68-15,930

FALLEY, Peter, 1936-
IDEALS OF OPERATORS.

The City University of New York, Ph.D., 1968
Mathematics

IDEALS OF OPERATORS

by

PETER FALLEY

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1968

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

April 26, 1968
date

Robert Schatten
Professor Robert Schatten
Chairman of Examining Committee

April 26, 1968
date

Eldon Dyer
Professor Eldon Dyer
Executive Officer

Professor Leo Zippin

Professor Richard Sacksteder

Professor Stanley Kaplan

Supervisory Committee

The City University of New York

ACKNOWLEDGEMENTS

This dissertation was written with the guidance and advice of Professor Robert Schatten, and I wish to express my deep gratitude to him. I am indebted to Professor Leo Zippin for his generous encouragement and support during the past three years. My thanks also go to Professor Richard V. Kadison, whose stimulating lectures at Columbia University lead to my interest in functional analysis, and to Professor Edwin Goldfarb for his advice and encouragement.

I am grateful for the financial support given me by The City University of New York, without which this dissertation could not have been written.

TABLE OF CONTENTS

INTRODUCTION

A characterization of ideals of operators on separable Hilbert space was first given by J. W. Calkin [1]. Subsequently, R. Schatten introduced the notion of a norm ideal of operators on Hilbert space [11], and succeeded in characterizing the precise class of minimal norm ideals (cross-spaces). The methods of both authors cannot be carried over to arbitrary Banach spaces, since they make use of the inner product, and the polar decomposition of an operator.

Thus, other methods must be found to discuss ideals of operators on arbitrary Banach spaces. There, certain well-known ideals may always be defined: The ideal of operators of finite rank, the ideal of completely continuous operators, and, of course, the trivial ideal of all bounded operators. Each of these may be described as the precise class of operators which map the unit ball of the space in question into a member of a prescribed saturated (saturé, gesättigt) class of bounded sets; the class of bounded finite-dimensional sets, the class of pre-compact sets, and the class of all bounded sets respectively. Also, each of these ideals may be viewed as a left exact functor of two variables from the category of Banach spaces into the category of complex linear spaces, contravariant in the first, and covariant in the second variable. These properties may be used to define the concept of a universal ideal $\mathcal{U}$, a procedure (functor) which, to every given pair of Banach spaces, E and F, associates a definite ideal $\mathcal{U}(E,F)$ of operators from E into F, called a realization of $\mathcal{U}$. A universal ideal $\mathcal{U}$ of completely continuous operators is completely determined (characterized) by its realization $\mathcal{U}(l^1, l^\infty)$. These realizations are precisely the ideals each of which consists of all those operators which map the unit ball of l^1 into a prescribed saturated class of pre-compact sets in l^∞.

It is gratifying to observe that every two-sided ideal of completely continuous operators on a Hilbert space is a realization of a universal ideal. In particular, this means the following: There is a construction $\mathcal{U}_2$ which associates with any pair of Banach spaces a well-defined ideal of operators, and whose realization for a Hilbert space $\mathcal{H}$ furnishes $\mathcal{U}_2(\mathcal{H},\mathcal{H})$, the Hilbert-Schmidt class of operators. Similarly, for some universal ideal $\mathcal{U}_1$, its realization $\mathcal{U}_1(\mathcal{H},\mathcal{H})$ represents the trace class. A similar remark applies to every two-sided ideal of completely continuous operators on a Hilbert space.

NOTATION AND TERMINOLOGY

We shall assume a familiarity with the basic theory of bounded operators on Banach spaces. In this connection, chapters II, IV and V of [3] will serve as a reference. The elementary properties of locally convex linear spaces may be found in [5] and [6]. For the theory of ideals of completely continuous operators on a Hilbert space the reader is referred to [11].

This paper will deal exclusively with Banach spaces over the field $\mathbb{C}$ of complex numbers. The space of continuous linear functionals on a Banach space E will be denoted by E*, and will be assumed to have the strong (bound) topology, unless otherwise indicated. If A is a bounded operator, we denote the adjoint of A by A*. The space of bounded linear operators from E to F will be denoted by $\mathcal{B}(E,F)$, and will always be assumed to have the uniform topology. We shall write $\mathcal{B}(E)$ instead of $\mathcal{B}(E,E)$.

A linear operator A from E to F is said to be compact or completely continuous if it maps the closed unit ball B_1 of E into a set whose closure $\overline{A(B_1)}$ is compact in the norm topology of F. If $\overline{A(B_1)}$ is compact in the weak topology of F, then A is said to be weakly compact. We shall denote the classes of compact and weakly compact operators from E to F by $\mathcal{C}(E,F)$ and $\mathcal{W}(E,F)$ respectively. The class of operators of finite rank from E to F is denoted by $\mathcal{R}(E,F)$.

A set M in a Banach space E is called absolutely convex if $x,y \in M$, $a,b \in \mathbb{C}$, and $|a|+|b| \leq 1$ implies that $ax+by \in M$. A set is absolutely convex if and only if it is convex and circled. The absolutely convex hull of a set M will be denoted by $\Gamma(M)$, and we define $\Gamma_{i=1}^{n}(M_i) = \Gamma(M_1,M_2,\ldots,M_n) = \Gamma(\bigcup_{i=1}^{n} M_i)$.

The symbol $\overset{n}{\oplus} E$ will represent the direct sum of E with itself taken n times. The topology of $\overset{n}{\oplus} E$ is given by the norm $\|(x_1,x_2,\ldots,x_n)\| = \sum_{i=1}^{n}\|x_i\|$.

We shall make frequent use of the natural embedding J_E of E in E^{**}. It is defined by $J_E(x)(f) = f(x)$ for all $f \in E^*$. We note that J_E is an isometry, and that $J_E^{*}J_{E^*}$ is the identity operator on E^*. It should be emphasized that J_E^{**} does not coincide with $J_{E^{**}}$ unless E is reflexive. (See [2]).

The topological tensor product of two Banach spaces E and F will be denoted by $E \otimes_\gamma F$. It is the completion of the algebraic tensor product $E \odot F$ of E and F with respect to the greatest cross-norm γ defined in [10, p.36]. We note that $E \otimes_\gamma F$ coincides with the completed projective tensor product $E \hat{\otimes} F$, and the completed inductive tensor product $E \bar{\otimes} F$, defined for locally convex linear spaces in [4].

With [10], we may view an element $\sum_{i=1}^{n} f_i \otimes y_i$ of $E^* \odot F$ as an operator of finite rank from E into F by setting $(\sum_{i=1}^{n} f_i \otimes y_i)(x) = \sum_{i=1}^{n} f_i(x)y_i$ for all $x \in E$. Conversely, every operator of finite rank from E to F may be represented in this way.

For $0 < p < \infty$ we shall denote by l^p_K the space of complex-valued functions φ on the set K for which the family $\{|\varphi(k)|^p\}_{k \in K}$ is summable. The space of bounded complex-valued functions on K will be denoted by l^∞_K. With the customary norms, the spaces l^p_K are Banach spaces for $1 \leq p \leq \infty$. If K is the set of natural numbers, the spaces l^p_K are just the sequence spaces l^p.

Prop. III.6 will refer to Proposition 6 in Section III. Throughout Section III, however, this proposition is simply referred to as Prop. 6.

I. IDEALS AND STRONG IDEALS OF OPERATORS

This section will be devoted to defining the notion of an ideal of operators from one Banach space into another, and to a discussion of some not commonly known properties of spaces of the form $\mathcal{B}(E,F^*)$. In particular, we define the concept of a strong ideal, which plays an important role in our subsequent work. Unless otherwise expressly stated, the Banach spaces under consideration will be assumed to be perfectly general.

<u>DEFINITION 1:</u> If E and F are Banach spaces, a linear subspace $\mathfrak{J}$ of $\mathcal{B}(E,F)$ is called a left (right) ideal if $A \in \mathfrak{J}$, $S \in \mathcal{B}(F)$ $(T \in \mathcal{B}(E))$ implies $SA \in \mathfrak{J}$ $(AT \in \mathfrak{J})$. If $\mathfrak{J}$ is both a left and a right ideal, it is called a two-sided ideal, or simply an ideal.[1]

It should be noted that each of the spaces $\mathcal{B}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{C}(E,F)$, and $\mathcal{R}(E,F)$ is a two-sided ideal in $\mathcal{B}(E,F)$. A simple example of a left ideal is provided by the set of operators in $\mathcal{B}(E,F)$ which annihilate a fixed closed subspace G of E. The set of operators in $\mathcal{B}(E,F)$ which map E into a fixed subspace H of F is a right ideal in $\mathcal{B}(E,F)$.

<u>NOTATION:</u> If $\mathfrak{X}$ is an arbitrary subset of $\mathcal{B}(E,F)$, we shall

---

[1] The use of the term ideal in this context is not new. See for example [10, p.61] and [12, p.673].

denote by $\mathfrak{X}^L$ (resp. $\mathfrak{X}^R$, $\mathfrak{X}^B$) the left (resp. right, two-sided) ideal in $\mathcal{B}(E,F)$ generated by $\mathfrak{X}$. For example, $\mathfrak{X}^L$ consists of all finite sums of the form $\sum_{i=1}^{n} S_i A_i$ with $S_i \in \mathcal{B}(F)$, $A_i \in \mathfrak{X}$ for $i = 1,2,\ldots,n$.

If E and F are reflexive spaces, the mapping $A \longrightarrow A^*$ is an isometric isomorphism from $\mathcal{B}(E,F)$ onto $\mathcal{B}(F^*,E^*)$. Since, in addition, $(SAT)^* = T^*A^*S^*$ for $S \in \mathcal{B}(F)$, $T \in \mathcal{B}(E)$, $A \in \mathcal{B}(E,F)$, this mapping establishes a one-to-one correspondence between the left (right) ideals in $\mathcal{B}(E,F)$, and the right (left) ideals in $\mathcal{B}(F^*,E^*)$. Such symmetry is absent if E and F are non-reflexive. We shall now show how a modified adjoint operation may be used to establish a similar symmetry for spaces of the form $\mathcal{B}(E,F^*)$, where E and F are non-reflexive.

It is shown in [10, p.47] that $\mathcal{B}(E,F^*)$ is isometrically isomorphic to the space $(E \otimes_\gamma F)^*$. By symmetry, the latter space is also isometrically isomorphic to $\mathcal{B}(F,E^*)$. This implies that there exists a norm-preserving isomorphism from $\mathcal{B}(E,F^*)$ onto $\mathcal{B}(F,E^*)$. We shall denote this isomorphism by ', and we shall use the same notation for the corresponding isomorphism from $\mathcal{B}(F,E^*)$ onto $\mathcal{B}(E,F^*)$. For $T \in \mathcal{B}(E,F^*)$, T' is the unique operator in $\mathcal{B}(F,E^*)$ with the property that $Tx(y) = T'y(x)$ for all $x \in E$, $y \in F$.[1] This may also be expressed by saying that, if $J_F: F \longrightarrow F^{**}$ is the

---

[1] T' is the dual of T with respect to the dual pairs $\langle E,E^*\rangle$ and $\langle F^*,F\rangle$ in the sense of [5, p.199].

natural embedding, then $T' = T^*J_F$. It then follows at once that $(T')' = (T^*J_F)' = J_F^*T^{**}J_E = T$, since T^{**} is an extension of T [3, p.479].

If G is a third Banach space, we can use the isomorphism ' to define a multiplication of operators $A \in \mathcal{B}(E,F^*)$ by operators T^* with $T \in \mathcal{B}(E^*,G^*)$ as follows:

$$A \circ T^* = (TA')'.$$

Clearly, $A \circ T^* \in \mathcal{B}(G,F^*)$, and we show in the following lemma that $\circ$ extends the customary multiplication of operators to pairs of operators for which composition is undefined.

<u>LEMMA 2:</u> With $A,B \in \mathcal{B}(E,F^*)$, $R,S \in \mathcal{B}(E^*,G^*)$, $T \in \mathcal{B}(G^*,H^*)$, $a,b \in \mathbb{C}$, and $\circ$ defined as above, we have

1) $(aA + bB)\circ R^* = a(A\circ R^*) + b(B\circ R^*)$
2) $A\circ(aR^* + bS^*) = a(A\circ R^*) + b(A\circ S^*)$
3) $(A\circ R^*)\circ T^* = A\circ(R^*T^*)$
4) $\|A\circ R^*\| \leq \|A\|\|R\|$
5) If $U \in \mathcal{B}(G,E)$, then $A\circ U^{**} = AU$, where the multiplication on the right is composition of operators.

PROOF: 1) and 2) are obvious.

3): $(A\circ R^*)\circ T^* = (T(A\circ R^*)')' = (TRA')' = A\circ(TR)^* = A\circ(R^*T^*)$.

4): Since ' is an isometry, we have

$$\|A\circ R^*\| = \|RA'\| \leq \|R\|\|A'\| = \|A\|\|R\|.$$

5): For $U \in \mathcal{B}(G,E)$,

$$A\circ U^{**} = (U^*A')' = (U^*A^*J_F)' = J_F^*A^{**}U^{**}J_G$$
$$= J_F^*(AU)^{**}J_G = AU, \text{ since}$$

$(AU)^{**}$ is an extension of AU to G^{**}.

REMARK: We observe that, in general, for $S \in \mathcal{B}(F^*,H^*)$, $T \in \mathcal{B}(E^*,G^*)$, $A \in \mathcal{B}(E,F^*)$, we have $(SA)\circ T^* \neq S(A\circ T^*)$. To construct an example of this, let E be a non-reflexive Banach space, and A the identity operator on E^*. It is shown in [2] that $E^{***} = J_{E^*}(E^*) \oplus E^{\perp}$, where $E^{\perp} = \{f \in E^{***}: f(J_E x) = 0 \text{ for all } x \in E\}$. Let $f \in E^*$, $f \neq 0$, $g \in E^{\perp}$, $g \neq 0$, and choose $z \in E^{**}$ so that $z(f) = 1$. If $T = g \otimes z \in \mathcal{B}(E^{**})$, then $T^*(J_{E^*}f) = (z \otimes g)(f) = z(f)g = g$. Since $g \neq 0$, there is a $w \in E^{**}$ such that $g(w) \neq 0$. Let $h \in E^*$, $h \neq 0$, and take $S = w \otimes h \in \mathcal{B}(E^*)$. Then a simple computation shows that $S^{**}(g) = g(w)J_{E^*}h$. Thus,

$$\begin{aligned}((SA)\circ T^*)(f) = (T(SA)')'(f) &= (TA^*S^*J_E)'(f)\\ &= J_E{}^*S^{**}A^{**}T^*J_{E^*}(f)\\ &= J_E{}^*S^{**}A^{**}(g)\\ &= J_E{}^*S^{**}(g)\\ &= J_E{}^*(g(w)J_{E^*}h) = g(w)h \neq 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}S(A\circ T^*)(f) = S(TA')'(f) = S(TA^*J_E)'(f) &= S(J_E{}^*A^{**}T^*J_{E^*})(f)\\ &= SJ_E{}^*(g) = 0, \text{ since } g \in E^{\perp}.\end{aligned}$$

This kind of pathology cannot occur if A is a weakly compact operator:

PROPOSITION 3: If $A \in \mathcal{B}(E,F^*)$ is weakly compact, then $S(A\circ T^*) = (SA)\circ T^*$ for all $S \in \mathcal{B}(F^*,H^*)$, $T \in \mathcal{B}(E^*,G^*)$.

