p-POTENTIAL THEORY ON GRAPHS p-PARABOLICITY AND p-HYPERBOLICITY

by

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Abstract

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The aim of this thesis is to present results in nonlinear potential theory mainly on infinite graphs with or without boundary. These objects are similar in many ways to Riemannian manifolds. To this end, we introduce a fundamental notion of p-capacity which allows us to classify finite graphs without boundary as p-parabolic and finite/infinite graphs with boundary as p-hyperbolic, to extend the divergence theorem and its consequences to p-Dirichlet spaces, to prove important analogues to the smooth case such as the Kelvin-Nevanlinna-Royden criterion for p-hyperbolicity, and the criteria of existence/non-existence of solutions to the p-Poisson Equation on p-hyperbolic and p-parabolic graphs.

Preface

In this dissertation, we intend to generalize several results of potential theory on infinite graphs and Riemann manifolds to p-potential theory on finite/infinite graphs with bounded geometry.

We begin in **Chapter 1** , geometrically, by introducing basic graph theoretic terms and a new concept of graph with boundary. Analytically, by defining the discrete p-Laplacian with its associated concepts as p-harmonicity, p-superharmonicity, etc on graphs with or without boundary. We also define the space of classes of functions of finite p-energy, that is, the p-Dirichlet spaces which perform a key role in this thesis.

In **Chapter 2** we extend the fundamental concepts of p-potential theory such as min/max principle, p-Harnack inequality, comparison principle to the categories of locally-finite infinite graphs with or without boundary. Also the solvability of p-Dirichlet problem for finite graphs with boundary is presented.

In **Chapter 3** we define p-capacity on graphs and examine their basic properties. We then define the notions of p-hyperbolicity and p-parabolicity of a graph in terms of p-capacity and proceed to give important examples of p-hyperbolic and p-parabolic graphs by given an explicit formulas to compute their capacities. Also a notion of isomorphism is developed which allows to have a natural notion of equivalence in this category.

In **Chapter 4** we show that the variational p-capacity can be represented through a unique p-superharmonic function with special properties which is called p-capacitory function. Importantly, we classify all finite graphs without boundary as p-parabolic and all graphs (finite or infinite) with boundary as p-hyperbolic. Finally, stronger versions of theorems like Green's theorems to p-Dirichlet spaces are proved.

In **Chapter 5** we give several characterizations of p-hyperbolic graphs. A key theorem, the Kelvin-Nevanlinna-Royden criterion is proved and some of its applications are given. Also, it is shown the existence of classes with bounded functions and finite p-Dirichlet energy on p-hyperbolic graphs that do not satisfy the divergence formula.

In **Chapter 6** we give a characterization of p-parabolic graphs in terms of non-existence of positive p-superharmonic functions and prove that positive or with finite p-energy p-harmonic functions must be constant.

In **Chapter 7** we prove that a p-Poisson equation with finite support source function on p-hyperbolic graph has solution, and it is unique. Also, we prove criteria that establish equivalences between the ambient conditions and existence/non-existence of solutions of p-Poisson equation with source of finite support.

Finally, in the **appendix**, we extend the divergence formula and its consequences, namely, convergence theorem and analogues of first, second Green's formulas which play a prominent role in this thesis. For the last, we generalize the Gauss' formula for graphs.

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Manhattan, NY - GC December 2003 Lúcio M-G Prado

To my daughter Emily Prado.

Contents

INTRODUCTION

In this thesis our main interest lies in the geometric-analysis on infinite graphs, using the machinery provided by functional analysis on L_p spaces and on p-Dirichlet spaces, all, of course, adapted to graphs. As main tools, we used p-capacity and p-Laplacian combined with p-energy formulas to classify the graphs and to examine their geometric-analytic properties. We seek to transfer concepts and notations from potential theory on Riemannian manifolds to graphs in such a way that we can easily make appropriate analogies.

Now, we proceed to give a brief historic view of the development of the subject.

As written in P. Soardi [**17**]:

" While there is a large body of mathematical literature devoted to finite electrical networks, infinite networks have received growing attention only in the last two decades. The reason for this attention is probably the increasing interest for the discrete methods in all branches of mathematics. Actually, there is strong indication that, at least from the point of view of potential theory, infinite networks are discrete model of noncompact Riemann manifolds".

The discrete Laplacian on square lattice has been studied by many authors. It was a source of motivation for the study of similar operators on arbitrary graphs. For the discrete Laplacian on graphs many properties of smooth Laplacian still hold, e.g., maximum (minimum) principle, Harnack inequality, Cheeger lower bound for the first eigenvalue, etc. Moreover, the discrete Laplacian governs random walks like smooth Laplacian governs Brownian motion. Thus, there is a natural interplay between *discrete stochastic processes* (Markov chains, martingales, recurrence/transience phenomena) and the *properties of discrete Laplacian*, (for example, existence of non-constant bounded (positive) harmonic functions with finite Dirichlet sum).

On the other hand, the study of "type problem" on noncompact Riemann surfaces makes use of graphs to answer questions about the existence of certain kinds of functions with prescribed properties (e.g., bounded harmonic functions with finite energy).

It is worth pointing out that Kanai [**11**] also gave a different way to associate a combinatorial structure (ϵ-nets) to Riemannian manifolds with bounded geometry; whose connection between the structures is established through a map called rough isometry which preserves the large scale geometry but not the local geometry or topology.

We remark that the theory of discrete Laplacians remains a very active field of study and a substantial part of it has already appeared in Soardi's book [**17**].

We also mention the earlier works of authors as Yamasaki [**16**],[**21**],[**22**] and Maeda [**14**] that address some similar problems, in network context, with different notations, hypotheses, definitions, techniques and proofs that are presented here.

Regarding Kelvin-Nevanlinna-Royden criterion, it is worth mentioning the earlier work of D.DeBaun [**2**] who proved, for $p = 2$, several similar results on triangulated surfaces by using cohomological methods. Later on D. Sullivan and T. Lyons mentioned the same criterion for noncompact manifolds, still for $p = 2$, on [**13**]. By the way, Sullivan and Lyons were the first to use the reference term Kelvin-Nevanlinna-Royden to name the criterion.

More recently, progress on the nonlinear potential theory ($p \neq 2$) has been made generalizing results from Euclidean spaces and compact manifolds to the setting of noncompact Riemann manifolds. In particular, Troyanov & Gold'shtein in [**7**] proved the Kelvin-Nevanlinna-Royden criterion for Riemannian manifolds. Also Troyanov [**19**] proved the existence of solutions of the p-Poisson equation on p-hyperbolic manifolds. An extensive account of p-parabolicity and p-hyperbolicity on Riemannian manifolds is given in [**18**].

Chapter 1

PRELIMINARIES

In this chapter, we present definitions, terminology, and collect facts that we will need in the sequel. A few constructions will be introduced later when appropriate. Some terminology, definitions, and theorems are new in the context of graph theory.

1.1 Basic graph-theoretic terms

Definition: A *simple graph* $G = (V, E)$ consists of a (nonempty) set V, whose elements are called *vertices* (or points, or nodes) and a (nonempty) set E of unordered pairs of distinct elements of V denoted by xy called *edges* or *links* with endpoints x and y. Thus, V is the vertex set and E the edge set of G.

A graph with vertex set V is said to be a graph *on* V.

Remark: For sake of completeness, a *empty graph* is one with no edges.

We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $x \in G$ (rather than $x \in V$), an edge $e \in G$, and so on.

Some terminology shall be used frequently. The endpoints x and y or simply ends of an edge are said to be *incident* with the edge, write $x \sim y$, and vice versa. Two vertices which are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex.

Definition: The *degree* or *valence* $m(x)$ of a vertex x in G is the number of edges of G incident with x.

Definition: If $m(x) < \infty$ for all $x \in V$, then G is called *locally finite.*

Assumption: We shall always assume tacitly that G is *countable*, *simple* (no loops and multiple edges), and *locally finite*, unless state otherwise.

Definitions:

- The *union* of graphs $G \cup G' := (V \cup V', E \cup E')$.
- The *intersection* of graphs $G \cap G' := (V \cap V', E \cap E')$.

  In this case, $V \cap V'$ must be nonempty.

  G and G' are disjoint if they have no vertex in common.

Definitions: A *subgraph* of $G = (V, E)$ given by U is a graph $G' = (U, E')$ such that $U \subseteq V$ and $E' \subseteq E$ which will sometimes be abbreviated as $G' \subseteq G$.

A *spanning subgraph* of G is a subgraph G' with $U = V$.

A subgraph $G' \subseteq G$ is called *induced* by U if every edge xy in E whose endpoints belong to U is in E' and we write $G[U]$ or, sometimes, simply F.

Definition: An *exhaustion* of G is a sequence of finite subgraphs $G_n = (V_n, E_n)$ $(n = 1, 2, 3, \ldots)$ such that $G_n \subseteq G_{n+1}$ and $G = \cup_{n=1}^{\infty} G_n$.

Definition: $G - U$ is the induced subgraph $G[V \setminus U]$, where U is any set of vertices (usually of G). In other words, $G - U$ is obtained by deleting all vertices in $U \cap V$ and their incident edges.

In particular, when U is the set of vertices of a subgraph of G' instead of $G - U$ we simply write $G - G'$.

Definitions: A *walk* in G is a *finite* non-null sequence $W = x_0 e_1 x_1 e_2 x_2 \ldots e_k x_k$ whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are x_{i-1} and x_i.

If the edges $e_1, e_2, \ldots, e_k$ of a walk W are distinct, W is called a *trail.* If, in addition, the vertices $x_0, x_1, \ldots, x_k$ are distinct, W is called a *path.* The *length* of this is k.

A closed trail whose the origin and internal vertices are distinct is a *cycle.*

Remark: The above definitions can be generalized in the way to include infinite walk, trail and path.

Definition: The graph G is said to be *connected* if any two vertices are joined by a path.

Now, we introduce a important class of graphs called Trees.

Tree An *acyclic* connected graph, one not containing any cycle, is called a *tree* which we denote by T. Vertices of degree 1 in a tree are its *leaves.* A special kind of tree is the *homogenous tree* which has all vertices have the same degree, say d. We denote it by T_d.

We now introduce the notion of a graph with boundary. We preferred to make more descriptive definitions to avoid technicalities of the standard graph notations.

Suppose that $G = (V, E)$ is a proper subgraph of a graph H, then the set of vertices of H that are adjacent to vertices of G but are not in G is called *exterior boundary* $\partial^e V$; the set of vertices in G adjacent to the vertices of $\partial^e V$ is called *interior boundary* $\partial^i V$.

Incidently, in this context a special nomenclature for V is used, namely, *interior.*

In the following, we define the *boundary graph* ∂G as well as its components, namely, the *tangent boundary* $\partial_T G$ and the *normal boundary* $\partial_N G$.

Definitions: Let $G = (V, E)$ be a subgraph of a graph H. Then,

- the *boundary graph* ∂G or simply the *boundary* is the spanning subgraph obtained from $H[\partial^i V \cup \partial^e V]$ by deleting those edges with *both endpoints* in $\partial^i V$.
- the *normal boundary graph* $\partial_N G$ or simply *normal boundary* is the bipartite subgraph of ∂G induced by the partition $(\partial^i V, \partial^e V)$, that is, $\partial G[(\partial^i V, \partial^e V)]$.
- the *tangent boundary graph* $\partial_T G$ or simply *tangent boundary* is $\partial G - \partial^i V$.

So $\partial G = \partial_T G \cup \partial_N G$.

Remark: Notice that, eventually, the tangent boundary is an empty graph.

Definition: The *graph with boundary* is the graph given by $G \cup \partial G$.

Definition: A *subgraph with boundary* $G' \cup \partial G'$ is a graph with boundary obtained from either G or $G \cup \partial G$. In particular, one can have $G' \subseteq G$ and $\partial G' \subseteq \partial G$.

Now, we introduce the concept of directed edges which will allow us first to define the geometric concept of orientation of a graph and hence, to draw a representative of a flow [1.2.3]; or, analytically, to write and simplify many of our formulas.

Definition: Let $G = (V, E)$ be a graph. For each edge $xy \in E$, we define two *directed oriented edges* $[x, y]$ and $[y, x]$, respectively, from x to y and from y to x. The vertex x is the *initial* point or *tail* and the vertex y the *terminal* point or *head* of $[x, y]$.

Definition: $\vec{E}$ is the set of *all* directed oriented edges $[x, y]$ obtained from E such that $[x, y] \in \vec{E}$ if and only if $[y, x] \in \vec{E}$.

Definition: A subset $\vec{X}$ of $\vec{E}$ such that, for all $x, y \in V$ $x \neq y$, $[x, y] \in \vec{X}$ if and only if $[y, x] \notin \vec{X}$ is called an *orientation* of G.

The notation $\vec{E}(x)$ means the set of all oriented edges $[x, y]$ emanating from x which are in $\vec{E}$.

Definitions: An *orientation* of a graph with boundary $G \cup \partial G$ will be always understood as G with orientation $\vec{X}$ given above; ∂G with orientation $\partial\vec{X}$, namely, $\partial\vec{X} = \partial_T\vec{X} \cup \partial_N\vec{X}$; where $\partial_N\vec{X}$ is the *outer orientation* of $\partial_N G$ and $\partial_T\vec{X}$ is an orientation of $\partial_T G$.

Definition: A graph (perhaps without boundary) with a particular orientation will be called *oriented graph.*

1.2 p-Laplacian on Graphs

In this section, we define certain real vector spaces (some of them are Banach spaces) as well as linear and nonlinear operators on them which will play important roles in the development of the theory.

Definition: Let $G = (V, E)$ be a graph. We denote by $C^0(V)$ the real vector space of all real-valued 0-*cochains*, i.e., real-valued functions u on vertices of G, $u : V \longrightarrow \mathbb{R}$.

For $p \in [1, \infty)$,

$$L^p(V) = \{u \in C^0(V) : \|u\|_p^p = \sum_{x \in V} |u(x)|^p < \infty\}. \tag{1.2.1}$$

For $p = \infty$,

$$L^\infty(V) = \{u \in C^0(V) : \|u\|_\infty = \sup_{x \in V} |u(x)| < \infty\}. \tag{1.2.2}$$

Remark: For $p \in [1, \infty)$. Let $C_0(V)$ be the vector subspace of $C^0(V)$ consisting of functions on vertices of G with *finite support.* It is a standard result in the theory of real variables that $C_0(V)$ is dense in $L^p(V)$.

Another important real normed space is the space of flows in G

Definitions: Let $G = (V, E)$ be a graph . Then the *space of flows* in G is

$$\Phi(\vec{E}) = \{\jmath : \vec{E} \longrightarrow \mathbb{R} : \jmath([x, y]) = -\jmath([y, x]), \forall\, [x, y] \in \vec{E}\}. \tag{1.2.3}$$

$\Phi(\vec{E})$ contains the subspaces $L^p(\vec{E})$ called spaces of the p-flows, namely, for $p \in [1, \infty)$

$$L^p(\vec{E}) = \{\jmath \in \Phi(\vec{E}) : \|\jmath\|_p^p = \frac{1}{2} \sum_{x \in V} \sum_{e \in \vec{E}(x)} |\jmath(e)|^p < \infty\}, \tag{1.2.4}$$

and for $p = \infty$

$$L^{\infty}(\vec{E}) = \{\jmath \in \Phi(\vec{E}) : \|\jmath\|_{\infty} = \sup_{x \in V}(\sup_{e \in \vec{E}(x)} |\jmath(e)|) < \infty\}. \tag{1.2.5}$$

Remarks:

- For $p \in [1, \infty]$, observe that the definition of the norm on $L^p(\vec{E})$ is independent of the choice of the representative determined by an orientation. Actually, if we would have chosen an orientation $\vec{X}$ for G, we would avoid $\frac{1}{2}$ on the definition of the norm.

- For $p \in [1, \infty]$, the spaces $L^p(V)$ and $L^p(\vec{E})$ are *Banach Spaces* by Riesz-Fisher Theorem.

- If $G \cup \partial G = (U \cup \partial U, \underbrace{E \cup \partial E}_{\vec{E}'})$ is a graph with boundary, then we write $L^p(U \cup \partial U)$ and $L^p(\vec{E} \cup \partial \vec{E})$.

Definition: The *divergence* operator div of a flow $\jmath$ is the linear operator given by

$$\operatorname{div} : \Phi(\vec{E}) \longrightarrow C^0(V)$$

$$\operatorname{div}(\jmath)(x) = \sum_{e \in \vec{E}(x)} \jmath(e). \tag{1.2.6}$$

Some terminology. If $\mathrm{div}(\jmath) \equiv 0$, then $\jmath$ is *divergence free.* A flow has *source* x_0 and *sink* y_0 when $\mathrm{div}(\jmath)(z) = 0$ for $z \neq x_0, y_0$, and $\mathrm{div}(\jmath)(x_0) > 0 > \mathrm{div}(\jmath)(y_0)$.

We now recall the definition of the coboundary operator.

Definition: The *coboundary* operator d is the linear operator given by

$$\mathrm{d}: C^0(V) \longrightarrow \Phi(\vec{E})$$

$$\mathrm{d}u([x,y]) = u(y) - u(x). \tag{1.2.7}$$

We now introduce the gradient as an important notation in this context to carry out the analogy with p-Laplacian on a Riemannian manifold.

The *gradient* ∇ is the linear operator that coincide with d, namely, for every $u \in C^0(V)$

$$\nabla u(e) = \mathrm{d}u(e), \quad e \in \vec{E}.$$

Actually, ∇u can be interpreted geometrically (or physically) as special flow on G generated by u.

We briefly review the definition of the p-Laplacian on a Riemannian manifold.

Let $(M, <,>)$ be a non-compact oriented n-dimensional ($n \geq 2$) Riemannian manifold of class C^∞.

The p-Laplacian $\Delta_p u$ of a C^2 function $u : M \longrightarrow \mathbb{R}$ is given by

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u), \quad 1 \leq p < \infty.$$

Actually, Δ_p is the Euler-Lagrange operator associated with the functional $u \longmapsto \int_M |du|^p$.

Remark: The book [8] is a good reference for p-Laplacian and related subjects in the continuous setting.

Along similar lines of the continuous settings, one can define the p-Laplacian on $G \cup \partial G$.

Definition: Let $G \cup \partial G$ be a graph with boundary, and let $u : U \cup \partial U \longrightarrow \mathbb{R}$ be a function. For each $p \in [1, \infty)$ the *discrete p-Laplacian of u* is the real function

$$\Delta_p u(x) = \Big[\mathrm{div}(|\nabla u|^{p-2}\nabla u)\Big](x), \quad \forall x \in U. \tag{1.2.8}$$

We now derive an explicit formula for $\Delta_p u$.

For each $x \in U$ and $p \geq 2$,

$$\begin{aligned}\Delta_p u(x) &= \mathrm{div}(|\nabla u|^{p-2}\nabla u)(x)\\ &= \sum_{[x,y]\in \vec{E}'(x)} (|\nabla u|^{p-2}\nabla u)([x,y])\\ &= \sum_{[x,y]\in \vec{E}'(x)} |\nabla u[x,y]|^{p-2}\nabla u([x,y])\\ &= \sum_{[x,y]\in \vec{E}'(x)} |u(y)-u(x)|^{p-2}\Big(u(y)-u(x)\Big).\end{aligned}$$

That is,

$$\Delta_p u(x) = \sum_{[x,y]\in \vec{E}'(x)} |u(y)-u(x)|^{p-2}\Big(u(y)-u(x)\Big). \tag{1.2.9}$$

For each $x \in U$ and $1 \leq p < 2$, we can deduce the same formula above

$$\Delta_p u(x) = \sum_{[x,y] \in \vec{E}'(x)} |u(y) - u(x)|^{p-2} \Big(u(y) - u(x)\Big); \tag{1.2.10}$$

where, we interpret $|u(y) - u(x)|^{p-2}\Big(u(y) - u(x)\Big)$ as equal to zero when $u(y) = u(x)$.

$\vec{E}'(x)$ denote all directed edges of $G \cup \partial G$ emanating from $x \in U$ with end-points in $U \cup \partial U$.

Remarks:

- If $x \in U$ is an isolated point, then we define the p-Laplacian at x as zero.

- We observe that the discrete p-Laplacian can be extended, naturally, to graphs without boundary $G = (V, E)$ with obvious adaptations.

1.3 p-Harmonic Functions on Graphs

For p-Laplacian, as in the continuous setting, we have automatically the fundamental concepts of p-harmonicity, p-superharmonicity, and p-subharmonicity on graphs.

Definition: Let $p \in [1, \infty)$, and let $G \cup \partial G$ be a graph with boundary. Then $u : U \cup \partial U \longrightarrow \mathbb{R}$ is *p-harmonic function* on G (or on U) if for every $x \in U$

$$\Delta_p u(x) = \sum_{[x,y] \in \vec{E}'(x)} |u(y) - u(x)|^{p-2} \Big(u(y) - u(x) \Big) = 0.$$

$\vec{E}'(x)$ denote all directed edges of $G \cup \partial G$ emanating from $x \in U$ with endpoints in $U \cup \partial U$.

In particular, for graphs without boundary $G = (V, E)$, u is simply called *p-harmonic.*

Definition: Let $p \in [1, \infty)$, and let $G \cup \partial G$ be a graph with boundary. Then $u : U \cup \partial U \longrightarrow \mathbb{R}$ is *p-superharmonic*(respectively, *p-subharmonic*)function on G (or on U) if for every $x \in U$

$$\Delta_p u(x) = \sum_{[x,y] \in \vec{E}'(x)} |u(y) - u(x)|^{p-2} \Big(u(y) - u(x) \Big) \leq 0 \quad \text{(respectively} \geq 0);$$

$\vec{E}'(x)$ denote all directed edges of $G \cup \partial G$ emanating from $x \in U$ with endpoints in $U \cup \partial U$.

In particular, for graphs without boundary $G = (V, E)$, u is simply called *p-superharmonic (*respectively, *p-subharmonic)* function.

1.4 p-Dirichlet Spaces on Graphs

We now introduce a very important class of Banach spaces $\mathcal{L}^{1,p}(U \cup \partial U)$ and $\mathcal{L}_0^{1,p}(U \cup \partial U)$ called p-Dirichlet spaces which shall be used largely throughout the thesis.

Definition: Let for $p \in [1, \infty]$, and let $G \cup \partial G$ be a graph with boundary. Then the *p-Dirichlet energy density* at $x \in U \cup \partial U$ induced by a function $u : U \cup \partial U \longrightarrow \mathbb{R}$ is

$$D_p u(x) = \sum_{[x,y] \in \vec{E}'(x)} |\nabla u([x,y])|^p; \tag{1.4.1}$$

in particular, for $p = \infty$

$$D_\infty u(x) = \sup_{[x,y] \in \vec{E}'(x)} |\nabla u([x,y])|,$$

where $\vec{E}'(x)$ denotes all directed edges of $G \cup \partial G$ emanating from $x \in U \cup \partial U$ with endpoints on $U \cup \partial^i U$.

Definition: Let for $p \in [1, \infty]$, and let $\Omega = G \cup \partial G$ be a graph with boundary. Then the *p-Dirichlet energy* of $u : U \cup \partial U \longrightarrow \mathbb{R}$ on Ω (or on $U \cup \partial U$) is

$$I_p(u, \Omega) = \frac{1}{2} \sum_{x \in U \cup \partial U} D_p u(x) \leq \infty; \tag{1.4.2}$$

in particular, for $p = \infty$

$$I_\infty(u, \Omega) = \sup_{x \in U \cup \partial U} D_\infty u(x) \leq \infty.$$

Remarks: For $p \in [1, \infty]$.

- Sometimes, we denote $I_p(u, \Omega)$ by $I_p(u)$ and call it "energy functional".

- When $I_p(u) < \infty$, we also say simply that u has finite p-energy or finite finite p-Dirichlet energy. Sometimes, u is called p-Dirichlet function.
- For *graph without boundary* the definitions of p-Dirichlet energy can be made on similar way.

In the following, without loss of generality, all graphs are considered connected.

Proposition 1.4.3 *For $p \in [1, \infty]$, let $\Omega = G \cup \partial G$ be a graph with boundary. Then,*

$$\ell^{1,p}(U \cup \partial U) = \{u \in C^0(U \cup \partial U) : \ ||u||_{1,p} < \infty\}, \tag{1.4.4}$$

where for $p \in [1, \infty)$, $||u||_{1,p} = I_p(u, \Omega)^{\frac{1}{p}}$ and for $p = \infty$, $||u||_{1,\infty} = I_\infty(u, \Omega)$ are seminormed vector spaces.

Proof. It is enough to observe that $||.||_{1,p} = ||.||_p \circ \nabla$ where ∇ is one linear operator and $||.||_p$ is a seminorm. ∎

Observe that $||u||_{1,p} = 0$ if and only if u =constant, consequently, to make $||.||_{1,p}$ a norm, we define, the equivalence relation in $\ell^{1,p}(U \cup \partial U)$ by

$$u \sim v \quad \text{if and only if} \quad u - v = \text{constant on } U \cup \partial U.$$

It is trivial to verify that $\sim$ is an equivalence relation, which is compatible with the operations of addition, and multiplication by real numbers. Consequently, by denoting the equivalence classes by $[u]$, we have the following definition.

Definition: Let $p \in [1, \infty]$, and let $\Omega = G \cup \partial G$ be a graph with boundary. The p-Dirichlet spaces on Ω (or $U \cup \partial U$) are the real normed spaces of equivalence classes

$$\mathcal{L}^{1,p}(U \cup \partial U) = \ell^{1,p}(U \cup \partial U)/\mathbb{R}$$

whose norms are $\|[u]\|_{1,p} = \|u\|_{1,p}$.

Remarks:

- If $v \in [u]$, then $u - v = c$ on $U \cup \partial U$. Hence $I_p(u) = I_p(c+v) = I_p(v)$, and, therefore, the norm is independent of the choice of representative. As usual, we omit the class notation $[\ .\ ]$ and write $\|u\|_{1,p}$ for $\|[u]\|_{1,p}$.

- Strictly, the elements of the spaces $\mathcal{L}^{1,p}(U \cup \partial U)$ or $\mathcal{L}_0^{1,p}(U \cup \partial U)$ are not functions but classes of functions such that in each nontrivial class any two functions differ by a constant. Since any two functions have the same norm, and the same energy over each subgraph, the distinction is not important for many purposes.

- Sometimes, we will write $u \in \mathcal{L}^{1,p}(U \cup \partial U)$ or $\mathcal{L}_0^{1,p}(U \cup \partial U)$ as an abbreviation for: u is function on the vertices contained in $U \cup \partial U$ with finite p-energy norm.

We now introduce the notion of uniformly rotund spaces which was first established by James Clarkson.

Definition: A normed space X is *uniformly rotund*, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if u, $v \in S_X$ and $\|u-v\| \geq \epsilon$, then $\|\frac{1}{2}(u+v)\| \leq 1-\delta$.

One very important class of uniformly rotund Banach spaces is that of the spaces $L^p(\Omega, \Sigma, \mu)$, where μ is a positive measure on the σ-algebra Σ of subsets of a set Ω and $p \in (1, \infty)$. For a proof see theorem in **[15]** [5.2.11]. With adequate adaptations, it follows that for $p \in (1, \infty)$ the spaces $L^p(\vec{E} \cup \partial \vec{E})$ are uniformly rotund Banach spaces. Incidently, it follows from Milman-Pettis theorem in **[15]**[5.2.15] that $L^p(\vec{E} \cup \partial \vec{E})$ are also reflexive spaces.

