

The Minimal Resultant and Conductor for Self Maps of the Projective Line

by

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Abstract

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Advisor: Lucien Szpiro

We develop and study the minimal resultant divisor of self-maps of the projective line over a number field or a function field and its relation to the conductor. The guiding focus is the exploration of a dynamical analog to Szpiro's Theorem (Theorem 5.1), which bounds the degree of the minimal discriminant of an elliptic curve over a function field in terms of the conductor. We also explore a question about Lattes maps: given an elliptic curve over a function field with semi-stable bad reduction, we show the associated Lattes maps for multiplication by n have unstable bad reduction for each n . We then study minimality and semi-stability, considering what conditions imply minimality and whether semi-stable models and presentations are minimal, proving results in the degree two case. We prove the singular reduction of a semi-stable presentation coincides with the bad reduction. Degree 2 maps in normal form with semi-stable bad reduction are used to construct a counterexample to a natural dynamical analog to Szpiro's Theorem. Finally, we con-

sider the notion of “critical bad reduction,” and show that a dynamical analog may still be possible using the locus of critical bad reduction to define the conductor.

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Chapter 1

Introduction

1.1 Preliminaries

Throughout, a **ring** will always mean a commutative ring with identity, and morphisms of rings take the identity to the identity. We will use k to denote an algebraically closed field. We will use F to denote an arbitrary field. We will use d to denote a non-negative integer; in some cases d may need to be 1 or greater, and this will be left unstated. We will use C to denote a Dedekind scheme (defined below).

1.2 Overview

Here we would just like to give an overview of the problem out of which this thesis grew, and attempt to summarize the basic ideas and train of thought that lead to the results in the pages that follow.

Let $K = k(C)$ be a function field over an algebraically closed field k . In 1978 Szpiro proved a theorem which bounds the degree of the minimal discriminant divisor of an ellip-

tic curve over a function field, in terms of the conductor, and improved these results later; Hindry and Silverman contributed also—for all of this, see [16], [9], [17] and [6]. We will discuss this further and more precisely in Chapter 5.

For now, the following intuitive description will hopefully suffice: the minimal discriminant is a divisor that is supported on the bad reduction of an elliptic curve over a function field or a number field. It is obtained by looking at the valuation of the discriminant form at each point, but with a minimality requirement. The minimality is with respect to choice of coordinates. To make this more explicit, one may choose a Weierstrass equation for the elliptic curve over the function field such that the coefficients are in the local ring at the point, and such that the valuation of the discriminant form is as small as possible with respect to this requirement. The valuation for the discriminant with respect to this choice is then taken to be the valuation of the minimal discriminant at that point (it is a non-negative integer).

The conductor is another divisor one can define and which is supported on the bad reduction. In fact, except in characteristic 2 and 3—where there are complications that we won't get into now—the conductor is “almost” the support of the minimal discriminant. It has a coefficient of 1 at the places of multiplicative bad reduction, and 2 at the places of additive bad reduction.

If \mathcal{D} is the minimal discriminant divisor of an elliptic curve E over a function field of

genus g , and f is the conductor, then Szpiro's result, in the characteristic 0 case, says:

$$\deg(\mathfrak{D}) \leq 6(2g - 2 + \deg(f)) \quad (1.1)$$

In the characteristic p case, a certain power of p shows up on the right hand side.

Our original goal was to consider a dynamical analogue to this result. For a dynamical system on the projective line over a function field or a number field, there is a concept of good and bad reduction, and this is measured by the “minimal resultant divisor.” We can then take the conductor to be the support of this divisor, and try to come up with a bound for the minimal resultant in terms of the conductor. We eventually showed this to be impossible in general, by constructing a family of examples, indexed by the natural numbers, in which the degree of the conductor is bounded, but the degree of the minimal resultant is unbounded. This is Theorem 6.1 below.

Nevertheless, the original idea we had in mind for proving a dynamical analogue did prove fruitful in another way. Szpiro's original proof of (1.1) involved considering the j -map:

$$j : C \rightarrow \mathbb{P}_k^1$$

obtained by evaluating the j -invariant at the points on the curve. If we consider the subset U of C where E has good reduction, this map lands on an affine subset of projective 1-space, the “ j -line,” which is a moduli space for elliptic curves. So, one way to view the map j is the following: it is obtained by looking at a map from an open subset of a curve to

a certain moduli space, and extending that map to the projective completion of the moduli space.

Szpiro's idea was to draw an analogy here with a map into the moduli space of dynamical systems, \mathcal{M}_d , to be defined below. That is, given a dynamical system over K , it is possible to define, on an open subset U (the “good reduction”) of C , a map:

$$U \rightarrow \mathcal{M}_d$$

This map originally arose in another context, in the proof of the equivalence of isotriviality and potential good reduction for self maps on \mathbb{P}^N over a function field (see [10]). The moduli space \mathcal{M}_d also has a projective completion \mathcal{M}_d^{ss} , and so we can extend the map:

$$C \rightarrow \mathcal{M}_d^{ss}$$

As remarked above, a general dynamical analogue to the formula 1.1 is not possible, and so we were unable to show such a result using this map! In Theorem 7.1 below, we do use this map, however, in order to show that the conductor is realized by a semi-stable presentation (a choice of basis for which all of the reductions are semi-stable). This can be considered as progress towards the conjecture (which we establish in the degree two case) that semi-stable presentations are minimal. This is also based on an analogy with the elliptic curves situation.

The section on Lattes maps grew out of the idea that semi-stable bad reduction of elliptic curves and semi-stable bad reduction of associated Lattes maps might be related

to each other. Interestingly, we were able to show that bad reduction of the Lattes map is *always* unstable.

The idea of evaluating the j -invariant at points on the curve also found a useful analogy in the dynamical setting. Note first that the j -invariant provides information about the discriminant precisely because it is “almost” the discriminant; it is a rational function in the coefficients of any Weierstrass model for the elliptic curve, and the denominator of this rational function is the discriminant form (up to a constant multiple). The minimality criterion, ultimately used to prove the minimality of the counterexample to the dynamical analogue of 1.1, involves evaluating a certain dynamical invariant (a symmetric function in the multipliers) whose denominator is a power of the resultant.

Initially, all of the results we obtained were done in the function field setting: we considered dynamical systems on the projective line over a function field $K = k(C)$ where k is algebraically closed and characteristic zero. Later, we extended most of these results to the case where K is the function field (i.e. fraction field of any affine open) of a Dedekind scheme of characteristic zero—this includes the case where K is a number field. It is this greater generality in which the results that follow will be presented.

1.3 Outline

The outline for this thesis is as follows. Chapters 2 through 4 provide mostly background and terminology that is used later on. Virtually all of the material specific to dynamics

can be found the work of Silverman in [15] and [13]; the geometric invariant theory background, which is also discussed in these works of Silverman, can be found in more detail in Mumford's foundational book [8], as well as in Dolgachev's book [1]. My hope with these chapters is to collect all of this material together in an order that is perhaps useful to others and that clarifies my own thinking on it. Chapters 5, 6 and 7 then build on this to present the work of the author's paper [19] with L. Szpiro and M. Tepper. The central results are in Chapters 5 and 7.

Chapter 2

Dedekind Schemes

2.1 Basic Properties

Recall that a dedekind domain is a one-dimensional Noetherian domain that is integrally closed. We begin with some well known results about dedekind domains that will be of use to us.

Theorem 2.1. *(Prime decomposition) Let A be a dedekind domain. Then the fractional ideals of A form an abelian group (under multiplication of fractional ideals) that is generated by the nonzero prime ideals of A .*

Theorem 2.2. *A one-dimensional noetherian domain A is a dedekind domain if and only if every local ring $A_{\mathfrak{p}}$ (\mathfrak{p} maximal) is a discrete valuation ring.*

Theorem 2.3. *Let A be a dedekind domain, K its field of fractions, and L a finite separable extension of K . Let B be the integral closure of A in L . Then B is a dedekind domain.*

Proofs of these results are available in standard texts on commutative algebra. In gen-

eral we will be concerned with objects defined over both function fields and number fields. One way to unify these concepts is to introduce the notion of a dedekind scheme.

Definition 2.1. *A **Dedekind scheme** C is a one dimensional integral Noetherian separated scheme, all of whose local rings discrete valuation rings. We will also sometimes call such a scheme a **curve**. We will denote the function field of C by K .*

The hypothesis that C be separated is made to ensure that the intersection of two affine opens is again affine. For simplicity, we will always assume K is characteristic 0, which implies extensions of K are separable. Really the two examples we have in mind throughout are $C = \text{Spec}(O_K)$, where O_K is the ring of integers of a number field, and C a complete non-singular curve over an algebraically closed field k of characteristic zero—also of interest are open subsets of such schemes. In the later case, $K = k(C)$ is then the function field of this curve; in general this is a transcendence degree 1 extension of k . When we talk about a point $p \in C$, we will always mean a closed point, unless otherwise specified.

For any $p \in C$, the fraction field of the local ring $\mathcal{O}_{C,p}$ is equal to K , and we denote by $v_p : K \rightarrow \mathbb{Z}$ the discrete valuation on K corresponding to $\mathcal{O}_{C,p}$. Given $f \in K$, the points p for which $v_p(f) > 0$ will be called **zeros** of f , and those for which $v_p(f) < 0$ will be called **poles**.

In the remainder of the chapter, we will prove two results which are needed chiefly in the proof of Theorem 7.1.

2.2 Finite extensions

In this section, we prove a proposition that is a familiar fact in the number field and function field settings described above.

Proposition 2.4. *Let C be a dedekind scheme, and K its function field. If K' is a finite, extension of K , then there exists a dedekind scheme C' with function field K' , and a finite to one map $C' \rightarrow C$ inducing the extension $K \hookrightarrow K'$. This scheme is formed by taking the integral closure in K' for the elements of any affine open cover of C .*

Proof. Let $C = \bigcup_i U_i$ where each $U_i = \text{Spec}(A_i)$ is affine. Given i, j , we have that, since C is separated, U_{ij} is then affine as well, equal to $\text{Spec}(A_{ij})$, and we have a map of rings

$$A_i \hookrightarrow A_{ij}$$

The map is injective since X is integral. In fact, the prime decomposition theorem implies that any open set of U_i is a co-finite collection of maximal ideals, along with the generic point, and that U_{ij} is of the form $D(f_{ij})$ for some $f_{ij} \in A_i$. Thus $A_{ij} = (A_i)_{f_{ij}}$. Let A'_i be the integral closure of A_i in K' . Since taking integral closures commutes with localization, the integral closure of A'_i in K' is $(A'_i)_{f_{ij}}$.

For each ordered pair i, j , we have the following diagram.

$$\begin{array}{ccccc}
 A'_i & \longrightarrow & (A'_i)_{f_{ij}} & \longrightarrow & K' \\
 \uparrow & & \uparrow & & \uparrow \\
 A_i & \longrightarrow & (A_i)_{f_{ij}} & \longrightarrow & K
 \end{array}$$

Thus each $\text{Spec}(A'_{ij})$ is naturally an open subscheme of both $\text{Spec}(A'_i)$ and $\text{Spec}(A'_j)$, so that the affine schemes $\text{Spec}(A'_i)$ may be patched together to form a scheme C' . By Theorem 2.3 and Theorem 2.2, C' is a dedekind scheme. Further, the maps on affine schemes induced by the inclusion maps $A_i \hookrightarrow A'_i$ obviously agree on intersections, and so patch together giving us a map of schemes $C' \rightarrow C$. The map is finite to one by the prime decomposition theorem again (applied to affine opens in C').

□

2.3 Valuative criterion

In this section, following [18], we give a proof of another basic property of dedekind schemes. Recall the notation for the functor of points applied to affine schemes: if A is a ring and X is a scheme over A , and B is an A -algebra, then $X(B) := \text{Hom}_A(\text{Spec}(B), X)$.

We also have the fundamental correspondence for morphisms to projective space:

Proposition 2.5. *Let A be a ring, and consider \mathbb{P}_A^n for some n . If X is any scheme over A , then there is a natural bijection (induced by pulling back $\mathcal{O}_{\mathbb{P}_A^n}$ and $n + 1$ global sections that are generators for the homogenous coordinate ring) between $\text{Hom}_A(X, \mathbb{P}_A^n)$ and the set of tuples $\{(L, s_0, \dots, s_n)\} / \sim$, where L is an invertible sheaf on X , s_0, \dots, s_n are global*

sections that generate L , and \sim is the equivalence relation given by: $(L, s_0, \dots, s_n) \sim (L', s'_0, \dots, s'_n)$ if and only if there is an isomorphism of line bundles $L \cong L'$ sending s_i to s'_i for each i .

Proof. See [18] (I.5.10) or [4] (II.7.1). □

Theorem 2.6. *Let A be a ring, B a dedekind domain that is an A -algebra, and K the field of fractions of B . Let X be a closed subscheme of \mathbb{P}_A^n for some n . Then there is natural bijection:*

$$\alpha : X(B) \rightarrow X(K)$$

More generally if C is a dedekind scheme over A with generic point $\text{Spec}(K)$, there is a natural bijection:

$$\alpha : X(C) \rightarrow X(K)$$

Proof. We do first the affine case; the general case will follow. The map α is obtained by composition with the map $\text{Spec}(K) \rightarrow \text{Spec}(B)$. We must show that α is a bijection.

To show surjectivity, let $f \in X(K)$. After composing with the inclusion $i : X \hookrightarrow \mathbb{P}_A^n$, by Proposition 2.5, this $i \circ f$ corresponds to an equivalence class under the described relation, and we can chose a representative, which is an invertible sheaf on $\text{Spec}(K)$ along with global sections that generate it. This is just a one dimensional vector space V over K , along with $n + 1$ elements (v_0, \dots, v_n) which are not all zero. Choose a basis for V and write each $v_i = r_i v$. Then since B has fraction field K , we can clear denominators to get an

$n+1$ -tuple (b_0, \dots, b_n) of elements in B . This collection of elements generates a nontrivial ideal I of B , and it follows from Theorem 2.1 that I is an *invertible* B module, which is the same thing as an invertible sheaf on $\text{Spec}(B)$; (b_0, \dots, b_n) are then global sections which obviously generate, and so this data gives us an element of $\mathbb{P}_A^n(B)$.

It is routine to check that this does not depend on the choices we made, and it is clear that this point corresponds to $i \circ f$ after composition with the inclusion $\text{Spec}(K) \hookrightarrow \text{Spec}(B)$. It remains to show that the morphism $\text{Spec}(B) \rightarrow \mathbb{P}_A^n$ factors through X . To show this we must verify that the coordinates (b_0, \dots, b_n) satisfy the homogenous equations defining X . But this is obvious since $(b_0, \dots, b_n) = (dr_0, \dots, dr_n)$ for some $d \in K$, and (r_0, \dots, r_n) satisfy these equations.

Now we show injectivity. Suppose $f, g \in X(B)$ are distinct morphisms. Then composing with the inclusion we obtain distinct morphisms in $\mathbb{P}_A^n(B)$, which we relabel to f, g . Then we may reduce to the case where B is a discrete valuation ring (dvr) as follows: if there is a prime \mathfrak{p} where f, g disagree, as maps of topological spaces, then pre-composing with the natural map $\text{Spec}(B_{\mathfrak{p}}) \hookrightarrow \text{Spec}(B)$ clearly gives two distinct morphisms. If there is no such prime, then we consider the maps on sheaves $f^\#, g^\# : \mathcal{O}_{\mathbb{P}_A^n} \rightarrow f_* \mathcal{O}_B$. Note that this makes sense because the push-forward sheaf of \mathcal{O}_B is the same for both f, g , since they agree as maps on topological spaces. These maps of sheaves must be different. This implies that on some open subset they are different, which in turn implies that they disagree on the stalk of a point whose pre-image contains a closed point \mathfrak{p} , and this will imply that

the maps that come again from pre composing with $\text{Spec}(B_{\mathfrak{p}}) \hookrightarrow \text{Spec}(B)$ are distinct.

Thus we have reduced to the case where B is a dvr. Now, the class group of a dvr is trivial, so every line bundle is isomorphic to the ring B itself. Consider our two distinct points $f, g \in \mathbb{P}_A^n(B)$. These correspond to the data $(A, x_0, \dots, x_n), (A, y_0, \dots, y_n)$, where x_i, y_j are elements of A . The nowhere vanishing condition means that at least one x_i and y_j is a unit. Now suppose they give us the same point; this means there is an isomorphism of K (as a one dimensional vector space) such that x_i maps to y_i . Such an isomorphism must be multiplication by an element λ of K^\times : $y_i = \lambda x_i$ for all i . Now either λ or its inverse is in B , since B is a dvr. Since we could have made our isomorphism go the other way, it is no loss of generality to suppose that $\lambda \in B$. But, in order for one of the x_i to be a unit, we must in fact have that λ is a unit. Thus λ also gives us an isomorphism of A sending x_i to y_i , implying that f and g are the same, a contradiction.

To get the last statement in the theorem, just apply the result to an affine open cover of C . □

Chapter 3

The Parameter Space of Dynamical Systems

3.1 Morphisms on the projective line

Let F be a field. We initially define the projective line in a way that is “basis independent” as follows: Let V be a two dimensional vector space, and set $\mathbb{P}_F^1 := \text{Proj}(\text{Sym}(V^*))$. Any choice of basis v_X, v_Y of V then induces a dual basis X, Y which generate the homogeneous coordinate ring $\text{Sym}(V^*) = F[X, Y]$ and give us coordinates $[X, Y]$ that allow us to concretely write down any point in \mathbb{P}_F^1 : if $p \in \mathbb{P}_F^1$, its representation with respect to these coordinates is the projective pair $[X(p), Y(p)]$.