PROOF: We note that $A \in \mathcal{B}(E,F^*)$ is weakly compact if and only if $A^{**}(E^{**}) \subset J_{F*}(F^*)$ [3, p.482]. Now,

$S(A \circ T^*) = S(TA')' = S(TA^*J_F)' = SJ_F^*A^{**}T^*J_G$, while

$(SA) \circ T^* = (T(SA)')' = (TA^*S^*J_H)' = J_H^*S^{**}A^{**}T^*J_G$

$= J_H^*S^{**}J_{F*}J_F^*A^{**}T^*J_G.$

The last equality holds because $A^{**}(E^{**}) \subset J_{F*}(F^*)$, and on $J_{F*}(F^*)$, $J_{F*}J_F^*$ is just the identity transformation. Thus the result follows at once from the fact that S^{**} is an extension of S, so that $J_H^*S^{**}J_{F*} = S$.

REMARK: If F is a reflexive space, then every operator $A \in \mathcal{B}(E,F^*)$ is weakly compact, so that in this case we always have $S(A \circ T^*) = (SA) \circ T^*$. In this situation, and whenever A is known to be weakly compact, we shall omit the parentheses, and simply write $SA \circ T^*$.

The preceding discussion suggests that one consider subspaces of $\mathcal{B}(E,F^*)$ which are not only right ideals in $\mathcal{B}(E,F^*)$, but are closed under the extended right multiplication by operators T^* with $T \in \mathcal{B}(E^*)$:

DEFINITION 4: A linear subspace $\mathfrak{J}$ of $\mathcal{B}(E,F^*)$ will be called a strong right ideal if $A \in \mathfrak{J}$, $T \in \mathcal{B}(E^*)$ implies $A \circ T^* \in \mathfrak{J}$. A strong ideal in $\mathcal{B}(E,F^*)$ is a subspace of $\mathcal{B}(E,F^*)$ which is both a strong right ideal and a left ideal in $\mathcal{B}(E,F^*)$.

We note that each of the spaces $\mathcal{B}(E,F^*)$, $\mathcal{W}(E,F^*)$,

$\mathcal{C}(E,F^*)$, and $\mathcal{R}(E,F^*)$ is a strong ideal in $\mathcal{B}(E,F^*)$.[1] Also, it is clear that if E is a reflexive space, then every right ideal in $\mathcal{B}(E,F^*)$ is a strong right ideal. The following example shows that this is not the case if E is non-reflexive: Let $\mathfrak{J}$ be the right ideal in $\mathcal{B}(E,E^{**})$ generated by J_E, and choose $x \in E$, $y \in E^{**}$, $y \notin J_E(E)$, with $x,y \neq 0$. Then there exists $f \in E^*$ such that $f(x) = 1$. Define $T \in \mathcal{B}(E^*)$ by $T = y \otimes f$. Then $T^*J_E(x) = f \otimes y(x) = f(x)y = y$. Now, $J_E \circ T^* = (TJ_E^{\ *}J_{E^*})' = T' = T^*J_E$. But this means that $J_E \circ T^*(x) \notin J_E(E)$, while $A \in \mathfrak{J}$ implies $A(x) \in J_E(E)$. Hence, $\mathfrak{J}$ is not a strong right ideal.

The isomorphism ' now provides a one-to-one correspondence between the left ideals in $\mathcal{B}(E,F^*)$ and the strong right ideals in $\mathcal{B}(F,E^*)$: The equality $(A \circ T^*)' = TA'$ shows that a subspace $\mathfrak{J}$ of $\mathcal{B}(F,E^*)$ is a strong right ideal in $\mathcal{B}(F,E^*)$ if and only if $\mathfrak{J}'$ is a left ideal in $\mathcal{B}(E,F^*)$, where $\mathfrak{J}' = \{A' \in \mathcal{B}(E,F^*) : A \in \mathfrak{J}\}$.

<u>NOTATION:</u> If $\mathfrak{X}$ is an arbitrary subset of $\mathcal{B}(E,F^*)$, we shall denote by $\mathfrak{X}^{\bar{R}}$ (resp. $\mathfrak{X}^{\bar{B}}$) the strong right ideal (resp. strong ideal) in $\mathcal{B}(E,F^*)$ generated by $\mathfrak{X}$.

<u>REMARKS:</u> 1) $\mathfrak{X}^{\bar{R}}$ consists of all finite sums of the form $\sum_{i=1}^{n} A_i \circ S_i^*$, with $A_i \in \mathfrak{X}$, $S_i \in \mathcal{B}(E^*)$, $i = 1,2,\ldots,n$. In general, $\mathfrak{X}^{\bar{B}}$ cannot be described in such a simple way, in view of

---

[1] In the case of $\mathcal{W}(E,F^*)$ (resp. $\mathcal{C}(E,F^*)$) the proof of this statement requires the use of Gantmacher's (resp.

the remark following Lemma 2. However, Prop. 3 shows that if $\mathfrak{X}$ consists of weakly compact operators, then $(\mathfrak{X}^L)^{\bar{R}} = (\mathfrak{X}^{\bar{R}})^L = \mathfrak{X}^{\bar{B}}$, and the latter then consists of all finite sums of the form $\sum_{i=1}^{n} T_i A_i \circ S_i^*$ with $T_i \in \mathcal{B}(F^*)$, $A_i \in \mathfrak{X}$, $S_i \in \mathcal{B}(E^*)$, $i = 1,2,\ldots,n$.

2) From the properties of the isomorphism ' it is clear that $(\mathfrak{X}')^L = (\mathfrak{X}^{\bar{R}})'$.

Of particular interest in our later discussion will be the case where $E = F$. In this case ' is an isomorphism of $\mathcal{B}(E,E^*)$ onto itself, and we shall frequently consider subsets of $\mathcal{B}(E,E^*)$ which are invariant under this isomorphism.

DEFINITION 5: A subset $\mathfrak{X}$ of $\mathcal{B}(E,E^*)$ will be said to be symmetric if $\mathfrak{X}' = \mathfrak{X}$. A symmetric set which is also a two-sided ideal in $\mathcal{B}(E,E^*)$ will be called a symmetric ideal.

PROPOSITION 6: If $\mathfrak{X}$ is a symmetric set in $\mathcal{B}(E,E^*)$, then $\mathfrak{X}^{\bar{B}}$ is a symmetric ideal in $\mathcal{B}(E,E^*)$.

PROOF: This is immediate from the above remarks if $\mathfrak{X}$ consists of weakly compact operators. If not, we observe that $\mathfrak{X}^{\bar{B}}$ will, in any case, consist of all finite sums of finite products of the following four types:

Schauder's) Theorem that an operator A is weakly compact (resp. compact) if and only if A* is weakly compact (resp. compact). See, for instance, [3, p.485].

$T_n(T_{n-1}(......(T_1(A \circ S_1^*)) \circ S_2^*)......) \circ S_n^*)$

$(T_{n-1}(......(T_1(A \circ S_1^*)) \circ S_2^*)........) \circ S_n^*$

$T_n(T_{n-1}(......(T_2((T_1 A) \circ S_1^*)) \circ S_2^*)......) \circ S_{n-1}^*)$

$(T_n(T_{n-1}(.....(T_2((T_1 A) \circ S_1^*)) \circ S_2^*).....) \circ S_{n-1}^*)) \circ S_n^*$, where in each case $A \in \mathfrak{X}$, and $S_i, T_i \in \mathcal{B}(E^*)$, $i = 1,2,....n$. By induction on the number of factors in such products, one sees easily that the set consisting of all such finite products is symmetric, and it follows that the set of all finite sums of such products is symmetric.

Finally, let us observe that each of the spaces $\mathcal{B}(E,E^*)$, $\mathcal{W}(E,E^*)$, $\mathcal{C}(E,E^*)$, and $\mathcal{R}(E,E^*)$ is symmetric.

II. FULL IDEALS

By analogy with the ideals $\mathcal{B}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{C}(E,F)$, and $\mathcal{R}(E,F)$, we now define the class of full (right) ideals, and we derive some of its elementary properties. We then consider full ideals in spaces of the form $\mathcal{B}(E,F^*)$. There, the full ideals of weakly compact operators turn out to be strong right ideals, and we obtain a characterization of these in terms of the left ideals in $\mathcal{B}(F,E^*)$ to which they correspond under the isomorphism '.

An examination of the four ideals mentioned above shows that each of these may be described as the set of all operators in $\mathcal{B}(E,F)$ which map the unit ball of E into a prescribed class of bounded sets in F. The corresponding classes of bounded sets share a number of properties, some of which are enumerated in the following definition:

DEFINITION 1: A class $\mathcal{M}$ of bounded sets in a Banach space E is said to be saturated [1] if it satisfies the following conditions: a) $M \in \mathcal{M}$, $N \subset M$ implies $N \in \mathcal{M}$.

b) $M \in \mathcal{M}$, $a \in \mathbb{C}$ implies $aM \in \mathcal{M}$.

c) $M,N \in \mathcal{M}$ implies $\overline{\Gamma(M,N)} \in \mathcal{M}$.

One can readily verify that the class of all bounded sets, the class of subsets of weakly compact absolutely

---

[1] The term saturated class is normally used with respect to a dual pair of linear spaces [6, p.257]. In this sense, our definition refers to the pair $\langle E^*,E \rangle$.

convex sets, the class of pre-compact sets, and the class of all bounded finite-dimensional sets in E are saturated classes.

PROPOSITION 2: If $\mathcal{M}$ is a saturated class of bounded sets in F, and B_1 the closed unit ball in E, then the set $\mathcal{A}(E,F;\mathcal{M}) = \{A \in \mathcal{B}(E,F): A(B_1) \in \mathcal{M}\}$ is a right ideal in $\mathcal{B}(E,F)$.

PROOF: It is clear that $\mathcal{A}(E,F;\mathcal{M})$ is closed under multiplication by scalars. Also, if $A,B \in \mathcal{A}(E,F;\mathcal{M})$, then $A(B_1), B(B_1) \in \mathcal{M}$, so that $(A+B)(B_1) = \frac{1}{2}(2A+2B)(B_1) \in \mathcal{M}$ by conditions (a) and (c) of Def. 1. Thus $A+B \in \mathcal{A}(E,F;\mathcal{M})$. Finally, if $A \in \mathcal{A}(E,F;\mathcal{M})$, $T \in \mathcal{B}(E)$, then $AT \in \mathcal{A}(E,F;\mathcal{M})$, since $AT(B_1) = A(T(B_1)) = \|T\| A(\frac{1}{\|T\|}T(B_1)) \subset \|T\| A(B_1)$.

Each of the ideals $\mathcal{B}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{C}(E,F)$, and $\mathcal{R}(E,F)$ is of the form $\mathcal{A}(E,F;\mathcal{M})$, where $\mathcal{M}$ is the class of all bounded sets, the class of subsets of weakly compact absolutely convex sets, the class of pre-compact sets, and the class of bounded finite-dimensional sets in F respectively. If $\mathcal{M}$ is the class of all bounded sets in a fixed closed subspace H of F, then $\mathcal{A}(E,F;\mathcal{M})$ is the right ideal of all operators in $\mathcal{B}(E,F)$ which map E into H.

REMARK: If $\mathcal{M}_0$ is a class of bounded sets in a Banach space E, the intersection $\mathcal{M}$ of all saturated classes of bounded sets in E containing $\mathcal{M}_0$ is called the saturated class generated by $\mathfrak{M}_o$. In most cases which will interest us, the class $\mathfrak{M}_o$ already satisfies condition (b) of Def. 1, and $\mathfrak{M}$ has then a particularly simple form:

LEMMA 3: If $\mathfrak{M}_o$ is a class of bounded sets in E satisfying condition (b) of Def. 1, then the saturated class $\mathfrak{M}$ generated by $\mathfrak{M}_o$ consists precisely of all subsets of sets of the form $\overline{\Gamma(M_1,M_2,\ldots,M_n)}$ with $M_i \in \mathfrak{M}_o$ for $i = 1,2,\ldots,n$.

PROOF: The class specified in the lemma clearly contains $\mathfrak{M}_o$ and is contained in $\mathfrak{M}$. Hence, it suffices to show that it is a saturated class. It certainly satisfies conditions (a) and (b) of Def. 1. To show that it satisfies (c), let $M \subset \overline{\Gamma(M_1,M_2,\ldots,M_n)}$, $N \subset \overline{\Gamma(N_1,N_2,\ldots,N_k)}$. Then $M \cup N \subset \overline{\Gamma(M_1,\ldots,M_n,N_1,\ldots,N_k)}$, and since $\overline{\Gamma(M,N)}$ is the smallest closed, absolutely convex set in E containing $M \cup N$, this implies $\overline{\Gamma(M,N)} \subset \overline{\Gamma(M_1,\ldots,M_n,N_1,\ldots,N_k)}$, and this completes the proof.

NOTATION: if $\mathfrak{X}$ is any set of operators in $\mathcal{B}(E,F)$, we shall denote by $\mathfrak{M}_{\mathfrak{X}}$ the saturated class of bounded sets in F generated by the class $\{A(B_1): A \in \mathfrak{X}\}$, where B_1 is again the closed unit ball in E.

DEFINITION 4: We shall say that a right ideal $\mathfrak{J}$ in $\mathcal{B}(E,F)$ is a full ideal if $\mathfrak{J} = \mathfrak{A}(E,F; \mathfrak{M}_{\mathfrak{J}})$.

