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Robust eigenstructure assignment with flight control application

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ROBUST EIGENSTRUCTURE ASSIGNMENT WITH FLIGHT CONTROL APPLICATION

BY

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Abstract

ROBUST EIGENSTRUCTURE ASSIGNMENT

WITH FLIGHT CONTROL APPLICATION

By

Wangling Yu

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Robust eigenstructure assignment algorithms are developed and applied to flight control design. The design problems which are considered include a stability augmentation system for the lateral dynamics of the L-1011 aircraft, the AFTI F-16 pitch pointing/vertical translation maneuver, and an autopilot for the lateral dynamics of the Extended Medium Range Air to Air Technology (EMRAAT) missile. The application of robust eigenstructure assignment to the EMRAAT missile is quite interesting because the lateral dynamics does not have a well defined dutch roll mode. Therefore, eigenstructure assignment is utilized not only for mode decoupling, but also for creating distinctly separate dutch roll and roll mode.

The design methods combine eigenstructure assignment with a condition for the robust stability of a linear time invariant system which is subject to linear time varying

structured state space uncertainty. Robust eigenstructure assignment design method I is based upon optimizing the robust sufficient condition while constraining the dominant eigenvalues of the system to lie within chosen regions. The robust stability condition is extended to a performance robustness condition in which the closed loop eigenvalues of the LTI system are guaranteed to lie within a chosen damping-settling time region for all specified time invariant structured state space uncertainty.

In collaboration with a colleague, a new sufficient condition is proposed for the robust stability of an LTI system subject to time varying structured state space uncertainty. A robust controller is designed for the lateral dynamics of the EMRAAT missile by minimizing the integrated roll rate with constraints on the real part of the dutch roll and roll mode eigenvalues, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This new robust design has an improved transient response as compared to an eigenstructure assignment design using an orthogonal projection.

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1. INTRODUCTION

1.1 BACKGROUND

Eigenstructure assignment is a method for incorporating classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework. However, this design approach does not guarantee stability when the plant is subject to structured state space uncertainty. In systems such as aircraft, this uncertainty may arise from errors in the identification method which is used to obtain the linearized models of the plant. Another source of uncertainty is caused by errors in the dynamic pressure computations which result in the gain schedule choosing the wrong linearized model.

The first result of this research considers eigenstructure assignment in which certain outputs are not fed back to certain inputs. This is equivalent to constraining some elements of the feedback gain matrix to be zero which is called gain suppression. We have derived a new method for determining the effect of gain suppression on the eigenvectors. We use the partial derivatives of the eigenvectors with respect to the feedback gains to determine how the feedback gain matrix should be simplified.

The second result of this research combines eigenstructure assignment with the sufficient condition obtained by Sobel et. al. [1] for the robust stability of a linear time invariant(LTI) system which is subject to linear time varying structured state space uncertainty. We develop a robust eigenstructure assignment design method in which the robust sufficient condition is optimized while constraining the dominant eigenvalues of the nominal system to lie within chosen regions in the complex plane. Conservatism of the robustness conditions is reduced by introducing a similarity transformation on the uncertain plant into the design procedure. One approach is to solve a nonlinear constrained optimization problem by using the sequential unconstrained minimization technique with an extended interior penalty function [2]. The parameters to be optimized include both the closed loop eigenvectors and the closed loop eigenvalues. The use of constraints on certain eigenvector entries and the effect of these constraints on robustness are considered. Solutions have been obtained for 1) output feedback, 2) dynamic compensators, and 3) constrained output feedback. Applications to the AFTI F16 and L-1011 aircraft are presented.

The third result of this research extends the robust stability sufficient condition of Sobel et. al. [1] to ensure that the closed loop eigenvalues lie within chosen performance regions when the linear time invariant(LTI) system is subjected to time invariant structured state space uncertainty. The conservatism of both the stability and performance robustness conditions has been reduced by simultaneously introducing a similarity transformation, a positive real diagonal weighting, and a unitary weighting into the design procedure.

The fourth result of this research extends the sufficient condition for robust stability of Sobel et. al. [1] to include simultaneous time varying structured state space uncertainty and output nonlinearities. Then, we develop an algorithm for robust eigenstructure assignment which utilizes a new sufficient condition for robust stability obtained by Piou [3] for systems with both structured state space uncertainty and unmodelled dynamics. Application of the algorithm to the bank-to-turn missile which was used by Lin and Lee [4] is presented.

In collaboration with Piou [3], we developed a new robust stability condition for an LTI system which is subject to linear time varying uncertainty. This new sufficient robustness result is shown to be less conservative than an earlier result proposed by Sobel et. al. [1]. We design a robust controller for the lateral dynamics of the Extended Medium Range Air to Air Technology (EMRAAT) missile by using the MATLAB™ Optimization Toolbox [5] to minimize the integrated roll rate with constraints on the real part of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This new design is compared to an earlier orthogonal projection eigenstructure assignment design proposed by Sobel and Cloutier [6] which does not satisfy the sufficient condition for robust stability. We conclude that this new design satisfies the new robustness condition while yielding an improved transient response.

In our final result, we consider time delay systems. We have obtained two robustness sufficient conditions for a linear system with both time delay and structured state space uncertainty. At this time, it is not clear which condition is better. However, a system is robust if it satisfies either condition.

1.2 HISTORICAL REVIEW

1.2.1 EIGENSTRUCTURE ASSIGNMENT FOR LINEAR SYSTEMS

Consider an LTI system described by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where (1) $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$;

(2) A, B, and C are real constant matrices;

(3) $\mathrm{rank}(B) = m \neq 0$, $\mathrm{rank}(C) = r \neq 0$.

The feedback control problem can be stated as follows: Given a set of desired eigenvalues $\{\lambda_i^d\}$, $i=1,2,\ldots,r$ and a corresponding set of desired eigenvectors $\{v_i^d\}$, $i=1,2,\ldots,r$, find a real $m\times r$ matrix F such that the eigenvalues of A+BFC contain $\{\lambda_i^d\}$ as a subset, and the corresponding eigenvectors of A+BFC are close to the respective members of the set $\{v_i^d\}$.

The freedom in eigenvalue and eigenvector assignability using constant gain output feedback is described by the following theorem which was first proposed by Srinathkumar [7].

Theorem [8]: Given the controllable and observable system described by (1)-(2), max(m,r) closed loop eigenvalues can be assigned and max(m,r) eigenvectors (or reciprocal vectors by duality) can be partially assigned with min(m,r) entries in each vector arbitrarily chosen using constant gain output feedback.

Andry et.al. [8] indicate that the i-th eigenvector v_i must be in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$ and this subspace is of dimension m which is equal to the number of independent control variables. Therefore, if we choose an eigenvector v_i which lies precisely in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$, it will be achieved exactly.

In general, however, a desired eigenvector v_i^d will not reside in the prescribed subspace and hence cannot be achieved. [8] describes a method for finding the "best possible" choice for an achievable eigenvector. This best possible eigenvector is the projection of v_i^d onto the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$ (in the least square sense).

In many practical situations, complete specification of v_i^d is neither required nor known, but rather the designer is interested only in certain elements of the eigenvectors.

[8] presents a technique for partial specification of eigenvectors. It also describes how the feedback gain matrix is computed for both constrained and unconstrained cases. Finally, [8] discusses how to choose desired eigenvectors and the discussion emphasizes mode decoupling. However, this approach does not guarantee stability when the plant is subject to structured state space uncertainty. This area has been studied in our research and is discussed in Chapter 3 and Chapter 6.

Recently, Sobel and Cloutier [6] applied eigenstructure assignment to the design of an autopilot for the EMRAAT missile. An important difference between this application and other eigenstructure assignment applications which have appeared in the literature is that the lateral dynamics of the EMRAAT missile does not have a well defined dutch roll mode. Therefore, eigenstructure assignment is utilized not only for mode decoupling, but also to create distinctly separate dutch roll and roll modes. Sobel and Cloutier [6] use the approach suggested by Andry et. al. [8] in which the i-th desired eigenvector v_i^d is chosen for mode decoupling. Then, the i-th achievable eigenvector v_i^a is chosen as the projection of v_i^d onto the so-called achievability subspace. Sobel and Cloutier [6] show that their design achieves improved decoupling between an initial sideslip angle and the integrated roll rate (which is approximately equal to the bank angle) when compared to a linear quadratic regulator design proposed by Bossi and Langehough [9]. However, the design of Sobel and Cloutier [6] does not consider that the missile's aerodynamic parameters are uncertain.

1.2.2 GAIN SUPPRESSION

Andry et. al. [8] present a method for computing the remaining active feedback gains once the entries of the gain matrix which are to be suppressed to zero are chosen. This method was used by Andry et.al. [8] to yield a simpler controller with only a small effect on performance. However, the choice of gains which are constrained to be zero is determined by computer simulation of many different feedback structures. It would be more desirable to obtain a mathematical approach for choosing the gains which are to be eliminated.

Calvo-Ramon [10] chose to eliminate those gains which have the smallest influence on the closed loop eigenvalues. This was done by computing the partial derivatives of the eigenvalues with respect to the gains and then computing the expected shift in the eigenvalues if the ij-th gain were eliminated. However, [10] does not consider the effect of gain suppression on the eigenvectors. Since the eigenvectors are important for mode decoupling, the effect of gain suppression on the eigenvectors must be determined. As presented in Chapter 2, our research established an improved design method for eigenstructure assignment with gain suppression by a priori choosing to eliminate those gains that have the smallest influence on both the eigenvalues and eigenvectors.

1.2.3 FEEDFORWARD CONTROL

O'Brien and Broussard [11] developed a feedforward control technique which enables a plant to perfectly track a model in the presence of a disturbance. [11] assumes that both the model and the disturbance have time varying inputs which are not known in advance and develops dynamic compensators which have as inputs only a finite number of derivatives of the model and disturbance inputs.

Sobel and Shapiro [12] developed a methodology for pitch pointing flight control systems which uses eigenstructure assignment and command generator tracking by applying the results of [11] to the special case in which the command is restricted to be a step. In [12], the feedback gains are computed by using eigenstructure assignment. The desired eigenvalues are chosen to obtain the desired damping and settling time, and the desired eigenvectors are chosen to decouple the pitch attitude and flight path angle of an aircraft. Feedforward gains are computed which ensure steady-state tracking of the pilot's command.

1.2.4 ROBUST CONTROL FOR LINEAR SYSTEMS WITH STATE SPACE UNCERTAINTY

Chen and Wong [13] introduced a robust linear controller designed in the time domain for multivariable systems with linear or nonlinear time varying uncertainties. Chen and Wong [13] use the Gronwall lemma to derive robust stability conditions which are based on the upper norm-bounds of the uncertainties. The parameters of a dynamic controller are selected to satisfy the requirements of robust stability under plant uncertainties. The contribution of [13] consists of two theorems.

For the first theorem, the system is described by

$$\dot{x} = Ax + Bu + \Delta A(x) + \Delta B(u) \tag{3}$$

$$y = Cx + Du + \Delta C(x) + \Delta D(u) \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector, and A, B, C, and D are constant matrices. $\Delta A(x)$, $\Delta B(u)$, $\Delta C(x)$, and $\Delta D(u)$ are nonlinear time varying parametric uncertainties with the following known upper norm-bonds.

$$\|\Delta A(x)\| \le \beta_1 \|x\|, \qquad \|\Delta B(u)\| \le \beta_2 \|u\| \tag{5),(6}$$

$$\|\Delta C(x)\| \le \beta_3 \|x\|, \qquad \|\Delta D(u)\| \le \beta_4 \|u\| \tag{7),(8}$$

and the dynamic controller has the following structure:

$$\dot{x}_r = A_r x_r + B_r y \tag{9}$$

$$u = K_1 x_r + K_2 x \tag{10}$$

where $x_r \in \mathbb{R}^{nr}$.

Note that this design method requires not only state feedback but also dynamic output feedback. With some definitions and derivations in [13], the closed loop system with parametric uncertainties can be represented as

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\,\bar{x} + \Delta\bar{A}(\bar{x}) \\ y &= \bar{C}\,\bar{x} + \Delta\bar{C}(\bar{x}). \end{aligned} \qquad \bar{x}(0) = \begin{bmatrix} x_0 \\ x_{r0} \end{bmatrix} \tag{11}$$

and the nominal closed loop feedback system is given by

$$\dot{\bar{x}} = \bar{A}\,\bar{x} \qquad \bar{x}(0) = \begin{bmatrix} x_0 \\ x_{r0} \end{bmatrix} \tag{12}$$
$$y = \bar{C}\,\bar{x}$$

Define the transition matrix $\Phi(t)$ of (10) as

$$\Phi(t) = \exp(\bar{A}t) \tag{13}$$

and suppose

$$\|\Phi(t)\| \le m\,\exp(-\alpha t) \tag{14}$$

for some constants $m > 0$, $\alpha > 0$.

Theorem [13]: For the system with nonlinear parametric uncertainty described by (3)-(8), if the control parameters of (9)-(10) are chosen such that the nominal closed loop system (12) is asymptotically stable and the following inequality is satisfied:

$$\alpha > m\{\beta_1 + (\beta_2 + \beta_4\|B_r\|)(\|[K_2\;\;K_1]\|) + \beta_3\|B_r\|\} \tag{15}$$

where $\qquad \alpha = -\max_i \mathrm{Re}[\lambda_i(\bar{A})]$,

then the nonlinear parametric perturbed closed loop system defined by (3)-(8) is also asymptotically stable.

For the second theorem the system is described by

$$\dot{x} = Ax + Bu \tag{16}$$

$$y = Cx + Du + \Delta h * u \tag{17}$$

where the operator $*$ denotes convolution. The structural model uncertainty Δh is linear time varying but bounded by the inequality

$$\|\Delta h(t)\| \leq \gamma \exp(-\beta t). \tag{18}$$

The second theorem of [13] states that if the nominal closed loop system (which is not shown here) is asymptotically stable and if

$$1 - \frac{m\ \gamma}{\alpha\ \beta} \|B_r\| \cdot \|[K_2\ K_1]\| > 0 \tag{19}$$

then the system described by (16)-(17) is also asymptotically stable.

The sufficient condition of (15) for the stability robustness of linear systems with time varying norm bounded state space uncertainty was extended by Sobel et. al. [1] to include the structure of the uncertainty. Note that, unlike [13], Sobel et. al. [1] do not use dynamic output feedback but use only constant gain output feedback. The result of [1] requires that the nominal eigenvalues lie to the left of a vertical line in the complex plane which is determined by a norm involving the structure of the uncertainty and the nominal closed loop eigenvector matrix. Therefore, this robustness result is especially well suited to the design of control systems using eigenstructure assignment. When the uncertainty is time invariant, the norm is also an upper bound on the incremental eigenvalue perturbations. [1] also considers the use of Perron weightings to reduce conservatism and the extension of the results to discrete time systems. The main result of [1] can be summarized as follows:

Consider a nominal linear time invariant multi-input multi-output system described by (1)-(2). Suppose that the nominal system is subject to linear time varying uncertainties in the entries of A and B described by $\Delta A(t)$ and $\Delta B(t)$, respectively. Then, the system with uncertainty is given by

$$\dot{x}(t) = Ax(t) + Bu(t) + \Delta A(t)x(t) + \Delta B(t)u(t) \qquad (20)$$

$$y(t) = Cx(t) \qquad (21)$$

Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$ and $\Delta B(t)$. That is,

$$|\Delta a_{ij}(t)| \le (a_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,n \tag{22}$$

$$|\Delta b_{ij}(t)| \le (b_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,m \tag{23}$$

Define $\Delta A^{*}(t)$ and $\Delta B^{*}(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$ and $\Delta B(t)$ by their absolute values. Also, define A_m and B_m as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively. Then,

$$\{ \Delta A(t) : \Delta A^{*}(t) \le A_m \} \tag{24}$$

$$\{ \Delta B(t) : \Delta B^{*}(t) \le B_m \} \tag{25}$$

where " $\le$ " is applied element by element to matrices and $A_m \in \mathbb{R}_+^{n \times n}$, $B_m \in \mathbb{R}_+^{n \times m}$, where $\mathbb{R}_+$ is the set of non-negative numbers.

Consider the constant gain output feedback control law described by

$$u(t) = Fy(t) \tag{26}$$

Then, the nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{27}$$

and the uncertain closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A(t) + \Delta B(t)FC]x(t) \tag{28}$$

Problem [1]. Given a feedback gain matrix $F \in \mathbb{R}^{m \cdot r}$ such that the nominal closed loop system exhibits desirable dynamic performance, determine if the uncertain closed loop system is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by (22)-(23).

Theorem [1]: Suppose that F is such that the nominal closed loop system described by (27) is asymptotically stable with a non-defective modal matrix. Then, the uncertain closed loop system given by (28) is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by (24)-(25) if

$$\alpha > \|(M^{-1})^{+}[A_m + B_m(FC)^{+}]M^{+}\|_2 \tag{29}$$

where $\alpha = -\max_i \mathrm{Re}[\lambda_i(A+BFC)]$ and M is a modal matrix of (A + BFC).

This theorem presents a sufficient condition for robust stability in terms of the eigenstructure of the nominal closed loop system. Robust stability is ensured provided that the nominal closed loop eigenvalues lie to the left of a vertical line in the complex plane which is determined by a norm involving the structure of the uncertainty and the nominal closed loop modal matrix.

The conservatism inherent in the sufficient condition has been reduced by Sobel et.al. [1]. [1] utilized a diagonal weighting matrix D whose real positive entries were chosen by using Perron weightings in order to reduce the conservatism. Note that given a nominal closed loop modal matrix M, another valid modal matrix is MD. This is because the matrix D just serves to scale the nominal closed loop eigenvectors. Therefore, (29) may be replaced by the sufficient condition given by

$$\alpha > \| D^{-1}(M^{-1})^{+}[A_m + B_m(FC)^{+}]M^{+}D \|_2 \tag{30}$$

or equivalently by the D-weighted 2-norm described by

$$\alpha > \| (M^{-1})^{+}[A_m + B_m(FC)^{+}]M^{+} \|_{2D} \tag{31}$$

[1] uses the following lemma which gives the infimum of the right hand side of (31) which has been attributed to Stoer and Witzgall [14].

Lemma [14]: Define the Perron eigenvalue of the non-negative matrix A^+, denoted by $\pi(A^+)$, to be the real non-negative eigenvalue $\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues $(i=1,\ldots,n)$ of A^+. Then

$$\inf_D \|A^+\|_{2D} = \pi(A^+) \tag{32}$$

Since the matrix inside the norm on the right hand side of (31) is a non-negative matrix, the sufficient condition becomes

$$\alpha > \inf_D \|(M^{-1})^+[A_m + B_m(FC)^+]M^+\|_{2D} \tag{33}$$

or, by using the lemma

$$\alpha > \pi[(M^{-1})^+[A_m + B_m(FC)^+]M^+] \tag{34}$$

By taking the infimum on the right hand side of (31), the vertical line in complex plane has been placed as close to the imaginary axis as possible with a real positive diagonal weighting. Recall that the sufficient condition for robust stability requires that the nominal eigenvalues lie to the left of this vertical line. In this sense, [1] has reduced the conservatism of the sufficient condition.

[1] also proves that when the uncertainty is time invariant, reducing $\|(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}\|_2$ in order to satisfy the sufficient condition for robust stability will also reduce an upper bound on the eigenvalue perturbation caused by the uncertainty. In our research, the stability robustness sufficient condition and optimal diagonal weighting of [1] have been combined with eigenstructure assignment to obtain a new robust eigenstructure assignment design method.

1.2.5 Performance Robustness

An important criterion for the design of a controller for an uncertain LTI system is performance robustness. A sufficient condition for performance robustness has been proposed by Juang, Hong and Wang [15] by using the Lyapunov approach to robustness. The closed loop eigenvalues are guaranteed to lie within chosen performance regions when the LTI system is subjected to time invariant structured state space uncertainty. However, this result requires the solution of a Lyapunov equation which contains complex matrices. In our research, we have extended the robustness condition of Sobel et. al. [1] to include performance robustness. The required computations for our result include the nominal closed loop eigenvalues/eigenvectors and a matrix norm involving the uncertainty structure and the nominal closed loop modal matrix.

1.3 CONTRIBUTIONS OF THE THESIS

1. Derived a new approach for determining which elements of the feedback gain matrix can be eliminated by using both eigenvalue and eigenvector derivatives.

2. Developed a robust eigenstructure assignment design method which minimizes the robustness condition of Sobel et. al. [1] subject to eigenvalue and eigenvector constraints. This method has been studied using (i) output feedback (ii) constrained output feedback and (iii) dynamic compensators.

3. Applied the new robust eigenstructure assignment method to flight control design using the linearized lateral dynamics of the L-1011 aircraft and the F-16 pitch pointing/vertical translation problem.

4. Extended the the robust stability condition of Sobel et. al. [1] to obtain a robust performance condition for LTI systems with time invariant structured state space uncertainty which guarantees that the performance robust system has damping $\zeta \geq \zeta_{min}$ and settling time $t_s \leq (t_s)_{max}$.

5. Reduced the conservatism of the sufficient conditions for both stability and performance robustness by introducing a new unitary weighting matrix.

6. Extended the sufficient condition for robust stability proposed by Sobel et. at. [1] to include simultaneous time varying structured state space uncertainty and output nonlinearities. Developed a robust eigenstructure assignment design method which uses a new result proposed by Piou [3] for systems with both state space uncertainty and unmodelled dynamics. Applied this method to the design of a controller for a bank-to-turn missile.

7. In collaboration with Piou [3], we developed a new robust eigenstructure assignment design method for LTI systems with linear time varying uncertainty by using a new sufficient robustness condition which is less conservative than the earlier result of Sobel et.al. [1]. We applied this method to the design of a robust controller for the lateral

dynamics of the EMRAAT missile. Our new design method minimizes the integrated roll rate with the new robustness condition as one of the constraints. This new design yields a robust controller with an improved transient response as compared to the design of Sobel and Cloutier [6].

8. Developed stability robustness sufficient conditions for a linear time invariant system with both time delay and time varying structured state space uncertainty.

1.4 OUTLINE

Chapter 2 presents a new method for eigenstructure assignment with gain suppression. This method determines how the feedback gain matrix should be simplified by determining the effect of gain suppression not only on eigenvalues but also on eigenvectors.

Chapter 3 combines eigenstructure assignment with the sufficient condition for robust stability proposed by Sobel et. al. [1] to develop a robust eigenstructure assignment design method. Applications to the AFTI F16 and L-1011 aircrafts are presented.

Chapter 4 presents a performance robustness condition for LTI systems with structured state space uncertainty. The conservatism of both the stability and performance robustness conditions have been reduced.

Chapter 5 extends the sufficient condition for robust stability of [1] to include simultaneous time varying structured state space uncertainty and output nonlinearities.

Chapter 6 develops an algorithm for robust eigenstructure assignment which uses a new result proposed by Piou [3] for systems with both state space uncertainty and unmodelled dynamics. Application of the algorithm to the bank-to-turn missile which was used in [4] is also presented.

Chapter 7 presents a robust eigenstructure assignment design method, which we developed in collaboration with Piou [3], for LTI systems with linear time varying uncertainty by using a new robustness sufficient condition. The method is applied to the design of a robust controller for the lateral dynamics of the EMRAAT missile.

Chapter 8 considers time delay systems. Two robustness sufficient conditions for a linear system with both time delay and structured state space uncertainty are presented.

2. GAIN SUPPRESSION

2.1 REVIEW OF EIGENSTRUCTURE ASSIGNMENT[8]

Consider the linear time invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{35}$$

$$y(t) = Cx(t) \tag{36}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector, and A, B, C are constant matrices. We shall refer to (35)-(36) as either the nominal plant or the design model. Also, as is usually the case in aircraft problems, it is assumed that $m < r < n$. If there are no pilot commands, the feedback control vector u equals a matrix times the output vector y:

$$u(t) = Fy(t) \tag{37}$$

and the nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{38}$$

The constant gain output feedback problem using eigenstructure assignment can be stated as follows:

Given a set of desired eigenvalues $\{\lambda_i^d\}$, $i = 1,\ldots,r$ and a corresponding set of desired eigenvectors, v_i^d, $i = 1,\ldots,r$, find a real $(m \times r)$ matrix F such that the eigenvalues of A + BFC contain λ_i^d as a subset and the corresponding eigenvectors of A + BFC are close to the respective members of the set $\{v_i^d\}$.

As mentioned earlier (section 1.2.1) the solution to these problems was shown by Andry et.al. [8] and begins with the closed loop system described by (38).

For an eigenvalue/eigenvector pair, λ_i and v_i

$$(A + BFC)v_i = \lambda_i v_i \tag{39}$$

or

$$v_i = (\lambda_i I - A)^{-1} BFCv_i \tag{40}$$

Define the $m \times 1$ vector m_i as

$$m_i = FCv_i \tag{41}$$

Then, (40) becomes

$$v_i = (\lambda_i I - A)^{-1} B m_i \tag{42}$$

In general, however, a desired eigenvector v_i^d will not reside in the prescribed subspace and, hence, cannot be achieved. Instead, a "best possible" choice for an achievable eigenvector is made. It is suggested by Andry et. al. [8] that this "best possible" eigenvector is the projection of v_i^d onto the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$. By using an orthogonal projection to obtain the achievable eigenvectors v_i^a , [8] minimizes the 2-norm error between a desired eigenvector and its corresponding achievable eigenvector. It is important to note that the concept of stability robustness was not considered in the design approach proposed by Andry et. al. [8].

To further complicate the situation, complete specification of v_i^d is neither required nor known in most practical situations. When the designer is interested in only certain elements of the eigenvector, [8] assumes that the desired eigenvector has a structure given by

$$v_i^d = [v_{i1}, x, x, x, x, v_{ij}, x, x, x, v_{in}]^T \tag{43}$$

where v_{ij} are the designer specified components and "x" the unspecified components. Define a reordering operator $\{\ \}^{R_i}$

as follows (see Harvey and Stein [16]):

$$\{v_i^d\}^{R_i} = \begin{bmatrix} l_i^d \\ d_i \end{bmatrix} \qquad (44)$$

where l_i^d is the vector of specified components of v_i^d and d_i is the vector of unspecified components of v_i^d. Begin the computation of an achievable eigenvector v_i^a by defining [8]

$$L_i = (\lambda_i I - A)^{-1} B \qquad (45)$$

Since an achievable eigenvector must lie in the subspace spanned by the columns of L_i, it is required that [8]

$$v_i^a = \lambda_i z_i \qquad (46)$$

where the vectors z_i are the eigenvector design parameters.

Andry et. al. [8] reorder the rows of L_i to conform with the reordered components of v_i^d. Then, L_i is partitioned as shown below:

$$\{L_i\}^{R_i} = \begin{bmatrix} \tilde{L}_i \\ D_i \end{bmatrix} \qquad (47)$$

It is suggested by Andry et.al. [8] that the value of the z_i be computed so that it corresponds to the orthogonal projection onto the so-called "achievability subspace". Hence, z_i is chosen to minimize

$$J = \| l_i^d - l_i^a \|_2^2 = \| l_i^d - L_i z_i \|_2^2 \tag{48}$$

Then, the i-th achievable eigenvector is given by (46). Once all r achievable eigenvectors are computed, the output feedback gain matrix is easily calculated, as shown in [8]. Andry et. al. [8] conclude with the remark that this approach should yield achievable eigenvectors v_i^a which are closest in the 2-norm sense to the desired eigenvectors v_i^d.

2.2 GAIN SUPPRESSION USING EIGENVECTOR DERIVATIVES

Calvo-Ramon [10] proposed a method for choosing a priori which gains should be set to zero based upon the sensitivities of the eigenvalues to changes in the feedback gains. The first order sensitivity of the h-th eigenvalue to changes in the ij-th entry of the matrix F is denoted by $\partial\lambda_h/\partial f_{ij}$. The expected shift in the eigenvalue λ_h when constraining feedback gain f_{ij} to zero is given by [10]

$$s^h_{ij} = (f_{ij})\frac{\partial\lambda_h}{\partial f_{ij}} \tag{49}$$

Next, [10] combines all the eigenvalue shifts which are related to the same feedback gain f_{ij} to form a decision matrix $D^\lambda = \{d_{ij}\}$, $D^\lambda \in \mathbb{R}^{m\times r}$, where

$$d^\lambda_{ij} = \frac{1}{n}\left[\sum_{h=1}^{n} (s^h_{ij})^2\right]^{\frac{1}{2}} \tag{50}$$

This last equation is (19) in [10]. However, note that since s^h_{ij} may be complex, the d_{ij} should really be computed by using the following:

$$d^\lambda_{ij} = \frac{1}{n}\left[\sum_{h=1}^{n} (\overline{s^h_{ij}})(s^h_{ij})\right]^{\frac{1}{2}} \tag{51}$$

where $(\bar{\cdot})$ denotes the complex conjugate of $(\cdot)$.

The decision matrix D^λ is used to determine which feedback gains f_{ij} should be set to zero. If d^λ_{ij} is "small", then setting f_{ij} to zero will have a small effect on the closed loop eigenvalues. Conversely, if d_{ij} is "large", then setting f_{ij} to zero will have a significant effect on the closed loop eigenvalues. The control system designer must determine which d^{λ}_{ij} are "small" and which are "large" for a particular problem. In this regard, it is assumed that the states, inputs, and outputs are scaled such that these variables are expressed in the same or equivalent units.

The decision matrix D^{λ} is used in [10] to design a constrained output feedback controller using eigenstructure assignment. However, the sensitivities of the eigenvectors with respect to the gains are not considered when deciding which feedback gains should be set to zero. Recall that the eigenvalues determine transient response characteristics such as overshoot and settling time, while the eigenvectors determine mode decoupling. This mode decoupling is related, for example, to the $|\phi/\beta|$ ratio in an aircraft lateral dynamics problem. This ratio must be small, as specified in [17] which implies that the closed loop aircraft should exhibit a significant degree of decoupling between the dutch roll mode and the roll mode. The approach of [10] may yield a constrained controller with acceptable overshoot and settling time, but the mode decoupling may be unacceptable. Thus, consideration of both eigenvalue and eigenvector sensitivities is important when choosing which feedback gains should be constrained to be zero.

The results of [10] are now extended to include both eigenvalue and eigenvector sensitivities to the feedback gains. To begin the development, consider the following result on eigenvalue and eigenvector derivatives which is credited to Lim et.al. [18]

Theorem [18]:

Let
$$Av_h = \lambda_h v_h; \ h=1,\ldots,n \tag{52}$$

and
$$w_q^T A = \lambda_q w_q^T; \ q=1,\ldots,n \tag{53}$$

Define

$$t_n = v_h^T v_h; \quad h=1,\ldots,n \tag{54}$$

$$s_h = w_h^T v_h; \quad h=1,\ldots,n \tag{55}$$

Then

$$\frac{\partial \lambda_h}{\partial \rho} = \left(\frac{1}{s_h}\right)\left[w_h^T\left(\frac{\partial A}{\partial \rho}\right)v_h\right] \tag{56}$$

where ρ is a scalar parameter. Also, let

$$\frac{\partial v_h}{\partial \rho} = \sum_{m=1}^{n} \alpha_{hm} v_m \tag{57}$$

Then

$$\alpha_{hq} = - \frac{1}{(\lambda_q - \lambda_h)s_q} \left[w_q^T \left(\frac{\partial A}{\partial \rho}\right) v_h \right]; \quad q \neq h \tag{58}$$

$$\alpha_{hh} = \left(- \frac{1}{t_h}\right) \sum_{\substack{m=1 \\ m \neq h}}^{n} \alpha_{hm} v_m^T v_h \qquad q=h \tag{59}$$

The result given by (56) describes the derivatives of the eigenvalues of the matrix A with respect to a scalar parameter ρ, whereas (57) describes the derivatives of the eigenvectors of the matrix A with respect to the scalar parameter ρ. The theorem will now be extended so that it applies to the matrix A+BFC which describes the dynamics of (35)-(36) when subjected to the feedback control law given by (37).

Corollary 1

Let
$$(A + BFC)v_h = \lambda_h v_h; \quad h=1,\ldots,n \tag{60}$$

$$w_q^T (A + BFC) = \lambda_q w_q^T; \quad q=1,\ldots,n \tag{61}$$

Define

$$t_h = v_h^T v_h; \qquad h=1,\ldots,n \tag{62}$$

$$s_h = w_h^T v_h; \qquad h=1,\ldots,n \tag{63}$$

Let f_{ij} be the ij-th entry of the matrix F, b_i is the i-th column of the matrix B, and c_j^T is the j-th row of the matrix C. Then, the derivative of the h-th eigenvalue of (A + BFC) with respect to the ij-th element of F is given by

$$\frac{\partial\lambda_h}{\partial f_{ij}} = \left(\frac{1}{s_h}\right)(w_h^T b_i c_j^T v_h) \tag{64}$$

and the derivative of the h-th eigenvector of A+BFC with respect to the ij-th element of F is given by

$$\frac{\partial v_h}{\partial f_{ij}} = \sum_{m=1}^{n} \alpha_{ijhm} v_m \tag{65}$$

where

$$\alpha_{ijhq} = -\frac{1}{(\lambda_q - \lambda_h)s_q}\left[w_q^T b_i c_j^T v_h\right]; \quad q\neq h \tag{66}$$

$$\alpha_{ijhh} = \left(-\frac{1}{t_h}\right)\sum_{\substack{m=1\\ m\neq h}}^{n} \alpha_{ijhm} v_m^T v_h; \qquad q=h \tag{67}$$

Outline of Proof:

Replace $w_q^T(\partial A/\partial\rho)v_h$ by $w_q^T[\partial(A+BFC)/\partial f_{ij}]v_h$. Note that

$\partial(A+BFC)/\partial f_{ij} = B(\partial F/\partial f_{ij})C$ and after some matrix manipulations, obtain $B(\partial F/\partial f_{ij})C = b_i c_j^T$. Thus,

$$w_q^T\left(\frac{\partial(A+BFC)}{\partial f_{ij}}\right)v_h = w_q^T b_i c_j^T v_h \tag{68}$$

The corollary then follows upon using (68) together with the theorem of [18].

The eigenvalue derivatives described by (64) are equivalent to the derivatives used by Calvo-Ramon [10]. However, the eigenvector derivatives, described by (65)-(67) are now available. The improved approach to gain suppression is to first calculate the eigenvalue decision matrix, denoted by D^{λ}, by using (49), (51) and (64). Then, the eigenvector derivatives are computed by using (65)-(67). These are used to compute the expected shift in eigenvector v_h when constraining feedback gain f_{ij} to be zero, which is given by [compare with (49)]

$$\tilde{s}_{ij}^{h} = (f_{ij})\left(\frac{\partial v_h}{\partial f_{ij}}\right) \tag{69}$$

Finally, all the eigenvector shifts which are related to the same feedback gain f_{ij} are combined to form an eigenvector decision matrix D^{v}, where [compare with (51)]

$$d^{v}_{ij} = \frac{1}{n} \left[\sum_{h=1}^{n} (\tilde{s}^{h}_{ij})^{*} (\tilde{s}^{h}_{ij}) \right]^{\frac{1}{2}} \tag{70}$$

where $(\cdot)^{*}$ denotes the complex conjugate transpose of $(\cdot)$. The gains which should be set to zero are determined by first eliminating those f_{ij} corresponding to entries of D^{λ} which are considered to be small. Then, those entries of D^{v} corresponding to those f_{ij} which were chosen to be set to zero based upon D^{λ} are reviewed. In this way, the designer can determine whether some of the f_{ij} which may be set to zero based upon eigenvalue considerations should not be constrained based upon eigenvector considerations.

