)}{n! (\gamma)_n} z^n \\
 & - \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{(-|\alpha_1|)_n (|\alpha_0|)_j (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (-|\alpha_1|)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} \\
 & \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{(2l + 1 + \gamma)}{(l + 1)(l + \gamma)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{(2(l + 2) + \gamma)}{(l + \frac{3}{2})(l + \gamma + \frac{1}{2})} \right) \right] z^n \\
 & + \sum_{n=|\alpha_1|+1}^{\infty} \sum_{j=1}^{|\alpha_0|} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \\
 & \times \frac{(-1)^{|\alpha_1|} (-|\alpha_0|)_j (|\alpha_1|)! \Gamma(n - |\alpha_1|) (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (-|\alpha_1|)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \\
 & + \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \\
 & \times \left. \frac{(-1)^{|\alpha_0|} (|\alpha_0|)! \Gamma(n - |\alpha_1|) \Gamma(j - |\alpha_0|) (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! \Gamma(j - |\alpha_1|) (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \right\}
 \end{aligned}$$

(2.463g)

(8) As $|\alpha_0| \geq 2$ & $|\alpha_1| \geq |\alpha_0| + 2$

$$\begin{aligned}
 & \mathcal{QW}_{|\alpha_0|, |\alpha_1|}^F(\gamma; z) \\
 &= (1 + \ln(x))F_{|\alpha_0|}(\gamma; z) + \sum_{n=1}^{|\alpha_0|} \frac{(-|\alpha_0|)_n (\gamma)_{|\alpha_0|}}{2 n! (\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(k - |\alpha_0|)} - \frac{(2k + \gamma + 1)}{(k + 1)(k + \gamma)} \right) z^n \\
 & - \frac{\varepsilon}{2} x \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \left\{ \ln x \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{n! (\gamma)_n} \right. \\
 & \times \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(n - |\alpha_1|)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})_k} z^k \\
 & + \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{n! (\gamma)_n} \frac{(n + \frac{1}{2})(n + \gamma - \frac{1}{2}) - (2n + \gamma)(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2 (n + \gamma - \frac{1}{2})^2} \\
 & \times \sum_{k=0}^{|\alpha_1| - n} \frac{(n - |\alpha_1|)_k}{(n + \frac{3}{2})_k (n + \gamma + \frac{1}{2})} z^k \\
 & + \left(\frac{\omega}{(\gamma - \frac{1}{2})} \frac{3(\gamma + \frac{1}{2}) + 2|\alpha_1|(2 + \gamma)}{3^2 (\gamma + \frac{1}{2})^2} + \frac{(2 + \omega) \gamma + |\alpha_0|(\gamma + 1)}{(\gamma + \frac{1}{2}) 2 \cdot \gamma^2} \right) z \\
 & - \frac{\omega}{2(\gamma - \frac{1}{2})} \left\{ \sum_{n=2}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{1}{(k + \frac{3}{2})} + \frac{1}{(k + \gamma + \frac{1}{2})} \right) z^n \right. \\
 & \left. - \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{(\frac{3}{2})_n (\gamma + \frac{1}{2})_n} z^n \right\}
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{n=2}^{|\alpha_0|} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(-|\alpha_0|)_n}{n! (\gamma)_n} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{1}{(k + 1)} + \frac{1}{(k + \gamma)} \right) z^n \\
 & + \sum_{n=|\alpha_0|+1}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \frac{(-1)^{|\alpha_1|} (\psi_1)! \Gamma(n - |\alpha_1|)}{n! (\gamma)_n} z^n \\
 & - \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \frac{(-|\alpha_1|)_n (|\alpha_0|)_j (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (-|\alpha_1|)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} \\
 & \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{(2l + 1 + \gamma)}{(l + 1)(l + \gamma)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{(2(l + 2) + \gamma)}{(l + \frac{3}{2})(l + \gamma + \frac{1}{2})} \right) \right] z^n \\
 & + \sum_{n=|\alpha_0|+2}^{|\alpha_1|} \sum_{j=|\alpha_0|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \\
 & \times \frac{(-1)^{|\alpha_0|} (|\alpha_0|)! \Gamma(j - |\alpha_0|) (-|\alpha_1|)_n (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (-|\alpha_1|)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \\
 & + \sum_{n=|\alpha_1|+1}^{\infty} \sum_{j=1}^{|\alpha_0|} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \\
 & \times \frac{(-1)^{|\alpha_1|} (|\alpha_1|)! \Gamma(n - |\alpha_1|) (-|\alpha_0|)_j (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! (-|\alpha_1|)_j (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \\
 & + \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \\
 & \times \frac{(-1)^{|\alpha_0|} (|\alpha_0|)! \Gamma(n - |\alpha_1|) \Gamma(j - |\alpha_0|) (\frac{3}{2})_j (\gamma + \frac{1}{2})_j}{j! \Gamma(j - |\alpha_1|) (\frac{3}{2})_n (\gamma + \frac{1}{2})_n (\gamma)_j} z^n \Big\}
 \end{aligned} \tag{2.463h}$$

(C) the case of $\nu = 1$

As ν is 1, then two solution are identical: ($\lambda_1 = \lambda_2$). Then two solutions can be

solved by Case B) in theorem 1, the first independent solution is

$$g_1(x) = G(x)\Big|_{\lambda=0, \gamma=1} = \mathcal{QW}_{|\alpha_0|, |\alpha_1|} \left(\gamma = 1; z = -\frac{1}{2} \mu x^2 \right) \quad (2.464)$$

The second independent solution is

$$g_2(x) = g_1(x) \ln x + x^{\lambda_1} \sum_{n=0}^{\infty} b_n(\lambda_1) x^n \quad (2.465)$$

To generate this solution, keep the recurrence formula in term of λ and use it to find to coefficients $C_n (n \geq 1)$ in terms of both λ and C_0 , where coefficient C_0 remains arbitrary. Substitute these C_n into $G(x) = x^\lambda \sum_{n=0}^{\infty} C_n x^n$ to obtain a function $G(\lambda, x)$ which depends on the variables λ & x . From the Case B) in theorem 1, the second independent solution is

$$g_2(x) = \frac{\partial G(\lambda, x)}{\partial \lambda} \Big|_{\lambda=0, \gamma=1} = \ln x \sum_{n=0}^{\infty} C_n x^{n+\lambda} \Big|_{\lambda=0, \gamma=1} + \sum_{n=0}^{\infty} \frac{\partial C_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=0, \gamma=1} \quad (2.466)$$

From eq(2.262), $G(x) = g(x)_{domin.} + g(x)_{small}$. Plug it into eq(2.466).

$$g_2(x) = \frac{\partial g(x)_{domin.}}{\partial \lambda} \Big|_{\lambda=0, \gamma=1} + \frac{\partial g(x)_{small}}{\partial \lambda} \Big|_{\lambda=0, \gamma=1} \quad (2.467)$$

The first term of r.h.s in eq(2.467) is

$$\begin{aligned}
 \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=0, \gamma=1} &= \left. \frac{\partial}{\partial \lambda} \left\{ \sum_{n=0}^{\infty} C_{2n}(\lambda, n) x^{2n+\lambda} \right\} \right|_{\lambda=0, \gamma=1} \\
 &= \left. \ln x \sum_{n=0}^{\infty} C_{2n}(\lambda, n) x^{2n+\lambda} \right|_{\lambda=0, \gamma=1} \\
 &\quad + \left. \sum_{n=0}^{\infty} \frac{\partial C_{2n}(\lambda, n)}{\partial \lambda} x^{2n+\lambda} \right|_{\lambda=0, \gamma=1} \tag{2.468a}
 \end{aligned}$$

The second term of r.h.s in eq(2.467) is

$$\begin{aligned}
 \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=0, \gamma=1} &= \left. \frac{\partial}{\partial \lambda} \left\{ \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \right\} \right|_{\lambda=0, \gamma=1} \\
 &= \left. \ln x \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \right|_{\lambda=0, \gamma=1} \\
 &\quad + \left. \sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \right|_{\lambda=0, \gamma=1} \tag{2.468b}
 \end{aligned}$$

We know that $g(x)_{domin.}$ as $\lambda = 0$ is equal to the first kind of confluent hypergeometric polynomial. So a solution for eq(2.468a) is similar to Remark 1 on appendix. Replace α and λ by $-|\alpha_0|$ and $\frac{\lambda}{2}$ in eq(2.154) on appendix, then differentiate it with respect to λ . After that, plug $\lambda = 0$ and $\gamma = 1$, and substitute it into we get in eq(2.169) on

appendix. Then we obtain

$$\begin{aligned} \left. \frac{\partial g(x)_{\text{domin.}}}{\partial \lambda} \right|_{\lambda=0, \gamma=1} &= \ln(x) F_{|\alpha_0|}(\gamma = 1; z) \\ &+ \sum_{n=1}^{|\alpha|} \frac{(-1)^n (|\alpha|!)^2}{2 (n!)^2 (|\alpha| - n)!} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k - |\alpha|)} - \frac{2}{(k + 1)} \right\} z^n \end{aligned} \quad (2.469)$$

The first summation of eq(2.468b) which is $\sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=0}$ is equivalent to eq(2.124) plugging $\gamma = 1$ in it. Then, we obtain

$$\begin{aligned} \ln x \sum_{n=0}^{\infty} C_{2n+1}(\lambda, n) x^{2n+1+\lambda} \Big|_{\lambda=0} &= -\frac{\varepsilon}{2} x \ln x \prod_{|\alpha_0|}^{|\alpha_1|} (\gamma = 1; z) \\ &= -x \ln x \frac{\varepsilon}{2} \Gamma(|\alpha_0| + 1) \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{(n!)^2} z^n \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2}) (\Gamma(n + \frac{1}{2}))^2 (n - |\alpha_1|)_k}{(\Gamma(k + n + \frac{3}{2}))^2} z^k \end{aligned} \quad (2.470)$$

The second summation of eq(2.468b) which is $\sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \Big|_{\lambda=0, \gamma=1}$ is equivalent to eq(2.454) plugging $\gamma = 1$ into it.

$$\sum_{n=0}^{\infty} \frac{\partial C_{2n+1}(\lambda, n)}{\partial \lambda} x^{2n+1+\lambda} \Big|_{\lambda=0, \gamma=1} = I_{|\alpha_0|, |\alpha_1|} \Big|_{\gamma=1} + K_{|\alpha_0|, |\alpha_1|} \Big|_{\gamma=1} \quad (2.471)$$

Plug $\gamma = 1$ into eq(2.455b) and eq(2.462).

$$\begin{aligned}
 I_{|\alpha_0|,|\alpha_1|} \Big|_{\gamma=1} &= -\frac{\varepsilon}{2} x \Gamma(|\alpha_0| + 1) \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{(n!)^2} \left(\frac{1}{2} - \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})} \right) \\
 &\quad \times \sum_{k=0}^{|\alpha_1|-n} \frac{(\Gamma(n + \frac{1}{2}))^2 (n - |\alpha_1|)_k}{(\Gamma(n + k + \frac{3}{2}))^2} z^k
 \end{aligned} \tag{2.472a}$$

$$\begin{aligned}
 K_{|\alpha_0|,|\alpha_1|} \Big|_{\gamma=1} &= \frac{\varepsilon}{2} x \Gamma(\psi_0 + 1) \left\{ \sum_{n=1}^{\infty} \left\{ \omega \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{2}{(k + \frac{3}{2})} \right) \right. \right. \\
 &\quad + \left. \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{2}{(k + 1)} \right) \right\} z^n \\
 &\quad + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})^2} \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)^2} \\
 &\quad \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{2}{(l + 1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{2}{(l + \frac{3}{2})} \right) \right] z^n \Big\}
 \end{aligned} \tag{2.472b}$$

Plug eq(2.472a) and eq(2.472b) into eq(2.471), and substitute it and eq(2.470) into eq(2.468b) after that, add eq(2.469) into eq(2.467):

$$\begin{aligned}
 g_2(x) &= \mathcal{QW}_{|\alpha_0|, |\alpha_1|}^S(\gamma = 1; z = -\frac{1}{2}\mu x^2) \\
 &= \ln(x)F_{|\alpha_0|}(\gamma = 1; z) + \sum_{n=1}^{|\alpha_0|} \frac{(-1)^n (|\alpha_0|!)^2}{2(n!)^2 (|\alpha_0| - n)!} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k - |\alpha_0|)} - \frac{2}{(k + 1)} \right\} z^n \\
 &\quad - x \frac{\varepsilon}{2} \Gamma(|\alpha_0| + 1) \left\{ \ln x \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n}{(n!)^2} z^n \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2}) (\Gamma(n + \frac{1}{2}))^2 (n - |\alpha_1|)_k}{(\Gamma(k + n + \frac{3}{2}))^2} z^k \right. \\
 &\quad + \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{(n!)^2} \left(\frac{1}{2} - \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})} \right) \sum_{k=0}^{|\alpha_1| - n} \frac{(\Gamma(n + \frac{1}{2}))^2 (n - |\alpha_1|)_k}{(\Gamma(n + k + \frac{3}{2}))^2} z^k \\
 &\quad - \sum_{n=1}^{\infty} \left\{ \omega \prod_{p=0}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{2}{(k + \frac{3}{2})} \right) \right. \\
 &\quad \left. + \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{(p - |\alpha_0|)}{(p + 1)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{2}{(k + 1)} \right) \right\} z^n \\
 &\quad - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \prod_{p=j}^{n-1} \frac{(p - |\alpha_1|)}{(p + \frac{3}{2})^2} \prod_{k=0}^{j-1} \frac{(k - |\alpha_0|)}{(k + 1)^2} \\
 &\quad \left. \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{2}{(l + 1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{2}{(l + \frac{3}{2})} \right) \right] z^n \right\}
 \end{aligned}$$

Table 2.15: The second independent solution of grand confluent hypergeometric function as $\nu = 1$

We can describe $\mathcal{QW}_{|\alpha_0|, |\alpha_1|}^S(\gamma = 1; z)$ more precisely in the following way.

(1) As $|\alpha_0| = 0$ & $|\alpha_1| = 0$

$$\begin{aligned}
 & \mathcal{QW}_{0,0}^S(\gamma = 1; z) \\
 &= \ln(x) - \frac{\varepsilon}{2}x \left\{ \left[2\omega \ln(x) + 2^2 \left(\frac{1}{2} - \omega \right) \right] \right. \\
 &+ \left. \sum_{n=1}^{\infty} \left[\omega \frac{\Gamma(n)}{\left(\left(\frac{3}{2} \right)_n \right)^2} + \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{1}{n n!} \right] z^n + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{\Gamma(n) \left(\left(\frac{3}{2} \right)_j \right)^2}{(j!)^2 \left(\left(\frac{3}{2} \right)_n \right)^2} z^n \right\} \\
 & \hspace{15em} (2.473a)
 \end{aligned}$$

(2) As $|\alpha_0| = 0$ & $|\alpha_1| = 1$

$$\begin{aligned}
 & \mathcal{QW}_{0,1}^S(\gamma = 1; z) \\
 &= \ln(x) - \frac{\varepsilon}{2}x \left\{ 2\omega \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{\left(\left(\frac{3}{2} \right)_n \right)^2} z^n + 2^2 \left(\frac{1}{2} - \omega \right) \sum_{n=0}^1 \frac{(-1)^n}{\left(\left(\frac{3}{2} \right)_n \right)^2} z^n + 2\omega \frac{2^2 \cdot 5(\gamma + \frac{11}{10})}{3^4} z \right. \\
 &- \omega \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{\left(\left(\frac{3}{2} \right)_n \right)^2} z^n + \sum_{n=1}^{\infty} \frac{(n + \omega)}{2(n + \frac{1}{2})^2} \frac{1}{n n!} z^n + \left. \sum_{n=3}^{\infty} \sum_{j=2}^{n-1} \frac{(j + \omega)}{2(j + \frac{1}{2})^2} \frac{\Gamma(n-1) \left(\left(\frac{3}{2} \right)_j \right)^2}{j \cdot j! \left(\left(\frac{3}{2} \right)_n \right)^2} z^n \right\} \\
 & \hspace{15em} (2.473b)
 \end{aligned}$$

(3) As $|\alpha_0| = 0$ & $|\alpha_1| \geq 2$

$$\begin{aligned}
 & \mathcal{QW}_{0,|\alpha_1|}^S(\gamma = 1; z) \\
 &= \ln(x) - \frac{\varepsilon}{2}x \left\{ \ln(x)(2\omega) \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{((\frac{3}{2})_n)^2} z^n + 2^2 \left(\frac{1}{2} - \omega\right) \sum_{n=0}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{((\frac{3}{2})_n)^2} z^n \right. \\
 & - \omega \left[\sum_{n=1}^{|\alpha_1|} \frac{(-\psi_1)_n}{((\frac{3}{2})_n)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{2}{(k + \frac{3}{2})} \right) z^n \right. \\
 & \left. \left. + \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{-|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{((\frac{3}{2})_n)^2} z^n \right] + \sum_{n=1}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{1}{n n!} z^n \right. \\
 & \left. + \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-|\alpha_1|)_n ((\frac{3}{2})_j)^2}{j j! (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} z^n \right. \\
 & \left. + \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{\Gamma(n - |\alpha_1|) ((\frac{3}{2})_j)^2}{j j! \Gamma(j - |\alpha_1|) ((\frac{3}{2})_n)^2} z^n \right\} \tag{2.473c}
 \end{aligned}$$

(4) As $|\alpha_0| = 1$ & $|\alpha_1| = 1$

$$\begin{aligned}
 & \mathcal{QW}_{1,1}^S(\gamma = 1; z) \\
 &= \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{n!} z^n + \frac{3}{2}z - x \frac{\varepsilon}{2} \gamma \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{n!} \sum_{k=0}^{1-n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})^2} \frac{(n-1)_k}{((n + \frac{3}{2})_k)^2} z^{n+k} \right. \\
 &+ \sum_{n=0}^1 \frac{(-1)^n}{n!} \left(\frac{(n + \frac{1}{2}) - 2(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^3} \right) \sum_{k=0}^{1-n} \frac{(n-1)_k}{((n + \frac{3}{2})_k)^2} z^{n+k} \\
 &+ \left\{ \omega \frac{2^2 \cdot 7}{3^3} + \frac{(2 + \omega)}{3} \right\} z - \omega \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{((\frac{3}{2})_n)^2} z^n \\
 &\left. - \sum_{n=2}^{\infty} \frac{(n + \frac{\omega}{2})}{2((n + \frac{1}{2})^2) n! \Gamma(n+1)} z^n - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{\Gamma(n-1) ((\frac{3}{2})_j)^2}{j! n! ((\frac{3}{2})_n)^2} z^n \right\} \\
 & \tag{2.473d}
 \end{aligned}$$

(5) As $|\alpha_0| = 1$ & $|\alpha_1| = 2$

$$\begin{aligned}
 & \mathcal{QW}_{1,2}^S(\gamma = 1; z) \\
 &= \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{n!} z^n + \frac{3}{2}z - x \frac{\varepsilon}{2} \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{n!} \sum_{k=0}^{2-n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})^2} \frac{(n-2)_k}{((n + \frac{3}{2})_k)^2} z^{n+k} \right. \\
 &+ \sum_{n=0}^1 \frac{(-1)^n}{n!} \left(\frac{(n + \frac{1}{2}) - 2(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^3} \right) \sum_{k=0}^{2-n} \frac{(n-2)_k}{((n + \frac{3}{2})_k)^2} z^{n+k} + \left(\omega \frac{2^2 \cdot 11}{3^3} + \frac{(2 + \omega)}{3} \right) z \\
 &- 2\omega \left[\frac{2^5}{3^2 \cdot 5^2} \sum_{k=0}^1 \left(\frac{1}{(2-k)} + \frac{2}{(k + \frac{3}{2})} \right) z^2 - \sum_{n=3}^{\infty} \frac{\Gamma(n-2)}{((\frac{3}{2})_n)^2} z^n \right] \\
 &- \sum_{n=2}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{\Gamma(n-1)}{n! \Gamma(n+1)} z^n - \left(1 + \frac{\omega}{2} \right) \frac{2^7}{3 \cdot 5^3} z^2 + \sum_{n=3}^{\infty} \frac{(1 + \frac{\omega}{2})}{2^2} \frac{\Gamma(n-2)}{((\frac{3}{2})_n)^2} z^n \\
 &- \left. \sum_{n=4}^{\infty} \sum_{j=3}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{\Gamma(j-1)\Gamma(n-2)((\frac{3}{2})_j)^2}{\Gamma(j+1)\Gamma(j-2)j! ((\frac{3}{2})_n)^2} z^n \right\} \tag{2.473e}
 \end{aligned}$$

(6) As $|\alpha_0| = 1$ & $|\alpha_1| \geq 3$

$$\begin{aligned}
 & \text{mathcal}QW_{1,|\alpha_1|}^S(\gamma = 1; z) \\
 &= \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{n!} z^n + \frac{3}{2}z - x \frac{\varepsilon}{2} \left\{ \ln(x) \sum_{n=0}^1 \frac{(-1)^n}{n!} \sum_{k=0}^{|\alpha_1|-n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})^2} \frac{(n - |\alpha_1|)_k}{((n + \frac{3}{2})_k)^2} z^{n+k} \right. \\
 &+ \sum_{n=0}^1 \frac{(-1)^n}{n!} \left(\frac{(n + \frac{1}{2}) - 2(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^3} \right) \sum_{k=0}^{|\alpha_1|-n} \frac{(n - |\alpha_1|)_k}{((n + \frac{3}{2})_k)^2} z^{n+k} \\
 &+ \left[2\omega \left(\left(\frac{2}{3} \right)^2 + |\alpha_1| \left(\frac{2}{3} \right)^3 \right) + \frac{(2 + \omega)}{3} \right] z \\
 &- \omega \left[\sum_{n=2}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{((\frac{3}{2})_n)^2} \sum_{k=0}^1 \left(\frac{1}{(|\alpha_1| - k)} + \frac{2}{(k + \frac{3}{2})} \right) \right] z^2 \\
 &- \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{((\frac{3}{2})_n)^2} z^n \left. - \sum_{n=2}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{\Gamma(n - 1)}{n! \Gamma(n + 1)} z^n \right. \\
 &- \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-|\alpha_1|)_n \Gamma(j - 1) ((\frac{3}{2})_j)^2}{(j!)^2 (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} z^n \\
 &- \sum_{n=|\alpha_1|+1}^{\infty} \frac{(1 + \frac{\omega}{2})}{2} \frac{(-1)_{|\alpha_1|} \Gamma(|\alpha_1|) \Gamma(n - |\alpha_1|)}{((\frac{3}{2})_n)^2} z^n \\
 &- \left. \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{\Gamma(j - 1) \Gamma(n - |\alpha_1|) ((\frac{3}{2})_j)^2}{j! \Gamma(j + 1) \Gamma(j - |\alpha_1|) ((\frac{3}{2})_n)^2} z^n \right\} \tag{2.473f}
 \end{aligned}$$

(7) As $|\alpha_0| \geq 2$ & $|\alpha_0| \leq |\alpha_1| \leq |\alpha_0| + 1$

$$\begin{aligned}
 & \mathcal{QW}_{|\alpha_0|, |\alpha_1|}^S(\gamma = 1; z) \\
 &= \ln(x)F_{|\alpha_0|}(\gamma = 1; z) + \sum_{n=1}^{|\alpha_0|} \frac{(-|\alpha_0|)_n (|\alpha_0|)!}{2 (n!)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(k - |\alpha_0|)} - \frac{2}{(k + 1)} \right) z^n \\
 & - \frac{\varepsilon}{2} x \Gamma(|\alpha_0| + 1) \left\{ \ln x \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{(n!)^2} \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})^2} \frac{(n - |\alpha_1|)_k}{((n + \frac{3}{2})_k)^2} z^k \right. \\
 & + \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{(n!)^2} \frac{(n + \frac{1}{2}) - 2(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^3} \sum_{k=0}^{|\alpha_1| - n} \frac{(n - |\alpha_1|)_k}{((n + \frac{3}{2})_k)^2} z^k \\
 & + \left[2\omega \left(\frac{2}{3^2} + \left(\frac{2}{3} \right)^3 \right) + (2 + \omega) \frac{(1 + 2|\alpha_0|)}{3} \right] z - \omega \left\{ \sum_{n=2}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{((\frac{3}{2})_n)^2} \right. \\
 & \times \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{2}{(k + \frac{3}{2})} \right) z^n - \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{((\frac{3}{2})_n)^2} z^n \left. \right\} \\
 & - \sum_{n=2}^{|\alpha_0|} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{(-|\alpha_0|)_n}{(n!)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{2}{(k + 1)} \right) z^n \\
 & + \sum_{n=|\alpha_0|+1}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{(-1)^{|\alpha_1|} (\psi_1)! \Gamma(n - |\alpha_1|)}{(n!)^2} z^n - \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \\
 & \times \frac{(-|\alpha_1|)_n (|\alpha_0|)_j ((\frac{3}{2})_j)^2}{(j!)^2 (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{2}{(l + 1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{2}{(l + \frac{3}{2})} \right) \right] z^n \\
 & + \sum_{n=|\alpha_1|+1}^{\infty} \sum_{j=1}^{|\alpha_0|} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-1)^{|\alpha_1|} (-|\alpha_0|)_j (|\alpha_1|)! \Gamma(n - |\alpha_1|) ((\frac{3}{2})_j)^2}{(j!)^2 (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} z^n \\
 & + \left. \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-1)^{|\alpha_0|} (|\alpha_0|)! \Gamma(n - |\alpha_1|) \Gamma(j - |\alpha_0|) ((\frac{3}{2})_j)^2}{(j!)^2 \Gamma(j - |\alpha_1|) ((\frac{3}{2})_n)^2} z^n \right\}
 \end{aligned}$$

(2.473g)

(8) As $|\alpha_0| \geq 2$ & $|\alpha_1| \geq |\alpha_0| + 2$

$$\begin{aligned}
 & \mathcal{QW}_{|\alpha_0|, |\alpha_1|}^S(\gamma = 1; z) \\
 &= \ln(x)F_{|\alpha_0|}(\gamma = 1; z) + \sum_{n=1}^{|\alpha_0|} \frac{(-|\alpha_0|)_n (|\alpha_0|)!}{2 (n!)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(k - |\alpha_0|)} - \frac{2}{(k + 1)} \right) z^n \\
 & - \frac{\varepsilon}{2} x \Gamma(|\alpha_0| + 1) \left\{ \ln x \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{(n!)^2} \sum_{k=0}^{|\alpha_1| - n} \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})^2} \frac{(n - |\alpha_1|)_k}{((n + \frac{3}{2})_k)^2} z^k \right. \\
 & + \sum_{n=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_n z^n}{(n!)^2} \frac{(n + \frac{1}{2}) - 2(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^3} \sum_{k=0}^{|\alpha_1| - n} \frac{(n - |\alpha_1|)_k}{((n + \frac{3}{2})_k)^2} z^k \\
 & + \left[2\omega \left(\frac{2}{3^2} + \left(\frac{2}{3} \right)^3 \right) + (2 + \omega) \frac{(1 + 2|\alpha_0|)}{3} \right] z \\
 & - \omega \left\{ \sum_{n=2}^{|\alpha_1|} \frac{(-|\alpha_1|)_n}{((\frac{3}{2})_n)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_1| - k)} + \frac{2}{(k + \frac{3}{2})} \right) z^n \right. \\
 & - \left. \sum_{n=|\alpha_1|+1}^{\infty} \frac{(-1)^{|\alpha_1|} \Gamma(|\alpha_1| + 1) \Gamma(n - |\alpha_1|)}{((\frac{3}{2})_n)^2} z^n \right\} \\
 & - \sum_{n=2}^{|\alpha_0|} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{(-|\alpha_0|)_n}{(n!)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(|\alpha_0| - k)} + \frac{2}{(k + 1)} \right) z^n \\
 & + \sum_{n=|\alpha_0|+1}^{\infty} \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \frac{(-1)^{|\alpha_1|} (\psi_1)! \Gamma(n - |\alpha_1|)}{(n!)^2} z^n - \sum_{n=2}^{|\alpha_1|} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \\
 & \times \frac{(-|\alpha_1|)_n (|\alpha_0|)_j ((\frac{3}{2})_j)^2}{(j!)^2 (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} \left[\sum_{l=0}^{j-1} \left(\frac{1}{(|\alpha_0| - l)} + \frac{2}{(l + 1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(|\alpha_1| - l)} + \frac{2}{(l + \frac{3}{2})} \right) \right] z^n
 \end{aligned} \tag{2.473h}$$

$$\begin{aligned}
 & + \sum_{n=|\alpha_0|+2}^{|\alpha_1|} \sum_{j=|\alpha_0|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-1)^{|\alpha_0|} (|\alpha_0|)! \Gamma(j - |\alpha_0|) (-|\alpha_1|)_n ((\frac{3}{2})_j)^2}{(j!)^2 (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} z^n \\
 & + \sum_{n=|\alpha_1|+1}^{\infty} \sum_{j=1}^{|\alpha_0|} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-1)^{|\alpha_1|} (-|\alpha_0|)_j (|\alpha_1|)! \Gamma(n - |\alpha_1|) ((\frac{3}{2})_j)^2}{(j!)^2 (-|\alpha_1|)_j ((\frac{3}{2})_n)^2} z^n \\
 & + \left. \sum_{n=|\alpha_1|+2}^{\infty} \sum_{j=|\alpha_1|+1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \frac{(-1)^{|\alpha_0|} (|\alpha_0|)! \Gamma(n - |\alpha_1|) \Gamma(j - |\alpha_0|) ((\frac{3}{2})_j)^2}{(j!)^2 \Gamma(j - |\alpha_1|) ((\frac{3}{2})_n)^2} z^n \right\}
 \end{aligned}$$

2.2 Infinite series

2.2.1 $\nu = \text{non-integer}$

In the previous section, we discussed the solutions of polynomial case. The 1st independent solution has two eigennumber which are

$$\left\{ \begin{array}{l} |\alpha_0| = -\frac{\Omega}{2\mu} \Rightarrow 0, 1, 2, 3, \dots \\ |\alpha_1| = -\frac{\Omega}{2\mu} - \frac{1}{2} \Rightarrow 0, 1, 2, 3, \dots \quad ; \text{ only if } |\alpha_1| \geq |\alpha_0| \end{array} \right. \quad (2.474)$$

Also, the 2nd independent solution has two eigennumber which are

$$\begin{cases} \psi_0 = -\frac{\Omega}{2\mu} + \gamma - 1 \Rightarrow 0, 1, 2, 3, \dots \\ \psi_1 = -\frac{\Omega}{2\mu} + \gamma - \frac{3}{2} \Rightarrow 0, 1, 2, 3, \dots \end{cases} \quad ; \text{ only if } \psi_1 \geq \psi_0 \quad (2.475)$$

These conditions make the solutions as finite series. However, if

$$-\frac{\Omega}{2\mu} \neq \begin{cases} 0, 1, 2, 3, \dots \\ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \end{cases} \quad -\frac{\Omega}{2\mu} \neq \begin{cases} 1 - \gamma, 2 - \gamma, 3 - \gamma, \dots \\ \frac{3}{2} - \gamma, \frac{5}{2} - \gamma, \frac{7}{2} - \gamma, \dots \end{cases} \quad (2.476)$$

Then, independent solutions turn to be infinite series. It means that as x goes to infinite, the function is divergent. Physically, this is means nothing. However, mathematically it can be an answer. From a mathematical point of view, the finite series is considered as the special case of infinite series. But it is totally a wrong approach. First of all, we need get a condition what situation makes polynomial as solution, after that, we need to expand it as infinite series. Polynomial case is totally different from the infinite series case. Polynomial case is not the special case of infinite series, they are independent with each other. Lets now consider the case of infinite series.

(A) As $\lambda_1 = 0$

From the 1st kind of grand confluent hypergeometric polynomial, let $|\alpha_0| = -\frac{\Omega}{2\mu}$ and $|\alpha_1| = -\frac{\Omega}{2\mu} - \frac{1}{2}$ in eq(2.125), eq(2.126a) and eq(2.126b) where $|\alpha_0|, |\alpha_1| \neq 0, 1, 2, 3, \dots$. Then the series turns to the infinite series.

$$\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) = \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} M\left(\frac{\Omega}{2\mu}, \gamma, z\right) - \frac{\varepsilon}{2} x \Pi(\gamma; z) \quad (2.477)$$

where,

$$M\left(\frac{\Omega}{2\mu}, \gamma, z\right) = \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} z^n \quad (2.478a)$$

$$\begin{aligned} \Pi(\gamma; z) &= \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} z^n \\ &\times \sum_{k=0}^{\infty} \frac{(n + \frac{\omega}{2})\Gamma(n + \frac{1}{2})\Gamma(n + \gamma - \frac{1}{2})(n + \frac{\Omega}{2\mu} + \frac{1}{2})_k}{\Gamma(k + n + \frac{3}{2})\Gamma(k + n + \gamma + \frac{1}{2})} z^k \end{aligned} \quad (2.478b)$$

Eq(2.478b) can be described in the following way.

$$\Pi(\gamma; z) = \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \Pi_n(\gamma; z) \quad (2.479)$$

where,

$$M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) = \frac{\left(\frac{\Omega}{2\mu}\right)_n}{n! (\gamma)_n} z^n \quad (2.480a)$$

$$\prod_n(\gamma; z) = \sum_{k=0}^{\infty} \frac{\left(n + \frac{\omega}{2}\right)\Gamma\left(n + \frac{1}{2}\right)\Gamma\left(n + \gamma - \frac{1}{2}\right)\left(n + \frac{\Omega}{2\mu} + \frac{1}{2}\right)_k}{\Gamma\left(k + n + \frac{3}{2}\right)\Gamma\left(k + n + \gamma + \frac{1}{2}\right)} z^k \quad (2.480b)$$

(a) Plug $n = 0$ into eq(2.480a) and eq(2.480b)

$$M_0\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_0(\gamma; z) = \frac{\omega}{2} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\gamma - \frac{1}{2}\right)\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right)_k}{\Gamma\left(k + \frac{3}{2}\right)\Gamma\left(k + \gamma + \frac{1}{2}\right)} z^k \quad (2.481)$$

By using eq(2.139) into the r.h.s of eq(2.481)

$$M_0\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_0(\gamma; z) = \frac{\omega}{2} \int_{-1}^1 dp \sum_{k=0}^{\infty} \frac{\Gamma\left(\gamma - \frac{1}{2}\right)\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right)_k}{\Gamma(k+1)\Gamma\left(k + \gamma + \frac{1}{2}\right)} [z(1-p^2)]^k \quad (2.482)$$

Replace p and q by $\gamma - \frac{1}{2}$ and $k + 1$ in eq(2.138). Substitute it into eq(2.482).

$$\begin{aligned} & M_0\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_0(\gamma; z) \\ &= \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right)_k}{\Gamma(k+1)\Gamma(k+1)} [z(1-t)(1-p^2)]^k \\ &= \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right)_k}{(1)_k \Gamma(k+1)} [z(1-t)(1-p^2)]^k \\ &= \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} M\left(\frac{\Omega}{2\mu} + \frac{1}{2}, 1, \lambda = z(1-t)(1-p^2)\right) \end{aligned} \quad (2.483)$$

(b) Plug $n = 1$ into eq(2.480a) and eq(2.480b)

$$M_1\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_1(\gamma; z) = \frac{\left(\frac{\Omega}{2\mu}\right)_1 z}{(\gamma)_1(1!)} \sum_{k=0}^{\infty} \frac{\left(1 + \frac{\omega}{2}\right) \Gamma\left(\frac{3}{2}\right) \Gamma\left(\gamma + \frac{1}{2}\right) \left(\frac{\Omega}{2\mu} + \frac{3}{2}\right)_k}{\Gamma\left(k + \frac{5}{2}\right) \Gamma\left(k + \gamma + \frac{3}{2}\right)} z^k \quad (2.484)$$

Replace k by $k+1$ in eq(2.139), then plug it into eq(2.484)

$$\begin{aligned} & M_1\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_1(\gamma; z) \quad (2.485) \\ &= \frac{1}{2} \left(1 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_1 z}{(\gamma)_1(1!)} \int_{-1}^1 dp (1-p^2) \sum_{k=0}^{\infty} \frac{\Gamma\left(\gamma + \frac{1}{2}\right) \left(\frac{\Omega}{2\mu} + \frac{3}{2}\right)_k}{\Gamma(k+2) \Gamma\left(k + \gamma + \frac{3}{2}\right)} [z(1-p^2)]^k \end{aligned}$$

Replace p and q by $\gamma + \frac{1}{2}$ and $k + 1$ in eq(2.138). Substitute it into eq(2.485).

$$\begin{aligned} & M_1\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_1(\gamma; z) \\ &= \frac{1}{2} \left(1 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_1 z}{(\gamma)_1(1!)} \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\gamma-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{3}{2}\right)_k}{\Gamma(k+2) \Gamma(k+1)} \lambda^k \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(1 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_1 z}{(\gamma)_1(1!)} \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\gamma-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{3}{2}\right)_k}{(2)_k k!} \lambda^k \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(1 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_1 z}{(\gamma)_1(1!)} \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\gamma-\frac{1}{2}} M\left(\frac{\Omega}{2\mu} + \frac{3}{2}, 2, \lambda\right) \end{aligned} \quad (2.486)$$

where $\lambda = z(1-t)(1-p^2)$

(c) Plug $n = 2$ into eq(2.480a) and eq(2.480b).

$$M_2\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_2(\gamma; z) = \frac{\left(\frac{\Omega}{2\mu}\right)_2 z^2}{(\gamma)_2(2!)} \sum_{k=0}^{\infty} \frac{\left(2 + \frac{\omega}{2}\right) \Gamma\left(\frac{5}{2}\right) \Gamma\left(\gamma + \frac{3}{2}\right) \left(\frac{\Omega}{2\mu} + \frac{5}{2}\right)_k}{\Gamma\left(k + \frac{7}{2}\right) \Gamma\left(k + \gamma + \frac{5}{2}\right)} z^k \quad (2.487)$$

Replace k by $k+2$ in eq(2.139), then plug it into eq(2.487)

$$\begin{aligned} & M_2\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_2(\gamma; z) \quad (2.488) \\ &= \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(2 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_2 z^2}{(\gamma)_2(2!)} \int_{-1}^1 dp (1-p^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma\left(\gamma + \frac{3}{2}\right) \left(\frac{\Omega}{2\mu} + \frac{5}{2}\right)_k}{\Gamma(k+3) \Gamma\left(k + \gamma + \frac{5}{2}\right)} [z(1-p^2)]^k \end{aligned}$$

Replace p and q by $\gamma + \frac{3}{2}$ and $k + 1$ in eq(2.138), substitute it into eq(2.488).

$$\begin{aligned} & M_2\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_2(\gamma; z) \\ &= \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(2 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_2 z^2}{(\gamma)_2(2!)} \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\gamma+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{5}{2}\right)_k}{\Gamma(k+1) \Gamma(k+3)} \lambda^k \\ &= \frac{1}{2!} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(2 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_2 z^2}{(\gamma)_2(2!)} \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\gamma+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{5}{2}\right)_k}{(3)_k k!} \lambda^k \\ &= \frac{1}{2!} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(2 + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu}\right)_2 z^2}{(\gamma)_2(2!)} \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\gamma+\frac{1}{2}} M\left(\frac{\Omega}{2\mu} + \frac{5}{2}, 3, \lambda\right) \end{aligned}$$

$$\text{where } \lambda = z(1-t)(1-p^2) \quad (2.489)$$

Add eq(2.483), eq(2.486) and eq(2.489) to each other.

$$\begin{aligned}
 & \sum_{n=0}^2 M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\
 &= \frac{\omega}{2} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} M\left(\frac{\Omega}{2\mu} + \frac{1}{2}, 1, \lambda = z(1-t)(1-p^2)\right) \\
 &+ \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} \left(1 + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu})_1 z}{(\gamma)_1 (1!)} \int_{-1}^1 dp (1-p^2) \int_0^1 dt t^{\gamma-\frac{1}{2}} M\left(\frac{\Omega}{2\mu} + \frac{3}{2}, 2, \lambda\right) \\
 &+ \frac{1}{2!} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} \left(2 + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu})_2 z^2}{(\gamma)_2 (2!)} \int_{-1}^1 dp (1-p^2)^2 \int_0^1 dt t^{\gamma+\frac{1}{2}} M\left(\frac{\Omega}{2\mu} + \frac{5}{2}, 3, \lambda\right) \\
 &\text{where } \lambda = z(1-t)(1-p^2) \tag{2.490}
 \end{aligned}$$

As index n goes up to infinity in eq(2.490), we obtain

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\
 &= \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n z^n \Gamma(n + \frac{1}{2})}{(\gamma)_n (n!) n! \Gamma(\frac{1}{2})} \left(n + \frac{\omega}{2}\right) \int_{-1}^1 dp (1-p^2)^n \\
 &\times \int_0^1 dt t^{\gamma-\frac{3}{2}+n} M\left(\frac{\Omega}{2\mu} + n + \frac{1}{2}, n+1, \lambda\right) \\
 &\text{where } \lambda = z(1-t)(1-p^2) \tag{2.491}
 \end{aligned}$$

Replace a,b, and z by $\frac{\Omega}{2\mu} + n + \frac{1}{2}$, $n+1$ and λ in eq(2.172)

$$M\left(\frac{\Omega}{2\mu} + n + \frac{1}{2}, n + 1, \lambda\right) = \frac{\Gamma(n + 1)}{\Gamma(-\frac{\Omega}{2\mu} + \frac{1}{2})\Gamma(\frac{\Omega}{2\mu} + \frac{1}{2} + n)} \int_0^1 du e^{\lambda u} u^{\frac{\Omega}{2\mu} + n - \frac{1}{2}} (1 - u)^{-\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right)} \quad (2.492)$$

Substitute eq(2.492) into eq(2.491).

$$\begin{aligned} & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\ &= \frac{1}{\Gamma(\frac{1}{2})\Gamma(-\frac{\Omega}{2\mu} + \frac{1}{2})\Gamma(\frac{\Omega}{2\mu})} \int_{-1}^1 dp \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_0^1 du e^{\lambda u} u^{\frac{\Omega}{2\mu} - \frac{1}{2}} (1 - u)^{-\left(\frac{\Omega}{2\mu} + \frac{1}{2}\right)} \\ & \times \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n B\left(n + \frac{1}{2}, \frac{\Omega}{2\mu}\right) \left(n + \frac{\omega}{2}\right)}{(\gamma)_n (n!)} [z(1 - p^2)tu]^n \\ & \text{where } \lambda = z(1 - t)(1 - p^2) \end{aligned} \quad (2.493)$$

Replace p and q by $n + \frac{1}{2}$ and $\frac{\Omega}{2\mu}$ into eq(2.138).

$$B\left(n + \frac{1}{2}, \frac{\Omega}{2\mu}\right) = \int_0^1 ds s^{n - \frac{1}{2}} (1 - s)^{\frac{\Omega}{2\mu} - 1} \quad (2.494)$$

Plug eq(2.494) into eq(2.493).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\
 &= \frac{1}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right)} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} \\
 & \times \int_0^1 du e^{\lambda u} u^{\frac{\Omega}{2\mu}-\frac{1}{2}} (1-u)^{-\left(\frac{\Omega}{2\mu}+\frac{1}{2}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \\
 & \times \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n(n-1)!} [z(1-p^2)tsu]^n + \frac{\omega}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n(n)!} [z(1-p^2)tsu]^n \right\} \\
 & \text{where } \lambda = z(1-t)(1-p^2) \tag{2.495}
 \end{aligned}$$

We know

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n(n-1)!} [z(1-p^2)tsu]^n + \frac{\omega}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n(n)!} [z(1-p^2)tsu]^n \\
 &= [z(1-p^2)tsu] \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n(n-1)!} [z(1-p^2)tsu]^{n-1} \\
 &+ \frac{\omega}{2} M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) \tag{2.496} \\
 &= w_1 \partial_{w_1} M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) + \frac{\omega}{2} M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right)
 \end{aligned}$$

Differentiate $M\left(\frac{\Omega}{2\mu}, \gamma, w_1\right)$ with respect w_1 and multiply w_1 on it.

$$w_1 \frac{\partial}{\partial w_1} M\left(\frac{\Omega}{2\mu}, \gamma, w_1\right) = (\gamma - 1) \left\{ M\left(\frac{\Omega}{2\mu}, \gamma - 1, w_1\right) - M\left(\frac{\Omega}{2\mu}, \gamma, w_1\right) \right\} \quad (2.497)$$

Plug eq(2.497) into eq(2.496).

