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PRESENTATIONS AND ISOPERIMETRIC FUNCTIONS OF FINITELY GENERATED METABELIAN GROUPS

by

CHING-FEN FUH

A dissertation submitted to the Graduate Faculty in Mathematics in partial
fulfillment of the requirements for the degree of Doctor of Philosophy,
the City University of New York

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Abstract

PRESENTATIONS AND ISOPERIMETRIC FUNCTIONS OF FINITELY GENERATED METABELIAN GROUPS

by

Ching-Fen Fuh

Advisor: Distinguished Professor Gilbert Baumslag

This work is motivated by Philip Hall's result in 1954 that finitely generated metabelian groups satisfy maximal condition for normal subgroups. It turns out that they have finite metabelian presentations. We investigate metabelian presentations for a varied collection of finitely generated metabelian groups and the relative isoperimetric functions associated with these presentations. The most general result in this thesis is that the isoperimetric function of the wreath product of a finitely generated abelian group by another is bounded above by a polynomial which depends only on the rank of the second group. We also introduce the notion of the isoperimetric function for a finitely generated module over a polynomial ring in finitely many variables. We do not have enough knowledge finding lower bounds for groups having been dealt with in this research but the relative centralized isoperimetric functions are discussed in general.

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1 Introduction

1.1 Free metabelian groups

This thesis is concerned with the presentations and isoperimetric functions of finitely generated metabelian groups. We recall that a group G is termed *metabelian* if its commutator subgroup is abelian (see §2.1 for details). The class of all metabelian groups forms a *variety*, in the sense of B.H. Neumann [N1] (see also H. Neumann [N2] for a general reference to this subject), that is it is closed under subgroups, epimorphic images and unrestricted direct products. There are, in each variety of groups, the so-called *free groups*. In particular, in the variety of metabelian groups, these "free groups" are termed *free metabelian groups*. More precisely, a metabelian group F is termed free metabelian if it comes equipped with a map θ from a set X into F such that for every metabelian group G and every set map $\sigma : X \to G$ there exists a unique homorphism $\phi : F \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\theta"] \arrow[dr, "\sigma"'] & F \arrow[d, "\phi"] \\
& G
\end{tikzcd}$$

It turns out that θ is monic and so we can identify X with a subset of F. We sometimes say that X *freely generates* F or that F *is free on* X. It is important to note that we have here omitted the word "metabelian". Indeed we will be concerned only with metabelian groups, unless otherwise stated.

These free metabelian groups can be easily identified. In fact, if now E is an absolutely free group; i.e., free on X in the above sense, in the variety of all groups, then E/E'' is a free metabelian group, free on X; here E'' denotes the second commutator subgroup of E (see §2.1 for the definition). Moreover, every free metabelian group takes this form.

1.2 Finitely generated metabelian groups

A group G is said to satisfy the *maximal condition* Max if every properly ascending chain of subgroups of G is finite. We shall be interested here in an analogous condition $Max-n$, introduced by P. Hall [H], the *maximal condition for normal subgroups.* More precisely, a group G satisfies $Max - n$ if every properly ascending chain of normal subgroups of G is finite. Equivalently, $G \in Max - n$ if and only if every normal subgroup N of G is the normal closure of a finite set Y, say. We denote the normal closure N of Y in G by $gp_G(Y)$ – so

$$gp_G(Y) = gp(g^{-1}yg | g \in G, y \in Y).$$

(see §2.1 for a discussion of this and other notation used here).

In 1954 P. Hall [H] proved, in particular, that finitely generated metabelian groups satisfy $Max - n$. This theorem will be central to my thesis. Observe, that if G is a finitely generated metabelian group, then

$$G \cong F/N \tag{1.1}$$

where F is a finitely generated free metabelian group free on $X = \{x_1, \ldots, x_p\}$ and

N is a normal subgroup of F. Since F is a finitely generated metabelian group, N is the normal closure of a finite set R. Here $R = \{r_1, \ldots, r_q\}$ is simply a finite set of elements r_i of F; i.e., a set of products of elements of X and their inverses. We express this isomorphism (1.1) by writing

$$G = << x_1, \ldots, x_p; r_1, \ldots, r_q >> \tag{1.2}$$

or

$$G = << X; R >> .$$

This notation should be compared with the usual presentation notation (see §2.1) in which the symbols $x_1, \ldots, x_p, r_1, \ldots, r_q$ are enclosed by single brackets. We emphasis that the notation (1.2) expresses the fact that there is a given homomorphism from the free metabelian group F on $\{x_1, \ldots, x_p\}$ onto G with kernel $gp_F(r_1, \ldots, r_q)$. We will refer to (1.2) as a *metabelian presentation* of G. Thus

$$F = << x_1, \ldots, x_p >>$$

is, as above, the free metabelian group on $\{x_1, \ldots, x_p\}$.

Notice that this discussion reveals that every finitely generated metabelian group G has a finite metabelian presentation. Henceforth, we shall simply say that G has a "finite presentation" since we will work only in the variety of metabelian groups.

1.3 Presenting metabelian groups

As we have seen every finitely generated metabelian group has a finite presentation. Our first result concerning such presentations is the following:

Theorem A *Let A and T be finitely generated abelian groups, say*

$$A =< a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+r}; a_{m+i}^{p_i} = 1 \quad (1 \leq i \leq r),$$

$$[a_i, a_j] = 1 \quad (1 \leq i < j \leq m+r) >$$

and

$$T =< t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+s}; t_{n+j}^{q_j} = 1 \quad (1 \leq j \leq s),$$

$$[t_i, t_j] = 1 \quad (1 \leq i < j \leq n+s) >$$

(see §2.2 for notation). Then the wreath product

$$W = A \wr T$$

can be presented in the form:

$$<< a_1, \ldots, a_{m+r}, t_1, \ldots, t_{n+s}; \ a_{m+i}^{p_i} = 1 \quad (1 \leq i \leq r), \quad t_{n+j}^{q_j} = 1 \quad (1 \leq j \leq s),$$

$$[a_i, a_j] = 1 \quad (1 \leq i < j \leq m+r), \ [t_i, t_j] = 1 \quad (1 \leq i < j \leq n+s),$$

$$[a_i, t_j]^{a_k} = [a_i, t_j] \quad (1 \leq i, k \leq m+r, \ 1 \leq j \leq n+s),$$

$$[a_i, t_j]^{t_k^{-1}} = [a_i, t_k]^{t_j^{-1}} \quad (1 \leq i \leq m+r, \ 1 \leq j \neq k \leq n+s) >>$$

(see §4.1 for the details).

Since we shall be concerned also with groups of the form

$$B_{m,n} =<< a, t; (a^m)^t = a^n >>,$$

we remark only that these groups are simply the quotients of the usual B-S-groups

$$< a, t; (a^m)^t = a^n >$$

by their second derived groups. We shall say more about this later on.

1.4 Isoperimetric functions for metabelian groups

Let G be a metabelian group presented as (1.2). We will need to separate the right-hand-side of presentations of the form (1.2) (see §1.2 for notation) from the group G itself. We will denote such presentation symbols simply by

$$<< x_1, \ldots, x_p; r_1, \ldots, r_q >> .$$

Let F be the free metabelian group free on $X = \{x_1, \ldots, x_p\}$. Then (1.2) gives rise to an isomorphism

$$G \cong F/N,$$

where

$$N = gp_F(r_1, \ldots, r_q).$$

If a "word" $w = w(x_1, \ldots, x_p) \in F$, then w can be written in the form

$$w = x_{i_1}^{\varepsilon_1} \cdots x_{i_n}^{\varepsilon_n} \quad (x_{i_j} \in X, \varepsilon_j \in \pm 1). \tag{1.3}$$

We define the *length* of w, which we denote by

$$\ell(w) \text{ or } \ell_X(w),$$

to be the minimum of all integers n satisfying (1.3). Notice that we sometimes call this function

$$\ell : F \to \mathbf{Z}^{\geq 0}$$

the *length function* on F and denote it by ℓ_F. If u and v are X-words, then we write

$$u =_G v$$

if u and v are equal in G. We shall make use of this notation throughout, irrespective of the variety of groups that we are concerned with.

We term a word $w = w(x_1, \ldots, x_p) \in F$ a *relator* for G if $w =_G 1$; i.e., w takes the value 1 in G, or, equivalently $w \in N$. Thus

$$w = \prod_{j=1}^{k} f_j^{-1} r_{i_j}^{\varepsilon_{i_j}} f_j \quad (r_{i_j} \in \{r_1, \ldots, r_q\},\ \varepsilon_{i_j} = \pm 1). \tag{1.4}$$

Let

$$\mathcal{P} = << x_1, \ldots, x_p; r_1, \ldots, r_q >> .$$

We define the *area* $\Delta_{\mathcal{P}}(w)$ of w to be the least k for which (1.4) holds. This allows us to formulate the following:

Definition 1.5 *Let G be a finitely generated metabelian group with presentation*

$$\mathcal{P} = << x_1, \ldots, x_p; r_1, \ldots, r_q >>;$$

i.e.,

$$G = << x_1, \ldots, x_p; r_1, \ldots, r_q >> .$$

We define

$$\Phi_{\mathcal{P}}(n) = \max\{\Delta_{\mathcal{P}}(w) | \ell(w) \leq n, w =_G 1\}.$$

It is easy to see that $\Phi_{\mathcal{P}}(n)$ is an increasing function; i.e.,

$$\text{if } \ n < m \ \text{ then } \ \Phi_{\mathcal{P}}(n) \leq \Phi_{\mathcal{P}}(m).$$

Notice that $\Phi_{\mathcal{P}}(n)$ clearly depends on the given presentation $\mathcal{P}$ of G; in fact, it is, in a sense, "independent" of $\mathcal{P}$. In order to explain this, consider the collection $\mathcal{C}$ of all functions

$$f : \mathbf{N} \to \mathbf{R}^{\geq 0}.$$

We introduce a relation $\preccurlyeq$ on $\mathcal{C}$ as follows: if $f_1, f_2 \in \mathcal{C}$, then we define

$$f_1 \preccurlyeq f_2$$

if there exist positive constants k_1, k_2, k_3 such that

$$f_1(n) \leq k_2 f_2(k_1 n) + k_3 n \quad \forall\, n \in \mathbf{N}.$$

Then we define

$$f_1 \simeq f_2 \ \text{ if } \ f_1 \preccurlyeq f_2 \;\&\; f_2 \preccurlyeq f_1.$$

"$\simeq$" is an equivalence relation on $\mathcal{C}$. Then it is not too hard to prove the following

Theorem B1 *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations of the finitely generated metabelian group G. Then*

$$\Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{Q}}.$$

So these functions, termed *isoperimetric functions*, are independent, in the above sense, of the given presentations. Henceforth, we choose any one of the functions in the equivalence class of $\Phi_{\mathcal{P}}(n)$ and denote it by Φ_G or $\Phi_G(n)$.

The proof of Theorem B1 follows that of the corresponding theorem for the usual presentations in the variety of all groups (cf. e.g. [BMS]). We shall give a complete proof of this theorem in §3.1.

It is, in general, difficult to compute upper bounds for Φ_G and even harder to compute lower bounds. Here we will prove the following

Theorem B2 *Let W be the wreath product of the groups A and T, as described in Theorem A. Then*

$$\Phi_W(\mathcal{N}) \preccurlyeq \begin{cases} \mathcal{N}^{4\cdot 3^{n-1}} & \text{if } r = s = 0, \\ \mathcal{N}^{8\cdot 3^{n+s-1}} & \text{otherwise} \end{cases}$$

(see §4.2 for a complete proof).

1.5 Lower bounds for isoperimetric functions

Let G be a metabelian group and let

$$G \cong F/N,$$

where F is a free metabelian group. Then we define

$$H_1(G,\mathbf{Z}) = G/G' = F/F'N \text{ and } H_2(G,\mathbf{Z}) = \frac{[F,F] \cap N}{[F,N]}$$

(see §2.1 for notation). This group is independent (up to isomorphism) of the choices of F and N above, depending only on G. We can, as in [BMS], define what we term, a *centralized isoperimetric function.* The definition closely follows that of $\Phi_{\mathcal{P}}$ discussed in §1.4. In more detail, suppose that

$$\mathcal{P} = \ll x_1, \ldots, x_p; r_1, \ldots, r_q \gg$$

is a presentation of the finitely generated metabelian group G:

$$G = \ll x_1, \ldots, x_p; r_1, \ldots, r_q \gg .$$

Then, as before, $G \cong F/N$. If $w = w(x_1, \ldots, x_p) =_G 1$, then $w \in N$. Then

$$w \equiv \prod_{j=1}^{k} f_j^{-1} r_{i_j}^{\varepsilon_{i_j}} f_j \text{ modulo } [F, N], \tag{1.6}$$

where $r_{i_j} \in \{r_1, \ldots, r_q\}$, $\varepsilon_{i_j} = \pm 1$, & $f_j \in F$.

We define the *centralized area* $\Delta_{\mathcal{P}}^{cent}(w)$ of w to be the least k for which (1.6) holds. Notice that modulo $[F, N]$

$$f_j^{-1} r_{i_j}^{\varepsilon_{i_j}} f_j = r_{i_j}^{\varepsilon_{i_j}}.$$

Then we have the following

Definition 1.7 *As in Definition 1.5, we define*

$$\Phi_{\mathcal{P}}^{cent}(n) = \max\{\Delta_{\mathcal{P}}^{cent}(w) | \ell(w) \leq n, w =_G 1\}.$$

Then $\Phi_{\mathcal{P}}^{cent}(n)$ is independent of the presentation $\mathcal{P}$ of G in the sense of the equivalence relation discussed in §1.4. We then denote any one of the functions in the relevant equivalence class by Φ_G^{cent} or $\Phi_G^{cent}(n)$.

Here our main observation is

Theorem C *$\Phi_G^{cent}(n) \simeq n$ if $H_2(G, \mathbf{Z})$ is finite.*

Theorem C is the analogous result to one proved in [BMS]. The main point of such centralized isoperimetric function is the obvious observation

$$\Phi_G^{cent} \preccurlyeq \Phi_G.$$

So Φ_G^{cent} provides us with a lower bound for Φ_G.

1.6 Isoperimetric functions for extensions

Suppose that G is a finitely generated metabelian group. Then G is the middle of a short exact sequence

$$1 \to A \to G \to T \to 1, \tag{1.8}$$

where A and T are abelian groups. We can choose $A = G'$ and $T = G/A$. But there are sometimes other choices available. The main point here is that given such a short exact sequence, we can view A as a module over the integral group ring $\mathbf{Z}T$ of T. To see how this comes about, we identify A with a normal subgroup of G and T with G/A. Now if $t \in T$, then $t = gA$, for some $g \in G$. We define an action of T on A by

$$a \bullet t = g^{-1}ag \quad (a \in A, g \in G).$$

Philip Hall has pointed out that this turns A into a finitely generated $\mathbf{Z}T$-module. We now think of A as a module over a polynomial ring $\Lambda = \mathbf{Z}[x_1, \ldots, x_p, y_1, \ldots, y_p]$ in twice as many variables as the number of generators as T (see §3.3 for more details). Observe that, by Hilbert's Basis Theorem, A can be expressed as a quotient of a finitely generated free Λ-module F by a finitely generated submodule N:

$$A \cong F/N$$

By mimicking the discussion above, we can now introduce the notion of an isoperimetric function for A denoted by Φ_A or $\Phi_A(n)$. The details closely resemble those discussed in §1.4 and will be left to §3.3. Our result in §3.3 is the following:

Theorem D *Assuming the notation above, if $A = G'$ and $T = G/G'$, then*

$$\Phi_G(n) \preccurlyeq \Phi_A(n^4).$$

Notice that if the short exact sequence (1.8) "splits" and T is free abelian, then we can prove somewhat more, namely that

$$\Phi_G(n) \preccurlyeq \max\{n^2 \cdot 3^n, \Phi_A(n^2)\}.$$

Indeed more is true, namely

Theorem E *Let T be a finitely generated free abelian group and let M be a finitely generated $\mathbb{Z}T$-module. Form the semidirect product*

$$G = M \rtimes T.$$

Then,

$$\Phi_G(n) \preccurlyeq \max\{n^2 \cdot 3^n, \Phi_M(n^2)\}.$$

Theorem E will be proved in §7.1.

2 Preliminaries

2.1 Notations and definitions

We express the fact that H is a subgroup of a group G by writing $H \leq G$; if H is a normal subgroup we write $H \trianglelefteq G$. Let Y be a subset of G. We denote the normal closure of Y in G by

$$gp_G(Y);$$

thus

$$gp_G(Y) = gp(g^{-1}yg \mid g \in G, y \in Y);$$

in other words, the normal subgroup of G generated by a set Y is simply $gp_G(Y)$. We write

$$G = < X; R >$$

if G has a presentation $< X; R >$; i.e., X comes equipped with a set map $\sigma : X \to G$ such that the extension of σ to a homomorphism from the free group E freely generated by X is onto G and has kernel, the normal closure $N = gp_E(R)$ of R in E. So, as usual, we have

$$F \cong E/N.$$

If $X = \{x_1, \ldots, x_p\}$ and $R = \{r_1, \ldots, r_q\}$ then we simply write

$$G = < x_1, \ldots, x_p; r_1, \ldots, r_q > .$$

The map σ described above, is usually given implicitly. As noted in the introduction we sometimes refer to E (above) as the absolutely free group on X or the free group freely generated by X.

If now G is any group and $x, y \in G$, we denote $x^{-1}y^{-1}xy$ by $[x, y]$ and $x^{-1}yx$ by y^x. If H and K are non-empty subsets of G, we define

$$[H, K] = gp([a, b] \mid a \in H, b \in K).$$

Then the commutator subgroup G' of G is defined by $G' = [G, G]$. The commutator subgroup of a group G is sometimes termed the derived group of G. We define the second derived group of G or second commutator subgroup, to be $(G')'$ which we denote by G''. A group is termed metabelian if $[G,G]$ is abelian or $G'' = 1$ or equivalently G is the middle of a short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow T \rightarrow 1, \tag{2.1}$$

where A and T are abelian. We will usually identify A with a normal subgroup of G and T with the quotient G/A. So (2.1) is essentially the sequence

$$A \hookrightarrow G \twoheadrightarrow T, \tag{2.2}$$

We denote the integral group ring of a group T by $\mathbb{Z}T$. Then A (in (2.2)) can be turned into a $\mathbb{Z}T$-module by defining

$$a \bullet \left(\sum_{i=1}^{m} n_i g_i A\right) = \prod_{i=1}^{m} \left(a^{g_i}\right)^{n_i}. \tag{2.3}$$

We will sometimes resort to additive notation under these circumstances and then (2.3) becomes

$$a \bullet \left(\sum_{i=1}^{m} n_i g_i A\right) = \sum_{i=1}^{m} n_i\left(a^{g_i}\right). \tag{2.4}$$

and (2.1) is replaced by

$$0 \rightarrow A \rightarrow G \rightarrow T \rightarrow 1. \tag{2.5}$$

A metabelian group F is termed free metabelian if it comes equipped with a map θ from a set X into F such that for every metabelian group G and every set map $\sigma : X \rightarrow G$ there exists a unique homomorphism $\phi : F \rightarrow G$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "\sigma"'] & F \arrow[d, "\phi"] \\ & G \end{tikzcd}$$

We say that F is a free metabelian group free on X or freely generated by X. It is not hard to show that if F is free metabelian on X and if E is the absolutely free group on X, then

$$F \cong E/E''.$$

If now G is any metabelian group then we write

$$G = << X; R >>$$

if X comes equipped with a map σ into G such that the extension of σ to a homomorphism from the free metabelian group F on X has kernel $gp_F(R)$. If $X = \{x_1, \ldots, x_p\}$ and $R = \{r_1, \ldots, r_q\}$ we simply write

$$G = << x_1, \ldots, x_p; r_1, \ldots, r_q >>$$

and term

$$<< x_1, \ldots, x_p; r_1, \ldots, r_q >>$$

a metabelian presentation of G.

2.2 *Max* and *Max* − *n*

A group G is said to satisfy the maximal condition if every properly ascending chain of subgroups is finite or equivalently every subgroup is finitely generated. We then write $G \in Max$. Similarly, a group G is said to satisfy the maximal condition for normal subgroups if every properly ascending chain of normal subgroups is finite or equivalently every normal subgroup is finitely generated as a normal subgroup; i.e., every normal subgroup is the normal closure of a finite set. We then write $G \in Max - n$. The condition $Max - n$ was introduced by P. Hall in [H], where he proved, in particular, the following

Theorem 2.6 *(P. Hall) A finitely generated metabelian group satisfies Max-n.*

It follows immediately that

Corollary 2.7 *If G is a finitely generated metabelian group then G has a metabelian presentation of the form*

$$G = << x_1, \ldots, x_p; r_1, \ldots, r_q >> \quad (p, q < \infty).$$

We will make use of this corollary throughout this thesis.

2.3 Wreath products

A group G is said to be a product of its subgroup A and T if

$$G = gp(A \cup T).$$

If W is a product of A and T we say that W is the wreath product of A and T if

$$B = gp_W(A) = \prod_{t \in T} A^t;$$

i.e., B is the restricted direct product of the conjugates of A by the elements $t \in T$. We then write

$$W = A \wr T.$$

Notice that

$$W = B \rtimes T,$$

where here we use the notation $W = B \rtimes T$ to express the fact W is the semidirect product of B and T; i.e., that

$$B \trianglelefteq W, \quad W = BT \quad \text{and} \quad B \cap T = 1.$$

Notice that if A and T are finitely generated then so too is $A \wr T$. Moreover if A and T are abelian, then $A \wr T$ is metabelian. We will be concerned with such metabelian groups in the sequel. In fact, many of the groups that we will work on take the form

$$B \rtimes T,$$

where B and T are abelian.

2.4 Word length in groups and modules

Suppose G is a group generated by a set $X = \{x_1, \ldots, x_p\}$. If $g \in G$, then g can be written in the form

$$g = x_{i_1}^{\varepsilon_1} \cdots x_{i_n}^{\varepsilon_n} \quad (\varepsilon_i = \pm 1, \quad x_i \in X).$$

We define the *length* of g

$$\ell(g) = \min\{n \mid g = x_{i_1}^{\varepsilon_1} \cdots x_{i_n}^{\varepsilon_n}, \ \varepsilon_i = \pm 1, \quad x_i \in X\}.$$

Notice that the definition is clearly dependent on the set X of generators. We then denote $\ell(g)$ also by $\ell_X(g)$ in order to avoid confusion. Notice that

$$\ell(1) = 0.$$

If E is the absolutely free group on X and if $g \in E$, then $\ell(g)$ is as usual the length of the unique freely reduced word equal to g.

Now let Λ be a commutative unitary ring generated by the finite set $Y = \{y_1, \ldots, y_m\}$. As usual, we term a product of the form

$$y_1^{e_1} \cdots y_m^{e_m} \quad (e_1 \geq 0, \ldots, e_m \geq 0)$$

a monomial. Then the unit element 1 is the trivial monomial in which the exponents of y_i are all zero. Every element $\lambda \in \Lambda$ can be expressed as a sum of integral multiples of monomials in $y_1, \ldots, y_m$:

$$\lambda = c_0 + \sum_{(e_1,\ldots,e_m)\neq(0,0,\cdots,0)} c_{e_1,\ldots,e_m}\, y_1^{e_1} \cdots y_m^{e_m} \tag{2.8}$$

where $c_0, c_{e_1,\ldots,e_m} \in \mathbf{Z},\ e_1, \ldots, e_m \geq 0$. Define the length of λ

$$\ell_Y(\lambda) = \min\, \{\ |c_0| + \sum_{(e_1,\ldots,e_m)\neq(0,0,\cdots,0)} |c_{e_1,\ldots,e_m}|\ (e_1 + \cdots + e_m)\}$$

where the sums involved range over all possible representations of λ in the form (2.8). Notice that if Λ is actually the usual polynomial ring in the variables $y_1, \ldots, y_m$; i.e.,

$$\Lambda = \mathbf{Z}[y_1, \ldots, y_m]$$

then λ has a unique expression of the kind given by (2.8). Under these circumstances we define the *width* $\omega(\lambda)$ of λ by

$$\omega(\lambda) = |c_0| + \sum_{(e_1,\ldots,e_m)\neq(0,0,\cdots,0)} |c_{e_1,\ldots,e_m}|;$$

i.e., the number of monomials involved in the unique representation of λ in the form (2.8). For instance,

$$\omega(y_1y_2 - 2y_2y_3 + 5) = 8.$$

The "width" of a *module word* will be used in defining the isoperimetric function of a module (see §2.5).

If M is a right Λ-module then we denote the submodule of M generated by the subset Ξ of M by

$$md_M(\Xi).$$

If $\Xi = \{\xi_1, \ldots, \xi_k\}$ then every element $\xi \in md_M(\Xi)$ can be written in the form

$$\xi = \xi_1 f_1 + \cdots + \xi_k f_k \tag{2.9}$$

where $f_i \in \Lambda$. If every $\xi \in md_M(\Xi)$ has a unique expression of the form (2.9), we call $md_M(\Xi)$ a free right Λ-module free on Ξ. We shall say more about this in the next section.

We define the length $\ell'(\xi)$ of the module word ξ by

$$\ell'(\xi) = \ell'_\Xi(\xi) = \min\{\sum_{i=1}^{k} \ell_Y(f_i)\}$$

where the minimum here is taken over all possible expressions for ξ of the form (2.9).

2.5 Presentations and isoperimetric functions for modules

As usual, we can also use the universal mapping property to define what is meant by a *free module.* Let Λ be a commutative unitary ring. A right Λ-module F is termed a free module if it comes equipped with a map θ from a set Ξ into F such that for every right Λ-module M and every set map $\sigma : \Xi \to M$ there exists a unique homomorphism $\phi : F \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
\Xi \arrow[r, "\theta"] \arrow[dr, "\sigma"'] & F \arrow[d, "\phi"] \\
& M
\end{tikzcd}$$

Given any finitely generated right Λ-module M, there exists a finitely generated free right Λ-module F such that M can be expressed as a quotient F/N of F for some submodule N of F; i.e.,

$$M \cong F/N,$$

where F is free on Ξ and $N = md_F(\Gamma)$. Adopting the terminology that we have introduced above, we find then that we can express this isomorphism in presentation form as follows:

$$M = < \Xi; \Gamma > .$$

Now $\Lambda = \mathbf{Z}[y_1, \ldots, y_m]$ is a polynomial ring in finitely many variables and F is finitely generated. Hence, by Hilbert's Basis Theorem [AM], F is Noetherian. Consequently N is finitely generated. It follows that Γ can be chosen finite. If now $\Xi = \{\xi_1, \ldots, \xi_k\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_l\}$ say, then we obtain a presentation $\mathcal{P}$ of M as a right Λ-module which takes the form

$$\mathcal{P} = < \xi_1, \ldots, \xi_k; \gamma_1, \ldots, \gamma_l >;$$

i.e.,

$$M = < \xi_1, \ldots, \xi_k; \gamma_1, \ldots, \gamma_l > .$$

We now are in the position to define what is meant by the isoperimetric function $\Phi_{\mathcal{P}}$ for this presentation $\mathcal{P}$. With this in mind we put $Y = \{y_1, \ldots, y_m\}$. Let w be a word of length at most n in M and suppose that $w =_M 0$; i.e., w evaluated in M is 0. Thus w has an expression of the form

$$w = \xi_1 f_1 + \cdots + \xi_k f_k \quad \text{with} \quad f_i \in \Lambda \ \& \ \sum_{i=1}^{k} \ell_Y(f_i) \leq n. \tag{2.10}$$

$\Phi_{\mathcal{P}}(n)$ is the "minimal number" of defining relators needed to deduce that $w =_M 0$ for all such w of length at most n. More precisely, let $w =_M 0$, where w is written in the form (2.10). Then, viewing w as an element of M, we can also express it in the form

$$w = \gamma_1 g_1 + \cdots + \gamma_l g_l, \quad \text{with} \quad g_j \in \Lambda \tag{2.11}$$
$$\& \ \sum_{j=1}^{l} \omega(g_j) \leq \Phi_{\mathcal{P}}(n).$$

We define the *area* of a module word w, denoted by $\Delta_{\mathcal{P}}(w)$, to be the least value of $\sum_{j=1}^{l} \omega(g_j)$ for which (2.11) holds.

Definition 2.12 *The isoperimetric function $\Phi_{\mathcal{P}}$ is defined by*

$$\Phi_{\mathcal{P}}(n) = \max \left\{ \Delta_{\mathcal{P}}(w) | \ell'(w) \ \leq n \text{ and } w =_M 0 \right\}.$$

Assuming that Λ is fixed once and for all, $\Phi_{\mathcal{P}}(n)$ turns out to be independent of the presentation of M in the same sense as that elaborated on in §1.4. We then denote any one of the functions in the equivalence class of $\Phi_{\mathcal{P}}(n)$ by Φ_M or $\Phi_M(n)$. We shall give a complete proof of the invariance of $\Phi_M(n)$ in §3.2. It depends on the analogue for modules of the notion of a retract in groups, which amounts to nothing more than a direct summand.

Definition 2.13 *A submodule L of a right Λ-module M is called a retract of M if there exists a submodule A of M such that M is the direct sum of L and A.*

3 Basic Structure

3.1 The invariance of Φ_G

In this section, we show that the isoperimetric function of a metabelian group is an invariant.

Let now F be a free metabelian group with a given set of free generators.

Lemma 3.1 *If u and v are two words in F then $\ell(uv) \leq \ell(u) + \ell(v)$.*

Proof. The proof is obvious and is omitted.

Definition 3.2 *A subgroup H of a group G is called a retract of G if there exists a normal subgroup A of G such that $G = AH$ and $A \cap H = 1$.*

We will, throughout this section, assume that

$$F = << a_1, \ldots, a_k >>, \quad N = gp_F(r_1, \ldots, r_l)$$

and

$$K = << b_1, \ldots, b_p >>, \quad P = gp_K(s_1, \ldots, s_q).$$

Lemma 3.3 *Suppose that G is a finitely generated metabelian group and H is a subgroup of G. If H is a retract of G and $G \cong F/N$ and $H \cong K/P$ where F and K are free metabelian groups and N and P are normal subgroups of F and K, respectively, then there are homomorphisms $\phi : K \to F$ and $\psi : F \to K$ which induce $\overline{\phi} : K/P \to F/N$ and $\overline{\psi} : F/N \to K/P$, respectively, such that $\overline{\psi} \circ \overline{\phi} : K/P \to K/P$ is the identity.*

Proof. Let

$$G \cong F/N = \ll a_1, \ldots, a_k; r_1, \ldots, r_l \gg$$

and let

$$H \cong K/P = \ll b_1, \ldots, b_p; s_1, \ldots, s_q \gg .$$

Since H is a retract of G, there are homomorphisms $\overline{\phi} : H \to G$ and $\overline{\psi} : G \to H$ such that $\overline{\psi} \circ \overline{\phi} = id_H$. Now, identify H with K/P and G with F/N and choose words v_i in F and u_j in E such that $v_i N = \overline{\phi}(b_i P)$, $i = 1, \ldots, p$ and $u_j P = \overline{\psi}(a_j N)$, $j = 1, \ldots, k$. Then, we define homomorphisms $\phi : K \to F$ by $b_i \mapsto v_i$, $i = 1, \ldots, p$ and $\psi : F \to K$ by $a_j \mapsto u_j$, $j = 1, \ldots, k$. This completes the proof. □

Lemma 3.4 *Suppose that G is a finitely generated metabelian group and that H is a retract of G. If $\mathcal{P}_G$ and $\mathcal{P}_H$ are finite metabelian presentations of G and H, respectively, then there are positive integer constants k_1, k_2, k_3 such that the following inequality holds:*

$$\Phi_{\mathcal{P}_H}(n) \leq k_2 \Phi_{\mathcal{P}_G}(k_1 n) + k_3 n \quad \forall\, n \in \mathbb{N};$$

i.e., $\Phi_{\mathcal{P}_H} \preccurlyeq \Phi_{\mathcal{P}_G}$.