This independence will be important for us later. For now, however, we suppose chosen already fixed generators X, Y for the homogeneous coordinate ring $F[X, Y]$. Let $\varphi : \mathbb{P}_F^1 \rightarrow \mathbb{P}_F^1$ be a morphism of schemes. By Proposition 2.5, the pullback $\varphi^*O(1)$ is a line bundle on \mathbb{P}_F^1 , and hence is isomorphic to $O(d)$ for some d ([4] 6.17 p.145). Note that, since φ^*X, φ^*Y are nowhere vanishing global sections, d is non-negative.

Looking at φ^*X, φ^*Y in $O(d)$ under this isomorphism we obtain two homogeneous polynomials of degree d in the variables X, Y . If we choose a different isomorphism with $O(d)$, the composition gives an automorphism of $O(d)$, which must be multiplication by a scalar (a non-zero element of F), since

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_F^1}}(O(d), O(d)) = F$$

(see [4] proof of 6.12 p. 143). Thus our sections are determined up to multiplication by a scalar, and so their coefficients are a well-defined point in $\mathbb{P}^{2d+1}(F)$, where $\mathbb{P}^{2d+1} := \mathrm{Proj}(\mathbb{Z}[A_0, \dots, A_d, B_0, \dots, B_d])$. Thus we may make the following definition.

Definition 3.1. *Let $\varphi : \mathbb{P}_F^1 \rightarrow \mathbb{P}_F^1$ be a morphism of degree d . A **presentation** Φ of φ , given by the choice of coordinates $[X, Y]$, is the point $[\mathbf{a}, \mathbf{b}] = [a_0, \dots, a_d, b_0, \dots, b_d] \in \mathbb{P}^{2d+1}(F)$ given by the coefficients of a pair of homogeneous polynomials $F_{\mathbf{a}} = a_0X^d + a_1X^{d-1}Y + \dots + a_dY^d$, $F_{\mathbf{b}} = b_0X^d + b_1X^{d-1}Y + \dots + b_dY^d$ defining the morphism φ .*

The condition that the polynomials $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ be non-vanishing sections means that, for every point in \mathbb{P}_F^1 , at least one of the polynomials is a unit when restricted to the stalk of $O(d)$ at that point. This is equivalent to the polynomials not sharing a common homogeneous factor. Another way to characterize this is that, as dehomogenized polynomials in the variable $x = X/Y$, they have no common factor, and also they have no common factor of the variable Y , as homogenous polynomials. Equivalent to this is the condition that they not have a common root over $\mathbb{P}^1(\bar{F})$. We can exactly characterize these equivalent

conditions by the resultant form, applied to the coefficients of these polynomials. We will do this below.

What this all means is that morphisms of degree d are parameterized by the projective coefficients of the defining polynomials, subject to the restriction that the resultant form not vanish. This perspective generalizes significantly, via the work of Silverman [15], as we will see below. First we must define the resultant.

3.1.1 The Resultant

Consider the ring $\mathbb{Z}[A_0, \dots, A_d, B_0, \dots, B_d]$, where A_i, B_j are indeterminants.

Definition 3.2. *The **resultant** is the polynomial given by taking the determinant of the $2d \times 2d$ matrix:*

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} & A_d & 0 & \cdots & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_{d-1} & A_d & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & \cdots & A_0 & A_1 & \cdots & \cdots & A_{d-1} & A_d \\ B_0 & B_1 & \cdots & B_{d-1} & B_d & 0 & \cdots & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_{d-1} & B_d & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & \cdots & \cdots & B_0 & B_1 & \cdots & \cdots & B_{d-1} & B_d \end{pmatrix}$$

We will denote this polynomial by ρ . We will sometimes need to refer to the variables of this polynomial, a tuple which we will denote by (\mathbf{A}, \mathbf{B}) and also possibly by (\mathbf{a}, \mathbf{b}) if it's clear from context that we are dealing with the general case and not a specific point.

The resultant can be defined more generally for two polynomials of different degrees, but we will not need this.

Theorem 3.1. *Let F be a field, and $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ two homogeneous polynomials of degree d with coefficients in F . Then the following statements are equivalent.*

1. $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have no common factor.
2. $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ do not vanish at a common point of $\mathbb{P}^1(\bar{F})$
3. $\rho(\mathbf{a}, \mathbf{b}) \neq 0$

Proof. See [13] Proposition 2.13 and its proof. □

There is a more intuitive characterization of the resultant that makes the above properties transparent.

Theorem 3.2. *Suppose $F_{\mathbf{a}}, F_{\mathbf{b}}$ have the factorizations*

$$F_{\mathbf{a}} = a_0 \left(\prod_i (X - \alpha_i Y) \right), F_{\mathbf{b}} = b_0 \left(\prod_j (X - \beta_j Y) \right)$$

over some algebraic extension of F . Then

$$\rho(\mathbf{a}, \mathbf{b}) = a_0^d b_0^d \prod_{i,j} (\alpha_i - \beta_j)$$

Proof. See [13] Proposition 2.13 (b). □

3.1.2 Parameterizing Morphisms

From the formula, it can be seen that the resultant ρ is homogeneous of degree $2d$ in the variables $A_0, \dots, A_d, B_0, \dots, B_d$, and that it has coefficients in \mathbb{Z} . Thus its complement defines an open subset of \mathbb{P}^{2d+1} :

Definition 3.3. *The space of rational maps of degree d is $\text{Rat}_d := \mathbb{P}^{2d+1} \setminus V(\rho)$.*

This affine scheme is isomorphic to $\text{Spec}(R)$, where

$$R = \mathbb{Z}[A_0, \dots, A_d, B_0, \dots, B_d]_{(\rho)}$$

is the subring of elements of degree zero in the localization $\mathbb{Z}[A_0, \dots, A_d, B_0, \dots, B_d]_{\rho}$.

In [15], it is shown that Rat_d parameterizes all rational maps of degree d . More precisely, for every scheme S , define a set:

$$\underline{\text{Rat}}_d(S) = \{S\text{-morphisms } \varphi : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^1 \mid \varphi^* \mathcal{O}(1) \cong \mathcal{O}(d)\}$$

This actually defines a functor from schemes to sets via the pullback functor. We then have:

Theorem 3.3. *(Silverman) The scheme Rat_d represents the functor $\underline{\text{Rat}}_d$.*

Proof. See [15], Theorem 3.1. □

We really only need this result for fields F , but it is worth having in mind the general fact.

3.1.3 Coordinate Choices

Recall that, above, we had fixed a basis X, Y for the dual space V^* , which allowed us to write down points in projective space by evaluating these functions at those points; thus a point $p \in \mathbb{P}_F^1$ can be represented as homogeneous pair $[X(p), Y(p)]$. Pulling back the sections X, Y of $O_{\mathbb{P}_F^1}(1)$ gave us two homogeneous polynomials $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$, and this gives us a concrete representation of our map φ with respect to this basis. So clearly the point (\mathbf{a}, \mathbf{b}) depends on this choice of basis—if we want to write the map concretely for a different choice of basis X', Y' , we will need different polynomials $F_{\mathbf{a}'}, F_{\mathbf{b}'}$. How are the coefficients $(\mathbf{a}', \mathbf{b}')$ related to (\mathbf{a}, \mathbf{b}) ?

To answer this, let $X = \alpha X' + \beta Y', Y = \gamma X' + \delta Y'$ where $\Delta = \alpha\delta - \beta\gamma \neq 0$. We know what φ does to $[X, Y]$, so we have

$$\begin{aligned} [X, Y] &\mapsto [F_{\mathbf{a}}(X, Y), F_{\mathbf{b}}(X, Y)] \\ &= [F_{\mathbf{a}}(\alpha X' + \beta Y', \gamma X' + \delta Y'), F_{\mathbf{b}}(\alpha X' + \beta Y', \gamma X' + \delta Y')] \end{aligned}$$

So if our *input* is in terms of X', Y' , then the right hand side here can tell us the image of φ in terms of the homogeneous coordinates $[X, Y]$. However, to write it with respect to $[X', Y']$ instead, we must take the inverse linear transformation. In fact, to avoid having to do any division, we will use the adjoint instead—since our coordinates are projective, this has the same effect. Thus with respect to the coordinates $[X', Y']$, φ is:

$$[F_{\mathbf{a}'}(X', Y'), F_{\mathbf{b}'}(X', Y')]$$

where

$$F_{\mathbf{a}'}(X', Y') = \delta F_{\mathbf{a}}(\alpha X' + \beta Y', \gamma X' + \delta Y') - \beta F_{\mathbf{b}}(\alpha X' + \beta Y', \gamma X' + \delta Y')$$

and

$$F_{\mathbf{b}'}(X', Y') = -\gamma F_{\mathbf{a}}(\alpha X' + \beta Y', \gamma X' + \delta Y') + \alpha F_{\mathbf{b}}(\alpha X' + \beta Y', \gamma X' + \delta Y')$$

If we look at what has happened here algebraically, we see that $(\mathbf{a}', \mathbf{b}')$ are the same (up to scalar multiple) as the coefficients for $\Gamma^{-1} \circ \varphi \circ \Gamma$ with respect to the coordinates $[X, Y]$, where $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. This means that the entire orbit of φ , with respect to the action of conjugation by degree 1 maps, is somehow intrinsic to φ . To put it another way, we might want to identify φ with a point in a moduli space under this action. To what extent we can do this is a central focus of [15]. These results have since been generalized to \mathbb{P}^N in [10] and [7].

We will now explore the case of \mathbb{P}^1 , beginning with a survey of the necessary results from geometric invariant theory.

Chapter 4

GIT and the Moduli Space of Dynamical Systems

Geometric invariant theory has to do with taking quotients in the algebraic category: we have a group acting on a scheme (what this means will be defined below), and we want to know if there is a suitable notion of a “quotient space” for this action. As noted above, we are concerned with the application of this machinery to a particular situation (the space of rational maps on \mathbb{P}^1), so in this chapter we will simply review the necessary definitions and theorems. A detailed treatment is available in [8] and [1].

4.1 Group Schemes and Actions

A group scheme over some base scheme S is a scheme which, via the functor of points, yields a group for each S -scheme T . The precise definition is as follows.

Definition 4.1. *A group scheme over S is a scheme G over S (with structure map $\pi : G \rightarrow S$) along with morphisms $\mu : G \times_S G \rightarrow G$, $\beta : G \rightarrow G$, and $e : S \rightarrow G$ such that:*

$$\begin{array}{ccc}
 G \times_S G \times_S G & \xrightarrow{\mu \times 1_G} & G \times_S G \\
 \downarrow 1_G \times \mu & & \downarrow \mu \\
 G \times_S G & \xrightarrow{\mu} & G
 \end{array}$$

commutes, the compositions:

$$G \xrightarrow{\Delta} G \times_S G \xrightarrow{1_G \times \beta} G \times_S G \xrightarrow{\mu} G$$

$$G \xrightarrow{\Delta} G \times_S G \xrightarrow{\beta \times 1_G} G \times_S G \xrightarrow{\mu} G$$

are equal to $\pi \circ e$, and the compositions:

$$G \xlongequal{\quad} S \times_S G \xrightarrow{e \times 1_G} G \times_S G \xrightarrow{\mu} G$$

$$G \xlongequal{\quad} G \times_S S \xrightarrow{1_G \times e} G \times_S G \xrightarrow{\mu} G$$

equal the identity map on G .

These three conditions correspond to the usual group axioms: associativity, inverses, and identity. It is a good exercise to check that the above definition makes $G(T)$ into a group for each S -scheme T . One can phrase this definition more elegantly in the language of category theory—see [2] p. 258.

4.1.1 Key examples: the linear groups

The groups schemes we are concerned with are linear groups schemes: GL_2 , PGL_2 and SL_2 . Since it is no more work to do so, let us define them for general n .

Definition 4.2. Let $GL_n = \text{Spec}(\mathbb{Z}[x_{ij}]_{\det(x_{ij})})$ where x_{ij} are indeterminants for all i, j between 1 and n . Similarly, let $PGL_n = \text{Proj}(\mathbb{Z}[x_{ij}] - V(\det(x_{ij})))$. Let SL_n be the closed subscheme of GL_n defined by the equation $\det(x_{ij}) = 1$.

Each of GL_n , PGL_n , and SL_n , when given the appropriate maps as above, is a group scheme. The maps are what we ought to expect them to be; all the operations in question are defined algebraically, and the right things are invertible, so that everything works out in terms of constructing these maps as morphisms of schemes.

4.1.2 Actions and Quotients

As with the construction of group schemes themselves, we define what it means for a group scheme to act on a scheme by providing diagrams which will induce an action of $G(T)$ on $X(T)$ for every S -scheme T , via the functor of points.

Definition 4.3. Let G be a group scheme over S , and let X be a scheme over S . We say that G acts on X if there is an S -morphism $\sigma : G \times_S X \rightarrow X$ such that the following diagram

commutes:

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times \sigma} & G \times_S X \\ \downarrow \mu \times 1_X & & \downarrow \sigma \\ G \times_S X & \xrightarrow{\sigma} & X \end{array}$$

and such that the composition:

$$X \xlongequal{\quad} S \times_S X \xrightarrow{e \times 1_X} G \times_S X \xrightarrow{\sigma} X$$

equals the identity on X .

The next step is to define a quotient by such an action. We will be concerned with two types of quotients. The first type, a categorical quotient, has a relatively simple and natural definition. The second, a geometric quotient, is in some ways “better” as a quotient, as we will see, but it has a more technical definition, so unfortunately its definition will seem to come out of nowhere. We will repair this, somewhat, by providing a description of some of the nice properties that a geometric quotient satisfies.

Definition 4.4. *Let G act on X (as schemes over S). A **categorical quotient** of X by G is an S -scheme Y and a morphism $\pi : X \rightarrow Y$ that is universal for the following diagram (which commutes):*

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow p_X & & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

where p_X is the projection to X .

Let now Ψ be the map $(\sigma, p_X) : G \times_S X \rightarrow X \times_S X$.

Definition 4.5. Let G act on X (as schemes over S). A **geometric quotient** of X by G is an S -scheme Y and a morphism $\pi : X \rightarrow Y$ that fits into a commutative diagram:

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow p_X & & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

where p_X is the projection to X . In addition we require:

1. π is surjective and the image of Ψ is $X \times_Y X$.
2. A subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open in X .
3. The structure sheaf \mathcal{O}_Y is the subsheaf of $\pi_*(\mathcal{O}_X)$ consisting of invariant functions.

Precisely, given an open subset U of Y , $\Gamma(U, \mathcal{O}_Y)$ consists of those $f \in \Gamma(U, \pi_*(\mathcal{O}_X))$

such that the following diagram commutes:

$$\begin{array}{ccc} G \times \pi^{-1}(U) & \xrightarrow{\sigma} & \pi^{-1}(U) \\ \downarrow p_2 & & \downarrow F \\ \pi^{-1}(U) & \xrightarrow{F} & \mathbb{A}^1 \end{array}$$

where F is the morphism defined by f .

In giving these definitions, Mumford notes ([8] p. 4) that for the right sort of schemes, the first condition guarantees that the orbits of geometric points over an algebraically closed field are in one to one correspondence with the geometric points of Y ; this is one of the key properties that makes a geometric quotient nice.

4.2 The Construction of the Moduli Space

We will now apply these definitions to explain Silverman’s construction of \mathcal{M}_d , the moduli space of rational maps of degree d . This will be a scheme defined over \mathbb{Z} that is a geometric quotient of Rat_d . We will also describe the “semistable completion” of this space, which is a categorical quotient. Stability and semi-stability are important notions in geometric invariant theory; we will not be able to do the general notions justice here, but we will supply concrete descriptions of what they mean in the example of concern for us, and these will play a central role in what follows.

4.2.1 The Action

Above, we described a natural $\text{PGL}_2(F)$ action on presentations that is induced by choice of basis for generators of the homogeneous coordinate ring. This action is defined over \mathbb{Z} , and also makes sense as a GL_2 and SL_2 action. Let us make all of this precise. We define a GL_2 action on the projective space \mathbb{P}^{2d+1} as follows. If $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_2$ and (X, Y) are coordinates for \mathbb{A}^2 , define $(X, Y) \cdot \Gamma$ to be $(\alpha X + \beta Y, \gamma X + \delta Y)$.

If $[\mathbf{a}, \mathbf{b}] \in \mathbb{P}^{2d+1}$, define a GL_2 action on the affine representative (\mathbf{a}, \mathbf{b}) by sending it to the new the coefficients $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ obtained from the following pair of polynomials:

$$(F_{\mathbf{a}}((X, Y) \cdot \Gamma), F_{\mathbf{b}}((X, Y) \cdot \Gamma)) \cdot \Gamma^{\text{adj}}$$

This is a group action on \mathbb{A}^{2d+2} . It descends to a well defined group action when passing to PGL_2 and the projective space \mathbb{P}^{2d+1} . We denote this action by $\Phi^\Gamma = [\mathbf{a}^\Gamma, \mathbf{b}^\Gamma]$. In light

of the remarks at the end of Chapter 3 we would expect this action to send Rat_d to itself. In fact, this follows also from the following formula, which is not difficult to verify, for how the resultant form transforms under the GL_2 action:

$$\rho(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma) = \det(\Gamma)^{d^2+d} \rho(\mathbf{a}, \mathbf{b}) \quad (4.1)$$

This formula will be of use to us later on.