2.3 EXAMPLE: F-18 HARV AIRCRAFT

Consider the lateral directional dynamics of the F-18 HARV aircraft linearized at a Mach number of 0.38, an altitude of 5000 feet, and an angle of attack of 5 degrees. The aerodynamic model is augmented with first order actuators and a yaw rate washout filter. The eight state variables are aileron deflection (δ_a), stabilator deflection (δ_s), rudder deflection (δ_r), sideslip angle (β), roll rate (p), yaw rate (r), bank angle (ϕ), and washout filter state (x_8). The three control variables are aileron command (δ_{ac}), stabilator command (δ_{sc}), and rudder command (δ_{rc}). The four

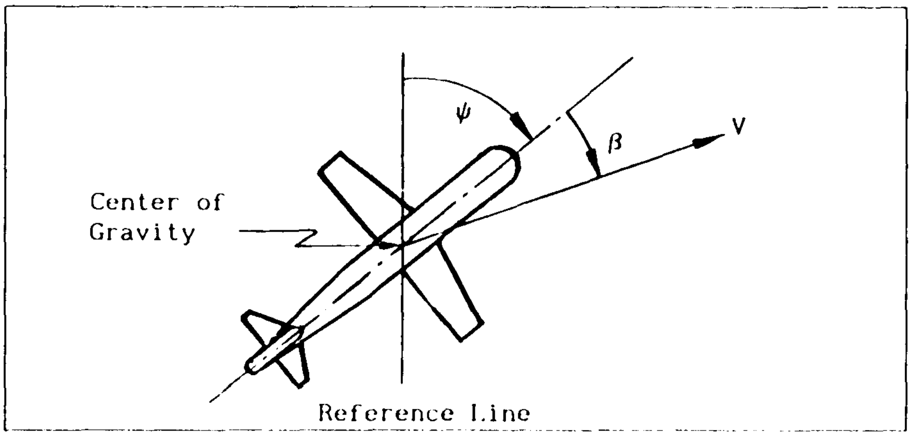

Figure 1. Aircraft Lateral Angle Definitions

measurements are r_{wo}, p, β, ϕ where r_{wo} is the washedout yaw rate. All units are in degrees or degrees per second. The relationship between the angles is shown in figure 1. The state space matrices A, B, and C which completely describe the model are shown below

$$A=\begin{bmatrix} -30.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -30.000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -30.000 & 0 & 0 & 0 & 0 & 0 \\ -0.0070 & -0.0140 & 0.0412 & -0.1727 & 0.0873 & -.9946 & .076 & 0 \\ 15.3225 & 12.0601 & 2.2022 & -11.0723 & -2.1912 & .7096 & 0 & 0 \\ -0.3264 & 0.2041 & -1.3524 & 2.1137 & -0.0086 & -.1399 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & .0875 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5000 & 0 & -.5 \end{bmatrix} \tag{71}$$

$$B=\begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{72}$$

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (73)$$

An unconstrained output feedback gain matrix is now computed by using eigenstructure assignment. The desired dutch roll eigenvalues are chosen to have a damping ratio of 0.707 and a natural frequency in the vicinity of 3 rad/sec. The roll subsidence and spiral modes are chosen to be merged into a complex roll mode, as suggested in [8]. The desired eigenvalues are

$$\text{Dutch Roll Mode:} \quad \lambda^{d}_{dr} = -2 \pm j2$$

$$\text{Roll Mode:} \quad \lambda^{d}_{roll} = -3 \pm j2$$

The desired eigenvectors are chosen to keep the quantity $|\phi/\beta|$ small. Therefore, the desired dutch roll eigenvectors will have zero entries in the rows corresponding to bank angle and roll rate. The desired roll mode eigenvectors will have zero entries in the rows corresponding to yaw rate, sideslip, and x_8 (which is filtered yaw rate). The desired eigenvectors are

$$v_{dr}^{d} = \begin{bmatrix} x \\ x \\ x \\ 1 \\ 0 \\ x \\ 0 \\ x \end{bmatrix} \pm j \begin{bmatrix} x \\ x \\ x \\ x \\ 0 \\ x \\ 0 \\ x \end{bmatrix} \qquad v_{roll}^{d} = \begin{bmatrix} x \\ x \\ x \\ 0 \\ 1 \\ 0 \\ x \\ 0 \end{bmatrix} \pm j \begin{bmatrix} x \\ x \\ x \\ 0 \\ x \\ 0 \\ x \\ 0 \end{bmatrix} \begin{matrix} \delta_a \\ \delta_s \\ \delta_r \\ \beta \\ p \\ r \\ \phi \\ x_8 \end{matrix}$$

The achievable eigenvectors are shown in table 1 where the underlined numbers indicate the small couplings between p, ϕ and the dutch roll mode and between β, r, x_8 and the roll mode. Hence, the ratio $|\phi/\beta|$ can be expected to be small.

The feedback gain matrix is shown in table 2. The closed loop state and control deflection responses to an initial sideslip angle of one degree are shown in figures 2a and 2b,

Table 1. Achievable Eigenvectors

(Unconstrained Output Feedback)

Dutch Roll Mode				Roll Mode			
0.4642		-0.400	δ_a	-0.0038		0.0731	δ_a
0.0872		1.6341	δ_s	0.0078		0.0754	δ_s
0.6703		-5.4587	δ_r	-0.0313		-0.0396	δ_r
0.9928	$\pm j$	0.0000	β	$\underline{-0.0152}$	$\pm j$	$\underline{-0.0007}$	β
$\underline{-0.0180}$		$\underline{0.0219}$	p	0.8103		-0.3970	p
1.8396		-2.2402	r	$\underline{0.0063}$		$\underline{-0.0121}$	r
$\underline{-0.0793}$		$\underline{0.0078}$	ϕ	-0.2484		-0.0329	ϕ
-0.5792		-0.0255	x_8	-0.0019		0.0009	x_8

Table 2. Comparison of Control Laws

Feedback Gain Matrix (deg/deg)				Gain and Phase Margins (at inputs $\delta_{ac}, \delta_{sc}, \delta_{rc}$)		$\mathrm{Max}\lvert\phi\rvert$ $\mathrm{Max}\lvert p\rvert$
unconstrained:						
r_{wo}	p	β	ϕ			
0.1518	-0.1370	0.0580	-0.4128	δ_a	[-5.50db, 19.12db]	0.0532deg
-0.6941	-0.1088	1.6227	-0.4824	δ_s	±52.41deg	0.2815deg/sec
2.2815	-0.0177	-4.5311	0.3890	δ_r		
constrained, D^{λ} only:						
r_{wo}	p	β	ϕ			
0	-0.4472	0	-1.7004	δ_a	[-4.87db, 12.11db]	0.2044deg
-0.3726	-0.5734	0	-2.3813	δ_s	±44.17deg	1.1082deg/sec
2.2790	0	-4.5534	0	δ_r		
constrained, D^{λ} and D^{v}:						
r_{wo}	p	β	ϕ			
0.1633	-0.1536	0	-0.4807	δ_a	[-5.34db, 16.46db]	0.0546deg
-0.6941	-0.1088	1.6227	-0.4824	δ_s	±50.28deg	0.3199deg/sec
2.2790	0	-4.5534	0	δ_r		

respectively. Observe that the maximum absolute values of the bank angle and roll rate are 0.056 degree and 0.278 degree/second, respectively.

The eigenvalue decision matrix D^{λ} and the eigenvector decision matrix D^{v} are shown in table 3. The entries of D^{λ} which are considered to be large are underlined in table 3. Observe that only seven of the twelve feedback gains are

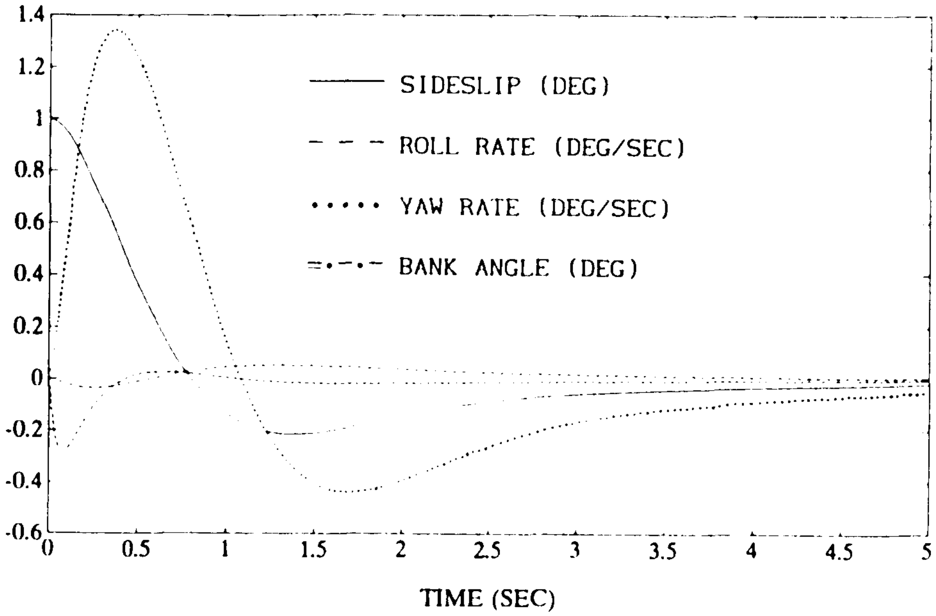

Figure 2a Closed Loop State Responses
(Unconstrained Feedback)

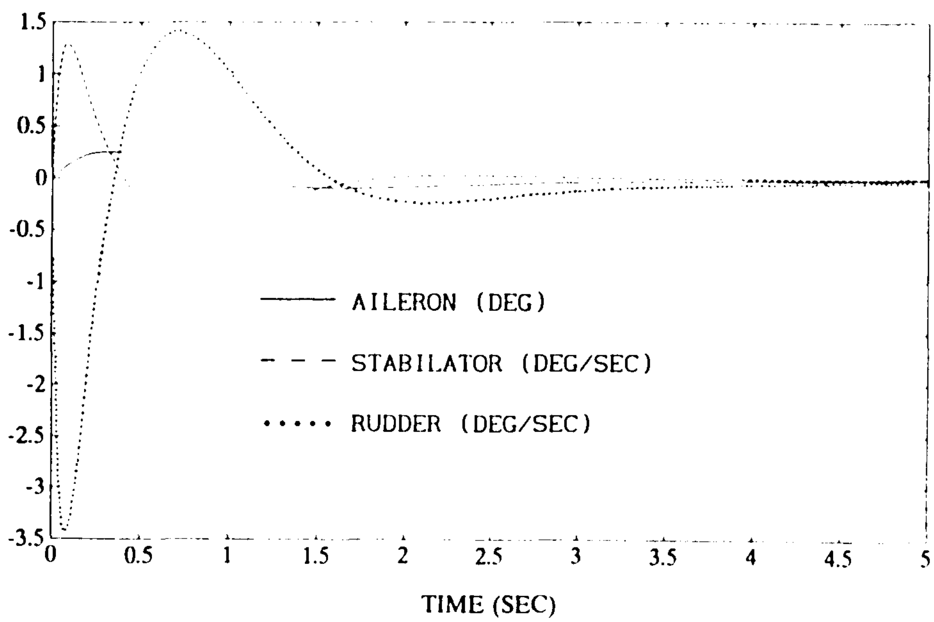

Figure 2b Closed Loop Control Deflections
(Unconstrained Feedback)

Table 3. Eigenvalue and Eigenvector Decision Matrices

Eigenvalue Decision Matrix D^{λ}					Eigenvector Decision Matrix D^{v}				
r_{wo}	p	β	ϕ		r_{wo}	p	β	ϕ	
.05390	.50540	.00398	.35896	δ_a	.26029	.97888	.02234	.12648	δ_a
.18236	.32765	.07076	.34145	δ_s	.94654	.63998	.49314	.11542	δ_s
.68059	.00771	.34195	.05448	δ_r	1.44353	.02638	.28115	.11096	δ_r

needed when only eigenvalue sensitivities are considered. The constrained output feedback gain matrix D^{λ} is shown in table 2. The state responses to an initial sideslip angle of one degree are shown in figure 3a. Observe the significantly increased coupling between sideslip and bank angle. The maximum absolute values of the bank angle and roll rate are now 0.178 degree and 1.028 degrees/second compared with 0.056 degree and 0.287 degree/second which were obtained with the unconstrained feedback gain matrix. The increased coupling is due to ignoring the eigenvector sensitivities and illustrates the importance of the eigenvectors in achieving adequate mode decoupling.

Next, consider the entries of D^{v} which correspond to those f_{ij} which were chosen to be set to zero based upon the eigenvalue decision matrix. The two largest d^{v}_{ij} which belong to this class are d^{v}_{11} and d^{v}_{23}. A new constrained output feedback gain matrix is computed in which f_{11} and f_{23} are not set to zero. Thus, nine gains now need to be unconstrained when utilizing both eigenvalue and eigenvector information. The new feedback gain matrix is shown in table 2 and the state responses for an initial condition of one degree sideslip angle are shown in figure 3b. Observe that these time responses are almost identical to the responses in figure 2a, which were obtained by using all twelve feedback gains. Thus, a simpler controller is obtained with a negligible change in the aircraft time responses. Finally, the multivariable gain and phase margins at the inputs are computed using the results of Lehtomaki et. al. [19].

Theorem [19]:

If there exists a bound $\alpha \le 1$ such that $\underline{\sigma}(\omega) \ge \alpha$ for all ω, where $\underline{\sigma}(\omega)$ is the minimum singular value of the return difference matrix at any given frequency ω, then

$$PM = \pm 2\sin^{-1}(\alpha/2) \tag{74a}$$

$$GM = 1/(1 \pm \alpha) \tag{74b}$$

where PM and GM are the phase and gain margins, respectively.

These margins are shown in table 2 for each of the three feedback gain matrices. The value of these margins were accepted in light of the conservatism inherent in singular value based multivariable stability margin computation.

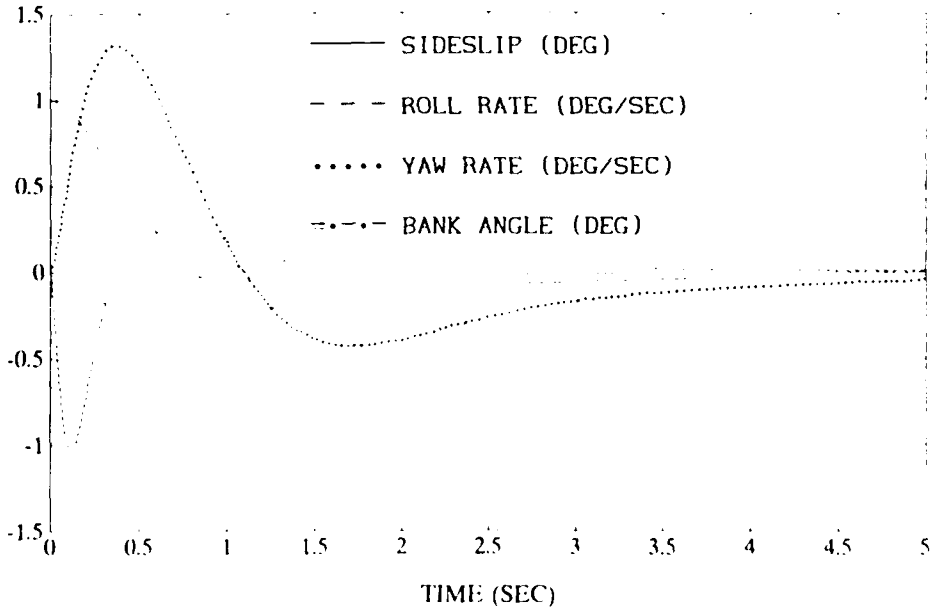

Figure 3a Closed Loop State Responses
(Constrained Using Only D^{λ})

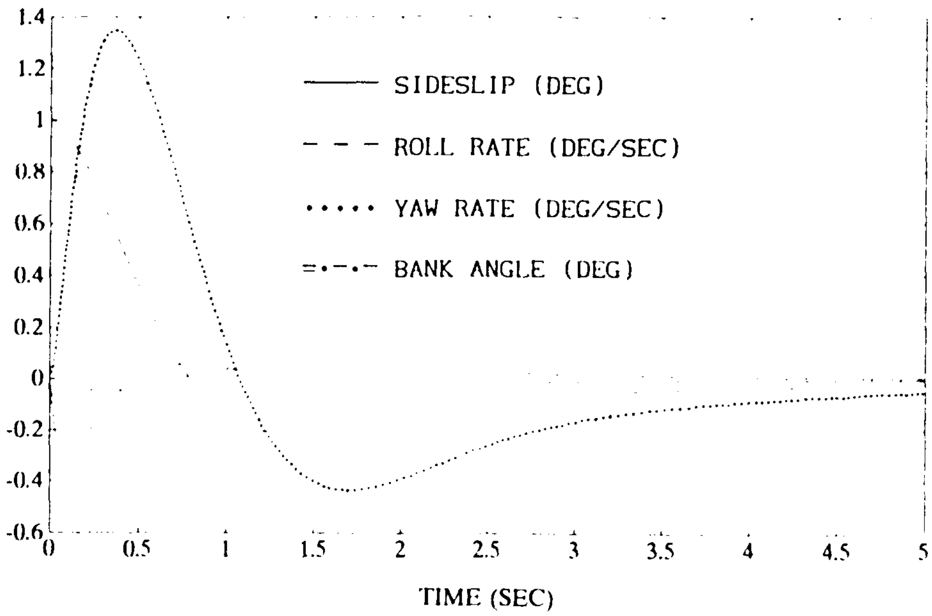

Figure 3b Closed Loop State Responses
(Constrained Using Both D^{λ} and D^{v})

3. ROBUST EIGENSTRUCTURE ASSIGNMENT DESIGN METHOD I

3.1 CONSTRAINED OPTIMIZATION

A sufficient condition for the robust stability of a linear time invariant system subject to linear time varying structured state space uncertainty has been proposed by Sobel et. al. [1] This result, which is based upon the Gronwall lemma, ensures robust stability if the nominal eigenvalues lie to the left of a vertical line in the complex plane. This line is determined by the maximum eigenvalue of a matrix which involves the product of the uncertainty structure, the nominal closed loop modal matrix, and the inverse of the nominal closed loop modal matrix.

We propose a robust eigenstructure assignment design method which optimizes the sufficient condition for stability robustness subject to constraints on the dominant eigenvalues. We also introduce a similarity transformation on the uncertain plant which can reduce the conservatism of the sufficient condition. In the first example, the sequential unconstrained minimization technique is used to solve the constrained optimization problem for a robust pitch pointing/vertical translation controller for the AFTI F-16 aircraft. The second example considers the linearized lateral dynamics of the L-1011 aircraft. Three different controllers are designed which include (1) state feedback, (2) gain suppression, and (3) dynamic compensation.

3.2 PROBLEM FORMULATION

Consider a nominal linear time invariant multi-input multi-output system described by (35)-(36). Suppose that the nominal system is subject to linear time varying structured state space uncertainty. The linear time varying state space uncertainty is described by uncertainties in the entries of A and B and is denoted by $\Delta A(t)$ and $\Delta B(t)$, respectively. We shall assume that the entries of $\Delta A(t)$ and $\Delta B(t)$ are continuous functions of time. Then, the plant with uncertainty is given by

$$\dot{x} = Ax + Bu + \Delta A(t)x + \Delta B(t)u \tag{75}$$

$$y = Cx \tag{76}$$

We shall refer to (75)-(76) as the uncertain plant or the true model. Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$ and $\Delta B(t)$. That is,

$$|\Delta a_{ij}(t)| \leq (a_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,n \tag{77}$$

$$|\Delta b_{ij}(t)| \leq (b_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,m \tag{78}$$

Define $\Delta A^{+}(t)$ and $\Delta B^{+}(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$ and $\Delta B(t)$ by their absolute values. Also, define A_m and B_m as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively. Then,

$$\{\Delta A(t):\ \Delta A^{+}(t) \leq A_m\} \tag{79}$$

$$\{\Delta B(t):\ \Delta B^{+}(t) \leq B_m\} \tag{80}$$

where "$\leq$" is applied element by element by element to matrices and $A_m \in \mathbb{R}_{+}^{n\times n}$, $B_m \in \mathbb{R}_{+}^{n\times m}$ where $\mathbb{R}_{+}$ is the set of non-negative real numbers.

Consider the constant gain output feedback control law described by (37). Then, the nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{81}$$

and the uncertain closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A(t) + \Delta B(t)FC]x(t) \tag{82}$$

Stability Robustness Problem: Given a feedback gain matrix $F \in \mathbb{R}^{m\times r}$ such that the nominal closed loop system exhibits desirable dynamic performance, determine if the uncertain closed loop system is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by (79)-(80).

3.3 REVIEW OF THE SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE (SEE REF.2)

The constrained optimization to be considered is to find a vector v^* of design variables that minimizes the function $m(v)$ subject to the constraints

$$g_i(v) \geq 0; \ i=1,\dots,n \tag{83}$$

It is assumed that at least one of the constraints of (83). is critical at the minimum v^*, that is, $g_i(v^*) = 0$ for some i. This constrained optimization problem may be transformed into a series of unconstrained minimization problems by introducing a penalty function associated with the constraints, and the transformed problem can be solved by the sequence of unconstrained minimizations technique. The resulting transformed problem is to find the minimum of a function P(r) as r goes to zero where

$$P(r) = m(v) + r \sum_{i=1}^{n} f_i(v) \tag{84}$$

and $f_i(v)$ is defined by

$$f_i(v) = 1/g_i(v) \quad \text{if } g_i(v) \geq 0 \tag{85}$$

The term $rf_i(v)$ represents the penalty associated with the i-th constraint, and is an interior penalty function in the sense that it is defined only if v is inside the feasible design domain. With v^r denoting the point in the design space where P(r) attains its minimum value for a given value of r, it may be shown that as r goes to zero

$$\begin{gathered} \min P(r) \to m(v^*) \\ v^r \to v^* \end{gathered} \tag{86}$$

Next, a quadratic extended interior penalty function is introduced to permit the use of design points that are outside the feasible domain. The definition of the f_i in (84) for the quadratic extended penalty function is [2]

$$f_i = \begin{cases} 1/g_i & \text{if } g_i \geq g_0 \\ 1/g_0[(g_i/g_0)^2 - 3(g_i/g_0) + 3] & \text{if } g_i \leq g_0 \end{cases} \quad (87)$$

To complete the definition of the quadratic extended penalty function, a relation that defines the transition point g_0 between the two constraint functions in (87) is required. This definition is described by [2]

$$g_0 = C\, r^p; \quad 1/2 \geq p \geq 1/3 \quad (88)$$

3.4 MATRICES D AND Q FOR REDUCING CONSERVATISM

In this section, we address the problem of the conservatism of the sufficient condition for robustness based upon the Gronwall lemma. Sobel et. al. [1] utilized a diagonal weighting matrix D whose real positive entries were chosen by using Perron weightings in order to reduce the conservatism. Here, we introduce an additional matrix Q which can further reduce the conservatism. Although a simple closed form solution for the optimal matrix Q is not yet known, we propose a choice based upon obtaining a unitary Q from the singular value decomposition of the product of the nominal closed loop eigenvector matrix and the weighting matrix D. We start with a preliminary lemma and then we present a theorem which ensures robust stability for LTI systems with time varying structured state space uncertainties.

Lemma 1

Consider the system described by (75)-(82). Let $\Delta A_c(t) = \Delta A(t) + \Delta B(t)FC$ and let M be a modal matrix of A_c. Then,

$$\|(MDQ)^{-1}\Delta A_c(t)MDQ\|_2 \le \|[(MDQ)^{-1}]^+[A_m + B_m(FC)^+](MDQ)^+\|_2 \tag{89}$$

Proof

Use the result given by Kouvaritakis and Latchman [20] that for any matrix $A \in \mathbb{C}^{n\times m}$

$$\|A\|_2 \le \|A^+\|_2 \tag{90}$$

Thus,

$$\|(MDQ)^{-1}\Delta A_c(t)(MDQ)\|_2 \le \|[(MDQ)^{-1}\Delta A_c(t)MDQ]^+\|_2$$

$$= \sup \frac{\|[(MDQ)^{-1}\Delta A_c(t)MDQ]^+x^+\|_2}{\|x^+\|_2}$$

$$= \sup \frac{\|\{(MDQ)^{-1}[\Delta A(t) + \Delta B(t)FC]MDQ\}^{+}x^{+}\|_2}{\|x^{+}\|_2}$$

$$\le \sup \frac{\|[(MDQ)^{-1}]^{+}[\Delta A(t) + \Delta B(t)FC]^{+}(MDQ)^{+}x^{+}\|_2}{\|x^{+}\|_2}$$

$$\le \sup \frac{\|[(MDQ)^{-1}]^{+}[A_m + B_m(FC)^{+}](MDQ)^{+}x^{+}\|_2}{\|x^{+}\|_2}$$

$$= \|[(MDQ)^{-1}]^{+}[A_m + B_m(FC)^{+}(MDQ)^{+}\|_2 \qquad (91)$$

End of proof

Theorem 1

Suppose that F is such that the nominal closed loop system described by (81) is asymptotically stable with a non-defective modal matrix, D is real positive diagonal and Q is nonsingular. Then, the closed loop system with uncertainties given by (82) is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by (79)-(80) if

$$\alpha > \kappa_2(Q)\cdot\|[(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}][MDQ]^{+}\|_2 \qquad (92)$$

where $\alpha = -\max_i[\mathrm{Re}(\lambda_i)]$ and $\kappa_2(Q)=\|Q^{-1}\|_2\cdot\|Q\|_2$ is the 2-norm condition number of the matrix Q.

Proof:

The closed loop system with time varying uncertainties is described by (82). Denote $A_c = A + BFC$ and $\Delta A_c(t) = \Delta A(t) + \Delta B(t)FC$, then

$$\dot{x} = A_c x(t) + \Delta A_c(t)x(t) \tag{93}$$

Let $x(t) = MDQz(t)$, where M and MD are both modal matrices of A_c, then,

$$\dot{z}(t) = Q^{-1}D^{-1}M^{-1}A_c MDQz(t) + Q^{-1}D^{-1}M^{-1}\Delta A_c(t)MDQz(t) \tag{94}$$

which has a solution given by

$$z(t) = \exp(Q^{-1}\Lambda Qt)z(0)$$
$$+ \int_0^t \exp[Q^{-1}\Lambda Q(t-\tau)]Q^{-1}D^{-1}M^{-1}\Delta A_c(\tau)MDQz(\tau)d\tau \tag{95}$$

where $\Lambda = D^{-1}M^{-1}A_c MD$ is a diagonal eigenvalue matrix. Use $\exp(Q^{-1}\Lambda Qt) = Q^{-1}\exp(\Lambda t)Q$ to obtain

$$z(t) = Q^{-1}\exp(\Lambda t)Qz(0)$$
$$+ \int_0^t Q^{-1}\exp[\Lambda(t-\tau)]QQ^{-1}D^{-1}M^{-1}\Delta A_c(\tau)MDQz(\tau)d\tau \tag{96}$$

In the remainder of the proof, the subscript "2" on the norms is omitted for notational convenience. Take norms on both sides of (96) to obtain

$$\|z(t)\| = \|Q^{-1}\|\cdot\|\exp(\Lambda t)\|\cdot\|Q\|\cdot\|z(0)\|$$

$$+ \int_0^t \|Q^{-1}\|\cdot\|\exp[\Lambda(t-\tau)]\|\cdot\|Q\|\cdot\|Q^{-1}D^{-1}M^{-1}\Delta A_c(\tau)MDQ\|\cdot\|z(\tau)\|d\tau \tag{97}$$

Use $\|\exp(\Lambda t)\| \le \exp(\alpha t)$ where $\alpha=-\max_i\{\mathrm{Re}[\lambda_i(A+BFC)]\}$ and also use $\kappa_2(Q)=\|Q^{-1}\|\cdot\|Q\|$ and $Q^{-1}D^{-1}M^{-1}=(MDQ)^{-1}$. Then

$$\|z(t)\| \le \kappa_2(Q)\exp(-\alpha t)\|z(0)\|$$

$$+ \int_0^t \kappa_2(Q)\exp[-\alpha(t-\tau)]\|(MDQ)^{-1}\Delta A_c(\tau)MDQ\|\cdot\|z(\tau)\|d\tau \tag{98}$$

or

$$\|z(t)\|\exp(\alpha t) \le \kappa_2(Q)\|z(0)\|$$

$$+ \int_0^t \kappa_2(Q)\|(MDQ)^{-1}\Delta A_c(\tau)MDQ\|\exp(\alpha\tau)\|z(\tau)\|d\tau \tag{99}$$

Use the Gronwall lemma to obtain

$$\|z(t)\|\exp(\alpha t) \le \kappa_2(Q)\|z(0)\|\exp\left[\int_0^t \kappa_2(Q)\|(MDQ)^{-1}\Delta A_c(\tau)MDQ\|d\tau\right] \tag{100}$$

Use lemma 1 to obtain

$$\|z(t)\|\exp(\alpha t)$$

$$\le \kappa_2(Q)\|z(0)\|\exp\{\kappa_2(Q)\|[(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}](MDQ)^{+}\|t\} \tag{101}$$

Thus,

$$\|z(t)\|$$

$$\leq \kappa_2(Q)\|z(0)\|\exp\{-\alpha+\kappa_2(Q)\|[(MDQ)^{-1}]^+[A_m+B_m(FC)^+](MDQ)^+\|\}t \tag{102}$$

Thus, $\|z(t)\| \to 0$ and hence $\|x(t)\| \to 0$ exponentially if

$$\alpha > \kappa_2(Q)\|[(MDQ)^{-1}]^+[A_m+B_m(FC)^+](MDQ)^+\| \tag{103}$$

End of proof

Now, we consider the optimal choice of the diagonal real positive matrix D in (92) when Q is the identity matrix. We start by stating a lemma which is attributed to Stoer and Witzgall [14].

Lemma [14]

Define the Perron eigenvalue of the non-negati:.e matrix A^+, denoted by $\pi(A^+)$, to be the real non-negative eigenvalue $\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues $(i = 1,\ldots,n)$ of A^+. Define the right and left Perron eigenvectors, denoted by x and y, respectively. Normalize the Perron eigenvectors such that $\pi x = Ax$ and $\pi y^T = y^TA$ with $\|x\|_\infty = \|y\|_\infty = 1$. Then, the matrix D_{opt} which yields the infimum of $\|D^{-1}A^+D\|_2$ is given by

$$D_{opt} = \text{diag}\{[y(1)/x(1)]^{1/2},\ldots,\ [y(n)/x(n)]^{1/2}\} \tag{104}$$

We seek D_{opt} which yields the infimum of

$$\| [(MD)^{-1}]^{+} [A_m + B_m(FC)^{+}] [MD]^{+} \|_2 \tag{105}$$

or equivalently

$$\| D^{-1}(M^{-1})^{+} [A_m + B_m(FC)^{+}] M^{+} D \|_2 \tag{106}$$

The lemma of Stoer and Witzgall [14] applies to the robustness problem because the matrix $(M^{-1})^{+}[A_m + B_m(FC)^{+}]M^{+}$ in (106) is non-negative. Also, (105) and (106) are equivalent because $[MD]^{+}=M^{+}D$ and $[(MD)^{-1}]^{+}=D^{-1}(M^{-1})^{+}$. These last two equalities hold because D is a real positive diagonal matrix. Therefore, the above lemma gives the optimal matrix D_{opt} which yields the infimum of (106). We now propose a conjecture for the choice of the matrix Q which will reduce the conservatism in the sufficient condition for robust stability.

Conjecture 1

A choice for the matrix Q which will reduce the norm in (92) is given by

$$Q = UV^{T}. \tag{107}$$

where $U\Sigma V^{T}$ is the singular value decomposition of the matrix $(MD)^{T}$.

An optimal choice for the matrix Q is not yet available. However, the choice of (107) results in $\kappa_2(Q)$ equal to its minimum value of unity because U and V are unitary due to the property of the singular value decomposition. Furthermore, the choice of (107) solves the following problem (see Golub and Van Loan [21]):

$$\min \|MDQ - I\|_F \quad \text{subject to } Q^TQ=I \tag{108}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. This provides some motivation for the conjecture that the choice of Q in (107) will reduce the value of the norm in (92).

The proposed method is to first compute D_{opt} using (104) and then compute Q using (107). Then, these matrices are used to compute the norm in the robustness condition given by (92).

3.5 EXAMPLE: AFTI F-16 PITCH POINTING/VERTICAL TRANSLATION DESIGN

Consider the LTI model of the AFTI F-16 aircraft described by

$$\begin{bmatrix} \dot{\gamma} \\ \dot{q} \\ \dot{\alpha} \\ \dot{\delta}_e \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} 0 & .00665 & 1.3411 & .16897 & .25183 \\ 0 & -.86939 & 43.223 & -17.251 & -1.5766 \\ 0 & .99335 & -1.3411 & -0.16897 & -.25183 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} \gamma \\ q \\ \alpha \\ \delta_e \\ \delta_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \delta_{ec} \\ \delta_{fc} \end{bmatrix} \tag{109}$$

where $\gamma=\theta-\alpha$ is the flight path angle, θ is pitch attitude, q is pitch rate, α is angle of attack, δ_e is elevator deflection, δ_f is flaperon deflection, δ_{ec} is elevator deflection command, and δ_{fc} is flaperon deflection command. The relationship between the angles is shown in figure 4.

A pitch pointing/vertical translation eigenstructure assignment controller was proposed by Sobel and Shapiro [12] to decouple the pitch attitude and flight path responses.

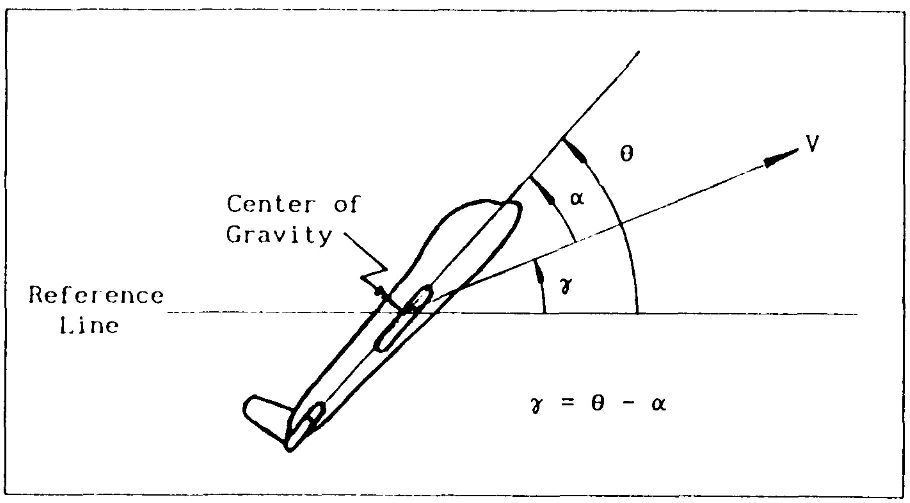

Figure 4. Aircraft Longitudinal Angle Definitions

The zero entries in the short period eigenvectors are for pitch pointing (θ command with no coupling to γ) while the zero entry in the gamma mode eigenvector is for vertical translation (γ command with no coupling to θ or q). The desired eigenvectors are shown in table 4 where we observe the zero entries which are chosen to obtain the required decoupling. The feedback gains described by Sobel and Shapiro [12] are for the outputs $y^T = [q, n_{zp}, \gamma, \delta_e, \delta_f]$, where n_{zp} is the normal acceleration at the pilot's station. However, for simplicity, in this thesis, we will use full state feedback. The full state feedback gain matrix is shown in table 5 and it is obtained by applying a transformation to the feedback gain matrix described in [12]. The achievable eigenvectors are shown in table 6 where we observe that the three zero entries have been exactly achieved. Thus, we expect excellent decoupling in both the pitch pointing and vertical translation responses. Finally, feedforward gains based upon Broussard's [11] command generator tracker are computed to obtain zero steady state error to a step command. The vertical translation and pitch pointing step responses are shown in figures 5 and 6, respectively. As expected, the decoupling is excellent for both cases. However, the design of [12] does not consider stability robustness when the aircraft is subject to linear time varying state space uncertainty.