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n (n-1)!} [z(1-p^2)tsu]^n + \frac{\omega}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_n}{(\gamma)_n (n)!} [z(1-p^2)tsu]^n \\ &= \left\{ w_1 \partial_{w_1} + \frac{\omega}{2} \right\} M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) \\ &= (\gamma - 1) M\left(\frac{\Omega}{2\mu}, \gamma - 1, w_1\right) + \left(1 - \gamma + \frac{\omega}{2}\right) M\left(\frac{\Omega}{2\mu}, \gamma, w_1\right) \end{aligned} \quad (2.498)$$

Substitute eq(2.498) into eq(2.495).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\
 &= \frac{1}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right)} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_0^1 du \frac{u^{\frac{\Omega}{2\mu}-\frac{1}{2}} e^{z(1-t)(1-p^2)u}}{(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\
 & \times \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \left\{ w_1 \partial_{w_1} + \frac{\omega}{2} \right\} M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) \\
 &= \frac{1}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right)} \int_{-1}^1 dp \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_0^1 du \frac{u^{\frac{\Omega}{2\mu}-\frac{1}{2}} e^{z(1-t)(1-p^2)u}}{(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\
 & \times \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \left\{ (\gamma-1)M\left(\frac{\Omega}{2\mu}, \gamma-1, w_1 = z(1-p^2)tsu\right) \right. \\
 & \left. + (1-\gamma + \frac{\omega}{2})M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) \right\} \tag{2.499}
 \end{aligned}$$

Eq(2.499) can be described as various integral forms of several different special function in the following way:

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\
 &= \frac{1}{\pi B\left(\gamma - \frac{\Omega}{2\mu}, \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_{-1}^1 dp \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu}-1}}{(1-v)^{2-\gamma+\frac{\Omega}{2\mu}}} \\
 & \times \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} \\
 & \times M\left(\frac{\Omega}{2\mu}, \frac{1}{2}, w_2 = z(1-p^2)\{1-t(1-sv)\}\right) \tag{2.500}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \Pi_n(\gamma; z) \\
 &= \frac{2}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right) B\left(\gamma - \frac{\Omega}{2\mu}, \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \\
 & \times \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu}-1}}{(1-v)^{2-\gamma+\frac{\Omega}{2\mu}}} \int_0^1 du \frac{u^{\frac{\Omega}{2\mu}-1}}{(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) \right. \\
 & \left. + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(1, \frac{3}{2}, w_3 = zu\{1 - t(1 - sv)\}\right)
 \end{aligned} \tag{2.501}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \Pi_n(\gamma; z) \\
 &= \frac{1}{\pi\left(\gamma - \frac{1}{2}\right) B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right) B\left(\gamma - \frac{\Omega}{2\mu}, \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_{-1}^1 dp \\
 & \times \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu}-1}}{(1-v)^{2-\gamma+\frac{\Omega}{2\mu}}} \int_0^1 du \frac{e^{zsvu(1-p^2)} u^{\frac{\Omega}{2\mu}-1}}{(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\
 & \times \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(1, \gamma + \frac{1}{2}, w_4 = zu(1-p^2)(1-sv)\right)
 \end{aligned} \tag{2.502}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \Pi_n(\gamma; z) \\
 &= \frac{1}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu} + \frac{1}{2}\right) B\left(\gamma - \frac{\Omega}{2\mu}, \frac{\Omega}{2\mu}\right)} \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_{-1}^1 dp \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu}-1}}{(1-v)^{2-\gamma+\frac{\Omega}{2\mu}}} \\
 & \times \int_0^1 du \frac{e^{zu(1-p^2)(1-t)} u^{\frac{\Omega}{2\mu}-1}}{(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} \\
 & \times M\left(\frac{1}{2}, \frac{\Omega}{2\mu} + \frac{1}{2}, w_5 = zu(1-p^2)tv\right) \tag{2.503}
 \end{aligned}$$

Replace a, b and z by $\frac{\Omega}{2\mu}$, γ and $w_1 = zu(1-p^2)st$ in eq(2.172).

$$\begin{aligned}
 M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) &= \frac{1}{B\left(\frac{\Omega}{2\mu}, \gamma - \frac{\Omega}{2\mu}\right)} \int_0^1 dv \frac{e^{zuv(1-p^2)st}}{v^{1-\frac{\Omega}{2\mu}}(1-v)^{1-\gamma+\frac{\Omega}{2\mu}}} \\
 & \text{where, } \operatorname{Re}(\gamma) > \operatorname{Re}\left(\frac{\Omega}{2\mu}\right) > 0 \tag{2.504}
 \end{aligned}$$

Substitute eq(2.504) into eq(2.499).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu}, \gamma, z\right) \prod_n(\gamma; z) \\
 &= \frac{1}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right) B\left(\frac{\Omega}{2\mu}, \gamma - \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \\
 & \times \int_{-1}^1 dp \int_0^1 \frac{du}{u^{\frac{1}{2}-\frac{\Omega}{2\mu}} (1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\
 & \times \int_0^1 \frac{dv}{v^{1-\frac{\Omega}{2\mu}} (1-v)^{1-\gamma+\frac{\Omega}{2\mu}}} \left\{ zuv(1-p^2)st + \frac{\omega}{2} \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \\
 &= \frac{1}{\pi B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right) B\left(\frac{\Omega}{2\mu}, \gamma - \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \\
 & \times \int_{-1}^1 dp \int_0^1 \frac{du}{u^{\frac{1}{2}-\frac{\Omega}{2\mu}} (1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\
 & \times \int_0^1 \frac{dv}{v^{1-\frac{\Omega}{2\mu}} (1-v)^{2-\gamma+\frac{\Omega}{2\mu}}} \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} e^{zu(1-p^2)\{1-t(1-sv)\}}
 \end{aligned} \tag{2.505}$$

We describe eq(2.500), eq(2.501), eq(2.502), eq(2.503) as integral formalism. It is exactly equivalent eq(2.505). Plug eq(2.499) through eq(2.503) into eq(2.479), and then, substitute them into eq(2.477):

$$\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) = \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} M\left(\frac{\Omega}{2\mu}, \gamma, z\right) - \frac{\varepsilon}{2} x \Pi(\gamma; z)$$

where,

$$\begin{aligned} M\left(\frac{\Omega}{2\mu}, \gamma, z\right) &= \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} z^n \\ \Pi(\gamma; z) &= \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} z^n \sum_{k=0}^{\infty} \frac{(n + \frac{\omega}{2})\Gamma(n + \frac{1}{2})\Gamma(n + \gamma - \frac{1}{2})(n + \frac{\Omega}{2\mu} + \frac{1}{2})_k}{\Gamma(k + n + \frac{3}{2})\Gamma(k + n + \gamma + \frac{1}{2})} z^k \\ &= \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\pi\Gamma(\gamma)B(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu})} \int_{-1}^1 dp \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_0^1 du \frac{u^{\frac{\Omega}{2\mu} - \frac{1}{2}} e^{z(1-t)(1-p^2)u}}{(1-u)^{\frac{\Omega}{2\mu} + \frac{1}{2}}} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu} - 1} \\ &\quad \times \left\{ w_1 \partial_{w_1} + \frac{\omega}{2} \right\} M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) \\ &= \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\pi\Gamma(\gamma)B(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu})} \int_{-1}^1 dp \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_0^1 du \frac{u^{\frac{\Omega}{2\mu} - \frac{1}{2}} e^{z(1-t)(1-p^2)u}}{(1-u)^{\frac{\Omega}{2\mu} + \frac{1}{2}}} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu} - 1} \\ &\quad \times \left\{ (\gamma - 1)M\left(\frac{\Omega}{2\mu}, \gamma - 1, w_1 = z(1-p^2)tsu\right) + (1 - \gamma + \frac{\omega}{2})M\left(\frac{\Omega}{2\mu}, \gamma, w_1 = z(1-p^2)tsu\right) \right\} \\ &= \frac{1}{\pi\Gamma(\frac{\Omega}{2\mu})} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu} - 1} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu} - 1}}{(1-v)^{2-\gamma + \frac{\Omega}{2\mu}}} \\ &\quad \times \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(\frac{\Omega}{2\mu}, \frac{1}{2}, w_2 = z(1-p^2)\{1-t(1-sv)\}\right) \\ &= \frac{2}{\pi\Gamma(\frac{\Omega}{2\mu})B(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu})} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu} - 1} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu} - 1}}{(1-v)^{2-\gamma + \frac{\Omega}{2\mu}}} \\ &\quad \times \int_0^1 du \frac{u^{\frac{\Omega}{2\mu} - 1}}{(1-u)^{\frac{\Omega}{2\mu} + \frac{1}{2}}} \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(1, \frac{3}{2}, w_3 = zu\{1-t(1-sv)\}\right) \\ &= \frac{(\gamma - \frac{1}{2})^{-1}}{\pi\Gamma(\frac{\Omega}{2\mu})B(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu})} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu} - 1} \int_{-1}^1 dp \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu} - 1}}{(1-v)^{2-\gamma + \frac{\Omega}{2\mu}}} \int_0^1 du \frac{e^{zsvu(1-p^2)} u^{\frac{\Omega}{2\mu} - 1}}{(1-u)^{\frac{\Omega}{2\mu} + \frac{1}{2}}} \\ &\quad \times \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(1, \gamma + \frac{1}{2}, w_4 = zu(1-p^2)(1-sv)\right) \\ &= \frac{1}{\pi\Gamma(\frac{\Omega}{2\mu})B(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu} + \frac{1}{2})} \int_0^1 dt t^{\gamma - \frac{3}{2}} \int_{-1}^1 dp \int_0^1 dv \frac{v^{\frac{\Omega}{2\mu} - 1}}{(1-v)^{2-\gamma + \frac{\Omega}{2\mu}}} \int_0^1 du \frac{e^{zu(1-p^2)(1-t)} u^{\frac{\Omega}{2\mu} - 1}}{(1-u)^{\frac{\Omega}{2\mu} + \frac{1}{2}}} \\ &\quad \times \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(\frac{1}{2}, \frac{\Omega}{2\mu} + \frac{1}{2}, w_5 = zu(1-p^2)tv\right) \end{aligned}$$

Table 2.16: 1st kind of grand confluent hypergeometric function of infinite series

Plug eq(2.505) into eq(2.479) and substitute it into eq(2.477). We obtain the integral form of the 1st kind of grand confluent hypergeometric function of infinite series in which is the following way:

$$\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) = \frac{1}{\Gamma\left(\frac{\Omega}{2\mu}\right)} \int_0^1 du \frac{e^{zu}}{u^{1-\frac{\Omega}{2\mu}}(1-u)^{1-\gamma+\frac{\Omega}{2\mu}}} - \frac{\varepsilon}{2} x \Pi(\gamma; z)$$

where,

$$\begin{aligned} \Pi(\gamma; z) &= \frac{1}{\pi\Gamma\left(\frac{\Omega}{2\mu}\right)B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_{-1}^1 dp \int_0^1 \frac{du}{u^{\frac{1}{2}-\frac{\Omega}{2\mu}}(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\ &\quad \times \int_0^1 \frac{dv}{v^{1-\frac{\Omega}{2\mu}}(1-v)^{1-\gamma+\frac{\Omega}{2\mu}}} \left\{ zuv(1-p^2)st + \frac{\omega}{2} \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \\ &= \frac{1}{\pi\Gamma\left(\frac{\Omega}{2\mu}\right)B\left(-\frac{\Omega}{2\mu} + \frac{1}{2}, \frac{\Omega}{2\mu}\right)} \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-1} \int_0^1 dt t^{\gamma-\frac{3}{2}} \int_{-1}^1 dp \int_0^1 \frac{du}{u^{\frac{1}{2}-\frac{\Omega}{2\mu}}(1-u)^{\frac{\Omega}{2\mu}+\frac{1}{2}}} \\ &\quad \times \int_0^1 \frac{dv}{v^{1-\frac{\Omega}{2\mu}}(1-v)^{2-\gamma+\frac{\Omega}{2\mu}}} \left\{ \left(\gamma - 1 - \frac{\Omega}{2\mu}\right) + (1-v)\left(1 - \gamma + \frac{\omega}{2}\right) \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \end{aligned}$$

Table 2.17: Integral form of the 1st kind of grand confluent hypergeometric function of infinite series

(B) As $\lambda_2 = 1 - \nu$

From the 2nd kind of grand confluent hypergeometric polynomial, let $\psi_0 = -\frac{\Omega}{2\mu} - 1 + \gamma$ and $\psi_1 = -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma$ in eq(2.280), eq(2.281a) and eq(2.281b) where $\psi_0, \psi_1 \neq$

0, 1, 2, 3, ... Then the series turns to be the infinite series.

$$\begin{aligned} & \mathcal{RW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \\ &= z^{1-\gamma} \left\{ \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\Gamma(2 - \gamma)} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) - \frac{\varepsilon}{2} x \Lambda(\gamma; z) \right\} \end{aligned} \quad (2.506)$$

where,

$$M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) = \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{n! (2 - \gamma)_n} z^n \quad (2.507a)$$

$$\begin{aligned} & \Lambda(\gamma; z) \\ &= \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{n! (2 - \gamma)_n} z^n \\ &\times \sum_{k=0}^{\infty} \frac{(n + 1 - \gamma + \frac{\omega}{2})\Gamma(n + \frac{1}{2})\Gamma(n + \frac{3}{2} - \gamma)(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma + n)_k}{\Gamma(k + n + \frac{3}{2})\Gamma(k + n + \frac{5}{2} - \gamma)} z^k \end{aligned} \quad (2.507b)$$

Eq(2.507b) can be described in the following way.

$$\Lambda(\gamma; z) = \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \Lambda_n(\gamma; z) \quad (2.508)$$

where,

$$M_n\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) = \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{n! (2 - \gamma)_n} z^n \quad (2.509a)$$

$$\Lambda_n(\gamma; z) = \sum_{k=0}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2})\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2}-\gamma)(\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma+n)_k}{\Gamma(k+n+\frac{3}{2})\Gamma(k+n+\frac{5}{2}-\gamma)} z^k \quad (2.509b)$$

(a) Plug $n = 0$ into eq(2.509a) and eq(2.509b), and multiply them with each other.

$$M_0\left(\frac{\Omega}{2\mu}+1-\gamma, 2-\gamma, z\right) \Lambda_0(\gamma; z) = (1-\gamma+\frac{\omega}{2}) \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-\gamma)(\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma)_k}{\Gamma(k+\frac{3}{2})\Gamma(k+\frac{5}{2}-\gamma)} z^k \quad (2.510)$$

By using eq(2.139) into the r.h.s of eq(2.510)

$$\begin{aligned} & M_0\left(\frac{\Omega}{2\mu}+1-\gamma, 2-\gamma, z\right) \Lambda_0(\gamma; z) \\ &= (1-\gamma+\frac{\omega}{2}) \int_{-1}^1 dp \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2}-\gamma)(\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma)_k}{\Gamma(k+1)\Gamma(k+\frac{5}{2}-\gamma)} [z(1-p^2)]^k \end{aligned} \quad (2.511)$$

Replace p and q by $\frac{3}{2} - \gamma$ and $k + 1$ in eq(2.138), substitute it into eq(2.511).

$$\begin{aligned}
 & M_0\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \mathcal{L}_0(\gamma; z) \\
 &= (1 - \gamma + \frac{\omega}{2}) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \sum_{k=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_k}{\Gamma(k+1)\Gamma(k+1)} [z(1-t)(1-p^2)]^k \\
 &= (1 - \gamma + \frac{\omega}{2}) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \sum_{k=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_k}{(1)_k \Gamma(k+1)} [z(1-t)(1-p^2)]^k \\
 &= (1 - \gamma + \frac{\omega}{2}) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} M\left(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, 1, \lambda\right) \\
 & \text{where } \lambda = z(1-t)(1-p^2) \tag{2.512}
 \end{aligned}$$

(b) Plug $n = 1$ into eq(2.509a) and eq(2.509b), and multiply them with each other.

$$\begin{aligned}
 & M_1\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \mathcal{L}_1(\gamma; z) \\
 &= \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_1 z}{1! (2 - \gamma)_1} \sum_{k=0}^{\infty} \frac{(2 - \gamma + \frac{\omega}{2}) \Gamma(\frac{3}{2}) \Gamma(\frac{5}{2} - \gamma) (\frac{\Omega}{2\mu} + \frac{5}{2} - \gamma)_k}{\Gamma(k + \frac{5}{2}) \Gamma(k + \frac{7}{2} - \gamma)} z^k \tag{2.513}
 \end{aligned}$$

Replace k by $k+1$ in eq(2.139). Then plug it into eq(2.513)

$$\begin{aligned}
 & M_1\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \bigwedge_1(\gamma; z) \\
 &= \frac{1}{2}\left(2 - \gamma + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)_1 z}{1! (2 - \gamma)_1} \\
 &\times \int_{-1}^1 dp (1 - p^2) \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{5}{2} - \gamma\right) \left(\frac{\Omega}{2\mu} + \frac{5}{2} - \gamma\right)_k}{\Gamma(k + 2) \Gamma\left(k + \frac{7}{2} - \gamma\right)} [z(1 - p^2)]^k
 \end{aligned} \tag{2.514}$$

Replace p and q by $\frac{5}{2} - \gamma$ and $k + 1$ in eq(2.138), substitute it into eq(2.514).

$$\begin{aligned}
 & M_1\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \bigwedge_1(\gamma; z) \\
 &= \frac{1}{2}\left(2 - \gamma + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)_1 z}{1! (2 - \gamma)_1} \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\frac{3}{2} - \gamma} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{5}{2} - \gamma\right)_k}{\Gamma(k + 2) \Gamma(k + 1)} \lambda^k \\
 &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(2 - \gamma + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)_1 z}{1! (2 - \gamma)_1} \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\frac{3}{2} - \gamma} \sum_{k=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{5}{2} - \gamma\right)_k}{(2)_k k!} \lambda^k \\
 &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(2 - \gamma + \frac{\omega}{2}\right) \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)_1 z}{1! (2 - \gamma)_1} \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\frac{3}{2} - \gamma} M\left(\frac{\Omega}{2\mu} + \frac{5}{2} - \gamma, 2, \lambda\right) \\
 &\text{where } \lambda = z(1 - t)(1 - p^2)
 \end{aligned} \tag{2.515}$$

c) Plug $n = 2$ into eq(2.509a) and eq(2.509b), and multiply them with each other.

$$\begin{aligned}
 & M_2\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \bigwedge_2(\gamma; z) \\
 &= \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_2 z^2}{2! (2 - \gamma)_2} \sum_{k=0}^{\infty} \frac{(3 - \gamma + \frac{\omega}{2})\Gamma(\frac{5}{2})\Gamma(\frac{7}{2} - \gamma)(\frac{\Omega}{2\mu} + \frac{7}{2} - \gamma)_k}{\Gamma(k + \frac{7}{2})\Gamma(k + \frac{9}{2} - \gamma)} z^k \quad (2.516)
 \end{aligned}$$

Replace k by $k+2$ in eq(2.139), then plug it into eq(2.516)

$$\begin{aligned}
 & M_2\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \bigwedge_2(\gamma; z) \\
 &= \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} \left(3 - \gamma + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_2 z^2}{2! (2 - \gamma)_2} \\
 &\times \int_{-1}^1 dp (1 - p^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(\frac{7}{2} - \gamma)(\frac{\Omega}{2\mu} + \frac{7}{2} - \gamma)_k}{\Gamma(k + 3)\Gamma(k + \frac{9}{2} - \gamma)} [z(1 - p^2)]^k \quad (2.517)
 \end{aligned}$$

Replace p and q by $\frac{7}{2} - \gamma$ and $k + 1$ in eq(2.138), substitute it into eq(2.517).

$$\begin{aligned}
 & M_2\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \Lambda_2(\gamma; z) \\
 &= \frac{1}{2!} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} \left(3 - \gamma + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_2 z^2}{2! (2 - \gamma)_2} \int_{-1}^1 dp (1 - p^2)^2 \int_0^1 dt t^{\frac{5}{2} - \gamma} \\
 &\times \sum_{k=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{7}{2} - \gamma)_k}{k! (3)_k} \lambda^k \\
 &= \frac{1}{2!} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} \left(3 - \gamma + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_2 z^2}{2! (2 - \gamma)_2} \\
 &\times \int_{-1}^1 dp (1 - p^2)^2 \int_0^1 dt t^{\frac{5}{2} - \gamma} M\left(\frac{\Omega}{2\mu} + \frac{7}{2} - \gamma, 3, \lambda\right) \\
 &\quad \text{where } \lambda = z(1 - t)(1 - p^2) \tag{2.518}
 \end{aligned}$$

Add eq(2.512), eq(2.515) and eq(2.518) to each other.

$$\begin{aligned}
 & \sum_{n=0}^2 M_n\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \Lambda_n(\gamma; z) \\
 &= (1 - \gamma + \frac{\omega}{2}) \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2} - \gamma} M\left(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, 1, \lambda\right) \\
 &+ \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} \left(2 - \gamma + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_1 z}{1! (2 - \gamma)_1} \int_{-1}^1 dp (1 - p^2) \int_0^1 dt t^{\frac{3}{2} - \gamma} M\left(\frac{\Omega}{2\mu} + \frac{5}{2} - \gamma, 2, \lambda\right) \\
 &+ \frac{1}{2!} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} \left(3 - \gamma + \frac{\omega}{2}\right) \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_2 z^2}{2! (2 - \gamma)_2} \int_{-1}^1 dp (1 - p^2)^2 \int_0^1 dt t^{\frac{5}{2} - \gamma} \\
 &\times M\left(\frac{\Omega}{2\mu} + \frac{7}{2} - \gamma, 3, \lambda\right) \quad \text{where } \lambda = z(1 - t)(1 - p^2) \tag{2.519}
 \end{aligned}$$

As index n goes up to infinity in eq(2.519), we obtain

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma \right)_n z^n \Gamma(n + \frac{1}{2})}{n! (2 - \gamma)_n n! \Gamma(\frac{1}{2})} \left(n + 1 - \gamma + \frac{\omega}{2} \right) \int_{-1}^1 dp (1 - p^2)^n \int_0^1 dt t^{n + \frac{1}{2} - \gamma} \\
 & \times M \left(\frac{\Omega}{2\mu} + n + \frac{3}{2} - \gamma, n + 1, \lambda \right) \\
 & \text{where } \lambda = z(1 - t)(1 - p^2) \tag{2.520}
 \end{aligned}$$

Replace $a, b,$ and z by $\frac{\Omega}{2\mu} + n + \frac{3}{2} - \gamma, n + 1$ and λ in eq(2.172)

$$\begin{aligned}
 & M \left(\frac{\Omega}{2\mu} + n + \frac{3}{2} - \gamma, n + 1, \lambda \right) \\
 &= \frac{\Gamma(n + 1)}{\Gamma(-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma) \Gamma(\frac{\Omega}{2\mu} + n + \frac{3}{2} - \gamma)} \int_0^1 du e^{\lambda u} \frac{u^{\frac{\Omega}{2\mu} + n + \frac{1}{2} - \gamma}}{(1 - u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \tag{2.521}
 \end{aligned}$$

Substitute eq(2.521) into eq(2.520).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{1}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right)} \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_0^1 du \frac{e^{z(1-p^2)(1-t)u}}{u^{-\frac{\Omega}{2\mu}-\frac{1}{2}+\gamma}(1-u)^{\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma}} \\
 \times & \sum_{n=0}^{\infty} \frac{(n+1-\gamma+\frac{\omega}{2}) B\left(n+\frac{1}{2}, \frac{\Omega}{2\mu}+1-\gamma\right) \left(\frac{\Omega}{2\mu}+1-\gamma\right)_n}{(2-\gamma)_n (n!)} [zu(1-p^2)t]^n \quad (2.522)
 \end{aligned}$$

Replace p and q by $n + \frac{1}{2}$ and $\frac{\Omega}{2\mu} + 1 - \gamma$ into eq(2.138).

$$B\left(n + \frac{1}{2}, \frac{\Omega}{2\mu} + 1 - \gamma\right) = \int_0^1 ds s^{n-\frac{1}{2}} (1-s)^{\frac{\Omega}{2\mu}-\gamma} \quad (2.523)$$

Plug eq(2.523) into eq(2.522).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{1}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right)} \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_0^1 du \frac{e^{z(1-p^2)(1-t)u}}{u^{-\frac{\Omega}{2\mu}-\frac{1}{2}+\gamma}(1-u)^{\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma}} \\
 \times & \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma-\frac{\Omega}{2\mu}}} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)_n}{(2-\gamma)_n (n-1)!} [zu(1-p^2)st]^n \right. \\
 & \left. + \left(1 - \gamma + \frac{\omega}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)_n}{(2-\gamma)_n n!} [zu(1-p^2)st]^n \right\} \quad (2.524)
 \end{aligned}$$

We know

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{(2 - \gamma)_n (n - 1)!} [zu(1 - p^2)st]^n + \left(1 - \gamma + \frac{\omega}{2}\right) \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{(2 - \gamma)_n n!} [zu(1 - p^2)st]^n \\
 &= k_1 \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{(2 - \gamma)_n (n - 1)!} k_1^{n-1} + \left(1 - \gamma + \frac{\omega}{2}\right) M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) \\
 &= k_1 \partial_{k_1} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) + \left(1 - \gamma + \frac{\omega}{2}\right) M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) \\
 & \quad \text{where } k_1 = zu(1 - p^2)st \tag{2.525}
 \end{aligned}$$

Differentiate $M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right)$ with respect k_1 and multiply k_1 on it:

$$k_1 \partial_{k_1} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) = (1 - \gamma) \left\{ M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \gamma, k_1\right) - M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) \right\} \tag{2.526}$$

Plug eq(2.526) into eq(2.525).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{(2 - \gamma)_n (n - 1)!} [zu(1 - p^2)st]^n + \left(1 - \gamma + \frac{\omega}{2}\right) \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{(2 - \gamma)_n n!} [zu(1 - p^2)st]^n \\
 &= \left\{ k_1 \partial_{k_1} + \left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) \\
 &= (1 - \gamma) M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \gamma, k_1\right) + \frac{\omega}{2} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1\right) \\
 & \quad \text{where } k_1 = zu(1 - p^2)st \tag{2.527}
 \end{aligned}$$

Substitute eq(2.527) into eq(2.524).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{1}{\pi B \left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma \right)} \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \\
 & \times \int_0^1 du \frac{e^{z(1-p^2)(1-t)u}}{u^{-\frac{\Omega}{2\mu}-\frac{1}{2}+\gamma}(1-u)^{\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma}} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma-\frac{\Omega}{2\mu}}} \\
 & \times \left\{ k_1 \partial_{k_1} + \left(1 - \gamma + \frac{\omega}{2} \right) \right\} M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1 = zu(1-p^2)st \right) \\
 &= \frac{1}{\pi B \left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma \right)} \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_0^1 du \frac{e^{z(1-p^2)(1-t)u}}{u^{-\frac{\Omega}{2\mu}-\frac{1}{2}+\gamma}(1-u)^{\frac{\Omega}{2\mu}+\frac{3}{2}-\gamma}} \\
 & \times \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma-\frac{\Omega}{2\mu}}} \left\{ (1-\gamma) M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \gamma, k_1 = zu(1-p^2)st \right) \right. \\
 & \left. + \frac{\omega}{2} M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1 = zu(1-p^2)st \right) \right\} \tag{2.528}
 \end{aligned}$$

Eq(2.528) can be described as various integral forms of several different special functions in the following way:

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{B\left(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right)}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right) B\left(-\frac{\Omega}{2\mu} + 1, \frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \\
 & \times \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \left\{ -\frac{\Omega}{2\mu} + (1-v)\frac{\omega}{2} \right\} \\
 & \times M\left(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, 1, k_2 = z(1-p^2)\{1 - (1-sv)t\}\right) \tag{2.529}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{2}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right) B\left(-\frac{\Omega}{2\mu} + 1, \frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \\
 & \times \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \\
 & \times \left\{ -\frac{\Omega}{2\mu} + (1-v)\frac{\omega}{2} \right\} M\left(1, \frac{3}{2}, k_3 = zu\{1 - (1-sv)t\}\right) \tag{2.530}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{\left(\frac{3}{2} - \gamma \right)^{-1}}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right) B\left(-\frac{\Omega}{2\mu} + 1, \frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \\
 & \times \int_{-1}^1 dp \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \int_0^1 \frac{du e^{zu(1-p^2)sv}}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \\
 & \times \left\{ -\frac{\Omega}{2\mu} + (1-v)\frac{\omega}{2} \right\} M\left(1, \frac{5}{2} - \gamma, k_4 = zu(1-p^2)(1-sv)\right)
 \end{aligned} \tag{2.531}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \Lambda_n (\gamma; z) \\
 &= \frac{B\left(\frac{1}{2}, \frac{\Omega}{2\mu} + 1 - \gamma\right)}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right) B\left(-\frac{\Omega}{2\mu} + 1, \frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 dt t^{\frac{1}{2} - \gamma} \\
 & \times \int_{-1}^1 dp \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \int_0^1 \frac{du e^{zu(1-p^2)(1-t)}}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \\
 & \times \left\{ -\frac{\Omega}{2\mu} + (1-v)\frac{\omega}{2} \right\} M\left(\frac{1}{2}, \frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, k_5 = zu(1-p^2)tv\right)
 \end{aligned} \tag{2.532}$$

Replace a, b and z by $\frac{\Omega}{2\mu} + 1 - \gamma$, $2 - \gamma$ and $k_1 = zu(1 - p^2)st$ in eq(2.172).

$$\begin{aligned}
 & M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1 = zu(1 - p^2)st\right) \\
 &= \frac{1}{B\left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \frac{\Omega}{2\mu}\right)} \int_0^1 dv \frac{e^{zuv(1-p^2)st}}{v^{\gamma - \frac{\Omega}{2\mu}}(1-v)^{\frac{\Omega}{2\mu}}} \\
 & \text{where, } \operatorname{Re}(2 - \gamma) > \operatorname{Re}\left(\frac{\Omega}{2\mu} + 1 - \gamma\right) > 0 \tag{2.533}
 \end{aligned}$$

Substitute eq(2.533) into eq(2.528).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} M_n\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \Lambda_n(\gamma; z) \\
 &= \frac{1}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right) B\left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \frac{\Omega}{2\mu}\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \\
 & \times \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \int_0^1 \frac{dv}{v^{-\frac{\Omega}{2\mu} + \gamma} (1-v)^{\frac{\Omega}{2\mu}}} \\
 & \times \left\{ zuv(1 - p^2)st + \left(1 - \gamma + \frac{\omega}{2}\right) \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \\
 &= \frac{1}{\pi B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right) B\left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \frac{\Omega}{2\mu}\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \\
 & \times \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \int_0^1 \frac{dv}{v^{-\frac{\Omega}{2\mu} + \gamma} (1-v)^{\frac{\Omega}{2\mu} + 1}} \\
 & \times \left\{ -\frac{\Omega}{2\mu} + (1-v)\frac{\omega}{2} \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \tag{2.534}
 \end{aligned}$$

Our description of describe eq(2.529), eq(2.530), eq(2.531), eq(2.532) as integral formalism, is exactly equivalent to eq(2.534). Plug eq(2.528) through eq(2.532) into eq(2.508), and then, substitute them into eq(2.506):

$$\begin{aligned}
 \mathcal{RW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) &= z^{1-\gamma} \left\{ \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\Gamma(2-\gamma)} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) - \frac{\varepsilon}{2} x \Lambda(\gamma; z) \right\} \\
 \text{where,} \\
 M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) &= \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{n! (2 - \gamma)_n} z^n \\
 \Lambda(\gamma; z) &= \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{n! (2 - \gamma)_n} z^n \sum_{k=0}^{\infty} \frac{(n + 1 - \gamma + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma + n)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \frac{5}{2} - \gamma)} z^k \\
 &= \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\pi \Gamma(2 - \gamma) B(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)} \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_0^1 du \frac{e^{z(1-p^2)(1-t)u}}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \\
 &\times \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \left\{ k_1 \partial_{k_1} + \left(1 - \gamma + \frac{\omega}{2}\right) \right\} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1 = zu(1-p^2)st\right) \\
 &= \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\pi \Gamma(2 - \gamma) B(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)} \int_{-1}^1 dp \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_0^1 du \frac{e^{z(1-p^2)(1-t)u}}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \\
 &\times \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \left\{ (1 - \gamma) M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 1 - \gamma, k_1 = zu(1-p^2)st\right) \right. \\
 &\left. + \frac{\omega}{2} M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, k_1 = zu(1-p^2)st\right) \right\} \\
 &= \frac{B(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)}{\pi \Gamma(\frac{\Omega}{2\mu} + 1 - \gamma) B(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \\
 &\times \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \left\{ -\frac{\Omega}{2\mu} + (1-v) \frac{\omega}{2} \right\} M\left(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, 1, k_2 = z(1-p^2)\{1 - (1-sv)t\}\right) \\
 &= \frac{2}{\pi \Gamma(\frac{\Omega}{2\mu} + 1 - \gamma) B(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \\
 &\times \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \left\{ -\frac{\Omega}{2\mu} + (1-v) \frac{\omega}{2} \right\} M\left(1, \frac{3}{2}, k_3 = zu\{1 - (1-sv)t\}\right) \\
 &= \frac{(\frac{3}{2} - \gamma)^{-1}}{\pi \Gamma(\frac{\Omega}{2\mu} + 1 - \gamma) B(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \int_{-1}^1 dp \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \\
 &\times \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} e^{zu(1-p^2)sv} \left\{ -\frac{\Omega}{2\mu} + (1-v) \frac{\omega}{2} \right\} M\left(1, \frac{5}{2} - \gamma, k_4 = zu(1-p^2)(1-sv)\right) \\
 &= \frac{1}{\sqrt{\pi} \Gamma(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma) B(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma)} \int_0^1 dt t^{\frac{1}{2}-\gamma} \int_{-1}^1 dp \int_0^1 \frac{dv}{v^{\gamma - \frac{\Omega}{2\mu}} (1-v)^{1 + \frac{\Omega}{2\mu}}} \\
 &\times \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} e^{zu(1-p^2)(1-t)} \left\{ -\frac{\Omega}{2\mu} + (1-v) \frac{\omega}{2} \right\} M\left(\frac{1}{2}, \frac{\Omega}{2\mu} + \frac{3}{2} - \gamma, k_5 = zu(1-p^2)tv\right)
 \end{aligned}$$

 Table 2.18: 2^{nd} kind of grand confluent hypergeometric function of infinite series

Plug eq(2.534) into eq(2.508) and substitute into eq(2.506). We obtain the integral representation of the second kind of grand confluent hypergeometric function:

$$\mathcal{RW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) = z^{1-\gamma} \left\{ \frac{1}{\Gamma\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 du e^{zu} u^{\frac{\Omega}{2\mu} - \gamma} (1-u)^{-\frac{\Omega}{2\mu}} - \frac{\varepsilon}{2} x \wedge(\gamma; z) \right\}$$

where,

$$\begin{aligned} \wedge(\gamma; z) &= \frac{1}{\pi\Gamma\left(1 - \frac{\Omega}{2\mu}\right)B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \int_0^1 \frac{dv}{v^{-\frac{\Omega}{2\mu} + \gamma} (1-v)^{\frac{\Omega}{2\mu}}} \\ &\times \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \left\{ zuv(1-p^2)st + \left(1 - \gamma + \frac{\omega}{2}\right) \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \\ &= \frac{1}{\pi\Gamma\left(1 - \frac{\Omega}{2\mu}\right)B\left(\frac{\Omega}{2\mu} + 1 - \gamma, -\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma\right)} \int_0^1 ds \frac{s^{-\frac{1}{2}}}{(1-s)^{\gamma - \frac{\Omega}{2\mu}}} \int_0^1 dt t^{\frac{1}{2} - \gamma} \int_{-1}^1 dp \\ &\times \int_0^1 \frac{du}{u^{-\frac{\Omega}{2\mu} - \frac{1}{2} + \gamma} (1-u)^{\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma}} \int_0^1 \frac{dv}{v^{-\frac{\Omega}{2\mu} + \gamma} (1-v)^{\frac{\Omega}{2\mu} + 1}} \left\{ -\frac{\Omega}{2\mu} + (1-v)\frac{\omega}{2} \right\} e^{zu(1-p^2)\{1-t(1-sv)\}} \end{aligned}$$

Table 2.19: Integral form of 2nd kind of grand confluent hypergeometric function of infinite series

2.2.2 $\nu = \text{integer}$

When ν is integer, one of two solution of the grand confluent hypergeometric equation does not have any meaning. Because $y_2(x) = \mathcal{RW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right)$ can be described as $\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right)$ as long as $|\lambda_1 - \lambda_2| = |\nu - 1| = \text{integer}$. If $\nu = 1$, the two solutions are identical. Then two solutions can be solved by Case B) of theorem 1 in the appendix. And if ν is any integer except 1, then it

can be calculated by Case C) and Case D) of theorem 1 in the appendix. Now let's investigate the second independent solution as ν an integer.

(i) As $\nu =$ negative integer, 0, and 1

As ν is smaller than 1, $\lambda_2 = 1 - \nu$ is positive integer. So $\lambda_1 = 0$ is smaller than λ_2 . By using the Case C) in Theorem 1, $g_1(x) = G(x)|_{\lambda=\lambda_1=0}$ is equal to $\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right)$. Also, as ν is equal to 1, λ_1 is equal to λ_2 . So by using the Case B) in Theorem 1, $g_1(x) = G(x)|_{\lambda=\lambda_1=0}$ is equal to $\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right)$. It is same as before. Only difference is that the previous case was ν is equal to 0 or negative integer and in this case ν is equal to 1. Now, lets assume that the second independent solution is

$$g_2(x) = u(x)\mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \quad (2.535)$$

Substitute eq(2.535) into the grand Confluent hypergeometric differential equation

$$x (u(x)\mathcal{QW}(z))'' + (\mu x^2 + \varepsilon x + \nu) (u(x)\mathcal{QW}(z))' + (\Omega x + \varepsilon \omega) (u(x)\mathcal{QW}(z)) = 0 \quad (2.536)$$

Then, we obtain

$$u(x) = C_1 \int \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}_{|\alpha_0|, |\alpha_1|}^2(z)} dx + C_0 \quad ; C_0, C_1 = \text{arbitrary constant} \quad (2.537)$$

Substitute eq(2.537) into eq(2.535)

$$g_2(x) = C_1 \mathcal{QW}(z) \int \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}^2(z)} dx + C_0 \mathcal{QW}(z) \quad (2.538)$$

For simplicity, let say $C_0 = 0$ & $C_1 = 1$. Then,

$$\boxed{g_2(x) = \mathcal{QW}(z) \int \frac{x^{-\nu} e^{-\left(\frac{1}{2}\mu x^2 + \varepsilon x\right)}}{\mathcal{QW}^2(z)} dx} \quad (2.539)$$

The eq(2.539) is called "The first kind integral formalism of the second independent solution of the grand confluent hypergeometric function as $\nu =$ negative integer, 0 and 1 for the case of infinite series."

(ii) Case of $\nu =$ positive integer except 1

As ν is greater than 1, $\lambda_2 = 1 - \nu$ is negative integer. So λ_2 is smaller than λ_1 .

By using the Case E) in theorem 1, $g_1(x) = G(x)|_{\lambda=\lambda_2=1-\nu}$ is equal to $\mathcal{RW}(\gamma =$

$\frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2$). Now, let assume that the second independent solution is

$$g_2(x) = u(x)g_1(x) = u(x)\mathcal{RW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \quad (2.540)$$

Substitute eq(2.540) into the grand confluent hypergeometric differential equation

$$x (u(x)\mathcal{RW}(z))'' + (\mu x^2 + \varepsilon x + \nu) (u(x)\mathcal{RW}(z))' + (\Omega x + \varepsilon \omega) (u(x)\mathcal{RW}(z)) = 0 \quad (2.541)$$

Then, we obtain

$$u(x) = Q_1 \int \frac{x^{-\nu} e^{-(\frac{1}{2}\mu x^2 + \varepsilon x)}}{\mathcal{RW}_{\psi_0, \psi_1}^2(z)} dx + Q_0 \quad ; Q_0, Q_1 = \text{arbitrary constant} \quad (2.542)$$

Substitute eq(2.542) into eq(2.540)

$$g_2(x) = Q_1 \mathcal{RW}(z) \int \frac{x^{-\nu} e^{-(\frac{1}{2}\mu x^2 + \varepsilon x)}}{\mathcal{RW}^2(z)} dx + Q_0 \mathcal{RW}(z) \quad (2.543)$$

For simplicity, let say $Q_0 = 0$ & $Q_1 = 1$. Then,

$$\boxed{g_2(x) = \mathcal{RW}(z) \int \frac{x^{-\nu} e^{-(\frac{1}{2}\mu x^2 + \varepsilon x)}}{\mathcal{RW}^2(z)} dx} \quad (2.544)$$

The eq(2.544) is called "The second kind integral formalism of the second independent solution of the grand confluent hypergeometric function as $\nu =$ positive integer except 1 for the case of infinite series." Eq(2.539) and eq(2.544) are inconvenient for direct calculation, as the integrand contains $[\mathcal{QW}(z)]^{-2}$ and $[\mathcal{RW}(z)]^{-2}$. Now let's investigate eq(2.539) and eq(2.544) more precisely how these two analytic functions have power series. These two functions can be described by the Frobenius method.