The coefficients $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ are given as homogeneous polynomials in terms of the coefficients (\mathbf{a}, \mathbf{b}) and $\alpha, \beta, \gamma, \delta$. It will be convenient for us to fix some notation for the generic case of this:

Notation 4.6. For the indeterminants A_0, \dots, B_d , recall our previous notation: $(\mathbf{A}, \mathbf{B}) = (A_0, \dots, B_d)$, and then set:

$$(A'_0, \dots, A'_d, B'_0, \dots, B'_d) = (\mathbf{A}^\Gamma, \mathbf{B}^\Gamma) \quad (4.2)$$

4.2.2 The Quotient Space of Rat_d

The action we have just described is an action of group schemes in the sense defined above. When we use SL_2 as the group we are acting with, Formula 4.1 shows us that the resultant form is invariant under this action. With respect to this action, there are the semi-stable and stable spaces $(\mathbb{P}^{2d+1})^s \subseteq (\mathbb{P}^{2d+1})^{ss}$; these are dense open subschemes of \mathbb{P}^{2d+1} , and each is fixed (as a set) with respect to the SL_2 action. Points which fall outside of $(\mathbb{P}^{2d+1})^s$ are sometimes called **not stable** and those which fall outside of $(\mathbb{P}^{2d+1})^{ss}$ are called **unstable**.

The invariance of the resultant under the SL_2 action implies that Rat_d is contained in $(\mathbb{P}^{2d+1})^s$. We will give a concrete characterization of the geometric points of the stable and semi-stable spaces below—this characterization will be our main tool in dealing with stability and semi-stability. First, however, let us state Silverman’s main results about the quotient spaces under these actions. Recall our notation above: $R = \mathbb{Z}[A_0, \dots, B_d]_{(\rho)}$, the elements of degree zero in $\mathbb{Z}[A_0, \dots, B_d]_{\rho}$.

Definition 4.7. *The ring of invariant functions for the SL_2 action on Rat_d is defined to be*

$$R^{\mathrm{SL}_2} = \{f \in R \mid f^{\Gamma} = f\}$$

where f^{Γ} is defined by replacing the variables A_i, B_j with forms A'_i, B'_j as in Formula 4.2 above. We also define

$$\mathbb{Z}[A_0, \dots, B_d]^{\mathrm{SL}_2}$$

similarly.

Theorem 4.1. (Silverman)

1. *There exists a geometric quotient \mathcal{M}_d of Rat_d under the SL_2 action defined over \mathbb{Z} .*

Also, there is a geometric quotient \mathcal{M}_d^s of $(\mathbb{P}^{2d+1})^s$, and a categorical quotient \mathcal{M}_d^{ss} of $(\mathbb{P}^{2d+1})^{ss}$, and the following chain of dense open inclusions

$$\mathcal{M}_d \subseteq \mathcal{M}_d^s \subseteq \mathcal{M}_d^{ss}$$

2. *\mathcal{M}_d is affine and \mathcal{M}_d^{ss} is projective.*

3. \mathcal{M}_d is given concretely as Spec of the ring of invariant functions under the action:

$$\mathcal{M}_d = \text{Spec}(\mathbb{R}^{\text{SL}_2}). \quad \mathcal{M}_d^{\text{SS}} \text{ is given concretely as } \text{Proj}(\mathbb{Z}[A_0, \dots, B_d]^{\text{SL}_2}).$$

4. If k is algebraically closed, then the points $\mathcal{M}_d^{\text{SS}}$ are in bijective correspondence with the $\text{SL}_2(k)$ orbits of the points of $\text{Rat}_d(k)$.

Proof. See [15] Theorem 1.3, the comments that follow, and Theorem 2.1. For the projectivity of $(\mathcal{M}_d)^{\text{SS}}$ see also [12], Theorem 4 (v). \square

When k is algebraically closed, the $\text{SL}_2(k)$ and $\text{PGL}_2(k)$ orbits of $\mathbb{P}^{2d+1}(k)$ coincide, and so we may work over $\text{PGL}_2(k)$ instead. In general we'd always like to work over PGL_2 , since this is the group whose action arises naturally in the study of dynamical systems.

4.2.3 The Numerical Criterion

There is a “numerical criterion” of Mumford which determines when a geometric point is stable or semi-stable. Often it is the case that this criterion works out to be something quite concrete; this is the case in our situation.

Theorem 4.2. (*Silverman*) *Let k be algebraically closed.*

(1) *A point of $\mathbb{P}^{2d+1}(k)$ is not in $(\mathbb{P}^{2d+1})^{\text{SS}}(k)$ if and only if, after a $\text{SL}_2(k)$ conjugation, it satisfies*

$$a_i = 0, \quad i \leq \frac{d-1}{2}, \quad b_j = 0, \quad j \leq \frac{d+1}{2}$$

(2) A point of $\mathbb{P}^{2d+1}(k)$ is not in $(\mathbb{P}^{2d+1})^s(k)$ if and only if, after an $SL_2(k)$ conjugation, it satisfies

$$a_i = 0, i < \frac{d-1}{2}, b_j = 0, j < \frac{d+1}{2}$$

Proof. See [15] Proposition 2.2. □

There is a reformulation of this numerical criterion that is sometimes more intuitive for applications. We first came across the statement of this criterion in [11], though it seems well known.

Theorem 4.3. *Let k be algebraically closed.*

Let $[\mathbf{a}, \mathbf{b}] \in \mathbb{P}^{2d+1}(k)$. Let ψ be the rational map on \mathbb{P}_k^1 obtained by cancelation of the greatest common factor of $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$. Note that ψ has degree $d - D$, where D is the degree of the greatest common factor.

(1) *Suppose $d = 2r$ is even. Then $[\mathbf{a}, \mathbf{b}]$ is unstable if and only if it is not stable, which happens if and only if either*

(a) *$F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have a common root in \mathbb{P}_k^1 of order $r + 1$ or*

(b) *$F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have a common root in \mathbb{P}_k^1 of order r which is also a fixed point of ψ .*

(2) *Suppose $d = 2r + 1$ is odd. Then $[\mathbf{a}, \mathbf{b}]$ is unstable if and only if either*

(a) *$F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have a common root in \mathbb{P}_k^1 of order $r + 2$ or*

(b) $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have a common root in \mathbb{P}_k^1 of order $r + 1$ which is also a fixed point of ψ .

Meanwhile, $[\mathbf{a}, \mathbf{b}]$ is not stable if and only if either

(a') $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have a common root in \mathbb{P}_k^1 of order $r + 1$ or

(b') $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ have a common root in \mathbb{P}_k^1 of order r which is also a fixed point of ψ .

Proof. (sketch)

We will sketch the idea for the case where d is even. The other cases are similar.

In the even case, the stable locus coincides with the semi-stable locus, and Theorem 4.2 says that a point $[\mathbf{a}, \mathbf{b}]$ is outside of this set if and only if one of its conjugates $[\mathbf{a}', \mathbf{b}']$, under the $PGL_2(k)$ action, has the coefficients $a'_0, \dots, a'_{r-1}, b'_0, \dots, b'_r$ vanishing. This happens if and only if Y is a common root of $F_{\mathbf{a}'}, F_{\mathbf{b}'}$ of order at least r . It is easy to see that a'_r is nonzero also if and only the common root Y is of order exactly r and $[1, 0]$ is a fixed point of the map obtained by cancelation. Checking that the group action preserves these properties (but changes the common root and fixed point in question), one can verify that statement (1) follows. \square

We will sometimes work with presentations defined over non-algebraically closed fields (for example, the residue fields of the ring of integers of a number field). Stability or semi-stability in this case is determined by considering these points over the algebraic

closure of the field in question and applying Theorem 4.2 or Theorem 4.3.

4.3 The Symmetric Functions of the Multipliers

The symmetric functions of the multipliers of the fixed points and periodic points of a dynamical system provide a way of finding regular functions on the space \mathcal{M}_d ; they are explored in [15] and [13]. There is a key result about these that we will need to prove some of the main theorems in Chapter 7, and we will review this result in this section.

Let φ be a morphism of degree d over a field F . Choosing coordinates $[X, Y]$ such that ∞ is not a fixed point, and dehomogenizing with $z = X/Y$, the multipliers of φ are defined to be the derivative $\varphi'(z)$ evaluated at each fixed point $\alpha_1, \dots, \alpha_{d+1}$ (counted with multiplicities):

$$\lambda_{\varphi,i} = \varphi'(\alpha_i)$$

A chain rule calculation shows that this does not depend on the original choice of coordinates, so the set of multipliers, counted by multiplicities, is $\mathrm{PGL}_2(F)$ invariant. Thus taking the elementary symmetric functions of these, we get another collection of elements that does not depend on the choice of coordinates.

Notation 4.8. We take $\sigma_i(\varphi)$ to be the i -th coefficient of the polynomial

$$\prod_j (T + \lambda_{\varphi,j}) = \sum_{i=0}^{d+1} \sigma_i(\varphi) T^{d+1-i}$$

We may do the same thing for the periodic points of φ . Let φ^n be the n -th iterate of φ .

Notation 4.9. Let $n \geq 1$. We denote by $\sigma_{i,n}(\varphi)$ the i coefficient of the polynomial

$$\prod_j (T + \lambda_{\varphi^n, j}) = \sum_{i=0}^{d^n+1} \sigma_{i,n}(\varphi) T^{d^n+1-i}$$

The nice thing about the $\sigma_{i,n}(\varphi)$, in addition to the fact that they don't depend on any coordinate choices, is that they can be written completely in terms of the coefficients of the map (here with respect to a presentation corresponding to a choice of basis), without having to take any roots. That is, generically they are actually defined over $\mathbb{Q}(\mathbf{A}, \mathbf{B})$, which is not usually the case for the multipliers themselves. In fact, they admit a nice form in terms of the resultant, and it is chiefly this fact that we will use in Chapter 6. This form is expressed in the following theorem.

Theorem 4.4. For each $i, n \in \mathbb{N}$, there is an element $\sigma_{i,n}$ of $\mathcal{O}_{\text{Rat}_d} = \mathbb{Z}[\mathbf{a}, \mathbf{b}]_{(\rho)}$ such that $\sigma_{i,n}(\varphi)$ is $\sigma_{i,n}$ evaluated at any coefficients (\mathbf{a}, \mathbf{b}) of any presentation $[\mathbf{a}, \mathbf{b}]$ of φ corresponding to a choice of coordinates $[X, Y]$. To put it another way,

$$\sigma_{i,n} = \frac{P_{i,n}(\mathbf{A}, \mathbf{B})}{\rho^m} \tag{4.3}$$

where $m \in \mathbb{N}$ and $P_{i,n}$ is homogeneous of degree $2dm$ ($2d$ is the degree of the resultant). Furthermore, $\sigma_{i,n}$, as a function, is constant on PGL_2 or GL_2 equivalence classes for their respective actions.

Because of the PGL_2 , GL_2 , and hence SL_2 invariance, the $\sigma_{i,n}$ are global sections of $\mathcal{O}_{\mathcal{M}_d}$.

Chapter 5

Elliptic Curves

This chapter will describe some old and new results on elliptic curves over function fields.

We begin with a summary of Szpiro's theorem, which is the inspiration for much of what follows in the rest of this thesis.

5.1 Szpiro's Theorem

Let k be an algebraically closed field of characteristic 0. In 1978 Szpiro proved a theorem which bounds the minimal discriminant of a semi-stable elliptic surface in terms of the conductor [16] (Lemma 3.2.2 and Proposition 4.2), [17].

Theorem 5.1. *Let $f : \mathcal{E} \rightarrow C$ be a proper and flat morphism of a projective surface \mathcal{E} , smooth over k , to a curve C , projective, smooth, of genus q and geometrically connected over k . Suppose that the generic fiber of f is an elliptic curve, E , smooth and geometrically connected over the function field of C . Suppose, further, that f is not isotrivial and the degenerate fibers are semi-stable. Then, if $\Delta_{\mathcal{E}}$ is the discriminant divisor of \mathcal{E} and if s is*

the number of geometric points of C where the fibers are not smooth, one has:

$$\deg(\Delta_{\mathcal{E}}) \leq 6(2q - 2 + s).$$

Theorem 5.1, as originally stated, actually gives a bound for the characteristic p case as well. The fact that the bad fibers are assumed to be semi-stable implies that the discriminant divisor, bounded by the theorem, is minimal, in the sense of the following definition.

Definition 5.1. Let $K = k(C)$ be a function field and E an elliptic curve over K , given by a Weierstrass equation $y^2 = x^3 + Ax + B$. For each $p \in C$, let v_p be the valuation associated to the discrete valuation ring $\mathcal{O}_{C,p}$. Let $\Delta = 4A^3 + 27B^2$. Let

$$n_p = v_p(\Delta) - 12 \left\lfloor \frac{\min(v_p(A^3), v_p(B^2))}{12} \right\rfloor$$

where $\lfloor \cdot \rfloor$ denotes the floor function. The minimal discriminant of E/K is defined to be the divisor:

$$\mathfrak{D}_{E/K} = \sum_{p \in C} n_p [p]$$

The generic fiber of $f: \mathcal{E} \rightarrow C$ is an elliptic curve, E over K , and if we take the minimal discriminant of this curve, we get the discriminant divisor of \mathcal{E} mentioned in the theorem. This divisor has support at precisely the bad reduction of the elliptic curve. Intuitively, these are the points on C where the discriminant vanishes and thus the fibers given by the resulting Weierstrass equation are singular. We define the conductor to be the divisor $\mathfrak{f}_{E/K}$ associated to these points, without any multiplicities (when reduction is semi-stable,

this definition agrees with the “classical” definition of the conductor). Then Theorem 5.1 basically asserts that, as the degree of the conductor goes to infinity, the ratio between the degree of these two divisors is bounded by a factor of 6. In [9], the result is strengthened to include the case where reduction is not assumed to be semi-stable.

The proof of Theorem 5.1 depends heavily on the properties of the relative dualizing sheaf of a family of nodal curves; see [5] for basic information. Another proof of the same inequality, for the characteristic zero case (and also valid for characteristic p not equal to 2 or 3), uses different techniques; see [6] (p. 434).

In both proofs, a key ingredient is the j map

$$j: C \rightarrow \mathbb{P}_k^1$$

obtained by evaluating the j -invariant. This gives us a map on a dense open subset U of the curve to $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$, and we can extend this map to the projective variety \mathbb{P}_k^1 uniquely to a map on the whole curve C . If we think of an elliptic curve over a function field or an associated elliptic surface as a family of curves, parameterized by C , and almost all of which are elliptic curves, then this map is obtained by extending the original map on U to a projective completion of the *moduli space* of elliptic curves. Below (Theorem 7.1), we will apply a similar idea to the dynamical setting via the moduli space of dynamical systems.

We will also explore a “dynamical analogue” to Theorem 5.1. In the elliptic curves setting, the minimal discriminant, described above, is a divisor on the curve C which

is supported on the bad reduction of the elliptic curve. It is obtained by looking at the valuation of the discriminant form in the coefficients of the Weierstrass equation at the different points of C , but possibly with some change of variables made in order to make the value minimal. For dynamical systems (on \mathbb{P}_1), the natural analogue to the discriminant is the resultant: it tells us when our coefficients *do not* give us a dynamical system of some fixed degree d . In a similar way to the above notion of bad reduction for elliptic curves, we can formulate what it means for a dynamical system over K , a number field or function field, to have bad reduction, and we can construct an associated divisor derived from this. Then we can ask whether there is a bound on the ratio of the degree of the minimal resultant divisor to the degree of a similarly constructed conductor divisor.

This will all be done in Chapter 6. In the remainder of this chapter, we will go on something of a tangent as we prove a result about the bad reduction of the Lattes maps associated to multiplication on an elliptic curve over of function field. These were initially of interest to us because of the relationship between the discriminant of these elliptic curves, and the resultant of the associated Lattes map for multiplication by 2 on the elliptic curve.

5.2 Lattes Maps

Throughout this section, we will assume K is a function field of a nonsingular curve over an algebraically closed field k of characteristic zero. We assume characteristic zero in order to avoid the problems that positive characteristic may cause in the calculations be-

low. It is possible that this assumption could be relaxed somewhat by making appropriate assumptions about the residue characteristic, obtaining more general statements than those that follow.

Under these assumptions, we can construct $\text{Rat}_d \subset \mathbb{P}_k^{2d+1}$ over the base field k . **We assume this construction in place for this section only.**

A family of maps that is useful to consider while studying dynamics in parallel to the theory of curves are the Lattes maps. If $E(K)$ is an elliptic curve given by $y^2 = x^3 + Ax + B$, then its discriminant is $4A^3 + 27B^2$. Interestingly, if φ_E is the Lattes map associated to the x coordinate of the multiplication by 2 map on the elliptic curve, then the resultant of this dynamical system is, up to multiplication by an integer, the square of this discriminant. We will show this below.