Theorem 1.4.5 *Suppose* $p \in (1, +\infty)$. *Then* $(\mathcal{L}^{1,p}(U \cup \partial U), \|\,.\,\|_{1,p})$ *is a uniformly rotund Banach space.*

Proof. Consider

$$\mathcal{L}^{1,p}(U \cup \partial U) \xrightarrow{\nabla} L^p(\vec{E} \cup \partial \vec{E}).$$

Then ∇ is an isometric isomorphism between $(\mathcal{L}^{1,p}(U \cup \partial U), \|\,.\,\|_{1,p})$ and a closed subspace of $L^p(\vec{E} \cup \partial \vec{E})$. So

a) $(\mathcal{L}^{1,p}(U \cup \partial U)$ is a Banach space.

b) Since subspace of a uniformly rotund space is uniformly rotund, see **[15]** [5.2.21] and uniformly rotundity is preserved under isometric isomorphism **[15]** [5.2.21], it follows that $\mathcal{L}^{1,p}(U \cup \partial U)$ is a uniformly rotund Banach space. ∎

A very important subspace of $\mathcal{L}^{1,p}(U \cup \partial U)$ to be considered in this context are $\mathcal{L}_0^{1,p}(U \cup \partial U)$, namely, the closure in $\mathcal{L}^{1,p}$-norm of the equivalence classes of $C_0(U \cup \partial U)$ in $\mathcal{L}^{1,p}(U \cup \partial U)$.

Thus, by [**15**] [5.2.21] $\mathcal{L}_0^{1,p}(U \cup \partial U)$ are uniformly rotund Banach spaces and reflexives.

The next theorem has more general scope. However, we have adapted it to our context.

Theorem 1.4.6 *For $p \in (1, +\infty)$, let Λ be a nonempty convex subset of $\mathcal{L}^{1,p}(U \cup \partial U)$ (or $\mathcal{L}_0^{1,p}(U \cup \partial U)$). Then,*

1. *there exists a unique element of $\overline{\Lambda}$ (closure of Λ in the $\mathcal{L}^{1,p}$-norm) with least norm;*

2. *any sequence $\{u_i\}_{i \geq 1}$ in Λ that minimizes the least norm is Cauchy.*

Proof. 1.) follows from [**15**] [5.2.17] and 2.) follows from [**15**] [5.3.20]. ∎

1.5 Euler-Lagrange Operator of $I_p(\cdot)$

Proposition 1.5.1 *Let $p \in [1, \infty)$, and let $G \cup \partial G$ be a graph with boundary. Then the p-Laplacians Δ_p are the Euler-Lagrange operators for the "energy functionals" $I_p(\cdot)$.*

Proof. Let u be a real-valued function on $U \cup \partial U$ satisfying $I_p(u) < \infty$, and x_0 be an arbitrary point of U. Consider the Dirac measure concentrated at $x = x_0$, that is,

$$\delta_{x_0}(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise.} \end{cases}$$

For each $x_0 \in U$, we define a function $u_{x_0}(x,t) = u(x) + t\delta_{x_0}(x)$ on $U \times (-\epsilon, \epsilon)$.

As $I_p(u) < \infty$ and $I_p(u_{x_0}(.,t)) < \infty$ differ only in a finite number of terms corresponding to x_0 and its neighbours, then $I_p(u_{x_0}(.,t)) < \infty$, $t \in (-\epsilon, \epsilon)$, that is, $I_p(u_{x_0}(.,t))$ is absolutely convergent on $(-\epsilon, \epsilon)$. So we can rearrange its terms corresponding to x_0 and its neighbours first, and then apply $\frac{d}{dt}$.

Indeed,

$$\begin{aligned}
\frac{d}{dt}\big(I_p(u_{x_0}(x,t))\big)\big|_{t=0} &= \frac{1}{2}\frac{d}{dt}\Bigg(\sum_{x\in V(x_0)}\sum_{[x,y]}|u_{x_0}(y,t) - u_{x_0}(x,t)|^p + \\
&\qquad \sum_{x\in V\setminus V(x_0)}\sum_{[x,y]}|u_{x_0}(y,t) - u_{x_0}(x,t)|^p\Bigg)\Big|_{t=0} \\
&= \frac{1}{2}\Bigg(\sum_{x\in V(x_0)}\sum_{[x,y]}\frac{d}{dt}|\big(u(y) - u(x)\big) + t(\delta_{x_0}(y) - \delta_{x_0}(x))|^p\Bigg)\Big|_{t=0} \\
&= \frac{p}{2}\Bigg(\sum_{x\in V(x_0)}\sum_{[x,y]}|u(y) - u(x)|^{p-2}(u(y) - u(x))(\delta_{x_0}(y) - \delta_{x_0}(x))\Bigg) \\
&= -p\sum_{[x_0,y]}|u(y) - u(x_0)|^{p-2}\big(u(y) - u(x_0)\big) \\
&= -p\Delta_p u(x_0).
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\Big(I_p(u_{x_0}(x,t))\Big)\Big|_{t=0} = -p\Delta_p u(x_0).$$

As x_0 was taken arbitrary in U, so the result holds for $G \cup \partial G$. ■

Chapter 2

p-SUPERHARMONIC FUNCTIONS

2.1 Min/ Max Principles

In this chapter, we present maximum and minimum principles (local and global), the p-Harnack inequality and some consequences.

In the following, $p \in [1, \infty)$ and $G \cup \partial G$ is a graph with boundary unless stated otherwise.

Proposition 2.1.1 (Local Minimum Principle) *Suppose $u \in C^0(U \cup \partial U)$ is p-superharmonic at $x \in U$. If for every neighbouring vertex y, $y \sim x$, $u(y) \geq u(x)$, then $u(y) = u(x)$ for all y, $y \sim x$.*

Proof. Suppose that u has a minimum at x and $u(y_0) \neq u(x)$ for $y_0 \sim x$. Then $u(y_0) > u(x)$ so that

$$0 < |u(y_0) - u(x)|^{p-2}(u(y_0) - u(x)) \leq \sum_{y \sim x} |u(y) - u(x)|^{p-2}(u(y) - u(x)) \leq 0.$$

This is a contradiction and, therefore $u(y) = u(x)$ for all $y \sim x$. ∎

Remark: Similarly a Local Maximum Principle holds for all p-subharmonic functions $u \in C^0(U \cup \partial U)$.

Theorem 2.1.2 (Global Minimum Principle) *Let $G \cup \partial G$ be a connected graph with boundary, and let $u \in C^0(U \cup \partial U)$ be a p-superharmonic on U. If u takes its minimum at a point of U, then u is a constant function.*

Proof. Suppose that there exists a point $x \in U$ such that $u(x) = m$ is the minimum of u on U. By Proposition [2.1.1] $u(y) = m$ for all neighbours y of x. Take an arbitrary vertex $z \in \partial U$, then by connectedness of $G \cup \partial G$ there exists a path with vertices $x_0 = x, x_1, ..., z = x_n$ of vertices on $U \cup \partial U$ such that $x_j \sim x_{j+1}$, $j = 0, 1, ..., n-1$, and x_j is in U for $0 \leq j < n$. Then $m = u(x_0) = u(x_1) = ... = u(x_n)$. ∎

Remarks:

- The Global Minimum Principle holds, with same proof given above, for a connected graph without boundary.

- It is immediate to check that the Global Maximum Principle holds for p-subharmonic functions on graphs with or without boundaries.

Corollary 2.1.3 *If G is finite graph without boundary, then every p-superharmonic (p-subharmonic) function on G is constant.*

Proof. Combine the first remark above with the fact that the functions assume minimum in G. ■

Theorem 2.1.4 *Let $G \cup \partial G$ be a connected graph. If $u \in C^0(U \cup \partial U)$ is p-harmonic and nonconstant on U, then u attains its maximum and minimum on ∂U.*

Proof. The proof follows, immediately, from the global minimum and the global maximum principles. ■

2.2 p-Harnack Inequality

We open this section by proving an important theorem for graphs which generalizes the linear case in [**4**, p.789].

Theorem 2.2.1 (p-Harnack Inequality) *Suppose that $p \in (1, \infty)$, $x \sim y$, and $u \in C^0(U \cup \partial U)$ is nonnegative function.*

If u is p-superharmonic at both vertices x and y, then

$$\frac{1}{\left(m(y)^{\frac{1}{p-1}} + 1\right)} u(x) \leq u(y) \leq \left(m(x)^{\frac{1}{p-1}} + 1\right) u(x).$$

Proof. 1) If $u(x) = 0$ (or $u(y) = 0$), then by the minimum principle $u(y) = 0$ (or $u(x) = 0$) for every y (or x), $y \sim x$, the Harnack inequality is trivially true.

2) If $u(x) > 0$, then

$$\Delta_p u(x) = \sum_{y\sim x} |u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big) = \sum_{y\sim x} \operatorname{sgn}\big(u(y)-u(x)\big)|u(y)-u(x)|^{p-1}.$$

We rewrite the sum above as

$$\sum_{\substack{y\sim x\\ u(y)>u(x)}} \big(u(y)-u(x)\big)^{p-1} - \sum_{\substack{y\sim x\\ u(y)<u(x)}} \big(u(y)-u(x)\big)^{p-1} \le 0.$$

Thus

$$\sum_{\substack{y\sim x\\ u(y)>u(x)}} \big(u(y)-u(x)\big)^{p-1} \le \sum_{\substack{y\sim x\\ u(y)<u(x)}} \big(u(x)-u(y)\big)^{p-1}.$$

Dividing by $u(x) > 0$, we obtain

$$\sum_{\substack{y\sim x\\ u(y)>u(x)}} \left(\frac{u(y)}{u(x)} - 1\right)^{p-1} \le \sum_{\substack{y\sim x\\ u(y)<u(x)}} \left(1 - \frac{u(y)}{u(x)}\right)^{p-1} \le m(x). \qquad (2.2.2)$$

Now, we have two cases to consider to complete the proof.

(i) For $u(y) \ge u(x)$ we have from (2.2.2) above

$$\frac{u(y)}{u(x)} \le \big(m(x)^{\frac{1}{p-1}} + 1\big)$$

or

$$u(y) \le \big(m(x)^{\frac{1}{p-1}} + 1\big)u(x).$$

Therefore, for $u(y) \geq u(x)$

$$u(x) \leq u(y) \leq \left(m(x)^{\frac{1}{p-1}} + 1\right)u(x). \tag{2.2.3}$$

(ii) For $u(y) < u(x)$, we consider

$$\Delta_p u(y) = \sum_{z \sim y} \operatorname{sgn}\big(u(z) - u(y)\big)|u(z) - u(y)|^{p-1} \leq 0.$$

Since $u(y) > 0$, we have

$$\sum_{\substack{z \sim y \\ u(z) > u(y)}} \big(u(z) - u(y)\big)^{p-1} \leq \sum_{\substack{z \sim y \\ u(z) < u(y)}} \big(u(y) - u(z)\big)^{p-1}$$

$$\sum_{\substack{z \sim y \\ u(z) > u(y)}} \left(\frac{u(z)}{u(y)} - 1\right)^{p-1} \leq \sum_{\substack{z \sim y \\ u(z) < u(y)}} \left(1 - \frac{u(z)}{u(y)}\right)^{p-1} \leq m(y),$$

so

$$\frac{u(z)}{u(y)} - 1 \leq m(y)^{\frac{1}{p-1}}$$

or

$$u(z) \leq \left(m(y)^{\frac{1}{p-1}} + 1\right)u(y).$$

Put $z = x$, which gives

$$u(x) \leq \left(m(y)^{\frac{1}{p-1}} + 1\right)u(y)$$

$$\frac{1}{\left(m(y)^{\frac{1}{p-1}} + 1\right)} u(x) \leq u(y).$$

Therefore, for $u(y) < u(x)$,

$$\frac{1}{\left(m(y)^{\frac{1}{p-1}} + 1\right)} u(x) \leq u(y) \leq u(x). \tag{2.2.4}$$

$$\text{Since} \quad \frac{1}{\left(m(y)^{\frac{1}{p-1}}+1\right)} < 1 \quad \text{and} \quad 1 < \left(m(x)^{\frac{1}{p-1}}+1\right),$$

(2.2.3) and (2.2.4) imply

$$\frac{1}{\left(m(y)^{\frac{1}{p-1}}+1\right)} u(x) \leq u(y) \leq \left(m(x)^{\frac{1}{p-1}}+1\right) u(x). \quad \blacksquare$$

In what follows, we establish Theorem [2.2.5] as an application of our p-Harnack inequality. We note that this is the graph version of a Harnack Principle in the continuous setting.

Theorem 2.2.5 *Suppose that* $p \in (1,\infty)$ *and* $\{v_j\}_{j\geq 1}$ *is a monotonically increasing (decreasing) sequence of* p*-harmonic functions on a connected graph* $G \cup \partial G$.

The function $v(x) = \lim_{j\to\infty} v_j(x)$ *(pointwise sense) is either* p*-harmonic on* U *or identically* $+\infty$ *(resp.* $-\infty$*).*

Proof. If $v_1 \leq v_2 \leq v_3 \leq ...v_j \leq ...$, then for every j, $u_j = v_j - v_1 \geq 0$. Suppose that there exists a vertex $x \in U$ such that $\lim_{j\to\infty} u_j(x) = u(x) < \infty$. Take an arbitrary $y \in U \cup \partial U$. Since $G \cup \partial G$ is connected, there is a (x,y)-path connecting x and y, i.e., $x = z_1 \sim z_2 \sim ... \sim z_k = y$. Then, by repeated application of the p-Harnack inequality along the (x,y)-path, each u_j satisfies

$$u_j(y) \leq \left(\prod_{i=1}^{k-1}\left(m(z_i)^{\frac{1}{p-1}}+1\right)\right) u_j(x)$$

so that

$$\lim_{j \to \infty} u_j(y) = u(y) \quad \text{is finite.}$$

As y is arbitrary, it follows that u is finite on all of $U \cup \partial U$.

Finally, u is p-harmonic on U. In fact, since each u_j is p-harmonic on U, so, for any $y \in U$, $0 = \Delta_p u_j(y) \to \Delta_p u(y)$ as $j \to \infty$ or $\Delta_p u(y) = 0$, that is, u is p-harmonic at all $y \in U$.

For a decreasing sequence $\{v_j\}_{j \geq 1}$, it is enough to consider the sequence $\{-v_j\}_{j \geq 1}$ and, then, apply the result above. ∎

Remarks:

- If we consider either a monotone sequence of p-superharmonic or p-subharmonic functions, then the conclusion is similar, that is, the limit function is either infinite or, p-superharmonic or p-subharmonic, respectively.

- We point out that the results above hold for graphs without boundaries.

Proposition 2.2.6 *Let $(G_i = (U_i, E_i))_{i \geq 1}$ be sequence of finite connected graphs such that $G_1 \subset G_2 \subset \ldots \subset G_i \ldots \subset \ldots G' = \cup_i G_i$. Let $\{u_i\}_{i \geq 1}$ be a sequence of functions such that $u(x) = \lim_{i \to \infty} u_i(x)$ exists for every $x \in G'$.*

If $\{u_i\}_{i \geq 1}$ is a sequence of p-harmonic (p-suharmonic, p-superharmonic) functions in G_i, then u is p-harmonic (resp., p-subharmonic, p-superharmonic) in G'.

Proof. We give the proof for p-harmonic functions. For p-subharmonic and p-superharmonic functions, the proofs are similar.

Take $x \in U := \cup U_i$, then there exists i_0 such that for all i, $i \geq i_0$, $x \in U_i$; but the u_i's are p-harmonics on U_i, consequently,

$$\Delta_p u_i(x) = \sum_{[x,y]\in \vec{E}_i} |u_i(y) - u_i(x)|^{p-2}\big(u_i(y) - u_i(x)\big) = 0$$

for all $i \geq i_0$.

Letting $i \to \infty$, we conclude that $\Delta_p u(x) = 0$; but x is arbitrary, therefore u is p-harmonic on G'. ∎

2.3 Comparison Principle

Definition: The real vector space of *positive test* functions is given by

$$\mathcal{D}_0^+(U) = \{\omega \in C_0(U \cup \partial U) : \omega \geq 0 \text{ on } U \text{ and } w \equiv 0 \text{ on } \partial U\}.$$

Lemma 2.3.1 *Let $G \cup \partial G$ be a oriented connected graph with boundary. Then, $u : U \cup \partial U \to \mathbb{R}$ is p-superharmonic (respectively, p-subharmonic) on U if and only if*

$$\sum_{x\in U} \sum_{[x,y]\in \vec{X}} (\nabla\omega\nabla u)|\nabla u|^{p-2} \geq 0 \textit{ (resp., } \leq 0), \quad \textit{for all } \omega \in \mathcal{D}_0^+(U).$$

Proof. By Theorem [A.1.1] in Appendix A, we have for all $\omega \in \mathcal{D}_0^+(U)$

$$\sum_{x\in U}\sum_{[x,y]\in\vec{X}}(\nabla\omega\nabla u)|\nabla u|^{p-2} = -\sum_{x\in U}\omega(x)\Delta_p u(x).$$

Take an arbitrary $x \in U$ and the Dirac measure $\omega = \delta_x$; then, it is immediate that

$$\Delta_p u(x) \leq 0, \quad \text{(respectively, } \Delta_p u(x) \geq 0\text{)},$$

that is, u is p-superharmonic at x. However, x is arbitrary, so u is p-superharmonic on all of Ω.

Conversely, suppose u is p-superharmonic (respectively, p-subharmonic), that is $\Delta_p u(x) \leq 0$ on Ω (respectively, $\Delta_p u(x) \geq 0$). Then, for all $\omega \in \mathcal{D}_0^+(U)$

$$\sum_{x\in U}\sum_{[x,y]\in\vec{X}}(\nabla\omega\nabla u)|\nabla u|^{p-2} = -\sum_{x\in U}\omega(x)\Delta_p u(x) \geq 0 \quad \text{(respectively, } \leq 0\text{)}. \quad \blacksquare$$

Theorem 2.3.2 (Comparison Principle) *Let $G \cup \partial G$ be a connected graph with boundary (non-necessarily finite), $u \in C^0(U \cup \partial U)$ p-subharmonic, and $v \in C^0(U \cup \partial U)$ p-superharmonic.*

If $u \leq v$ on ∂U, then $u \leq v$ on $U \cup \partial U$.

Proof. We consider the following notations

$$A = \{x \in U / u(x) \leq v(x)\}$$

$$B = \{x \in U / u(x) > v(x)\} \quad \text{and}$$

$$A_x = \{y \in U\cup\partial U / y \sim x \;\; \textit{and} \; u(y) \leq v(y)\}$$

$$B_x = \{y \in U\cup\partial U / y \sim x \; \textit{and} \; u(y) > v(y)\}.$$

Let ω be an auxiliary function on $U \cup \partial U$ defined by $\omega = \max(0, u - v)$. Then, $\omega \in \mathcal{D}_0^+(U)$.

Let W be an arbitrary finite subset of U which induces a connected subgraph of Ω. Let $\omega^\sharp = \omega\chi_W$ be a restriction of ω to W, so $\omega^\sharp \in \mathcal{D}_0^+(U)$. By taking a suitable orientation, it follows from the previous lemma that for the p-subharmonic function u

$$\sum_{x \in U} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla u|\nabla u|^{p-2})([x,y]) \leq 0;$$

and for the p-superharmonic function v that

$$\sum_{x \in U} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla v|\nabla v|^{p-2})([x,y]) \geq 0.$$

Thus,

$$\begin{aligned}
0 &\geq \sum_{x \in U} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla u|\nabla u|^{p-2} - \nabla v|\nabla v|^{p-2})([x,y]) \\
&= \sum_{\substack{x \in A \\ y \in A_x}} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla u|\nabla u|^{p-2} - \nabla v|\nabla v|^{p-2})([x,y]) && (S_1) \\
&+ \Big(\sum_{\substack{x \in A \\ y \in B_x}} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla u|\nabla u|^{p-2} - \nabla v|\nabla v|^{p-2})([x,y]) \\
&+ \sum_{\substack{x \in B \\ y \in A_x}} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla u|\nabla u|^{p-2} - \nabla v|\nabla v|^{p-2})([x,y]) \Big) && (S_2) \\
&+ \sum_{\substack{x \in B \\ y \in B_x}} \sum_{[x,y] \in \vec{X}} \nabla\omega^\sharp(\nabla u|\nabla u|^{p-2} - \nabla v|\nabla v|^{p-2})([x,y]) && (S_3).
\end{aligned}$$

For every vertices $x \in W$, we have the cases:

$S_1 = 0$ since if $x \in A$ and $y \in A_x$, then $\omega^\sharp(x) = \omega^\sharp(y) = 0$.

We next consider S_3. If $x \in B$ and $y \in B_x$, then $\omega^\sharp(x) = (u-v)(x)$ and $\omega^\sharp(y) = (u-v)(y)$ so that $\nabla\omega^\sharp = \nabla u - \nabla v$. But the identity $|a|^{p-2}a(a-b) > |b|^{p-2}b(a-b)$ for all real a, b implies immediately that

$$(|\nabla u|^{p-2}\nabla u)(\nabla u - \nabla v) \geq (|\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v) \quad \text{at } [x,y].$$

and we conclude that $S_3 \geq 0$.

We claim that if there were at least one edge $[x,y]$ such that either $x \in A$ and $y \in B_x$ or $x \in B$ and $y \in A_x$, then $S_2 > 0$.

We prove this claim for $x \in B$ and $y \in A_x$, for $x \in A$ and $y \in B_x$ the proof is similar.

Indeed, suppose $x \in W \cap B$ and $y \in A_x$. Then,

$$\nabla\omega^\sharp[x,y] < 0 \quad \text{since}$$

$$\nabla\omega^\sharp[x,y] = \omega^\sharp(y) - \omega^\sharp(x) = -(u(x) - v(x)) = v(x) - u(x) < 0.$$

$$\nabla u < \nabla v \quad \text{since}$$

$$v(x) - u(x) < 0 \leq v(y) - u(y).$$

To simplify, we omit $[x,y]$.

1) Suppose $\nabla v > 0$,

if $\nabla u > 0$, then

$$(|\nabla u|^{p-1} - |\nabla v|^{p-1})\nabla\omega^\sharp > 0; \qquad (2.3.3)$$

if $\nabla u \leq 0$, then

$$(-|\nabla u|^{p-1} - |\nabla v|^{p-1})\nabla\omega^{\sharp} > 0. \tag{2.3.4}$$

2) Suppose $\nabla v \leq 0$, then $\nabla u < \nabla v \leq 0$ which by its turn implies

$$|\nabla u| > |\nabla v| \tag{2.3.5}$$

$$(-|\nabla u|^{p-1} + |\nabla v|^{p-1})\nabla\omega^{\sharp} > 0. \tag{2.3.6}$$

So, by (2.3.3), (2.3.4) and (2.3.6), it follows that $S_2 > 0$.

Finally, suppose, by contradiction, $B \neq \emptyset$, then, eventually, by considering a path leading to the boundary ∂U, we would find $x_0 \in B$ and $y_0 \in A_{x_0}$ which would yield $S_2 > 0$, hence, $0 \geq S_1 + S_2 + S_3 > 0$, a contradiction.

Therefore, $B = \emptyset$ and the proof is finished. ∎

Essentially, the argument we have used is the same as in [**9**, p.100]. The main point is that we extend the proof to *infinite* graphs with boundary.

Corollary 2.3.7 *Let $G \cup \partial G$ be a connected graph with boundary (non-necessarily finite), and let $u, v \in C^0(U \cup \partial U)$ be p-harmonics on U such that $u = v$ on ∂U. Then*

$$u = v \quad \text{on } U \cup \partial U.$$

Proof. Apply the comparison theorem twice. ∎

2.4 p-Dirichlet Problem

Now, we will consider the solution of Dirichlet problems in the context of p-Dirichlet spaces.

Theorem 2.4.1 *Suppose that* $p \in [1, \infty)$ *and* $G \cup \partial G = (U \cup \partial U, E \cup \partial E)$ finite *graph with boundary. Let* u *be a function defined on* ∂U*. Then there exists a unique* p*-harmonic function* h *on* U *with* $h = u$ *on* ∂U*.*

Proof. Let's prove first the existence.

If $G \cup \partial G$ has n vertices, then $\mathcal{L}^{1,p}(U \cup \partial U)$ is n-dimensional real vector space, so there exists an isomorphism ϕ from $\mathcal{L}^{1,p}(U \cup \partial U)$ to $\mathbb{R}^n$.

Defining in $\mathbb{R}^n$ a norm $\|x\| = \|\phi^{-1}(x)\|_{1,p}$, then ϕ turns out an isometric isomorphism and, consequently, the space $\mathcal{L}^{1,p}(U \cup \partial U)$ is isometrically isomorphic to $\mathbb{R}^n$.

Consider

$$A = \{v \in \mathcal{L}^{1,p}(U \cup \partial U) : u(x_m) \leq v(x) \leq u(x_M)\,, all\, x \in U\, and\, v \equiv u\, on\, \partial U\}$$

where x_m and x_M are the vertices in ∂U that u assumes, respectively, the minimum and the maximum.

It is easy to verify that A is *non-empty*, *closed*, and *bounded* in $\mathcal{L}^{1,p}(U \cup \partial U)$. Now, by the isometric isomorphism $\phi(A)$ *non-empty*, *closed*, and *bounded* in $\mathbb{R}^n$, so by Heine-Borel property $\phi(A)$ must be a compact set. By using now ϕ^{-1}, it follows that A is also a compact set.

On the other hand, $I_p(.) = \|.\|_{1,p}^p$ is a continuous nonlinear functional on $\mathcal{L}^{1,p}(U \cup \partial U)$, consequently, when restricted to A must attain a minimum $u^* \in A$. Then, by standard variational arguments like in [1.5.1], it follows that u^* is p-harmonic on U.

Secondly, the uniqueness is immediate consequence of the Corollary [2.3.7].

Finally, to complete the proof, denote u^* by h. ■

Remark: Notice, we can guarantee that the solution of a Dirichlet Problem exists only if U is finite. For the infinite case, that is, U is infinite, we need extra conditions to solve it.

Chapter 3

p-CAPACITY ON GRAPHS

In this chapter, we construct a basic tool of p-potential theory for graphs with or without boundaries, namely, p-capacities. Important properties of the p-capacities are established as well as the definitions of p-hyperbolic and p-parabolic graphs. As examples, we determine the p-parabolicity and p-hyperbolicity of $\mathbb{Z}^n$ and T_d through formulas that allow us to evaluate, precisely, their p-capacities.

Also, the concept of morphisms between graphs with and without boundaries is presented. This concept leads, naturally, to the notion of isomorphism and, consequently, what equivalence means in this category.

Finally, we estimate p-capacities of subgraphs and graphs with or without boundaries through the operations of cutting and shorting.

Many definitions and proofs are new in the context of graphs.

3.1 Basic Definitions

It is well known that *capacity* is a natural measure in the context of linear and nonlinear potential theory on Riemannian manifolds. But, it is not really a measure; in fact it is not even finitely additive on disjoint sets. However, many problems can be given a complete solution in terms of capacity, and this makes it a powerful tool.

In this section, our aim is to define a similar notion of capacity for graphs, namely, *p-capacity*; and with that, to create a potential theory on graphs similar to that (linear or non-linear) on Riemannian manifolds.