(A)the case of $\nu = 0$ or negative integer

As ν is 0 or negative integer, $\lambda_2 = 1 - \nu$ is positive integer. So $\lambda_1 = 0$ is smaller than λ_2 . By using Case C) in theorem 1, the first independent solution is

$$g_1(x) = G(x)|_{\lambda=\lambda_1=0} = \mathcal{QW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \quad (2.545)$$

The second independent solution is

$$g_2(x) = g_1(x)\ln x + x^{\lambda_2} \sum_{n=0}^{\infty} b_n(\lambda_2)x^n \quad (2.546)$$

To generate this solution, keep the recurrence formula in term of λ and use it to find the coefficients $C_n(n \geq 1)$ in terms of both λ and C_0 , where coefficient C_0 remains

arbitrary. Substitute these C_n into $G(x) = x^\lambda \sum_{n=0}^{\infty} C_n x^n$ to obtain a function $G(\lambda, x)$ which depends on the variables λ & x . From the Case C) in theorem 1, the second independent solution is

$$g_2(x) = \left. \frac{\partial G(\lambda, x)}{\partial \lambda} \right|_{\lambda=\lambda_2} = \ln x \sum_{n=0}^{\infty} C_n x^{n+\lambda} \Big|_{\lambda=\lambda_2} + \sum_{n=0}^{\infty} \frac{\partial C_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=\lambda_2} \quad (2.547)$$

From eq(2.262), $G(x) = g(x)_{domin.} + g(x)_{small}$. Plug it into eq(2.547).

$$g_2(x) = \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=\lambda_2} + \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=\lambda_2} \quad (2.548)$$

Replace ψ_0 and ψ_1 by $-\frac{\Omega}{2\mu} - 1 + \gamma$ and $-\frac{\Omega}{2\mu} - \frac{3}{2} + \gamma$ in Table 2.13, and Change the maximum index number n from ψ_0 to ∞ in it. Also, change maximum index number k from $\psi_1 - n$ to ∞ . Then the second independent solution of grand confluent hypergeometric function as $\nu = 0$ or negative integer for the case of infinite series is exactly equivalent to eq(2.548) in the following way:

$$\begin{aligned}
 g_2(x) &= \mathcal{RW}^F\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \\
 &= z^{1-\gamma} \frac{\Gamma(-\frac{\Omega}{2\mu} + 1)}{\Gamma(2-\gamma)} \left\{ (1 + \ln(x)) M\left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z\right) \right. \\
 &\quad + \sum_{n=1}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{2 n! (2 - \gamma)_n} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k+1 + \frac{\Omega}{2\mu} - \gamma)} - \frac{(2k+3-\gamma)}{(k+1)(k+2-\gamma)} \right\} z^n \\
 &\quad - \frac{\varepsilon}{2} x \left\{ \ln x \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n z^n}{n! (2 - \gamma)_n} \sum_{k=0}^{\infty} \frac{(n+1-\gamma + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma + n)_k}{\Gamma(k+n + \frac{3}{2}) \Gamma(k+n + \frac{5}{2} - \gamma)} z^k \right. \\
 &\quad + \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n z^n}{n! (2 - \gamma)_n} \left(\frac{1}{2} - \frac{(n+1 - \frac{\gamma}{2})(n+1 - \gamma + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \right) \\
 &\quad \times \sum_{k=0}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{2} - \gamma) (\frac{\Omega}{2\mu} + \frac{3}{2} + n - \gamma)_k}{\Gamma(n+k + \frac{3}{2}) \Gamma(n+k + \frac{5}{2} - \gamma)} z^k \\
 &\quad - \sum_{n=1}^{\infty} \left\{ \frac{(1 - \gamma + \frac{\omega}{2})}{(\frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} + p - \gamma)}{(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - \frac{3}{2} - k + \gamma)} + \frac{(2k+4-\gamma)}{(k + \frac{3}{2})(k + \frac{5}{2} - \gamma)} \right) \right. \\
 &\quad + \frac{(n+1 - \gamma + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \frac{3}{2} - \gamma)} \prod_{p=0}^{n-1} \frac{(\frac{\Omega}{2\mu} + 1 + p - \gamma)}{(p+1)(p+2-\gamma)} \\
 &\quad \times \sum_{k=0}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - 1 - k + \gamma)} + \frac{(2k+3-\gamma)}{(k+1)(k+2-\gamma)} \right) \left. \right\} z^n \\
 &\quad - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j+1 - \gamma + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \frac{3}{2} - \gamma)} \prod_{p=j}^{n-1} \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} + p - \gamma)}{(p + \frac{3}{2})(p + \frac{5}{2} - \gamma)} \prod_{k=0}^{j-1} \frac{(\frac{\Omega}{2\mu} + 1 + k - \gamma)}{(k+1)(k+2-\gamma)} \\
 &\quad \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - 1 - l + \gamma)} + \frac{(2l+3-\gamma)}{(l+1)(l+2-\gamma)} \right) \right. \\
 &\quad \left. + \sum_{l=j}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - \frac{3}{2} - l + \gamma)} + \frac{(2l+4-\gamma)}{(l + \frac{3}{2})(l + \frac{5}{2} - \gamma)} \right) \right] z^n \left. \right\}
 \end{aligned}$$

Table 2.20: 2^{nd} independent solution of grand confluent hypergeometric function as $\nu = 0$ or negative integer for the case of infinite series.

(B) the case of $\nu =$ positive integer except 1

As ν is positive integer except 1, $\lambda_2 = 1 - \nu$ is negative integer. So $\lambda_2 = 1 - \nu$ is

smaller than $\lambda_1 = 0$. By using Case E) in theorem 1, the first independent solution is

$$g_1(x) = G(x)\Big|_{\lambda=\lambda_2=1-\nu} = \mathcal{RW}\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \quad (2.549)$$

The second independent solution is

$$g_2(x) = g_1(x)\ln x + x^{\lambda_1} \sum_{n=0}^{\infty} b_n(\lambda_1)x^n \quad (2.550)$$

To generate this solution, keep the recurrence formula in term of λ and use it to find to coefficients $C_n(n \geq 1)$ in terms of both λ and C_0 , where coefficient C_0 remains arbitrary. Substitute these C_n into $G(x) = x^\lambda \sum_{n=0}^{\infty} C_n x^n$ to obtain a function $G(\lambda, x)$ which depends on the variables λ & x . From the Case E) in theorem 1, the second independent solution is

$$g_2(x) = \frac{\partial G(\lambda, x)}{\partial \lambda} \Big|_{\lambda=\lambda_1} = \ln x \sum_{n=0}^{\infty} C_n x^{n+\lambda} \Big|_{\lambda=\lambda_1} + \sum_{n=0}^{\infty} \frac{\partial C_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=\lambda_1} \quad (2.551)$$

From eq(2.262), $G(x) = g(x)_{domin.} + g(x)_{small}$. Plug it into eq(2.551).

$$g_2(x) = \frac{\partial g(x)_{domin.}}{\partial \lambda} \Big|_{\lambda=\lambda_1} + \frac{\partial g(x)_{small}}{\partial \lambda} \Big|_{\lambda=\lambda_1} \quad (2.552)$$

Replace $|\alpha_0|$ and $|\alpha_1|$ by $-\frac{\Omega}{2\mu}$ and $-\frac{\Omega}{2\mu} - \frac{1}{2}$ in Table 2.14, and Change the maximum index number n from $|\alpha_0|$ to ∞ in it. Also, change maximum index number k from $|\alpha_1| - n$ to ∞ . Then the second independent solution of grand confluent hypergeometric function as $\nu =$ positive integer except 1 for the case of infinite series is exactly equivalent to eq(2.552) in the following way:

$$\begin{aligned}
 g_2(x) &= \mathcal{QW}^F\left(\gamma = \frac{1}{2}(1 + \nu); z = -\frac{1}{2}\mu x^2\right) \\
 &= (1 + \ln(x)) \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} M\left(\frac{\Omega}{2\mu}, \gamma, z\right) \\
 &+ \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} \sum_{n=1}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{2^n n! (\gamma)_n} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k + \frac{\Omega}{2\mu})} - \frac{(2k + 1 + \gamma)}{(k + 1)(k + \gamma)} \right\} z^n \\
 &- x \frac{\varepsilon}{2} \frac{\Gamma(-\frac{\Omega}{2\mu} + \gamma)}{\Gamma(\gamma)} \left\{ \ln x \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} z^n \sum_{k=0}^{\infty} \frac{(n + \frac{\omega}{2}) \Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (\frac{\Omega}{2\mu} + \frac{1}{2} + n)_k}{\Gamma(k + n + \frac{3}{2}) \Gamma(k + n + \gamma + \frac{1}{2})} z^k \right. \\
 &+ \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} \left(\frac{1}{2} - \frac{(n + \frac{\gamma}{2})(n + \frac{\omega}{2})}{(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \right) \sum_{k=0}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) (\frac{\Omega}{2\mu} + \frac{1}{2} + n)_k}{\Gamma(n + k + \frac{3}{2}) \Gamma(n + k + \gamma + \frac{1}{2})} z^k \\
 &- \sum_{n=1}^{\infty} \left\{ \frac{\frac{\omega}{2}}{(\gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + p)}{(p + \frac{3}{2})(p + \gamma + \frac{1}{2})} \sum_{k=0}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - \frac{1}{2} - k)} + \frac{(2(k + 1) + \gamma)}{(k + \frac{3}{2})(k + \gamma + \frac{1}{2})} \right) \right. \\
 &+ \left. \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})(n + \gamma - \frac{1}{2})} \prod_{p=0}^{n-1} \frac{(\frac{\Omega}{2\mu} + p)}{(p + 1)(p + \gamma)} \sum_{k=0}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - k)} + \frac{(2k + 1 + \gamma)}{(k + 1)(k + \gamma)} \right) \right\} z^n \\
 &- \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})(j + \gamma - \frac{1}{2})} \prod_{p=j}^{n-1} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + p)}{(p + \frac{3}{2})(p + \frac{1}{2} + \gamma)} \prod_{k=0}^{j-1} \frac{(\frac{\Omega}{2\mu} + k)}{(k + 1)(k + \gamma)} \\
 &\times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - l)} + \frac{(2l + 1 + \gamma)}{(l + 1)(l + \gamma)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - \frac{1}{2} - l)} + \frac{(2(k + 1) + \gamma)}{(l + \frac{3}{2})(l + \frac{1}{2} + \gamma)} \right) \right] z^n \Big\}
 \end{aligned}$$

Table 2.21: 2^{nd} independent solution of grand confluent hypergeometric function as $\nu =$ positive integer except 1 for the case of infinite series

(C)the case of $\nu = 1$

As ν is 1, then two solution are identical: ($\lambda_1 = \lambda_2$). Then two solutions can be solved by Case B) in theorem 1. The first independent solution is

$$g_1(x) = G(x)|_{\lambda=0, \gamma=1} = \mathcal{QW}\left(\gamma = 1; z = -\frac{1}{2}\mu x^2\right) \quad (2.553)$$

The second independent solution is

$$g_2(x) = g_1(x)\ln x + x^{\lambda_1} \sum_{n=0}^{\infty} b_n(\lambda_1)x^n \quad (2.554)$$

To generate this solution, keep the recurrence formula in term of λ and use it to find to coefficients $C_n(n \geq 1)$ in terms of both λ and C_0 , where coefficient C_0 remains arbitrary. Substitute these C_n into $G(x) = x^\lambda \sum_{n=0}^{\infty} C_n x^n$ to obtain a function $G(\lambda, x)$ which depends on the variables λ & x . From the Case B) in theorem 1, the second independent solution is

$$g_2(x) = \frac{\partial G(\lambda, x)}{\partial \lambda} \Big|_{\lambda=0, \gamma=1} = \ln x \sum_{n=0}^{\infty} C_n x^{n+\lambda} \Big|_{\lambda=0, \gamma=1} + \sum_{n=0}^{\infty} \frac{\partial C_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=0, \gamma=1} \quad (2.555)$$

From eq(2.262), $G(x) = g(x)_{domin.} + g(x)_{small}$. Plug it into eq(2.555).

$$g_2(x) = \left. \frac{\partial g(x)_{domin.}}{\partial \lambda} \right|_{\lambda=0, \gamma=1} + \left. \frac{\partial g(x)_{small}}{\partial \lambda} \right|_{\lambda=0, \gamma=1} \quad (2.556)$$

Replace $|\alpha_0|$ and $|\alpha_1|$ by $-\frac{\Omega}{2\mu}$ and $-\frac{\Omega}{2\mu} - \frac{1}{2}$ in Table 2.15, and Change the maximum index number n from $|\alpha_0|$ to ∞ in it. Also, change maximum index number k from $|\alpha_1| - n$ to ∞ . Then the second independent solution of grand confluent hypergeometric function as $\nu = 1$ for the case of infinite series is exactly equivalent to eq(556) in the following way:

$$\begin{aligned}
 g_2(x) &= \mathcal{QW}^S\left(\gamma = 1; z = -\frac{1}{2}\mu x^2\right) \\
 &= \ln(x)\Gamma\left(-\frac{\Omega}{2\mu} + 1\right) M\left(\frac{\Omega}{2\mu}, 1, z\right) + \Gamma\left(-\frac{\Omega}{2\mu} + 1\right) \sum_{n=1}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{2(n!)^2} \sum_{k=0}^{n-1} \left\{ \frac{1}{(k + \frac{\Omega}{2\mu})} - \frac{2}{(k+1)} \right\} z^n \\
 &\quad - x \frac{\varepsilon}{2} \Gamma\left(-\frac{\Omega}{2\mu} + 1\right) \left\{ \ln x \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{(n!)^2} z^n \sum_{k=0}^{\infty} \frac{(n + \frac{\omega}{2})(\Gamma(n + \frac{1}{2}))^2 (\frac{\Omega}{2\mu} + \frac{1}{2} + n)_k}{(\Gamma(k + n + \frac{3}{2}))^2} z^k \right. \\
 &\quad + \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n z^n}{(n!)^2} \left(\frac{1}{2} - \frac{(n + \frac{\omega}{2})}{(n + \frac{1}{2})} \right) \sum_{k=0}^{\infty} \frac{(\Gamma(n + \frac{1}{2}))^2 (\frac{\Omega}{2\mu} + \frac{1}{2} + n)_k}{(\Gamma(n + k + \frac{3}{2}))^2} z^k \\
 &\quad - \sum_{n=1}^{\infty} \left\{ \omega \prod_{p=0}^{n-1} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + p)}{(p + \frac{3}{2})^2} \sum_{k=0}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - \frac{1}{2} - k)} + \frac{2}{(k + \frac{3}{2})} \right) \right. \\
 &\quad \left. + \frac{(n + \frac{\omega}{2})}{2(n + \frac{1}{2})^2} \prod_{p=0}^{n-1} \frac{(\frac{\Omega}{2\mu} + p)}{(p+1)^2} \sum_{k=0}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - k)} + \frac{2}{(k+1)} \right) \right\} z^n \\
 &\quad - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{(j + \frac{\omega}{2})}{2(j + \frac{1}{2})^2} \prod_{p=j}^{n-1} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + p)}{(p + \frac{3}{2})^2} \prod_{k=0}^{j-1} \frac{(\frac{\Omega}{2\mu} + k)}{(k+1)^2} \\
 &\quad \times \left[\sum_{l=0}^{j-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - l)} + \frac{2}{(l+1)} \right) + \sum_{l=j}^{n-1} \left(\frac{1}{(-\frac{\Omega}{2\mu} - \frac{1}{2} - l)} + \frac{2}{(l + \frac{3}{2})} \right) \right] z^n \Big\}
 \end{aligned}$$

 Table 2.22: 2^{nd} independent solution of grand confluent hypergeometric function as

$\nu = 1$ for the case of infinite series.

Chapter 3

Three-term Recurrence Formula

Unfortunately, there is no exact solution for linear ordinary second order differential equation consisting of three term recurrence relations: some of the examples are Lamé function, the generalized Lamé function, Heun's equation, Mathieu function, grand confluent hypergeometric function, etc. The reason why these functions are unsolved currently is that they consist of three term recurrence formulas. Two term recurrence formula are easy to solve: some of examples are Legendre function, hypergeometric function, Kummer function, Bessel function, etc. Let's think about one of the examples which is unsolved in detail.

$$\frac{\partial^2 y}{\partial t^2} + \frac{1}{2} \left(\frac{1}{t-a} + \frac{1}{t-b} + \frac{1}{t-c} \right) \frac{\partial y}{\partial t} - \frac{\alpha(\alpha+1)t + \beta}{t(t-a)(t-b)(t-c)} y = 0 \quad (3.1)$$

Eq(3.1) is Lamé differential equation. If α is not positive integer, the solution of it is called as the generalized Lamé function. Replace t by $x+a$ in eq(3.1).

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x+a-b} + \frac{1}{x+a-c} \right) \frac{\partial y}{\partial x} - \frac{\{\alpha(\alpha+1)a + \beta\} + \alpha(\alpha+1)x}{4x(x+a-b)(x+a-c)} y = 0 \quad (3.2)$$

Assume that

$$y(x) = \sum_{n=0}^{\infty} c_n (t-a)^{n+\lambda} = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \quad (3.3)$$

Plug eq(3.3) into eq(3.2).

$$c_{n+1} = A_n c_n + B_n c_{n-1} \quad ; n \geq 1 \quad (3.4)$$

Divide c_n in eq(3.4).

$$K_n = A_n + \frac{B_n}{K_{n-1}} \quad ; n \geq 1 \quad (3.5)$$

where,

$$K_n = \frac{c_{n+1}}{c_n} \quad (3.6a)$$

$$K_{n-1} = \frac{c_n}{c_{n-1}} \quad (3.6b)$$

$$A_n = \frac{\{\alpha(\alpha+1)a + \beta\} - 2^2(2a-b-c)(n+\lambda)^2}{2^2(a-b)(a-c)(n+\lambda+1)(n+\lambda+\frac{1}{2})} \quad (3.6c)$$

$$B_n = \frac{\{\alpha + 2(n+\lambda) - 1\}\{\alpha - 2(n+\lambda) + 2\}}{2^2(a-b)(a-c)(n+\lambda+1)(n+\lambda+\frac{1}{2})} \quad (3.6d)$$

All other unsolved differential equations can be described as in eq(3.4). If we get a

formula of eq(3.4) type, we can apply it to all other unsolved functions: Lamé function, generalized Lamé function, Mathieu function, Heun's equation, grand confluent hypergeometric function, etc.

3.1 Infinite series

Assume that

$$c_1 = A_0 c_0 \quad (3.7)$$

Eq(3.7) is a necessary boundary condition. The three term recurrence formula in all unsolved differential equations follow eq(3.7).

$$\prod_{n=a_i}^{a_i-1} B_n = 1 \quad \text{where } a_i \text{ is positive integer including } 0 \quad (3.8)$$

Eq(3.8) is also a necessary condition. Every unsolved differential equation is also take satisfied with eq(3.8). In general, B_n is described as Pochhammer symbol. For

example, let

$$\begin{aligned}\prod_{n=a_i}^{b_i} B_n &= \prod_{n=a_i}^{b_i} \frac{(n + \beta)}{(n + 1)(n + \lambda)} \\ &= \frac{(\beta)_{b_i+1}(\lambda)_{a_i}(1)_{a_i}}{(\beta)_{a_i}(\lambda)_{b_i+1}(1)_{b_i+1}}\end{aligned}\tag{3.9}$$

replace b_i by $a_i - 1$ in eq(3.9).

$$\prod_{n=a_i}^{a_i-1} B_n = \frac{(\beta)_{a_i}(\lambda)_{a_i}(1)_{a_i}}{(\beta)_{a_i}(\lambda)_{a_i}(1)_{a_i}} = 1\tag{3.10}$$

Our definition of $B_{i,j,k,l}$ refer to $B_i B_j B_k B_l$. Also, $A_{i,j,k,l}$ refer to $A_i A_j A_k A_l$. For $n = 0, 1, 2, 3, \dots$, eq(3.4) gives

$$c_0$$

$$c_1 = A_0 c_0$$

$$c_2 = (A_{0,1} + B_1) c_0$$

$$c_3 = (A_{0,1,2} + A_2 B_1 + A_0 B_2) c_0$$

$$c_4 = (A_{0,1,2,3} + A_{2,3} B_1 + A_{0,3} B_2 + A_{0,1} B_3 + B_{1,3}) c_0$$

$$c_5 = (A_{0,1,2,3,4} + A_{2,3,4} B_1 + A_{0,3,4} B_2 + A_{0,1,4} B_3 + A_4 B_{1,3} + A_{0,1,2} B_4 \\ + A_2 B_{1,4} + A_0 B_{2,4}) c_0$$

$$c_6 = (A_{0,1,2,3,4,5} + A_{2,3,4,5} B_1 + A_{0,3,4,5} B_2 + A_{0,1,4,5} B_3 + A_{4,5} B_{1,3} + A_{0,1,2,5} B_4 + A_{2,5} B_{1,4} \\ + A_{0,5} B_{2,4} + A_{0,1,2,3} B_5 + A_{2,3} B_{1,5} + A_{0,3} B_{2,5} + A_{0,1} B_{3,5} + B_{1,3,5}) c_0$$

$$c_7 = (A_{0,1,2,3,4,5,6} + A_{2,3,4,5,6} B_1 + A_{0,3,4,5,6} B_2 + A_{0,1,4,5,6} B_3 + A_{4,5,6} B_{1,3} + A_{0,1,2,5,6} B_4 \\ + A_{2,5,6} B_{1,4} + A_{0,5,6} B_{2,4} + A_{0,1,2,3,6} B_5 + A_{2,3,6} B_{1,5} + A_{0,3,6} B_{2,5} \\ + A_{0,1,6} B_{3,5} + A_6 B_{1,3,5} + A_{0,1,2,3,4} B_6 + A_{2,3,4} B_{1,6} + A_{0,3,4} B_{2,6} + A_{0,1,4} B_{3,6} \\ + A_4 B_{1,3,6} + A_{0,1,2} B_{4,6} + A_2 B_{1,4,6} + A_0 B_{2,4,6}) c_0$$

$$\begin{aligned}
c_8 = & (A_{0,1,2,3,4,5,6,7} + A_{2,3,4,5,6,7}B_1 + A_{0,3,4,5,6,7}B_2 + A_{0,1,4,5,6,7}B_3 + A_{4,5,6,7}B_{1,3} \\
& + A_{0,1,2,5,6,7}B_4 + A_{2,5,6,7}B_{1,4} + A_{0,5,6,7}B_{2,4} + A_{0,1,2,3,6,7}B_5 + A_{2,3,6,7}B_{1,5} \\
& + A_{0,3,6,7}B_{2,5} + A_{0,1,6,7}B_{3,5} + A_{6,7}B_{1,3,5} + A_{0,1,2,3,4,7}B_6 + A_{2,3,4,7}B_{1,6} + A_{0,3,4,7}B_{2,6} \\
& + A_{0,1,4,7}B_{3,6} + A_{4,7}B_{1,3,6} + A_{0,1,2,7}B_{4,6} + A_{2,7}B_{1,4,6} + A_{0,7}B_{2,4,6} + A_{0,1,2,3,4,5}B_7 \\
& + A_{2,3,4,5}B_{1,7} + A_{0,3,4,5}B_{2,7} + A_{0,1,4,5}B_{3,7} + A_{4,5}B_{1,3,7} + A_{0,1,2,5}B_{4,7} + A_{2,5}B_{1,4,7} \\
& + A_{0,5}B_{2,4,7} + A_{0,1,2,3}B_{5,7} + A_{2,3}B_{1,5,7} + A_{0,3}B_{2,5,7} + A_{0,1}B_{3,5,7} + B_{1,3,5,7})c_0 \\
c_9 = & (A_{0,1,2,3,4,5,6,7,8} + A_{2,3,4,5,6,7,8}B_1 + A_{0,3,4,5,6,7,8}B_2 + A_{0,1,4,5,6,7,8}B_3 + A_{4,5,6,7,8}B_{1,3} \\
& + A_{0,1,2,5,6,7,8}B_4 + A_{2,5,6,7,8}B_{1,4} + A_{0,5,6,7,8}B_{2,4} + A_{0,1,2,3,6,7,8}B_5 + A_{2,3,6,7,8}B_{1,5} \\
& + A_{0,3,6,7,8}B_{2,5} + A_{0,1,6,7,8}B_{3,5} + A_{6,7,8}B_{1,3,5} + A_{0,1,2,3,4,7,8}B_6 + A_{2,3,4,7,8}B_{1,6} \\
& + A_{0,3,4,7,8}B_{2,6} + A_{0,1,4,7,8}B_{3,6} + A_{4,7,8}B_{1,3,6} + A_{0,1,2,7,8}B_{4,6} + A_{2,7,8}B_{1,4,6} \\
& + A_{0,7,8}B_{2,4,6} + A_{0,1,2,3,4,5,8}B_7 + A_{2,3,4,5,8}B_{1,7} + A_{0,3,4,5,8}B_{2,7}
\end{aligned}$$

$$\begin{aligned}
& +A_{0,1,4,5,8}B_{3,7} + A_{4,5,8}B_{1,3,7} \\
& +A_{0,1,2,5,8}B_{4,7} + A_{2,5,8}B_{1,4,7} + A_{0,5,8}B_{2,4,7} + A_{0,1,2,3,8}B_{5,7} + A_{2,3,8}B_{1,5,7} \\
& +A_{0,3,8}B_{2,5,7} + A_{0,1,8}B_{3,5,7} + A_8B_{1,3,5,7} + A_{0,1,2,3,4,5,6}B_8 + A_{2,3,4,5,6}B_{1,8} \\
& +A_{0,3,4,5,6}B_{2,8} + A_{0,1,4,5,6}B_{3,8} + A_{4,5,6}B_{1,3,8} + A_{0,1,2,5,6}B_{4,8} + A_{2,5,6}B_{1,4,8} \\
& +A_{0,5,6}B_{2,4,8} + A_{0,1,2,3,6}B_{5,8} + A_{2,3,6}B_{1,5,8} + A_{0,3,6}B_{2,5,8} + A_{0,1,6}B_{3,5,8} \\
& +A_6B_{1,3,5,8} + A_{0,1,2,3,4}B_{6,8} + A_{2,3,4}B_{1,6,8} + A_{0,3,4}B_{2,6,8} + A_{0,1,4}B_{3,6,8} \\
& +A_4B_{1,3,6,8} + A_{0,1,2}B_{4,6,8} + A_2B_{1,4,6,8} + A_0B_{2,4,6,8})c_0 \\
& \vdots \qquad \qquad \qquad \vdots
\end{aligned} \tag{3.11}$$

Surprisingly, as we see in eq(3.11), individual number of sequences in every c_n 's term are followed by Fibonacci number: 1,1,2,3,5,8,13,21,34,55,⋯. The sequence of each c_n consists of combinations A_n and B_n in eq(3.11). First of all, let; look at the sequence of each c_n in which does not include A_n 's

(1) Zero term of A_n 's

$$\begin{aligned}
 c_0 & \\
 c_2 &= B_1 c_0 \\
 c_4 &= B_{1,3} c_0 \\
 c_6 &= B_{1,3,5} c_0 \\
 c_8 &= B_{1,3,5,7} c_0 \\
 c_{10} &= B_{1,3,5,7,9} c_0 \\
 \vdots & \quad \vdots
 \end{aligned} \tag{3.12}$$

When a function $y(x)$, analytic at $x=0$, is expanded in a power series we have

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \tag{3.13}$$

λ is the indicial root, and put eq(3.12) in eq(3.13).

$$y_0(x) = c_0 x^\lambda + c_0 \sum_{n=1}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n+\lambda} \tag{3.14}$$

By using eq(3.8), eq(3.14) becomes

$$y_0(x) = c_0 \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n+\lambda} \quad (3.15)$$

Now, let see the sequence of each c_n which include one term of A_n 's in eq(3.11).

(2) One term of A_n 's

$$\begin{aligned} c_1 &= A_0 c_0 \\ c_3 &= \left\{ A_0 \left(\frac{B_2}{1} \right) 1 + A_2 \left(\frac{B_2}{B_2} \right) B_1 \right\} c_0 \\ c_5 &= \left\{ A_0 \left(\frac{B_{2,4}}{1} \right) 1 + A_2 \left(\frac{B_{2,4}}{B_2} \right) B_1 + A_4 \left(\frac{B_{2,4}}{B_{2,4}} \right) B_{1,3} \right\} c_0 \\ c_7 &= \left\{ A_0 \left(\frac{B_{2,4,6}}{1} \right) 1 + A_2 \left(\frac{B_{2,4,6}}{B_2} \right) B_1 + A_4 \left(\frac{B_{2,4,6}}{B_{2,4}} \right) B_{1,3} + A_6 \left(\frac{B_{2,4,6}}{B_{2,4,6}} \right) B_{1,3,5} \right\} c_0 \\ c_9 &= \left\{ A_0 \left(\frac{B_{2,4,6,8}}{1} \right) 1 + A_2 \left(\frac{B_{2,4,6,8}}{B_2} \right) B_1 + A_4 \left(\frac{B_{2,4,6,8}}{B_{2,4}} \right) B_{1,3} + A_6 \left(\frac{B_{2,4,6,8}}{B_{2,4,6}} \right) B_{1,3,5} \right. \\ &\quad \left. + A_8 \left(\frac{B_{2,4,6,8}}{B_{2,4,6,8}} \right) B_{1,3,5,7} \right\} c_0 \\ c_{11} &= \left\{ A_0 \left(\frac{B_{2,4,6,8,10}}{1} \right) 1 + A_2 \left(\frac{B_{2,4,6,8,10}}{B_2} \right) B_1 + A_4 \left(\frac{B_{2,4,6,8,10}}{B_{2,4}} \right) B_{1,3} + A_6 \left(\frac{B_{2,4,6,8,10}}{B_{2,4,6}} \right) B_{1,3,5} \right. \\ &\quad \left. + A_8 \left(\frac{B_{2,4,6,8,10}}{B_{2,4,6,8}} \right) B_{1,3,5,7} + A_{10} \left(\frac{B_{2,4,6,8,10}}{B_{2,4,6,8,10}} \right) B_{1,3,5,7,9} \right\} c_0 \\ &\quad \vdots \qquad \qquad \qquad \vdots \end{aligned} \quad (3.16)$$

Eq(3.16) is simply

$$c_{2n+1} = \sum_{i_1=0}^n \left(A_{2i_1} \frac{\prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=0}^{n-1} B_{2i_3+2}}{\prod_{i_3=0}^{i_1-1} B_{2i_3+2}} \right) c_0 \quad (3.17)$$

And

$$\frac{\prod_{i_a=0}^{i_b-1} B_{i_a}}{\prod_{i_a=0}^{i_c-1} B_{i_a}} = \prod_{i_a=i_c}^{i_b-1} B_{i_a} \quad (3.18)$$

Substitute eq(3.18) into eq(3.17).

$$c_{2n+1} = \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} c_0 \quad (3.19)$$

Put eq(3.19) in eq(3.13) and use eq(3.8).

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1+\lambda} \quad (3.20)$$

Lets now see the sequence of each c_n which includes two terms of A_n 's in eq(3.11):

(3) Two terms of A_n 's

$$\begin{aligned}
c_2 &= A_{0,1}c_0 \\
c_4 &= \left\{ A_0 \left\{ A_1 \left(\frac{1}{1} \right) \left(\frac{B_{1,3}}{B_1} \right) 1 + A_3 \left(\frac{B_2}{1} \right) \left(\frac{B_{1,3}}{B_{1,3}} \right) 1 \right\} + A_2 \left\{ A_3 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{1,3}}{B_{1,3}} \right) B_1 \right\} \right\} c_0 \\
c_6 &= \left\{ A_0 \left\{ A_1 \left(\frac{1}{1} \right) \left(\frac{B_{1,3,5}}{B_1} \right) 1 + A_3 \left(\frac{B_2}{1} \right) \left(\frac{B_{1,3,5}}{B_{1,3}} \right) 1 + A_5 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{1,3,5}}{B_{1,3,5}} \right) 1 \right\} \right. \\
&\quad + A_2 \left\{ A_3 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{1,3,5}}{B_{1,3}} \right) B_1 + A_5 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{1,3,5}}{B_{1,3,5}} \right) B_1 \right\} \\
&\quad \left. + A_4 \left\{ A_5 \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{1,3,5}}{B_{1,3,5}} \right) B_{1,3} \right\} \right\} c_0 \\
c_8 &= \left\{ A_0 \left\{ A_1 \left(\frac{1}{1} \right) \left(\frac{B_{1,3,5,7}}{B_1} \right) 1 + A_3 \left(\frac{B_2}{1} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3}} \right) 1 + A_5 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5}} \right) 1 \right. \right. \\
&\quad + A_7 \left(\frac{B_{2,4,6}}{1} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5,7}} \right) 1 \left. \right\} \\
&\quad + A_2 \left\{ A_3 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3}} \right) B_1 + A_5 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5}} \right) B_1 \right. \\
&\quad + A_7 \left(\frac{B_{2,4,6}}{B_2} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5,7}} \right) B_1 \left. \right\} \\
&\quad + A_4 \left\{ A_5 \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5}} \right) B_{1,3} + A_7 \left(\frac{B_{2,4,6}}{B_{2,4}} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5,7}} \right) B_{1,3} \right\} \\
&\quad \left. + A_6 \left\{ A_7 \left(\frac{B_{2,4,6}}{B_{2,4,6}} \right) \left(\frac{B_{1,3,5,7}}{B_{1,3,5,7}} \right) B_{1,3,5} \right\} \right\} c_0
\end{aligned}$$

$$\begin{aligned}
 c_{10} = & \left\{ A_0 \left\{ A_1 \left(\frac{1}{1} \right) \left(\frac{B_{1,3,5,7,9}}{B_1} \right) 1 + A_3 \left(\frac{B_2}{1} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3}} \right) 1 + A_5 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5}} \right) 1 \right. \right. \\
 & + A_7 \left(\frac{B_{2,4,6}}{1} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7}} \right) 1 + A_9 \left(\frac{B_{2,4,6,8}}{1} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7,9}} \right) 1 \left. \right\} \\
 & + A_2 \left\{ A_3 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3}} \right) B_1 + A_5 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5}} \right) B_1 \right. \\
 & + A_7 \left(\frac{B_{2,4,6}}{B_2} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7}} \right) B_1 + A_9 \left(\frac{B_{2,4,6,8}}{B_2} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7,9}} \right) B_1 \left. \right\} \\
 & + A_4 \left\{ A_5 \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5}} \right) B_{1,3} + A_7 \left(\frac{B_{2,4,6}}{B_{2,4}} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7}} \right) B_{1,3} \right. \\
 & + A_9 \left(\frac{B_{2,4,6,8}}{B_{2,4}} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7,9}} \right) B_{1,3} \left. \right\} \\
 & + A_6 \left\{ A_7 \left(\frac{B_{2,4,6}}{B_{2,4,6}} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7}} \right) B_{1,3,5} + A_9 \left(\frac{B_{2,4,6,8}}{B_{2,4,6}} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7,9}} \right) B_{1,3,5} \right\} \\
 & + A_8 \left\{ A_9 \left(\frac{B_{2,4,6,8}}{B_{2,4,6,8}} \right) \left(\frac{B_{1,3,5,7,9}}{B_{1,3,5,7,9}} \right) B_{1,3,5,7} \right\} \left. \right\} c_0 \\
 & \vdots \qquad \qquad \qquad \vdots
 \end{aligned} \tag{3.21}$$

Eq(3.21) is simply

$$c_{2n+2} = \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \frac{\prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=0}^{i_2-1} B_{2i_4+2} \prod_{i_5=0}^{n-1} B_{2i_5+3}}{\prod_{i_4=0}^{i_1-1} B_{2i_4+2} \prod_{i_5=0}^{i_2-1} B_{2i_5+3}} \right\} \right\} c_0 \tag{3.22}$$

By using eq(3.18), Eq(3.22) turns out to be

$$c_{2n+2} = \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} c_0 \quad (3.23)$$

Put eq(3.23) in eq(3.13) and use eq(3.8).

$$y_2(x) = c_0 \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} \right\} x^{2n+2+\lambda} \quad (3.24)$$

Lets now see the sequence of each c_n which includes three terms of A_n 's in eq(3.11):

(3) Three terms of A_n 's

$$c_3 = A_{0,1,2} c_0$$

$$c_5 = \left\{ A_0 \left\{ A_1 \left[A_2 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{1}{1} \right) \left(\frac{B_4}{1} \right) + A_4 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_3}{1} \right) \left(\frac{B_4}{B_4} \right) \right] \right. \right. \\ \left. \left. + A_3 A_4 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_4}{B_4} \right) \right\} + A_2 \left\{ A_3 \left[A_4 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_4}{B_4} \right) \right] \right\} \right\} c_0$$

$$c_7 = \left\{ A_0 \left\{ A_1 \left[A_2 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{1}{1} \right) \left(\frac{B_{4,6}}{1} \right) + A_4 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_3}{1} \right) \left(\frac{B_{4,6}}{B_4} \right) \right] \right. \right. \\ \left. \left. + A_6 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_{3,5}}{1} \right) \left(\frac{B_{4,6}}{B_{4,6}} \right) \right] \right. \\ \left. + A_3 \left[A_4 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_{4,6}}{B_4} \right) + A_6 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5}}{B_3} \right) \left(\frac{B_{4,6}}{B_{4,6}} \right) \right] \right. \\ \left. + A_5 \left[A_6 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6}}{B_{4,6}} \right) \right] \right\} \\ \left. + A_2 \left\{ A_3 \left[A_4 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_{4,6}}{B_4} \right) + A_6 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{3,5}}{B_3} \right) \left(\frac{B_{4,6}}{B_{4,6}} \right) \right] \right. \right. \\ \left. \left. + A_5 \left[A_6 B_1 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6}}{B_{4,6}} \right) \right] \right\} \right. \\ \left. + A_4 \left\{ A_5 \left[A_6 B_{1,3} \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6}}{B_{4,6}} \right) \right] \right\} \right\} c_0$$

$$\begin{aligned}
c_9 = & \left\{ A_0 \left\{ A_1 \left[A_2 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{1}{1} \right) \left(\frac{B_{4,6,8}}{1} \right) + A_4 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_3}{1} \right) \left(\frac{B_{4,6,8}}{B_4} \right) \right. \right. \right. \\
& + A_6 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_{3,5}}{1} \right) \left(\frac{B_{4,6,8}}{B_{4,6}} \right) + A_8 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_{3,5,7}}{1} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \left. \right] \\
& + A_3 \left[A_4 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_{4,6,8}}{B_4} \right) + A_6 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5}}{B_3} \right) \left(\frac{B_{4,6,8}}{B_{4,6}} \right) \right. \\
& + A_8 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5,7}}{B_3} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \left. \right] + A_5 \left[A_6 \cdot 1 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6,8}}{B_{4,6}} \right) \right. \\
& + A_8 \cdot 1 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{3,5,7}}{B_{3,5}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \left. \right] + A_7 \left[A_8 \cdot 1 \left(\frac{B_{2,4,6}}{1} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \right] \left. \right\} \\
& + A_2 \left\{ A_3 \left[A_4 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_{4,6,8}}{B_4} \right) + A_6 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{3,5}}{B_3} \right) \left(\frac{B_{4,6,8}}{B_{4,6}} \right) \right. \right. \\
& + A_8 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{3,5,7}}{B_3} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \left. \right] \\
& + A_5 \left[A_6 B_1 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6,8}}{B_{4,6}} \right) + A_8 B_1 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{3,5,7}}{B_{3,5}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \right] \\
& + A_7 \left[A_8 B_1 \left(\frac{B_{2,4,6}}{B_2} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \right] \left. \right\} \\
& + A_4 \left\{ A_5 \left[A_6 B_{1,3} \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6,8}}{B_{4,6}} \right) + A_8 B_{1,3} \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{3,5,7}}{B_{3,5}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \right] \right. \\
& + A_7 \left[A_8 B_{1,3} \left(\frac{B_{2,4,6}}{B_{2,4}} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \right] \left. \right\} \\
& + A_6 \left\{ A_7 \left[A_8 B_{1,3,5} \left(\frac{B_{2,4,6}}{B_{2,4,6}} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8}}{B_{4,6,8}} \right) \right] \right\} \left. \right\} c_0
\end{aligned}$$

$$\begin{aligned}
c_{11} = & \left\{ A_0 \left\{ A_1 \left[A_2 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{1}{1} \right) \left(\frac{B_{4,6,8,10}}{1} \right) + A_4 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_3}{1} \right) \left(\frac{B_{4,6,8,10}}{B_4} \right) \right. \right. \\
& + A_6 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_{3,5}}{1} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6}} \right) \\
& + A_8 \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_{3,5,7}}{1} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} \cdot 1 \left(\frac{1}{1} \right) \left(\frac{B_{3,5,7,9}}{1} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \left. \right] \\
& + A_3 \left[A_4 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_4} \right) + A_6 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5}}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6}} \right) \right. \\
& + A_8 \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5,7}}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} \cdot 1 \left(\frac{B_2}{1} \right) \left(\frac{B_{3,5,7,9}}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \left. \right] \\
& + A_5 \left[A_6 \cdot 1 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6}} \right) \right. \\
& + A_8 \cdot 1 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{3,5,7}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} \cdot 1 \left(\frac{B_{2,4}}{1} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \left. \right] \\
& + A_7 \left[A_8 \cdot 1 \left(\frac{B_{2,4,6}}{1} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} \cdot 1 \left(\frac{B_{2,4,6}}{1} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right. \\
& + A_9 \left[A_{10} \cdot 1 \left(\frac{B_{2,4,6,10}}{1} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7,9}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \left. \right] \left. \right\} \\
& + A_2 \left\{ A_3 \left[A_4 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_3}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_4} \right) + A_6 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{3,5}}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6}} \right) \right. \right. \\
& + A_8 B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{3,5,7}}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} B_1 \left(\frac{B_2}{B_2} \right) \left(\frac{B_{3,5,7,9}}{B_3} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \left. \right] \\
& + A_5 \left[A_6 B_1 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6}} \right) + A_8 B_1 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{3,5,7}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) \right. \\
& + A_{10} B_1 \left(\frac{B_{2,4}}{B_2} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \left. \right] \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& +A_7 \left[A_8 B_1 \left(\frac{B_{2,4,6}}{B_2} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} B_1 \left(\frac{B_{2,4,6}}{B_2} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \\
& +A_9 \left[A_{10} B_1 \left(\frac{B_{2,4,6,8}}{B_2} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7,9}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \} \\
& +A_4 \left\{ A_5 \left[A_6 B_{1,3} \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{3,5}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6}} \right) + A_8 B_{1,3} \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{3,5,7}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) \right] \right. \\
& \left. +A_{10} B_{1,3} \left(\frac{B_{2,4}}{B_{2,4}} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \\
& +A_7 \left[A_8 B_{1,3} \left(\frac{B_{2,4,6}}{B_{2,4}} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) + A_{10} B_{1,3} \left(\frac{B_{2,4,6}}{B_{2,4}} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \\
& +A_9 \left[A_{10} B_{1,3} \left(\frac{B_{2,4,6,8}}{B_{2,4}} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7,9}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \} \\
& +A_6 \left\{ A_7 \left[A_8 B_{1,3,5} \left(\frac{B_{2,4,6}}{B_{2,4,6}} \right) \left(\frac{B_{3,5,7}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8}} \right) \right] \right. \\
& \left. +A_{10} B_{1,3,5} \left(\frac{B_{2,4,6}}{B_{2,4,6}} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \\
& +A_9 \left[A_{10} B_{1,3,5} \left(\frac{B_{2,4,6,8}}{B_{2,4,6}} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7,9}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \} \\
& +A_8 \left\{ A_9 \left[A_{10} B_{1,3,5,7} \left(\frac{B_{2,4,6,8}}{B_{2,4,6,8}} \right) \left(\frac{B_{3,5,7,9}}{B_{3,5,7,9}} \right) \left(\frac{B_{4,6,8,10}}{B_{4,6,8,10}} \right) \right] \right\} c_0 \\
& \vdots \qquad \qquad \qquad \vdots
\end{aligned}$$

(3.25)

Eq(3.25) is simply

$$c_{2n+3} = \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \times \sum_{i_3=i_2}^n \left\{ A_{2i_3+2} \frac{\prod_{i_4=0}^{i_1-1} B_{2i_4+1} \prod_{i_5=0}^{i_2-1} B_{2i_5+2} \prod_{i_6=0}^{i_3-1} B_{2i_6+3} \prod_{i_7=0}^{n-1} B_{2i_7+4}}{\prod_{i_5=0}^{i_1-1} B_{2i_5+2} \prod_{i_6=0}^{i_2-1} B_{2i_6+3} \prod_{i_7=0}^{i_3-1} B_{2i_7+4}} \right\} \right\} \right\} c_0 \quad (3.26)$$

By using eq(3.18), Eq(3.26) turns out to be

$$c_{2n+3} = \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \sum_{i_3=i_2}^n \left\{ A_{2i_3+2} \prod_{i_4=0}^{i_1-1} B_{2i_4+1} \prod_{i_5=i_1}^{i_2-1} B_{2i_5+2} \times \prod_{i_6=i_2}^{i_3-1} B_{2i_6+3} \prod_{i_7=i_3}^{n-1} B_{2i_7+4} \right\} \right\} \right\} c_0 \quad (3.27)$$

Put eq(3.27) in eq(3.13) and use eq(3.8).

$$y_3(x) = c_0 \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \sum_{i_3=i_2}^n \left\{ A_{2i_3+2} \prod_{i_4=0}^{i_1-1} B_{2i_4+1} \prod_{i_5=i_1}^{i_2-1} B_{2i_5+2} \times \prod_{i_6=i_2}^{i_3-1} B_{2i_6+3} \prod_{i_7=i_3}^{n-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2n+3+\lambda} \quad (3.28)$$

Eq(3.29) is simply

$$\begin{aligned}
 y(x) = & c_0 \left\{ \sum_{n=0}^{\infty} \left(\prod_{i_1=0}^{n-1} B_{2i_1+1} \right) x^{2n+\lambda} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1+\lambda} \right. \\
 & + \sum_{N=2}^{\infty} \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{k=1}^{N-1} \left(\sum_{i_k=i_{k-1}}^n A_{2i_k+(k-1)} \right) \sum_{i_N=i_{N-1}}^n \left\{ A_{2i_N+(N-1)} \right. \right. \right. \\
 & \times \left. \left. \prod_{l=0}^{N-1} \left(\prod_{i_{l+1+N}=i_l}^{i_{l+1}-1} B_{2i_{l+1+N}+(l+1)} \right) \prod_{i_{2N+1}=i_N}^{n-1} B_{2i_{2N+1}+(N+1)} \right\} \right\} \left. \right\} x^{2n+N+\lambda} \Bigg\} \\
 & \text{where, } i_0 = 0
 \end{aligned} \tag{3.30}$$

3.2 Polynomial which makes B_n term terminated

Now, let's look at the polynomial case of eq(3.30). Assume that B_n is terminated at certain value of n. Then, each $y_i(x)$ where $i = 0, 1, 2, \dots$ will be polynomial. Examples of these are Heun's equation, Lamé function, grand confluent hypergeometric function, etc. First of all, B_{2k+1} will be terminated at certain value of k. And we choose eigenvalue β_0 in which B_{2k+1} is terminated where $\beta_0 = 0, 1, 2, \dots$. Then, we choose $B_{2\beta_0+1} = 0$. And B_{2k+2} will be terminated at certain value of k. We choose eigenvalue β_1 in which B_{2k+2} is terminated where $\beta_1 = 0, 1, 2, \dots$. again choose $B_{2\beta_1+2} = 0$. Also, B_{2k+3} will be terminated at certain value of k. Choosing eigenvalue

β_2 in which B_{2k+3} is terminated where $\beta_2 = 0, 1, 2, \dots$. Then choose $B_{2\beta_2+3} = 0$. By repeating this process, we obtain

$$B_{2\beta_i+(i+1)} = 0 \quad \text{where } i = 0, 1, 2, \dots \quad \text{and } \beta_i = 0, 1, 2, \dots \quad (3.31)$$

In general, the two term recurrence formula for polynomial has only one eigenvalue, for example, the Laguerre function, confluent hypergeometric function, Legendre function, etc. But the three term recurrence formula has infinite eigenvalues which are β_i , where $i = 0, 1, 2, \dots$ and $\beta_i = 0, 1, 2, \dots$.