Let E be an elliptic curve over K . Denote by $[n]$ the multiplication by n map with respect to the group structure on E . The quotient $E/\{\pm 1\}$ is isomorphic to \mathbb{P}_K^1 and so we have a map:

$$\pi : E \rightarrow \mathbb{P}_K^1$$

The map $[n]$ descends to a map φ_n on \mathbb{P}_K^1 via this projection:

$$\begin{array}{ccc} E & \xrightarrow{[n]} & E \\ \downarrow & & \downarrow \\ \mathbb{P}_K^1 & \xrightarrow{\varphi_n} & \mathbb{P}_K^1 \end{array}$$

Now fix variables X, Y , and Z giving a Weierstrass equation for E . Let $x = X/Z, y = Y/Z$,

and let $y^2 = x^3 + Ax + B$ be that Weierstrass equation. Let $P(x) = x^3 + Ax + B$. With respect to the coordinates $[x, y, 1]$, the map π is then:

$$[x, y, 1] \mapsto [x, 1].$$

Further, with respect to the coordinate $[x, 1]$ on \mathbb{P}_K^1 , φ_n is then given explicitly as a rational function in terms of x, A , and B ,

$$\varphi_n([x, 1]) = \left[\frac{P(A, B, x)}{Q(A, B, x)}, 1 \right].$$

This formula admits the form for $n = 2$,

$$\varphi_2([x, 1]) = \left[\frac{(P'(x))^2 - 8xP(x)}{4P(x)}, 1 \right].$$

From this we obtain the following proposition.

Proposition 5.2. *With the notation above, if D is the discriminant of E (which is the discriminant of $P(x)$) and R is the resultant of φ_2 , then $R = 256D^2$.*

Proof. This can be calculated directly using the formula for the resultant. Alternatively, we may observe the following. The discriminant is itself the resultant

$$D = \text{Res}(P'(x), P(x)). \tag{5.1}$$

If one looks at the matrix whose determinant yields the resultant, it is clear that, in general, $\text{Res}(f(x), g(x)) = \text{Res}(f(x) + h(x)g(x), g(x))$ for any polynomials f, g, h . Thus we have

$$R = \text{Res}((P'(x))^2 - 8xP(x), 4P(x)) = \text{Res}((P'(x))^2, 4P(x)).$$

From the difference of roots formula for the resultant,

$$\text{Res}((P'(x))^2, 4P(x)) = \text{Res}(P'(x), 4P(x))^2.$$

Writing this as a product of differences of roots and factoring out a power of 4^2 from the power of the leading coefficient gives

$$\text{Res}(P'(x), 4P(x))^2 = (16\text{Res}(P'(x), P(x)))^2.$$

The result follows from (5.1). □

An interesting question we have not yet explored is whether the Lattes $[n]$ maps yield similar nice forms for the resultant in terms of the discriminant.

Semi-stability in the parameter space of elliptic curves and semi-stability in the parameter space of rational maps are related notions in that each is a special case of the more general GIT notion of semi-stability, discussed above. Each can be shown to be equivalent to a condition involving the multiplicities of roots of polynomials. We've seen how this is so in the dynamical setting: semi-stability of an element of \mathbb{P}_k^{2d+1} , in the GIT sense, is equivalent to a condition involving a bound on the order of vanishing of common roots of the two polynomials; this is expressed in Theorem 4.3 above. It turns out that, in the elliptic curves setting, curves that are given by Weierstrass equations with a double root, but not a triple root, are semi-stable in the GIT sense, with respect to the group action of change variables on the defining equation. Asking whether semi-stable bad reduction of

an elliptic curve over a function field implies that the reduced Lattes map is semi-stable yields the following negative answer:

Theorem 5.3. *Let E be given as above, let $y^2 = x^3 + Ax + B$ and let $p \in C$ be a closed point such that evaluation of A and B at p is defined, and such that the resulting reduced Weierstrass equation defines a singular curve (over k) with a node:*

$$y^2 = x^3 + A(p)x + B(p) = (x - \lambda_1)(x - \lambda_2)^2$$

For each $n \geq 2$, evaluation of the coefficients of the map φ_n , described above, defines a point in $\mathbb{P}_k^{2n^2+1}$, and this point lies outside of the semi-stable locus.

Proof. Here, n^2 is the degree of the map over the function field. We may now prove Theorem 5.3. To ease notation, let $A(p) = a, B(p) = b$. Let $x^3 + ax + b = q(x)$. Write $\varphi_{n,p}$ for the reduced map at p . We first handle the case $n = 2$. We have, by the representation of φ_2 above

$$\varphi_{2,p} = \frac{(q'(x))^2 - 8xq(x)}{4q(x)} = \frac{(2(x - \lambda_1)(x - \lambda_2) + (x - \lambda_2)^2)^2 - 8x(x - \lambda_1)(x - \lambda_2)^2}{4(x - \lambda_1)(x - \lambda_2)^2}.$$

One sees that λ_2 is at least a common double root of the numerator and denominator. Since $\lambda_1 \neq \lambda_2$ by assumption, it is clear that it is not a common triple root, and that canceling $(x - \lambda_2)^2$ gives us a rational map of degree 2. To apply the above proposition, part 1 (b), we must have that λ_2 is a fixed point of this map. This is a straightforward calculation. First note that, since the x^2 term of the Weierstrass equation is 0, $\lambda_1 = -2\lambda_2$. Then by

plugging in to the above and after cancelation, we have

$$\varphi_{2,p}(\lambda_2) = \frac{(2(\lambda_2 + 2\lambda_2))^2 - 8\lambda_2(\lambda_2 + 2\lambda_2)}{4(\lambda_2 + 2\lambda_2)} = 3\lambda_2 - 2\lambda_2 = \lambda_2.$$

Thus λ_2 is a fixed point, so $\varphi_{2,p}$ is unstable.

The case $n = 2$ is the only case that requires verification of this fixed point criteria. For $n \geq 3$, we will use Theorem 4.3, parts 1 (a) and 2 (a). Following [14], (p. 105 ex. 3.7), we can describe the x coordinate of multiplication by n in the group law. Given our elliptic curve with Weierstrass equation $y^2 = x^3 + Ax + B$, we define

$$\Psi_1 = 1$$

$$\Psi_2 = 2y$$

$$\Psi_3 = 3x^4 + 6Ax^2 + 12Bx - A^2$$

$$\Psi_4 = 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3)$$

and for $m \geq 2$

$$\Psi_{2m+1} = \Psi_{m+2}\Psi_m^3 - \Psi_{m-1}\Psi_{m+1}^3$$

and

$$\Psi_{2m} = \frac{\Psi_m(\Psi_{m+2}\Psi_{m-1}^2 - \Psi_{m-2}\Psi_{m+1}^2)}{2y}.$$

Then

$$\frac{x\Psi_n^2 - \Psi_{n+1}\Psi_{n-1}}{\Psi_n^2}$$

is a rational function in x of degree n^2 that gives us the x coordinate of multiplication by n in the group law.

For each $n \geq 1$, let ψ_n be the polynomial in $k(x, y)$ obtained by evaluation of the coefficients A and B in Ψ_n at p . Our goal is to see that the reduction

$$\phi_{n,p} = \frac{x\psi_n^2 - \psi_{n+1}\psi_{n-1}}{\psi_n^2}$$

has an appropriately high power of $(x - \lambda_2)$ dividing the numerator and denominator. For each n , let M_n be $\min(v(x\psi_n^2 - \psi_{n+1}\psi_{n-1}), v(\psi_n^2))$, where v is the order of vanishing of $(x - \lambda_2)$. We are looking for a lower bound on M_n . The following table outlines the situation for the first few n . Here d is the degree of the map ϕ_n , and r is, in line with Theorem 4.3, chosen so that $2r = d$ for d even and $2r + 1 = d$ for d odd.

n	2	3	4	5	6	7	8	9	10
d	4	9	16	25	36	49	64	81	100
r	2	4	8	12	18	24	32	40	50
$r+1$	3		9		19		33		51
$r+2$		6		14		26		42	

Separating the even and odd cases, the bottom two rows give us the lower bound on M_n that we wish to show. When $n = 2m$ is even, the sequence of $r + 1$ is given by $2m^2 + 1$. When $n = 2m + 1$ is odd, the sequence of $r + 2$ is given by $2(m^2 + m + 1)$.

Again taking into account that $\lambda_1 = -2\lambda_2$, we have that $a = -3\lambda_2^2$ and $b = 2\lambda_2^3$. From this, one can compute

$$\psi_3 = 3(x - \lambda_2)^3(x + 3\lambda_2)$$

$$\psi_4 = 4y(x - \lambda_2)^5(x + 5\lambda_2).$$

Thus we may calculate

$$v(\psi_1) = 0$$

$$v(\psi_2) = 1$$

$$v(\psi_3) = 3$$

$$v(\psi_4) = 6$$

This will allow us to inductively compute a lower bound on $2v(\psi_n)$ for each n . For reasons explained below, we will do this by hand for the next few ψ . Using these, we will then calculate a lower bound on the numerator as well, and thus on M_n for the first few n .

n	lower bound on $2v(\psi_n)$	lower bound on M_n	$r+1$	$r+2$
2	2	2	3	
3	6	6		6
4	12	12	9	
5	18	18		14
6	26	26	19	
7	36	36		26
8	48	48	42	
9	60	60		51

The above table establishes the result for $3 \leq n \leq 9$. It is computed as follows. First we write

$$\psi_5 = \psi_4 \psi_2^3 - \psi_1 \psi_3^3$$

and using elementary properties of valuations, deduce that $v(\psi_5) \geq 9$. We then write ψ_6 in terms of the lower ψ_n , and use the lower bounds already established on them to make a similar deduction. The rest follows. This gives us the leftmost column of the

table, and then the column next to it is a simple calculation of a lower bound on each $x\psi_n^2 + \psi_{n-1}\psi_{n+1}$. Note that these two columns are actually equal so far. It may be the case that the lower bound is actually an equality and that the numerator and denominator always have the same power of $(x - \lambda_2)$ in them. This is not required for what we want to show, however.

Lemma 5.4. *Suppose $m \geq 2$. Then:*

$$(1) \ 2v(\psi_{2m}) \geq 2m^2 + 1 \text{ and}$$

$$(2) \ 2v(\psi_{2m+1}) \geq 2(m^2 + m + 1).$$

The right hand side of these inequalities is $r + 1$ and $r + 2$ in the even and odd cases $n = 2m$ and $n = 2m + 1$ respectively. The left hand side is v of the denominator of $\varphi_{n,p}$. Thus Lemma 5.4 will imply the result if we show that the numerator has at least as high a valuation as well. We now show that this also follows from Lemma 5.4. Our reasoning is valid for $m - 1 \geq 2$ in the even case and $m \geq 2$ in the odd case, establishing the result for $n \geq 5$. The result for lower n is given by the above table.

Even case ($n = 2m$):

$$v(x\psi_{2m}^2 - \psi_{2m+1}\psi_{2m-1}) \geq \min(2m^2 + 1, v(\psi_{2m+1}\psi_{2m-1}))$$

and we have

$$v(\psi_{2m+1}\psi_{2m-1}) \geq m^2 + m + 2 + (m - 1)^2 + (m - 1) + 1 = 2m^2 + 2$$

yielding a common root of order at least $2m^2 + 1 = r + 1$.

Odd case ($n = 2m + 1$):

$$v(x\psi_{2m+1}^2 - \psi_{2m+2}\psi_{2m}) \geq \min(2v(\psi_{2m+1}), v(\psi_{2m+2}) + v(\psi_{2m}))$$

and we have

$$2v(\psi_{2m+1}) \geq 2(m^2 + m + 1).$$

Then

$$v(\psi_{2m+2}) + v(\psi_{2m}) \geq \frac{2(m+1)^2 + 1}{2} + \frac{2m^2 + 1}{2} = 2(m^2 + m + 1)$$

yielding a common root of order at least $2(m^2 + m + 1) = r + 2$.

It remains to prove Lemma 5.4. We must split into even and odd cases

$$m = 2s,$$

$$m = 2s + 1$$

and verify a number of early cases by consulting the above table in order to get the induction going. We will be doing strong induction applied to values $2 \leq k \leq m$. In the even case we will need $s - 1, s, s + 1$ to be in this range. In the odd case, we will need $s, s + 1, s + 2$ to be in this range. Thus we will have to assume $m \geq 5$. Again, the result for lower indices m is given by the table above.

Assume (1) and (2) hold for $2 \leq k \leq m$. We want to show

$$(a) \ 2v(\psi_{2(m+1)}) \geq 2(m+1)^2 + 1$$

$$(b) \ 2v(\psi_{2(m+1)+1}) \geq 2((m+1)^2 + (m+1) + 1).$$

Even case ($m = 2s$)

For (a),

$$\begin{aligned} 2v(\psi_{2m+1}) &= 2v\left(\frac{\psi_{m+1}(\psi_{m+3}\psi_m^2 - \psi_{m-1}\psi_{m+2}^2)}{2y}\right) \\ &\geq 2(v(\psi_{m+1}) + \min(v(\psi_{m+3}) + 2v(\psi_m), v(\psi_{m-1}) + 2v(\psi_{m+2}) - 1)) \end{aligned}$$

and

$$\begin{aligned} 2(v(\psi_{m+1}) + v(\psi_{m+3}) + 2v(\psi_m) - 1) &= 2(v(\psi_{2s+1}) + v(\psi_{2s+3}) + 2v(\psi_{2s}) - 1) \\ &\geq 2(s^2 + s + 1 + (s+1)^2) \\ &\quad + (s+1) + 1 + 2s^2 + 1 - 1) \\ &= 2(4s^2 + 4s + 4) \\ &= 2(m^2 + 2m + 4) \\ &= 2(m+1)^2 + 6 \end{aligned}$$

while

$$\begin{aligned}
2(v(\varphi_{m+1}) + v(\varphi_{m-1}) + 2v(\varphi_{m+2}) - 1) &= 2(v(\varphi_{2s+1}) + v(\varphi_{2s-1}) + 2v(\varphi_{2s+2}) - 1) \\
&\geq 2((s^2 + s + 1) + (s - 1)^2 \\
&\quad + (s - 1) + 1 + 2(s + 1)^2 + 1 - 1) \\
&= 2(m + 1)^2 + 6
\end{aligned}$$

This proves (a).

For (b),

$$\begin{aligned}
2v(\varphi_{2(m+1)+1}) &= 2v(\varphi_{m+3}\varphi_{m+1}^3 - \varphi_m\varphi_{m+2}^3) \\
&\geq 2(\min(v(\varphi_{m+3}) + 3v(\varphi_{m+1}), v(\varphi_m) + 3v(\varphi_{m+2})))
\end{aligned}$$

and

$$\begin{aligned}
2(v(\varphi_{m+3}) + 3v(\varphi_{m+1})) &= 2(v(\varphi_{2s+3}) + 3v(\varphi_{2s+1})) \\
&= 2(4s^2 + 6s + 6) \\
&= 2((m + 1)^2 + (m + 1) + 4)
\end{aligned}$$

while

$$\begin{aligned}
2(v(\varphi_m) + 3v(\varphi_{m+2})) &= 2(v(\varphi_{2s}) + 3v(\varphi_{2s+2})) \\
&\geq (2s^2 + 1 + 3(2(s+1)^2 + 1)) \\
&= 2((m+1)^2 + (m+1) + 3).
\end{aligned}$$

Odd case ($m = 2s + 1$)

For (a),

$$\begin{aligned}
2v(\psi_{2m+1}) &= 2v\left(\frac{\psi_{m+1}(\psi_{m+3}\psi_m^2 - \psi_{m-1}\psi_{m+2}^2)}{2y}\right) \\
&\geq 2(v(\psi_{m+1}) + \min(v(\psi_{m+3}) \\
&\quad + 2v(\psi_m), v(\psi_{m-1}) + 2v(\psi_{m+2}) - 1)
\end{aligned}$$

and

$$\begin{aligned}
2(v(\psi_{m+1}) + v(\psi_{m+3}) + 2v(\psi_m)) - 1 &= 2(v(\psi_{2s+2}) + v(\psi_{2s+4}) + 2v(\psi_{2s+1})) - 1 \\
&\geq (2(s+1)^2 + 1 + 2(s+2)^2 \\
&\quad + 1 + 4(s^2 + s + 1) - 2) \\
&= 2(4s^2 + 8s + 7) \\
&= 2(m+1)^2 + 6
\end{aligned}$$

while

$$\begin{aligned}
 2(v(\psi_{m+1}) + v(\psi_{m-1}) + 2v(\psi_{m+2}) - 1) &= 2(v(\psi_{2s+2}) + v(\psi_{2s}) + 2v(\psi_{2s+3}) - 1) \\
 &\geq 2(4s^2 + 8s + 7) \\
 &= 2(m+1)^2 + 6
 \end{aligned}$$

Taking care of (a).

Finally, for (b),

$$2v(\psi_{2(m+1)+1}) = 2v(\psi_{m+3}\psi_{m+1}^3 - \psi_m\psi_{m+2}^3)$$

and

$$\begin{aligned}
 2v(\psi_{m+3}\psi_{m+1}^3) &= 2v(\psi_{2s+4}\psi_{2s+2}^3) \\
 &= (2(s+2)^2 + 1 + 3(2(s+1)^2 + 1)) \\
 &= 8s^2 + 20s + 17 \\
 &= 2((m+1)^2 + (m+1) + 1) + 3
 \end{aligned}$$

while

$$\begin{aligned}
 2v(\psi_m\psi_{m+2}^3) &= 2v(\psi_{2s+1}\psi_{2s+3}^3) \\
 &= 2(s^2 + s + 1 + 3((s+1)^2 + (s+1) + 1)) \\
 &= 2((m+1)^2 + (m+1) + 4)
 \end{aligned}$$

□

5.3 An Analogy

To conclude this chapter, let us push an analogy, mentioned in the introduction, that we found useful. When one changes variables in such a way as to preserve the Weierstrass form, the effect on the discriminant is multiplication by a power of a scalar. Something similar is true for the resultant, in the dynamical setting, as suggested by Formula 4.1. In this way, both the discriminant and the resultant are “almost invariant” under a natural group action.

