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DAWSON'S CHESS, SNORT ON GRAPHS AND GRAPH INVOLUTIONS

by

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Abstract

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We show that many 2-person graph vertex coloring games are equivalent to Achievement, a vertex coloring game introduced by Frank Harary and Zsolt Tuza. We then study classes of graphs that satisfy certain graph involutions and show that these involutions can be used to develop winning strategies for these games. Many classes of graphs possess such involutions including Cayley graphs and generalized Kneser graphs. We also study misere versions of these games for special classes of graphs.

Acknowledgments

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Contents

Introduction

In [6] Frank Harary and Zsolt Tuza introduce two graph-coloring games. Starting with a finite graph G, players A and B take turns assigning colors from a set $\{1, \ldots, m\}$ to the vertices of G with the proviso that adjacent vertices are to receive different colors. Player A goes first and the game ends when no legal moves are possible. The game of **Achievement** (respectively **Avoidance**) is won (respectively lost) by the player who makes the last move. Since we are dealing with finite graphs it is clear that in either of the games either player A or B has a winning strategy. We denote the player with the winning strategy in Achievement (respectively Avoidance) played on the graph G with color set $\{1, \ldots, m\}$ by $\mathbf{W}(\mathbf{G}, \mathbf{m})$ (respectively $\mathbf{W}'(\mathbf{G}, \mathbf{m})$). Note that the Avoidance game is what is usually referred to as the misère version of the Achievement game.

In [14], Thomas J. Schaefer shows that the problem of determining whether there is a winning strategy for the first player in one color achievement, which he refers to as (NODE) KAYLES, is PSPACE-complete. (The existence of a polynomial-time algorithm for deciding a PSPACE-complete game would imply the existence of polynomial-time algorithms for deciding all other PSPACE-complete games as well as all NP-complete problems.) Hans L. Bodlaender observes in [2] that, by transformation, the problem of determining whether there is a winning strategy for the first player in two color Achievement (which he calls the COLORING GAME with two colors) is PSPACE-complete as well. He also points out, however, that for more than two colors the complexity of the COLORING GAME is open.

H. L. Bodlaender also introduces a variant of Achievement which he calls the SEQUENTIAL COLORING GAME. Here the vertices are ordered and players must alternately color the lowest numbered uncolored vertex. Bodlaender shows in [2] that for more than two colors the complexity of this game is PSPACE-complete, but that for two colors it is solvable in polynomial time. As a variant of this we can consider ordering the set of colors instead. In fact, some of our results involving graph involutions apply equally well to this game.

Contrary to Harary and Tuza's assertion in [6, p.149], the analysis of one color Achievement played on paths has already been undertaken. Dawson's Chess [1] is played on a chessboard consisting of 3 rows and n columns with n white pawns occupying the first row and n black pawns occupying the last row. The game is played like regular chess except that capturing is mandatory. As usual, if you can't move you lose. A moment's thought reveals that this game is equivalent to one color Achievement played on a path of length n.

In Winning Ways v.1 [1], Berlekamp, Conway and Guy work out the nim sequence of Dawson's Chess showing that it has a periodicity of 34. The nim sequence in this case is a function $\nu : \mathbf{N}^* \longrightarrow \mathbf{N}$, where $\mathbf{N}$ is the set of nonnegative integers and $\mathbf{N}^*$ is the set of positive integers. Player A wins Dawson's Chess played on a 3 by n chessboard if and only if $\nu(n) > 0$. In fact, player B wins if $n = 14$, $n = 34$ and, if $n > 34$, for $n \equiv 4 \pmod{34}$, $n \equiv 8 \pmod{34}$, $n \equiv 20 \pmod{34}$ or $n \equiv 28$

(mod 34). We should also observe that the corresponding nim sequence for Achievement played on cycles with one color is immediately obtained from the sequence for Dawson's Chess since player A wins one color Achievement on a cycle with t vertices, $t \geq 4$, if and only if player B wins one color Achievement on a path with $t-3$ vertices. However, as far as I know, one color Avoidance for paths and cycles has not been analyzed. By the observation just made, the analysis of this game on cycles reduces to that on paths, the latter being equivalent to misère Dawson's Chess.

Harary and Tuza are more successful on paths and cycles if the number of colors is at least two. The following results for path and cycles appear in [6].

Theorem: *(Theorem 2, p.146) Let P_t be the path with t vertices. Then*

$$W(P_t, 2) = W'(P_t, 2) = \begin{cases} \text{A} & \textit{for } t \textit{ odd,} \\ \text{B} & \textit{for } t \textit{ even.} \end{cases}$$

Theorem: *(Theorem 3, p.148) If C_t is the cycle with t vertices, $t \geq 2$, then $W(C_t, 2) = B$ and $W'(C_t, 2) = A$.*

(We will prove analogous results for $W_S(P_t, 2)$, $W_S(C_t, 2)$, $W_S'(P_t, 2)$ and $W_S'(C_t, 2)$, where $W_S(G, m)$ and $W_S'(G, m)$ for a graph G and positive integers m will be defined on page 8 below.)

On the other hand, if $m \geq 2$, then

$$W(P_t, m) = W(C_t, m) = \begin{cases} \text{A} & \text{for } t \text{ odd,} \\ \text{B} & \text{for } t \text{ even.} \end{cases}$$

and

$$W'(P_t, m) = W'(C_t, m) = \begin{cases} \text{B} & \text{for } t \text{ odd,} \\ \text{A} & \text{for } t \text{ even.} \end{cases}$$

This follows form the simple observation of Harary and Tuza [6] that if $m > \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G, then no vertex can be rendered uncolorable during the course of the game and consequently the winner is determined solely by the parity of $|V(G)|$, the number of vertices of G. If G is any tree then we may obtain the same conclusion by requiring that $m > 3$, while for planar graphs we need to require that $m > 32$.

This follows from the work of U. Faigle and W. Kern who prove that the game chromatic number of a tree is at most four [4], and that of H. A Kierstead and W. T. Trotter, who prove that the game chromatic number of a planar graph is at most thirty-three [10]. (The latter result makes use of the four color theorem. The game chromatic number is defined below.) The subject of these articles is a two person game played on the vertices of a graph G with m colors. Players alternatively assign colors to vertices with the proviso that adjacent vertices receive different colors. The first player's goal is to color all the vertices, the second player's goal is to render at least one vertex uncolorable. (Bodlaender calls this the COLORING CONSTRUCTION GAME in [2] and states that its complexity is an open problem. He is, however, able to determine the complexity of the SEQUENTIAL COLORING CONSTRUCTION GAME, a version of the game where the vertices are ordered and players must alternately color the lowest numbered uncolored vertex.)

For $|V(G)|$ odd, it is clear that a first player win in the COLORING CONSTRUCTION GAME produces a first player win in $m-$color Achievement. The game chromatic number $\chi_g(G)$ of G is defined (in [2]) to be the smallest m for which the first player has a winning strategy in the COLORING CONSTRUCTION GAME. Hence if the number of colors $m \geq \chi_g(G)$, then the first player has a winning strategy in $m-$color Achievement if $|V(G)|$ is an odd number.

In order to be able to make the analogous statement when $|V(G)|$ is even, we need to defined the game and game chromatic number slightly differently. More specifically, the game is played like before except that it is now the second player's goal to color all the vertices, and the first player's goal to prevent this from happening. The game chromatic number $\chi_g'(G)$ of a graph G is then defined to be smallest m for which the second player has a winning strategy. In particular if the number of colors $m \geq \chi_g'(G)$ then the second player has a winning strategy in $m-$color Achievement if $|V(G)|$ is even.

If one can show that the bounds of [4] and [10] hold equally well for $\chi_g'(G)$ (and we do this for G a tree), then one would have shown that for $m > 3$ and G a tree or for $m > 32$ and G a planar graph, the winner of the $m-$achievement game on G is determined by the parity of G.

Implicit in the proof of the first theorem above of Harary and Tuza (and to some extent in the proof of the second) is the use of "graph involutions" in describing the winning strategies. We will make this explicit thereby allowing us to extend the techniques in the proof to other classes of graphs, for example Cayley graphs. (In fact our general results involving "graph involutions" extend easily to hypergraphs. But we don't pursue such generalizations in this paper.)

In [6], the second theorem above appears as a corollary to the first. We give a simple independent proof which allows us to generalize the result, via Cartesian products, to other classes of graphs such "web" graphs and "cube-like" graphs. (The **Cartesian product $\mathbf{G} \boxtimes \mathbf{H}$** of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that $(u,v)(u',v') \in E(G \boxtimes H)$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.) Cartesian products also allow us to reduce $m-$color Achievement (respectively Avoidance) to $1-$color Achievement. In fact, $W(G,m) = W(G \boxtimes \mathbf{K}_m, 1)$ and $W'(G,m) = W'(G \boxtimes \mathbf{K}_m, 1)$.

In addition to paths and cycles, F. Harary and Z. Tuza also study the Petersen graph P. There is a well known representation of the Petersen graph as $\overline{L(\mathbf{K}_5)}$, the complement of the line graph of the complete graph on five vertices. They introduce what they refer to as $r-$MIC's (maximal intersection colorings with r colors) in order to study this representation and determine $W(P,m)$ and $W'(P,m)$ for $m = 1,2,3$. We determine $W\left(\overline{L(\mathbf{K}_n)}, m\right)$ and $W'\left(\overline{L(\mathbf{K}_n)}, m\right)$ for $n \geq 3$ and $m = 1,2,3$.

More generally, we have $\overline{L(\mathbf{K}_n)} = K(n,2,1)$, where $\mathbf{K}(\mathbf{n},\mathbf{k},\mathbf{t})$ is the **general Kneser graph** with integer parameters $n > k > t > 0$. For positive integers i, j and k with $j,k > i$ we have an isomorphism

$$K(k+j,\, k,\, k-i) \cong K(k+j,\, j,\, j-i)\,.$$

In particular,

$$K(k+2,\, k,\, k-1) \cong K(k+2,\, 2, 1)\,.$$

So we will in fact be determining $W(K(n,k,k-1),2)$ for all values of $n > k$. We also show, using graph involution techniques, that

$$W(K(n,k,t),m) = B$$

for odd integers k and even integers m and n and that

$$W(K(k+j,k,k-i),2) = B$$

for $j,k \geq 2i+1$. In addition, we study Avoidance and various other graph coloring games on Kneser graphs. We introduce these games next.