Proof. We adopt here the notation used in the proof of Lemma 3.3.

Now let

$$\mathcal{P}_G = \ll a_1, \ldots, a_k; r_1, \ldots, r_l \gg$$

and let

$$\mathcal{P}_H = \ll b_1, \ldots, b_p; s_1, \ldots, s_q \gg .$$

Observe that $b_i^{-\varepsilon_i}\psi(\phi(b_i))^{\varepsilon_i} \in P$, $\varepsilon_i = \pm 1$ for all $1 \le i \le p$ and that $\psi(r_j) \in P$ for all $1 \le j \le l$. Let ℓ_F be the length function on F and let ℓ_K be the length function on K. Then, we define three integer constants k_1, k_2, k_3 :

$$\begin{aligned} k_1 &= \max\left\{\ell_F(\phi(b_i)) \mid 1 \le i \le p\right\}, \\ k_2 &= \max\left\{\Delta_{\mathcal{P}_H}(\psi(r_i)) \mid 1 \le i \le l\right\}, \\ k_3 &= \max\left\{\Delta_{\mathcal{P}_H}(b_i^{-\varepsilon_i}\psi(\phi(b_i))^{\varepsilon_i}) \mid 1 \le i \le p,\ \varepsilon_i = \pm 1\right\}. \end{aligned}$$

Clearly, $k_1, k_2, k_3 \ge 1$. Given any n, let w be any word in K with

$$w =_K \prod_{j=1}^{m} b_{i_j}^{e_j}, \tag{3.5}$$

where $e_j = \pm 1$ and $\ell_K(w) = m \le n$. Moreover, if $w \in P$, then $\phi(w) \in N$. Therefore, $\phi(w)$ has the following expression:

$$\phi(w) =_F \prod_{j=1}^{\mathcal{N}} u_j^{-1} r_{i_j}^{\varepsilon_{i_j}} u_j \quad (\varepsilon_{i_j} = \pm 1), \tag{3.6}$$

such that

$$\mathcal{N} = \Delta_{\mathcal{P}_G}(\phi(w)).$$

We claim that the following inequalities hold:

i) $\Delta_{\mathcal{P}_H}(w\ \psi(\phi(w))^{-1}) \le k_3 \ell_K(w)$;

ii) $\mathcal{N} \le \Phi_{\mathcal{P}_G}(k_1 \ell_K(w))$;

iii) $\Delta_{\mathcal{P}_H}(\psi(\phi(w))) \le k_2 \mathcal{N}$.

To prove i),

$$\begin{aligned} w\ \psi(\phi(w))^{-1} &= \left(\textstyle\prod_{j=1}^{m} b_{i_j}^{e_j}\right)\psi(\phi(\textstyle\prod_{j=1}^{m} b_{i_j}^{e_j}))^{-1} \\ &= b_{i_1}^{e_1}\cdots b_{i_m}^{e_m}\psi(\phi(b_{i_m}))^{-e_m}\cdots\psi(\phi(b_{i_1}))^{-e_1} \\ &= b_{i_1}^{e_1}\cdots b_{i_{m-1}}^{e_{m-1}}b_{i_m}^{e_m}\psi(\phi(b_{i_m}))^{-e_m}b_{i_{m-1}}^{-e_{m-1}}\cdots b_{i_1}^{-e_1}\cdot \\ &\quad\ b_{i_1}^{e_1}\cdots b_{i_{m-2}}^{e_{m-2}}b_{i_{m-1}}^{e_{m-1}}\psi(\phi(b_{i_{m-1}}))^{-e_{m-1}}b_{i_{m-2}}^{-e_{m-2}}\cdots b_{i_1}^{-e_1}\cdot \\ &\quad\ \cdots b_{i_1}^{e_1}b_{i_2}^{e_2}\psi(\phi(b_{i_2}))^{-e_2}b_{i_1}^{-e_1}\cdot b_{i_1}^{e_1}\psi(\phi(b_{i_1}))^{-e_1} \\ &= \textstyle\prod_{\alpha} v_{\alpha}^{-1}b_{\alpha}^{-\varepsilon_\alpha}\psi(\phi(b_{\alpha}))^{\varepsilon_\alpha}v_{\alpha}, \end{aligned}$$

where the product ranges over all of the $b_{\alpha}^{-\varepsilon_\alpha}\psi(\phi(b_{\alpha}))^{\varepsilon_\alpha}$ involved in the expression for $w\psi(\phi(w))^{-1}$ and there are exactly $\ell_K(w) = m$ conjugates in the product. Hence,

$$\Delta_{\mathcal{P}_H}(w\ \psi(\phi(w))^{-1}) \le k_3\ell_K(w).$$

To prove ii), by (3.5), we have

$$\phi(w) =_K \prod_{j=1}^{m}\phi(b_{i_j})^{e_j}.$$

Hence,

$$\ell_F(\phi(w)) \le \sum_{j=1}^{m}\ell_F(\phi(b_{i_j})) \le k_1 m = k_1\ell_K(w).$$

On the other hand, $\mathcal{N} \le \Phi_{\mathcal{P}_G}(k_1\ell_K(w))$, since $\mathcal{N} = \Delta_{\mathcal{P}_G}(\phi(w))$.

To prove (iii), by (3.6), we have

$$\psi(\phi(w)) =_K \prod_{j=1}^{\mathcal{N}}\psi(u_j)^{-1}\psi(r_{i_j})^{\epsilon_{i_j}}\psi(u_j).$$

Hence,

$$\Delta_{\mathcal{P}_H}(\psi(\phi(w))) \le k_2\mathcal{N}.$$

Finally, if $w =_H 1$ and $w = w\ \psi(\phi(w))^{-1} \cdot \psi(\phi(w))$, then

$$\begin{aligned} \Delta_{\mathcal{P}_H}(w) &\leq \Delta_{\mathcal{P}_H}(\psi(\phi(w))) + \Delta_{\mathcal{P}_H}(w\ \psi(\phi(w))^{-1}) \\ &\leq k_2\mathcal{N} + k_3\ell_K(w) && \text{(by i) \& iii))} \\ &\leq k_2\Phi_{\mathcal{P}_G}(k_1\ell_K(w)) + k_3\ell_K(w) && \text{(by ii))} \end{aligned}$$

Since, by definition, $\Phi_{\mathcal{P}_G}$ is an increasing function and k_1, k_2, k_3 are positive, we have

$$\Delta_{\mathcal{P}_H}(w) \leq k_2\Phi_{\mathcal{P}_G}(k_1 n) + k_3 n \quad \forall\ w \text{ with } \ell(w) \leq n.$$

Then,

$$\Phi_{\mathcal{P}_H}(n) \leq k_2\Phi_{\mathcal{P}_G}(k_1 n) + k_3 n \quad \forall\ n;$$

i.e., $\Phi_{\mathcal{P}_H} \preccurlyeq \Phi_{\mathcal{P}_G}$ □

Theorem B1 *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations of the finitely generated metabelian group G. Then*

$$\Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{Q}}.$$

Proof. Since every group can be viewed as a retract of the group itself, we have $\Phi_{\mathcal{P}} \preccurlyeq \Phi_{\mathcal{Q}}$ and $\Phi_{\mathcal{Q}} \preccurlyeq \Phi_{\mathcal{P}}$. This completes the proof of the theorem. □

Combining Lemma 3.4 and Theorem B1 we have the following:

Corollary 3.7 *If H is a retract of G, then $\Phi_H \preccurlyeq \Phi_G$.*

We complete this section by showing how to compute an upper bound for the isoperimetric function of the wreath product of one infinite cyclic group by another. The methods involved in this computation are typical of some of the more elaborate ones that we shall develop in the sequel.

Theorem 3.8 *Let G be the wreath product of the infinite cyclic group on a by the infinite cyclic group on t:*

$$G = < a > \wr < t > .$$

Then $\Phi_G(n) \leq n^3$.

A key step in the proof of Theorem 3.8 is the following:

Lemma 3.9 *Let*

$$G = < a > \wr < t >,$$

the wreath product of one infinite cyclic group by another. Then

i) G is a one-relator group with presentation $\mathcal{P} = << a, t; [a, a^t] = 1 >>$;

ii) $\Delta_{\mathcal{P}}([a, a^{t^n}]) \leq n \quad \forall n \in \mathbf{N}$.

Proof. It is easy to see, and well-known, that $G = < a > \wr < t >$ can be presented as follows:

$$G = < a, t; [a, a^{t^i}] = 1 \quad (i = 1, 2, \ldots) > .$$

It follows then (see Theorem 2.6 & Corollary 2.7), that G has a finite metabelian presentation. Indeed our objective is to prove i). In the course of proving i) we shall keep track of the number of uses of the given relator, which will give rise to the proof of ii).

Let

$$H = << a, t; [a, a^t] = 1 >> .$$

Then

$$H \cong F/N,$$

where F is the free metabelian group free on $\{a, t\}$ and

$$N = gp_F([a, a^t]).$$

In proving $G = H$, it suffices to prove that

$$[a, a^{t^i}] \in N = gp_F([a, a^t]) \quad \forall\, i \in \mathbf{N};$$

i.e., all relations $[a, a^{t^i}] = 1$ can be deduced from $[a, a^t] = 1$.

To see this, we first make use of the following identities:

$$[xy, z] = [x, z]^y [y, z],$$

$$[x, yz] = [x, z][x, y]^z.$$

It is then easy to verify that

$$\text{if } [x, y] = 1 \text{ then } [x^{\varepsilon_1}, y^{\varepsilon_2}] = 1 \ \forall\, \varepsilon_1 = \pm 1, \ \varepsilon_2 = \pm 1.$$

Hence,

$$\text{if } [a, a^t] = 1 \text{ then } [a^{\varepsilon_1}, a^{\varepsilon_2 t}] = 1 \ \forall\, \varepsilon_1 = \pm 1, \ \varepsilon_2 = \pm 1. \tag{3.10}$$

Next, we claim, by induction, that $\forall\, n \in \mathbf{N}$

if $[a, a^t] = 1$ then $[a, a^{t^n}] = 1$;

there are n uses of the defining relation needed to make this deduction. (3.11)

If $n = 2$, observe that $aa^{-t} = at^{-1}a^{-1}t = [a^{-1}, t] \in F'$ and $a^t a^{-t^2} = (aa^{-t})^t = [a^{-1}, t]^t \in F'$. Since F is metabelian, aa^{-t} and $a^t a^{-t^2}$ commute; i.e., $[aa^{-t}, a^t a^{-t^2}] = 1$. Then,

$$\begin{aligned}
1 &= [aa^{-t}, a^t a^{-t^2}] \\
&= a^t a^{-1} a^{t^2} a^{-t} a \underbrace{a^{-t} a^t}_{\text{cancel}} a^{-t^2} \\
&= a^t a^{-1} \underbrace{a^{t^2} a^{-t}}_{\text{commute}} a a^{-t^2} && (1) \\
&= a^t \underbrace{a^{-1} a^{-t}}_{\text{commute}} a^{t^2} a a^{-t^2} && (2) \\
&= a^{-1} a^{t^2} a a^{-t^2} \\
&= [a, a^{-t^2}]
\end{aligned}$$

Notice that (1) needs the defining relator once since it is the conjugate of $a^t a^{-1}$ by t and a^t commutes with a^{-1} by (3.10). Moreover, (2) needs it once as well. Hence, (3.11) holds when $n = 2$.

Now suppose that (3.11) holds for $n = k$. If $n = k + 1$, then

$$\begin{aligned}
1 &= [\underbrace{aa^{-t^k}}_{\in F'}, \underbrace{a^{t^k} a^{-t^{k+1}}}_{\in F'}] \\
&= a^{t^k} a^{-1} a^{t^{k+1}} a^{-t^k} a \underbrace{a^{-t^k} a^{t^k}}_{\text{cancel}} a^{-t^{k+1}} \\
&= a^{t^k} a^{-1} \underbrace{a^{t^{k+1}} a^{-t^k}}_{\text{commute}} a a^{-t^{k+1}} && (3) \\
&= a^{t^k} \underbrace{a^{-1} a^{-t^k}}_{\text{commute}} a^{t^{k+1}} a a^{-t^{k+1}} && (4) \\
&= a^{-1} a^{t^{k+1}} a a^{-t^{k+1}} \\
&= [a, a^{-t^{k+1}}].
\end{aligned}$$

In (3), since $[a^t, a^{-1}] = 1$ and $a^{t^{k+1}} a^{-t^k}$ is the conjugate of $a^t a^{-1}$ by t^k we have $[a^{t^{k+1}}, a^{-t^k}] = 1$; i.e., they commute. Here we use the defining relator only once. In

(4), since $[a, a^{t^k}] = 1$ we have $[a^{-1}, a^{-t^k}] = 1$, by (3.10). Moreover, by the inductive hypothesis, there are k uses of the defining relator needed. Hence, the total number of uses is $k+1$. This completes the proof of (3.11)and hence also that of i) and ii). □

Lemma 3.12 *Let F be the free metabelian group on $\{a, t\}$ and let w be a word in F with $\ell(w) \leq n$; i.e.,*

$$w = a^{d_1} t^{e_1} \cdots a^{d_l} t^{e_l} \quad \text{with} \quad \sum_{i=1}^{l} |d_i| + \sum_{i=1}^{l} |e_i| \leq n.$$

Then w can be re-expressed in the form

$$a_{i_0}^{\alpha_0} a_{i_1}^{\alpha_1} \cdots a_{i_k}^{\alpha_k} t^{e_1 + \cdots + e_l} \quad (i_0 < i_1 < \cdots < i_k, \ k \geq 0, \ \alpha_i \in \mathbf{Z}).$$

where $a_i = a^{t^i}$, $i \in \mathbf{Z}$ with the aid of at most n^3 uses of the relator $[a, a^t]$.

Proof. Observe that

$$\begin{aligned} w &= a^{d_1} t^{e_1} a^{d_2} t^{e_2} \cdots a^{d_l} t^{e_l} \\ &= a^{d_1} \underbrace{t^{e_1} a^{d_2} t^{-e_1}} t^{e_1+e_2} a^{d_3} t^{e_3} \cdots a^{d_l} t^{e_l} \\ &\ \ \vdots \\ &= a^{d_1} \underbrace{t^{e_1} a^{d_2} t^{-e_1}} \underbrace{t^{e_1+e_2} a^{d_3} t^{-(e_1+e_2)}} \cdots \underbrace{t^{e_1+\cdots+e_{l-1}} a^{d_l} t^{-(e_1+\cdots+e_{l-1})}} t^{e_1+\cdots+e_l} \\ &= a_0^{d_1} a_{-e_1}^{d_2} a_{-(e_1+e_2)}^{d_3} \cdots a_{-(e_1+\cdots+e_{l-1})}^{d_l} t^{e_1+\cdots+e_l}. \qquad (*) \end{aligned}$$

Let

$$w' = a_0^{d_1} a_{-e_1}^{d_2} a_{-(e_1+e_2)}^{d_3} \cdots a_{-(e_1+\cdots+e_{l-1})}^{d_l}.$$

We shall count the number of uses of the relator $[a, a^t]$ in rewriting w' in ascending order. Before the counting process, we observe that there are at most $|i-j|$ uses in switching a_i with a_j. This follows readily on invoking part ii) of Lemma 3.9 and will left to the reader. It is not hard to see that we can assume, without any loss of generality, that $d_i > 0$, $\forall i$. Notice that the worst case is

$$0 > -e_1 > -(e_1 + e_2) > \cdots > -(e_1 + \cdots + e_{l-1})$$

since in this case the a_i's will travel the most places from right to left. We shall move every $a_{-(e_1+\cdots+e_{l-1})}$ to the left most place across w', involving the d_l occurrences of $a_{-(e_1+\cdots+e_{l-1})}$. Next we move every $a_{-(e_1+\cdots+e_{l-2})}$ to the place between $a^{d_l}_{-(e_1+\cdots+e_{l-1})}$ and $a_0^{d_1}$ and so on. In this way we are able to transform w' to a word in ascending order, as detailed below:

$$w' = a_0^{d_1} a_{-e_1}^{d_2} a_{-(e_1+e_2)}^{d_3} \cdots a_{-(e_1+\cdots+e_{l-1})}^{d_l}$$

$$= a_0^{d_1} a_{-e_1}^{d_2} a_{-(e_1+e_2)}^{d_3} \cdots a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}-1} a_{-(e_1+\cdots+e_{l-1})} a_{-(e_1+\cdots+e_{l-2})} a_{-(e_1+\cdots+e_{l-1})}^{d_l-1}$$

(need $|e_{l-1}|$ uses)

$$= a_0^{d_1} a_{-e_1}^{d_2} \cdots a_{-(e_1+\cdots+e_{l-3})}^{d_{l-2}} a_{-(e_1+\cdots+e_{l-1})} a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}} a_{-(e_1+\cdots+e_{l-1})}^{d_l-1}$$

(accumulatively, need $|d_{l-1}||e_{l-1}|$ uses)

$$= a_0^{d_1} a_{-e_1}^{d_2} \cdots a_{-(e_1+\cdots+e_{l-4})}^{d_{l-3}} a_{-(e_1+\cdots+e_{l-1})} a_{-(e_1+\cdots+e_{l-3})}^{d_{l-2}} a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}} a_{-(e_1+\cdots+e_{l-1})}^{d_l-1}$$

(accumulatively, need $|d_{l-1}||e_{l-1}| + |d_{l-2}||e_{l-1} + e_{l-2}|$ uses)

$$\vdots$$

$$= a_{-(e_1+\cdots+e_{l-1})} a_0^{d_1} a_{-e_1}^{d_2} \cdots a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}} a_{-(e_1+\cdots+e_{l-1})}^{d_l-1}$$

(accumulatively, need $|d_{l-1}||e_{l-1}| + |d_{l-2}||e_{l-1} + e_{l-2}| + \cdots$

$+|d_1||e_{l-1} + \cdots + e_2 + e_1| \leq |d_{l-1}|n + |d_{l-2}|n + \cdots + |d_1|n \leq n^2$ uses)

$$\vdots$$

$$= a_{-(e_1+\cdots+e_{l-1})}^{d_l} a_0^{d_1} a_{-e_1}^{d_2} \cdots a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}}$$

(accumulatively, need $\leq |d_l|n^2$ uses)

$$= a_{-(e_1+\cdots+e_{l-1})}^{d_l} a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}} a_0^{d_1} a_{-e_1}^{d_2} \cdots a_{-(e_1+\cdots+e_{l-3})}^{d_{l-2}}$$

(accumulatively, need $\leq |d_l|n^2 + |d_{l-1}|n^2$ uses)

$$\vdots$$

$$= a_{-(e_1+\cdots+e_{l-1})}^{d_l} a_{-(e_1+\cdots+e_{l-2})}^{d_{l-1}} \cdots a_{-e_1}^{d_2} a_0^{d_1}$$

(totally, need $\leq (|d_l| + \cdots + |d_1|)n^2 \leq n^3$ uses) □

Proof of Theorem 3.8. By part i) of Lemma 3.9, we have a presentation

$$<< a, t; [a, a^t] = 1 >>$$

for $G =< a > \wr < t >$. Since here G is the semidirect product of the base group B and $T =< t >$; i.e.,

$$G = B \rtimes T \text{ with } B = \prod_{t \in T} < a >^t,$$

any element $g \in G$ has a normal form as follows:

$$g = a_{i_0}^{\alpha_0} a_{i_1}^{\alpha_1} \cdots a_{i_k}^{\alpha_k} t^{\beta} \quad (i_0 < i_1 < \cdots < i_k,\ k \geq 0,\ \alpha_i,\ \beta \in \mathbf{Z}).$$

Hence $g = 1$ if and only if $\alpha_0 = \alpha_1 = \cdots = \alpha_k = \beta = 0$. Let $w =_G 1$ with $\ell(w) \leq n$; i.e.,

$$w = a^{d_1} t^{e_1} \cdots a^{d_l} t^{e_l} \text{ with } \sum_{i=1}^{l} |d_i| + \sum_{i=1}^{l} |e_i| \leq n$$

(see Lemma 3.12). Rewriting w in the form $(*)$ described in the proof of Lemma 3.12, we find that

$$w' = a_0^{d_1} a_{-e_1}^{d_2} a_{-(e_1+e_2)}^{d_3} \cdots a_{-(e_1+\cdots+e_{l-1})}^{d_l} =_G 1 \tag{3.13}$$

and

$$t^{e_1+\cdots+e_l} =_G 1. \tag{3.14}$$

Notice that no relators are needed to prove (3.14), since $e_1 + \cdots + e_l$ must be 0. Hence only the number of uses of the defining relator for (3.13) counts. By Lemma 3.12, the cost of rewriting w' in normal form is at most n^3. Furthermore, the exponent sum of each a_i in the normal form for w' must be zero. Thus the number of uses of the given defining relator for G needed to prove that $w =_G 1$ is at most n^3. □

3.2 The invariance of Φ_M

Let

$$\Lambda = \mathbf{Z}[y_1, \dots, y_m]$$

be the polynomial ring over the integers in the variables $y_1, \dots, y_m$. Then, as we mentioned in §2.4, every element $\lambda \in \Lambda$ can be written uniquely in the form (2.8); i.e.,

$$\lambda = c_0 + \sum_{(e_1,\dots,e_m)\neq(0,0,\cdots,0)} c_{e_1,\dots,e_m}\, y_1^{e_1}\cdots y_m^{e_m}$$

where $c_0, c_{e_1,\dots,e_m} \in \mathbf{Z}$, $e_1, \dots, e_m \geq 0$. Let $Y = \{y_1, \dots, y_m\}$. Then the length of λ

$$\ell(\lambda) = \ell_Y(\lambda) = |c_0| + \sum_{(e_1,\dots,e_m)\neq(0,0,\cdots,0)} |c_{e_1,\dots,e_m}|\,(e_1 + \cdots + e_m) \tag{3.15}$$

Recall the definition of the width $\omega(\lambda)$ of λ

$$\omega(\lambda) = |c_0| + \sum_{(e_1,\dots,e_m)\neq(0,0,\cdots,0)} |c_{e_1,\dots,e_m}|; \tag{3.16}$$

thus $\omega(\lambda)$ is the number of monomials involved in the unique representation of λ in the form (2.8).

Lemma 3.17 *For any λ and μ in Λ, $\ell(\lambda+\mu) \leq \ell(\lambda) + \ell(\mu)$.*

Proof. For any λ and μ in Λ, there exists an integer k large enough such that

$$\lambda = c_0 + \sum_{(e_1,\dots,e_m)\in D} c_{e_1,\dots,e_m}\, y_1^{e_1}\cdots y_m^{e_m}$$

and

$$\mu = d_0 + \sum_{(e_1,\dots,e_m)\in D} d_{e_1,\dots,e_m}\, y_1^{e_1}\cdots y_m^{e_m},$$

where c_0, $c_{e_1,\dots,e_m}$, d_0, $d_{e_1,\dots,e_m} \in \mathbf{Z}$ and

$$D = \{(e_1,\dots,e_m) \mid 0 \le e_1,\dots,e_m \le k \ \ \& \ \ (e_1,\dots,e_m) \ne (0,0,\cdots,0)\}.$$

Then

$$\ell(\lambda) = |c_0| + \sum_{(e_1,\dots,e_m)\in D} |c_{e_1,\dots,e_m}| \, (e_1 + \cdots + e_m)$$

and

$$\ell(\mu) = |d_0| + \sum_{(e_1,\dots,e_m)\in D} |d_{e_1,\dots,e_m}| \, (e_1 + \cdots + e_m).$$

Hence,

$$\lambda + \mu = c_0 + d_0 + \sum_{(e_1,\dots,e_m)\in D} (c_{e_1,\dots,e_m} + d_{e_1,\dots,e_m}) \, y_1^{e_1} \cdots y_m^{e_m}$$

and

$$\begin{aligned} \ell(\lambda+\mu) &= |c_0 + d_0| + \textstyle\sum_{(e_1,\dots,e_m)\in D} |c_{e_1,\dots,e_m} + d_{e_1,\dots,e_m}|(e_1 + \cdots + e_m) \\ &\le |c_0| + \textstyle\sum_{(e_1,\dots,e_m)\in D} |c_{e_1,\dots,e_m}|(e_1 + \cdots + e_m) + \\ &\quad\ |d_0| + \textstyle\sum_{(e_1,\dots,e_m)\in D} |d_{e_1,\dots,e_m}|(e_1 + \cdots + e_m) \\ &= \ell(\lambda) + \ell(\mu). \ \square \end{aligned}$$

We define the *degree* of a monomial to be the sum of its exponents. Then, as usual, the degree of a polynomial is defined to be the degree of a monomial in the above representation of the highest degree. Moreover, the *lowest term* of a polynomial is the sum of the monomials of the lowest degree. In general

$$\ell(\lambda\mu) \le \ell(\lambda)\ell(\mu)$$

does not always hold. For instance, if $\lambda = y_1$ and $\mu = y_2$, then, $\ell(\lambda) = \ell(\mu) = 1$ and $\ell(\lambda\mu) = \ell(y_1 y_2) = 2 > \ell(\lambda)\ell(\mu)$. However, given the right conditions, $\ell(\lambda\mu) \le \ell(\lambda)\ell(\mu)$:

Lemma 3.18 *If the lowest terms of λ and μ are of degree ≥ 2 then $\ell(\lambda\mu) \leq \ell(\lambda)\ell(\mu)$.*

Proof. Following the procedure adopted before in Lemma 3.17, observe that if λ and μ of degree ≥ 2, there exists an integer k sufficiently large such that

$$\lambda = \sum_{(e_1,\ldots,e_m)\in D} c_{e_1,\ldots,e_m}\, y_1^{e_1}\cdots y_m^{e_m}$$

and

$$\mu = \sum_{(e_1,\ldots,e_m)\in D} d_{e_1,\ldots,e_m}\, y_1^{e_1}\cdots y_m^{e_m},$$

where $c_{e_1,\ldots,e_m}$, $d_{e_1,\ldots,e_m} \in \mathbf{Z}$ and

$$D = \{(e_1,\ldots,e_m) \mid 0 \leq e_1,\ldots,e_m \leq k \ \ \& \ \ e_1+\cdots+e_m \geq 2\}.$$

Then

$$\ell(\lambda) = \sum_{(e_1,\ldots,e_m)\in D} |c_{e_1,\ldots,e_m}|\ (e_1+\cdots+e_m)$$

and

$$\ell(\mu) = \sum_{(e_1,\ldots,e_m)\in D} |d_{e_1,\ldots,e_m}|\ (e_1+\cdots+e_m).$$

Moreover,

$$\lambda\mu = \sum_{(e_1,\ldots,e_m)\in D}\ \sum_{(e'_1,\ldots,e'_m)\in D} c_{e_1,\ldots,e_m} d_{e'_1,\ldots,e'_m}\, y_1^{e_1+e'_1}\cdots y_m^{e_m+e'_m}.$$

Hence,

$$\begin{aligned}
&\ell(\lambda\mu)\\
\leq\ & \textstyle\sum_{(e_1,\ldots,e_m)\in D}\sum_{(e'_1,\ldots,e'_m)\in D} |c_{e_1,\ldots,e_m} d_{e'_1,\ldots,e'_m}|((e_1+e'_1)+\cdots+(e_m+e'_m))\\
=\ & \textstyle\sum_{(e_1,\ldots,e_m)\in D}\sum_{(e'_1,\ldots,e'_m)\in D} |c_{e_1,\ldots,e_m} d_{e'_1,\ldots,e'_m}|((e_1+\cdots+e_m)+(e'_1+\cdots+e'_m))\\
\leq\ & \textstyle\sum_{(e_1,\ldots,e_m)\in D}\sum_{(e'_1,\ldots,e'_m)\in D} |c_{e_1,\ldots,e_m} d_{e'_1,\ldots,e'_m}|(e_1+\cdots+e_m)(e'_1+\cdots+e'_m)\\
& (\text{since } e_1+\cdots+e_m \ \&\ e'_1+\cdots+e'_m \geq 2)\\
=\ & \ell(\lambda)\ell(\mu).\ \square
\end{aligned}$$

We shall also need the following two lemmas. Their proofs are straightforward and hence are omitted.

Lemma 3.19 *For any* λ *and* μ *in* Λ, $\omega(\lambda) \le \ell(\lambda)$, $\omega(\lambda+\mu) \le \omega(\lambda)+\omega(\mu)$ *and* $\omega(\lambda\mu) \le \omega(\lambda)\omega(\mu)$.

Lemma 3.20 *If* μ *is a monomial then*

$$\ell(\lambda\mu) = \ell(\lambda) + d(\mu)\omega(\lambda) - |c(\lambda)|,$$

where $d(\mu)$ *is the degree of* μ *and* $c(\lambda)$ *is the constant term of* λ.

Lemma 3.21 *For any* λ_i *in* Λ,

$$\ell\Big(\prod_{i=1}^{k} \lambda_i\Big) \le 3^k \prod_{i=1}^{k} \ell(\lambda_i).$$

Proof. Let $\Sigma = \Lambda[y]$; i.e., $\Sigma = \mathbf{Z}[y_1, \ldots, y_m, y]$, and let $\mu_i = \lambda_i y^2\ \forall i$. Since the lowest term of each μ_i is of degree ≥ 2, we have the following inequality:

$$\ell\Big(\prod_{i=1}^{k} \mu_i\Big) \le \prod_{i=i}^{k} \ell(\mu_i).$$

Now

$$\ell\Big(\prod_{i=1}^{k} \lambda_i\Big) \le \ell\Big(y^{2k} \prod_{i=1}^{k} \lambda_i\Big) = \ell\Big(\prod_{i=1}^{k} \mu_i\Big).$$

Moreover, by Lemma 3.20,

$$\ell(\lambda\mu) = \ell(\lambda) + d(\mu)\omega(\lambda) - |c(\lambda)| \le \ell(\lambda) + d(\mu)\omega(\lambda).$$

Putting these observations together, we find that

$$\ell\Big(\prod_{i=1}^{k} \lambda_i\Big) \le \prod_{i=1}^{k} \big(\ell(\lambda_i) + 2\,\omega(\lambda_i)\big) \le 3^k \prod_{i=1}^{k} \ell(\lambda_i). \square$$

We will, throughout this section, assume that F, K are free right Λ-modules and that N, P are submodules of F, K, respectively where

$$\Xi = \{\xi_1, \ldots, \xi_k\}, \quad F =< \Xi >, \quad \Gamma = \{\gamma_1, \ldots, \gamma_l\}, \quad N = md_\Lambda(\Gamma),$$

$$\Pi = \{\pi_1, \ldots, \pi_p\}, \quad K =< \Pi >, \quad \Delta = \{\delta_1, \ldots, \delta_q\}, \quad P = md_\Lambda(\Delta).$$

We adopt this notation in the following lemma.

Lemma 3.22 *Suppose that M is a finitely generated Λ-module and that L is a submodule of M. If L is a retract of M, if $M \cong F/N$ and if $L \cong K/P$, then there are homomorphisms $\phi : K \to F$ and $\psi : F \to K$ which induce $\overline{\phi} : K/P \to F/N$ and $\overline{\psi} : F/N \to K/P$ such that $\overline{\psi} \circ \overline{\phi} : K/P \to K/P$ is the identity.*

Proof. Let

$$M \cong F/N =< \xi_1, \ldots, \xi_k; \gamma_1, \ldots, \gamma_l >$$

and let

$$L \cong K/P =< \pi_1, \ldots, \pi_p; \delta_1, \ldots, \delta_q > .$$

Since L is a retract of M, there are homomorphisms $\overline{\phi} : L \to M$ and $\overline{\psi} : M \to L$ such that $\overline{\psi} \circ \overline{\phi} = id_L$. Now, identify L with K/P and M with F/N and choose words v_i in F and u_j in K such that $v_i + N = \overline{\phi}(\pi_i + P)$, $i = 1, \ldots, p$ and let $u_j + P = \overline{\psi}(\xi_j + N)$, $j = 1, \ldots, k$. Then we define the homomorphism $\phi : K \to F$ by $\phi : \pi_i \mapsto v_i$, $i = 1, \ldots, p$ and the homomorphism $\psi : F \to K$ by $\psi : \xi_j \mapsto u_j$, $j = 1, \ldots, k$. Then ϕ and ψ have the desired properties. □

Proposition 3.23 *Suppose that M is a finitely generated Λ-module and L is a retract of M. If $\mathcal{P}_M$ and $\mathcal{P}_L$ are finite module presentations of M and L, respectively, then* *there are integer constants* k_1*,* k_2*,* k_3 *such that the inequality*

$$\Phi_{\mathcal{P}_L}(n) \le k_2 \Phi_{\mathcal{P}_M}(k_1 n) + k_3 n$$

holds for all n; i.e., $\Phi_{\mathcal{P}_L} \preccurlyeq \Phi_{\mathcal{P}_M}$.