The following lemma shows that each of the examples given above is a full ideal:

LEMMA 5: If $\mathfrak{M}$ is a saturated class of bounded sets in F, then $\mathfrak{A}(E,F;\mathfrak{M})$ is a full ideal in $\mathcal{B}(E,F)$.

PROOF: Let $\mathfrak{A}(E,F;\mathfrak{M}) = \mathfrak{J}$. Then, clearly $\mathfrak{M}_{\mathfrak{J}} \subset \mathfrak{M}$, so that $\mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{J}}) \subset \mathfrak{J}$. On the other hand, if $A \in \mathfrak{J}$, then $A(B_1) \in \mathfrak{M}_{\mathfrak{J}}$, so that $A \in \mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{J}})$, i.e. $\mathfrak{J} \subset \mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{J}})$, and this completes the proof.

It will be shown in the next section that not every right ideal of operators is a full ideal. In particular, if $\mathfrak{X}$ is any set of operators in $\mathcal{B}(E,F)$, then $\mathfrak{X}^R$ is not, in general, a full ideal. We shall call the smallest full ideal containing $\mathfrak{X}$ the full ideal generated by $\mathfrak{X}$. It must clearly coincide with the ideal $\mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{X}})$. The following proposition shows that $\mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{X}})$ may also be viewed as the intersection of all full ideals containing $\mathfrak{X}$:

PROPOSITION 6: If $\{\mathfrak{J}_i\}_{i \in I}$ is a family of full ideals in $\mathcal{B}(E,F)$, then $\mathfrak{J} = \bigcap_{i \in I} \mathfrak{J}_i$ is a full ideal in $\mathcal{B}(E,F)$.

PROOF: Let $\mathfrak{M}_i = \mathfrak{M}_{\mathfrak{J}_i}$, and $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. We show that $\mathfrak{J} = \mathfrak{A}(E,F;\mathfrak{M})$, and this will complete the proof, in view of Lemma 5. If $A \in \mathfrak{J}$, then $A(B_1) \in \mathfrak{M}_i$ for each i, so that $A(B_1) \in \mathfrak{M}$. Hence $\mathfrak{J} \subset \mathfrak{A}(E,F;\mathfrak{M})$. On the other hand, if $A \in \mathfrak{A}(E,F;\mathfrak{M})$, then $A(B_1) \in \mathfrak{M}_i$ for each i, so that $A \in \mathfrak{J}_i$ for each i, and $A \in \mathfrak{J}$. Thus, $\mathfrak{A}(E,F;\mathfrak{M}) \subset \mathfrak{J}$ and we have equality as claimed.

Proposition 6 also allows us to show that full ideals can arise in yet another manner. We shall need the following lemma:

LEMMA 7: If M and N are bounded sets in E, and $T \in \mathcal{B}(E,F)$, then $T(\overline{\Gamma(M,N)}) \subset \overline{\Gamma(T(M),T(N))}$.

PROOF: Let $x_1, x_2, \ldots, x_n \in M \cup N$, $a_1, a_2, \ldots, a_n \in C$, with $\sum_{i=1}^{n} |a_i| \leq 1$. Then $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i) \in \Gamma(T(M),T(N))$. Thus, $T(\Gamma(M,N)) \subset \Gamma(T(M),T(N))$, and since T is continuous, this implies the desired result.

PROPOSITION 8: If $\mathfrak{J}$ is a full ideal in $\mathcal{B}(E,G)$, and $\mathfrak{X}$ an arbitrary set of operators in $\mathcal{B}(F,G)$, then the set $\mathfrak{A}(\mathfrak{J}, \mathfrak{X}) = \{A \in \mathcal{B}(E,F): SA \in \mathfrak{J} \text{ for all } S \in \mathfrak{X}\}$ is a full ideal in $\mathcal{B}(E,F)$.

PROOF: In view of Prop. 6, it suffices to consider the case where $\mathfrak{X} = \{S\}$, for it is clear that, in general, $\mathfrak{A}(\mathfrak{J}, \mathfrak{X}) = \bigcap_{S \in \mathfrak{X}} \mathfrak{A}(\mathfrak{J}, \{S\})$. Now, let $\mathfrak{M}$ be the class of all bounded sets M in F such that $S(M) \in \mathfrak{M}_{\mathfrak{J}}$. It follows from Lemma 7 that $\mathfrak{M}$ is a saturated class. Also, we have $A \in \mathfrak{A}(\mathfrak{J}, \{S\})$ if and only if $A(B_1) \in \mathfrak{M}$. Hence $\mathfrak{A}(\mathfrak{J}, \{S\}) = \mathfrak{A}(E,F;\mathfrak{M})$, so that $\mathfrak{A}(\mathfrak{J}, \{S\})$ is a full ideal by Lemma 5.

If $\mathfrak{J} = \{0\}$, then $\mathfrak{A}(\mathfrak{J}, \mathfrak{X}) = \{A \in \mathcal{B}(E,F): SA = 0 \text{ for all } S \in \mathfrak{X}\}$ is just the space of all operators $A \in \mathcal{B}(E,F)$ which map E into the closed subspace $\bigcap_{S \in \mathfrak{X}} S^{-1}(0)$ of F. Ideals of this type are discussed by Yood in [12].

We next turn our attention again to spaces of the form $\mathcal{B}(E,F^*)$. Here the isomorphism ' can be used to facilitate the discussion of full ideals. We shall need some further notation:

If $\mathfrak{X}$ is a subset of $\mathcal{B}(E,F)$, we shall denote by $\mathcal{T}_{\mathfrak{X}}$ the weakest locally convex topology on E such that all the operators in $\mathfrak{X}$ are continuous from E with the topology $\mathcal{T}_{\mathfrak{X}}$ to F with the norm topology. The topology $\mathcal{T}_{\mathfrak{X}}$ is generated by the set of seminorms $\{p_S\colon S \in \mathfrak{X}\}$, where $p_S(x) = \|Sx\|$ for all $x \in E$. In other words, for $x \in E$, the sets $N(x;S_1,S_2,\ldots,S_k,\rho) = \{y \in E\colon p_{S_i}(x-y) < \rho,\ i = 1,\ldots,k\}$ form a fundamental system of neighborhoods of x for the topology $\mathcal{T}_{\mathfrak{X}}$. If $\mathcal{T}$ is any locally convex topology on E, we shall denote by $\mathfrak{A}(E,F;\mathcal{T})$ the set of operators $A \in \mathcal{B}(E,F)$ such that A is continuous from E with the topology $\mathcal{T}$ to F with the norm topology. One sees at once that $\mathfrak{A}(E,F;\mathcal{T})$ is a left ideal in $\mathcal{B}(E,F)$. We note that for $\mathfrak{X} \subset \mathcal{B}(E,F)$, we always have $\mathfrak{X} \subset \mathfrak{A}(E,F;\mathcal{T}_{\mathfrak{X}})$. We shall see later that the opposite inclusion may fail to hold even if $\mathfrak{X}$ is a left ideal in $\mathcal{B}(E,F)$.

<u>PROPOSITION 9:</u> If $\mathfrak{J}$ is a left ideal in $\mathcal{B}(E,F^*)$ such that $\mathfrak{J} = \mathfrak{A}(E,F^*;\mathcal{T}_{\mathfrak{J}})$, then $\mathfrak{J}'$ is a full ideal in $\mathcal{B}(F,E^*)$.

PROOF: We first note that if M is any bounded set in E^*, and $x \in E$, then $\sup\{|f(x)|\colon f \in M\} = \sup\{|f(x)|\colon f \in \overline{\Gamma(M)}\}$: Let $f \in \Gamma(M)$. Then there exist $f_1,f_2,\ldots,f_k \in M$, and

$a_1, a_2, \ldots, a_k \in \mathbb{C}$, with $\sum_{i=1}^{k} |a_i| \leq 1$, such that $f = \sum_{i=1}^{k} a_i f_i$. But then $|f(x)| = |\sum_{i=1}^{k} a_i f_i(x)| \leq \sum_{i=1}^{k} |a_i| \, |f_i(x)| \leq \max_{1 \leq i \leq k} |f_i(x)|$. Hence, $\sup\{|f(x)| : f \in \Gamma(M)\} \leq \sup\{|f(x)| : f \in M\}$, and since x is a continuous linear functional on E*, this implies $\sup\{|f(x)| : f \in \overline{\Gamma(M)}\} \leq \sup\{|f(x)| : f \in M\}$, and the opposite inequality is obvious.

Now let $A \in \mathfrak{A}(F, E^*; \mathfrak{M}_{\mathfrak{J}'})$. Then, by Lemma 3, there exist $A_1, A_2, \ldots, A_n \in \mathfrak{J}$ such that $A(B_1) \subset \overline{\Gamma_{i=1}^{n}(A_i'(B_1))}$, where B_1 is the unit ball of F. Then, for $x \in E$, we have

$$\begin{aligned} \|A'x\| = \sup\{|A'x(y)| : y \in B_1\} &= \sup\{|Ay(x)| : y \in B_1\} \\ &= \sup\{|f(x)| : f \in A(B_1)\} \\ &\leq \sup\{|f(x)| : f \in \overline{\Gamma_{i=1}^{n}(A_i'(B_1))}\} \\ &= \sup\{|f(x)| : f \in \textstyle\bigcup_{i=1}^{n}(A_i'(B_1))\} \\ &= \max_{1 \leq i \leq n} \sup\{|f(x)| : f \in A_i'(B_1)\} \\ &= \max_{1 \leq i \leq n} \sup\{|A_i'y(x)| : y \in B_1\} \\ &= \max_{1 \leq i \leq n} \sup\{|A_i x(y)| : y \in B_1\} \\ &= \max_{1 \leq i \leq n} \|A_i x\| = \max_{1 \leq i \leq n} p_{A_i}(x). \end{aligned}$$

This inequality implies that A' is continuous from E with the topology $\mathcal{T}_{\mathfrak{J}}$ to F with the norm topology. Thus, $A' \in \mathfrak{J} = \mathfrak{A}(E, F^*; \mathcal{T}_{\mathfrak{J}})$, so that $A = A'' \in \mathfrak{J}'$, and this completes the proof.

The converse of Proposition 9 does not always hold, as may be seen by considering the full ideal $\mathfrak{J}$ in $\mathcal{B}(E, E^{**})$ generated by the natural embedding $J_E : E \to E^{**}$, where E is a non-reflexive space. The remarks following Def. I.4 show that $\mathfrak{J}$ is not a strong right ideal, so that

$\mathcal{J}'$ is not a left ideal, and hence $\mathcal{J}' \neq \mathfrak{A}(E^*,E^*; \mathcal{T}_{\mathcal{J}'})$. The converse of Proposition 9 does hold, however, for ideals of weakly compact operators:

PROPOSITION 10: If $\mathcal{J}$ is a full ideal of weakly compact operators in $\mathcal{B}(E,F^*)$, then $\mathcal{J}' = \mathfrak{A}(F,E^*; \mathcal{T}_{\mathcal{J}'})$.

PROOF: Let $A \in \mathfrak{A}(F,E^*; \mathcal{T}_{\mathcal{J}'})$. Then there exist operators $A_1, A_2, \ldots, A_n \in \mathcal{J}$ such that $\|Ax\| \leq \max_{1 \leq i \leq n} \|A_i' x\|$ for all $x \in F$. By the proof of Prop. 9, this inequality is equivalent to $\sup\{|f(x)| : f \in A'(B_1)\} \leq \sup\{|f(x)| : f \in \overline{\Gamma_{i=1}^{n}(A_i(B_1))}\}$. Now let us assume that $A'(B_1) \not\subset \overline{\Gamma_{i=1}^{n}(A_i(B_1))}$. We note that $\Gamma_{i=1}^{n}(A_i(B_1)) \subset \Gamma_{i=1}^{n}(\overline{A_i(B_1)}) \subset \overline{\Gamma_{i=1}^{n}(A_i(B_1))}$, and that the absolutely convex hull of a finite family of absolutely convex compact sets in a locally convex space is compact [6,p.244]. This implies that $\overline{\Gamma_{i=1}^{n}(A_i(B_1))} = \Gamma_{i=1}^{n}(\overline{A_i(B_1)})$, and that this is a weakly compact set, since $\overline{A_i(B_1)}$ is weakly compact by assumption. Hence, $\overline{\Gamma_{i=1}^{n}(A_i(B_1))}$ is also w*-compact (since the w* topology on F* is weaker than the weak topology), and hence w*-closed. Let $x \in B_1$ such that $A'x \notin \overline{\Gamma_{i=1}^{n}(A_i(B_1))}$. Then, by the Separation Theorem [3, p.417], there exists $y \in F$, and a real number c, such that

$$\sup\{\operatorname{Re} f(y) : f \in \overline{\Gamma_{i=1}^{n}(A_i(B_1))}\} \leq c < \operatorname{Re} A'x(y)$$
$$\leq \sup\{|f(y)| : f \in A'(B_1)\}.$$

But, since $\overline{\Gamma_{i=1}^{n}(A_i(B_1))}$ is a circled set, we have $\sup\{\operatorname{Re} f(y) : f \in \overline{\Gamma_{i=1}^{n}(A_i(B_1))}\} = \sup\{|f(y)| : f \in \overline{\Gamma_{i=1}^{n}(A_i(B_1))}\}$, and this clearly contradicts our original inequality. Thus, we must have $A'(B_1) \subset \overline{\Gamma_{i=1}(A_i(B_1))}$, which means that $A' \in \mathfrak{J}$, since $\mathfrak{J}$ is a full ideal. Hence, $A = A'' \in \mathfrak{J}'$, which proves that $\mathfrak{J}' = \mathfrak{A}(F, E^*; \mathcal{T}_{\mathfrak{J}'})$.

COROLLARY 11: Every full ideal of weakly compact operators in $\mathcal{B}(E, F^*)$ is a strong right ideal.

If E and F are reflexive spaces, Propositions 9 and 10 take the following form:

COROLLARY 12: If E and F are reflexive spaces, a right ideal $\mathfrak{J}$ in $\mathcal{B}(E,F)$ is a full ideal if and only if $\mathfrak{J}^* = \mathfrak{A}(F^*, E^*; \mathcal{T}_{\mathfrak{J}^*})$.