(a) As $\beta_0=0$, then $B_1=0$ in eq(3.12).

$$c_0 \quad (3.32)$$

(b) As $\beta_0=1$, then $B_3=0$ in eq(3.12).

$$\begin{aligned} c_0 \\ c_2 = B_1 c_0 \end{aligned} \quad (3.33)$$

(c) As $\beta_0=2$, then $B_5=0$ in eq(3.12).

$$\begin{aligned} c_0 & \\ c_2 &= B_1 c_0 \\ c_4 &= B_{1,3} c_0 \end{aligned} \tag{3.34}$$

(d) As $\beta_0=3$, then $B_7=0$ in eq(3.12).

$$\begin{aligned} c_0 & \\ c_2 &= B_1 c_0 \\ c_4 &= B_{1,3} c_0 \\ c_6 &= B_{1,3,5} c_0 \end{aligned} \tag{3.35}$$

Plug eq(3.32), eq(3.33), eq(3.34) and eq(3.35) into eq(3.15).

$$y_0(x) = c_0 \sum_{n=0}^{\beta_0} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n+\lambda} \tag{3.36}$$

Now, let's look at the polynomial case of one of the terms of A_n 's.

(a) As $\beta_0=0$, then $B_1=0$ in eq(3.16).

$$\begin{aligned}
 c_1 &= A_0 c_0 \\
 c_3 &= A_0 B_2 c_0 \\
 c_5 &= A_0 B_{2,4} c_0 \\
 c_7 &= A_0 B_{2,4,6} c_0 \\
 c_9 &= A_0 B_{2,4,6,8} c_0 \\
 &\vdots \quad \vdots
 \end{aligned}
 \tag{3.37}$$

As $i=1$ in eq(3.31),

$$B_{2\beta_1+2} = 0 \quad \text{where } \beta_1 = 0, 1, 2, \dots \tag{3.38}$$

Substitute eq(3.37) into eq(3.13) by using eq(3.38).

$$y_1^0(x) = c_0 A_0 \sum_{n=0}^{\beta_1} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+2} \right\} x^{2n+1+\lambda} \tag{3.39}$$

(b) As $\beta_0=1$, then $B_3=0$ in eq(3.16).

$$\begin{aligned}
 c_1 &= A_0c_0 \\
 c_3 &= \{A_0B_2 \cdot 1 + A_2 \cdot 1B_1\}c_0 \\
 c_5 &= \{A_0B_{2,4} \cdot 1 + A_2B_4B_1\}c_0 \\
 c_7 &= \{A_0B_{2,4,6} \cdot 1 + A_2B_{4,6}B_1\}c_0 \\
 c_9 &= \{A_0B_{2,4,6,8} \cdot 1 + A_2B_{4,6,8}B_1\}c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned} \tag{3.40}$$

The first term in the bracket on coefficient of each c_n in eq(3.40) is same as eq(3.37).

Then, its solution is equal to eq(3.39). Then substitute eq(3.38) into the second term in the bracket on coefficient of c_n in eq(3.40), and plug eq(3.39) into eq(3.13):

$$y_1^1(x) = c_0 \left\{ A_0 \sum_{n=0}^{\beta_1} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+2} \right\} + A_2B_1 \sum_{n=1}^{\beta_1} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+2} \right\} \right\} x^{2n+1+\lambda} \tag{3.41}$$

(c) As $\beta_0=2$, then $B_5=0$ in eq(3.16).

$$\begin{aligned}
 c_1 &= A_0c_0 \\
 c_3 &= \{A_0B_2 \cdot 1 + A_2 \cdot 1B_1\}c_0 \\
 c_5 &= \{A_0B_{2,4} \cdot 1 + A_2B_4B_1 + A_4 \cdot 1B_{1,3}\}c_0 \\
 c_7 &= \{A_0B_{2,4,6} \cdot 1 + A_2B_{4,6}B_1 + A_4B_6B_{1,3}\}c_0 \\
 c_9 &= \{A_0B_{2,4,6,8} \cdot 1 + A_2B_{4,6,8}B_1 + A_4B_{6,8}B_{1,3}\}c_0 \\
 c_{11} &= \{A_0B_{2,4,6,8} \cdot 1 + A_2B_{4,6,8}B_1 + A_4B_{6,8,10}B_{1,3}\}c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned}
 \tag{3.42}$$

The polynomial of x for first and second term in the bracket on the coefficients of each of c_n in eq(3.42) is same as eq(3.41). Substitute eq(3.38) into the third term in the bracket on coefficients of c_n in eq(3.42), and then plug eq(3.41) into eq(3.13).

$$\begin{aligned}
 y_1^2(x) &= c_0 \left\{ A_0 \sum_{n=0}^{\beta_1} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+2} \right\} + A_2 B_1 \sum_{n=1}^{\beta_1} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+2} \right\} \right. \\
 &\quad \left. + A_4 B_{1,3} \sum_{n=2}^{\beta_1} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+2} \right\} \right\} x^{2n+1+\lambda}
 \end{aligned}
 \tag{3.43}$$

According to eq(3.39), eq(3.41) and eq(3.43), the general expression for all β_0 of $y_1(x)$

is

$$y_1(x) = c_0 \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1+\lambda} \quad (3.44)$$

Now, let's look at the polynomial case of two term of A_n 's.

(a) As $\beta_0=0$, then $B_1=0$ in eq(3.21).

$$c_2 = A_{0,1}c_0$$

$$c_4 = A_0\{A_1 \cdot 1 \cdot B_3 \cdot 1 + A_3B_2 \cdot 1 \cdot 1\}c_0$$

$$c_6 = A_0\{A_1 \cdot 1 \cdot B_{3,5} \cdot 1 + A_3B_2B_5 \cdot 1 + A_5B_{2,4} \cdot 1 \cdot 1\}c_0$$

$$c_8 = A_0\{A_1 \cdot 1 \cdot B_{3,5,7} \cdot 1 + A_3B_2B_{5,7} \cdot 1 + A_5B_{2,4}B_7 \cdot 1 + A_7B_{2,4,6} \cdot 1 \cdot 1\}c_0 \quad (3.45)$$

$$c_{10} = A_0\{A_1 \cdot 1 \cdot B_{3,5,7,9} \cdot 1 + A_3B_2B_{5,7,9} \cdot 1 + A_5B_{2,4}B_{7,9} \cdot 1$$

$$+ A_7B_{2,4,6}B_9 \cdot 1 + A_9B_{2,4,6,8} \cdot 1 \cdot 1\}c_0$$

⋮

⋮

(i) As $\beta_1=0$, then $B_2=0$ in eq(3.45).

$$\begin{aligned}
 c_2 &= A_{0,1}c_0 \\
 c_4 &= A_0A_1 \cdot 1 \cdot B_3 \cdot 1c_0 \\
 c_6 &= A_0A_1 \cdot 1 \cdot B_{3,5} \cdot 1c_0 \\
 c_8 &= A_0A_1 \cdot 1 \cdot B_{3,5,7} \cdot 1c_0 \\
 c_{10} &= A_0A_1 \cdot 1 \cdot B_{3,5,7,9} \cdot 1c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned}
 \tag{3.46}$$

As $i=2$ in eq(3.31),

$$B_{2\beta_2+3} = 0 \quad \text{where } \beta_2 = 0, 1, 2, \dots \tag{3.47}$$

Substitute eq(3.46) into eq(3.13) by using eq(3.47).

$$y_2^{0,0}(x) = c_0A_0A_1 \sum_{n=0}^{\beta_2} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+3} \right\} x^{2n+2+\lambda} \tag{3.48}$$

(ii) As $\beta_1=1$, then $B_4=0$ in eq(3.45).

$$\begin{aligned}
 c_2 &= A_{0,1}c_0 \\
 c_4 &= A_0\{A_1 \cdot 1 \cdot B_3 \cdot 1 + A_3B_2 \cdot 1 \cdot 1\}c_0 \\
 c_6 &= A_0\{A_1 \cdot 1 \cdot B_{3,5} \cdot 1 + A_3B_2B_5 \cdot 1\}c_0 \\
 c_8 &= A_0\{A_1 \cdot 1 \cdot B_{3,5,7} \cdot 1 + A_3B_2B_{5,7} \cdot 1\}c_0 \\
 c_{10} &= A_0\{A_1 \cdot 1 \cdot B_{3,5,7,9} \cdot 1 + A_3B_2B_{5,7,9} \cdot 1\}c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned} \tag{3.49}$$

The polynomial of x for first term in the bracket on coefficient of each c_n in eq(3.49) is the same as in eq(3.48). Substitute eq(3.47) into the second term in the bracket on coefficient of c_n in eq(3.49), then plug eq(3.48) into eq(3.13).

$$y_2^{0,1}(x) = c_0A_0 \left\{ A_1 \sum_{n=0}^{\beta_2} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+3} \right\} + A_3B_2 \sum_{n=1}^{\beta_2} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+3} \right\} \right\} x^{2n+2+\lambda} \tag{3.50}$$

(iii) As $\beta_1=2$, then $B_6=0$ in eq(3.45).

$$\begin{aligned}
c_2 &= A_{0,1}c_0 \\
c_4 &= A_0\{A_1 \cdot 1 \cdot B_3 \cdot 1 + A_3B_2 \cdot 1 \cdot 1\}c_0 \\
c_6 &= A_0\{A_1 \cdot 1 \cdot B_{3,5} \cdot 1 + A_3B_2B_5 \cdot 1 + A_5B_{2,4} \cdot 1 \cdot 1\}c_0 \\
c_8 &= A_0\{A_1 \cdot 1 \cdot B_{3,5,7} \cdot 1 + A_3B_2B_{5,7} \cdot 1 + A_5B_{2,4}B_7 \cdot 1\}c_0 \\
c_{10} &= A_0\{A_1 \cdot 1 \cdot B_{3,5,7,9} \cdot 1 + A_3B_2B_{5,7,9} \cdot 1 + A_5B_{2,4}B_{7,9} \cdot 1\}c_0 \\
&\vdots \qquad \qquad \qquad \vdots
\end{aligned} \tag{3.51}$$

The polynomial of x for first and second term in the bracket on coefficient of each c_n in eq(3.51) is same as eq(3.50). Substitute eq(3.47) into the third term in the bracket on coefficient of c_n in eq(3.51), then plug those coefficient of c_6, c_8, c_{10}, \dots and eq(3.50) into eq(3.13):

$$\begin{aligned}
y_2^{0,2}(x) &= c_0A_0 \left\{ A_1 \sum_{n=0}^{\beta_2} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+3} \right\} + A_3B_2 \sum_{n=1}^{\beta_2} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+3} \right\} \right. \\
&\quad \left. + A_5B_{2,4} \sum_{n=2}^{\beta_2} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+3} \right\} \right\} x^{2n+2+\lambda} \tag{3.52}
\end{aligned}$$

According to eq(3.48), eq(3.50) and eq(3.52), the general expression for all $\beta_0 = 0$ of

The polynomial of x for first term in the bracket on coefficient of each c_n including A_0 in eq(3.54) is equal to eq(3.53). Now, let's look at the second term in the bracket on coefficient of each c_n including A_2 in eq(3.54).

(i) As $\beta_1=1$, then $B_4=0$ in second term in the bracket on coefficient of each c_n including A_2 in eq(3.54).

$$\begin{aligned}
 c_4 &= A_2 B_1 \{ A_3 \cdot 1 \cdot 1 \} c_0 \\
 c_6 &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_5 \} c_0 \\
 c_8 &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_{5,7} \} c_0 \\
 c_{10} &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_{5,7,9} \} c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned}
 \tag{3.55}$$

Substitute eq(3.55) into eq(3.13) and use eq(3.47).

$$y_2^{1,1}(x) = c_0 A_2 B_1 A_3 \sum_{n=1}^{\beta_2} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+3} \right\} x^{2n+2+\lambda}
 \tag{3.56}$$

(ii) As $\beta_1=2$, then $B_6=0$ in second term in the bracket on coefficient of each c_n

including A_2 in eq(3.54).

$$\begin{aligned}
 c_4 &= A_2 B_1 \{ A_3 \cdot 1 \cdot 1 \} c_0 \\
 c_6 &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_5 + A_5 B_4 \cdot 1 \} c_0 \\
 c_8 &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_{5,7} + A_5 B_4 B_7 \} c_0 \\
 c_{10} &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_{5,7,9} + A_5 B_4 B_{7,9} \} c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned} \tag{3.57}$$

The polynomial of x for first term in the bracket on coefficient of each c_n in eq(3.57) is same as eq(3.56). Substitute eq(3.47) into the second term in the bracket on coefficient of c_n in eq(3.57), then plug it into eq(3.13) and add to eq(3.56).

$$y_2^{1,2}(x) = c_0 A_2 B_1 \left\{ A_3 \sum_{n=1}^{\beta_2} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+3} \right\} + A_5 B_4 \sum_{n=2}^{\beta_2} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+3} \right\} \right\} x^{2n+2+\lambda} \tag{3.58}$$

(iii) As $\beta_1=3$, then $B_8=0$ in second term in the bracket on coefficient of each c_n

including A_2 in eq(3.54).

$$\begin{aligned}
c_4 &= A_2 B_1 \{ A_3 \cdot 1 \cdot 1 \} c_0 \\
c_6 &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_5 + A_5 B_4 \cdot 1 \} c_0 \\
c_8 &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_{5,7} + A_5 B_4 B_7 + A_7 B_{4,6} \cdot 1 \} c_0 \\
c_{10} &= A_2 B_1 \{ A_3 \cdot 1 \cdot B_{5,7,9} + A_5 B_4 B_{7,9} + A_7 B_{4,6} B_9 \} c_0 \\
&\vdots \qquad \qquad \qquad \vdots
\end{aligned} \tag{3.59}$$

The polynomial of x for first and second term in the bracket on coefficient of each c_n in eq(3.59) is same as eq(3.58). Substitute eq(3.47) into the third term in the bracket on coefficient of c_n in eq(3.59), then plug it into eq(3.13) and add to eq(3.58).

$$\begin{aligned}
y_2^{1,3}(x) &= c_0 A_2 B_1 \left\{ A_3 \sum_{n=1}^{\beta_2} \left\{ \prod_{i_1=1}^{n-1} B_{2i_1+3} \right\} + A_5 B_4 \sum_{n=2}^{\beta_2} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+3} \right\} \right. \\
&\quad \left. + A_7 B_{4,6} \sum_{n=3}^{\beta_2} \left\{ \prod_{i_1=3}^{n-1} B_{2i_1+3} \right\} \right\} x^{2n+2+\lambda} \tag{3.60}
\end{aligned}$$

According to eq(3.53), eq(3.56), eq(3.58) and eq(3.60), the general expression for all

$\beta_0 = 1$ of $y_2^1(x)$ replacing the index n by i_0 is

$$\begin{aligned}
 y_2^1(x) = & c_0 \left\{ A_0 \sum_{i_0=0}^{\beta_1} \left\{ A_{2i_0+1} \prod_{i_1=0}^{i_0-1} B_{2i_1+2} \sum_{i_2=i_0}^{\beta_2} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+3} \right\} \right\} \right. \\
 & \left. + A_2 B_1 \sum_{i_0=1}^{\beta_1} \left\{ A_{2i_0+1} \prod_{i_1=1}^{i_0-1} B_{2i_1+2} \sum_{i_2=i_0}^{\beta_2} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+3} \right\} \right\} \right\} x^{2i_2+2+\lambda} \quad (3.61)
 \end{aligned}$$

(i) As $\beta_1=2$, then $B_6=0$ in third term in the bracket on coefficient of each c_n including $A_4B_{1,3}$ in eq(3.62).

$$\begin{aligned}
 c_6 &= A_4B_{1,3}\{A_5 \cdot 1 \cdot 1\}c_0 \\
 c_8 &= A_4B_{1,3}\{A_5 \cdot 1 \cdot B_7\}c_0 \\
 c_{10} &= A_4B_{1,3}\{A_5 \cdot 1 \cdot B_{7,9}\}c_0 \\
 \vdots & \quad \quad \quad \vdots
 \end{aligned}
 \tag{3.63}$$

Substitute eq(3.63) into eq(3.13), and by using eq(3.47).

$$y_2^{2,2}(x) = c_0A_4B_{1,3}A_5 \sum_{n=2}^{\beta_2} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+3} \right\} x^{2n+2+\lambda}
 \tag{3.64}$$

(ii) As $\beta_1=3$, then $B_8=0$ in third term in the bracket on coefficient of each c_n including

$A_4B_{1,3}$ in eq(3.62).

$$\begin{aligned}
 c_6 &= A_4B_{1,3}\{A_5 \cdot 1 \cdot 1\}c_0 \\
 c_8 &= A_4B_{1,3}\{A_5 \cdot 1 \cdot B_7 + A_7B_6 \cdot 1\}c_0 \\
 c_{10} &= A_4B_{1,3}\{A_5 \cdot 1 \cdot B_{7,9} + A_7B_6B_9\}c_0 \\
 &\vdots \qquad \qquad \qquad \vdots
 \end{aligned}
 \tag{3.65}$$

The polynomial of x for first term in the bracket on coefficient of each c_n in eq(3.65) is same as eq(3.64). Substitute eq(3.47) into the second term in the bracket on coefficient of c_n in eq(3.65), then plug it into eq(3.13) and add to eq(3.64):

$$y_2^{2,3}(x) = c_0A_4B_{1,3} \left\{ A_5 \sum_{n=2}^{\beta_2} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+3} \right\} + A_7B_6 \sum_{n=3}^{\beta_2} \left\{ \prod_{i_1=3}^{n-1} B_{2i_1+3} \right\} \right\} x^{2n+2+\lambda}
 \tag{3.66}$$

(iii) As $\beta_1=4$, then $B_{10}=0$ in third term in the bracket on coefficient of each c_n

including A_2 in eq(3.62).

$$\begin{aligned}
c_6 &= A_4 B_{1,3} \left\{ A_5 \cdot 1 \cdot 1 \right\} c_0 \\
c_8 &= A_4 B_{1,3} \left\{ A_5 \cdot 1 \cdot B_7 + A_7 B_6 \cdot 1 \right\} c_0 \\
c_{10} &= A_4 B_{1,3} \left\{ A_5 \cdot 1 \cdot B_{7,9} + A_7 B_6 B_9 + A_9 B_{6,8} \cdot 1 \right\} c_0 \\
c_{12} &= A_4 B_{1,3} \left\{ A_5 \cdot 1 \cdot B_{7,9,11} + A_7 B_6 B_{9,11} + A_9 B_{6,8} B_{11} \right\} c_0 \\
&\vdots \qquad \qquad \qquad \vdots
\end{aligned} \tag{3.67}$$

The polynomial of x for first and second term in the bracket on coefficient of each c_n in eq(3.67) is same as eq(3.66). Substitute eq(3.47) into the third term in the bracket on coefficient of c_n in eq(3.67), then plug it into eq(3.13) and add to eq(3.66).

$$\begin{aligned}
y_2^{2,4}(x) &= c_0 A_4 B_{1,3} \left\{ A_5 \sum_{n=2}^{\beta_2} \left\{ \prod_{i_1=2}^{n-1} B_{2i_1+3} \right\} + A_7 B_6 \sum_{n=3}^{\beta_2} \left\{ \prod_{i_1=3}^{n-1} B_{2i_1+3} \right\} \right. \\
&\quad \left. + A_9 B_{6,8} \sum_{n=4}^{\beta_2} \left\{ \prod_{i_1=4}^{n-1} B_{2i_1+3} \right\} \right\} x^{2n+2+\lambda}
\end{aligned} \tag{3.68}$$

According to eq(3.53), eq(3.61), eq(3.64), eq(3.66) and eq(3.68), the general expression for all $\beta_0 = 2$ of $y_2^2(x)$ replacing index n by i_0 is

$$\begin{aligned}
y_2^2(x) = & c_0 \left\{ A_0 \sum_{i_0=0}^{\beta_1} \left\{ A_{2i_0+1} \prod_{i_1=0}^{i_0-1} B_{2i_1+2} \sum_{i_2=i_0}^{\beta_2} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+3} \right\} \right\} \right. \\
& + A_2 B_1 \sum_{i_0=1}^{\beta_1} \left\{ A_{2i_0+1} \prod_{i_1=1}^{i_0-1} B_{2i_1+2} \sum_{i_2=i_0}^{\beta_2} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+3} \right\} \right\} \\
& \left. + A_4 B_{1,3} \sum_{i_0=2}^{\beta_1} \left\{ A_{2i_0+1} \prod_{i_1=2}^{i_0-1} B_{2i_1+2} \sum_{i_2=i_0}^{\beta_2} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+3} \right\} \right\} \right\} x^{2i_2+2+\lambda} \quad (3.69)
\end{aligned}$$

We obtain general expression for all β_0 of $y_2(x)$ of two term of A_i 's from eq(3.53), eq(3.61) and eq(3.69).

$$y_2(x) = c_0 \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\beta_2} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2+\lambda} \quad (3.70)$$

By using similar process for the previous case of zero, one and two term of A_i 's, the general expression for all β_0 of $y_3(x)$ for the case of three term of A_i 's is

$$\begin{aligned}
y_3(x) = & c_0 \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right. \\
& \left. \left. \times \sum_{i_4=i_2}^{\beta_2} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{\beta_3} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2i_6+3+\lambda} \quad (3.71)
\end{aligned}$$

We have a general expression of the power series of $y(x)$ for the polynomial case in the three term recurrence formula according to eq(3.36), eq(3.44), eq(3.70), and eq(3.71) is

$$\begin{aligned}
y(x) = & c_0 \left\{ \sum_{i_0=0}^{\beta_0} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} \right. \\
& + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \\
& + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right. \right. \\
& \times \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\beta_k} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \\
& \left. \left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\beta_N} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N+\lambda} \left. \right\} \quad (3.72)
\end{aligned}$$

For infinite series, replacing $\beta_0, \beta_1, \beta_k$ and β_N by ∞ in eq(3.72):

$$\begin{aligned}
y(x) = & c_0 \left\{ \sum_{i_0=0}^{\infty} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} \right. \\
& + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \\
& + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right. \right. \\
& \times \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \\
& \left. \left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\infty} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N+\lambda} \left. \right\} \quad (3.73)
\end{aligned}$$

Eq(3.73) is exactly equivalent to eq(3.30). Eq(3.73) is another general expression of power series of $y(x)$ for the infinite series of eq(3.30).

3.3 Polynomial which makes A_n term terminated

Earlier we talked about the polynomial case as B_n term terminated at certain value of n . Now, let's think about the polynomial case as A_n term being terminated at certain value of n . Then, some of $y_i(x)$ where $i = 0, 1, 2, \dots$ will be zero at specific eigenvalues as we see eq(3.29) and eq(3.30). First of all, let's say A_{2i_1} is zero at every

value of i_1 at certain eigenvalue. We choose this eigenvalue as α_0 in which A_{2i_1} is zero where $\alpha_0 = 0, 1, 2, \dots$. Then, we further choose $A_{2\alpha_0} = 0 = A_0 = A_2 = A_4 = \dots$. As we see from eq(3.29), $y_1(x) = y_2(x) = y_3(x) = \dots$ is zero which satisfies $A_{2i_1} = 0$. Then we obtain the function $y(x)$

$$y(x) = c_0 x^\lambda \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} \quad \text{where } \alpha_0 = 0, 1, 2, \dots \quad (3.74)$$

Let A_{2i_2+1} be zero at every value of i_2 as $\alpha_1 = 0, 1, 2, \dots$. Then we choose $A_{2\alpha_1+1} = A_1 = A_3 = A_5 = \dots = 0$. We see in eq(3.29), $y_2(x) = y_3(x) = y_4(x) = \dots$ is zero, which satisfies $A_{2i_2+1} = 0$. Then we obtain the function $y(x)$

$$\begin{aligned} y(x) = & c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} \right. \\ & \left. + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right\} \\ & \text{where } \alpha_1 = 0, 1, 2, \dots \end{aligned} \quad (3.75)$$

Let A_{2i_3+2} is zero at every value of i_3 as $\alpha_2 = 0, 1, 2, \dots$. Then we choose $A_{2\alpha_2+2} = A_2 = A_4 = A_6 = \dots = 0$. We see eq(3.29), $y_3(x) = y_4(x) = y_5(x) = \dots$ is zero, which

satisfies $A_{2i_3+2} = 0$. Then we obtain the function $y(x)$

$$\begin{aligned}
y(x) &= c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right. \\
&+ \left. \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} \right\} x^{2n+2} \right\} \\
&\text{where } \alpha_2 = 0, 1, 2, \dots \tag{3.76}
\end{aligned}$$

Let A_{2i_4+3} is zero at every value of i_4 as $\alpha_3 = 0, 1, 2, \dots$. Again we choose $A_{2\alpha_3+3} = A_3 = A_5 = A_7 = \dots = 0$. As we see eq(3.29), $y_4(x) = y_5(x) = y_6(x) = \dots$ is zero, which satisfies $A_{2i_4+3} = 0$. Then we obtain the function $y(x)$

$$\begin{aligned}
y(x) &= c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right. \\
&+ \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} \right\} x^{2n+2} \\
&+ \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \sum_{i_3=i_2}^n \left\{ A_{2i_3+2} \prod_{i_4=0}^{i_1-1} B_{2i_4+1} \prod_{i_5=i_1}^{i_2-1} B_{2i_5+2} \right. \right. \right. \right. \\
&\times \left. \left. \left. \prod_{i_6=i_2}^{i_3-1} B_{2i_6+3} \prod_{i_7=i_3}^{n-1} B_{2i_7+4} \right\} \right\} \right\} x^{2n+3} \left. \right\} \text{ where } \alpha_3 = 0, 1, 2, \dots \tag{3.77}
\end{aligned}$$

Using the previous cases we obtain the necessary condition which is

$$A_{2\alpha_m+m} = 0 \quad \text{where } m = 0, 1, 2, \dots \text{ and } \alpha_m = 0, 1, 2, \dots \quad (3.78)$$

According to eq(3.74), eq(3.75), eq(3.76) and eq(3.77), the general expression of $y(x)$ for the polynomial which makes A_n term terminated is

$$\begin{aligned} y(x) &= c_0 x^\lambda \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} \quad \text{where } \alpha_0 = 0, 1, 2, \dots \\ &= c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right\} \quad \text{where } \alpha_1 = 0, 1, 2, \dots \\ &= c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right. \\ &\quad \left. + \sum_{N=2}^m \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{k=1}^{N-1} \left(\sum_{i_k=i_{k-1}}^n A_{2i_k+(k-1)} \right) \right. \right. \right. \\ &\quad \times \sum_{i_N=i_{N-1}}^n \left\{ A_{2i_N+(N-1)} \prod_{l=0}^{N-1} \left(\prod_{i_{l+1}+N+(l+1)}^{i_{l+1}-1} B_{2i_{l+1}+N+(l+1)} \right) \right. \right. \\ &\quad \left. \left. \left. \times \prod_{i_{2N+1}=i_N}^{n-1} B_{2i_{2N+1}+(N+1)} \right\} \right\} x^{2n+N} \right\} \\ &\quad \text{where } i_0 = 0, \alpha_m = 0, 1, 2, \dots \text{ and } m \geq 2 \end{aligned} \quad (3.79)$$

By putting eq(3.78) into eq(3.73), we obtain another expression of $y(x)$ for the polynomial which makes A_n term be terminated.

$$\begin{aligned}
 y(x) &= c_0 x^\lambda \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \quad \text{where } \alpha_0 = 0, 1, 2, \dots \\
 &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \right. \\
 &\quad \left. + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right\} \quad \text{where } \alpha_1 = 0, 1, 2, \dots \\
 &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right. \\
 &\quad \left. + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right\} \right. \right. \\
 &\quad \left. \left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\infty} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N} \\
 &\quad \text{where } \alpha_m = 0, 1, 2, \dots \text{ and } m \geq 2
 \end{aligned} \tag{3.80}$$

Eq(3.80) is exactly equivalent to eq(3.79).

3.4 Polynomial which makes A_n and B_n terms terminated

Now, let's think about the polynomial case where A_n and B_n terms are terminated at certain value of n . Put $m = 0$ in eq(3.78). We have

$$A_{2\alpha_0} = 0 = A_0 = A_2 = A_4 = \dots \quad (3.81)$$

As we plug eq(3.81) into eq(3.72), we obtain $y_1(x) = y_2(x) = y_3(x) = \dots = 0$, and the maximum value of α_0 should be equal to or greater than β_0 . If it doesn't, the analytic function $y(x)$ can not be polynomial any more. We define this condition as

$$\text{Max}(\alpha_0) \geq \beta_0 \quad \text{where } \alpha_0, \beta_0 = 0, 1, 2, \dots \quad (3.82)$$

Then, the function $y(x)$ is

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \quad \text{where } \text{Max}(\alpha_0) \geq \beta_0 \quad (3.83)$$

Putting $m = 1$ in eq(3.78). We have

$$A_{2\alpha_1+1} = 0 = A_1 = A_3 = A_5 = \dots \quad (3.84)$$

As we plug eq(3.84) into eq(3.72), we obtain $y_2(x) = y_3(x) = y_4(x) = \dots = 0$. And the maximum value of α_1 should be equal to or greater than β_1 . If it doesn't the analytic function $y(x)$ can not be polynomial any longer. We define this condition as

$$\text{Max}(\alpha_1) \geq \beta_1 \quad \text{where } \alpha_1, \beta_1 = 0, 1, 2, \dots \quad (3.85)$$

Then, the function $y(x)$ is

$$\begin{aligned} y(x) &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \right. \\ &\quad \left. + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right\} \\ &\text{where } \text{Max}(\alpha_1) \geq \beta_1 \end{aligned} \quad (3.86)$$

Put $m = 2$ in eq(3.78). We have

$$A_{2\alpha_2+2} = 0 = A_2 = A_4 = A_6 = \dots \quad (3.87)$$

As we plug eq(3.87) into eq(3.72), we obtain $y_3(x) = y_4(x) = y_5(x) = \dots = 0$. And the maximum value of α_2 should be equal to or greater than β_2 . If it doesn't the analytic function $y(x)$ can not be polynomial any longer. We define this condition as

$$\text{Max}(\alpha_2) \geq \beta_2 \quad \text{where } \alpha_2, \beta_2 = 0, 1, 2, \dots \quad (3.88)$$

Then, the function $y(x)$ is

$$\begin{aligned} y(x) &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \right. \\ &\quad + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \\ &\quad + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right. \\ &\quad \left. \left. \times \sum_{i_4=i_2}^{\beta_2} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2} \left. \right\} \\ &\text{where } \text{Max}(\alpha_2) \geq \beta_2 \end{aligned} \quad (3.89)$$

Put $m = 3$ in eq(3.78). We have

$$A_{2\alpha_3+3} = 0 = A_3 = A_5 = A_7 = \dots \quad (3.90)$$

As we plug eq(3.90) into eq(3.72), we obtain $y_4(x) = y_5(x) = y_6(x) = \dots = 0$. And the maximum value of α_3 should be equal to or greater than β_3 . If it doesn't, the analytic function $y(x)$ can not be polynomial any longer. We define this condition as

$$\text{Max}(\alpha_3) \geq \beta_3 \quad \text{where } \alpha_3, \beta_3 = 0, 1, 2, \dots \quad (3.91)$$

Then, the function $y(x)$ is

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \right. \\
 & + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \\
 & + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\beta_2} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2} \\
 & + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right. \\
 & \left. \left. \times \sum_{i_4=i_2}^{\beta_2} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{\beta_3} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2i_6+3} \left. \right\} \\
 & \text{where } \text{Max}(\alpha_3) \geq \beta_3 \tag{3.92}
 \end{aligned}$$

By using the previous cases, we obtain the necessary condition which is

$$\text{Max}(\alpha_m) \geq \beta_m \quad \text{where } m = 0, 1, 2, \dots \quad \text{and } \alpha_m, \beta_m = 0, 1, 2, \dots \tag{3.93}$$

According to eq(3.83), eq(3.86), eq(3.89) and eq(3.92), the general expression of $y(x)$ for the polynomial which makes A_n and B_n terms terminated where $\alpha_m, \beta_m =$

$0, 1, 2, \dots$ is

$$\begin{aligned}
 y(x) &= c_0 x^\lambda \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \quad \text{where } \text{Max}(\alpha_0) \geq \beta_0 \\
 &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \right. \\
 &\quad \left. + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right\} \quad \text{where } \text{Max}(\alpha_1) \geq \beta_1 \\
 &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \right. \\
 &\quad \left. + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right. \\
 &\quad \left. + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right. \right. \right. \\
 &\quad \times \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\beta_k} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \\
 &\quad \left. \left. \left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\beta_N} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N} \right\} \\
 &\quad \text{where } \text{Max}(\alpha_m) \geq \beta_m \text{ and } m \geq 2
 \end{aligned}$$

The Fibonacci numbers are the numbers in the following integer sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots \tag{3.94}$$

the sequence c_n of Fibonacci numbers is defined by the recurrence relation

$$c_{n+1} = c_n + c_{n-1} \quad : n \geq 1 \quad (3.95)$$

with seed values

$$c_0 = 0 \quad c_1 = 1 \quad (3.96)$$

The generating function of the Fibonacci sequence is the power series.

$$\sum_{n=0}^{\infty} c_n x^n = \frac{x}{1 - x - x^2} \quad (3.97)$$

We know, the three term recurrence relation is

$$c_{n+1} = A_n c_n + B_n c_{n-1} \quad ; n \geq 1 \quad (3.98)$$

with seed values

$$c_1 = A_0 c_0 \quad (3.99)$$

As we see the numbers of each of sequence of c_n in eq(3.11) is followed by a Fibonacci number, and if $A_n = B_n = 1$ in eq(3.99), it is exactly equivalent to the recurrence

relation of Fibonacci numbers. Because of this reason, we call each sequence of c_n in eq(3.11) as ' n^{th} Hyper-Fibonacci generator'. And eq(3.99) is called as the recurrence relation of Hyper-Fibonacci generator. Also eq(3.30), eq(3.72) and eq(3.73) correspond to eq(3.77). Because of it, we call eq(3.72) as the generating function of Hyper-Fibonacci generator for the polynomial. Eq(3.30) and eq(3.73) are called the generating function of Hyper-Fibonacci generators for the case of infinite series.

There is an algebraic number series in which is

$$1, 1, 1, 1, 1, 1, \dots \quad (3.100)$$

We call eq(3.101) as the identity number, and its recurrence equation is

$$c_{n+1} = c_n \quad : n \geq 0 \quad (3.101)$$

with seed values

$$c_0 = 1 \quad (3.102)$$

The generating function of the identity sequence is a power series

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1}{1-x} \quad (3.103)$$

If $B_n=0$ in eq(3.99), then it turns to be two term recurrence relation.

$$c_{n+1} = A_n c_n \quad ; n \geq 0 \quad (3.104)$$

Some of the examples are the Legendre function, Kummer function, hypergeometric function, Bessel function, etc. And the number of each of sequence c_n in eq(3.105) is

$$1, 1, 1, 1, 1, 1, \dots \quad (3.105)$$

Eq(3.106) is equivalent to eq(3.101). As we put $A_n=1$ in eq(3.105), it becomes eq(3.102). Because of this, we call eq(3.105) the recurrence relation of Hyper-Identity generator, and each sequence of c_n in eq(3.105) as ' n^{th} Hyper-Identity generator'.

The power series of eq(3.105) is

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda} = c_0 \sum_{n=0}^{\infty} \left(\prod_{i=0}^{n-1} A_i \right) x^{n+\lambda} \quad (3.106)$$

And polynomial case of eq(3.86) is

$$y(x) = \sum_{n=0}^{\alpha_0} c_n x^{n+\lambda} = c_0 \sum_{n=0}^{\alpha_0} \left(\prod_{i=0}^{n-1} A_i \right) x^{n+\lambda} \quad (3.107)$$

Eq(3.107) is called the generating function of Hyper-Identity generator for infinite series. And eq(3.108) is called the generating function of Hyper-Identity generator for polynomial.

Now, let's think about four term recurrence relation in ordinary differential equation. The four term recurrence formula is

$$c_{n+1} = A_n c_n + B_n c_{n-1} + C_n c_{n-2} \quad ; n \geq 2 \quad (3.108)$$

with seed values

$$c_1 = A_0 c_0 \quad c_2 = (A_0 A_1 + B_1) c_0 \quad (3.109)$$

And the number of each of sequence c_n in eq(3.109) is the following way:

$$1, 1, 2, 4, 7, 13, 24, 44, \dots \quad (3.110)$$

Eq(3.111) is Tribonacci number. And it's recurrence equation is

$$c_{n+1} = c_n + c_{n-1} + c_{n-2} \quad ; n \geq 2 \quad (3.111)$$

with seed values

$$c_0 = 0 \quad c_1 = 1 \quad c_2 = 1 \quad (3.112)$$

If $A_n = B_n = C_n = 1$ in eq(3.109), it's exactly equivalent to Tribonacci number.

Then we call eq(3.109) the recurrence relation of Hyper-Tribonacci generator.

And five term recurrence formula in ordinary differential equation is

$$c_{n+1} = A_n c_n + B_n c_{n-1} + C_n c_{n-2} + D_n c_{n-3} \quad ; n \geq 3 \quad (3.113)$$

with seed values

$$c_1 = A_0 c_0 \quad c_2 = (A_0 A_1 + B_1) c_0 \quad c_3 = (A_0 A_1 A_2 + A_0 B_2 + A_2 B_1 + C_2) c_0 \quad (3.114)$$

And the number of each of sequence c_n in eq(3.114) is the following way:

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, \dots \quad (3.115)$$

Eq(3.116) is Tetranacci number. And it's recurrence equation is

$$c_{n+1} = c_n + c_{n-1} + c_{n-2} + c_{n-3} \quad ; n \geq 3 \quad (3.116)$$

with seed values

$$c_0 = 0 \quad c_1 = 1 \quad c_2 = 1 \quad c_3 = 2 \quad (3.117)$$

If $A_n = B_n = C_n = D_n = 1$ in eq(3.114), it becomes exactly equivalent to Tetranacci number. Then we call eq(3.114) the recurrence relation of Hyper-Tetranacci generator.

3.5 Summary

<p>(1) As $\alpha_0 = 0, 1, 2, \dots$ that makes $A_0 = A_2 = A_4 = \dots = 0$</p> $y(x) = c_0 x^\lambda \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n}$ $= c_0 x^\lambda \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0}$ <p>(2) As $\alpha_1 = 0, 1, 2, \dots$ that makes $A_1 = A_3 = A_5 = \dots = 0$</p> $y(x) = c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right\}$ $= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right\}$ <p>(3) As $\alpha_2 = 0, 1, 2, \dots$ that makes $A_2 = A_4 = A_6 = \dots = 0$</p> $y(x) = c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right.$ $\left. + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} \right\} x^{2n+2} \right\}$ $= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right.$ $\left. + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\infty} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2} \right\}$ <p>(4) As $\alpha_3 = 0, 1, 2, \dots$ that makes $A_3 = A_5 = A_7 = \dots = 0$</p> $y(x) = c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right.$ $+ \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} \right\} x^{2n+2}$ $+ \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \sum_{i_3=i_2}^n \left\{ A_{2i_3+2} \prod_{i_4=0}^{i_1-1} B_{2i_4+1} \prod_{i_5=i_1}^{i_2-1} B_{2i_5+2} \right. \right. \right.$ $\left. \left. \times \prod_{i_6=i_2}^{i_3-1} B_{2i_6+3} \prod_{i_7=i_3}^{n-1} B_{2i_7+4} \right\} \right\} \right\} x^{2n+3} \left. \right\}$ $= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right.$ $+ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\infty} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2}$ $+ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right.$ $\left. \times \sum_{i_4=i_2}^{\infty} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{\infty} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} x^{2i_6+3} \left. \right\}$
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Table 3.1: The power series of $y(x)$ for three different eigenvalues; α_0 , α_1 and α_2 that only makes A_n term terminated for polynomial case

(1) As $\alpha_0 = 0, 1, 2, \dots$ that makes $A_0 = A_2 = A_4 = \dots = 0$

$$\begin{aligned} y(x) &= c_0 x^\lambda \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} \\ &= c_0 x^\lambda \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} \end{aligned}$$

(2) As $\alpha_1 = 0, 1, 2, \dots$ that makes $A_1 = A_3 = A_5 = \dots = 0$

$$\begin{aligned} y(x) &= c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right\} \\ &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right\} \end{aligned}$$

(3) As $\alpha_m = 0, 1, 2, \dots$ that makes $A_{2\alpha_m+m} = 0$ where $m \geq 2$

$$\begin{aligned} y(x) &= c_0 x^\lambda \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1} \right. \\ &\quad + \sum_{N=2}^m \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{k=1}^{N-1} \left(\sum_{i_k=i_{k-1}}^n A_{2i_k+(k-1)} \right) \sum_{i_N=i_{N-1}}^n \right. \right. \\ &\quad \times \left. \left. \left\{ A_{2i_N+(N-1)} \prod_{l=0}^{N-1} \left(\prod_{i_{l+1}+N=i_l}^{i_{l+1}-1} B_{2i_{l+1}+N+(l+1)} \right) \right. \right. \right. \\ &\quad \times \left. \left. \left. \prod_{i_{2N+1}=i_N}^{n-1} B_{2i_{2N+1}+(N+1)} \right\} \right\} x^{2n+N} \right\} \quad \text{where, } i_0 = 0 \\ &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1} \right. \\ &\quad + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right. \right. \\ &\quad \times \left. \left. \sum_{i_{2N}=i_{2(N-1)}}^{\infty} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} x^{2i_{2N}+N} \right\} \end{aligned}$$

Table 3.2: General expression of power series of $y(x)$ for Table 1

(1) zero term of A'_n s
$y_0(x) = c_0 \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0+\lambda}$
(2) one term of A'_n s
$y_1(x) = c_0 \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1+\lambda}$
(3) two term of A'_n s
$y_2(x) = c_0 \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\beta_2} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2+\lambda}$
(4) three term of A'_n s
$y_3(x) = c_0 \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right. \\ \left. \left. \times \sum_{i_4=i_2}^{\beta_2} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{\beta_3} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2i_6+3+\lambda}$

Table 3.3: Summary of the zero, one, two and three term of A'_n 's that only makes B_n term terminated for polynomial case.

$$\begin{aligned}
y(x) = c_0 & \left\{ \sum_{i_0=0}^{\beta_0} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \right. \\
& + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\beta_k} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right. \right. \\
& \left. \left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\beta_N} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N+\lambda} \left. \right\}
\end{aligned}$$

Table 3.4: General expression of power series of $y(x)$ for Table 3

(1) As $\text{Max}(\alpha_0) \geq \beta_0$ where $\alpha_0, \beta_0 = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0}$$

(2) As $\text{Max}(\alpha_1) \geq \beta_1$ where $\alpha_1, \beta_1 = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} x^{2i_2+1} \right\} \right\}$$

(3) As $\text{Max}(\alpha_2) \geq \beta_2$ where $\alpha_2, \beta_2 = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \left\{ \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} x^{2i_2+1} \right\} \right\} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\beta_2} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} \right\} x^{2i_4+2}$$

(4) As $\text{Max}(\alpha_3) \geq \beta_3$ where $\alpha_3, \beta_3 = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \left\{ \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} x^{2i_2+1} \right\} \right\} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\beta_2} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} \right\} x^{2i_4+2} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\beta_2} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{\beta_3} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} \right\} \right\} x^{2i_6+3}$$

Table 3.5: The power series of $y(x)$ for three different eigenvalues; α_0, α_1 and α_2 that makes A_n and B_n terms terminated for polynomial case.