The "graph involution" techniques also apply, to varying degrees, to other variations of Harary and Tuza's Achievement and Avoidance games. For linguistic convenience we will refer to these two games as the **Different Colors Vertex Coloring Game (DC-VCG)** and the **Misère Different Colors Vertex Coloring Game (Misère DC-VCG)**. We can also consider partisan versions of the games by fixing the color that each player is allowed to use (the two colors being different.) We denote these games by **PDC-VCG** and **Misère PDC-VCG**, respectively. The player with the winning strategy for these games played on a graph G will be denoted by $\mathbf{W_P}(\mathbf{G})$ and $\mathbf{W'_P}(\mathbf{G})$, respectively. These games reduce to partisan versions of one color Achievement and Avoidance, respectively, played on $G \boxtimes \mathbf{K}_2$ with half of the vertices reserved for one player and half for the other player. More specifically, if we denote the two vertices of $\mathbf{K}_2$ by 1 and 2, then player A can only choose vertices of $G \boxtimes \mathbf{K}_2$ having second coordinate 1, while player B can only choose vertices of $G \boxtimes \mathbf{K}_2$ having second coordinate 2.

Another variation results if we change the proviso that adjacent vertices must receive different colors to the proviso that adjacent vertices must receive the *same* color. This will be referred to as the **Same Color Vertex Coloring Game** (**SC-VCG**). The partisan version, denoted by **PSC-VCG**, is what is referred to in [14] as **SNORT ON GRAPHS**. (The usual game of SNORT is a face coloring game on planar graphs. However, the game can be construed by dualization as a vertex coloring game on planar graphs.) In [14] Thomas J. Schaefer shows that the problem of recognizing winning positions in SNORT ON GRAPHS is complete in PSPACE. The player with the winning strategy for SC-VCG (respectively PSC-VCG) played on a graph G with m colors will be denoted by $\mathbf{W_S}\left(\mathbf{G},\mathbf{m}\right)$ (respectively $\mathbf{W_{PS}}\left(\mathbf{G}\right)$). If we let G' denote the graph obtained from G by adding a loop at every vertex, then we have that $W_S\left(G,m\right) = W\left(G' \times \mathbf{K}_m, 1\right)$, where "$\times$" denotes the direct product. (The purpose of the loops is to insure that (v,i) and (v,j) be adjacent in $G' \times \mathbf{K}_m$ for any vertex $v \in V\left(G'\right) = V\left(G\right)$ and any two "colors" i and j. This guarantees that at most one color be assigned to any particular vertex of G in the corresponding $m-$color game on G. In other words, one color Achievement on $G \times \mathbf{K}_m$ (as opposed to $G' \times \mathbf{K}_m$) corresponds to the game on G which is played just like the Same Color Vertex Coloring Game except that vertices may be assigned more than one color.) Similarly, PSC-VCG reduces to the partisan version of one color Achievement which we considered in the last paragraph. [It may be worthwhile to investigate one color Achievement on $G \times \mathbf{L}_m$, where $\mathbf{L}_m$ consists of m loops on m vertices. This game is equivalent to a vertex coloring game on G played just like $m-$color Achievement except that vertices may be assigned more than one color.]

Finally, we can also consider the Misère versions of these two games, **Misère SC-VCG** and **Misère PSC-VCG**, respectively. Let $\mathbf{W}'_{\mathbf{S}}(\mathbf{G},\mathbf{m})$ (respectively $\mathbf{W}'_{\mathbf{PS}}(\mathbf{G})$) denote the player with the winning strategy for Misère SC-VCG (respectively Misère PSC-VCG) played on a graph G with m (respectively 2) colors. By the usual trick, these games reduce to unbiased and partisan Avoidance games, respectively.

More generally, one color Achievement or (Avoidance) on $G \boxtimes H$ can be thought of either as a coloring game played on the vertices of G using the vertices of H as colors or as a coloring game played on the vertices of H using the vertices of G as colors. In the first case, a player may assign (new) colors to previously colored vertices of G with the provisos that adjacent vertices can only receive different colors (that is, vertices of H) and that the set of all colors assigned to any vertex form an independent set (of vertices) in H. It may be worth noting that other products (direct products, wreath products, etc.) will result in other such classes of games which reduce to one color Achievement or variants thereof.

Another alternative is to define an $H-$coloring game on a graph G as a game where again players assign vertices of H to vertices of G only this time a vertex of G may be assigned at most one "color" and adjacent vertices of G may only be assigned adjacent vertices of H. The special case $H = \mathbf{K}_m$ results in $m-$color Achievement and Avoidance (DC-VCG and misère DC-VCG with m colors), while letting H consist exclusively of loops on m vertices results in SC-VCG and misère DC-SCG on m colors. However, this game can also be described in terms of one color Achievement (and Avoidance). More precisely, playing the $H-$coloring game on the graph G is equivalent to playing one color Achievement (or Avoidance). More precisely, playing the $H-$coloring game on the graph G is equivalent to playing one color Achievement (or Avoidance) on the "product" $\mathbf{G} \star \bar{\mathbf{H}}$ defined by

$$V\left(G \star \overline{H}\right) = V(G) \times V\left(\bar{H}\right) = V(G) \times V(H)$$

with

$$((g,h),(g',h')) \in E(G \star H)$$

if and only if $g = g'$ and $h \neq h'$ or $(g,g') \in E(G)$ and $(h,h') \in E\left(\overline{H}\right)$ (that is (h,h') is *not* an edge in H), where $\bar{H}$ is the complement of H.

Chapter 1
Graphs with Involutions

Denote the set of vertices of a graph G by $V(G)$, the set of edges of G by $E(G)$ and the distance between vertices v and w by $D(v,w)$. An **involution** is a graph automorphism $\varphi : G \longrightarrow G$ with $\varphi^2 = 1$. We will require that the involution be either fixed-point free (when $|V(G)|$ is even) or possess exactly one fixed point $v_0 \in V(G)$ (when $|V(G)|$ is odd).

We will be considering two additional properties of involutions, the second of which only applies to involutions $\varphi : G \longrightarrow G$ having exactly one fixed point $v_0 \in V(G)$.

1.1 Definitions of the Properties ($\mathbf{P}_1$) and ($\mathbf{Q}_1$)

The property ($\mathbf{P}_1$) says that if $v \in V(G)$, then $v\varphi(v) \notin E(G)$ and the property ($\mathbf{Q}_1$) says that if $vv_0 \notin E(G)$, then $v\varphi(v) \notin E(G)$.

Observe that if an involution φ with a unique fixed point $v_0 \in V(G)$ satisfies ($\mathbf{P}_1$) then it satisfies property ($\mathbf{Q}_1$) as well. Moreover, if $v = v_0$, then ($\mathbf{Q}_1$) reduces to a tautology. (So in checking condition ($\mathbf{Q}_1$) we will always assume $v \neq v_0$.)

1.2 Theorem

(a) Let G be a graph with $|V(G)|$ even. If there exists a fixed-point free involution $\varphi : G \longrightarrow G$, then $W_S(G,m) = B$ for all integers $m \geq 1$, $W(G,m) = B$ for all even integers $m \geq 2$ and $W_P(G) = B$.

If the involution satisfies property $(\mathbf{P_1})$ as well, then $W(G,m) = B$ for all integers $m \geq 1$ and $W_{PS}(G) = B$.

(b) Let G be graph with $|V(G)|$ odd. Suppose there exists an involution $\varphi : G \longrightarrow G$ with unique fixed point $v_0 \in V(G)$. Then $W_S(G,m) = A$ for all integers $m \geq 1$. If, in addition, the involution satisfies property $(\mathbf{Q_1})$, then $W(G,m) = A$ for all odd integers $m \geq 1$ and $W_{PS}(G) = A$.

If we replace property $(\mathbf{Q_1})$ by $(\mathbf{P_1})$ in the hypothesis, we then get $W(G,m) = A$ for all integers $m \geq 1$.

Proof. (a) If SC-VCG is the game in question (in which case adjacent vertices must be assigned the same color) and player A assigns the color i to the vertex v, then player B should assign the same color i to the vertex $\varphi(v)$. Since φ is a fixed-point free involution, player B's move is legal if and only if player A's move is.

Now assume the number of colors is even. Denote the set of colors by $\mathbf{C}$ and choose a fixed-point free involution $\kappa : \mathbf{C} \longrightarrow \mathbf{C}$. (Here $\mathbf{C}$ is to be thought of as the trivial graph with $|\mathbf{C}|$ vertices.) Suppose we are dealing with either the DC-VCG (that is the Achievement game) or the PDC-VCG (in which case each player has his own fixed color, say 1 for player A and 2 for player B, and adjacent vertices must be colored differently). If player A assigns the color i to the vertex v, then player B should assign the color $\kappa(i)$ to the vertex $\varphi(v)$. Once again since φ is a fixed-point free involution, player B's move is legal if and only if player A's move is.

Now assume further that the involution satisfies property $(\mathbf{P_1})$. In the DC-VCG (that is Achievement game), if player A assigns the color i to the vertex v, then player B should respond by assigning the same color i to the vertex $\varphi(v)$. Once again since φ is a fixed-point free involution, player B's move is legal if and only if player A's move is.

In the PSC-VCG, if player A assigns the color 1 to v, then player B should respond by assigning the color 2 to $\varphi(v)$. (The vertices v and $\varphi(v)$ cannot be adjacent since the involution satisfies property $(\mathbf{P_1})$.)

(b) In the SC-VCG, player A should assign any color to the vertex v_0. Then if player B assigns the color i to any other vertex v, then player A should respond by assigning the same color i to the vertex $\varphi(v)$. Since φ is an involution with unique fixed point v_0, player B's move is legal if and only if player A's move is.

Now we assume that φ satisfies property $(\mathbf{Q_1})$ and that the number of colors is odd. Hence we can choose an involution $\kappa : \mathbf{C} \longrightarrow \mathbf{C}$ on the set of colors having precisely one fixed point, say 1. First we consider the DC-VCG. Player A should begin by assigning the color 1 to v_0. If in any subsequent move player B assigns the color i to a vertex v, then player A should assign the color $\kappa(i)$ to the vertex $\varphi(v)$. Thus player A's choice of color will be different from B's when $i \neq 1$ and the same as B's when $i = 1$. Since φ is an involution with unique fixed point v_0 and satisfies property $(\mathbf{Q_1})$, player A's move is legal if and only if player B's move is.

In the PSC-VCG, we assume that φ satisfies property $(\mathbf{Q}_1)$ and player A starts off by assigning the color 1 to v_0. If in any subsequent move player B assigns the color 2 to some vertex v, then player A responds by assigning the color 1 to $\varphi(v)$. Once again player A's move is legal if and only if player B's move is.