Considering the elliptic curves situation, we do have an object which is truly invariant under change of variables: the j -invariant. The j -invariant is closely related to the discriminant, and studying it gives us information about the discriminant. In the dynamical setting, the same is true about the symmetric functions in the multipliers: they are truly PGL_2 -invariant, and are closely related to the resultant, as we have seen above. So studying them ought to give us information about the resultant.

Chapter 6

The Minimal Resultant and the Conductor

In this chapter we will consider bad reduction of dynamical systems defined over K , where K is the function field of a Dedekind scheme. We will develop the minimal resultant and the conductor, and prove some initial properties. The two examples to have in mind for K are a number field (in which case $C = \text{Spec}(O_K)$), and the function field of a non-singular projective curve C .

6.1 Singular and Bad Reduction of Rational maps

Let us consider morphism φ and a presentation $\Phi \in \text{Rat}_d(K)$. For each p , we may obtain a point in $\mathbb{P}^{2d+1}(\kappa(p))$ in a natural way, as follows. We first define the notion of a model.

Definition 6.1. *Let A be a ring with fraction field K and let Φ be a presentation of a morphism φ over K . A **model** of Φ is an affine representative (\mathbf{a}, \mathbf{b}) for the projective point $\Phi = [\mathbf{a}, \mathbf{b}]$. A **model of Φ over A** is a model of Φ with coefficients in A . If $p \in C$, a p -model*

of φ is a model (\mathbf{a}, \mathbf{b}) of φ over $\mathcal{O}_{C,p}$, such that at least one of the coefficients of (\mathbf{a}, \mathbf{b}) is a unit.

By choosing a p -model, we can reduce the coefficients of our map modulo the maximal ideal of the local ring at p . If we are given any model, it is easy to turn it into a p -model:

Notation 6.2. Let (\mathbf{a}, \mathbf{b}) be a model of Φ . Set $c_i = a_i$ for $0 \leq i \leq d$ and $c_i = b_{i-d-1}$ for $i \leq d+1 \leq 2d+1$, and set $(\mathbf{c}) = (c_0, \dots, c_{2d+1})$. Let $p \in C$. Set

$$n_p(\mathbf{a}, \mathbf{b}) := \min_j(v_p(c_j)) \quad (6.1)$$

Clearly (\mathbf{a}, \mathbf{b}) is a p -model of Φ if and only if $n_p = 0$. Thus for any model (\mathbf{a}, \mathbf{b}) of Φ , if we chose a scalar u_p such that $v_p(u_p) = -n_p$, then $(u_p\mathbf{a}, u_p\mathbf{b})$ is a p -model. Equipped with a p -model, we may find the reduction of Φ at p .

Definition 6.3. Let $p \in C$ and let (\mathbf{a}, \mathbf{b}) be a model of Φ over $\mathcal{O}_{C,p}$ such that $n_p(\mathbf{a}, \mathbf{b}) = 0$ (i.e. a p -model). **The reduction of Φ at p** is the point $\Phi_p = [\mathbf{a}(p), \mathbf{b}(p)] \in \mathbb{P}^{2d+1}(\kappa(p))$ obtained by evaluating each coefficient of (\mathbf{a}, \mathbf{b}) at p (i.e. looking at the image of that coefficient in the residue field).

It is easy to verify that this definition does not depend on the choice of the p -model (it does, however, depend on the presentation—an important point for later). For most points, the reduction of Φ at p will be a presentation of a morphism over $\kappa(p)$ of degree d . However, for a finite set of points, this will fail: the reduced coefficients will describe

two polynomials with non-trivial common zeros, and thus will not describe a morphism of degree d (We can, however, cancel the common zeros and obtain a morphism of lower degree). Where this happens is captured by evaluating the resultant form of a p -model. We can construct a divisor that captures all this information at once: given any model (\mathbf{a}, \mathbf{b}) and the choices above, an elementary calculation shows:

$$v_p(\rho(u_p \mathbf{a}, u_p \mathbf{b})) = v_p(\rho(\mathbf{a}, \mathbf{b})) - 2dn_p(\mathbf{a}, \mathbf{b}) \quad (6.2)$$

It can be verified that this number depends only on the presentation $[\mathbf{a}, \mathbf{b}]$.

Notation 6.4. Let $\Phi = [\mathbf{a}, \mathbf{b}]$ be a presentation of φ . Set:

$$N_{\Phi, p} := v_p(\rho(\mathbf{a}, \mathbf{b})) - 2dn_p(\mathbf{a}, \mathbf{b})$$

Definition 6.5. Let $\varphi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ and let Φ be a presentation of φ with a model (\mathbf{a}, \mathbf{b}) . The resultant divisor of the presentation Φ is

$$R_{\Phi} = R_{[\mathbf{a}, \mathbf{b}]} = \sum_{p \in C} N_{\Phi, p} [p]$$

We refer to the support of R_{Φ} as the **singular reduction locus of Φ** , and if p is in this support, we say that Φ has **singular reduction** at p . If p is not in the support of R_{Φ} , then we say Φ has **non-singular reduction** at p .

Another way to define the resultant divisor of a presentation is the following: for each $p \in C$, the reduction of Φ at p is an element of $\mathbb{P}^{2d+1}(\kappa(p))$ which may or may not be in

$\text{Rat}_d(\kappa(p))$. In fact, we may construct a morphism of schemes,

$$F_\Phi : C \rightarrow \mathbb{P}^{2d+1}$$

for which when we compose with the natural inclusion morphism $\kappa(p) \hookrightarrow C$ the resulting point of $\mathbb{P}^{2d+1}(\kappa(p))$ is precisely this reduction of Φ at p . The resultant divisor of the presentation Φ is then the divisor of zeros of the section of $F_\Phi^*O(2d)$ given by pulling back the resultant form.

6.2 Minimality

Given a morphism φ , the resultant divisors of two different presentations may certainly be different. We may, in particular, be able to act in such a way that singular reduction becomes non-singular reduction. Therefore it is useful to look instead for a notion of a resultant divisor which is invariant under the $\text{PGL}_2(K)$ action, and thus depends only on the morphism φ . To obtain this, we consult the discussion in [13], section 4.11, which develops the *minimal* resultant in the number field case. The definition of the minimal resultant, as well as Proposition 6.2 and Proposition 6.3, are taken from this discussion.

Definition 6.6. *Let φ be a morphism and $\Phi = [\mathbf{a}, \mathbf{b}]$ a presentation of φ . Let*

$$\varepsilon_p(\varphi) = \min_{\Gamma \in \text{PGL}_2(K)} (R_{\Phi\Gamma, p})$$

*The **minimal resultant** of φ is*

$$R_\varphi = \sum_{p \in C} \varepsilon_p(\varphi)[p].$$

This divisor is invariant under the $PGL_2(K)$ action, and has support on the points of C for which every $PGL_2(K)$ conjugate of Φ (that is, every presentation of φ) has singular reduction.

Definition 6.7. Let φ be a morphism. The **conductor** of φ is the divisor:

$$\mathfrak{C}_\varphi = \text{Support}(R_\varphi)$$

If p has a nonzero coefficient in \mathfrak{C}_φ , then we say φ has **bad reduction** at p . If the coefficient of p in \mathfrak{C}_φ is zero, then we say φ has **good reduction** at p .

Notation 6.8. Let D be a divisor on C . Then $(D)_p$ is the coefficient of D at p .

A dynamical analog to Theorem 5.1 would bound the degree of the minimal resultant in terms of the degree of the conductor. Without additional assumptions, however, the dynamical analog to Theorem 5.1 is not always true, at least for degree 2 maps, as we see in the following theorem.

Theorem 6.1. Let $K = k(t)$. For each $N \in \mathbb{Z}^+$, let φ_n be the degree 2 morphism given by:

$$\varphi_n = \frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

where $\lambda_1 = a + bt^N$, $\lambda_2 = a^{-1} + b't^N$, $a, b, b' \neq 0, 1$, and $ab' + b/a = 0$. Then the degree of the conductor of φ_n is at most 2, and the degree of the minimal resultant is at least $2N$.

Proof. This follows immediately from Corollary 7.5 and Corollary 7.8 in Section 4. \square

There is a similar example for the number field case, in Theorem 7.14 below.

6.2.1 Conjugation

The difficulty in calculating R_φ lies in understanding, for a given p , which presentation Φ of φ truly realizes the minimal value for $N_{\Phi,p}$.

Let now $\Gamma \in GL_2(K)$, $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. As above, we write $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ for the new coefficients under the action of Γ on a model of Φ . Recall that this is a group action that descends to the conjugation action under the projection to PGL_2 and projective space of the coefficients. Applying valuations to formula (4.1) above, we have:

$$v_p(\rho(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)) = v_p(\rho(\mathbf{a}, \mathbf{b})) + (d^2 + d)v_p(\det(\Gamma)) \quad (6.3)$$

This tells us what the action does to the valuation of the resultant form. We can also say something about what happens to $n_p(\mathbf{a}, \mathbf{b})$ under the same action. Let

$$v_p(\Gamma) = \min(v_p(\alpha), v_p(\beta), v_p(\gamma), v_p(\delta))$$

Then we have an inequality

$$n_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma) \geq n_p(\mathbf{a}, \mathbf{b}) + (d+1)v_p(\Gamma) \quad (6.4)$$

To see this, observe that each coefficient in $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ is a sum of terms of the form,

$$(\text{coefficient of } (\mathbf{a}, \mathbf{b})) \cdot (\text{homogeneous polynomial of degree } d+1 \text{ in } \mathbb{Z}[\alpha, \beta, \gamma, \delta]).$$

If we write c_i^Γ for the i -th coefficient of $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$, for each i , we have some polynomial

p and a j such that,

$$\begin{aligned} v_p(c_i^\Gamma) &\geq v_p(c_j \cdot p(\alpha, \beta, \gamma, \delta)) \\ &\geq n_p(\mathbf{a}, \mathbf{b}) + (d+1)v_p(\Gamma). \end{aligned}$$

Another useful observation is the following. Suppose $\Gamma_p \in GL_2(\mathcal{O}_{C,p})$. Then

$$v_p(\rho(\mathbf{a}, \mathbf{b})) = v_p(\rho(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p})) \quad (6.5)$$

and

$$n_p(\mathbf{a}, \mathbf{b}) = n_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p}) \quad (6.6)$$

These quickly follow from the above formulas when it is observed that being in $GL_2(\mathcal{O}_{C,p})$ is equivalent to having $v_p(\det(\Gamma_p)) = 0$ and $v_p(\Gamma_p) \geq 0$. Thus we see immediately (6.5) follows from (6.3). For (6.6), $v_p(\Gamma_p) \geq 0$ implies $n_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p}) \geq n_p(\mathbf{a}, \mathbf{b})$ by (6.4). But since $\Gamma_p^{-1} \in GL_2(\mathcal{O}_{C,p})$ as well, $n_p(\mathbf{a}, \mathbf{b}) \geq n_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p})$.

Notice also that (6.5) and (6.6) together imply:

$$N_{\Phi,p} = N_{\Phi^{\Gamma_p},p}.$$

The above formulas are some basic tools for understanding how the valuation of the resultant changes under the action of K^\times and the actions of $PGL_2(K)$ and $GL_2(K)$ on presentations $[\mathbf{a}, \mathbf{b}]$ and models (\mathbf{a}, \mathbf{b}) respectively. Let us collect them in the following proposition.

Proposition 6.2. *Let φ be a morphism, let Φ be a presentation of φ , and let (\mathbf{a}, \mathbf{b}) be a model of Φ . If $\Gamma \in GL_2(K)$, then*

- (1) $v_p(\rho(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)) = v_p(\rho(\mathbf{a}, \mathbf{b})) + (d^2 + d)v_p(\det(\Gamma))$
- (2) $n_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma) \geq n_p(\mathbf{a}, \mathbf{b}) + (d + 1)v_p(\Gamma)$
- (3) If $\Gamma_p \in \mathrm{GL}_2(\mathcal{O}_{C,p})$ then $v_p(\rho(\mathbf{a}, \mathbf{b})) = v_p(\rho(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p}))$,
- (4) $n_p(\mathbf{a}, \mathbf{b}) = n_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p})$ and
- (5) $N_{\Phi,p} = N_{\Phi^{\Gamma_p},p}$.

6.2.2 Minimal Presentations and Models

Is it possible to find a global model or presentation that realizes the minimal resultant at each point? If C is affine, it is reasonable to look for a *model* that does this; for example, in the case of a number field K , Silverman asks if there Γ such that Φ^Γ can be written as $[\mathbf{a}', \mathbf{b}']$ where $(\mathbf{a}', \mathbf{b}')$ has coefficients in the ring of integers \mathcal{O}_K of the number field, and where

$$R_\varphi = \sum_{\mathfrak{p} \in \mathrm{Specmax}(\mathcal{O}_K)} v_p(\rho(\mathbf{a}, \mathbf{b}))[\mathfrak{p}] \quad (6.7)$$

This is an analogy of a minimal model of an elliptic curve over a number field. More generally, if C is affine, such a definition makes sense. However, if C is a complete non-singular curve, if we could even find a presentation and a model that gave us the value of R_φ for all points of C , it would follow that R_φ is trivial, being an effective divisor that is also principal. Likewise, requiring the coefficients to be in the global sections of C would imply triviality. So this definition is not very useful in this case. In regards to the number field situation, Silverman mentions a notion in [13] (Exercise 4.46b) that will be useful,

which we may formulate in our setting as follows.

Definition 6.9. *Let U be an open subset of C and let S be the complement of U . An S -minimal global model of φ is a model (\mathbf{a}, \mathbf{b}) over $\mathcal{O}_C(U)$ of a presentation Φ of φ such that $(R_\varphi)_p = v_p(\rho(\mathbf{a}, \mathbf{b}))$ for every p in U .*

In addition to a minimal model, we might look for a minimal presentation.

Definition 6.10. *We say that a presentation $\Phi = [\mathbf{a}, \mathbf{b}]$ is a **minimal presentation** of φ if:*

$$R_\varphi = \sum_{p \in C} N_{\Phi,p}[p].$$

For number fields, Silverman proves a necessary condition for having a minimal model in the sense of (6.7). Here we show that this condition works in our setting, when instead we consider S -minimal models.

Proposition 6.3. *Let $\Phi \in \text{Rat}_d(K)$ be a presentation of φ .*

(1) *If d is odd, then for each model (\mathbf{a}, \mathbf{b}) of Φ , there is a divisor $A_{(\mathbf{a}, \mathbf{b})}$ satisfying,*

$$R_\varphi = \text{div}(\rho(\mathbf{a}, \mathbf{b})) + 2dA_{(\mathbf{a}, \mathbf{b})}.$$

(2) *If d is even, then for each model (\mathbf{a}, \mathbf{b}) of Φ , there is a divisor $A_{(\mathbf{a}, \mathbf{b})}$ satisfying,*

$$R_\varphi = \text{div}(\rho(\mathbf{a}, \mathbf{b})) + dA_{(\mathbf{a}, \mathbf{b})}.$$

The image of this divisor in $\text{Pic}(C)$ is independent of both the chosen model, and of the chosen presentation Φ . In the even case, the divisors associated to two different models of φ differ by a principal divisor coming from the square of an element in K .