As an application, we record the following occasionally useful criterion for complete continuity:

COROLLARY 13: Let E and F be reflexive Banach spaces, and $A \in \mathcal{B}(E,F)$. If there exists a finite set $A_1, A_2, \ldots, A_n$ of completely continuous operators in $\mathcal{B}(E,F)$ such that $\|Ax\| \leq \max_{1 \leq i \leq n} \|A_i x\|$ for all $x \in E$, then A is completely continuous.

PROOF: The assertion of the corollary is clearly equivalent to the statement that if $\mathfrak{C}$ is the ideal of completely continuous operators in $\mathcal{B}(E,F)$, then $\mathfrak{C} = \mathfrak{A}(E, F; \mathcal{T}_{\mathfrak{C}})$. But, by Schauder's Theorem [3, p.485], $\mathfrak{C}^*$ is just the ideal of completely continuous operators in $\mathcal{B}(F^*, E^*)$, and thus a full ideal. Hence, our result follows from Corollary 12.

III. SOME SPECIAL CASES AND A COUNTEREXAMPLE

In the preceding section we saw that every full ideal of weakly compact operators in $\mathcal{B}(E,F^*)$ is a strong right ideal. An example at the end of this section will show that, in general, the converse of this assertion is false. However, we will see that, if E satisfies cerain special conditions, then every strong right ideal in $\mathcal{B}(E,F^*)$ is a full ideal. In particular, this is the case if $E = l^1_K$, and this fact will permit us to obtain another characterization of the class of full ideals of weakly compact operators in $\mathcal{B}(E,F^*)$ for arbitrary E and F.

Proposition II.9 enables us to use a construction due to E. Michael [7] to obtain the following result:

<u>PROPOSITION 1:</u> Let E be a Banach space with the property that, for every natural number n, every bounded operator from a closed subspace G of $\overset{n}{\oplus} E^*$ to E^* has a continuous extension to all of $\overset{n}{\oplus} E^*$ with range in E^*. Then every strong right ideal in $\mathcal{B}(E,F^*)$ is a full ideal (F an arbitrary Banach space).

Proof: By Prop. II.9, it suffices to show that if $\mathfrak{J}$ is a left ideal in $\mathcal{B}(F,E^*)$, then $\mathfrak{J} = \mathcal{O}\!\ell(F,E^*; \mathcal{T}_{\mathfrak{J}})$. We now use Michael's argument: Let $A \in \mathcal{O}\!\ell(F,E^*; \mathcal{T}_{\mathfrak{J}})$. Then there exist $A_1, A_2, \ldots, A_n \in \mathfrak{J}$ such that $\|Ax\| \leq \max_{1 \leq i \leq n} \|A_i x\|$ for all $x \in F$. For each i, $i = 1,2,\ldots,n$, let I_i be the natural embedding of E^* in the i-th factor of $\overset{n}{\oplus} E^*$, and let $J: F \longrightarrow \overset{n}{\oplus} E^*$ be defined by $J = \sum_{i=1}^{n} I_i A_i$. Next, for $x \in F$, let $S(Jx) = Ax$. We claim that S is a well-defined bounded linear operator from J(F) into E^*. To show that S is well-defined, we note that if $Jx = Jy$, then $J(x-y) = 0$, so that $A_i(x-y) = 0$, $i = 1,2,..,n$, and hence $A(x-y) = 0$. Thus $Ax = Ay$, so that $S(Jx) = S(Jy)$. One sees easily that S is linear, and that it is bounded follows from the inequality $\|S(Jx)\| = \|Ax\| \leq \max_{1 \leq i \leq n} \|A_i x\| \leq \sum_{i=1}^{n} \|A_i x\| = \|Jx\|$. Thus S has a continuous linear extension $\overline{S}$ to $\overline{J(F)}$ with range in E^*, and by hypothesis $\overline{S}$ has a continuous linear extension $\tilde{S}$ to all of $\overset{n}{\oplus} E^*$ with range in E^*. But then, $A = \tilde{S}J = \sum_{i=1}^{n} \tilde{S} I_i A_i$. Thus, since for each i, $\tilde{S} I_i \in \mathcal{B}(E^*)$, and $\mathcal{J}$ is a left ideal in $\mathcal{B}(F,E^*)$, we have $A \in \mathcal{J}$, and this completes the proof.

COROLLARY 2: If $\mathcal{H}$ is a Hilbert space, and F an arbitrary Banach space, then every right ideal in $\mathcal{B}(\mathcal{H},F^*)$ is a full ideal.

PROOF: Since $\overset{n}{\oplus} \mathcal{H}^*$ is topologically isomorphic to a Hilbert space, it is clear that $\mathcal{H}$ satisfies the condition of Prop. 1. Hence, it suffices to recall that every right ideal in $\mathcal{B}(\mathcal{H},F^*)$ is a strong right ideal.

COROLLARY 3: If F is an arbitrary Banach space, then every strong right ideal in $\mathcal{B}(l^1_K,F^*)$ is a full ideal.

PROOF: It is shown in [9] that every bounded operator from a closed subspace G of a Banach space E into l^{∞}_K has a continuous linear extension to all of E with range in l^{∞}_K. Thus, since $(l^1_K)^* = l^{\infty}_K$, it is clear that l^1_K satisfies the condition of Prop. 1.

This corollary leads to a particularly simple proof of the following known result (see, for instance [4, p.168 Corollaire, and p.185 Prop. 41]):

<u>COROLLARY 4:</u> If F is an arbitrary Banach space, then every completely continuous operator in $\mathcal{B}(l^1_K, F^*)$ is the limit, in the norm of $\mathcal{B}(l^1_K, F^*)$, of operators of finite rank.

PROOF: Let $\mathcal{C}_o$ be the closure in $\mathcal{B}(l^1_K, F^*)$ of $\mathcal{R}(l^1_K, F^*)$. It follows from Lemma I.2(4) that $\mathcal{C}_o$ is a strong right ideal, and hence a full ideal in $\mathcal{B}(l^1_K, F^*)$. We show that $\mathcal{M}_{\mathcal{C}_o}$ is precisely the class of pre-compact sets in F*, and this will clearly imply that $\mathcal{C}_o = \mathcal{C}(l^1_K, F^*)$ as claimed. Since $\mathcal{C}_o \subset \mathcal{C}(l^1_K, F^*)$, we know that every set in $\mathcal{M}_{\mathcal{C}_o}$ is pre-compact. On the other hand, let M be a pre-compact set in F*. Then M is contained in the closed, absolutely convex hull of a sequence $\{x_n\}$ of elements of F*, with $x_n \to 0$ as $n \to \infty$ [6, p.250]. Thus, $M = \{\sum_{i=1}^{\infty} a_i x_i : \sum_{i=1}^{\infty} |a_i| \le 1\}$ [5, p.250]. Now let $k_1, k_2, \ldots$ be a sequence of elements of K, and define $\varphi_{k_i} \in l^{\infty}_K$ by $\varphi_{k_i}(h) = 1$ if $h = k_i$, and $\varphi_{k_i}(h) = 0$ if $h \neq k_i$. Then the operators of finite rank $\sum_{i=1}^{n} \varphi_{k_i} \otimes x_i$ converge in norm to the operator $T = \sum_{i=1}^{\infty} \varphi_{k_i} \otimes x_i \in \mathcal{B}(l^1_K, F^*)$, and we see that $T(B_1) = \{\sum_{i=1}^{\infty} a_i x_i : \sum_{i=1}^{\infty} |a_i| \le 1\} \supset M$. Hence $M \in \mathcal{M}_{\varepsilon_0}$, and this completes the proof.

Corollary 3 also leads to a new characterization of the full ideals of weakly compact operators:

We recall that every Banach space E may be viewed as a quotient space of l^1_K for some appropriate set K. In particular, there exists an operator $P: l^1_K \longrightarrow E$ which maps the unit ball of l^1_K onto that of E [6, p.283], so that $P^*: E^* \longrightarrow l^\infty_K$ is an isometry. If $P_1: l^1_K \longrightarrow E$ and $P_2: l^1_K \longrightarrow F$ are two such operators (K may be chosen so that both E and F are quotient spaces of l^1_K), then the map $\Delta: \mathcal{B}(E,F^*) \longrightarrow \mathcal{B}(l^1_K, l^\infty_K)$ defined by $\Delta(A) = P_2^* A P_1$ is an isometric embedding of $\mathcal{B}(E,F^*)$ in $\mathcal{B}(l^1_K, l^\infty_K)$. Similarly, $\mathcal{B}(F,E^*)$ may be embedded in $\mathcal{B}(l^1_K, l^\infty_K)$ by means of the map Δ' defined by $\Delta'(A) = P_1^* A P_2$. It should be noted that the operators P_1 and P_2 are in no way canonical. However, we shall see that most of our considerations are independent of their choice, and for the moment, we shall assume that P_1, P_2, and thus Δ and Δ', are fixed. We note that Δ and Δ' are related by $\Delta'(A') = (\Delta(A))'$, for we have

$$\Delta'(A') = P_1^* A' P_2 = P_1^* A^* J_F P_2 = P_1^* A^* P_2^{**} J l^1_K = (P_2^* A P_1)'$$
$$= (\Delta(A))'.$$

<u>PROPOSITION 5:</u> A set $\mathcal{J}$ of weakly compact operators in $\mathcal{B}(E,F^*)$ is a full ideal in $\mathcal{B}(E,F^*)$ if and only if

$$\Delta(\mathcal{J}) = \Delta(\mathcal{B}(E,F^*)) \cap (\Delta(\mathcal{J}))^{\overline{R}}.$$

PROOF: a) Assume $\mathfrak{J}$ is a full ideal. The strong right ideal $(\Delta(\mathfrak{J}))^{\bar{R}}$ is a full ideal in $\mathcal{B}(l_K^1, l_K^\infty)$ by Cor. 3. Since $\Delta(\mathfrak{J})$ consists of weakly compact operators, the full ideal $\mathfrak{A}(l_K, l_K; \mathcal{M}_{\Delta(\mathfrak{J})})$ generated by $\Delta(\mathfrak{J})$ is a strong right ideal in $\mathcal{B}(l_K^1, l_K^\infty)$ by Cor. II.11, and this implies that it must coincide with $(\Delta(\mathfrak{J}))^{\bar{R}}$. Also, it is clear that $\mathcal{M}_{\Delta(\mathfrak{J})} = \{P_2{*}(M) : M \in \mathcal{M}_{\mathfrak{J}}\}$. Now let S be an element of $\Delta(\mathcal{B}(E,F^*)) \cap (\Delta(\mathfrak{J}))^{\bar{R}}$. Then $S = \Delta(A) = P_2{*}AP_1$ for some $A \in \mathcal{B}(E,F^*)$, and $S(B_1) \in \mathcal{M}_{\Delta(\mathfrak{J})}$. Thus, $AP_1(B_1) = (P_2{*})^{-1}(S(B_1)) \in \mathcal{M}_{\mathfrak{J}}$. But, since P_1 maps the unit ball of l_K^1 onto that of E, this implies that $A(B_1) \in \mathcal{M}_{\mathfrak{J}}$, where B_1 is now the unit ball of E. Thus $A \in \mathfrak{J}$, since $\mathfrak{J}$ is a full ideal, and this implies that $\Delta(\mathcal{B}(E,F^*)) \cap (\Delta(\mathfrak{J}))^{\bar{R}} \subset \Delta(\mathfrak{J})$, and the opposite inclusion holds in any case.

b) Assume that $\mathfrak{J}$ satisfies the equalitv of the Proposition, and let $A \in \mathfrak{A}(E,F^*; \mathcal{M}_{\mathfrak{J}})$. As in (a), we have $\mathcal{M}_{\Delta(\mathfrak{J})} = \{P_2{*}(M) : M \in \mathcal{M}_{\mathfrak{J}}\}$, and this implies that $\Delta(A)$ maps the unit ball of l_K^1 into an element of $\mathcal{M}_{\Delta(\mathfrak{J})}$. Thus, $\Delta(A) \in \mathfrak{A}(l_K^1, l_K^\infty; \mathcal{M}_{\Delta(\mathfrak{J})})$. The latter is contained in $(\Delta(\mathfrak{J}))^{\bar{R}}$, since $(\Delta(\mathfrak{J}))^{\bar{R}}$ is a full ideal by Cor. 3. Hence $\Delta(A) \in (\Delta(\mathfrak{J}))^{\bar{R}}$, and this implies, by assumption that $\Delta(A) \in \Delta(\mathfrak{J})$, so that $A \in \mathfrak{J}$, and $\mathfrak{J}$ is a full ideal.

The second part of the foregoing proof did not require the assumption that $\mathfrak{J}$ consisted of weakly compact operators, so that we immediately have the following corollary:

<u>COROLLARY 6:</u> If $\mathcal{L}$ is a strong right ideal in $\mathcal{B}(l^1_K, l^\infty_K)$, then $\Delta^{-1}(\mathcal{L})$ is a full ideal in $\mathcal{B}(E,F^*)$.

PROOF: $\Delta(\mathcal{B}(E,F^*)) \cap (\Delta(\Delta^{-1}(\mathcal{L})))^{\bar{R}} \subset \Delta(\mathcal{B}(E,F^*)) \cap \mathcal{L}$
$= \Delta(\Delta^{-1}(\mathcal{L}))$, and

since the opposite inclusion holds in any case, the corollary follows from Prop. 5.

<u>COROLLARY 7:</u> If $\mathfrak{X}$ is any set of weakly compact operators in $\mathcal{B}(E,F^*)$, then $\mathfrak{A}(E,F^*;\mathfrak{M}_{\mathfrak{X}}) = \Delta^{-1}(\Delta(\mathfrak{X})^{\bar{R}})$.

PROOF: If $\mathfrak{J}$ is any full ideal of weakly compact operators in $\mathcal{B}(E,F^*)$ containing $\mathfrak{X}$, then $\mathfrak{J} = \Delta^{-1}(\Delta(\mathfrak{J})^{\bar{R}})$ by Prop. 5, and clearly $\Delta^{-1}(\Delta(\mathfrak{J})^{\bar{R}}) \supset \Delta^{-1}(\Delta(\mathfrak{X})^{\bar{R}})$, Since the last is a full ideal by Cor. 6, this completes the proof.