Proof. First of all, let

$$\mathcal{P}_M = < \xi_1, \ldots, \xi_k; \gamma_1, \ldots, \gamma_l >$$

let F be free on $\Xi = \{\xi_1, \ldots, \xi_k\}$ and let $N = md_F(\gamma_1, \ldots, \gamma_l)$. Furthermore let

$$\mathcal{P}_L = < \pi_1, \ldots, \pi_p; \delta_1, \ldots, \delta_q >$$

let K be free on $\Pi = \{\pi_1, \ldots, \pi_p\}$ and let $P = md_K(\delta_1, \ldots, \delta_q)$. By Lemma 3.22 , there are homomorphisms $\phi : K \to F$ and $\psi : F \to K$ such that $-\pi_i + \psi(\phi(\pi_i)) \in P$ for all $i = 1, \ldots, p$. Let ℓ'_Ξ = the length function on F and let ℓ'_Π = the length function on K. Then define the four constants k_1, k'_1, k_2, k_3 :

$$\begin{aligned} k_1 &= 9k'_1, \\ k'_1 &= \max\{\ell'_\Xi(\phi(\pi_i)), i = 1, \ldots, p\}, \\ k_2 &= \max\{\Delta_{\mathcal{P}_L}(\psi(\gamma_i)), i = 1, \ldots, l\}, \\ k_3 &= \max\{\Delta_{\mathcal{P}_L}(-\pi_i + \psi(\phi(\pi_i))), i = 1, \ldots, p\}. \end{aligned}$$

Given any n, let w be any word in K with

$$w = \sum_{i=1}^{p} \pi_i f_i, \quad f_i \in \Lambda, \quad \ell'_\Pi(w) = \sum_{i=1}^{p} \ell(f_i) \le n.$$

Moreover, if $w \in P$, then $\phi(w) \in N$. We then have

$$\phi(w) = \sum_{i=1}^{l} \gamma_i g_i \quad \text{for some } g_i \in \Lambda$$

such that

$$\sum_{i=1}^{l}\omega(g_i) = \Delta_{\mathcal{P}_M}(\phi(w)).$$

Then

$$\sum_{i=1}^{l}\omega(g_i) = \Delta_{\mathcal{P}_M}(\phi(w)) \leq \Phi_{\mathcal{P}_M}(\ell'_{\Xi}(\phi(w))).$$

We claim that the following three inequalities hold:

i) $\Delta_{\mathcal{P}_L}(-w+\psi(\phi(w))) \leq k_3\ell'_{\Pi}(w)$;

ii) $\sum_{i=1}^{l}\omega(g_i) \leq \Phi_{\mathcal{P}_M}(k_1\ell'_{\Pi}(w))$;

iii) $\Delta_{\mathcal{P}_L}(\psi(\phi(w))) \leq k_2\sum_{i=1}^{l}\omega(g_i)$.

To prove i), notice that

$$\begin{aligned}-w+\psi(\phi(w)) &= -\sum_{i=1}^{p}\pi_i f_i + \psi\Big(\phi\Big(\sum_{i=1}^{p}\pi_i f_i\Big)\Big)\\ &= -\sum_{i=1}^{p}\pi_i f_i + \sum_{i=1}^{p}\psi(\phi(\pi_i))f_i = \sum_{i=1}^{p}(-\pi_i+\psi(\phi(\pi_i)))f_i.\end{aligned}$$

Let

$$-\pi_i+\psi(\phi(\pi_i)) = \sum_{j=1}^{q}\delta_j g_{i,j},$$

for some $g_{i,j}\in\Lambda$ such that

$$\Delta_{\mathcal{P}_L}(-\pi_i+\psi(\phi(\pi_i))) = \sum_{j=1}^{q}\omega(g_{i,j})$$

and so

$$\sum_{j=1}^{q}\omega(g_{i,j}) \leq k_3.$$

Moreover,

$$-w+\psi(\phi(w)) = \sum_{i=1}^{p}\Big(\sum_{j=1}^{q}\delta_j g_{i,j}\Big)f_i = \sum_{j=1}^{q}\delta_j\sum_{i=1}^{p}g_{i,j}f_i.$$

So,

$$\Delta_{\mathcal{P}_L}(-w+\psi(\phi(w))) \le \sum_{j=1}^{q}\omega\Big(\sum_{i=1}^{p} g_{i,j}f_i\Big)$$

$$\le \sum_{j=1}^{q}\sum_{i=1}^{p}\omega(g_{i,j})\omega(f_i) = \sum_{i=1}^{p}\Big(\sum_{j=1}^{q}\omega(g_{i,j})\Big)\omega(f_i)$$

$$\le k_3\sum_{i=1}^{p}\omega(f_i) \le k_3\sum_{i=1}^{p}\ell(f_i) = k_3\ell'_{\Pi}(w).$$

This proves i).

To prove ii), recall that

$$w = \sum_{i=1}^{p}\pi_i f_i, \quad \ell'_{\Pi}(w) = \sum_{i=1}^{p}\ell(f_i) \le n.$$

Then,

$$\phi(w) = \sum_{i=1}^{p}\phi(\pi_i)f_i.$$

Let

$$\phi(\pi_i) = \sum_{j=1}^{k}\xi_j h_{i,j}, \quad h_{i,j}\in\Lambda.$$

Then,

$$\ell'_{\Xi}(\phi(\pi_i)) = \sum_{j=1}^{k}\ell(h_{i,j}) \le k'_1.$$

So

$$\phi(w) = \sum_{i=1}^{p}\Big(\sum_{j=1}^{k}\xi_j h_{i,j}\Big)f_i = \sum_{j=1}^{k}\xi_j\sum_{i=1}^{p}h_{i,j}f_i.$$

Hence,

$$\ell'_{\Xi}(\phi(w)) = \sum_{j=1}^{k}\ell\Big(\sum_{i=1}^{p}h_{i,j}f_i\Big) \le \sum_{j=1}^{k}\sum_{i=1}^{p}\ell(h_{i,j}f_i)$$

$$\le 9\sum_{j=1}^{k}\sum_{i=1}^{p}\ell(h_{i,j})\ell(f_i) = 9\sum_{i=1}^{p}\Big(\sum_{j=1}^{k}\ell(h_{i,j})\Big)\ell(f_i)$$

$$\leq 9k_1' \sum_{i=1}^{p} \ell(f_i) = k_1 \ell_\Pi'(w).$$

So we have

$$\sum_{i=1}^{l} \omega(g_i) \leq \Phi_{\mathcal{P}_M}(\ell_\Xi'(\phi(w))) \leq \Phi_{\mathcal{P}_M}(k_1 \ell_\Pi'(w)).$$

This proves ii).

To prove iii), recall

$$k_2 = \max \{\Delta_{\mathcal{P}_L}(\psi(\gamma_i)), i = 1, \ldots, l\}$$

and

$$\phi(w) = \sum_{i=1}^{l} \gamma_i g_i.$$

Then,

$$\psi(\phi(w)) = \sum_{i=1}^{l} \psi(\gamma_i) g_i.$$

Let

$$\psi(\gamma_i) = \sum_{j=1}^{q} \delta_j p_{i,j}, \quad p_{i,j} \in \Lambda$$

such that

$$\Delta_{\mathcal{P}_L}(\psi(\gamma_i)) = \sum_{j=1}^{q} \omega(p_{i,j}).$$

So,

$$\sum_{j=1}^{q} \omega(p_{i,j}) \leq k_2, \quad \forall i.$$

Then,

$$\psi(\phi(w)) = \sum_{i=1}^{l} \Big(\sum_{j=1}^{q} \delta_j p_{i,j} \Big) g_i = \sum_{j=1}^{q} \delta_j \sum_{i=1}^{l} p_{i,j} g_i.$$

Hence,

$$\Delta_{\mathcal{P}_L}(\psi(\phi(w)) \leq \sum_{j=1}^{q} \omega \Big(\sum_{i=1}^{l} p_{i,j} g_i \Big) \leq \sum_{j=1}^{q} \sum_{i=1}^{l} \omega(p_{i,j} g_i)$$

$$\leq \sum_{j=1}^{q}\sum_{i=1}^{l}\omega(p_{i,j})\omega(g_i) = \sum_{i=1}^{l}\omega(g_i)\Big(\sum_{j=1}^{q}\omega(p_{i,j})\Big) \leq k_2\sum_{i=1}^{l}\omega(g_i).$$

This proves iii).

Now if $w =_L 0$ and $w = \psi(\phi(w)) + (w - \psi(\phi(w)))$, then

$$\begin{aligned}
\Delta_{\mathcal{P}_L}(w) &\leq \Delta_{\mathcal{P}_L}(\psi(\phi(w))) + \Delta_{\mathcal{P}_L}(w - \psi(\phi(w))) \\
&\leq k_2 \textstyle\sum_{i=1}^{l} \omega(g_i) + k_3\ell'_{\Pi}(w) && \text{(by i) \& iii))} \\
&\leq k_2\Phi_{\mathcal{P}_M}(k_1\ell'_{\Pi}(w)) + k_3\ell'_{\Pi}(w) && \text{(by ii))}
\end{aligned}$$

Hence, by the same argument as in the proof of Lemma 3.3, we have

$$\Phi_{\mathcal{P}_L}(n) \leq k_2\Phi_{\mathcal{P}_M}(k_1 n) + k_3 n \quad \forall\ n;$$

i.e., $\Phi_{\mathcal{P}_L} \preccurlyeq \Phi_{\mathcal{P}_M}$. □

Theorem 3.24 *If $\mathcal{P}$ and $\mathcal{Q}$ are two presentations of the finitely generated Λ-module M, then*

$$\Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{Q}}.$$

Proof. The module M can be thought of a retract of itself. Then, by Proposition 3.23 $\Phi_{\mathcal{P}} \preccurlyeq \Phi_{\mathcal{Q}}$ and $\Phi_{\mathcal{Q}} \preccurlyeq \Phi_{\mathcal{P}}$. Then, $\Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{Q}}$. This proves the theorem. □

As before we term any one of the functions in the equivalence class of, say $\Phi_{\mathcal{Q}}$, the isoperimetric function of M and denote it simply by Φ_M.

Combining Proposition 3.23 and Theorem 3.24, we obtain the following corollary:

Corollary 3.25 *If L is a retract of M, then $\Phi_L \preccurlyeq \Phi_M$.*

3.3 Connections between the isoperimetric function of a finitely metabelian group and that of the underlying module

Suppose that G is a finitely generated metabelian group generated by $a_1, \ldots, a_m$. Let $A = G'$ and $T = G/G'$. Then G is the middle of a short exact sequence

$$A \hookrightarrow G \twoheadrightarrow T \tag{3.26}$$

and A can be viewed as a module over the integral group ring $\mathbf{Z}T$ of T, where the action of T on A is defined as follows:

$$a \bullet t = g^{-1}ag \quad (a \in A, g \in G, t = gA).$$

Throughout this section, we will denote $a \bullet t$ by a^t and write elements in the module A multiplicatively (see (2.3) in §2.1). It is easy then to prove the following lemma.

Lemma 3.27 *G', viewed as a $\mathbf{Z}T$-module, is generated by the elements $\beta_{i,j}$, where $\beta_{i,j} = [a_i, a_j]$, $1 \leq i < j \leq m$.*

Notice that $\mathbf{Z}T$ is a finitely generated commutative ring and hence is isomorphic to a quotient of a polynomial ring in finitely many variables. For instance, suppose that

$$T =< t_1, \ldots, t_m, t_{m+1}, \ldots, t_{m+s}; t_{m+i}^{p_i} = 1 \ (1 \leq i \leq s), [t_j, t_k] = 1 \ (1 \leq j, k \leq m+s) > .$$

Let $\Sigma = \mathbf{Z}T$ and let

$$\Lambda = \mathbf{Z}[x_1, \ldots, x_{m+s}, y_1, \ldots, y_{m+s}].$$

Then

$$\Sigma \cong \Lambda / md_\Lambda(x_j y_j - 1 \ (1 \leq j \leq m+s), \ x_{m+i}^{p_i} - 1 \ (1 \leq i \leq s)).$$

In other words, given any finitely generated abelian group T, its group ring $\mathbf{Z}T$ can be lifted to a polynomial ring Λ; i.e., there is a homomorphism φ from Λ onto $\mathbf{Z}T$. Hence any $\mathbf{Z}T$-module can be viewed as a Λ-module, for example by defining an action of Λ on G' by

$$a \bullet f = a^{\varphi(f)} \quad \forall\, a \in G',$$

So by Lemma 3.27, G' is generated as a Λ-module by the $\beta_{i,j}$. Consequently (see §2.5), G' is a finitely presented Λ-module.

In order to study the connection between $\Phi_{G'}$ and Φ_G, we will compare the length function ℓ on the free metabelian group $<< a_1, \ldots, a_m >>$ and the length function ℓ' on the free Λ-module $< \beta_{i,j}\ (1 \le i < j \le m) >$, where, as above,

$$\Lambda = \mathbf{Z}[x_1, \ldots, x_m, y_1, \ldots, y_m].$$

Let $t_i = a_i G'$. Then, if $z \in G'$,

$$z^{x_i} = z^{t_i} = z^{a_i} \;\&\; z^{y_j} = z^{t_j^{-1}} = z^{a_j^{-1}} \quad \forall i, j.$$

Now if $w = w(a_1, \ldots, a_m)$ is a word in the given generators of G and if $w \in G'$, then w can be re-expressed as a "module word" $w' = w'(\beta_{1,2}, \ldots, \beta_{m-1,m})$ in the generators $\beta_{i,j}$ $(1 \le i < j \le m)$ of the Λ-module G'. Notice that if $w =_G 1$, then $w' =_{G'} 1$. We compare the module length $\ell'(w')$ of w' with $\ell(w)$. We will adopt this notation in the proposition that follows.

Proposition 3.28 *Let G be a finitely generated metabelian group, generated by the elements $a_1, \ldots a_m$ and let w be a word in the generators of G of length n. If $w \in G'$,* *then w can be rewritten as a Λ-module word w' in the $\beta_{i,j}$ with*

$$\ell'(w') \leq O(n^4).$$

To prove Proposition 3.28, we need the following:

Lemma 3.29 *Suppose that p and q are any two positive integers. Then*

i) $[a_i^p, a_j^q] = \beta_{i,j}{}^{\sum_{k=0}^{p-1} x_i^k \sum_{l=0}^{q-1} x_j^l}$;

ii) $[a_i^{-p}, a_j^q] = \beta_{i,j}{}^{-\sum_{k=1}^{p} y_i^k \sum_{l=0}^{q-1} x_j^l}$;

iii) $[a_i^p, a_j^{-q}] = \beta_{i,j}{}^{-\sum_{k=0}^{p-1} x_i^k \sum_{l=1}^{q} y_j^l}$;

iv) $[a_i^{-p}, a_j^{-q}] = \beta_{i,j}{}^{\sum_{k=1}^{p} y_i^k \sum_{l=1}^{q} y_j^l}$.

Proof. Recall:

$$[ab, c] = [a, c]^b [b, c],$$

$$[a, bc] = [a, c][a, b]^c.$$

First of all, we claim that

$$[a_i^p, a_j] = \beta_{i,j}{}^{\sum_{k=0}^{p-1} x_i^k} \quad \forall\, p \in \mathbf{N}.$$

It is easy to see

$$[a_i^2, a_j] = [a_i, a_j]^{a_i}[a_i, a_j] = \beta_{i,j}{}^{x_i}\beta_{i,j} = \beta_{i,j}{}^{x_i+1}.$$

Suppose that

$$[a_i^p, a_j] = \beta_{i,j}{}^{\sum_{k=0}^{p-1} x_i^k} \quad \text{for some } p.$$

Then,

$$[a_i^{p+1}, a_j] = [a_i^p, a_j]^{a_i}[a_i, a_j] = \left(\beta_{i,j}{}^{\sum_{k=0}^{p-1} x_i^k}\right)^{x_i} \beta_{i,j} = \beta_{i,j}{}^{\sum_{k=0}^{p} x_i^k}.$$

So, by induction, the claim holds.

Similarly, we have

$$[a_i, a_j^q] = \beta_{i,j}{}^{\sum_{l=0}^{q-1} x_j^l} \quad \forall\, q \in \mathbf{N}.$$

Hence,

$$[a_i^p, a_j^q] = [a_i, a_j^q]^{\sum_{k=0}^{p-1} x_i^k} = \beta_{i,j}{}^{\sum_{k=0}^{p-1} x_i^k \sum_{l=0}^{q-1} x_j^l} \quad \forall p, q \in \mathbf{N}.$$

Secondly, we claim that

$$[a_i^{-p}, a_j] = \beta_{i,j}{}^{-\sum_{k=1}^{p} y_i^k} \quad \forall\, p \in \mathbf{N}.$$

Since

$$1 = [a_i a_i^{-1}, a_j] = [a_i, a_j]^{a_i^{-1}} [a_i^{-1}, a_j],$$

we have

$$[a_i^{-1}, a_j] = [a_i, a_j]^{-a_i^{-1}} = \beta_{i,j}{}^{-t_i^{-1}} = \beta_{i,j}{}^{-y_i}.$$

Similarly, since

$$1 = [a_i^2 a_i^{-2}, a_j] = [a_i^2, a_j]^{a_i^{-2}} [a_i^{-2}, a_j]$$

and $x_i y_i$ acts as the identity, we have

$$[a_i^{-2}, a_j] = [a_i^2, a_j]^{-a_i^{-2}} = (\beta_{i,j}{}^{x_i+1})^{-y_i^2} = \beta_{i,j}{}^{-y_i - y_i^2}.$$

Suppose that

$$[a_i^{-p}, a_j] = \beta_{i,j}{}^{-\sum_{k=1}^{p} y_i^k} \quad \text{for some } p.$$

Then

$$[a_i^{-p-1}, a_j] = [a_i^{-p}, a_j]^{a_i^{-1}} [a_i^{-1}, a_j] = (\beta_{i,j}{}^{-\sum_{k=1}^{p} y_i^k})^{y_i} \beta_{i,j}{}^{-y_i}$$

$$= (\beta_{i,j}{}^{-\sum_{k=1}^{p} y_i^{k+1}}) \beta_{i,j}{}^{-y_i} = \beta_{i,j}{}^{-\sum_{k=1}^{p+1} y_i^k}.$$

So the claim follows by induction.

Then, for all $p, q \in \mathbf{N}$,

$$[a_i^{-p}, a_j^q] = [a_i^{-p}, a_j]^{\sum_{l=0}^{q-1} x_j^l} = ({\beta_{i,j}}^{-\sum_{k=1}^{p} y_i^k})^{\sum_{l=0}^{q-1} x_j^l} = {\beta_{i,j}}^{-\sum_{k=1}^{p} y_i^k \sum_{l=0}^{q-1} x_j^l}.$$

This completes the proofs of i) and ii). The proofs of iii) and iv) are similar and are consequently omitted. □

Next, we calculate the module length of these typical commutators.

Lemma 3.30

$$\ell'([a_i^p, a_j^q]) = 1 + \frac{pq}{2}(p + q - 2)$$

$$\ell'([a_i^{-p}, a_j^q]) = \frac{pq}{2}(p + q)$$

$$\ell'([a_i^p, a_j^{-q}]) = \frac{pq}{2}(p + q)$$

$$\ell'([a_i^{-p}, a_j^{-q}]) = \frac{pq}{2}(p + q + 2)$$

Proof. According to Lemma 3.29, we first find that

$$\begin{aligned}
\ell'([a_i^p, a_j^q]) &= \ell(\textstyle\sum_{k=0}^{p-1} x_i^k \sum_{l=0}^{q-1} x_j^l) \\
&= \ell(1 + x_i + x_i^2 + \cdots + x_i^{p-1}) + \ell(x_j + x_i x_j + x_i^2 x_j + \cdots + x_i^{p-1} x_j) \\
&\quad + \cdots + \ell(x_j^{q-1} + x_i x_j^{q-1} + x_i^2 x_j^{q-1} + \cdots + x_i^{p-1} x_j^{q-1}) \\
&= (1 + 1 + 2 + \cdots + (p-1)) + (1 + 2 + \cdots + p) \\
&\quad + \cdots + ((q-1) + q + \cdots + (p + q - 2)) \\
&= 1 + \tfrac{p(p-1)}{2} + \tfrac{p(p+1)}{2} + \cdots + \tfrac{p(p+2q-3)}{2} \\
&= 1 + \tfrac{p}{2}[(p-1) + (p+1) + \cdots + (p + 2q - 3)] \\
&= 1 + \tfrac{pq}{4}(2p + 2q - 4) \\
&= 1 + \tfrac{pq}{2}(p + q - 2)
\end{aligned}$$

Next,

$$\begin{aligned}
\ell'([a_i^{-p}, a_j^q]) &= \ell(\textstyle\sum_{k=1}^{p} y_i^k \sum_{l=0}^{q-1} x_j^l) \\
&= \ell(y_i + y_i^2 + \cdots + y_i^p) + \ell(y_i x_j + y_i^2 x_j + \cdots + y_i^p x_j) \\
&\quad + \cdots + \ell(y_i x_j^{q-1} + y_i^2 x_j^{q-1} + \cdots + y_i^p x_j^{q-1}) \\
&= (1 + 2 + \cdots + p) + (2 + 3 + \cdots + (p+1)) \\
&\quad + \cdots + (q + (q+1) + \cdots + (p+q-1)) \\
&= \tfrac{p(p+1)}{2} + \tfrac{p(p+3)}{2} + \cdots + \tfrac{p(p+2q-1)}{2} \\
&= \tfrac{pq}{2}(p+q).
\end{aligned}$$

Similar calculation for the other two cases. □

Lemma 3.31 *Let G be a finitely generated metabelian group, generated by the elements $a_1, \ldots a_m$ and let w be a word in the generators of G of length n. Then w can be rewritten in the form*

$$a_1^{d_1} \ldots a_m^{d_m} w',$$

where $w' \in G'$ is expressible as a Λ-module word with

$$\ell'(w') \leq O(n^4).$$

Proof. Let w be a word in the generators of G of length n.

$$w = a_{i_1}^{e_{i_1}} a_{i_2}^{e_{i_2}} \cdots a_{i_k}^{e_{i_k}}$$

where $i_j \neq i_{j+1}$ for all $1 \leq j \leq k-1$ and $|e_{i_1}| + |e_{i_2}| + \cdots + |e_{i_k}| = n$.

Notice that w can be rewritten in the form:

$$a_1^{d_1} a_2^{d_2} \cdots a_m^{d_m} \prod u_j^{-1} V_j u_j \quad \text{or} \quad a_1^{d_1} a_2^{d_2} \cdots a_m^{d_m} \prod V_j^{u_j},$$

where d_i are integers, V_j are commutators and u_j are words in $<< a_1, \ldots, a_m >>$.

We now rewrite w in the form above, keeping track of the least number of commutators that arise in the process. Compare i_k with $i_{k-1}, i_{k-2}, \ldots, i_1$. Notice that the case which gives rise to the greatest number of commutators is that in which $i_k < i_{k-1}, i_{k-2}, \ldots, i_1$, since in this case each a_{i_k} will need to be moved to the most places from right to left. So $a_{i_k}^{e_{i_k}}$ has to be moved to the front by adding one appropriate commutator at a time; i.e.,

$$\begin{aligned}
w &= a_{i_1}^{e_{i_1}} \cdots a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_k}^{e_{i_k}} a_{i_{k-1}}^{e_{i_{k-1}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}] \\
&= a_{i_1}^{e_{i_1}} \cdots a_{i_{k-3}}^{e_{i_{k-3}}} a_{i_k}^{e_{i_k}} a_{i_{k-2}}^{e_{i_{k-2}}} [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_k}^{e_{i_k}}] a_{i_{k-1}}^{e_{i_{k-1}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}] \\
&= a_{i_1}^{e_{i_1}} \cdots a_{i_{k-3}}^{e_{i_{k-3}}} a_{i_k}^{e_{i_k}} a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_{k-1}}^{e_{i_{k-1}}} [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-1}}^{e_{i_{k-1}}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}] \\
&= a_{i_1}^{e_{i_1}} \cdots a_{i_{k-4}}^{e_{i_{k-4}}} a_{i_k}^{e_{i_k}} a_{i_{k-3}}^{e_{i_{k-3}}} [a_{i_{k-3}}^{e_{i_{k-3}}}, a_{i_k}^{e_{i_k}}] \cdot a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_{k-1}}^{e_{i_{k-1}}} [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-1}}^{e_{i_{k-1}}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}] \\
&= a_{i_1}^{e_{i_1}} \cdots a_{i_{k-4}}^{e_{i_{k-4}}} a_{i_k}^{e_{i_k}} a_{i_{k-3}}^{e_{i_{k-3}}} a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_{k-1}}^{e_{i_{k-1}}} [a_{i_{k-3}}^{e_{i_{k-3}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_{k-1}}^{e_{i_{k-1}}}} \cdot \\
&\quad [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-1}}^{e_{i_{k-1}}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}] \\
&\vdots \\
&= a_{i_k}^{e_{i_k}} a_{i_1}^{e_{i_1}} a_{i_2}^{e_{i_2}} \cdots a_{i_{k-1}}^{e_{i_{k-1}}} [a_{i_1}^{e_{i_1}}, a_{i_k}^{e_{i_k}}]^{a_{i_2}^{e_{i_2}} a_{i_3}^{e_{i_3}} \cdots a_{i_{k-1}}^{e_{i_{k-1}}}} \cdots \\
&\quad [a_{i_{k-3}}^{e_{i_{k-3}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_{k-1}}^{e_{i_{k-1}}}} [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-1}}^{e_{i_{k-1}}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}].
\end{aligned}$$

Let

$$\begin{aligned}
w_k &= [a_{i_1}^{e_{i_1}}, a_{i_k}^{e_{i_k}}]^{a_{i_2}^{e_{i_2}} a_{i_3}^{e_{i_3}} \cdots a_{i_{k-1}}^{e_{i_{k-1}}}} \cdots [a_{i_{k-3}}^{e_{i_{k-3}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-2}}^{e_{i_{k-2}}} a_{i_{k-1}}^{e_{i_{k-1}}}} \cdot \\
&\quad [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_k}^{e_{i_k}}]^{a_{i_{k-1}}^{e_{i_{k-1}}}} [a_{i_{k-1}}^{e_{i_{k-1}}}, a_{i_k}^{e_{i_k}}].
\end{aligned}$$

Now compare i_{k-1} with $i_{k-2}, i_{k-3}, \ldots$ etc. Again suppose that, the worst case, $i_{k-1} < i_{k-2}, i_{k-3}, \ldots, i_1$. Then

$$\begin{aligned}
w &= a_{i_k}^{e_{i_k}} a_{i_{k-1}}^{e_{i_{k-1}}} a_{i_1}^{e_{i_1}} a_{i_2}^{e_{i_2}} \cdots a_{i_{k-2}}^{e_{i_{k-2}}} [a_{i_1}^{e_{i_1}}, a_{i_{k-1}}^{e_{i_{k-1}}}]^{a_{i_2}^{e_{i_2}} a_{i_3}^{e_{i_3}} \cdots a_{i_{k-2}}^{e_{i_{k-2}}}} \cdots \\
&\quad [a_{i_{k-4}}^{e_{i_{k-4}}}, a_{i_{k-1}}^{e_{i_{k-1}}}]^{a_{i_{k-3}}^{e_{i_{k-3}}} a_{i_{k-2}}^{e_{i_{k-2}}}} [a_{i_{k-3}}^{e_{i_{k-3}}}, a_{i_{k-1}}^{e_{i_{k-1}}}]^{a_{i_{k-2}}^{e_{i_{k-2}}}} [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_{k-1}}^{e_{i_{k-1}}}] w_k.
\end{aligned}$$

Let

$$\begin{aligned} w_{k-1} &= [a_{i_1}^{e_{i_1}}, a_{i_{k-1}}^{e_{i_{k-1}}}]^{a_{i_2}^{e_{i_2}} a_{i_3}^{e_{i_3}} \cdots a_{i_{k-2}}^{e_{i_{k-2}}}} \cdots [a_{i_{k-4}}^{e_{i_{k-4}}}, a_{i_{k-1}}^{e_{i_{k-1}}}]^{a_{i_{k-3}}^{e_{i_{k-3}}} a_{i_{k-2}}^{e_{i_{k-2}}}} \cdot \\ & [a_{i_{k-3}}^{e_{i_{k-3}}}, a_{i_{k-1}}^{e_{i_{k-1}}}]^{a_{i_{k-2}}^{e_{i_{k-2}}}} [a_{i_{k-2}}^{e_{i_{k-2}}}, a_{i_{k-1}}^{e_{i_{k-1}}}]. \end{aligned}$$

Continue the process and suppose the worst case all the time. We find

$$w = a_{i_k}^{e_{i_k}} a_{i_{k-1}}^{e_{i_{k-1}}} \cdots a_{i_1}^{e_{i_1}} w_2 \cdots w_k$$

where

$$\begin{aligned} w_l &= [a_{i_1}^{e_{i_1}}, a_{i_l}^{e_{i_l}}]^{a_{i_2}^{e_{i_2}} a_{i_3}^{e_{i_3}} \cdots a_{i_{l-1}}^{e_{i_{l-1}}}} \cdots [a_{i_{l-3}}^{e_{i_{l-3}}}, a_{i_l}^{e_{i_l}}]^{a_{i_{l-2}}^{e_{i_{l-2}}} a_{i_{l-1}}^{e_{i_{l-1}}}} \cdot \\ & [a_{i_{l-2}}^{e_{i_{l-2}}}, a_{i_l}^{e_{i_l}}]^{a_{i_{l-1}}^{e_{i_{l-1}}}} [a_{i_{l-1}}^{e_{i_{l-1}}}, a_{i_l}^{e_{i_l}}], \end{aligned}$$

for all $2 \leq l \leq k$. Let $w' = \prod_{l=2}^{k} w_l$. Then

$$\ell'(w') \leq \sum_{l=2}^{k} \ell'(w_l).$$

Finally, let's compute $\ell'(w_l)$, $l = 2, \ldots, k$. By Lemma 3.30, we get

$$\ell'([a_i^p, a_j^q]) \leq \frac{|pq|}{2}(|p| + |q| + 2) \ \forall p, q \in \mathbf{Z}.$$

By Lemma 3.29, $[a_i^p, a_j^q] = \beta_{i,j}^f$ for some $f \in \Lambda$ with $\omega(f) = |pq|$. Furthermore, $\ell(fg) \leq \ell(f) + d(g)\omega(f)$ for any monomial $g \in \Lambda$, by Lemma 3.20. We then obtain

$$\ell'([a_i^p, a_j^q]^g) \leq \frac{|pq|}{2}(|p| + |q| + 2) + d(g)\ |pq|. \tag{3.32}$$

So,

$$\begin{aligned} \ell'(w_l) \leq & \left(\tfrac{1}{2}|e_{i_{l-1}} e_{i_l}|(|e_{i_{l-1}}| + |e_{i_l}| + 2)\right) + \\ & \left(\tfrac{1}{2}|e_{i_{l-2}} e_{i_l}|(|e_{i_{l-2}}| + |e_{i_l}| + 2) + |e_{i_{l-2}} e_{i_l}|\ |e_{i_{l-1}}|\right) + \\ & \left(\tfrac{1}{2}|e_{i_{l-3}} e_{i_l}|(|e_{i_{l-3}}| + |e_{i_l}| + 2) + |e_{i_{l-3}} e_{i_l}|\ (|e_{i_{l-2}}| + |e_{i_{l-1}}|)\right) + \\ & \cdots + \left(\tfrac{1}{2}|e_{i_1} e_{i_l}|(|e_{i_1}| + |e_{i_l}| + 2) + |e_{i_1} e_{i_l}|\ (|e_{i_2}| + \cdots + |e_{i_{l-1}}|)\right). \end{aligned}$$