Finally, assume that the number of colors is even and φ satisfies property $(\mathbf{P}_1)$. In the DC-VCG player A once more should begin by assigning the color 1 to v_0. If in any subsequent move player B assigns the color i to a vertex v, then player A should respond by assigning the same color i to the vertex $\varphi(v)$. This time the fact that φ is an involution with unique fixed point v_0 and satisfies property $(\mathbf{P}_1)$ (in the Achievement game) guarantees that player A's move is legal if and only if player B's move is. (We need the stronger condition $(\mathbf{P}_1)$ for $i \neq 1$; when $i = 1$ property $(\mathbf{Q}_1)$ suffices as was noted above.) ■

Let us now consider the H–coloring game mentioned in the introduction. Recall that the H–coloring game on a graph G is the game where players assign vertices of a graph H to vertices of a graph G with adjacent vertices of G receiving adjacent vertices of H.

1.3 Corollary

Let $\varphi : G \longrightarrow G$ and $\psi : H \longrightarrow H$ be graph involutions. (Denote by $\bar{\psi} : \bar{H} \longrightarrow \bar{H}$ the involution induced by φ on the complement $\bar{H}$ of H .)

(a) If at least one of the numbers $|V(G)|$ and $|V(H)|$ is even, and φ is fixed-point free and satisfies property $(\mathbf{P}_1)$, then player B wins the H–coloring game.

(b) If $|V(G)|$ and $|V(H)|$ are both odd, both φ and $\bar{\psi}$ have exactly one fixed point, denoted by g_0 and h_0, respectively, and both satisfy property $(\mathbf{Q}_1)$, then player A wins the H–coloring game.

Proof. We pointed out in the introduction that playing the H–coloring game on a graph G is equivalent to playing one color Achievement on the "product" $G \star \bar{H}$, where $V(G \star H) = V(G) \times V(\bar{H}) = V(G) \times V(H)$ with (g,h) and (g',h') adjacent in $G \star \bar{H}$ if and only if $g = g'$ and $h \neq h'$ or $(g,g') \in E(G)$ and $(h,h') \in E(\overline{H})$ (that is (h,h') is *not* an edge in H).

(a) Observe first that $|V(G \star \bar{H})| = |V(G)|\,|V(\bar{H})|$ is even. Define $\Phi = (\varphi, \bar{\psi})$: $G \star \bar{H} \longrightarrow G \star \bar{H}$ by

$$\Phi(g,h) = (\varphi(g), \bar{\psi}(h)).$$

Since φ and ψ are involutions, it follows from the definition of $G \star \bar{H}$ that Φ is also an involution. Clearly Φ is fixed-point free since φ is. Moreover, Φ satisfies $(\mathbf{P}_1)$: the vertices (g,h) and $(\varphi(g), \bar{\psi}(h))$ are not adjacent in $G \star \bar{H}$ since $g \neq \varphi(g)$ and, because φ satisfies property $(\mathbf{P}_1)$, g and $\varphi(g)$ cannot be adjacent in G either. The desired result now follows from Theorem 1.2(a).

(b) This time $|V(G \star H)|$ is odd. Define $\Phi = (\varphi, \bar{\psi}) : G \star \bar{H} \longrightarrow G \star \bar{H}$ as above. Clearly Φ has exactly one fixed point, namely (g_0, h_0). We will show that if

(g,h) is not adjacent to (g_0,h_0) in $G \star \bar{H}$ then (g,h) cannot be adjacent to $(\varphi(g), \bar{\psi}(h))$ either.

The vertices (g,h) and $(\varphi(g), \bar{\psi}(h))$ are adjacent in $G \star \bar{H}$ if either $g = \varphi(g)$ and $h \neq \bar{\psi}(h)$ or g is adjacent to $\varphi(g)$ in G and h is adjacent to $\bar{\psi}(h)$ in $\bar{H}$. If the first is the case, then $g = g_0$ and $h \neq h_0$. But that implies that (g,h) and (g_0,h_0) are adjacent in $G \star \bar{H}$ which is a contradiction. If the second is the case, then $g \neq g_0$ and $h \neq h_0$. Moreover, since both φ and $\bar{\psi}$ are assumed to satisfy property $(\mathbf{Q}_1)$, we must also have that g and g_0 are adjacent in G and h and h_0 are adjacent in $\bar{H}$. But this implies that (g,h) and (g_0,h_0) are adjacent in $G \star \bar{H}$, again a contradiction. ■

We can now restate the proof and statement of Theorem 2 of [6] in terms of graph involutions. First let P_t be the path with t vertices.

1.4 Corollary

For t even, $W(P_t, 2) = B$. For t odd, we have $W(P_t, m) = A$ for $m = 1, 2$.

Proof. Let $V(P_t) = \{0, 1, \ldots, t-1\}$ and

$$E(P_t) = \{\{i, i+1\} \mid 0 \leq i \leq t-2\}.$$

Define the involution $\varphi : P_t \longrightarrow P_t$ by $\varphi(i) = t - i - 1$ for $0 \leq i \leq t-1$. It is fixed-point free for t even and has exactly one fixed point, namely $\frac{t-1}{2}$, for t odd. The involution φ does not satisfy property $(\mathbf{P}_1)$ for even t. However it does satisfy it for odd t since in this case

$$D(i, \varphi(i)) = 2D\left(i, \frac{t-1}{2}\right) \geq 2$$

for all $0 \leq i \leq t-1$, $i \neq \frac{t-1}{2}$. ∎

In fact we have proved more.

1.5 Corollary

For t even,

$$W_S(P_t, 1) = W_S(P_t, 2) = W_P(P_t) = B.$$

For t odd,

$$W_S(P_t, 1) = W_S(P_t, 2) = W_{PS}(P_t) = A.$$

We also have the following analogues of Theorem 2 and Theorem 3 in [6].

1.6 Theorem

Let P_t be the path with $t \geq 4$ vertices. Then

$$W_S'(P_t, 2) = \begin{cases} \text{A} & \text{for } t \text{ odd,} \\ \text{B} & \text{for } t \text{ even.} \end{cases}$$

1.7 Theorem

If C_t is a cycle with $t \geq 1$ vertices then

$$W'_S(C_t, 2) = \begin{cases} \text{A} & \text{for } t \text{ even,} \\ \text{B} & \text{for } t \text{ odd.} \end{cases}$$

In order to prove these results we need an analogue of Lemma 1 in [6] for the 2−color SC-VCG.

1.8 Lemma

Let P_t be the path with t vertices and assume that its endpoints (and only its endpoints) are colored. If the endpoints have distinct (respectively the same) colors, then each player can force an odd (respectively even) number of uncolored vertices in P_t.

Proof. First observe that if player A colors a vertex next to one of the colored endpoints, we end up with a shorter uncolored path with the "same" colors on the endpoints as the original path. Hence player A can force the conclusion of the lemma on path with t vertices if player B can force the conclusion of the lemma on path with on $t-1$ vertices. So it suffices to show that player B can force the conclusion of the lemma on any path consisting of at least two vertices.

The proof is by induction on t with the lemma clearly holding for $t \leq 5$. Assume that $t > 5$ and that the lemma holds for any path having less than t vertices. We need to show that it holds for a path having t vertices.

Let us consider a path having t vertices. Every internal node x has a neighbor y whose other neighbor z is not an endpoint. If player A assigns the color i to x, then player B should assign the color i to y. This results in two uncolored subpaths. There are essentially three different cases to look at, as shown in Figure 1.1.

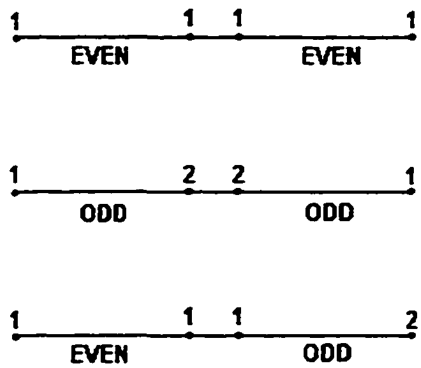


Figure 1.1.

In each case we have indicated the parity of the number of uncolored vertices in each subpath that player B can force using the inductive hypothesis. The inductive step is now easily verified to complete the proof. ■

Proof of Theorem 1.6. Suppose that $t \geq 4$. Set $V(P_t) = \{v_0, v_1, \ldots, v_{t-1}\}$, $L = \{v_0, v_1\}$ and $R = \{v_{t-2}, v_{t-1}\}$. If t is odd, then player A should start by assigning the same color 1 to $v_{\frac{t-1}{2}}$. Then whenever player B assigns color i to v_r, player A should assign the same color i to v_{t-r-1}, as long as $v_r \notin L \cup R$. If t is even, then whenever player A assigns same the color i to v_r player B should assign the same color i to v_{t-r-1}.

Since the proof for t even and t odd are almost identical, from this point on we will only consider the case where t is even. If at some point player A assigns the color

i to a vertex $v_{r'} \in L \cup R$, then player B should assign the color $3 - i$ to the vertex $v_{t-r'-1}$, if possible.

Figure 1.2.

Each uncolored subpath in Figure 1.2 is labeled with the parity of the number of uncolored vertices player B can force. Observe that the parities of the subpaths to the right and left of the "middle" subpath are in one-to-one correspondence except for the "outermost" subpaths which have opposite parity. Since the parity of the "middle" subpath is even, then by applying the previous lemma to each uncolored subpath we conclude that player B can force an odd number of uncolored vertices on the whole path. Since the path is assumed to have an even number of vertices, this is equivalent to saying that player B can force an odd number of colored vertices. (A similar argument holds for paths of odd length. The only difference is that we no longer have a "middle" subpath.)

The only time player B won't be able to respond in the above fashion is when v_2 and v_{t-3} are already colored and player A colors v_1 or v_{t-2}. For argument's sake, assume that player A has assigned the color 2 to v_1 (and v_2 and v_{t-3} have been assigned the color 2 in previous steps). See Figure 1.3. Player B cannot assign the color 1 to v_{t-2}; however he can assign the color 1 to v_{t-1}.

Figure 1.3.

This renders v_{t-2} uncolorable. Once again player B forces an odd number of uncolored vertices which translates into a win in Avoidance. The situation for odd t is analogous. Consequently, we have provided a winning strategy for Avoidance for player A (respectively B) for odd (respectively even) values of t. ■

Proof. of **Theorem 1.7.** Assume first that t is an odd number and player A assigns a color to a vertex v of C_t. Then player B should assign the same color to an adjacent vertex w. The path from v to w has an odd number of vertices and its endpoints have been assigned the same color. By Lemma 1.8, player B can force an even number of uncolored vertices in C_t. Equivalently, he can force an odd number of colored vertices and so he wins the Avoidance game.