Finally, it is easy to see that Proposition 5 may also be expressed as follows:

<u>COROLLARY 8:</u> A set $\mathfrak{J}$ of weakly compact operators in $\mathcal{B}(E,F^*)$ is a full ideal in $\mathcal{B}(E,F^*)$ if and only if

$$\Delta'(\mathfrak{J}') = \Delta'(\mathcal{B}(F,E^*)) \cap (\Delta'(\mathfrak{J}'))^L.$$

We have seen in Cor. 2 that every right ideal of operators on a Hilbert space is a full ideal. We now show that this is not the case for arbitrary reflexive spaces. Since every right ideal of operators on a reflexive space is a strong right ideal, this will show that, in general, not every strong right ideal in $\mathcal{B}(E,F^*)$ is a full ideal. We shall need the following lemma:

LEMMA 9: Let F be a closed subspace of l^p, $1 \leq p < \infty$. Then there exists an operator $T: l^p \longrightarrow F$ such that $T(l^p)$ is dense in F.

PROOF: Since l^p is separable, so is F. Thus, there is a countable set $\{x_1, x_2, \ldots\}$ which is dense in the unit ball of F. For $i = 1, 2, \ldots$, define $\varphi_i \in l^q$, $\frac{1}{p} + \frac{1}{q} = 1$, by $\varphi_i(j) = \delta_{ij}$. Then $T = \sum_{i=1}^{\infty} (\frac{1}{2})^i \varphi_i \otimes x_i$ defines a bounded linear operator from l^p into F, and $T(l^p)$ is dense in F.

PROPOSITION 10: There exists a reflexive Banach space E, and a right ideal $\mathfrak{I}$ in $\mathcal{B}(E)$ such that $\mathfrak{I}$ is not a full ideal.

PROOF: Let $p \neq 2$, $1 < p < \infty$. Then there is a closed subspace F of l^p such that the identity operator I_F on F has no continuous linear extension to all of l^p with range in F. (Murray has shown that there exists a closed subspace F of l^p such that there is no continuous projection of l^p onto F [8]. An extension of I_F to all of l^p with range in F would be a projection of l^p onto F.) Now let $E^* = F \oplus l^p$ ($F \oplus l^p$ is a reflexive space), and let $T: l^p \longrightarrow F$ be an operator such that $T(l^p)$ is dense in F. Next, define an operator $A: F \oplus l^p \longrightarrow F \oplus l^p$ by $A = \begin{pmatrix} I_F & 0 \\ 0 & T \end{pmatrix}$, where T is now considered as an operator from l^p into l^p. Then it is clear that $A(F \oplus l^p)$ is dense in the subspace $F \oplus F$ of $F \oplus l^p$. Now let $\mathfrak{J} = \{A^*\}^{\bar{R}}$. We claim that $\mathfrak{J}^*$ does not coincide with $\mathfrak{A}(E^*, E^*; \mathcal{T}_{\mathfrak{J}^*})$, which would imply that $\mathfrak{J}$ is not a full ideal by Cor. II.12. Note that $\mathfrak{J}^* = \{A\}^L$. We define an operator $S: F \oplus F \longrightarrow F \oplus l^p$ by

$$S = \begin{pmatrix} 0 & I_F \\ J & 0 \end{pmatrix},$$

where $J: F \rightarrow l^p$ is the inclusion map. It is obvious that $\|S\| = 1$, so that, if we let $B = SA$, we have $\|Bx\| \leq \|Ax\|$ for all $x \in E^*$. Thus $B \in \mathfrak{A}(E^*, E^*; \mathcal{T}_{\mathfrak{J}^*})$. If $B = CA$, with $C \in \mathcal{B}(E^*)$, then $SA - CA = (S - C)A = 0$, so that $S - C$ annihilates the dense linear manifold $A(F \oplus l^p)$ of $F \oplus F$. Thus, S and C must agree on all of $F \oplus F$, which means that C is an extension of S to all of $F \oplus l^p$. Now, let p_1 be the natural projection of $F \oplus l^p$ onto F, i_2 the natural embedding of l^p in $F \oplus l^p$, and j_2 the natural embedding of F in the second factor of $F \oplus F$. Then $p_1 C i_2 J = p_1 S j_2 = I_F$, and this means that $p_1 C i_2$ is an extension of I_F to all of l^p with range in F, contrary to the choice of F. Hence $\mathfrak{J}^* \neq \mathfrak{A}(E^*, E^*; \mathcal{T}_{\mathfrak{J}^*})$, and this completes the proof.

IV. INVARIANT AND MINIMAL SATURATED CLASSES

In this section we examine some further aspects of the relationship between saturated classes and full ideals. In particular, we consider saturated classes which give rise to two-sided ideals, and we show that the correspondence between saturated classes of precompact sets in a Banach space E and the full ideals of completely continuous operators in $\mathcal{B}(l^1_K,E)$ is a lattice isomorphism. This will imply that, for any infinite set K, the lattice of strong ideals of completely continuous operators in $\mathcal{B}(l^1_K,l^\infty_K)$ is isomorphic to the lattice of strong ideals of completely continuous operators in $\mathcal{B}(l^1,l^\infty)$.

Each of the ideals $\mathcal{B}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{C}(E,F)$, and $\mathcal{R}(E,F)$ is a two-sided ideal. The saturated classes of bounded sets in F which give rise to these ideals share the following property:

DEFINITION 1: We shall say that a class $\mathcal{M}$ of subsets of a Banach space E is invariant if $M \in \mathcal{M}$, $T \in \mathcal{B}(E)$ implies that $T(M) \in \mathcal{M}$.

LEMMA 2: If $\mathcal{M}_0$ is an invariant class of bounded sets in E, then the saturated class $\mathcal{M}$ generated by $\mathcal{M}_0$ is invariant.

PROOF: We note that if $\mathcal{M}_0$ is invariant, then it is closed under multiplication by scalars. Thus, the result follows directly from Lemmas II.3 and II.7.

PROPOSITION 3: If $\mathfrak{J}$ is a left ideal in $\mathcal{B}(E,F)$, then $\mathfrak{M}_{\mathfrak{J}}$ is an invariant saturated class of bounded sets in F. On the other hand, if $\mathfrak{M}$ is an invariant saturated class of bounded sets in F, then $\mathfrak{A}(E,F;\mathfrak{M})$ is a two-sided ideal in $\mathcal{B}(E,F)$.

PROOF: It is clear that if $\mathfrak{J}$ is a left ideal in $\mathcal{B}(E,F)$, then the class $\{A(B_1): A \in \mathfrak{J}\}$, is invariant. Thus the first part of the proposition follows from Lemma 2, and the second part is obvious.

COROLLARY 4: If $\mathfrak{J}$ is a left ideal in $\mathcal{B}(E,F)$, then the full ideal $\mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{J}})$ generated by $\mathfrak{J}$ is a two-sided ideal in $\mathcal{B}(E,F)$.

It should be noted that, in order for $\mathfrak{A}(E,F;\mathfrak{M})$ to be a two-sided ideal in $\mathcal{B}(E,F)$, it is not necessary that $\mathfrak{M}$ be invariant, for the correspondence between saturated classes of bounded sets in F and full ideals in $\mathcal{B}(E,F)$ is, in general, not one-to-one. As a simple example of the latter assertion one might consider the case where E is a finite-dimensional space, F is infinite-dimensional, $\mathfrak{M}$ is the class of all bounded sets in F, and $\mathfrak{N}$ the class of all bounded finite-dimensional sets in F. Then $\mathfrak{M}$ and $\mathfrak{N}$ are clearly distinct, while $\mathfrak{A}(E,F;\mathfrak{M}) = \mathfrak{A}(E,F;\mathfrak{N}) = \mathcal{B}(E,F)$. We note, however, that if $\mathfrak{J}$ is a full ideal in $\mathcal{B}(E,F)$, then there always exists a smallest saturated class $\mathfrak{M}$ of bounded sets in F such that $\mathfrak{J} = \mathfrak{A}(E,F;\mathfrak{M})$, namely $\mathfrak{M} = \mathfrak{M}_{\mathfrak{J}}$. This suggests the following definition:

DEFINITION 5: A saturated class $\mathfrak{M}$ of bounded sets in F will be called E-minimal if $\mathfrak{M} = \mathfrak{M}_{\mathfrak{A}(E,F;\mathfrak{M})}$.

The following simple lemma will be useful in the sequel:

LEMMA 6: If $\mathfrak{X}$ is any set of operators in $\mathcal{B}(E,F)$, then the saturated class $\mathfrak{M}_{\mathfrak{X}}$ of bounded sets in F is E-minimal.

PROOF: Clearly $\mathfrak{X} \subset \mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{X}})$, so that we have $\mathfrak{M}_{\mathfrak{X}} \subset \mathfrak{M}_{\mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{X}})}$. On the other hand, $A \in \mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{X}})$ implies $A(B_1) \in \mathfrak{M}_{\mathfrak{X}}$, so that $\mathfrak{M}_{\mathfrak{A}(E,F;\mathfrak{M}_{\mathfrak{X}})} \subset \mathfrak{M}_{\mathfrak{X}}$, and this proves the assertion.

COROLLARY 7: There is a one-to-one correspondence between the full ideals in $\mathcal{B}(E,F)$ and the E-minimal saturated classes of bounded sets in F.

PROOF: The one-to-one correspondence is clearly given by $\mathfrak{J} \longleftrightarrow \mathfrak{M}_{\mathfrak{J}}$.

PROPOSITION 8: If E is an arbitrary Banach space, then every saturated class of pre-compact sets in E is l^1_K-minimal for every infinite set K.

PROOF: If $\mathfrak{M}$ is a saturated class of pre-compact sets in

E, then the class $\mathcal{M}_o$ of all closed, absolutely convex sets in $\mathcal{M}$ generates $\mathcal{M}$. Now, let $M \in \mathcal{M}_o$. Since M is compact, there exists a countable set $\{x_1, x_2, \ldots\ldots\}$ which is dense in M. Let $k_1, k_2, \ldots\ldots$ be a sequence of elements of K, and define $\varphi_{k_i} \in l^{\infty}_K$ by $\varphi_{k_i}(h) = 1$ if $h = k_i$, and $\varphi_{k_i}(h) = 0$ if $h \neq k_i$. If we define $T: l^1_K \longrightarrow E$ by $T = \sum_{i=1}^{\infty} \varphi_{k_i} \otimes x_i$, it is easy to see that T is a bounded linear operator, and that $\overline{T(B_1)} = M$. This shows that $\mathcal{M} = \mathcal{M}_{\mathcal{X}}$, where $\mathcal{X} = \{T \in \mathcal{B}(l^1_K, E) : \overline{T(B_1)} \in \mathcal{M}_o\}$, so that $\mathcal{M}$ is l^1_K-minimal by Lemma 6.

<u>COROLLARY 9:</u> If E is any Banach space, there exists a one-to-one correspondence between the strong right ideals of completely continuous operators in $\mathcal{B}(l^1_K, E^*)$ and the saturated classes of pre-compact sets in E^*.

PROOF: This follows directly from Cor. III.3, Cor. 7, and Prop. 8.

<u>COROLLARY 10:</u> A saturated class $\mathcal{M}$ of pre-compact sets in a Banach space E is invariant if and only if $\mathfrak{A}(l^1, E; \mathcal{M})$ is a two-sided ideal in $\mathcal{B}(l^1, E)$.

In the next section, the strong ideals of completely continuous operators in $\mathcal{B}(l^1_K, l^{\infty}_K)$ will play a key role. Proposition 8 implies that, for our purposes, the study of these ideals can be reduced to the study of strong ideals of completely continuous operators in $\mathcal{B}(l^1, l^{\infty})$. To demonstrate this, we shall use the following lemma, which, as we shall see later, implies Schauder's Theorem, that a bounded operator A is completely continuous if and only if A* is completely continuous.

LEMMA 11: If K, K' are arbitrary sets, an operator A in $\mathcal{B}(l^1_K, l^\infty_{K'})$ is completely continuous if and only if A' is completely continuous.

PROOF: It is easy to see that an operator $R \in \mathcal{B}(l^1_K, l^\infty_{K'})$ is of finite rank if and only if R' is of finite rank. Since ' is an isometry, the lemma now follows directly from Cor. III.4.

We remark that if E and F are any two Banach spaces, then the strong ideals of completely continuous operators in $\mathcal{B}(E,F^*)$ form a complete lattice with respect to the operations $\wedge$ and $\vee$ defined by $\bigwedge_{\gamma\in\Gamma} \mathfrak{J}_\gamma = \bigcap_{\gamma\in\Gamma} \mathfrak{J}_\gamma$, and $\bigvee_{\gamma\in\Gamma} \mathfrak{J}_\gamma = (\bigcup_{\gamma\in\Gamma} \mathfrak{J}_\gamma)^{\bar{B}}$ respectively, and it follows from Lemma 11 that the mapping $\mathfrak{J} \longrightarrow \mathfrak{J}'$ is an isomorphism from the lattice of strong ideals of completely continuous operators in $\mathcal{B}(E,F^*)$ onto the lattice of strong ideals of completely continuous operators in $\mathcal{B}(F,E^*)$. We also observe that the invariant saturated classes of pre-compact sets in a Banach space E form a complete lattice with respect to the operations $\wedge$ and $\vee$ defined by $\bigwedge_{\gamma\in\Gamma} \mathfrak{M}_\gamma = \bigcap_{\gamma\in\Gamma} \mathfrak{M}_\gamma$, and $\bigvee_{\gamma\in\Gamma} \mathfrak{M}_\gamma =$ the saturated class generated by $\bigcup_{\gamma\in\Gamma} \mathfrak{M}_\gamma$. Moreover, it follows from Prop. II.6 that the one-to-one correspondence between strong ideals of completely continuous operators in $\mathcal{B}(l^1_K, E^*)$ and invariant saturated classes of pre-compact sets in E* is a lattice isomorphism. We can now assert the following:

<u>PROPOSITION 12:</u> If K is an arbitrary infinite set, then there exists an isomorphism from the lattice of strong ideals of completely continuous operators in $\mathcal{B}(l^1, l^\infty)$ onto the lattice of strong ideals of completely continuous operators in $\mathcal{B}(l^1_K, l^\infty_K)$.