Then, we have

$$\sum_{l=2}^{k}\ell'(w_l) \le \sum_{1\le s<t\le k} |e_{i_s}e_{i_t}| + \frac{1}{2}\sum_{1\le s<t\le k} |e_{i_s}|^2|e_{i_t}|$$
$$+\frac{1}{2}\sum_{1\le s<t\le k} |e_{i_s}||e_{i_t}|^2 + \sum_{1\le r<s<t\le k} |e_{i_r}e_{i_s}e_{i_t}|.$$

Notice that

$$\sum_{1\le s<t\le k} |e_{i_s}e_{i_t}| \le \sum_{1\le s\le k} |e_{i_s}|(|e_{i_1}|+\cdots+|e_{i_k}|) \le n^2; \tag{3.33}$$

$$\sum_{1\le s<t\le k} |e_{i_s}|^2|e_{i_t}| \le \sum_{1\le s<k} |e_{i_s}|^2(|e_{i_1}|+\cdots+|e_{i_k}|) \le \sum_{1\le s<k} |e_{i_s}|^2 n \le n^4; \tag{3.34}$$

$$\sum_{1\le r<s<t\le k} |e_{i_r}e_{i_s}e_{i_t}| \le \sum_{1\le r<k} |e_{i_r}| \sum_{r<s<t\le k} |e_{i_s}e_{i_t}| \le \sum_{1\le r<k} |e_{i_r}|n^2 \le n^3. \tag{3.35}$$

Hence

$$\ell'(w') \le \sum_{l=2}^{k}\ell'(w_l) \le n^2 + \frac{n^4}{2} + \frac{n^4}{2} + n^3 \approx O(n^4). \ \square$$

Proof of Proposition 3.28. By the hypothesis, $w \in G'$ with $\ell(w) = n$. We let

$$w = b_1^{e_1}b_2^{e_2}\cdots b_k^{e_k},$$

where $b_i \in \{a_1, a_2, \ldots, a_m\}$, $0 \neq e_i \in \mathbf{Z}$, and $\sum_{i=1}^{k}|e_i| = n$ and the exponent sum of each b_i must be zero. Use the same stratege in Lemma 3.31, all b_i will cancel out after adding appropriate commutators; i.e., we can express w as w', a product of commutators $\beta_{i,j}$ and their conjugates and keep track of the least number of commutors arising in the following procedure:

First of all, we begin with b_k. Suppose that $b_{i_1}, b_{i_2}, \ldots, b_{i_l}$ are letters which occur in w and are actually the letter b_k. We then view w as

$$w = u_0\, b_{i_1}^{e_{i_1}}\, u_1\, b_{i_2}^{e_{i_2}}\, u_2 \cdots b_{i_l}^{e_{i_l}}\, u_l\, b_k^{e_k},$$

where

$$\begin{aligned}
u_0 &= b_1^{e_1} b_2^{e_2} \cdots b_{i_1-1}^{e_{i_1-1}}, \\
u_1 &= b_{i_1+1}^{e_{i_1+1}} b_{i_1+2}^{e_{i_1+2}} \cdots b_{i_2-1}^{e_{i_2-1}}, \\
&\vdots \\
u_{l-1} &= b_{i_{l-1}+1}^{e_{i_{l-1}+1}} b_{i_{l-1}+2}^{e_{i_{l-1}+2}} \cdots b_{i_l-1}^{e_{i_l-1}}, \\
u_l &= b_{i_{l+1}+1}^{e_{i_{l+1}+1}} b_{i_{l+1}+2}^{e_{i_{l+1}+2}} \cdots b_{k-1}^{e_{k-1}}.
\end{aligned}$$

Move $b_k^{e_k}$ to the right hand side of $b_{i_l}^{e_{i_l}}$ across u_l by adding the commutor $[u_l, b_k^{e_k}]$. If $|e_{i_l} + e_k| = 0$, work on the previous b_k in the same way. If not, then move $b_k^{e_{i_l}+e_k}$ to the right hand side of $b_{i_{l-1}}^{e_{i_{l-1}}}$ across u_{l-1} by adding the commutor $[u_{l-1}, b_k^{e_{i_l}+e_k}]$, and so on. Therefore, without any loss of generality, w can be rewritten as

$$w_1 = u_0 u_1 \cdots u_l v_1^{u_2 \cdots u_l} v_2^{u_3 \cdots u_l} \cdots v_{l-2}^{u_{l-1} \cdots u_l} v_{l-1}^{u_l} v_l,$$

where

$$v_{j-1} = [u_{j-1}, b_k^{e_{i_j}+e_{i_{j+1}}+\cdots+e_{i_l}+e_k}] \quad (j = 2, \ldots, l) \;\;\&\;\; v_l = [u_l, b_k^{e_k}].$$

Now let

$$\begin{aligned}
P_j &= \{i_j, i_{j+1}, \ldots, i_l, k\}; \\
D_j &= \{\alpha \in \mathrm{N} \mid i_j < \alpha < k \;\&\; \alpha \text{ is not in } P_j\}; \\
p_j &= \textstyle\sum_{\alpha \in P_j} |e_\alpha|; \\
d_j &= \textstyle\sum_{\alpha \in D_j} |e_\alpha|.
\end{aligned}$$

Next, we shall see how $v_{j-1}^{u_j u_{j+1} \cdots u_l}$ being presented as a module word. Clearly, $u_j u_{j+1} \cdots u_l$ is just a monomial with degree $\leq d_j$. Moreover, let $p = \sum_{\alpha \in P_j} e_\alpha$. Then

$$\begin{aligned}
v_{j-1} &= [b_{i_{j-1}+1}^{e_{i_{j-1}+1}} b_{i_{j-1}+2}^{e_{i_{j-1}+2}} \cdots b_{i_j-1}^{e_{i_j-1}}, b_k^{e_{i_j}+e_{i_{j+1}}+\cdots+e_{i_l}+e_k}] \\
&= [b_{i_{j-1}+1}^{e_{i_{j-1}+1}} \cdots b_{i_j-2}^{e_{i_j-2}}, b_k^p]^{b_{i_j-1}^{e_{i_j-1}}} [b_{i_j-1}^{e_{i_j-1}} b_k^p] \\
&= [b_{i_{j-1}+1}^{e_{i_{j-1}+1}} \cdots b_{i_j-3}^{e_{i_j-3}}, b_k^p]^{b_{i_j-2}^{e_{i_j-2}} b_{i_j-1}^{e_{i_j-1}}} [b_{i_j-2}^{e_{i_j-2}}, b_k^p]^{b_{i_j-1}^{e_{i_j-1}}} [b_{i_j-1}^{e_{i_j-1}}, b_k^p] \\
&\;\;\vdots \\
&= [b_{i_{j-1}+1}^{e_{i_{j-1}+1}}, b_k^p]^{b_{i_{j-1}+2}^{e_{i_{j-1}+2}} \cdots b_{i_j-1}^{e_{i_j-1}}} \cdots [b_{i_j-2}^{e_{i_j-2}}, b_k^p]^{b_{i_j-1}^{e_{i_j-1}}} [b_{i_j-1}^{e_{i_j-1}}, b_k^p].
\end{aligned}$$

As we did in Lemma 3.31 (see (3.32)),

$$\begin{aligned}
&\ell(v_{j-1}^{u_j u_{j+1} \cdots u_l}) \\
\leq\; & \tfrac{1}{2}|e_{i_{j-1}+1}||p|(|e_{i_{j-1}+1}| + |p| + 2) + |e_{i_{j-1}+1}||p|(|e_{i_{j-1}+2}| + \cdots + |e_{i_j-1}| + d_j) \\
& +\tfrac{1}{2}|e_{i_{j-1}+2}||p|(|e_{i_{j-1}+2}| + |p| + 2) + |e_{i_{j-1}+2}||p|(|e_{i_{j-1}+3}| + \cdots + |e_{i_j-1}| + d_j) \\
& + \cdots + \tfrac{1}{2}|e_{i_j-1}||p|(|e_{i_j-1}| + |p| + 2) + |e_{i_j-1}||p|d_j) \\
=\; & |p|(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) + \tfrac{1}{2}|p|(|e_{i_{j-1}+1}|^2 + \cdots + |e_{i_j-1}|^2) \\
& +\tfrac{1}{2}|p|^2(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) + |p|d_j(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) \\
& +|p| \textstyle\sum_{\alpha,\beta \in D_{j-1}-D_j \;\&\; \alpha<\beta} |e_\alpha||e_\beta| \\
\leq\; & p_j(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) + \tfrac{1}{2}p_j(|e_{i_{j-1}+1}|^2 + \cdots + |e_{i_j-1}|^2) \\
& +\tfrac{1}{2}p_j^2(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) + p_j d_j(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) \\
& +p_j \textstyle\sum_{\alpha,\beta \in D_{j-1}-D_j \;\&\; \alpha<\beta} |e_\alpha||e_\beta| \\
=\; & p_j(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) + \tfrac{1}{2}p_j(|e_{i_{j-1}+1}|^2 + \cdots + |e_{i_j-1}|^2) \\
& +\tfrac{1}{2} \textstyle\sum_{\gamma \in P_j} |e_\gamma|^2(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) \\
& + \textstyle\sum_{\gamma,\delta \in P_j \;\&\; \gamma<\delta} |e_\gamma e_\delta|(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) \\
& +p_j d_j(|e_{i_{j-1}+1}| + \cdots + |e_{i_j-1}|) + p_j \textstyle\sum_{\alpha,\beta \in D_{j-1}-D_j \;\&\; \alpha<\beta} |e_\alpha||e_\beta|.
\end{aligned}$$

We see that $\ell'(v_{j-1}^{u_j u_{j+1} \cdots u_l})$ is bounded above by a number which is pretty much dealing with $D_{j-1} - D_j$. It is then not too hard to check that

$$\ell'(v_1^{u_2 \cdots u_l} v_2^{u_3 \cdots u_l} \cdots v_{l-2}^{u_{l-1} \cdots u_l} v_{l-1}^{u_l} v_l)$$

does not exceed $n^4 + n^3 + n^2$ (see (3.33), (3.34), (3.35) in Lemma 3.31). Now, let

$$w_2 = u_0 u_1 \cdots u_l.$$

We again begin with the very end letter and find the occurrences of the same letter across w_2 just as we did to w_1. Clearly, all b_k no longer occur in w_2 and so e_γ $(\gamma \in P_j)$ will not happen in the calculation. Hence, accumulatively, the module length so far is still bounded by $n^4 + n^3 + n^2$. Continue the process and use the same argument, we have $\ell'(w') \leq O(n^4)$. □

Theorem D will then follow almost immediately.

Theorem D *Assuming the notation above, if $A = G'$ and $T = G/G'$, then*

$$\Phi_G(n) \preccurlyeq \Phi_A(n^4).$$

Proof. There exists a "preferred presentation" of the metabelian group G which is obtained from the modul presentation of G' and the group presentation of G/G' (see [BCM] for details). Since G is finitely generated, we should have a finite metabelian presentation by dropping lots of relations in the preferred presentation. By Proposition 3.28, we get

$$\Phi_G(n) \preccurlyeq \Phi_{G'}(n^4). \ \square$$

4 Isoperimetric functions for wreath products

4.1 Preferred metabelian presentations

In this section, we shall show, by examples on wreath products, how one can obtain a "preferred metabelian presentation" of a finitely generated metabelian group by using the idea first introduced in [BCM], so-called *preferred presentation.* First of all, we list all the examples we will be focusing on:

- Let W_{11} be the wreath product $< a > \wr < t >$ of the infinitely cyclic groups on a and t, respectively. It will be presented as:

$$W_{11} =<< a, t; [a,t]^a = [a,t] >> .$$

- Let W_{12} be the wreath product $< a > \wr T$, where $< a >$ is the infinitely cyclic group on a and T is the free abelian group on t_1 and t_2. It will be presented as:

$$W_{12} =<< a, t_1, t_2; [t_1, t_2] = 1, [a, t_1]^a = [a, t_1],$$
$$[a, t_2]^a = [a, t_2], [a, t_1]^{t_2 - 1} = [a, t_2]^{t_1 - 1} >> .$$

- Let W_{21} be the wreath product $A \wr < t >$, where A is the free abelian group on a_1 and a_2, and $< t >$ is the infinitely cyclic group on t. It's presentation will be:

$$W_{21} =<< a_1, a_2, t; [a_1, a_2] = 1, [a_1, t]^{a_1} = [a_1, t],$$
$$[a_1, t]^{a_2} = [a_1, t], [a_2, t]^{a_1} = [a_2, t], [a_2, t]^{a_2} = [a_2, t] >> .$$

- Let W_{22} be the wreath product $A \wr T$, where A and T are the free abelian groups on a_1 and a_2, and t_1 and t_2 respectively. It has a presentation as follows:

$$W_{22} =<< a_1, a_2, t_1, t_2; [a_1, a_2] = 1, [t_1, t_2] = 1,$$

$$[a_1, t_1]^{a_1} = [a_1, t_1], [a_1, t_1]^{a_2} = [a_1, t_1], [a_1, t_2]^{a_1} = [a_1, t_2], [a_1, t_2]^{a_2} = [a_1, t_2],$$

$$[a_2, t_1]^{a_1} = [a_2, t_1], [a_2, t_1]^{a_2} = [a_2, t_1], [a_2, t_2]^{a_1} = [a_2, t_2], [a_2, t_2]^{a_2} = [a_2, t_2],$$

$$[a_1, t_1]^{t_2 - 1} = [a_1, t_2]^{t_1 - 1}, [a_2, t_1]^{t_2 - 1} = [a_2, t_2]^{t_1 - 1} >> .$$

- Let W_{mn} be the wreath product of $A \wr T$, where A and T are free abelian groups of rank m and n, respectively. Then, W_{mn} has a metabelian presentation with generators:

$$a_1, \ldots, a_m, t_1, \ldots, t_n,$$

and with defining relations:

$$[a_i, a_j] = 1 \quad (1 \le i < j \le m),$$

$$[t_i, t_j] = 1 \quad (1 \le i < j \le n),$$

$$[a_i, t_j]^{a_k} = [a_i, t_j] \quad (1 \le i, k \le m, \ 1 \le j \le n),$$

$$[a_i, t_j]^{t_k - 1} = [a_i, t_k]^{t_j - 1} \quad (1 \le i \le m, \ 1 \le j \ne k \le n).$$

- Let W be the wreath product of $A \wr T$, where

$$A =< a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+r}; a_{m+i}^{p_i} = 1 \quad (1 \le i \le r),$$

$$[a_i, a_j] = 1 \quad (1 \le i < j \le m + r) >$$

and

$$T = < t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+s}; t_{n+j}^{q_j} = 1 \quad (1 \le j \le s),$$

$$[t_i, t_j] = 1 \quad (1 \le i < j \le n+s) > .$$

Then, W has a preferred metabelian presentation with generators:

$$a_1, \ldots, a_{m+r}, t_1, \ldots, t_{n+s},$$

and with defining relations:

$$a_{m+i}^{p_i} = 1 \quad (1 \le i \le r), \quad t_{n+j}^{q_j} = 1, \quad (1 \le j \le s),$$

$$[a_i, a_j] = 1 \quad (1 \le i < j \le m+r),$$

$$[t_i, t_j] = 1 \quad (1 \le i < j \le n+s),$$

$$[a_i, t_j]^{a_k} = [a_i, t_j] \quad (1 \le i, k \le m+r, \ 1 \le j \le n+s),$$

$$[a_i, t_j]^{t_k - 1} = [a_i, t_k]^{t_j - 1} \quad (1 \le i \le m+r, \ 1 \le j \ne k \le n+s).$$

Secondly, we introduce what is meant by a *preferred metabelian presentation*. Let G be a finitely generated metabelian group. Choose a finite set of generators

$$g_1, g_2, \ldots, g_p, u_1, u_2, \ldots, u_q$$

in such a way that $g_1, g_2, \ldots, g_p$ generate G modulo its derived group G', $u_1, u_2, \ldots, u_q$ are contained in G' and their normal closure in G is G'; i.e., G' is generated by the conjugates

$$u_i^w$$

where i ranges from 1 to q and w ranges over all products of the form

$$w = g_1^{r_1} g_2^{r_2} \cdots g_p^{r_p},$$

where the exponents $r_1, r_2, \ldots, r_p$ are integers. Since G' is a finitely generated $\mathbf{Z}(G/G')$-module and so, by Hilbert's Basis Theorem, is finitely related. Each of the relators, written multiplicatively, can be written as products of the form

$$(u_1^{n_1})^{w_1} \cdots (u_q^{n_q})^{w_q},$$

where the n_i are integers and w_i are words of the form taken by w. This means that we can present G, as a metabelain group, as follows:

$$<< g_1, g_2, \ldots, g_p, u_1, u_2, \ldots, u_q; R_1, R_2, R_3 >>, \tag{4.1}$$

where

i) R_1 consists of the relations that express the fact that G/G' is a finitely presented abelian group; i.e., take all of the commutators of all of the generators $g_1, g_2, \ldots, g_p$ with each other and express them as products of conjugates of the module generators $u_1, u_2, \ldots, u_q$.

ii) R_2 consists of the relations which define the fact that G/G' is finitely presented as an abelian group. So these take the form

$$w = \text{a product of conjugates of } u_1, u_2, \ldots, u_q.$$

iii) R_3 consists of the defining relators for G' as a finitely presented $\mathbf{Z}(G/G')$-module.

The presentation of (4.1) is called a ***preferred metabelian presentation*** of G.

We now try this out with the wreath product $W_{11} =< a > \wr < t >$ of the infinite cyclic groups on a and t, respectively. Notice that W_{11}/W'_{11} is a free abelian group on a and t, modulo W'_{11} and W'_{11}, viewed as a $\mathbf{Z}(W_{11}/W'_{11})$-module, is generated by $u_1 = [a, t]$ and defined by the single relation

$$u_1^a = u_1 .$$

The procedure above then gives us the following metabelian presentation for W_{11}:

$$<< a, t, u_1; [a, t] = u_1, u_1^a = u_1 >> .$$

Notice that since W_{11}/W'_{11} is free abelian, we do not have any relations to add to the "abelian presentation" for W_{11}/W'_{11} - we need only the first type of relation, ensuring that W_{11}/W'_{11} is abelian. And we need only one $\mathbf{Z}(W_{11}/W'_{11})$-module relation. So, by a Tietze transformation, we can throw out the generator u_1 and the relation $[a, t] = u_1$ yielding the presentation

$$\mathcal{P}_{11} =<< a, t; [a, t]^a = [a, t] >> .$$

To see that $\mathcal{P}_{11}$ is really a presentation of W_{11}, with the result

$$W_{11} =<< a, t; [a, a^t] = 1 >>$$

in part i) of Lemma 3.9, we claim that

$$<< a, t; [a, t]^a = [a, t] >> \cong << a, t; [a, a^t] = 1 >> .$$

It suffices to show that $[a,t]^a[a,t]^{-1}$ is a conjugate of $[a,a^t]$:

$$\begin{aligned} [a,t]^a[a,t]^{-1} &= a^{-1}a^{-1}\underbrace{t^{-1}at}a\cdot\underbrace{t^{-1}a^{-1}t}a \\ &= a^{-1}[a,t^{-1}a^{-1}t]a \\ &= a^{-1}[a,a^{-t}]a \\ &= a^{-1}[a,a^t]^{-a^{-t}}a \\ &= [a,a^t]^{-a^{1-t}}. \end{aligned}$$

On the other hand, we present W'_{11} as a finitely presented $\mathbf{Z}(W_{11}/W'_{11})$-module in the following way:

$$W'_{11} =< u_1; u_1^a = u_1 > .$$

Next, we try

$$W_{12} =< a > \wr < t_1, t_2; [t_1,t_2] = 1 > .$$

Then W_{12}/W'_{12} is free abelian on the cosets of a, t_1, t_2 and W'_{12} is the normal closure of $u_1 = [a,t_1], u_2 = [a,t_2]$ presented as a $\mathbf{Z}(W_{12}/W'_{12})$-module as follows:

$$W'_{12} =< u_1, u_2; u_1^a = u_1, u_2^a = u_2, u_1^{t_2-1} = u_2^{t_1-1} > .$$

Then we find a metabelian presentation for W_{12}:

$$<< a, t_1, t_2, u_1, u_2; [a,t_1] = u_1, [a,t_2] = u_2, [t_1,t_2] = 1,$$

$$u_1^a = u_1, u_2^a = u_2, u_1^{t_2-1} = u_2^{t_1-1} >> .$$

By using two Tietze transformations, it can be re-written as

$$\mathcal{P}_{12} =<< a, t_1, t_2; [t_1,t_2] = 1, [a,t_1]^a = [a,t_1],$$

$$[a,t_2]^a = [a,t_2], [a,t_1]^{t_2-1} = [a,t_2]^{t_1-1} >> .$$

To see that $\mathcal{P}_{12}$ is a presentation of W_{12}, we observe that

$$\begin{aligned}
W_{12} &= \langle a \rangle \wr \langle t_1, t_2; [t_1, t_2] = 1 \rangle \\
&= \langle a, t_1, t_2; [t_1, t_2] = 1, [a, a^{t_1^i t_2^j}] = 1 \ (\forall\, i, j \in \mathbf{Z}) \rangle \\
&= \langle\langle a, t_1, t_2; [t_1, t_2] = 1, [a, a^{t_1}] = 1, [a, a^{t_2}] = 1 \rangle\rangle .
\end{aligned}$$

This observation is not difficult to prove and so is omitted. The thing we shall prove is that

$$\mathcal{P}_{12} \cong \langle\langle a, t_1, t_2; [t_1, t_2] = 1, [a, a^{t_1}] = 1, [a, a^{t_2}] = 1 \rangle\rangle .$$

Clearly, $[a, t_1]^{a-1}$ and $[a, t_2]^{a-1}$ are conjugates of $[a, a^{t_1}]$ and $[a, a^{t_2}]$, respectively. Hence, it suffices to show that $[a, t_1]^{t_2-1}[a, t_2]^{1-t_1}$ is a conjugate of $[t_1, t_2]$ which is obtained by the reasoning below:

$$\begin{aligned}
[a, t_1]^{t_2-1}[a, t_2]^{1-t_1} &= [a, t_1]^{t_2}[a, t_1]^{-1}[a, t_2][a, t_2]^{-t_1} \\
&= t_2^{-1}a^{-1}t_1^{-1}at_1t_2 \cdot t_1^{-1}a^{-1}t_1a \cdot a^{-1}t_2^{-1}at_2 \cdot t_1^{-1}t_2^{-1}a^{-1}t_2at_1 \\
&= a^{-1}t_2^{-1}at_2 \cdot t_2^{-1}a^{-1}t_1^{-1}at_1t_2 \cdot t_1^{-1}t_2^{-1}a^{-1}t_2at_1 \cdot t_1^{-1}a^{-1}t_1a \\
&= a^{-1}t_2^{-1}t_1^{-1}at_1t_2 \cdot t_1^{-1}t_2^{-1}a^{-1}t_2t_1a \\
&= a^{-1}t_2^{-1}t_1^{-1}a[t_1^{-1}, t_2^{-1}]a^{-1}t_2t_1a \\
&= a^{-1}t_2^{-1}t_1^{-1}a[t_1^{-1}, t_2^{-1}]a^{-1}t_1t_2t_2^{-1}t_1^{-1}t_2t_1a \\
&= a^{-1}[t_1^{-1}, t_2^{-1}]^{a^{-1}t_1t_2}aa^{-1}t_2^{-1}t_1^{-1}t_2t_1a \\
&= [t_1^{-1}, t_2^{-1}]^{a^{-1}t_1t_2a}[t_2, t_1]^{a} \\
&= [t_1, t_2]^{t_1^{-1}t_2^{-1}a^{-1}t_1t_2a}[t_2, t_1]^{a} \\
&= [t_1, t_2]^{t_1^{-1}t_2^{-1}a^{-1}t_1t_2a}[t_1, t_2]^{-a} \\
&= [t_1, t_2]^{t_1^{-1}t_2^{-1}a^{-1}t_1t_2a-a}.
\end{aligned}$$

Similarly, in the case of

$$W_{21} = \langle a_1, a_2; [a_1, a_2] = 1 \rangle \wr \langle t \rangle,$$

again W_{21}/W'_{21} is free abelian on the cosets of a_1, a_2, t. Let $u_1 = [a_1, t]$ and $u_2 = [a_2, t]$. W'_{21} is the normal closure of u_1 and u_2 and has a $\mathbf{Z}(W_{21}/W'_{21})$-module presentation

$$W'_{21} =< u_1, u_2; u_1^{a_1} = u_1, u_2^{a_1} = u_2, u_1^{a_2} = u_1, u_2^{a_2} = u_2 > .$$

W_{21} is then presented by

$$<< a_1, a_2, t, u_1, u_2; [a_1, a_2] = 1, u_1 = [a_1, t], u_2 = [a_2, t]$$

$$u_1^{a_1} = u_1, u_2^{a_1} = u_2, u_1^{a_2} = u_1, u_2^{a_2} = u_2 > .$$

By using two Tietze transformations, we obtain

$$\mathcal{P}_{21} =<< a_1, a_2, t; [a_1, a_2] = 1, [a_1, t]^{a_1} = [a_1, t],$$

$$[a_2, t]^{a_1} = [a_2, t], [a_1, t]^{a_2} = [a_1, t], [a_2, t]^{a_2} = [a_2, t] >> .$$

Now we need to indicate the following two remarks:

Remark 4.2 *Given a group G,*

$$[a, b^{t^{-1}}] = [b, a^t]^{-t^{-1}a^2} \quad \forall\ a, b, t \in G.$$

The proof is straightforward and so is omitted.

Remark 4.3 *When G is metabelian, for any $a, b, t \in G$, if $[a, b] = 1$ and $[a, b^t] = 1$ then $[a, b^{t^i}] = 1\ \forall\ i \in \mathbf{N}$.*

The proof can be done by induction and is omitted.

Observing, by the remarks above,

$$\begin{aligned} W_{21} &= < a_1, a_2; [a_1, a_2] = 1 > \wr < t > \\ &= < a_1, a_2, t; [a_1, a_2] = 1, [a_1, a_1^{t^i}] = 1, [a_2, a_2^{t^i}] = 1, [a_1, a_2^{t^i}] = 1 \ (\ \forall\ i \in \mathbf{Z}) > \\ &= << a_1, a_2, t; [a_1, a_2] = 1, [a_1, a_1^t] = 1, [a_2, a_2^t] = 1, [a_1, a_2^t] = 1, [a_2, a_1^t] = 1 >> \end{aligned}$$

one has the isomorphism

$$\mathcal{P}_{21} \cong << a_1, a_2, t; [a_1, a_2] = 1, [a_1, a_1^t] = 1, [a_2, a_2^t] = 1, [a_1, a_2^t] = 1, [a_2, a_1^t] = 1 >>$$

by showing

$$\begin{aligned} [a_1, a_2^t] &= a_1^{-1} a_2^{-t} a_1 a_2^t \\ &= a_1^{-1} t^{-1} a_2^{-1} t a_1 t^{-1} a_2 t \\ &= a_1^{-1} t^{-1} a_2^{-1} t a_1 a_2 a_2^{-1} t^{-1} a_2 t \\ &= [t a_1, a_2][a_2, t] \\ &= [t, a_2]^{a_1} [a_1, a_2][a_2, t] \\ &= [a_1, a_2][a_2, t]^{1-a_1}. \end{aligned}$$

In this way, one can try W_{22} as well as all torsion free cases; i.e., W_{mn} $\forall m, n \in \mathbf{N}$. In Theorem A, we will work on the most general case.

Theorem A *Let A and T be finitely generated abelian groups, say*

$$A =< a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+r}; a_{m+i}^{p_i} = 1 \quad (1 \le i \le r),$$

$$[a_i, a_j] = 1 \quad (1 \le i < j \le m + r) >$$

and

$$T =< t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+s}; t_{n+j}^{q_j} = 1 \quad (1 \le j \le s),$$

$$[t_i, t_j] = 1 \quad (1 \le i < j \le n+s) > .$$

Then the wreath product

$$W = A \wr T$$

can be presented in the form:

$$<< a_1, \ldots, a_{m+r}, t_1, \ldots, t_{n+s};\ a_{m+i}^{p_i} = 1 \quad (1 \le i \le r), \quad t_{n+j}^{q_j} = 1, \quad (1 \le j \le s),$$

$$[a_i, a_j] = 1 \quad (1 \le i < j \le m+r), \ [t_i, t_j] = 1 \quad (1 \le i < j \le n+s),$$

$$[a_i, t_j]^{a_k} = [a_i, t_j] \quad (1 \le i, k \le m+r, \ 1 \le j \le n+s),$$

$$[a_i, t_j]^{t_k - 1} = [a_i, t_k]^{t_j - 1} \quad (1 \le i \le m+r, \ 1 \le j \ne k \le n+s) >> .$$

Proof. By the procedure in presenting W in the form of (4.1) we let

$$u_{i,j} = [a_i, t_j] \ (1 \le i \le m+r, \ 1 \le j \le n+s).$$

Then the generators are

$$a_i \ (1 \le i \le m+r), \ t_i \ (1 \le j \le n+s), u_{i,j} \ (1 \le i \le m+r, \ 1 \le j \le n+s).$$

Clearly, R_1 consistis of

$$u_{i,j} = [a_i, t_j] \ (1 \le i \le m+r, \ 1 \le j \le n+s).$$

Since W/W' is an abelian group generated by the cosets of

$$a_i \ (1 \le i \le m+r), t_j \ (1 \le j \le n+s),$$

where $a_iW' \ (1 \le i \le m), t_jW' \ (1 \le j \le n)$ are of infinite order and

$a_{m+i}W'$ is of order p_i $(1 \le i \le r)$, and

$$t_{n+j}W' \text{ is of order } q_j \ (1 \leq j \leq s),$$

eventually, R_2 only consists of

$$[a_i, a_j] = 1 \ (1 \leq i \leq m+r), \ [t_i, t_j] = 1 \ (1 \leq j \leq n+s),$$

$$a_{m+i}^{p_i} = 1 \ (1 \leq i \leq r) \ \text{ and } \ t_{n+j}^{q_j} = 1 \ (1 \leq j \leq s).$$

Now while viewing W' as a $Z(W/W')$-module, we have the relations in R_3 as follows:

$$u_{m+i,j}^{p_i} = 1 \qquad (1 \leq i \leq r, \ 1 \leq j \leq n+s) \qquad (1)$$

$$u_{m+i,j}^{a_{m+i}^{p_i-1}+a_{m+i}^{p_i-2}+\cdots+a_{m+i}+1} = 1 \qquad (1 \leq i \leq r, \ 1 \leq j \leq n+s) \qquad (2)$$

$$u_{m+i,j}^{a_{m+i}^{-(p_i-1)}+a_{m+i}^{-(p_i-2)}+\cdots+a_{m+i}^{-1}+1} = 1 \qquad (1 \leq i \leq r, \ 1 \leq j \leq n+s) \qquad (3)$$

$$u_{i,n+j}^{t_{n+j}^{q_j-1}+t_{n+j}^{q_j-2}+\cdots+t_{n+j}+1} = 1 \qquad (1 \leq i \leq m+r, \ 1 \leq j \leq s) \qquad (4)$$

$$u_{i,n+j}^{t_{n+j}^{-(q_j-1)}+t_{n+j}^{-(q_j-2)}+\cdots+t_{n+j}^{-1}+1} = 1 \qquad (1 \leq i \leq m+r, \ 1 \leq j \leq s) \qquad (5)$$

$$u_{i,j}^{a_k} = u_{i,j} \qquad (1 \leq i, k \leq m+r, \ 1 \leq j \leq n+s) \qquad (6)$$

$$u_{i,j}^{t_k-1} = u_{i,k}^{t_j-1} \qquad (1 \leq i \leq m+r, \ 1 \leq j \neq k \leq n+s) \qquad (7)$$

The relations in (1) are easy to obtain by the following

Remark 4.4 *Given any group G, for any $a, t \in G$, if $[a, a^t] = 1$ and $a^p = 1$ for some $p \in \mathbf{N}$, then $[a, t]^p = 1$.*

The relations in (2), (3), (4), (5) are given from the following

Remark 4.5 *Given any metabelian group G, for any $a, t \in G$, $p \in \mathbf{N}$, one has the two identities below:*

(i) $[a^p, t] = [a, t]^{a^{p-1}+a^{p-2}+\cdots+a+1}$;

(ii) $[a^p, t]^{a^{1-p}} = [a, t]^{a^{-(p-1)}+a^{-(p-2)}+\cdots+a^{-1}+1}$.