Now assume that t is an even number. Let us say that players A and B have colored the vertices v and w, respectively. If v and w are adjacent then they must have been assigned the same color. By Lemma 1.8, player A can force an even number of uncolored vertices in C_t. Since the path from v to w has an even number of vertices, that results in an even number of colored vertices as well and so this time player A wins the Avoidance game.

If, on the other hand, vertices v and w are *not* adjacent, then there are two paths from v to w. The lengths of the two paths have the same parity. By Lemma 1.8, the number of uncolored vertices player A can force in one path has the same parity as the number of uncolored vertices he can force in the other path. Consequently, the number of colored vertices in one path will have the same parity as the number of colored vertices. Altogether there will be an even number colored vertices and so player A wins the Avoidance game. ■

Chapter 2
Involutions on Cayley Graphs

Let S be a set of generators for a finite group $\mathcal{F}$ which satisfies the following two conditions:

(a) The identity element $e \notin S$.

(b) If $s \in S$, then $s^{-1} \in S$.

The **Cayley graph** $G = G(\mathcal{F}, S)$ is defined by

$$V(G) = \mathcal{F} \quad \text{and} \quad E(G) = \{\{g, h\} \mid g^{-1}h \in S\}.$$

2.1 Theorem

Suppose $\mathcal{F}$ is a finite group of even order and S is a set of generators as above. Then $W(G(\mathcal{F}, S), m) = B$ for all even integers $m \geq 2$. If, furthermore, there exists an element a of order two with $a \notin gSg^{-1}$ for any $g \in \mathcal{F}$, then $W(G(\mathcal{F}, S), m) = B$ for all integers $m \geq 1$.

Proof. For each element $a \in \mathcal{F}$, there is a bijection of sets $\varphi_a : \mathcal{F} \longrightarrow \mathcal{F}$ defined by left translation: $\varphi_a(g) = ag$ for all $g \in \mathcal{F}$. Observe that $\{g, h\} \in E(G)$ if and only if $\{ag, ah\} \in E(G)$ since $(ag)^{-1}(ah) = g^{-1}h$. Consequently the bijection φ_a induces an automorphism $\bar{\varphi}_a : G \longrightarrow G$ of the Cayley graph.

If $a \in \mathcal{F}$ is an element of order two, then $\bar{\varphi}_a$ is a fixed-point free involution. There is a theorem due to Cauchy which states that every group of even order has an element of order two. If we let a be any such element then the first part of our theorem follows from the first part of Theorem 1.2(a).

The involution satisfies property $(\mathbf{P}_1)$ if for all $g \in \mathcal{F}$ we have $g\bar{\varphi}_a(g) \notin E(G)$. But g and $\bar{\varphi}_a(g) = ag$ are adjacent precisely when $g^{-1}ag \in S$. Consequently, the involution $\bar{\varphi}_a$ satisfies property $(\mathbf{P}_1)$ if and only if $a \notin gSg^{-1}$ for any $g \in \mathcal{F}$.

The last part of our theorem then follows from the last part of Theorem 1.2(a).

■

2.2 Corollary

Suppose $\mathcal{F}$ is finite abelian group of even order. Then

$$\mathcal{F} \cong Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}$$

for some positive integer k, where p_i, $1 \leq i \leq k$, are (not necessarily distinct) prime numbers and s_i, $1 \leq i \leq k$, are positive integers. Let S consist of the set of k generators of $\mathcal{F}$ which correspond under the isomorphism to the standard basis $\{x_1, x_2, \ldots, x_k\}$ of

$$Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}.$$

If either

(a) for some $1 \leq i \leq k$ we have $p_i = 2$ and $s_i \geq 2$, or

(b) for some $1 \leq i \leq k$, $i \neq j$, $p_i = p_j = 2$,

then $W(G(\mathcal{F}, S), m) = B$ for all integers $m \geq 1$.

Proof. Observe that since $\mathcal{F}$ is abelian $a \notin gSg^{-1}$ for any $g \in \mathcal{F}$ if and only if $a \notin S$. The corollary will follow from the last part of Theorem 2.1 once we show that condition (a) or (b) guarantees the existence of an element $a \in \mathcal{F} \setminus S$ of order two.

If the group satisfies condition (a) then let $a \in \mathcal{F}$ be the element which corresponds to $x_i^{2^{s_i-1}}$ under the isomorphism

$$\mathcal{F} \cong Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}.$$

Clearly the element a has order two. Since $s_i \geq 2$, then $x_i^{2^{s_i-1}} \neq x_i$ and $x_i^{2^{s_i-1}} \neq x_i^{-1}$. Consequently $a \notin S$. If, on the other hand, the group satisfies condition (b), then let $a \in \mathcal{F}$ be the element which corresponds to $x_i x_j$ under the isomorphism

$$\mathcal{F} \cong Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}.$$

Since $(x_i x_j)^2 = x_i^2 x_j^2 = e$, then a has order two. Moreover, $a \notin S$. ■

The **dihedral group $\mathbf{D}_n$** is the group of order $2n$ generated by elements a and b subject to the relations $a^n = 1$, $b^2 = 1$ and $ba = a^{-1}b$.

2.3 Corollary

Let $S = \{a, a^{2k-1}, b\}$. Then $W(G(\mathbf{D}_{2k}, S), m) = B$ for all positive integers k and m.

Proof. Since

$$(ab)^2 = abab = aa^{-1}bb = b^2 = e,$$

the elements ab has order two. The second part of Theorem 2.1 will apply once we show that for all $g \in \mathbf{D}_{2k} g^{-1}(ab)$ we have $g \notin S$. Note that $g = a^i b^j$, where $0 \leq i \leq 2k-1$ and $0 \leq j \leq 1$.

Assume first that $j = 0$, that is that $g = a^i$ with $0 \leq i \leq 2k-1$. Then

$$g^{-1}(ab)\,g = a^{-i}(ab)\,a^i = a^{1-i}ba^i = a^{1-2i}b.$$

But $a^{1-2i}b \in S$ if and only if $2k|\,(1-2i)$, which is impossible.

If, on the other hand, $j = 1$, that is if $g = a^i b$, $0 \leq i \leq 2k-1$, then

$$g^{-1}(ab)\,g = \left(a^i b\right)^{-1}(ab)\left(a^i b\right) = ba^{-i}aba^i b = ba^{1-i}ba^i b = a^{2i-1}b.$$

But once again $a^{2i-1}b \in S$ if and only if $2k \mid (1-2i)$, which is impossible.

Since we have shown that $g^{-1}(ab)\,g \notin S$, for all $g \in \mathbf{D}_{2k}$, we are done. ■

2.4 Remark on Symmetric Groups

Let $\mathbf{S}_n$ be the **symmetric group** on n letters. The group $\mathbf{S}_n$ has order $n!$ which is even if $n \geq 2$. Let $x = (1\,2\,3 \cdots n-1\,n)$ and $y = (1\,n\,n-1 \cdots 4\,3)$. Then $\mathbf{S}_n$ is generated by x and y with $(yx)^2 = e$.

2.5 Corollary

Let $S = \{x, x^{-1}, y, y^{-1}\}$ and that $n \geq 4$. Then $W(G(\mathbf{S}_n, S), m) = B$ for all positive integers m.

Proof. Let $a = yx = (12)$. Then $a^2 = e$. In fact, $g^{-1}ag = (g^{-1}(1), g^{-1}(2))$ has order two for any $g \in \mathbf{S}_n$. But x has order n and y has order $n-1$. Since we are assuming that $n \geq 4$, $g^{-1}ag \notin S$ for all $g \in \mathbf{S}_n$. The corollary then follows from the second part of Theorem 2.1. ■

2.6 Example

Figure 2.4 shows the graph of $G(\mathbf{S}_4, S)$. The "rule" for labelling the vertices is as follows: as you proceed around a triangular region you multiply vertices by y and as you proceed around the "darkened" rectangular regions you multiply vertices by x. Then for any element $g \in \mathbf{S}_4$ the involution maps the elements g and ag to each other where $a = (12) = yx = x^3y^2$. For example, $e \longleftrightarrow yx = x^3y^2$, $y \longleftrightarrow x^3$, $y^2 \longleftrightarrow x^3y$, $x \longleftrightarrow yx^2$ and $x^2 \longleftrightarrow yx^3$.

2.7 Theorem

Suppose $\mathcal{F}$ is a finite abelian group of odd order and S is a set of generators as in Theorem 2.1. If $g^2 \notin S$ for all elements $g \notin S$, then $W(G(\mathcal{F}, S), m) = A$ for all odd integers $m \geq 1$.

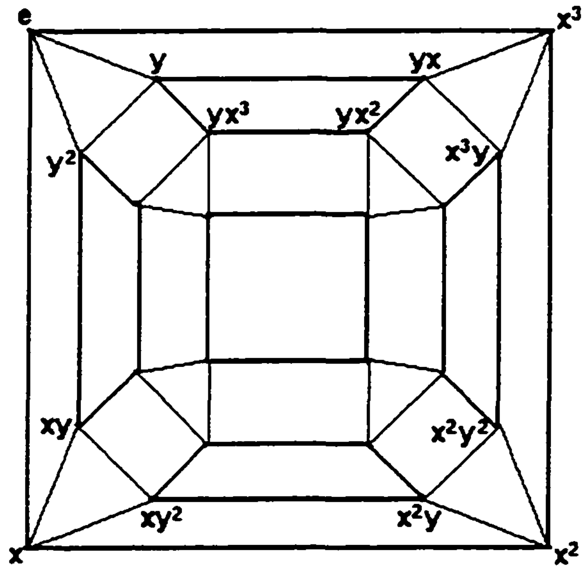


Figure 2.4.

Proof. For any abelian group $\mathcal{F}$ we have a group automorphism $\psi : \mathcal{F} \longrightarrow \mathcal{F}$ defined by $\psi(g) = g^{-1}$ for all $g \in \mathcal{F}$. Then for any set S as described above, ψ induces a graph involution $\bar{\psi}$ on the Cayley graph $G(\mathcal{F}, S)$ with unique fixed point e, the unit element of $\mathcal{F}$.