Table 4. Desired Eigenvectors

Short Period		Gamma Mode	Actuator Mode	Actuator Mode	
0	0	1	x	x	γ
1	x	0	x	x	q
x	1	x	x	x	α
x	x	x	1	x	δ_e
x	x	x	x	1	δ_f

Table 5. Closed Loop Eigenvalues and Control Gains

Closed Loop Eigenvalues	Feedforward Gains (γ_c, θ_c)	Feedback Gains (γ, q, α, δ_e, δ_f)
Non-Robust Design:		
-5.61±j4.19 -1.00 -19.0 -19.5	$\begin{bmatrix} -.375 & -2.87 \\ 4.12 & 1.98 \end{bmatrix}$	$\begin{array}{c} \begin{array}{ccccc} \gamma & q & \alpha & \delta_e & \delta_f \end{array} \\ \begin{bmatrix} 3.25 & .891 & 7.11 & -.526 & -.0840 \\ -6.10 & -.898 & -10.02 & .420 & .102 \end{bmatrix} \end{array} \begin{array}{c} \delta_{ec} \\ \delta_{fc} \end{array}$
Robust Design:		
-3.63±j5.18 -2.66 -19.0 -19.5	$\begin{bmatrix} -.992 & -1.62 \\ 10.93 & -6.47 \end{bmatrix}$	$\begin{bmatrix} 2.61 & .581 & 5.31 & -.288 & -.0555 \\ -4.47 & -.0642 & -1.11 & -.0131 & -.0204 \end{bmatrix}$
Robust Design with v_i Constraint:		
-3.63±j5.14 -3.00 -19.0 -19.5	$\begin{bmatrix} -1.11 & -2.01 \\ 12.37 & -1.83 \end{bmatrix}$	$\begin{bmatrix} 3.12 & .606 & 5.72 & -.291 & -.0543 \\ -10.54 & -.361 & -5.97 & .0239 & -.0342 \end{bmatrix}$
Robust Design with Matrix Q:		
-3.62±j5.18 -2.40 -19.0 -19.5	$\begin{bmatrix} -.890 & -1.68 \\ 9.90 & -5.76 \end{bmatrix}$	$\begin{bmatrix} 2.57 & .552 & 5.29 & -.259 & -.0540 \\ -4.14 & .243 & -.925 & -.331 & -.0363 \end{bmatrix}$

Table 6. Closed Loop Eigenvectors (normalized $\|x\| = 1$)

Short Period		Gamma Mode	Actuator Mode	Actuator Mode
Non-Robust Design:				
0.0	0.0	.3089	-.0054	-.0135
.5954	-.7679	.0001	1.0	.0343
-.1339	.0369	-.3090	-.0472	.0118
-.4502	-.2629	-.8656	.9317	-.0249
1.0	0.0	1.0	.0081	1.0
Robust Design:				
-.0434	.0301	-.2668	-.0052	-.0137
1.0	0.0	-.0038	1.0	.0631
-.0472	-.1598	.2682	-.0475	.0104
.0215	-.7004	.5802	.9316	.0029
.2153	-.0044	1.0	-.0060	1.0
Robust Design with v_i Constraint:				
-.0200	.0077	-.1911	-.0053	-.0137
1.0	0.0	.0010	1.0	.0599
-.0716	-.1378	.1908	-.0474	.0106
-.0685	-.7063	.3868	.9312	-.0002
.5301	.6895	1.0	-.0018	1.0
Robust Design with Matrix Q:				
-.0418	.0119	-.3711	-.0053	-.0137
1.0	0.0	.0008	1.0	.0587
-.0486	-.1419	.3708	-.0473	.0106
-.0177	-.6719	.8377	.9325	-.0013
.5985	.1743	1.0	.0018	1.0

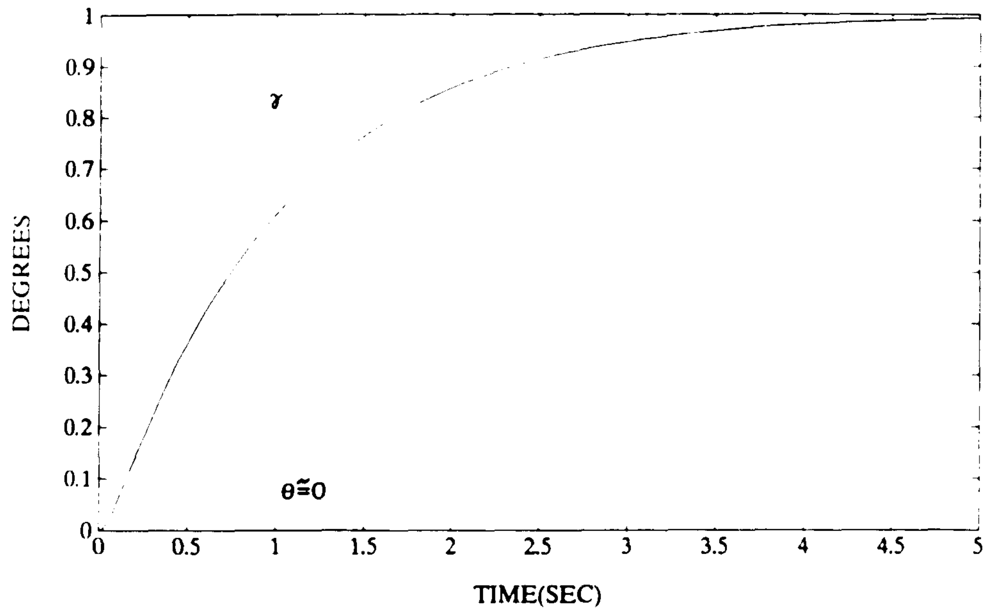

Figure 5. AFTI F-16 Non-Robust Design (Vertical Translation)

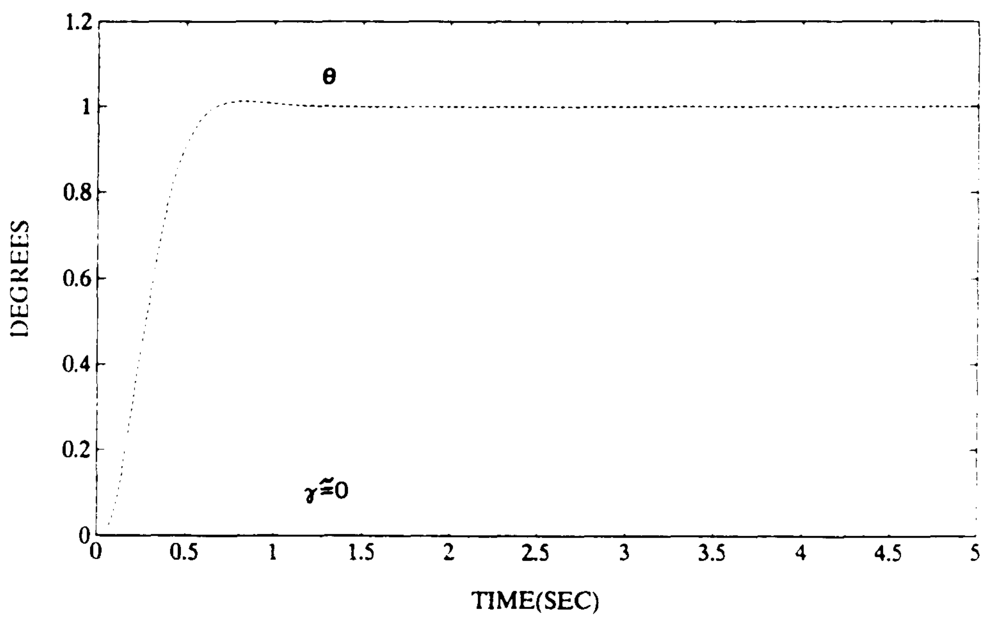

Figure 6. AFTI F-16 Non-Robust Design (Pitch Pointing)

Next, we consider a robust eigenstructure assignment pitch pointing/vertical translation control law. First, to reduce the conservatism in (34), we use a constant similarity transformation to obtain the transformed state given by

$$\tilde{x}(t) = [\theta,\ q,\ \alpha,\ \delta_e,\ \delta_f]^T \tag{110}$$

where θ is the pitch attitude in degrees.

The utilization of such a transformation is valid because the stability of a linear time varying system is preserved when subjected to a constant similarity transformation [22]. The use of such a similarity transformation to reduce conservatism was used by Yedavalli and Liang [23] in connection with a Lyapunov approach to robustness.

In the transformed coordinates, the equation $\dot{\theta}=q$ is a physical relationship without uncertain parameters. Thus, the first row of A_m will contain all zeros. The matrices A_m and B_m for the transformed system are chosen to be

$$A_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0869 & 1.08 & 0.4313 & 0.158 \\ 0 & 0 & 0.1341 & 0.0169 & 0.0252 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} \tag{111}$$

$$B_m = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \tag{112}$$

This choice of A_m and B_m is for illustrative purposes. The A_m and B_m chosen correspond to maximum uncertainty of 10% in a_{22}, a_{25}, a_{33}, a_{34}, and a_{35}; 2.5% in a_{23} and a_{24}; and 0.5% in the actuator parameters a_{44}, a_{55}, b_{41}, and b_{52}.

We emphasize that the transformed system is used only for computation of (34), while the original system of (109) is used for eigenvalue/eigenvector selection and gain computation. For the design of Sobel and Shapiro [12], α=1.0 while the right hand side of (34) equals 4.28 which indicates that the sufficient condition for robust stability is not satisfied. We use a constrained optimization to solve

$$\min \left[\lambda_{max}[(M^{-1})^{+}(A_m+B_mF^{+})M^{+}] - \alpha\right] \tag{113}$$

while constraining some of the eigenvalues and eigenvectors. First, we choose to assign the actuator eigenvalues and eigenvectors to the same values as in the design presented by Sobel and Shapiro [12]. Then, an optimization is performed over a subset of the short period eigenvector parameters, short period eigenvalues, and the gamma mode eigenvalue. This choice of optimization parameters is chosen by examining the gradient at the initial point corresponding to the design of [12]. The term "eigenvector parameters" is the vector z_i, where an eigenvector v_i is defined by $v_i = L_i z_i$ and where the columns of L_i are a basis for the i-th eigenvector subspace. So during the optimization, the first entry (both real and imaginary parts) of the vector z_{sp} for the short period mode is allowed to change, the short period eigenvalues are allowed to change subject to the constraints $-7.6 \le Re[\lambda_{sp}] \le -3.6$, $3.2 \le Im[\lambda_{sp}] \le 5.2$, and the gamma mode eigenvalue is allowed to change subject to the constraint $-2.65 \le \lambda_{\gamma} \le -0.5$. Each time λ_{γ} is changed, its eigenvector is computed as the orthogonal projection of the desired gamma mode eigenvector onto the current achievable gamma mode subspace. In this way, the zero entry in the gamma mode eigenvector might be achieved in which case the vertical translation decoupling may be nearly the same as in the design of [12].

When the optimization is complete $\alpha = 2.65$ and the right hand side of (34) equals 2.57; thus, the sufficient condition for robust stability is satisfied. The closed loop eigenvectors are shown in table 6 from which we observe that the zero entry in the gamma mode eigenvector has been achieved. This suggests that the vertical translation decoupling will be the same in [12]. However, the entries in the short period eigenvectors which were zero in [12], are now non-zero. This suggests that the pitch pointing decoupling will not be as good as in [12]. The closed loop eigenvalues and feedback gains are shown in table 5. Observe that the short period eigenvalues have moved to the boundary of the constraint region corresponding to the largest settling time and smallest damping ratio. The gamma mode eigenvalue has moved to the leftmost point in its constraint interval which is to be expected since λ_γ appears explicitly in (34). Also observe that the magnitude of the feedback gains has been reduced. The vertical translation and pitch pointing responses are shown in figures 7 and 8, respectively. As expected, the vertical translation decoupling is virtually identical to the [12] design while the pitch pointing decoupling has been degraded. Table 7 shows that the new robust eigenstructure assignment design exhibits a significantly improved minimum singular value of the return difference matrix at the inputs and an improved condition number of the closed loop modal matrix as compared to the [12] design.

Table 7. Comparison of Robustness Measures

	$\text{Max}\lvert\gamma\rvert$ $\theta_c=1$	Min $\underline{\sigma}(I+FG)$	Cond(M)
Non-Robust Design	$6.66\text{x}10^{-4}$	0.1983	56.00
Robust Design	0.2171	0.5021	27.30
Robust Design with v_i constraint	0.0787	0.2802	36.80
Robust Design with Matrix Q	0.1617	0.4244	27.11

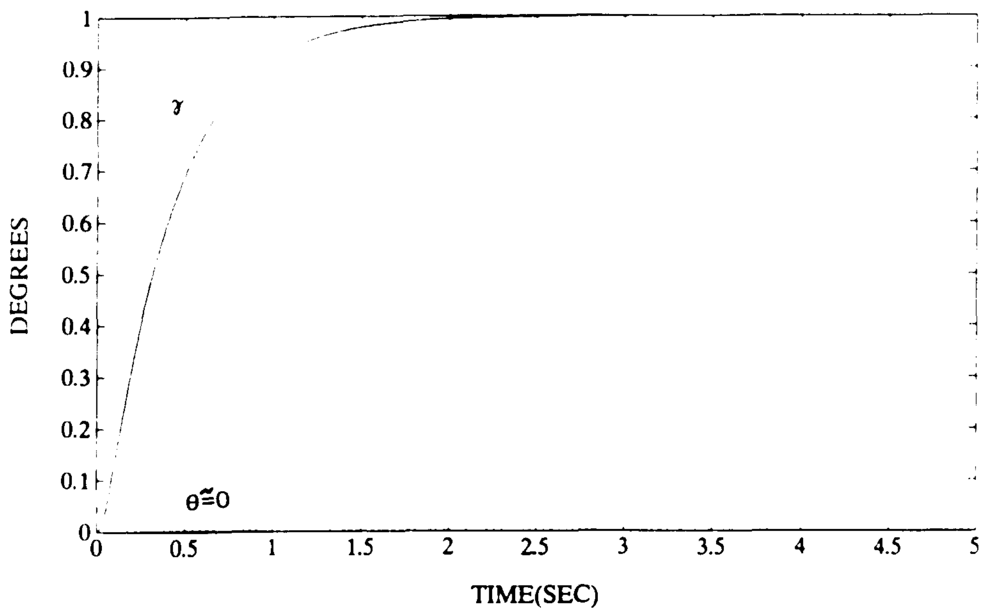

Figure 7. AFTI F-16 Robust Design (Vertical Translation)

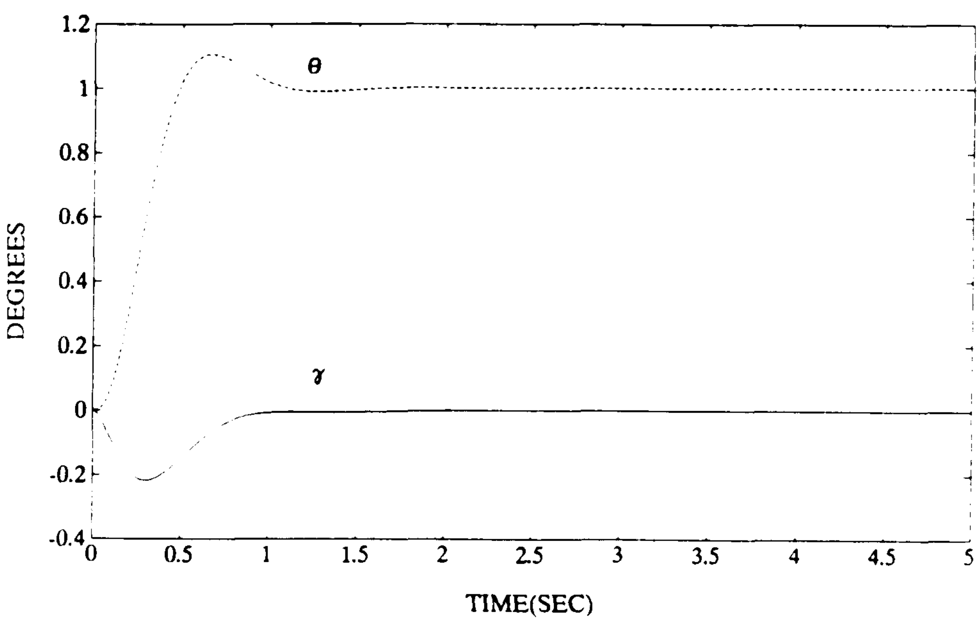

Figure 8. AFTI F-16 Robust Design (Pitch Pointing)

Next, we place a constraint on the first entry of the complex conjugate short period eigenvectors. This will allow us to place additional emphasis on the mode decoupling which is required for the pitch pointing maneuver. The constraints are chosen to be $|Re(v_{sp})| \leq 0.01$ and $|Im(v_{sp})| \leq 0.01$ where the eigenvectors are normalized to unit length in the 2-norm sense. A solution could not be obtained for the eigenvalue constraints which were used in the previous design. Therefore, the gamma mode eigenvalue constraint is relaxed to become $-3.0 \leq \lambda_{\gamma} \leq -0.5$ which will have the effect of moving the gamma mode eigenvalue farther into the left half of the complex plane. The new closed loop eigenvectors are shown in table 6 where we observe that the real and imaginary parts of the first entry of the short period eigenvectors are significantly smaller than in the previous design. Thus, we should expect that the pitch pointing response will be improved. The vertical translation and pitch pointing responses are shown in figures 9 and 10, respectively. We observe that the coupling between γ and θ has been reduced by approximately 50% as compared to the previous design. However, we observe from table 6 that the feedback gains are larger in magnitude. Also, from table 7 we observe that we no longer have a significant improvement in the minimum singular value of the return difference matrix.

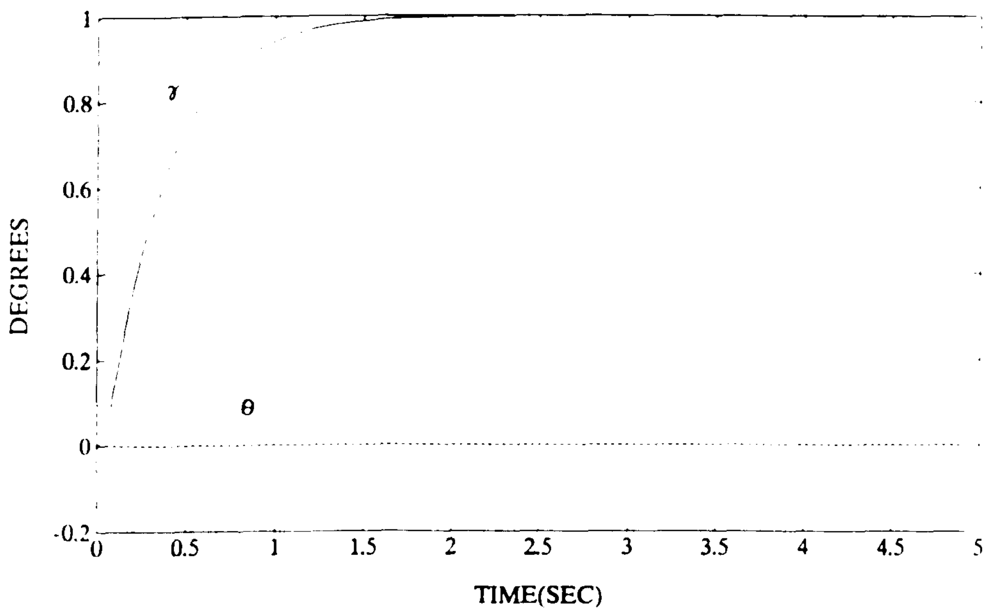

Figure 9. AFTI F-16 Robust Design with v_i Constraints (Vertical Translation)

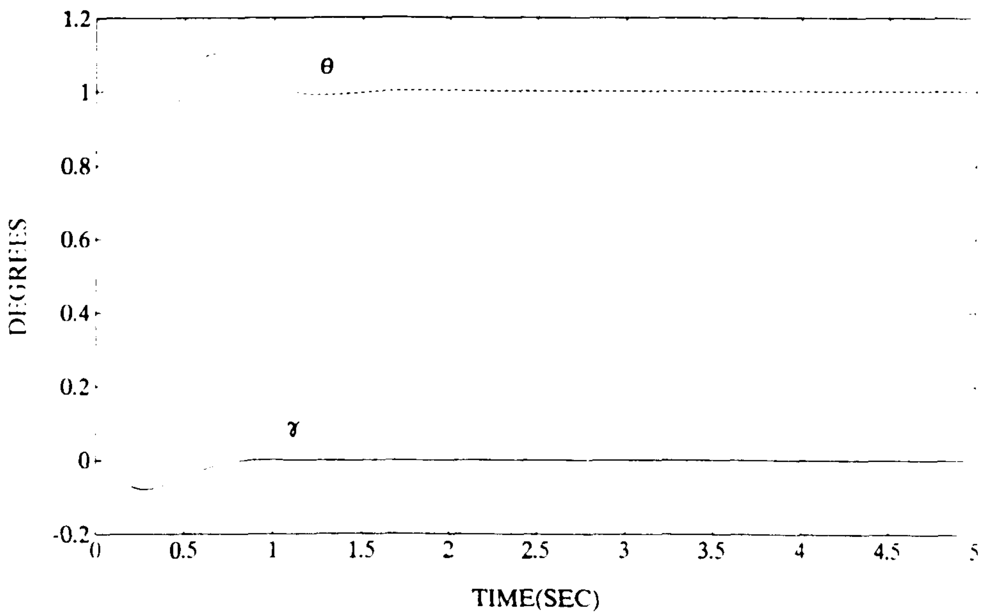

Figure 10. AFTI F-16 Robust Design with v_i Constraints (Pitch Pointing)

Next, we remove the additional short period eigenvector constraints, but we introduce the unitary weighting matrix Q in order to reduce the conservatism in the robust stability condition. Due to the use of the matrix Q, we can tighten the constraint on the gamma mode eigenvalue to $-2.4 \le \lambda_\gamma \le -0.5$. Thus, we obtain a solution with $\lambda_\gamma = -2.4$ which is to compared with the previous solution with $\lambda_\gamma = -2.65$. So we do not need to move the gamma mode eigenvalue as far left in the complex plane as before. The vertical translation and pitch pointing responses are shown in figures 11 and 12, respectively. The vertical translation response exhibits slightly less coupling than the first robust design which did not utilize either the unitary matrix Q or the explicit eigenvector constraints on $|Re[v_{sp}(1)]|$ and $|Im[v_{sp}(1)]|$.

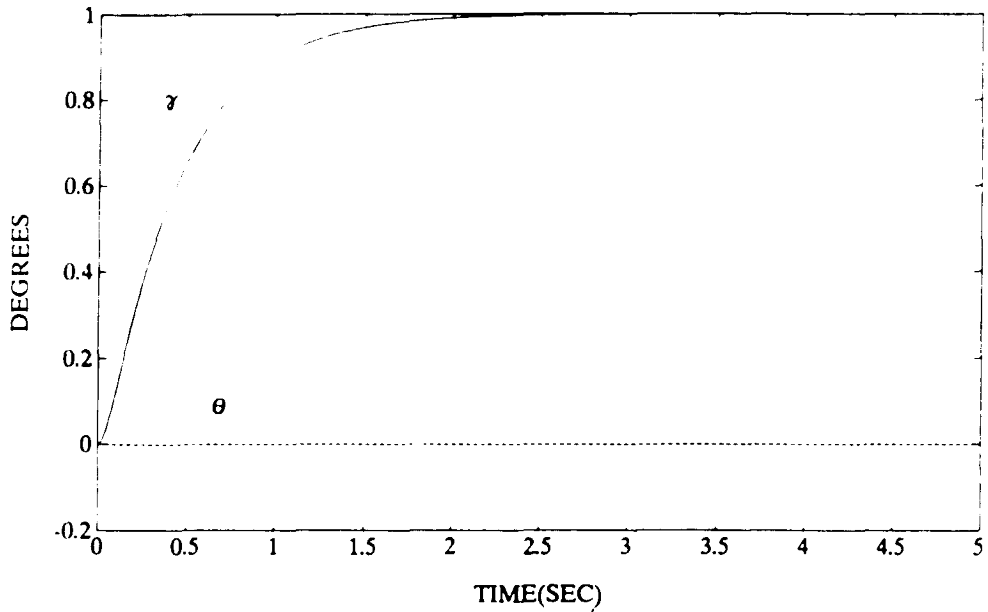

Figure 11. AFTI F-16 Robust Design with Matrix Q (Vertical Translation)

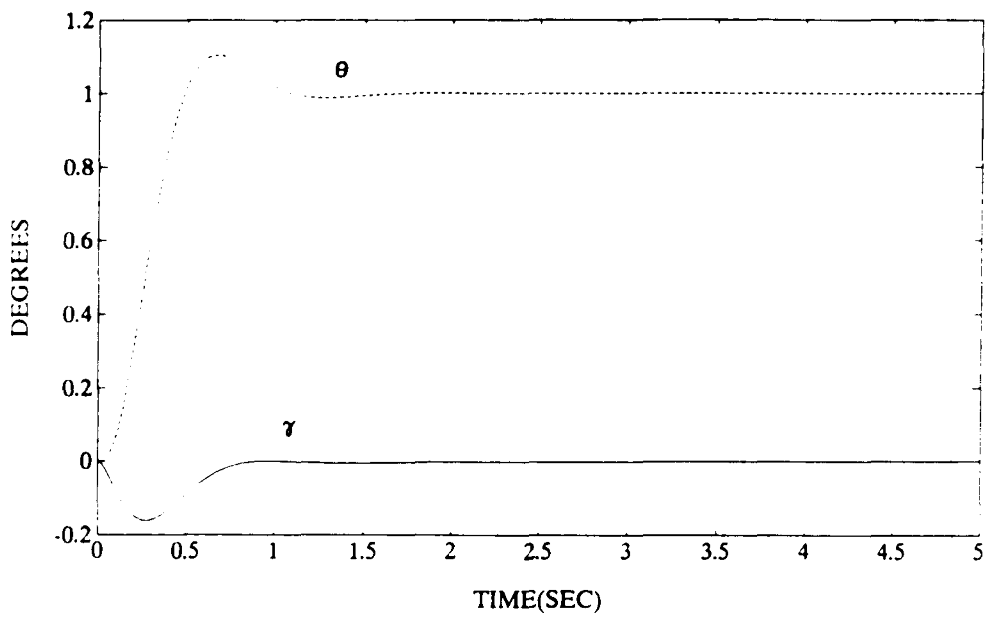

Figure 12. AFTI F-16 Robust Design with Matrix Q (Pitch Pointing)

3.6 EXAMPLE: L-1011 AIRCRAFT DESIGN

3.6.1 L-1011 AIRCRAFT WITH STATE FEEDBACK

Consider the linearized lateral dynamics of the L-1011 aircraft described by

$$\begin{bmatrix} \dot{\phi} \\ \dot{r} \\ \dot{p} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & N_r & N_p & N_\beta \\ 0 & L_r & L_p & L_\beta \\ g/V & (Y_r/V)-1 & Y_p/V & Y_\beta/V \end{bmatrix} \begin{bmatrix} \phi \\ r \\ p \\ \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ N_{\delta_r} & N_{\delta_a} \\ L_{\delta_r} & L_{\delta_a} \\ Y_{\delta_r}/V & Y_{\delta_a}/V \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \tag{114}$$

with stability and control derivatives shown in table 8. Where ϕ is bank angle(deg), r is yaw rate(deg/sec), p is roll rate(deg/sec), β is sideslip(deg), δ_r is rudder deflection, and δ_a is aileron deflection. An eigenstructure assignment controller was designed by Sobel and Shapiro [24] which decouples the dutch roll mode from the roll mode while assigning the eigenvalues as shown below:

$$\lambda_{dr} = -1.5\pm j1.5 \tag{115}$$

$$\lambda_{roll} = -2.0\pm j1.0 \tag{116}$$

The full state feedback gain matrix is given by [24]

$$F = \begin{bmatrix} -.367 & 3.52 & -.163 & -3.64 \\ 4.35 & 1.28 & 2.63 & -5.68 \end{bmatrix} \qquad (117)$$

Table 8. Stability and Control Derivatives for L-1011 Aircraft

Parameter	Norminal Value	Maximum of Absolute Value of Uncertainty
N_r	-0.154	0.01925(12.5%)
L_r	0.249	0.031125(12.5%)
$(Y_r/V)-1$	-0.996	0.0
N_p	-0.0042	0.0021(50%)
L_p	-1.00	0.125(12.5)
Y_p/V	-0.000295	0.0001475(50%)
N_β	1.54	0.1155(7.5%)
L_β	-5.20	0.26(5%)
Y_β/V	-0.117	0.0146(12.5%)
g/V	0.0386	0.0
N_{δ_r}	-0.744	0.093(12.5%)
N_{δ_a}	-0.032	0.004(12.5%)
L_{δ_r}	0.337	0.042125(12.5%)
L_{δ_a}	-1.12	0.084(7.5%)
Y_{δ_r}/V	0.02	0.01(50%)
Y_{δ_a}/V	0.0	0.0

We evaluate the stability robustness condition of (34) where α=1.5 and the right hand side of (34) equals 1.46. Therefore, the stability robustness sufficient condition is satisfied. Nonetheless, we choose to optimize (113) in order to increase the stability margin. That is, to guarantee stability for larger uncertainty than that shown in table 8. The gradient is computed at the design described by (117) to determine which parameters should be varied during the optimization. The chosen parameters are $Re[z_{dr}(1)]$, $Im[z_{dr}(1)]$, $Re[\lambda_{dr}]$, $Re[z_{roll}(2)]$, $Im[z_{roll}(2)]$, $Re[\lambda_{roll}]$, and $Im[\lambda_{roll}]$. The constraints are

$$-1.4 \leq Re[\lambda_{dr}] \leq -1.9 \tag{118}$$

$$-1.95 \leq Re[\lambda_{roll}] \leq -3.0 \tag{119}$$

$$0.5 \leq Im[\lambda_{roll}] \leq 1.45 \tag{120}$$

The optimal solution has eigenvalues given by

$$\lambda_{dr} = -1.87 \pm j1.5 \tag{121}$$

$$\lambda_{roll} = -1.95 \pm j1.45 \tag{122}$$

with feedback gain matrix given by

$$F = \begin{bmatrix} -.411 & 4.47 & -.161 & -5.07 \\ 5.16 & 1.57 & 2.54 & -6.17 \end{bmatrix} \tag{123}$$

We find that α=1.87 while the right hand side of (34) equals 1.50. Thus, the system is robustly stable for larger uncertainty than that shown in table 8. Also of interest is that the design of (117) has a closed loop modal matrix with condition number equal to 5.83 as compared with 4.56 for the design of (123). Thus, as we have also seen in the F-16 example, optimizing (113) seems to provide other desirable system characteristics. The time responses to an initial sideslip of one degree are shown for the designs of (117) and (123) in figure 13 and 14 respectively. We observe that the new design has a slightly larger peak in yaw rate and a slightly faster response.

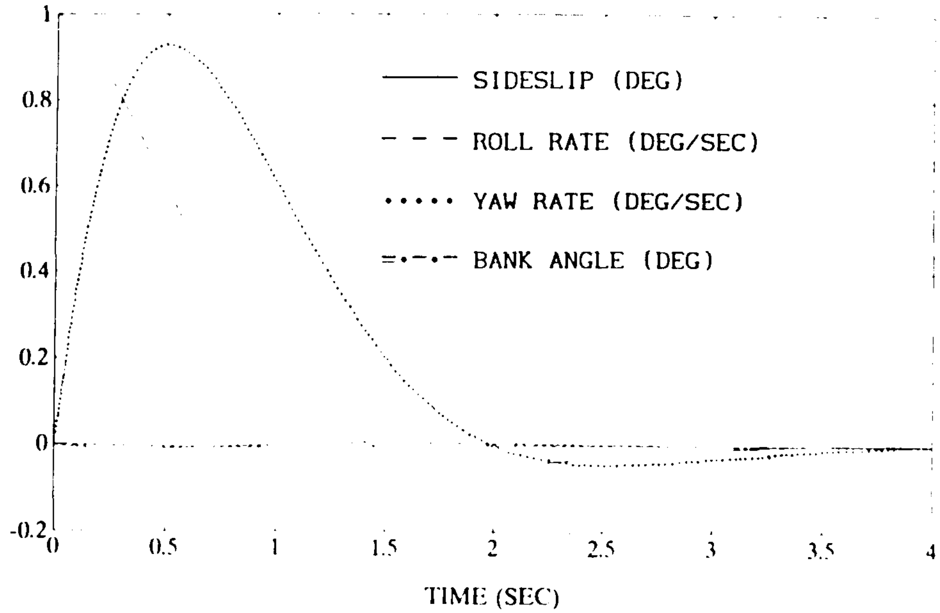

Figure 13. L-1011 State Feedback Initial Design

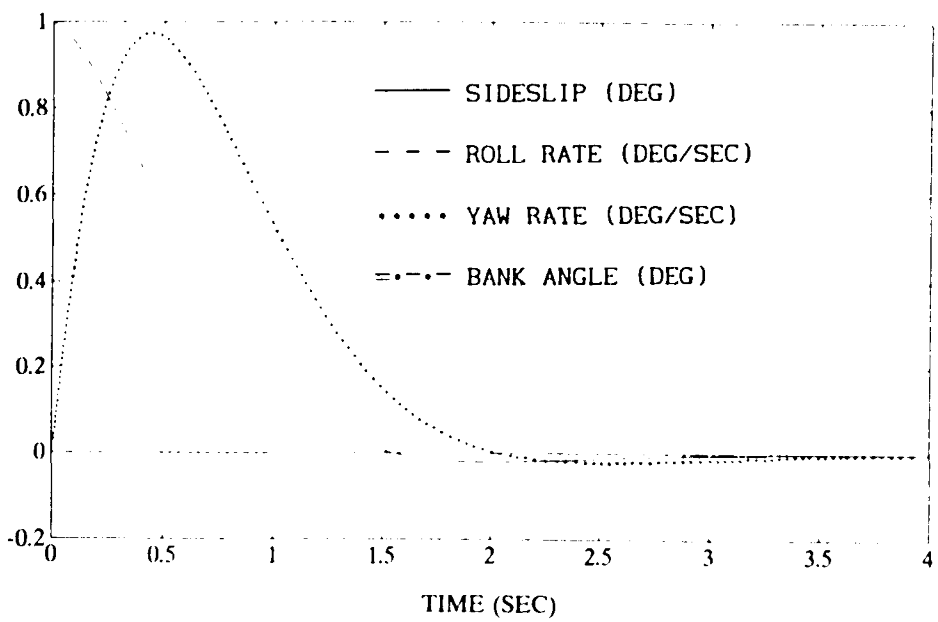

Figure 14. L-1011 State Feedback Robust Design

3.6.2 L-1011 AIRCRAFT WITH GAIN SUPPRESSION

We consider computing a robust eigenstructure assignment feedback gain matrix where certain outputs are not fed back to certain inputs. This is equivalent to constraining some elements of the feedback gain matrix to be zero. By suppressing certain gains to zero, we reduce controller complexity and improve reliability. Mathematically, for the control system described by (35)-(37), the output feedback gain matrix F using eigenstructure assignment satisfies (see Andry et.al. [8])

$$F\Omega = \psi \tag{124}$$

where Ω and ψ depend upon the open loop system, the closed loop eigenvalues, and the closed loop eigenvectors. As shown by Andry et.al. [8], each row of feedback gains (f_i) can be computed independently of all other rows. That is,

$$f_i = \psi_i \Omega^{-1} \tag{125}$$

If we were to constrain f_{ij} to be zero, then we delete f_{ij} from f_i^T and delete the j-th column of Ω^T. We now solve the reduced problem given by

$$\tilde{\Omega}^T \, \tilde{f}_i^T = \psi_i^T \tag{126}$$

where $\tilde{\Omega}^T$ is the matrix Ω^T with its j-th column deleted and $\tilde{f}_i^T$ is the vector f_i^T with its j-th element deleted. (126) may be solved for $\tilde{f}_i$, the remaining active gains in the i-th row, as shown by Andry et.al. [8]. If more than one gain in a row of F is to be set to zerc, then (126) should be appropriately modified.