(1) As $\text{Max}(\alpha_0) \geq \beta_0$ where $\alpha_0, \beta_0 = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0}$$

(2) As $\text{Max}(\alpha_1) \geq \beta_1$ where $\alpha_1, \beta_1 = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} x^{2i_2+1} \right\} \right\}$$

(3) As $\text{Max}(\alpha_m) \geq \beta_m$ where $\alpha_m, \beta_m = 0, 1, 2, \dots$ and $m \geq 2$

$$\begin{aligned} y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\beta_0} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0} + \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} x^{2i_2+1} \right\} \right. \\ & + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\beta_0} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\beta_k} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right. \right. \\ & \left. \left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\beta_N} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N} \end{aligned}$$

Table 3.6: General expression of power series of $y(x)$ for Table 5

(1) zero term of A'_n s

$$\begin{aligned} y_0(x) &= c_0 \sum_{i_0=0}^{\infty} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0+\lambda} \\ &= c_0 \sum_{n=0}^{\infty} \left\{ \prod_{i_1=0}^{n-1} B_{2i_1+1} \right\} x^{2n+\lambda} \end{aligned}$$

(2) one term of A'_n s

$$\begin{aligned} y_1(x) &= c_0 \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1+\lambda} \\ &= c_0 \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1+\lambda} \end{aligned}$$

(3) two term of A'_n s

$$\begin{aligned} y_2(x) &= c_0 \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{\infty} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2+\lambda} \\ &= c_0 \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \prod_{i_3=0}^{i_1-1} B_{2i_3+1} \prod_{i_4=i_1}^{i_2-1} B_{2i_4+2} \prod_{i_5=i_2}^{n-1} B_{2i_5+3} \right\} \right\} \right\} x^{2n+2+\lambda} \end{aligned}$$

(4) three term of A'_n s

$$\begin{aligned} y_3(x) &= c_0 \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right. \\ &\quad \left. \left. \times \sum_{i_4=i_2}^{\infty} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{\infty} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2i_6+3+\lambda} \\ &= c_0 \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \sum_{i_2=i_1}^n \left\{ A_{2i_2+1} \sum_{i_3=i_2}^n \left\{ A_{2i_3+2} \prod_{i_4=0}^{i_1-1} B_{2i_4+1} \prod_{i_5=i_1}^{i_2-1} B_{2i_5+2} \right. \right. \right. \right. \\ &\quad \left. \left. \times \prod_{i_6=i_2}^{i_3-1} B_{2i_6+3} \prod_{i_7=i_3}^{n-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2n+3+\lambda} \end{aligned}$$

Table 3.7: Summary of the zero, one, two and three term of A_n 's for infinite series

$$\begin{aligned}
y(x) &= c_0 \left\{ \sum_{i_0=0}^{\infty} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \right. \\
&\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right. \right. \\
&\quad \times \left. \left. \sum_{i_{2N}=i_{2(N-1)}}^{\infty} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N+\lambda} \Big\} \\
&= c_0 \left\{ \sum_{n=0}^{\infty} \left(\prod_{i_1=0}^{n-1} B_{2i_1+1} \right) x^{2n+\lambda} + \sum_{n=0}^{\infty} \left\{ \sum_{i_1=0}^n \left\{ A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \prod_{i_3=i_1}^{n-1} B_{2i_3+2} \right\} \right\} x^{2n+1+\lambda} \right. \\
&\quad + \sum_{N=2}^{\infty} \left\{ \sum_{n=0}^{\infty} \left\{ \prod_{k=1}^{N-1} \left(\sum_{i_k=i_{k-1}}^n A_{2i_k+(k-1)} \right) \right. \right. \\
&\quad \times \sum_{i_N=i_{N-1}}^n \left\{ A_{2i_N+(N-1)} \prod_{l=0}^{N-1} \left(\prod_{i_{l+1}+N=i_l}^{i_{l+1}-1} B_{2i_{l+1}+N+(l+1)} \right) \right. \\
&\quad \times \left. \left. \prod_{i_{2N+1}=i_N}^{n-1} B_{2i_{2N+1}+(N+1)} \right\} \right\} \Big\} x^{2n+N+\lambda} \quad \text{where, } i_0 = 0
\end{aligned}$$

Table 3.8: General expression of power series of $y(x)$ for Table 7

Chapter 4

Analytic solution for GCH equation

$$x \frac{\partial^2 y}{\partial x^2} + (\mu x^2 + \varepsilon x + \nu) \frac{\partial y}{\partial x} + (\Omega x + \varepsilon \omega) y = 0 \quad (4.1)$$

Eq(4.1) is called as grand confluent hypergeometric differential equation where μ , ε , ν , Ω and ω are real parameters . Assume that its solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \quad (4.2)$$

Plug eq(4.2) into eq(4.1).

$$c_{n+1} = A_n c_n + B_n c_{n-1} \quad ; n \geq 1 \quad (4.3)$$

where,

$$A_n = -\frac{\varepsilon(n + \omega + \lambda)}{(n + 1 + \lambda)(n + \nu + \lambda)} \quad (4.4a)$$

$$B_n = -\frac{\Omega + \mu(n - 1 + \lambda)}{(n + 1 + \lambda)(n + \nu + \lambda)} \quad (4.4b)$$

$$c_1 = A_0 c_0 \quad (4.4c)$$

We have two indicial roots which are $\lambda_1 = 0$ and $\lambda_2 = 1 - \nu$

4.0.1 Power series of polynomial in which B_n term is terminated:

The polynomial of x which makes B_n term terminated for zero, one, two and three term of A_n 's are

(1) Zero term of A_n 's

$$y_0(x) = c_0 \sum_{i_0=0}^{|\alpha_0|} \left\{ \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right\} x^{2i_0+\lambda} \quad (4.5a)$$

(2) One term of A_n 's

$$y_1(x) = c_0 \sum_{i_0=0}^{|\alpha_0|} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{|\alpha_1|} \left\{ \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right\} \right\} x^{2i_2+1+\lambda} \quad (4.5b)$$

(3) Two term of A_n 's

$$y_2(x) = c_0 \sum_{i_0=0}^{|\alpha_0|} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{|\alpha_1|} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \sum_{i_4=i_2}^{|\alpha_2|} \left\{ \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \right\} \right\} \right\} x^{2i_4+2+\lambda} \quad (4.5c)$$

(4) Three term of A_n 's

$$y_3(x) = c_0 \sum_{i_0=0}^{|\alpha_0|} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{|\alpha_1|} \left\{ A_{2i_2+1} \prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right. \right. \\ \left. \left. \times \sum_{i_4=i_2}^{|\alpha_2|} \left\{ A_{2i_4+2} \prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} \sum_{i_6=i_4}^{|\alpha_3|} \left\{ \prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} \right\} \right\} \right\} \right\} x^{2i_6+3+\lambda} \quad (4.5d)$$

the general expression of power series of $y(x)$ for polynomial is

$$y(x) = c_0 \left\{ \sum_{i_0=0}^{|\alpha_0|} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} \right. \\ + \sum_{i_0=0}^{|\alpha_0|} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{|\alpha_1|} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \\ + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_2(k-1)}^{|\alpha_k|} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_2(k-1)}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right. \right. \\ \left. \left. \times \sum_{i_{2N}=i_2(N-1)}^{|\alpha_N|} \left(\prod_{i_{2N+1}=i_2(N-1)}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \right\} \right\} x^{2i_{2N}+N+\lambda} \quad (4.6)$$

In the above, α_i is 0, -1, -2, -3, \dots where $i = 0, 1, 2, 3, \dots$. Then, $|\alpha_i| = 0, 1, 2, 3, \dots$.

For a polynomial, we need a condition, which is:

$$B_{2|\alpha_i|+(i+1)} = 0 \quad \text{where } i = 0, 1, 2, 3, \dots \quad (4.7)$$

replace n by $2|\alpha_0| + 1$ in eq(4.4b), and it must to be equal to zero by eq(4.7).

$$B_{2|\alpha_0|+1} = -\frac{\Omega + \mu(2|\alpha_0| + \lambda)}{(2|\alpha_0| + 2 + \lambda)(2|\alpha_0| + 1 + \nu + \lambda)} = 0 \quad (4.8)$$

We argue that

$$\Omega = -\mu(2|\alpha_0| + \lambda) \quad (4.9)$$

replace n by $2i_1 + 1$ and plug eq(4.9) into eq(4.4b).

$$B_{2i_1+1} = \left(-\frac{1}{2}\mu\right) \frac{(i_1 - |\alpha_0|)}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \quad \text{where} \quad \gamma = \frac{1}{2}(1 + \nu) \quad (4.10)$$

We can expand B_{2i_1+1} in a finite product in an interval $[0, i_0 - 1]$

$$\prod_{i_1=0}^{i_0-1} B_{2i_1+1} = \left(-\frac{1}{2}\mu\right)^{i_0} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0}(\gamma + \frac{\lambda}{2})_{i_0}} \quad (4.11)$$

Substitute eq(4.11) into eq(4.5a).

$$y_0(x) = c_0 x^\lambda \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0}(\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad \text{where} \quad z = -\frac{1}{2}\mu x^2 \quad (4.12)$$

replace n by $2|\alpha_1| + 2$ in eq(4.4b), and it must to be equal to zero by eq(4.7):

$$B_{2|\alpha_1|+2} = -\frac{\Omega + \mu(2|\alpha_1| + 1 + \lambda)}{(2|\alpha_1| + 3 + \lambda)(2|\alpha_1| + 2 + \nu + \lambda)} = 0 \quad (4.13)$$

We argue that

$$\Omega = -\mu(2|\alpha_1| + 1 + \lambda) \quad (4.14)$$

replace n by $2i_3 + 2$ and plug eq(4.14) into eq(4.4b).

$$B_{2i_3+2} = \left(-\frac{1}{2}\mu\right) \frac{(i_3 - |\alpha_1|)}{(i_3 + \frac{3}{2} + \frac{\lambda}{2})(i_3 + \frac{1}{2} + \gamma + \frac{\lambda}{2})} \quad (4.15)$$

We can expand B_{2i_3+2} in a finite product in an interval $[i_0, i_2 - 1]$

$$\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} = \left(-\frac{1}{2}\mu\right)^{i_2-i_0} \frac{(-|\alpha_1|)_{i_2} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_2} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_2}} \quad (4.16)$$

replace n by $2i_0$ in eq(4.4a).

$$A_{2i_0} = -\left(\frac{1}{2}\varepsilon\right) \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \quad (4.17)$$

Substitute eq(4.11), eq(4.16) and eq(4.17) into eq(4.5b) and replace i_2 by i_1 on it:

$$\begin{aligned}
y_1(x) &= c_0 x^\lambda \left(-\frac{1}{2} \varepsilon x \right) \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
&\quad \left. \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \\
&\quad \text{where } z = -\frac{1}{2} \mu x^2
\end{aligned} \tag{4.18}$$

replace n by $2|\alpha_2| + 3$ in eq(4.4b), and it must be equal to zero by eq(4.7).

$$B_{2|\alpha_2|+3} = -\frac{\Omega + \mu(2|\alpha_2| + 2 + \lambda)}{(2|\alpha_2| + 4 + \lambda)(2|\alpha_2| + 3 + \nu + \lambda)} = 0 \tag{4.19}$$

We argue that

$$\Omega = -\mu(2|\alpha_2| + 2 + \lambda) \tag{4.20}$$

replace n by $2i_5 + 3$ and plug eq(4.20) into eq(4.4b).

$$B_{2i_5+3} = \left(-\frac{1}{2} \mu \right) \frac{(i_5 - |\alpha_2|)}{(i_5 + 2 + \frac{\lambda}{2})(i_5 + 1 + \gamma + \frac{\lambda}{2})} \tag{4.21}$$

We can expand B_{2i_5+3} in a finite product in an interval $[i_2, i_4 - 1]$

$$\prod_{i_5=i_2}^{i_4-1} B_{2i_5+3} = \left(-\frac{1}{2}\mu\right)^{i_4-i_2} \frac{(-|\alpha_2|)_{i_4}(2+\frac{\lambda}{2})_{i_2}(\gamma+1+\frac{\lambda}{2})_{i_2}}{(-|\alpha_2|)_{i_2}(2+\frac{\lambda}{2})_{i_4}(\gamma+1+\frac{\lambda}{2})_{i_4}} \quad (4.22)$$

replace n by $2i_2 + 1$ in eq(4.4a):

$$A_{2i_2+1} = -\left(\frac{1}{2}\varepsilon\right) \frac{(i_2 + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_2 + 1 + \frac{\lambda}{2})(i_2 + \gamma + \frac{\lambda}{2})} \quad (4.23)$$

Substitute eq(4.11), eq(4.16), eq(4.17), eq(4.22) and eq(4.23) and replaced i_2 and i_4 by i_1 and i_2 on it, to get

$$\begin{aligned} y_2(x) = & c_0 x^\lambda \left(-\frac{1}{2}\varepsilon x\right)^2 \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0}(\gamma + \frac{\lambda}{2})_{i_0}} \right. \\ & \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(i_1 + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_1|)_{i_1}(\frac{3}{2} + \frac{\lambda}{2})_{i_0}(\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0}(\frac{3}{2} + \frac{\lambda}{2})_{i_1}(\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\ & \left. \left. \times \sum_{i_2=i_1}^{|\alpha_2|} \left\{ \frac{(-|\alpha_2|)_{i_2}(2 + \frac{\lambda}{2})_{i_1}(\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1}(2 + \frac{\lambda}{2})_{i_2}(\gamma + 1 + \frac{\lambda}{2})_{i_2}} z^{i_2} \right\} \right\} \right\} \quad (4.24) \end{aligned}$$

replace n by $2|\alpha_3| + 4$ in eq(4.4b), and it must be equal to zero by eq(4.7).

$$B_{2|\alpha_3|+4} = -\frac{\Omega + \mu(2|\alpha_3| + 3 + \lambda)}{(2|\alpha_3| + 5 + \lambda)(2|\alpha_3| + 4 + \nu + \lambda)} = 0 \quad (4.25)$$

We argue that

$$\Omega = -\mu(2|\alpha_3| + 3 + \lambda) \quad (4.26)$$

replace n by $2i_7 + 4$ and plugging eq(4.26) into eq(4.4b):

$$B_{2i_7+4} = \left(-\frac{1}{2}\mu\right) \frac{(i_7 - |\alpha_3|)}{(i_7 + \frac{5}{2} + \frac{\lambda}{2})(i_7 + \frac{3}{2} + \gamma + \frac{\lambda}{2})} \quad (4.27)$$

We can expand B_{2i_7+4} in a finite product in an interval $[i_4, i_6 - 1]$

$$\prod_{i_7=i_4}^{i_6-1} B_{2i_7+4} = \left(-\frac{1}{2}\mu\right)^{i_6-i_4} \frac{(-|\alpha_3|)_{i_6}(\frac{5}{2} + \frac{\lambda}{2})_{i_4}(\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_4}}{(-|\alpha_3|)_{i_4}(\frac{5}{2} + \frac{\lambda}{2})_{i_6}(\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_6}} \quad (4.28)$$

replace n by $2i_4 + 2$ in eq(4.4a):

$$A_{2i_4+2} = -\left(\frac{1}{2}\varepsilon\right) \frac{(i_4 + 1 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_4 + \frac{3}{2} + \frac{\lambda}{2})(i_4 + \frac{1}{2} + \gamma + \frac{\lambda}{2})} \quad (4.29)$$

Substitute eq(4.11), eq(4.16), eq(4.17), eq(4.22), eq(4.23), eq(4.28) and eq(4.29) and replace i_2 , i_4 and i_6 by i_1 , i_2 and i_3 on it:

$$\begin{aligned}
y_3(x) = & c_0 x^\lambda \left(-\frac{1}{2} \varepsilon x \right)^3 \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(i_1 + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \sum_{i_2=i_1}^{|\alpha_2|} \left\{ \frac{(i_2 + 1 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_2 + \frac{3}{2} + \frac{\lambda}{2})(i_2 + \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} \right. \\
& \times \left. \left. \left. \sum_{i_3=i_2}^{|\alpha_3|} \frac{(-|\alpha_3|)_{i_3} (\frac{5}{2} + \frac{\lambda}{2})_{i_2} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_2}}{(-|\alpha_3|)_{i_2} (\frac{5}{2} + \frac{\lambda}{2})_{i_3} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_3}} z^{i_3} \right\} \right\} \right\} \quad (4.30)
\end{aligned}$$

According to eq(4.12), eq(4.18), eq(4.24) and eq(4.30), the general expression of power series of $y(x)$ for polynomial is

$$\begin{aligned}
y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
& + \tilde{\varepsilon} \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \left. \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \\
& + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \right. \\
& \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{|\alpha_k|} \frac{(i_k + \frac{\lambda}{2} + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{\lambda}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2} + \frac{\lambda}{2})} \right. \\
& \times \left. \frac{(-|\alpha_k|)_{i_k} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{k}{2} + \gamma + \frac{\lambda}{2})_{i_{k-1}}}{(-|\alpha_k|)_{i_{k-1}} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{k}{2} + \gamma + \frac{\lambda}{2})_{i_k}} \right\} \\
& \times \left. \left. \sum_{i_N=i_{N-1}}^{|\alpha_N|} \frac{(-|\alpha_N|)_{i_N} (1 + \frac{N}{2} + \frac{\lambda}{2})_{i_{N-1}} (\frac{N}{2} + \gamma + \frac{\lambda}{2})_{i_{N-1}} z^{i_N}}{(-|\alpha_N|)_{i_{N-1}} (1 + \frac{N}{2} + \frac{\lambda}{2})_{i_N} (\frac{N}{2} + \gamma + \frac{\lambda}{2})_{i_N}} \right\} \right\} \tilde{\varepsilon}^N \Big\} (4.31)
\end{aligned}$$

where

$$\left\{ \begin{array}{l} z = -\frac{1}{2}\mu x^2 \\ \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x \\ \gamma = \frac{1}{2}(1 + \nu) \\ \Omega = -\mu(2|\alpha_i| + i + \lambda) \text{ as } i = 0, 1, 2, 3, \dots \text{ and } |\alpha_i| = 0, 1, 2, 3, \dots \\ \text{if } i \leq j \rightarrow |\alpha_i| \leq |\alpha_j| \end{array} \right. \quad (4.32)$$

Where $\lambda=0$ and $c_0 = \frac{\Gamma(\gamma+|\alpha_0|)}{\Gamma(\gamma)}$ in eq(4.31),

$$\begin{aligned}
 y(x) &= QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \tilde{\varepsilon} \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\quad \times \left. \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \\
 &\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{|\alpha_k|} \frac{(i_k + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(-|\alpha_k|)_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + \gamma)_{i_{k-1}}}{(-|\alpha_k|)_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + \gamma)_{i_k}} \right\} \\
 &\quad \times \left. \left. \sum_{i_N=i_{N-1}}^{|\alpha_N|} \frac{(-|\alpha_N|)_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + \gamma)_{i_{N-1}} z^{i_N}}{(-|\alpha_N|)_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + \gamma)_{i_N}} \right\} \right\} \tilde{\varepsilon}^N \quad (4.33)
 \end{aligned}$$

Eq(4.33) is called as the 1st kind of independent solution of grand confluent hypergeometric polynomial as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ where $|\alpha_i|, i = 0, 1, 2, \dots$.

replace $|\alpha_i|$ by ψ_i in eq(4.31), also, put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = (-\frac{1}{2}\mu)^{1-\gamma} \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)}$

on it:

$$\begin{aligned}
y(x) &= RW_{\psi_i} \left(\psi_i = -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} z^{i_0} \right. \\
&\quad + \tilde{\varepsilon} \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(-\psi_0)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \sum_{i_1=i_0}^{\psi_1} \left\{ \frac{(-\psi_1)_{i_1} (\frac{3}{2})_{i_0} (\frac{5}{2} - \gamma)_{i_0}}{(-\psi_1)_{i_0} (\frac{3}{2})_{i_1} (\frac{5}{2} - \gamma)_{i_1}} z^{i_1} \right\} \right\} \\
&\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(-\psi_0)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \right. \\
&\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\psi_k} \frac{(i_k + 1 - \gamma + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k + \frac{3}{2} - \gamma + \frac{k}{2})} \frac{(-\psi_k)_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (2 - \gamma + \frac{k}{2})_{i_{k-1}}}{(-\psi_k)_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (2 - \gamma + \frac{k}{2})_{i_k}} \right\} \\
&\quad \left. \left. \times \sum_{i_N=i_{N-1}}^{\psi_N} \frac{(-\psi_N)_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (2 - \gamma + \frac{N}{2})_{i_{N-1}}}{(-\psi_N)_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (2 - \gamma + \frac{N}{2})_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \quad (4.34)
\end{aligned}$$

Eq(4.34) is called as the 2nd kind of independent solution of grand confluent hypergeometric polynomial as $\Omega = -2\mu(\psi_i + 1 - \gamma + \frac{i}{2})$ where $\psi_i, i = 0, 1, 2, \dots$

4.0.2 infinite series

Replacing the finite summation in intervals $[0, |\alpha_0|], [i_0, |\alpha_1|], [i_{k-1}, |\alpha_k|]$ and $[i_{N-1}, |\alpha_N|]$ by infinite summation with an intervals $[0, \infty], [i_0, \infty], [i_{k-1}, \infty]$ and $[i_{N-1}, \infty]$ in

eq(4.33), also, replacing $|\alpha_i|$ by $-\frac{\Omega}{2\mu} - \frac{i}{2}$ on it.

$$\begin{aligned}
 y(x) &= QW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \tilde{\varepsilon} \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\quad \times \left. \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \\
 &\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + \gamma)_{i_{k-1}}}{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + \gamma)_{i_k}} \right\} \\
 &\quad \times \left. \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + \gamma)_{i_{N-1}}}{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \Bigg\} \quad (4.35)
 \end{aligned}$$

Eq(4.35) is called as the 1st kind of independent solution of grand confluent hypergeometric infinite series.

Replacing the finite summation in an intervals $[0, \psi_0]$, $[i_0, \psi_1]$, $[i_{k-1}, \psi_k]$ and $[i_{N-1}, \psi_N]$ by infinite summation with intervals $[0, \infty]$, $[i_0, \infty]$, $[i_{k-1}, \infty]$ and $[i_{N-1}, \infty]$ in eq(4.34),

and, replacing ψ_i by $-\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}$ on it:

$$\begin{aligned}
 y(x) &= RW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} z^{i_0} \right. \\
 &\quad + \tilde{\varepsilon} \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \\
 &\quad \times \left. \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_0} (\frac{5}{2} - \gamma)_{i_0}}{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_1} (\frac{5}{2} - \gamma)_{i_1}} z^{i_1} \right\} \right\} \\
 &\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \right. \\
 &\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + 1 - \gamma + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k + \frac{3}{2} - \gamma + \frac{k}{2})} \right. \\
 &\quad \times \left. \frac{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (2 - \gamma + \frac{k}{2})_{i_{k-1}}}{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (2 - \gamma + \frac{k}{2})_{i_k}} \right\} \\
 &\quad \times \left. \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{N}{2})_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (2 - \gamma + \frac{N}{2})_{i_{N-1}}}{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{N}{2})_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (2 - \gamma + \frac{N}{2})_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \left. \right\} \quad (4.36)
 \end{aligned}$$

Eq(4.36) is called as the 2nd kind of independent solution of grand confluent hypergeometric infinite series. As we see from eq(4.33), eq(4.34), eq(4.35) and eq(4.36), it is required that $\gamma \neq 0, -1, -2, \dots$ for the first kind of independent solution of grand confluent hypergeometric function for the polynomial and infinite series. Also

$\gamma \neq 2, 3, 4, \dots$ is required for the second kind of independent solution of grand confluent hypergeometric function for both the polynomial and infinite series.

4.0.3 Integral formalism of polynomial which makes B_n term terminated:

Now, let's investigate the integral formalism of eq(4.31). First of all, there is a generalized hypergeometric function which is

$$\begin{aligned}
 I_j &= \sum_{i_j=i_{j-1}}^{|\alpha_j|} \frac{(-|\alpha_j|)_{i_j} (1 + \frac{j}{2} + \frac{\lambda}{2})_{i_{j-1}} (\frac{j}{2} + \gamma + \frac{\lambda}{2})_{i_{j-1}}}{(-|\alpha_j|)_{i_{j-1}} (1 + \frac{j}{2} + \frac{\lambda}{2})_{i_j} (\frac{j}{2} + \gamma + \frac{\lambda}{2})_{i_j}} z^{i_j} \\
 &= z^{i_{j-1}} \sum_{l=0}^{\infty} \frac{(i_{j-1} - |\alpha_j|)_l}{(i_{j-1} + 1 + \frac{j}{2} + \frac{\lambda}{2})_l (i_{j-1} + \frac{j}{2} + \gamma + \frac{\lambda}{2})_l} z^l \quad (4.37) \\
 &= z^{i_{j-1}} \sum_{l=0}^{\infty} \frac{B(i_{j-1} + \frac{j}{2} + \frac{\lambda}{2}, l+1) B(i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{\lambda}{2}, l+1) (-|\alpha_j| + i_{j-1})_l}{(i_{j-1} + \frac{j}{2} + \frac{\lambda}{2})^{-1} (i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{\lambda}{2})^{-1} (1)_l l!} z^l
 \end{aligned}$$

By using integral form of beta function,

$$B\left(i_{j-1} + \frac{j}{2} + \frac{\lambda}{2}, l+1\right) = \int_0^1 dt_j t_j^{i_{j-1} + \frac{j}{2} - 1 + \frac{\lambda}{2}} (1-t_j)^l \quad (4.38a)$$

$$B\left(i_{j-1} + \gamma - 1 + \frac{j}{2} + \frac{\lambda}{2}, l+1\right) = \int_0^1 du_j u_j^{i_{j-1} + \gamma - 2 + \frac{j}{2} + \frac{\lambda}{2}} (1-u_j)^l \quad (4.38b)$$

Substitute eq(4.38a) and eq(4.38b) into eq(4.37), and divide $(i_{j-1} + \frac{j}{2} + \frac{\lambda}{2})(i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{\lambda}{2})$ into I_j .

$$\begin{aligned}
& \frac{1}{(i_{j-1} + \frac{j}{2} + \frac{\lambda}{2})(i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{\lambda}{2})} \\
& \times \sum_{i_j=i_{j-1}}^{|\alpha_j|} \frac{(-|\alpha_j|)_{i_j} (1 + \frac{j}{2} + \frac{\lambda}{2})_{i_{j-1}} (\frac{j}{2} + \gamma + \frac{\lambda}{2})_{i_{j-1}}}{(-|\alpha_j|)_{i_{j-1}} (1 + \frac{j}{2} + \frac{\lambda}{2})_{i_j} (\frac{j}{2} + \gamma + \frac{\lambda}{2})_{i_j}} z^{i_j} \\
& = \int_0^1 dt_j t_j^{\frac{j}{2}-1+\frac{\lambda}{2}} \int_0^1 du_j u_j^{\gamma-2+\frac{j}{2}+\frac{\lambda}{2}} (zt_j u_j)^{i_{j-1}} \\
& \times \sum_{l=0}^{\infty} \frac{(-(|\alpha_j| - i_{j-1}))_l}{(1)_l l!} [z(1-t_j)(1-u_j)]^l \tag{4.39}
\end{aligned}$$

The first kind of confluent hypergeometric function is

$$F_{|\alpha_0|}(\gamma; z) = \frac{|\alpha_0|!}{2\pi i} \oint dv \frac{e^{-\frac{zv}{(1-v)}}}{v^{|\alpha_0|+1}(1-v)^\gamma} = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(-|\alpha_0|)_n}{(\gamma)_n n!} z^n \tag{4.40}$$

replace $|\alpha_0|$, γ , v and z by $|\alpha_j| - i_{j-1}$, 1 , v_j and $z(1 - t_j)(1 - u_j)$ in eq(4.40), and divide $\Gamma(|\alpha_j| + 1 - i_{j-1})$ on it:

$$\begin{aligned} \frac{F_{|\alpha_j|-i_{j-1}}(\gamma = 1; z(1 - t_j)(1 - u_j))}{\Gamma(|\alpha_j| + 1 - i_{j-1})} &= \frac{1}{2\pi i} \oint dv_j \frac{e^{-\frac{v_j}{(1-v_j)}z(1-t_j)(1-u_j)}}{v_j^{|\alpha_j|+1-i_{j-1}}(1-v_j)} \\ &= \sum_{l=0}^{|\alpha_j|-i_{j-1}} \frac{(-(|\alpha_j| - i_{j-1}))_l}{(1)_l l!} [z(1 - t_j)(1 - u_j)]^l \\ &= \sum_{l=0}^{\infty} \frac{(-(|\alpha_j| - i_{j-1}))_l}{(1)_l l!} [z(1 - t_j)(1 - u_j)]^l \end{aligned} \quad (4.41)$$

Substitute eq(4.41) into eq(4.39).

$$\begin{aligned} &\frac{1}{(i_{j-1} + \frac{j}{2} + \frac{\lambda}{2})(i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{\lambda}{2})} \\ &\times \sum_{i_j=i_{j-1}}^{|\alpha_j|} \frac{(-|\alpha_j|)_{i_j} (1 + \frac{j}{2} + \frac{\lambda}{2})_{i_{j-1}} (\frac{j}{2} + \gamma + \frac{\lambda}{2})_{i_{j-1}} z^{i_j}}{(-|\alpha_j|)_{i_{j-1}} (1 + \frac{j}{2} + \frac{\lambda}{2})_{i_j} (\frac{j}{2} + \gamma + \frac{\lambda}{2})_{i_j}} \quad (4.42) \\ &= \int_0^1 dt_j t_j^{\frac{j}{2}-1+\frac{\lambda}{2}} \int_0^1 du_j u_j^{\gamma-2+\frac{j}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_j \frac{e^{-\frac{v_j}{(1-v_j)}z(1-t_j)(1-u_j)}}{v_j^{|\alpha_j|+1}(1-v_j)} (zt_j u_j v_j)^{i_{j-1}} \end{aligned}$$

Put $j = 1$ in eq(4.42).

$$\begin{aligned}
& \frac{1}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \gamma - \frac{1}{2} + \frac{\lambda}{2})} \sum_{i_1=i_0}^{|\alpha_1|} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\frac{1}{2} + \gamma + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\frac{1}{2} + \gamma + \frac{\lambda}{2})_{i_1}} z^{i_1} \\
= & \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} z(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} (zt_1 u_1 v_1)^{i_0}
\end{aligned} \tag{4.43}$$

Plug eq(4.43) into eq(4.18).

$$\begin{aligned}
y_1(x) &= c_0 x^\lambda \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} z(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} \\
&\quad \times \sum_{i_0=0}^{|\alpha_0|} \left(i_0 + \frac{\omega}{2} + \frac{\lambda}{2} \right) \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} (zt_1 u_1 v_1)^{i_0} \\
= & c_0 x^\lambda \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} z(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} \\
&\quad \times \left\{ w_{1,1} \partial_{w_{1,1}} + \left(\frac{\omega}{2} + \frac{\lambda}{2} \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,1}^{i_0}
\end{aligned} \tag{4.44}$$

$$\text{where } w_{1,1} = z \prod_{l=1}^1 t_l u_l v_l$$

Put $j = 2$ in eq(4.42).

$$\begin{aligned}
 & \frac{1}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \sum_{i_2=i_1}^{|\alpha_2|} \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (1 + \gamma + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (1 + \gamma + \frac{\lambda}{2})_{i_2}} z^{i_2} \\
 = & \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}z(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} (zt_2u_2v_2)^{i_1} \quad (4.45)
 \end{aligned}$$

Plug eq(4.45) into eq(4.24).

$$\begin{aligned}
 y_2(x) = & c_0 x^\lambda \tilde{\varepsilon}^2 \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}z(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} \\
 & \times \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \left. \left\{ w_{2,2} \partial_{w_{2,2}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} w_{2,2}^{i_1} \right\} \right\} \\
 \text{where } & w_{2,2} = z \prod_{l=2}^2 t_l u_l v_l \quad (4.46)
 \end{aligned}$$

replace z by $w_{2,2}$ in eq(4.43).

$$\begin{aligned}
& \frac{1}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \gamma - \frac{1}{2} + \frac{\lambda}{2})} \sum_{i_1=i_0}^{|\alpha_1|} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\frac{1}{2} + \gamma + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\frac{1}{2} + \gamma + \frac{\lambda}{2})_{i_1}} w_{2,2}^{i_1} \\
= & \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,2} (1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} w_{1,2}^{i_0} \quad (4.47) \\
& \text{where } w_{1,2} = z \prod_{l=1}^2 t_l u_l v_l
\end{aligned}$$

plug eq(4.47) into eq(4.46).

$$\begin{aligned}
y_2(x) = & c_0 x^\lambda \tilde{\varepsilon}^2 \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)} z (1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1} (1-v_2)} \\
& \times \left\{ w_{2,2} \partial_{w_{2,2}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
& \times \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,2} (1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} \\
& \times \left\{ w_{1,2} \partial_{w_{1,2}} + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0} \quad (4.48)
\end{aligned}$$

Put $j = 3$ in eq(4.42).

$$\begin{aligned}
& \frac{1}{(i_2 + \frac{3}{2} + \frac{\lambda}{2})(i_2 + \gamma + \frac{1}{2} + \frac{\lambda}{2})} \sum_{i_3=i_2}^{|\alpha_3|} \frac{(-|\alpha_3|)_{i_3} (\frac{5}{2} + \frac{\lambda}{2})_{i_2} (\frac{3}{2} + \gamma + \frac{\lambda}{2})_{i_2}}{(-|\alpha_3|)_{i_2} (\frac{5}{2} + \frac{\lambda}{2})_{i_3} (\frac{3}{2} + \gamma + \frac{\lambda}{2})_{i_3}} z^{i_3} \\
= & \int_0^1 dt_3 t_3^{\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma - \frac{1}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)} z(1-t_3)(1-u_3)}}{v_3^{|\alpha_3|+1} (1-v_3)} (w_{3,3})^{i_2} \quad (4.49) \\
& \text{where } w_{3,3} = z \prod_{l=3}^3 t_l u_l v_l
\end{aligned}$$

Plug eq(4.49) into eq(4.30).

$$\begin{aligned}
y_3(x) = & c_0 x^\lambda \tilde{\varepsilon}^3 \int_0^1 dt_3 t_3^{\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma - \frac{1}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)} z(1-t_3)(1-u_3)}}{v_3^{|\alpha_3|+1} (1-v_3)} \\
& \times \left\{ w_{3,3} \partial_{w_{3,3}} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
& \times \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(i_2 + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_2 + 1 + \frac{\lambda}{2})(i_2 + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \left. \left. \sum_{i_2=i_1}^{|\alpha_2|} \left\{ \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} w_{3,3}^{i_2} \right\} \right\} \right\} \quad (4.50)
\end{aligned}$$

replace z by $w_{3,3}$ in eq(4.45).

$$\begin{aligned}
& \frac{1}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \sum_{i_2=i_1}^{|\alpha_2|} \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (1 + \gamma + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (1 + \gamma + \frac{\lambda}{2})_{i_2}} w_{3,3}^{i_2} \\
= & \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)} w_{3,3}(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} w_{2,3}^{i_1} \quad (4.51) \\
& \text{where } w_{2,3} = z \prod_{l=2}^3 t_l u_l v_l
\end{aligned}$$

plug eq(4.51) into eq(4.50).

$$\begin{aligned}
y_3(x) = & c_0 x^\lambda \tilde{\varepsilon}^3 \int_0^1 dt_3 t_3^{\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma-\frac{1}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)} z(1-t_3)(1-u_3)}}{v_3^{|\alpha_3|+1}(1-v_3)} \\
& \times \left\{ w_{3,3} \partial_{w_{3,3}} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
& \times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)} w_{3,3}(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} \\
& \times \left\{ w_{2,3} \partial_{w_{2,3}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
& \times \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \left. \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} w_{2,3}^{i_1} \right\} \right\} \quad (4.52)
\end{aligned}$$

replace z by $w_{2,3}$ in eq(4.43).

$$\begin{aligned}
& \frac{1}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 + \gamma - \frac{1}{2} + \frac{\lambda}{2})} \sum_{i_1=i_0}^{|\alpha_1|} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\frac{1}{2} + \gamma + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\frac{1}{2} + \gamma + \frac{\lambda}{2})_{i_1}} w_{2,3}^{i_1} \\
= & \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,3} (1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} w_{1,3}^{i_0} \quad (4.53) \\
& \text{where } w_{1,3} = z \prod_{l=1}^3 t_l u_l v_l
\end{aligned}$$

Plug eq(4.53) into eq(4.52).

$$\begin{aligned}
y_3(x) = & c_0 x^\lambda \tilde{\varepsilon}^3 \int_0^1 dt_3 t_3^{\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma - \frac{1}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)} z (1-t_3)(1-u_3)}}{v_3^{|\alpha_3|+1} (1-v_3)} \\
& \times \left\{ w_{3,3} \partial_{w_{3,3}} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
& \times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma - 1 + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)} w_{3,3} (1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1} (1-v_2)} \\
& \times \left\{ w_{2,3} \partial_{w_{2,3}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
& \times \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,3} (1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1} (1-v_1)} \\
& \times \left\{ w_{1,3} \partial_{w_{1,3}} + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,3}^{i_0} \quad (4.54)
\end{aligned}$$

According to eq(4.12), eq(4.44), eq(4.48) and eq(4.54), the integral formalism of the function $y(x)$ for the polynomial is

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 & + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1+\frac{\lambda}{2}} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2+\frac{\lambda}{2}} \right. \right. \\
 & \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1}(1-v_{n-j})} \\
 & \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1+\omega+\lambda) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,n}^{i_0} \right\} \right\} \tilde{\varepsilon}^n \left. \right\}
 \end{aligned} \tag{4.55}$$

where

$$w_{a,b} = \begin{cases} z \prod_{l=a}^b t_l u_l v_l \\ z \text{ only if } a > b \end{cases} \tag{4.56}$$

Let $\lambda = 0$ and $c_0 = \frac{\Gamma(\gamma+|\alpha_0|)}{\Gamma(\gamma)}$ in eq(4.55).

$$\begin{aligned}
y(x) &= F_{|\alpha_0|}(\gamma; z) + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1}(1-v_{n-j})} \\
&\quad \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2}(n-j-1+\omega) \right\} F_{|\alpha_0|}(\gamma; w_{1,n}) \right\} \right\} \tilde{\varepsilon}^n \quad (4.57)
\end{aligned}$$

Substitute eq(4.40) into eq(4.57).

$$\begin{aligned}
y(x) &= QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, \gamma = \frac{1}{2}(1+\nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1}(1-v_0)^\gamma} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1}(1-v_{n-j})} \\
&\quad \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2}(n-j-1+\omega) \right\} \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1,n}v_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1}(1-v_0)^\gamma} \right\} \right\} \tilde{\varepsilon}^n \quad (4.58)
\end{aligned}$$

Eq(4.58) is called as the integral formalism of the first kind of grand confluent hypergeometric function for polynomial as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ where $i, |\alpha_i| = 0, 1, 2, \dots$.

replace $|\alpha_i|$ by ψ_i in eq(4.55) Also, put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = \left(-\frac{1}{2}\mu\right)^{1-\gamma} \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)}$ on it.

$$\begin{aligned}
 y(x) = & z^{1-\gamma} \left\{ A_{\psi_0}(\gamma; z) + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \right. \\
 & \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{\psi_{n-j}+1}(1-v_{n-j})} \\
 & \left. \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j+1-2\gamma+\omega) \right\} A_{\psi_0}(\gamma; w_{1,n}) \right\} \right\} \tilde{\varepsilon}^n \right\} \quad (4.59)
 \end{aligned}$$

The second kind of confluent hypergeometric function is

$$A_{\psi_0}(\gamma; z) = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n = \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1} (1-v_0)^{2-\gamma}} \quad (4.60)$$

Substitute eq(4.60) into eq(4.59).

$$\begin{aligned}
y(x) &= RW_{\psi_i} \left(\psi_i = -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \left\{ \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right. \\
&\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{\psi_{n-j}+1}(1-v_{n-j})} \quad (4.61) \\
&\quad \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} \left(n - j + 1 - 2\gamma + \omega \right) \right\} \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1,n} v_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \left. \right\}
\end{aligned}$$

4.0.4 Integral formalism of infinite series

For infinite series, we replace $|\alpha_i|$ and ψ_i by $-\frac{\Omega}{2\mu} - \frac{i}{2}$ and $-\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}$ in eq(4.58) and eq(4.61). Now, replace $|\alpha_0|$ by $-\frac{\Omega}{2\mu}$ in the first kind of confluent hypergeometric function:

$$F_{|\alpha_0|}(\gamma; z) \rightarrow \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_n}{n! (\gamma)_n} z^n = \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} M \left(\frac{\Omega}{2\mu}, \gamma, z \right) \quad (4.62)$$

The integral formalism of the first kind of Kummer function is

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 dv_0 e^{zv_0} v_0^{a-1} (1-v_0)^{b-a-1} \quad \text{where } \operatorname{Re}(b) > \operatorname{Re}(a) > 0 \quad (4.63)$$

By using eq(4.63), eq(4.62) turns to be

$$F_{|\alpha_0|}(\gamma; z) \rightarrow \frac{1}{\Gamma\left(\frac{\Omega}{2\mu}\right)} \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{\gamma-\frac{\Omega}{2\mu}-1} \quad \text{where } \operatorname{Re}\left(\frac{\Omega}{2\mu}\right) > \operatorname{Re}(\gamma) > 0 \quad (4.64a)$$

And,

$$F_{|\alpha_0|}(\gamma; w_{1,n}) \rightarrow \frac{1}{\Gamma\left(\frac{\Omega}{2\mu}\right)} \int_0^1 dv_0 e^{w_{1,n}v_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{\gamma-\frac{\Omega}{2\mu}-1} \quad (4.64b)$$

$$|\alpha_{n-j}| \rightarrow -\frac{\Omega}{2\mu} - \frac{1}{2}(n-j) \quad (4.64c)$$

Substitute eq(4.64a), eq(4.64b) and eq(4.64c) into eq(4.57).

$$\begin{aligned}
 y(x) &= QW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{1}{\Gamma(\frac{\Omega}{2\mu})} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{\gamma-\frac{\Omega}{2\mu}-1} \right. \\
 &\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\frac{\Omega}{2\mu}-\frac{1}{2}(n-j)+1} (1-v_{n-j})} \\
 &\quad \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1+\omega) \right\} \\
 &\quad \left. \left. \left. \times \int_0^1 dv_0 e^{w_{1,n}v_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{\gamma-\frac{\Omega}{2\mu}-1} \right\} \right\} \tilde{\varepsilon}^n \right\} \quad (4.65)
 \end{aligned}$$

Eq(4.65) is called the integral formalism of the first kind of grand confluent hypergeometric function for infinite series.