Proof. For each $p \in C$, there is some $\Gamma_p \in \text{GL}_2(K)$ such that coefficient of R_φ at p is given by:

$$v_p(\rho(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p})) - 2dn_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p}) = v_p(\rho(\mathbf{a}, \mathbf{b})) + (d^2 + d)v_p(\det(\Gamma_p)) - 2dn_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p})$$

In the odd case, we can factor $2d$ out of the terms on the right. In the even case, we can factor d out. Thus in the odd case we simply define,

$$A_{(\mathbf{a}, \mathbf{b})} = \sum_p \left[\frac{(d+1)}{2} v_p(\det(\Gamma_p)) - n_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p}) \right] [p].$$

And in the even case we define,

$$A_{(\mathbf{a}, \mathbf{b})} = \sum_p \left[(d+1)v_p(\det(\Gamma_p)) - 2n_p(\mathbf{a}^{\Gamma_p}, \mathbf{b}^{\Gamma_p}) \right] [p].$$

Now recall that R_φ is invariant under the group action. So keeping in mind how the valuation of the resultant changes with respect to the GL_2 action (Proposition 6.2), and with respect to scalar multiplication, it is a calculation to see that the divisor class is invariant under both scalar multiplication and the $\text{PGL}_2(K)$ action. For the even case, let

$M \in \mathrm{GL}_2(K)$. We have

$$\begin{aligned} \mathrm{div}(\rho(\mathbf{a}, \mathbf{b})) + dA_{(\mathbf{a}, \mathbf{b})} &= R_\varphi \\ &= \mathrm{div}(\rho(\mathbf{a}^M, \mathbf{b}^M)) + dA_{(\mathbf{a}^M, \mathbf{b}^M)} \\ &= \mathrm{div}(\rho(\mathbf{a}, \mathbf{b})) + (d^2 + d)\det(M) + dA_{(\mathbf{a}^M, \mathbf{b}^M)}. \end{aligned}$$

Then

$$A_{(\mathbf{a}, \mathbf{b})} = \mathrm{div}(\det(M)^{(d+1)}) + A_{(\mathbf{a}^M, \mathbf{b}^M)}.$$

A similar calculation can be made for scalar multiplication of the coefficients and the odd case. \square

Corollary 6.4. *Suppose φ has an S -minimal global model. Then the image of $A_{(\mathbf{a}, \mathbf{b})}$ in the restricted group $\mathrm{Pic}(U)$ (where $U = C - S$) is trivial.*

Proof. By assumption there exists some Γ that allows us to write

$$R_\varphi|_U = \mathrm{div}(\rho(\mathbf{a}', \mathbf{b}'))|_U$$

where $[\mathbf{a}', \mathbf{b}'] = [\mathbf{a}^\Gamma, \mathbf{b}^\Gamma]$. Then $A_{(\mathbf{a}', \mathbf{b}')}|_U = 0$ in $\mathrm{Div}(U)$. So its class in $\mathrm{Pic}(U)$ is also 0. However, this equals the image of $A_{(\mathbf{a}, \mathbf{b})}|_U$ in $\mathrm{Pic}(U)$, by the above and basic properties of the restriction homomorphism on Div and Pic . \square

Corollary 6.5. *Suppose d is odd and $U \subset C$ is open. Then if (\mathbf{a}, \mathbf{b}) is a model of Φ and $A_{(\mathbf{a}, \mathbf{b})}$ is trivial in $\mathrm{Pic}(U)$, then there is $c \in K$ such that $v_p(\rho(c\mathbf{a}, c\mathbf{b})) = (R_\varphi)_p$ for all $p \in U$. In particular, if $U = \mathrm{Spec}(A)$ with A a principal ideal domain, then such c always exists.*

Proof. By assumption, we can write $A_{(\mathbf{a}, \mathbf{b})}|_U = \text{div}(c)|_U$, where $c \in K$. Then

$$\begin{aligned} R_\varphi|_U &= (\text{div}(\rho(\mathbf{a}, \mathbf{b})) + (2d)\text{div}(c))|_U \\ &= \text{div}(\rho(\mathbf{ca}, \mathbf{cb}))|_U \end{aligned}$$

□

Notice that this is not saying that the resultant divisor of the presentation $[\mathbf{a}, \mathbf{b}]$ has the value of R_φ for each $p \in U$. For that to happen, we would have to be able to evaluate $(\mathbf{ca}, \mathbf{cb})$ at each $p \in U$. We don't know that we can do this, because some of these coefficients may have poles in U .

6.2.3 Local Minimality Conditions

Even locally, finding the value of the minimal resultant is difficult, since we have to take into consideration all conjugates of a given map. One thing we can say initially is the following.

Proposition 6.6. *Suppose $U = \text{Spec}(A)$ is a PID and φ has a presentation $\Phi = [\mathbf{a}, \mathbf{b}]$ where the model (\mathbf{a}, \mathbf{b}) is over $\mathcal{O}_C(U)$, and the coefficients have no common factors (We can always do this in a PID). Let $p \in U$. Then if $v_p(\rho(\mathbf{a}, \mathbf{b})) < 2d$, $v_p(\rho(\mathbf{a}, \mathbf{b})) = (R_\varphi)_p$ at p .*

Proof. Just write $R_\varphi|_U = \text{div}(\rho(\mathbf{ca}, \mathbf{cb}))$ as in the above proof. The conditions on the coefficients imply that $N_{\Phi, p} = v_p(\rho(\mathbf{a}, \mathbf{b}))$. Thus if $v_p(c) > 0$, we contradict the minimality

of R_φ , and if $v_p(c) < 0$, we contradict the fact that R_φ is an effective divisor. \square

Let $p \in C$. Any $[\Gamma] \in PGL_2(K)$ is a presentation of an associated degree 1 map, and we may always take our $\Gamma \in GL_2(K)$ to be p -model of $[\Gamma]$. This is the same as requiring that $v_p(\Gamma) = 0$, as defined above. Under this assumption, the valuation at p of the determinant of $[\Gamma]$ is well defined. In fact, the determinant is just the degree 1 case of the resultant, and so this is just the value at p of the resultant divisor of the presentation $[\Gamma]$ of the associated degree one map. We will denote this number by $v_p(\det[\Gamma])$.

Proposition 6.7. *Let $p \in C$. The presentation Φ realizes the minimal resultant at p if and only if for every $[\Gamma] \in PGL_2(K)$ such that $v_p(\det[\Gamma]) > 0$, we have*

$$\frac{n_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)}{v_p(\det[\Gamma])} \leq \frac{d+1}{2}$$

where (\mathbf{a}, \mathbf{b}) is a p -model of Φ .

Proof. The assumption that (\mathbf{a}, \mathbf{b}) is a p -model means $n_p(\mathbf{a}, \mathbf{b}) = 0$. Thus, recalling the definition of $R_{[\mathbf{a}, \mathbf{b}]}$ we have that

$$(R_{[\mathbf{a}, \mathbf{b}]})_p \leq (R_{[\mathbf{a}^\Gamma, \mathbf{b}^\Gamma]})_p$$

$$\iff$$

$$v_p(\rho(\mathbf{a}, \mathbf{b})) \leq v_p(\rho(\mathbf{a}, \mathbf{b})) + (d^2 + d)v_p(\det[\Gamma]) - 2dn_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$$

and the statement follows by canceling and dividing. \square

To check this condition, it may be helpful to know that it suffices to check it for only certain sorts of conjugates. In the function field case, we have the following:

Proposition 6.8. *Let $K = k(C)$ where C is a nonsingular projective curve over k . To check minimality at p , it suffices to check that the condition of Proposition 6.7 holds for*

$[\Gamma] \in PGL_2(K)$ where $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a p -model of one of the following 3 forms:

$$(1) \alpha(p), \beta(p) = 0; \gamma(p), \delta(p) \neq 0$$

$$(2) \alpha(p), \beta(p) \neq 0; \gamma(p), \delta(p) = 0$$

$$(3) \alpha(p), \beta(p), \gamma(p), \delta(p) \neq 0$$

Proof. First note that by Proposition 6.2 (5), any conjugation by a matrix in $GL_2(k)$ will have no affect on the value of $R_{[\mathbf{a}, \mathbf{b}]}$. Thus if $M \in GL_2(k)$, we have

$$(R_{[\mathbf{a}^\Gamma, \mathbf{b}^\Gamma]})_p = (R_{[\mathbf{a}^{\Gamma \cdot M}, \mathbf{b}^{\Gamma \cdot M}]})_p.$$

So it is no loss of generality to multiply the Γ we start with *on the right* by such a matrix.

By the same proposition, we need only consider conjugations for which

$$v_p(\det(\Gamma)) > 0.$$

The possible configurations for the vanishing of $\alpha, \beta, \gamma, \delta$ that yield this are

$$A = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, B = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, C = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, D = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, E = \begin{pmatrix} * & * \\ * & * \end{pmatrix},$$

$$F = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, H = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, J = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, \text{ and } L = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

where a * indicates that the coefficient has positive valuation. Then since,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha + \beta \\ \gamma & \gamma + \delta \end{pmatrix},$$

we see by the basic addition inequalities of valuations that we can reduce the case of B into case of L , the case of E into the case of J , and the case of H into the case of A . Likewise, we have that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha + \beta & \beta \\ \gamma + \delta & \delta \end{pmatrix}.$$

This reduces the case of C to L , D to J , and F to A . Thus only the cases of A , J , and L remain. □

To reduce the forms in cases J and L to the form in case A , we could multiply on the left by some M in an analogous way as above. However since we are acting first by M instead of by Γ , it is not immediate as before that $(R_{[\mathbf{a}^\Gamma, \mathbf{b}^\Gamma]})_p = (R_{[\mathbf{a}^{M\Gamma}, \mathbf{b}^{M\Gamma}]})_p$.

Below, we will prove another minimality criteria having to do with the symmetric functions of the multipliers of periodic points—it is this criterion that will prove most useful.

Chapter 7

Semi-stability, Minimality, Examples

In this chapter we will explore further the notions of semi-stability, minimality, and their relationship. We will show a counterexample to the dynamical analogue of Szpiro's theorem, in both the number field and function field case. We will also consider some directions for further research in regards to these topics.

7.1 Semi-stable Presentations and the Conductor

One of our initial inspirations, to formulate a dynamical analog to Theorem 5.1, comes from the natural map

$$C \rightarrow \mathbb{P}^{2d+1}$$

used in [10], and mentioned above, that arises from a given morphism $\varphi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ and a presentation Φ . Let us outline precisely how this map is obtained. A presentation of a morphism Φ is a K -valued point of the scheme Rat_d —i.e. it is a morphism $\text{Spec}(K) \rightarrow \text{Rat}_d$. Composing with the inclusion into projective space, we get a map $\text{Spec}(K) \rightarrow \mathbb{P}^{2d+1}$. Then

the valuative criterion for properness (Theorem 2.6) tells us that this map now extends uniquely to give us a map $C \rightarrow \mathbb{P}^{2d+1}$.

If this map lands in the semi-stable locus $(\mathbb{P}^{2d+1})^{ss}$, then we say that the presentation Φ is a semi-stable presentation:

Definition 7.1. *Let Φ be a presentation of φ . If the morphism $C \rightarrow \mathbb{P}^{2d+1}$ factors through $(\mathbb{P}^{2d+1})^{ss}$, then Φ is a **semi-stable presentation** of φ .*

If Φ is a semi-stable presentation, then for all $p \in C$, the reduction of Φ at p is in $(\mathbb{P}^{2d+1})^{ss}(\kappa(p))$. The following theorem says that if Φ is a semi-stable presentation of φ , then the locus of non-singular points for Φ is equal to the locus of points of good reduction of φ .

Theorem 7.1. *Let $\varphi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ be a morphism with degree $d > 1$ and \mathfrak{f}_φ be the conductor of φ . Let $\mathfrak{L}_\varphi = C \setminus \mathfrak{f}_\varphi$ be the set of points of C where φ has good reduction and $\Phi = [\mathbf{a}, \mathbf{b}]$ be a semi-stable presentation of φ . Let U_Φ be the open set of C such that Φ has non-singular reduction. Then $U_\Phi = \mathfrak{L}_\varphi$.*

Proof. First we will show, without any loss in generality, we may replace the field K with a finite extension K' . That is, it is sufficient to prove the theorem for the map induced by a finite extension K' , $\varphi_{K'} : \mathbb{P}_{K'}^1 \rightarrow \mathbb{P}_{K'}^1$. Considering this extension is necessary to use techniques from [10]. Under the assumptions on C , one can construct a curve C' and a finite morphism $C' \rightarrow C$ (see Theorem 2.4).

Suppose $U_{\Phi_{K'}} = \mathfrak{L}_{\varphi_{K'}}$, where $U_{\Phi_{K'}}$ and $\mathfrak{L}_{\varphi_{K'}}$ are the subsets of C' defined above associated to the morphism $\varphi_{K'}$ and presentation $\Phi_{K'}$ defined over the extension field K' . Suppose the locus of singular reduction $T' = C' \setminus U_{\Phi_{K'}}$ and the conductor $\mathfrak{f}_{\varphi_{K'}} = C' \setminus \mathfrak{L}_{\varphi_{K'}}$ are equal. If we restrict the finite map $C' \rightarrow C$ to $T' \subseteq C'$, we obtain a surjection $\varpi : T' \rightarrow T$, where $T = C \setminus U_{\Phi}$. If $p \in T$ then there exists a $p' \in T'$ such that $\varpi(p') = p$. Since $T' = \mathfrak{f}_{\varphi_{K'}}$ it follows that $p = \varpi(p') \in \mathfrak{f}_{\varphi}$, because the conductor over K' will land in the conductor over K via the map $C' \rightarrow C$. This shows, $T \subseteq \mathfrak{f}_{\varphi}$. We have $\mathfrak{f}_{\varphi} \subseteq T$ by construction. Therefore $\mathfrak{f}_{\varphi} = T$ and $U_{\Phi} = \mathfrak{L}_{\varphi}$.

Now, by the discussion above there exists a map $U_{\Phi} \rightarrow \text{Rat}_d$, and composing with the quotient we have

$$U_{\Phi} \rightarrow \text{Rat}_d \rightarrow \mathcal{M}_d.$$

The first half of this can be trivially extended to $C \rightarrow \mathbb{P}^{2d+1}$. Given that Φ is a semi-stable presentation we know that this extension factors through the semi-stable locus $(\mathbb{P}^{2d+1})^{ss}$.

Therefore composing with the quotient map we obtain

$$C \xrightarrow{F_{\Phi}} (\mathbb{P}^{2d+1})^{ss} \xrightarrow{\pi} (\mathcal{M}_d)^{ss}$$

and denote this composition by $f : C \rightarrow (\mathcal{M}_d)^{ss}$.

We will now be applying the methods in [10]; to do so we will have to consider the “good reduction loci” of models of presentations, rather than of the presentations themselves, as the argument there requires. So for any presentation Ψ and associated model

(\mathbf{a}, \mathbf{b}) we define $U_{(\mathbf{a}, \mathbf{b})}$ be the open subset of C consisting of those points p where (\mathbf{a}, \mathbf{b}) is a p -model over $\mathcal{O}_{C,p}$ and the resultant $\rho(\mathbf{a}, \mathbf{b})$ is a unit. This is a subset of the non-singular reduction locus of Ψ , and also yields a map $U_{(\mathbf{a}, \mathbf{b})} \rightarrow \text{Rat}_d$. Composing with the quotient map as above gives

$$U_{(\mathbf{a}, \mathbf{b})} \rightarrow \text{Rat}_d \rightarrow \mathcal{M}_d.$$

Considering now all possible models of presentations of φ , we note that each open set $U_{(\mathbf{a}, \mathbf{b})}$ is contained in \mathfrak{U}_φ and we can take a collection of these open sets $\{U_{(\mathbf{a}, \mathbf{b})_i}\}$ to create a cover of \mathfrak{U}_φ . Choose a cover that contains $U = U_{(\mathbf{a}_0, \mathbf{b}_0)}$ where now $(\mathbf{a}_0, \mathbf{b}_0)$ is a model of the presentation Φ . Using the arguments from [10] §3.4 we can construct a morphism

$$\mathfrak{U}_\varphi \rightarrow \mathcal{M}_d$$

from the morphisms $U_{(\mathbf{a}, \mathbf{b})_i} \rightarrow \text{Rat}_d \rightarrow \mathcal{M}_d$. Due to the methods in [10] we have to consider a base extension of K . However, we have shown that this will not affect our argument. Now we have the composition

$$\mathfrak{U}_\varphi \rightarrow \mathcal{M}_d \hookrightarrow (\mathcal{M}_d)^{ss}.$$

By Theorem 2.6, since $(\mathcal{M}_d)^{ss}$ is projective, the composition of these morphisms, $\mathfrak{U}_\varphi \rightarrow (\mathcal{M}_d)^{ss}$ extends uniquely to C ,

$$f' : C \rightarrow (\mathcal{M}_d)^{ss}$$

Observe from our two constructions that $f'|_U = f|_U$. The equality, $f' = f$, follows again from Theorem 2.6.

Now we must note that the (scheme theoretic) points of $(\mathbb{P}^{2d+1})^{ss}$ that are not in Rat_d go to points of $(\mathcal{M}_d)^{ss} \setminus \mathcal{M}_d$ under π . This is because π is the restriction to the semi-stable locus of the canonical rational map induced by the inclusion of graded rings $\mathbb{Z}[\mathbf{a}, \mathbf{b}]^{SL_2} \hookrightarrow \mathbb{Z}[\mathbf{a}, \mathbf{b}]$.

With this in mind, as we stated earlier, $U_\Phi \subset \mathcal{L}_\Phi$. Suppose $p \in \mathcal{L}_\Phi$ and p not in U_Φ . Then $f(p) \in (\mathcal{M}_d)^{ss} \setminus \mathcal{M}_d$ by our construction of f and the fact p is not in U_Φ . On the other hand, by our construction of f' , $f'(p) \in \mathcal{M}_d$ which is a contradiction. Therefore, $\mathcal{L}_\Phi \setminus U_\Phi$ must be empty. \square

An interesting question that we have not explored is *when* a morphism has a semi-stable presentation.

The above theorem says that semi-stable presentations realize the conductor as their singular reduction locus, a property that any minimal presentation certainly must have. So one might hope that semi-stable presentations are also minimal. We have two partial results in this direction. One treats the degree two case: Theorems 7.9 and 7.10 below. The other is the following.

Proposition 7.2. *Let $\Phi \in \text{Rat}_d(K)$ be a semi-stable presentation. Suppose $d = 2r$ is even.*

Let $p \in C$. Then $N_{\Phi,p}$ is minimal for all $\text{PGL}_2(K)$ conjugates of Φ by diagonal elements

$$[\Gamma] = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}.$$

Proof. We will show that

$$\frac{n_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)}{v_p(\det(\Gamma))} \leq \frac{d+1}{2}$$

for any $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ in $\mathrm{GL}_2(K)$ with $v_p(u)$ and $v_p(v)$ both greater than or equal to 0. The result will follow by Proposition 6.7 (it is easy to see that this proposition can be appropriately restricted to any subset of $\mathrm{PGL}_2(K)$). By Theorem 4.2, we know that at least one of the coefficients

$$a_0, \dots, a_{r-1}, b_0, \dots, b_r \tag{7.1}$$

doesn't vanish at p . By conjugation, we may say the same about the coefficients,

$$a_r, \dots, a_d, b_{r+1}, \dots, b_d.$$

Now conjugate by $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$. The new coefficients are

$$vu^d a_0, v^2 u^{d-1} a_1, \dots, v^r u^{r+1} a_{r-1}, u^{d+1} b_0, vu^d b_1, \dots, v^r u^{r+1} b_r \tag{7.2}$$

and

$$v^{r+1} u^r a_r, v^{r+2} u^{r-1}, \dots, v^{d+1} a_d, v^{r+1} u^r b_{r+1}, v^{r+2} u^{r-1} b_{r+2}, \dots, v^d u b_d. \tag{7.3}$$

In (7.2), the power of v that shows up is at most r , and in (7.3) the power of u that shows up is at most r . Since the patterns of powers of v that show up in (7.2) and (7.3)

are identical when we permute u and v in one of them, it will be no loss of generality to suppose that $n = v_p(u) \leq v_p(v) = m$, and consider the coefficients in (7.2).