First, we determine which gains may be set to zero while having the least impact on the closed loop eigenvalues and eigenvectors. We compute the eigenvalue decision matrix D^λ and the eigenvector decision matrix D^v as described in Chapter 2. The eigenvalue decision matrix is given by

$$D^\lambda = \begin{bmatrix} .02185 & \underline{.64226} & .02173 & \underline{.29302} \\ \underline{.86204} & .00786 & \underline{1.16413} & .02358 \end{bmatrix} \tag{127}$$

and the eigenvector decision matrix is given by

$$D^v = \begin{bmatrix} .05228 & .51731 & .05201 & .23612 \\ .15345 & \underline{.60016} & .20723 & \underline{1.23568} \end{bmatrix} \tag{128}$$

The larger entries in (127), denoted by an underline, are the elements of F which need to be non-zero based upon eigenvalue considerations. Thus, we could constrain f_{11}, f_{13}, f_{22}, f_{24} to be zero based solely upon their small effect on the closed loop eigenvalues. However, from (128) we observe that f_{22} and f_{24} are important for eigenvector considerations. Therefore, we choose to constrain only f_{11} and f_{13} to be zero.

An eigenstructure assignment feedback gain matrix with $f_{11}=f_{13}=0$ is computed using the same choice for the desired eigenvalues and eigenvectors which were used in section 3.6.1. The closed loop eigenvalues are given by

$$\lambda_{dr} = -1.5\pm j1.5 \tag{129}$$

$$\lambda_{roll} = -1.97\pm j0.993 \tag{130}$$

and the feedback gain matrix is given by

$$F = \begin{bmatrix} 0.0 & 3.52 & 0.0 & -3.46 \\ 4.45 & 1.28 & 2.63 & -5.68 \end{bmatrix} \tag{131}$$

We evaluate the stability robustness condition of (34) where A_m and B_m are given in table 8, α=1.50 and the right hand side of (34) equals 1.67. Therefore, the stability robustness sufficient condition is not satisfied. We optimize (113) with the additional parameter of $Im[\lambda_{dr}]$ as compared with the full state robust design. We also include the constraint given by $1.0 \le Im[\lambda_{dr}] \le 2.0$ in addition to the constraints which were used in the full state robust design. However, an important difference is that now the constraints apply only to the eigenvalues used for subspace computation and not to the actual closed loop eigenvalues. The optimized design has eigenvalues for subspace computation given by

$$\lambda_{dr} = -1.74 \pm j1.96 \tag{132}$$

$$\lambda_{roll} = -2.00 \pm j0.958 \tag{133}$$

and closed loop eigenvalues given by

$$\lambda_{dr} = -1.77 \pm j1.85 \tag{134}$$

$$\lambda_{roll} = -1.77 \pm j1.01 \tag{135}$$

The robust feedback gain matrix is given by

$$F = \begin{bmatrix} 0.0 & 4.62 & 0.0 & -6.11 \\ 3.69 & 1.47 & 2.27 & -6.48 \end{bmatrix} \tag{136}$$

We now find that α=1.77 while the right hand side of (34) equals 1.63. Therefore, robust stability is ensured for the uncertainty described in table 8. Also of interest is that we have reduced the condition number of the closed loop modal matrix from 5.78 to 4.99 which indicates that optimizing (113) yields other desirable closed loop system characteristics. The non-robust and robust time responses for an initial sideslip of one degree are shown in figures 15 and 16, respectively. We observe that both designs exhibit perfect decoupling between the dutch roll mode and the roll mode.

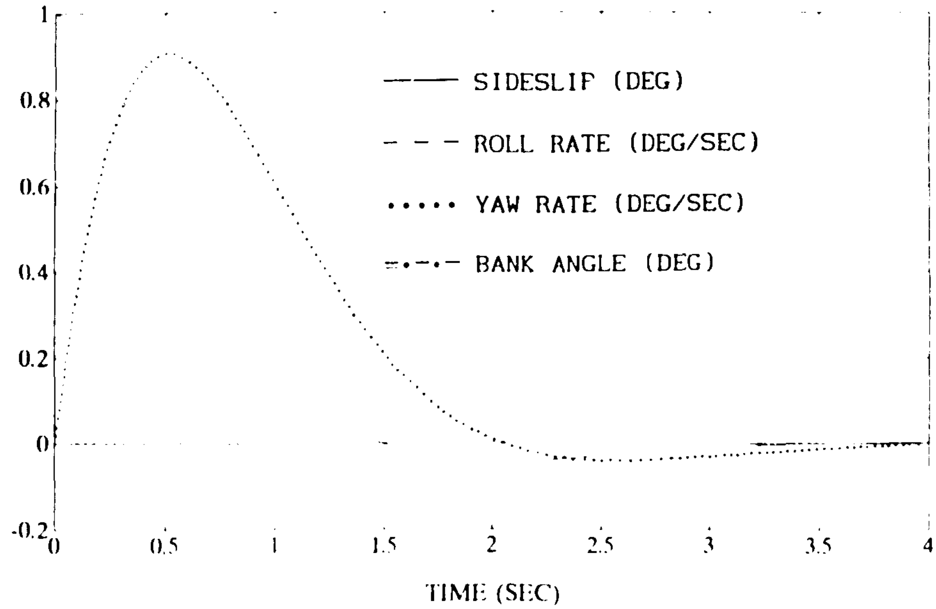

Figure 15. L-1011 Gain Suppression Initial Design

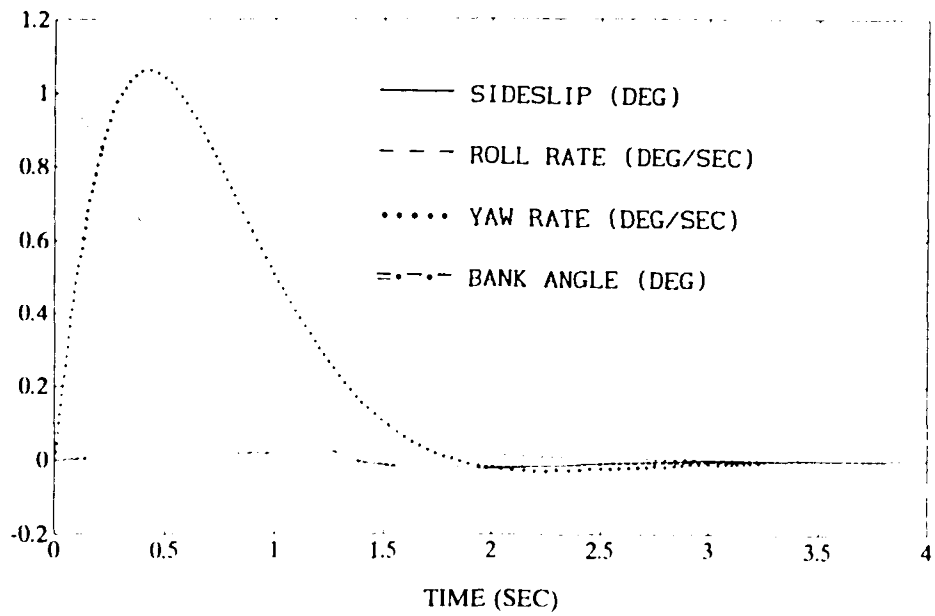

Figure 16. L-1011 Gain Suppression Robust Design

3.6.3 L-1011 AIRCRAFT WITH DYNAMIC COMPENSATION

The use of constant gain output feedback is desirable because of the simplicity of the resulting control structure. However, it may not always be possible to achieve the dual goals of performance and robustness with a constant gain controller using only output feedback. In fact, if the number of outputs is too small, it may not even be possible to achieve the performance specifications using constant gain output feedback.

Sobel and Shapiro [25] have considered flight control design using eigenstructure assignment with low order dynamic compensators. This allows the performance specifications to be achieved in the case of limited measurements. Mathematically, the linear time invariant dynamic controller is described by

$$\dot{z} = Dz + Ey \tag{137}$$

$$u = Fz + Gy \tag{138}$$

where the controller state z is of dimension ℓ, $0 \le \ell \le n-r$ It is convenient to model the plant and compensator by the composite system originally proposed by Johnson and Athans [26]. Thus, define

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,\bar{u} \tag{139}$$

$$\bar{y} = \bar{C}\,\bar{x} \tag{140}$$

$$\bar{u} = \bar{F}\,\bar{y} \tag{141}$$

where

$$\bar{x} = \begin{bmatrix} x \\ z \end{bmatrix}; \tag{142}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; \tag{143}$$

$$\bar{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}; \tag{144}$$

$$\bar{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}; \tag{145}$$

$$\bar{F} = \begin{bmatrix} G & F \\ E & D \end{bmatrix} \tag{146}$$

For a second order dynamic compensator, the structure is described by

$$\bar{A}=\begin{bmatrix} A_{4x4} & 0 \\ 0 & 0_{2x2} \end{bmatrix}; \tag{147}$$

$$\bar{B}=\begin{bmatrix} B_{4x2} & 0 \\ 0 & I_{2x2} \end{bmatrix}; \tag{148}$$

$$\bar{C}=\begin{bmatrix} C_{3x4} & 0 \\ 0 & I_{2x2} \end{bmatrix}; \tag{149}$$

$$\bar{A}_m=\begin{bmatrix} A_m & 0 \\ 0 & 0_{2x2} \end{bmatrix}; \tag{150}$$

$$\bar{B}_m=\begin{bmatrix} B_m & 0 \\ 0 & 0_{2x2} \end{bmatrix} \tag{151}$$

where A, B, A_m, and B_m are the same as before (given in table 8) and

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{152}$$

corresponding to $y^T = [r, p, \phi]$.

The desired closed loop eigenvalues are (five can be chosen because we have five outputs; 3 measurements plus 2 compensator states)

$$\lambda_{dr} = -1.5 \pm j1.5 \tag{153}$$

$$\lambda_{roll} = -2.0 \pm j1.0 \tag{154}$$

$$\lambda_5 = -3.0 \text{ (compensator eigenvalue)} \tag{155}$$

The achievable closed loop eigenvalues are

$$\lambda_{dr} = -1.5 \pm j1.5 \tag{156}$$

$$\lambda_{roll} = -2.0 \pm j1.0 \tag{157}$$

$$\lambda_5 = -3.0 \text{ (compensator eigenvalue)} \tag{158}$$

$$\lambda_6 = -3.64 \text{ (compensator eigenvalue)} \tag{159}$$

The dynamic compensator is described by

$$\begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} 2.90 & 2.00 & 1.86 \\ 0.26 & 6.18 & 8.02 \end{bmatrix} \begin{bmatrix} r \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 2.13 & -2.13 \\ 3.51 & -3.51 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{160}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2.46 & -0.54 \\ -1.50 & -1.50 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0.96 & 1.54 & 3.54 \\ 0.00 & -0.50 & -0.50 \end{bmatrix} \begin{bmatrix} r \\ p \\ \phi \end{bmatrix} \qquad (161)$$

We evaluate the stability robustness condition to find $\alpha=1.5$ and the right hand side of (34) equals 13.6 which indicates that the sufficient condition is not satisfied.

A robust design is computed by optimizing (113) over all 25 possible parameters. These include the eigenvalues $\lambda_1,\ldots,\lambda_5$ and their corresponding eigenvector parameter vectors $z_1,\ldots,z_5$. The constraints on the eigenvalues are $-\infty < \mathrm{Re}[\lambda_i] \le -0.5$ for $i=1,\ldots,5$ which are chosen to ensure some degree of nominal relative stability. The optimized design has closed loop eigenvalues given by

$$\lambda \quad = -1.03 \pm j1.08 \qquad (162)$$

$$\lambda_{roll} = -1.98 \pm j1.13 \qquad (163)$$

$$\lambda_5 = -6.61 \text{ (compensator eigenvalue)} \qquad (164)$$

$$\lambda \quad = -5.01 \text{ (compensator eigenvalue)} \qquad (165)$$

and the dynamic compensator is described by

$$\begin{bmatrix}\delta_r\\ \delta_a\end{bmatrix}=\begin{bmatrix}1.92 & .070 & -.005\\ -2.57 & 4.23 & .456\end{bmatrix}\begin{bmatrix}r\\ p\\ \phi\end{bmatrix}+\begin{bmatrix}.405 & -.0005\\ .083 & -3.94\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \tag{166}$$

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}-7.45 & -3.71\\ .618 & -2.87\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\begin{bmatrix}5.12 & .442 & .758\\ -1.41 & 1.28 & -5.41\end{bmatrix}\begin{bmatrix}r\\ p\\ \phi\end{bmatrix} \tag{167}$$

Unfortunately, we find $\alpha=1.03$ while the right hand side of (34) equals 1.97 which indicates that the sufficient condition for robust stability is not satisfied. However, note that the difference between the right and left hand sides of (34) is significantly smaller than for the initial design. We conclude that the sufficient condition cannot be satisfied for the A_m and B_m described in table 8. However, if we let A_m and B_m be replaced by $A_m/2$ and $B_m/2$ then we find that $\alpha=1.03 > 0.986$ so that robust stability can be ensured for uncertainty which equals one-half the uncertainty shown in table 8. The time responses for the initial and optimized design for an initial one degree sideslip are shown in figures 17 and 18, respectively. We observe that both designs exhibit similar and acceptable time responses.

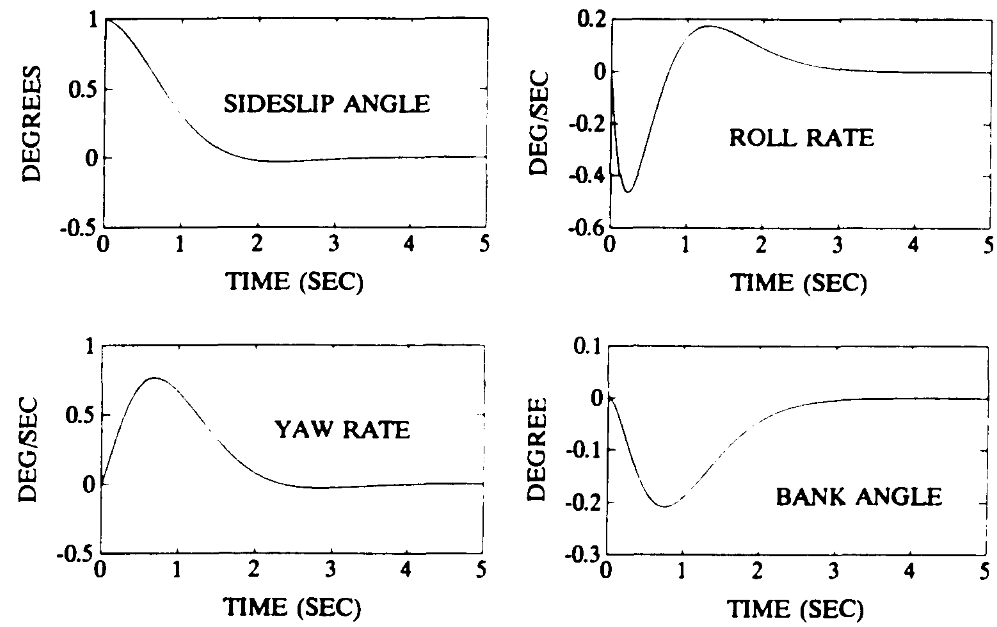

Figure 17. L-1011 Compensator Initial Design

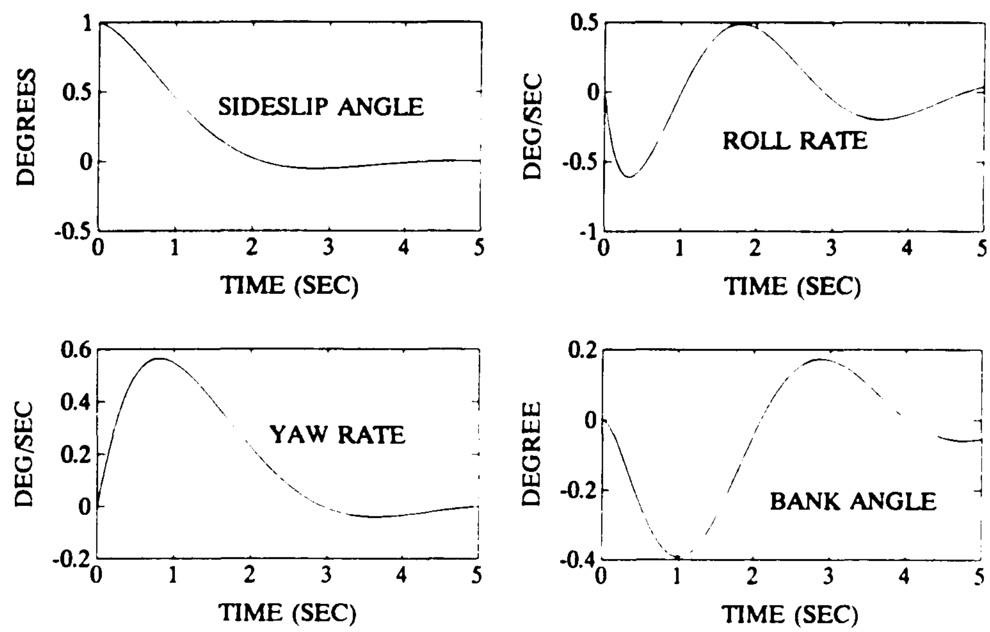

Figure 18. L-1011 Compensator Robust Design

4. PERFORMANCE ROBUSTNESS FOR LTI SYSTEMS WITH STRUCTURED STATE SPACE UNCERTAINTY

4.1 OVERVIEW

An important criterion for the design of uncertain LTI systems is performance robustness. A sufficient condition for performance robustness has been proposed by Juang, Hong, and Wang [15] by using a Lyapunov approach. The closed loop eigenvalues are guaranteed to lie within chosen performance regions when the LTI system is subjected to time invariant structured state space uncertainty. However, this approach requires the solution of a Lyapunov matrix equation involving complex matrices.

In this section, we use a lemma of [15] to obtain a sufficient condition for performance robustness by using the Gronwall lemma approach. The closed loop eigenvalues are guaranteed to lie within chosen performance regions when the LTI system is subjected to time invariant structured state space uncertainty. If the performance region is chosen as the entire left half plane, then the new result reduces to the stability robustness condition described by Sobel et.al. [1]. The required computations include the nominal eigenvalues/eigenvectors and a matrix norm involving the uncertainty structure and the closed loop eigenvector matrix.

Two examples are presented to illustrate the use of the sufficient condition for performance robustness. The chosen performance region corresponds to a minimum damping ratio of 0.15 and a maximum time constant of 4 seconds. These examples also illustrate the need for the unitary matrix Q in order to satisfy the sufficient condition for performance robustness.

4.2 PROBLEM FORMULATION

Consider the robust eigenstructure assignment problem described in Chapter 3. The nominal plant is described by (35)-(36). Suppose that the nominal system is subject to linear time invariant uncertainties in the entries of A and B described by ΔA and ΔB, respectively. Then, the system with uncertainty is given by

$$\dot{x}(t) = Ax(t) + Bu(t) + \Delta Ax(t) + \Delta Bu(t) \tag{168}$$

$$y(t) = Cx(t) \tag{169}$$

Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of ΔA and ΔB. That is,

$$|\Delta a_{ij}| \le (a_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,n \tag{170}$$

$$|\Delta b_{ij}| \le (b_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,m \tag{171}$$

Define ΔA^+ and ΔB^+ as the matrices obtained by replacing the entries of ΔA and ΔB by their absolute values. Also, define A_m and B_m as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively. Then,

$$\{\Delta A\colon\ \Delta A^+ \le A_m\} \tag{172}$$

$$\{\Delta B\colon\ \Delta B^+ \le B_m\} \tag{173}$$

where "$\le$" is applied element by element by element to matrices and $A_{max} \in \mathbb{R}_+^{n\times n}$, $B_{max} \in \mathbb{R}_+^{n\times m}$ where $\mathbb{R}_+$ is the set of non-negative real numbers.

Consider the constant gain output feedback control law described by (37). Then, the nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{174}$$

and the uncertain closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A + \Delta BFC]x(t) \tag{175}$$

Performance Robustness Problem:

A feedback gain matrix $F \in \mathbb{R}^{m \times r}$ is chosen such that all of the eigenvalues of the nominal closed loop system are inside a region R. Determine if all of the eigenvalues of the uncertain closed loop system are inside the region R for all time invariant ΔA and ΔB described by (172)-(173).

4.3 PRELIMINARY RESULTS

First, we review the following result which is attributed to Juang, Hong, and Wang [15]: Consider a line L which separates the complex plane into two open half planes, namely H and $\bar{H}$ as shown in figure 19. The line L intersects the real axis at point "a" and makes an angle θ with respect to the positive imaginary axis, where θ is assumed positive in a counterclockwise sense and $-\pi < \theta \leq \pi$. Denote a specific region H by $H(a,\theta)$.

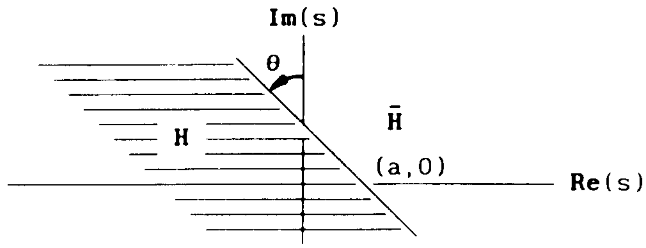

Figure 19. Description of H Region

Lemma [15]

All the eigenvalues of a constant matrix E lie in the H region if and only if the eigenvalues of the matrix $e^{-j\theta}(E-aI)$ lie in the open left half plane.

Next we present two new definitions which relate the so-called transformed uncertain system to the uncertain system described by (175). Then, we propose a lemma which shows that the transformed uncertain system is asymptotically stable if and only if the eigenvalues of the uncertain system of (175) are in the H region.

Note: When we refer to the uncertain system in the context of performance robustness, we assume that the uncertainty is time invariant.

Definition 1

The transformed uncertain system is given by

$$\dot{\tilde{x}}(t) = e^{-j\theta}(A + BFC - aI)\tilde{x}(t) + e^{-j\theta}(\Delta A + \Delta BFC)\tilde{x}(t) \quad (176)$$

Definition 2

The nominal closed loop system matrix A_c, the uncertainty matrix ΔA_c, the transformed closed loop system matrix $\tilde{A}_c$, and the transformed uncertainty matrix $\Delta\tilde{A}_c$ are defined by

$$A_c = A + BFC \quad (177)$$

$$\Delta A_c = \Delta A + \Delta BFC \quad (178)$$

$$\tilde{A}_c = e^{-j\theta}(A_c - aI) \quad (179)$$

$$\Delta\tilde{A}_c = e^{-j\theta}\Delta A_c \quad (180)$$

Then, the uncertain closed loop system is given by

$$\dot{x}(t) = A_c x(t) + \Delta A_c x(t) \quad (181)$$

and the transformed uncertain closed loop system is given by

$$\dot{\tilde{x}}(t) = \tilde{A}_c \tilde{x}(t) + \Delta\tilde{A}_c \tilde{x}(t) \quad (182)$$

Lemma 2

The eigenvalues of (181) are in the H region for all ΔA,

ΔB described by (172)-(173) if and only if the eigenvalues of (182) are in open left half complex plane for all ΔA, ΔB described by (172)-(173).

Proof

The uncertain closed loop system may be written as

$$\dot{x}(t) = (A_c + \Delta A_c)x(t) \tag{183}$$

and the transformed uncertain closed loop system may be written as

$$\dot{\tilde{x}}(t) = e^{-j\theta}(A_c + \Delta A_c - aI)\tilde{x}(t) \tag{184}$$

Then, the result follows by using the lemma of [15] with $E = A_c + \Delta A_c$.

End of proof

Now suppose that the real matrix A_c has distinct eigenvalues. Then, the next lemma shows that the modal matrix of A_c will also diagonalize the complex matrix $\tilde{A}_c$.

Lemma 3

Suppose that the matrix A_c has distinct eigenvalues. Let M be a modal matrix of A_c such that $\Lambda = M^{-1}A_cM$ is a diagonal matrix with the eigenvalues of A_c on the diagonal of Λ.

Then, $\tilde{\Lambda} = M^{-1}\tilde{A}_c M$ is a diagonal matrix with the eigenvalues of $\tilde{A}_c$ on the diagonal of $\tilde{\Lambda}$.

Proof

Consider the transformed uncertain closed loop system which is described by

$$\dot{\tilde{x}}(t) = \tilde{A}_c \tilde{x}(t) + \Delta\tilde{A}_c \tilde{x}(t) \tag{185}$$

where

$$\tilde{A}_c = e^{-j\theta}(A_c - aI) \text{ and } \Delta\tilde{A}_c = e^{-j\theta}\Delta A_c. \tag{186}$$

Let λ_i be the i-th eigenvalue of A_c and let $\tilde{\lambda}_i$ be the corresponding eigenvalue of $\tilde{A}_c$. Then,

$$\det[\tilde{\lambda}_i I - \tilde{A}_c] = 0 \tag{187}$$

$$\det[\tilde{\lambda}_i I - e^{-j\theta}(A_c - aI)] = 0 \tag{188}$$

$$\det[(\tilde{\lambda}_i I + e^{-j\theta}a)I - e^{-j\theta}A_c] = 0 \tag{189}$$

$$\det[e^{-j\theta}I]\det[e^{j\theta}(\tilde{\lambda}_i + e^{-j\theta}a)I - A_c] = 0 \tag{190}$$

$$\lambda_i = e^{j\theta}\tilde{\lambda}_i + a \tag{191}$$

$$\tilde{\lambda}_i = e^{-j\theta}(\lambda_i - a) \tag{192}$$

$$\tilde{\Lambda} = e^{-j\theta}(\Lambda - aI) \tag{193}$$

Let $A_c = M\Lambda M^{-1}$

Since $\tilde{A}_c = e^{-j\theta}(A_c - aI)$, it follows that

$$\tilde{A}_c = e^{-j\theta}(M\Lambda M^{-1} - aI)$$

$$= e^{-j\theta}M(\Lambda - aI)M^{-1}$$

$$= M[e^{-j\theta}(\Lambda - aI)]M^{-1}$$

$$= M\tilde{\Lambda}M^{-1} \tag{194}$$

Therefore, $$\tilde{A}_c = M\tilde{\Lambda}M^{-1} \tag{195}$$

and $$\tilde{\Lambda} = M^{-1}\tilde{A}_c M \tag{196}$$

End of proof

The final preliminary result (lemma 4) will establish an upper bound on $\|M^{-1}\Delta\tilde{A}_c M\|_2$ which will be required in the proof of the main performance robustness result. We observe that the upper bound of the norm of the complex matrix

$M^{-1}\Delta\tilde{A}_c M$ is given by the norm of a real matrix.

Lemma 4

Let $\Delta\tilde{A}_c$ be given by definition 2, M is the modal matrix of A_c, and A_m, B_m are given by (172)-(173). Then,

$$\|M^{-1}\Delta\tilde{A}_c M\|_2 \le \|(M^{-1})^+[A_m + B_m(FC)^+]M^+\|_2 \tag{197}$$

Proof

Use the result given by Kouvaritakis and Latchman [20] that for any matrix $A \in \mathbb{C}^{n\times m}$

$$\|A\|_2 \le \|A^+\|_2 \tag{198}$$

Thus,

$$\|M^{-1}\Delta\tilde{A}_c M\|_2 \le \|[M^{-1}\Delta\tilde{A}_c M]^+\|_2$$

$$= \sup \frac{\|[M^{-1}\Delta\tilde{A}_c M]^+x^+\|_2}{\|x^+\|_2}$$

$$= \sup \frac{\|[M^{-1}e^{-j\theta}\Delta A_c M]^+x^+\|_2}{\|x^+\|_2}$$

$$= \sup \frac{\|\{M^{-1}e^{-j\theta}[\Delta A + \Delta BFC]M\}^{+}x^{+}\|_2}{\|x^{+}\|_2}$$

$$\le \sup \frac{|e^{-j\theta}| \cdot \|\{(M^{-1})^{+}[\Delta A + \Delta BFC]^{+}M^{+}x^{+}\|_2}{\|x^{+}\|_2}$$

$$\le \sup \frac{\|(M^{-1})^{+}[A_m + B_m(FC)^{+}]M^{+}x^{+}\|_2}{\|x^{+}\|_2}$$

$$= \|(M^{-1})^{+}[A_m + B_m(FC)^{+}M^{+}\|_2 \qquad (199)$$

End of proof

4.4 PERFORMANCE ROBUSTNESS SUFFICIENT CONDITION

Theorem 2

Suppose that F is such that the nominal closed loop system described by (174) has only distinct eigenvalues all of which lie in some $H(a,\theta)$ region. Let M be a modal matrix of A_c, let D be a diagonal matrix with positive entries, and let Q be a nonsingular matrix. Then, the eigenvalues of the uncertain closed loop system given by (175) will be in the

$H(a,\theta)$ region for all ΔA and ΔB described by (172)-(173) if

$$\alpha \geq \kappa_2(Q)\cdot\|[(MDQ)^{-1}]^{+}[A_m + B_m(FC)^{+}][MDQ]^{+}\|_2 \tag{200}$$

where

$$\alpha = -\max_i \mathrm{Re}[\lambda_i(\tilde{A}_c)]$$

$$= -\max_i \mathrm{Re}[e^{-j\theta}(\lambda_i - a)] \tag{201}$$

and where $\kappa_2(Q) = \|Q^{-1}\|_2\cdot\|Q\|_2$ is the 2-norm condition number of the matrix Q.

Proof

The transformed uncertain closed loop system is described by

$$\dot{\tilde{x}}(t) = \tilde{A}_c\tilde{x}(t) + \Delta\tilde{A}_c\tilde{x}(t) \tag{202}$$

Let $\tilde{x}(t) = MDQz(t)$ where M and MD are both modal matrices of A_c.

$$z(t) = Q^{-1}D^{-1}M^{-1}\tilde{A}_cMDQz(t) + Q^{-1}D^{-1}M^{-1}\Delta\tilde{A}_cMDQz(t) \tag{203}$$

Use lemma 3 to obtain

$$z(t) = Q^{-1}\tilde{\Lambda}Qz(t) + Q^{-1}D^{-1}M^{-1}\Delta\tilde{A}_c MDQz(t) \tag{204}$$

which has a solution described by

$$z(t)=\exp(Q^{-1}\tilde{\Lambda}Qt)z(0)+\int_0^t \exp(Q^{-1}\tilde{\Lambda}(t-\tau)Q)Q^{-1}D^{-1}M^{-1}\Delta\tilde{A}_c MDQz(\tau)d\tau \tag{205}$$

Use $\exp(Q^{-1}\tilde{\Lambda}Qt) = Q^{-1}\exp(\tilde{\Lambda}t)Q$ to obtain

$$z(t)=Q^{-1}\exp(\tilde{\Lambda}t)Qz(t)+\int_0^t Q^{-1}\exp[\Lambda(t-\tau)]QQ^{-1}D^{-1}M^{-1}\Delta\tilde{A}_c MDQz(\tau)d\tau \tag{206}$$

In the remainder of the proof, the subscript "2" on the norms is omitted for notational convenience. Take norms on both sides of the (206) to obtain

$$\|z(t)\| \le \|Q^{-1}\|\cdot\|\exp(\tilde{\Lambda}t)\|\cdot\|Q\|\cdot\|z(0)\|$$
$$+ \int_0^t \|Q^{-1}\|\cdot\|\exp[\tilde{\Lambda}(t-\tau)]\|\cdot\|Q\|\cdot\|(Q^{-1}D^{-1}M^{-1}\Delta\tilde{A}_c MDQ\|\cdot\|z(\tau)\|d\tau \tag{207}$$

Use $\|\exp(\tilde{\Lambda}t)\| \le \exp(-\alpha t)$,

where $\alpha = -\max_i \{Re[\lambda_i(\tilde{A}_c)]\} = -\max_i \{Re[e^{-j\theta}(\lambda_i - a)]$.

Also, use $\kappa_2(Q) = \|Q\| \cdot \|Q^{-1}\|$ and $Q^{-1}D^{-1}M^{-1} = (MDQ)^{-1}$

Then, $\|z(t)\| \le \kappa_2(Q)\exp(-\alpha t)\|z(0)\| +$

$$+ \int_0^t \kappa_2(Q)\exp[-\alpha(t-\tau)]\,\|(MDQ)^{-1}\Delta\tilde{A}_c MDQ\| \cdot \|z(\tau)\|\, d\tau \tag{208}$$

Multiply both sides by $\exp(\alpha t)$ to obtain

$$\|z(t)\|\exp(\alpha t) \le \kappa_2(Q)\cdot\|z(0)\| +$$

$$+ \int_0^t \kappa_2(Q)\cdot\|(MDQ)^{-1}\Delta\tilde{A}_c MDQ\|\cdot\exp(\alpha t)\cdot\|z(\tau)\|d\tau \tag{209}$$

use the Gronwall lemma to obtain

$$\|z(t)\|\exp(\alpha t) \le \kappa_2(Q)\cdot\|z(0)\|\cdot\exp[\int_0^t \kappa_2(Q)\cdot\|(MDQ)^{-1}\Delta\tilde{A}_c MDQ\|d\tau] \tag{210}$$

use lemma 4 to obtain

$$\|z(t)\|\exp(\alpha t)$$

$$\le \kappa_2(Q)\cdot\|z(0)\|\cdot\exp\{\kappa_2(Q)\cdot\|[(MDQ)^{-1}]^+[A_m+B_m(FC)^+][MDQ]^+\|t\} \tag{211}$$

or

$$\|z(t)\| \le \kappa_2(Q)\|z(0)\|\exp\{-\alpha+\kappa_2(Q)\|[(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}][MDQ]^{+}\|t\} \tag{212}$$

Thus, $\|z(t)\| \to 0$ if $\alpha > \|\kappa_2(Q)[(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}][MDQ]^{+}\|$

Then, z(t) is asymptotically stable

$\Rightarrow$ $\tilde{x}(t)$ is asymptotically stable

$\Rightarrow$ $\tilde{A}_c+\Delta\tilde{A}_c$ has all its eigenvalues in the open left half complex plane.