Moreover, if $a^p = 1$ then the right hand sides in both (i) and (ii) are actually equal to the identity.

The relations in (6) and (7) are as the same as those we did in W_{12} and W_{21}, respectively.

Finally, putting everything together and using a few Tietze transformations we have the presentation as required. □

It is worth to point out here that the commutator subgroup W' of W can be viewed as a Λ-module, by lifting $\mathbf{Z}(W/W')$ to Λ, where

$$\Lambda = \mathbf{Z}[x_1, \ldots, x_{m+r}, y_1, \ldots, y_{m+r}, \xi_1, \ldots, \xi_{n+s}, \eta_1, \ldots, \eta_{n+s}],$$

with the homomorphism ϕ from Λ onto $\mathbf{Z}(W/W')$, by

$$x_i \mapsto a_i,\ y_i \mapsto a_i^{-1},\ \xi_j \mapsto t_j \text{ and } \eta_j \mapsto t_j^{-1} \quad (1 \le i \le m+r,\ 1 \le j \le n+s),$$

and has the following module presentation with generators:

$$u_{ij} \quad (1 \le i \le m+r,\ 1 \le j \le n+s)$$

and with defining relations:

$$
\begin{array}{lll}
u_{m+i,j}^{p_i} = 1 & (1 \le i \le r,\ 1 \le j \le n+s) & (1)' \\
u_{m+i,j}^{x_{m+i}^{p_i-1}+x_{m+i}^{p_i-2}+\cdots+x_{m+i}+1} = 1 & (1 \le i \le r,\ 1 \le j \le n+s) & (2)' \\
u_{m+i,j}^{y_{m+i}^{p_i-1}+y_{m+i}^{p_i-2}+\cdots+y_{m+i}+1} = 1 & (1 \le i \le r,\ 1 \le j \le n+s) & (3)' \\
u_{i,n+j}^{\xi_{n+j}^{q_j-1}+\xi_{n+j}^{q_j-2}+\cdots+\xi_{n+j}+1} = 1 & (1 \le i \le m+r,\ 1 \le j \le s) & (4)' \\
u_{i,n+j}^{\eta_{n+j}^{q_j-1}+\eta_{n+j}^{q_j-2}+\cdots+\eta_{m+j}+1} = 1 & (1 \le i \le m+r,\ 1 \le j \le s) & (5)' \\
u_{i,j}^{x_k} = u_{i,j} & (1 \le i,k \le m+r,\ 1 \le j \le n+s) & (6)' \\
u_{i,j}^{y_k} = u_{i,j} & (1 \le i,k \le m+r,\ 1 \le j \le n+s) & (6)'' \\
u_{i,j}^{\xi_l-1} = u_{i,l}^{\xi_j-1} & (1 \le i \le m+r,\ 1 \le j \ne l \le n+s) & (7)' \\
u_{i,j}^{1-\eta_l} = u_{i,l}^{(\xi_j-1)\eta_l} & (1 \le i \le m+r,\ 1 \le j \ne l \le n+s) & (7)'' \\
u_{i,j}^{x_k y_k} = u_{i,j} & (1 \le i,k \le m+r,\ 1 \le j \le n+s) & (8) \\
u_{i,j}^{\xi_l \eta_l} = u_{i,j} & (1 \le i \le m+r,\ 1 \le j,l \le n+s) & (9)
\end{array}
$$

Notice that (1)′ is exactly (1) in the proof of Theorem A; (2)′ through (7)′ are related to (2) through (7), respectively; (6)″ and (7)″ can be obtained from (6)′ and (7)′, respectively; (8) and (9) indicate that $x_k y_k$ and $\xi_l \eta_l$ act as the identity, respectively. This presentation will be used in the next section and more detail will be provided there.

4.2 A polynomial upper bound

It is very interesting that the isoperimetric function of the wreath product of a finitely generated abelian group by another is bounded above by a polynomial. More precisely,

Theorem B2 *Let W be the wreath product of A by T, as described in Theorem A.*

Then

$$\Phi_W(\mathcal{N}) \preccurlyeq \begin{cases} \mathcal{N}^{4\cdot 3^{n-1}} & \textit{if } r = s = 0, \\ \mathcal{N}^{8\cdot 3^{n+s-1}} & \textit{otherwise.} \end{cases}$$

In order to see the main idea in proving Theorem B2, we start with the following example:

Let G be W_{12}, the wreath product $< a > \wr T$, where $< a >$ is the infinitely cyclic group on a, and T is the free abelian group on t_1 and t_2. G has a preferred metabelian presentation as follows:

$$G =<< a, t_1, t_2; [t_1, t_2] = 1, [a, t_1]^a = [a, t_1], [a, t_2]^a = [a, t_2], [a, t_1]^{t_2 - 1} = [a, t_2]^{t_1 - 1} >> .$$

Notice that G/G' is free abelian on aG', t_1G', t_2G' and that G' is finitely presented as a $\mathbf{Z}(G/G')$-module. Moreover, by lifting $\mathbf{Z}(G/G')$ to the commutative ring

$$\Lambda = \mathbf{Z}[x_1, \xi_1, \xi_2, y_1, \eta_1, \eta_2],$$

G' has a module presentation over Λ as follows:

$$G' =< u_1, u_2; u_1^{x_1} = u_1, u_2^{x_1} = u_2, u_1^{\xi_2 - 1} = u_2^{\xi_1 - 1},$$

$$u_i^{x_1 y_1} = u_i (i = 1, 2), u_i^{\xi_j \eta_j} = u_i (i = 1, 2 \ \& \ j = 1, 2),$$

$$u_1^{y_1} = u_1, u_2^{y_1} = u_2, u_1^{1 - \eta_2} = u_2^{(\xi_1 - 1)\eta_2}, u_2^{1 - \eta_1} = u_1^{(\xi_2 - 1)\eta_1} > .$$

Here, one can view u_1 as $[a, t_1]$, u_2 as $[a, t_2]$, x_1 as aG', ξ_1 as t_1G', ξ_2 as t_2G', y_1 as $a^{-1}G'$, η_1 as $t_1^{-1}G'$ and η_2 as $t_2^{-1}G'$. Since $x_1 y_1$, $\xi_i \eta_i$ $(i = 1, 2)$ act on the module as the identity, the following relations should be added to the module presentation:

$$u_i^{x_1 y_1} = u_i (i = 1, 2), u_i^{\xi_j \eta_j} = u_i (i = 1, 2 \ \& \ j = 1, 2).$$

Furthermore, for our convenience, we add

$$u_1^{y_1} = u_1, u_2^{y_1} = u_2, u_1^{1-\eta_2} = u_2^{(\xi_1-1)\eta_2}, u_2^{1-\eta_1} = u_1^{(\xi_2-1)\eta_1}$$

which are deduced from previous relations.

For given any word $w =_G 1$ with $\ell(w) = \mathcal{N}$, w can be viewed as a module word $w =_{G'} 1$ with $\ell'(w) = \mathcal{L}$, where $\mathcal{L} \leq O(\mathcal{N}^4)$, by Proposition 3.28. Suppose that

$$w = u_1^{f_1} u_2^{f_2},$$

where $f_i \in \Lambda$ and $\ell(f_1) + \ell(f_2) = \mathcal{L}$. By using relations:

$$u_i^{x_1 y_1} = u_i (i = 1, 2), u_i^{\xi_j \eta_j} = u_i (i = 1, 2 \ \& \ j = 1, 2),$$

$$u_1^{x_1} = u_1, u_2^{x_1} = u_2, u_1^{y_1} = u_1, u_2^{y_1} = u_2$$

w is reduced to

$$w' = u_1^{g_1} u_2^{g_2}$$

where $g_i \in \mathbf{Z}[\xi_1, \xi_2, \eta_1, \eta_2]$ and ξ_j and η_j do not occur simultaneously at every involving monomial in the expression of g_i. We call all such g_i *reduced polynomials*. Here, there are at most $\mathcal{L}$ uses of defining relations needed to reduce w to w'.

Clearly, $\ell'(w') \leq \mathcal{L}$. Since $w' =_{G'} 1$ we have $u_1^{g_1} = u_2^{-g_2}$. So, g_1 is in the ideal $md_\Lambda(\xi_2 - 1, 1 - \eta_2)$. Suppose that

$$g_1 = \xi_2^n p_n + \xi_2^{n-1} p_{n-1} + \cdots + \xi_2^2 p_2 + \xi_2 p_1 + p_0,$$

where n is a positive integer, p_0 is a reduced polynomial in $\mathbf{Z}[\xi_1, \eta_1, \eta_2]$ and p_i, $i = 1, \ldots, n$, are reduced polynomials in $\mathbf{Z}[\xi_1, \eta_1]$. Then

$$g_1 = (\xi_2 - 1) h_1 + r_1,$$

where

$$r_1 = p_n + p_{n-1} + \cdots + p_1 + p_0$$

and

$$h_1 = \sum_{i=0}^{n-1} \xi_2^i \sum_{j=i+1}^{n} p_j,$$

since

$$(\xi_2 - 1)h_1 = g_1 - r_1 = (\xi_2^n - 1)p_n + (\xi_2^{n-1} - 1)p_{n-1} + \cdots + (\xi_2 - 1)p_1$$

$$= (\xi_2 - 1)((\xi_2^{n-1} + \cdots + \xi_2 + 1)p_n + (\xi_2^{n-2} + \cdots + \xi_2 + 1)p_{n-1} + \cdots + (\xi_2 + 1)p_2 + p_1)$$

$$= (\xi_2 - 1)(\xi_2^{n-1}p_n + \xi_2^{n-2}(p_n + p_{n-1}) + \cdots + \xi_2(p_n + \cdots + p_2) + (p_n + \cdots + p_1)).$$

Observing the expression of h_1 above, we see that h_1 is in $\mathbf{Z}[\xi_1, \xi_2, \eta_1]$. Since g_1 is in $md_\Lambda(\xi_2 - 1, 1 - \eta_2)$, we have

$$r_1 = (1 - \eta_2)h_2,$$

where h_2 is in $\mathbf{Z}[\xi_1, \eta_1, \eta_2]$. Hence

$$g_1 = (\xi_2 - 1)h_1 + (1 - \eta_2)h_2.$$

Notice that

$$\ell(r_1) \le \sum_{i=1}^{n} \ell(p_i) \le \ell(g_1) \le \mathcal{L}$$

and, by Lemma 3.20,

$$\ell(h_1) = \sum_{i=0}^{n-1} \ell\Big(\xi_2^i \sum_{j=i+1}^{n} p_j\Big) \le \sum_{i=0}^{n-1} \ell\Big(\sum_{j=i+1}^{n} p_j\Big) + \sum_{i=0}^{n-1} i\,\omega\Big(\sum_{j=i+1}^{n} p_j\Big)$$

$$\le n\mathcal{L} + \frac{n(n-1)}{2}\mathcal{L} = \frac{n(n+1)}{2}\mathcal{L} \le \frac{1}{2}\mathcal{L}^2(\mathcal{L} + 1).$$

In this way, we also obtain

$$\ell(h_2) \leq \frac{1}{2}\mathcal{L}^2(\mathcal{L}+1).$$

Similarly, there exist $k_1 \in \mathbf{Z}[\xi_1, \xi_2, \eta_2]$ and $k_2 \in \mathbf{Z}[\xi_2, \eta_1, \eta_2]$ such that

$$g_2 = (\xi_1 - 1)k_1 + (1 - \eta_1)k_2,$$

where for each i,

$$\ell(k_i) \leq \frac{1}{2}\mathcal{L}^2(\mathcal{L}+1).$$

Hence, w' has the following expression:

$$w' = u_1^{(\xi_2-1)h_1+(1-\eta_2)h_2} u_2^{(\xi_1-1)k_1+(1-\eta_1)k_2}$$

Notice that, by using the defining relations:

$$u_1^{\xi_2-1} = u_2^{\xi_1-1}, u_1^{1-\eta_2} = u_2^{(\xi_1-1)\eta_2}, u_2^{1-\eta_1} = u_1^{(\xi_2-1)\eta_1}$$

one can rewrite w' as a cyclic module word on either u_1 or u_2. So, suppose that, the second case, it can be rewritten as

$$w'' = u_2^{(\xi_1-1)h_1+(\xi_1-1)\eta_2h_2+(\xi_1-1)k_1+(1-\eta_1)k_2}.$$

Here, there are $\omega(h_1) + \omega(h_2) \leq \mathcal{L}^2(\mathcal{L}+1)$ uses of the defining relations needed to transform w' to w''. Let

$$q = (\xi_1 - 1)h_1 + (\xi_1 - 1)\eta_2 h_2 + (\xi_1 - 1)k_1 + (1 - \eta_1)k_2.$$

Then, by Lemma 3.20,

$$\ell(q) \leq \ell((\xi_1 - 1)h_1) + \ell((\xi_1 - 1)\eta_2 h_2) + \ell((\xi_1 - 1)k_1 + (1 - \eta_1)k_2)$$

$$\leq \ell(\xi_1 h_1) + \ell(h_1) + \ell(\xi_1 \eta_2 h_2) + \ell(\eta_2 h_2) + \ell((\xi_1 - 1)k_1 + (1 - \eta_1)k_2)$$

$$\leq 3\ell(h_1) + 5\ell(h_2) + \ell(g_2) \leq 4\mathcal{L}^2(\mathcal{L}+1) + \mathcal{L} = 4\mathcal{L}^3 + 4\mathcal{L}^2 + \mathcal{L}.$$

Since $w'' =_{G'} 1$, q can be reduced to 0, by the given relations

$$u_2^{\xi_i \eta_i} = u_2 \ (i = 1, 2)$$

and the obvious cancellation among monomials in the polynomial g, and hence there will be at most $4\mathcal{L}^3 + 4\mathcal{L}^2 + \mathcal{L}$ uses of the defining relations needed. Therefore, it costs at most

$$\underbrace{\mathcal{L}}_{w \text{ to } w'} + \underbrace{\mathcal{L}^2(\mathcal{L}+1)}_{w' \text{ to } w''} + \underbrace{4\mathcal{L}^3 + 4\mathcal{L}^2 + \mathcal{L}}_{w'' \text{ to } 1} = 5\mathcal{L}^3 + 5\mathcal{L}^2 + 2\mathcal{L} \simeq O(\mathcal{L}^3)$$

uses of the defining relations to prove that, as a module word, $w =_{G'} 1$. Recall: $\mathcal{L} \leq O(\mathcal{N}^4)$. Totally, there are at most $O(\mathcal{N}^{12})$ uses of defining relations to prove that $w =_G 1$.

Next, we try the example $G = W_{mn}$. Recall that W_{mn} is the wreath product $A \wr T$, where A and T are free abelian groups of rank m and n, respectively. Then, W_{mn} has a metabelian presentation with generators:

$$a_1, \ldots, a_m, t_1, \ldots, t_n$$

and with defining relations:

$$[a_i, a_j] = 1 \quad (1 \leq i < j \leq m),$$

$$[t_i, t_j] = 1 \quad (1 \leq i < j \leq n),$$

$$[a_i, t_j]^{a_k} = [a_i, t_j] \quad (1 \leq i, k \leq m, \ 1 \leq j \leq n),$$

$$[a_i, t_j]^{t_k - 1} = [a_i, t_k]^{t_j - 1} \quad (1 \le i \le m, \ 1 \le j \ne k \le n).$$

G' can be viewed as a Λ-module, where

$$\Lambda = \mathbf{Z}[x_1, \ldots, x_m, y_1, \ldots, y_m, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n],$$

by $x_i \mapsto a_i, y_i \mapsto a_i^{-1}, \xi_j \mapsto t_j$, and $\eta_j \mapsto t_j^{-1}$ for $1 \le i \le m, 1 \le j \le n$, and has the following module presentation with generators:

$$u_{i,j} \quad (1 \le i \le m, \ 1 \le j \le n)$$

and with defining relations:

$$u_{i,j}^{x_k y_k} = u_{i,j}, \ u_{i,j}^{\xi_l \eta_l} = u_{i,j}(1 \le i, k \le m, \ 1 \le j, l \le n),$$

$$u_{i,j}^{x_k} = u_{i,j}, u_{i,j}^{y_k} = u_{i,j} \quad (1 \le i, k \le m, \ 1 \le j \le n),$$

$$u_{i,j}^{\xi_l - 1} = u_{i,l}^{\xi_j - 1} \quad (1 \le i \le m, \ 1 \le j \ne l \le n),$$

$$u_{i,j}^{1 - \eta_l} = u_{i,l}^{(\xi_j - 1)\eta_l} \quad (1 \le i \le m, \ 1 \le j \ne l \le n).$$

For given any word $w =_G 1$ with $\ell(w) = \mathcal{N}$, as usual, w can be viewed as a module word $w =_{G'} 1$ with $\ell'(w) = \mathcal{L}$, where $\mathcal{L} \le O(\mathcal{N}^4)$. Suppose that

$$w = \prod_{i=1}^{m} \prod_{j=1}^{n} u_{i,j}^{f_{i,j}},$$

where $f_{i,j} \in \Lambda$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} \ell(f_{i,j}) = \mathcal{L}$. By using relations:

$$u_{i,j}^{x_k y_k} = u_{i,j}, \ u_{i,j}^{\xi_l \eta_l} = u_{i,j}(1 \le i, k \le m, \ 1 \le j, l \le n),$$

$$u_{i,j}^{x_k} = u_{i,j}, u_{i,j}^{y_k} = u_{i,j} \quad (1 \le i, k \le m, \ 1 \le j \le n),$$

w can be reduced to

$$w' = \prod_{i=1}^{m}\prod_{j=1}^{n} u_{i,j}^{g_{i,j}},$$

where $g_{i,j}$ are reduced in $\mathbf{Z}[\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n]$; i.e., ξ_l and η_l do not occur simultaneously at every involving monomial in the expression of $g_{i,j}$. Here, there are at most $\mathcal{L}$ uses of the defining relations needed to reduce w to w'. According to the defining relations:

$$u_{i,j}^{\xi_l-1} = u_{i,l}^{\xi_j-1} \quad (1 \le i \le m,\ 1 \le j \ne l \le n),$$

$$u_{i,j}^{1-\eta_l} = u_{i,l}^{(\xi_j-1)\eta_l} \quad (1 \le i \le m,\ 1 \le j \ne l \le n),$$

for each i,

$$\prod_{j=1}^{n} u_{i,j}^{g_{i,j}} =_{G'} 1$$

since $w' =_{G'} 1$. Hence, it suffices to count the aid of uses of the defining relations for the subword

$$w_i' = \prod_{j=1}^{n} u_{i,j}^{g_{i,j}}.$$

Since $w_i' =_{G'} 1$,

$$u_{i,1}^{g_{i,1}} = \prod_{j=2}^{n} u_{i,j}^{-g_{i,j}}.$$

Hence, $g_{i,1}$ is in the ideal $md_\Lambda(\xi_l - 1, 1 - \eta_l\ (2 \le l \le n))$. Let

$$g_{i,1} = (\xi_2 - 1)h_2 + r_2$$

for some $h_2 \in \mathbf{Z}[\xi_1,\ldots,\xi_n,\eta_1,\eta_3,\ldots,\eta_n]$ and $r_2 \in \mathbf{Z}[\xi_1,\xi_3,\ldots,\xi_n,\eta_1,\ldots,\eta_n]$. So, r_2 is in the ideal $md_\Lambda(\xi_k - 1 (3 \le k \le n), 1 - \eta_l\ (2 \le l \le n))$. Same as before

$\ell(h_2) \leq O(\ell(g_{i,1})^3) \leq O(\mathcal{L}^3)$ and $\ell(r_2) \leq O(\ell(g_{i,1})) \leq O(\mathcal{L})$. Then let

$$r_2 = (1-\eta_2)k_2 + s_2$$

for some $k_2 \in \mathbf{Z}[\xi_1, \xi_3, \ldots, \xi_n, \eta_1, \ldots, \eta_n]$ and $s_2 \in \mathbf{Z}[\xi_1, \xi_3, \ldots, \xi_n, \eta_1, \eta_3, \ldots, \eta_n]$. So, s_2 is in the ideal $md_\Lambda(\xi_k - 1\ (3 \leq k \leq n), 1-\eta_l\ (3 \leq l \leq n))$. Moreover $\ell(k_2) \leq O(\ell(r_2)^3) \leq O(\mathcal{L}^3)$ and $\ell(s_2) \leq O(\ell(r_2)) \leq O(\mathcal{L})$. In this way, after finitely many steps, we can express $g_{i,1}$ as follows:

$$g_{i,1} = \sum_{l=2}^{n}(\xi_l - 1)h_l + \sum_{l=2}^{n}(1-\eta_l)k_l,$$

where

$$h_l \in \mathbf{Z}[\xi_1, \xi_l, \ldots, \xi_n, \eta_1, \eta_{l+1}, \ldots, \eta_n]$$

and

$$k_l \in \mathbf{Z}[\xi_1, \xi_{l+1}, \ldots, \xi_n, \eta_1, \eta_l, \ldots, \eta_n]$$

and $\ell(h_l), \ell(k_l) \leq O(\mathcal{L}^3)$. Observe that

$$u_{i,1}^{\sum_{l=2}^{n}(\xi_l-1)h_l+\sum_{l=2}^{n}(1-\eta_l)k_l} = \left(\prod_{l=2}^{n} u_{i,l}^{(\xi_1-1)h_l}\right)\left(\prod_{l=2}^{n} u_{i,l}^{(\xi_1-1)\eta_l k_l}\right)$$

then there are at most $\sum_{l=2}^{n}\omega(h_l)+\sum_{l=2}^{n}\omega(k_l) \leq O(\mathcal{L}^3)$ uses of the defining relations needed. Hence w_i' becomes

$$w_i'' = u_{i,1}^{g_{i,1}} \prod_{j=2}^{n} u_{i,j}^{g_{i,j}} = \left(\prod_{l=2}^{n} u_{i,l}^{(\xi_1-1)h_l}\right)\left(\prod_{l=2}^{n} u_{i,l}^{(\xi_1-1)\eta_l k_l}\right)\left(\prod_{j=2}^{n} u_{i,j}^{g_{i,j}}\right)$$

$$= \prod_{l=2}^{n} u_{i,l}^{(\xi_1-1)h_l+(\xi_1-1)\eta_l k_l+g_{i,l}}.$$

Notice that

$$\ell'(w_i'') \leq \sum_{l=2}^{n} \ell((\xi_1-1)h_l + (\xi_1-1)\eta_l k_l + g_{i,l}).$$

Moreover,

$$\ell((\xi_1 - 1)h_l) \leq \ell(\xi_1 h_l) + \ell(h_l) \leq \ell(h_l) + \omega(h_l) + \ell(h_l) \leq 3\ell(h_l) \leq O(\mathcal{L}^3),$$

$$\ell((\xi_1 - 1)\eta_l k_l) \leq \ell(\xi_1 \eta_l k_l) + \ell(\eta_l k_l) \leq \ell(k_l) + 2\omega(k_l) + \ell(k_l) + \omega(k_l) \leq 5\ell(k_l) \leq O(\mathcal{L}^3),$$

and

$$\ell(g_{i,l}) \leq \mathcal{L}.$$

So,

$$\ell'(w_i'') \leq \sum_{l=2}^{n} O(\mathcal{L}^3) \simeq O(\mathcal{L}^3)$$

since n is fixed. Let

$$w_i'' = \prod_{l=2}^{n} u_{i,l}^{\hat{g}_{i,l}}.$$

So, $\ell(\hat{g}_{i,l}) \leq O(\mathcal{L}^3)$. Since $w' =_{G'} 1$, again we have

$$u_{i,2}^{\hat{g}_{i,2}} = \prod_{l=3}^{n} {u_{i,l}}^{-\hat{g}_{i,l}}.$$

Therefore, $\hat{g}_{i,2} \in md_{\Lambda}(\xi_l - 1, 1 - \eta_l(3 \leq l \leq n))$. As we did before, $\hat{g}_{i,2}$ can be expressed as

$$\hat{g}_{i,2} = \sum_{l=3}^{n}(\xi_l - 1)\hat{h}_l + \sum_{l=3}^{n}(1 - \eta_l)\hat{k}_l,$$

where

$$\hat{h}_l \in \mathbf{Z}[\xi_1, \xi_2, \xi_l, \ldots, \xi_n, \eta_1, \eta_2, \eta_{l+1}, \ldots, \eta_n]$$

and

$$\hat{k}_l \in \mathbf{Z}[\xi_1, \xi_2, \xi_{l+1}, \ldots, \xi_n, \eta_1, \eta_2, \eta_l, \ldots, \eta_n]$$

and $\ell(\hat{h}_l), \ell(\hat{k}_l) \leq O(\ell(\hat{g}_{i,l})^3) \leq O(\mathcal{L}^9)$. Then,

$${u_{i,2}}^{\sum_{l=3}^{n}(\xi_l - 1)\hat{h}_l + \sum_{l=3}^{n}(1-\eta_l)\hat{k}_l} = \left(\prod_{l=3}^{n} u_{i,l}^{(\xi_2 - 1)\hat{h}_l}\right)\left(\prod_{l=3}^{n} u_{i,l}^{(\xi_2 - 1)\eta_l k_l}\right).$$

Hence, w_i'' becomes

$$w_i^{(3)} = \Big(\prod_{l=3}^{n} u_{i,l}^{(\xi_2-1)\hat{h}_l}\Big)\Big(\prod_{l=3}^{n} u_{i,l}^{(\xi_2-1)\eta_l\hat{k}_l}\Big)\Big(\prod_{l=3}^{n} u_{i,l}^{\hat{g}_{i,l}}\Big).$$

Same as before, there are at most $O(\mathcal{L}^9)$ uses of defining relations needed and

$$\ell(w_i^{(3)}) \leq \sum_{l=3}^{n}(\ell((\xi_2-1)\hat{h}_l) + \ell((\xi_2-1)\eta_l\hat{k}_l) + \ell(\hat{g}_{i,l})) \leq O(\mathcal{L}^9).$$

Repeating the process, at j-th step, there are at most

$$O(\mathcal{L}^{3^{j-1}})$$

uses of the defining relations needed where j runs from 1 to $n-1$. Therefore, the number of uses of the defining relations for each w_i $(1 \leq i \leq m)$ is bounded above by

$$O(\mathcal{L}^{3^{n-1}}).$$

Since $\mathcal{L} \leq O(\mathcal{N}^4)$, the total cost is at most

$$O(\mathcal{N}^{4\cdot 3^{n-1}})$$

in proving $w =_G 1$.

Now before giving a complete proof of Theorem B2, we need the following

Lemma 4.6 *Let*

$$\Lambda = \mathbf{Z}[\xi, \xi_2, \ldots, \xi_m]$$

and let f be a polynomial over Λ with $\ell(f) = \mathcal{L}$. For given any integer p, $2 \leq p < \mathcal{L}$, there exist h, $r \in \Lambda$ such that

$$f = (\xi^{p-1} + \xi^{p-2} + \cdots + \xi + 1)h + r$$

with $\omega(h) \le O(\mathcal{L}^2)$, $\ell(h) \le O(\mathcal{L}^3)$, $\ell(r) \le O(\mathcal{L})$ *and* $d_\xi(r) \le p-2$, *where* $d_\xi(r)$ *is the degree of* r *when viewed as a polynomial of* ξ *over* $\mathbf{Z}[\xi_2, \ldots, \xi_m]$.

Proof. Let

$$f = P_k\xi^k + P_{k-1}\xi^{k-1} + \cdots + P_1\xi + P_0,$$

where $P_k, P_{k-1}, \ldots, P_1, P_0 \in \mathbf{Z}[\xi_2, \ldots, \xi_m]$. By hypothesis, we have $k \le \mathcal{L}$ and $\sum_{i=0}^{k} \ell(P_i) \le \mathcal{L}$. Let $g = (\xi - 1)f$; i.e.,

$$g = P_k\xi^{k+1} + (P_{k-1} - P_k)\xi^k + \cdots + (P_1 - P_2)\xi^2 + (P_0 - P_1)\xi - P_0.$$

Then, there exist $h, r \in \mathbf{Z}[\xi, \xi_2, \ldots, \xi_m]$ such that

$$g = (\xi^p - 1)h + (\xi - 1)r.$$

We claim that h and r have the desired property if they are chosen by the following procedure:

Let $Q_{k+1} = P_k, Q_k = P_{k-1} - P_k, \ldots, Q_1 = P_0 - P_1, Q_0 = -P_0$; i.e.,

$$g = \sum_{i=0}^{k+1} Q_i\xi^i.$$

By using long division algorithm, dividing g by $\xi^p - 1$, we have the quotient

$$h = T_{k+1}\xi^{k+1-p} + T_k\xi^{k-p} + \cdots + T_{p+1}\xi + T_p,$$

and the remainder

$$(\xi - 1)r = T_{p-1}\xi^{p-1} + T_{p-2}\xi^{p-2} + \cdots + T_1\xi + T_0$$

with

$$r = T_{p-1}\xi^{p-2} + (T_{p-2} + T_{p-1})\xi^{p-3} + (T_{p-3} + T_{p-2} + T_{p-1})\xi^{p-4} + \cdots +$$

$$(T_3 + T_4 + \cdots + T_{p-1})\xi^2 + (T_2 + T_3 + \cdots + T_{p-1})\xi + (T_1 + T_2 + \cdots + T_{p-1}),$$

where

$$T_j = Q_j + Q_{j+p} + Q_{j+2p} + \cdots + Q_{j+lp}$$

with $0 \le j \le k+1$ and l depends on i such that $k - p + 2 \le j + lp \le k+1$. Notice that each P_i occurs at most once in every T_j. We then have

$$\ell(T_j) = \ell(Q_j + Q_{j+p} + Q_{j+2p} + \cdots + Q_{j+lp}) \le \sum_{i=0}^{k} \ell(P_i) \le \mathcal{L}.$$

By Lemma 3.20,

$$\begin{aligned}
\ell(r) &= \ell\Big(\sum_{i=1}^{p-1}\sum_{j=i}^{p-1} T_j\xi^{i-1}\Big) \le \sum_{i=1}^{p-1} \ell\Big(\xi^{i-1}\sum_{j=i}^{p-1} T_j\Big) \\
&\le \sum_{i=1}^{p-1}\Big(\ell\Big(\sum_{j=i}^{p-1} T_j\Big) + (i-1)\omega\Big(\sum_{j=i}^{p-1} T_j\Big)\Big) \\
&\le \sum_{i=1}^{p-1} i\ell\Big(\sum_{j=i}^{p-1} T_j\Big) \le \mathcal{L}\sum_{i=1}^{p-1} i(p-i) \\
&= \mathcal{L}\Big(p\sum_{i=1}^{p-1} i - \sum_{i=1}^{p-1} i^2\Big) \le O(\mathcal{L})
\end{aligned}$$

since p is a constant. Similarly,

$$\begin{aligned}
\ell(h) &= \ell\Big(\sum_{i=p}^{k+1} T_i\xi^{i-p}\Big) \le \sum_{i=p}^{k+1} \ell\big(T_i\xi^{i-p}\big) \le \mathcal{L}\sum_{i=p}^{k+1}(i-p+1) \\
&= \mathcal{L}\cdot\frac{(k+3-p)(k+2-p)}{2} \le O(\mathcal{L}^3)
\end{aligned}$$

and

$$\omega(h) = \sum_{i=p}^{k+1}\omega(T_i) \le \sum_{i=p}^{k+1}\ell(T_i) \le (k+2-p)\mathcal{L} \le \mathcal{L}^2$$

since $k \le \mathcal{L}$ □.