We are assuming that the group $\mathcal{F}$ has odd order. It follows by a theorem of Lagrange that $\mathcal{F}$ has no element of order two. Consequently, $\bar{\psi}$ satisfies property $(\mathbf{Q}_1)$ if whenever $\{g, e\} \notin E(G)$ then $\{g, g^{-1}\} \notin E(G)$. This is equivalent to saying that if $g \notin S$ then $g^2 \notin S$. The theorem now follows from the first part of Theorem 1.2(b). ■

If $\mathcal{F}$ is a finite abelian group of odd order, then

$$\mathcal{F} \cong Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}$$

for some positive integer k, where p_i, $1 \leq i \leq k$, are odd primes and s_i, $1 \leq i \leq k$, are positive integers. Suppose that $\{a_1, \ldots, a_k\}$ is the set of generators of $\mathcal{F}$ that corresponds to the standard basis of

$$Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}$$

under the isomorphism.

2.8 Corollary

Suppose that $\mathcal{F}$ is a finite abelian group of odd order

as above and let

$$S = \left\{a_1, a_1^2, \ldots, a^{p_1^{s_1}-1}, a_2, a_2^2, \ldots, a_2^{p_2^{s_2}-1}, \ldots, a_k, a_k^2, \ldots, a_k^{p_k^{s_k}-1}\right\}.$$

Then $W(G(\mathcal{F}, S), m) = A$ for all odd integers $m \geq 1$.

Proof. We just need to show that $g^2 \notin S$ for all elements $g \in \mathcal{F} \setminus S$. Let

$$g = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k} \in \mathcal{F},$$

where $0 \leq r_i < p_i^{s_i}$ for all $1 \leq i \leq k$. Assume $g \notin S$. So there exist i and j, $i \neq j$, such that $r_i \neq 0$ and $r_j \neq 0$. Since p_i and p_j are odd primes, $2r_i \not\equiv 0 \pmod{p_i}$ and $2r_j \not\equiv 0 \pmod{p_j}$. Consequently,

$$g^2 = a_1^{2r_1} a_2^{2r_2} \cdots a_k^{2r_k} \notin S. \blacksquare$$

Chapter 3
Graph Constructions

In this chapter we will study how the involutions introduced in the first section behave with respect to several graph operations.

The **power** d of a graph G, denoted by G^d, is the graph obtained from G by adding edges between nonadjacent vertices that are distance d or less apart. (Observe that we can generalize the game of Achievement by strengthening the notion of a legal move: vertices that are distance d or less apart cannot be assigned the same color. Then playing this more general game of Achievement on the vertices of a graph G is tantamount to playing the usual game of Achievement on the vertices of G^d.)

Clearly this operation is "functorial" with respect to graph homomorphisms. Consequently an involution $\varphi : G \longrightarrow G$ induces an involution $\varphi_d : G^d \longrightarrow G^d$ for all integers $d \geq 1$. It is obvious that all these involutions have the same set of fixed points. Moreover, φ_d satisfies property $(\mathbf{P}_1)$ or $(\mathbf{Q}_1)$ if and only if φ satisfies properties $(\mathbf{P}_d)$ or $(\mathbf{Q}_d)$, respectively, where $(\mathbf{P}_d)$ and $(\mathbf{Q}_d)$ are as follows:

$(\mathbf{P_d})$ $D(\varphi(v), v) > d$, for all $v \in V(G)$

$(v \neq v_0$ if φ has a unique fixed point $v_0)$,

and

$(\mathbf{Q_d})$ If $D(v, v_0) > d$, then $D(\varphi(v), v) > d$.

(For future reference observe that any involution with at most one fixed point satisfies properties $(\mathbf{P}_0)$ or $(\mathbf{Q}_0)$.)

As a consequence of these observations we have the following immediate corollary of Theorem 1.2.

3.1 Theorem

(a) Let G be a graph with $|V(G)|$ even. If there exists a fixed-point free involution $\varphi : G \longrightarrow G$, then $W(G^d, m) = B$ for all integers $d \geq 1$ and all even integers $m \geq 2$.

If in addition, φ satisfies property $(\mathbf{P}_d)$ for some integer $d \geq 1$, then $W(G^d, m) = B$ for all integers $m \geq 1$.

(b) Let G be a graph with $|V(G)|$ odd. Suppose that there exists an involution $\varphi : G \longrightarrow G$ with unique fixed point $v_0 \in V(G)$. If the involution satisfies property $(\mathbf{Q}_d)$ for some integer $d \geq 1$, then $W(G^d, m) = A$ for all odd integers $m \geq 1$.

If we replace property $(\mathbf{Q}_d)$ by $(\mathbf{P}_d)$ in the hypothesis, we then get $W(G^d, m) = A$ for all integers $m \geq 1$.

3.2 Corollary

(a) If t is an even integer, then $W\left(P_t^d, m\right) = B$ for all integers $d \geq 1$ and all even integers $m \geq 2$.

(b) If t is an odd integer, then $W\left(P_t^d, m\right) = A$ for all integers $d \geq 1$ and all odd integers $m \geq 1$ as well as for $d = 1$ and all even integers $m \geq 2$.

Proof. Let the involution $\varphi : P_t \longrightarrow P_t$ be defined as in Corollary 1.4.

(a) For even t, φ is fixed-point free and so the first part of Theorem 3.1(a) applies.

(b) For odd t we have already checked that φ satisfies property $(\mathbf{P}_1)$. In doing so we used that fact that $D(i, \varphi(i)) = 2D\left(i, \frac{t-1}{2}\right)$ for all $0 \leq i \leq t-1$, $i \neq \frac{t-1}{2}$. But this fact also implies that if $D\left(i, \frac{t-2}{2}\right) > d$ then $D(i, \varphi(i)) > 2d$, for $0 \leq i \leq t-1$. Hence, for odd t, φ also satisfies property $(\mathbf{Q}_d)$ for all integers $d \geq 1$. Thus the first part of Theorem 3.1(b) applies for all $d \geq 1$, while the second part applies only for $d = 1$. ■

3.3 Cartesian Product of Graphs

The **Cartesian product** $G \boxtimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that $(u,v)(u',v') \in E(G \boxtimes H)$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. Similarly, if we are given graphs G_i, $1 \leq i \leq r$, we can define their Cartesian product $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r$ inductively.

Clearly the Cartesian product is a bifunctor and so given involutions $\varphi_i : G_i \longrightarrow G_i$, $1 \leq i \leq r$, their product

$$\varphi_1 \boxtimes \varphi_2 \boxtimes \cdots \boxtimes \varphi_r : G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r \longrightarrow G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r$$

is also an involution.

Observe that

$$\varphi \triangleq \varphi_1 \boxtimes \varphi_2 \boxtimes \cdots \boxtimes \varphi_r$$

is fixed-point free unless each φ_i has fixed points. In this case, the number of fixed points of φ equals $\prod_{i=1}^{r} n_i$, where n_i equals the number of fixed points of φ_i, $1 \leq i \leq r$, respectively. In particular, if each φ_i has a fixed point v_i, then $v_0 = (v_i, v_2, \ldots, v_r)$ is the unique fixed point of φ.

3.4 Theorem

Let $\psi : G \longrightarrow G$ be a fixed-point free involution and H any graph. Then

$$W\left((G \boxtimes H)^d, m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$.

If ψ satisfies $(\mathbf{P}_d)$ for some integer $d \geq 1$, then

$$W\left((G \boxtimes H)^d, m\right) = B$$

for all integers $m \geq 1$.

Proof. Clearly ψ induces another fixed-point free involution

$$\psi' \triangleq \psi \boxtimes 1_H : G \boxtimes H \longrightarrow G \boxtimes H.$$

Therefore the first part of Theorem 3.1 (a) applies. The second part will also apply once we show that ψ' satisfies property $(\mathbf{P}_d)$. But

$$D((u,v)\ \psi'(u,v)) = D_G(u, \psi(u))$$

for all $(u,v) \in V(G \boxtimes H)$, where D is the distance function on $G \boxtimes H$ and D_G is the distance function on G. Consequently ψ' satisfies property $(\mathbf{P}_d)$ if and only if ψ does. ■

3.5 Lemma

Let $\varphi_i : G_i \longrightarrow G_i$, $1 \le i \le r$, be involutions. If each φ_i, $1 \le i \le r$, has a unique fixed point and satisfies property $(\mathbf{Q}_d)$ for all $d \ge 0$, then φ has a unique fixed point and satisfies property $(\mathbf{Q}_d)$ for all $d \ge 0$.

Proof. We have already observed that φ has a unique fixed point if each φ_i has one and this is tantamount to saying that φ satisfies property $(\mathbf{Q}_0)$ if each φ_i satisfies it. Let $v_0 = (v_i, v_2, \ldots, v_r)$ be the unique fixed point of φ.

If

$$u = (u_1, u_2, \ldots, u_r) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r),$$

then

$$D(u,v) = \sum_{j=1}^{r} D_j(u_j, v_j),$$

where D is the distance function on $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r$, D_j is the distance function on G_j, $j \in J$, and

$$v_0 = (v_1, v_2, \ldots, v_r) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)$$

is the unique fixed point of φ. Since each φ_i satisfies property $(\mathbf{Q}_d)$ for all $d \geq 0$, $D_j(u_j, \varphi_j(u_j)) \geq D_j(u_j, v_j)$ for all $1 \leq j \leq r$. Hence,

$$D(u, \varphi(u)) = \sum_{j=1}^{r} D_j(u_j, \varphi_j(u_j)) \geq \sum_{j=1}^{r} D_j(u_j, v_j) = D(u, v)$$

for all $u \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)$. It follows from this that φ satisfies property $(\mathbf{Q}_d)$ for all $d \geq 0$. ■

3.6 Theorem

(a) Let $\varphi_i : G_i \longrightarrow G_i$, $1 \leq i \leq r$, be involutions, at least one of which is fixed-point free. Then $W\left((G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)^d, m\right) = B$ for all integers $d \geq 1$ and all even integers $m \geq 2$.

Let $J = \left\{j \mid \varphi_j \text{ is fixed-point free}\right\}$ and say that in addition φ_j, $j \in J$, satisfies property $(\mathbf{P}_d)$ for some $d_j \geq 0$. If

$$d' = \sum_{j \in J} d_j + |J| > 1,$$

then

$$W\left((G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)^d, m\right) = B$$

for all integers $1 \leq d < d'$ and all integers $m \geq 1$.