Then, by using lemma 2, it follows that $A_c+\Delta A_c$ has all its eigenvalues in the H region.

End of proof

Theorem 2 presents a sufficient condition for robust performance in terms of the eigenstructure of the nominal closed loop system. Robust performance is ensured provided that the eigenvalues of the transformed system matrix $\tilde{A}$ lie to the left of a vertical line in the complex plane. The phrase "robust performance", as used here, means that the eigenvalues of the matrix $A_c + \Delta A_c$ lie inside the $H(a,\theta)$ region for all time invariant ΔA and ΔB described by (172)-(173).

Consider the special case where the H region is the entire open left half complex plane which corresponds to $a=\theta=0$ and let Q be the identity matrix. Then (200) reduces to

$$\alpha > \|D^{-1}(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}D\|_2 \;;\; \alpha=-\max_i[\mathrm{Re}(\lambda_i)] \qquad (213)$$

which is the result shown by Sobel et.al. [1] for stability robustness. Thus, Theorem 2 includes the earlier stability robustness sufficient condition as a special case. However, the stability robustness result given by (213) is applicable to time varying uncertainty while the performance robustness result of Theorem 2 requires that the uncertainty be time invariant.

Corollary 2: Multiple H-regions

Let the region R be the intersection of $H_k(a_k,\theta_k)$ for $k=1,2,\ldots,q$. Suppose F is such that the nominal closed loop system described by (174) has distinct eigenvalues all of which are inside R. Let M and Q be defined as in Theorem 2. Then, the eigenvalues of the uncertain closed loop system given by (175) will be in R for all ΔA and ΔB described by (172)-(173) if

$$\min_k \alpha_k > \kappa_2(Q)\cdot\|[(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}][MDQ]^{+}\|_2 \qquad (214)$$

where $\alpha_k = -\max_i \mathrm{Re}[e^{-j\theta_k}(\lambda_i - a_k)]$

Proof

Using Theorem 2, it follows that the eigenvalues of the system matrix of (175) are in H_k if

$$\alpha_k > \kappa_2(Q)\cdot\|[(MDQ)^{-1}]^+[A_m+B_m(FC)^+][MDQ]^+\|_2 \tag{215}$$

The eigenvalues of the system matrix of (175) are in R if they are simultaneously in each H_k; k=1,2,...,q. This will be true if (214) is satisfied.

End of proof

Corollary 3: Special Case

Let the region R be the intersection of $H_1(a_1,0)$, $H_2(0,\theta_2)$, and $H_3(0,-\theta_2)$ as shown in figure 20. Suppose that the matrices F, M, and Q satisfy the conditions of Theorem 2. Then, the eigenvalues of the uncertain closed loop system given by (175) will be in R for all ΔA and ΔB described by (172)-(173) if

$$\min[\alpha_1,\alpha_2] > \kappa_2(Q)\cdot\|[(MDQ)^{-1}]^+[A_m+B_m(FC)^+][MDQ]^+\|_2 \tag{216}$$

where $\alpha_1 = -\max_i [Re(\lambda_i)] + a_1$

and $\alpha_2 = -\max_i [Re(e^{-j\theta_2} \lambda_i)]$

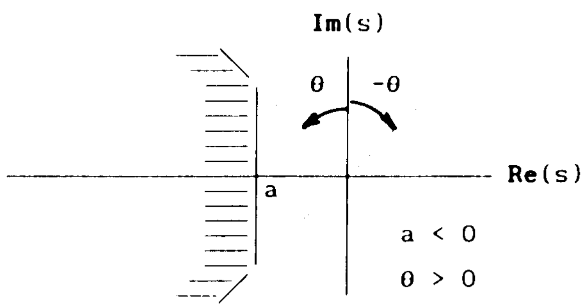

Figure 20. Region for Special Case of Corollary

Proof

From Corollary 2 we have

$$\min[\alpha_1, \alpha_2, \alpha_3] > \kappa_2(Q) \cdot \| [(MDQ)^{-1}]^{+} [A_m + B_m (FC)^{+}] [MDQ]^{+} \|_2 \quad (217)$$

where

$$\alpha_k = -\max_i \{Re[e^{-j\theta_k}(\lambda_i - a_k)]\}$$

Thus,

$$\alpha_1 = -\max_i\{\mathrm{Re}[e^{-j0}(\lambda_i - a_1)]\}$$

$$= -\max_i\{\mathrm{Re}[\lambda_i - a_1]\}$$

$$= -\max_i\{\mathrm{Re}(\lambda_i)\} + a_1 \tag{218}$$

Also,

$$\alpha_2 = -\max_i\{\mathrm{Re}[e^{-j\theta_2}(\lambda_i - 0)]\}$$

$$= -\max_i\{\mathrm{Re}[e^{-j\theta_2}\lambda_i]\}$$

$$= -\max_i[\mathrm{Re}(\lambda_i)\cos\theta + \mathrm{Im}(\lambda_i)\sin\theta] \tag{219}$$

and

$$\alpha_3 = -\max_i\{\mathrm{Re}[e^{j\theta_2}(\lambda_i - 0)]\}$$

$$= -\max_i\{\mathrm{Re}[e^{j\theta_2}\lambda_i]\}$$

$$= -\max_i[\mathrm{Re}(\lambda_i)\cos\theta - \mathrm{Im}(\lambda_i)\sin\theta] \tag{220}$$

Since the λ_i are the eigenvalues of the real matrix A_c, the λ_i are either real or they occur in complex conjugate pairs. If λ_i is real, then $\mathrm{Im}(\lambda_i) = 0$ and $\alpha_2 = \alpha_3$. Consider the complex pair $\lambda_i = a_i + b_j$; $\lambda_{i+1} = \bar{\lambda}_i = a_i - b_j$. In this case,

$$\alpha_2 = -\max_i\{a_i\cos\theta + b_i\sin\theta,\ a_i\cos\theta - b_i\sin\theta\} \tag{221}$$

$$\alpha_3 = -\max_i\{a_i\cos\theta - b_i\sin\theta,\ a_i\cos\theta + b_i\sin\theta\} \tag{222}$$

So it follows that $\alpha_2 = \alpha_3$. In general, α_2 and α_3 are the maximum of sets with the same members which appear in a different order. Thus, the maximum of the two sets is equal and $\alpha_2 = \alpha_3$.

End of proof

The region R in Corollary 3 ensures that a system with a dominant pair of complex conjugate eigenvalues has a damping ratio $\zeta \geq \zeta_{min}$ and a settling time $t_s \leq (t_s)_{max}$.

4.5 EXAMPLE: L-1011 AIRCRAFT

Consider the linearized lateral dynamics of the L-1011 aircraft which was described in section 3.6. However, it is assumed here that the uncertainty is time invariant. The feedback gain matrix F using state feedback is obtained by minimizing (216) with constraints on the closed loop eigenvalues. The optimization is initialized at the design described by the feedback gain matrix in (117). This approach ensures that the nominal closed loop eigenvalues are in the region R which is the intersection of $H_1(-0.25,0)$, $H_2(0,8.63\text{deg})$, and $H_3(0,-8.63\text{deg})$. This region guarantees that the damping ratio of the dominant mode is $\zeta \geq \zeta_{min} = 0.15$ and that the time constant of the dominant mode is less than or equal to 4 seconds. First, we optimize (216) with Q=I (the identity matrix) and D_{opt} which is computed using (104) and is updated during the optimization. The final D_{opt} is given by

$$D_{opt} = \text{diag}(1.9291,\ 1.9291,\ 0.4861,\ 0.4861) \tag{223}$$

The feedback gain matrix which is obtained upon optimizing (216) is given by

$$F = \begin{array}{c} \begin{array}{cccc} \quad\phi & \quad r & \quad p & \quad\beta \end{array} \\ \begin{bmatrix} -0.426 & 4.13 & -0.176 & -5.57 \\ 4.24 & 1.46 & 2.63 & -6.32 \end{bmatrix} \end{array} \begin{array}{c} \\ \delta_r \\ \delta_a \end{array} \tag{224}$$

The nominal closed loop eigenvalues using the gain matrix in (224) are given by

$$\lambda_{1,2} = -1.75 \pm j1.75 \qquad \text{(dutch roll mode)} \tag{225}$$

$$\lambda_{3,4} = -2.0 \pm j1.0 \qquad \text{(roll mode)} \tag{226}$$

Then, evaluating the performance robustness sufficient condition of (216) according to Corollary 3, we have

$$\alpha_1 = -\max_i [\mathrm{Re}(\lambda_i)] - 0.25$$

$$= -\max[-1.75,\ -2.0] - 0.25$$

$$= 1.5 \tag{227}$$

$$\alpha_2 = -\max_i [\mathrm{Re}(e^{-j8.63\bullet\pi/180}\lambda_i)]$$

$$= -\max_i [\mathrm{Re}(e^{-j0.1506}\lambda_i)]$$

$$= -\max[-1.4676,\ -1.8273]$$

$$= 1.4647 \tag{228}$$

Then, using Corollary 3, we require that

$$\min[\alpha_1,\alpha_2] > \kappa_2(Q)\cdot\| [(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}][MDQ]^{+}\|_2 \tag{229}$$

and we obtain from (227)-(229), with the Q and D_{opt} described above, that

$$\min[\alpha_1,\alpha_2] = 1.4676$$
$$< \kappa_2(Q)\cdot\| [(MDQ)^{-1}]^{+}[A_m+B_m(FC)^{+}][MDQ]^{+}\|_2 = 1.5843$$

Therefore, the robust performance sufficient condition is not satisfied.

Next, we optimize (216) with the matrix D_{opt} obtained from (104) and a unitary matrix Q obtained from (107). D_{opt} and Q are both updated during the optimization. The final matrix Q is given by

$$Q=\begin{bmatrix} -.2238+j.6706 & -.0025+j.0029 & .6708+j.2238 & .0021-j.0123 \\ -.2238-j.6706 & -.0025-j.0029 & .6708-j.2238 & .0021+j.0123 \\ -.0010-j.0128 & .6935+j.1382 & .0018-j.0015 & .1383-j.6933 \\ -.0010+j.0128 & .6935-j.1382 & .0018+j.0015 & .1383+j.6933 \end{bmatrix} \tag{230}$$

Then, (229) yields

$$\min[\alpha_1,\alpha_2] = 1.4676$$

$$> \kappa_2(Q)\cdot\|[(MDQ)^{-1}]^+[A_m+B_m(FC)^+][MDQ]^+\|_2 = 1.4572$$

and the performance robustness is guaranteed for the chosen region R. That is, the eigenvalues of the uncertain closed loop system will remain within R for all possible combinations of the time invariant uncertainty described by ΔA and ΔB. This illustrates the importance of our new unitary matrix Q in satisfying the sufficient condition for robustness.

4.6 EXAMPLE: AFTI F-16 AIRCRAFT

Consider the LTI model of the AFTI F-16 aircraft described in section 3.5. Here we assume that the uncertainty is time invariant. We need to design a controller such that the model will satisfy the performance robustness sufficient condition given by corollary 3. We choose a=-0.25 and θ=8.63 deg which corresponds to a region with $\zeta \geq 0.15$ and $t_s \leq 4$ second. We compute the feedback gain matrix F by optimizing the performance robustness condition of (216) with the optimization initialized at the design used by Sobel and Shapiro [12]. We use (104) to compute D_{opt} and then (107) to compute the unitary matrix Q. D_{opt} and Q are updated during the optimization. The constraint for the gamma mode eigenvalue is chosen to be $-2.7 \leq \lambda_\gamma \leq -0.5$. The feedback gain matrix F is given by

$$F = \begin{array}{c} \begin{array}{ccccc} \gamma & q & \alpha & \delta_e & \delta_f \end{array} \\ \left[\begin{array}{ccccc} 2.69 & 0.557 & 5.31 & -0.262 & -0.0529 \\ -5.47 & 0.183 & -1.23 & -0.290 & -0.0476 \end{array}\right] \end{array} \begin{array}{c} \\ \delta_{ec} \\ \delta_{fc} \end{array} \qquad (231)$$

The nominal closed loop eigenvalues using the gain matrix in (231) are given by

$$\lambda_{1,2} = -3.6 \pm j5.2 \quad \text{(short period)} \qquad (232)$$

$$\lambda_3 = -2.7 \quad \text{(gamma mode)} \qquad (233)$$

$$\lambda_4 = -19.0 \qquad \text{(actuator)} \tag{234}$$

$$\lambda_5 = -19.5 \qquad \text{(actuator)} \tag{235}$$

Then, according to corollary 3,

$$\alpha_1 = -\max_i[\mathrm{Re}(\lambda_i)] - 0.25$$

$$= -\max[-3.61,\ -2.71,\ -19.0,\ -19.5] - 0.25$$

$$= 2.46 \tag{236}$$

$$\alpha_2 = -\max_i[\mathrm{Re}(e^{-j8.63*\pi/180}\lambda_i)]$$

$$= -\max_i[\mathrm{Re}(e^{-j0.1506}\lambda_i)]$$

$$= -\max[-2.79,\ -4.35,\ -2.68,\ -18.8,\ -19.3]$$

$$= 2.68 \tag{237}$$

We find that

$$\min(\alpha_1,\alpha_2) = 2.46 \tag{238}$$

while the right hand side of (216) equals 2.393. Thus, the sufficient condition for performance robustness is satisfied. The vertical translation and pitch pointing responses are shown in figures 21 and 22, respectively. We observe that these responses are almost identical to the robust stability responses shown in figures 11 and 12 of section 3.5. However, we now guarantee that the closed loop eigenvalues remain in the performance region for all time invariant uncertainty that satisfies the A_m and B_m bounds.

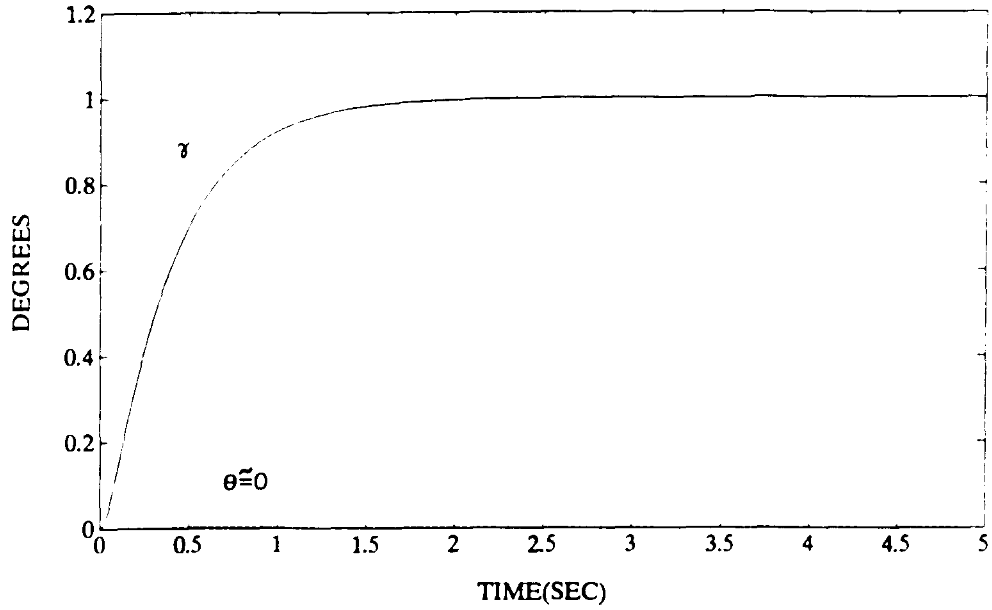

Figure 21. AFTI F-16 Performance Robust Design
(Vertical Translation)

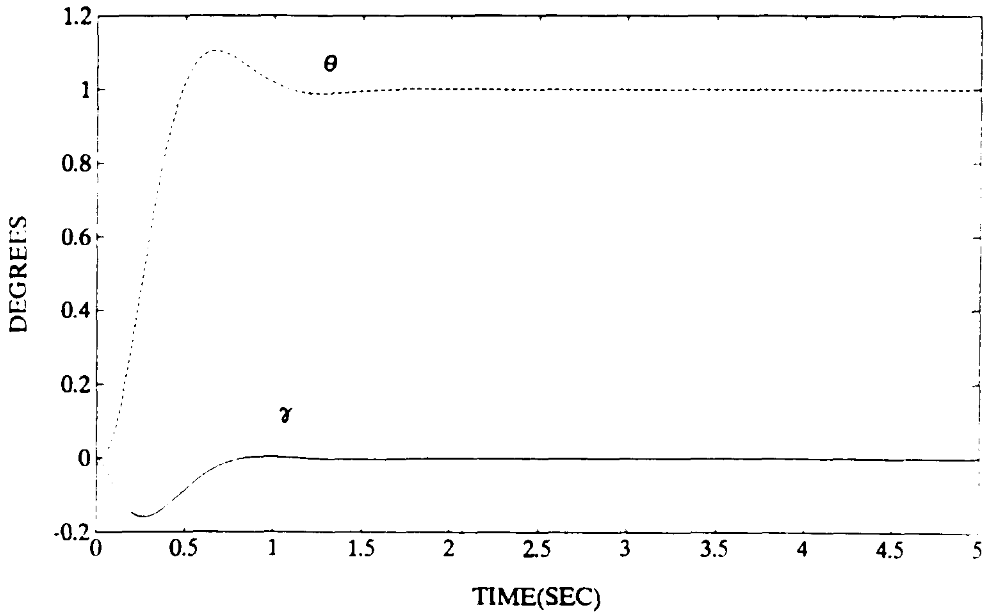

Figure 22. AFTI F-16 Performance Robust Design
(Pitch Pointing)

5. ROBUST STABILITY FOR SYSTEMS WITH STRUCTURED STATE SPACE UNCERTAINTY AND OUTPUT NONLINEARITIES

We extend the robustness sufficient condition proposed by Sobel et. al. [1] to include simultaneous time varying structured state space uncertainty and output nonlinearities for an LTI system with constant gain feedback. Consider a nominal LTI multi-input multi-output plant described by (35)-(36). Suppose that the nominal plant is subject to linear time varying uncertainties $\Delta A(t)$ and $\Delta B(t)$, as shown in (75). Furthermore, suppose that the output includes an unknown nonlinear function $\Delta h(u)$. Then, the uncertain plant may be described by

$$\dot{x}(t) = Ax(t) + Bu(t) + \Delta A(t)x(t) + \Delta B(t)u(t) \tag{239}$$

$$y(t) = Cx(t) + \Delta h(u) \tag{240}$$

Further, suppose that $\|\Delta h(u)\|$ is bound by

$$\|\Delta h(u)\| \leq \gamma \cdot \|u(t)\| \tag{241}$$

Consider the constant gain output feedback control law described by

$$u(t)=Fy(t) \tag{242}$$

Theorem 3

Suppose F is such that the nominal closed loop system described by

$$\dot{x}(t) = (A + BFC)x(t) \tag{243}$$

is asymptotically stable with a non-defective modal matrix. Then, the uncertain closed loop system is given by

$$\dot{x}(t) = (A+BFC)x(t) + [\Delta A(t)+\Delta B(t)FC]x(t) + [B+\Delta B(t)]F\cdot\Delta h(u) \tag{244}$$

is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by (79)-(80) and all $\Delta h(u)$ described by (241) if

$$\alpha > \|(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}\|_2 + \beta\|(M^{-1})^{+}(B^{+}+B_m)F^{+}\|_2 \tag{245}$$

where $\qquad \alpha = -\max_i \mathrm{Re}[\lambda_i(A+BFC)]$

$$\beta = \frac{\gamma\cdot\|F\|\cdot\|CM\|}{1 - \gamma\cdot\|F\|}$$

and $\qquad \|F\| < 1/\gamma$

Proof

Combine (239), (240), and (242) to obtain

$$\dot{x}(t)=Ax(t)+BFCx(t)+\Delta A(t)x(t)+\Delta B(t)FCx(t)+[B+\Delta B(t)]F\cdot\Delta h(u)$$

$$=(A+BFC)x(t)+[\Delta A(t)+\Delta B(t)FC]x(t)+[B+\Delta B(t)]F\cdot\Delta h(u)$$

$$=A_c x(t)+\Delta A_c(t)x(t)+[B+\Delta B(t)]F\cdot\Delta h(u) \qquad (246)$$

where $A_c = A + BFC$

and $\Delta A_c(t) = \Delta A(t) + \Delta B(t)FC$

Let $x(t) = Mz(t)$

where M is a modal matrix for A_c.

Then,

$$\dot{z}(t) = M^{-1}A_c Mz(t) + M^{-1}\Delta A_c(t)Mz(t) + M^{-1}[B+\Delta B(t)]F\cdot\Delta h(u) \qquad (247)$$

which has a solution given by

$$z(t)=\exp(\Lambda t)z(0)$$

$$+ \int_0^t \exp[\Lambda(t-\tau)]\{M^{-1}\Delta A_c(\tau)Mz(\tau) + M^{-1}[B+\Delta B(\tau)]F\Delta h(u)\}d\tau \qquad (248)$$

use $\|\exp(\Lambda t)\| \le \exp(\alpha t)$ where $\alpha = -\max_i \mathrm{Re}[\lambda_i(A+BFC)]$

$$\|z(t)\| \le \|\exp(\Lambda t)\| \cdot \|z(0)\|$$

$$+ \int_0^t \|\exp[\Lambda(t-\tau)]\| \cdot \|M^{-1}\Delta A_c(\tau)M\| \cdot \|z(\tau)\| d\tau$$

$$+ \int_0^t \|\exp[\Lambda(t-\tau)]\| \cdot \|M^{-1}[B+\Delta B(\tau)]F\Delta h(u)\| d\tau \tag{249}$$

Consider the second integral in (249) which satisfies

$$\int_0^t \|\exp[\Lambda(t-\tau)]\| \cdot \|M^{-1}[B+\Delta B(\tau)]F\Delta h(u)\| d\tau$$

$$\le \int_0^t \exp[-\alpha(t-\tau)] \|M^{-1}[B+\Delta B(\tau)]F\| \cdot \|\Delta h(u)\| d\tau \tag{250}$$

Recall from (241) that $\|\Delta h(u)\| \le \gamma \cdot \|u(t)\|$

Therefore,

$$\|\Delta h(u)\| \le \gamma \cdot \|u(t)\| \le \gamma \cdot \|F\| \cdot \|y(t)\| = \gamma \cdot \|F\| \cdot \|CMz(t) + \Delta h(u)\|$$

$$\le \gamma \cdot \|F\| \left(\|CMz(t)\| + \|\Delta h(u)\| \right) \tag{251}$$

$$(1 - \gamma\|F\|)\|\Delta h(u)\| \le \gamma\|F\| \cdot \|CM\| \cdot \|z(t)\| \tag{252}$$

which yields the following bound on $\Delta h(u)$

$$\|\Delta h(u)\| \le \beta\|z(t)\| \qquad \text{if } 1 - \gamma\|F\| > 0 \tag{253}$$

where
$$\beta = \frac{\gamma\|F\|\cdot\|CM\|}{1 - \gamma\|F\|}$$

Therefore, the second integral in (249) satisfies the following:

$$\int_0^t \|\exp[\Lambda(t-\tau)]\|\cdot\|M^{-1}[B+\Delta B(\tau)]F\Delta h(u)\|d\tau$$

$$\le \int_0^t \exp[-\alpha(t-\tau)]\beta\cdot\|M^{-1}[B+\Delta B(\tau)]F\|\cdot\|z(\tau)\|d\tau \tag{254}$$

Substitute (254) into (249) to obtain

$$\|z(t)\| \le \exp(-\alpha t)\|z(0)\|$$
$$+ \int_0^t \exp[-\alpha(t-\tau)]\{\|M^{-1}\Delta A_c(\tau)M\|+\beta\|M^{-1}[B+\Delta B(\tau)]F\|\}\cdot\|z(\tau)\|d\tau \tag{255}$$

or

$$\|z(t)\|\exp(\alpha t) \le \|z(0)\|$$
$$+ \int_0^t \{\|M^{-1}\Delta A_c(\tau)M\|+\beta\|M^{-1}[B+\Delta B(\tau)]F\|\}\exp(\alpha\tau)\|z(\tau)\|d\tau \tag{256}$$

Use the Gronwall lemma to obtain

$$\|z(t)\|\exp(\alpha t)\le\|z(0)\|\exp\left[\int_0^t\{\|M^{-1}\Delta A_c(\tau)M\|+\beta\|M^{-1}[B+\Delta B(\tau)]F\|\}d\tau\right] \tag{257}$$

$$\|z(t)\|\exp(\alpha t)$$

$$\le\|z(0)\|\exp\left[\int_0^t\{\|(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}\|+\beta\|(M^{-1})^{+}(B^{+}+B_m)F^{+}\|\}d\tau\right]$$

$$\le\|z(0)\|\exp\left[\{\|(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}\|+\beta\|(M^{-1})^{+}(B^{+}+B_m)F^{+}\|\}t\right] \tag{258}$$

Thus,

$$\|z(t)\|$$

$$\le\ \|z(0)\|\exp\left[\{-\alpha+\|(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}\|+\beta\|(M^{-1})^{+}(B^{+}+B_m)F^{+}\|\}t\right] \tag{259}$$

Thus, $\|z(t)\|$ goes to zero, and hence $\|x(t)\|$ goes to zero if

$$\alpha > \|(M^{-1})^{+}[A_m+B_m(FC)^{+}]M^{+}\|+\beta\|(M^{-1})^{+}(B^{+}+B_m)F^{+}\| \tag{260}$$

where $\beta = \dfrac{\gamma\|F\|\cdot\|CM\|}{1-\gamma\|F\|}$; $\gamma < \|F\|^{-1}$ or $\|F\| < 1/\gamma$

End of proof

6. ROBUST EIGENSTRUCTURE ASSIGNMENT FOR SYSTEM WITH STRUCTURED UNCERTAINTY AND UNMODELLED DYNAMICS

6.1 INTRODUCTION

In Chapter 3, we used eigenstructure assignment to design a robust constant gain controller for the pitch pointing/vertical translation maneuver of the AFTI F-16 aircraft. This design method utilized a sufficient condition for robust stability which was proposed by Sobel et. al. [1]. However, this sufficient condition does not consider the problem of unmodelled dynamics and it only applies to constant gain feedback. A new sufficient condition for robust stability has been obtained recently by Piou [3] which includes (1) simultaneous time varying structured state space uncertainty and unmodelled dynamics, (2) non-strictly proper plants, (3) structured state space uncertainty in the output equation in addition to the earlier uncertainty in the state equation, and (4) output feedback using dynamic compensators. We develop an algorithm for robust eigenstructure assignment which utilizes the new sufficient condition for robust stability [3]. This algorithm is applied to the design of a robust autopilot for a bank-to-turn missile.

6.2 PROBLEM FORMULATION

Consider the LTI plant with time varying structured state space uncertainty and unmodelled dynamics described by

$$\dot{x}(t) = (A+\Delta A(t))x(t)+(B+\Delta B(t))u(t) \tag{261}$$

$$y(t) = (C+\Delta C(t))x(t)+(D+\Delta D(t))u(t)+\Delta H(t)\ast u(t) \tag{262}$$

where $\ast$ denotes the convolution operator.

Let a dynamic compensator be described by

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{263}$$

$$u(t) = C_c x_c(t) + D_c y(t) \tag{264}$$

and let the unmodelled dynamics $\Delta H(t)$ satisfy

$$\Delta H(t) = C_\Delta e^{A_\Delta t} B_\Delta + D_\Delta \delta(t) \tag{265}$$

and let the time varying structured state space uncertainty $A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ satisfy

$$\{\Delta A(t):\ \Delta A^{+}(t) \le A_m\} \tag{266}$$

$$\{\Delta B(t):\ \Delta B^{+}(t) \le B_m\} \tag{267}$$

$$\{\Delta C(t):\ \Delta C^{+}(t) \le C_m\} \tag{268}$$

$$\{\Delta D(t):\ \Delta D^{+}(t) \le D_m\} \tag{269}$$

where $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ are continuous functions of time.

The plant and unmodelled dynamics satisfy the following assumptions:

Assumption (i): $\bar{\sigma}(C_\Delta e^{A_\Delta t} B_\Delta) \le r_1 e^{-\beta t}$

Assumption (ii): $\bar{\sigma}(D_\Delta) \le r_o$

Theorem [3]

Suppose (A_c, B_c, C_c, D_c) is such that the nominal closed loop system is asymptotically stable with a non-defective modal matrix. Then, the uncertain closed loop system described by (261)-(262) is asymptotically stable for all $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ which satisfy (266)-(269) and $\Delta H(t)$ which satisfies assumptions (i)-(ii) if

$$
\begin{aligned}
&\left(\frac{1}{\alpha\cdot\underline{\sigma}(Q)}\right)\Bigg\{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix} A_m+B_m(D_cC)^{+}+(B^{+}+B_m)D_c^{+}C_m & B_mC_c^{+} \\ B_cC_m & 0 \end{pmatrix}(MD_1Q)^{+}\right] \\
&+\frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix} (B^{+}+B_m)D_c^{+}(D^{+}+D_m) \\ B_c^{+}(D^{+}+D_m) \end{pmatrix}\right]\cdot\bar{\sigma}\left[[D_c^{+}(C^{+}+C_m),C_c^{+}](MD_1Q)^{+}\right]}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_m)]-\bar{\sigma}(D_c)\cdot(r_1/\beta+r_0)} \\
&+\frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix} (B^{+}+B_m)D_c^{+} \\ B_c^{+} \end{pmatrix}\right]\bar{\sigma}\left[[D_c^{+}(C^{+}+C_m),C_c^{+}](MD_1Q)^{+}\right](r_1/\beta+r_0)}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_m)]-\bar{\sigma}(D_c)\cdot(r_1/\beta+r_0)}\Bigg\}<1
\end{aligned}
\tag{270}
$$

and if

$$
\bar{\sigma}[D_c^{+}(D^{+}+D_m)] + \bar{\sigma}(D_c)\cdot(r_1/\beta + r_0) < 1 \tag{271}
$$

where

$$
\alpha = -\max_i \mathrm{Re}[\lambda_i(\tilde{A}_{cl})]
$$

and

$$
\tilde{A}_{cl} = \begin{pmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{pmatrix}
$$

and M is the modal matrix of $\tilde{A}_{cl}$.

Corollary: Constant Gain Output Feedback [3]

For constant gain output feedback $u = D_c y$, the uncertain closed loop system is asymptotically stable if

$$\left(\frac{1}{\alpha\cdot\underline{\sigma}(Q)}\right)\left\{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}[A_m+B_m(D_cC)^{+}+(B^{+}+B_m)D_c^{+}C_m](MD_1Q)^{+}\right]\right.$$

$$+\ \frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}(B^{+}+B_m)D_c^{+}(D^{+}+D_m)\right]\ \bar{\sigma}\left[D_c^{+}(C^{+}+C_m)(MD_1Q)^{+}\right]}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_m)]-\bar{\sigma}(D_c)\cdot(r_1/\beta+r_o)}$$

$$\left.+\ \frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}(B^{+}+B^{m})D_c^{+}\right]\bar{\sigma}\left[D_c^{+}(C^{+}+C_m)(MD_1Q)^{+}\right](r_1/\beta+r_o)}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_m)]-\bar{\sigma}(D_c)\cdot(r_1/\beta+r_o)}\right\} < 1 \qquad (272)$$

and if

$$\bar{\sigma}[D_c^{+}(D^{+}+D_m)] + \bar{\sigma}(D_c)\cdot(r_1/\beta + r_o) < 1 \qquad (273)$$

where

$$\alpha = -\max_i \mathrm{Re}[\lambda_i(A+BD_cC)]$$

and M is the modal matrix of $(A+BD_cC)$.

6.3 ROBUST EIGENSTRUCTURE ASSIGNMENT ALGORITHM

We propose an algorithm for robust eigenstructure assignment which utilizes the new sufficient condition for robust stability obtained by Piou [3]. This algorithm is applied to the design of a robust autopilot using output feedback with dynamic compensation for a bank-to-turn missile. This missile is characterized by uncertainty in its aerodynamic coefficients and unmodelled high frequency bending modes. The direct design of low order dynamic compensators using eigenstructure assignment is an alternative to other methods which design a high order dynamic compensator and then use model order reduction techniques to obtain a lower order compensator.

6.3.1 COMPUTATION OF THE COMPENSATOR FOR A PLANT WITH A MATRIX D

We now show how to compute the compensator (A_c, B_c, C_c, D_c) for the plant (A, B, C, D) by using the existing eigenstructure assignment algorithm for the strictly proper plant (A, B, C). Consider the plant (A, B, C) with the compensator described by (263)-(264). The output equation is given by

$$y = Cx \tag{274}$$

and the input equation is given by

$$u = C_c x_c + D_c Cx \tag{275}$$

First, we compute the compensator gains A_c, B_c, C_c, and D_c for the plant described by the triple (A, B, C). Then, consider the plant described by the quadruple (A, B, C, D) with output equation given by

$$y = Cx + Du \tag{276}$$

and input equation given by

$$\begin{aligned} u &= C_c x_c + D_c Cx \\ &= C_c x_c + D_c (y-Du) \end{aligned} \tag{277}$$

Solve (277) for u(t) to obtain

$$u = (I+D_c D)^{-1} C_c x_c + (I+D_c D)^{-1} D_c y \tag{278}$$

Let

$$C_c^D = (I+D_cD)^{-1}C_c \tag{279}$$

and let

$$D_c^D = (I+D_cD)^{-1}D_c \tag{280}$$

Then, the compensator for the plant described by (A, B, C, D) is described by

$$\dot{x}_c = A_cx_c + B_cy \tag{281}$$

$$u = C_c^Dx_c + D_c^Dy \tag{282}$$

6.3.2 COMPUTATION OF THE MATRICES D_1 AND Q FOR THE SPECIAL CASE WHEN D_1 AND Q ARE REAL, POSITIVE, AND DIAGONAL

Lemma [28]

Define the Perron-Frobenius eigenvalue of the non-negative matrix A, denoted by $\pi(A)$, to be the real non-negative eigenvalue $\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues $(i=1,...,n)$ of A^+. Let R and L be real, positive, and diagonal matrices and let A and B be positive matrices. Then,

$$\inf_{L,R} \{\bar{\sigma}(LAR)\cdot\bar{\sigma}(R^{-1}BL^{-1})\} = \pi(AB) \tag{283}$$

$$L = \mathrm{diag}[y_i/x_i]^{1/2};\ R = \mathrm{diag}[u_i/v_i]^{1/2} \tag{284}$$

where x and y are the right and left Perron-Frobenius eigenvectors of AB and where u and v are the right and left Perron-Frobenius eigenvectors of BA.