Definitions:

1. Let $G = (V, E)$ be a graph (without boundary), and let K be finite subset of V. Then the set of admissible functions to measure the capacity of K relative to V is

$$M(K, V) = \{u \in C_0(V) : u \geq 1 \, on \, K\}.$$

2. Let $G \cup \partial G$ be a graph with boundary, and let K be finite subset of U. Then the set of admissible functions to measure the capacity of K relative U is

$$M(K, U \cup \partial U) = \{u \in C_0(U \cup \partial U) : u \geq 1 \, on \, K, u \equiv 0 \, on \, \partial U\}.$$

Definitions:

1. Let $G = (V, E)$ be a graph and K a finite subset of V. Then, for $p \in [1, \infty]$,

$$\mathrm{Cap}_p(K, G) = \inf\{I_p(u) : u \in M(K, V)\} \tag{3.1.1}$$

is called *p-capacity* of the *condenser* (K, G), or simply, *p-capacity* of K.

2. Let $\Omega = G \cup \partial G$ be a graph with boundary and K a finite subset of U. Then, for $p \in [1, \infty]$,

$$\mathrm{Cap}_p(K, \Omega) = \inf\{I_p(u, \Omega) : u \in M(K, U \cup \partial U)\} \tag{3.1.2}$$

is called *p-capacity* of the condenser (K, Ω), or simply, *p-capacity* of K *relative* to Ω.

We do the definitions, the statements and the proofs for condensers (K, Ω); however, it is very simple matter to adapt them to condensers without boundaries like (K, G).

Now, we want to extend the definition of p-capacity to *infinite* subsets D of U.

Definition: Let D be any subset of U either finite or infinite . Then, the (inner) p-capacity of the condenser (D, Ω) is given by

$$_*\mathrm{Cap}_p(D, \Omega) = \sup\{\mathrm{Cap}_p(K, \Omega) : K \subset D,\ K \text{ finite}\}. \tag{3.1.3}$$

Proposition 3.1.4 ${}_{*}\mathrm{Cap}_p$ *is an extension of* Cap_p *for* K *finite.*

Proof. This is obvious. ∎

Also the (inner) p-capacity allows us to define the (outer) p-capacity.

Definition: Let A be an arbitrary subset of U. Then the (outer) p-capacity of the condenser (A, Ω) is given by

$$^{*}\mathrm{Cap}_p(A,\Omega) = \inf\{{}_{*}\mathrm{Cap}_p(D,\Omega) : D \supseteq A,\ D \text{ arbitrary}\}.$$

Proposition 3.1.5 *If* A *an arbitrary subset of* U *either finite or infinite , then* $^{*}\mathrm{Cap}_p(A,\Omega) = {}_{*}\mathrm{Cap}_p(A,\Omega)$.
In particular, for finite K, $\mathrm{Cap}_p(K,\Omega) = {}^{*}\mathrm{Cap}_p(K,\Omega) = {}_{*}\mathrm{Cap}_p(K,\Omega)$.

Proof. It is immediate that

$$^{*}\mathrm{Cap}_p(A,\Omega) \le {}_{*}\mathrm{Cap}_p(A,\Omega). \tag{3.1.6}$$

On the other hand, take arbitrary $\epsilon > 0$. Then there exists $D \supseteq A$ such that ${}_{*}\mathrm{Cap}_p(D,\Omega) < {}^{*}\mathrm{Cap}_p(A,\Omega) + \epsilon$. But as for all $K \subseteq D$ $\mathrm{Cap}_p(K,\Omega) \le {}_{*}\mathrm{Cap}_p(D,\Omega)$, in particular, the inequality holds for all $K \subseteq A$ so by taking the sup over all finite K, we have ${}_{*}\mathrm{Cap}_p(A,\Omega) \le {}_{*}\mathrm{Cap}_p(D,\Omega)$. Consequently, ${}_{*}\mathrm{Cap}_p(A,\Omega) < {}^{*}\mathrm{Cap}_p(A,\Omega) + \epsilon$.
Since ϵ was chosen arbitrarily, we have

$${}_{*}\mathrm{Cap}_p(A,\Omega) \le {}^{*}\mathrm{Cap}_p(A,\Omega). \tag{3.1.7}$$

From [3.1.6] and [3.1.7], the result follows. ∎

As consequence of the above propositions, we define p-capacity as (inner) p-capacity, and omit the star in the notation, that is, we write $\mathrm{Cap}_p(A, \Omega)$. When the context is clear, we simply write $\mathrm{Cap}_p(A)$.

3.2 Capacity as Outer Measure

Finally, as we have mentioned, for each fixed $p \in [1, \infty]$ Cap_p is not measure but we can prove that Cap_p is an *exterior measure* on subsets of U.

Proposition 3.2.1 *Let $p \in [1, \infty]$, and let $\Omega = G \cup \partial G$ be a graph with boundary. Then,*

$$\mathrm{Cap}_p : \mathcal{P}(U) \longrightarrow [0, \infty]$$
$$D \longmapsto \mathrm{Cap}_p(D, \Omega)$$

(i) : $\mathrm{Cap}_p(\emptyset) = 0$;

(ii): *(Monotonicity relative to a set being measured) If D_1, $D_2 \in \mathcal{P}(U)$, $D_1 \subseteq D_2$, then $\mathrm{Cap}_p(D_1, \Omega) \leq \mathrm{Cap}_p(D_2, \Omega)$.*

(iii) : *(σ- subaditivity) Let $D_i \in \mathcal{P}(U)$, $i = 1, 2, \ldots$, and $D = \bigcup_{i=1}^{\infty} D_i$, then*

$$\mathrm{Cap}_p(D, \Omega) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(D_i, \Omega);$$

Proof. (i) is obvious.

(ii) For the proof, we consider the cases:

a) If D_1, D_2 are finite subsets of U, then the proof follows easily from $M(D_2, U \cup \partial U) \subseteq M(D_1, U \cup \partial U)$.

b) If D_1 is finite and D_2 infinite, then it is enough to use the definition of capacity for D_2.

c) If both D_1 and D_2 are infinity, then for every finite $K \subset D_1 \subseteq D_2 \subseteq U$ we have

$$\{\mathrm{Cap}_p(K,\Omega) : K \subset D_1, K\, finite\} \subseteq \{\mathrm{Cap}_p(K,\Omega) : K \subset D_2, K\, finite\}.$$

Hence, $\mathrm{Cap}_p(D_1,\Omega) \leq \mathrm{Cap}_p(D_2,\Omega)$.

(iii) To prove we divide in cases:

a) If $\sum_i^\infty \mathrm{Cap}_p(D_i) = \infty$ the result is obvious.

b) If $\sum_i^\infty \mathrm{Cap}_p(D_i) < \infty$

b1) Suppose we have finite number of $D_i \in \mathcal{P}(U), i = 1, ..., k$ and each D_i is a finite set. Let $\epsilon > 0$. Then, by definition, there exist functions $u_i \in C_0$, $u_i \geq 1$ on D_i, $i = 1, ..., k$ such that

$$I_p(u_i) < \mathrm{Cap}_p(D_i) + \frac{\epsilon}{k}, \quad i = 1, 2, ..., k. \tag{3.2.2}$$

For $x \in V$ define the function

$$u(x) = \sup\{u_i(x) : 1 \leq i \leq k\}.$$

Obviously, by construction, the function u has finite support since each u_i has, and $u \geq 1$ on $\bigcup_{i=1}^k D_i$. Also

$$I_p(u) \leq \sum_{i=1}^k I_p(u_i)$$

so, we have

$$I_p(u) \leq \sum_{i=1}^{k} I_p(u_i) < \sum_{i=1}^{k} \mathrm{Cap}_p(D_i) + \sum_{i=1}^{k} \frac{\epsilon}{k}$$

thus,

$$I_p(u) < \sum_{i=1}^{k} \mathrm{Cap}_p(D_i) + \epsilon.$$

Since ϵ is arbitrary, so, by definition of p-capacity, we have

$$\mathrm{Cap}_p(\bigcup_{i=1}^{k} D_i) \leq \sum_{i=1}^{k} \mathrm{Cap}_p(D_i).$$

b2) Suppose we have infinitely many $D_i \in \mathcal{P}(U), i = 1, 2, \ldots$ where every D_i is finite. Let's call $D = \bigcup_{i=1}^{\infty} D_i$. Then, by definition, given $\epsilon > 0$ there exists K finite $K \subset D$ such that

$$\mathrm{Cap}_p(D) - \epsilon < \mathrm{Cap}_p(K).$$

Now, since K is finite, there exist $D_i, i = 1, 2, \ldots m$ such that $K \subset \bigcup_{i=1}^{m} D_i$ so, by monotonicity and part b1) above, we have

$$\mathrm{Cap}_p(K) \leq \mathrm{Cap}_p(\bigcup_{i=1}^{m} D_i) \leq \sum_{i=1}^{m} \mathrm{Cap}_p(D_i) < \sum_{i=1}^{\infty} \mathrm{Cap}_p(D_i).$$

Therefore,

$$\mathrm{Cap}_p(D) - \epsilon < \mathrm{Cap}_p(K) < \sum_{i=1}^{\infty} \mathrm{Cap}_p(D_i).$$

Since ϵ is arbitrary, we have

$$\mathrm{Cap}_p(D) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(D_i).$$

b3) Suppose now that $D_i \in \mathcal{P}(D), i = 1, 2, ...$ but not necessarily all D_i finite. As before, consider $D = \bigcup_{i=i}^{\infty} D_i$. Then, by definition, for any $\epsilon > 0$ there exists K finite such that

$$\mathrm{Cap}_p(D) - \epsilon < \mathrm{Cap}_p(K).$$

Evidently, we have $K = \bigcup_{i=i}^{\infty}(K \cap D_i)$ where each $K \cap D_i$ is finite so by part b2) it follows that

$$\mathrm{Cap}_p(D) - \epsilon < \mathrm{Cap}_p(K) = \mathrm{Cap}_p(\bigcup_{i=i}^{\infty}(K \cap D_i) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(K \cap D_i).$$

Finally, since $K \cap D_i \subset D_i$ by monotonicity

$$\mathrm{Cap}_p(D) - \epsilon < \sum_{i=1}^{\infty} \mathrm{Cap}_p(K \cap D_i) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(D_i)$$

but ϵ is arbitrary, so we have

$$\mathrm{Cap}_p(D) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(D_i).$$

From a) and b), the result follows. ∎

We have just proved that the $\mathrm{Cap}_p(., \Omega)$ satisfies properties i),ii), and iii), i.e., $\mathrm{Cap}_p(., \Omega)$ is an *outer measure.*

Corollary 3.2.3 *If a set $K \subseteq U$ has zero p-capacity, then any of its subsets also has zero p-capacity.*

Proof. It follows immediately from monotonicity property. ∎

3.3 Simple Properties

We alert that the following proposition is valid only for *induced subgraphs with induced boundaries*, that is, $A_k = G[U_k] \cup G[(\partial^i U_k, \partial^e U_k)] \subseteq G$. We might say that they are analogues of open sets in topology.

Proposition 3.3.1 *Let $p \in [1, \infty]$, and let G be a graph. Then p-capacities satisfy the following properties:*

(i) If A_1 and A_2 are induced subgraphs with induced boundaries of G and $D \subset U_1 \subseteq U_2$, then

$$\mathrm{Cap}_p(D, A_1) \geq \mathrm{Cap}_p(D, A_2);$$

(ii) If $D \subset A_1 \subset A_2 \subset \ldots \subset \bigcup_{i=1}^{\infty} A_i = A$, then

$$\mathrm{Cap}_p(D, A) = \lim_{i \to \infty} \mathrm{Cap}_p(D, A_i). \tag{3.3.2}$$

Proof.

(*i*) Two cases to consider.

a) Let $D \subset U_1$ be finite and take an arbitrary $\epsilon > 0$. Then, there exists $u \in M(D, U_1 \cup \partial U_1)$ such that $I_p(u) < \mathrm{Cap}_p(D, A_1) + \epsilon$. Define $\tilde{u}$, the natural extension of u to A_2, by assigning zeros for the new vertices. It follows that $\tilde{u} \in M(D, U_2 \cup \partial U_2)$ and $I_p(\tilde{u}) = I_p(u)$.

Hence, $\mathrm{Cap}_p(D, A_2) \leq I_p(\tilde{u}) < \mathrm{Cap}_p(D, A_1) + \epsilon$. But ϵ is arbitrary, so it follows $\mathrm{Cap}_p(D, A_1) \geq \mathrm{Cap}_p(D, A_2)$.

b) Let D be infinite. For each $\epsilon > 0$ and arbitrary there exists finite, $K \subseteq D$ such that $\text{Cap}_p(D, A_2) - \epsilon \leq \text{Cap}_p(K, A_2)$. By part a) $\text{Cap}_p(K, A_2) \leq \text{Cap}_p(K, A_1)$ so that by the definition of capacity in A_1, it follows $\text{Cap}_p(K, A_1) \leq \text{Cap}_p(D, A_1)$. Combining the steps with arbitrariness of ϵ, we get $\text{Cap}_p(D, A_1) \geq \text{Cap}_p(D, A_2)$, and the proof of i) is complete.

To prove (ii) we again consider two steps:

Step 1) For all $j \in \mathbb{N}_*$, by (i), we have

$$\text{Cap}_p(D, A_1) \geq \ldots \geq \text{Cap}_p(D, A_j) \geq \ldots \geq \text{Cap}_p(D, A);$$

so

$$\lim_{i \to \infty} \text{Cap}_p(D, A_j) \geq \text{Cap}_p(D, A). \tag{3.3.3}$$

Step 2) a) Let D be *finite* and $\epsilon > 0$ and arbitrary. Then, there exists $u \in M(D, U \cup \partial U)$ such that

$$I_p(u) < \text{Cap}_p(D, A) + \epsilon$$

but for some $j_0 \in \mathbb{N}*$, *supp* $u \subset A_{j_0}$, that is, u is admissible for (D, A_{j_0}), so

$$\text{Cap}_p(D, A_{j_0}) < I_p(u) < \text{Cap}_p(D, A) + \epsilon;$$

since ϵ is arbitrary

$$\text{Cap}_p(D, A_{j_0}) \leq \text{Cap}_p(D, A);$$

by (i), for $j \geq j_0$

$$\text{Cap}_p(D, A_j) \leq \text{Cap}_p(D, A). \tag{3.3.4}$$

Taking the limit of (3.3.4), which exists by step 1, it follows that

$$\lim_{j\to\infty} \mathrm{Cap}_p(D, A_j) \leq \mathrm{Cap}_p(D, A). \tag{3.3.5}$$

b) Let D be *infinite* and $\epsilon > 0$ and arbitrary. Then, for all $j \in \mathbb{N}*$, there exists $K_j \subset D$ finite such that

$$\mathrm{Cap}_p(D, A_j) - \epsilon < \mathrm{Cap}_p(K_j, A_j)$$

but by step 1,

$$\mathrm{Cap}_p(K_j, A_j) \leq \mathrm{Cap}_p(K_j, A)$$

$$\mathrm{Cap}_p(D, A_j) - \epsilon < \mathrm{Cap}_p(K_j, A). \tag{3.3.6}$$

Now, by definition of p-capacity of D

$$\mathrm{Cap}_p(K_j, A) \leq \mathrm{Cap}_p(D, A)$$

which combined with (3.3.6) leads to

$$\mathrm{Cap}_p(D, A_j) - \epsilon < \mathrm{Cap}_p(D, A);$$

but ϵ is arbitrary, so for all $j \in \mathbb{N}_*$

$$\mathrm{Cap}_p(D, A_j) \leq \mathrm{Cap}_p(D, A). \tag{3.3.7}$$

Taking the limit of (3.3.7) , since it exists by step 1, it follows that

$$\lim_{j\to\infty} \mathrm{Cap}_p(D, A_j) \leq \mathrm{Cap}_p(D, A). \tag{3.3.8}$$

The proof of (ii) follows from (3.3.3),(3.3.5), and (3.3.8). ∎

Remark: The example of homogeneous trees T_d, in the coming section, shows that we cannot replace $\geq$ by $=$ on the item 1. In fact, if $D_{R_1} \hookrightarrow D_{R_2}$, then $\text{Cap}_p(D_r, D_{R_1}) > \text{Cap}_p(D_r, D_{R_2})$ for $R_1 < R_2$.

The next proposition shows, without loss of generality, that we may restrict ourselves to the conditions $0 \leq u \leq 1$ and $u \equiv 1$ on K.

First, for finite K, we define new set of admissible functions

$$\mathcal{M}(K, U \cup \partial U) = \{u \in C_0(U \cup \partial U) : \ 0 \leq u \leq 1 \ , \ u \equiv 1 \text{ on } K \text{ and } u \equiv 0 \text{ on } \partial U\}.$$

To simplify, we only consider finite $K \subseteq U$ in the following.

Proposition 3.3.9 *Let $p \in [1, \infty]$, $\Omega = G \cup \partial G$ a graph with boundary and K be a finite set on U. Then*

$$\text{Cap}_p(K, \Omega) = \inf\{I_p(u, \Omega) : u \in \mathcal{M}(K, U \cup \partial U)\}.$$

Proof. a) It is immediate $M(K, U \cup \partial U) \supset \mathcal{M}(K, U \cup \partial U)$, so

$$\text{Cap}_p(K, \Omega) \leq \inf\{I_p(u, \Omega) : u \in \mathcal{M}(K, U \cup \partial U)\}. \tag{3.3.10}$$

b) Take $\epsilon > 0$ and arbitrary. Then, there exists $u \in M(K, U \cup \partial U)$ such that

$$I_p(u) < \text{Cap}_p(K, \Omega) + \epsilon.$$

Consider a truncation of u, that is,

$$\tilde{u} = \max(0, \min(1, u))$$

so, it is easy to see that $\tilde{u} \in \mathcal{M}(K, U \cup \partial U)$, and, by dividing in cases, to show that $I_p(\tilde{u}) \leq I_p(u)$. Hence,

$$\inf\{I_p(u) : u \in \mathcal{M}(K, U \cup \partial U)\} \leq I_p(\tilde{u}) \leq I_p(u) < \operatorname{Cap}_p(K, \Omega) + \epsilon.$$

But by arbitrariness of ϵ, we conclude that

$$\inf\{I_p(u) : u \in \mathcal{M}(K, U \cup \partial U)\} \leq \operatorname{Cap}_p(K, \Omega).$$

Then, from the inequality above and (3.3.10), the claim follows . ∎

Corollary 3.3.11 *Let $p \in [1, \infty]$, $\Omega = G \cup \partial G$ a graph with boundary and $K \cup \partial K$ be a finite set on U. Then*

$$\operatorname{Cap}_p\big(K \cup \partial K, \Omega\big) = \operatorname{Cap}_p\big(\partial K, \Omega\big).$$

Proof. We prove it in two steps.

a) By Monotonicity, we have

$$\operatorname{Cap}_p\big(\partial K, \Omega\big) \leq \operatorname{Cap}_p\big(K \cup \partial K, \Omega\big).$$

b) For the opposite inequality, let ϵ be positive and arbitrary. Then, by previous proposition, there exists $u \in \mathcal{M}(\partial K, U \cup \partial U)$

$$I_p(u) < \operatorname{Cap}_p\big(\partial K, \Omega\big) + \epsilon.$$

Define an extension of u, namely, $u^* = \begin{cases} u & \text{on } (U \backslash K) \cup \partial U \\ 1 & \text{on } K. \end{cases}$ which is admissible for $K \cup \partial K$, so

$$\operatorname{Cap}_p\big(K \cup \partial K, \Omega\big) \leq I_p(u^*) \leq I_p(u) < \operatorname{Cap}_p\big(\partial K, \Omega\big) + \epsilon;$$

but, by arbitrariness of ϵ, we have

$$\mathrm{Cap}_p\big(K \cup \partial K, \Omega\big) \leq \mathrm{Cap}_p\big(\partial K, \Omega\big).$$

Combining a) and b), yields the proof of the corollary. ■

Now, we use p-capacity to produce definitions that play important roles in the classification of finite/infinite graphs, and which allow us to deduce their potential theoretic properties.

Definitions: For $p \in [1, \infty]$, a graph G is called *p-parabolic* if $\mathrm{Cap}_p(K) = 0$ for all *finite* subsets $K \subseteq V$ and *p-hyperbolic* otherwise.

In the following, we extend the previous definitions to graphs with boundaries.

Definitions: For $p \in [1, \infty]$, and let $\Omega = G \cup \partial G$ be a graph with boundary. Then, Ω is called *p-parabolic* if $\mathrm{Cap}_p(K, \Omega) = 0$ for all finite subsets $K \subset U$ and *p-hyperbolic* otherwise.

Remark: Actually, we will see latter on, as a consequence of Proposition [4.1.9], that for a p-hyperbolic graph $\mathrm{Cap}_p(K) > 0$ for all *finite* subsets $K \subset U$ (respectively V).

3.4 The n-dimensional Lattices - $\mathbb{Z}^n$

This section has as goal to show nontrivial examples of p-hyperbolic and p-parabolic graphs. For that purpose an *explicit formula* for evaluating p-capacities on $\mathbb{Z}^n$ is determined.

To view $\mathbb{Z}^n$ as graph, as usual, for every pair of integers points at unit distance (Manhattan distance) is connected by an edge parallel to the coordinates axes.

Let $\{e_1, e_2, \ldots, e_n\}$ the canonical basis of the module $\mathbb{Z}^n$. Then for $n \geq 1$ the set of vertices $S^{(n-1)} = \left\{(-1)^j e_i\right\}_{\substack{j=0,1 \\ i=1,2,\ldots,n}}$ is called *radial n-sphere* in $\mathbb{Z}^n$. Obviously, the radial n-sphere has volume $2n$ and it determines $2n$ (i,j) directions, namely, n i-directions each with two j-senses.

Example: $S^{(1)} = \left\{(-1)^j e_i\right\}_{\substack{j=0,1 \\ i=1,2}} = \{(1,0);(-1,0);(0,1);(0,-1)\}$ is the 1-sphere in $\mathbb{Z}^2$ with volume 4 and 4 directions.

To compute the p-capacities, we shall split $\mathbb{Z}^n$ in $2n$ *sectors*, namely, the radial (i,j)-*sectors.*

Definition: An edge of $\mathbb{Z}^n$ that is incident with the vertices that are at the same distance from the origin (0) is called *transversal edge* and *radial edge* otherwise.

Definition: The *radial subgraph* $\mathbb{Z}^n_{rad}$ is the graph obtained by deleting all transversal edges from $\mathbb{Z}^n$.

In the following, we describe the radial (i,j)-sectors of radius r and the disk of radius r.

First, we define auxiliary sets.

For $t = 0, 1, 2, ...,$ let $A_t = \{x \in \mathbb{Z} / -t \leq x \leq t\}$ be a family of sets on $\mathbb{Z}$. Then, for each $i = 1, 2, ..., n$ define a family of sets on $\mathbb{Z}^n$ by

$$B_t^i = A_t \times A_t \times \ldots \times A_t \times \{0\} \times A_t \times \ldots \times A_t$$

$$= \{x = (x_1, x_2, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) : -t \leq x_k \leq t, k = 1, 2, ..., \hat{\imath}, ..., n\}$$

which satisfies the following properties:

- $B_0^i \subset B_1^i \subset B_2^i \subset \cdots \subset B_t^i \subset \cdots$.
- $\#(B_t^i) = \prod_{\substack{k=1 \\ k \neq i}}^n \#(A_t) = (2t+1)^{n-1}$, since $\#(A_t) = 2t + 1$.

Secondly, the set of vertices in the (i, j)-sector at distance t from the origin is describe by

$$V_t^{(i,j)} = B_t^i + t(-1)^j e_i$$

$$= \{x_t^{(i,j)} = x + t(-1)^j e_i \in \mathbb{Z}^n : x = (x_1, x_2, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \in B_t^i\}.$$

Definition: For $i = 1, 2, ..., n$, $j = 0, 1$ and $r \in \mathbb{N}$, the *radial (i, j)-sector* of radius r, is the induced subgraph of $\mathbb{Z}^n_{rad}$ given by

$$\mathcal{S}_r^{(i,j)} = \mathbb{Z}^n_{rad}[\cup_{t=0}^r V_t^{(i,j)}].$$

whose set of radial edges $[x_t^{(i,j)}, x_{t+1}^{(i,j)}]$ on the stage t is denoted by $E_t^{(i,j)}$. By the way, when the context is clear, we use instead the full notation for a vertex $x_t^{(i,j)}$ its shortened variant x_t.

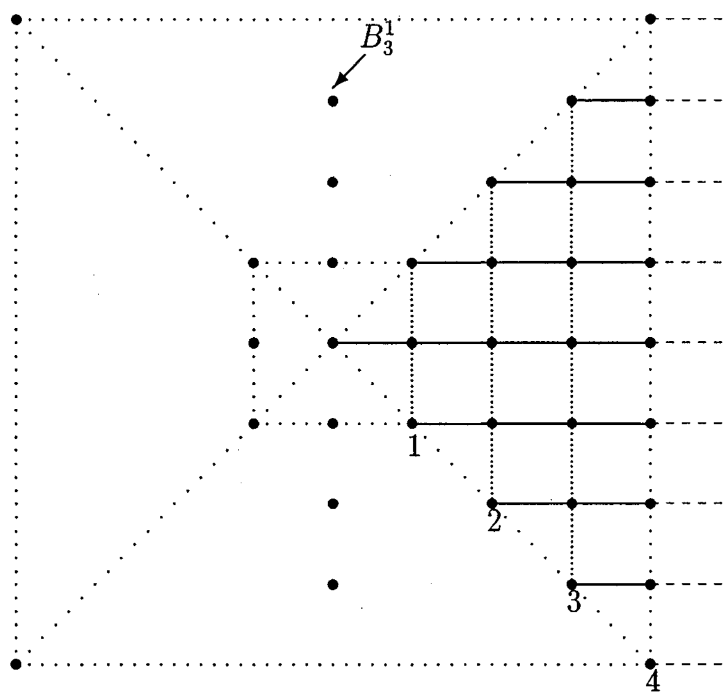

Figure 3.1 : The sector $S_3^{(1,0)}$ of $\mathbb{Z}^2_{rad}$

Definition: The *disk* of $\mathbb{Z}^n$ centered at origin and radius r is the induced subgraph

$$D_r = \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \mathbb{Z}^n[\cup_{t=0}^{r} V_t^{(i,j)}].$$

Also note that $\{x \in \mathbb{Z}^n : d(0,x) \leq r\}$, where d is the Manhattan distance, is the set of vertices of D_r.

In the following, we need a kind of averaging function on $\mathbb{Z}^n$. Also, observe that the referred distance on the definition is the Manhattan distance.

Definition: The set of vertices at distance t from the origin (0) is called *t-layer* of vertices of either $\mathbb{Z}^n_{rad}$ or $\mathbb{Z}^n$, that is,

$$L_t = \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} V_t^{(i,j)}.$$

Notice that $\#(L_t) = (2t+1)^n - \sum_{r=0}^{t-1} \#(L_r)$.

Definition: Let $u \in C^0(\mathbb{Z}^n)$. The *transversal average* of u on $\mathbb{Z}^n$ is the function $\overline{u} \in C^0(\mathbb{Z}^n)$ that assigns to the vertices of each layer L_t, $t = 0, 1, 2, \ldots.$, respectively, the value $\frac{\sum_{x \in L_t} u(x)}{\#(L_t)}$.