Replace ψ_0 by $-\frac{\Omega}{2\mu} + \gamma - 1$ in eq(4.60).

$$\begin{aligned}
 A_{\psi_0}(\gamma; z) &= \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n}{n! (2 - \gamma)_n} z^n \quad (4.66) \\
 &\rightarrow \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \sum_{n=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_n}{n! (2 - \gamma)_n} z^n = \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right)
 \end{aligned}$$

By using eq(4.63), eq(4.66) turns to be

$$A_{\psi_0}(\gamma; z) \rightarrow \frac{1}{\Gamma\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \quad (4.67a)$$

where $\operatorname{Re}(2 - \gamma) > \operatorname{Re}\left(\frac{\Omega}{2\mu} + 1 - \gamma\right) > 0$

And,

$$A_{\psi_0}(\gamma; w_{1,n}) \rightarrow \frac{1}{\Gamma\left(\frac{\Omega}{2\mu} + 1 - \gamma\right)} \int_0^1 dv_0 e^{w_{1,n}v_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \quad (4.67b)$$

$$\psi_{n-j} \rightarrow -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{1}{2}(n - j) \quad (4.67c)$$

Substitute eq(4.67a), eq(4.67b) and eq(4.67c) into eq(4.59).

$$\begin{aligned}
 y(x) &= RW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{z^{1-\gamma}}{\Gamma(\frac{\Omega}{2\mu} + 1 - \gamma)} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-\gamma} (1-v_0)^{-\frac{\Omega}{2\mu}} \right. \\
 &\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\frac{\Omega}{2\mu} + \gamma - \frac{1}{2}(n-j)} (1-v_{n-j})} \\
 &\quad \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j+1 - 2\gamma + \omega) \right\} \\
 &\quad \left. \left. \left. \times \int_0^1 dv_0 e^{w_{1,n} v_0} v_0^{\frac{\Omega}{2\mu}-\gamma} (1-v_0)^{-\frac{\Omega}{2\mu}} \right\} \right\} \tilde{\varepsilon}^n \right\} \quad (4.68)
 \end{aligned}$$

Eq(4.68) is called as the integral formalism of the second kind of grand confluent hypergeometric function for infinite series.

4.0.5 Power series of polynomial in which makes A_n term terminated

Again, let's think about the polynomial case which makes A_n term terminated. First of all, replace the finite summation in intervals $[0, |\alpha_0|]$, $[i_0, |\alpha_1|]$, $[i_{k-1}, |\alpha_k|]$ and

$[i_{N-1}, |\alpha_N|]$ by infinite summation in intervals $[0, \infty]$, $[i_0, \infty]$, $[i_{k-1}, \infty]$ and $[i_{N-1}, \infty]$ in eq(4.31). Also, replaced $-|\alpha_i|$ by $\frac{\Omega}{2\mu} + \frac{i}{2} + \frac{\lambda}{2}$ on it where $i = 0, 1, 2, \dots$. Then, we obtain the infinite series of function $y(x)$

The infinite series of x of zero, one, two and three term of A_n 's are

(1) Zero term of A_n 's

$$y_0(x) = c_0 x^\lambda \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad (4.69a)$$

(2) One term of A_n 's

$$y_1(x) = c_0 x^\lambda \tilde{\varepsilon} \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\ \left. \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \quad (4.69b)$$

(3) Two term of A_n 's

$$\begin{aligned}
y_2(x) = & c_0 x^\lambda \tilde{\varepsilon}^2 \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(i_1 + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \left. \left. \sum_{i_2=i_1}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} z^{i_2} \right\} \right\} \right\} \quad (4.69c)
\end{aligned}$$

(4) Three term of A_n 's

$$\begin{aligned}
y_3(x) = & c_0 x^\lambda \tilde{\varepsilon}^3 \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(i_1 + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \sum_{i_2=i_1}^{\infty} \left\{ \frac{(i_2 + 1 + \frac{\lambda}{2} + \frac{\omega}{2})}{(i_2 + \frac{3}{2} + \frac{\lambda}{2})(i_2 + \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} \right. \\
& \times \left. \left. \sum_{i_3=i_2}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} + \frac{\lambda}{2})_{i_3} (\frac{5}{2} + \frac{\lambda}{2})_{i_2} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_2}}{(\frac{\Omega}{2\mu} + \frac{3}{2} + \frac{\lambda}{2})_{i_2} (\frac{5}{2} + \frac{\lambda}{2})_{i_3} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_3}} z^{i_3} \right\} \right\} \right\} \quad (4.69d)
\end{aligned}$$

the general expression of power series of $y(x)$ for polynomial is

$$\begin{aligned}
 y(x) &= \sum_{m=0}^{\infty} y_m(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\
 &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \right. \\
 &\quad + \tilde{\varepsilon} \sum_{i_0=0}^{\infty} \left\{ \frac{\left(i_0 + \frac{\lambda}{2} + \frac{\omega}{2}\right)}{\left(i_0 + \frac{1}{2} + \frac{\lambda}{2}\right) \left(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2}\right)} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} \right. \\
 &\quad \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_1} \left(\frac{3}{2} + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_0}}{\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_0} \left(\frac{3}{2} + \frac{\lambda}{2}\right)_{i_1} \left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_1}} z^{i_1} \right\} \left. \right\} \\
 &\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{\left(i_0 + \frac{\lambda}{2} + \frac{\omega}{2}\right)}{\left(i_0 + \frac{1}{2} + \frac{\lambda}{2}\right) \left(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2}\right)} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} \right. \right. \\
 &\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{\left(i_k + \frac{\lambda}{2} + \frac{\omega}{2} + \frac{k}{2}\right)}{\left(i_k + \frac{1}{2} + \frac{\lambda}{2} + \frac{k}{2}\right) \left(i_k - \frac{1}{2} + \gamma + \frac{k}{2} + \frac{\lambda}{2}\right)} \right. \\
 &\quad \times \left. \frac{\left(\frac{\Omega}{2\mu} + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_k} \left(1 + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_{k-1}} \left(\frac{k}{2} + \gamma + \frac{\lambda}{2}\right)_{i_{k-1}}}{\left(\frac{\Omega}{2\mu} + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_{k-1}} \left(1 + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_k} \left(\frac{k}{2} + \gamma + \frac{\lambda}{2}\right)_{i_k}} \right\} \left. \right. \\
 &\quad \times \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_N} \left(1 + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_{N-1}} \left(\frac{N}{2} + \gamma + \frac{\lambda}{2}\right)_{i_{N-1}}}{\left(\frac{\Omega}{2\mu} + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_{N-1}} \left(1 + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_N} \left(\frac{N}{2} + \gamma + \frac{\lambda}{2}\right)_{i_N}} z^{i_N} \right\} \left. \right\} \tilde{\varepsilon}^N \left. \right\} \quad (4.70)
 \end{aligned}$$

As $\frac{\omega}{2} + \frac{\lambda}{2} = -|\beta_0|$ where $|\beta_0| = 0, 1, 2, \dots$ in eq(4.70), $y_1(x) = y_2(x) = y_3(x) = \dots = 0$.

We obtain

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \quad \text{where } \omega = -2 \left(|\beta_0| + \frac{\lambda}{2} \right) \quad (4.71)$$

As $\frac{\omega}{2} + \frac{1}{2} + \frac{\lambda}{2} = -|\beta_1|$ where $|\beta_1| = 0, 1, 2, \dots$ in eq(4.70), $y_2(x) = y_3(x) = y_4(x) = \dots = 0$. We obtain

$$\begin{aligned}
 y(x) &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 &\quad + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{1}{2} + |\beta_1|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 &\quad \times \left. \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \left. \right\} \\
 &\quad \text{where } \omega = -2 \left(|\beta_1| + \frac{1}{2} + \frac{\lambda}{2} \right) \tag{4.72}
 \end{aligned}$$

As $\frac{\omega}{2} + 1 + \frac{\lambda}{2} = -|\beta_2|$ where $|\beta_2| = 0, 1, 2, \dots$ in eq(4.70), $y_3(x) = y_4(x) = y_5(x) =$

$\dots = 0$. We obtain

$$\begin{aligned}
y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
& + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (1 + |\beta_2|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} \\
& + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (1 + |\beta_2|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(i_1 + \frac{1}{2} - (1 + |\beta_2|))}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \sum_{i_2=i_1}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} z^{i_2} \right\} \left. \right\} \tilde{\varepsilon}^2 \Big\} \\
& \text{where } \omega = -2 \left(|\beta_2| + 1 + \frac{\lambda}{2} \right) \tag{4.73}
\end{aligned}$$

As $\frac{\omega}{2} + \frac{3}{2} + \frac{\lambda}{2} = -|\beta_3|$ where $|\beta_3| = 0, 1, 2, \dots$ in eq(4.70), $y_4(x) = y_5(x) = y_6(x) =$

$\dots = 0$. We obtain

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 & + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{3}{2} + |\beta_3|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} \\
 & + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{3}{2} + |\beta_3|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(i_1 + \frac{1}{2} - (\frac{3}{2} + |\beta_3|))}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
 & \times \sum_{i_2=i_1}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} z^{i_2} \right\} \tilde{\varepsilon}^2 \\
 & + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{3}{2} - |\beta_3|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(i_1 + \frac{1}{2} - (\frac{3}{2} - |\beta_3|))}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
 & \times \sum_{i_2=i_1}^{\infty} \left\{ \frac{(i_2 + 1 - (\frac{3}{2} - |\beta_3|))}{(i_2 + \frac{3}{2} + \frac{\lambda}{2})(i_2 + \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2})_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} \right. \\
 & \times \left. \left. \left. \sum_{i_3=i_2}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} + \frac{\lambda}{2})_{i_3} (\frac{5}{2} + \frac{\lambda}{2})_{i_2} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_2}}{(\frac{\Omega}{2\mu} + \frac{3}{2} + \frac{\lambda}{2})_{i_2} (\frac{5}{2} + \frac{\lambda}{2})_{i_3} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_3}} z^{i_3} \right\} \right\} \tilde{\varepsilon}^3 \left. \right\} \\
 & \text{where } \omega = -2 \left(|\beta_3| + \frac{3}{2} + \frac{\lambda}{2} \right) \tag{4.74}
 \end{aligned}$$

According to eq(4.71), eq(4.72), eq(4.73) and eq(4.74), the general expression of the function $y(x)$ for the polynomial in which makes $A'_n s$ term terminated at certain eigenvalue is

(1) As $\omega = -2 \left(|\beta_0| + \frac{\lambda}{2} \right)$ where $|\beta_0| = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \quad (4.75)$$

(2) As $\omega = -2 \left(|\beta_1| + \frac{1}{2} + \frac{\lambda}{2} \right)$ where $|\beta_1| = 0, 1, 2, \dots$

$$\begin{aligned} y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \right. \\ & + \sum_{i_0=0}^{\infty} \left\{ \frac{\left(i_0 - \left(\frac{1}{2} + |\beta_1|\right)\right)}{\left(i_0 + \frac{1}{2} + \frac{\lambda}{2}\right) \left(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2}\right)} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} \right. \\ & \left. \left. \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_1} \left(\frac{3}{2} + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_0}}{\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_0} \left(\frac{3}{2} + \frac{\lambda}{2}\right)_{i_1} \left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \quad (4.76) \end{aligned}$$

(3) As $\omega = -2 \left(|\beta_m| + \frac{m}{2} + \frac{\lambda}{2} \right)$ where $|\beta_m| = 0, 1, 2, \dots$ only if $m \geq 2$

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \right. \\
 & + \sum_{i_0=0}^{\infty} \left\{ \frac{\left(i_0 - \left(\frac{m}{2} + |\beta_m|\right)\right)}{\left(i_0 + \frac{1}{2} + \frac{\lambda}{2}\right) \left(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2}\right)} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_1} \left(\frac{3}{2} + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_0}}{\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_0} \left(\frac{3}{2} + \frac{\lambda}{2}\right)_{i_1} \left(\gamma + \frac{1}{2} + \frac{\lambda}{2}\right)_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} \\
 & + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{\left(i_0 - \left(\frac{m}{2} + |\beta_m|\right)\right)}{\left(i_0 + \frac{1}{2} + \frac{\lambda}{2}\right) \left(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2}\right)} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} \right. \right. \\
 & \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{\left(i_k + \frac{k}{2} - \left(\frac{m}{2} + |\beta_m|\right)\right)}{\left(i_k + \frac{1}{2} + \frac{\lambda}{2} + \frac{k}{2}\right) \left(i_k - \frac{1}{2} + \gamma + \frac{k}{2} + \frac{\lambda}{2}\right)} \right. \\
 & \times \frac{\left(\frac{\Omega}{2\mu} + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_k} \left(1 + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_{k-1}} \left(\frac{k}{2} + \gamma + \frac{\lambda}{2}\right)_{i_{k-1}}}{\left(\frac{\Omega}{2\mu} + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_{k-1}} \left(1 + \frac{k}{2} + \frac{\lambda}{2}\right)_{i_k} \left(\frac{k}{2} + \gamma + \frac{\lambda}{2}\right)_{i_k}} \left. \right\} \\
 & \times \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_N} \left(1 + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_{N-1}} \left(\frac{N}{2} + \gamma + \frac{\lambda}{2}\right)_{i_{N-1}} z^{i_N}}{\left(\frac{\Omega}{2\mu} + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_{N-1}} \left(1 + \frac{N}{2} + \frac{\lambda}{2}\right)_{i_N} \left(\frac{N}{2} + \gamma + \frac{\lambda}{2}\right)_{i_N}} \right\} \tilde{\varepsilon}^N \left. \right\} \quad (4.77)
 \end{aligned}$$

Put $\lambda = 0$ and $c_0 = \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)}$ in eq(4.75), eq(4.76) and eq(4.77).

(1) As $\omega = -2|\beta_0|$ where $|\beta_0| = 0, 1, 2, \dots$

$$\begin{aligned}
 & QW_{|\beta_0|} \left(|\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & = \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu}\right)_{i_0}}{(1)_{i_0} (\gamma)_{i_0}} z^{i_0} \quad (4.78)
 \end{aligned}$$

(2) As $\omega = -2(|\beta_1| + \frac{1}{2})$ where $|\beta_1| = 0, 1, 2, \dots$

$$\begin{aligned}
 & QW_{|\beta_1|} \left(|\beta_1| = -\frac{1}{2}(1 + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{1}{2} + |\beta_1|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \left. \left. \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \quad (4.79)
 \end{aligned}$$

(3) As $\omega = -2(|\beta_m| + \frac{m}{2})$ where $|\beta_m| = 0, 1, 2, \dots$ only if $m \geq 2$

$$\begin{aligned}
 & QW_{|\beta_m|} \left(|\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \left. \left. \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right. \\
 &+ \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + |\beta_m|))}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \right. \\
 &\times \left. \left. \frac{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + \gamma)_{i_{k-1}}}{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + \gamma)_{i_k}} \right\} \right. \\
 &\times \left. \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + \gamma)_{i_{N-1}}}{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \quad (4.80)
 \end{aligned}$$

Eq(4.78), eq(4.79) and eq(4.80) are called as 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2(|\beta_i| + \frac{i}{2})$ where $i, |\beta_i| = 0, 1, 2, \dots$.

Again, put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = (-\frac{1}{2}\mu)^{1-\gamma} \frac{\Gamma(1-\frac{\Omega}{2\mu})}{\Gamma(2-\gamma)}$ in eq(4.75), eq(4.76) and eq(4.77). Also, replace $|\beta_i|$ by ϕ_i where $i = 0, 1, 2, \dots$.

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ where $\phi_0 = 0, 1, 2, \dots$

$$\begin{aligned} & RW_{\phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ &= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} \end{aligned} \quad (4.81)$$

(2) As $\omega = -2(\phi_1 + \frac{3}{2} - \gamma)$ where $\phi_1 = 0, 1, 2, \dots$

$$\begin{aligned} & RW_{\phi_1} \left(\phi_1 = -\frac{\omega}{2} - \frac{3}{2} + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ &= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{1}{2} + \phi_1))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right. \\ &\quad \left. \left. \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \end{aligned} \quad (4.82)$$

(3) As $\omega = -2(\phi_m + \frac{m}{2} + 1 - \gamma)$ where $\phi_m = 0, 1, 2, \dots$ only if $m \geq 2$

$$\begin{aligned}
 & RW_{\phi_m} \left(\phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right. \\
 &\times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} \\
 &+ \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right. \\
 &\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + \phi_m))}{(i_k + \frac{3}{2} + \frac{k}{2} - \gamma)(i_k + \frac{1}{2} + \frac{k}{2})} \right. \\
 &\times \left. \frac{(\frac{\Omega}{2\mu} + \frac{k}{2} + 1 - \gamma)_{i_k} (2 + \frac{k}{2} - \gamma)_{i_{k-1}} (1 + \frac{k}{2})_{i_{k-1}}}{(\frac{\Omega}{2\mu} + \frac{k}{2} + 1 - \gamma)_{i_{k-1}} (2 + \frac{k}{2} - \gamma)_{i_k} (1 + \frac{k}{2})_{i_k}} \right\} \\
 &\times \left. \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{N}{2} + 1 - \gamma)_{i_N} (2 + \frac{N}{2} - \gamma)_{i_{N-1}} (1 + \frac{N}{2})_{i_{N-1}}}{(\frac{\Omega}{2\mu} + \frac{N}{2} + 1 - \gamma)_{i_{N-1}} (2 + \frac{N}{2} - \gamma)_{i_N} (1 + \frac{N}{2})_{i_N}} z^{i_N} \right\} \tilde{\varepsilon}^N \right\} \quad (4.83)
 \end{aligned}$$

Eq(4.81), eq(4.82) and eq(4.83) are called as 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2(\phi_i + \frac{i}{2} + 1 - \gamma)$ where $i, \phi_i = 0, 1, 2, \dots$

4.0.6 Integral formalism of polynomial in which makes A_n term terminated

Let's see integral formalism about the polynomial case which makes A_n term terminated. First of all, replace the finite summation with an interval $[0, |\alpha_0|]$ by infinite summation with an interval $[0, \infty]$ in eq(4.44), eq(4.48), eq(4.54) and eq(4.55). Also, replace $-|\alpha_i|$ by $\frac{\Omega}{2\mu} + \frac{i}{2} + \frac{\lambda}{2}$ on it where $i = 0, 1, 2, \dots$. Then, we obtain the integral formalism of infinite series of function $y(x)$

The integral formalism of infinite series of x of zero, one, two and three term of A_n 's are

(1) Zero term of A_n 's

$$y_0(x) = c_0 x^\lambda \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \quad (4.84a)$$

(2) One term of A_n 's

$$\begin{aligned} y_1(x) &= c_0 x^\lambda \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} z(1-t_1)(1-u_1)}}{v_1^{-\left(\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2}\right) + 1} (1-v_1)} \\ &\times \left\{ w_{1,1} \partial_{w_{1,1}} + \left(\frac{\omega}{2} + \frac{\lambda}{2}\right) \right\} \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} w_{1,1}^{i_0} \end{aligned} \quad (4.84b)$$

(3) Two term of A_n 's

$$\begin{aligned}
y_2(x) &= c_0 x^\lambda \tilde{\varepsilon}^2 \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}z(1-t_2)(1-u_2)}}{v_2^{-\left(\frac{\Omega}{2\mu}+1+\frac{\lambda}{2}\right)+1}(1-v_2)} \\
&\quad \times \left\{ w_{2,2} \partial_{w_{2,2}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
&\quad \times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,2}(1-t_1)(1-u_1)}}{v_1^{-\left(\frac{\Omega}{2\mu}+\frac{1}{2}+\frac{\lambda}{2}\right)+1}(1-v_1)} \\
&\quad \times \left\{ w_{1,2} \partial_{w_{1,2}} + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} w_{1,2}^{i_0} \quad (4.84c)
\end{aligned}$$

(4) Three term of A_n 's

$$\begin{aligned}
y_3(x) &= c_0 x^\lambda \tilde{\varepsilon}^3 \int_0^1 dt_3 t_3^{\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma-\frac{1}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)}z(1-t_3)(1-u_3)}}{v_3^{-\left(\frac{\Omega}{2\mu}+\frac{3}{2}+\frac{\lambda}{2}\right)+1}(1-v_3)} \\
&\quad \times \left\{ w_{3,3} \partial_{w_{3,3}} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
&\quad \times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}w_{3,3}(1-t_2)(1-u_2)}}{v_2^{-\left(\frac{\Omega}{2\mu}+1+\frac{\lambda}{2}\right)+1}(1-v_2)} \\
&\quad \times \left\{ w_{2,3} \partial_{w_{2,3}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
&\quad \times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,3}(1-t_1)(1-u_1)}}{v_1^{-\left(\frac{\Omega}{2\mu}+\frac{1}{2}+\frac{\lambda}{2}\right)+1}(1-v_1)} \\
&\quad \times \left\{ w_{1,3} \partial_{w_{1,3}} + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{i_0=0}^{\infty} \frac{\left(\frac{\Omega}{2\mu} + \frac{\lambda}{2}\right)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} w_{1,3}^{i_0} \quad (4.84d)
\end{aligned}$$

According to eq(4.84a), eq(4.84b), eq(4.84c) and eq(4.84d), the general expression of power series of $y(x)$ for infinite series is

$$\begin{aligned}
 y(x) &= \sum_{m=0}^{\infty} y_m(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \dots \\
 &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 &\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1+\frac{\lambda}{2}} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2+\frac{\lambda}{2}} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-(\frac{\Omega}{2\mu} + \frac{n-j}{2} + \frac{\lambda}{2})+1} (1-v_{n-j})} \right. \\
 &\quad \left. \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1 + \omega + \lambda) \right\} \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,n}^{i_0} \right\} \right\} \tilde{\varepsilon}^n \right\}
 \end{aligned} \tag{4.85}$$

As we put $\omega = -2(|\beta_i| + \frac{i}{2} + \frac{\lambda}{2})$ where $i = 0, 1, 2, \dots$ in eq(4.85), we obtain

(1) As $\omega = -2(|\beta_0| + \frac{\lambda}{2})$ where $|\beta_0| = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \tag{4.86a}$$

(2) As $\omega = -2(|\beta_m| + \frac{m}{2} + \frac{\lambda}{2})$ where $|\beta_m| = 0, 1, 2, \dots$ only if $m \geq 1$

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 & + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1+\frac{\lambda}{2}} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2+\frac{\lambda}{2}} \right. \right. \\
 & \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-(\frac{\Omega}{2\mu} + \frac{n-j}{2} + \frac{\lambda}{2})+1} (1-v_{n-j})} \\
 & \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1-m) - |\beta_m| \right\} \\
 & \left. \left. \times \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,n}^{i_0} \right\} \right\} \tilde{\varepsilon}^n \left. \right\} \quad (4.86b)
 \end{aligned}$$

Put $\lambda = 0$ and $c_0 = \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)}$ in eq(4.86a) and eq(4.86b).

(1) As $\omega = -2|\beta_0|$ where $|\beta_0| = 0, 1, 2, \dots$

$$\begin{aligned}
 & QW_{|\beta_0|} \left(|\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & = \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0} (\gamma)_{i_0}} z^{i_0} = \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} M \left(\frac{\Omega}{2\mu}, \gamma, z \right) \quad (4.87)
 \end{aligned}$$

(2) As $\omega = -2(|\beta_m| + \frac{m}{2})$ where $|\beta_m| = 0, 1, 2, \dots$ only if $m \geq 1$

$$\begin{aligned}
 & QW_{|\beta_m|} \left(|\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ M \left(\frac{\Omega}{2\mu}, \gamma, z \right) + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma + \frac{1}{2}(n-j)-2} \right. \right. \right. \\
 &\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\left(\frac{\Omega}{2\mu} + \frac{n-j}{2}\right)+1} (1-v_{n-j})} \\
 &\times \left. \left. \left. \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} \left(n - j - 1 - m \right) - |\beta_m| \right\} M \left(\frac{\Omega}{2\mu}, \gamma, w_{1,n} \right) \right\} \right\} \tilde{\varepsilon}^n \right\} \quad (4.88)
 \end{aligned}$$

By using eq(4.63), eq(4.87) and eq(4.88) can be described as

(1) As $\omega = -2|\beta_0|$ where $|\beta_0| = 0, 1, 2, \dots$

$$\begin{aligned}
 & QW_{|\beta_0|} \left(|\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{1}{\Gamma\left(\frac{\Omega}{2\mu}\right)} \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{-\frac{\Omega}{2\mu}-1+\gamma} \quad (4.89)
 \end{aligned}$$

(2) As $\omega = -2 \left(|\beta_m| + \frac{m}{2} \right)$ where $|\beta_m| = 0, 1, 2, \dots$ only if $m \geq 1$

$$\begin{aligned}
 & QW_{|\beta_m|} \left(|\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{1}{\Gamma\left(\frac{\Omega}{2\mu}\right)} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{-\frac{\Omega}{2\mu}-1+\gamma} \right. \\
 &+ \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
 &\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\left(\frac{\Omega}{2\mu} + \frac{n-j}{2}\right)+1} (1-v_{n-j})} \\
 &\times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} \left(n-j-1-m \right) - |\beta_m| \right\} \\
 &\times \left. \left. \int_0^1 dv_0 e^{w_{1,n} v_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{-\frac{\Omega}{2\mu}-1+\gamma} \right\} \right\} \tilde{\varepsilon}^n \left. \right\} \quad (4.90)
 \end{aligned}$$

Eq(4.89) and eq(4.90) are called as the integral formalism of 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2 \left(|\beta_i| + \frac{i}{2} \right)$ where $i, |\beta_i| = 0, 1, 2, \dots$.

Again, put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = \left(-\frac{1}{2}\mu\right)^{1-\gamma} \frac{\Gamma\left(1-\frac{\Omega}{2\mu}\right)}{\Gamma(2-\gamma)}$ in eq(4.86a) and eq(4.86b). Also, replace $|\beta_i|$ by ϕ_i where $i = 0, 1, 2, \dots$.

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ where $\phi_0 = 0, 1, 2, \dots$

$$\begin{aligned}
& RW_{\phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} \\
&= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \tag{4.91}
\end{aligned}$$

(2) As $\omega = -2(\phi_m + \frac{m}{2} + 1 - \gamma)$ where $\phi_m = 0, 1, 2, \dots$ only if $m \geq 1$

$$\begin{aligned}
& RW_{\phi_m} \left(\phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, z \right) \right. \\
&+ \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \\
&\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-(\frac{\Omega}{2\mu} + \frac{n-j}{2} - \gamma)} (1 - v_{n-j})} \\
&\times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2}(n - j - 1 - m) - \phi_m \right\} \\
&\times \left. \left. \left. M \left(\frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, w_{1,n} \right) \right\} \right\} \tilde{\varepsilon}^n \right\} \tag{4.92}
\end{aligned}$$

By using eq(4.63), eq(4.91) and eq(4.92) can be described as

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ where $\phi_0 = 0, 1, 2, \dots$

$$\begin{aligned}
 & RW_{\phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{z^{1-\gamma}}{\Gamma(\frac{\Omega}{2\mu} + 1 - \gamma)} \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}}
 \end{aligned} \tag{4.93}$$

(2) As $\omega = -2(\phi_m + \frac{m}{2} + 1 - \gamma)$ where $\phi_m = 0, 1, 2, \dots$ only if $m \geq 1$

$$\begin{aligned}
 & RW_{\phi_m} \left(\phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{z^{1-\gamma}}{\Gamma(\frac{\Omega}{2\mu} + 1 - \gamma)} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \right. \\
 &+ \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j) - \gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j) - 1} \right. \right. \\
 &\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1, n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-(\frac{\Omega}{2\mu} + \frac{n-j}{2} - \gamma)} (1 - v_{n-j})} \\
 &\times \left\{ w_{n-j, n} \partial_{w_{n-j, n}} + \frac{1}{2} (n - j - 1 - m) - \phi_m \right\} \\
 &\left. \left. \left. \times \int_0^1 dv_0 e^{w_{1, n} v_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \right\} \right\} \tilde{\varepsilon}^n \left. \right\}
 \end{aligned} \tag{4.94}$$

Eq(4.93) and eq(4.94) are called the integral formalism of 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega =$

$-2(\phi_i + \frac{i}{2} + 1 - \gamma)$ where $i, \phi_i = 0, 1, 2, \dots$.

4.0.7 Power series of polynomial in which makes A_n and B_n term terminated

Finally, let's think about the polynomial case where A_n and B_n terms terminated at certain value of n. First of all, if $\frac{\Omega}{2\mu} + \frac{\lambda}{2} = -|\beta_0|$ where $|\beta_0| = 0, 1, 2, \dots$ in eq(4.12), eq(4.18), eq(4.24), eq(4.30) and eq(4.31), we obtain $y_1(x) = y_2(x) = y_3(x) = \dots = 0$. And the maximum value of $|\beta_0|$ should be equal to or greater than $|\alpha_0|$. If it doesn't, the function $y(x)$ can not be polynomial any more. We define this condition as $\text{Max}(|\beta_0|) \geq |\alpha_0|$ where $|\alpha_0|, |\beta_0| = 0, 1, 2, \dots$. Then, the analytic function $y(x)$ is

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad \text{where } \text{Max}(|\beta_0|) \geq |\alpha_0| \quad (4.95)$$

If $\frac{\Omega}{2\mu} + \frac{1}{2} + \frac{\lambda}{2} = -|\beta_1|$ where $|\beta_1| = 0, 1, 2, \dots$ in eq(4.12), eq(4.18), eq(4.24), eq(4.30) and eq(4.31), we obtain $y_2(x) = y_3(x) = y_4(x) = \dots = 0$. Then, the analytic function

$y(x)$ is

$$\begin{aligned}
y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
& + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{1}{2} + |\beta_1|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \left. \left. \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \quad \text{where } \text{Max}(|\beta_1|) \geq |\alpha_1|
\end{aligned} \tag{4.96}$$

If $\frac{\Omega}{2\mu} + 1 + \frac{\lambda}{2} = -|\beta_2|$ where $|\beta_2| = 0, 1, 2, \dots$ in eq(4.12), eq(4.18), eq(4.24), eq(4.30) and eq(4.31), we obtain $y_3(x) = y_4(x) = y_5(x) = \dots = 0$. Then, the analytic function

$y(x)$ is

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 & + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (1 + |\beta_2|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \left. \right\} \tilde{\varepsilon} \\
 & + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (1 + |\beta_2|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(i_1 + \frac{1}{2} - (1 + |\beta_2|))}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
 & \times \sum_{i_2=i_1}^{|\alpha_2|} \left\{ \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} z^{i_2} \right\} \left. \right\} \left. \right\} \tilde{\varepsilon}^2 \Big\} \\
 & \text{where } \text{Max}(|\beta_2|) \geq |\alpha_2| \tag{4.97}
 \end{aligned}$$

If $\frac{\Omega}{2\mu} + \frac{3}{2} + \frac{\lambda}{2} = -|\beta_3|$ where $|\beta_3| = 0, 1, 2, \dots$ in eq(4.12), eq(4.18), eq(4.24), eq(4.30) and eq(4.31), we obtain $y_4(x) = y_5(x) = y_6(x) = \dots = 0$. Then, the analytic function

$y(x)$ is

$$\begin{aligned}
y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
& + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{3}{2} + |\beta_3|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \left. \right\} \tilde{\varepsilon} \\
& + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{3}{2} + |\beta_3|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(i_1 + \frac{1}{2} - (\frac{3}{2} + |\beta_3|))}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \sum_{i_2=i_1}^{|\alpha_2|} \left\{ \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} z^{i_2} \right\} \left. \right\} \tilde{\varepsilon}^2 \\
& + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{3}{2} + |\beta_3|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
& \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(i_1 + \frac{1}{2} - (\frac{3}{2} + |\beta_3|))}{(i_1 + 1 + \frac{\lambda}{2})(i_1 + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} \right. \\
& \times \sum_{i_2=i_1}^{|\alpha_2|} \left\{ \frac{(i_2 + 1 - (\frac{3}{2} + |\beta_3|))}{(i_2 + \frac{3}{2} + \frac{\lambda}{2})(i_2 + \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_2|)_{i_2} (2 + \frac{\lambda}{2})_{i_1} (\gamma + 1 + \frac{\lambda}{2})_{i_1}}{(-|\alpha_2|)_{i_1} (2 + \frac{\lambda}{2})_{i_2} (\gamma + 1 + \frac{\lambda}{2})_{i_2}} \right. \\
& \times \sum_{i_3=i_2}^{|\alpha_3|} \left\{ \frac{(-|\alpha_3|)_{i_3} (\frac{5}{2} + \frac{\lambda}{2})_{i_2} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_2}}{(-|\alpha_3|)_{i_2} (\frac{5}{2} + \frac{\lambda}{2})_{i_3} (\gamma + \frac{3}{2} + \frac{\lambda}{2})_{i_3}} z^{i_3} \right\} \left. \right\} \left. \right\} \tilde{\varepsilon}^3 \left. \right\}
\end{aligned}$$

where $\text{Max}(|\beta_3|) \geq |\alpha_3|$ (4.98)

According to eq(4.95), eq(4.96), eq(4.97) and eq(4.98), the general expression of the function $y(x)$ for the polynomial in which makes A'_n s and B'_n s terms terminated at certain value is

(1) As $\omega = -2(|\beta_0| + \frac{\lambda}{2})$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$ where $|\alpha_0|, |\beta_0| = 0, 1, 2, \dots$

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad (4.99)$$

(2) As $\omega = -2(|\beta_1| + \frac{1}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$ where $|\alpha_1|, |\beta_1| = 0, 1, 2, \dots$

$$\begin{aligned} y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\ & + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{1}{2} + |\beta_1|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\ & \left. \left. \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \quad (4.100) \end{aligned}$$

(3) As $\omega = -2(|\beta_m| + \frac{m}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ where $|\alpha_m|, |\beta_m| = 0, 1, 2, \dots$

only if $m \geq 2$

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
 & + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \\
 & \times \sum_{i_1=i_0}^{|\alpha_1|} \left. \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2} + \frac{\lambda}{2})_{i_0} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2} + \frac{\lambda}{2})_{i_1} (\gamma + \frac{1}{2} + \frac{\lambda}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \\
 & + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2} + \frac{\lambda}{2})(i_0 - \frac{1}{2} + \gamma + \frac{\lambda}{2})} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} \right. \right. \\
 & \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{|\alpha_k|} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + |\beta_m|))}{(i_k + \frac{1}{2} + \frac{\lambda}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2} + \frac{\lambda}{2})} \right. \\
 & \left. \left. \frac{(-|\alpha_k|)_{i_k} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{k}{2} + \gamma + \frac{\lambda}{2})_{i_{k-1}}}{(-|\alpha_k|)_{i_{k-1}} (1 + \frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{k}{2} + \gamma + \frac{\lambda}{2})_{i_k}} \right\} \right\} \\
 & \times \left. \sum_{i_N=i_{N-1}}^{|\alpha_N|} \frac{(-|\alpha_N|)_{i_N} (1 + \frac{N}{2} + \frac{\lambda}{2})_{i_{N-1}} (\frac{N}{2} + \gamma + \frac{\lambda}{2})_{i_{N-1}}}{(-|\alpha_N|)_{i_{N-1}} (1 + \frac{N}{2} + \frac{\lambda}{2})_{i_N} (\frac{N}{2} + \gamma + \frac{\lambda}{2})_{i_N}} z^{i_N} \right\} \tilde{\varepsilon}^N \Big\} \quad (4.101)
 \end{aligned}$$

Put $\lambda = 0$ and $c_0 = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)}$ in eq(4.99), eq(4.100) and eq(4.101).

(1) As $\omega = -2|\beta_0|$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$\begin{aligned}
 & QW_{|\alpha_0|, |\beta_0|} \left(|\alpha_0| = -\frac{\Omega}{2\mu}, |\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0} (\gamma)_{i_0}} z^{i_0} \quad (4.102)
 \end{aligned}$$

(2) As $\omega = -2(|\beta_1| + \frac{1}{2})$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$

$$\begin{aligned}
 & QW_{|\alpha_i|,|\beta_1|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_1| = -\frac{1}{2}(1 + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{1}{2} + |\beta_1|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \left. \left. \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1}(\frac{3}{2})_{i_0}(\gamma + \frac{1}{2})_{i_0}}{(-|\alpha_1|)_{i_0}(\frac{3}{2})_{i_1}(\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \quad (4.103)
 \end{aligned}$$

(3) As $\omega = -2(|\beta_m| + \frac{m}{2})$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ only if $m \geq 2$

$$\begin{aligned}
 & QW_{|\alpha_i|,|\beta_m|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1}(\frac{3}{2})_{i_0}(\gamma + \frac{1}{2})_{i_0}}{(-|\alpha_1|)_{i_0}(\frac{3}{2})_{i_1}(\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{|\alpha_k|} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + |\beta_m|))}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(-|\alpha_k|)_{i_k}(1 + \frac{k}{2})_{i_{k-1}}(\frac{k}{2} + \gamma)_{i_{k-1}}}{(-|\alpha_k|)_{i_{k-1}}(1 + \frac{k}{2})_{i_k}(\frac{k}{2} + \gamma)_{i_k}} \right\} \\
 &\times \left. \left. \sum_{i_N=i_{N-1}}^{|\alpha_N|} \frac{(-|\alpha_N|)_{i_N}(1 + \frac{N}{2})_{i_{N-1}}(\frac{N}{2} + \gamma)_{i_{N-1}}}{(-|\alpha_N|)_{i_{N-1}}(1 + \frac{N}{2})_{i_N}(\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \tilde{\varepsilon}^N \right\} \quad (4.104)
 \end{aligned}$$

Eq(4.102), eq(4.103) and eq(4.104) are called the 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ and $\omega = -2(|\beta_j| + \frac{j}{2})$ where $|\alpha_i|, |\beta_j| = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$.

Put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = \left(-\frac{1}{2}\mu\right)^{1-\gamma} \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)}$ in eq(4.99), eq(4.100) and eq(4.101). Also, replace $|\alpha_i|$ and $|\beta_i|$ by ψ_i and ϕ_i where $i = 0, 1, 2, \dots$ in them.

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ and $\text{Max}(\phi_0) \geq \psi_0$

$$\begin{aligned} RW_{\psi_0, \phi_0} & \left(\psi_0 = -\frac{\Omega}{2\mu} - 1 + \gamma, \phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ & = z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} \end{aligned} \quad (4.105)$$

(2) As $\omega = -2(\phi_1 + \frac{3}{2} - \gamma)$ and $\text{Max}(\phi_1) \geq \psi_1$

$$\begin{aligned} RW_{\psi_i, \phi_1} & \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_1 = -\frac{\omega}{2} - \frac{3}{2} + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ & = z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} \right. \\ & \left. + \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 - (\frac{1}{2} + \phi_1))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \sum_{i_1=i_0}^{\psi_1} \left\{ \frac{(-\psi_1)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(-\psi_1)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\} \end{aligned} \quad (4.106)$$

(3) As $\omega = -2(\phi_m + \frac{m}{2} + 1 - \gamma)$ and $\text{Max}(\phi_m) \geq \psi_m$ only if $m \geq 2$

$$\begin{aligned}
 & RW_{\psi_i, \phi_m} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right. \\
 &\times \sum_{i_1=i_0}^{\psi_1} \left\{ \frac{(-\psi_1)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(-\psi_1)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} \\
 &+ \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right. \\
 &\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\psi_k} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + \phi_m))}{(i_k + \frac{3}{2} - \gamma + \frac{k}{2})(i_k + \frac{1}{2} + \frac{k}{2})} \frac{(-\psi_k)_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + 2 - \gamma)_{i_{k-1}}}{(-\psi_k)_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + 2 - \gamma)_{i_k}} \right\} \\
 &\times \left. \left. \sum_{i_N=i_{N-1}}^{\psi_N} \frac{(-\psi_N)_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + 2 - \gamma)_{i_{N-1}}}{(-\psi_N)_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + 2 - \gamma)_{i_N}} z^{i_N} \right\} \tilde{\varepsilon}^N \right\} \quad (4.107)
 \end{aligned}$$

Eq(4.105), eq(4.106) and eq(4.107) are called as 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu(\psi_i + \frac{i}{2} + 1 - \gamma)$ and $\omega = -2(\phi_j + \frac{j}{2} + 1 - \gamma)$ where $\psi_i, \phi_j = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$.

4.0.8 Integral formalism of polynomial in which makes A_n and B_n term terminated

Let's see integral formalism about the polynomial case which makes A_n and B_n terms terminated. First of all, as we put $\omega = -2(|\beta_0| + \frac{\lambda}{2})$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$ in eq(4.55),

we obtain $y_1(x) = y_2(x) = y_3(x) = \dots = 0$. And the analytic function $y(x)$ is

$$y(x) = c_0 x^\lambda \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad (4.108)$$

As we put $\omega = -2(|\beta_1| + \frac{1}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$ in eq(4.55), we obtain $y_2(x) = y_3(x) = y_4(x) = \dots = 0$. And the analytic function $y(x)$ is

$$\begin{aligned} y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\ & + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{zv_1}{(1-v_1)}(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1}(1-v_1)} \\ & \left. \times \left\{ w_{1,1} \partial_{w_{1,1}} - \left(\frac{1}{2} + |\beta_1| \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,1}^{i_0} \right\} \quad (4.109) \end{aligned}$$

As we put $\omega = -2(|\beta_2| + 1 + \frac{\lambda}{2})$ and $\text{Max}(|\beta_2|) \geq |\alpha_2|$ in eq(4.55), we obtain $y_3(x) = y_4(x) = y_5(x) = \dots = 0$. And the analytic function $y(x)$ is

$$\begin{aligned}
 y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \right. \\
 & \times \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{zv_1}{(1-v_1)}(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1}(1-v_1)} \\
 & \times \left\{ w_{1,1} \partial_{w_{1,1}} - (1 + |\beta_2|) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,1}^{i_0} \\
 & + \tilde{\varepsilon}^2 \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma - 1 + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}z(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} \\
 & \times \left\{ w_{2,2} \partial_{w_{2,2}} - \left(\frac{1}{2} + |\beta_2| \right) \right\} \\
 & \times \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,2}(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1}(1-v_1)} \\
 & \left. \times \left\{ w_{1,2} \partial_{w_{1,2}} - (1 + |\beta_2|) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0} \right\} \quad (4.110)
 \end{aligned}$$

As we put $\omega = -2(|\beta_3| + \frac{3}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_3|) \geq |\alpha_3|$ in eq(4.55), we obtain $y_4(x) = y_5(x) = y_6(x) = \dots = 0$. And the analytic function $y(x)$ is

$$\begin{aligned}
y(x) = & c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
& + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{zv_1}{(1-v_1)}(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1}(1-v_1)} \\
& \times \left\{ w_{1,1} \partial_{w_{1,1}} - \left(\frac{3}{2} + |\beta_3| \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,1}^{i_0} \\
& + \tilde{\varepsilon}^2 \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}z(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} \left\{ w_{2,2} \partial_{w_{2,2}} - (1 + |\beta_3|) \right\} \\
& \times \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,2}(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1}(1-v_1)} \\
& \times \left\{ w_{1,2} \partial_{w_{1,2}} - \left(\frac{3}{2} + |\beta_3| \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0} \\
& + \tilde{\varepsilon}^3 \int_0^1 dt_3 t_3^{\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma - \frac{1}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)}z(1-t_3)(1-u_3)}}{v_3^{|\alpha_3|+1}(1-v_3)} \\
& \times \left\{ w_{3,3} \partial_{w_{3,3}} - \left(\frac{1}{2} + |\beta_3| \right) \right\} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \\
& \times \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}w_{3,3}(1-t_2)(1-u_2)}}{v_2^{|\alpha_2|+1}(1-v_2)} \left\{ w_{2,3} \partial_{w_{2,3}} - (1 + |\beta_3|) \right\} \\
& \times \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,3}(1-t_1)(1-u_1)}}{v_1^{|\alpha_1|+1}(1-v_1)} \\
& \times \left\{ w_{1,3} \partial_{w_{1,3}} - \left(\frac{3}{2} + |\beta_3| \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,3}^{i_0} \left. \right\} \tag{4.111}
\end{aligned}$$

As we put $\omega = -2(|\beta_m| + \frac{m}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ only if $m \geq 1$ in eq(4.55), we obtain

$$\begin{aligned}
y(x) &= c_0 x^\lambda \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
&\quad + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1+\frac{\lambda}{2}} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2+\frac{\lambda}{2}} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1} (1-v_{n-j})} \\
&\quad \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1-m) - |\beta_m| \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} w_{1,n}^{i_0} \right\} \right\} \tilde{\varepsilon}^n \left. \right\} \quad (4.112)
\end{aligned}$$

Put $\lambda = 0$ and $c_0 = \frac{\Gamma(|\alpha_0|+\gamma)}{\Gamma(\gamma)}$ in eq(4.108) and eq(4.112) by using eq(4.40).