Now, $v_p(\det(\Gamma)) = n + m$. Take any of the coefficients in (7.2) such that the corresponding original coefficient from (7.1) doesn't vanish. Call this coefficient c . Thus the valuation of c at p will come entirely from the contributions of the powers of u and v . Further, we know that the power v makes at most a contribution of rm to the valuation. Set $k \leq r$ to be the exponent on v in c . We have

$$\begin{aligned}
 n_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma) &\leq v_p(c) \\
 &= (d+1-k)n + km \\
 &= (d+1)n - kn + km \\
 &= (d+1)n - 2kn + kn + km \\
 &= (d+1-2k)n + k(n+m).
 \end{aligned}$$

Since $2k \leq 2r \leq d$, we have that $d+1-2k$ is positive. Since $n \leq m$ implies that $n \leq \frac{m+n}{2}$,

we have

$$\begin{aligned}
 (d+1-2k)n + k(n+m) &\leq (d+1-2k) \left(\frac{m+n}{2} \right) + 2k \left(\frac{m+n}{2} \right) \\
 &= (d+1) \left(\frac{m+n}{2} \right).
 \end{aligned}$$

Thus

$$\frac{n_p(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)}{m+n} \leq \frac{d+1}{2}.$$

□

The above argument does not work for d odd. However, with Proposition 6.2 (5), we may strengthen this result.

Corollary 7.3. *In the above setting, $N_{\Phi,p}$ is minimal among all conjugates of Φ of the form*

$$[\Gamma] = [D][\Gamma'],$$

where $[D] = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ as above and $\Gamma' \in \mathrm{GL}_2(\mathcal{O}_{C,p})$.

Proof. Proposition 6.2 (5) implies directly that $v_p(\rho([\mathbf{a}^{D\Gamma'}, \mathbf{b}^{D\Gamma'}])) = v_p(\rho([\mathbf{a}^D, \mathbf{b}^D]))$, and then we apply Proposition 7.2. □

7.2 Building A Counterexample

In this section, we will prove Theorem 6.1.

In what follows, by “a counterexample to a dynamical analog to Theorem 5.1,” we mean an infinite collection of maps for which the degree of the conductor is bounded, and for which the degree of the minimal resultant is unbounded.

7.2.1 Two Examples

Example 1

The following example is due originally to Patrick Ingram, who gave it in the number field case (with t replaced by p a prime number, below). It gives a collection of presentations of rational maps with only one point of singular reduction, but of unbounded degree of the resultant for that presentation. Since we do not know the presentation is minimal at the point of singular reduction, this does not give us an immediate counterexample to the dynamical analog of Theorem 5.1, but it is a good starting point for addressing the question of how to construct one.

Let $C = \mathbb{P}_k^1$, so $K = k(t)$. Consider the polynomial map

$$x^2 + t^{-N}$$

where $N > 0$. Written in projective coordinates, this is

$$\varphi = \frac{X^2 + t^{-N}Y^2}{Y^2}$$

Let $\Phi = [\mathbf{a}, \mathbf{b}]$ be the associated presentation. We can see immediately that Φ has good reduction at ∞ , so that $(R_{[\mathbf{a}, \mathbf{b}]})_\infty = 0$. For the reduction at the other points, we'll multiply throughout by t^N

$$\varphi = \frac{t^N X^2 + Y^2}{t^N Y^2}.$$

It is then a simple matrix calculation to see that the resultant of this is t^{4N} . This means that the only point of singular reduction with respect to this presentation is $t = 0$. Notice

here that the reduction is unstable, by Theorem 4.3 and $(R_{[\mathbf{a}, \mathbf{b}]})_0 = 4N$. Thus if we could show that this presentation is minimal for $t = 0$, we'd have a family of rational maps with the degree of the conductor equal to 1, and degree of the minimal resultant unbounded in terms of N .

In fact, this presentation is not minimal for $t = 0$. At least for $N = 2M$, there is a conjugate that lowers the degree of the resultant by a factor of 4. This obtained by conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & t^M \end{pmatrix}$, yielding

$$\frac{t^{3M}X^2 + t^{3M}Y^2}{t^{4M}Y^2} = \frac{X^2 + Y^2}{t^M Y^2}.$$

The resultant of this is computed to be t^N , so the degree of the resultant divisor at $t = 0$ has been lowered. However, it is still unbounded in terms of N .

Example 2

In analogy with hypotheses of Theorem 5.1, one might expect that having a semi-stable presentation would be a necessary or helpful condition for formulating and proving a dynamical analog. It is easy to see, however, that Example 1 has unstable reduction at $t = 0$, so we don't know that this example qualifies. We are thus led to seek a counterexample to a dynamical analog of Theorem 5.1 that also admits a semi-stable presentation.

Suppose now φ is a degree 2 map that can be written in the form:

$$\frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

Rational maps of this form are said to be in *normal form*. In this form, the coefficients

λ_1 and λ_2 are two of the multipliers of φ . For the presentations corresponding to such forms, we prove the following necessary and sufficient criterion for semi-stability.

Proposition 7.4. *Let φ be a morphism of degree 2 over K that can be written in normal form:*

$$\varphi = \frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

Then the corresponding presentation $\Phi = [1, \lambda_1, 0, 0, \lambda_2, 1]$ is a semi-stable presentation if and only if:

- (1) *any poles of λ_1 and λ_2 occur at exactly the same points, where moreover they have the same multiplicity and*
- (2) *λ_1 and λ_2 never evaluate simultaneously to 1.*

Proof. By Theorem 4.3, for singular reduction of a degree 2 rational map to be unstable, it is necessary and sufficient that a common root showing up in reduction is either of order 2, or of order 1 while also being a fixed point of the map obtained after canceling all the common roots.

Suppose now Φ is a semi-stable presentation. Then if λ_1 has a pole of higher negative order than λ_2 at some $p \in C$, in order to take the reduction of Φ at p we multiply the coefficients throughout by an appropriate power of the uniformizer to cancel this pole. Since the pole of λ_1 is of higher negative order, this causes the reduced map to look like

$$\frac{aXY}{0}.$$

Thus, when we cancel common factors, the resulting map is the constant map that sends everything to ∞ . Thus ∞ is a fixed point of this map. Since Y was one of the roots that we canceled, this map is unstable. A similar argument applies if λ_2 has a pole of higher negative order. This shows **(1)**.

For **(2)**, we see easily that, if $\lambda_1(p) = \lambda_2(p) = 1$, the reduced map looks like

$$\frac{X^2 + XY}{XY + Y^2}.$$

After canceling $X + Y$, we get

$$\frac{X}{Y},$$

This is the identity map, and so it has $[-1, 1]$, the canceled root, as a fixed point. Thus the reduction is unstable. This shows **(2)**.

Conversely, if **(1)** and **(2)** hold, first note that the resultant of Φ is $1 - \lambda_1 \lambda_2$, by a simple matrix calculation. Thus when the product $\lambda_1 \lambda_2$ evaluates to 1, we will have singular reduction. By **(2)**, the reduction must be of the form

$$\frac{X^2 + aXY}{a^{-1}XY + Y^2}$$

where $a \neq 1$. After cancelation, this is

$$\frac{aX}{Y}$$

and $[-a, 1]$, the canceled root, is not a fixed point. Hence the reduction is semi-stable.

The other possibility for singular reduction is when some of the coefficients have poles, which in this case, by **(1)**, can only happen at the common poles of λ_1 and λ_2 . For these, we see that the reduction is of the form

$$\frac{aXY}{bXY}$$

where a and b are not 0. Thus after canceling the common roots X and Y , we get a constant map that is not 0 or ∞ , so that neither 0 nor ∞ is a fixed point, and hence the reduction is semi-stable. \square

Corollary 7.5. *Suppose $a, b, b' \neq 1$ and let, for each $N \in \mathbb{Z}$ with $N \geq 1$*

$$\begin{aligned}\lambda_1 &= a + bt^N \\ \lambda_2 &= a^{-1} + b't^N.\end{aligned}$$

where $a, b, b' \in k$ satisfy $a, b, b' \neq 0, 1$, and $ab' + b/a = 0$. Then $\Phi = [1, \lambda_1, 0, 0, \lambda_2, 1]$ is a semi-stable presentation. In addition, for each N , the non-singular reduction locus of Φ contains 2 points, and the degree of the resultant divisor of Φ is at least $2N$.

Proof. The corresponding Φ , by construction, satisfies **(1)** and **(2)** and is therefore a semi-stable presentation by Proposition 7.4. The only pole of λ_1, λ_2 is ∞ , and so we have one

point of singular reduction there. The resultant is

$$\begin{aligned} 1 - \lambda_1 \lambda_2 &= 1 - (a + bt^N)(a^{-1} + b't^N) \\ &= -(ab' + b/a)t^N - bb't^{2N} \\ &= -bb't^{2N} \end{aligned}$$

Thus we have only one other point of singular reduction (at $t = 0$), whose multiplicity in the resultant divisor for this presentation is $2N$. Thus the non-singular reduction locus has degree 2, and the degree of the resultant divisor is unbounded in terms of N . \square

Corollary 7.5 implies that the degree of the resultant divisor of a semi-stable presentation cannot be bounded in terms of the conductor. In order for this to be a counterexample to a dynamical analog of Theorem 5.1, there is still the question of the minimality of this presentation at the unbounded point.

7.2.2 Proving Minimality

In attempting to show the minimality of these two examples at the unbounded points, one line of attack is to act by the group and then analyze the order of vanishing at t of the coefficients, and apply Proposition 6.7. Doing this for Example 2 leads to the following partial result.

Proposition 7.6. *The above example is minimal at $t = 0$ with respect to conjugations of the form $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$ where $v_0(\alpha) > 0$.*

Proof. We apply the criteria of Proposition 6.7, but with respect to the restricted subset of $\mathrm{GL}_2(K)$ stated in the theorem. First, by the inner substitution

$$\begin{aligned} \frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2} &\rightarrow \frac{(\alpha X + \beta Y)^2 + \lambda_2(\alpha X + \beta Y)Y}{\lambda_2(\alpha X + \beta Y)Y + Y^2} \\ &= \frac{\alpha^2 X^2 + (2\alpha\beta + \lambda_1\alpha)XY + (\beta^2 + \lambda_1\beta)Y^2}{\lambda_2\alpha XY + (\lambda_2\beta + 1)Y^2}. \end{aligned}$$

Now performing the outer substitution yields

$$\frac{\alpha^2 X^2 + (2\alpha\beta + \lambda_1\alpha - \beta\lambda_2\alpha)XY + (\beta^2 + \lambda_1\beta - \lambda_2\beta^2 - \beta)Y^2}{\lambda_2\alpha^2 XY + (\alpha\lambda_2\beta + \alpha)Y^2}. \quad (7.4)$$

Suppose minimality fails. This means we can find α, β such that minimal valuation at 0 of the coefficients of (7.4), which we will call n , satisfies

$$\frac{n}{v_0(\alpha)} > \frac{3}{2}.$$

This will certainly imply

$$n > v_0(\alpha).$$

This imposes three restrictions

- (a) $v_0(\alpha\lambda_2\beta + \alpha) > v_0(\alpha)$,
- (b) $v_0(2\alpha\beta + \lambda_1\alpha - \beta\lambda_2\alpha) > v_0(\alpha)$,
- (c) $v_0(\beta(\beta + \lambda_1 - \lambda_2\beta - 1)) > v_0(\alpha)$.

Factoring α in (a) and (b), and observing that by assumption $v_0(\alpha) > 0$ in 3, we reduce these to

- (a) $v_0(\lambda_2\beta + 1) > 0$,
- (b) $v_0(2\beta + \lambda_1 - \beta\lambda_2) > 0$,
- (c) $v_0(\beta(\beta + \lambda_1 - \lambda_2\beta - 1)) > 0$.

Now, since λ_2 doesn't vanish at 0, (a) implies that $v_0(\beta) = 0$, and so we already have a contradiction if $v_0(\beta) \neq 0$, proving minimality among those. If $v_0(\beta) = 0$ then (c) reduces to

$$v_0(\beta + \lambda_1 - \lambda_2\beta - 1) > 0. \quad (7.5)$$

Let β_0 be the evaluation of β at 0. Then it follows from (7.5) and (b) respectively that

$$\beta_0 + a - a^{-1}\beta_0 - 1 = 0 \quad (7.6)$$

$$2\beta_0 + a - a^{-1}\beta_0 = 0. \quad (7.7)$$

Subtracting yields

$$-\beta_0 - 1 = 0,$$

so $\beta_0 = -1$. But then by (7.6), $-2 + a + a^{-1} = 0$, which implies $a = 1$, a contradiction. \square

The brute force methods used in the above proof—using the basic properties of valuations in an effort to find a contradiction—appear to be insufficient to test for minimality in general. But a different approach may be taken. We are able to show the minimality of the above example via a minimality criteria that is proven using the PGL_2 invariance of the

symmetric functions of the periodic points of the multipliers of a rational map, discussed above.

For a given φ , we defined $\sigma_{i,n}(\varphi)$ be the i -th symmetric function in the multipliers of these periodic points. Each $\sigma_{i,n}(\varphi)$ is PGL_2 invariant and thus depends only on φ and not on the presentation and model we choose to represent it. Theorem 4.3 tells us we can find $\sigma_{i,n}$ of ϕ by evaluating a generic form for any choice of presentation and model:

$$\sigma_{i,n}(\phi) = \frac{P_{i,n}(\mathbf{a}, \mathbf{b})}{(\rho(\mathbf{a}, \mathbf{b}))^m} \quad (7.8)$$

where $m \geq 0$ is an integer (depending on i and n) and $P_{i,n}(\mathbf{a}, \mathbf{b})$ is a homogeneous polynomial of degree $2dm$ (so that the fraction is degree zero). We may use this to prove a criterion for minimality.

Theorem 7.7. *Let $\Phi = [\mathbf{a}, \mathbf{b}]$ be a presentation of a degree d rational map φ over a field K . Let $p \in C$ be a point of singular reduction, and suppose that (\mathbf{a}, \mathbf{b}) is a p -model of Φ . Consider any of the $\sigma_{i,n}(\varphi)$, and let $P_{i,n}(\mathbf{a}, \mathbf{b})$ and be as above, and suppose $m \geq 1$. If $v_p(P_{i,n}(\mathbf{a}, \mathbf{b})) = 0$, then $(R_\varphi)_p = (R_\Phi)_p$ (i.e. R_Φ is minimal at p).*

Proof. Recall that, in general, $(R_\Phi)_p$ is simply the valuation of $\rho(u_p \mathbf{a}, u_p \mathbf{b}) \in K$, where u_p is chosen so that $(u_p \mathbf{a}, u_p \mathbf{b})$ is a p -model of Φ . Since we have assumed this already holds, we can take $u_p = 1$, i.e.

$$S := (R_\Phi)_p = v_p(\rho(\mathbf{a}, \mathbf{b})) \quad (7.9)$$

Since (\mathbf{a}, \mathbf{b}) is a p model, all of the coefficients of (\mathbf{a}, \mathbf{b}) are in $\mathcal{O}_{C,p}$. Let $[\Gamma] \in PGL_2(K)$, where the representative $\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a p -model of $[\Gamma]$. Let $(\mathbf{a}'', \mathbf{b}'')$ be the new coefficients obtained via the action of Γ on (\mathbf{a}, \mathbf{b}) . Viewed on the affine cone, recall that this action is pre-composition by Γ and post-composition by the adjoint of Γ (it descends to the conjugation action when we pass to projective space). Because of this, the new coefficients $(\mathbf{a}'', \mathbf{b}'')$ are in $\mathcal{O}_{C,p}$.

We now cancel the greatest common power of the uniformizer occurring in each coefficient; this gives us new coefficients $(\mathbf{a}', \mathbf{b}')$, still all in $\mathcal{O}_{C,p}$, and now $(\mathbf{a}', \mathbf{b}')$ is a p -model of $[\mathbf{a}^\Gamma, \mathbf{b}^\Gamma]$. Now we have that

$$S' := (R_{\Phi^\Gamma})_p = v_p(\rho(\mathbf{a}', \mathbf{b}')) \quad (7.10)$$

We need to show that $S \leq S'$. Now, by assumption, $v_p(P_{i,n}(\mathbf{a}, \mathbf{b})) = 0$. This implies, by (7.8) and (7.9), that $v_p(\sigma_{i,n}(\varphi)) = -Sm$. Let $r = v_p(P_{i,n}(\mathbf{a}', \mathbf{b}'))$. Since the coefficients $(\mathbf{a}', \mathbf{b}')$ are in $\mathcal{O}_{C,p}$, we have that $r \geq 0$. Thus by (7.8), (7.10), and $PGL_2(K)$ invariance:

$$\begin{aligned} -Sm &= v_p(\sigma_{i,n}(\varphi^\Gamma)) \\ &= v_p\left(\frac{P_{i,n}(\mathbf{a}', \mathbf{b}')}{\rho(\mathbf{a}', \mathbf{b}')}\right) \\ &= r - S'm \\ &\geq -S'm \end{aligned}$$

Hence $S \leq S'$. □

Corollary 7.8. *Let $K = k(t)$. Let φ be the degree 2 morphism given by:*

$$\frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

where $\lambda_1 = a + bt^N$, $\lambda_2 = a^{-1} + b't^N$, $a, b, b' \neq 0, 1$, and $ab' + b/a = 0$. Let Φ be the corresponding presentation. Then $(R_\Phi)_0 = (R_\varphi)_0$ (i.e. R_Φ is minimal at $0 \in \mathbb{A}^1$).