Proposition 3.4.1 *Let $\mathbb{Z}^n, n \geq 1$, be n-dimensional lattice. Then, for $p > 1$ and $0 \leq r \leq R \leq \infty$,*

$$\operatorname{Cap}_p(D_r, D_R) = Vol(S^{(n-1)})\Big(\sum_{t=r}^{R}(2t+1)^{\frac{1-n}{p-1}}\Big)^{1-p}. \tag{3.4.2}$$

Proof. First we prove that $\leq$.

To simplify the notation, we define the function $\phi(t)$ to denote the number of radial edges on each (i,j)-sector on the stage t, $\#(E_t^{(i,j)})$, that is,

$$\phi(t) = (2t+1)^{n-1}.$$

Now, we define a new function on $\mathbb{Z}^n$ that is admissible for D_r by assigning its values on each t-layer. Namely, for all $t, r, R \in \mathbb{N}$ and $0 \leq r < R \leq \infty$

$$u_{r,R}(x_t) = \begin{cases} 1 & \text{if } 0 \leq t \leq r, \\ 1 - \gamma_{r,R}\Big(\sum_{\rho=r}^{t-1} \phi(\rho)^{\frac{1}{1-p}}\Big) & \text{if } r+1 \leq t \leq R+1, \\ 0 & \text{if } t \geq R+1. \end{cases}$$

where $\gamma_{r,R} = \left(\sum_{\rho=r}^{R} \phi(\rho)^{\frac{1}{1-p}}\right)^{-1}$.

We have

1. For $0 \leq t \leq r-1$: $\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big) = 0$.

2. For $t = r$ and $t = r+1$:

$$\begin{aligned}
&\nabla u_{r,R}\Big([x_r, x_{r+1}]\Big) = \\
&= u_{r,R}(x_{r+1}) - u_{r,R}(x_r) \\
&= 1 - \gamma_{r,R}\phi(r)^{\frac{1}{1-p}} - 1 \\
&= -\gamma_{r,R}\phi(r)^{\frac{1}{1-p}}.
\end{aligned}$$

3. For $r+1 \leq t \leq R$:

$$\begin{aligned}
&\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big) = \\
&= u_{r,R}(x_{t+1}) - u_{r,R}(x_t) \\
&= -\gamma_{r,R}\Big[\sum_{\rho=r}^{t} \phi(\rho)^{\frac{1}{1-p}} - \sum_{\rho=r}^{t-1} \phi(\rho)^{\frac{1}{1-p}}\Big] \\
&= -\gamma_{r,R}\phi(t)^{\frac{1}{1-p}}.
\end{aligned}$$

4. For $t \geq R+1$:

$$\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big) = 0.$$

Thus,

$$\left|\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big)\right| = \begin{cases} \gamma_{r,R}\phi(t)^{\frac{1}{1-p}} & \text{if } r \leq t \leq R, \\ 0 & \text{else} \end{cases}$$

We sum up over the whole region to get:

$$\sum_{\substack{1\leq i\leq n\\ 0\leq j\leq 1}}\sum_{t=r}^{R}\sum_{[x_t,x_{t+1}]\in E_t^{(i,j)}}\left|\nabla u_{r,R}\Big([x_t,x_{t+1}]\Big)\right|^p =$$

$$= \sum_{\substack{1\leq i\leq n\\ 0\leq j\leq 1}}\sum_{t=r}^{R}\sum_{[x_t,x_{t+1}]\in E_t^{(i,j)}} \gamma_{r,R}^p \phi(t)^{\frac{p}{1-p}}$$

$$= Vol(S^{(n-1)})\gamma_{r,R}^p \sum_{t=r}^{R}\phi(t)\phi(t)^{\frac{p}{1-p}}$$

$$= Vol(S^{(n-1)})\gamma_{r,R}^p \Big(\sum_{t=r}^{R}\phi(t)^{\frac{1}{1-p}}\Big)$$

$$= Vol(S^{(n-1)})\gamma_{r,R}^{p-1}.$$

As $u_{r,R}$ is an admissible function for D_r relative to D_R, so for $0 \leq r < R \leq \infty$

$$Cap_p(D_r, D_R) \leq I_p(u_{r,R}) =$$

$$= Vol(S^{(n-1)})\gamma_{r,R}^{p-1} = Vol(S^{(n-1)})\Big(\sum_{t=r}^{R}(2t+1)^{\frac{1-n}{p-1}}\Big)^{1-p}. \tag{3.4.3}$$

We now prove the reverse inequality.

Let u be an arbitrary admissible function for D_r, that is, $u \equiv 1$ in D_r and $0 \leq u \leq 1$ in D_R. Let $\overline{u}$ be the transversal average of u on D_R. Then, select a (i, j)-sector, we have

$$1 \leq |\sum_{t=r}^{R} \nabla\bar{u}[0_t, 0_{t+1}]| \leq \sum_{t=r}^{R} |\nabla\bar{u}[0_t, 0_{t+1}]|$$
$$= \sum_{t=r}^{R} \Big(|\nabla\bar{u}[0_t, 0_{t+1}]|(2t+1)^{\frac{(n-1)}{p}}\Big)\Big((2t+1)^{\frac{(1-n)}{p}}\Big)$$
$$\leq \Big(\sum_{t=r}^{R} |\nabla\bar{u}[0_t, 0_{t+1}]|^p (2t+1)^{(n-1)}\Big)^{\frac{1}{p}} \Big(\sum_{t=r}^{R} (2t+1)^{\frac{(1-n)}{p-1}}\Big)^{\frac{(p-1)}{p}}.$$

After raising to p and since $\#(B_t^i) = (2t+1)^{n-1}$, we get

$$\Big(\sum_{t=r}^{R} \sum_{x_t \in B_t^i} |\nabla\bar{u}[x_t, x_{t+1}]|^p\Big)\Big(\sum_{t=r}^{R} (2t+1)^{\frac{(1-n)}{p-1}}\Big)^{(p-1)} \geq 1$$

consequently,

$$\sum_{t=r}^{R} \sum_{x_t \in B_t^i} |\nabla\bar{u}[x_t, x_{t+1}]|^p \geq \Big(\sum_{t=r}^{R} (2t+1)^{\frac{(1-n)}{p-1}}\Big)^{(1-p)}.$$

Now, we sum up over all radial (i, j)-sectors to get

$$\sum_{\substack{1 \leq i \leq n \\ 0 \leq j \leq 1}} \sum_{t=r}^{R} \sum_{x_t \in B_t^i} |\nabla\bar{u}[x_t^{(i,j)}, x_{t+1}^{(i,j)}]|^p \geq Vol(S^{(n-1)})\Big(\sum_{t=r}^{R} (2t+1)^{\frac{(1-n)}{p-1}}\Big)^{(1-p)}.$$

or

$$I_p(\bar{u}) \geq Vol(S^{(n-1)})\Big(\sum_{t=r}^{R} (2t+1)^{\frac{(1-n)}{p-1}}\Big)^{(1-p)}.$$

As averaging transversally a function can only decrease its p-energy, namely, $I_p(u) \geq I_p(\bar{u})$. Then,

$$I_p(u) \geq I_p(\bar{u}) \geq Vol(S^{(n-1)})\Big(\sum_{t=r}^{R} (2t+1)^{\frac{(1-n)}{p-1}}\Big)^{(1-p)}.$$

But u is an arbitrary admissible function of the condenser (D_r, D_R), so

$$Cap_p(D_r, D_R) \geq Vol(S^{(n-1)})\Big(\sum_{t=r}^{R}(2t+1)^{\frac{(1-n)}{p-1}}\Big)^{(1-p)}. \tag{3.4.4}$$

From [3.4.3 and 3.4.4], it follows the proof. ∎

Corollary 3.4.5 *For $n \geq 1$ and $p > 1$, the lattice $\mathbb{Z}^n$ is p-parabolic if and only if* $\sum_{t=0}^{\infty} \frac{1}{(2t+1)^{\frac{n-1}{p-1}}} = \infty$.

Proof. Combining the Proposition [3.4.1] and (3.3.2), we have

$$\begin{aligned} Cap_p(0, \mathbb{Z}^n) &= Vol(S^{(n-1)})\Big(\sum_{t=0}^{\infty}(2t+1)^{(\frac{1-n}{p-1})}\Big)^{(1-p)} \\ &= Vol(S^{(n-1)})\left(\sum_{t=0}^{\infty}\frac{1}{(2t+1)^{(\frac{n-1}{p-1})}}\right)^{(1-p)}. \end{aligned}$$

Now it is immediate the proof. ∎

Proposition 3.4.6 *For $p = 1$ and $n \geq 1$. Let $\mathbb{Z}^n$ be lattice. Then, $Cap_1(D_r, D_R) \geq 1$, for all r, R, $0 \leq r < R \leq \infty$.*

Proof. Let u be an arbitrary admissible function for 1-capacity. Fix a path on a direction μ, then, by "telescoping ", we have

$$\begin{aligned} I_1(u) &\geq \sum_{t=r}^{R}\Big|\nabla u\Big([x_t, x_{t+1}]\Big)\Big| \\ &\geq \Big|\sum_{t=r}^{R}\nabla u\Big([x_t, x_{t+1}]\Big)\Big| = 1 \end{aligned}$$

so, we conclude that $Cap_1(D_r, D_R) \geq 1$, for $0 \leq r < R \leq \infty$. ∎

Corollary 3.4.7 *For $p = 1$ and $n \geq 1$, $\mathbb{Z}^n$ is 1-hyperbolic.*

Proof. The proof follows immediately from the above Proposition and Proposition [3.3.1]. ■

Proposition 3.4.8 *For $p = \infty$ and $n \geq 1$. $\mathbb{Z}^n$ is ∞-parabolic.*

Proof. Consider a condenser (D_r, D_R), $0 \leq r < R \leq \infty$. Define

$$u_{r,R}(x_t) = \begin{cases} 1 & \text{if } 0 \leq t \leq r, \\ 1 - (R-r)^{-1}(t-r) & \text{if } r \leq t \leq R, \\ 0 & \text{if } t \geq R, \end{cases}$$

$u_{r,R}$ is an admissible function for (D_r, D_R).

It is easy to verify that

$$\left|\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big)\right| = \begin{cases} (R-r)^{-1} & \text{if } t = r \\ (R-r)^{-1}r & \text{if } r+1 \leq t \leq R-1. \\ 0 & \text{else} \end{cases}$$

hence, $I_\infty(u_{r,R}) = \max\{\frac{1}{R-r}, \frac{r}{R-r}\}$. But $u_{r,R}$ is an admissible function, thus

$$Cap_\infty(D_r, D_R) \leq I_\infty(u_{r,R}).$$

By Proposition [3.3.1- (2)]

$$Cap_\infty(D_r, \mathbb{Z}^n) = \lim_{R\to\infty} Cap_\infty(D_r, D_R) = 0.$$

Take an arbitrary finite $K \subset \mathbb{Z}^n$ so, there exists some D_{r_0} such that $K \subseteq D_{r_0}$. Then, by monotonicity of ∞-capacity, it follows that

$$Cap_\infty(K, \mathbb{Z}^n) \leq Cap_\infty(D_{r_0}, \mathbb{Z}^n) = 0.$$

Thus, $\mathbb{Z}^n$ is ∞-parabolic. ■

Theorem 3.4.9

1. *For $n \geq 2$, $\mathbb{Z}^n$ is p-parabolic if and only if $n \leq p \leq \infty$.*

2. *For $n = 1$, $\mathbb{Z}$ is p-parabolic if and only if $n = 1 < p \leq \infty$.*

Proof.

1. For $n \geq 2$. If $p \geq n$, then $\alpha = \frac{n-1}{p-1} \leq 1$. As $\sum_{t=1}^{\infty} \frac{1}{t^\alpha}$ diverges for $\alpha \leq 1$, then by limit comparison test $\sum_{t=1}^{\infty} \frac{1}{(2t+1)^\alpha}$ also diverges. Then, by Corollary [3.4.5], it follows that $\mathbb{Z}^n$ is p-parabolic.

   Conversely, we suppose, by contradiction, that $p < n$, then $\alpha = \frac{n-1}{p-1} > 1$. As $\sum_{t=1}^{\infty} \frac{1}{t^\alpha}$ converges, then by limit comparison test $\sum_{t=1}^{\infty} \frac{1}{(2t+1)^\alpha}$ also converges which is in contradiction with Corollary [3.4.5] since by hypothesis $\mathbb{Z}^n$ is p-parabolic.

   Therefore, p-parabolicity of $\mathbb{Z}^n$ implies $p \geq n$.

2. If $p > n = 1$, then, by Corollary [3.4.5], it follows that $\mathbb{Z}$ is p-parabolic.

   Conversely, suppose, by contradiction, that $1 \leq p \leq n = 1$, that is, $p = n = 1$. Then, by Corollary [3.4.7] $\mathbb{Z}$ is 1-hyperbolic which contradicts the hypothesis.

   Therefore, p-parabolicity of $\mathbb{Z}$ implies $p > n = 1$.

3. For $p = \infty$. It is the Proposition [3.4.8] above. ∎

Corollary 3.4.10

1. *For $n \geq 2$, $\mathbb{Z}^n$ is p-hyperbolic if and only if* $1 \leq p < n$.

2. *For $n = 1$, $\mathbb{Z}$ is p-hyperbolic if and only if* $n = p = 1$.

Proof. The proofs are contrapositive of those given on the previous theorem. ■

In the article [**14**, p.148-152], it was proved the Theorem [3.4.9] and Corollary [3.4.10]. It is worth to mention that our proofs are completely different of those given by Maeda and, additionally, cover some omitted cases.

3.5 Homogeneous Tree - T_d

The purpose of this section is to determine a formula for evaluating p-capacities of homogeneous trees T_d. This formulas allows to determine whether T_d are p-parabolic or p-hyperbolic.

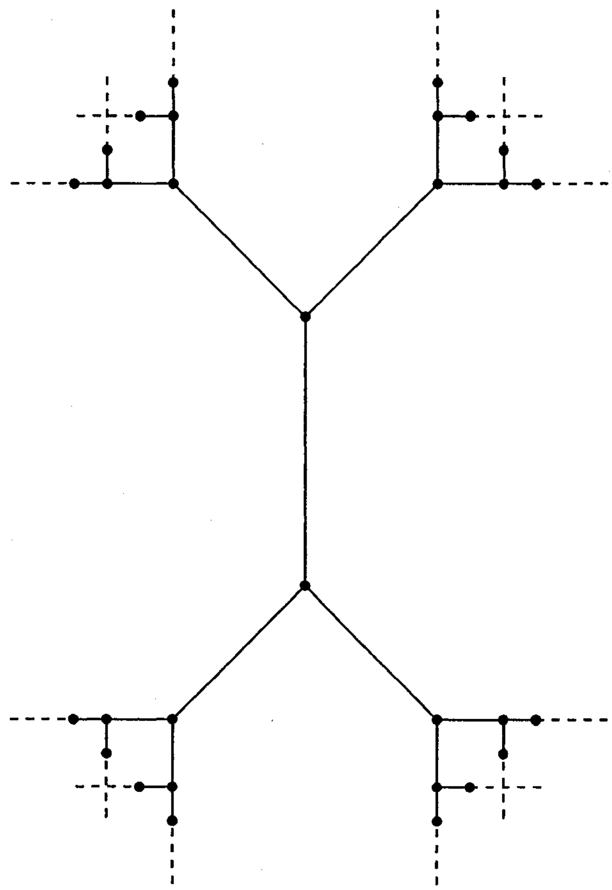

Figure 3.2 Homogenous tree of degree 3

Let T_d be a homogeneous tree of degree $d \geq 3$ with no vertices of degree 1 (*without terminals*). Then T_d has d *branches* (copies) with respect a fixed point a called *root.*

For each branch $k = 1, 2, ..., d$, the set of vertices in the generation t, $(t = 1, 2, ...)$ is

$$V_t^k = \{x_t^k \in T_d / d(a, x_t^k) = t\}.$$

where $d(a, x_t^k)$ represents the geodesic distance from the root a to the vertex

x_t^k. Throughout a proof, when the context is clear, we drop out k of x_t^k.

Definition: The *disk* of center a and radius r is the induced subgraph

$$D_r = \bigcup_{k=1}^{d} T_d\Big[\bigcup_{t=1}^{r} V_t^k\Big].$$

Remark:

In each branch k, the number of edges that come out from the vertices of the generation t to generation $t+1$ is $f(t) = (d-1)^t$.

Proposition 3.5.1 *For $p > 1$ and $0 \le r < R \le \infty$, let T_d be a homogenous tree of degree $d \ge 3$. Then*

$$\mathrm{Cap}_p(D_r, D_R) = d\Big(\sum_{t=r}^{R}(d-1)^{\frac{t}{1-p}}\Big)^{1-p}. \tag{3.5.2}$$

Proof.

We define an admissible function for (D_r, D_R) by

$$u_{r,R}(x_t) = \begin{cases} 1 & \text{if } 0 \le t \le r, \\ 1 - \gamma_{r,R}\Big(\sum_{\rho=r}^{t-1} f(\rho)^{\frac{1}{1-p}}\Big) & \text{if } r+1 \le t \le R+1, \\ 0 & \text{if } t \ge R+2, \end{cases}$$

where $\gamma_{r,R} = \Big(\sum_{\rho=r}^{R} f(\rho)^{\frac{1}{1-p}}\Big)^{-1}$.

It is easy to verify that

$$\Big|\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big)\Big| = \begin{cases} \gamma_{r,R} f(t)^{\frac{1}{1-p}} & \text{if } r \le t \le R. \\ 0 & \text{else} \end{cases}$$

By the fact that $(\#V_t).(\#E(x_t)) = (d-1)^{t-1}(d-1) = f(t)$,

$$\begin{aligned} I_p(u_{r,R}) &= \sum_{k=1}^{d}\sum_{t=r}^{R}\sum_{x_t\in V_t^k}\sum_{[x_t,x_{t+1}]\in E(x_t)} \left|\nabla u_{r,R}\Big([x_t,x_{t+1}]\Big)\right|^p \\ &= \sum_{t=r}^{R}\sum_{k=1}^{d}\sum_{x_t\in V_t^k}\sum_{[x_t,x_{t+1}]\in E(x_t)} \gamma_{r,R}^p f(t)^{\frac{p}{1-p}} \\ &= d\gamma_{r,R}^p \sum_{t=r}^{R} f(t) f(t)^{\frac{p}{1-p}} \\ &= d\gamma_{r,R}^p \Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big) = d\gamma_{r,R}^{p-1}. \end{aligned}$$

But $u_{r,R}$ is an admissible function for (D_r, D_R) so,

$$\begin{aligned} Cap_p(D_r, D_R) &\le I_p(u_{r,R}) = \\ &= d\gamma_{r,R}^{p-1} = d\Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{1-p}. \end{aligned} \qquad (3.5.3)$$

We now prove the reverse inequality.

Throughout the proof k shall be removed from all notations.

Let (D_r, D_R) be a condenser, and let $x_r^0, x_{r+1}^0, \ldots, x_{R+1}^0$ an arbitrary path on a k branch.

Let u be an arbitrary admissible function on (D_r, D_R), that is, $u \equiv 1$ on D_r and $u \equiv 0$ on ∂D_R. Consider $\overline{u}$ the transversal average of u on D_R with similar definition given for the $\mathbb{Z}^n$ case. Then, for $p > 1$,

$$\begin{aligned} 1 \le \sum_{t=r}^{R}\left|\nabla\overline{u}\Big([x_t^0, x_{t+1}^0]\Big)\right| &= \sum_{t=r}^{R}\left(\left|\nabla\overline{u}\Big([x_t^0, x_{t+1}^0]\Big)\right| f(t)^{\frac{1}{p}}\right)\Big(f(t)^{\frac{-1}{p}}\Big) \\ &\le \Big(\sum_{t=r}^{R}\left|\nabla\overline{u}\Big([x_t^0, x_{t+1}^0]\Big)\right|^p f(t)\Big)^{\frac{1}{p}}\Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{\frac{p-1}{p}}. \end{aligned}$$

Thus,

$$\sum_{t=r}^{R}\left|\nabla\overline{u}\Big([x_t^0, x_{t+1}^0]\Big)\right|^p f(t) \geq \Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{1-p}.$$

Since $(\#V_t).(\#E(x_t)) = (d-1)^{t-1}(d-1) = f(t)$, it follows

$$\sum_{t=r}^{R}\sum_{x_t\in V_t}\sum_{[x_t,x_{t+1}]\in E(x_t)}\left|\nabla\overline{u}\Big([x_t, x_{t+1}]\Big)\right|^p \geq \Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{1-p}.$$

We now sum up over the whole D_R, precisely

$$\begin{aligned} I_p(\overline{u}) &= \sum_{k=1}^{d}\sum_{t=r}^{R}\sum_{x_t\in V_t}\sum_{[x_t,x_{t+1}]\in E(x_t)}\left|\nabla\overline{u}\Big([x_t, x_{t+1}]\Big)\right|^p \\ &\geq d\Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{1-p}. \end{aligned}$$

As averaging transversally can only decreases the p-energy, so

$$I_p(u) \geq I_p(\overline{u}) \geq d\Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{1-p}.$$

But u is an arbitrary admissible function, so by definition of p-capacity,

$$Cap_p(D_r, D_R) \geq d\Big(\sum_{t=r}^{R} f(t)^{\frac{1}{1-p}}\Big)^{1-p}. \qquad (3.5.4)$$

Therefore, from (3.5.4) and (3.5.3), it follows the proof. ■

Proposition 3.5.5 *For $d \geq 3$ and $p = 1$, let T_d be a homogenous tree then $Cap_1(D_r, D_R) \geq 1$, for all r, R, $0 \leq r < R \leq \infty$.*

Proof. Let u be an arbitrary admissible function for 1-capacity. We take a branch of the tree and a path in it from generation r to generation R, then

$$\begin{aligned} I_1(u) &\geq \sum_{t=r}^{R} \left|\nabla u\Big([x_{t-1}, x_t]\Big)\right| \\ &\geq \left|\sum_{t=r}^{R} \nabla u\Big([x_{t-1}, x_t]\Big)\right| = 1, \end{aligned}$$

so, that $Cap_1(D_r, D_R) \geq 1$, for $1 \leq r < R \leq \infty$ ■

Corollary 3.5.6 *For $d \geq 3$ and $p = 1$, T_d are 1-hyperbolic.*

Proof. It is immediately consequence of the above Proposition and Proposition [3.3.1]. ■

Proposition 3.5.7 *For $p = \infty$. If T_d ($d \geq 2$) is a homogenous tree, then T_d is ∞-parabolic.*

Proof. Consider a condenser (D_r, D_R), $0 \leq r < R \leq \infty$. Define

$$u_{r,R}(x_t) = \begin{cases} 1 & \text{if } 0 \leq t \leq r, \\ 1 - (R-r)^{-1}(t-r) & \text{if } r \leq t \leq R, \\ 0 & \text{if } t \geq R, \end{cases}$$

$u_{r,R}$ is an admissible function for (D_r, D_R).

It is easy to verify that

$$\left|\nabla u_{r,R}\Big([x_t, x_{t+1}]\Big)\right| = \begin{cases} (R-r)^{-1} & \text{if } t = r \\ (R-r)^{-1}r & \text{if } r+1 \leq t \leq R-1. \\ 0 & \text{else} \end{cases}$$

hence, $I_\infty(u_{r,R}) = \max\{\frac{1}{R-r}, \frac{r}{R-r}\}$. But $u_{r,R}$ is an admissible function, thus

$$Cap_\infty(D_r, D_R) \leq I_\infty(u_{r,R}).$$

By Proposition [3.3.1- (ii)]

$$Cap_\infty(D_r, T_d) = \lim_{R\to\infty} Cap_\infty(D_r, D_R) = 0.$$

Take an arbitrary finite $K \subset T_d$ so, there exists some D_{r_0} such that $K \subseteq D_{r_0}$. Then, by monotonicity of p-capacity, it follows that

$$Cap_\infty(K, T_d) \leq Cap_\infty(D_{r_0}, T_d) = 0.$$

Thus, T_d is ∞-parabolic. ■

Theorem 3.5.8 *If T_d homogenous tree of degree $d \geq 3$, then*

- *T_d is p-hyperbolic for $1 \leq p < \infty$.*
- *T_d is ∞-parabolic.*

Proof. It is enough to combine (3.3.2) with above Propositions [3.5.1], [3.5.5], and [3.5.7]. ■

Next proposition shows a pathological example of tree.

Proposition 3.5.9 *For $d = 2$, T_2 satisfies*

1. *if $p = 1$ is 1-hyperbolic.*

2. *if $1 < p \leq \infty$ is p-parabolic.*

Proof. It is enough to observe that a homogenous tree of degree 2 coincides with the lattice $\mathbb{Z}$. ∎

3.6 p-Capacity and Graphs' Morphisms

We first define the concept of graphs' morphisms to establish a geometric relationship between graphs.

Secondly, having a morphism between two graphs, we can deduce a relationship of their p-capacities. They will provide us with very useful tools to compare the p-potential of the graphs. For example, to determine the p-hyperbolicity of certain types of non-homogeneous trees by comparing their capacities with capacities of the homogeneous tree T_3.

Finally, we can formulate the concept of isomorphism, from which we will get a natural notion of equivalence on the p-potential graph structures. In the following, $p \in [1, \infty]$ unless state otherwise.

Definitions: Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be graphs without boundaries. A map $\varphi : V_1 \longrightarrow V_2$ *preserves adjacency* if $x \sim y$ implies

$\varphi(x) \sim \varphi(y)$. Naturally, φ induces a map $\Phi = \varphi \times \varphi : E_1 \to E_2$, defined by $\Phi([x, y]) = [\varphi(x), \varphi(y)]$.

The pair (φ, Φ) is called a *morphism* from G to H and the subgraph $\varphi(G) = (\varphi(V_1), \Phi(E_1)) \subseteq H$ is called *image* of G in H.

Definition: A morphism (φ, Φ) is a *monomorphism* if φ and Φ are one-to-one. In particular, a monomorphism *onto* is called an *isomorphism.*

Remarks:

- If φ is an *monomorphism* if φ is one-to-one map that preserves adjacency, then $m_G(x) \leq m_H(\varphi(x))$, for all $x \in G$. In particular, a monomorphism *onto* is called an *isomorphism* if $m_G(x) = m_H(\varphi(x))$, for all $x \in G$.
- Sometimes, we will write φ as abbreviation for a morphism (φ, Φ).
- The *type problem* is the problem of determining whether a graph is p-parabolic or p-hyperbolic. Incidently, we adopted this terminology from similar one used on *random walks.*

Theorem 3.6.1 *Let G and H be graphs without boundaries. If φ is an isomorphism from G to H, then G and H have the same type.*

Proof. Let a be an arbitrary vertex at G. Consider an arbitrary $\epsilon > 0$. Then, there exists an admissible function for $\varphi(a)$ such that

$$I_p(u) < \mathrm{Cap}_p\big(\varphi(a), H\big) + \epsilon.$$

Define $\tilde{u} = u \circ \varphi^{-1}$. It is easy to verify that $\tilde{u}$ is admissible for G and $I_p(\tilde{u}) = I_p(u)$. Hence,

$$I_p(\tilde{u}) = I_p(u) < \mathrm{Cap}_p\big(\varphi(a), H\big).$$

By definition of p-capacity on G and arbitrariness of ϵ, we have

$$\mathrm{Cap}_p\big(a, G\big) \leq \mathrm{Cap}_p\big(\varphi(a), H\big).$$

Similarly, by symmetry,

$$\mathrm{Cap}_p\big(\varphi(a), H\big) \leq \mathrm{Cap}_p\big(a, G\big).$$

Together, the two inequalities give

$$\mathrm{Cap}_p\big(\varphi(a), H\big) = \mathrm{Cap}_p\big(a, G\big).$$

Now, it is easy to complete the proof. ∎

Definition: Let G and H be graphs without boundaries. A monomorphism φ is called an *embedding* from G to H if $\varphi(G)$ is taken without boundary induced by H, that is, $\partial\varphi(G) = \emptyset$.