Proof of Theorem B2. Same as usual, for any word $w =_W 1$ with $\ell(w) = \mathcal{N}$, by Proposition 3.28, $\ell'(w) \leq O(\mathcal{N}^4)$ when viewed as a module word in W'. Let $\mathcal{L} = \ell'(w)$. We have

$$w = \prod_{i=1}^{m+r} \prod_{j=1}^{n+s} u_{i,j}^{f_{i,j}} \quad \text{for some } f_{i,j} \in \Lambda$$

with

$$\sum_{i=1}^{m+r} \sum_{j=1}^{n+s} \ell(f_{i,j}) = \mathcal{L}$$

(see §4.1 for the given module presentation). Using relations $(6)', (6)'', (8)$, and (9), at a cost of $O(\mathcal{L})$, w is transformed to

$$w' = \prod_{i=1}^{m+r} \prod_{j=1}^{n+s} u_{i,j}^{g_{i,j}}$$

with $g_{i,j}$ reduced and $\ell'(w') \leq \mathcal{L}$. According to the given relations, there is no cancellations between $u_{i,j}$ and $u_{k,l}$ when $i \neq k$. Hence we can break w' into $m+r$ subwords:

$$w'_i = \prod_{j=1}^{n+s} u_{i,j}^{g_{i,j}} \quad 1 \leq i \leq m+r.$$

Obviously, $w'_i =_W 1 \ \forall i$. When $1 \leq i \leq m$, as usual, working on w'_i from left to right. Since

$$u_{i,1}^{g_{i,1}} = \prod_{j=2}^{n+s} u_{i,j}^{-g_{i,j}},$$

$g_{i,1}$ is in the ideal $md_\Lambda(\xi_j - 1 \ (j \neq 1),\ 1 - \eta_k \ (k \neq 1))$. Use the same procedure as for W_{mn}, we are able to transform $u_{i,1}^{g_{i,1}}$ to a word over $u_{i,j} \ (j \neq 1)$, by using $(7)', (7)''$. Then, at a cost of $O(\mathcal{L}^3)$, w'_i becomes

$$w''_i = \prod_{j=2}^{n+s} u_{i,j}^{g'_{i,j}}$$

with $\ell'(w_i'') \leq O(\mathcal{L}^3)$. Continue the process, at n-th step, at a cost of

$$O(\mathcal{L}^{3^n}),$$

w_i' is transformed to

$$w_i^{(n+1)} = \prod_{j=n+1}^{n+s} u_{i,j}^{g_{i,j}^{(n)}}$$

with

$$\ell'(w_i^{(n+1)}) \leq O(\mathcal{L}^{3^n}).$$

Now we shall apply $(4)', (5)'$ to $u_{i,n+1}^{g_{i,n+1}^{(n)}}$ before the next step. By Lemma 4.6, the length does not increase, since $\ell(r)$ is linear, and the cost does not affect to the order of numbers when the $(n+1)$-step is done, since $\omega(h)$ is quadratic. That is, at a cost of $O(\mathcal{L}^{3^{n+1}})$, $w_i^{(n+1)}$ is transformed to a word, say $w_i^{(n+2)}$, with

$$\ell'(w_i^{(n+2)}) \leq O(\mathcal{L}^{3^{n+1}}).$$

Therefore, as we did for W_{mn}, continue the process until the $(n+s-1)$-step. At a cost of

$$O(\mathcal{L}^{3^{n+s-1}}),$$

w_i' is transformed to

$$w_i^{(n+s)} = u_{i,n+s}^{g_{i,n+s}^{(n+s-1)}}$$

with

$$\ell'(w_i^{(n+s)}) \leq O(\mathcal{L}^{3^{n+s-1}}).$$

If the word $w_i^{(n+s)}$ is not obviously equal to the identity, we might still need $(4)', (5)'$. Again, by Lemma 4.6, we should raise the order to the second power while applying

$(4)', (5)'$; i.e., the number of uses of the defining relations is at most

$$O(\mathcal{L}^{2\cdot 3^{n+s-1}}).$$

By Proposition 3.28, the aid of the number of uses is at most

$$O(\mathcal{N}^{8\cdot 3^{n+s-1}}).$$

Finally, we discuss w_i' when $m+1 \leq i \leq m+r$. The procedure is almost the same as before but using additional relations in $(1)', (2)', (3)'$ at each step. Notice that when use relations in $(1)'$ it costs linear time and does not increase the length. Moreover, when use relations in $(2)', (3)'$ is exactly the same discussion as in $(4)', (5)'$. Therefore, the totality is still at most

$$O(\mathcal{N}^{8\cdot 3^{n+s-1}}). \square$$

5 Lower bounds

5.1 The second homology groups

In order to prove some facts in this section, we will deal with absolutely free groups pretty much and so for our convenience we use E for an absolutely free group and F for the metabelianization of E; i.e.,

$$F \cong E/E''.$$

Every metabelian group G can be observed as a factor group of a free metabelian group F; i.e., there exists a normal subgroup N of F such that

$$G \cong F/N.$$

Recall: the second homology group $H_2(G, \mathbf{Z})$ of G is defined by

$$H_2(G, \mathbf{Z}) = \frac{[F, F] \cap N}{[F, N]}.$$

Remark 5.1 *$H_2(G, \mathbf{Z})$ is an invariant.*

Hint. If we express the finitely generated metabelian group G in the form

$$G \cong F/N \quad \text{or} \quad G \cong K/P,$$

where F & K are finitely generated free metabelian groups, then

$$\frac{F' \cap N}{[F, N]} \cong \frac{K' \cap P}{[K, P]}.$$

This can be proved by using Tietze transformations and we then term either

$$\frac{F' \cap N}{[F, N]} \quad \text{or} \quad \frac{K' \cap P}{[K, P]}$$

the second homology group of G, which we denoted by $H_2(G, \mathbf{Z})$.

5.2 Centralized isoperimetric functions

Given any finitely generated metabelian group G with the following presentation:

$$\mathcal{P} = << a_1, \ldots, a_p; r_1, \ldots, r_q >>$$

and $G \cong F/N$, where $F = << a_1, \ldots, a_p >>$ and $N = gp_F(r_1, \ldots, r_q)$, one can analyze isoperimetric behaviour by using the quotient $N/[F, N]$ (see §1.5).

Remark 5.2 $\Phi_{\mathcal{P}}^{cent}(n) \preccurlyeq \Phi_{\mathcal{P}}(n) \quad \forall\, n \in \mathbf{N}$.

Remark 5.3 *When G fixed, $\Phi_{\mathcal{P}}^{cent}$ is independent of the choices of $\mathcal{P}$ and hence denoted by Φ_G^{cent}.*

In order to prove the principal Proposition 5.5, we need the following lemma which is proved by Mostowski in an old paper and is not easy to obtain. However, refer to another one (see [M]), we reorganize the proof as follows:

Lemma 5.4 *Let $X = \{x_1, \dots, x_p\}$ and let $E =< X >$, the free group of rank p with basis X. Suppose that $z_1, \dots, z_p$ are X-words such that $\{z_1E', \dots, z_pE'\}$ forms a basis of the free abelian group E/E'. Then, there exist X-words $y_1, \dots, y_p$ such that E is freely generated by $y_1, \dots, y_p$; i.e., $E =< y_1, \dots, y_p >$, with $y_i \equiv z_i \pmod{E'}$, $1 \leq i \leq p$.*

Proof. By the hypothesis, $\{x_1E', \dots, x_pE'\}$ is also a basis of E/E'. We then have

$$\begin{aligned} z_1 &\equiv x_1^{e_{1,1}} x_2^{e_{1,2}} \cdots x_p^{e_{1,p}} \pmod{E'}, \\ z_2 &\equiv x_1^{e_{2,1}} x_2^{e_{2,2}} \cdots x_p^{e_{2,p}} \pmod{E'}, \\ &\vdots \\ z_p &\equiv x_1^{e_{p,1}} x_2^{e_{p,2}} \cdots x_p^{e_{p,p}} \pmod{E'}, \end{aligned}$$

where $e_{i,j} \in \mathbf{Z}$, $1 \leq i, j \leq p$. Notice that the $p \times p$ matrix

$$\mathcal{A} = [e_{i,j}]_{p \times p}$$

must be invertible and so the determinant of $\mathcal{A}$ is ± 1. Therefore, any two distinct $e_{i,k}$ and $e_{j,k}$ on the same colmn are co-prime. This enables us to diagonalize the matrix $\mathcal{A}$. In other words, by using finitely many elementary row operations on integer-valued matrices, $\mathcal{A}$ can be transformed to the identity matrix $\mathcal{I}$, where

$$\mathcal{I} = [\delta_{i,j}]_{p\times p}, \quad \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}; \end{cases}$$

i.e., there are finitely many matrices $\mathcal{E}_1, \dots, \mathcal{E}_k$ of the following types:

a) $\mathcal{P}_{i,j}$: the matrix after permuting i-th row with j-th row in $\mathcal{I}$

b) $\mathcal{V}_i$: the matrix after changing the sign of i-th row in $\mathcal{I}$

c) $\mathcal{W}_{i,j}$: the matrix after replacing i-th row by its sum with j-th row in $\mathcal{I}$

such that $\mathcal{E}_k\mathcal{E}_{k-1}\cdots\mathcal{E}_1\mathcal{A} = \mathcal{I}$. Hence, $\mathcal{A} = \mathcal{E}_1^{-1}\mathcal{E}_2^{-1}\cdots\mathcal{E}_k^{-1}$. Moreover,

$$\mathcal{A}_{p\times p} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}_{p\times 1} = \begin{bmatrix} x_1^{e_{1,1}} & \cdots & x_p^{e_{1,p}} \\ \vdots & \ddots & \vdots \\ x_1^{e_{p,1}} & \cdots & x_p^{e_{p,1}} \end{bmatrix}_{p\times p} \quad \text{(written multiplicatively).}$$

Notice that each elementary row operation corresponds to an elememtary Nielsen transformation (see [MSK]). Let $\mathcal{N}_{\beta_i}$ be the Nielsen transformation corresponds to $\mathcal{E}_i$, where β_i, $i = 1, \dots, k$, are the automorphisms of E induced by a free substitution. Let

$$\beta = \beta_k^{-1} \circ \beta_{k-1}^{-1} \circ \cdots \circ \beta_1^{-1}.$$

Then,

$$\mathcal{N}_\beta = \mathcal{N}_{\beta_1^{-1}}\mathcal{N}_{\beta_2^{-1}}\cdots\mathcal{N}_{\beta_k^{-1}} = \mathcal{N}_{\beta_1}^{-1}\mathcal{N}_{\beta_2}^{-1}\cdots\mathcal{N}_{\beta_k}^{-1}.$$

Let $(y_1, \dots, y_p) = \mathcal{N}_\beta(x_1, \dots, x_p)$. Then, $E = < y_1, \dots, y_p >$. We now claim that

$$y_i \equiv z_i \ (\mathrm{mod}\ E'), \ 1 \leq i \leq p.$$

When we are dealing with free abelian groups, every elementary Nielsen transformation can be represented by a matrix of types a), b), or c). Recall:

$$\begin{bmatrix} x_1^{e_{1,1}} & \cdots & x_p^{e_{1,p}} \\ \vdots & \ddots & \vdots \\ x_1^{e_{p,1}} & \cdots & x_p^{e_{p,p}} \end{bmatrix} = \mathcal{A} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \mathcal{E}_1^{-1} \cdots \mathcal{E}_k^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$

and

$$(y_1, \ldots, y_p) = \mathcal{N}_\beta(x_1, \ldots, x_p) = \mathcal{N}_{\beta_1}^{-1}\mathcal{N}_{\beta_2}^{-1} \cdots \mathcal{N}_{\beta_k}^{-1}(x_1, \ldots, x_p).$$

We then see that $y_i \equiv x_1^{e_{1,1}} \cdots x_p^{e_{1,p}} \equiv z_i \pmod{E'}$, $1 \le i \le p$. □

Proposition 5.5 *Let G be a finitely generated metabelian group such that $H_2(G, \mathbf{Z})$ is finite. Then Φ_G^{cent} is linear; i.e., $\Phi_G^{cent}(n) \simeq n$.*

Proof. Suppose that G has a finite metabelian presentation:

$$G = << a_1, \ldots, a_p; r_1, \ldots, r_q >>,$$

where $r_j = r_j(a_1, \ldots, a_p)$ are words in $\{a_1, \ldots, a_p\}$ with $1 \le j \le q$. Then, G can be viewed as a factor group of a free metabelian group. More precisely, let E be the absolutely free group freely generated by $X = \{x_1, \ldots, x_p\}$ and let F be the metabelization of E; i.e., $F = E/E''$. So, $G \cong F/N$, where

$$F = gp(x_1E'', \ldots, x_pE'') \cong << a_1, \ldots, a_p >>$$

by $x_iE'' \mapsto a_i$, $i = 1, \ldots, p$ and

$$N = gp_{E/E''}(r_1(x_1, \ldots, x_p)E'', \ldots, r_q(x_1, \ldots, x_p)E'').$$

Moreover, there is a canonical map

$$\varphi : E/E' \longrightarrow F/F'$$

defined by

$$x_i E' \mapsto a_i F' = (x_i E'')F'.$$

Obviously, φ gives the isomorphism $E/E' \cong F/F'$. Let $H_1 G = H_1(G, \mathbf{Z})$ and let $H_2 G = H_2(G, \mathbf{Z})$. Now, consider the following 5 terms exact sequence (see [R]):

$$\begin{tikzcd}
H_2F \arrow[r, "o"] \arrow[d, equal] & H_2G \arrow[r, "\iota"] \arrow[d, equal] & N/[F,N] \arrow[r, "\alpha"] \arrow[d, equal] & H_1F \arrow[r] \arrow[d, equal] & H_1G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \cap N/[F,N] \arrow[r] & N/[F,N] \arrow[r] & F/F' \arrow[r] & F/F'N \arrow[r] & 0,
\end{tikzcd}$$

where

$$\iota : r[F,N] \mapsto r[F,N], \ \forall\, r \in F' \cap N$$

and

$$\alpha : r[F,N] \mapsto rF', \ \forall\, r \in N.$$

By the exactness of the sequence, $H_2G = Im(\iota) = Ker(\alpha)$. Since H_1F is free abelian of rank p, $Im(\alpha)$ is also free abelian of rank k for some $k \leq p$. Hence, $N/[F,N]$ splits as the direct sum

$$N/[F,N] \cong H_2G \oplus \mathbf{Z}^k.$$

There exist words $\bar{v}_1, \ldots, \bar{v}_k$ in $F \cap N$ such that

$$\alpha \circ \varphi^{-1} : \bar{v}_i[F,N] \mapsto v_i[E,E]$$

where $\bar{v}_i = \bar{v}_i(a_1, \dots, a_p)$, $\bar{v}_i = v_iE''$ and $v_i \in E$ are representatives obtained by $v_i = \bar{v}_i(x_1, \dots, x_p)$, $1 \leq i \leq k$. Notice that $\{v_i[E,E] | 1 \leq i \leq k\}$ generate the free abelian group $Im(\alpha \circ \varphi^{-1})$. It is illustrated by the following diagram:

$$\begin{tikzcd}
N/[F,N] \arrow[r, "\alpha"] \arrow[dr, "\alpha \circ \varphi^{-1}"'] & F/[F,F] \geq Im(\alpha) \cong \mathbf{Z}^k \arrow[d, "\varphi^{-1}"] \\
 & E/[E,E] \geq Im(\alpha \circ \varphi^{-1}) \cong \mathbf{Z}^k
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{v}_i[F,N] \arrow[r, "\alpha"] \arrow[dr, "\alpha \circ \varphi^{-1}"'] & \bar{v}_i[F,F] \in Im(\alpha) \arrow[d, "\varphi^{-1}"] \\
 & v_i[E,E] \in Im(\alpha \circ \varphi^{-1}).
\end{tikzcd}$$

Recall that E/E' is free abelian freely genertated by $\{x_1E', \dots, x_pE'\}$ and $Im(\alpha\circ\varphi^{-1})$ is a subgroup of E/E', freely generated by $\{v_1E', \dots, v_kE'\}$. One can find a new basis $\{z_1E', \dots, z_pE'\}$ for E/E', where $z_i, 1 \leq i \leq p$ are words in E for which there exist $d_i > 0, 1 \leq i \leq k$ such that $\{z_1^{d_1}E', \dots, z_k^{d_k}E'\}$ freely generate the subgroup $Im(\alpha\circ\varphi^{-1})$ (see [K]). Let $u_i = z_i^{d_i}, 1 \leq i \leq k$. Then, $\bar{u}_i[F,F] \in Im(\alpha)$ and $\bar{u}_i[F,N] \in N/[F,N]$. By Lemma 5.4, there exist $y_i \in z_iE', 1 \leq i \leq p$ such that $\{y_1, \dots, y_p\}$ forms a basis of the free group E; i.e., $E =< x_1, \dots, x_p >=< y_1, \dots, y_p >$.

Before we carry out the connection between H_2G and $\Phi_G^{cent}(n)$, there are some observations to notice:

1. There are two bases for E. Let $Y = \{y_1, \ldots, y_p\}$. We can observe E as $< x_1, \ldots, x_p >$ or $< y_1, \ldots, y_p >$; i.e., $E =< X >=< Y >$.

2. E/E' has two bases $\{x_1E', \ldots, x_pE'\}$ and $\{y_1E', \ldots, y_pE'\}$.

3. E/E'' has two bases $\{x_1E'', \ldots, x_pE''\}$ and $\{y_1E'', \ldots, y_pE''\}$.

   Let $b_i = y_iE''$, $\forall i$. Then,

   $$E/E'' \cong << a_1, \ldots, a_p >> \cong << b_1, \ldots, b_p >> .$$

4. Suppose $\mathcal{N}_\beta$ is a Nielsen transformation of rank p such that

   $$\mathcal{N}_\beta(x_1, \ldots, x_p) = (y_1, \ldots, y_p),$$

   where $y_i \equiv \tau_i(x_1, \ldots, x_p)$ $\forall i$. Moreover,

   $$\mathcal{N}_\beta^{-1}(y_1, \ldots, y_p) = (x_1, \ldots, x_p),$$

   where $x_i \equiv \sigma_i(y_1, \ldots, y_p)$ $\forall i$. Let ρ_Y be the rewriting process from X-words to Y-words and let ρ_X be the rewriting process from Y-words to X-words. Recall G has the presentation:

   $$\mathcal{P} = << a_1, \cdots, a_p; r_1, \cdots, r_q >>,$$

   where $r_j \equiv r_j(a_1, \cdots, a_p)$ $\forall j$. Let

   $$s_j(y_1, \ldots, y_p) = \rho_Y(r_j(x_1, \ldots, x_p)) \; \forall j.$$

   Since $b_i = y_iE''$, we then find another presentation:

   $$\mathcal{Q} =<< b_1, \ldots, b_p; s_1, \cdots, s_q >>$$

   for G, where $s_j \equiv s_j(b_1, \ldots, b_p)$ $\forall j$.

5. Observe E as $< y_1, \dots, y_p >$ and let

$$P = gp_{E/E''}(s_1(y_1, \dots, y_p)E'', \dots, s_q(y_1, \dots, y_p)E'').$$

Then, $N = P$. They are not just isomorphic but exactly "the same" subgroup of E/E''. Hence, we can rewrite the five terms exact sequence in the following way:

$$\begin{tikzcd}
H_2F \arrow[r, "o"] \arrow[d, equal] & H_2G \arrow[r, "\iota"] \arrow[d, equal] & P/[F,P] \arrow[r, "\alpha"] \arrow[d, equal] & H_1F \arrow[r] \arrow[d, equal] & H_1G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \cap P/[F,P] \arrow[r] & P/[F,P] \arrow[r] & F/F' \arrow[r] & F/F'P \arrow[r] & 0,
\end{tikzcd}$$

where

$$\iota : s[F,P] \mapsto s[F,P], \ \forall\, s \in F' \cap P$$

and

$$\alpha : s[F,P] \mapsto F'P, \ \forall\, s \in P.$$

6. $Im(\alpha)$ stays the same and so does $Im(\alpha \circ \varphi^{-1})$.

Now, we are able to use presentation Q of G in item 4 and the exact sequence in item 5 to work out the centralized isoperimetric function of G, since Φ_G^{cent} is independent of the choices of presentations. Suppose that

$$\bar{w} = b_{i_1}^{\epsilon_1} b_{i_2}^{\epsilon_2} \cdots b_{i_n}^{\epsilon_n},$$

where

$$w = y_{i_1}^{\epsilon_1} y_{i_2}^{\epsilon_2} \cdots y_{i_n}^{\epsilon_n}$$

and $\epsilon_i = 0,\ \pm 1$ such that $\bar{w} =_G 1$ with $\ell(\bar{w}) \leq n$. Let $\ell_{ab}(\bar{w})$ = the length of $\bar{w}$ when abelianized. Clearly, $\ell_{ab}(\bar{w}) \leq \ell(\bar{w}) \leq n$. Recall:

$$u_i = z_i^{d_i} \text{ and } z_i \equiv y_i \pmod{F'},\ 1 \leq i \leq k.$$

If u_i is viewed as $\rho_Y(u_i)$, then

$$u_i \equiv y_i^{d_i} \pmod{E'},\ 1 \leq i \leq k.$$

Moreover, let

$$\bar{u}_i = (\rho_Y(u_i))(b_1, \ldots, b_p).$$

Since $\bar{w} = wE'' \in P$ and $P/[F,P] \cong H_2G \oplus \mathbf{Z}^k$, we see

$$\bar{w} \equiv \bar{w}' \prod_{i=1}^{k} \bar{u}_i^{e_i} \pmod{[F,P]},$$

where

$$\bar{w}'[F,P] \in H_2G,\quad \prod_{i=1}^{k} \bar{u}_i^{e_i}[F,P] \in \mathbf{Z}^k,\quad e_i \in \mathbf{Z},\ \forall i.$$

Hence,

$$\bar{w} \equiv \prod_{i=1}^{k} \bar{u}_i^{e_i} \pmod{[F,F]}.$$

Since $F/F' \cong E/E'$, we find

$$w \equiv \prod_{i=1}^{k} u_i^{e_i} \pmod{[E,E]},$$

and so

$$w \equiv \prod_{i=1}^{k} y_i^{d_i e_i} \pmod{[E,E]}$$

and then

$$\bar{w} \equiv \prod_{i=1}^{k} b_i^{d_i e_i} \pmod{[F,F]}.$$

Therefore,

$$\sum_{i=1}^{k} |e_i| \leq \sum_{i=1}^{k} |d_i e_i| = \ell_{ab}(\bar{w}) \leq n.$$

Since $w' \in H_2G = F' \cap P/[F,P]$,

$$w' \equiv \prod_{j=1}^{l} s_{i_j}^{\epsilon_j} \pmod{[F,P]},\ \ \epsilon_j = \pm 1.$$

We define the ***homological area*** of w', denoted by $\Delta_Q^{H_2}(\bar{w}')$, to be the least value of l for which there is such a representation of w'. By the hypothesis, H_2G is a finite set consisting of 1 and products of finintely many $s_i[F,P]$. Hence, $\Delta_Q^{H_2}(\bar{w}')$ is bounded above by a constant, say c_1. Since $\bar{u}_i \in S$,

$$\bar{u}_i \equiv \prod_{j=1}^{m_i} s_{i_j}^{\epsilon_{i,j}} \pmod{[F,P]},\ \ \epsilon_{i,j} = \pm 1,\ \ 1 \leq i \leq k.$$

Let

$$c_2 = \max\{m_i | 1 \leq i \leq k\}.$$

Notice that $u_i, 1 \leq i \leq k$ are chosen independently on w. We then have

$$\begin{aligned}
\Delta_Q^{cent}(\bar{w}) &\leq \Delta_Q^{cent}\left(\textstyle\prod_{i=1}^{k} \bar{u}_i^{e_i}\right) + \Delta_Q^{H_2}(\bar{w}') \\
&\leq \textstyle\sum_{i=1}^{k} m_i |e_i| + c_1 \\
&\leq c_2 \textstyle\sum_{i=1}^{k} |e_i| + c_1 \\
&\leq c_2 \cdot n + c_1.
\end{aligned}$$

Thus, $\Phi_G^{cent}(n) \simeq n$. □

6 Baumslag-Solitar groups made metabelian

6.1 Presentations

Using ordinary group presentation, lots of B-S groups (see [BGSS] & [G]) have exponential isoperimetric functions. However, in the variety of metabelian groups, it turns out that more have a polynomial upper bound. Let

$$B(n,m) = << a, t; (a^n)^t = a^m >> \quad \forall\, m, n \in \mathbf{Z}.$$

Notice that we denote a^t, a conjugated by t; i.e.,

$$a^t = t^{-1}at.$$

It is not easy to find preferred metabelian presentations for these groups except the case when $n = 1$ and $m = 2$. However, we have found another way to get upper bounds in some of these groups. In fact, $B(1,2)$ is a linear case (see §6.2) and $B(n, n+1)$ $(n \geq 2)$ has a cubic upper bound (see §6.3). Surely, for these special cases, we no longer need preferred metabelian presentations since they are already better than $O(\mathcal{N}^4)$ according to the result in §4.2. It will be interesting to know the upper bounds and, perhaps, the preferred metabelian presentations of those B-S groups being left out in this thesis which is one part of my future study.

6.2 A linear case

Theorem 6.1 *Let*

$$G = B(1,2) = << a, t; a^t = a^2 >> .$$

Then, $\Phi_G(\mathcal{N}) \simeq \mathcal{N}$.

In order to prove this theorem, we first show a few lemmas:

Lemma 6.2 *Suppose that* α, β *are two positive numbers such that* $\alpha = 2^j$ *and* $\beta \geq j+1$ *for some nonnegative integer* j. *Let* w *be a word in* $<< a, t >>$ *with* $w = t^\beta a^\alpha t^{-\beta}$. *Then, there are* $2^j - 1$ *uses of the defining relation* $a^t = a^2$ *needed in proving that* $w =_G t^{\beta - j} a t^{-\beta + j}$.

Proof. By hypothesis, we have

$$\begin{array}{rll}
t^\beta a^\alpha t^{-\beta} & =_G \; t^\beta a^{2^j} t^{-\beta} & \\
& =_G \; (t^\beta a^2 t^{-\beta})^{a^{2^{j-1}}} & (1) \\
& =_G \; (t^{\beta-1} a t^{-\beta+1})^{a^{2^{j-1}}} & (2) \\
& =_G \; (t^{\beta-1} a^2 t^{-\beta+1})^{a^{2^{j-2}}} & (3) \\
& =_G \; (t^{\beta-2} a t^{-\beta+2})^{a^{2^{j-2}}} & (4) \\
& \vdots & \vdots \\
& =_G \; t^{\beta-j+1} a^2 t^{-\beta+j-1} & (2j-1) \\
& =_G \; t^{\beta-j} a t^{-\beta+j}. & (2j)
\end{array}$$

So, there are 2^{j-1} uses from (1) to (2), 2^{j-2} from (3) to (4), ..., and 1 from $(2j-1)$ to $(2j)$. Totally, the number of uses of the defining relation needed here is equal to

$$2^{j-1} + 2^{j-2} + \cdots + 2 + 1 = 2^j - 1. \; \square$$

A trick in proving the theorem is to write a word $w \in << a, t >>$ of the form

$$w = t^\beta a^\alpha t^{-\beta}$$

in *binary notation*, where $\alpha, \beta \geq 1$ $(\alpha, \beta \in \mathbf{Z})$; i.e., if

$$\alpha = c_i 2^i + c_{i-1} 2^{i-1} + \cdots + c_1 2 + c_0,$$

with $c_i, c_{i-1}, \ldots, c_1, c_0 = 0, 1$, then we express w as follows:

$$\underbrace{0.00\cdots0\underbrace{c_ic_{i-1}\cdots c_1c_0}_{i\text{ digits}}}_{\beta\text{ digits}}$$

Actually,

$$w = t^{\beta-i}a^{c_i}t^{-\beta+i} \cdot t^{\beta-i+1}a^{c_{i-1}}t^{-\beta+i-1} \cdots t^{\beta-1}a^{c_1}t^{-\beta+1} \cdot t^{\beta}a^{c_0}t^{-\beta}.$$

Lemma 6.3 *Let $i, j, k, l > 0$ and let $l < k$. Suppose that u, v are two words in $<< a, t >>$ with*

$$u = t^k a^{2^i+2^{i-1}+\cdots+2+1} t^{-k}, \quad v = t^l a^{2^j+2^{j-1}+\cdots+2+1} t^{-l}.$$

Then, the number of uses of the defining relation in writing vu in binary notation can be described in the following three cases:

1. *If $i < k - l$, then there are at most $2^{i+1} + 2^{j+1} - 4 - i - j$ uses.*
2. *If $k - l \leq i \leq j + k - l$, then there are at most $2^{i+1} + 2^{j+2}$ uses.*
3. *If $i > j + k - l$, then there are at most 2^{i+2} uses.*

Proof.

Case 1:

$$\begin{aligned} u &= t^k a^{2^i+2^{i-1}+\cdots+2+1} t^{-k} \\ &= t^k a^{2^i} t^{-k} \cdot t^k a^{2^{i-1}} t^{-k} \cdots t^k a t^{-k} \\ &= t^{k-i} a t^{-k+i} \cdot t^{k-i+1} a t^{-k+i-1} \cdots t^k a t^{-k}; \end{aligned}$$

i.e., u can be written in binary notation as follows:

$$u = 0.\underbrace{\underbrace{00\cdots0\underbrace{00\cdots0}_{j\text{ digits}}}_{l\text{ digits}}00\cdots0\underbrace{11\cdots1}_{i\text{ digits}}}_{k\text{ digits}}$$

By Lemma 6.2, there are

$$(2^i - 1) + (2^{i-1} - 1) + \cdots + (2 - 1) = (2^i + 2^{i-1} + \cdots + 2) - i = 2^{i+1} - 2 - i$$

uses needed. Similarly, to write

$$v = t^{l-j}at^{-l+j} \cdot t^{l-j+1}at^{-l+j-1} \cdots t^{l}at^{-l};$$

i.e.,

$$v = 0.\underbrace{\underbrace{00\cdots0\underbrace{11\cdots1}_{j\text{ digits}}}_{l\text{ digits}}00\cdots0\underbrace{00\cdots0}_{i\text{ digits}}}_{k\text{ digits}}$$

there are $2^{j+1} - 2 - j$ uses needed. Since $i < k - l$, the nonzero digits of u do not meet those of v when written in binary notation; i.e.,

$$vu = 0.\underbrace{\underbrace{00\cdots0\underbrace{11\cdots1}_{j\text{ digits}}}_{l\text{ digits}}00\cdots0\underbrace{11\cdots1}_{i\text{ digits}}}_{k\text{ digits}}$$

and so totally there are $2^{i+1} + 2^{j+1} - 4 - i - j$ uses needed.