PROOF: If H is a countable subset of K, then l^1_K is topologically isomorphic to $l^1 \oplus l^1_{K\sim H}$, so that there exists a projection $P: l^1_K \longrightarrow l^1$ mapping the unit ball of l^1_K onto that of l^1. If E is an arbitrary Banach space, and we let $\mathfrak{X} = \{TP: T \in \mathfrak{A}(l^1, E^*; \mathfrak{M})\}$, where $\mathfrak{M}$ is a saturated class of pre-compact sets in E*, then it is easily seen that $\mathfrak{A}(l_K, E^*; \mathfrak{M}) = \mathfrak{X}^{\bar{R}}$. This, together with our above remarks concerning the map $\mathfrak{J} \longrightarrow \mathfrak{J}'$, shows that the mapping $\mathfrak{J} \longrightarrow (\mathfrak{J}'P)^{\bar{R}}$ is an isomorphism from the lattice of strong ideals of completely continuous operators in $\mathcal{B}(l^1, l^\infty)$ onto the lattice of strong ideals of completely continuous operators in $\mathcal{B}(l^1_K, l^\infty)$. Repeating this argument, we see that the desired isomorphism is given by the map $\mathfrak{J} \longrightarrow (((\mathfrak{J}'P)^{\bar{R}})'P)^{\bar{R}} = (P^*\mathfrak{J}P)^{\bar{B}}$.

<u>NOTATION:</u> If $\mathfrak{M}$ is an invariant saturated class of pre-compact sets in l^∞, we shall denote by $\mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$ the

ideal corresponding to $\mathfrak{A}(l^1,l^\infty;\mathfrak{M})$ under the isomorphism defined in the preceding proposition. (Then, of course, $\mathfrak{C}(l^1,l^\infty;\mathfrak{M}) = \mathfrak{A}(l^1,l^\infty;\mathfrak{M})$.)

REMARKS: 1) We can give a relatively simple description of $\mathfrak{C}(l^1_K,l^\infty_K;\mathfrak{M})$: Let $\mathfrak{J} = \mathfrak{A}(l^1,l^\infty;\mathfrak{M})$. Then $\mathfrak{C}(l^1_K,l^\infty_K;\mathfrak{M}) = (P^*\mathfrak{J}\, P)^{\overline{B}}$ consists of finite sums of elements of the form $SP^*AP\circ T^*$, with $S,T \in \mathcal{B}(l^\infty_K)$, $A \in \mathfrak{J}$, and P as in the proof of Prop. 12. Such an element may also be written as $QA\circ R^*$, with $Q,R \in \mathcal{B}(l^\infty,l^\infty_K)$. Thus, the elements of $\mathfrak{C}(l^1_K,l^\infty_K;\mathfrak{M})$ are of the form $\sum_{i=1}^{n} Q_iA_i\circ R_i^*$, with $Q_i,R_i \in \mathcal{B}(l^\infty,l^\infty_K)$, $A_i \in \mathfrak{J}$. Now, l^∞ is topologically isomorphic to $\overset{n}{\oplus}\, l^\infty$. Let $I_i : l^\infty \longrightarrow \overset{n}{\oplus}\, l^\infty$ be the embedding in the i-th factor, and $P_i : \overset{n}{\oplus}\, l^\infty \longrightarrow l^\infty$ the projection on the i-th factor. Then we can write $\sum_{i=1}^{n} Q_iA_i\circ R_i^* = (\sum_{i=1}^{n} Q_iP_i)(\sum_{i=1}^{n} I_iA_i\circ I_i^*)\circ(\sum_{i=1}^{n} R_iP_i)^*$, where $Q = \sum_{i=1}^{n} Q_iP_i$ and $R = \sum_{i=1}^{n} R_iP_i$ are in $\mathcal{B}(l^\infty, l^\infty_K)$, and $A = \sum_{i=1}^{n} I_iA_i\circ I_i^* \in \mathfrak{J}$. On the other hand, if T is of the form $T = QA\circ R^*$, with $Q,R \in \mathcal{B}(l^\infty,l^\infty_K)$, $A \in \mathfrak{J}$, then we can write $T = (Q(P^*)^{-1})P^*AP\circ(R(P^*)^{-1})^*$. Now, $Q(P^*)^{-1}$ and $R(P^*)^{-1}$ are defined on a closed subspace of l^∞_K, and have continuous extensions $\tilde{Q}$ and $\tilde{R}$ respectively to all of l^∞_K with range in l^∞_K. Thus, $T = \tilde{Q}P^*AP\circ\tilde{R}^* \in \mathfrak{C}(l^1_K,l^\infty_K;\mathfrak{M})$. Thus, we have shown that $T \in \mathfrak{C}(l^1_K,l^\infty_K;\mathfrak{M})$ if and only if there exist $Q,R \in \mathcal{B}(l^\infty,l^\infty_K)$, $A \in \mathfrak{A}(l^1,l^\infty;\mathfrak{M})$ such that $T = QA\circ R^*$.

2) If $\mathfrak{M}$ is an invariant saturated class of pre-

compact sets in l^{∞}, and we let $\mathfrak{m}' = \mathfrak{m}_{\mathfrak{J}'}$, where $\mathfrak{J} = \mathfrak{A}(l^1, l^{\infty}; \mathfrak{m})$, then it follows from the preceding remark that $(\mathfrak{C}(l^1_K, l^{\infty}_K; \mathfrak{m}))' = \mathfrak{C}(l^1_K, l^{\infty}_K; \mathfrak{m}')$.

Finally, it will be convenient to make the following definition:

DEFINITION 13: We shall say that a saturated class $\mathfrak{m}$ of pre-compact sets in l^{∞} is symmetric if $\mathfrak{m} = \mathfrak{m}'$ in the notation of the preceding remark.

REMARKS: 1) If $\mathfrak{m}$ is symmetric, then it is clearly invariant.

2) The class of all pre-compact sets in l^{∞} is symmetric, in view of Lemma 11. Also, the class of all bounded finite-dimensional sets in l^{∞} is symmetric. We shall see other examples of symmetric classes in the next section.

V. UNIVERSAL IDEALS OF OPERATORS

To this point, we have dealt with ideals of operators in $\mathcal{B}(E,F)$, with E and F fixed. The ideals $\mathcal{B}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{C}(E,F)$, and $\mathcal{R}(E,F)$, however, are defined for every pair of Banach spaces E and F, and they may, in fact, be interpreted as left exact functors of two variables from the category of Banach spaces into the category of complex linear spaces, contravariant in the first, and covariant in the second variable. Based on this observation, we now introduce the concept of a universal ideal of operators. We then restrict our attention to universal ideals of completely continuous operators, and we show that these are completely determined by the invariant saturated classes of pre-compact sets in l^{∞}. We also show that every two-sided ideal of completely continuous operators on a Hilbert space arises from a universal ideal of operators.

To symmetrize the discussion, we again deal with ideals in spaces of the form $\mathcal{B}(E,F^*)$.

<u>DEFINITION 1:</u> Suppose that, for every pair of Banach spaces E and F, a strong ideal $\mathcal{U}(E,F^*)$ in $\mathcal{B}(E,F^*)$ is given, and that the following condition is satisfied:

U) If J_1 is a topological linear embedding of G^* in E^*, and J_2 a topological linear embedding of H^* in F^*, where G and H are Banach spaces, then an operator

$A \in \mathcal{B}(E,H^*)$ is in $\mathcal{U}(E,H^*)$ if and only if $J_2 A \in \mathcal{U}(E,F^*)$, and an operator $B \in \mathcal{B}(G,F^*)$ is in $\mathcal{U}(G,F^*)$ if and only if $B \circ J_1^* \in \mathcal{U}(E,F^*)$.

We then say that $\mathcal{U}$ is a universal ideal of operators, and that $\mathcal{U}(E,F^*)$, for fixed E and F, is a realization of $\mathcal{U}$.

One can show without difficulty that the ideals $\mathcal{B}(E,F^*)$, $\mathcal{W}(E,F^*)$, $\mathcal{C}(E,F^*)$, and $\mathcal{R}(E,F^*)$ satisfy condition (U), so that $\mathcal{B}$, $\mathcal{C}$, $\mathcal{W}$, and $\mathcal{R}$ are universal ideals. An example of a strong ideal in $\mathcal{B}(E,F^*)$, which is defined for every pair of Banach spaces E and F, but which does not satisfy condition (U) is provided by the trace class. (The trace class is the set of operators in $\mathcal{B}(E,F^*)$ which are images of elements of $E^* \otimes_\gamma F^*$ under the natural mapping $E^* \otimes_\gamma F^* \longrightarrow \mathcal{B}(E,F^*)$. See [4, p.80 and p.88].)

We observe that if $\mathcal{U}$ is a universal ideal of operators, then condition (U) implies that $\mathcal{U}(E,F^*) = \{A \in \mathcal{B}(E,F^*) : J_{F^*} A \in \mathcal{U}(E,F^{***})\}$. Thus, we are justified in adopting the following

<u>NOTATION:</u> If $\mathcal{U}$ is a universal ideal of operators, we denote by $\mathcal{U}(E,F)$ the space of operators $A \in \mathcal{B}(E,F)$ such that $J_F A \in \mathcal{U}(E,F^{**})$.

We remark that if $\mathcal{U}$ is a universal ideal of operators, then $\mathcal{U}(E,F)$ is a two-sided ideal in $\mathcal{B}(E,F)$. For if

$S \in \mathcal{B}(E)$, $A \in \mathcal{U}(E,F)$, then $J_F AS \in \mathcal{U}(E,F^{**})$, since the latter is a two-sided ideal in $\mathcal{B}(E,F^{**})$. Also, if $T \in \mathcal{B}(F)$, then $J_F TA = T^{**}J_F A$, so that $TA \in \mathcal{U}(E,F)$.

DEFINITION 2: If $\mathcal{U}$ is a universal ideal of operators, we define $\mathcal{U}'$ by $\mathcal{U}'(E,F^*) = \{A \in \mathcal{B}(E,F^*): A' \in \mathcal{U}(F,E^*)\}$. The universal ideal $\mathcal{U}$ is called symmetric if $\mathcal{U}(E,F^*) = \mathcal{U}'(E,F^*)$ for every pair of Banach spaces E and F.

We note that it follows directly from Def. 1 that, if $\mathcal{U}$ is a universal ideal, then $\mathcal{U}'$ is a universal ideal.

PROPOSITION 3: If $\mathcal{U}$ is a symmetric universal ideal, and $A \in \mathcal{B}(E,F)$, then $A \in \mathcal{U}(E,F)$ if and only if $A^* \in \mathcal{U}(F^*,E^*)$.

PROOF: If $\mathcal{U}$ is symmetric, then $A \in \mathcal{U}(E,F)$ if and only if $J_F A \in \mathcal{U}(E,F^{**})$ if and only if $(J_F A)' = A^* J_F^{\,*} J_{F^*} \in \mathcal{U}(F^*,E^*)$, and this completes the proof, since $J_F^{\,*} J_{F^*}$ is the identity operator on F^*.

We now turn to an explicit construction of the universal ideals of completely continuous operators, that is, universal ideals $\mathcal{U}$ such that $\mathcal{U}(E,F^*) \subset \mathcal{C}(E,F^*)$ for every pair of Banach spaces E and F.

If E and F are arbitrary Banach spaces, we may view both as quotient spaces of l^1_K for an appropriate K, as in Section III. In particular, we have operators $P_1 : l^1_K \longrightarrow E$, and $P_2 : l^1_K \longrightarrow F$ mapping the unit ball of l^1_K onto that of E and F respectively, and we again define the operator $\Delta : \mathcal{B}(E,F^*) \longrightarrow \mathcal{B}(l^1_K, l^\infty_K)$ by $\Delta(A) = P_2{}^*AP_1$.

<u>DEFINITION 4:</u> If $\mathfrak{M}$ is an invariant saturated class of pre-compact sets in l^∞, we shall say that an operator $A \in \mathcal{B}(E,F^*)$ is of type $\mathfrak{M}$ if $\Delta(A) \in \mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$. We denote the set of operators of type $\mathfrak{M}$ in $\mathcal{B}(E,F^*)$ by $\mathcal{C}(E,F^*;\mathfrak{M})$

We must, of course, check that Def. 4 is independent of the choices of K and Δ . This can best be done by use of the following result:

<u>PROPOSITION 5:</u> With K and Δ as above, and $A \in \mathcal{B}(E,F^*)$, we have $\Delta(A) \in \mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$ if and only if $TA \circ S^*$ is in $\mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$ for all $T \in \mathcal{B}(F^*, l^\infty_K)$, $S \in \mathcal{B}(E^*, l^\infty_K)$.

PROOF: It is clear that, if the condition of the proposition is satisfied, then $\Delta(A) \in \mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$. On the other hand, if $\Delta(A) \in \mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$, and $T \in \mathcal{B}(F^*, l^\infty_K)$, $S \in \mathcal{B}(E^*, l^\infty_K)$, then $T(P_2{}^*)^{-1}$ and $S(P_1{}^*)^{-1}$ are operators mapping closed subspaces of l^∞_K into l^∞_K, and hence have continuous extensions $\tilde{T} \in \mathcal{B}(l^\infty_K)$ and $\tilde{S} \in \mathcal{B}(l^\infty_K)$ respectively. Then $TA \circ S^* = \tilde{T}P_2{}^*AP_1 \circ \tilde{S}^*$. Since, by assumption, $P_2{}^*AP_1$ is in $\mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$, and the latter is a strong ideal in $\mathcal{B}(l^1_K, l^\infty_K)$, it follows that $TA \circ S^* \in \mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$ as claimed.

It is immediate from Prop. 5 that the definition of $\mathcal{C}(E,F^*;\mathfrak{M})$ is independent of the choice of Δ if K is fixed. We next show that it is also independent of the choice of K. Thus, let K' be another infinite set such that E and F are quotient spaces of $l^1_{K'}$, and let us assume that card K' $\leq$ card K. Then there exists a projection $P: l^1_K \longrightarrow l^1_{K'}$ mapping the unit ball of l^1_K onto that of $l^1_{K'}$. Now assume that $A \in \mathcal{B}(E,F^*)$ is of type $\mathcal{M}$ with respect to K, and let $T \in \mathcal{B}(F^*, l^\infty_{K'})$, $S \in \mathcal{B}(E^*, l^\infty_{K'})$. Then, by Prop. 5, $P^*TA\circ(P^*S)^* \in \mathcal{C}(l^1_K, l^\infty_K; \mathcal{M})$, and in view of our earlier description of the latter ideal, this means that there are operators $Q \in \mathcal{B}(l^\infty, l^\infty_K)$, $R \in \mathcal{B}(l^\infty, l^\infty_K)$, and $B \in \mathcal{O}(l^1, l^\infty; \mathcal{M})$ such that $P^*TA\circ(P^*S)^* = QB\circ R^*$. Now let $\tilde{P}$ be an extension of $(P^*)^{-1}$ to all of l^∞_K with range in $l^\infty_{K'}$. Then $TA\circ S^* = \tilde{P}P^*TA\circ(P^*S)^*\circ\tilde{P}^* = \tilde{P}QB\circ R^*\circ\tilde{P}^* \in \mathcal{C}(l^1_{K'}, l^\infty_{K'}; \mathcal{M})$, and this shows that A is of type $\mathcal{M}$ with respect to K'. A similar argument shows the converse.