Proposition 3.6.2 *Let G and H be graphs. If φ is an embedding, then for some arbitrary $a \in G$*

$$\mathrm{Cap}_p\big(a, G\big) = \mathrm{Cap}_p\big(\varphi(a), \varphi(G)\big) \leq \mathrm{Cap}_p\big(\varphi(a), H\big).$$

Proof. Since φ is one-to-one, obviously, φ is isomorphism over its image so, by the previous theorem,

$$\mathrm{Cap}_p\big(a, G\big) = \mathrm{Cap}_p\big(\varphi(a), \varphi(G)\big). \tag{3.6.3}$$

Take $\epsilon > 0$ and arbitrary. Then, there exists $u \in M(\varphi(a), V_2)$ such that

$$I_p(u) \le \mathrm{Cap}_p\big(\varphi(a), H\big) + \epsilon.$$

Evidently, $u\mid_{\varphi(V_1)}$ is admissible for $(\varphi(a), \varphi(G))$, i.e., $u\mid_{\varphi(V_1)} \in M(\varphi(a), \varphi(V_1))$ and

$$I_p(u\mid_{\varphi(V_1)}) \le I_p(u) \le \mathrm{Cap}_p\big(\varphi(a), H\big) + \epsilon.$$

Thus, by definition of $\mathrm{Cap}_p\big(\varphi(a), \varphi(G)\big)$ and arbitrariness of ϵ,

$$\mathrm{Cap}_p\big(\varphi(a), \varphi(G)\big) \le \mathrm{Cap}_p\big(\varphi(a), H\big).$$

By combining (3.6.3) and the above inequality the proof follows. ∎

In the following, the example shows that the embedding $\imath$ in this case may be named *cut.*

Example: Let $G^\flat$ be an arbitrary subgraph of G. Consider the inclusion $\imath$ in place of φ on the previous proposition. Then,

$$\mathrm{Cap}_p\big(a, G^\flat\big) \le \mathrm{Cap}_p\big(\imath(a), G\big). \tag{3.6.4}$$

In this case, one says that $G^\flat$ is cut from G. Notice that cutting can only decreases (or does not affect) the p-capacity.

Corollary 3.6.5 *Let G be a p-hyperbolic graph.*

If G can be embedding by φ on H, then H is p-hyperbolic.

Proof. It is enough to apply the Proposition [3.6.2] for φ. ■

Theorem 3.6.6 *Let $p \in [1, \infty)$. If T is a tree without terminals with $d = \min_{x \in T} m(x) \geq 3$, then T is p-hyperbolic.*

Proof. By Theorem [3.5.8] T_3 is p-hyperbolic for $p \in [1, \infty)$. As T_3 can be embedded in T, it follows from the corollary above that T is p-hyperbolic. ■

Corollary 3.6.7 *If a graph G is p-parabolic, then G has no homogeneous trees of degree $d \geq 3$ embedded in it.*

Proof. Obvious. ■

Definition: Let $G = (V, E)$ be graph and $U \subset V$. If φ is a morphism from G to G defined by $\varphi(x) = x$ on U and there exists $b \in \partial^i U$ such that $\varphi(x) = b$ for all $x \notin U$, then φ induces a morphism from G to $\tilde{G} = \varphi(G) = (V', \tilde{E}')$, a *multisubgraph without boundary*, where $V' = \varphi(V) = U$ and $\tilde{E}'$ consists of all edges xy where x and y are neighbours in U, all edges xb where $x \in \partial^i U$, and the self-loop bb. This φ we called a *shorting morphism.*

Proposition 3.6.8 *Let $G = (V, E)$ be a graph and $U \subset V$. If φ is a shorting morphism, then for some arbitrary $a \in U$*

$$\operatorname{Cap}_p(a, G) \leq \operatorname{Cap}_p(\varphi(a), \varphi(G))$$

Proof. Let $a \in U$ and $\varphi(a) \in V'$. For $\epsilon > 0$ and arbitrary there exists $u' \in M(\varphi(a), U \cup \{b\})$ such that, $I_p(u') < \text{Cap}_p\big(\varphi(a), \varphi(G)\big) + \epsilon$.

Define $u(x) = \begin{cases} u'(\varphi(x)) & ,x \in U \\ u'(b) & ,x \in V \setminus U. \end{cases}$

Now $u \in M(a, V)$. Indeed, $u(a) = u'(\varphi(a)) = 1$; since u' has finite support, automatically, u has finite support.

We have

- if $x, y \in U$, then $\nabla u[x, y] = \nabla u'[x, y]$.
- if $x \in \partial^i U$ and $y \in \partial^e U$, then $\nabla u[x, y] = u(y) - u(x) = u'(b) - u'(x) = \nabla u'[x, b]$.
- if $x, y \in V \setminus U$, then $\nabla u[x, y] = u(y) - u(x) = u'(b) - u'(b) = 0$.

Consequently,

$$I_p(u) = I_p(u') < \text{Cap}_p\big(\varphi(a), \varphi(G)\big) + \epsilon,$$

and, then, by definition of p-capacity and arbitrariness of ϵ we get

$$\text{Cap}_p\big(a, G\big) \leq \text{Cap}_p\big(\varphi(a), \varphi(G)\big). \quad \blacksquare$$

Remark: Notice that we have used multiple edges on the previous proposition.

Finally, to complete the section, we need to deal with graphs with boundaries.

Definition: Let $\Omega_G = G \cup \partial G$ and $\Omega_H = H \cup \partial H$ be graphs with boundaries.

A *morphism of graphs with boundaries* is a map between Ω_G and Ω_H that preserves adjacency and boundary, that is, $\varphi(\partial G) \subseteq \partial H$. In particular, one-to-one morphism is called *monomorphism of graphs with boundaries.*

Definition: An *embedding* φ between graphs with boundaries is an monomorphism of graphs with boundaries where $\partial\varphi(G) = \varphi(\partial G)$.

Proposition 3.6.9 *Let φ be an embedding from Ω_G to Ω_H. Then, for some $a \in \Omega_G$*

$$\mathrm{Cap}_p\big(a, \Omega_G\big) = \mathrm{Cap}_p\big(\varphi(a), \varphi(\Omega_G)\big) \leq \mathrm{Cap}_p\big(\varphi(a), \Omega_H\big).$$

Proof. The proof is exactly the same that is given on Proposition [3.6.2]. ∎

Chapter 4

VARIATIONAL p-CAPACITY

In this chapter, it shall be showed that each p-capacity can be represented in terms of a unique positive p-superharmonic function called p-capacitory function (Theorem [4.1.1]).

Additionally, from the existence of p-capacitory functions, classifications of graphs relative to p-parabolicity and p-hyperbolicity shall be obtained. As an application of that classification, we present a much stronger version of the convergence theorem as well as of its corollaries that were originally given in the appendix.

4.1 Variational p-Capacity

We obtain below a different characterization of p-capacity in terms of a unique function called the p-capacitory function.

Theorem 4.1.1 (Variational Capacity) *For $p \in (1, \infty)$, let G be a graph (perhaps with boundary). Then for every finite $K \subseteq V$ there exists a unique function u^*, called p-capacitory function, that satisfies the following properties:*

1. $u^* \in \mathcal{L}_0^{1,p}(V)$;
2. $u^* \equiv 1$ *on* K;
3. $0 \leq u^* \leq 1$ *on* V;
4. $\Delta_p u^* \equiv 0$ *on* $V \setminus K$;
5. $\text{Cap}_p(K) = I_p(u^*)$.

Proof. Let G be a graph (perhaps with boundary), and let $K \subseteq V$ finite arbitrary set of vertices. Consider an exhaustion $(G_i)_{i \geq 1}$ of G of *finite induced subgraphs* with *induced boundaries* induced by G such that $K \subset G_i$. Solving the Dirichlet problem [2.4.1] on each G_i under the conditions $u_i^* \equiv 1$ on K, $u_i^* \equiv 0$ on ∂V_i, and for $i \geq 2$, $u_i^* \equiv u_{i-1}^*$ on $\partial^i V_{i-1}$, we have

1. $u_i^* \in C_0(V_i \cup \partial V_i)$ by construction.
2. $u_i^* \equiv 1$ on K also by construction.
3. $\Delta_p u_i^* \equiv 0$ on $V_i \setminus K$

4. $0 \leq u_i^* \leq 1$ by Max/Min principle. since u_i^* is solution of the Dirichlet problem.

5. $\text{Cap}_p(K, G_i \cup \partial G_i) = I_p(u_i^*, G_i \cup \partial G_i)$.

5.) requires some argument. Indeed, in one hand 2.) and 4.) imply that u_i^* is an admissible function for p-capacity of (K, G_i), hence, by definition of p-capacity

$$\text{Cap}_p(K, G_i \cup \partial G_i) \leq I_p(u_i^*, G_i \cup \partial G_i). \tag{4.1.2}$$

On the other hand, for an arbitrary $\epsilon > 0$, there exists $u \in \mathcal{M}(K, V_i \cup \partial V_i)$ such that

$$I_p(u, G_i \cup \partial G_i) < \text{Cap}_p(K, G_i \cup \partial G_i) + \epsilon \tag{4.1.3}$$

We claim that $I_p(u_i^*) \leq I_p(u)$. In fact, from 3.) u_i^* is p-harmonic on $V_i \setminus K$ and from 2.) $u_i^* \equiv 1$ on K, so

$$I_p(u_i^*, G_i \cup \partial G_i) \leq I_p(u, G_i \cup \partial G_i). \tag{4.1.4}$$

Now, taking in account the arbitrariness of ϵ, from (4.1.3) and (4.1.4)

$$I_p(u_i^*, G_i \cup \partial G_i) \leq \text{Cap}_p(K, G_i \cup \partial G_i) \tag{4.1.5}$$

Together (4.1.2) and (4.1.5) complete the proof of the claim and the first step of the proof.

In the second step, it is the limit case, that is, the p-capacity of the condenser (K, G).

First, we extend the functions u_i^*, still denoted by u_i^*, to G by assigning zeros to the vertices on $V \setminus V_i$.

Secondly, we observe that, by construction, the sequence $\{u_i^*\}_{i\geq 1}$ on $C_0(V)$ is increasing and bounded, consequently, converges pointwisely on V to a function u^*.

Thus, from 2.) $u^* \equiv 1$ on K, from 4.) $0 \leq u^* \leq 1$ on V and from 3.) combined with Harnack principle [2.2.6] $\Delta_p u^* \equiv 0$ on $V \setminus K$.

Remained to prove 1.) and 5.)

In fact, by Proposition [3.3.1-ii]

$$\mathrm{Cap}_p(K, G) = \lim_{i\to\infty} \mathrm{Cap}_p(K, G_i \cup \partial G_i) = \lim_{i\to\infty} I_p(u_i^*),$$

that is, $\{[u_i^*]\}_{i\geq 1}$ is a minimizing sequence for the p-capacity of (K, G) on $\mathcal{L}_0^{1,p}(V)$. Now, first we observe that the $\mathcal{L}^{1,p}$-norm and I_p have the same minimizers, so $\{[u_i^*]\}_{i\geq 1}$ is also a minimizing sequence for the $\mathcal{L}^{1,p}$-norm. Since $p \in (1, \infty)$, hence, by Theorem [1.4.6-2] the sequence is a Cauchy sequence in $\mathcal{L}_0^{1,p}(V)$. However, $\mathcal{L}_0^{1,p}(V)$ are Banach spaces, so there exists $[u^\sharp] \in \mathcal{L}_0^{1,p}(V)$ such that $[u_i^*] \to [u^\sharp]$ in $\mathcal{L}^{1,p}$-norm. So

$$\mathrm{Cap}_p(K, G) = \lim_{i\to\infty} I_p(u_i^*) = I_p(u^\sharp)$$

Now, by Fatou's Lemma

$$I_p(u^*) \leq \lim_{i\to\infty} I_p(u_i^*) = I_p(u^\sharp)$$

which implies that $u^* \in \overline{\mathcal{M}(K,V)}$ (closure in $\mathcal{L}_0^{1,p}(V)$ of the admissible functions of K).

Hence, by definition of p-capacity, it follows

$$I_p(u^\sharp) = \mathrm{Cap}_p(K,G) \leq I_p(u^*) \leq I_p(u^\sharp),$$

that is,

$$I_p(u^*) = I_p(u^\sharp).$$

So $u^* = u^\sharp$ by Theorem [1.4.6-1].

Incidently, in the case of a graph with boundary an extra property is obtained, namely, 6) $u^* \equiv 0$ on ∂U. ■

Remark: For G *finite* (perhaps with boundary), the case $p = 1$ is included in the first part of the above proof, that is, for every $K \subseteq V$, there exists a unique 1-capacitory function satisfying all properties above mentioned, inclusively, $\mathrm{Cap}_1(K,G) = I_1(u^*)$.

The following proposition generalize the concept of capacity as it was first defined by Wiener [**20**] for compact sets in $\mathbb{R}^n$, ($n \geq 3$ case).

Proposition 4.1.6 *For $p \in (1,\infty)$, let G be a graph (perhaps with boundary). Then for every finite $K \subseteq V$*

$$\mathrm{Cap}_p(K) = -\sum_{y \in \partial K} D_{N,p} u^*(y), \tag{4.1.7}$$

where u^* is the p-capacitory function for the condenser (K,G).

Proof. a) First graphs without boundary.

For $u^* \in \mathcal{L}_0^{1,p}(V)$, $U = V \backslash \partial^i K$ and $\partial U = \partial^i K$, it follows that $(D_{N,p}u^*)^+ \equiv 0$ on ∂U and $(\Delta_p u^*)^+ \equiv 0$ on U, so we can apply the Corollary [A.1.14], that is,

$$\mathrm{Cap}_p(K) = \sum_{y \in \partial^i K} D_{N,p}u^*(y) = -\sum_{y \in \partial K} D_{N,p}u^*(y). \quad \blacksquare$$

b) Secondly, graphs with boundary.

The proof is exactly the same as a); but we must apply the Corollary [A.1.14] with $u^* \equiv 0$ on the vertices of the exterior boundary.

Proposition 4.1.8 *For $p \in (1, \infty)$.*

If G is a connected p-parabolic graph and K a finite set of vertices of G, then the capacitory function u^* for K is $u^* \equiv 1$ on G.

Proof. G is p-parabolic, then for every finite $K \subseteq V$ $\mathrm{Cap}_p(K) = 0$. On the other hand, by the variational theorem, there exists a unique p-capacitory function u^* such that $I_p(u^*) = \mathrm{Cap}_p(K)$.

Thus, $I_p(u^*) = 0$, that is, u^* is constant. However, $u^* \equiv 1$ on K, consequently, $u^* \equiv 1$ on G. $\blacksquare$

Proposition 4.1.9 *For $p \in (1, \infty)$, let $G = (V, E)$ be a graph. If there exists a finite subset of vertices of $G = (V, E)$ with p-capacity zero, then the p-capacity is zero for every finite subset of vertices.*

Proof. Let K be a such finite subset of vertices of $G = (V, E)$ with $Cap_p(K) = 0$. Take one vertex x of K. By monotonicity of p-capacity $Cap_p(\{x\}) = 0$. By the variational theorem [4.1.1] of p-capacity, there exists the p-capacitory function u^* which satisfies $I_p(u^*) = Cap_p(\{x\}) = 0$ and, consequently, $u^* = 1$. But we have shown before on the variational capacity theorem that $u^* = 1$ is limit of a Cauchy's sequence $\{u_i\}_{i\geq 1}$ on $\mathcal{L}^{1,p}(V)$-norm, such that $u_i \to 1$ pointwise on G and $I_p(u_i) \to 0$ as $i \to \infty$.
Let L be any other finite subset of vertices of G . Consider $\min_{x\in L}(u_i(x)) = a_i$, $i \in I\!N_*$, as L is finite we can get a well defined sequence $\{a_i\}_{i\geq 1}$ such that $a_i \to 1$ as $i \to \infty$
Now construct a new sequence $\{v_i\}_{i\geq 1}$ where $v_i = a_i^{-1} u_i$. Clearly, $v_i \geq 1$ on L, and $I_p(v_i) = a_i^{-p} I_p(u_i) \to 0$ when $i \to \infty$. On the other hand, from definition of p-capacity we obtain $Cap_p(L) \leq I_p(v_i)$, $i \in \mathbb{N}*$ which combined with the previous paragraph result gives $Cap_p(L) = 0$. ∎

For the corollary below let $p \in (1, \infty)$.

Corollary 4.1.10

1. *If G is p-hyperbolic, then for* every finite $K \subseteq V$ $\mathrm{Cap}_p(K) > 0$.

2. *If $G \cup \partial G$ is p-hyperbolic, then for* every finite $K \subseteq U$ $\mathrm{Cap}_p(K) > 0$.

Proof. 1) G is p-hyperbolic, so there exists a finite $K_0 \subseteq V$ such that $\mathrm{Cap}_p(K_0) > 0$.

Suppose to the contrary that there exists finite $K \subseteq V$ such that $\mathrm{Cap}_p(K) = 0$. Then, by the proposition above $\mathrm{Cap}_p(K_0) = 0$; a contradiction.

Therefore, for every finite $K \subseteq V$, $\mathrm{Cap}_p(K) > 0$.

2) Suppose to the contrary that there exists a finite $K \subseteq U$ such that $\mathrm{Cap}_p(K) = 0$. Then $I_p(u^*) = 0$, where u^* is the p-capacitory function. Hence, u^* is constant; a contradiction however, since $u^* \equiv 0$ on ∂U and $u^* \equiv 1$ on K.

Therefore, every finite $K \subseteq U$, $\mathrm{Cap}_p(K) > 0$. ■

Remark: The meaning of the results above is that the existence/non-existence of a finite set of vertices with *positive* p-capacity is a property of the graph not of the set.

Proposition 4.1.11 *For* $p \in (1, \infty)$.

1) If G is a connected p-hyperbolic graph without boundary and K a finite set of vertices of G, then the capacitory function u^* satisfies $0 < u^*(x) < 1$ for all x in at least one unbounded connected component C of $G - K$.

2) If $G \cup \partial G$ is a connected p-hyperbolic graph with boundary and K a finite set of vertices of G, then the capacitory function u^* satisfies $0 < u^*(x) < 1$ for all x at least one connected component C of $(G - K) \cup \partial G$ whose boundary ∂C contains vertices of ∂U.

Proof. 1.) will be proved in two steps.

a) Let $K \subseteq V$ be finite set of vertices. By variational theorem there exists a unique p-capacitory function u^* such that

$$\text{Cap}_p(K) = I_p(u^*).$$

On the other hand, G is p-hyperbolic, so by Corollary [4.1.10] $\text{Cap}_p(K) > 0$.

Thus, $I_p(u^*) > 0$.

Now consider the components C determined by $\partial^i K$. First, suppose that C is a bounded component, hence finite. We claim that $u \equiv 1$ on C. Indeed, as $\partial C \subseteq \partial^i K$ and u^* is p-harmonic on C, then necessarily $u \equiv 1$ on C by Theorem [2.4.1].

Secondly, if also u^* were identically 1 on every unbounded component, then u^* would be identically 1 on the whole graph. That does not happen however, since $I_p(u^*) > 0$.

Thus, there exists at least an unbounded component C and a point x_0 such that $u^*(x_0) < 1$. But u^* is p-harmonic on C, so the Maximum Principle [2.1.4] combined with the fact that $\partial C \subseteq \partial^i K$ imply that $u^*(x) < 1$ for all x in C.

b) Let C be an arbitrary unbounded component determined by $\partial^i K$. Suppose that there were $x_0 \in C$ such that $u(x_0) = 0$, then, x_0 would be a minimum point of u^*. However, u^* is p-harmonic on C, so by Minimum Principle u^*

would be identically zero on $C \cup \partial C$; this is a contradiction however, since $\partial C \subseteq \partial^i K$ and $u^* \mid_{\partial^i K} \equiv 1$.

Thus, $u^*(x) > 0$ for all $x \in C$.

By combining a) and b), it follows that there exists at least one unbounded connected component where $0 < u^*(x) < 1$ for all x in C.

2) By using the same arguments given on 1.), it follows that $I_p(u^*) > 0$. As before, consider all connected components determined by $\partial^i K$, then there exists at least a component C whose boundary ∂C contains vertices of ∂U and $\partial^i K$. Hence, u^* assumes only 0 and 1 at that boundary, but u^* is p-harmonic on C however, so, necessarily, by Max/Min principle $0 < u^*(x) < 1$ for all x in C. ∎

Remark: Notice that in both cases, the proposition above shows more, namely, $u^*(x) > 0$ for all x in G.

4.2 Classification

Proposition 4.2.1 *Suppose that $p \in [1, \infty)$ and G a finite connected graph without boundary. Then G is p-parabolic.*

Proof. Let $K \subseteq V$ be an arbitrary set of vertices. Then, by the variational theorem and the remark that follows it, there exists the p-capacitory function u^* such that $\mathrm{Cap}_p(K, G) = I_p(u^*)$. However, u^* is p-superharmonic, so by the Corollary [2.2.2] u^* is constant. Hence, $\text{Cap}_p(K,G) = 0$, that is, G is p-parabolic. ∎

Proposition 4.2.2 *Suppose that $p = \infty$ and G (non-necessarily finite)connected graph without boundary. Then G is ∞-parabolic.*

Proof. a) Suppose G is infinite. Choose a reference vertex o as origin and, similarly, as it was done for $\mathbb{Z}^n$ consider the condenser (D_r, D_R) with $0 \leq r < R \leq \infty$. Define

$$u_{r,R}(x_t) = \begin{cases} 1 & \text{if } 0 \leq t \leq r \\ 1 - (R-r)^{-1}(t-r) & \text{if } r+1 \leq t \leq R \\ 0 & \text{if } t \geq R \end{cases}$$

It is easy to see that $u_{r,R}$ is an admissible function for (D_r, D_R) and

$$|\nabla u_{r,R}[x_t, x_{t+1}]| = \begin{cases} (R-r)^{-1} & \text{if } t = r \\ (R-r)^{-1}r & \text{if } r+1 \leq t \leq R-1 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\text{Cap}_\infty(D_r, D_R) \leq I_\infty(u_{r,R}) = \max\{\frac{1}{R-r}, \frac{r}{R-r}\}.$$

Let $K \subseteq V$ be an arbitrary finite set of vertices, then there exists r_0 such that $K \subseteq D_{r_0}$. Now by monotonicity of the p-capacity, it follows that , for all $R > r_0$,

$$\text{Cap}_\infty(K, D_R) \leq \text{Cap}_\infty(D_{r_0}, D_R) \leq \max\{\frac{1}{R-r_0}, \frac{r_0}{R-r_0}\}.$$

Letting $R \longrightarrow \infty$, it follows that

$$\mathrm{Cap}_{\infty}(K, G) = 0$$

that is, G is ∞-parabolic.

b) G is finite. Obviously, G can be embedding into a infinite graph without boundary $\widetilde{G}$.

Consider K an arbitrary set of vertices on G. Then by Proposition [3.6.2]

$$\mathrm{Cap}_{\infty}(K, G) \leq \mathrm{Cap}_{\infty}(K, \widetilde{G}).$$

Now by part a) $\mathrm{Cap}_{\infty}(K, G) = 0$, that is, G is ∞-parabolic. ∎

Theorem 4.2.3 (Classification I) *Suppose $p \in [1, \infty]$.*

If G is a finite graph without boundary, then G is p-parabolic.

Proof. It follows from the Propositions [4.2.1] and [4.2.2]. ∎

It is a beautiful consequence of the theory of p-capacity combined with energy formula that any graph with boundary is, necessarily, a p-hyperbolic graph. We now begin to prove that.

Proposition 4.2.4 *Suppose that $p \in (1, \infty)$.*

If $G \cup \partial G$ (non-necessarily finite) graph with boundary, then $G \cup \partial G$ is p-hyperbolic.

Proof. Let $a \in U$ be an arbitrary vertex. By the variational theorem there exists a unique p-capacitory function u^*, that is, p-harmonic on $G - a$, $u^* \equiv 0$ on ∂U, and $u^*(a) = 1$. Then, by Max/Min principle on $G \cup \partial G$, it follows that $0 < u^*(x) < 1$ on $G - a$. Now, by applying the Proposition [4.1.6], it is easy to verify that

$$\mathrm{Cap}_p(a, G \cup \partial G) = -\sum_{y \in V(a)} D_{N,p} u^*(y) > 0.$$

Thus, by definition, $G \cup \partial G$ is p-hyperbolic. ■

Proposition 4.2.5 *Suppose that $p = 1$.*

$G \cup \partial G$ (non-necessarily finite) a graph with boundary, then $G \cup \partial G$ is 1-hyperbolic.

Proof. a) Suppose that $\Omega = G \cup \partial G$ is finite. Let $a \in U$, and let u be an arbitrary admissible function for 1-capacity for a. Fixing a (a-b)-path $a = x_0.x_1...x_n = b$ where $b \in \partial U$, by cancelling out, we have

$$I_1(u) \geq \sum_{t=0}^{n} |\nabla u[x_t, x_{t+1}]| \geq |\sum_{t=0}^{n} \nabla u[x_t, x_{t+1}]| = 1,$$
$$\mathrm{Cap}_1(a, G \cup \partial G) \geq 1.$$

Thus, by definition, $G \cup \partial G$ is 1-hyperbolic.

b) Suppose that $\Omega = G \cup \partial G$ is infinite. By [3.6.4] "cutting" $\Omega^\flat$ finite with boundary from Ω such that $\partial \Omega^\flat \subseteq \partial G$, we have

$$\mathrm{Cap}_1(a, \Omega^\flat) \leq \mathrm{Cap}_1(a, \Omega)$$

but by part a) $\mathrm{Cap}_1(a, \Omega^\flat) > 0$. Thus, by definition, Ω is 1-hyperbolic. ∎

Proposition 4.2.6 *Suppose that $p = \infty$.*

If $G \cup \partial G$(non-necessarily finite) is a graph with boundary, then $G \cup \partial G$ is ∞-hyperbolic.

Proof. a) Suppose that $G \cup \partial G$ is finite and by contradiction that $G \cup \partial G$ were ∞-parabolic. Take an arbitrary $a \in U$ and $\epsilon > 0$. Then, there exists $u_\epsilon \in \mathcal{M}(a, U \cup \partial U)$ such that

$$I_\infty(u_\epsilon) < \epsilon$$

consequently, $|\nabla u_\epsilon| < \epsilon$ for all edges on $G \cup \partial G$.

Now

$$I_1(u_\epsilon) = \frac{1}{2} \sum_{x \in U} \sum_{[x,y] \in \vec{E}'(x)} |\nabla u_\epsilon| < (\sum_{U \cup \partial U} m(x))\epsilon$$

By definition of capacity

$$\mathrm{Cap}_1(a, G \cup \partial G) \leq (\sum_{U \cup \partial U} m(x))\epsilon$$

which combined with arbitrariness of ϵ implies that $\mathrm{Cap}_1(a, G \cup \partial G) = 0$.