In general, the matrices D_1 and Q in (270) may be arbitrary, complex, non-singular matrices. However, for the special case when these matrices are real, positive, and diagonal we can easily determine a choice which yields the infimum of the first term in (270).

Corollary 4

Let D_1 and Q be real, positive, diagonal matrices. Then,

$$\inf_{L,R}\left(\frac{1}{\alpha\cdot\underline{\sigma}(Q)}\right)\left\{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix} A_m+B_m(D_cC)^{+}+(B^{+}+B_m)D_c^{+}C_m & B_mC_c^{+} \\ B_c^{+}C_m & 0 \end{pmatrix}(MD_1Q)^{+}\right]\right\}$$

$$=\left(\frac{1}{\alpha}\right)\pi\left[(M^{-1})^{+}\begin{pmatrix} A_m+B_m(D_cC)^{+}+(B^{+}+B_m)D_c^{+}C_m & B_mC_c^{+} \\ B_c^{+}C_m & 0 \end{pmatrix}M^{+}\right] \tag{285}$$

where $\pi(\cdot)$ denotes the Perron-Frobenius eigenvalue of $(\cdot)$, $D_1=L^{-1}$, and $Q=D_1^{-1}R$.

Proof

With the substitution of $D_1 = L^{-1}$ and $Q = D_1^{-1}R$, where L and R are chosen by (284), the LHS of (285) becomes

$$\inf_{L,R}\left(\frac{1}{\alpha\cdot\underline{\sigma}(LR)}\right)\left\{\bar{\sigma}\left[(LM^{-1})^{+}\begin{pmatrix} A_m+B_m(D_cC)^{+}+(B^{+}+B_m)D_c^{+}C_m & B_mC_c^{+} \\ B_c^{+}C_m & 0 \end{pmatrix}(MR)^{+}\right]\right\}$$

$$=\inf_{L,R}\left(\frac{\bar{\sigma}(LR)}{\alpha}\right)\left\{\bar{\sigma}\left[L(M^{-1})^{+}\begin{pmatrix} A_m+B_m(D_cC)^{+}+(B^{+}+B_m)D_c^{+}C_m & B_mC_c^{+} \\ B_c^{+}C_m & 0 \end{pmatrix}M^{+}R\right]\right\} \tag{286}$$

Then, the proof of Corollary 4 follows from the application of Maciejowski's lemma [28] to (286).

End of proof

6.3.3 ALGORITHMS FOR ROBUST EIGENSTRUCTURE ASSIGNMENT

The nominal plant or design model is described by (35)-(36) and the compensator is described by (263)-(264). The composite system used for computing A_c, B_c, C_c, and D_c for the plant (A,B,C) is given by (Johnson and Athans [26])

$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \; \bar{B} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}, \; \bar{C} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}, \; \bar{F} = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix} \tag{287}$$

This composite system was used by Sobel and Shapiro [25] to design eigenstructure assignment dynamic compensators. The composite system which is used in (270) to determine robustness is given by

$$\tilde{A}_{cl} = \begin{pmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{pmatrix} \tag{288}$$

The objective function to be minimized, which is denoted by ρ, is the left hand side of (270).

Algorithm 1 (Optimal Solution)

$$\underset{M_s, \Lambda_s, D_1, Q}{\text{Minimize}} \quad \rho(M, \Lambda, D_1, Q) \tag{289}$$

subject to $\bar{\sigma}[D_c^+(D^+ + D_m)] + \bar{\sigma}(D_c)\cdot(r_1/\beta + r_0) < 1$ and $\lambda_i \;\varepsilon\; C_i$

where

M_s is a matrix whose columns are v_i; $i=1,\ldots,s$

Λ_s is a diagonal matrix with entries λ_i; $i=1,\ldots,s$

and C_i is a region of the complex plane.

The optimization may be over a subset of the eigenvalues and eigenvectors of $\tilde{A}_{cl}$ where $s \le n+n_c$ with $\dim(A)=n$ and $\dim(A_c)=n_c$.

Algorithm 2 (Suboptimal Solution)

$$\underset{M_s,\Lambda_s}{\text{Minimize}}\ \rho(M,\Lambda,D_1,Q) \tag{290}$$

subject to $\bar{\sigma}[D_c^+(D^+ + D_m)] + \bar{\sigma}(D_c)\cdot(r_1/\beta + r_0) < 1$ and $\lambda_i \in C_i$

where D_1 and Q are both real, diagonal, and positive matrices which are chosen from Corollary 4 to yield the minimum of the first term in (270). We remark that the matrices D_1 and Q must be updated at each function evaluation during the optimization.

Computations Required for Each Function Evaluation in Algorithms 1 and 2

Initialization: If D_c is non-zero, then set $\bar{F}$ equal to the orthogonal projection solution using eigenstructure assignment (Sobel and Shapiro [25]). If D_c equals zero, then use eigenstructure assignment with gain suppression.

For each function evaluation:

1. Obtain the composite matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ which are defined by (287).
2. If D_c is non-zero, then use eigenstructure assignment to obtain $\bar{F}$.

   If D_c is zero, then use eigenstructure assignment with gain suppression to compute

$$\bar{F} = \begin{pmatrix} 0 & C_c \\ B_c & A_c \end{pmatrix} \tag{291}$$

3. If D is non-zero, then compute C_c^D and D_c^D using (279)-(280).
4. Obtain the composite matrix $\tilde{A}_{cl}$ from (288).
5. Compute the eigenvalues of $\tilde{A}_{cl}$ to obtain α and compute the eigenvectors of $\tilde{A}_{cl}$ to obtain the closed loop modal matrix M.
6. If using algorithm 2, compute D_1 and Q by using Corollary 4.
7. Evaluate ρ.

The eigenvector assignment in the robust eigenstructure assignment algorithms is achieved via constraints on a subset of the entries of the eigenvectors. This approach is different from the more conventional approach of Andry et. al. [8] who suggest that the i-th eigenvector v_i should be chosen as the orthogonal projection of a desired eigenvector v_i^d onto the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$.

The constrained optimization algorithms are not guaranteed to converge to the global optimum. Thus, the designer might consider initializing the algorithms at different points in the parameter space and comparing the various solutions. An alternative approach, which is adopted by IMSL [27] subroutine ZXMWD, is to perform four iterations on many (perhaps 100) points in a user specified region. Then, an optimization to convergence is performed on the five points which have the smallest objective function value after the initial four iterations. However, in the problem considered here, we believe that a good approach is to initialize the constrained optimization at the orthogonal projection eigenstructure assignment solution. This should keep the performance of the robust solution from moving too far away from the performance of the orthogonal projection design. If the sufficient condition is not satisfied with this initialization, then the uncertainty bounds can be reduced.

6.4 EXAMPLE: BANK-TO-TURN MISSILE

Consider the bank-to-turn missile which was used by Lin and Lee [4] with state vector, input vector, and output vector given by $x = [\delta_r, \delta_p, r, n_y, p]^T$, $u = [\delta_{rc}, \delta_{pc}]^T$, and $y = [r, n_y, p]^T$, respectively. The definitions of the states and inputs are as follows: yaw control surface deflection (δ_r), roll control surface deflection (δ_p), yaw rate (r), yaw acceleration (n_y), roll rate (p), yaw control command (δ_{rc}), and roll control command (δ_{pc}). The state space matrices A, B, and C are:

$$A = \begin{bmatrix} -180 & 0 & 0 & 0 & 0 \\ 0 & -180 & 0 & 0 & 0 \\ -21.23 & 0 & -.6888 & -14.07 & 0 \\ 256.7 & 0 & 122.6 & -1.793 & 0 \\ -52.33 & 304.7 & 0 & 36.7 & -9.661 \end{bmatrix} \tag{292}$$

$$B = \begin{bmatrix} 180 & 0 \\ 0 & 180 \\ 0 & 0 \\ 256.7 & 0 \\ 0 & 0 \end{bmatrix} \tag{293}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{294}$$

The open loop eigenvalues are $\lambda_{dr} = -1.24 \pm j41.53$, $\lambda_{roll} = -9.66$, $\lambda_{act} = -180$, and $\lambda_{act} = -180$. The desired closed loop eigenvalues are chosen as $\lambda^{d}_{dr} = -20 \pm j20$ and $\lambda_{roll} = -20$ which corresponds to the three eigenvalues which can be assigned with output feedback using three measurements. The desired closed loop eigenvectors are chosen as

$$v^{d}_{dr} = \begin{bmatrix} x \\ x \\ 1 \\ x \\ 0 \end{bmatrix} \pm j \begin{bmatrix} x \\ x \\ x \\ 1 \\ 0 \end{bmatrix}; \qquad v^{d}_{roll} = \begin{bmatrix} x \\ x \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{295}$$

and the achievable closed loop eigenvectors obtained from orthogonal projections onto the achievability subspaces are given by

$$v^{a}_{dr} = \begin{bmatrix} -.4662 \\ -.5907 \\ 1.00 \\ 4.2398 \\ 0.00 \end{bmatrix} \pm j \begin{bmatrix} -.2201 \\ -.1583 \\ 1.5223 \\ 1.00 \\ 0.00 \end{bmatrix}; \qquad v^{a}_{roll} = \begin{bmatrix} 0.0 \\ -.0339 \\ 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \tag{296}$$

The feedback gain matrix for $u = Fy$ is given by

$$F = \begin{bmatrix} -.121 & -.0635 & 0.0 \\ -.0673 & -.104 & -.0302 \end{bmatrix} \tag{297}$$

The uncertainty matrices A_m and B_m are chosen to be

$$A_m = \begin{bmatrix} 1.35 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.35 & 0.0 & 0.0 & 0.0 \\ 1.06 & 0.0 & .0344 & .748 & 0.0 \\ 1.93 & 0.0 & 6.13 & 0.0 & 0.0 \\ 2.62 & 15.24 & 0.0 & 1.84 & .483 \end{bmatrix} \qquad (298)$$

$$B_m = \begin{bmatrix} 1.35 & 0.0 \\ 0.0 & 1.35 \\ 0.0 & 0.0 \\ 1.93 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \qquad (299)$$

This choice of A_m and B_m is for illustrative purposes. The A_m and B_m chosen correspond to maximum uncertainty of 0.75% in a_{11}, a_{22}, a_{41}, b_{11}, b_{22}, and b_{41}; 5% in a_{31}, a_{33}, a_{34}, a_{43}, a_{44}, a_{51}, a_{52}, a_{54}, and a_{55}. The bounds on the unmodelled dynamics are chosen to be $r_0 + r_1/\beta = 1.5$. We compute the LHS of (272) which is found to be equal to $6.23\text{x}10^7$. Thus, the sufficient condition for robust stability is not satisfied. Next, we minimize the LHS of (272) with respect to the the dutch roll and roll subsidence eigenvalues and the free parameters of their corresponding eigenvectors. The real part of the dutch roll eigenvalues and the roll subsidence eigenvalue are constrained to the interval [-50,-7] and the imaginary part of the dutch roll eigenvalues is constrained to the interval [0,45]. In addition, the denominator in (272) is constrained to be greater than 0.01 and the maximum of the real parts of all five closed loop eigenvalues are constrained to be less than -0.01. This nonlinear constrained optimization is performed on a personal computer by using function "constr" from the MATLAB™ Optimization Toolbox [5]. When the constraint on the denominator in (272) is violated during the optimization process it causes the function to become negative which results in divergence of the algorithm. Therefore, we added an ad hoc fix which sets the function to 10^{50} whenever the denominator in (272) becomes negative. We initialize the optimization at the orthogonal projection solution and we obtain a minimum function value of 0.835. The optimal gains and corresponding dutch roll and roll subsidence eigenvalues are shown in table 9 with the designation of case #1. We observe that the settling time is increased and the dutch roll damping is decreased when compared with the orthogonal projection design. The yaw rate, yaw acceleration, and roll rate time responses are shown in figures 23, 24, and 25, respectively, for a one g yaw acceleration initial condition. We observe that the responses are highly oscillatory with strong coupling to the roll rate response and thus we conclude that this design is unsatisfactory.

Table 9. Comparison of Constant Gain Designs

	Feedback Gain Matrices				Closed Loop Eigenvalues*
Initial	r	n_y	p		
Design (orthogonal	-.1209	-.0635	.0000	δ_{rc}	λ_{dr} = -20±j20
projection)	-.0673	-.1038	-.0302	δ_{pc}	λ_{roll} = -20
Case #1	.0002	-.0311	-.0000		λ_{dr} = -9.66±j40.6
$\mathrm{Im}\lambda_{dr} \in [0,45]$	.0257	-.0000	.0000		λ_{roll} = -9.66
Case #2	.0012	-.0311	-.0000		λ_{dr} = -9.66±j40.7
$\mathrm{Im}\lambda_{dr} \in [0,45]$ eigenvector constraints	-.0365	-.1073	-.0000		λ_{roll} = -9.66
Case #3	-.0286	-.0948	.0014		λ_{dr} = -31.7±j25.0
$\mathrm{Im}\lambda_{dr} \in [0,25]$ eigenvector constraints	-.1140	-.0836	-.0497		λ_{roll} = -27.4

*Actuator eigenvalues are not shown

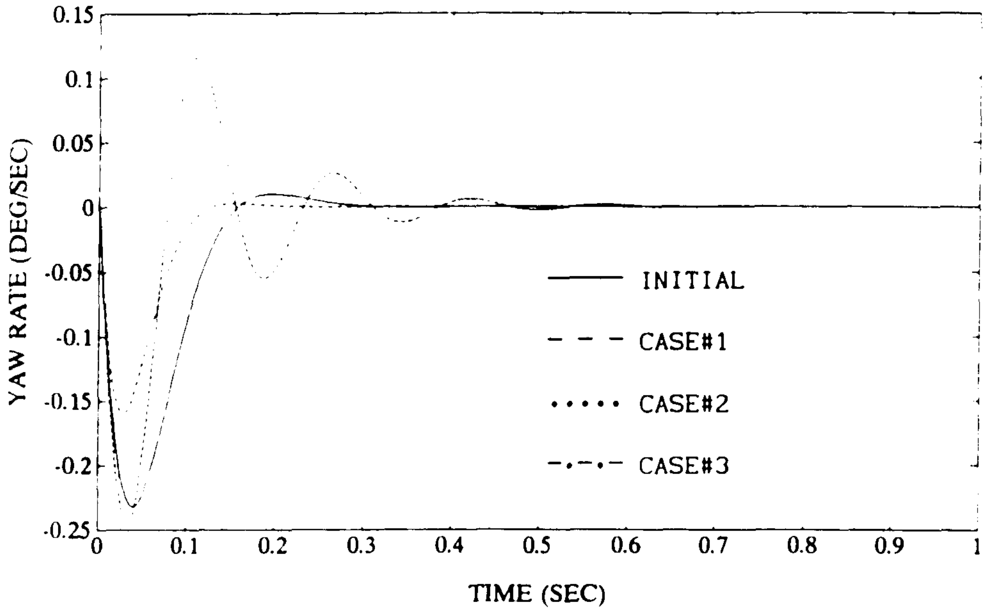

Figure 23. Bank-to-turn Missile Yaw Rate for Constant Gain Controller

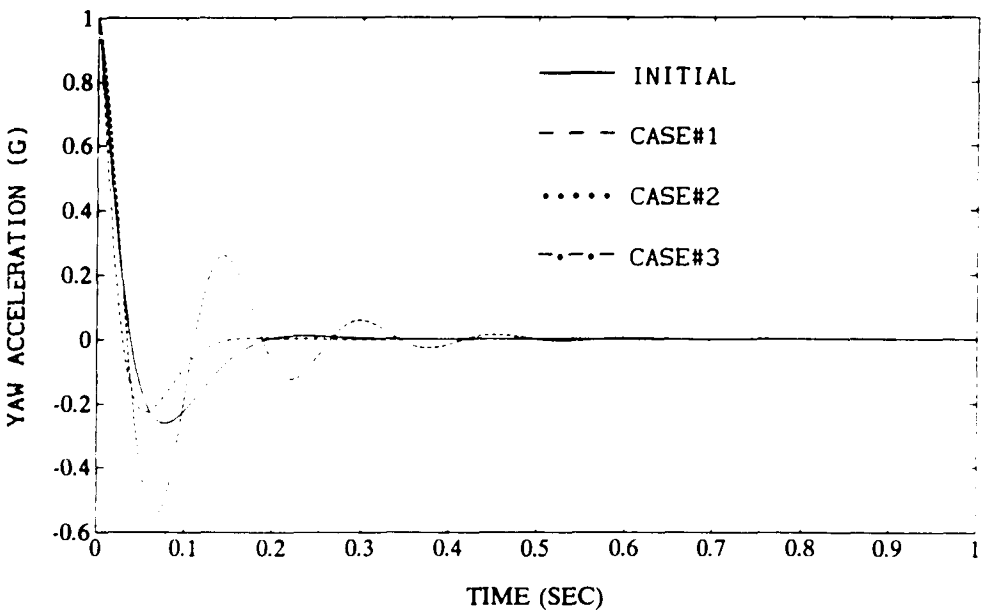

Figure 24. Bank-to-turn Missile Yaw Acceleration for Constant Gain Controller

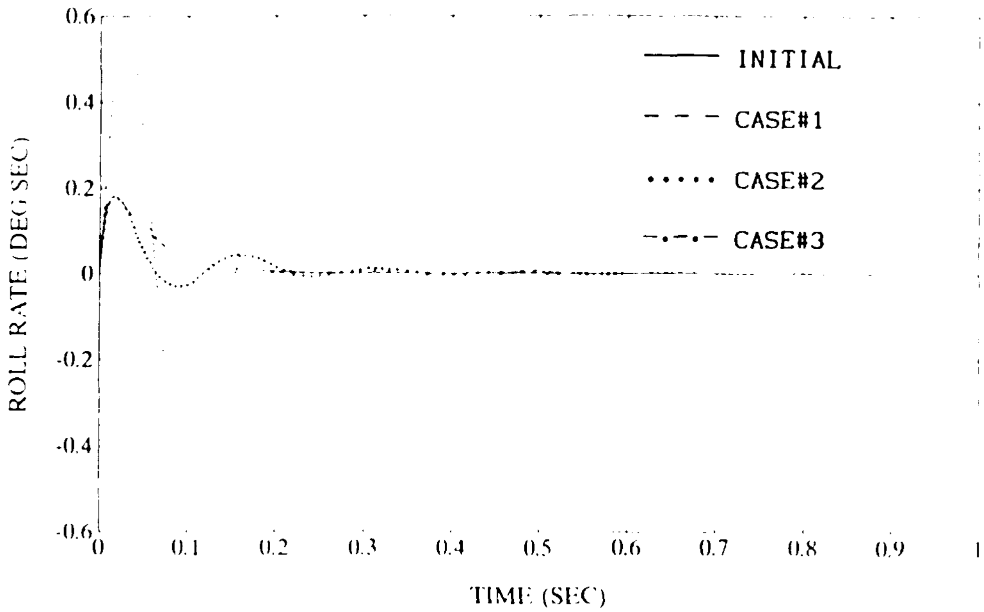

Figure 25. Bank-to-turn Missile Roll Rate
for Constant Gain Controller

The next design, which is designated to be case #2, is characterized by the addition of constraints on the real and imaginary parts of the last entry in the dutch roll eigenvectors. Recall that in the orthogonal projection design we desired that these entries be zero so that the coupling between the roll subsidence mode and dutch roll mode would be small. The optimized solution in design #1 caused these components to be large which explains the strong coupling from yaw acceleration to roll rate which was observed in the time responses for case #1. Thus, we repeat the optimization with the constraints that the absolute values of both $Re[v_{dr}(5)]$ and $Im[v_{dr}(5)]$ be less than 0.1 where the eigenvectors are normalized to unity one-norm before the constraints are calculated. However, because of the additional constraints we must reduce the uncertainty bounds in order to obtain an optimal solution which satisfies the robust stability condition of (272). In this regard we choose A_m and B_m to be one half the values shown in (298)-(299) and $r_0+r_1/\beta = 0.75$. Once again the optimization is initialized with the gains from the orthogonal projection solution. The optimal solution which minimizes the LHS of (272) is shown in table 9. We observe that the dutch roll and roll subsidence eigenvalues are at the same locations as in the case #1 design. However, the components of the dutch roll eigenvectors which are important for mode decoupling now satisfy the additional constraints. We observe from figures 23 and 24 that this design exhibits the same large overshoot and settling time as the case #1 design. However, we observe from figure 25 that the coupling to the roll rate has been significantly reduced with the peak roll rate being identical to the orthogonal projection design. All that remains in order to obtain an acceptable design is to reduce the overshoot and settling time.

The final design, which is designated as case #3, is characterized by replacing the constraint on the imaginary part of the dutch roll eigenvalues with the constraint that $0 \le \mathrm{Im}[\lambda_{dr}] \le 25$. Otherwise, all the other constraints of design #2 remain in effect. The uncertainty bounds for A_m and B_m are the same as in case #2, but the bound on the unmodelled dynamics is reduced to $r_0 + r_1/\beta = 0.25$. The optimization is initialized with the optimal gain matrix which was obtained for the case #2 design. The optimal feedback gain matrix together with the dutch roll and roll subsidence eigenvalues are shown in table 9. We observe that the imaginary part of the dutch roll eigenvalues is at the constraint boundary of 25 while the real part of the dutch roll eigenvalues and the roll subsidence eigenvalues are much farther into the left half plane than was the case in the earlier robust designs. We note that the damping ratio is much improved when compared with the other two robust designs. We observe from figures 23 and 24 that the yaw rate and yaw acceleration responses are now well behaved and are close to the time responses of the orthogonal projection (or initial design). Furthermore, we observe from figure 25 that the coupling from yaw acceleration to roll rate is approximately the same as the orthogonal projection design. Thus, we have obtained a robust design which satisfies the sufficient condition described by (272) while simultaneously yielding time responses which are close to the orthogonal projection design.

Next, we consider the same missile with only two measurements given by the output vector $y = [n_y,\ p]^T$. We use eigenstructure assignment to design a second order compensator with desired eigenvalues given by $\lambda^d_{dr}=-20\pm j20$, $\lambda^d_{roll}=-20$, and $\lambda_{comp}=-25$. These eigenvalues correspond to the four eigenvalues which may be assigned with two measurements together with a second order compensator. The desired eigenvectors are chosen as shown below:

$$
\begin{array}{c}
\\ \delta_r \\ \delta_p \\ r \\ n_y \\ p \\ x_{c1} \\ x_{c2}
\end{array}
\begin{array}{c}
\text{Dutch Roll Mode} \\
\begin{bmatrix} x \\ x \\ 1 \\ x \\ 0 \\ 1 \\ x \end{bmatrix} \pm j \begin{bmatrix} x \\ x \\ x \\ 1 \\ 0 \\ x \\ 1 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\text{Roll Subsidence} \\
\begin{bmatrix} x \\ x \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\text{Compensator Mode} \\
\begin{bmatrix} x \\ x \\ x \\ x \\ x \\ 1 \\ 1 \end{bmatrix}
\end{array}
$$

The compensator gain matrices are computed by using eigenstructure assignment with the augmented system described by (287). These gain matrices and the closed loop eigenvalues are shown in table 10 where we observe that the four desired eigenvalues are exactly achieved.

We consider computing a robust eigenstructure assignment controller by minimizing the LHS of (270) with A_m, B_m, and $r_0 + r_1/\beta$ chosen to be the same as in the case #3 design. However, the computation is extremely time consuming on the personal computer due to the higher order of the dynamic compensator problem. Thus, we stop the optimization when the LHS of (270) is less than or equal to 0.98 with the solution inside the feasible set. This minimization is performed with respect to the dutch roll, roll subsidence, and compensator mode #1 eigenvalues and the free parameters of their corresponding eigenvectors. The real part of the dutch roll

Table 10. Comparison of Dynamic Compensator Designs

	Feedback Gain Matrices	Closed Loop Eigenvalues
Orthogonal Projection (initial design)	$A_c = \begin{bmatrix} -40.2 & 15.2 \\ -7.86 & -17.1 \end{bmatrix}$	$\lambda_{dr} = -20.0 \pm j20.0$
		$\lambda_{roll} = -20.0$
	$B = \begin{bmatrix} 4.77 & 5.00 \\ -2.86 & 5.00 \end{bmatrix}$	$\lambda_{comp} = -25.0$
	$C_c = \begin{bmatrix} .126 & -.126 \\ .0701 & -.0701 \end{bmatrix}$	$\lambda_5 = -169.7$
		$\lambda_{6,7} = -103.1 \pm j21.1$
	$D_c = \begin{bmatrix} -.122 & 0.00 \\ -.136 & -.0302 \end{bmatrix}$	
Robust Design	$A_c = \begin{bmatrix} -40.2 & 22.3 \\ 3.56 & -37.8 \end{bmatrix}$	$\lambda_{dr} = -30.25 \pm j30.00$
		$\lambda_{roll} = -27.65$
	$B_c = \begin{bmatrix} 28.1 & -2.89 \\ -11.0 & 1.10 \end{bmatrix}$	$\lambda_{comp} = -29.91$
	$C_c = \begin{bmatrix} .0025 & -.0001 \\ .0082 & -.0200 \end{bmatrix}$	$\lambda_5 = -50.64$
		$\lambda_6 = -164.57$
	$D_c = \begin{bmatrix} -.0940 & -.0025 \\ -.1180 & -.0497 \end{bmatrix}$	$\lambda_7 = -140.99$

eigenvalues, the roll subsidence eigenvalue, and the compensator mode #1 eigenvalue are constrained to the interval [-50, -7] and the imaginary part of the dutch roll eigenvalue is constrained to the interval [0, 30]. In addition, we add constraints on the denominator in (270), on the maximum of the real part of all seven closed loop eigenvalues, and on the absolute values of the dutch roll eigenvector entries $Re[v_{dr}(5)]$ and $Im[v_{dr}(5)]$ exactly as was done in the constant gain output feedback problem. These constraint values are denominator greater than 0.01, maximum real part less than -0.01, and absolute eigenvector entries less than 0.1. To prevent divergence due to the function becoming negative during the optimization, we set the function to 10^{50} if either the denominator in (270) is negative or if the function itself becomes negative. We initialize the optimization at the orthogonal projection solution with initial and final function values of 5.09 and 0.957, respectively. The closed loop eigenvalues are shown in table 10 where we observe that the dutch roll damping is the same in both the initial and robust designs, but the time constants are smaller for the robust design. The yaw rate, yaw acceleration, and roll rate time responses are shown in figure 26, 27, and 28, respectively. We observe that the time responses for the robust design are close to the time responses of the orthogonal projection (or initial) design. Thus, the robust design satisfies the sufficient condition of (270) while simultaneously yielding acceptable overshoot, settling time, and mode decoupling.

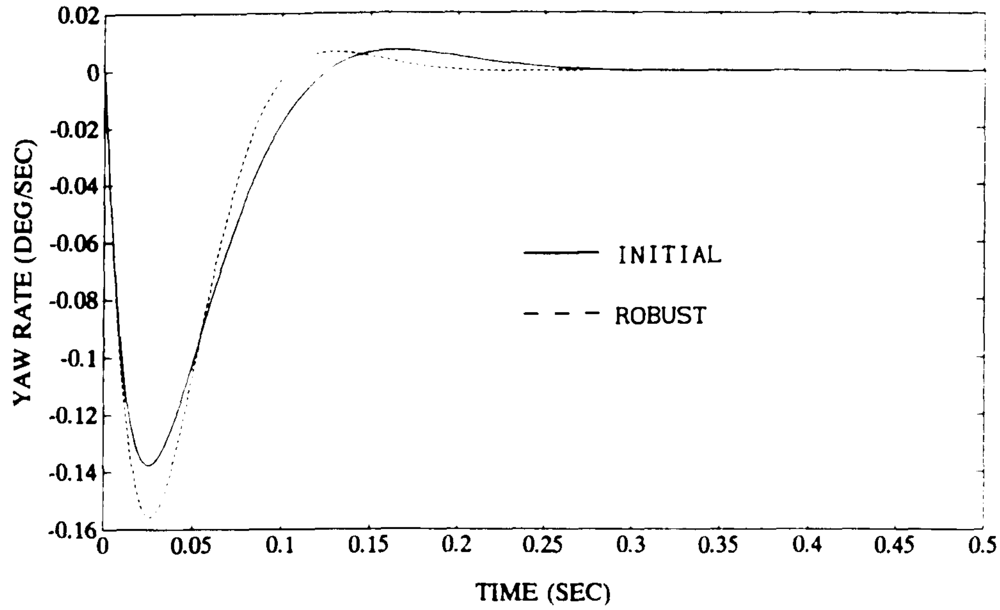

Figure 26. Bank-to-turn Missile Yaw Rate for Dynamic Compensator

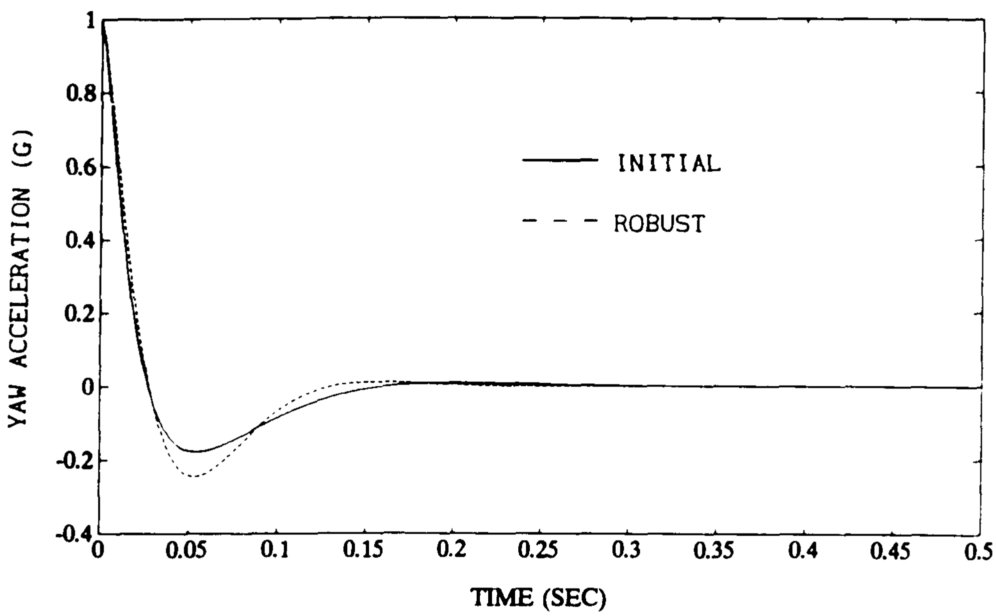

Figure 27. Bank-to-turn Missile Yaw Acceleration for Dynamic Compensator

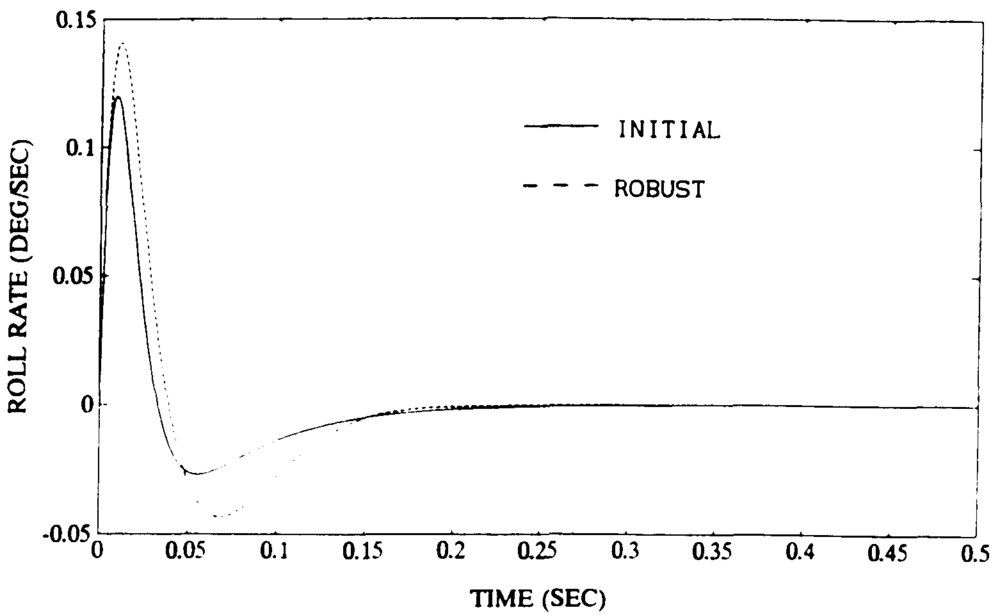

Figure 28. Bank-to-turn Missile Roll Rate
for Dynamic Compensator

7. ROBUST EIGENSTRUCTURE DESIGN METHOD II

7.1 NEW ROBUSTNESS SUFFICIENT CONDITION

In collaboration with Piou [3], we derived a new sufficient condition for the robust stability of an LTI system subject to linear time varying structured state space uncertainty. This new robustness condition is a sum of terms each of which involves the i-th right eigenvector, the i-th left eigenvector, and the real part of the i-th eigenvalue. We also show that this new robustness sufficient condition is less conservative than the earlier result of Sobel et. al. [1]. In section 7.2, we designed a robust controller for the lateral dynamics of Extended Medium Range Air to Air Technology (EMRAAT) missile and this new design was compared to the earlier orthogonal projection eigenstructure assignment design. The earlier design, which was proposed by Sobel and Cloutier [6], does not satisfy either sufficient condition for robust stability. The new design method uses the MATLAB™ Optimization Toolbox [5] to minimize the integrated roll rate with constraints on the real part of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This design satisfies the new robustness condition while also yielding an improved transient response as compared to the design of Sobel and Cloutier [6].

The sufficient condition proposed by Sobel et. al. [1] for the stability robustness problem is described by (34) which is repeated below.

$$\alpha > \lambda_{max}[(M^{-1})^{+}\{A_m + B_m(FC)^{+}\}M^{+}] \tag{300}$$

where $$\alpha = -\max_i\{Re[\lambda_i(A+BFC)]\}$$

and where M is a modal matrix of (A+BFC) and $\lambda_{max}(\cdot)$ of a non-negative matrix denotes the real non-negative eigenvalue $\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues.

The above theorem describes a sufficient condition for the robust stability in terms of the eigenstructure of the nominal closed loop system. Robust stability is ensured provided that the nominal closed loop eigenvalues lie to the left of a vertical line in the complex plane which is determined by a norm involving the structure of the uncertainty and the nominal closed loop modal matrix.

We remark that the sufficient condition given by (300) can be rewritten as shown below.

$$\left(\frac{1}{\alpha}\right)\cdot\lambda_{max}[(M^{-1})^{+}\{A_m + B_m(FC)^{+}\}M^{+}] < 1 \tag{301}$$

This alternate form of the stability robustness condition will be useful later when we compare it with the new robustness condition which is derived in this thesis.