1) As $\omega = -2|\beta_0|$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$\begin{aligned}
&QW_{|\alpha_0|,|\beta_0|} \left(|\alpha_0| = -\frac{\Omega}{2\mu}, |\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1+\nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= \frac{\Gamma(|\alpha_0|+\gamma)}{\Gamma(\gamma)} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0} (\gamma)_{i_0}} z^{i_0} = \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1} (1-v_0)^\gamma} \quad (4.113)
\end{aligned}$$

(2) As $\omega = -2(|\beta_m| + \frac{m}{2})$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ only if $m \geq 1$

$$\begin{aligned}
& QW_{|\alpha_i|, |\beta_m|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1} (1-v_0)^\gamma} + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
&\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1, n} (1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1} (1-v_{n-j})} \\
&\times \left. \left. \left\{ w_{n-j, n} \partial_{w_{n-j, n}} + \frac{1}{2}(n-j-1-m) - |\beta_m| \right\} \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1, n} v_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1} (1-v_0)^\gamma} \right\} \right\} \varepsilon^n \quad (4.114)
\end{aligned}$$

Eq(4.113) and eq(4.114) are called the integral formalism of 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ and $\omega = -2(|\beta_j| + \frac{j}{2})$ where $|\alpha_i|, |\beta_j| = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$.

Put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = (-\frac{1}{2}\mu)^{1-\gamma} \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)}$ in eq(4.108) and eq(4.112) by using eq(4.60). Also, replace $|\alpha_i|$ and $|\beta_i|$ by ψ_i and ϕ_i where $i = 0, 1, 2, \dots$ in them.

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ and $\text{Max}(\phi_0) \geq \psi_0$

$$\begin{aligned}
& RW_{\psi_0, \phi_0} \left(\psi_0 = -\frac{\Omega}{2\mu} - 1 + \gamma, \phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2-\gamma)_{i_0} (1)_{i_0}} z^{i_0} = z^{1-\gamma} \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1} (1-v_0)^{2-\gamma}} \quad (4.115)
\end{aligned}$$

(2) As $\omega = -2\left(\phi_m + \frac{m}{2} + 1 - \gamma\right)$ and $\text{Max}(\phi_m) \geq \psi_m$ only if $m \geq 1$

$$\begin{aligned}
& RW_{\psi_i, \phi_m} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \left\{ \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \right. \\
&\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1, n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{\psi_{n-j}+1}(1-v_{n-j})} \left. \left. \left\{ w_{n-j, n} \partial_{w_{n-j, n}} + \frac{1}{2}(n-j-1-m) - \phi_m \right\} \right. \right. \\
&\times \left. \left. \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1, n} v_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \quad (4.116)
\end{aligned}$$

Eq(4.115) and eq(4.116) are called the integral formalism of 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as

$\Omega = -2\mu\left(\psi_i + \frac{i}{2} + 1 - \gamma\right)$ and $\omega = -2\left(\phi_j + \frac{j}{2} + 1 - \gamma\right)$ where $\psi_i, \phi_j = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$. The boundary condition for x in eq(4.87), eq(4.88), eq(4.91), eq(4.92), eq(4.113), eq(4.114), eq(4.115) and eq(4.116) is that $|z|, |\tilde{\varepsilon}| < \infty$. Because the maximum value of index n is m. So $\tilde{\varepsilon}$ will terminate at certain value of m.

4.0.9 Generating function of polynomial in which makes B_n term terminated

Now, let's investigate the generating function of the first and second kind of grand confluent hypergeometric functions. Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ on eq(4.55) where $|s_i| < 1$ as $i = 0, 1, 2, \dots$.

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y(x) \tag{4.117} \\ &= \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} \{y_0(x) + y_1(x) + y_2(x) + y_3(x) + \dots\} \end{aligned}$$

We define

$$\begin{cases} s_{a,b} = s_a \cdot s_{a+1} \cdot s_{a+2} \cdots s_{b-2} \cdot s_{b-1} \cdot s_b \\ w_{a,b}^* = z s_{a,\infty} \prod_{l=a}^b t_l u_l \end{cases} \tag{4.118}$$

And, we have

$$\sum_{|\alpha_i|=|\alpha_j|}^{\infty} s_i^{|\alpha_i|} = \frac{s_i^{|\alpha_j|}}{(1 - s_i)} \tag{4.119}$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ on eq(4.12) by using eq(4.118) and eq(4.119).

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_0(x) \\ &= \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,\infty}^{|\alpha_0|}}{(|\alpha_0|)!} \left\{ c_0 x^\lambda \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} z^{i_0} \right\} \end{aligned} \quad (4.120)$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ on eq(4.44) by using eq(4.118) and eq(4.119).

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_1(x) \\ &= c_0 x^\lambda \tilde{\varepsilon} \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}z(1-t_1)(1-u_1)}}{(1-v_1)(v_1-s_{1,\infty})} \\ & \times \left\{ w_{1,1} \partial_{w_{1,1}} + \left(\frac{\omega}{2} + \frac{\lambda}{2} \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{1}{(|\alpha_0|)!} \left(\frac{s_{0,\infty}}{v_1} \right)^{|\alpha_0|} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} w_{1,1}^{i_0} \end{aligned} \quad (4.121)$$

Let $v_1 = s_{1,\infty}$ in eq(4.121).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_1(x) \\
&= c_0 x^\lambda \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \\
&\times \left\{ w_{1,1}^* \partial_{w_{1,1}^*} + \left(\frac{\omega}{2} + \frac{\lambda}{2} \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} (w_{1,1}^*)^{i_0} \quad (4.122)
\end{aligned}$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ on eq(4.48) by using eq(4.118) and eq(4.119).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_2(x) \\
&= c_0 x^\lambda \tilde{\varepsilon}^2 \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \\
&\times \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)} z(1-t_2)(1-u_2)}}{(1-v_2)(v_2-s_{2,\infty})} \left\{ w_{2,2} \partial_{w_{2,2}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
&\times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,2}(1-t_1)(1-u_1)}}{v_1(1-v_1)} \quad (4.123) \\
&\times \sum_{|\alpha_1|=|\alpha_0|}^{\infty} \left(\frac{s_{1,\infty}}{v_1 v_2} \right)^{|\alpha_1|} \left\{ w_{1,2} \partial_{w_{1,2}} + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0}
\end{aligned}$$

Let $v_2 = s_{2,\infty}$ in eq(4.123).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_2(x) \\
&= c_0 x^\lambda \varepsilon^2 \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \\
&\times e^{-\frac{s_{2,\infty}}{(1-s_{2,\infty})} z(1-t_2)(1-u_2)} \left\{ w_{2,2}^* \partial w_{2,2}^* + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
&\times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,2}^*(1-t_1)(1-u_1)}}{(1-v_1)(v_1-s_1)} \\
&\times \left\{ w_{1,2} \partial w_{1,2} + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{1}{(|\alpha_0|)!} \left(\frac{s_{0,1}}{v_1} \right)^{|\alpha_0|} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0}
\end{aligned} \tag{4.124}$$

Let $v_1 = s_1$ in eq(4.124).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_2(x) \\
&= c_0 x^\lambda \varepsilon^2 \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \\
&\times e^{-\frac{s_{2,\infty}}{(1-s_{2,\infty})} z(1-t_2)(1-u_2)} \left\{ w_{2,2}^* \partial w_{2,2}^* + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \\
&\times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{e^{-\frac{s_1}{(1-s_1)} w_{2,2}^*(1-t_1)(1-u_1)}}{(1-s_1)} \\
&\times \left\{ w_{1,2}^* \partial w_{1,2}^* + \left(\frac{\lambda}{2} + \frac{\omega}{2} \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} (w_{1,2}^*)^{i_0}
\end{aligned} \tag{4.125}$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ on eq(4.54) by using eq(4.118) and eq(4.119).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_3(x) \\
&= c_0 x^\lambda \tilde{\varepsilon}^3 \prod_{k=4}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_3 t_3^{\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma-\frac{1}{2}+\frac{\lambda}{2}} \\
&\quad \times \frac{1}{2\pi i} \oint dv_3 \frac{e^{-\frac{v_3}{(1-v_3)}z(1-t_3)(1-u_3)}}{(1-v_3)(v_3-s_{3,\infty})} \left\{ w_{3,3} \partial_{w_{3,3}} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\quad \times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)}w_{3,3}(1-t_2)(1-u_2)}}{v_2(1-v_2)} \\
&\quad \times \sum_{|\alpha_2|=|\alpha_1|}^{\infty} \left(\frac{s_{2,\infty}}{v_2 v_3} \right)^{|\alpha_2|} \left\{ w_{2,3} \partial_{w_{2,3}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\quad \times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,3}(1-t_1)(1-u_1)}}{v_1(1-v_1)} \\
&\quad \times \sum_{|\alpha_1|=|\alpha_0|}^{\infty} \left(\frac{s_1}{v_1} \right)^{|\alpha_1|} \left\{ w_{1,3} \partial_{w_{1,3}} + \left(\frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\quad \times \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} w_{1,3}^{i_0}
\end{aligned} \tag{4.126}$$

Let $v_3 = s_{3,\infty}$ in eq(4.126).

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|} s_n^{|\alpha_n|} \right\} y_3(x) \\
 &= c_0 x^\lambda \tilde{\varepsilon}^3 \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_3 t_3^{\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma-\frac{1}{2}+\frac{\lambda}{2}} \\
 & \times e^{-\frac{s_{3,\infty}}{(1-s_{3,\infty})} z(1-t_3)(1-u_3)} \left\{ w_{3,3}^* \partial_{w_{3,3}^*} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
 & \times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_2 \frac{e^{-\frac{v_2}{(1-v_2)} w_{3,3}^*(1-t_2)(1-u_2)}}{(1-v_2)(v_2-s_2)} \left\{ w_{2,3} \partial_{w_{2,3}} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
 & \times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)} w_{2,3}(1-t_1)(1-u_1)}}{v_1(1-v_1)} \tag{4.127} \\
 & \times \sum_{|\alpha_1|=|\alpha_0|}^{\infty} \left(\frac{s_{1,2}}{v_1 v_2} \right)^{|\alpha_1|} \left\{ w_{1,3} \partial_{w_{1,3}} + \left(\frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} w_{1,3}^{i_0}
 \end{aligned}$$

Let $v_2 = s_2$ in eq(4.127).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_3(x) \\
&= c_0 x^\lambda \varepsilon^3 \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_3 t_3^{\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma-\frac{1}{2}+\frac{\lambda}{2}} \\
&\times e^{-\frac{s_{3,\infty}}{(1-s_{3,\infty})}z(1-t_3)(1-u_3)} \left\{ w_{3,3}^* \partial_{w_{3,3}^*} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{e^{-\frac{s_2}{(1-s_2)}w_{3,3}^*(1-t_2)(1-u_2)}}{(1-s_2)} \left\{ w_{2,3}^* \partial_{w_{2,3}^*} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{1}{2\pi i} \oint dv_1 \frac{e^{-\frac{v_1}{(1-v_1)}w_{2,3}^*(1-t_1)(1-u_1)}}{(1-v_1)(v_1-s_1)} \quad (4.128) \\
&\times \left\{ w_{1,3} \partial_{w_{1,3}} + \left(\frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{1}{(|\alpha_0|)!} \left(\frac{s_{0,1}}{v_1}\right)^{|\alpha_0|} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0}(\gamma+\frac{\lambda}{2})_{i_0}} w_{1,3}^{i_0}
\end{aligned}$$

Let $v_1 = s_1$ in eq(4.128).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y_3(x) \\
&= c_0 x^\lambda \varepsilon^3 \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_3 t_3^{\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma-\frac{1}{2}+\frac{\lambda}{2}} \\
&\times e^{-\frac{s_{3,\infty}}{(1-s_{3,\infty})}z(1-t_3)(1-u_3)} \left\{ w_{3,3}^* \partial_{w_{3,3}^*} + \left(1 + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\times \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} \frac{e^{-\frac{s_2}{(1-s_2)}w_{3,3}^*(1-t_2)(1-u_2)}}{(1-s_2)} \left\{ w_{2,3}^* \partial_{w_{2,3}^*} + \left(\frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \\
&\times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} \frac{e^{-\frac{s_1}{(1-s_1)}w_{2,3}^*(1-t_1)(1-u_1)}}{(1-s_1)} \\
&\times \left\{ w_{1,3}^* \partial_{w_{1,3}^*} + \left(\frac{\lambda}{2} + \frac{\omega}{2}\right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,3}^*)^{i_0} \quad (4.129)
\end{aligned}$$

Substitute eq(4.120), eq(4.122), eq(4.125) and eq(4.129) into eq(4.117).

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y(x) \\
&= c_0 x^\lambda \left\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,\infty}^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
&+ \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \\
&\times \left\{ w_{1,1}^* \partial_{w_{1,1}^*} + \left(\frac{\omega}{2} + \frac{\lambda}{2} \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} (w_{1,1}^*)^{i_0} \\
&+ \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n t_n^{\frac{n}{2}-1+\frac{\lambda}{2}} \int_0^1 du_n u_n^{\gamma-2+\frac{n}{2}+\frac{\lambda}{2}} e^{-\frac{s_{n,\infty}}{(1-s_{n,\infty})} z(1-t_n)(1-u_n)} \right. \\
&\times \left\{ w_{n,n}^* \partial_{w_{n,n}^*} + \left(\frac{1}{2}(n-1) + \frac{\omega}{2} + \frac{\lambda}{2} \right) \right\} \\
&\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1+\frac{\lambda}{2}} \int_0^1 du_{n-j} u_{n-j}^{\gamma-2+\frac{1}{2}(n-j)+\frac{\lambda}{2}} e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n+1-j,n}^* (1-t_{n-j})(1-u_{n-j})} \right. \\
&\times \left\{ w_{n-j,n}^* \partial_{w_{n-j,n}^*} + \left(\frac{1}{2}(n-j-1) + \frac{\omega}{2} + \frac{\lambda}{2} \right) \right\} \\
&\times \left. \left. \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} (w_{1,n}^*)^{i_0} \right\} \right\} \tilde{\varepsilon}^n \quad (4.130)
\end{aligned}$$

Let $\lambda = 0$ and $c_0 = \frac{\Gamma(\gamma+|\alpha_0|)}{\Gamma(\gamma)}$ in eq(4.130).

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|} s_n^{|\alpha_n|} \right\} QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,\infty}^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(\gamma; z) \\
 &+ \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}} e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \left\{ w_{1,1}^* \partial_{w_{1,1}^*} + \frac{\omega}{2} \right\} \\
 &\times \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(\gamma; w_{1,1}^*) \\
 &+ \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n t_n^{\frac{n}{2}-1} \int_0^1 du_n u_n^{\gamma-2+\frac{n}{2}} e^{-\frac{s_{n,\infty}}{(1-s_{n,\infty})} z(1-t_n)(1-u_n)} \right. \\
 &\times \left. \left\{ w_{n,n}^* \partial_{w_{n,n}^*} + \left(\frac{1}{2}(n-1) + \frac{\omega}{2} \right) \right\} \right. \\
 &\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma-2+\frac{1}{2}(n-j)} e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n+1-j,n}^* (1-t_{n-j})(1-u_{n-j})} \right. \\
 &\times \left. \left. \left\{ w_{n-j,n}^* \partial_{w_{n-j,n}^*} + \left(\frac{1}{2}(n-j-1) + \frac{\omega}{2} \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(\gamma; w_{1,n}^*) \right\} \right\} \tilde{\varepsilon}^n \\
 &\text{where } \gamma = \frac{1}{2}(1+\nu) \tag{4.131}
 \end{aligned}$$

The generating function of the first kind of confluent hypergeometric function is

$$\frac{e^{-\frac{zt}{(1-t)}}}{(1-t)^\gamma} = \sum_{|\alpha_0|=0}^{\infty} \frac{t^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(\gamma; z) \tag{4.132}$$

Apply eq(4.132) into eq(4.131).

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \frac{e^{-\frac{zs_{0,\infty}}{(1-s_{0,\infty})}}}{(1-s_{0,\infty})^\gamma} \\
 &+ \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}} e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \\
 &\times \left\{ w_{1,1}^* \partial_{w_{1,1}^*} + \frac{\omega}{2} \right\} \frac{e^{-\frac{w_{1,1}^* s_0}{(1-s_0)}}}{(1-s_0)^\gamma} + \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n t_n^{\frac{n}{2}-1} \int_0^1 du_n u_n^{\gamma-2+\frac{n}{2}} \right. \\
 &\times e^{-\frac{s_{n,\infty}}{(1-s_{n,\infty})} z(1-t_n)(1-u_n)} \left\{ w_{n,n}^* \partial_{w_{n,n}^*} + \left(\frac{(n-1)}{2} + \frac{\omega}{2} \right) \right\} \\
 &\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma-2+\frac{1}{2}(n-j)} \frac{e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n+1-j,n}^* (1-t_{n-j})(1-u_{n-j})}}{(1-s_{n-j})} \right. \\
 &\times \left. \left. \left\{ w_{n-j,n}^* \partial_{w_{n-j,n}^*} + \left(\frac{1}{2}(n-j-1) + \frac{\omega}{2} \right) \right\} \frac{e^{-\frac{w_{1,n}^* s_0}{(1-s_0)}}}{(1-s_0)^\gamma} \right\} \right\} \tilde{\varepsilon}^n \\
 &\text{where } \gamma = \frac{1}{2}(1+\nu)
 \end{aligned} \tag{4.133}$$

Eq(4.133) is called the generating function of the first kind of grand confluent hypergeometric function as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ where $i, |\alpha_i| = 0, 1, 2, \dots$.

Replace $|\alpha_0|$, $|\alpha_{n-1}|$ and $|\alpha_n|$ by ψ_0 , ψ_{n-1} and ψ_n in eq(4.130), also, put $\lambda =$

is

$$\frac{e^{-\frac{zt}{(1-t)}}}{(1-t)^{2-\gamma}} = \sum_{\psi_0=0}^{\infty} \frac{t^{\psi_0}}{\psi_0!} A_{\psi_0}(\gamma; z) \quad (4.135)$$

Apply eq(4.135) into eq(4.134).

$$\begin{aligned} & \sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\psi_n=\psi_{n-1}}^{\infty} s_n^{\psi_n} \right\} RW_{\psi_i} \left(\psi_i = -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ &= z^{1-\gamma} \left\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \frac{e^{-\frac{zs_{0,\infty}}{(1-s_{0,\infty})}}}{(1-s_{0,\infty})^{2-\gamma}} + \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{\frac{1}{2}-\gamma} \int_0^1 du_1 u_1^{-\frac{1}{2}} \right. \\ & \times e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \left\{ w_{1,1}^* \partial_{w_{1,1}^*} + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} \frac{e^{-\frac{w_{1,1}^* s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \\ & + \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n t_n^{\frac{n}{2}-\gamma} \int_0^1 du_n u_n^{-1+\frac{n}{2}} e^{-\frac{s_{n,\infty}}{(1-s_{n,\infty})} z(1-t_n)(1-u_n)} \right. \\ & \times \left\{ w_{n,n}^* \partial_{w_{n,n}^*} + \left(\frac{1}{2}(n+1+\omega) - \gamma \right) \right\} \\ & \times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \frac{e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n+1-j,n}^*(1-t_{n-j})(1-u_{n-j})}}{(1-s_{n-j})} \right. \\ & \left. \left. \times \left\{ w_{n-j,n}^* \partial_{w_{n-j,n}^*} + \left(\frac{1}{2}(n-j+1+\omega) - \gamma \right) \right\} \frac{e^{-\frac{w_{1,n}^* s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \left. \right\} \\ & \text{where } \gamma = \frac{1}{2}(1+\nu) \end{aligned} \quad (4.136)$$

Eq(4.136) is called the generating function of the second kind of grand confluent hypergeometric function as $\Omega = -2\mu(\psi_i + 1 - \gamma + \frac{i}{2})$ where $i, \psi_i = 0, 1, 2, \dots$.

4.0.10 Generating function of polynomial in which makes A_n and B_n terms terminated

Again, let's investigate the generating functions for the case as A_n and B_n terms are terminated First of all, We define that

$$\left\{ \begin{array}{l} s_{i,j} = s_i \cdot s_{i+1} \cdot s_{i+2} \cdots s_{j-2} \cdot s_{j-1} \cdot s_j \\ \\ = s_i \text{ only if } i = j \\ \\ w_{i,j}^\dagger = z \prod_{l=i}^j t_l u_l s_l \text{ only if } j = m \\ \\ = z s_{i,m} \prod_{l=i}^j t_l u_l \text{ only if } j < m \end{array} \right. \quad (4.137)$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!}$ in eq(4.108).

(1) As $\omega = -2(|\beta_0| + \frac{\lambda}{2})$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} y(x) = c_0 x^\lambda \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad (4.138)$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} s_1^{|\alpha_1|}$ in eq(4.109).

(2) As $\omega = -2 \left(|\beta_1| + \frac{1}{2} + \frac{\lambda}{2} \right)$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} s_1^{|\alpha_1|} y(x) \\
 &= c_0 x^\lambda \left\{ \frac{1}{(1-s_1)} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,1}^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \right. \\
 &+ \tilde{\varepsilon} \frac{1}{(1-s_1)} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} e^{-\frac{zs_1}{(1-s_1)}(1-t_1)(1-u_1)} \\
 &\left. \times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{1}{2} + |\beta_1| \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,1}^\dagger)^{i_0} \right\} \quad (4.139)
 \end{aligned}$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^2 \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ in eq(4.110).

(3) As $\omega = -2(|\beta_2| + 1 + \frac{\lambda}{2})$ and $\text{Max}(|\beta_2|) \geq |\alpha_2|$

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^2 \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y(x) \\
&= c_0 x^\lambda \left\{ \prod_{k=1}^2 \frac{1}{(1-s_{k,2})} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,2}^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\
&+ \tilde{\varepsilon} \prod_{k=1}^2 \frac{1}{(1-s_{k,2})} \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} e^{-\frac{zs_{1,2}}{(1-s_{1,2})}(1-t_1)(1-u_1)} \\
&\times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - (1+|\beta_2|) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} (w_{1,1}^\dagger)^{i_0} \\
&+ \tilde{\varepsilon}^2 \prod_{k=1}^2 \frac{1}{(1-s_k)} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1+\frac{\lambda}{2}} e^{-\frac{zs_2}{(1-s_2)}(1-t_2)(1-u_2)} \left\{ w_{2,2}^\dagger \partial_{w_{2,2}^\dagger} - \left(\frac{1}{2} + |\beta_2|\right) \right\} \\
&\times \int_0^1 dt_1 t_1^{-\frac{1}{2}+\frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}+\frac{\lambda}{2}} e^{-\frac{s_1}{(1-s_1)} w_{2,2}^\dagger (1-t_1)(1-u_1)} \left\{ w_{1,2}^\dagger \partial_{w_{1,2}^\dagger} - (1+|\beta_2|) \right\} \\
&\times \left. \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1+\frac{\lambda}{2})_{i_0} (\gamma+\frac{\lambda}{2})_{i_0}} (w_{1,2}^\dagger)^{i_0} \right\} \tag{4.140}
\end{aligned}$$

Multiply $\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^3 \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\}$ in eq(4.111).

(4) As $\omega = -2 \left(|\beta_3| + \frac{3}{2} + \frac{\lambda}{2} \right)$ and $\text{Max}(|\beta_3|) \geq |\alpha_3|$

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^3 \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} y(x) \\
&= c_0 x^\lambda \left\{ \prod_{k=1}^3 \frac{1}{(1-s_{k,3})} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,3}^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \right. \\
&+ \tilde{\varepsilon} \prod_{k=1}^3 \frac{1}{(1-s_{k,3})} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} e^{-\frac{zs_{1,3}}{(1-s_{1,3})}(1-t_1)(1-u_1)} \\
&\times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{3}{2} + |\beta_3| \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,1}^\dagger)^{i_0} \\
&+ \tilde{\varepsilon}^2 \frac{1}{(1-s_1)} \prod_{k=1}^3 \frac{1}{(1-s_{k,3})} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1 + \frac{\lambda}{2}} e^{-\frac{zs_{2,3}}{(1-s_{2,3})}(1-t_2)(1-u_2)} \\
&\times \left\{ w_{2,2}^\dagger \partial_{w_{2,2}^\dagger} - \left(1 + |\beta_3| \right) \right\} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} e^{-\frac{s_1}{(1-s_1)} w_{2,2}^\dagger (1-t_1)(1-u_1)} \\
&\times \left\{ w_{1,2}^\dagger \partial_{w_{1,2}^\dagger} - \left(\frac{3}{2} + |\beta_3| \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,2}^\dagger)^{i_0} \\
&+ \tilde{\varepsilon}^3 \prod_{k=1}^3 \frac{1}{(1-s_k)} \int_0^1 dt_3 t_3^{\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_3 u_3^{\gamma - \frac{1}{2} + \frac{\lambda}{2}} e^{-\frac{s_3}{(1-s_3)} z(1-t_3)(1-u_3)} \\
&\times \left\{ w_{3,3}^\dagger \partial_{w_{3,3}^\dagger} - \left(\frac{1}{2} + |\beta_3| \right) \right\} \int_0^1 dt_2 t_2^{\frac{\lambda}{2}} \int_0^1 du_2 u_2^{\gamma-1 + \frac{\lambda}{2}} e^{-\frac{s_2}{(1-s_2)} w_{3,3}^\dagger (1-t_2)(1-u_2)} \\
&\times \left\{ w_{2,3}^\dagger \partial_{w_{2,3}^\dagger} - \left(1 + |\beta_3| \right) \right\} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} e^{-\frac{s_1}{(1-s_1)} w_{2,3}^\dagger (1-t_1)(1-u_1)} \\
&\times \left. \left\{ w_{1,3}^\dagger \partial_{w_{1,3}^\dagger} - \left(\frac{3}{2} + |\beta_3| \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,3}^\dagger)^{i_0} \right\} \quad (4.141)
\end{aligned}$$

According to eq(4.138), eq(4.139), eq(4.140) and eq(4.141), the general expression of the generating functions as A_n and B_n are terminated is

(1) As $\omega = -2(|\beta_0| + \frac{\lambda}{2})$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} y(x) = c_0 x^\lambda \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \quad (4.142)$$

(2) As $\omega = -2(|\beta_1| + \frac{1}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} s_1^{|\alpha_1|} y(x) \\ &= c_0 x^\lambda \left\{ \frac{1}{(1-s_1)} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,1}^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} z^{i_0} \right. \\ &+ \tilde{\varepsilon} \frac{1}{(1-s_1)} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} e^{-\frac{zs_1}{(1-s_1)}(1-t_1)(1-u_1)} \\ &\left. \times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{1}{2} + |\beta_1| \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1 + \frac{\lambda}{2})_{i_0} (\gamma + \frac{\lambda}{2})_{i_0}} (w_{1,1}^\dagger)^{i_0} \right\} \end{aligned} \quad (4.143)$$

(3) As $\omega = -2 \left(|\beta_m| + \frac{m}{2} + \frac{\lambda}{2} \right)$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ only if $m \geq 2$

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^m \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|} s_n^{|\alpha_n|} \right\} y(x) \\
 &= c_0 x^\lambda \left\{ \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \sum_{|\alpha_0|=0}^{\infty} \frac{s_{0,m}^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} z^{i_0} \right. \\
 &+ \tilde{\varepsilon} \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \int_0^1 dt_1 t_1^{-\frac{1}{2} + \frac{\lambda}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2} + \frac{\lambda}{2}} e^{-\frac{zs_{1,m}}{(1-s_{1,m})}(1-t_1)(1-u_1)} \\
 &\times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{m}{2} + |\beta_m| \right) \right\} \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,1}^\dagger)^{i_0} \\
 &+ \sum_{n=2}^m \left\{ \prod_{k=1}^{n-1} \frac{1}{(1-s_k)} \prod_{l=n}^m \frac{1}{(1-s_{l,m})} \int_0^1 dt_n t_n^{\frac{n}{2}-1 + \frac{\lambda}{2}} \int_0^1 du_n u_n^{\gamma + \frac{n}{2} - 2 + \frac{\lambda}{2}} \right. \\
 &\times e^{-\frac{s_{n,m}}{(1-s_{n,m})} z(1-t_n)(1-u_n)} \left\{ w_{n,n}^\dagger \partial_{w_{n,n}^\dagger} + \frac{1}{2} (n-1-m) - |\beta_m| \right\} \\
 &\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1 + \frac{\lambda}{2}} \int_0^1 du_{n-j} u_{n-j}^{\gamma + \frac{1}{2}(n-j)-2 + \frac{\lambda}{2}} \right. \\
 &\times e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n-j+1,n}^\dagger (1-t_{n-j})(1-u_{n-j})} \left\{ w_{n-j,n}^\dagger \partial_{w_{n-j,n}^\dagger} + \frac{1}{2} (n-j-1-m) - |\beta_m| \right\} \\
 &\left. \left. \times \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{\left(1 + \frac{\lambda}{2}\right)_{i_0} \left(\gamma + \frac{\lambda}{2}\right)_{i_0}} (w_{1,n}^\dagger)^{i_0} \right\} \right\} \tilde{\varepsilon}^n \left. \right\} \quad (4.144)
 \end{aligned}$$

Put $\lambda = 0$ and $c_0 = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)}$ in eq(4.142), eq(4.143) and eq(4.144) by using eq(4.132).

(1) As $\omega = -2|\beta_0|$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} QW_{|\alpha_0|,|\beta_0|} \left(|\alpha_0| = -\frac{\Omega}{2\mu}, |\beta_0| = -\frac{\omega}{2}, \gamma; \tilde{\varepsilon}; z \right) \\ &= \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} F_{|\alpha_0|}(z) = \frac{e^{-\frac{zs_0}{(1-s_0)}}}{(1-s_0)^\gamma} \end{aligned} \quad (4.145)$$

(2) As $\omega = -2(|\beta_1| + \frac{1}{2})$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$ where $i = 0, 1$

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} s_1^{|\alpha_1|} QW_{|\alpha_i|,|\beta_1|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_1| = -\frac{1}{2}(1+\omega), \gamma; \tilde{\varepsilon}; z \right) \\ &= \left\{ \frac{e^{-\frac{zs_{0,1}}{(1-s_{0,1})}}}{(1-s_1)(1-s_{0,1})^\gamma} + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}} e^{-\frac{zs_1}{(1-s_1)}(1-t_1)(1-u_1)} \right. \\ & \quad \left. \times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{1}{2} + |\beta_1| \right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_1)(1-s_0)^\gamma} \right\} \end{aligned} \quad (4.146)$$

(3) As $\omega = -2(|\beta_m| + \frac{m}{2} + \frac{\lambda}{2})$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ where $i = 0, 1, 2, \dots$ only if

$m \geq 2$

$$\begin{aligned}
& \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^m \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} QW_{|\alpha_i|, |\beta_m|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_m| = -\frac{1}{2}(m + \omega), \gamma; \tilde{\varepsilon}; z \right) \\
&= \prod_{k=1}^m \frac{1}{(1 - s_{k,m})} \left\{ \frac{e^{-\frac{zs_{0,m}}{(1-s_{0,m})}}}{(1 - s_{0,m})^\gamma} \right. \\
&+ \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \int_0^1 du_1 u_1^{\gamma - \frac{3}{2}} e^{-\frac{zs_{1,m}}{(1-s_{1,m})}(1-t_1)(1-u_1)} \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{m}{2} + |\beta_m| \right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1 - s_0)^\gamma} \left. \right\} \\
&+ \sum_{n=2}^m \left\{ \prod_{k=1}^{n-1} \frac{1}{(1 - s_k)} \prod_{l=n}^m \frac{1}{(1 - s_{l,m})} \int_0^1 dt_n t_n^{\frac{n}{2}-1} \int_0^1 du_n u_n^{\gamma + \frac{n}{2}-2} \right. \\
&\times e^{-\frac{s_{n,m}}{(1-s_{n,m})}z(1-t_n)(1-u_n)} \left\{ w_{n,n}^\dagger \partial_{w_{n,n}^\dagger} + \frac{1}{2}(n - 1 - m) - |\beta_m| \right\} \\
&\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma + \frac{1}{2}(n-j)-2} \right. \\
&\times e^{-\frac{s_{n-j}}{(1-s_{n-j})}w_{n-j+1,n}^\dagger(1-t_{n-j})(1-u_{n-j})} \left\{ w_{n-j,n}^\dagger \partial_{w_{n-j,n}^\dagger} + \frac{1}{2}(n - j - 1 - m) - |\beta_m| \right\} \\
&\left. \left. \times \frac{e^{-\frac{w_{1,n}^\dagger s_0}{(1-s_0)}}}{(1 - s_0)^\gamma} \right\} \right\} \tilde{\varepsilon}^n \tag{4.147}
\end{aligned}$$

where,

$$\begin{cases} \gamma = \frac{1}{2}(1 + \nu) \\ \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x \\ z = -\frac{1}{2}\mu x^2 \end{cases} \tag{4.148}$$

Eq(4.145), eq(4.146) and eq(4.147) are called the generating function of 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu (|\alpha_i| + \frac{i}{2})$ and $\omega = -2 (|\beta_j| + \frac{j}{2})$ where $|\alpha_i|, |\beta_j| = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$.

Put $\lambda = 1 - \nu = 2(1 - \gamma)$ and $c_0 = (-\frac{1}{2}\mu)^{1-\gamma} \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)}$ in eq(4.142), eq(4.143) and eq(4.144). Also, replace $|\alpha_i|$ and $|\beta_i|$ by ψ_i and ϕ_i where $i = 0, 1, 2, \dots$ in them by using eq(4.135).

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ and $\text{Max}(\phi_0) \geq \psi_0$

$$\begin{aligned} & \sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} RW_{\psi_0, \phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma; \tilde{\epsilon}; z \right) \\ &= z^{1-\gamma} \sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} A_{\psi_0}(z) = z^{1-\gamma} \frac{e^{-\frac{zs_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \end{aligned} \quad (4.149)$$

(2) As $\omega = -2(\phi_1 + \frac{3}{2} - \gamma)$ and $\text{Max}(\phi_1) \geq \psi_1$ where $i = 0, 1$

$$\begin{aligned}
& \sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} \sum_{\psi_1=\psi_0}^{\infty} s_1^{\psi_1} RW_{\psi_i, \phi_1} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_1 = -\frac{\omega}{2} - \frac{3}{2} + \gamma, \gamma; \tilde{\varepsilon}; z \right) \\
&= z^{1-\gamma} \left\{ \frac{e^{-\frac{zs_{0,1}}{(1-s_{0,1})}}}{(1-s_1)(1-s_{0,1})^{2-\gamma}} + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{\frac{1}{2}-\gamma} \int_0^1 du_1 u_1^{-\frac{1}{2}} e^{-\frac{zs_1}{(1-s_1)}(1-t_1)(1-u_1)} \right. \\
&\quad \left. \times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{1}{2} + \phi_1 \right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_1)(1-s_0)^{2-\gamma}} \right\} \tag{4.150}
\end{aligned}$$

(3) As $\omega = -2(\phi_m + \frac{m}{2} + 1 - \gamma)$ and $\text{Max}(\phi_m) \geq \psi_m$ where $i = 0, 1, 2, \dots$ only if

$m \geq 2$

$$\begin{aligned}
 & \sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} \prod_{n=1}^m \left\{ \sum_{\psi_n=\psi_{n-1}}^{\infty} s_n^{\psi_n} \right\} RW_{\psi_i, \phi_m} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma; \tilde{\varepsilon}; z \right) \\
 &= z^{1-\gamma} \left\{ \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \frac{e^{-\frac{zs_{0,m}}{(1-s_{0,m})}}}{(1-s_{0,m})^{2-\gamma}} \right. \\
 &+ \tilde{\varepsilon} \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \int_0^1 dt_1 t_1^{\frac{1}{2}-\gamma} \int_0^1 du_1 u_1^{-\frac{1}{2}} e^{-\frac{zs_{1,m}}{(1-s_{1,m})}(1-t_1)(1-u_1)} \\
 &\times \left\{ w_{1,1}^\dagger \partial w_{1,1}^\dagger - \left(\frac{m}{2} + \phi_m \right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \\
 &+ \sum_{n=2}^m \left\{ \prod_{k=1}^{n-1} \frac{1}{(1-s_k)} \prod_{l=n}^m \frac{1}{(1-s_{l,m})} \int_0^1 dt_n t_n^{\frac{n}{2}-\gamma} \int_0^1 du_n u_n^{\frac{n}{2}-1} \right. \\
 &\times e^{-\frac{s_{n,m}}{(1-s_{n,m})} z(1-t_n)(1-u_n)} \left\{ w_{n,n}^\dagger \partial w_{n,n}^\dagger + \frac{1}{2}(n-1-m) - \phi_m \right\} \\
 &\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \\
 &\times e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n-j+1,n}^\dagger (1-t_{n-j})(1-u_{n-j})} \left\{ w_{n-j,n}^\dagger \partial w_{n-j,n}^\dagger + \frac{1}{2}(n-j-1-m) - \phi_m \right\} \\
 &\left. \left. \left. \times \frac{e^{-\frac{w_{1,n}^\dagger s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \right\} \tag{4.151}
 \end{aligned}$$

Eq(4.149), eq(4.150) and eq(4.151) are called the generating function of 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu \left(\psi_i + \frac{i}{2} + 1 - \gamma \right)$ and $\omega = -2 \left(\phi_j + \frac{j}{2} + 1 - \gamma \right)$ where $\psi_i, \phi_j = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$.

As we see all cases of generating function of Grand confluent hypergeometric equation, We only get rid of $|\alpha_i|$ and ψ_i eigenvalues where $i = 0, 1, 2, \dots$. But if you want, you can get rid of $|\beta_j|$ and ϕ_j eigenvalue: for where $j = 0, 1, 2, \dots$. First of all, let's think about the polynomial case which makes A_n terms terminated at certain eigenvalues. If you multiply $\sum_{|\beta_m|=0}^{\infty} t^{|\beta_m|}$ on eq(4.90) and $\sum_{\phi_m=0}^{\infty} t^{\phi_m}$ on eq(4.94), you will get rid of all $|\beta_m|$ and ϕ_m terms and obtain the generating function of it. By same reason, if we multiply $\sum_{|\beta_1|=0}^{\infty} t^{|\beta_1|}$ on eq(4.146) and $\sum_{|\beta_m|=0}^{\infty} t^{|\beta_m|}$ on eq(4.147), we can get rid of $|\beta_1|$ and $|\beta_m|$ terms of the polynomial case in which A_n and B_n terms terminated at certain eigenvalues as well. Also, as we multiply $\sum_{\phi_1=0}^{\infty} t^{\phi_1}$ on eq(4.150) and $\sum_{\phi_m=0}^{\infty} t^{\phi_m}$ on eq(4.151), we can get rid of ϕ_1 and ϕ_m terms. The reason why we skip this process is that it is not relevant to physical problems. But only if you are interested in making mathematical rigor, you can try it that way.

4.0.11 Asymptotic behavior of the function $y(x)$ and the boundary condition for x in the case of B_n term terminated

As $n \gg 1$, eq(4.4a) and eq(4.4b) are

$$\lim_{n \gg 1} A_n = A = -\frac{\varepsilon}{n} \quad (4.152a)$$

And,

$$\lim_{n \gg 1} B_n = B = -\frac{\mu}{n} \quad (4.152b)$$

As B_n term terminated at certain eigenvalue, then eq(4.3) gives

$$c_{n+1} \approx A_n c_n \quad (4.153)$$

Put eq(4.152a) in eq(4.153).

$$c_{n+1} \approx -\frac{\varepsilon}{n} c_n \quad (4.154)$$

For $n = 0, 1, 2, \dots$, eq(4.154) gives

$$\begin{aligned}
 c_2 &= -\frac{\varepsilon}{1}c_1 \\
 c_3 &= (-1)^2 \frac{\varepsilon^2}{2!}c_1 \\
 c_4 &= (-1)^3 \frac{\varepsilon^3}{3!}c_1 \\
 c_5 &= (-1)^4 \frac{\varepsilon^4}{4!}c_1 \\
 &\vdots \quad \quad \quad \vdots
 \end{aligned}
 \tag{4.155}$$

From eq(4.155), we have

$$c_{n+1} = (-1)^n \frac{\varepsilon^n}{n!}c_1 \quad \text{where } n \geq 1 \tag{4.156}$$

Assume that it's solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \tag{4.157}$$

Plug eq(4.156) in eq(4.157).

$$\lim_{n \gg 1} y(x) = c_1 \sum_n (-1)^n \frac{\varepsilon^n}{n!} x^{n+1} \approx c_1 x e^{-\varepsilon x} \tag{4.158}$$

For simplicity, let say $c_1 = 0$. Then,

$$\lim_{n \gg 1} y(x) \approx x e^{-\varepsilon x} \quad \text{where } -\infty < x < \infty \quad (4.159)$$

4.0.12 Asymptotic behavior of the function $y(x)$ and the boundary condition for x in the case of A_n term terminated

As B_n term terminated at certain eigenvalue, then eq(4.3) gives

$$c_{n+1} \approx B_n c_{n-1} \quad (4.160)$$

Put eq(4.152b) in eq(4.160).

$$c_{n+1} \approx -\frac{\mu}{n} c_{n-1} \quad (4.161)$$

We can classify c_n as to even and odd terms from eq(4.161).

$$\begin{aligned}
 c_2 &= \frac{-\mu}{1} c_0 & c_3 &= \frac{-\mu}{2} c_1 \\
 c_4 &= \frac{(-1)^2 \mu^2}{1 \cdot 3} c_0 & c_5 &= \frac{-(-1)^2 \mu^2}{2 \cdot 4} c_1 \\
 c_6 &= \frac{(-1)^3 \mu^3}{1 \cdot 3 \cdot 5} c_0 & c_7 &= \frac{-(-1)^3 \mu^3}{2 \cdot 4 \cdot 6} c_1 \\
 c_8 &= \frac{(-1)^4 \mu^4}{1 \cdot 3 \cdot 5 \cdot 7} c_0 & c_9 &= \frac{-(-1)^4 \mu^4}{2 \cdot 4 \cdot 6 \cdot 8} c_1 \\
 c_{10} &= \frac{(-1)^5 \mu^5}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} c_0 & c_{11} &= \frac{-(-1)^5 \mu^5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} c_1 \\
 &\vdots & &\vdots
 \end{aligned} \tag{4.162}$$

From eq(4.162), we have

$$c_{2n} = \frac{1}{(n - \frac{1}{2})!} \left(-\frac{1}{2}\mu\right)^n c_0 \quad c_{2n+1} = \frac{1}{n!} \left(-\frac{1}{2}\mu\right)^n c_1 \quad \text{where } n \geq 1 \tag{4.163}$$

Put eq(4.163) in eq(4.157)

$$\lim_{n \gg 1} y(x) = c_0 \sum_n (-1)^n \frac{1}{(n - \frac{1}{2})!} \left(-\frac{1}{2}\mu x^2\right)^n + c_1 x \sum_n (-1)^n \frac{1}{n!} \left(-\frac{1}{2}\mu x^2\right)^n \tag{4.164}$$

By definition, error function is

$$\text{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y dt e^{-t^2} \quad (4.165)$$

Put $c_0 = \Gamma(\frac{1}{2})$ and $c_1 = 1$ in eq(4.164), and By using eq(4.165) into eq(4.164), we obtain

$$\lim_{n \gg 1} y(x) = 1 + \left\{ \sqrt{-\frac{\pi}{2} \mu x^2} \text{Erf} \left(\sqrt{-\frac{1}{2} \mu x^2} \right) + x \right\} e^{-\frac{1}{2} \mu x^2} \quad \text{where } -\infty < x < \infty \quad (4.166)$$

4.0.13 Asymptotic behavior of the function $y(x)$ and the boundary condition for x in the case of infinite series

The general expression of $y(x)$ for infinite series of three term recurrence formula is

$$\begin{aligned} y(x) = & c_0 \left\{ \sum_{i_0=0}^{\infty} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} + \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \right. \\ & + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{n-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \right\} \right. \\ & \left. \left. \times \sum_{i_{2n}=i_{2(n-1)}}^{\infty} \left(\prod_{i_{2n+1}=i_{2(n-1)}}^{i_{2n}-1} B_{2i_{2n+1}+(n+1)} \right) \right\} \right\} x^{2i_{2n}+n+\lambda} \quad (4.167) \end{aligned}$$

By applying eq(4.152a) and eq(4.152b) into eq(4.167) and putting $c_0 = 1$, we obtain

$$\begin{aligned}
\lim_{n \gg 1} y(x) &= y\left(z = -\frac{1}{2}\mu x^2; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x\right) \\
&= \sum_{i_0=0}^{\infty} \frac{1}{\left(\frac{1}{2}\right)_{i_0}} z^{i_0} + \tilde{\varepsilon} \sum_{i_0=1}^{\infty} \frac{1}{i_0 \left(\frac{1}{2}\right)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(1)_{i_0}}{(1)_{i_1}} z^{i_1} \\
&\quad + \sum_{n=2}^{\infty} \left\{ \sum_{i_0=1}^{\infty} \frac{1}{i_0 \left(\frac{1}{2}\right)_{i_0}} \prod_{k=1}^{n-1} \left(\sum_{i_k=i_{k-1}}^{\infty} \frac{1}{\left(i_k + \frac{k}{2}\right) \left(\frac{k}{2} + \frac{1}{2}\right)_{i_k}} \right) \right. \\
&\quad \left. \times \sum_{i_n=i_{n-1}}^{\infty} \frac{\left(\frac{n}{2} + \frac{1}{2}\right)_{i_{n-1}}}{\left(\frac{n}{2} + \frac{1}{2}\right)_{i_n}} z^{i_n} \right\} \tilde{\varepsilon}^n \quad \text{where } -\infty < x < \infty
\end{aligned} \tag{4.168}$$

As we see in eq(4.168), the minimum value of the index i_0 is 1. The reason why it starts from 1 is that if $i_0 = 0$, then $\frac{1}{i_0}$ term will be divergent. Mathematically, that has no meaning. So the index i_0 starts from 1, because we are interested with the asymptotic behavior of the function $y(x)$ as $n \gg 1$.