Let $K \subseteq U$ finite and arbitrary. Then, by using subadditivity, we conclude that $\mathrm{Cap}_1(K, G \cup \partial G) = 0$. Hence, $G \cup \partial G$ would be 1-parabolic which is a contradiction with Proposition [4.2.5]. Hence, $\mathrm{Cap}_\infty(K, G \cup \partial G) > 0$.

b) Suppose that $\Omega = G \cup \partial G$ is infinite with boundary. By [3.6.4] "cutting" Ω^b finite with boundary from Ω such that $\partial\Omega^b \subseteq \partial G$, we have for $a \in \Omega^b$

$$\mathrm{Cap}_\infty(a, \Omega^b) \le \mathrm{Cap}_\infty(a, \Omega)$$

but by part a) $\mathrm{Cap}_\infty(a, \Omega^b) > 0$, so

$$\mathrm{Cap}_\infty(a, \Omega) > 0,$$

so by definition, Ω is ∞-hyperbolic. ■

Theorem 4.2.7 (Classification II) *Suppose that* $p \in [1, \infty]$.
If $G \cup \partial G$ *(non-necessarily finite) graph with boundary, then* $G \cup \partial G$ *is* p*-hyperbolic.*

Proof. It follows from the Propositions [4.2.4], [4.2.5], and [4.2.6]. ■

4.3 Application

In this section, as consequence that *all graphs with boundary are p-hyperbolic* a much stronger versions of the convergence formula and its corollaries shall be get by removing certain hypotheses of the original ones in the appendix.

Corollary 4.3.1 (Convergence Formula) *Let* $G \cup \partial G$ *a graph with boundary (finite or infinite) with orientation* $(\vec{X} \cup \partial_N \vec{X})$, *where* $\partial_N \vec{X}$ *is the outer orientation of the boundary. Then, for any* $v = [v] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$, *and*

$\jmath \in L^q(\vec{E} \cup \partial\vec{E})$ *such that* $v\mathrm{div}(\jmath) \in L^1(U)$ *and* $v\mathrm{conv}(\jmath) \in L^1(\partial U)$ *the following formula holds*

$$\sum_{x\in U} \sum_{[x,y]\in(\vec{X}\cup\partial_N\vec{X})} (\nabla v.\jmath)([x,y]) = -\sum_{x\in U} v(x)\mathrm{div}(\jmath)(x) + \sum_{y\in\partial U} v(y)\mathrm{conv}(\jmath)(y). \tag{4.3.2}$$

Proof. By the Theorem [4.2.7] $G\cup\partial G$ is p-hyperbolic, consequently, by Theorem [5.1.7] each equivalence class of $\mathcal{L}_0^{1,p}(U\cup\partial U)$ reduces to only one representative. Hence, the extra hypotheses that were needed in the p-parabolic case can be removed. See [A.1.10]. ∎

Now, from the corollary above, we can deduce the analogue of Green's Identity for graphs.

Theorem 4.3.3 *$G\cup\partial G$ a graph with boundary with orientation $(\vec{X}\cup\partial_N\vec{X})$ where $\partial_N\vec{X}$ is the outer orientation of the boundary. Then, for any $u\in[u]\in\mathcal{L}^{1,p}(U\cup\partial U)$ and $v=[v]\in\mathcal{L}_0^{1,p}(U\cup\partial U)$ such that $v\Delta_p u\in L^1(U)$ and $vD_{N,p}u\in L^1(\partial U)$ the following formula holds*

$$\sum_{x\in U} \sum_{[x,y]\in(\vec{X}\cup\partial_N\vec{X})} (\nabla v.\nabla u)|\nabla u|^{p-2}([x,y]) = -\sum_{x\in U} v(x)\Delta_p u(x) + \sum_{y\in\partial U} v(y)D_{N,p}u(y).$$

Proof. The proof is the same of [A.1.13], however, we apply the Convergence Formula above. ∎

Corollary 4.3.4 *$G\cup\partial G$ a graph with boundary with orientation $(\vec{X}\cup\partial_N\vec{X})$ where $\partial_N\vec{X}$ is the outer orientation of the boundary. Then, for any*

$u = [u] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$ such that with $u\Delta_p u \in L^1(U)$ and $uD_{N,p}u \in L^1(\partial U)$ the following formula holds

$$\sum_{x \in U} \sum_{[x,y] \in (\vec{X} \cup \partial_N \vec{X})} |\nabla u[x,y]|^p = -\sum_{x \in U} u(x)\Delta_p u(x) + \sum_{y \in \partial U} u(y) D_{N,p} u(y).$$

Proof. Put $v = u$ in the formula on the above theorem. ∎

Corollary 4.3.5 *$G \cup \partial G$ a graph with boundary with orientation $(\vec{X} \cup \partial_N \vec{X})$ where $\partial_N \vec{X}$ is the outer orientation of the boundary.*
Let $u = [u], v = [v] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$ such that $v\Delta_p u, u\Delta_p v \in L^1(U)$ and $vD_{N,p}u, uD_{N,p}v \in L^1(\partial U)$. Then

$$\sum_{x \in U} \big(u(x)\Delta_p v(x) - v(x)\Delta_p u(x)\big) = \sum_{x \in U} \sum_{[x,y] \in (\vec{X} \cup \partial_N \vec{X})} (\nabla u . \nabla v)\left[|\nabla u|^{p-2} - |\nabla v|^{p-2}\right] + \sum_{y \in \partial U} \big(u(y) D_{N,p} v(y) - v(y) D_{N,p} u(y)\big).$$

Proof. It is enough to apply twice the previous corollary. ∎

Chapter 5

p-HYPERBOLIC GRAPHS

In this Chapter, it shall be presented characterizations of p-hyperbolic graphs in terms of existence of non-constant positive p-superharmonic functions. Estimates that allow to show that nonzero constants do not belong to the p-Dirichlet spaces $\mathcal{L}_0^{1,p}(V)$. Also, it shall be proved an important necessary and sufficient condition for a graph to be p-hyperbolic (Kelvin-Nevanlinna-Royden criterion) in terms of existence of a flow with certain special properties (Theorem [5.2.8]). Finally, in the last section, examples of application of the Kelvin-Nevanlinna-Royden criterion shall be given. Namely, the homogenous trees of degree $d \geq 3$ are p-hyperbolic, and finite Cartesian product of p-hyperbolic graphs is a p-hyperbolic graph (Theorem [5.4.1]).

5.1 Characterizations of p-Hyperbolic Graphs

In the following, $p \in (1, \infty)$ unless state otherwise.

Theorem 5.1.1 *Let $G = (V, E)$ be (connected) p-hyperbolic graph (perhaps with boundary). Then, there exists a function $v : V \longrightarrow \mathbb{R}$ such that*

1. $v \in \mathcal{L}_0^{1,p}(V)$;
2. *v is nonconstant (strictly) positive p-superharmonic function;*
3. *$\Delta_p v$ has finite support.*

Proof. The proof is applicable for both: graphs with boundary or without boundary.

Let a be an arbitrary vertex in V. Then, by the variational theorem [4.1.1], there exists a unique capacitory function u^*. Denoting u^* by v, we shall proved that v has all the required properties, precisely,

1. $v \in \mathcal{L}_0^{1,p}(V)$ from the variational theorem.
2. v is nonconstant, since by Proposition [4.1.10] $I_p(v) = \text{Cap}_p(a) > 0$.
3. v is strictly positive by the remark that follows the Proposition [4.1.11].
4. By the variational theorem, v is p-harmonic on $G - a$. It remains to show that $\Delta_p v(a) < 0$.

Indeed, by Proposition [4.1.11]

$$\Delta_p v(a) = \sum_{[a,y]\in\vec{E}(a)} |v(y) - v(a)|^{p-2}\big(v(y) - v(a)\big)$$
$$= \sum_{[a,y]\in\vec{E}(a)} |v(y) - 1|^{p-2}\big(v(y) - 1\big) < 0.$$

Thus, v is p-superharmonic on V.

5. Evidently, the support of $\Delta_p v$ is $\{a\}$, hence, finite. ■

Theorem 5.1.2 *A graph $G = (V, E)$ is p-hyperbolic if and only if G carries a non-constant positive p-superharmonic function u with finite p-Dirichlet energy.*

Proof. The necessity is a consequence of Theorem [5.1.1].

The sufficiency follows, by contradiction, from Theorem [6.1.1]. ■

Theorem 5.1.3 *All $G \cup \partial G$ carry a nonconstant positive p-superharmonic function with finite p-Dirichlet energy.*

Proof. In fact, by the classification theorem [4.2.7] $G \cup \partial G$ is p-hyperbolic, consequently, by the Variational Theorem [4.1.1] there exists a positive p-superharmonic function with finite p-energy. ■

Remark: Notice that also for $G\cup\partial G$ finite and $p = 1$ the result above holds.

The goal now is to obtain, in the context of infinite graphs, results analogous to those on Sobolev Spaces, namely, the Poincaré inequality and Rellich theorem.

Theorem 5.1.4 *For a graph G to be p-hyperbolic it is necessary and sufficient that for every finite set of vertices $K \subset V$ there exists a constant $C(K,p)$ such that*

$$\sum_{x \in K} |u(x)| \leq C(K,p)\|u\|_{1,p}, \quad \text{for all } u \in \mathcal{L}_0^{1,p}(V) \tag{5.1.5}$$

Proof. To prove sufficiency, suppose on the contrary that G is p-parabolic and (5.1.5) holds. By parabolicity, there exists $K \subset V$ finite such that $Cap_p(K) = 0$. But by the definition of p-capacity there exists a sequence of functions $\{u_i\}_{i \geq 1}$ on V with the following properties: for all $i \in \mathbb{N}_*$

a) $u_i \geq 1$ on K;

b) $u_i \in C_0(V)$;

c) $I_p(u_i) < \frac{1}{i}$.

We now apply the inequality (5.1.5) to this sequence to get

$$1 < \sum_{x \in K} |u_i(x)| \leq C(K,p)\|u_i\|_{1,p} \quad \text{for all } i,\, i \in \mathbb{N}_*$$

but then by c), and the fact that $\|u_i\|_{1,p} = I_p(u_i)^{\frac{1}{p}}$ we have

$$1 < C(K,p)I_p(u_i) < C(K,p)\frac{1}{i^{\frac{1}{p}}}, \quad i \in \mathbb{N}_*.$$

Now, by taking the limit as $i \to \infty$ we get a contradiction. Therefore, the condition above is sufficient for p-hyperbolicity.

We prove necessity in two steps.

It is enough to show the inequality holds for every K with single vertex. In fact, if the inequality holds for every K with single vertex, consider $K^\sharp = \{x_1, x_2, ..., x_m\}$ an arbitrary set of vertices of G. By denoting $K_i = \{x_i\}$ and the constant $C(K_i, p)$ such that, for all $u \in \mathcal{L}_0^{1,p}(V)$

$$|u(x_i)| \leq C(K_i, p)\|u\|_{1,p}.$$

Adding the inequalities above, we have

$$\sum_{i=1}^{m} |u(x_i)| \leq C(K^\sharp, p)\|u\|_{1,p}.$$

which shows the claim.

Now, let's show that the inequality holds for every K with single vertex. Suppose on the contrary that the inequality is false, i.e., there exists $K = \{x_0\}$ such that for every $k \in \mathbb{N}_*$ there exists $u_k \in \mathcal{L}_0^{1,p}(V)$:

$$1 = |u_k(x_0)| > k\|u_k\|_{1,p}. \tag{5.1.6}$$

We observe that $\{u_k\}$ is a sequence on $\overline{M(K,V)}$ (the closure in $\mathcal{L}_0^{1,p}(V)$ of the admissible functions of K), so that if u^* is the capacitory function of K it follows that

$$\frac{1}{k} > \|u_k\|_{1,p} \geq \|u^*\|_{1,p}.$$

Taking the limit, we get $||u^*||_{1,p} = 0$, consequently, u^* is constant. As $u^* = 1$ on K, necessarily u^* is identically 1 on V so that G is p-parabolic which is a contradiction.

Therefore, as K is an arbitrary finite set of vertices of G, the proof of the necessary condition follows. ■

Theorem 5.1.7 *The following conditions are equivalent*

1. *G is p-hyperbolic;*

2. *For every finite K there exists a constant $C(K,p)$ such that for all $u \in \mathcal{L}_0^{1,p}(V)$*

$$||u||_{L^p(K)} \leq C(K,p)||u||_{1,p}\,; \tag{5.1.8}$$

3. $1 \notin \mathcal{L}_0^{1,p}(V)$.

Proof. Notice that we are using $Vol(K)$ to denote the numbers of vertices of K.

First **1.)** implies **2.)**.

We have

$$||u||^p_{L^p(K)} = \sum_{x \in K} |u(x)|^p \leq \big(\sup_{x\in K}|u(x)|^p\big)Vol(K).$$

Taking the p^{th}-root gives

$$||u||_{L^p(K)} \leq \big(\sup_{x \in K}|u(x)|\big)Vol(K)^{\frac{1}{p}}.$$

Since $\sup_{x\in K}|u(x)| \leq \sum_{x\in K}|u(x)|$, it follows that

$$\|u\|_{L^p(K)} \leq \big(\sum_{x\in K}|u(x)|\big)\big(Vol(K)\big)^{\frac{1}{p}}.$$

As G is p-hyperbolic Theorem [5.1.4] yields a constant denoted by $\tilde{C}(K,p)$ such that for all $u \in \mathcal{L}_0^{1,p}(V)$,

$$\sum_{x\in K}|u(x)| \leq \tilde{C}(K,p)\|u\|_{1,p}.$$

Thus, for all $u \in \mathcal{L}_0^{1,p}(V)$,

$$\|u\|_{L^p(K)} \leq \tilde{C}(K,p)Vol(K)^{\frac{1}{p}}\|u\|_{1,p}$$

and to complete the proof, we denote $\tilde{C}(K,p)Vol(K)^{\frac{1}{p}}$ by $C(K,p)$.

2.) implies **3.)**

Suppose on contrary $1 \in \mathcal{L}_0^{1,p}(V)$, then inequality (5.1.8) is false.

3.) implies **1.)**

Suppose on contrary that G is p-parabolic then there exists K finite $\subset V$ such that $Cap_p(K) = 0$. Now, for every $\epsilon > 0$ there exists $u_\epsilon \in C_0(V)$, $u_\epsilon \equiv 1$ on K such that $I_p(u_\epsilon) < \epsilon$. Consequently, we have

$$\|u_\epsilon - 1\|_{1,p}^p = I_p(u_\epsilon - 1) = I_p(u_\epsilon) < \epsilon, \quad \text{for all } \epsilon > 0.$$

Thus, $\|u_\epsilon - 1\|_{1,p} \to 0$ as $\epsilon \to 0$; that is, $1 \in \mathcal{L}_0^{1,p}(V)$ which contradicts the hypothesis.

Therefore, if $1 \notin \mathcal{L}_0^{1,p}(V)$ then G is p-hyperbolic. ∎

Remark: It follows, from the theorem above, that if G is p-parabolic, then for all $c \in \mathbb{R}$, $[c] \in \mathcal{L}_0^{1,p}(V)$, consequently, (5.1.5) does not hold.

5.2 Kelvin-Nevanlinna-Royden Criterion

Lemma 5.2.1 *For $p \in (1, \infty)$.*

If G is p-hyperbolic (perhaps with boundary), then there exists a q-flow $\jmath$ on G such that

a) $\jmath \in L_1^q(\vec{E})$ where $(\frac{1}{p} + \frac{1}{q} = 1)$;

b) $\operatorname{div}(\jmath) \geq 0$ and $\operatorname{div}(\jmath) \not\equiv 0$;

c) $\operatorname{div}(\jmath)$ has finite support.

Proof. G is p-hyperbolic, then there exists $v : V \longrightarrow \mathbb{R}$ that satisfies the properties 1),2), and 3) of the Theorem [5.1.1].

Set $\jmath = -|\nabla v|^{p-2}\nabla v$. Then, we have $\operatorname{div}(\jmath) = \operatorname{div}(-|\nabla v|^{p-2}\nabla v) = -\Delta_p v$. So by 2) $\operatorname{div}(\jmath) = -\Delta_p v \geq 0$ and by 3) $\operatorname{div}(\jmath)$ has nonempty finite support.

Finally, to show that $\jmath \in L^q(\vec{E})$, we have to apply 1).

$$|\jmath|^q = ||\nabla v|^{p-2}\nabla v|^q = |\nabla v|^{q(p-1)} = |\nabla v|^p.$$

So, by the definition of the norm

$$\|\jmath\|_q^q = \frac{1}{2}\sum_{x\in V}\sum_{e\in\vec{E}(x)} |\jmath|^q(e) = \frac{1}{2}\sum_{x\in V}\sum_{e\in\vec{E}(x)} |\nabla v|^p(e) = I_p(v) < \infty$$

,that is, $\jmath \in L^q(\vec{E})$. ■

Lemma 5.2.2 (flow-capacity estimate) *Suppose that $p \in [1, \infty]$, and let G be a connected graph (perhaps with boundary).*

If $\jmath$ is a q-flow that satisfies

1. $\jmath \in L^q(\vec{E})$ (*where* $\frac{1}{p}+\frac{1}{q}=1$);

2. $\left(\operatorname{div}(\jmath)\right)^{-} \in L^1(V)$ *where* $\left(\operatorname{div}(\jmath)\right)^{-} = min\{0, \operatorname{div}(\jmath)\}$;

3. $0 < \sum_{x\in V} \operatorname{div}(\jmath)(x) \leq \infty$,

then there exists a finite $K \subseteq$ *supp* $\left(\operatorname{div}(\jmath)\right)$ *and a constant* $C(K)$, $0<C(K)<1$ *such that*

$$0 < C(K)\min_K(\operatorname{div}(\jmath))vol(K) \leq \operatorname{Cap}_p(K)^{\frac{1}{p}} \|\jmath\|_q. \tag{5.2.3}$$

Proof. For $p \in [1,\infty)$. First some notations would help the manipulations of the formulas. Consider $h = \operatorname{div}(\jmath)$, $h^+ = \max\{0, \operatorname{div}(\jmath)\}$, and $h^- = \min\{0, \operatorname{div}(\jmath)\}$, so that, $h = h^+ + h^-$.

a) We claim that there exists a finite subgraph $K \subset V$ such that $\gamma = \min_K\{h\} > 0$ and

$$\sum_{x\in K} h(x) > \left|\sum_{x\in V} h^-(x)\right|. \tag{5.2.4}$$

Indeed, by the assumptions

$$0 \leq \left|\sum_{x\in V} h^-(x)\right| = -\sum_{x\in V} h^-(x) < \sum_{x\in V} h^+(x),$$

which implies, easily, that there exists a finite set of vertices $K \subset V$ such that $h(x) = h^+(x) > 0$ $x \in K$, and

$$\left|\sum_{x\in V} h^-(x)\right| < \sum_{x\in K} h(x).$$

b) Since the inequality in a) is strict, there exists a $c \in \mathbb{R}$, $0 < c < 1$ such that

$$0 \geq \sum_V h^- > -c \sum_K h > -\sum_K h. \tag{5.2.5}$$

c) Take an arbitrary $\epsilon > 0$, from the definition of p-capacity of K, there exists an admissible function v with finite support such that

(i) $0 \leq v \leq 1$;

(ii) $v \equiv 1$ on K;

(iii) $I_p(v) < \mathrm{Cap}_p(K) + \epsilon$.

d) By (i) and (5.2.5)

$$0 \geq \sum_V vh^- \geq \sum_V h^- > -c \sum_K h = -c \sum_K vh;$$

so

$$\begin{aligned}(1-c)\sum_K vh &\leq \sum_K vh + \sum_V vh^- \\ &\leq \sum_K vh + \sum_V vh^- + \sum_{V \setminus K} vh^+ = \sum_V vh.\end{aligned}$$

e) Hence, it follows from $(1-c)\sum_K vh < \sum_V vh$ that

$$(1-c)(\min_K h)\mathrm{Vol}(K) \leq \sum_V vh. \tag{5.2.6}$$

f) On the other hand, $\nabla v \in L^p(\vec{E})$, since v has finite support, and $\jmath \in L^q(\vec{E})$ by hypothesis, so it follows from Hölder inequality that for $p \in (1, \infty)$

$$\sum_V \sum_{\vec{E}} |\nabla v||\jmath| \leq \Big(\sum_V \sum_{\vec{E}} |\nabla v|^p\Big)^{\frac{1}{p}} \Big(\sum_V \sum_{\vec{E}} |\jmath|^q\Big)^{\frac{1}{q}} = \|\nabla v\|_p \|\jmath\|_q. \tag{5.2.7}$$

Notice that for $p = 1$ the result follows the same way, except that the q-root is replace by sup.

Then, by (5.2.6), (5.2.7), and the divergence formula:

$$\begin{aligned}(1-c)\min_K(\mathrm{div}(\jmath))vol(K) &\leq \sum_{x\in V} v(x)\mathrm{div}(\jmath)(x)\\ &\underset{[A.1.1]}{=} -\frac{1}{2}\sum_V \sum_{[x,y]\in\vec{E}} (\nabla v.\jmath)[x,y]\\ &\leq \frac{1}{2}\sum_V \sum_{[x,y]\in\vec{E}} (|\nabla v||\jmath|)[x,y] \leq \frac{1}{2}\|\nabla v\|_p\|\jmath\|_q.\end{aligned}$$

From c), it follows that $\|\nabla v\|_p \leq (\mathrm{Cap}_p(K)+\epsilon)^{\frac{1}{p}}$ which combined with the inequality above, and taking into account the arbitrariness of ϵ, yields the following estimate

$$0 < C(K)\min_K(\mathrm{div}(\jmath))\mathrm{Vol}(K) \leq \|\jmath\|_q(\mathrm{Cap}_p(K))^{\frac{1}{p}},$$

where $C(K)$ denotes $(1-c)$.

Finally, for $p = \infty$ the procedure is practically the same. ∎

Remark: Notice that we have here a kind of dual of flow-capacity estimate, namely, the same hypothesis about G and with 1) $\jmath \in L^q(\vec{E})$, 2) $(\mathrm{div}(\jmath))^+ \in L^1(V)$, and 3) $-\infty \leq \sum_{x\in V}\mathrm{div}(\jmath)(x) < 0$, then there exists $K \subseteq supp(\mathrm{div}(\jmath))$ such that

$$0 < (c-1)\min_K(\mathrm{div}(\jmath)^-)\mathrm{Vol}(K) \leq \|\jmath\|_q(\mathrm{Cap}_p(K))^{\frac{1}{p}}.$$

Theorem 5.2.8 (Kelvin-Nevanlinna-Royden Criterion) *Let $p \in (1, \infty)$ and let be q its conjugate. Then, the graph $G = (V, E)$ is p-hyperbolic if and only if there exists a q-flow $\jmath$ on G satisfying the following conditions:*

1. $\jmath \in L^q(\vec{E})$ *(where $\frac{1}{p} + \frac{1}{q} = 1$);*

2. $\left(\mathrm{div}(\jmath)\right)^{-} \in L^1(V)$ *where* $\left(\mathrm{div}(\jmath)\right)^{-} = min\{0, \mathrm{div}(\jmath)\}$;

3. $0 < \sum_{x \in V} \mathrm{div}(\jmath)(x) \leq \infty$.

Proof. The necessity of the condition is exactly the Lemma [5.2.1]. The sufficiency is direct application of the flow-capacity estimate [5.2.2]. Indeed, because of the hypotheses, it follows that

$$0 < C(K) \min_K (\mathrm{div}(\jmath)) vol(K) \leq \mathrm{Cap}_p(K)^{\frac{1}{p}} \|\jmath\|_q.$$

which implies that $\mathrm{Cap}_p(K) > 0$ and, consequently, that G is p-hyperbolic. ∎

Proposition 5.2.9 *Let G be a graph (perhaps with boundary).*

If there exists a flow $\jmath$ that satisfies

a) $\jmath \in L^1(\vec{E})$ *(or $\jmath \in L^\infty(\vec{E})$) ;*

b) $\mathrm{div}(\jmath) \in L^1(V)$;

c) $\sum_{x \in V} \mathrm{div}(\jmath) \neq 0$.

Then, G is ∞-hyperbolic (resp. 1-hyperbolic).

Proof. For both cases, it is enough to use the flow-capacity estimate [5.2.2] and proceed on the same way it was done for the sufficient condition of Kelvin-Nevanlinna-Royden criterion. ∎

Remark: We can get a stronger proposition than above by considering on b) and c), respectively, either $\mathrm{div}(\jmath)^- \in L^1(V)$ and $0 < \sum_{x\in V} \mathrm{div}(\jmath) \leq \infty$ or their "duals".

The following theorem shows that on a p-hyperbolic graph can be assigned a unitary q-flow with a prescribed finite support. The theorem already appeared in [**21**, p.142] for infinite graphs without boundary. Here, however, it is presented with a new proof which also extends to graphs with boundary.

Theorem 5.2.10 *Suppose that $p \in (1, \infty)$, and G be a graph(perhaps with boundary).*
The following conditions are equivalents.
a) G is p-hyperbolic.
b) For every $K \subseteq V$ non-empty finite, there exists a flow $\jmath_K$ such that

1. $\jmath_K \in L^q(\vec{E})$.
2. *supp*$\{\mathrm{div}(\jmath_K)\} = K$.
3. $\mathrm{div}(\jmath_K) = 1$ *on* K.

Proof. First a) implies b). In fact, let $K = \{a\} \subseteq G$ be an arbitrary vertex. As G is p-hyperbolic, then, exactly as on the proof of Theorem [5.1.1], there exists u^* the p-capacitory function for K that allows to define the flow $\jmath = |\nabla u^*|^{p-2}\nabla u^*$ which satisfies 1), 2), and, after multiplying $\jmath$ by an adequate constant, a new flow $\jmath_a$ can be defined that satisfies 1), 2), and also 3), that is, $\mathrm{div}(\jmath_a)(a) = 1$.

In general, if $K = \{a_1, a_2, ..., a_k\}$ is an arbitrary set of vertices of G, then, from above part, there exist q-flows $\jmath_{a_1}, \jmath_{a_2}, ..., \jmath_{a_k}$ that satisfy all the properties given on b).

Now, defining a new flow by superposition

$$\jmath_K = \jmath_{a_1} + \jmath_{a_2} + ... + \jmath_{a_k},$$

it is easy to verify, that it has all the required properties on b).

Conversely, to show that b) implies a), it is enough to apply the sufficient condition of the Kelvin-Nevanlinna-Royden criterion [5.2.8]. ∎

5.3 q-flows

In this section, the divergence theorem [A.1.5] is restated to emphasize that we need less hypotheses on the hyperbolic case.

Theorem 5.3.1 *If G is p-hyperbolic, then $\forall u = [u] \in \mathcal{L}_0^{1,p}(V)$, $\forall \jmath \in L^q(\vec{E})$ such that* $u\mathrm{div}(\jmath) \in L^1(V)$ *imply*

$$\sum_{x \in V} \sum_{[x,y] \in \vec{X}} (\nabla u.\jmath)([x,y]) = -\sum_{x \in V} u(x)\mathrm{div}(\jmath)(x),$$

where $\vec{X}$ is an orientation of G.

Proof. It is enough to follow the same argument used to prove the divergence theorem see [A.1.5] and noting that the above hypotheses are enough to complete the proof. ∎

The following theorem shows that on a p-hyperbolic graph we can already find out functions in the class of bounded functions with finite p-energy $B\mathcal{L}^{1,p}(V)$ that do not satisfy the divergence formula.