We now present a new sufficient condition for robust stability of a LIT system subject to linear time varying structured state space uncertainty which is described by the following theorem which we obtained in collaboration with Piou [3].

Theorem 4

Suppose that F is such that the nominal closed loop system described by (81) is asymptotically stable with a non-defective modal matrix. Then, the uncertain closed loop system given by (82) is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by (79)-(80) if

$$\lambda_{max}\left\{\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_m + B_m(FC)^+]\right\} < 1 \tag{302}$$

where $$\alpha_i = -Re[\lambda_i(A+BFC)]$$

and where λ_i is the i-th eigenvalue of (A+BFC) with v_i and w^* the corresponding right and left eigenvectors, respectively; and where $(\cdot)^*$ denotes the complex conjugate transpose.

Proof

The uncertain closed loop plant may be written as

$$\dot{x}(t) = A_c x(t) + \Delta A_c(t)x(t) \tag{303}$$

where

$$A_c = A + BFC$$

and

$$\Delta A_c(t) = \Delta A(t) + \Delta B(t)FC$$

which has a solution given by

$$x(t) = \exp(A_c t)x(0) + \int_0^t \exp[A_c(t-\tau)]\Delta A_c(\tau)d\tau \tag{304}$$

Next, use the real, positive, diagonal transformation

$$x(t) = D^{-1}z(t) \tag{305}$$

and the property that

$$\exp(DA_c D^{-1}t) = D\exp(A_c t)D^{-1} \tag{306}$$

$$\exp(A_c t) = M\exp(\Lambda t)M^{-1} = \sum_{i=1}^{n} v_i w_i^* \, e^{\lambda_i t} \tag{307}$$

to obtain

$$z(t) = D\sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} D^{-1} z(0) +$$

$$+ \int_0^t D\sum_{i=1}^{n} v_i w_i^* e^{\lambda_i (t-\tau)} \Delta A_c(\tau) D^{-1} z(\tau) d\tau \qquad (308)$$

where M is a modal matrix of A_c; λ_i is the i-th eigenvalue of A_c with v_i and w_i^* the corresponding right and left eigenvectors, respectively; Λ is a diagonal matrix with the λ_i on the diagonal; and $(\cdot)^*$ denotes complex conjugate transpose.

Note that

$$\|z(t)\| \to 0 \text{ implies that } \|x(t)\| \to 0 \qquad (309)$$

Next, apply the absolute value operator, denoted by $(\cdot)^+$, to both sides of (308) where "+" and "≤" are applied element by element to vectors and matrices.

$$z^+(t) = \left[D\sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} D^{-1} z(0) \right]^+ +$$

$$+ \left[\int_0^t D\sum_{i=1}^{n} v_i w_i^* e^{\lambda_i (t-\tau)} \Delta A_c(\tau) D^{-1} z(\tau) d\tau \right]^+ \qquad (310)$$

$$\le D\sum_{i=1}^{n} (v_i w_i^*)^+ \; e^{-\alpha_i t} D^{-1} z^+(0) \; +$$

$$+ \int_0^t \left[D \sum_{i=1}^{n} v_i w_i^* \; e^{\lambda_i (t-\tau)} \Delta A_c(\tau) D^{-1} z(\tau) \right]^+ d\tau \qquad (311)$$

$$\le D\sum_{i=1}^{n} (v_i w_i^*)^+ \; e^{-\alpha_i t} D^{-1} z^+(0) \; +$$

$$+ \int_0^t D \sum_{i=1}^{n} (v_i w_i^*)^+ \; e^{-\alpha_i (t-\tau)} A_{cm} D^{-1} z^+(\tau) d\tau \qquad (312)$$

where $\alpha_i = -\mathrm{Re}(\lambda_i)$, $A_{cm} = A_m + B_m(FC)^+$ and where we have used the property that $[\exp(\lambda_i t)]^+ = \exp(-\alpha_i t)$.

Next, integrate both sides of (312) to obtain

$$\int_0^\infty z^+(t)dt \le D \sum_{i=1}^{n} (v_i w_i^*)^+ \int_0^\infty e^{-\alpha_i t} dt \cdot D^{-1} z^+(0)$$

$$+ \int_{t=0}^\infty \int_{\tau=0}^t D \sum_{i=1}^{n} (v_i w_i^*)^+ \; e^{-\alpha_i (t-\tau)} A_{cm} D^{-1} z^+(\tau) d\tau dt$$

$$(313)$$

Consider the double integral in (313) which may be written as

$$\lim_{R\to\infty}\left[\int_{t=0}^{R}\int_{\tau=0}^{t} D\sum_{i=1}^{n}(v_i w_i^*)^+ \; e^{-\alpha_i(t-\tau)} A_{cm} D^{-1} z^+(\tau)\,d\tau\,dt\right];\quad 0\le\tau\le t$$

(314)

The order of integration in (314) may be interchanged because all of the functions inside the integrals are continuous functions of t and τ (Churchill [29]). Thus, (314) is equal to

$$\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{t=\tau}^{R} D\sum_{i=1}^{n}(v_i w_i^*)^+ \; e^{-\alpha_i(t-\tau)} A_{cm} D^{-1} z^+(\tau)\,dt\,d\tau\right];\quad 0\le\tau\le t$$

(315)

Now use the change of variables given by $\gamma=t-\tau$, $d\gamma=dt$, $\gamma\ge0$. Then, (315) is equal to

$$\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{\gamma=0}^{R-\tau} D\sum_{i=1}^{n}(v_i w_i^*)^+ \; e^{-\alpha_i\gamma} A_{cm} D^{-1} z^+(\tau)\,d\gamma\,d\tau\right];\quad 0\le\tau\le t,\gamma\ge0$$

(316)

$$\le \lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^*)^+ \; e^{-\alpha_i\gamma} A_{cm} D^{-1} z^+(\tau)\,d\gamma\,d\tau\right];\quad \alpha_i>0$$

(317)

$$= \lim_{R\to\infty}\left[\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^*)^+ \; e^{-\alpha_i\gamma} A_{cm}D^{-1}d\gamma\int_{\tau=0}^{R} z^+(\tau)d\tau\right]; \quad \alpha_i>0$$

$$(318)$$

$$\le \lim_{R\to\infty}\left[D\sum_{i=1}^{n}\frac{(v_i w_i^*)^+}{-\alpha_i}\;(e^{-\alpha_i R}-1)A_{cm}D^{-1}\int_0^{\infty} z^+(\tau)d\tau\right]; \quad \alpha_i>0 \qquad (319)$$

Evaluating the limit of the term outside the integral in (319), note that τ is now a dummy variable of integration, recall that the α_i's are positive, and substitute the result into (313) to obtain

$$\int_0^{\infty} z^+(t)dt \le D\sum_{i=1}^{n}\frac{(v_i w_i^*)^+}{\alpha_i}\;D^{-1}z^+(0) \; +$$

$$+ \; D\sum_{i=1}^{n}\frac{(v_i w_i^*)^+}{\alpha_i}\;A_{cm}D^{-1}\int_0^{\infty} z^+(t)dt \qquad (320)$$

Take norms in (320) and rearrange to obtain

$$\left\|\int^{\infty} z^+(t)dt\right\| \le \frac{\left\| D\sum_{i=1}^{n}\frac{(v_i w_i^*)^+}{\alpha_i}\;D^{-1}z^+(0)\right\|}{1-\left\| D\sum\frac{(v_i w_i^*)^+}{\alpha_i}\;A_{cm}D^{-1}\right\|} \qquad (321)$$

Thus, $\left\| \int_0^\infty z^+(t)dt \right\| < \infty$ which implies that $\int_0^\infty \|z^+(t)\| dt < \infty$ if

$$\left\| D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_m + B_m (FC)^+] D^{-1} \right\| < 1 \tag{322}$$

Note that $x(t)$ and $z(t)$ are continuous because of the linearity of the uncertain closed loop plant which together with (322) implies that $\|z^+(t)\| \to 0$ as $t \to \infty$ (Hsu and Meyer [30]). This implies that $\|x(t)\| \to 0$ as $t \to \infty$ which proves that the linear uncertain closed loop plant is asymptotically stable.

Finally, Perron weightings may be used for the matrix D in (322) in the same manner as shown by Sobel et. al. [1] for an earlier robustness result. Thus, to reduce conservatism, (322) may be replaced by

$$\lambda_{max} \left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_m + B_m (FC)^+] \right\} < 1 \tag{323}$$

where $\lambda_{max}(\cdot)$ of a non-negative matrix denotes the real non-negative eigenvalue $\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues λ_i.

End of Proof

We remark that the quantity $|w_i^* v_i|$ is defined by Golub and Van Loan [21] to be the condition number of the i-th eigenvalue. This condition number is a measure of the sensitivity of the i-th eigenvalue to incremental perturbations in the matrix A+BFC. The quantity $v_i w_i^*$, which appears in (302), is weakly related to this eigenvalue condition number. Thus, the new stability robustness condition of (302) seems to have the heuristic interpretation that robustness can be achieved by having the sensitive eigenvalues far into the left half complex plane and by having the eigenvalues which are near the imaginary axis to be insensitive. Such an interpretation was not possible for the result of Sobel et. al. [1] because their stability robustness condition, which is given by (301), is in terms of the spectrum as a whole rather than the individual eigenvalues.

The new stability robustness condition of (302) is less conservative than the earlier condition of Sobel et. al. [1]. This property is described by the following theorem which is attributed to Piou [3].

Theorem [3]

The left hand side of (302) is less than or equal to the left hand side of (301). Mathematically,

$$\lambda_{max}\left\{\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i}[A_m+B_m(FC)^+]\right\}$$

$$\le \left(\frac{1}{\alpha}\right)\lambda_{max}\left\{(M^{-1})^+[A_m+B_m(FC)^+]M^+\right\} \tag{324}$$

7.2 EXAMPLE: ROBUST CONTROL DESIGN FOR THE EMRAAT MISSILE

Consider the Extended Medium Range Air to Air Technology (EMRAAT) bank-to-turn missile which is described by Bossi and Langehough [9]. A sixth order model of the yaw/roll dynamics at a 10 degree angle of attack is considered with state vector, control vector, and measurement vector given by $x=[\beta, r, p, p_I, \delta_r, \delta_a]^T$, $u=[\delta_r, \delta_a]^T$, and $y=[\beta, r, p, p_I]^T$, respectively. Here β is sideslip angle (deg), r is yaw rate (deg/sec), p is roll rate (deg/sec), p_I is integrated roll rate (deg), δ_r is rudder deflection (deg), and δ_a is aileron deflection (deg). The state space matrices A, B, and C are shown below.

$$A = \begin{bmatrix} -.5007 & -.9945 & .1736 & 0 & .109 & .00691 \\ 16.83 & -.5748 & .01233 & 0 & -132.8 & 27.19 \\ -322.7 & .3208 & -2.099 & 0 & -1620 & -1240 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -179 & 0 \\ 0 & 0 & 0 & 0 & 0 & -179 \end{bmatrix} \tag{325}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 179 & 0 \\ 0 & 179 \end{bmatrix} \tag{326}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{327}$$

Sobel and Cloutier [6] applied eigenstructure assignment to the design of an autopilot for the EMRAAT missile. An important difference between this application and other eigenstructure assignment applications which have appeared in the literature is that the lateral dynamics of the EMRAAT missile does not have a well defined dutch roll mode. Therefore, eigenstructure assignment is utilized not only for mode decoupling, but also for creating distinctly separate dutch roll and roll modes. Sobel and Cloutier [6] use eigenstructure assignment with an orthogonal projection. The desired dutch roll and roll mode eigenvalues are achieved exactly because four measurements are available for feedback. The desired dutch roll eigenvectors are chosen to yield a complex mode which is composed of sideslip angle and yaw rate with no coupling to roll rate and integrated roll rate. The desired roll mode eigenvectors are chosen to yield a complex mode which is composed of roll rate and integrated roll rate with no coupling to sideslip angle and yaw rate.

Then, the achievable eigenvectors are computed by using the orthogonal projection of the i-th desired eigenvector v_i^d onto the subspace which is spanned by the columns of $(\lambda_i I-A)^{-1}B$. The closed loop eigenvalues and the feedback gain matrix are shown in table 11. Sobel and Cloutier [6] show that their design achieves improved decoupling between an initial sideslip angle and the integrated roll rate (which is approximately equal to the bank angle) when compared to a linear quadratic regulator design proposed by Bossi and Langehough [9]. However, the design of Sobel and Cloutier [6] does not consider that the missile's aerodynamic parameters are uncertain.

In collaboration with Piou [3], we propose a new robust design which minimizes the integrated roll rate (which is approximately equal to the bank angle) subject to constraints on the real part of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. Mathematically, the objective function to be minimized is given by

$$J = \sum_{k=1}^{30} [p_I(kT_1)]^2 \tag{328}$$

where T_1=0.01 sec.

Table 11. Comparison of EMRAAT Designs

	Closed Loop Eigenvalues*	Feedback Gain Matrix
Eigen-structure Assignment Design of [6]	$\lambda_{dr}=-23.98\pm j17.99$ $\lambda_{roll}=-10.01\pm j9.98$ $\lambda_{act} = -132.1$ $\lambda_{act} = -161.1$	$\begin{array}{cccc|c} \beta & r & p & p_I & \\ -4.19 & .233 & .00374 & .731 & \delta_{rc} \\ 2.89 & -.290 & .00631 & -.812 & \delta_{ac} \end{array}$
Robust Design	$\lambda_{dr}=-14.94\pm j16.68$ $\lambda_{roll}=-40.55\pm j92.90$ $\lambda_{act} = -150.3$ $\lambda_{act} = -99.93$	$\begin{array}{cccc|c} \beta & r & p & p_I & \\ -2.82 & .162 & .0127 & 1.18 & \delta_{rc} \\ .770 & -.180 & .062 & 3.14 & \delta_{ac} \end{array}$

*Eigenvalues are computed by using feedback gains which are rounded to three significant digits.

The values of T_1 and k are chosen to include the time interval [0,0.3] during which most of the transient response occurs. Of course, computation of (328) requires that a linear simulation be performed during each function evaluation of the optimization. The constraints are shown below where ζ is the damping ratio.

$$\text{Re } \lambda_{dr} \in [-50,-6] \tag{329}$$

$$\text{Re } \lambda_{roll} \in [-50,-6] \tag{330}$$

$$\text{Re } \lambda_{rudder} \leq -100 \tag{331}$$

$$\text{Re } \lambda_{aileron} \leq -100 \tag{332}$$

$$\zeta_{dr} \in [0.4,0.8] \tag{333}$$

$$\zeta_{roll} \in [0.4,0.8] \tag{334}$$

$$|\dot{\delta}_a| \leq 275 \text{ deg/sec} \tag{335}$$

$$|\dot{\delta}_r| \leq 275 \text{ deg/sec} \tag{336}$$

$$\lambda_{max}\left\{\sum_{i=1}^{6} \frac{(v_i w_i^*)^+}{\alpha_i}[A_m+B_m(FC)^+]\right\} \leq 0.999 \tag{337}$$

where for illustrative purposes we have chosen $A_m=0.1\cdot A^+$ and $B_m=0$. The actuator deflection rates are computed from the slopes of the time responses of the deflections during the time interval [0,0.03]. This interval is chosen because the slopes of the deflections are largest during this time interval. Mathematically,

$$|\dot{\delta}_r| = \frac{\max|\delta_r(mT_2)|}{mT_2} \tag{338}$$

$$|\dot{\delta}_a| = \frac{\max|\delta_a(mT_2)|}{mT_2} \tag{339}$$

where T_2=0.001 sec and m=0,1,...,30. The maximum deflection rates chosen for the constraints are well within the expected 400 deg/sec limit for the advanced state of the art electromechanical actuator described by Langehough and Simons [31].

The parameter vector contains the quantities which may be varied by the optimization. This twelve dimensional vector includes Re λ_{dr}, Im λ_{dr}, Re λ_{roll}, Im λ_{roll}, Re $z_1(1)$, Re $z_1(2)$, Im $z_1(1)$, Im $z_1(2)$, Re $z_3(1)$, Re $z_3(2)$, Im $z_3(1)$, Im $z_3(2)$. Here, the two dimensional complex vectors z_i contain the free eigenvector parameters. That is, the i-th eigenvector v_i may be written as

$$v_i = L_i z_i \tag{340}$$

where the columns of $L_i=(\lambda_i I-A)^{-1}B$ are a basis for the subspace in which the i-th eigenvector must reside. Thus, the free parameters are the vectors z_i rather than the eigenvectors v .

The optimization is performed by using subroutine *constr* from the MATLAB™ Optimization Toolbox [5] on a 486™ 25MHz personal computer. The optimization is initialized with the design proposed by Sobel and Cloutier [6] which yields an initial value of 5.0214 for the objective function of (328), a value of 4.0472 for the left hand side (LHS) of the earlier robustness condition of (301), and a value of 2.0686 for the LHS of the new robustness condition of (302). The optimization is complete after 2698 function evaluations which requires approximately one hour of computation and yields an optimal objective function of 0.0284, a value of 2.4432 for the LHS of the earlier robustness condition of (301), and a value of 0.999 for the LHS of the new robustness condition of (302). In table 11, we observe that the dutch roll mode is dominant in the robust design whereas the roll mode was chosen to be dominant in the design of Sobel and Cloutier [6]. Furthermore, the optimization has assigned the roll mode damping to be the smallest value which is allowed by the constraint of (334).

The time histories of sideslip angle, yaw rate, roll rate, integrated roll rate, rudder deflection, and aileron deflection to a one degree initial sideslip are shown in figures 29, 30, 31, 32, 33, and 34, respectively. We observe a significant improvement in the integrated roll rate response (which is desired to be zero) when compared to the earlier design of Sobel and Cloutier [6]. The earlier design of Sobel and Cloutier [6] has a minimum $p_I(t)$ of -0.646 deg but the new design of this thesis has a minimum $p_I(t)$ of -0.112 deg which is an improvement of approximately 80%. We note that this improved response is obtained with both smaller aileron and rudder deflections. Furthermore, it is interesting that the aileron in the design of Sobel and Cloutier [6] exhibits an initial positive deflection of approximately two degrees before becoming negative, whereas this large positive initial aileron deflection does not appear in the new robust design.

Next, the response to a roll rate step of 5 deg/sec is considered. The roll rate, rudder deflection, and aileron deflection are shown in figures 35, 36, and 37, respectively. The new robust design exhibits a roll rate response with a significantly smaller settling time which is achieved by using larger aileron deflection rates. Nevertheless, these deflection rates of approximately 25 deg/sec are well within the allowable limits.

Algorithm:

The new robust design method presented above can be expressed by the following flow chart:

Initialization:

Input: 1. System matrices $A(n\times n)$, $B(n\times m)$, $C(r\times n)$;
2. Maximum uncertainty matrices $A_m(n\times n)$, $B_m(n\times m)$
3. Desired eigenvalues $\lambda_i (i=1,\ldots,r)$
and desired eigenvectors.

Use orthogonal projection to obtain achievable eigenvector parameters z_i $(i=1,\ldots,r)$

Set initial optimization parameters to λ_i and z_i. $(i=1,\ldots,r)$

Optimization:

$$\underset{\lambda_i,\ z_i}{\text{Minimize}} \sum_{k=1}^{30} [p_i(kT_i)]^2$$ by calling MATLAB™ subroutine CONSTR

subject to the constraints described by (329)-(337)

The flow chart for the function which is called by MATLAB™ subroutine CONSTR is shown below.

Use λ_i and z_i to obtain eigenvector v_i $(i=1,\ldots,r)$

Calculate feedback gain matrix F

Obtain time responses by calling MATLAB™ subroutine LSIM

Calculate function value by using (328)

Calculate constraints given by LHS of (333)-337)

Return function value and constraint values

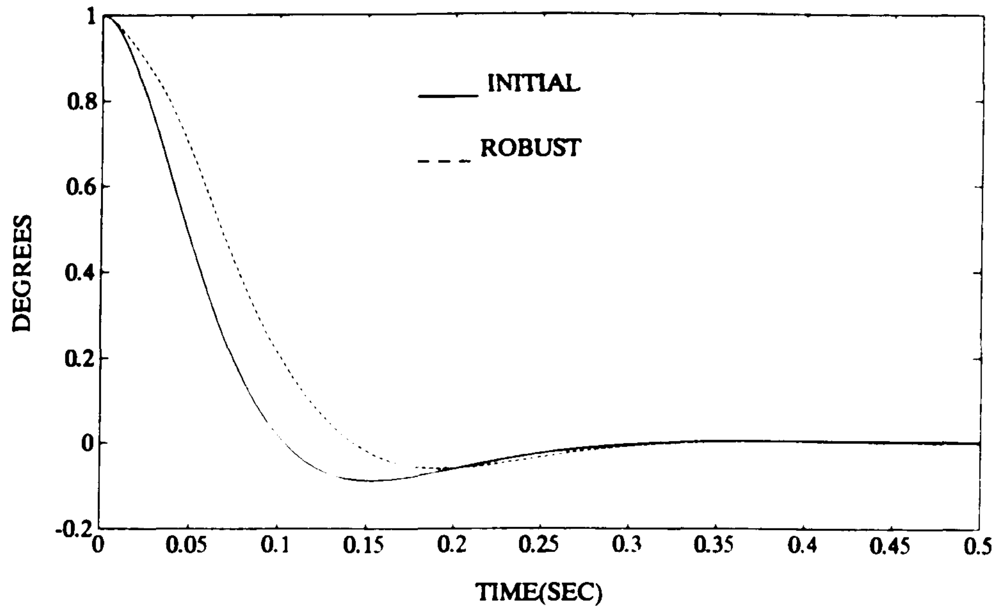

Figure 29. EMRAAT Missile Sideslip Angle

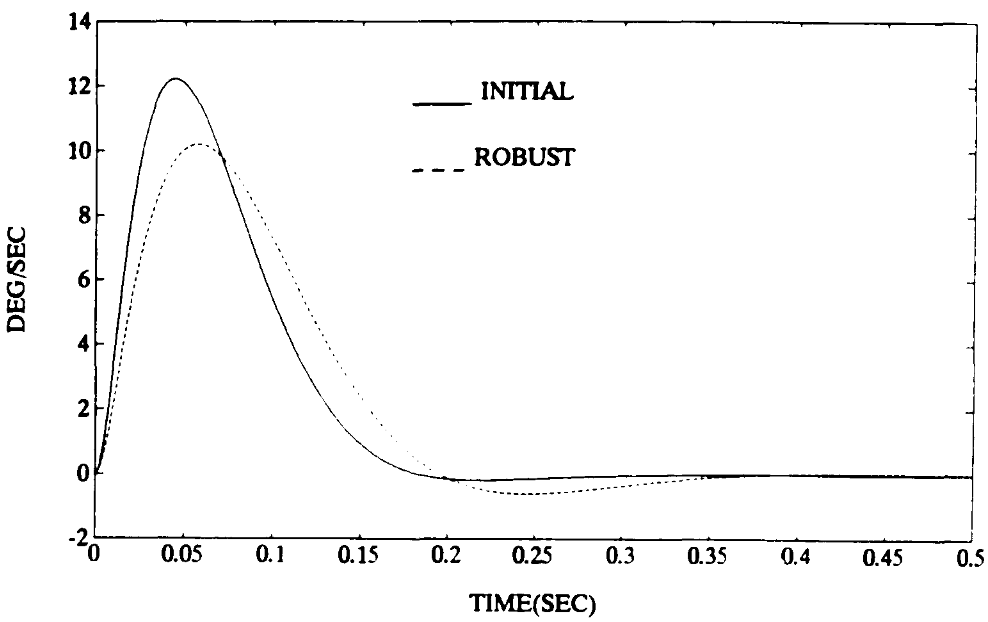

Figure 30. EMRAAT Missile Yaw Rate

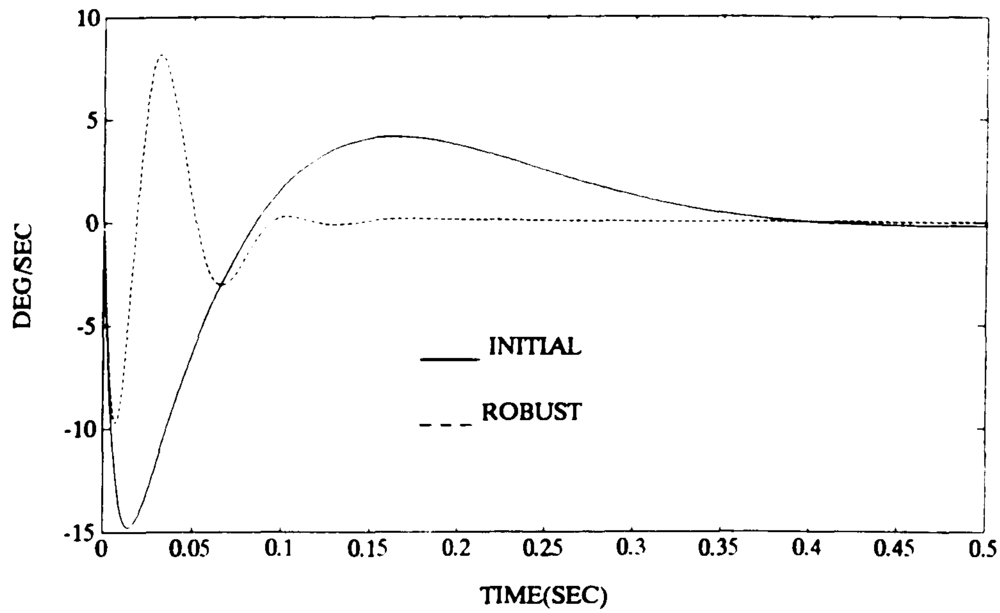

Figure 31. EMRAAT Missile Roll Rate

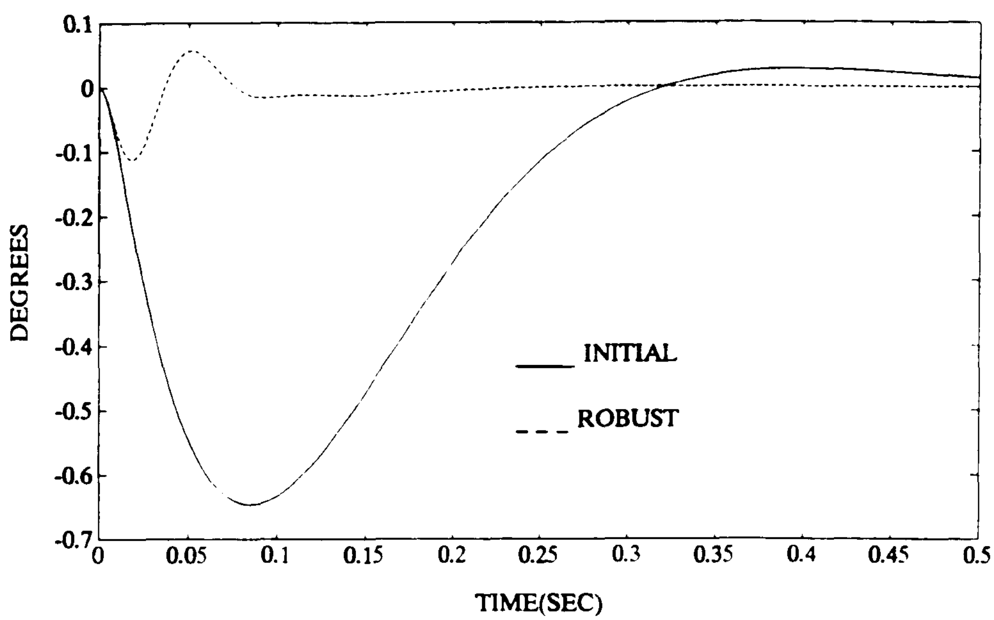

Figure 32. EMRAAT Missile Integrated Roll Rate

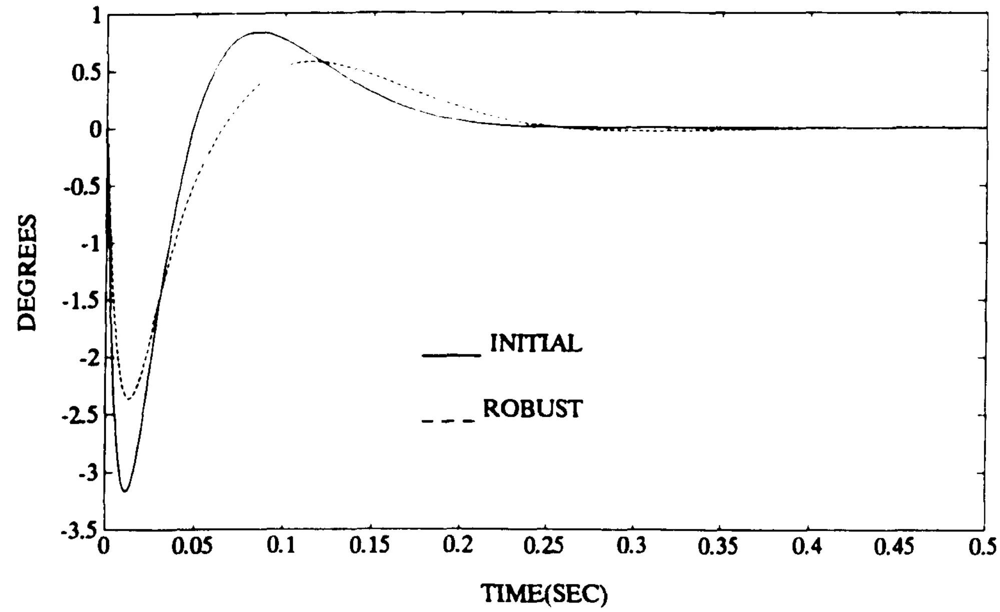

Figure 33. EMRAAT Missile Rudder Deflection

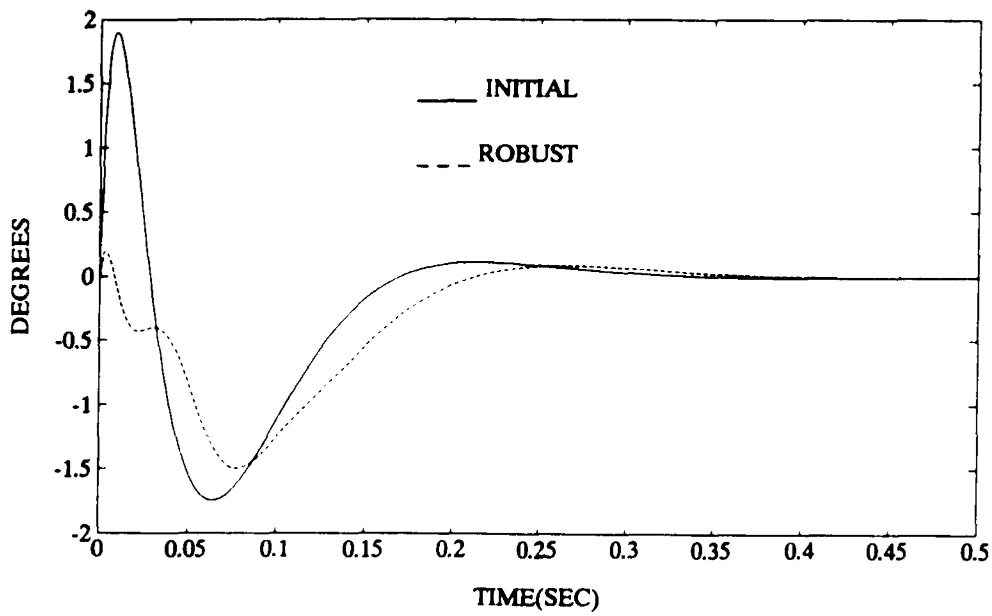

Figure 34. EMRAAT Missile Aileron Deflection

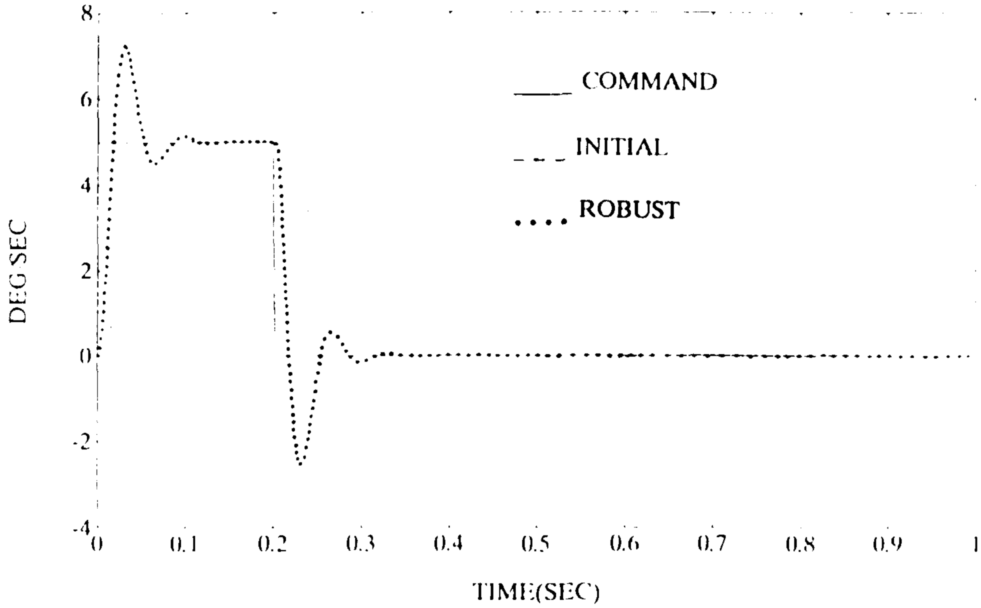

Figure 35. EMRAAT Missile Roll Rate for Step Response

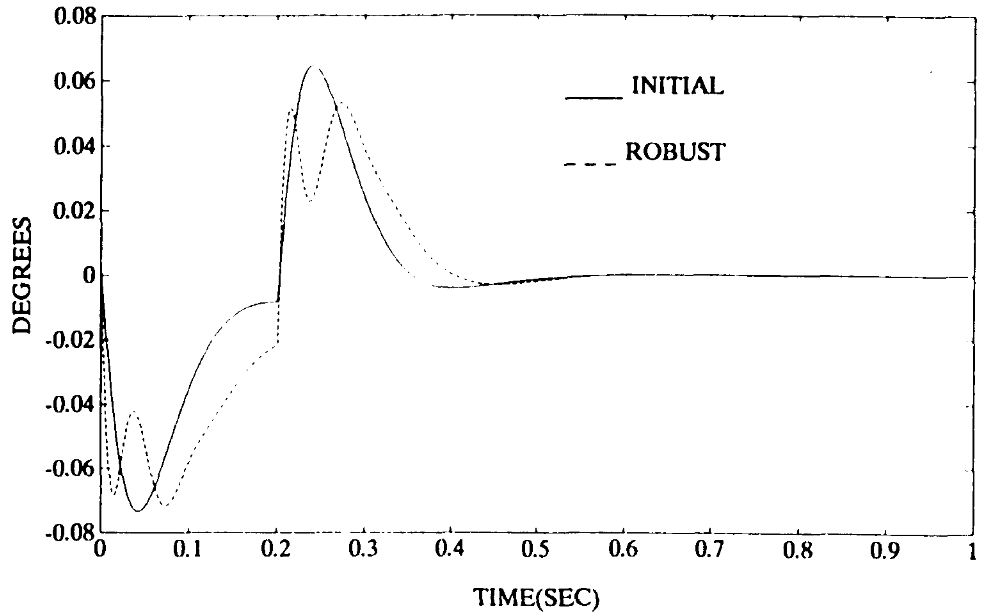

Figure 36. EMRAAT Missile Rudder Deflection for Step Response

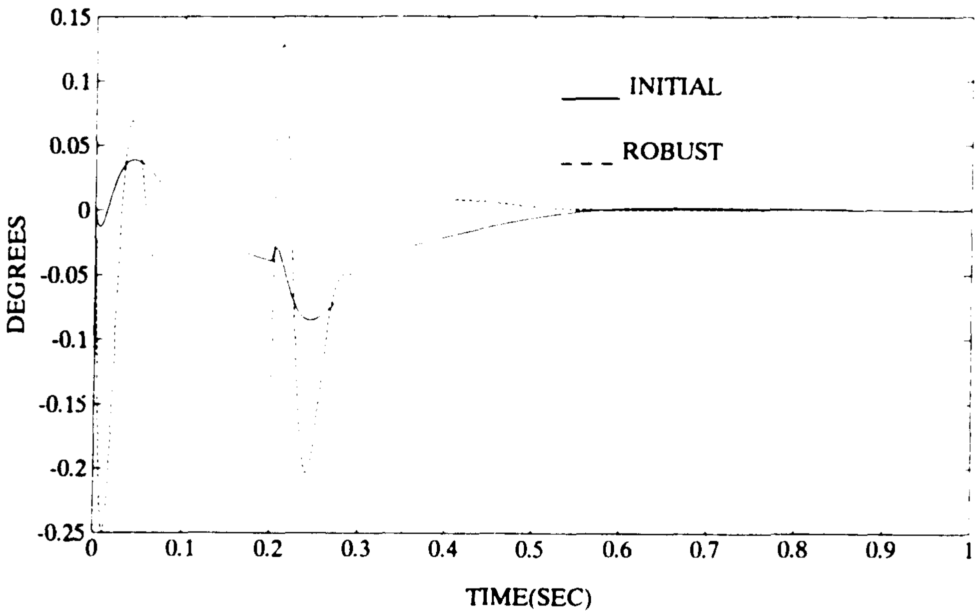

Figure 37. EMRAAT Missile Aileron Deflection for Step Response

8. ROBUST EIGENSTRUCTURE ASSIGNMENT FOR TIME DELAY SYSTEMS

8.1 INTRODUCTION

In this Chapter, we present two theorems which extend the results of [1] and Chapter 7 of this thesis to time delay systems and give an illustrative example to show the application of the second theorem.