(b) Let $\varphi_i : G_i \longrightarrow G_i$, $1 \leq i \leq r$, be involutions each of which has a unique fixed point and satisfies property $(\mathbf{Q}_d)$ for all integers $d \geq 1$. Then

$$W\left((G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)^d, m\right) = A$$

for all integers $d \geq 1$ and for all odd integers $m \geq 1$.

If instead each involution φ_i, $1 \leq i \leq r$, satisfies property $(\mathbf{P}_{\mathrm{d_i}})$ for some $d_i \geq 1$, then

$$W\left((G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)^d, m\right) = A$$

for all integers $1 \leq d \leq d'$ and all integers $m \geq 1$, where $d' = \min\{d_i \mid 1 \leq i \leq r\}$.

Proof.

(a) (a) We have already noted that φ is fixed-point free and so the first part of Theorem 3.1 (a) applies. In order to apply the second part, we need to show that φ satisfies property $(\mathbf{P}_{\mathrm{d}})$ for $1 \leq d < d' = \sum_{j \in J} d_j + |J|$, assuming each φ_j satisfies property $(\mathbf{P}_{\mathrm{d_j}})$. But if

$$u = (u_1, u_2, \ldots, u_r) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r),$$

then

$$D(u, \varphi(u)) = \sum_{i=1}^{r} D(u_i, \varphi_i(u_i)) \geq \sum_{j \in J} D_j(u_j, \varphi_j(u_j))$$

$$\geq \sum_{j \in J} (d_j + 1) = \sum_{j \in J} d_j + |J| = d' > d.$$

(b) We have already observed that φ has a unique fixed point. Each φ_i satisfies property $(\mathbf{Q}_{\mathrm{d}})$ for all integers $d \geq 1$ and so by Lemma 3.5 so does φ. Now we can apply the first part of Theorem 3.1(b). Let

$$u = (u_1, u_2, \ldots, u_r) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_r)$$

with $u \neq v_0$, the fixed point. Then for some $1 \leq i \leq r$ we have $u_i \neq v_i$. So

$$D(u, \varphi(u)) \geq D_i(u_i, \varphi_i(u_i)) > d_i \geq d' \geq d,$$

since each involution φ_i, $1 \leq i \leq r$, satisfies property $(\mathbf{P}_{d_j})$. So the hypothesis of the second part of Theorem 3.1(b) is satisfied and the desired conclusion follows.

■

Chapter 4
Examples: Grids, Web Graphs and Cube-Like Graphs

Let

$$G = P_{t_1} \boxtimes P_{t_2} \boxtimes \cdots \boxtimes P_{t_r},$$

the Cartesian product of r paths of possible different lengths. Also let $\varepsilon = |\{i \mid t_i \text{ is even}\}|$, the number of even length paths.

4.1 Theorem

(a) If $\varepsilon \geq 1$, then $W(G^d, m) = B$ for all integers $d \geq 1$ and all even integers $m \geq 2$. If $\varepsilon > 1$, then $W(G^d, m) = B$ for all integers $1 \leq d < \varepsilon$ and all integers $m \geq 1$.

(b) If $\varepsilon = 0$, then $W(G^d, m) = A$ for all integers $d \geq 1$ and all odd integers $m \geq 1$ or for $d = 1$ and all integers $m \geq 1$.

Proof.

(a) Define the involutions $\varphi_i : P_{t_i} \longrightarrow P_{t_i}$, $1 \leq i \leq r$, as in the proof of Corollary 1.4. Recall that they are fixed-point free for t_i even and have exactly one fixed point for t_i odd. So if $\varepsilon \geq 1$, then φ is fixed-point free and the first part of Theorem

3.6(a) applies. For any t_i, φ_i satisfies property $(\mathbf{P}_0)$ so if $\varepsilon > 1$ then the second part of Theorem 3.6(a) (with $d' = \varepsilon$) applies as well.

(b) As we observed in the proof of Corollary 3.2, if t_i is odd then φ_i satisfies property $(\mathbf{Q}_d)$ for all integers $d \geq 1$. Therefore we can now apply the first part of Theorem 3.6(b). We have already observed that for odd t_i, φ_i satisfies property $(\mathbf{P}_1)$. Consequently the second part of Theorem 3.6(b) also applies. ■

4.2 Grids

The special case $P_s \boxtimes P_t$ is the $s \times t$ **grid**. It is the graph with vertex set

$$\{(i,j) \mid 1 \leq i \leq s,\ 1 \leq j \leq t\}$$

and edge set

$$\{(i,j)(i',j') \mid i = i' \text{ and } |j - j'| = 1 \text{ or } j = j' \text{ and } |i - i'| = 1\}.$$

4.3 Corollary

(a) For all integers $d \geq 1$ and $s \cdot t$ even, we have $W\left((P_s \boxtimes P_t)^d, m\right) = B$ for even integers m. If in addition, both s and t are even, then $W(P_s \boxtimes P_t, m) = B$ for all integers $m \geq 1$.

(b) For all integers $d \geq 1$ and $s \cdot t$ odd, we have $W\left((P_s \boxtimes P_t)^d, m\right) = A$ for odd integers m. Moreover, $W(P_s \boxtimes P_t, m) = A$ for all integers $m \geq 1$.

In the examples shown in Figure 4.5, Figure 4.6 and Figure 4.7, the vertices i and i' are identified by the involution. The last example involves a fixed point which is circled.

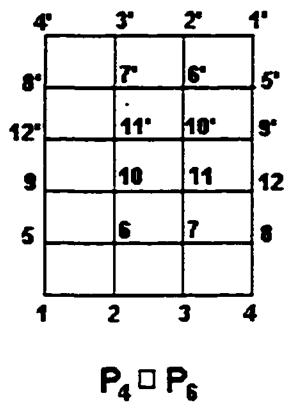


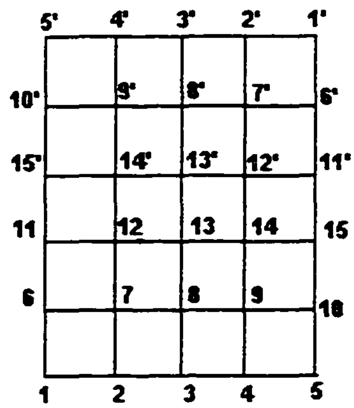
Figure 4.5.

4.4 (r, s) −Web Graph

The (r, s) −**web graph** G is the graph with vertex set

$$V(G) = \{(i, j) \mid 0 \leq i \leq r-1,\ 0 \leq j \leq s-1\}$$

and edge set



$P_5 \square P_6$

Figure 4.6.

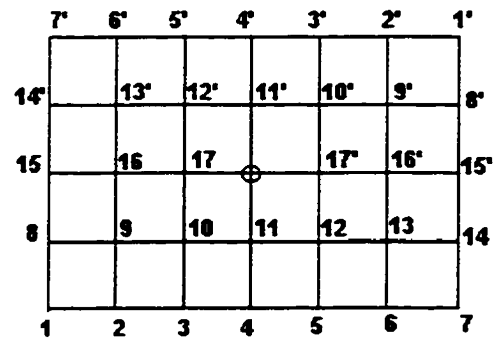
$$\begin{aligned} E(G) &= \{(i,j)(i,j+1) \mid 0 \le i \le r-1,\ 0 \le j \le s-2\} \\ &\cup \{(i,s-1)(i,0) \mid 0 \le i \le r-1\} \\ &\cup \{(i,j)(i+1,j) \mid 0 \le i \le r-2,\ 0 \le j \le s-1\}. \end{aligned}$$

Observe that (r,s) −web graph is isomorphic to $P_r \boxtimes C_s$.

4.5 Corollary

Let G be an (r,s) −web graph.

(a) If r is even and s is odd, then $W(G^d, m) = B$ for all integers $d \ge 1$ and all even integers $m \ge 2$.



$P_7 \square P_5$.

Figure 4.7.

(b) If r is odd and s is even, then $W(G^d, m) = B$ for all integers $d \geq 1$ and all even integers $m \geq 2$ or for all integers $1 \leq d < \frac{s}{2}$ and all integers $m \geq 1$.

(c) If r and s are both even, then $W(G^d, m) = B$ for all integers $d \geq 1$ and all even integers $m \geq 2$ or for all integers $1 \leq d \leq \frac{s}{2}$ and all integers $m \geq 1$.

Proof. Let the involution $\varphi_r : P_r \longrightarrow P_r$ be defined as in the proof of Corollary 1.4. For s even, we can define an involution $\varphi_s : C_s \longrightarrow C_s$ by setting $\varphi_s(i) = i + \frac{s}{2}$, $0 \leq i \leq s-1$. Let $\varphi : G \longrightarrow G$ be defined as follows.

If r is even and s is odd, let $\varphi = \varphi_r \boxtimes id$. If r is odd and s is even, let $\varphi = id \boxtimes \varphi_s$. Finally, if r and s are both even, let $\varphi = \varphi_r \boxtimes \varphi_s$. Parts (a) and (b) now follow from Theorem 3.4 while part (c) follows from Theorem3.6(a). ■

Observe that for r and s both odd, we don't have a clear cut pattern. For example, if G is the $(3,3)$ −web graph shown in Figure 4.8, then $W(G, 1) = A$.

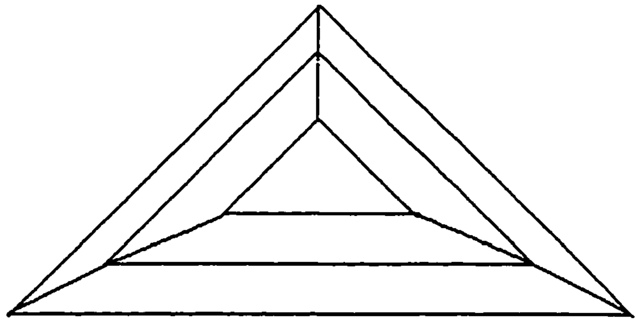


Figure 4.8.

Every game ends with precisely three vertices chosen, one from each triangle. On the other hand, if G is a $(3,5)$ −web graph, then $W(G,1) = B$. Figure 4.9 and Figure 4.10 indicate (with black dots) the two essentially different first moves for player A together with player B's responses. (Vertices rendered uncolorable by players A and B's first moves are "crossed out".)

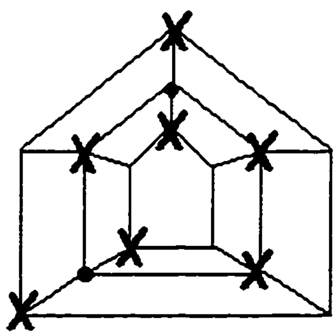


Figure 4.9.