Case 2: Let $d = k - l$. Then,

$$\begin{aligned}
u &= t^k a^{2^i + 2^{i-1} + \cdots + 2 + 1} t^{-k} \\
&= t^k a^{2^i + 2^{i-1} + \cdots + 2^d} t^{-k} \cdot t^k a^{2^{d-1} + 2^{d-2} + \cdots + 2 + 1} t^{-k} \\
&= t^{k-d} a^{2^{i-d} + 2^{i-1-d} + \cdots + 2 + 1} t^{-k+d}. \\
&\quad\; t^{k-d+1} a t^{-k+d-1} \cdot t^{k-d+2} a t^{-k+d-2} \ldots t^{k-1} a t^{-k+1} \cdot t^k a t^{-k} \\
&= \underbrace{t^l a^{2^{i-d} + 2^{i-1-d} + \cdots + 2 + 1} t^{-l}}_{(1)} . \\
&\quad\; \underbrace{t^{l+1} a t^{-l-1} \cdot t^{l+2} a t^{-l-2} \ldots t^{k-1} a t^{-k+1} \cdot t^k a t^{-k}}_{(2)};
\end{aligned}$$

i.e., express u from k-th place to $(l+1)$-th place backwords as part (2) and leave the rest as part (1) with v. So far, by lemma 6.2 and its proof, there are

$$\begin{aligned}
& (2^{i-1} + 2^{i-2} + \cdots + 2^{i-d}) + (2^{i-2} + 2^{i-3} + \cdots + 2^{i-d-1}) \\
& + \cdots + (2^{d-1} + 2^{d-2} + \cdots + 1) \\
= \; & 2^{i-d}(2^{d-1} + 2^{d-2} + \cdots + 1) + 2^{i-d-1}(2^{d-1} + 2^{d-2} + \cdots + 1) \\
& + \cdots + (2^{d-1} + 2^{d-2} + \cdots + 1) \\
= \; & (2^{i-d} + 2^{i-d-1} + \cdots + 2 + 1)(2^{d-1} + 2^{d-2} + \cdots + 1) \\
= \; & (2^{i-d+1} - 1)(2^d - 1) \\
= \; & 2^{i+1} - 2^{i-d+1} - 2^d + 1
\end{aligned}$$

uses needed for (1) and

$$\begin{aligned}
& (2^{d-1} - 1) + (2^{d-2} - 1) + \cdots + (2 - 1) \\
= \; & 2^{d-1} + 2^{d-2} + \cdots + 2 - (d - 1) \\
= \; & (2^{d-1} + 2^{d-2} + \cdots + 2 + 1) - d \\
= \; & 2^d - 1 - d
\end{aligned}$$

uses needed for (2). So, $2^{i+1} - 2^{i-d+1} - d$ for (1) and (2). Next we combine (1) with v; i.e.,

$$\begin{aligned} & t^l a^{2^{i-d}+2^{i-1-d}+\cdots+2+1} t^{-l} \cdot t^l a^{2^j+2^{j-1}+\cdots+2+1} t^{-l} \\ = \ & t^l a^{(2^{i-d}+2^{i-1-d}+\cdots+2+1)+(2^j+2^{j-1}+\cdots+2+1)} t^{-l}. \end{aligned}$$

In this case, $j \geq i-d$. So, when we re-express $(2^{i-d}+2^{i-1-d}+\cdots+2+1)+(2^j+2^{j-1}+\cdots+2+1)$ as the sum of some distinct powers of 2 in descending order, the leading term will have power $\leq j+1$. Again, by using Lemma 6.2, there are at most 2^{j+2} uses of the defining relation needed. So, totally, the number of uses of the defining relation is

$$\leq 2^{i+1} - 2^{i-d+1} - d + 2^{j+2} \leq 2^{i+1} + 2^{j+2}.$$

Case 3: Notice that $j < i - (k - l) = i - d$. Mostly the discussion remains the same as case 2 except the final step; i.e., when we re-express $(2^{i-d} + 2^{i-1-d} + \cdots + 2 + 1) + (2^j + 2^{j-1} + \cdots + 2 + 1)$ as the sum of distinct powers of 2, the exponent of the leading term turns out to be $\leq i - d + 1$. Same argument, we find that the total number of uses of the defining relation is

$$\leq 2^{i+1} - 2^{i-d+1} - d + 2^{i-d+2} \leq 2^{i+1} + 2^{i-d+2} \leq 2^{i+2}. \ \square$$

Lemma 6.4 *Suppose that α_i, β_i $(1 \leq i \leq k)$ are positive integers such that $\sum_1^k \alpha_i \leq \mathcal{N}$ and $\beta_1 < \beta_2 < \cdots < \beta_k$. Let w be a word in $<< a, t >>$ with*

$$w = t^{\beta_1} a^{\alpha_1} t^{-\beta_1} \cdot t^{\beta_2} a^{\alpha_2} t^{-\beta_2} \cdots t^{\beta_k} a^{\alpha_k} t^{-\beta_k}.$$

Then, the number of uses of the defining relation in writing w in binary notation is less than $4\mathcal{N}$.

Proof. Let $2^{p_i} \geq 0$ $(1 \leq i \leq k)$ be the leading term of α_i when written as the sum of some distinct powers of 2 in descending order; i.e.,

$$\begin{aligned}
\alpha_1 &= 2^{p_1} + c_{1,1}\, 2^{p_1-1} + c_{1,2}\, 2^{p_1-2} + \cdots \\
\alpha_2 &= 2^{p_2} + c_{2,1}\, 2^{p_1-1} + c_{2,2}\, 2^{p_1-2} + \cdots \\
&\vdots \\
\alpha_k &= 2^{p_k} + c_{k,1}\, 2^{p_k-1} + c_{k,2}\, 2^{p_k-2} + \cdots,
\end{aligned}$$

where $c_{i,j} = 0, 1$ $\forall i, j$ and let $d_i = \beta_{k-i+1} - \beta_{k-i} > 0$ $(1 \leq i \leq k-1)$. First of all, let's re-express $t^{\beta_k} a^{\alpha_k} t^{-\beta_k}$ from β_k -th place to β_{k-1} -th place in the following way:

$$\begin{aligned}
t^{\beta_k} a^{\alpha_k} t^{-\beta_k} &= t^{\beta_k} a^{2^{p_k} + c_{k,1} 2^{p_k-1} + c_{k,2} 2^{p_k-2} + \cdots} t^{-\beta_k} \\
&= \underbrace{t^{\beta_k} a^{\theta} t^{-\beta_k}}_{(1)} \cdot \underbrace{t^{\beta_k} a^{\lambda} t^{-\beta_k}}_{(2)},
\end{aligned}$$

where

$$\begin{aligned}
\theta &= 2^{p_k} + c_{k,1}\, 2^{p_k-1} + c_{k,2}\, 2^{p_k-2} + \cdots + c_{k,p_k-d_1}\, 2^{d_1}, \\
\lambda &= c_{k,p_k-d_1+1}\, 2^{d_1-1} + c_{k,p_k-d_1+2}\, 2^{d_1-2} + \cdots
\end{aligned}$$

Notice that (1) will be rewritten as

$$t^{\beta_{k-1}} a^{\theta'} t^{-\beta_{k-1}},$$

where

$$\theta' = 2^{p_k-d_1} + c_{k,1}\, 2^{p_k-d_1-1} + c_{k,2}\, 2^{p_k-d_1-2} + \cdots + c_{k,p_k-d_1}$$

and (2) will be rewritten as

$$t^{\beta_{k-1}+1} a t^{-\beta_{k-1}-1} \cdot t^{\beta_{k-1}+2} a t^{-\beta_{k-1}-2} \cdots t^{\beta_k} a t^{-\beta_k},$$

where, depending on $c_{i,j}$, there might be some missing terms. According to the proof of Lemma 6.3, there are at most

$$\underbrace{2^{p_k+1} - 2^{d_1} - 2^{p_k-d_1+1} + 1}_{\text{for (1)}} + \underbrace{2^{d_1} - 1 - d_1}_{\text{for (2)}} \leq 2^{p_k+1}$$

uses needed. Noting that $2^{p_k} \leq \alpha_k$.

Secondly, we combine (1) with $t^{\beta_{k-1}} a^{\alpha_{k-1}} t^{-\beta_{k-1}}$. Recall:

$$\begin{aligned} \alpha_{k-1} &= 2^{p_{k-1}} + c_{k-1,1}\, 2^{p_{k-1}-1} + c_{k-1,2}\, 2^{p_{k-1}-2} + \cdots, \\ \theta' &= 2^{p_k-d_1} + c_{k,1}\, 2^{p_k-d_1-1} + c_{k,2}\, 2^{p_k-d_1-2} + \cdots + c_{k,p_k-d_1}. \end{aligned}$$

Since $\theta' \leq 2^{-d_1}\alpha_k$, the sum

$$\alpha_{k-1} + \theta' \leq 2^{-d_1}\alpha_k + \alpha_{k-1}.$$

Let $2^{\mathcal{K}_1}$ be the leading term of $\alpha_{k-1} + \theta'$ when written as the sum of some distinct powers of 2 in descending order. Clearly,

$$2^{\mathcal{K}_1} \leq 2^{-d_1}\alpha_k + \alpha_{k-1}.$$

Now we re-express the new combination from β_{k-1}-th place to β_{k-2}-th place by breaking it into part (1) and part (2) as before. Since $2^{\mathcal{K}_1}$ is the leading term in the exponent of a, there are at most $2^{\mathcal{K}_1+1}$ uses needed as we discuss before.

Similarly, combining the updated (1) with $t^{\beta_{k-2}} a^{\alpha_{k-2}} t^{-\beta_{k-2}}$. The exponent of a in the resulting expression is

$$\leq 2^{-d_2}(\alpha_{k-1}+\theta') + \alpha_{k-2} \leq 2^{-d_2}(2^{-d_1}\alpha_k + \alpha_{k-1}) + \alpha_{k-2} = 2^{-d_1-d_2}\alpha_k + 2^{-d_2}\alpha_{k-1} + \alpha_{k-2}.$$

Let $2^{\mathcal{K}_2}$ be the leading term of the exponent of a. We then have

$$2^{\mathcal{K}_2} \leq 2^{-d_1-d_2}\alpha_k + 2^{-d_2}\alpha_{k-1} + \alpha_{k-2}.$$

Then, re-express the updated combination from β_{k-2}-th place to β_{k-3}-th place as before. Same discussion, there are at most $2^{\mathcal{K}_2+1}$ uses needed. Continue the process until the whole word w are rewritten in binary notation. Hence, there are at most

$$2^{p_k+1} + 2^{\mathcal{K}_1+1} + 2^{\mathcal{K}_2+1} + \cdots + 2^{\mathcal{K}_{k-1}+1} \tag{6.5}$$

uses needed. Furthermore,

$$\begin{aligned}
2^{p_k} &\leq \alpha_k && (1)\\
2^{\mathcal{K}_1} &\leq 2^{-d_1}\alpha_k + \alpha_{k-1} && (2)\\
2^{\mathcal{K}_2} &\leq 2^{-d_1-d_2}\alpha_k + 2^{-d_2}\alpha_{k-1} + \alpha_{k-2} && (3)\\
&\ \vdots && \vdots\\
2^{\mathcal{K}_{k-1}} &\leq 2^{-d_1-d_2-\cdots-d_{k-1}}\alpha_k + 2^{-d_2-d_3-\cdots-d_{k-1}}\alpha_{k-1} + \cdots\\
&\quad +2^{-d_{k-2}-d_{k-1}}\alpha_3 + 2^{-d_{k-1}}\alpha_2 + \alpha_1. && (k)
\end{aligned}$$

Since $d_i \geq 1$ $(1 \leq i \leq k)$, we see that

$$\begin{aligned}
\text{the right hand side of (1)} &\leq \alpha_k && (1)'\\
\text{the right hand side of (2)} &\leq 2^{-1}\alpha_k + \alpha_{k-1} && (2)'\\
\text{the right hand side of (3)} &\leq 2^{-2}\alpha_k + 2^{-1}\alpha_{k-1} + \alpha_{k-2} && (3)'\\
&\ \vdots && \vdots\\
\text{the right hand side of } (k) &\leq 2^{-(k-1)}\alpha_k + 2^{-(k-2)}\alpha_{k-1} + \cdots\\
&\quad +2^{-2}\alpha_3 + 2^{-1}\alpha_2 + \alpha_1. && (k)'
\end{aligned}$$

Hence, the sum of the right hand sides from (1)′ to (k)′ is equal to

$$
\begin{aligned}
& (1+2^{-1}+\cdots+2^{-(k-1)})\alpha_k+(1+2^{-1}+\cdots+2^{-(k-2)})\alpha_{k-1}+\cdots \\
& +(1+2^{-1}+2^{-2})\alpha_3+(1+2^{-1})\alpha_2+\alpha_1 \\
= \; & (2-2^{-(k-1)})\alpha_k+(2-2^{-(k-2)})\alpha_{k-1}+\cdots \\
& +(2-2^{-2})\alpha_3+(2-2^{-1})\alpha_2+(2-1)\alpha_1 \\
< \; & 2(\alpha_k+\alpha_{k-1}+\cdots+\alpha_2+\alpha_1) \\
\leq \; & 2\mathcal{N}.
\end{aligned}
$$

Therefore, by (6.5), there are at most $4\mathcal{N}$ uses of the defining relation needed. □

Proof of Theorem 6.1 Suppose that w is a word in $<< a,t >>$ such that $w =_G 1$ with $\ell(w)=\mathcal{N}$. Then,

$$w = t^{m_1}a^{n_1}t^{m_2}a^{n_2}\cdots t^{m_k}a^{n_k}, \quad \text{with } \sum_1^k |m_i| + \sum_1^k |n_i| = \mathcal{N},$$

where $m_1, n_1, \ldots, m_k, n_k \in \mathbf{Z}$.

Observe that $t^{-1}at = a^2$. We see that $[a,t]=a$ and so $a \in G'$. Hence,

$$a^{t^i} \in G' \;\; \forall i \in \mathbf{Z}$$

and then

$$[a, a^{t^i}] = 1 \;\; \forall i \in \mathbf{Z},$$

since G is metabelian. Let's rewrite w as follows:

$$
\begin{aligned}
w \; &= \; t^{m_1}a^{n_1}t^{m_2}a^{n_2}\ldots t^{m_{k-1}}a^{n_{k-1}}t^{m_k}a^{n_k} \\
&= \; t^{m_1}a^{n_1}t^{m_2}a^{n_2}\ldots t^{m_{k-1}}t^{m_k}\cdot \underbrace{t^{-m_k}a^{n_{k-1}}t^{m_k}}_{}\cdot a^{n_k} \\
&\;\;\vdots \\
&= \; t^{m_1+\cdots+m_k}\cdot \underbrace{t^{-(m_2+\cdots+m_k)}a^{n_1}t^{m_2+\cdots+m_k}}_{}\cdots \underbrace{t^{-m_k}a^{n_{k-1}}t^{m_k}}_{}\cdot a^{n_k}.
\end{aligned}
$$

Since $w =_G 1$, $t^{m_1+\cdots+m_k} = 1$ and so $\sum_1^k m_i = 0$. Hence,

$$w = \underbrace{t^{-(m_2+\cdots+m_k)}a^{n_1}t^{m_2+\cdots+m_k}}\cdots\underbrace{t^{-m_k}a^{n_{k-1}}t^{m_k}}\cdot a^{n_k}.$$

Since $a^{t^i} \in G'$ $\forall i \in \mathbf{Z}$, we need at most $\mathcal{N}$ uses of the defining relation to re-express w as follows:

$$w =_G t^{-\tau_1}a^{\alpha_1}t^{\tau_1} \cdot t^{-\tau_2}a^{\alpha_2}t^{\tau_2} \cdots t^{-\tau_l}a^{\alpha_l}t^{\tau_l},$$

where $\tau_1 > \tau_2 > \cdots > \tau_l$, $|\tau_i| \le \mathcal{N}$, $\forall\, i \in \{1, \ldots, l\}$, and $\sum_1^l |\alpha_i| \le \mathcal{N}$.

Without any loss of generality, we suppose that $\tau_1 > 0$. Now conjugate w by $t^{-\tau_1}$:

$$w' = t^{\tau_1}wt^{-\tau_1} = a^{\alpha_1} \cdot t^{\tau_1-\tau_2}a^{\alpha_2}t^{-\tau_1+\tau_2} \cdots t^{\tau_1-\tau_l}a^{\alpha_l}t^{\tau_1+\tau_l}.$$

Put $\beta_1 = 0$, $\beta_i = -\tau_1 + \tau_i$, $2 \le i \le l$. We then have

$$w' = t^{\beta_1}a^{\alpha_1}t^{-\beta_1} \cdot t^{\beta_2}a^{\alpha_2}t^{-\beta_2} \cdots t^{\beta_l}a^{\alpha_l}t^{-\beta_l}$$

for which $w' =_G 1$, $\ell(w') \le 3\mathcal{N}$, $0 = \beta_1 < \beta_2 < \cdots < \beta_l$, $\beta_i \le 2\mathcal{N}$, $\sum_1^l |\alpha_i| \le \mathcal{N}$.

Since

$$t^{\beta_1}a^{\alpha_1}t^{-\beta_1} \cdot t^{\beta_2}a^{\alpha_2}t^{-\beta_2} \cdots t^{\beta_l}a^{\alpha_l}t^{-\beta_l} = 1,$$

we can move $t^{\beta_i}a^{\alpha_i}t^{-\beta_i}$ to the right hand side for all $\alpha_i < 0$. Then,

$$t^{\gamma_1}a^{\delta_1}t^{-\gamma_1} \cdots t^{\gamma_r}a^{\delta_r}t^{-\gamma_r} = t^{\eta_1}a^{\mu_1}t^{-\eta_1} \cdots t^{\eta_s}a^{\mu_s}t^{-\eta_s}, \tag{6.6}$$

where

$$\begin{array}{lll} \delta_i > 0, & 1 \le i \le r, & \gamma_1 < \gamma_2 < \cdots < \gamma_r \le 2\mathcal{N}, \\ \mu_j > 0, & 1 \le j \le s, & \eta_1 < \eta_2 < \cdots < \eta_s \le 2\mathcal{N}, \\ \text{and} & \sum_1^r \delta_i + \sum_1^s \mu_j \le \mathcal{N}. & \end{array}$$

Now let's rewrite both sides of (6.6) in binary notation. Suppose that q is the positive integer such that $2^q \leq \mathcal{N} < 2^{q+1}$ and that

$$\delta_i = c_{i,q}2^q + c_{i,q-1}2^{q-1} + \cdots + c_{i,2}2^2 + c_{i,1}2 + c_{i,0},$$

where $c_{i,j} = 0, 1$ ($1 \leq i \leq r,\ i \leq j \leq q$). Hence,

$$t^{\gamma_i}a^{\delta_i}t^{-\gamma_i} = t^{\gamma_i}a^{c_{i,q}2^q}t^{-\gamma_i} \cdots t^{\gamma_i}a^{c_{i,2}2^2}t^{-\gamma_i} \cdot t^{\gamma_i}a^{c_{i,1}2}t^{-\gamma_i} \cdot t^{\gamma_i}a^{c_{i,0}}t^{-\gamma_i}.$$

Then, there are three possible cases in expressing $t^{\gamma_i}a^{\delta_i}t^{-\gamma_i}$.

When $\gamma_i > q+1$, we write $t^{\gamma_i}a^{\delta_i}t^{-\gamma_i}$ as:

$$0.\underbrace{\underbrace{0\ 0\cdots 0}_{\gamma_i-q-1}c_{i,q}\ c_{i,q-1}\cdots c_{i,2}\ c_{i,1}\ c_{i,0}}_{\gamma_i \text{ digits}} \tag{6.7}$$

When $\gamma_i = q+1$, we write $t^{\gamma_i}a^{\delta_i}t^{-\gamma_i}$ as:

$$0.\underbrace{c_{i,q}\ c_{i,q-1}\cdots c_{i,2}\ c_{i,1}\ c_{i,0}}_{\gamma_i \text{ digits}} \tag{6.8}$$

When $\gamma_i < q+1$, we write $t^{\gamma_i}a^{\delta_i}t^{-\gamma_i}$ as:

$$\underbrace{c_{i,q}\ c_{i,q-1}\cdots c_{i,\gamma_i}}_{q+1-\gamma_i}\cdot\underbrace{c_{i,\gamma_i-1}\cdots c_{i,1}\ c_{i,0}}_{\gamma_i} \tag{6.9}$$

Notice that $\sum_1^r \delta_i \leq \mathcal{N}$. According to Lemma 6.4, we need at most $4\mathcal{N}$ uses of the defining relation in rewriting the left hand side of (6.6) in binary notation. Similarly, there are at most $4\mathcal{N}$ uses for the right hand side of (6.6). Hence, $\Phi_G(\mathcal{N}) \preccurlyeq \mathcal{N}$; i.e., $\Phi_G(\mathcal{N}) \simeq \mathcal{N}$. □

6.3 A case with a cubic upper bound

Theorem 6.10 *Let*

$$G = B(n,m) =<< a, t; (a^n)^t = a^m >> \quad (m > 2,\ m = n+1).$$

Then, $\Phi_G(\mathcal{N}) \preccurlyeq \mathcal{N}^3$.

In order to prove the theorem above, there are a few lemmas needed to be proved first:

Lemma 6.11 *If* $m, n \in \mathbf{N}$ & $1 < n < m$, *then*

a) there exists $p \in \mathbf{N}$ *such that* $n^{p+1} < m^p$;

b) $n^{q+1} < m^q \quad \forall\, q \geq p$;

c) $n^{q+1} - n < m^q - 1 \quad \forall\, q \geq p$.

Proof. Since $\lim_{r\to\infty} n^{1+\frac{1}{r}} = n$ and $n < m$, there exists $p \in \mathbf{N}$ such that

$$n^{1+\frac{1}{q}} < n + \frac{1}{2}(m-n) \quad \forall q \geq p.$$

So,

$$n^{1+\frac{1}{q}} < \frac{1}{2}(m+n) < m \quad \forall q \geq p.$$

Hence,

$$n^{\frac{q+1}{q}} < m \quad \forall q \geq p.$$

When taking the q-th power of both sides, the inequality above becomes

$$n^{q+1} < m^q \quad \forall q \geq p.$$

Hence, we have a) and b). Then, obviously we have

$$n^{q+1} - n < m^q - 1 \quad \forall q \geq p.$$

Hence, we have c). □

Lemma 6.12 *Assuming m and n as indicated in Theorem 6.10 and p as in Lemma 6.11. If there are c_1 steps needed to rewrite $t^{\beta}a^{m^p}t^{-\beta}$ in m-nary notation. Then, there will be at most $qm^q + c_1$ steps needed to rewrite $t^{\beta}a^{m^q}t^{-\beta}$ in m-nary notation for all $q > p$.*

Proof. For given any $q > p$,

$$\begin{aligned}
t^{\beta}a^{m^q}t^{-\beta} &= (t^{\beta-1}a^{n}t^{-\beta+1})^{m^{q-1}} && (m^{q-1} \text{ steps needed}) \\
&= (t^{\beta-2}a^{n}t^{-\beta+2})^{nm^{q-2}} && (nm^{q-2} \text{ steps needed}) \\
&\ \vdots && \quad\vdots \\
&= (t^{\beta-q}a^{n}t^{-\beta+q})^{n^{q-1}} && (n^{q-1} \text{ steps needed}) \\
&= t^{\beta-q}a^{n^q}t^{-\beta+q}.
\end{aligned}$$

Hence, from $t^{\beta}a^{m^q}t^{-\beta}$ to $t^{\beta-q}a^{n^q}t^{-\beta+q}$, we need $m^{q-1}+nm^{q-2}+\cdots+n^{q-1}$ steps. Similarly, from $t^{\beta}a^{m^{q-1}}t^{-\beta}$ to $t^{\beta-q+1}a^{n^{q-1}}t^{-\beta+q-1}$, we need $m^{q-2}+nm^{q-3}+\cdots+n^{q-2}$ steps. Continue the process, there are $m^p + nm^{p-1} + \cdots + n^p$ steps needed from $t^{\beta}a^{m^{p+1}}t^{-\beta}$ to $t^{\beta-p-1}a^{n^{p+1}}t^{-\beta+p+1}$. Since $n^q < m^{q-1}, n^{q-1} < m^{q-2}, n^{q-2} < m^{q-3}, \ldots, n^{p+1} < m^p$, to write $t^{\beta}a^{m^q}t^{-\beta}$ in m-nary notation, the number of steps needed is bounded above by

$$\begin{aligned} & (m^{q-1} + nm^{q-2} + \cdots\cdots\cdots\cdots\cdots\cdots + n^{q-2}m + n^{q-1}) \\ + \; & (m^{q-2} + nm^{q-3} + \cdots\cdots\cdots\cdots + n^{q-3}m + n^{q-2}) \\ + \; & (m^{q-3} + nm^{q-4} + \cdots + n^{q-4}m + n^{q-3}) \\ \vdots \; & \\ + \; & (m^p + nm^{p-1} + \cdots + n^p) + c_1 \\ < \; & qm^{q-1} + (q-1)m^{q-2} + \cdots + (p+1)m^p + c_1 \\ < \; & q(m^{q-1} + m^{q-2} + \cdots + m^p) + c_1 \\ \le \; & q(m^q - 1) + c_1 \\ < \; & qm^q + c_1. \end{aligned}$$

Hence, there are at most $qm^q + c_1$ steps needed. □

Lemma 6.13 *Assuming m, n, p as indicated in Lemma 6.12. If there are c_2 steps needed to rewrite $t^\beta a^{m^p-1}t^{-\beta}$ in m-nary notation. Then, there will be at most $qm^q + c_2$ steps needed to rewrite $t^\beta a^{m^q-1}t^{-\beta}$ in m-nary notation for all $q > p$.*

Proof. Since $m^q - 1 = m(m^{q-1} - 1) + (m-1)$, there are $m^{q-1} - 1$ steps needed from $t^\beta a^{m^q-1}t^{-\beta}$ to $t^{\beta-1}a^{n(m^{q-1}-1)}t^{-\beta+1} \cdot t^\beta a^{m-1}t^{-\beta}$. Since $n(m^{q-1} - 1) = m(nm^{q-2} - 1) + (m-n)$, there are $nm^{q-2} - 1$ steps needed from $t^{\beta-1}a^{n(m^{q-1}-1)}t^{-\beta+1}$ to $t^{\beta-2}a^{n(m^{q-2}-1)}t^{-\beta+2} \cdot t^{\beta-1}a^{m-1}t^{-\beta+1}$. Continue the process, we see that from $t^{\beta-q+1}a^{n(n^{q-2}m-1)}t^{-\beta+q-1}$ to $t^{\beta-q}a^{n(n^{q-1}-1)}t^{-\beta+q} . t^{\beta-q+1}a^{m-n}t^{-\beta+q-1}$, there are $n^{q-1} - 1$ steps needed since $n(n^{q-2}m - 1) = m(n^{q-1} - 1) + (m-n)$. So, totally, from $t^\beta a^{m^q-1}t^{-\beta}$ to

$$t^{\beta-q}a^{n^q-n}t^{-\beta+q} . t^{\beta-q+1}a^{m-n}t^{-\beta+q-1} . t^{\beta-q+2}a^{m-n}t^{-\beta+q-2} \ldots t^{\beta-1}a^{m-n}t^{-\beta+1} . t^\beta a^{m-1}t^{-\beta},$$

there are $(m^{q-1} + nm^{q-2} + \cdots + n^{q-1}) - q$ steps needed. Same argument, from $t^{\beta}a^{m^{q-1}-1}t^{-\beta}$ to

$$t^{\beta-q+1}a^{n^{q-1}-n}t^{-\beta+q-1} \cdot t^{\beta-q+2}a^{m-n}t^{-\beta+q-2} \cdots t^{\beta-1}a^{m-n}t^{-\beta+1} \cdot t^{\beta}a^{m-1}t^{-\beta},$$

there are $(m^{q-2} + nm^{q-3} + \cdots + n^{q-2}) - (q-1)$ steps needed. Continue the process, from $t^{\beta}a^{m^{p+1}-1}t^{-\beta}$ to

$$t^{\beta-p-1}a^{n^{p+1}-n}t^{-\beta+p+1} \cdot t^{\beta-p}a^{m-n}t^{-\beta+p} \cdots t^{\beta-1}a^{m-n}t^{-\beta+1} \cdot t^{\beta}a^{m-1}t^{-\beta},$$

there are $(m^{p} + nm^{p-1} + \cdots + n^{p}) - (p+1)$ steps needed. Now same as we did in Lemma 6.12, since $n^q - n < m^{q-1} - 1, n^{q-1} - n < m^{q-2} - 1, \ldots, n^{p+1} - n < m^p - 1$, to write $t^{\beta}a^{m^q-1}t^{-\beta}$ in m-nary notation, the number of uses of the defining relator is bounded above by

$$\begin{aligned} & (m^{q-1} + nm^{q-2} + \cdots\cdots\cdots\cdots\cdots\cdots + n^{q-2}m + n^{q-1} - q) \\ + & \ (m^{q-2} + nm^{q-3} + \cdots\cdots\cdots\cdots + n^{q-3}m + n^{q-2} - (q-1)) \\ & \vdots \\ + & \ (m^{p} + nm^{p-1} + \cdots + n^{p-1}m + n^{p} - (p+1)) \\ + & \ c_2. \end{aligned}$$

Same calculation as we did in Lemma 6.12, there are at most $qm^q + c_2$ uses of the defining relation needed. □

Lemma 6.14 *Let $i, j, k, l > 0$ and let $l < k$. Assume that m, n, p as indicated in Lemma 6.12 and c_2 as in Lemma 6.13 and that u, v are two words in $<< a, t >>$ with*

$$u = t^{k}a^{m^{i}-1}t^{-k}, \quad v = t^{l}a^{m^{j}-1}t^{-l}.$$

Let $q = \max \{j, i, p\}$ *and let* $b = \max \{d\, m^{i-1}, i\, m^i + c_2\}$. *Then, the number of uses of the defining relation in writing vu in binary notation is bounded above by*

$$b + (q+1)m^{q+1} + c_2.$$

Proof. Let $d = k - l$. Then view d as

$$d = \begin{cases} k - l & \text{if } k - l < i \text{ or } i \leq p \\ i + (i-1) + \cdots + (i-h) + e & \text{if } \begin{cases} k - l \geq i \;\&\; i - h > p + 1 \;\&\; h \geq 0 \\ \&\; e = 0 \text{ or } i - h - 1 < e < i - h \end{cases} \\ i + (i-1) + \cdots + (p+1) + e' & \text{if } k - l \geq i \;\&\; e' \geq 0. \end{cases}$$

Case 1: If $i \leq p$ then the number of uses of the defining relation to write $t^k a^{m^i - 1} t^{-k}$ in m-nary notation from k-th place to l-th place is clearly $\leq c_2$, by the hypothesis in Lemma 6.13. The thing left in the l-th place, say θ is clearly $< m^i - 1$. Combining $m^j - 1$ with θ, we have

$$m^j - 1 + \theta < (m^j - 1) + (m^i - 1) = m^j + m^i - 2.$$

Let $c\, m^{\mathcal{K}}$ $(0 < c < m)$ be the leading term in the expression of $m^j - 1 + \theta$ when written in m-nary notation. Then

$$m^{\mathcal{K}} \leq m^j - 1 + \theta \leq m^{\mathcal{K}+1} - 1; \quad (i)$$

$$m^{\mathcal{K}} < m^j + m^i - 2. \qquad (ii)$$

Again, by Lemma 6.13, to write $t^l a^{m^j - 1 + \theta} t^{-l}$ in m-nary notation we need at most $(\mathcal{K} + 1)m^{\mathcal{K}+1} + c_2$ uses by (i).

If $j < p$, then $\mathcal{K} \leq p$ by (ii). So, there are at most

$$(\mathcal{K} + 1)m^{\mathcal{K}+1} + c_2 \leq (p+1)m^{p+1} + c_2$$

uses needed to write $t^l a^{m^j-1+\theta} t^{-l}$ in m-nary notation.