8.2 PROBLEM FORMULATION

Consider a nominal linear time invariant system with time delay described by

$$\dot{x}(t) = Ax(t) + A_0x(t-\tau) + Bu(t) \tag{341}$$

$$y(t) = Cx(t) \tag{342}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector, and A, B, C, and A_0 are constant matrices.

Suppose that the nominal time delay system is subject to linear time varying parametric uncertainty in the entries of A, B, C, and A_0 described by $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta A_0(t)$, respectively. We shall assume that the entries of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta A_0(t)$ are continuous functions of time. Then, the uncertain time delay system is given by

$$\dot{x}(t) = [A+\Delta A(t)]x(t) + [A_0+\Delta A_0(t)]x(t-\tau) + [B+\Delta B(t)]u(t) \tag{343}$$

$$y(t)=[C+\Delta C(t)]x(t) \tag{344}$$

Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta A_0(t)$. That is,

$$|\Delta a_{ij}(t)| \le (a_{ij})_{max}; \quad i=1,\dots,n; \quad j=1,\dots,n \tag{345}$$

$$|\Delta b_{ij}(t)| \le (b_{ij})_{max}; \quad i=1,\dots,n; \quad j=1,\dots,m \tag{346}$$

$$|\Delta c_{ij}(t)| \le (c_{ij})_{max}; \quad i=1,\dots,r; \quad j=1,\dots,n \tag{347}$$

$$|\Delta a_{oij}(t)| \le (a_{oij})_{max}; \quad i=1,\dots,n; \quad j=1,\dots,n \tag{348}$$

Define $\Delta A^+(t)$, $\Delta B^+(t)$, $\Delta C^+(t)$, and $\Delta A_0^+(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta A_0(t)$ by their absolute values. Also, define A_m, B_m, C_m, and A_{mo} as the matrices with entries $(a_{ij})_{max}$, $(b_{ij})_{max}$,

$(c_{ij})_{max}$, and $(a_{oij})_{max}$, respectively. Then,

$$\{\Delta A(t):\ \Delta A^{+}(t) \le A_m\} \tag{349}$$

$$\{\Delta B(t):\ \Delta B^{+}(t) \le B_m\} \tag{350}$$

$$\{\Delta C(t):\ \Delta C^{+}(t) \le C_m\} \tag{351}$$

$$\{\Delta A_o(t):\ \Delta A_o^{+}(t) \le A_{mo}\} \tag{352}$$

where "$\le$" is applied element by element to matrices and $A_m \in \mathbb{R}_+^{n\times n}$, $B_m \in \mathbb{R}_+^{n\times m}$, $C_m \in \mathbb{R}_+^{r\times n}$, $A_{mo} \in \mathbb{R}_+^{n\times n}$.

Consider the constant gain output feedback control law described by (37). Then, the nominal closed loop system is given by

$$\dot{x}(t) = (A+BFC)x(t) + A_o x(t-\tau) \tag{353}$$

and the uncertain closed loop system is given by

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A)x(t)+(A_o+\Delta A_o)x(t-\tau)+(B+\Delta B)F(C+\Delta C)x(t)\\ &=(A+BFC)x(t)+(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)x(t)+(A_o+\Delta A_o)x(t-\tau)\end{aligned} \tag{354}$$

Finally, the stability robustness problem can be stated as follows: Given a feedback gain matrix $F \in \mathbb{R}^{m\times r}$ such that the nominal closed loop system with time delay exhibits desirable dynamic performance, determine if the uncertain closed loop system is asymptotically stable for all $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta A_0(t)$ described by (349)-(352).

8.3 ROBUSTNESS RESULTS

We now present the first robust stability theorem which extends the result of Sobel et. al. [1] to time delay systems.

Theorem 5

Suppose that F is such that the nominal closed loop system without time delay described by $\dot{x}(t)=(A+BFC)x(t)$ is asymptotically stable with a non-defective modal matrix. Then, the uncertain closed loop system with time delay given by (354) is asymptotically stable for all $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta A_0(t)$ described by (349)-(352) if

$$\alpha \geq \|(M^{-1})^{+}[A_m+B_m(FC)^{+}+(BF)^{+}C_m+B_mF^{+}C_m]M^{+}\| +$$

$$+ \exp(\alpha\tau)\|(M^{-1})^{+}(A_0^{+}+A_{mo})M^{+}\| \tag{355}$$

where

$$\alpha = -\max_i \mathrm{Re}[\lambda_i(A+BFC)]$$

and where M is a modal matrix of (A+BFC).

Proof

From (343), (344), and (37), we have

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A)x(t)+(A_0+\Delta A_0)x(t-\tau)+(B+\Delta B)F(C+\Delta C)x(t)\\ &= (A+BFC)x(t)+(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)x(t)+(A_0+\Delta A_0)x(t-\tau)\end{aligned} \tag{356}$$

We begin with two preliminary lemmas.

Lemma 5

$$\begin{aligned}&\|M^{-1}(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)M\|_2\\ &\le \|(M^{-1})^+\{A_m+B_m(FC)^+ + (BF)^+C_m+B_mF^+C_m\}^+M^+\|\end{aligned} \tag{357}$$

Proof: Use the result given by Kouvaritakis and Latchman [20] that for any matrix $A \in \mathbb{C}^{n\times m}$

$$\|A\|_2 \le \|A^+\|_2 \tag{358}$$

Thus

$$\|M^{-1}(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)M\|_2$$

$$\leq \|\{M^{-1}(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)M\}^+\|_2 \tag{359}$$

$$= \sup \frac{\|\{M^{-1}(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)M\}^+x\|_2}{\|x\|_2} \tag{360}$$

$$= \sup \frac{\|\{M^{-1}(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)M\}^+x^+\|_2}{\|x^+\|_2} \tag{361}$$

$$= \sup \frac{\|(M^{-1})^+(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)^+M^+x^+\|_2}{\|x^+\|_2} \tag{362}$$

$$= \sup \frac{\|(M^{-1})^+\{A_m+B_m(FC)^++(BF)^+C_m+B_mF^+C_m\}M^+x^+\|_2}{\|x^+\|_2} \tag{363}$$

$$= \|(M^{-1})^+\{A_m+B_m(FC)^++(BF)^+C_m+B_mF^+C_m\}^+M^+\| \tag{364}$$

End of proof

Lemma 6

$$\|M^{-1}(A_0+\Delta A_0)M\|_2 \leq \|(M^{-1})^+(A_0^++A_{mo})M^+\|_2 \tag{365}$$

Proof

$$\|M^{-1}(A_0+\Delta A_0)M\|_2 \leq \|\{M^{-1}(A_0+\Delta A_0)M\}^+\|_2 \tag{366}$$

$$= \sup \frac{\| \{M^{-1}(A_0+\Delta A_0)M\}^+ x \|_2}{\|x\|_2} \tag{367}$$

$$= \sup \frac{\| \{M^{-1}(A_0+\Delta A_0)M\}^+ x^+ \|_2}{\|x^+\|_2} \tag{368}$$

$$= \sup \frac{\| (M^{-1})^+ (A_0+\Delta A_0)^+ M^+ x^+ \|_2}{\|x^+\|_2} \tag{369}$$

$$= \sup \frac{\| (M^{-1})^+ (A_0^+ + A_{mo}) M^+ x^+ \|_2}{\|x^+\|_2} \tag{370}$$

$$= \| (M^{-1})^+ (A_0^+ + A_{mo}) M^+ \|_2 \tag{371}$$

End of proof

Returning to the proof of theorem 5, we let $x(t) = Mz(t)$

then $z(t) = M^{-1}x(t)$, $\Lambda = M^{-1}(A+BFC)M$

$$\begin{aligned} z(t) &= \exp(\Lambda t)z(0) \\ &+ \int_0^t \exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)ds \\ &+ \int_0^t \exp\{\Lambda(t-s)\}M^{-1}[A_0+\Delta A_0(s)]Mz(s-\tau)ds \end{aligned} \tag{372}$$

Since z(t) is causal, the second integral should start from

τ,

$$z(t)=\exp(\Lambda t)z(0)$$

$$+\int_0^t \exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)ds$$

$$+\int_\tau^t \exp\{\Lambda(t-s)\}M^{-1}[A_0+\Delta A_0(s)]Mz(s-\tau)ds \tag{373}$$

Let $u=s-\tau$, i.e. $s=u+\tau$, $ds=du$, then

$$z(t)=\exp(\Lambda t)z(0)$$

$$+\int_0^t \exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)ds$$

$$+\int_0^{t-\tau} \exp\{\Lambda(t-u-\tau)\}M^{-1}[A_0+\Delta A_0(u+\tau)]Mz(u)du \tag{374}$$

Take the 2-norm on both sides of the above equation to obtain

$$\|z(t)\|=\|\exp(\Lambda t)z(0)+$$

$$\int_0^t \exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)ds$$

$$+\int_0^{t-\tau} \exp\{\Lambda(t-u-\tau)\}M^{-1}[A_0+\Delta A_0(u+\tau)]Mz(u)du\| \tag{375}$$

$$\le\|\exp(\Lambda t)z(0)\|$$

$$+\left\|\int_0^t \exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)ds\right\|$$

$$+\left\|\int_0^{t-\tau}\exp\{\Lambda(t-u-\tau)\}M^{-1}[A_0+\Delta A_0(u+\tau)]Mz(u)du\right\| \tag{376}$$

$$\leq\|\exp(\Lambda t)z(0)\|$$

$$+\int_0^t\|\exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)\|ds$$

$$+\int_0^{t-\tau}\|\exp\{\Lambda(t-u-\tau)\}M^{-1}[A_0+\Delta A_0(u+\tau)]Mz(u)\|du \tag{377}$$

$$\leq\|\exp(\Lambda t)z(0)\|$$

$$+\int_0^t\|\exp\{\Lambda(t-s)\}M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]Mz(s)\|ds$$

$$+\int_0^t\|\exp\{\Lambda(t-u-\tau)\}M^{-1}[A_0+\Delta A_0(u+\tau)]Mz(u)\|du \tag{378}$$

Note that u is a dummy variable of integration. Therefore, we may substitute s for u in (378). Therefore,

$$\|z(t)\| \leq \|\exp(\Lambda t)\|\cdot\|z(0)\|+$$

$$\int_0^t\|\exp\{\Lambda(t-s)\}\|\cdot\|M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]M\|$$

$$\cdot\|z(s)\|ds$$

$$+\int_0^t\|\exp\{\Lambda(t-s-\tau)\}\|\cdot\|M^{-1}[A_0+\Delta A_0(s+\tau)]M\|$$

$$\cdot\|z(s)\|ds \tag{379}$$

Using $\|\exp(\Lambda t)\| \leq \exp(-\alpha t)$, where $\alpha=-\max[\lambda_i(A+BFC)]$, we obtain

$$\|z(t)\| \le \exp(-\alpha t)\cdot\|z(0)\|$$

$$+ \int_0^t \exp\{-\alpha t+\alpha s\}\cdot\|M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]M\|$$

$$\cdot\|z(s)\|ds$$

$$+ \int_0^t \exp\{-\alpha t+\alpha s+\alpha\tau\}\cdot\|M^{-1}[A_0+\Delta A_0(s+\tau)]M\|\cdot\|z(s)\|ds \tag{380}$$

$$\|z(t)\|\exp(\alpha t) \le \|z(0)\|$$

$$+ \int_0^t \|M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]M\|$$

$$\cdot\|z(s)\|\exp(\alpha s)ds$$

$$+ \int_0^t \exp(\alpha\tau)\|M^{-1}[A_0+\Delta A_0(s+\tau)]M\|\cdot\|z(s)\|\exp(\alpha s)ds \tag{381}$$

Use the Gronwall lemma to obtain

$$\|z(t)\|\exp(\alpha t)\le \|z(0)\|$$

$$\cdot\exp\{\int_0^t[\|M^{-1}[\Delta A(s)+\Delta B(s)FC+BF\Delta C(s)+\Delta B(s)F\Delta C(s)]M\|$$

$$+\exp(\alpha\tau)\|M^{-1}[A_0+\Delta A_0(s+\tau)]M\|]ds\} \tag{382}$$

Use lemmas 5 and 6 to obtain

$$\|z(t)\|\exp(\alpha t) \le \|z(0)\|\cdot\exp\{[\|(M^{-1})^+A_{cm}M^+\|$$

$$+ \exp(\alpha\tau)\|(M^{-1})^+A_{cmo}M^+\|]t\} \tag{383}$$

where

$$A_{cm} = A_m + B_m(FC)^+ + (BF)^+C_m + B_mF^+C_m$$

and $$A_{cmo} = A_o^+ + A_{mo}$$

or

$$\|z(t)\| \le \|z(0)\|$$

$$\cdot\exp\left[\left(\{\|(M^{-1})^+A_{cm}M^+\|+\exp(\alpha\tau)\|(M^{-1})^+A_{cmo}M^+\|\}-\alpha\right)t\right] \qquad (384)$$

Thus $\|z(t)\| \to 0$ if

$$\alpha \ge \|(M^{-1})^+[A_m+B_m(FC)^++(BF)^+C_m+B_mF^+C_m]M^+\|$$

$$+ \exp(\alpha\tau)\|(M^{-1})^+(A_o^++A_{mo})M^+\| \qquad (385)$$

End of proof

We next present the second stability robustness theorem which extends the result of Yu, Piou and Sobel [32] to time delay systems.

Theorem 6

Suppose that F is such that the nominal closed loop system without time delay described by $\dot{x}(t)=(A+BFC)x(t)$ is asymptotically stable with a non-defective modal matrix. Then, the uncertain closed loop system with time delay given by (354) is asymptotically stable for all $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta A_0(t)$ described by (349)-(352) if

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} A_{cm} + \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} e^{\alpha_i \tau} A_{cmo} \right\} < 1 \qquad (386)$$

where

$$A_{cm} = A_m + B_m(FC)^+ + (BF)^+ C_m + B_m F^+ C_m ,$$

$$A_{cmo} = A_0^+ + A_{mo} ,$$

and $$\alpha_i = -\mathrm{Re}[\lambda_i(A+BFC)] ,$$

and where λ_i is the i-th eigenvalue of $(A+BFC)$ with v_i and w_i^* the corresponding right and left eigenvectors, respectively; and where $(\cdot)^*$ denotes the complex conjugate transpose.

Proof

The uncertain closed loop plant may be written as

$$\dot{x}(t)=(A+\Delta A)x(t)+(A_0+\Delta A_0)x(t-\tau)+(B+\Delta B)F(C+\Delta C)x(t)$$

$$=(A+BFC)x(t)+(\Delta A+\Delta BFC+BF\Delta C+\Delta BF\Delta C)x(t)+(A_0+\Delta A_0)x(t-\tau)$$

$$=A_c x(t)+\Delta A_c x(t)+\Delta A_{co} x(t-\tau) \qquad (387)$$

where

$$A_c = A + BFC$$

$$\Delta A_c = \Delta A(t) + \Delta B(t)FC + BF\Delta C(t) + \Delta B(t)F\Delta C(t)$$

$$\Delta A_{co} = A_0 + \Delta A_0(t)$$

which has a solution given by

$$x(t) = \exp(A_c t)x(0)$$
$$+ \int_0^t \exp[A_c(t-s)]\Delta A_c(s)x(s)ds$$
$$+ \int_0^t \exp[A_c(t-s)]\Delta A_{co}(s)x(s-\tau)ds \qquad (388)$$

Next, use the real, positive, diagonal transformation

$$x(t) = D^{-1}z(t) \qquad (389)$$

and the property that

$$\exp(DA_cD^{-1}t) = D\exp(A_c t)D^{-1} \qquad (390)$$

and

$$\exp(A_c t) = M\exp(\Lambda t)M^{-1} = \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} \qquad (391)$$

to obtain

$$z(t) = D \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} D^{-1} z(0)$$

$$+ \int_0^t D \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i (t-s)} \Delta A_c(s) D^{-1} z(s) ds$$

$$+ \int_0^t D \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i (t-s)} \Delta A_{co}(s) D^{-1} z(s-\tau) ds \qquad (392)$$

where M is a modal matrix of A_c; λ_i is the i-th eigenvalue of A_c with v_i and w_i^* the corresponding right and left eigenvectors, respectively; Λ is a diagonal matrix with the λ_i on the diagonal; and $(\cdot)^*$ denotes complex conjugate transpose. Note that

$$\|z(t)\| \to 0 \text{ implies that } \|x(t)\| \to 0$$

So it is sufficient to prove that $\|z(t)\| \to 0$.

Since z(t) is causal, the second integral in (392) should start from τ.

$$z(t) = D \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} D^{-1} z(0)$$

$$+ \int_0^t D \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i (t-s)} \Delta A_c(s) D^{-1} z(s) ds$$

$$+\int_{\tau}^{t} D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i (t-s)}\, \Delta A_{co}(s) D^{-1} z(s-\tau)\,ds \tag{393}$$

In the second integral of (393) let $u=s-\tau$, i.e. $s=u+\tau$, $ds=du$, then

$$z(t) = D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i t} D^{-1} z(0)$$

$$+\int_{0}^{t} D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i (t-s)}\, \Delta A_{c}(s) D^{-1} z(s)\,ds$$

$$+\int_{0}^{t-\tau} D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i (t-u-\tau)}\, \Delta A_{co}(u+\tau) D^{-1} z(u)\,du \tag{394}$$

Note that u is a dummy variable of integration. Therefore, we may substitute s for u in (394).

$$z^{+}(t) \le \left[D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i t} D^{-1} z(0) \right]^{+}$$

$$+\left[\int_{0}^{t} D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i (t-s)}\, \Delta A_{c}(s) D^{-1} z(s)\,ds \right]^{+}$$

$$+\left[\int_{0}^{t-\tau} D\sum_{i=1}^{n} v_i w_i^{*}\, e^{\lambda_i (t-s-\tau)}\, \Delta A_{co}(s+\tau) D^{-1} z(s)\,ds \right]^{+} \tag{395}$$

$$\le D\sum_{i=1}^{n} (v_i w_i^{*})^{+}\, e^{-\alpha_i t} D^{-1} z^{+}(0)$$

$$+ \int_0^t \left[D \sum_{i=1}^n v_i w_i^* \, e^{\lambda_i (t-s)} \Delta A_c(s) D^{-1} z(s) \right]^+ ds$$

$$+ \int_0^{t-\tau} \left[D \sum_{i=1}^n v_i w_i^* \, e^{\lambda_i (t-s-\tau)} \Delta A_{co}(s+\tau) D^{-1} z(s) \right]^+ ds \qquad (396)$$

$$\le D \sum_{i=1}^n (v_i w_i^*)^+ \, e^{-\alpha_i t} D^{-1} z^+(0)$$

$$+ \int_0^t D \sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i (t-s)} A_{cm} D^{-1} z^+(s) ds$$

$$+ \int_0^t D \sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i (t-s-\tau)} A_{cmo} D^{-1} z^+(s) ds \qquad (397)$$

where $\alpha_i = -\mathrm{Re}[\lambda_i(A+BFC)]$,

$$A_{cm} = A_m + B_m (FC)^+$$

$$A_{cmo} = A_o^+ + A_{mo}$$

and where we have used the property that $[\exp(\lambda_i t)]^+ = \exp(-\alpha_i t)$. Next, integrate both sides of (397) to obtain

$$\int_0^\infty z^+(t)dt$$

$$\leq D\sum_{i=1}^{n}(v_i w_i^*)^+\int_0^\infty e^{-\alpha_i t}dt\cdot D^{-1}z^+(0)$$

$$+\int_0^\infty\int_0^t D\sum_{i=1}^{n}(v_i w_i^*)^+ e^{-\alpha_i(t-s)}A_{cm}D^{-1}z^+(s)dsdt$$

$$+\int_0^\infty\int_0^t D\sum_{i=1}^{n}(v_i w_i^*)^+ e^{-\alpha_i(t-s-\tau)}A_{cmo}D^{-1}z^+(s)dsdt \qquad (398)$$

$$= D\sum_{i=1}^{n}(v_i w_i^*)^+\int_0^\infty e^{-\alpha_i t}dt\cdot D^{-1}z^+(0)$$

$$+\lim_{R\to\infty}\left[\int_0^R\int_0^t D\sum_{i=1}^{n}(v_i w_i^*)^+ e^{-\alpha_i(t-s)}A_{cm}D^{-1}z^+(s)dsdt\right]$$

$$+\lim_{R\to\infty}\left[\int_0^R\int_0^t D\sum_{i=1}^{n}(v_i w_i^*)^+ e^{-\alpha_i(t-s-\tau)}A_{cmo}D^{-1}z^+(s)dsdt\right];0\leq s\leq t$$

$$(399)$$

The order of integration in (399) may be interchanged because all of the functions inside the integrals are continuous functions of t and s [29]. Thus, the second and third terms of (399) become

$$\lim_{R\to\infty}\left[\int_0^R\int_0^t D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i(t-s)} A_{cm} D^{-1} z^+(s)\,ds\,dt\right]$$

$$+\lim_{R\to\infty}\left[\int_0^R\int_0^t D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i(t-s-\tau)} A_{cmo} D^{-1} z^+(s)\,ds\,dt\right]; \quad 0\le s\le t$$

$$= \lim_{R\to\infty}\left[\int_{s=0}^R\int_{t=s}^R D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i(t-s)} A_{cm} D^{-1} z^+(s)\,dt\,ds\right]$$

$$+\lim_{R\to\infty}\left[\int_{s=0}^R\int_{t=s}^R D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i(t-s-\tau)} A_{cmo} D^{-1} z^+(s)\,dt\,ds\right];$$

$$0\le s\le t \tag{400}$$

Now use the change of variables given by $\gamma=t-s$, $d\gamma=dt$, $\gamma\ge0$. Then, (400) is equal to

$$\lim_{R\to\infty}\left[\int_{s=0}^R\int_{\gamma=0}^{R-s} D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i\gamma} A_{cm} D^{-1} z^+(s)\,d\gamma\,ds\right]$$

$$+\lim_{R\to\infty}\left[\int_{s=0}^R\int_{\gamma=0}^{R-s} D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i\gamma} e^{\alpha_i\tau} A_{cmo} D^{-1} z^+(s)\,d\gamma\,ds\right];$$

$$0\le s\le t,\ \gamma\ge0$$

$$\le \lim_{R\to\infty}\left[\int_{s=0}^R\int_{\gamma=0}^R D\sum_{i=1}^n (v_i w_i^*)^+ e^{-\alpha_i\gamma} A_{cm} D^{-1} z^+(s)\,d\gamma\,ds\right]$$

$$+\lim_{R\to\infty}\left[\int_{s=0}^{R}\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^{\bullet})^{+}\, e^{-\alpha_i\gamma} e^{\alpha_i\tau} A_{cmo} D^{-1} z^{+}(s)\,d\gamma\, ds\right];$$

$$0\le s\le t,\ \gamma\ge 0 \qquad (401)$$

$$= \lim_{R\to\infty}\left[\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^{\bullet})^{+}\, e^{-\alpha_i\gamma} A_{cm} D^{-1} d\gamma \int_{s=0}^{R} z^{+}(s)ds\right]$$

$$+\lim_{R\to\infty}\left[\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^{\bullet})^{+}\, e^{-\alpha_i\gamma} e^{\alpha_i\tau} A_{cmo} D^{-1} d\gamma \int_{s=0}^{R} z^{+}(s)ds\right];\ \alpha_i>0$$

$$(402)$$

$$\le \lim_{R\to\infty}\left[D\sum_{i=1}^{n}\frac{(v_i w_i^{\bullet})^{+}}{-\alpha_i}\,(e^{-\alpha_i R}-1)A_{cm} D^{-1}\int_{0}^{\infty} z^{+}(s)ds\right]$$

$$+\lim_{R\to\infty}\left[D\sum_{i=1}^{n}\frac{(v_i w_i^{\bullet})^{+}}{-\alpha_i}\,(e^{-\alpha_i R}-1)e^{\alpha_i\tau} A_{cmo} D^{-1}\int_{0}^{\infty} z^{+}(s)ds\right];\ \alpha_i>0$$

$$(403)$$

Evaluate the limit of the term outside the integral in (403), note that s is now a dummy variable of integration, recall that the α_i's are positive, and substitute the result into (399) to obtain

$$\int_{0}^{\infty} z^{+}(t)dt \le D\sum_{i=1}^{n}\frac{(v_i w_i^{\bullet})^{+}}{\alpha_i}\, D^{-1} z^{+}(0)$$

$$+ D\left(\sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} A_{cm} + \sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} e^{\alpha_i \tau} A_{cmo} \right) D^{-1} \int_0^{\infty} z^+(t)dt$$

(404)

Take norms in (404) and rearrange to obtain

$$\left\| \int_0^{\infty} z^+(t)dt \right\| \le \frac{\left\| D \sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} D^{-1} z^+(0) \right\|}{1 - \left\| D\left(\sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} A_{cm} + \sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} e^{\alpha_i \tau} A_{cmo} \right) D^{-1} \right\|}$$

(405)

Thus, $\left\| \int_0^{\infty} z^+(t)dt \right\| < \infty$ if

$$\left\| D\left(\sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} A_{cm} + \sum_{i=1}^{n} \frac{(v_i w_i^{\bullet})^+}{\alpha_i} e^{\alpha_i \tau} A_{cmo} \right) D^{-1} \right\| < 1 \qquad (406)$$

Note that

$$\left\| \int_0^{\infty} z^+(t)dt \right\| < \infty \text{ implies that } \int_0^{\infty} \|z^+(t)\| dt < \infty \qquad (407)$$

Also note that $x(t)$ and $z(t)$ are continuous because of the linearity of the uncertain closed loop plant and the continuity of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, $\Delta A_0(t)$ which together with (406)-(407) implies that $\|z^{+}(t)\| \to 0$ as $t \to 0$ (Hsu and Meyer [30]). This implies that $\|x(t)\| \to 0$ as $t \to 0$ which proves that the linear uncertain closed loop plant is asymptotically stable.

Finally, Perron weightings may be used for the matrix D in (406) in the same manner as shown by Sobel et. al. [1] for an earlier robustness result. Thus, to reduce conservatism, (406) may be replaced by

$$\lambda_{max}\left\{\sum_{i=1}^{n}\frac{(v_i w_i^{*})^{+}}{\alpha_i}A_{cm}+\sum_{i=1}^{n}\frac{(v_i w_i^{*})^{+}}{\alpha_i}e^{\alpha_i\tau}A_{cmo}\right\} < 1 \qquad (408)$$

where $\lambda_{max}(\cdot)$ of a non-negative matrix denotes the real non-negative eigenvalue $\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues λ_i.

End of proof

We remark that the term $\exp(\alpha_i\tau)$ appears in (406). Therefore, it is more difficult to satisfy the sufficient condition as the time delay τ becomes larger. If $\tau \to \infty$, the norm in (406) goes to infinity because α is positive for a stable system. This implies that the sufficient condition for robust stability cannot be satisfied for a system with infinite time delay.

8.4 EXAMPLE

We now present an example to illustrate the result of Theorem 6. Consider the L-1011 aircraft example which was considered by Sobel et. al. [1] and described in section 3.6. For illustrative purposes, we assume that there is a time delay in the sideslip angle β, which is related to the last column of A_0, and we assume that the last column of A_0 has 10% of the value of the last column of A in (114). Thus, we split the last column of A in (114) and to obtain

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & N_r & N_p & 0.9N_\beta \\ 0 & L_r & L_p & 0.9L_\beta \\ g/V & (Y_r/V)-1 & Y_p/V & 0.9Y_\beta/V \end{bmatrix} \tag{409}$$

and

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & N_r & N_p & 0.1N_\beta \\ 0 & L_r & L_p & 0.1L_\beta \\ g/V & (Y_r/V)-1 & Y_p/V & 0.1Y_\beta/V \end{bmatrix} \tag{410}$$

The input matrix B is the same as the original matrix B which is given by (114) and table 8. The output matrix C is a 4×4 identity matrix. The state feedback gain matrix F is the same as obtained by Sobel and Shapiro [24] and is shown in (117). Bounds for uncertainties A_m, B_m, and A_{mo}, are obtained from table 8 and (409)-(410). We assume that C_m is zero.

Using theorem 6, we found that if the time delay constant τ is less than 0.037, then the system described by (341)-(342) will be asymptotically stable for all uncertainties described by (349)-(352).

9. CONCLUDING REMARKS

9.1 CONCLUSIONS

This research has proposed theorems and developed algorithms for the robust eigenstructure assignment design of controllers for multi-input multi-output linear systems with structured state space uncertainties.

Six robust stability theorems were presented for linear time invariant systems with uncertainties. Theorem 1 presents a robustness sufficient condition which is less conservative than an earlier one proposed by Sobel et. al. [1]. Theorem 2 provides a performance robustness sufficient condition which ensures that the closed loop eigenvalues lie within chosen performance regions when the linear time invariant system is subjected to time invariant structured state space uncertainties. Theorem 3 extends the sufficient condition for robust stability of Sobel et. at. [1] to include simultaneous time varying structured state space uncertainty and output nonlinearities. Theorem 4, which was obtained in collaboration with Piou [3], presents a new robust stability condition for a linear time invariant system which is subject to linear time varying uncertainty. Theorems 5 and 6 present robustness sufficient conditions for a linear system with both time delay and structured state space uncertainty.

Four algorithms were developed and applied to aircraft and missile flight control:

- An approach to gain suppression using eigenvector derivatives was presented in Chapter 2. This approach was applied to the design of a flight control law for the F-18 HARV aircraft. We use the partial derivatives of both the eigenvalues and eigenvectors with respect to the feedback gains to determine how the feedback gain matrix should be simplified. Our result shows that, with a simpler controller, the aircraft time responses are almost identical to the responses without gain suppression.

- In Chapter 3, a robust eigenstructure assignment design method was developed. Flight control laws were designed for the F-16 pitch pointing/vertical translation maneuver and for an L-1011 stability augmentation system. In these designs, we optimized the robustness condition obtained by Sobel et. al. [1] over the eigenvalues and eigenvectors with related constraints. Therefore, we obtained feedback controllers which not only yield acceptable time responses but also ensure robust stability. We also applied dynamic compensation to the design of a control law for the L-1011 aircraft to allow the performance specifications to be achieved in the case of limited measurements.

- In Chapter 6, we proposed an algorithm for robust eigenstructure assignment which utilizes the new sufficient condition for robust stability obtained by Piou [3]. This algorithm is applied to the design of a robust autopilot for a bank-to-turn missile. In our new robust eigenstructure assignment algorithm, the eigenvector assignment is achieved via constraints on a subset of the entries of the eigenvectors. This approach is different from the more conventional approach of Andry et. al. [8] who suggest that the i-th eigenvector v_i should be chosen as the orthogonal projection of a desired eigenvector v_i^d onto the subspace spanned by the column of $(\lambda_i I-A)^{-1}B$. We have obtained a robust design which satisfies the robustness condition of Piou [3] while simultaneously yielding time responses which are close to the orthogonal projection design.

- In Chapter 7, we presented a new robust design method which was obtained in collaboration with Piou [3]. This method is applied to the design of a robust controller for the lateral dynamics of the Extended Medium Range Air to Air Technology missile and the design is compared to the earlier orthogonal projection eigenstructure assignment design which was proposed by Sobel and Cloutier [6]. In the new design method, we choose to minimize the integrated roll rate with constraints on the real part of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This design satisfies the new robustness condition while also yielding an improved transient response as compared to the design of Sobel and Cloutier [6].

The work mentioned above is documented in the following papers:

K.M. Sobel, W. Yu, and E.Y. Shapiro, "A Systematic Approach to Gain Suppression Using Eigenstructure Assignment", Proceedings of the 1989 American Control Conference, Pittsburgh, PA, 6/21-23/1989

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9.2 PROBLEMS AND RECOMMENDATIONS

Robustness sufficient conditions have been studied for systems with time varying structured state space uncertainties. Output nonlinearities are also addressed in Chapter 5. However, nonlinear state space uncertainties have not been considered. A possible extension to the robustness sufficient condition will be to consider linear time invariant systems with nonlinear state space uncertainties.

We have obtained a performance robustness sufficient condition for linear time invariant systems with time invariant structured state space uncertainties. Recently, Zhu and Johnson [33] presented new spectral canonical realizations for time varying linear systems using a unified eigenvalue concept. They showed that the new spectral canonical realizations are the natural time varying counterparts of spectral canonical realizations traditionally used for time invariant linear systems. Further research should extend the performance robustness sufficient condition to include time varying uncertainties.

In this research, we have applied robust eigenstructure assignment to obtain controllers for linearized models which are valid only on specific operating conditions. Further research should consider theorems and applications for a gain scheduled controller which is valid over a wide range of operating conditions.

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