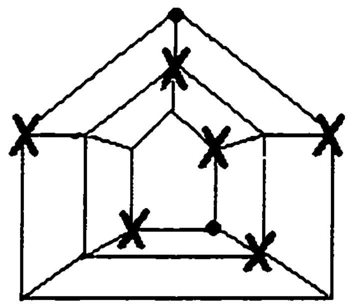


Figure 4.10.

4.6 Theorem

Let C_s be the cycle with s vertices and $\mathbf{K}_s$ the complete graph on s vertices, $s \geq 2$. If H is any graph and $G = H \boxtimes C_s$ or $G = H \boxtimes \mathbf{K}_s$, then $W\left(G^d, 2\right) = B$ for all integers $d \geq 1$.

Proof. Label the vertices of $\mathbf{K}_s$ with integers mod s (as we did with C_s). Player B's strategy is as follows. Whenever player A assigns the color 1 (respectively the color 2) to some vertex (v, i) of G, player B should assign the color 2 (respectively the color 1) to the vertex $(v, i+1)$ (respectively $(v, i-1)$). Here addition in the second coordinate is modulo s.

Assume this strategy does not work. Say player A has assigned the color 1 to the vertex (v, i), but player B cannot assign the color 2 to $(v, i+1)$. (The other case is treated analogously.) Further assume that this is the first instance in the game in which player B cannot respond. We can assume that the vertex $(v, i+1)$ is uncolored. (If $(v, i+1)$ had been assigned the color 2 in a previous move, then, in accordance with player B's strategy, (v, i) would have been assigned the color 1 in some previous move.) For $(v, i+1)$ to be uncolorable there must exist a vertex $(v', i'+1)$ which has been assigned the color 2 with

$$D((v, i+1)(v', i'+1)) \leq d.$$

Since we are assuming that up to this point player B has been able to follow his strategy, the vertex (v', i') must have been assigned the color 1. However,

$$D((v, i)(v', i')) = D((v, i+1)(v', i'+1)) \leq d,$$

which means that player A cannot assign the color 1 to the vertex (v, i), a contradiction. ■

4.7 Corollary

If G is any (r, s)−web graph, then $W(G^d, 2) = B$ for all integers $d \geq 1$.

4.8 Corollary

$W(C_s^d, 2) = B$ for all integers $d \geq 1$ and all integers $s \geq 2$.

4.9 Note

For $d = 1$, which is Theorem 3 of [6], it makes no difference what moves players A and B make. Player B always wins, which is to say all games end after an even number of moves. In showing this we will thus be providing an independent proof of Theorem 3 of [6].

Suppose that a subset of the vertices of any cycle C_s have been assigned the colors 1 and 2 and that no more legal moves are possible. Let i be any vertex which has been assigned a color, say 1. Then either $i+1$ has been assigned the color 2 or it is an uncolorable vertex. The latter can only occur if the vertex $i+2$ has been assigned the color 2. Either way, we have shown that if we start with any colored vertex and proceed around the cycle clockwise then the colors alternate. Therefore we must have an even number of colored vertices. (In fact the number of vertices colored 1 equals the number of vertices colored 2.)

4.10 Note

As a trivial corollary of Theorem 4.6, we have that $W(\mathbf{K}_s, 2) = B$ for all integers $s \geq 2$. More generally, if $d \geq \operatorname{diam}(G)$, the diameter of G, then $G^d = \mathbf{K}_n$, where $n = |V(G)|$. Consequently

$$W\left(G^d, m\right) = W(\mathbf{K}_n, m)$$

is determined by the parity of m when $m \leq |V(G)|$ and by the parity of $|V(G)|$ when $m \geq |V(G)|$.

There is another similar observation to be made, namely that if $m > \Delta(G)$, the maximum degree of G, then again $W(G, m)$ is determined by the parity of $|V(G)|$.

If the maximal degree $\Delta(G) > 2$, then the analogous statement for G^d is obtain by letting

$$m > \Delta(G) \cdot \left\lceil \frac{(\Delta(G) - 1)^d - 1}{\Delta(G) - 2} \right\rceil$$

since

$$\Delta(G) \cdot \left[\frac{(\Delta(G)-1)^d - 1}{\Delta(G)-2}\right]$$

$$= \Delta(G) + \Delta(G)(\Delta(G)-1) + \cdots + \Delta(G)(\Delta(G)-1)^{d-1}$$

gives an upper bound on the number of vertices whose distance from a fixed vertex is d or less, that is on $\Delta\left(G^d\right)$. If $\Delta(G) = 2$, then

$$\Delta(G) + \Delta(G)(\Delta(G)-1) + \cdots + \Delta(G)(\Delta(G)-1)^{d-1} = 2d.$$

So in this case the analogous statement is obtained by letting $m > 2d$.

If we restrict our attention to trees T, then we can in fact replace $m > \Delta(T)$ by $m > 3$ in the above observation.

4.11 Theorem

If the number of colors $m > 3$, then

$$W(T,m) = \begin{cases} \text{A} & \text{for } t \text{ odd,} \\ \text{B} & \text{for } t \text{ even,} \end{cases}$$

where t is the number of vertices in T.

Proof. We elaborate on the rather terse proof of U. Faigle, W. Kern, H. Kierstead and W. T. Trotter [4], noting that, in fact, their strategy can be applied by either player to completely color the tree T. Assume first that player A is the one who wants to force the coloring of all the vertices of T. He starts by assigning some color to some vertex v_0 of T. Let $T_0 = \{v_0\}$. We expand the tree T_0 after each move in such a way as to include all the vertices colored so far. In fact, we will prove inductively that after each move all the uncolored vertices of T_0 will have at most three colored neighbors in T_0. Since we are assuming that $m \geq 4$, all these vertices are colorable. Moreover, since any vertex v of $T \setminus T_0$ has at most one neighbor in T_0, any vertex outside of T_0 is colorable as well. Thus eventually we must end up with $T = T_0$ with all of its vertices colored and that is how player A wins.

Suppose player B colors a vertex v of T. Let P be the unique shortest path from v to the tree T_0. In other words, P is a path from v to w, where w is the only vertex on the path that lies in T_0. We "connect the branch" P to T_0 at the "juncture" w obtaining a larger tree. This tree will be our new T_0. (if v is a vertex of T_0, then we have $v = w$ and we do not need to expand T_0.)

By induction, all the uncolored vertices of the new tree, except possibly w, will have at most two colored neighbors. So if w is uncolored player A can color it. (Again we assume that $m \geq 4$.) That way the new tree will satisfy the inductive hypothesis.

If w has already been colored, then player A can color any other vertex of the tree T_0. (Clearly the inductive hypothesis will still hold.) If all of the vertices of the tree have already been colored, then player A can color any vertex adjacent to the tree T_0. Join this new vertex of the tree to produce a new T_0. (As in the original tree, all of the vertices of T_0 are colored.) If there are no vertices adjacent to T_0 then $T = T_0$ with all of its vertices colored and player A has won.

We have shown inductively that player A can always respond to player B's move in such a way that no uncolored vertex of T_0, in fact of T, will have more than two colored neighbors in T. Hence since T has a finite number of vertices, eventually we must end up with $T = T_0$ and all of its vertices colored.

If it were player B's rather than player A's objective to force a complete colored tree, then he should proceed in a similar fashion. Say player A starts by assigning some color to some vertex v_0 of T. Then player B can color any adjacent vertex v_1 of T. Set $T_0 = \{v_0, v_1\}$. The inductive hypothesis is satisfied and we can proceed as before. So no matter who starts either player can force the coloring of all the vertices and the winner of the Achievement game is determined by the parity of the number of vertices of the tree T. ■

4.12 Note

Zsolt Tuza has informed me that he and Hal Kierstead found the following more general result a couple of years ago. (The result is still unpublished and the paper in which it is to appear is in process of being refereed).

If G has treewidth at most t, then the game chromatic number of G is at most $6t - 2$. Moreover, their winning strategy for trees ($t = 1$) is simpler (and different) than the strategy above. However for $t > 1$ their strategy is more complicated. Tuza postulates the following open problem: determine the smallest c such that a graph of treewidth t has game chromatic number at most $ct + o(t)$ as t tends to infinity. Kierstead and Tuza have determined that $1 < c < 6$.

Suppose that l, n and q are positive integers with $1 \leq l \leq n$ and $q \geq 2$. Let the **cube-like** graph $Q_l^n(q)$ be the graph whose vertices are $n-$tuples on a set of $q-$elements, say

$$\{(x_1, x_{2,\ldots,}x_n) \mid 0 \leq x_i \leq q-1,\ 1 \leq i \leq n\}.$$

Two vertices $(x_1, x_{2,\ldots,}x_n) \neq (y_1, y_{2,\ldots,}y_n)$ are adjacent if and only if they differ in at most l coordinates. (See p.157 of [9].) Thus

$$Q_l^n(q) = (Q_1^n(q))^l.$$

In particular, $Q_1^n(2)$ is the graph of the $n-$dimensional cube and so $Q_l^n(2)$ can be described as the graph obtained from the graph of the $n-$cube by joining all pairs of vertices that are at most distance l apart.

Observe that

$$Q_n^n(q) = \mathbf{K}_{q^n},$$

the complete graph on q^n vertices. So in the sequel we assume that $1 \leq l < n$. Also observe that

$$Q_1^n(q) = \mathbf{K}_q \boxtimes \cdots \boxtimes \mathbf{K}_q$$

(n copies of $\mathbf{K}_q$).

The graphs of

$$Q_1^3(2) = \mathbf{K}_2 \boxtimes \mathbf{K}_2 \boxtimes \mathbf{K}_2$$

and

$$Q_1^2(3) = \mathbf{K}_3 \boxtimes \mathbf{K}_3$$

are given in Figure 4.11 and Figure 4.12.

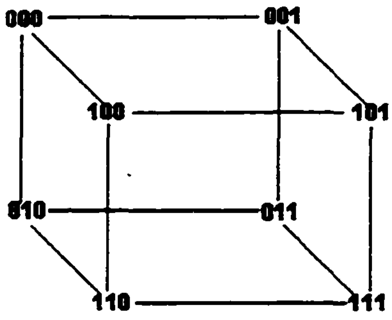


Figure 4.11.$\mathbf{Q}_1^3(2)$

4.13 Corollary

For q even and $n \geq 2$,

$$W\left(Q_l^n(q), m\right) = B$$

for all integers $m \geq 1$. For q odd,

$$W\left(Q_l^n(q), m\right) = A$$

for all odd integers $m \geq 1$.