Before discussing the relationship between the universal ideals of completely continuous operators and the classes $\mathcal{C}(E,F^*;\mathcal{M})$, we derive some of the elementary properties of the latter.

PROPOSITION 6: If $\mathcal{M}$ is the class of all pre-compact sets in l^∞, then $\mathcal{C}(E,F^*;\mathcal{M}) = \mathcal{C}(E,F^*)$. Also, if $\{\mathcal{M}_i\}_{i\in I}$ is any family of invariant saturated classes of pre-compact sets in l^∞, then $\bigcap_{i\in I} \mathcal{C}(E,F^*;\mathcal{M}_i) = \mathcal{C}(E,F^*; \bigcap_{i\in I}\mathcal{M}_i)$.

PROOF: It is clear from Prop. IV.12 and the definition of $\mathcal{C}(l^1_K, l^\infty_K;\mathcal{M})$ that, if $\mathcal{M}$ is the class of all pre-compact sets in l^∞, then $\mathcal{C}(l^1_K, l^\infty_K;\mathcal{M}) = \mathcal{C}(l^1_K, l^\infty_K)$. Also, with K and

Δ as in Def. 4, the properties of P_1 and P_2 imply at once that $A \in \mathcal{B}(E,F^*)$ is compact if and only if $\Delta(A)$ is compact, and this yields the first part of the proposition. The second part follows directly from the definitions and Prop. IV.12.

PROPOSITION 7: $A \in \mathcal{B}(E,F^*)$ is of type $\mathcal{M}$ if and only if A' is of type $\mathcal{M}'$. In particular, if $\mathcal{M}$ is symmetric, then A is of type $\mathcal{M}$ if and only if A' is of type $\mathcal{M}$.

PROOF: Note that for $A \in \mathcal{W}(E,F^*)$, $S \in \mathcal{B}(E^*, l^\infty_K)$, $T \in \mathcal{B}(F^*, l^\infty_K)$, we have $SA' \circ T^* = (TA \circ S^*)'$. Thus, the proposition follows from Prop. 5, and our earlier observation that $(\mathcal{C}(l^1_K, l^\infty_K; \mathcal{M}))' = \mathcal{C}(l^1_K, l^\infty_K; \mathcal{M}')$.

As a corollary, we have Schauder's Theorem:

COROLLARY 8: An operator $A \in \mathcal{B}(E,F)$ is compact if and only if A^* is compact.

PROOF: Clearly, $A \in \mathcal{B}(E,F)$ is compact if and only if $J_F A \in \mathcal{B}(E,F^{**})$ is compact. Thus, by the same argument as in the proof of Prop. 3, the result follows from Propositions 6 and 7, and our observation at the end of Section IV that the class of all pre-compact sets in l^∞ is symmetric.

PROPOSITION 9: If $A \in \mathcal{B}(E,F^*)$ is of type $\mathcal{M}$, and $Q \in \mathcal{B}(E^*,G^*)$, $R \in \mathcal{B}(F^*,H^*)$, where G and H are arbitrary Banach spaces, then $RA \circ Q^* \in \mathcal{B}(G,H^*)$ is of type $\mathcal{M}$. Thus, in particular, $\mathcal{C}(E,F^*;\mathcal{M})$ is a strong ideal in $\mathcal{B}(E,F^*)$.

PROOF: Choose the set K so that each of the spaces E,F, G, and H may be viewed as quotient spaces of l^1_K. Then, if $S \in \mathcal{B}(H^*, l^\infty_K)$, $T \in \mathcal{B}(G^*, l^\infty_K)$, we have by Prop. 5 that $SRA \circ Q^* \circ T^* = (SR)A \circ (TQ)^* \in \mathcal{C}(l^1_K, l^\infty_K; \mathcal{M})$, since $SR \in \mathcal{B}(F^*, l^\infty_K)$, and $TQ \in \mathcal{B}(E^*, l^\infty_K)$. Thus, $RA \circ Q^*$ is of type $\mathcal{M}$ by Prop. 5.

<u>REMARK:</u> Cor. III.6 implies that $\mathcal{C}(E,F^*;\mathcal{M})$ is not only a strong ideal in $\mathcal{B}(E,F^*)$, but, in fact, a full ideal.

<u>COROLLARY 10:</u> If $A \in \mathcal{B}(E,F^*)$ is of type $\mathcal{M}$, and $B \in \mathcal{B}(F^*,H^*)$ is of type $\mathcal{N}$, then BA is of type $\mathcal{M} \cap \mathcal{N}$. Similarly, if $C \in \mathcal{B}(E^*,G^*)$ is of type $\mathcal{N}$, then $A \circ C^*$ is of type $\mathcal{M} \cap \mathcal{N}'$.

PROOF: This follows easily from Propositions 6 and 9.

We are now ready to give the following characterization of the universal ideals of completely continuous operators:

<u>THEOREM 11:</u> If $\mathcal{M}$ is an invariant saturated class of precompact sets in l^∞, and we define $\mathcal{U}_\mathcal{M}$ by $\mathcal{U}_\mathcal{M}(E,F^*) = \mathcal{C}(E,F^*;\mathcal{M})$, then $\mathcal{U}_\mathcal{M}$ is a universal ideal of completely continuous operators. Conversely, if $\mathcal{U}$ is a universal ideal of completely continuous operators, then there exists a unique invariant saturated class of precompact sets in l^∞ such that $\mathcal{U}(E,F^*) = \mathcal{C}(E,F^*;\mathcal{M})$ for every pair of Banach spaces E and F.

PROOF: We have already seen that the $\mathcal{C}(E,F^*;\mathcal{M})$ are strong ideals in $\mathcal{B}(E,F^*)$. Thus, to prove the first part of the theorem it only remains to be shown that, if $\mathcal{U}_{\mathfrak{M}}$ is defined as in the theorem, then it satisfies condition (U). Let $J_1:G^* \longrightarrow E^*$ and $J_2:H^* \longrightarrow F^*$ be as in Def. 1. It is clear from Prop. 9 that, if $A \in \mathcal{U}_{\mathfrak{M}}(E,H^*)$, then $J_2A \in \mathcal{U}_{\mathfrak{M}}(E,F^*)$, and if $B \in \mathcal{U}_{\mathfrak{M}}(G,F^*)$, then $B \circ J_1 \in \mathcal{U}_{\mathfrak{M}}(E,F^*)$. Now choose K so that E,F,G, and H are quotient spaces of l^1_K by means of projections $P_1:l^1_K \longrightarrow E$, $P_2:l^1_K \longrightarrow F$, $Q_1:l^1_K \longrightarrow G$, and $Q_2:l^1_K \longrightarrow H$ respectively. By definition, $J_2A \in \mathcal{U}_{\mathfrak{M}}(E,F^*)$ means that $P_2{}^*J_2AP_1 \in \mathcal{C}(l^1_K,l^\infty_K;\mathfrak{M})$. Now, $P_2{}^*J_2AP_1 = P_2{}^*J_2(Q_2{}^*)^{-1}Q_2{}^*AP_1$. Let $S = (P_2{}^*J_2(Q_2{}^*)^{-1})^{-1}$. S maps a closed subspace of l^∞_K into l^∞_K, and hence has an extension $\tilde{S} \in \mathcal{B}(l^\infty_K)$. Then it is clear that $Q_2{}^*AP_1 = \tilde{S}P_2{}^*J_2(Q_2{}^*)^{-1}Q_2{}^*AP_1 \in \mathcal{C}(l^1_K,l^\infty_K;\mathfrak{M})$. Thus, we have shown that, if $J_2A \in \mathcal{U}_{\mathfrak{M}}(E,F^*)$, then $A \in \mathcal{U}_{\mathfrak{M}}(E,H^*)$. The symmetric argument shows that if $B \circ J_1{}^* \in \mathcal{U}_{\mathfrak{M}}(E,F^*)$, then $B \in \mathcal{U}_{\mathfrak{M}}(G,F^*)$, so that $\mathcal{U}_{\mathfrak{M}}$ satisfies condition (U), and is a universal ideal as claimed.

To prove the second half of the theorem, we note that, if $\mathcal{U}$ is a universal ideal of completely continuous operators, then $\mathcal{U}(l^1,l^\infty)$ is a strong ideal of completely continuous operators in $\mathcal{B}(l^1,l^\infty)$. We have seen that this means that there is a unique invariant saturated class of pre-compact sets in l^∞ such that $\mathcal{U}(l^1,l^\infty) = \mathcal{C}(l^1,l^\infty;\mathfrak{M})$. Condition (U), and the definition of $\mathcal{C}(E,F^*;\mathfrak{M})$ then imply at once that $\mathcal{U}(E,F^*) = \mathcal{C}(E,F^*;\mathfrak{M})$ for every pair of Banach spaces E and F, and this completes the proof.

<u>REMARK:</u> Theorem 11 shows, in particular, that the universal ideals of completely continuous operators form a set. Moreover, this set is a complete lattice with respect to the operations $\wedge$ and $\vee$ defined by

$(\bigwedge_{i\in I}\mathcal{U}_i)(E,F^*) = \bigcap_{i\in I}\mathcal{U}_i(E,F^*)$ and

$\bigvee_{i\in I}\mathcal{U}_i = \bigwedge\{\mathcal{U} : \mathcal{U}_i(E,F^*)\subset\mathcal{U}(E,F^*)$ for all $i\in I$ and every pair of Banach spaces E and F$\}$.

Proposition 6 shows that this lattice is isomorphic to the lattice of invariant saturated classes of pre-compact sets in l^∞. We also note that the operation ' is a lattice isomorphism.

<u>NOTATION:</u> In accordance with the convention already adopted for universal ideals, we denote by $\mathcal{C}(E,F;\mathcal{M})$ the ideal $\{A\in\mathcal{B}(E,F)\colon J_F A\in\mathcal{C}(E,F^{**};\mathcal{M})\}$, and we say that an operator $A\in\mathcal{B}(E,F)$ is of type $\mathcal{M}$ if $A\in\mathcal{C}(E,F;\mathcal{M})$.

Analogs of Propositions 6 and 9 and Cor. 10 may be proven for the ideals $\mathcal{C}(E,F;\mathcal{M})$. In particular, we note that if $A\in\mathcal{B}(E,F)$ is of type $\mathcal{M}$, and $S\in\mathcal{B}(F,H)$, $T\in\mathcal{B}(G,E)$, then $SAT\in\mathcal{B}(G,H)$ is of type $\mathcal{M}$. This follows directly from Prop. 9, and the equality $J_H SAT = S^{**}J_F A\circ T^{**}$.

To conclude our discussion, we now show that every two-sided ideal of completely continuous operators on a Hilbert space is a realization of a symmetric universal ideal of operators. It is useful to begin with the following simple lemma:

<u>LEMMA 12:</u> Let E and F be arbitrary Banach spaces, and $\Delta : \mathcal{B}(E,F^*) \longrightarrow \mathcal{B}(l^1_K, l^\infty_K)$ as in Def. 4. Then a set $\mathfrak{J}$ of completely continuous operators in $\mathcal{B}(E,F^*)$ is a realization of a universal ideal of completely continuous operators if and only if

$$\Delta(\mathfrak{J}) = \Delta(\mathcal{B}(E,F^*)) \cap (\Delta(\mathfrak{J}))^{\overline{B}}.$$

PROOF: We note that $(\Delta(\mathfrak{J}))^{\overline{B}}$ is a strong ideal in $\mathcal{B}(l^1_K, l^\infty_K)$, and hence of the form $\mathcal{C}(l^1_K, l^\infty_K; \mathfrak{M})$ for some $\mathfrak{M}$. Thus, it is clear that, if $\mathfrak{J}$ satisfies the equality of the lemma, then $\mathfrak{J} = \mathcal{C}(E,F^*;\mathfrak{M})$. On the other hand, if $\mathfrak{J}$ is a realization of a universal ideal $\mathcal{U}$ of completely continuous operators, then $\mathfrak{J} = \Delta^{-1}(\mathcal{U}(l^1_K, l^\infty_K))$, and this implies the equality of the lemma as in the proof of Cor. III.6.

We now let $\mathcal{H}$ be a Hilbert space of arbitrary dimension. If $A \in \mathcal{B}(\mathcal{H})$, we shall denote the Banach space adjoint of A by A', to distinguish it from the Hilbert space adjoint A* of A.

<u>THEOREM 13:</u> Every two-sided ideal of completely continuous operators in $\mathcal{B}(\mathcal{H})$ is a realization of a universal ideal of completely continuous operators.