Theorem 5.3.2 *G is p-hyperbolic if and only if $\exists u \in [u] \in B\mathcal{L}^{1,p}(V) \setminus B\mathcal{L}_0^{1,p}(V)$, $\exists \jmath \in L^q(\vec{E})$ such that* $\mathrm{div}(\jmath) \in L^1(V)$ *imply*

$$\sum_{x \in V} \sum_{[x,y] \in \vec{X}} (\nabla u.\jmath)([x,y]) \neq -\sum_{x \in V} u(x)\mathrm{div}(\jmath)(x),$$

where $\vec{X}$ is an orientation of G.

Proof. It is the contrapositive of Theorem [6.3.4] combined with theorem above. ∎

5.4 Examples

In this section, we shall give some examples of the criterion of Kelvin-Nevanlinna-Royden in the context of graphs. We are assuming $p \in (1, \infty)$.

The first application of the Kelvin-Nevanlinna-Royden criterion is to show that the homogenous trees T_d of degree d are p-hyperbolic.

For ease of notation, we prove the claim for T_3.

Let t be the geodesic distance from a vertex to the root a. We define a representative of the flow $\jmath$ as it is shown on the figure below.

It is easy to see that for $a \neq b$,

$$\mathrm{div}(\jmath)(a) = 1;$$

$$\mathrm{div}(\jmath)(b) = 0.$$

Thus, the flow satisfies, trivially, hypotheses 2) and 3) of the Kelvin-Nevanlinna-Royden criterion. It remains to prove 1), that is, $\jmath \in L^q(\vec{E})$, $q \in (1, \infty)$.

Indeed,

$$\begin{aligned}
\|\jmath\|_q^q &= 3\left(\frac{1}{3}\right)^q + \sum_{t=1}^{\infty} 3(2^t)\left[\left(\frac{1}{2}\right)^t\left(\frac{1}{3}\right)\right]^q \\
&= \left(\frac{1}{3}\right)^{q-1} + \sum_{t=1}^{\infty}\left(\frac{1}{3}\right)^{q-1}\left(\frac{1}{2}\right)^{tq-t} \\
&= \left(\frac{1}{3}\right)^{q-1} + \sum_{t=1}^{\infty}\left(\frac{1}{3}\right)^{q-1}\left(\frac{1}{2^{q-1}}\right)^{t} \\
&= \left(\frac{1}{3}\right)^{q-1} + \left(\frac{1}{3}\right)^{q-1}\sum_{t=1}^{\infty}\left(\frac{1}{2^{q-1}}\right)^{t} < \infty,
\end{aligned}$$

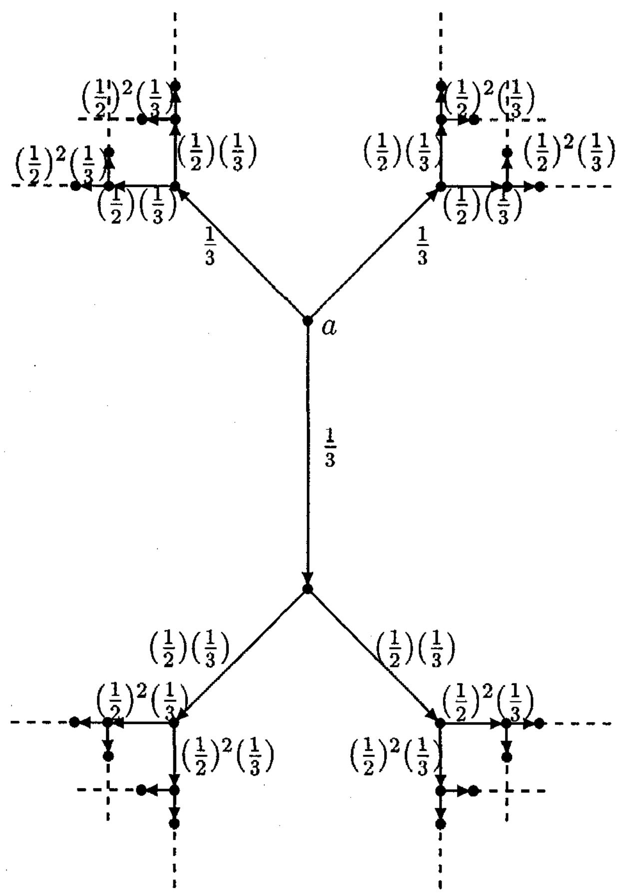

Figure 5.1 An oriented flow in a homogenous tree of degree 3

that is, $\jmath \in L^q(\vec{E})$. Therefore, by the Kelvin-Nevanlinna-Royden criterion, T_3 must be p-hyperbolic.

Remark: For the general case, it is enough to replace 3 by d and 2 by $d-1$ to get the proof.

The second application of the Kelvin-Nevanlinna-Royden criterion is to the Cartesian product of p-hyperbolic graphs.

Definition: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We define the Cartesian product of G_1 and G_2 denoted by $G_1 \times G_2$ as

• vertices : $V(G_1 \times G_2) = V_1 \times V_2$ and

•• edges : $(x_1, x_2) \sim (y_1, y_2) \iff \{x_1 y_1 \in E_1\}$ and $\{x_2 = y_2\}$

or

$\{x_1 = y_1\}$ and $\{x_2 y_2 \in E_2\}$.

Theorem 5.4.1 *For $p \in (1, \infty)$, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be p-hyperbolic graphs and $G = G_1 \times G_2$ their Cartesian product. Then G is p-hyperbolic.*

Proof. To simplify the notations, we shall omit the summation symbols on the vertices of each graph involved, that is, $\sum_{V(G)}$ etc...

Since by hypothesis G_1 and G_2 are p-hyperbolic graphs there exist a p-flow $\jmath_1$ for G_1 and a p-flow $\jmath_2$ for G_2 which satisfy the conditions of the Kelvin-Nevanlinna-Royden criterion.

Now, we define a flow in G as following.

$$J : \vec{E}(G) \longrightarrow \mathbb{R}$$

$$e = [(x_1, x_2), (y_1, y_2)] \longmapsto J(e) = \jmath_1([x_1, y_1]) + \jmath_2([x_2, y_2]).$$

where $\jmath_i([x_i, y_i]) \equiv 0$ if $x_i = y_i$.

Then J satisfies all three properties of of the Kelvin-Nevanlinna-Royden criterion. Indeed,

(i) J is a flow in G;

$$\begin{aligned} J\Big([(x_1,x_2),(y_1,y_2)]\Big) &= \jmath_1([x_1,y_1]) + \jmath_2([x_2,y_2]) \\ &= -\Big(\jmath_1([y_1,x_1]) + \jmath_2([y_2,x_2])\Big) \\ &= -J\Big([(y_1,y_2),(x_1,x_2)]\Big). \end{aligned}$$

(ii) $J \in L^q(\vec{E}(G))$;

Let $\vec{e} = [(x_1,x_2),(y_1,y_2)]$ be an edge of G, then

$$\begin{aligned} \|J\|_q^q &= \frac{1}{2}\sum_{\vec{e}\in\vec{E}(G)} |J(e)|^q \\ &= \frac{1}{2}\sum_{[(x_1,y_1),(x_2,y_2)]\in\vec{E}(G)} |\jmath_1([x_1,y_1]) + \jmath_2([x_2,y_2])|^q \\ &\le \frac{1}{2}\Bigg(\sum_{[x_1,y_1]\in\vec{E}(G_1)} |\jmath_1([x_1,y_1])|^q + \sum_{[x_2,y_2]\in\vec{E}(G_2)} |\jmath_2([x_2,y_2])|^q\Bigg) \\ &= \|\jmath_1\|_q^q + \|\jmath_2\|_q^q < \infty. \end{aligned}$$

Thus, $J \in L^q\left(\vec{E}(G)\right)$.

(iii)

$$\begin{aligned} \operatorname{div}(J)\big((x_1,x_2)\big) &= \sum_{[(x_1,x_2),(y_1,y_2)]\in\vec{E}((x_1,x_2))} J\big([(x_1,x_2),(y_1,y_2)]\big) \\ &= \sum_{\substack{y_1\sim x_1 \\ y_2\sim x_2}} \jmath_1([x_1,y_1]) + \jmath_2([x_2,y_2]) \\ &= \operatorname{div}(\jmath_1)(x_1) + \operatorname{div}(\jmath_2)(x_2). \end{aligned}$$

Thus, $\operatorname{div}(J)\left((x_1,x_2)\right) = \operatorname{div}(\jmath_1)(x_1) + \operatorname{div}(\jmath_2)(x_2)$ for every vertex (x_1,x_2) in $V(G)$.

We also have

$$\begin{aligned}\sum_{(x_1,x_2)\in V_1\times V_2} \left|(\operatorname{div}(J))^-((x_1,x_2))\right| &= \sum_{(x_1,x_2)\in V_1\times V_2} \left|\min\{\operatorname{div}(J)((x_1,x_2)),0\}\right| \\ &\le \sum_{x_1\in V_1} |\min\{\operatorname{div}(\jmath_1)(x_1),0\}| + \sum_{x_2\in V_2} |\min\{\operatorname{div}(\jmath_2)(x_2),0\}| \\ &= \|\operatorname{div}(\jmath_1)^-\|_1 + \|\operatorname{div}(\jmath_2)^-\|_1 < \infty.\end{aligned}$$

Thus, $(\operatorname{div}(J))^- \in L^1(V(G))$.

(iv)

$$\begin{aligned}\sum_{(x_1,x_2)\in V(G)} \operatorname{div}(J)((x_1,x_2)) &= \sum_{(x_1,x_2)\in V(G)} \operatorname{div}(\jmath_1)(x_1) + \operatorname{div}(\jmath_2)(x_2) \\ &= \sum_{x_1\in V_1} \operatorname{div}(\jmath_1)(x_1) + \sum_{x_2\in V_2} \operatorname{div}(\jmath_2)(x_2) > 0.\end{aligned}$$

Hence J satisfies the hypotheses of the Kelvin-Nevanlinna-Royden criterion [5.2.8], and we conclude that the graph $G=G_1\times G_2$ is p-hyperbolic. ■

Remarks:

- It follows, inductively, that the finite Cartesian product $G=G_1\times G_2\times \ldots \times G_n$ of p-hyperbolic graphs is also a p-hyperbolic graph.
- Note that p-parabolicity is not preserved by Cartesian products. For example, $\mathbb{Z}^2$ is parabolic. However, $\mathbb{Z}^2\times\mathbb{Z}^2$ is hyperbolic.

Chapter 6

p-PARABOLIC GRAPHS

6.1 A Characterization of p-Parabolic Graphs

The proofs of the Theorems [6.1.1] and [6.1.3] below are modelled, respectively, on the proofs of Theorems 5.2 and 5.7 in [**10**]. We assume that $p \in (1, \infty)$ unless stated otherwise.

Theorem 6.1.1 *A graph $G = (V, E)$ is p-parabolic if and only if every positive p-superharmonic function on G is constant.*

Proof. First of all the sufficiency. Suppose that every positive p-superharmonic function is constant.

Let K be an arbitrary finite subset of vertices on V. Then, by the variational capacity theorem [4.1.1], there exists a unique p-capacitory function

u^* that is p-superharmonic and

$$\mathrm{Cap}_p(K) = I_p(u^*),$$

However, by hypothesis u^* must be constant, consequently, $\mathrm{Cap}_p(K) = 0$, that is, G is p-parabolic.

Conversely, if G is p-parabolic graph, we prove that every positive p-superharmonic function is constant.

Suppose to the contrary that there were a *nonconstant* positive p-superharmonic function u on G. Let x_0 be an arbitrary vertex in V such that $u(x_0) < \infty$. Choose an arbitrary $\epsilon > 0$ and an exhaustion $G = (G_i)_{i \geq 1}$ of G by finite induced subgraphs with induced boundary such that $\{x_0\} \subset G_1$, $G_i \subset G_{i+1}$. Then, by induction and Theorem [2.4.1], there exists a unique p-harmonic function h_i which is equal to $u(x_0) - \epsilon$ at x_0, $h_i|_{\partial^i V_{i-1}} \equiv h_{i-1}|_{\partial^i V_{i-1}}$, $h_i|_{G-G_i} \equiv 0$ $(h_i \in C_0(V))$, and h_i is p-harmonic on $G_i - \{x_0\}$.

Now, define the sequence $\{u_i\}_{i \geq 1} = \left\{\frac{h_i}{u(x_0)-\epsilon}\right\}_{i \geq 1}$. It has the following properties:

1. $u_i \in C_0(V)$;

2. $u_i(x_0) = 1$;

3. $0 \leq u_i \leq 1$ on V;

4. $u_i \equiv 0$ on $G - G_i$;

5. $\{u_i\}_{i\geq 1}$ is an increasing sequence on V, by Min/Max principle;

6. Each u_i is p-harmonic on $G_i - \{x_0\}$.

Restricting each u_i to G_i, we note that u_i satisfies all properties of the p-capacitory function for (K, G_i), so by uniqueness

$$\mathrm{Cap}_p(K, G_i) = I_p(u_i)$$

On the other hand, the sequence $\{u_i\}_{i\geq 1}$ is an increasing and bounded sequence, so there exists $\tilde{u} \in C^0(V)$ defined by $\tilde{u} = \lim_{i\to\infty} u_i$ (pointwise convergence on V), which has the following properties:

1. $\tilde{u}(x_0) = 1$;

2. $0 \leq \tilde{u} \leq 1$ on V

3. $\tilde{u}$ is p-harmonic on $V \setminus \{x_0\}$, e.g., by Harnack principle [2.2.5].

However, to conclude that $\tilde{u}$ is the p-capacitory function of the condenser (K, G) it remains to prove that $\tilde{u} \in \mathcal{L}_0^{1,p}(V)$.

For, we use the same argument used in the variational capacity Theorem [4.1.1]. In fact, by Proposition [3.3.1-(ii)]

$$\mathrm{Cap}_p(K, G) = \lim_{i\to\infty} \mathrm{Cap}_p(K, G_i) = \lim_{i\to\infty} I_p(u_i),$$

In other words, the sequence $\{[u_i]\}_{i\geq 1}$ is a minimizing sequence of p-capacity for the condenser (K, G). The sequence is Cauchy by Theorem [1.4.6-2] and hence the sequence converges to $[u^\sharp]$ by completeness of $\mathcal{L}_0^{1,p}(V)$.

$$\mathrm{Cap}_p(K,G) = \lim_{i\to\infty} I_p(u_i) = I_p(u^\sharp)$$

Now by using the same arguments in the variational theorem [4.1.1], we have

$$\mathrm{Cap}_p(K,G) = I_p(u^\sharp) = I_p(\tilde{u})$$

Thus, $\tilde{u}$ satisfies

1. $\tilde{u} \in \mathcal{L}_0^{1,p}(V)$;

2. $\tilde{u}(x_0) = 1$;

3. $0 \leq \tilde{u} \leq 1$ on V;

4. $\tilde{u}$ is p-harmonic on $V \setminus \{x_0\}$

5. $\mathrm{Cap}_p(K,G) = \lim_{i\to\infty} \mathrm{Cap}_p(K,G_i) = \lim_{i\to\infty} I_p(u_i) = I_p(\tilde{u})$

But since only the p-capacitory function satisfies the above five properties, then, necessarily, $\tilde{u}$ must be the p-capacitory function of the condenser (K,G).

Since by hypothesis G is p-parabolic, it follows by Corollary [4.1.8-1)] that $\tilde{u} = 1$ on V, that is, $\lim_{i\to\infty} h_i(x) = u(x_0) - \epsilon$, for all $x \in V$. On the other hand, $u \geq h_i$ on the boundary of G_i, so by the comparison principle [2.3.2] $u \geq h_i$ on $G_i \setminus \{x_0\}$ and, hence, $u \geq h_i$ on V, for all i. Taking the limit, it follows that $u \geq u(x_0) - \epsilon$ on V and, consequently, by the arbitrariness of ϵ

that $u \geq u(x_0)$ on V. However, it was supposed that u were not constant, so there exists $x_1 \in V$ such that $u(x_1) > u(x_0)$.

Repeating the same argument given above, but now with $x_1 \in V$ in the place of x_0, we obtain $u \geq u(x_1)$ on V, which would be, evidently, a contradiction.

Therefore, if G is p-parabolic, then any positive p-superharmonic function is constant. ■

Corollary 6.1.2 *If G is a p-parabolic graph, then every bounded p-superharmonic function on G is constant.*

Proof. Suppose that u is bounded, that is, there exists a constant $M > 0$ such that $|u| < M$. Then, it is enough to apply the Theorem above to $u + M > 0$. ■

We now prove a *Liouville type* property which we call p-Liouville property.

Theorem 6.1.3 *If G is a p-parabolic graph, then:*

1. *every positive p-harmonic function on G is constant.*
2. *if u is p-harmonic function on G and either bounded above or bounded below, then u is constant.*

Proof. 1.) Suppose that u is a positive p-harmonic function on G, then u is, trivially, a positive p-superharmonic function on G, which by Theorem [6.1.1] is constant.

2.) Since $-u$ is p-harmonic if u is p-harmonic, we can assume that u is bounded below. As sum of a p-harmonic function and a constant λ is p-harmonic, we have $u + \lambda \geq 0$ for some λ and, hence, by part 1) $u + \lambda$ and, consequently, u are constant. ■

6.2 Harmonic Functions with p-Energy

In this section, we present a new proof that a p-harmonic function with finite p-energy on p-parabolic graph must be constant.

Theorem 6.2.1 *The following conditions are equivalent:*

1. *G is p-parabolic.*
2. $1 \in \mathcal{L}_0^{1,p}(V)$.
3. $\mathcal{L}^{1,p}(V) = \mathcal{L}_0^{1,p}(V)$.

Proof. 1) is equivalent to 2) follows, directly, by contradiction from the Theorem [5.1.7].

3) implies 2) trivial.

2) implies 3) is proved in [**21**, p.139]. ■

The theorem above is well known [**21**, p.139].

The following theorem also holds for Riemannian manifolds [**10**]. However, the proof below is new even for Riemannian manifolds.

A warning here is necessary about abused language. Instead functions with finite p-energy, actually, we are dealing with their classes on $\mathcal{L}_0^{1,p}(V)$.

Theorem 6.2.2 *If G is p-parabolic graph, then every p-harmonic function (class) in $\mathcal{L}_0^{1,p}(V)$ is constant.*

Proof. Let $u \in [u] \in \mathcal{L}_0^{1,p}(V)$ be a p-harmonic representative, then, trivially, $\Delta_p u, u\Delta_p u \in L^1(V)$. Now, by Corollary [A.1.14] with $\partial U = \emptyset$, it follows that

$$I_p(u) = -\sum_{x \in V} u(x)\Delta_p u(x) = 0,$$

which implies that u must be constant, consequently, its class is the class of the constants. ∎

6.3 q-flows

In this section, it shall be given some consequences of the Kelvin-Nevanlinna-Royden criterion for p-parabolic graphs.

Theorem 6.3.1 *Let $p \in (1, \infty)$, q its conjugate, and let G be a p-parabolic graph .*

If $\jmath \in L^q(\vec{E})$ such that either $\operatorname{div}(\jmath)^-$ or $\operatorname{div}(\jmath)^+ \in L^1(V)$, then $\sum_V \operatorname{div}(\jmath) = 0$.

Proof. It follows immediately from Kelvin-Nevanlinna-Royden criterion [5.2.8] by contraposition. ∎

The next theorem is inspired on a theorem of Gaffney for manifolds [**6**]. "If X is a vector field on a complete Riemannian manifold M such that $X \in L^1$ and $\operatorname{div}(X) \in L^1$, then $\int_M \operatorname{div}(X) = 0$".

Theorem 6.3.2 *Let G be a ∞-parabolic graph.*
If $\jmath \in L^1(\vec{E})$ such that either $\operatorname{div}(\jmath)^-$ or $\operatorname{div}(\jmath)^+ \in L^1(V)$, then $\sum_V \operatorname{div}(\jmath) = 0$.

Proof. It is enough to argue by contradiction, that is, suppose that there were a flow $\jmath \in L^1(\vec{E})$ such that $\operatorname{div}(\jmath)^-$ or $\operatorname{div}(\jmath)^+ \in L^1(V)$ such that $\sum_V \operatorname{div}(\jmath) \neq 0$. Then, by Proposition [5.2.9], G would be ∞-hyperbolic, a contradiction. ∎

The next theorem it is a kind of dual of the above theorem.

Theorem 6.3.3 *Let G be a 1-parabolic graph.*
If $\jmath \in L^\infty(\vec{E})$ such that either $\operatorname{div}(\jmath)^-$ or $\operatorname{div}(\jmath)^+ \in L^1(V)$, then $\sum_V \operatorname{div}(\jmath) = 0$.

Proof. It is enough to argue by contradiction, that is, suppose that there were a flow $\jmath \in L^\infty(\vec{E})$ such that $\operatorname{div}(\jmath)^-$ or $\operatorname{div}(\jmath)^+ \in L^1(V)$ such that $\sum_V \operatorname{div}(\jmath) \neq 0$. Then, by Proposition [5.2.9], G would be 1-hyperbolic, a contradiction. ∎

The following theorem gives a condition in terms of divergence (dissipation energy) formula [A.1.5] for a graph to be p-parabolic.

Theorem 6.3.4 *G is p-parabolic if and only if* $\forall u \in [u] \in B\mathcal{L}^{1,p}(V)$, $\forall \jmath \in L^q(\vec{E})$ *such that* $\mathrm{div}(\jmath) \in L^1(V)$ *imply*

$$\sum_{x \in V} \sum_{[x,y] \in \vec{X}} (\nabla u.\jmath)([x,y]) = -\sum_{x \in V} u(x)\mathrm{div}(\jmath)(x), \tag{6.3.5}$$

where $\vec{X}$ is an orientation of G.

Proof. Suppose that G is p-parabolic, then $B\mathcal{L}^{1,p}(V) = B\mathcal{L}_0^{1,p}(V)$ by Theorem [6.2.1]. From $u \in [u] \in B\mathcal{L}_0^{1,p}(V)$ and $\mathrm{div}(\jmath) \in L^1(V)$ result that $u\mathrm{div}(\jmath) \in L^1(V)$. Hence, by applying [A.1.5], it follows the proof of the claim.

Conversely, suppose, by contradiction, that G were p-hyperbolic, then by Theorem [5.2.10], with a slight modification, there exists a q-flow $\jmath$ such that $supp\{\mathrm{div}(\jmath)\} = K$ is nonempty finite, and $\sum_K \mathrm{div}(\jmath) = 1$. On the other hand, by substituting $u = 1 \in [c] \in B\mathcal{L}^{1,p}(V)$ and the above q-flow into the formula (6.3.5), it follows that $\sum_K \mathrm{div}(\jmath) = 0$, which would be a contradiction.

Therefore, G is p-parabolic. ∎

6.4 Examples

The following theorem indicates that the existence of nonconstant p-harmonic functions with a certain property depends essentially on p.

Theorem 6.4.1 *For $p \geq n \geq 2$.*

1) Every bounded p-harmonic function on $\mathbb{Z}^n$ is constant.

2) Every p-harmonic function on $\mathbb{Z}^n$ with finite p-energy is constant.

Proof. In fact, $\mathbb{Z}^n$ is p-parabolic for these p-values by [3.4.9], so 1) follows from Theorem [6.1.3], and 2) from Theorem [6.2.2]. ■

Chapter 7

p-POISSON EQUATIONS

In this Chapter, it shall be proved that on a p-hyperbolic graph a p-Poisson equation with source of finite support has unique solution in $\mathcal{L}_0^{1,p}(V)$. In contrast, on a p-parabolic graph a p-Poisson equation with unbalance source of finite support, has no solution in $\mathcal{L}_0^{1,p}(V)$. Their proofs except the uniqueness are modelled on the cases for manifolds that appeared, respectively, in [**19**] and [**7**].

In the last section of this Chapter which is called Criteria, it shall be proved a new criterion for a graph to be p-hyperbolic in terms of solution in $\mathcal{L}_0^{1,p}(V)$ of p-Poisson equations with sources of finite support. Similarly, its counterpart for p-parabolic graph shall be established.

In the same token another point of view can be obtained, namely, what is the condition the ambient must satisfy for which a p-Poisson equation with source of finite support either has solution or not. All these criteria are new even for manifolds.

7.1 Existence and Uniqueness

In this section, it shall be proved that on p-hyperbolic graphs a p-Poisson equation with *source of finite support* h has a solution in $\mathcal{L}_0^{1,p}(V)$, and it is unique.

Theorem 7.1.1 *For $p \in (1,\infty)$. Suppose that G is a p-hyperbolic graph and $h \in C_0(V)$. Then, $\Delta_p u + h = 0$ has a solution $u \in \mathcal{L}_0^{1,p}(V)$.*

Proof. We begin by choosing a finite set of vertices $K \subset V$ such that *supp* $h \subset K$.

We define a functional

$$\mathcal{I} : \mathcal{L}_0^{1,p}(V) \longrightarrow \mathbb{R}$$
$$u \longrightarrow \mathcal{I}(u)$$

where

$$\mathcal{I}(u) = \frac{1}{p}\|u\|_{1,p}^p - \sum_{x\in V}(hu)(x).$$

As G is p-hyperbolic we have, for all $u \in \mathcal{L}_0^{1,p}(V)$

$$\begin{aligned}\mathcal{I}(u) &\geq \frac{1}{p}\|u\|_{1,p}^p - |\sum_{x\in V}(hu)(x)| \\ &\geq \frac{1}{p}\|u\|_{1,p}^p - \|h\|_{L^\infty(V)}\|u\|_{L^1(K)} \\ &\geq \frac{1}{p}\|u\|_{1,p}^p - C(K,p)\|h\|_{L^\infty(V)}\|u\|_{1,p}\,.\end{aligned}$$

Since the function $f(x) = |x|^p - ax$ of real variable x is bounded below, we conclude that the functional $\mathcal{I}$ is bounded below on $\mathcal{L}_0^{1,p}(V)$.

Consider $m = \inf\{\mathcal{I}(u) : u \in \mathcal{L}_0^{1,p}(V)\}$ and the minimizing sequence $\{u_i\}_{i\geq 1}$ for $\mathcal{I}$ (that is, $\mathcal{I}(u_i) \to m$). Then, for all $i \in \mathbb{N}*$,

$$\begin{aligned}\mathcal{I}(u_i) &\geq \frac{1}{p}\|u_i\|_{1,p}^p - C(K,p)\|h\|_{L^\infty}\|u_i\|_{1,p} \\ &= \left[\frac{1}{p}\|u_i\|_{1,p}^{p-1} - C(K,p)\|h\|_{L^\infty}\right]\|u_i\|_{1,p}\end{aligned}$$

now taking the lim sup, it follows that $\{u_i\}_{\geq 1}$ is a bounded sequence in $\mathcal{L}_0^{1,p}(V)$.