Proof. The symmetric function $\sigma_1(\varphi)$ is simply

$$\lambda_{P_1} + \lambda_{P_2} + \lambda_{P_3},$$

where the λ_{P_i} are the multipliers of the fixed points P_i of φ . Silverman provides, in [13], the coefficients for the general formula for σ_1 mentioned above, and this makes it easy for us to calculate $\sigma_1(\varphi)$ (alternatively, we could make the calculation as in Theorem 7.9 below). The formula is as follows. First set

$$\begin{aligned} P(\mathbf{a}, \mathbf{b}) = & a_1^3 b_0 - 4a_0 a_1 a_2 b_0 - 6a_2^2 b_0^2 - a_0 a_1^2 b_1 + 4a_1 a_2 b_0 b_1 \\ & - 2a_0 a_2 b_1^2 + a_2 b_1^3 - 2a_1^2 b_0 b_2 + 4a_0 a_2 b_0 b_2 \\ & - 4a_2 b_0 b_1 b_2 - a_1 b_1^2 b_2 + 2a_0^2 b_2^2 + 4a_1 b_0 b_2^2 \end{aligned}$$

Then we have

$$\sigma_1(\varphi) = \frac{P(\mathbf{a}, \mathbf{b})}{a_2^2 b_0^2 - a_1 a_2 b_0 b_1 + a_0 a_2 b_1^2 + a_1^2 b_0 b_2 - 2a_0 a_2 b_0 b_2 - a_0 a_1 b_1 b_2 + a_0^2 b_2^2} \quad (7.11)$$

Since the given model is a p -model for $p = 0 \in \mathbb{A}^1$, all we must show is that the form $P(\mathbf{a}, \mathbf{b})$ doesn't vanish at $t = 0$ for these coefficients. Thus we must show that the constant term is nonzero; this is a simple calculation by plugging in to the expression for $P(\mathbf{a}, \mathbf{b})$ above. Most of the terms vanish, and we are left with the expression:

$$-(a + bt^N)^2(a^{-1} + b't^N) - (a + bt^N)(a^{-1} + b't^N)^2 + 2$$

The constant term of this expression is $-a - a^{-1} + 2$, which can never be zero, by the assumption that $a \neq 1$. □

By generalizing the above proof, we can show that semi-stable presentations of degree two maps in normal form are minimal at certain places.

Theorem 7.9. *Let Φ be a normal form presentation of a degree 2 morphism φ :*

$$\varphi = \frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

Suppose Φ has everywhere semi-stable reduction, and let P be the common poles of λ_1 and λ_2 , as described in Proposition 7.4. Then $(1, \lambda_1, 0, 0, \lambda_2, 1)$ is a P -minimal global model for Φ .

Proof. The singular reduction of Φ occurs either at the common poles of λ_1 and λ_2 , or where $\lambda_1 \lambda_2$ evaluates to 1. We must show that in the latter case $(R_\Phi)_p$ is minimal. Let p be such a point of singular reduction. It is known (see [13]) that in general $\sigma_1 = \sigma_3 + 2$.

Thus if we let λ_3 be the third multiplier of φ (in normal form, the first two multipliers are λ_1 and λ_2), we have:

$$\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 \lambda_2 \lambda_3 + 2$$

Thus

$$\lambda_3 = \frac{2 - \lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2}$$

so that

$$\sigma_1 = \lambda_1 + \lambda_2 + \frac{2 - \lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2}$$

From this we can calculate $P(\mathbf{a}, \mathbf{b})$ directly in terms of λ_1 and λ_2 :

$$\begin{aligned} \text{Res}(\Phi)\sigma_1 &= (1 - \lambda_1 \lambda_2) \left(\lambda_1 + \lambda_2 + \frac{2 - \lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \right) \\ &= -\lambda_1^2 \lambda_2 - \lambda_1 \lambda_2^2 + 2 \end{aligned}$$

Let $a = \lambda_1(p)$. Then since the resultant vanishes at p , $\lambda_2(p) = a^{-1}$. Further, we see from Proposition 7.4 that $a \neq 1$. Hence we have:

$$-\lambda_1(p)^2 \lambda_2(p) - \lambda_1(p) \lambda_2(p)^2 + 2 = -a - a^{-1} + 2 \neq 0$$

By Theorem 7.7, R_Φ is minimal at p . □

Using a slightly different approach, we now show that, in the function field situation, semi-stability implies minimality for all degree two maps.

Theorem 7.10. *Let K be a function field over an algebraically closed field k . Let φ be a morphism of degree two, and let Φ be a presentation of φ . Let $p \in C$. If the reduction of Φ at p is semi-stable, then $(R_\varphi)_p = (R_\Phi)_p$.*

Proof. First we need a lemma.

Lemma 7.11. *Let K be a function field over an algebraically closed field k . Let φ be a morphism, $\Phi \in \text{Rat}_d(K)$ a presentation of φ , and $p \in C$. Let $\Gamma \in GL_2(k)$. Then $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ is a p -model of Φ^Γ and $(\Phi^\Gamma)_p = \Phi_p^\Gamma$.*

Proof. Let (\mathbf{a}, \mathbf{b}) be a p -model of Φ . Each coefficient of $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ is just a polynomial in $\alpha, \beta, \gamma, \delta, a_0, \dots, a_d, b_0, \dots, b_d$, and so it is clear that $(\mathbf{a}^\Gamma(p), \mathbf{b}^\Gamma(p)) = (\mathbf{a}(p)^\Gamma, \mathbf{b}(p)^\Gamma)$. Since the right hand side descends to a point of projective space, $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$ must be a p -model of Φ^Γ , and we have the desired equality. \square

Let now (\mathbf{a}, \mathbf{b}) be a p -model of our Φ . It follows from Lemma 6.2 of [15] that we may find a $\Gamma \in GL_2(k)$ such that $\Phi_p^\Gamma = [0, A, 0, 0, 1, B]$ where A, B are not both zero. Since $\Gamma \in GL_2(\mathcal{O}_{C,p})$, it follows from Proposition 6.2 (5) that it suffices to show minimality of Φ^Γ .

We have formulas for $\rho(\mathbf{a}, \mathbf{b})\sigma_1(\varphi)$ and $\rho(\mathbf{a}, \mathbf{b})\sigma_2(\varphi)$ (the former is given above and both appear together on p.17 of [15]) and they are polynomials in the coefficients (\mathbf{a}, \mathbf{b}) . Applying these formulas to the coefficients $(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)$, we may show the minimality of Φ^Γ at p by showing that at least one does not evaluate to zero at p , by Theorem 7.7.

By Lemma 7.11, we know that $(\mathbf{a}^\Gamma(p), \mathbf{b}^\Gamma(p)) = (0, A, 0, 0, 1, B)$. Plugging in thus yields:

1. $\rho(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)\sigma_1(\varphi)(p) = -AB$
2. $\rho(\mathbf{a}^\Gamma, \mathbf{b}^\Gamma)\sigma_2(\varphi)(p) = -A^2 - B^2$

Clearly (1), (2) cannot simultaneously be zero, and so we are done. \square

The above argument relies on the fact that the residue field of p is embedded in $\mathcal{O}_{C,p}$, which fails, for example, in the number field situation.

7.2.3 The Critical Conductor

In [20], the L. Szpiro and T. Tucker propose an alternative definition of good reduction, called **critical good reduction**. Here we show that it is at least possible that the complement of the locus of the critical good reduction increases without bound for Example 2 above, implying that this example is not necessarily a counterexample to the dynamical analog of Theorem 5.1 under this alternative definition of the conductor.

Following [20], Let φ be a morphism of degree at least 2 over K . Let $[X, Y]$ be coordinates on \mathbb{P}_K^1 . Let $R(\varphi)$ be the ramification divisor of φ over the algebraic closure \bar{K} of K , and let K' be an extension of K such that the points in the support of the ramification divisor are in $\mathbb{P}_{K'}^1$. Let C' be the corresponding curve. Then for $P \in \mathbb{P}_{K'}^1$, we can define the reduction of P at $p \in C'$ by choosing coefficients $(\alpha, \beta) \in \mathcal{O}_{C',p}^2$ with at least

one coefficient a unit, such that $[\alpha, \beta] = P$, and then evaluating these coefficients:

$$r_p(P) := [\alpha(p), \beta(p)]$$

Definition 7.2. *With the notations and assumptions as above, φ has **critical good reduction** at $p \in C$ if:*

1. *If $P, Q \in \text{Supp}(R(\varphi))$ and $P \neq Q$, then $r_p(P) \neq r_p(Q)$.*
2. *If $P, Q \in \varphi(\text{Supp}(R(\varphi)))$ and $P \neq Q$, then $r_p(P) \neq r_p(Q)$.*

*If these conditions are not satisfied, then φ has **critical bad reduction** at p .*

Definition 7.3. *The **critical conductor** of φ is the divisor*

$$f_{cr} = \sum_{p \in S} [p]$$

where S is the set of critical bad reduction.

This definition depends on the choice of coordinates on \mathbb{P}_K^1 . What we will now show is that, with respect to the coordinates $[X, Y]$ in Example 2 above, φ has many (unbounded in terms of N) points of critical bad reduction.

Proposition 7.12. *Let φ be given by:*

$$\varphi = \frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

where $\lambda_1 = a + bt^N$, $\lambda_2 = a^{-1} + b't^N$, $a, b, b' \neq 0, 1$, and $ab' + b/a = 0$. Then, with respect to the coordinates $[X, Y]$, φ has at least $N + 1$ points of critical bad reduction.

Proof. A simple derivative calculation shows that the ramification divisor is defined over K and consists of exactly two points:

$$P = \frac{-1 + i\sqrt{bb't^N}}{a^{-1} + b't^N}, Q = \frac{-1 - i\sqrt{bb't^N}}{a^{-1} + b't^N}$$

P and Q will specialize to the same point over \mathbb{P}_k^1 when $t = 0$, and also when the (common) denominator of these two fractions evaluates to 0. Since a^{-1} and b' are not zero, the equation $a^{-1} + b't^N$ has N distinct roots, yielding an additional N points of critical bad reduction. \square

Combining this with Theorem 6.1, we have:

Theorem 7.13. *For each $N \in \mathbb{Z}^+$, let φ be the degree 2 morphism given by:*

$$\varphi = \frac{X^2 + \lambda_1 XY}{\lambda_2 XY + Y^2}$$

where $\lambda_1 = a + bt^N$, $\lambda_2 = a^{-1} + b't^N$, $a, b, b' \neq 0, 1$, and $ab' + b/a = 0$. Then the degree of the conductor of φ is at most 2, and the degree of the minimal resultant is $2N$, and the degree of the critical conductor is at least $N+1$.

7.2.4 The Number Field Case

In the number field setting, the standard notion of the degree of a divisor is the norm of the associated ideal, made additive by applying log. Thus the degree of the prime divisor \mathfrak{p} on $\text{Spec}(O_K)$ is $f \log p$ where f is the residue degree of \mathfrak{p} over the prime integer p .

Examples 1 and 2 above can be given instead for number fields if we replace the variable t by a prime $p \in \mathbb{Z}$. It is then possible to construct a similar counter example, in the number field setting, to the dynamical analog of theorem 5.1.

Theorem 7.14. *Let K be a number field of degree n over \mathbb{Q} , let $p \in \mathbb{Z}$ be prime, and let $\{\mathfrak{p}_i\}$ be the primes of \mathcal{O}_K lying over p . Suppose a, b, b' are units of \mathcal{O}_K and let, for each $N \in \mathbb{Z}$ with $N \geq 1$*

$$\begin{aligned}\lambda_1 &= a + bp^N \\ \lambda_2 &= a^{-1} + b'p^N.\end{aligned}$$

where $a, b, b' \in K$ also satisfy $a, b, b' \neq 0, 1$, and $ab' + b/a = 0$. Then for each N , the degree of the conductor is at most $n(\log p)$, and the degree of the minimal resultant divisor is $2nN \log p$.

Proof. Let e_i and f_i be the ramification indexes and residue degrees of the \mathfrak{p}_i respectively.

We will use the basic fact that

$$\sum_i e_i f_i = n \tag{7.12}$$

The presentation given is a \mathfrak{q} model for every $\mathfrak{q} \in \text{Spec}(\mathcal{O}_K)$. The resultant for the given presentation, as before, is $bb'p^N$, and since b and b' are units, we therefore have the singular reduction occurring exactly at the primes \mathfrak{p}_i dividing p . The degree of the conductor is therefore $\sum_i f_i \log p$ (the f_i being the residue degrees of the \mathfrak{p}_i), which is at

most $n(\log p)$ by (7.12). For each i , $v_{\mathfrak{p}_i}(bb'p^N) = 2Ne_i$, which, by (7.12) again, gives the desired degree for the resultant divisor of this presentation.

It follows from Theorem 7.9 and the assumption that $a \neq 1$ that the given presentation is minimal at each \mathfrak{p}_i , and so we are done. \square

A concrete example realizing the conditions of Theorem 7.14 is the following: Let $K = \mathcal{Q}(\zeta)$, where ζ is a primitive m -th root of unity for some m . Then take $a = \zeta, b = -\zeta$, and $b' = \zeta^{m-1}$.

The question of what happens to the critical bad reduction for this example, in the number field case, is interesting. It turns out that if we assume the following conjectural version of Theorem 5.1 for number fields we can show that the critical bad reduction for this example grows with N .

Conjecture 7.15. (*Szpiro's Conjecture*) *Let E be an elliptic curve over a number field K of discriminant $D(K)$, let $\Delta(E)$ be the norm of the minimal discriminant of E , and $\mathfrak{f}(E)$ the norm of its conductor. Then for every $\varepsilon > 0$ one has :*

$$\Delta(E) \leq_{\varepsilon} (D(K)\mathfrak{f}(E))^{6+\varepsilon}$$

This conjecture is closely related to the ABC conjecture when applied to elliptic curves of the form:

$$y^2 = x(x-A)(x+B)$$

(cf. [3]).

Theorem 7.16. *Under the conditions of Theorem 7.14, Szpiro's conjecture implies that the norm of the critical conductor is bounded below by a linear increasing function of p^N .*

Proof. Let $\varepsilon > 0$, and let C_ε be the implied constant in the discriminant conjecture. The same calculation as given in Proposition 7.12 shows the critical bad reduction occurs at the usual bad reduction of the given presentation Φ of φ , and also where $\lambda_2 = a^{-1} + b'p^N$ vanishes. The elliptic curve with equation:

$$y^2 = x(x - a^{-1})(x + b'p^N)$$

has minimal discriminant $2^{-8}a^{-2}b'^2p^{2N}$. The discriminant conjecture gives

$$\text{Norm}(f_{cr})^{6+\varepsilon} = \text{Norm}((\lambda_2 p))^{6+\varepsilon} \geq C'_\varepsilon p^{2N}$$

□

7.3 Conjectures for further research

Since the degree of the resultant divisor for the above example, as we have seen, is a linear, increasing function of N , the above theorem (as well as Proposition 7.12) could be rephrased as saying that, for this particular family, we have a bound for the degree of the resultant in terms of the degree of the critical conductor. This means that, in both the function field and number field case, if we consider the critical conductor, a dynamical analogue to Theorem 5.1 may still be possible.

Thus in the spirit of passing from elliptic curves, and their Lattes maps, to dynamical systems on the sphere, we present two conjectures.

Conjecture 7.17. (*Resultant conjecture for a function field*) If K is a function field (over a field k), and φ is a self map of degree d of \mathbb{P}_K^1 with minimal resultant $R(\varphi)$ and critical conductor $f_{cr}(\varphi)$, then there exists a constant $C(K)$ and integer $s(d)$ such that

$$\deg(R(\varphi)) \leq p^e s(d) (C(K) + \deg(f_{cr}(\varphi)))$$

where p is the characteristic of k and e is the inseparability degree of the canonical map, induced by φ , from $\text{Spec}(K)$ to the moduli space \mathcal{M}_d .

Conjecture 7.18. (*Resultant conjecture for a number field*) If K is a number field, and φ is a self map of degree d of \mathbb{P}_K^1 with minimal resultant of norm $R(\varphi)$ and critical conductor of norm $f_{cr}(\varphi)$, then there exists a constant $C(K)$ and integer $s(d)$ such that for every $\varepsilon > 0$:

$$(R(\varphi)) \leq_{\varepsilon} (C(K) f_{cr}(\varphi))^{s(d)+\varepsilon}$$

Another conjecture we are hopeful about for further research is that semi-stable presentations are minimal:

Conjecture 7.19. Let Φ be a semi-stable presentation of φ . Then Φ is a minimal presentation of φ .

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