4.1 Summary

Summary of all independent solutions of grand confluent hypergeometric function:

$$\begin{aligned}
 y(x) &= QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{\Gamma(\gamma + |\alpha_0|)}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \tilde{\varepsilon} \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \sum_{i_1=i_0}^{|\alpha_1|} \left. \left\{ \frac{(-|\alpha_1|)_{i_1}(\frac{3}{2})_{i_0}(\gamma + \frac{1}{2})_{i_0}}{(-|\alpha_1|)_{i_0}(\frac{3}{2})_{i_1}(\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \\
 &+ \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
 &\times \prod_{k=1}^{N-1} \left. \left\{ \sum_{i_k=i_{k-1}}^{|\alpha_k|} \frac{(i_k + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(-|\alpha_k|)_{i_k}(1 + \frac{k}{2})_{i_{k-1}}(\frac{k}{2} + \gamma)_{i_{k-1}}}{(-|\alpha_k|)_{i_{k-1}}(1 + \frac{k}{2})_{i_k}(\frac{k}{2} + \gamma)_{i_k}} \right\} \right. \\
 &\times \left. \left. \sum_{i_N=i_{N-1}}^{|\alpha_N|} \frac{(-|\alpha_N|)_{i_N}(1 + \frac{N}{2})_{i_{N-1}}(\frac{N}{2} + \gamma)_{i_{N-1}}}{(-|\alpha_N|)_{i_{N-1}}(1 + \frac{N}{2})_{i_N}(\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \left. \right\}
 \end{aligned}$$

Table 4.1: The 1st kind of independent solution of grand confluent hypergeometric function for polynomial as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ where $i, |\alpha_i| = 0, 1, 2, \dots$

$$\begin{aligned}
y(x) &= RW_{\psi_i} \left(\psi_i = -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} z^{i_0} \right. \\
&\quad + \tilde{\varepsilon} \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(-\psi_0)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \\
&\quad \times \left. \sum_{i_1=i_0}^{\psi_1} \left\{ \frac{(-\psi_1)_{i_1} (\frac{3}{2})_{i_0} (\frac{5}{2} - \gamma)_{i_0}}{(-\psi_1)_{i_0} (\frac{3}{2})_{i_1} (\frac{5}{2} - \gamma)_{i_1}} z^{i_1} \right\} \right\} \\
&\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(-\psi_0)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \right. \\
&\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\psi_k} \frac{(i_k + 1 - \gamma + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k + \frac{3}{2} - \gamma + \frac{k}{2})} \frac{(-\psi_k)_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (2 - \gamma + \frac{k}{2})_{i_{k-1}}}{(-\psi_k)_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (2 - \gamma + \frac{k}{2})_{i_k}} \right\} \\
&\quad \times \left. \left. \sum_{i_N=i_{N-1}}^{\psi_N} \frac{(-\psi_N)_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (2 - \gamma + \frac{N}{2})_{i_{N-1}}}{(-\psi_N)_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (2 - \gamma + \frac{N}{2})_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \left. \right\}
\end{aligned}$$

Table 4.2: The 2nd kind of independent solution of grand confluent hypergeometric function for polynomial as $\Omega = -2\mu(\psi_i + 1 - \gamma + \frac{i}{2})$ where $i, \psi_i = 0, 1, 2, \dots$

<p>(1) As $\omega = -2 \beta_0$</p> $y(x) = QW_{ \beta_0 } \left(\beta_0 = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$ $= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0}$ <p>(2) As $\omega = -2 \left(\beta_1 + \frac{1}{2} \right)$</p> $y(x) = QW_{ \beta_1 } \left(\beta_1 = -\frac{1}{2}(1 + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$ $= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{1}{2} + \beta_1))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right.$ $\left. \left. \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\}$ <p>(3) As $\omega = -2 \left(\beta_m + \frac{m}{2} \right)$ only if $m \geq 2$</p> $y(x) = QW_{ \beta_m } \left(\beta_m = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$ $= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + \beta_m))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right.$ $\times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + \beta_m))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right.$ $\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + \beta_m))}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + \gamma)_{i_{k-1}}}{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + \gamma)_{i_k}} \right\}$ $\left. \left. \times \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + \gamma)_{i_{N-1}}}{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \tilde{\varepsilon}^N \right\}$

Table 4.3: 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2 \left(|\beta_i| + \frac{i}{2} \right)$ where $i, |\beta_i| = 0, 1, 2, \dots$

<p>(1) As $\omega = -2(\phi_0 + 1 - \gamma)$</p> $y(x) = RW_{\phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$ $= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0}$ <p>(2) As $\omega = -2 \left(\phi_1 + \frac{3}{2} - \gamma \right)$</p> $y(x) = RW_{\phi_1} \left(\phi_1 = -\frac{\omega}{2} - \frac{3}{2} + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$ $= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{1}{2} + \phi_1))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right.$ $\left. \left. \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\}$ <p>(3) As $\omega = -2 \left(\phi_m + \frac{m}{2} + 1 - \gamma \right)$ only if $m \geq 2$</p> $y(x) = RW_{\phi_m} \left(\phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$ $= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right.$ $\left. \left. \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right.$ $+ \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right.$ $\left. \left. \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + \phi_m))}{(i_k + \frac{3}{2} + \frac{k}{2} - \gamma)(i_k + \frac{1}{2} + \frac{k}{2})} \right. \right.$ $\left. \left. \times \frac{(\frac{\Omega}{2\mu} + \frac{k}{2} + 1 - \gamma)_{i_k} (2 + \frac{k}{2} - \gamma)_{i_{k-1}} (1 + \frac{k}{2})_{i_{k-1}}}{(\frac{\Omega}{2\mu} + \frac{k}{2} + 1 - \gamma)_{i_{k-1}} (2 + \frac{k}{2} - \gamma)_{i_k} (1 + \frac{k}{2})_{i_k}} \right\} \right.$ $\left. \left. \times \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{N}{2} + 1 - \gamma)_{i_N} (2 + \frac{N}{2} - \gamma)_{i_{N-1}} (1 + \frac{N}{2})_{i_{N-1}}}{(\frac{\Omega}{2\mu} + \frac{N}{2} + 1 - \gamma)_{i_{N-1}} (2 + \frac{N}{2} - \gamma)_{i_N} (1 + \frac{N}{2})_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \right\}$

Table 4.4: 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2 \left(\phi_i + \frac{i}{2} + 1 - \gamma \right)$ where $i, \phi_i = 0, 1, 2, \dots$

(1) As $\omega = -2|\beta_0|$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$y(x) = QW_{|\alpha_0|,|\beta_0|} \left(|\alpha_0| = -\frac{\Omega}{2\mu}, |\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0}$$

(2) As $\omega = -2 \left(|\beta_1| + \frac{1}{2} \right)$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$ where $i = 0, 1$

$$y(x) = QW_{|\alpha_i|,|\beta_1|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_1| = -\frac{1}{2}(1 + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{1}{2} + |\beta_1|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right.$$

$$\left. \left. \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\}$$

(3) As $\omega = -2 \left(|\beta_m| + \frac{m}{2} \right)$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ only if $m \geq 2$ where $i = 0, 1, 2, \dots$

$$y(x) = QW_{|\alpha_i|,|\beta_m|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{|\alpha_0|} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right.$$

$$\left. \left. \times \sum_{i_1=i_0}^{|\alpha_1|} \left\{ \frac{(-|\alpha_1|)_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(-|\alpha_1|)_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{|\alpha_0|} \left\{ \frac{(i_0 - (\frac{m}{2} + |\beta_m|))}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(-|\alpha_0|)_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right.$$

$$\left. \left. \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{|\alpha_k|} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + |\beta_m|))}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(-|\alpha_k|)_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + \gamma)_{i_{k-1}}}{(-|\alpha_k|)_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + \gamma)_{i_k}} \right\} \right. \right.$$

$$\left. \left. \times \sum_{i_N=i_{N-1}}^{|\alpha_N|} \frac{(-|\alpha_N|)_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + \gamma)_{i_{N-1}}}{(-|\alpha_N|)_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \tilde{\varepsilon}^N \right\}$$

Table 4.5: 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu \left(|\alpha_i| + \frac{i}{2} \right)$ and $\omega = -2 \left(|\beta_j| + \frac{j}{2} \right)$ where $|\alpha_i|, |\beta_j| = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ and $\text{Max}(\phi_0) \geq \psi_0$

$$y(x) = RW_{\psi_0, \phi_0} \left(\psi_0 = -\frac{\Omega}{2\mu} - 1 + \gamma, \phi_0 = -\frac{\omega}{2} - 1 + \gamma; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0}$$

(2) As $\omega = -2\left(\phi_1 + \frac{3}{2} - \gamma\right)$ and $\text{Max}(\phi_1) \geq \psi_1$ where $i = 0, 1$

$$y(x) = RW_{\psi_i, \phi_1} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_1 = -\frac{\omega}{2} - \frac{3}{2} + \gamma; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 - (\frac{1}{2} + \phi_1))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right.$$

$$\left. \left. \times \sum_{i_1=i_0}^{\psi_1} \left\{ \frac{(-\psi_1)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(-\psi_1)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \right\} \tilde{\varepsilon} \right\}$$

(3) As $\omega = -2\left(\phi_m + \frac{m}{2} + 1 - \gamma\right)$ and $\text{Max}(\phi_m) \geq \psi_m$ only if $m \geq 2$ where $i = 0, 1, 2, \dots$

$$y(x) = RW_{\psi_i, \phi_m} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\psi_0} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} z^{i_0} + \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right.$$

$$\times \sum_{i_1=i_0}^{\psi_1} \left\{ \frac{(-\psi_1)_{i_1} (\frac{5}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_0}}{(-\psi_1)_{i_0} (\frac{5}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_1}} z^{i_1} \right\} \tilde{\varepsilon} + \sum_{N=2}^m \left\{ \sum_{i_0=0}^{\psi_0} \left\{ \frac{(i_0 - (\frac{m}{2} + \phi_m))}{(i_0 + \frac{3}{2} - \gamma)(i_0 + \frac{1}{2})} \frac{(-\psi_0)_{i_0}}{(2 - \gamma)_{i_0} (1)_{i_0}} \right. \right.$$

$$\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\psi_k} \frac{(i_k + \frac{k}{2} - (\frac{m}{2} + \phi_m))}{(i_k + \frac{3}{2} - \gamma + \frac{k}{2})(i_k + \frac{1}{2} + \frac{k}{2})} \frac{(-\psi_k)_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + 2 - \gamma)_{i_{k-1}}}{(-\psi_k)_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + 2 - \gamma)_{i_k}} \right\}$$

$$\left. \left. \times \sum_{i_N=i_{N-1}}^{\psi_N} \frac{(-\psi_N)_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + 2 - \gamma)_{i_{N-1}} z^{i_N}}{(-\psi_N)_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + 2 - \gamma)_{i_N}} \right\} \tilde{\varepsilon}^N \right\}$$

where $\gamma = \frac{1}{2}(1 + \nu)$

Table 4.6: 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu(\psi_i + \frac{i}{2} + 1 - \gamma)$ and $\omega = -2(\phi_j + \frac{j}{2} + 1 - \gamma)$ where $\psi_i, \phi_j = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$

$$\begin{aligned}
y(x) &= QW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= \frac{\Gamma(\gamma - \frac{\Omega}{2\mu})}{\Gamma(\gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} z^{i_0} + \tilde{\varepsilon} \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
&\times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_1} (\frac{3}{2})_{i_0} (\gamma + \frac{1}{2})_{i_0}}{(\frac{\Omega}{2\mu} + \frac{1}{2})_{i_0} (\frac{3}{2})_{i_1} (\gamma + \frac{1}{2})_{i_1}} z^{i_1} \right\} + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \frac{(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}(\gamma)_{i_0}} \right. \right. \\
&\times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k - \frac{1}{2} + \gamma + \frac{k}{2})} \frac{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (\frac{k}{2} + \gamma)_{i_{k-1}}}{(\frac{\Omega}{2\mu} + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (\frac{k}{2} + \gamma)_{i_k}} \right\} \\
&\times \left. \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (\frac{N}{2} + \gamma)_{i_{N-1}}}{(\frac{\Omega}{2\mu} + \frac{N}{2})_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (\frac{N}{2} + \gamma)_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N
\end{aligned}$$

Table 4.7: The 1st kind of independent solution of grand confluent hypergeometric function for infinite series

$$\begin{aligned}
y(x) &= RW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= z^{1-\gamma} \frac{\Gamma(1 - \frac{\Omega}{2\mu})}{\Gamma(2 - \gamma)} \left\{ \sum_{i_0=0}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} z^{i_0} + \tilde{\varepsilon} \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \right. \\
&\quad \times \sum_{i_1=i_0}^{\infty} \left\{ \frac{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_1} (\frac{3}{2})_{i_0} (\frac{5}{2} - \gamma)_{i_0}}{(\frac{\Omega}{2\mu} + \frac{3}{2} - \gamma)_{i_0} (\frac{3}{2})_{i_1} (\frac{5}{2} - \gamma)_{i_1}} z^{i_1} \right\} \left. \right\} \\
&\quad + \sum_{N=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \left\{ \frac{(i_0 + 1 - \gamma + \frac{\omega}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2} - \gamma)} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma)_{i_0}}{(1)_{i_0}(2 - \gamma)_{i_0}} \right. \right. \\
&\quad \times \prod_{k=1}^{N-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k + 1 - \gamma + \frac{\omega}{2} + \frac{k}{2})}{(i_k + \frac{1}{2} + \frac{k}{2})(i_k + \frac{3}{2} - \gamma + \frac{k}{2})} \right. \\
&\quad \times \left. \frac{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{k}{2})_{i_k} (1 + \frac{k}{2})_{i_{k-1}} (2 - \gamma + \frac{k}{2})_{i_{k-1}}}{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{k}{2})_{i_{k-1}} (1 + \frac{k}{2})_{i_k} (2 - \gamma + \frac{k}{2})_{i_k}} \right\} \\
&\quad \times \left. \left. \sum_{i_N=i_{N-1}}^{\infty} \frac{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{N}{2})_{i_N} (1 + \frac{N}{2})_{i_{N-1}} (2 - \gamma + \frac{N}{2})_{i_{N-1}}}{(\frac{\Omega}{2\mu} + 1 - \gamma + \frac{N}{2})_{i_{N-1}} (1 + \frac{N}{2})_{i_N} (2 - \gamma + \frac{N}{2})_{i_N}} z^{i_N} \right\} \right\} \tilde{\varepsilon}^N \left. \right\}
\end{aligned}$$

Table 4.8: The 2nd kind of independent solution of grand confluent hypergeometric function for infinite series.

$$\begin{aligned}
 y(x) &= QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1}(1-v_0)^\gamma} + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1}(1-v_{n-j})} \left. \left. \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2}(n-j-1+\omega) \right\} \right. \right. \\
 &\quad \left. \left. \times \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1,n}v_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1}(1-v_0)^\gamma} \right\} \right\} \tilde{\varepsilon}^n \quad \text{where } w_{a,b} = z \prod_{l=a}^b t_l u_l v_l \\
 &\quad \quad \quad w_{a,b} = z \text{ only if } a > b
 \end{aligned}$$

Table 4.9: Integral formalism of 1st kind of independent solution of grand confluent hypergeometric function for polynomial as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ where $i, |\alpha_i| = 0, 1, 2, \dots$

$$\begin{aligned}
 y(x) &= RW_{\psi_i} \left(\psi_i = -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \left\{ \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right. \\
 &\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{\psi_{n-j}+1}(1-v_{n-j})} \\
 &\quad \left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} \left(n-j+1 - 2\gamma + \omega \right) \right\} \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1,n}v_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \left. \right\}
 \end{aligned}$$

Table 4.10: Integral formalism of 2nd kind of independent solution of grand confluent hypergeometric function for polynomial as $\Omega = -2\mu(\psi_i + 1 - \gamma + \frac{i}{2})$ where $i, \psi_i = 0, 1, 2, \dots$

$$\begin{aligned}
 & (1) \text{As } \omega = -2|\beta_0| \\
 & y(x) = QW_{|\beta_0|} \left(|\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & \quad = \frac{1}{\Gamma(\frac{\Omega}{2\mu})} \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{-\frac{\Omega}{2\mu}-1+\gamma} \\
 & (2) \text{As } \omega = -2 \left(|\beta_m| + \frac{m}{2} \right) \text{ only if } m \geq 1 \\
 & y(x) = QW_{|\beta_m|} \left(|\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & \quad = \frac{1}{\Gamma(\frac{\Omega}{2\mu})} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{-\frac{\Omega}{2\mu}-1+\gamma} \right. \\
 & \quad + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
 & \quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\left(\frac{\Omega}{2\mu} + \frac{n-j}{2}\right)+1} (1-v_{n-j})} \right. \\
 & \quad \times \left. \left. \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} \left(n-j-1-m \right) - |\beta_m| \right\} \right. \right. \\
 & \quad \left. \left. \times \int_0^1 dv_0 e^{w_{1,n} v_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{-\frac{\Omega}{2\mu}-1+\gamma} \right\} \right\} \tilde{\varepsilon}^n \left. \right\}
 \end{aligned}$$

Table 4.11: Integral formalism of 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2 \left(|\beta_i| + \frac{i}{2} \right)$ where $i, |\beta_i| = 0, 1, 2, \dots$

$$\begin{aligned}
 & (1) \text{As } \omega = -2(\phi_0 + 1 - \gamma) \\
 & y(x) = RW_{\phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & \quad = \frac{z^{1-\gamma}}{\Gamma(\frac{\Omega}{2\mu} + 1 - \gamma)} \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \\
 & (2) \text{As } \omega = -2 \left(\phi_m + \frac{m}{2} + 1 - \gamma \right) \text{ only if } m \geq 1 \\
 & y(x) = RW_{\phi_m} \left(\phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 & \quad = \frac{z^{1-\gamma}}{\Gamma(\frac{\Omega}{2\mu} + 1 - \gamma)} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \right. \\
 & \quad + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j) - \gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j) - 1} \right. \right. \\
 & \quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\left(\frac{\Omega}{2\mu} + \frac{n-j}{2} - \gamma\right)} (1 - v_{n-j})} \\
 & \quad \times \left. \left. \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n - j - 1 - m) - \phi_m \right\} \right. \right. \\
 & \quad \times \left. \left. \int_0^1 dv_0 e^{w_{1,n} v_0} v_0^{\frac{\Omega}{2\mu} - \gamma} (1 - v_0)^{-\frac{\Omega}{2\mu}} \right\} \right\} \tilde{\varepsilon}^n
 \end{aligned}$$

Table 4.12: Integral formalism of 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\omega = -2(\phi_i + \frac{i}{2} + 1 - \gamma)$ where $i, \phi_i = 0, 1, 2, \dots$

$$\begin{aligned}
 & (1) \text{As } \omega = -2|\beta_0| \\
 y(x) &= QW_{|\alpha_0|,|\beta_0|} \left(|\alpha_0| = -\frac{\Omega}{2\mu}, |\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{z v_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1} (1-v_0)^\gamma} \\
 & (2) \text{As } \omega = -2 \left(|\beta_m| + \frac{m}{2} \right) \text{ and } \text{Max}(|\beta_m|) \geq |\alpha_m| \text{ only if } m \geq 1 \\
 y(x) &= QW_{|\alpha_i|,|\beta_m|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_m| = -\frac{1}{2}(m + \omega), \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{z v_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1} (1-v_0)^\gamma} \\
 &+ \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
 &\times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{|\alpha_{n-j}|+1} (1-v_{n-j})} \\
 &\left. \left. \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1-m) - |\beta_m| \right\} \frac{|\alpha_0|!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1,n} v_0}{(1-v_0)}}}{v_0^{|\alpha_0|+1} (1-v_0)^\gamma} \right\} \right\} \tilde{\varepsilon}^n
 \end{aligned}$$

Table 4.13: Integral formalism of 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu \left(|\alpha_i| + \frac{i}{2} \right)$ and $\omega = -2 \left(|\beta_j| + \frac{j}{2} \right)$ where $|\alpha_i|, |\beta_j| = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$

$$\begin{aligned}
& (1) \text{As } \omega = -2(\phi_0 + 1 - \gamma) \text{ and } \text{Max}(\phi_0) \geq \psi_0 \\
& y(x) = RW_{\psi_0, \phi_0} \left(\psi_0 = -\frac{\Omega}{2\mu} - 1 + \gamma, \phi_0 = -\frac{\omega}{2} - 1 + \gamma; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
& \quad = z^{1-\gamma} \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \\
& (2) \text{As } \omega = -2\left(\phi_m + \frac{m}{2} + 1 - \gamma\right) \text{ and } \text{Max}(\phi_m) \geq \psi_m \text{ only if } m \geq 1 \\
& y(x) = RW_{\psi_i, \phi_m} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
& \quad = z^{1-\gamma} \left\{ \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{zv_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right. \\
& \quad + \sum_{n=1}^m \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \\
& \quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1, n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{\psi_{n-j}+1}(1-v_{n-j})} \\
& \quad \left. \left. \times \left\{ w_{n-j, n} \partial_{w_{n-j, n}} + \frac{1}{2}(n-j-1-m) - \phi_m \right\} \frac{\psi_0!}{2\pi i} \oint dv_0 \frac{e^{-\frac{w_{1, n} v_0}{(1-v_0)}}}{v_0^{\psi_0+1}(1-v_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \left. \right\} \\
& \text{where } \gamma = \frac{1}{2}(1 + \nu)
\end{aligned}$$

Table 4.14: Integral formalism of 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu\left(\psi_i + \frac{i}{2} + 1 - \gamma\right)$ and $\omega = -2\left(\phi_j + \frac{j}{2} + 1 - \gamma\right)$ where $\psi_i, \phi_j = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$

$$\begin{aligned}
 y(x) &= QW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \frac{1}{\Gamma(\frac{\Omega}{2\mu})} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{\gamma-\frac{\Omega}{2\mu}-1} \right. \\
 &\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \right. \right. \\
 &\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\frac{\Omega}{2\mu}-\frac{1}{2}(n-j)+1} (1-v_{n-j})} \\
 &\quad \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j-1+\omega) \right\} \\
 &\quad \left. \left. \left. \times \int_0^1 dv_0 e^{w_{1,n} v_0} v_0^{\frac{\Omega}{2\mu}-1} (1-v_0)^{\gamma-\frac{\Omega}{2\mu}-1} \right\} \right\} \tilde{\varepsilon}^n \right\}
 \end{aligned}$$

Table 4.15: Integral formalism of 1st kind of independent solution of grand confluent hypergeometric function for infinite series

$$\begin{aligned}
y(x) &= RW \left(\gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
&= \frac{z^{1-\gamma}}{\Gamma(\frac{\Omega}{2\mu} + 1 - \gamma)} \left\{ \int_0^1 dv_0 e^{zv_0} v_0^{\frac{\Omega}{2\mu}-\gamma} (1-v_0)^{-\frac{\Omega}{2\mu}} \right. \\
&\quad + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \right. \right. \\
&\quad \times \frac{1}{2\pi i} \oint dv_{n-j} \frac{e^{-\frac{v_{n-j}}{(1-v_{n-j})} w_{n-j+1,n} (1-t_{n-j})(1-u_{n-j})}}{v_{n-j}^{-\frac{\Omega}{2\mu} + \gamma - \frac{1}{2}(n-j)} (1-v_{n-j})} \\
&\quad \times \left\{ w_{n-j,n} \partial_{w_{n-j,n}} + \frac{1}{2} (n-j+1 - 2\gamma + \omega) \right\} \\
&\quad \left. \left. \left. \times \int_0^1 dv_0 e^{w_{1,n} v_0} v_0^{\frac{\Omega}{2\mu}-\gamma} (1-v_0)^{-\frac{\Omega}{2\mu}} \right\} \right\} \tilde{\varepsilon}^n \right\}
\end{aligned}$$

Table 4.16: Integral formalism of 2nd kind of independent solution of grand confluent hypergeometric function for infinite series

$$\begin{aligned}
 & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^{\infty} \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} QW_{|\alpha_i|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \frac{e^{-\frac{zs_{0,\infty}}{(1-s_{0,\infty})}}}{(1-s_{0,\infty})^\gamma} + \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \\
 &\times \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}} e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \\
 &\times \left\{ w_{1,1}^* \partial_{w_{1,1}^*} + \frac{\omega}{2} \right\} \frac{e^{-\frac{w_{1,1}^* s_0}{(1-s_0)}}}{(1-s_0)^\gamma} + \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n t_n^{\frac{n}{2}-1} \int_0^1 du_n u_n^{\gamma-2+\frac{n}{2}} \right. \\
 &\times e^{-\frac{s_{n,\infty}}{(1-s_{n,\infty})} z(1-t_n)(1-u_n)} \left. \left\{ w_{n,n}^* \partial_{w_{n,n}^*} + \left(\frac{(n-1)}{2} + \frac{\omega}{2} \right) \right\} \right\} \\
 &\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma-2+\frac{1}{2}(n-j)} \frac{e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n+1-j,n}^* (1-t_{n-j})(1-u_{n-j})}}{(1-s_{n-j})} \right. \\
 &\times \left. \left\{ w_{n-j,n}^* \partial_{w_{n-j,n}^*} + \left(\frac{1}{2}(n-j-1) + \frac{\omega}{2} \right) \right\} \frac{e^{-\frac{w_{1,n}^* s_0}{(1-s_0)}}}{(1-s_0)^\gamma} \right\} \tilde{\varepsilon}^n \\
 &\text{where } s_{a,b} = s_a \cdot s_{a+1} \cdot s_{a+2} \cdots s_{b-2} \cdot s_{b-1} \cdot s_b \\
 &\quad w_{a,b}^* = z s_{a,\infty} \prod_{l=a}^b t_l u_l \\
 &\text{and } \gamma = \frac{1}{2}(1 + \nu)
 \end{aligned}$$

Table 4.17: The generating function of 1st kind of grand confluent hypergeometric function as $\Omega = -2\mu(|\alpha_i| + \frac{i}{2})$ where $i, |\alpha_i| = 0, 1, 2, \dots$

$$\begin{aligned}
 & \sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\psi_n=\psi_{n-1}}^{\infty} s_n^{\psi_n} \right\} RW_{\psi_i} \left(\psi_i = -\frac{\Omega}{2\mu} + \gamma - 1 - \frac{i}{2}; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\
 &= z^{1-\gamma} \left\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \frac{e^{-\frac{zs_{0,\infty}}{(1-s_{0,\infty})}}}{(1-s_{0,\infty})^{2-\gamma}} + \tilde{\varepsilon} \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1 t_1^{\frac{1}{2}-\gamma} \int_0^1 du_1 u_1^{-\frac{1}{2}} \right. \\
 &\quad \times e^{-\frac{s_{1,\infty}}{(1-s_{1,\infty})} z(1-t_1)(1-u_1)} \left\{ w_{1,1}^* \partial w_{1,1}^* + \left(\frac{\omega}{2} + 1 - \gamma \right) \right\} \frac{e^{-\frac{w_{1,1}^* s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \\
 &\quad + \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n t_n^{\frac{n}{2}-\gamma} \int_0^1 du_n u_n^{-1+\frac{n}{2}} e^{-\frac{s_{n,\infty}}{(1-s_{n,\infty})} z(1-t_n)(1-u_n)} \right. \\
 &\quad \times \left\{ w_{n,n}^* \partial w_{n,n}^* + \left(\frac{1}{2}(n+1+\omega) - \gamma \right) \right\} \\
 &\quad \times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} \frac{e^{-\frac{s_{n-j}}{(1-s_{n-j})} w_{n+1-j,n}^*(1-t_{n-j})(1-u_{n-j})}}{(1-s_{n-j})} \right. \\
 &\quad \left. \left. \times \left\{ w_{n-j,n}^* \partial w_{n-j,n}^* + \left(\frac{1}{2}(n-j+1+\omega) - \gamma \right) \right\} \frac{e^{-\frac{w_{1,n}^* s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n \Bigg\} \\
 &\text{and } \gamma = \frac{1}{2}(1+\nu)
 \end{aligned}$$

Table 4.18: The generating function of 2nd kind of grand confluent hypergeometric function as $\Omega = -2\mu(\psi_i + \frac{i}{2} + 1 - \gamma)$ where $i, \psi_i = 0, 1, 2, \dots$

(1) As $\omega = -2|\beta_0|$ and $\text{Max}(|\beta_0|) \geq |\alpha_0|$

$$\sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} QW_{|\alpha_0|,|\beta_0|} \left(|\alpha_0| = -\frac{\Omega}{2\mu}, |\beta_0| = -\frac{\omega}{2}, \gamma = \frac{1}{2}(1+\nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) = \frac{e^{-\frac{zs_0}{(1-s_0)}}}{(1-s_0)^\gamma}$$

(2) As $\omega = -2\left(|\beta_1| + \frac{1}{2}\right)$ and $\text{Max}(|\beta_1|) \geq |\alpha_1|$ where $i = 0, 1$

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \sum_{|\alpha_1|=|\alpha_0|}^{\infty} s_1^{|\alpha_1|} QW_{|\alpha_i|,|\beta_1|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_1| = -\frac{1}{2}(1+\omega), \gamma = \frac{1}{2}(1+\nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ &= \left\{ \frac{e^{-\frac{zs_{0,1}}{(1-s_{0,1})}}}{(1-s_1)(1-s_{0,1})^\gamma} + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}} e^{-\frac{zs_1}{(1-s_1)}(1-t_1)(1-u_1)} \right. \\ & \quad \left. \times \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{1}{2} + |\beta_1| \right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_1)(1-s_0)^\gamma} \right\} \end{aligned}$$

(3) As $\omega = -2\left(|\beta_m| + \frac{m}{2}\right)$ and $\text{Max}(|\beta_m|) \geq |\alpha_m|$ only if $m \geq 2$ where $i = 0, 1, 2, \dots$

$$\begin{aligned} & \sum_{|\alpha_0|=0}^{\infty} \frac{s_0^{|\alpha_0|}}{(|\alpha_0|)!} \prod_{n=1}^m \left\{ \sum_{|\alpha_n|=|\alpha_{n-1}|}^{\infty} s_n^{|\alpha_n|} \right\} QW_{|\alpha_i|,|\beta_m|} \left(|\alpha_i| = -\frac{\Omega}{2\mu} - \frac{i}{2}, |\beta_m| = -\frac{1}{2}(m+\omega), \gamma; \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right) \\ &= \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \left\{ \frac{e^{-\frac{zs_{0,m}}{(1-s_{0,m})}}}{(1-s_{0,m})^\gamma} \right. \\ & \quad + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{-\frac{1}{2}} \int_0^1 du_1 u_1^{\gamma-\frac{3}{2}} e^{-\frac{zs_{1,m}}{(1-s_{1,m})}(1-t_1)(1-u_1)} \left\{ w_{1,1}^\dagger \partial_{w_{1,1}^\dagger} - \left(\frac{m}{2} + |\beta_m| \right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_0)^\gamma} \left. \right\} \\ & \quad + \sum_{n=2}^m \left\{ \prod_{k=1}^{n-1} \frac{1}{(1-s_k)} \prod_{l=n}^m \frac{1}{(1-s_{l,m})} \int_0^1 dt_n t_n^{\frac{n}{2}-1} \int_0^1 du_n u_n^{\gamma+\frac{n}{2}-2} e^{-\frac{s_{n,m}}{(1-s_{n,m})}z(1-t_n)(1-u_n)} \right. \\ & \quad \times \left\{ w_{n,n}^\dagger \partial_{w_{n,n}^\dagger} + \frac{1}{2}(n-1-m) - |\beta_m| \right\} \\ & \quad \times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} e^{-\frac{s_{n-j}}{(1-s_{n-j})}w_{n-j+1,n}^\dagger(1-t_{n-j})(1-u_{n-j})} \right. \\ & \quad \left. \left. \times \left\{ w_{n-j,n}^\dagger \partial_{w_{n-j,n}^\dagger} + \frac{1}{2}(n-j-1-m) - |\beta_m| \right\} \frac{e^{-\frac{w_{1,n}^\dagger s_0}{(1-s_0)}}}{(1-s_0)^\gamma} \right\} \right\} \tilde{\varepsilon}^n \end{aligned}$$

where $w_{i,j}^\dagger = z \prod_{l=i}^j t_l u_l s_l$ only if $j = m$

$$= z s_{i,m} \prod_{l=i}^j t_l u_l \text{ only if } j < m$$

Table 4.19: The generating function of 1st kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu\left(|\alpha_i| + \frac{i}{2}\right)$ and $\omega = -2\left(|\beta_j| + \frac{j}{2}\right)$ where $|\alpha_i|, |\beta_j| = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$

(1) As $\omega = -2(\phi_0 + 1 - \gamma)$ and $\text{Max}(\phi_0) \geq \psi_0$

$$\sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} RW_{\psi_0, \phi_0} \left(\phi_0 = -\frac{\omega}{2} - 1 + \gamma, \phi_0 = -\frac{\omega}{2} - 1 + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= z^{1-\gamma} \frac{e^{-\frac{zs_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}}$$

(2) As $\omega = -2\left(\phi_1 + \frac{3}{2} - \gamma\right)$ and $\text{Max}(\phi_1) \geq \psi_1$ where $i = 0, 1$

$$\sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} \sum_{\psi_1=\psi_0}^{\infty} s_1^{\psi_1} RW_{\psi_i, \phi_1} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_1 = -\frac{\omega}{2} - \frac{3}{2} + \gamma, \gamma = \frac{1}{2}(1 + \nu); \tilde{\varepsilon} = -\frac{1}{2}\varepsilon x; z = -\frac{1}{2}\mu x^2 \right)$$

$$= z^{1-\gamma} \left\{ \frac{e^{-\frac{zs_{0,1}}{(1-s_{0,1})}}}{(1-s_1)(1-s_{0,1})^{2-\gamma}} + \tilde{\varepsilon} \int_0^1 dt_1 t_1^{\frac{1}{2}-\gamma} \int_0^1 du_1 u_1^{-\frac{1}{2}} e^{-\frac{zs_1}{(1-s_1)}(1-t_1)(1-u_1)} \right.$$

$$\left. \times \left\{ w_{1,1}^\dagger \partial w_{1,1}^\dagger - \left(\frac{1}{2} + \phi_1\right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_1)(1-s_0)^{2-\gamma}} \right\}$$

(3) As $\omega = -2\left(\phi_m + \frac{m}{2} + 1 - \gamma\right)$ and $\text{Max}(\phi_m) \geq \psi_m$ only if $m \geq 2$ where $i = 0, 1, 2, \dots$

$$\sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} \prod_{n=1}^m \left\{ \sum_{\psi_n=\psi_{n-1}}^{\infty} s_n^{\psi_n} \right\} RW_{\psi_i, \phi_m} \left(\psi_i = -\frac{\Omega}{2\mu} - 1 - \frac{i}{2} + \gamma, \phi_m = -\frac{\omega}{2} - \frac{m}{2} - 1 + \gamma, \gamma; \tilde{\varepsilon}; z \right)$$

$$= z^{1-\gamma} \left\{ \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \frac{e^{-\frac{zs_{0,m}}{(1-s_{0,m})}}}{(1-s_{0,m})^{2-\gamma}} \right.$$

$$+ \tilde{\varepsilon} \prod_{k=1}^m \frac{1}{(1-s_{k,m})} \int_0^1 dt_1 t_1^{\frac{1}{2}-\gamma} \int_0^1 du_1 u_1^{-\frac{1}{2}} e^{-\frac{zs_{1,m}}{(1-s_{1,m})}(1-t_1)(1-u_1)} \left\{ w_{1,1}^\dagger \partial w_{1,1}^\dagger - \left(\frac{m}{2} + \phi_m\right) \right\} \frac{e^{-\frac{w_{1,1}^\dagger s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}}$$

$$+ \sum_{n=2}^m \left\{ \prod_{k=1}^{n-1} \frac{1}{(1-s_k)} \prod_{l=n}^m \frac{1}{(1-s_{l,m})} \int_0^1 dt_n t_n^{\frac{n}{2}-\gamma} \int_0^1 du_n u_n^{\frac{n}{2}-1} e^{-\frac{sn,m}{(1-s_{n,m})}z(1-t_n)(1-u_n)} \right.$$

$$\times \left\{ w_{n,n}^\dagger \partial w_{n,n}^\dagger + \frac{1}{2}(n-1-m) - \phi_m \right\}$$

$$\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{2}(n-j)-1} e^{-\frac{s_{n-j}}{(1-s_{n-j})}w_{n-j+1,n}^\dagger(1-t_{n-j})(1-u_{n-j})} \right.$$

$$\left. \left. \times \left\{ w_{n-j,n}^\dagger \partial w_{n-j,n}^\dagger + \frac{1}{2}(n-j-1-m) - \phi_m \right\} \frac{e^{-\frac{w_{1,n}^\dagger s_0}{(1-s_0)}}}{(1-s_0)^{2-\gamma}} \right\} \right\} \tilde{\varepsilon}^n$$

Table 4.20: Generating function of 2nd kind of independent solution of grand confluent hypergeometric function for the polynomial as $\Omega = -2\mu\left(\psi_i + \frac{i}{2} + 1 - \gamma\right)$ and $\omega = -2\left(\phi_j + \frac{j}{2} + 1 - \gamma\right)$ where $\psi_i, \phi_j = 0, 1, 2, \dots$ and $i, j = 0, 1, 2, \dots$

Chapter 5

Conclusion

We see from the generalization of the three-term recurrence formula in chapter 3, we can obtain the exact solution of grand confluent hypergeometric function shown in chapter 4, the power series of infinite and polynomial case, and their integral forms as well as their generating functions. We have shown other examples such as the Lamé, Mathieu, and Heun functions (these are some of unsolved problems in physics and in mathematics). But those examples are skipped otherwise thesis will reach over 900 pages. We are going to publish these examples soon, then one can clearly see how they work extremely well analytically, our subject is also intermediately related with number theory in mathematics. For example, a generalized continued fraction is a generalization of regular continued fractions in canonical form in which the partial numerators and partial denominators can assume arbitrary real or complex values. A generalized continued fraction is an expression of the form

$$x = a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \frac{b_3}{a_4 + \ddots}}}} \quad (5.1)$$

where the b_n ($n \geq 0$) are partial numerators, the a_n are the partial denominators. And the leading term a_0 is called the integer part of the continued fraction. an example of it is

$$\begin{aligned}
 \pi &= 3 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)(2n+1)} = 3 + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{2 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 7} - + \dots \\
 &= 3 + \frac{1^2}{6 + \frac{3^2}{6 + \frac{5^2}{6 + \dots}}}
 \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
 \sqrt[12]{27} = 1 + & \frac{7}{12 + \frac{5}{2 + \frac{19}{36 + \frac{17}{2 + \frac{31}{60 + \frac{29}{2 + \frac{43}{84 + \frac{41}{2 + \ddots}}}}}}}}
 \end{aligned} \tag{5.3}$$

Using eq(3.5) in chapter 3,

$$\begin{aligned}
 K_n &= \frac{c_{n+1}}{c_n} = A_n + \frac{B_n}{K_{n-1}} \\
 &= A_n + \frac{B_n}{A_{n-1} + \frac{B_{n-1}}{A_{n-2} + \frac{B_{n-3}}{A_{n-4} + \frac{B_{n-4}}{\vdots}}}} \\
 &= A_n + \frac{B_n}{A_{n-1} + \frac{B_{n-1}}{A_{n-2} + \frac{B_{n-3}}{A_{n-4} + \frac{B_{n-4}}{A_4 + \frac{B_4}{A_3 + \frac{B_3}{A_2 + \frac{B_2}{A_1 + \frac{B_1}{A_0}}}}}}} \quad (5.4)
 \end{aligned}$$

If n goes to infinity in eq(5.4), it exactly corresponds to eq(5.1). Index n starts from zero on top of fraction and then goes to infinity at the bottom in eq(5.1), and index n starts from infinity on the top of fraction and then goes to zero at the bottom in eq(5.4). It is exactly the same story. One further note, the index n starts from zero,

and the latter one starts from infinity. There, we argue

$$x = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \quad (5.5)$$

The r.h.s. of eq(5.5) exactly corresponds to ratio test for the convergence of a series. Surprisingly, it means that all such numbers must equivalent to the ratio tests in any kind of linear differential equations with three term recurrence formulas, to be exact. Actually, we can clearly show how they correspond to the numbers related in solution of our differential equations. Furthermore, we know irrational numbers are equivalent to generalized continued fraction. And rational numbers are equivalent to finite generalized continued fraction in general. It means that irrational numbers correspond to infinite series in linear differential equation, also rational numbers correspond to polynomial in it. The summary is

$$\begin{aligned} \text{irrational numbers} &\longleftrightarrow \text{infinite series in linear differential equation} \\ \text{rational numbers} &\longleftrightarrow \text{polynomial in linear differential equation} \end{aligned} \quad (5.6)$$

We see in all above examples, everything is connect to each other in an arrogant way. This would entail a beautiful application of number theory into mathematical physics realm. We hope that our small efforts may lead into a new clues for connection between theory of numbers in mathematics and physics. This kind of analysis is just a beginning in such hopes of producing new connections. We hope to be able to apply such analysis to many diverse areas in engineering, architecture, quantum field theories, SUSY theories, string theories, and so on.

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