PROOF: Let $P_1 : l^1_K \longrightarrow \mathcal{H}$ and $P_2 : l^1_K \longrightarrow \mathcal{H}^*$ be projections mapping the unit ball of l^1_K onto the unit ball of $\mathcal{H}$ and $\mathcal{H}^*$ respectively, and let $\Delta : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(l^1_K, l^\infty_K)$ be defined as usual. We must show that if $\mathfrak{J}$ is a two-sided ideal of completely continuous operators in $\mathcal{B}(\mathcal{H})$, then $\mathfrak{J}$ satisfies the equality of Lemma 12. Let $\mathfrak{X}$ be the set consisting of all finite sums of the form $\sum_{i=1}^{n} T_i P_1{}^* A_i{}'$, with $A_i \in \mathfrak{J}$, $T_i \in \mathcal{B}(l_K^\infty)$. It is clear that $\mathfrak{X}$ is a right ideal in $\mathcal{B}(\mathcal{H}^*, l_K^\infty)$, and hence $\mathfrak{X}$ is a full ideal by Cor. III.2, so that $\mathfrak{X} = \mathfrak{A}(\mathcal{H}^*, l_K; \mathfrak{M}_{\mathfrak{X}})$. Let $\mathcal{E} = \{SP_2 : S \in \mathfrak{X}\}$, and observe that $\mathfrak{M}_{\mathcal{E}} = \mathfrak{M}_{\mathfrak{X}}$. Thus, $\mathcal{E}^{\overline{R}} = \mathfrak{A}(l_K^1, l_K^\infty; \mathfrak{M}_{\mathfrak{X}})$. Also, $\mathcal{E} = \Delta'(\mathfrak{J}')^L$, so that $\mathcal{E}^{\overline{R}} = (\Delta'(\mathfrak{J}'))^{\overline{B}}$. Now, let $A \in \mathcal{B}(\mathcal{H})$ be such that $\Delta(A) \in (\Delta(\mathfrak{J}))^{\overline{B}}$. Then we have $\Delta'(A') = P_1{}^* A' P_2 \in (\Delta'(\mathfrak{J}'))^{\overline{B}} = \mathfrak{A}(l_K^1, l_K^\infty; \mathfrak{M}_{\mathfrak{X}})$. But this clearly implies that $P_1{}^* A'$ maps the unit ball of $\mathcal{H}^*$ into an element of $\mathfrak{M}_{\mathfrak{X}}$, so that $P_1{}^* A' \in \mathfrak{X}$, and this means that $\Delta'(A') \in \mathcal{E}$. Since $\mathcal{E} = (\Delta'(\mathfrak{J}'))^L$, and $\mathfrak{J}$ is a full ideal, this implies that $A' \in \mathfrak{J}'$ by Cor. III.8. Thus $A \in \mathfrak{J}$, and we have shown that $\Delta(\mathcal{B}(\mathcal{H})) \cap (\Delta(\mathfrak{J}))^{\overline{B}} \subset \Delta(\mathfrak{J})$, and since the opposite inclusion holds in any case, the proof is complete.

While we cannot assert that there is a unique universal ideal $\mathcal{U}$ such that $\mathcal{U}(\mathcal{H}, \mathcal{H}) = \mathfrak{J}$, it is clear that there is a smallest such universal ideal (with respect to the above mentioned lattice structure on the set of universal ideals of completely continuous operators), namely, the unique universal ideal $\mathcal{U}$ such that $\mathcal{U}(l_K^1, l_K^\infty) = (\Delta(\mathfrak{J}))^{\overline{B}}$. It is possible to give quite a complete description of these universal ideals. We shall use the following definition due to R. Schatten [11,p.26]:

<u>DEFINITION 14:</u> A set Σ of non-increasing sequences of non-negative real numbers is called a characteristic set, if i) $\{a_n\} \in \Sigma$ implies $a_n \to 0$ as $n \to \infty$.

ii) $\{a_1, a_2, \ldots\} \in \Sigma$ implies $\{a_1, a_1, a_2, a_2, \ldots\} \in \Sigma$.

iii) $\{a_n\}, \{b_n\} \in \Sigma$ implies $\{a_n + b_n\} \in \Sigma$.

iv) $\{a_n\} \in \Sigma$, $a_n \geq b_n$, $b_n \geq b_{n+1}$, $n = 1, 2, \ldots$ implies $\{b_n\} \in \Sigma$.

If A is a completely continuous operator on the Hilbert space $\mathcal{H}$, then the characteristic sequence of A is defined to be the sequence of eigenvalues of $(A^*A)^{\frac{1}{2}}$ arranged in non-increasing order of magnitude [11, p25]. The characteristic sequence of a completely continuous operator $A \in \mathcal{B}(\mathcal{H})$ is always a sequence of non-negative real numbers converging to zero. If Σ is a characteristic set, we define $\mathcal{C}_\Sigma(\mathcal{H})$ to be the set of completely continuous operators on $\mathcal{H}$ whose characteristic sequences lie in Σ. It is shown in [11] that $\mathcal{C}_\Sigma(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, and that, if $\mathcal{H}$ is infinite-dimensional, the map $\Sigma \longrightarrow \mathcal{C}_\Sigma(\mathcal{H})$ establishes a one-to-one correspondence between the characteristic sets, and the two-sided ideals of completely continuous operators in $\mathcal{B}(\mathcal{H})$. This correspondence is, in fact, a lattice isomorphism. (See also [1].)

Now, let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let the notation be as in Theorem 13.

<u>DEFINITION 15:</u> If Σ is a characteristic set, we define $\mathcal{C}_\Sigma$ to be the (unique) universal ideal of completely continuous operators such that $\mathcal{C}_\Sigma(l^1_K, l^\infty_K) = (\Delta(\mathcal{C}_\Sigma(\mathcal{H})))^{\bar{B}}$. If Σ is the set of all non-increasing sequences of non-negative real numbers in l^p, with $0 < p < \infty$ (resp. all non-increasing sequences of non-negative real numbers converging to zero), we write $\mathcal{C}_p$ (resp. $\mathcal{C}_\infty$) instead of $\mathcal{C}_\Sigma$.

The following lemma shows that the definition of $\mathcal{C}_\Sigma$ is independent of the (infinite) dimension of $\mathcal{H}$:

<u>LEMMA 16:</u> Let $\mathcal{H}_0$ be separable Hilbert space. Then $\mathcal{C}_\Sigma(l^1_K, l^\infty_K) = \{SAT\colon S \in \mathcal{B}(\mathcal{H}_0, l^\infty_K),\ A \in \mathcal{C}_\Sigma(\mathcal{H}_0),\ T \in \mathcal{B}(l^1_K, \mathcal{H}_0)\}$.

PROOF: Using the polar representation of compact operators, it may easily be seen that if the equality of the lemma holds with $\mathcal{H}_0$ replaced by $\mathcal{H}$, then it holds for $\mathcal{H}_0$. Now, Prop. 5 shows that if $S \in \mathcal{B}(\mathcal{H}, l^\infty_K)$, $A \in \mathcal{C}_\Sigma(\mathcal{H})$, and $T \in \mathcal{B}(l^1_K, \mathcal{H})$, then $SAT \in \mathcal{C}_\Sigma(l^1_K, l^\infty_K)$. We next observe that, if $Q, R \in \mathcal{B}(l^\infty_K)$, then for $A \in \mathcal{C}_\Sigma(\mathcal{H})$ we have $RP_2^*AP_1 \circ Q^* = (RP_2^*)A \circ (QP_1^*)^* = (RP_2^*)A(P_1^{**}Q^*J_{l^1_K}) = SAT$, where $S = RP_2^* \in \mathcal{B}(\mathcal{H}, l^\infty_K)$, and $T = P_1^{**}Q^*J_{l^1_K} \in \mathcal{B}(l^1_K, \mathcal{H})$. Thus, $\mathcal{C}_\Sigma(l^1_K, l^\infty_K) = (\Delta(\mathcal{C}_\Sigma(\mathcal{H})))^{\bar{B}}$ consists of all finite sums of the form $\sum_{i=1}^n S_i A_i T_i$, with $S_i \in \mathcal{B}(\mathcal{H}, l^\infty_K)$, $A_i \in \mathcal{C}_\Sigma(\mathcal{H})$, and $T_i \in \mathcal{B}(l^1_K, \mathcal{H})$. Now, $\mathcal{H}$ is topologically isomorphic to $\overset{n}{\oplus}\mathcal{H}$. Let $I_i\colon \mathcal{H} \longrightarrow \overset{n}{\oplus}\mathcal{H}$ be the injection in the i-th factor, and $P_i\colon \overset{n}{\oplus}\mathcal{H} \longrightarrow \mathcal{H}$ the projection on the i-th factor. Then $\sum_{i=1}^n S_i A_i T_i = (\sum_{i=1}^n S_i P_i)(\sum_{i=1}^n I_i A_i P_i)(\sum_{i=1}^n I_i T_i) = SAT$, where

$S = \sum_{i=1}^{n} S_i P_i \in \mathcal{B}(\mathcal{H}, l_K^\infty)$, $A = \sum_{i=1}^{n} I_i A_i P_i \in \mathcal{C}_{\Sigma'}(\mathcal{H})$, and $T = \sum_{i=1}^{n} I_i T_i \in \mathcal{B}(l_K^1, \mathcal{H})$, and this yields the desired result.

Finally, we have the following

THEOREM 17: a) The universal ideals $\mathcal{C}_{\Sigma'}$ are symmetric.

b) If Σ and Σ' are arbitrary characteristic sets, and $A \in \mathcal{C}_\Sigma(E,F)$, $B \in \mathcal{C}_{\Sigma'}(F,G)$, then $BA \in \mathcal{C}_{\Sigma \cap \Sigma'}(E,G)$. More particularly, if $0 < p,q,r < \infty$, $1/p + 1/q = 1/r$, and $A \in \mathcal{C}_p(E,F)$, $B \in \mathcal{C}_q(F,G)$, then $BA \in \mathcal{C}_r(E,G)$.

PROOF: One can readily see that if A is a completely continuous operator on $\mathcal{H}$, then there are linear operators $U: \mathcal{H}^* \longrightarrow \mathcal{H}$, and $V: \mathcal{H} \longrightarrow \mathcal{H}^*$, such that $A' = VA^*U$. Thus, part (a) of the theorem follows easily from Lemma 16, and the fact that $A \in \mathcal{C}_{\Sigma'}(\mathcal{H})$ if and only if $A^* \in \mathcal{C}_{\Sigma'}(\mathcal{H})$. To prove part (b), we begin by noting that the first part of the proof of Theorem 13 shows that if $Q \in \mathcal{C}_{\Sigma'}(\mathcal{H}, l_K^\infty)$, then Q has the form $\sum_{i=1}^{n} R_i D_i$, with $R_i \in \mathcal{B}(\mathcal{H}, l_K^\infty)$, $D_i \in \mathcal{C}_\Sigma(\mathcal{H})$. Now let A and B be as in the theorem, and choose K so that there are appropriate projections $P_1: l_K^1 \longrightarrow E$, $P_2: l_K^1 \longrightarrow F^*$, and $P_3: l_K^1 \longrightarrow G^*$. Consider the operator $P_3^* J_G B A P_1 = P_3^* J_G B (P_2^* J_F)^{-1} P_2^* J_F A P_1$. By definition of $\mathcal{C}_\Sigma(E,F)$ and Lemma 16, $P_2^* J_F A P_1 = SCT$ for some $S \in \mathcal{B}(\mathcal{H}, l_K)$, $C \in \mathcal{C}_\Sigma(\mathcal{H})$, $T \in \mathcal{B}(l_K^1, \mathcal{H})$, and by a simple argument, we may assume that the range of S is contained in $P_2^* J_F(F)$. Thus, we may write $P_3^* J_G B A P_1 = P_3^* J_G B (P_2^* J_F)^{-1} SCT$. The assumption that $B \in \mathcal{C}_{\Sigma'}(F,G)$ implies that $P_3^* J_G B (P_2^* J_F)^{-1} S \in \mathcal{C}_{\Sigma'}(\mathcal{H}, l_K^\infty)$, so that, by our earlier remark, this operator has the form $\sum_{i=1}^{n} R_i D_i$, with $R_i \in \mathcal{B}(\mathcal{H}, l_K^{\infty})$, $D_i \in \mathcal{C}_{\Sigma'}(\mathcal{H})$. Hence, we have $P_3{}^* J_G BAP_1 = \sum_{i=1}^{n} R_i D_i CT$. Since $\mathcal{C}_{\Sigma}(\mathcal{H})$ and $\mathcal{C}_{\Sigma'}(\mathcal{H})$ are two-sided ideals in $\mathcal{B}(\mathcal{H})$, it follows that the $D_i C$ are elements of $\mathcal{C}_{\Sigma \cap \Sigma'}(\mathcal{H})$. Thus, the definition of $\mathcal{C}_{\Sigma \cap \Sigma'}(E,G)$ and Lemma 16 imply that $BA \in \mathcal{C}_{\Sigma \cap \Sigma'}(E,G)$. Finally, if $A \in \mathcal{C}_p(E,F)$, $B \in \mathcal{C}_q(F,G)$, then the D_i are in $\mathcal{C}_q(\mathcal{H})$, and $C \in \mathcal{C}_p(\mathcal{H})$, and it is shown in [3, p.1093] that this implies that the $D_i C$ are elements of $\mathcal{C}_r(\mathcal{H})$. Thus, by the definition of $\mathcal{C}_r(E,G)$ and Lemma 16, we have $BA \in \mathcal{C}_r(E,G)$ as claimed, and the proof is complete.

BIBLIOGRAPGY

1. J. W. Calkin
Two-sided Ideals and Congruences in the Ring of Bounded Operators in Hilbert Space.
Annals of Math., vol. 42 (1941) pp. 839-873

2. J. Dixmier
Sur Un Theorem De Banach
Duke Math. J., vol. 15 (1948) pp.1057-1071

3. N. Dunford and J. T. Schwartz
Linear Operators, Parts I and II
Interscience, New York 1958 and 1963

4. A. Grothendieck
Produits Tensoriels Topologiques Et Espaces Nucléaires
Memoirs Amer. Math. Soc. no. 16 (1955)

5. J. L. Kelley and I. Namioka
Linear Topological Spaces
D. van Nostrand Co. , New York 1963

6. G. Köthe
Topologische Lineare Räume, Part I, Second Edition
Springer Verlag, Berlin, 1966

7. E. Michael
Transformations from a Linear Space with Weak Topology
Pro. Amer. Math. Soc. 3 (1952) pp.671-676

8. F. J. Murray
On Complementary Manifolds and Projections in Spaces L_p and l_p.
Trans. Amer. Math. Soc. 41 (1937) pp. 138-152

9. R. S. Phillips
On Linear Transformations
Trans. Amer. Math. Soc. 48 (1940) pp. 516-541

10. R. Schatten
A Theory of Cross-Spaces
Annals of Math. Studies 26 (1950), Princeton Univ. Press, Princeton.

11. R. Schatten
Norm Ideals of Completely Continuous Operators
Ergebnisse der Math., Neue Folge 27 (1960), Springer Verlag, Berlin.

12. B. Yood
Additive Groups and Linear Manifolds of Transformations Between Banach Spaces
Amer. J. Math. 71 (1949) 663-677.

AUTOBIOGRAPHICAL STATEMENT

Peter Falley was born in Brandenburg, Germany, on January 30, 1936. He immigrated to the United States in 1953. After completing his secondary education in New York City, he entered Columbia University, where he received the Bachelor of Science and Master of Arts degrees in Mathematics in 1960 and 1962 respectively. He pursued further graduate study at Columbia University until 1964. He was employed in the insurance industry from 1953 until 1961, when he was appointed Lecturer of Mathematics at The City College of New York. He continued in this capacity until 1966. Since 1965 he has been a student in the Department of Mathematics of The City University of New York.