If $j \geq p$, then $\mathcal{K} \leq j$. There are at most

$$(\mathcal{K}+1)m^{\mathcal{K}+1} + c_2 \leq (j+1)m^{j+1} + c_2$$

uses needed. Let $q_1 = \max\{j, p\}$. Totally, the number of uses is bounded above by

$$(q_1+1)m^{q_1+1} + 2c_2. \tag{6.15}$$

Case 2: If $i > p$ with d written as the second expression, then as we did in Lemma 6.13, from k-th place to $(k-i)$-th place, there are

$$m^{i-1} + nm^{i-2} + \cdots + n^{i-1} - i$$

uses needed. From $(k-i)$-th place to $(k-i-(i-1))$-th place, there are

$$m^{i-2} + nm^{i-3} + \cdots + n^{i-2} - (i-1)$$

uses needed. Similarly, from $(k-i-(i-1)-\cdots-(i-h+1))$-th place to $(k-i-(i-1)-\cdots-(i-h))$-th place, there are

$$m^{i-h-1} + nm^{i-h-2} + \cdots + n^{i-h-1} - (i-h)$$

uses needed. Finally, from $(k-i-(i-1)-\cdots-(i-h))$-th place to $(k-i-(i-1)-\cdots-(i-h)-e)$-th place, there are

$$m^{i-h-2} + nm^{i-h-3} + \cdots + n^{e-1}m^{i-h-1-e} - e$$

uses needed. Add all of them together,

$$\begin{aligned}
& (m^{i-1}+nm^{i-2}+\cdots\cdots\cdots\cdots\cdots\cdots+n^{i-2}m+n^{i-1}-i) \\
+ \; & (m^{i-2}+nm^{i-3}+\cdots\cdots\cdots\cdots+n^{i-3}m+n^{i-2}-(i-1)) \\
\vdots \; & \\
+ \; & (m^{i-h-1}+nm^{i-h-2}+\cdots+n^{i-h-1}-(i-h)) \\
+ \; & (m^{i-h-2}+nm^{i-h-3}+\cdots+n^{e-1}m^{i-h-1-e}-e) \\
< \; & im^{i-1}+(i-1)m^{i-2}+\cdots+(i-h)m^{i-h-1}+em^{i-h-2} \\
< \; & (i+(i-1)+\cdots+(i-h)+e)m^{i-1} \\
= \; & d\,m^{i-1}.
\end{aligned}$$

Moreover, the thing left in the l-th place, say θ' is $\leq m^{i-h-2}-1$. Combining m^j-1 with θ' we have

$$m^j-1+\theta' \leq (m^j-1)+(m^{i-h-2}-1)=m^j+m^{i-h-2}-2.$$

Let $c'\ m^{\mathcal{K}'}$ $(0<c'<m)$ be the leading term of $m^j-1+\theta'$ when written in m-nary notation. Then

$$m^{\mathcal{K}'} \leq m^j-1+\theta' \leq m^{\mathcal{K}'+1}-1; \quad (iii)$$

$$m^{\mathcal{K}'} < m^j+m^{i-h-2}-2. \qquad (iv)$$

By (iii), to write $t^l a^{m^j-1+\theta'}t^{-l}$ in m-nary notation, we need at most $(\mathcal{K}'+1)m^{\mathcal{K}'+1}+c_2$ uses. Let $q_2=\max\{j,i-h-2\}$. So, by (iv), there are at most $(q_2+1)m^{q_2+1}+c_2$ uses needed. Hence, the total number of uses is bounded above by

$$dm^{i-1}+(q_2+1)m^{q_2+1}+c_2. \tag{6.16}$$

Case 3: If $i>p$ with d written as the third expression, we use the same procedure until the $(k-i-(i-1)-\cdots-(p+1))$-th place. By Lamma 6.13, there are at most

$im^i + c_2$ uses needed. The thing left in the l-th place, say θ'' is $\leq m^p - 1$. Combining θ'' with $m^j - 1$ we have

$$m^j - 1 + \theta'' \leq (m^j - 1) + (m^p - 1) = m^j + m^p - 2.$$

Same discussion as before, let $c''m^{\mathcal{K}''}$ $(0 < c'' < m)$ be the leading term of $m^j - 1 + \theta''$ when written in m-nary notation. Then

$$m^{\mathcal{K}''} \leq m^j - 1 + \theta'' \leq m^{\mathcal{K}''+1} - 1; \quad (v)$$

$$m^{\mathcal{K}''} \leq m^j + m^p - 2. \qquad (vi)$$

Same discussion, to write $t^l a^{m^j - 1 + \theta''} t^{-l}$ in m-nary notation, there are at most $(\mathcal{K}'' + 1)m^{\mathcal{K}''+1} + c_2$ uses needed. Moreover, $\mathcal{K}'' \leq q_1$ and so there are at most $(q_1 + 1)m^{q_1+1} + c_2$ uses needed. Totally, the number of uses is bounded above by

$$im^i + (q_1 + 1)m^{q_1+1} + 2c_2. \tag{6.17}$$

Recall: $q = \max\{j, i, p\}$ and $b = \max\{d\, m^{i-1}, i\, m^i + c_2\}$. According to (6.15), (6.16), (6.17), the upper bound $b + (q + 1)m^{q+1} + c_2$ fits all cases. □

Observation. *The item b in Lemma 6.14 is an upper bound of the number of uses when written from k-th place to l-th place and will play an important role in proving the Theorem 6.10.*

Proof of Theorem 6.10. Since $m = n+1$, we get $a = a^{-n}t^{-1}a^n t \in G'$. So, $a^{t^i} \in G'$ since G' is normal in G. Hence, a commutes with a^{t^i}, $\forall\, i \in \mathbf{Z}$; i.e.,

$$[a, a^{t^i}] = 1.$$

Same as theorem 6.1, we suppose that w is a word in $<< a, t >>$ such that $w =_G 1$ with $\ell(w) = \mathcal{N}$. Then, same argument, with at most $\mathcal{N}$ uses of the defining relation,

w can be re-expressed as:

$$w =_G t^{-\tau_1}a^{\alpha_1}t^{\tau_1}\cdot t^{-\tau_2}a^{\alpha_2}t^{\tau_2}\cdots t^{-\tau_l}a^{\alpha_l}t^{\tau_l},$$

where $\tau_1 > \tau_2 > \cdots > \tau_l$, $|\tau_i| \le \mathcal{N}$, $\forall\ i \in \{1,\ldots,l\}$, and $\sum_1^l |\alpha_i| \le \mathcal{N}$. Same arrangement, we obtain a new word $w' \in << a, t >>$, such that

$$w' = t^{\beta_1}a^{\alpha_1}t^{-\beta_1}\cdot t^{\beta_2}a^{\alpha_2}t^{-\beta_2}\cdots t^{\beta_l}a^{\alpha_l}t^{-\beta_l}$$

for which $w' =_G 1$, $\ell(w') \le 3\mathcal{N}$, $0 = \beta_1 < \beta_2 < \cdots < \beta_l$, $\beta_i \le 2\mathcal{N}$, $\sum_1^l |\alpha_i| \le \mathcal{N}$. Again, since

$$t^{\beta_1}a^{\alpha_1}t^{-\beta_1}\cdot t^{\beta_2}a^{\alpha_2}t^{-\beta_2}\cdots t^{\beta_l}a^{\alpha_l}t^{-\beta_l} = 1,$$

we can move $t^{\beta_i}a^{\alpha_i}t^{-\beta_i}$ to the right hand side for all $\alpha_i < 0$. Then,

$$t^{\gamma_1}a^{\delta_1}t^{-\gamma_1}\cdots t^{\gamma_r}a^{\delta_r}t^{-\gamma_r} = t^{\eta_1}a^{\mu_1}t^{-\eta_1}\cdots t^{\eta_s}a^{\mu_s}t^{-\eta_s}, \tag{6.18}$$

where

$$\begin{array}{lll} \delta_i > 0, & 1 \le i \le r, & \gamma_1 < \gamma_2 < \cdots < \gamma_r \le 2\mathcal{N}, \\ \mu_j > 0, & 1 \le j \le s, & \eta_1 < \eta_2 < \cdots < \eta_s \le 2\mathcal{N}, \\ \text{and} & \sum_1^r \delta_i + \sum_1^s \mu_j \le \mathcal{N}. & \end{array}$$

After both sides of (6.18) being written in m-nary notation their expressions must be exactly the same. In doing that, we begin with some observations as follows:

Let $d_i = \gamma_{r-i+1} - \gamma_{r-i} > 0$ $(1 \le i \le r-1)$ and let $c_{i,0}m^{p_i} > 0$ $(1 \le i \le r)$ be the leading term of δ_i when written in m-nary notation; i.e.,

$$\begin{aligned} \delta_1 &= c_{1,0}m^{p_1} + c_{1,1}\, m^{p_1-1} + c_{1,2}\, m^{p_1-2} + \cdots \\ \delta_2 &= c_{2,0}m^{p_2} + c_{2,1}\, m^{p_1-1} + c_{2,2}\, m^{p_1-2} + \cdots \\ &\ \ \vdots \\ \delta_r &= c_{r,0}m^{p_r} + c_{r,1}\, m^{p_r-1} + c_{r,2}\, m^{p_r-2} + \cdots, \end{aligned}$$

where $c_{i,j} \in \{0,1,\ldots,m-1\}$ & $c_{i,0} \neq 0\ \forall i,j$. Then, we have

$$\delta_1 \le m^{p_1+1}-1,\ \delta_2 \le m^{p_2+1}-1,\ldots,\ \delta_r \le m^{p_r+1}-1.$$

Notice that $\sum_{i=1}^{r} \delta_i \le \mathcal{N}$. Let $\mathcal{K}_1$ be the least number such that $\mathcal{N} \le m^{\mathcal{K}_1}-1$. Then, we have

$$\sum_{i=1}^{r} \delta_i \le m^{\mathcal{K}_1}-1. \tag{6.19}$$

Similarly, let $\mathcal{K}_j$ be the least number such that

$$\sum_{i=j}^{r} \delta_i \le m^{\mathcal{K}_j}-1 \quad (2 \le j \le r). \tag{6.20}$$

Clearly, $\mathcal{K}_r \le \mathcal{K}_{r-1} \le \cdots \le \mathcal{K}_2 \le \mathcal{K}_1$. We now write $t^{\gamma_1}a^{\delta_1}t^{-\gamma_1}\cdots t^{\gamma_r}a^{\delta_r}t^{-\gamma_r}$ in m-nary notation by the following procedure:

First we write $t^{\gamma_r}a^{\delta_r}t^{-\gamma_r}$ in m-nary notation from γ_r-th place to γ_{r-1}-th place. By Lemma 6.14, the number of uses is bounded above by

$$\max\{d_1\, m^{\mathcal{K}_r-1}, \mathcal{K}_r m^{\mathcal{K}_r}+c_2\}.$$

The thing left in the γ_{r-1}-th place, say θ_1 is clearly $< \delta_r$. Then, by (6.20),

$$\delta_{r-1}+\theta_1 < \delta_{r-1}+\delta_r \le m^{\mathcal{K}_{r-1}}-1.$$

Then we write $t^{\gamma_{r-1}}a^{\delta_{r-1}+\theta_1}t^{-\gamma_{r-1}}$ in m-nary notation from γ_{r-1}-th place to γ_{r-2}-th place. Again, by Lemma 6.14, the number of uses is bounded above by

$$\max\{d_2\, m^{\mathcal{K}_{r-1}-1}, \mathcal{K}_{r-1} m^{\mathcal{K}_{r-1}}+c_2\}.$$

The thing left in the γ_{r-2}-th place, say θ_2 is clearly $< \delta_{r-1}+\delta_r$. Then, again by (6.20),

$$\delta_{r-2}+\theta_2 < \delta_{r-2}+\delta_{r-1}+\delta_r \le m^{\mathcal{K}_{r-2}}-1.$$

Then, to write $t^{\gamma_{r-2}}a^{\delta_{r-2}+\theta_2}t^{-\gamma_{r-2}}$ in m-nary notation from γ_{r-2}-th place to γ_{r-3}-th place we need at most

$$\max\,\{d_3\;m^{\mathcal{K}_{r-2}-1}, \mathcal{K}_{r-2}m^{\mathcal{K}_{r-2}} + c_2\}$$

uses of the given relation. Continue the process, from γ_2-th place to γ_1-th place, there are at most

$$\max\,\{d_{r-1}\;m^{\mathcal{K}_2-1}, \mathcal{K}_2 m^{\mathcal{K}_2} + c_2\}$$

uses needed. Finally, the thing left in the γ_1-th place, say θ_{r-1} is certainly $< \sum_{i=1}^{r} \delta_i \leq m^{\mathcal{K}_1} - 1$ by (6.19). Moreover, by Lemma 6.13, there are at most $\mathcal{K}_1 m^{\mathcal{K}_1} + c_2$ steps needed to write $t^{\gamma_1}a^{\theta_{r-1}}t^{-\gamma_1}$ in m-nary notation from γ_1-th place. Notice that

$$\sum_{i=1}^{r-1} d_i m^{\mathcal{K}_{r+1-i}-1} \leq \Big(\sum_{i=1}^{r-1} d_i\Big) m^{\mathcal{K}_2-1} \leq 2\mathcal{N} m^{\mathcal{K}_2-1} \leq 2\mathcal{N}^2;$$

$$\sum_{i=2}^{r} (\mathcal{K}_i m^{\mathcal{K}_i} + c_2) \leq (r-1)(\mathcal{K}_2 m^{\mathcal{K}_2} + c_2) \leq \mathcal{N}(\mathcal{N}^2 + c_2).$$

Therefore, there are at most $O(\mathcal{N}^3)$ uses to write both sides of (6.18) in m-nary notation; i.e., $\Phi_G(\mathcal{N}) \preccurlyeq \mathcal{N}^3$. □

7 Further study

7.1 A split extension

Theorem E *Let T be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}T$-module. Form the semidirect product*

$$G = M \rtimes T.$$

Then,

$$\Phi_G(n) \preccurlyeq \max\{n^2 \cdot 3^n, \Phi_M(n^2)\}.$$

Proof. Let

$$T =< t_1, t_2, \ldots, t_m; [t_i, t_j] = 1 \ (1 \le i, j \le m) >,$$

and let

$$M =< \alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q >,$$

where

$$\gamma_k = \prod_{l=1}^{p} \alpha_l^{f_{k,l}}, \quad f_{k,l} \in \mathbf{Z}T \quad \text{(written multiplicatively)}.$$

Then $G = M \rtimes T$ has presentations as follows:

$$\begin{aligned} G &= < t_1, \ldots, t_m, \alpha_1, \ldots, \alpha_p; \ [t_i, t_j] = 1 \ (1 \le i, j \le m), \\ &\quad \textstyle\prod_{l=1}^{p} \alpha_l^{f_{k,l}} = 1 \ (1 \le k \le q), \ [\alpha_k^{\delta_i}, \alpha_l^{\nu_j}] = 1 \ (\delta_i, \nu_j \in \mathbf{Z}T) > \\ &= << t_1, \ldots, t_m, \alpha_1, \ldots, \alpha_p; \ [t_i, t_j] = 1, \ [\alpha_k, \alpha_l] = 1, \ \textstyle\prod_{l=1}^{p} \alpha_l^{f_{k,l}} = 1, \\ &\quad [\alpha_k^{t_i}, \alpha_l] = 1, \ (1 \le i, j \le m, \ 1 \le k, l \le p) >> . \end{aligned}$$

Notice that the finite metabelian presentation above is easy to obtain by the fact that G is metabelian. More precisely, all relations of the form

$$[\alpha_k^{\delta_i}, \alpha_l^{\nu_j}] = 1 \ (1 \le k, l \le p, \ \delta_i, \nu_j \in \mathbf{Z}T)$$

are deducible from

$$[t_i, t_j] = 1, \ [\alpha_k, \alpha_l] = 1 \ \& \ [\alpha_k^{t_i}, \alpha_l] = 1, \ (1 \le i, j \le m, \ 1 \le k, l \le p),$$

by induction. We omit the proof here. Moreover, as we did in §3.3, by lifting $\mathbf{Z}T$ to

$$\Lambda = \mathbf{Z}[x_1, \ldots, x_m, y_1, \ldots, y_m],$$

M is viewed as a Λ-module and has the following presentation;

$$M = < \alpha_1, \ldots, \alpha_p; \prod_{l=1}^{p} \alpha_l^{h_{k,l}} \ (1 \le k \le q),\ \alpha_l^{x_i y_i} = \alpha_l \ (1 \le i \le m,\ 1 \le l \le p) >,$$

where $h_{k,l}$ is an expression of $f_{k,l}$ by substituting x_i for t_i and y_i for t_i^{-1}.

If $w =_G 1$ with $\ell(w) = n$, then

$$w = u_1(\underset{\sim}{t})v_1(\underset{\sim}{\alpha})u_2(\underset{\sim}{t})v_2(\underset{\sim}{\alpha}) \cdots u_s(\underset{\sim}{t})v_s(\underset{\sim}{\alpha}),$$

$u_i(\underset{\sim}{t})$ and $v_i(\underset{\sim}{\alpha})$ are subwords involved letters only in $\{t_1, \ldots, t_m\}$ and $\{\alpha_1, \ldots, \alpha_p\}$, respectively, such that

$$\sum_{i=1}^{s} \ell(u_i(\underset{\sim}{t})) + \sum_{i=1}^{s} \ell(v_i(\underset{\sim}{\alpha})) = n. \tag{7.1}$$

Observing that

$$w = v_1(\underset{\sim}{\alpha})^{u_1(\underset{\sim}{t})^{-1}} v_2(\underset{\sim}{\alpha})^{u_2(\underset{\sim}{t})^{-1}u_1(\underset{\sim}{t})^{-1}} \cdots v_s(\underset{\sim}{\alpha})^{u_s(\underset{\sim}{t})^{-1} \cdots u_1(\underset{\sim}{t})^{-1}} u_1(\underset{\sim}{t})u_2(\underset{\sim}{t}) \cdots u_s(\underset{\sim}{t})$$

we can only focus on the subword

$$w' = v_1(\underset{\sim}{\alpha})^{u_1(\underset{\sim}{t})^{-1}} v_2(\underset{\sim}{\alpha})^{u_2(\underset{\sim}{t})^{-1}u_1(\underset{\sim}{t})^{-1}} \cdots v_s(\underset{\sim}{\alpha})^{u_s(\underset{\sim}{t})^{-1} \cdots u_1(\underset{\sim}{t})^{-1}}$$

since $u_1(\underset{\sim}{t})u_2(\underset{\sim}{t}) \cdots u_s(\underset{\sim}{t})$ must have value 1 and it costs at most

$$\ell(u_1(\underset{\sim}{t})u_2(\underset{\sim}{t}) \cdots u_s(\underset{\sim}{t}))^2 \le \Big(\sum_{i=1}^{s} \ell(u_i(\underset{\sim}{t}))\Big)^2 \le n^2$$

uses to prove it. Similarly, each $v_i(\underset{\sim}{\alpha})$, at a cost of order

$$\ell(v_i(\underset{\sim}{\alpha}))^2 \le n^2,$$

can be rewitten as

$$v_i(\underset{\sim}{\alpha}) = \prod_{j=1}^{p} \alpha_j^{e_{i,j}} \quad \text{with} \quad \sum_{j=1}^{p} |e_{i,j}| \le \ell(v_i(\underset{\sim}{\alpha})). \tag{7.2}$$

Then w' becomes

$$\begin{aligned} w'' &= \left(\alpha_1^{e_{1,1}}\cdots\alpha_p^{e_{1,p}}\right)^{u_1^{-1}}\left(\alpha_1^{e_{2,1}}\cdots\alpha_p^{e_{2,p}}\right)^{u_2^{-1}u_1^{-1}}\cdots\left(\alpha_1^{e_{s,1}}\cdots\alpha_p^{e_{s,p}}\right)^{u_s^{-1}\cdots u_1^{-1}} \\ &= \left(\alpha_1^{e_{1,1}u_1^{-1}}\cdots\alpha_p^{e_{1,p}u_1^{-1}}\right)\left(\alpha_1^{e_{2,1}u_2^{-1}u_1^{-1}}\cdots\alpha_p^{e_{2,p}u_2^{-1}u_1^{-1}}\right)\cdots \\ &\quad \left(\alpha_1^{e_{s,1}u_s^{-1}\cdots u_1^{-1}}\cdots\alpha_p^{e_{s,p}u_s^{-1}\cdots u_1^{-1}}\right), \end{aligned}$$

where u_i is the abbreviation of $u_i(\underset{\sim}{t})$ $\forall i = 1, 2, \ldots, s$. We have to write w'' as a module word by commuting α_i. More precisely, move each α_1 within the second pair of parentheses to the left of all α_2 in the first pair and then move α_1 within the third pair of parentheses to the left of all α_2 in the first pair and so on. Then move every α_2 across the word to the the left of all α_3 in the first pair. Continue the process until w'' is transformed to

$$w^{(3)} = \prod_{k=1}^{p} \alpha_k^{\sum_{i=1}^{s} e_{i,k}\, u_i^{-1}\cdots u_1^{-1}}.$$

There are two things need to know:

(i) How many uses are needed to transform w' to $w^{(3)}$?

(ii) What is $\ell'(w^{(3)})$?

Clearly,

$$\begin{aligned} \ell'(w^{(3)}) &= \textstyle\sum_{k=1}^{p} \ell(\sum_{i=1}^{s} e_{i,k}\, u_i^{-1}\cdots u_1^{-1}) \\ &\le \textstyle\sum_{k=1}^{p}\sum_{i=1}^{s} \ell(e_{i,k}\, u_i^{-1}\cdots u_1^{-1}) \\ &= \textstyle\sum_{k=1}^{p}\sum_{i=1}^{s} |e_{i,k}|\ell(u_i^{-1}\cdots u_1^{-1}) \\ &\le \textstyle\sum_{k=1}^{p}\sum_{i=1}^{s} |e_{i,k}| n && \text{(by (7.1))} \\ &= n\textstyle\sum_{i=1}^{s}\left(\sum_{k=1}^{p} |e_{i,k}|\right) \\ &\le n\textstyle\sum_{i=1}^{s} \ell(v_i(\underset{\sim}{\alpha})) && \text{(by (7.2))} \\ &\le n^2 && \text{(by (7.1))} \end{aligned}$$

To answer (i), we shall apply the following

Lemma 7.3 *Let $u(\underset{\sim}{t})$ be any word in $<< t_1, \ldots, t_m >>$. Then, at a cost of at most*

$$2 \cdot 3^{\ell(u(\underset{\sim}{t}))} - 1$$

uses of the defining relations, one can deduce

$$[\alpha_k, \alpha_l^{u(\underset{\sim}{t})}] = 1.$$

(The proof is provided later on.)

Recall:

$$\begin{aligned} w'' \;=\; & \left(\alpha_1^{e_{1,1}u_1^{-1}} \cdots \alpha_p^{e_{1,p}u_1^{-1}}\right)\left(\alpha_1^{e_{2,1}u_2^{-1}u_1^{-1}} \cdots \alpha_p^{e_{2,p}u_2^{-1}u_1^{-1}}\right) \cdots \\ & \left(\alpha_1^{e_{s,1}u_s^{-1} \cdots u_1^{-1}} \cdots \alpha_p^{e_{s,p}u_s^{-1} \cdots u_1^{-1}}\right). \end{aligned}$$

Each $\alpha_1^{u_2^{-1}u_1^{-1}}$ within the second pair of parentheses has to be moved to the left of $\alpha_2^{e_{1,2}u_1^{-1}}$ in the first pair of parentheses. Notice that

$$[\alpha_k^{u_1^{-1}}, \alpha_1^{u_2^{-1}u_1^{-1}}] = [\alpha_k, \alpha_1^{u_2^{-1}}]^{u_1^{-1}} \quad \forall k = 2, \ldots, p.$$

Each of such commutators costs at most

$$2 \cdot 3^{\ell(u_2^{-1})} - 1$$

uses of the defining relations by Lemma 7.3. Hence there are at most

$$2|e_{2,1}|(|e_{1,2}| + \cdots + |e_{1,p}|)3^{\ell(u_2^{-1})} \leq 2|e_{2,1}|n \cdot 3^n \quad \text{(by (7.1) \& (7.2))}$$

uses needed to move all $\alpha_1^{u_2^{-1}u_1^{-1}}$ to the desired place. Similarly, to move all $\alpha_1^{u_3^{-1}u_2^{-1}u_1^{-1}}$ to the left of $\alpha_2^{e_{1,2}u_1^{-1}}$ there are at most

$$2|e_{3,1}|(|e_{2,2}| + \cdots + |e_{2,p}|)3^{\ell(u_3^{-1})} + 2|e_{3,1}|(|e_{1,2}| + \cdots + |e_{1,p}|)3^{\ell(u_3^{-1}u_2^{-1})} \leq 2|e_{3,1}|n \cdot 3^n$$

uses needed. Continue the process until every α_1 is moved to the desired place. The number of uses so far is bounded above by

$$2(|e_{2,1}| + |e_{3,1}| + \cdots + |e_{s,1}|)n \cdot 3^n.$$

We then follow the same procedure for α_2 and the number of uses is, as we expect, bounded by

$$2(|e_{2,2}| + |e_{3,2}| + \cdots + |e_{s,2}|)n \cdot 3^n.$$

Similarly, for α_{p-1}, the number is bounded above by

$$2(|e_{2,p-1}| + |e_{3,p-1}| + \cdots + |e_{s,p-1}|)n \cdot 3^n.$$

Totally, there are at most

$$2n \cdot 3^n \sum_{j=1}^{p-1} \sum_{i=2}^{s} |e_{i,j}| = 2n \cdot 3^n \sum_{i=2}^{s} \Big(\sum_{j=1}^{p-1} |e_{i,j}| \Big) \leq 2n \cdot 3^n \sum_{i=2}^{s} \ell(v_i(\underset{\sim}{\alpha})) \leq O(n^2 \cdot 3^n),$$

by (7.1) & (7.2).

All in all, we find that the number of uses of the defining relations to convert w' to the expression $w^{(3)}$ as a module word is less than $O(n^2 \cdot 3^n)$ and its length $\ell'(w^{(3)})$ is less than n^2. Therefore,

$$\Phi_G(n) \preccurlyeq \max\{n^2 \cdot 3^n, \Phi_M(n^2)\}. \ \square$$

Proof of Lemma 7.3. We prove this lemma by induction on $\ell(u(\underset{\sim}{t}))$. Clearly, when $\ell(u(\underset{\sim}{t})) = 1$ there is only one relation needed. Let C_r be the number of uses when $\ell(u(\underset{\sim}{t})) = r$. So $C_1 = 1$. When $\ell(u(\underset{\sim}{t})) = 2$, Without any loss of generality, say

$u(\underset{\sim}{t}) = t_i t_j$. Notice that we are working on groups in the variety of finitely generated metabelian groups. Hence

$$\begin{aligned}
1 &= [\alpha_k \alpha_k^{-t_j}, \alpha_l^{-t_j} \alpha_l^{t_i t_j}] \\
&= \alpha_k^{t_j} \alpha_k^{-1} \alpha_l^{-t_i t_j} \underbrace{\alpha_l^{t_j}}_{} \alpha_k \alpha_k^{-t_j} \underbrace{\alpha_l^{-t_j}}_{} \alpha_l^{t_i t_j} && (1) \\
&= \underbrace{\alpha_k^{t_j}}_{} \alpha_k^{-1} \alpha_l^{-t_i t_j} \alpha_k \underbrace{\alpha_k^{-t_j}}_{} \alpha_l^{t_i t_j} && (2) \\
&= \alpha_k^{-1} \alpha_l^{-t_i t_j} \alpha_k \alpha_l^{t_i t_j} \\
&= [\alpha_k, \alpha_l^{t_i t_j}]
\end{aligned}$$

In (1), $\alpha_l^{t_j}$ should move across $\alpha_k \alpha_k^{-t_j}$ to cancel out with $\alpha_l^{-t_j}$. Observing that each of $[\alpha_l^{t_j}, \alpha_k] = 1$ and $[\alpha_l^{t_j}, \alpha_k^{-t_j}] = 1$ need one relation. In (2), $\alpha_k^{t_j}$ should move across $\alpha_k^{-1} \alpha_l^{-t_i t_j} \alpha_k$ to cancel out with $\alpha_k^{-t_j}$. Observing that each of $[\alpha_k^{t_j}, \alpha_k^{-1}] = 1$ $[\alpha_k^{t_j}, \alpha_l^{-t_i t_j}] = 1$ and $[\alpha_k^{t_j}, \alpha_k] = 1$ need one relation. So, we have $C_2 = 5$ and $C_2 = 3C_1 + 2$. Now assuming that there are at most C_{r-1} uses of the defining relations needed to deduce

$$[\alpha_k, \alpha_l^{u(\underset{\sim}{t})}] = 1 \quad \forall\ \ell(u(\underset{\sim}{t})) = r - 1,$$

and $C_{r-1} = 3C_{r-2} + 2$. If $\ell(u(\underset{\sim}{t})) = r$, say

$$u(\underset{\sim}{t}) = t_i v(\underset{\sim}{t}),$$

where t_i is the letter in the very beginning of $u(\underset{\sim}{t})$ and $v(\underset{\sim}{t})$ is the remaining segment. So, $\ell(v(\underset{\sim}{t})) = r - 1$. Let u and v be the abbreviation of $u(\underset{\sim}{t})$ and $v(\underset{\sim}{t})$, respectively.

Observing that, again since G is metabelian,

$$\begin{aligned} 1 &= [\alpha_k\alpha_k^{-v}, \alpha_l^{-v}\alpha_l^u] \\ &= \alpha_k^v\alpha_k^{-1}\alpha_l^{-u}\,\underbrace{\alpha_l^v}\,\alpha_k\alpha_k^{-v}\,\underbrace{\alpha_l^{-v}}\,\alpha_l^u && (1)' \\ &= \underbrace{\alpha_k^v}\,\alpha_k^{-1}\alpha_l^{-u}\alpha_k\,\underbrace{\alpha_k^{-v}}\,\alpha_l^u && (2)' \\ &= \alpha_k^{-1}\alpha_l^{-u}\alpha_k\alpha_l^u \\ &= [\alpha_k, \alpha_l^u], \end{aligned}$$

there are at most $C_{r-1}+1$ uses needed in $(1)'$ and at most $2C_{r-1}+1$ uses needed in $(2)'$; i.e., $C_r = 3C_{r-1}+2$. By induction,

$$C_r = 3C_{r-1}+2 \quad \forall r \in \mathbf{N}.$$

Furthermore, $C_r = 2\cdot 3^r - 1 \quad \forall r \in \mathbf{N}$. □

References

[AM] M.F. Atiyah & I.G. Macdonald, *An Introduction to Commutative Algebra*, Addison-Wesley, Reading, Mass., 1969

[B1] G. Baumslag, *Topics in Combinatorial Group Theory*, Lectures in Mathematics, ETH Zurich, Birkhauser, New York, 1993

[B2] G. Baumslag, *Some Reflections on Finitely Generated Metabelian Groups*, Contemporary Mathematics Volume **109**, 1990

[BCM] G. Baumslag, Frank B. Cannonito and Charles F. Miller III, *Infinitely Generated Subgroups of Finitely Presented Groups. Part II*, Mathematatische Zeitschrift **172**, pp. 97-105, 1980

[BGSS] G. Baumslag, S.M. Gersten, M. Shapiro, H. Short, *Automatic groups and amalgams* J. Pure Appl.Algebra **76**, pp. 229-316, 1991

[BMS] G. Baumslag, C.F. Miller III, and H. Short, *Isoperimetric inequalities and the homology of groups*, Inventiones Mathematicae **113**, pp. 531-560, 1993

[G] S.M. Gersten, *Dehn functions and ℓ_1 – norms of finite presentations in: G. Baumslag and C.F. Miller III, eds., Algorithms and Classification in Combinatorial Group Theory*, Mathematical Sciences Research Institute Publications **23**, Springer, Berlin, 1991

[H] P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. **(3) 4**, pp. 419-436, 1954

[K] A.G. Kurosh, *The Theory of Groups, Volume I,* Shelsea Publishing Company, New York, 1960

[M] A. Wlodzimierz Mostowski, *On automorphisms of relatively free groups,* Fundamenta Mathematicae, pp. 403-411, L 1962

[MSK] Magnus-Karrass-Solitar, *Combinatorial Group Theory,* Pure and Applied Mathematics, Volume XIII, Interscience Publisher, New York

[N1] B.H. Neumann, *Identical relations in groups I,* Math. Ann. **114**, 506-525, 1937

[N2] H. Neumann, *Varieties of Groups,* Ergebnisse der Mathematik und ihrer Grenzgebiete. New Series. Vol.**37**, 1967

[R] Robinson, *A Course in the Theory of Groups, 2nd ed,* Graduate Texts in Mathematics: # **80** Springer