We have seen that for each $p \in (1,\infty)$ $\mathcal{L}_0^{1,p}(V)$ is a reflexive Banach space, hence there exists a weakly convergent subsequence, still denoted by $\{u_i\}$, which converges weakly to u^* in $\mathcal{L}_0^{1,p}(V)$. Moreover,

$$\|u^*\|_{1,p} \leq \liminf \|u_i\|_{1,p} \tag{7.1.2}$$

From (5.1.5) one has $\|.\|_{L^1(K)} \leq c\|.\|_{1,p}$ which implies that $\mathcal{L}_0^{1,p}(V) \hookrightarrow L^1(K)$, consequently, the sequence $u_i\,|_K$ converges to $u^*\,|_K$ in $L^1(K)$, in particular,

$$\sum_K hu_i \to \sum_K hu^*. \tag{7.1.3}$$

From (7.1.2) and (7.1.3), we have

$$\mathcal{I}(u^*) = \frac{1}{p}\|u^*\|_{1,p}^p - \sum_V hu^* \leq \lim_{i\to\infty} \mathcal{I}(u_i) = m.$$

Therefore, $\mathcal{I}(u^*) = m$.

We can now consider small perturbations of the minimizer u^* and, then, derive the Euler-Lagrange equation $\Delta_p u^* = -h$. ∎

The above theorem is proved in [**22**, p.7] by using the technique of flows.

Theorem 7.1.4 (Uniqueness of the Solutions on $\mathcal{L}_0^{1,p}(V)$)

Let $p \in (1, \infty)$ and suppose that G is a p-hyperbolic graph.

If $u \in \mathcal{L}_0^{1,p}(V)$ is a solution of a Poisson Equation [7.1.1], then u is unique.

Proof. Suppose the equation has two solutions u^* and $\tilde{u}$ both on $\mathcal{L}_0^{1,p}(V)$. Then, as $\Delta_p u^*$ and $\Delta_p \tilde{u}$ have finite support, we can apply the analog of Green's identity [A.1.13] with $\partial U = \emptyset$ to get

$$-\sum_{supp\, h\subset K} (u^* - \tilde{u})\Delta_p u^* = \sum_{x\in V}\sum_{[x,y]\in\vec{X}} \nabla(u^* - \tilde{u})\nabla u^* |\nabla u^*|^{p-2}$$

and

$$-\sum_{supp\, h\subset K} (u^* - \tilde{u})\Delta_p \tilde{u} = \sum_{x\in V}\sum_{[x,y]\in\vec{X}} \nabla(u^* - \tilde{u})\nabla \tilde{u} |\nabla \tilde{u}|^{p-2};$$

so

$$\sum_{x\in V}\sum_{[x,y]\in\vec{X}} |\nabla u^*|^p + |\nabla \tilde{u}|^p - (\nabla u^* \nabla \tilde{u})(|\nabla u^*|^{p-2} + |\nabla \tilde{u}|^{p-2}) = 0. \qquad (7.1.5)$$

Now, as $g(x,y) = |x|^p + |y|^p - xy(|x|^{p-2} + |y|^{p-2}) \geq 0$ and $g(x,y) = 0$ if, only if $x = y$; we conclude from [7.1.5] that

$$\nabla u^* = \nabla \tilde{u}$$

which implies

$$u^* = \tilde{u} + c,$$

that is, the solutions u^* and u are in the same equivalence class. ■

The above theorem is also proved in [**22**, p.7], but here we present a new proof.

Note that the following theorem holds for both p-parabolic and p-hyperbolic graphs. In this case, we may say that the source h is balance and with finite support.

Theorem 7.1.6 *Let $p \in (1, \infty)$ and $G = (V, E)$ a graph.*
If $h \in C_0(V)$ satisfies $\sum_{x \in V} h(x) = 0$. Then $\Delta_p u + h = 0$ has a unique solution in $\mathcal{L}_0^{1,p}(V)$.

Proof. It is proved in [**22**, p.8]. ■

7.2 Non-Existence

Theorem 7.2.1 *Let $G = (V, E)$ be a p-parabolic graph and h a function such that $h^- \in L^1(V)$ (or $h^+ \in L^1(V)$) $\sum_{x \in V} h(x) > 0$ (resp. $\sum_{x\in V} h(x) < 0$). Then the Poisson equation $\Delta_p u + h = 0$ has no solution $u \in [u] \in \mathcal{L}_0^{1,p}(V)$.*

Proof. Assume that a solution u exists and, then, define the flow $\jmath = -|\nabla u|^{p-2}\nabla u$.

We shall check now that $\jmath$ satisfies the following conditions:

1. $\jmath \in L^q(\vec{E})$;

$$\sum_{e\in E} |\jmath(e)|^q = \sum_{e\in E} |\nabla u|^{(p-2)q} |\nabla u|^q = \sum_{e\in E} |\nabla u|^{pq-q}$$
$$= \sum_{e\in E} |\nabla u|^{(p-1)q} = \sum_{e\in E} |\nabla u|^p < \infty.$$

2. $\operatorname{div}(\jmath) = \operatorname{div}(-|\nabla u|^{p-2}\nabla u) = -\Delta_p u = h$ with $\operatorname{div}(\jmath)^- = h^- \in L^1(V)$ $\big($resp. $\operatorname{div}(\jmath)^+ = h^+ \in L^1(V)\big)$.

3. $\sum_{x\in V} \operatorname{div}(\jmath(x)) = \sum_{x\in V} h(x) > 0$. (resp. < 0).

Thus, $\jmath$ satisfies the hypotheses of the Kelvin-Nevanlinna-Royden criterion, consequently, G is p-hyperbolic which contradicts the hypothesis. ∎

7.3 Criteria

The theorems below give a complete answer whether or not a p-Poisson equation has solution in terms of the type of graph (ambient) and source function with finite support.

Theorem 7.3.1 (Criterion)

A graph G (perhaps with boundary) is p-hyperbolic graph if and only if the Poisson equation $\Delta_p u + h = 0$ has unique solution in $\mathcal{L}_0^{1,p}(V)$ for all $h \in C_0(V)$.

Proof. For necessity of the p-hyperbolicity, suppose that G is p-hyperbolic, then from the Theorems [7.1.1] and [7.1.4] follow that for each $h \in C_0(V)$ the Poisson Equation has a unique solution.

For sufficiency of the p-hyperbolicity, it is enough to argue by contradiction. In fact, suppose that G were p-parabolic, then from Theorem [7.2.1], we would get a contradiction with the hypotheses above. ∎

Theorem 7.3.2 (Criterion)

A graph G is p-parabolic if and only if the p-Poisson equation $\Delta_p u + h = 0$ has no solution in $\mathcal{L}_0^{1,p}(V)$ for all (sources) $h \in C_0(V)$ such that $\sum_{x \in V} h(x) \neq 0$.

Proof. The condition is necessary for the p-parabolicity of G. Indeed, it is Theorem [7.2.1] since, trivially, $h \in L^1(V)$.

To prove that the condition is sufficient for p-parabolicity, it is enough to argue by contradiction on the Theorem [7.3.1] above . ■

Remark: Note that Theorem [7.1.6] shows that we cannot extend the above Theorem to include the cases of balanced sources with finite support.

A Physics interpretation of Theorem [7.3.2] together with Theorem [7.1.6] is that on a p-parabolic graph/manifold have a q-flow $\jmath$ with sources and/or sinks and $\mathrm{div}(\jmath)$ with finite support, only if the conservation of energy law holds, namely, $\sum_V \mathrm{div}(\jmath) \equiv 0$.

Appendix A

THE DIVERGENCE FORMULA

A.1 Divergence and Green type formulas

In this appendix, it is presented some fundamental theorems of the p-potential theory. They related the dissipation of energy on the vertices (nodes) with dissipation of the energy on the edges (branches) of a graph (network).

The first theorem is a extension of a version of the *principle of conservation of energy* in [5, p.61-62]. It is based on a generalization of the concept of flows and number of poles. By the way, we could also say that the formula is the discrete analog of the divergence formula in the integral case.

Theorem A.1.1 *Let $G = (V, E)$ be a graph with a orientation $\vec{X}$.*
Let $v \in C_0(V)$ or $\jmath \in \Phi(\vec{E})$ with finite support. Then, $\mathrm{supp}\big(v\mathrm{div}(\jmath)\big) = \{a_1, ..., a_k\}$ *is finite and the following formula holds*

$$\sum_{x \in V} \sum_{[x,y] \in \vec{X}} (\nabla v \jmath)\big([x, y]\big) = -\sum_{i=1}^{k} v(a_i)\mathrm{div}(\jmath)(a_i). \tag{A.1.2}$$

Proof. It is enough to write explicitly the definition of the gradient on the left hand side and use the fact that summation shall be of finite number of terms, hence they can be reordered (*parts*) under the light of the orientation to get the right hand side. ■

Remark: Notice that the condition $\mathrm{supp}\big(v\mathrm{div}(\jmath)\big) = \{a_1, ..., a_k\}$ be finite is not enough to guarantee that the theorem above holds.

In fact, consider $v = c$ and $\jmath$ from Theorem [5.2.10] on a p-hyperbolic graph. It is easy to see that $\mathrm{supp}\big(v\mathrm{div}(\jmath)\big) = \{a_1, ..., a_k\}$ is finite, however the *principle of conservation of energy* above does not hold.

In the following, we consider $p \in [1, \infty)$ unless state otherwise.

Lemma A.1.3 *Let G be a connected graph. For every $x \in V$, for every $[u] \in \mathcal{L}^{1,p}(V)$ there exist a constant $M(x) > 0$ and a representative of $[u]$, still denoted by u, such that*

$$|u(x)| \leq M(x)\|u\|_{1,p}.$$

Proof. Fix x_0. Without loss of generality, we may assume that $u(x_0) = 0$. By assumption, for every edge $[x, y]$

$$|\nabla u[x, y]|^p \leq I_p(u) < \infty,$$

from which it follows that

$$|\nabla u| \leq \|u\|_{1,p}.$$

Then, if $d(x_0, x) = k$, consider a path $x_0 \sim x_1 ... \sim x_k = x$, so that

$$|u(x)| \leq k \max|\nabla u| \leq k\|u\|_{1,p}$$

where we denote $M(x) = k$. ∎

Lemma A.1.4 *Let $G = (V, E)$ be a graph.*

For every $v \in [v] \in \mathcal{L}_0^{1,p}(V)$, there exists a representative v and a sequence $\{v_i\}_{i\geq 1}$ with the following properties:

1. *$v_i \in C_0(V)$, $|v_i| \leq |v|$ on V;*

2. *$\{v_i\}_{i\geq 1}$ converges* pointwisely *and in the* norm *to v.*

Proof.

1) First, suppose that G is p-hyperbolic. Let $v = [v]$ be an arbitrary class in $\mathcal{L}_0^{1,p}(V)$. Then, by definition, there exists a sequence $\{u_i\}_{i\geq 1}$ on $C_0(V)$ that converges to v in the norm and by Theorem [5.1.4] also pointwise.

2) Secondly, suppose that G is p-parabolic. Let $[v]$ be an arbitrary class in $\mathcal{L}_0^{1,p}(V)$. Then, by Lemma [A.1.3] and the definition of $[v]$, there exists a

representative v and a sequence $\{u_i\}_{i\geq 1}$ on $C_0(V)$ that converges to v in the norm and pointwise.

To obtain the desired sequence for both cases, it is enough to truncate $\{u_i\}_{i\geq 1}$ by v, precisely,

$$v_i(x) = \begin{cases} u_i(x) & \text{if } |u_i(x)| \leq |v(x)| \\ v(x) & \text{if } |u_i(x)| > |v(x)|. \end{cases}$$

Now, it is easy to see that $\{v_i\}_{i\geq 1}$ has all the required properties.

Indeed, by construction, it follows that $|v_i| \leq |v|$ on V and $v_i \to v$ pointwisely. To show that $\{v_i\}_{i\geq 1}$ converges in the norm to v, it is enough to note that on V

$$|v_i - v|^p \leq |u_i - v|^p,$$

thus,

$$0 \leq I_p(v_i - v) \leq I_p(u_i - v) \to 0 \text{ as } i \to \infty. \quad \blacksquare$$

Theorem A.1.5 (Divergence) *Let $G = (V, E)$ be an oriented graph with an orientation $\vec{X}$. For $v \in [v] \in \mathcal{L}_0^{1,p}(V)$ and flow $\jmath \in L^q(\vec{E})$ such that either* $\mathrm{div}(\jmath)^-$ *or* $\mathrm{div}(\jmath)^+ \in L^1(V)$ *and* $v\mathrm{div}(\jmath) \in L^1(V)$ *the following formula holds*

$$\sum_{x\in V} \sum_{[x,y]\in\vec{X}} \left(\nabla v.\jmath\right)\left([x,y]\right) = -\sum_{x\in V} v(x)\mathrm{div}(\jmath)(x). \tag{A.1.6}$$

Proof. For $p \in (1, \infty)$. Let $[v]$ be an arbitrary class in $\mathcal{L}_0^{1,p}(V)$. Then, by the previous Lemma [A.1.4], there exists a representative, still denoted by v, and a sequence $\{v_i\}_{i\geq 1}$ in $C_0(V)$ such that

1. $|v_i| \leq |v|$ on V.

2. $\{v_i\}_{i\geq 1}$ converges to v in the norm and pointwise.

Thus, we have,

$$|\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v.\jmath) - \sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v_i.\jmath)| = |\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v - \nabla v_i)\jmath|$$
$$\leq \sum_{x\in V}\sum_{[x,y]\in\vec{X}}|\nabla v - \nabla v_i||\jmath| \leq \Big(\sum_{x\in V}\sum_{[x,y]\in\vec{X}}|\jmath|^q\Big)^{\frac{1}{q}}\Big(\sum_{x\in V}\sum_{[x,y]\in\vec{X}}|\nabla v - \nabla v_i|^p\Big)^{\frac{1}{p}} \to 0$$

when $i \to \infty$,

that is,

$$\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v.\jmath) = \lim_{i\to\infty}\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v_i.\jmath). \tag{A.1.7}$$

Since v_i has finite support, we sum by parts, obtaining

$$\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v_i.\jmath) = -\sum_{x\in V} v_i \mathrm{div}(\jmath),$$

which combined with (A.1.7) yields

$$\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v.\jmath) = -\lim_{i\to\infty}\sum_{x\in V} v_i \mathrm{div}(\jmath). \tag{A.1.8}$$

As $\{v_i\}_{i\geq 1}$ satisfies $|v_i| \leq |v|$ on V, we have

$$|v_i \mathrm{div}(\jmath)| \leq |v \mathrm{div}(\jmath)| \in L^1(V).$$

and, consequently,

$$\lim_{i\to\infty}\sum_{x\in V} v_i \mathrm{div}(\jmath) = \sum_{x\in V} v \mathrm{div}(\jmath). \tag{A.1.9}$$

From (A.1.8) and (A.1.9), it follows

$$\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v.\jmath) = -\sum_{x\in V} v\mathrm{div}(\jmath).$$

For $p = 1$ and $q = \infty$ the same proof holds with slit change, namely, rather q-root the sup should be used on the inequalities like above.

Finally, we need to show that, for both cases, the divergence formula does not depend on a particular representative.

In fact, if G is p-hyperbolic, then nothing to prove, since from [5.1.7] its classes are trivial, that is, $v = [v]$.

For $p \in [1, \infty]$, if G is p-parabolic, then by applying Theorems [6.3.1] and [6.3.3] we have $\sum_V \mathrm{div}(\jmath) \equiv 0$. Now, we take an arbitrary $\hat{v} \in [v]$, precisely, $\hat{v} = v - c$, $c \in \mathbb{R}$

$$\begin{aligned}\sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla\hat{v}.\jmath)([x,y]) &= \sum_{x\in V}\sum_{[x,y]\in\vec{X}}(\nabla v.\jmath)([x,y])\\ &= -\sum_{x\in V} v(x)\mathrm{div}(\jmath)(x) = -\sum_{x\in V} v(x)\mathrm{div}(\jmath) + c\underbrace{\sum_{x\in V}\mathrm{div}(\jmath)(x)}_{\equiv 0}\\ &= -\sum_{x\in V}\hat{v}(x)\mathrm{div}(\jmath),\end{aligned}$$

that is, the divergence formula does not depend on a particular representative. ■

We now define the analogue of outer normal derivative for manifolds.

In the following, we denote by U the sets of vertices of the graph G, and by $\partial^i U \cup \partial U$ the set of vertices of the boundary graph ∂G.

Definition: Let $G \cup \partial G$ be an oriented graph with boundary (finite or infinite). For $u \in C^0(U \cup \partial U)$,

1. the *normal p-derivative* of u at $y \in \partial U$ is

   - for $p \geq 2$

$$D_{N,p}u(y) = \sum_{\substack{x \in \partial^i U \\ [x,y] \in \partial_N \vec{X}}} |u(y) - u(x)|^{p-2}(u(y) - u(x)),$$

   - for $1 \leq p < 2$, the same formula that is on the previous item with the interpretation that when $u(y) - u(x) = 0$ the corresponding terms in the summation are zeros;

2. the *tangential p-derivative* of u at $y \in \partial U$ is

   - for $p \geq 2$

$$D_{T,p}u(y) = \sum_{\substack{x \in \partial U \\ [x,y] \in \partial_T \vec{X}}} |u(x) - u(y)|^{p-2}(u(x) - u(y)).$$

   - for $1 \leq p < 2$, the same formula that is on the previous item with the interpretation that when $u(x) - u(y) = 0$ the corresponding terms in the summation are zeros.

Remark: The idea of normal p-derivative already appear in [**12**, p.58] for $p = 2$ on finite subnetworks but with a restriction that the boundary vertices are grounded.

Definition: Let ∂G be a boundary graph. The *convergence operator* is the linear operator given by

$$\text{conv} : \phi(\partial\vec{E}) \longrightarrow C^0(\partial U)$$
$$\jmath \longmapsto \text{conv}(\jmath)$$

such that for each $y \in \partial U$

$$\text{conv}(\jmath)(y) := \sum_{e\in\partial\vec{X}[y]} \jmath(e),$$

where $\partial\vec{X}[y]$ represents the set of oriented edges of $\partial\vec{X}$ that have the terminal vertex (head) at y.

Corollary A.1.10 (Convergence Formula) *Let $G \cup \partial G$ be an oriented graph with boundary (finite or infinite) with orientation $(\vec{X} \cup \partial\vec{X})$, $\partial\vec{X} = \partial_T\vec{X} \cup \partial_N\vec{X}$ where $\partial_N\vec{X}$ is outer orientation. Then, for any $v \in [v] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$, and $\jmath \in L^q(\vec{E} \cup \partial\vec{E})$ such that $\text{div}(\jmath)^- \in L^1(U)$, (or $\text{div}(\jmath)^+ \in L^1(U)$), $\text{conv}(\jmath)^- \in L^1(\partial U)$ (resp. $\text{conv}(\jmath)^+ \in L^1(\partial U)$), $v\text{div}(\jmath) \in L^1(U)$ and $v\text{conv}(\jmath) \in L^1(\partial U)$ the following formulas hold*

1.

$$\sum_{x\in U}\sum_{[x,y]\in(\vec{X}\cup\partial_N\vec{X})} (\nabla v.\jmath)([x,y]) = -\sum_{x\in U} v(x)\text{div}(\jmath)(x) + \sum_{y\in\partial U} v(y)\text{conv}(\jmath_N)(y). \tag{A.1.11}$$

2.

$$\sum_{x\in\partial U}\sum_{[x,y]\in\partial_T\vec{X}} (\nabla v.\jmath)([x,y]) = -\sum_{y\in\partial U} v(y)\text{div}(\jmath_T)(y). \tag{A.1.12}$$

where $\jmath_T = \jmath|_{\partial_T G} =$ and $\jmath_N = \jmath|_{\partial_N G}$.

Proof.

1) It is enough to apply the divergence formula [A.1.5] to graph $G \cup \partial_N G$ which turns out to be considered a graph without boundary. Indeed, from the hypotheses it follows that $v\text{div}(\jmath) \in L^1(U \cup \partial U)$ and also either $\text{div}(\jmath)^- \in L^1(U \cup \partial U)$ (or resp. $\text{div}(\jmath)^+ \in L^1(U \cup \partial U)$). Thus, we can write, since the boundary has outer orientation,

$$\sum_{x \in U} \sum_{[x,y] \in (\vec{X} \cup \partial_N \vec{X})} (\nabla v.\jmath)([x,y]) = \sum_{x \in U \cup \partial U} \sum_{[x,y] \in (\vec{X} \cup \partial_N \vec{X})} (\nabla v.\jmath)([x,y])$$
$$= - \sum_{x \in U \cup \partial U} v(x)\text{div}(\jmath)(x) = - \sum_{x \in U} v(x)\text{div}(\jmath)(x) + \sum_{y \in \partial U} v(y)\text{conv}(\jmath_N)(y).$$

2) The proof is totally similar to 1). ■

Now, from the corollary above, we can deduce the analogue of Green's Identity for graphs.

Theorem A.1.13 *$G \cup \partial G$ a graph with boundary with orientation $(\vec{X} \cup \partial_N \vec{X})$ where $\partial_N \vec{X}$ is the outer orientation of the boundary. Then, for any $u \in [u] \in \mathcal{L}^{1,p}(U \cup \partial U)$ and $v \in [v] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$ such that $(\Delta_p u)^- \in L^1(U)$, (or $(\Delta_p u)^+ \in L^1(U)$), $(D_{N,p} u)^- \in L^1(\partial U)$ (resp. $(D_{N,p} u)^+ \in L^1(\partial U)$), $v\Delta_p u \in L^1(U)$ and $vD_{N,p} u \in L^1(\partial U)$ the following formula holds*

$$\sum_{x \in U} \sum_{[x,y] \in (\vec{X} \cup \partial_N \vec{X})} (\nabla v.\nabla u)|\nabla u|^{p-2}([x,y]) = - \sum_{x \in U} v(x)\Delta_p u(x) + \sum_{y \in \partial U} v(y) D_{N,p} u(y).$$

Proof. The proof is a consequence of the convergence formula. Define the flow $\jmath = |\nabla u|^{p-2}\nabla u$. Then all hypotheses of the convergence theorem hold.

In fact, $v \in [v] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$, $\|\jmath\|_q^q = \sum_U \sum_{[x,y]} |\nabla u|^{(p-1)q} = \|u\|_{1,p}^p < \infty$, that is, $\jmath \in L^q(\vec{E} \cup \partial\vec{E})$, $\sum_U |v\text{div}(\jmath)| = \sum_U |v\Delta_p u| < \infty$, that is, $v\text{div}(\jmath) \in L^1(U)$, and, finally, $\sum_{\partial U} |v\text{conv}(\jmath)| = \sum_{\partial U} |vD_{N,p}u| < \infty$, that is, $v\text{conv}(\jmath) \in L^1(\partial U)$. ■

The next two corollaries are analogues of the 1^{st} and 2^{nd} Green's formula on manifolds, respectively. For finite graphs without boundary and $p = 2$, in the same spirit of the below 1^{st} corollary, see [4, p.789].

Corollary A.1.14 *$G \cup \partial G$ a graph with boundary with orientation $(\vec{X} \cup \partial_N \vec{X})$ where $\partial_N \vec{X}$ is the outer orientation of the boundary. Then, $u \in [u] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$ such that $(\Delta_p u)^- \in L^1(U)$ (or $(\Delta_p u)^+ \in L^1(U)$), $(D_{N,p}u)^- \in L^1(\partial U)$ (resp. $(D_{N,p}u)^+ \in L^1(\partial U)$), $u\Delta_p u \in L^1(U)$ and $uD_{N,p}u \in L^1(\partial U)$ the following formula holds*

$$\sum_{x \in U} \sum_{[x,y] \in (\vec{X} \cup \partial_N \vec{X})} |\nabla u[x,y]|^p = -\sum_{x \in U} u(x)\Delta_p u(x) + \sum_{y \in \partial U} u(y) D_{N,p} u(y).$$

Proof. Put $v = u$ in the formula on the above theorem. ■

To simplify, we state the next corollary with hypotheses stronger than those on the previous theorems and corollaries.

Corollary A.1.15 *$G \cup \partial G$ a graph with boundary with orientation $(\vec{X} \cup \partial_N \vec{X})$ where $\partial_N \vec{X}$ is the outer orientation of the boundary.*

Let $u \in [u], v \in [v] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$ such that $\Delta_p u, \Delta_p v, v\Delta_p u, u\Delta_p v \in L^1(U)$ and $D_{N,p}u, D_{N,p}v, vD_{N,p}u, uD_{N,p}v \in L^1(\partial U)$. Then

$$\sum_{x \in U}\big(u(x)\Delta_p v(x) - v(x)\Delta_p u(x)\big) = \sum_{x \in U}\sum_{[x,y]\in(\vec{X}\cup\partial_N\vec{X})}(\nabla u.\nabla v)\big[|\nabla u|^{p-2} - |\nabla v|^{p-2}\big] + \sum_{y\in\partial U}\big(u(y)D_{N,p}v(y) - v(y)D_{N,p}u(y)\big).$$

Proof. It is enough to apply twice the previous corollary. ∎

Example Let $p = 2$. Then, we get a formula similar to classical integral formula, namely,

$$\sum_{x \in U}\big(u(x)\Delta v(x) - v(x)\Delta u(x)\big) = \sum_{y\in\partial U}\big(u(y)D_N v(y) - v(y)D_N u(y)\big).$$

A.2 Application

The following theorem is analogous to Gauss' integral theorem for a ball in $\mathbb{R}^n$ with $p = 2$.

Theorem A.2.1 *Let $G \cup \partial G$ a graph with boundary (finite or infinite). If $u \in [u] \in \mathcal{L}_0^{1,p}(U \cup \partial U)$, p-harmonic on U, and $D_{N,p}u \in L^1(\partial U)$, then*

$$\sum_{y\in\partial U} D_{N,p}u(y) = 0.$$

Proof. Let $(G_n \cup \partial G_n)_{n\geq 1}$ be an arbitrary exhaustion of finite subgraphs with boundary, that is, $G_n \subseteq G_{n+1}$ and $\partial G_n \subseteq G_{n+1}$. Explicitly, $G_n = (U_n, E_n)$ and $\partial G_n = ((\partial^i U_n, \partial U_n), \partial E_n)$. Define $v_n \equiv 1$ on $U_n \cup \partial U_n$, so $v_n \in C_0(U_n \cup \partial U_n)$.

Applying the Theorem [A.1.13] with orientation $\vec{X_n} \cup \partial_N \vec{X_n}$, we get

$$\sum_{x\in U_n} \sum_{[x,y]\in(\vec{X_n}\cup\partial_N\vec{X_n})} (\nabla v_n . \nabla u)|\nabla u|^{p-2}([x,y]) = -\sum_{x\in U_n} v_n(x)\Delta_p u(x) + \sum_{y\in\partial U_n} v_n(y) D_{N,p}u(y).$$

But $v_n \equiv 1$ and u is p-harmonic, so

$$\sum_{y\in\partial U_n} D_{N,p}u(y) = 0.$$

Since the sequence $\{\sum_{y\in\partial U_n} D_{N,p}u(y)\}_{n\geq 1}$ is the zero sequence and the exhaustion is arbitrary, it follows that

$$\sum_{y\in\partial U} D_{N,p}u(y) = 0. \quad \blacksquare$$

Remarks: Under the same hypotheses of the Theorem [A.2.1].

- If u is a p-superharmonic function with $\Delta_p u \in L^1(U)$ and $D_{N,p}u \in L^1(\partial U)$, then

$$\sum_{y\in\partial U} D_{N,p}u(y) \leq 0.$$

- If u is a p-subharmonic function with $\Delta_p u \in L^1(U)$ and $D_{N,p}u \in L^1(\partial U)$, then

$$\sum_{y \in \partial U} D_{N,p}u(y) \geq 0.$$

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