Proof. Label the vertices of $\mathbf{K}_q$ with the elements $\{0, 1, \ldots, q-1\}$. Define an involution $\varphi : \mathbf{K}_q \longrightarrow \mathbf{K}_q$ by $\varphi(i) = q - i - 1$, $0 \leq i \leq q-1$. It is fixed-point free for even integers q and has exactly one fixed point for odd integers q. Since

$$Q_1^n(q) = \mathbf{K}_q \boxtimes \cdots \boxtimes \mathbf{K}_q$$

(n copies of $\mathbf{K}_q$), φ induces an involution

$$\varphi \boxtimes \varphi \boxtimes \cdots \boxtimes \varphi : Q_1^n(q) \longrightarrow Q_1^n(q).$$

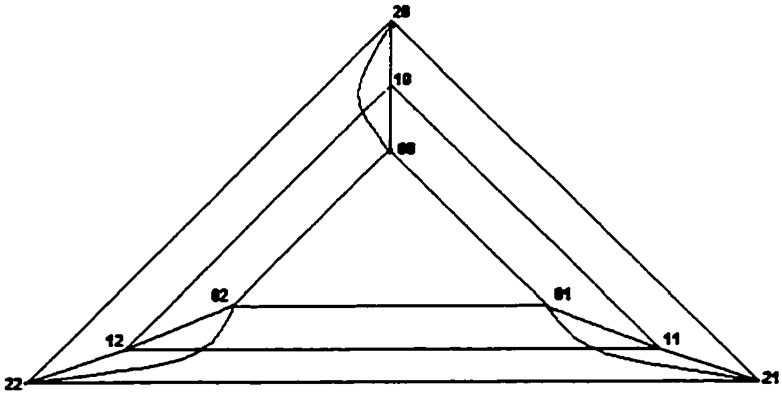


Figure 4.12.$\mathbf{Q}_1^2\,(3)$

Observe that for q even the involution φ satisfies the property $(\mathbf{P_0})$. Since $n \geq 2$, the hypotheses of the second part of Theorem 3.6(a) is satisfied (with $d' = n$ and $d = l$) and we get the desired result for even q. (Recall that we are assuming that $1 \leq l < n$.) For odd q, φ (vacuously) satisfies property $(\mathbf{Q}_d)$ for all $d \geq 1$. In fact, for $i \neq \frac{q-1}{2}$,

$$D\left(i, \frac{q-1}{2}\right) = D\left(i, \varphi\left(i\right)\right) = 1.$$

So the hypothesis of the first part of Theorem 3.6(b) is satisfied and we get the desired result for odd q as well. ■

The following is another consequence of Theorem 4.6.

4.14 Corollary

$$W\left(Q_l^n\left(q\right), 2\right) = B$$

for all integers $q \geq 2$ and $1 \leq l < n$.

4.15 Cube-Like Graphs

Let S be a set of n elements and $\Im \subseteq 2^S$ a family of subsets of S. The **"cube-like" graph** $Q_S(\Im)$ is the graph with vertex set 2^S, where two vertices $x, y \subseteq S$ are adjacent if and only if their symmetric difference

$$x \bigtriangleup y \triangleq (x \setminus y) \cup (y \setminus x)$$

belongs to $\Im$. (See p.156 of [9].)

For example, take $1 \leq l \leq n$ and let $\Im_l$ denote the family of subsets of S of size at most l. Then $Q_S(\Im) \cong Q_l^n(2)$.

4.16 Note

An involution $\varphi : 2^S \longrightarrow 2^S$, where 2^S is to be thought of as the trivial graph with 2^n vertices, induces an involution

$$\varphi : Q_S(\Im) \longrightarrow Q_S(\Im)$$

if and only if the following property is satisfied.

(S) If $x, y \subseteq S$ and $x \bigtriangleup y \in \Im$, then $\varphi(x) \bigtriangleup \varphi(y) \in \Im$.

And if this is the case then the first part of Theorem 3.1(a) implies that

$$W\left((Q_S(\Im))^d, m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$.

4.17 Theorem

$$W\left(\left(Q_S\left(\Im_l\right)\right)^d, m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$.

Proof. Let $\varphi : 2^S \longrightarrow 2^S$ be the map that sends a set to its complement. It is easy to see that this map is a fixed-point free involution. (In fact, the map corresponds to the involution on $Q_l^n(2)$ we defined above.) For any $x, y \subseteq S$, we have

$$\varphi(x) \triangle \varphi(y) = \varphi(x) \setminus \varphi(y) \cup \varphi(y) \setminus \varphi(x) = (y \setminus x) \triangle (x \setminus y) = x \triangle y.$$

Thus property **(S)** is satisfied and the first part of Theorem 3.1(a) applies. ■

Chapter 5
Generalized Petersen and Permutation Graphs

Start with graphs G and H together with a partial map $\sigma : V(G) \longrightarrow V(H)$, that is a map $\sigma : X \longrightarrow V(H)$ for some $X \subseteq V(G)$. The graph $P_\sigma(G, H)$ consists of G and H along with edges obtained by joining each $v \in X$ with $\varphi(v) \in V(H)$. In particular for $X = \emptyset$ there is a unique map $\emptyset : \emptyset \longrightarrow V(H)$ and $P_\emptyset(G, H) = G \cup H$, the disjoint union of G and H. We also have the following special cases of the definition.

5.1 Sigma and Cycle Permutation Graphs

Let G be a graph with n vertices. Suppose

$$V(G) = \{0, 1, \ldots, n-1\}$$

$(n \geq 4)$. Choose $\sigma \in \mathbf{S}_n$, where $\mathbf{S}_n$ is the group of permutations of $V(G)$. The graph

$$P_\sigma(G) \triangleq P_\sigma(G, G)$$

is referred to as the **σ−permutation graph of G**. If the original graph G is an n−cycle, then $P_\sigma(G)$ is called a **cycle permutation graph** and we denote it by $C(n, \sigma)$. (See pages 316 and 317 of [8].)

As an example take $C(6, \sigma)$, where $\sigma = (012)(345)$. See Figure 5.13.

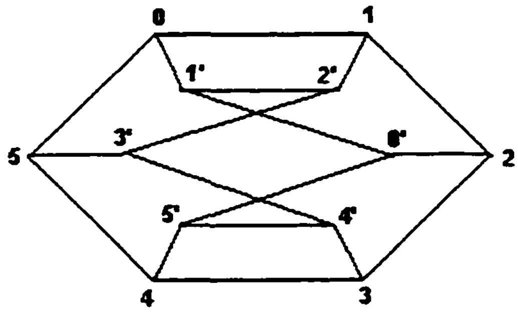


Figure 5.13.

5.2 Corollary

Let G, H and $\sigma : V(G) \longrightarrow V(H)$ be as above. Suppose that $\varphi_G : G \longrightarrow G$ and $\varphi_H : H \longrightarrow H$ are fixed-point free involutions. If $\varphi_G(X) = X$ and

$$\sigma \circ \varphi_G = \varphi_H \circ \sigma : V(G) \longrightarrow V(H),$$

then

$$W\left(P_\sigma^d(G,H), m\right) = W\left(G^d, m\right) = W\left(H^d, m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$, where $P_\sigma^d(G,H)$ is the dth power of $P_\sigma(G,H)$.

If we furthermore assume that σ is injective and that φ_G and φ_H satisfy properties $(\mathbf{P}_{d_G})$ and $(\mathbf{P}_{d_H})$, respectively, where d_G and d_H are positive integers, then

$$W\left(P_\sigma^d(G,H), m\right) = W\left(G^d, m\right) = W\left(H^d, m\right) = \mathrm{B}$$

for $1 \leq d \leq \min\{d_G, d_H, 3\}$ and all integers $m \geq 1$.

Proof. Since V

$$(P_\sigma(G,H)) = V(G) \cup V(H),$$

we can define

$$\varphi : V(P_\sigma(G,H)) \longrightarrow V(P_\sigma(G,H))$$

by letting $\varphi = \varphi_G \cup \varphi_H$. Clearly φ is fixed-point free because φ_G and φ_H are both fixed-point free and $\varphi^2 = 1$ because $\varphi_G^2 = 1$ and $\varphi_H^2 = 1$.

To complete the proof that φ is an involution, we need to show that $vw \in E(P_\sigma(G,H))$ if and only if

$$\varphi(vw) = \varphi(v)\varphi(w) \in E(P_\sigma(G,H)).$$

If

$$vw \in E(G) \subset E(P_\sigma(G,H))$$

or

$$vw \in E(H) \subset E(P_\sigma(G,H)),$$

then this follows from the fact that both φ_G and φ_H are graph automorphisms. so we assume that

$$v \in V(G) \subset V(P_\sigma(G,H))$$

and

$$w = \sigma(v) \in V(H) \subset V(P_\sigma(G,H)).$$

Then

$$\varphi(vw) = \varphi(v)\varphi(w) = \varphi(v)\varphi(\sigma(v)) = \varphi(v)\sigma(\varphi(v)) \in E(P_\sigma(G,H)).$$

On the other hand, if

$$\varphi(vw) = \varphi(v)\varphi(w) \in E(P_\sigma(G,H)),$$

then

$$vw = \varphi^2(vw) = \varphi(\varphi(vw)) \in E(P_\sigma(G,H)).$$

It now follows from the first part of Theorem 3.1(a), that

$$W\left(P_\sigma^d(G,H), m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$. In order to show that $W\left(P_\sigma^d(G,H), m\right) = B$ for $1 \leq d \leq \min\{d_G, d_H, 3\}$.

Let

$$v \in V(G) \subset V(P_\sigma(G,H))$$

(respectively $v \in V(H) \subset V(P_\sigma(G,H))$). Then $D(v,\varphi(v)) \leq D_G(v,\varphi(v))$ (respectively $D(v,\varphi(v)) \leq D_H(v,\varphi(v))$), where D_G (respectively D_H) is the distance function on G (respectively H) and D is the distance function $P_\sigma(G,H)$. Any path from v to $\varphi(v)$ of length shorter than $D_G(v,\varphi(v)) > d_G$ (respectively $D_H(v,\varphi(v)) > d_H$) would have to travel through $H \subset P_\sigma(G,H)$ (respectively $G \subset P_\sigma(G,H)$). Since $\varphi(v) \in V(G) \subset V(P_\sigma(G,H))$ (respectively $\varphi(v) \in V(H) \subset V(P_\sigma(G,H))$), such a path would have to have length at least 4. Therefore, φ satisfies $(\mathbf{P}_d)$ for all $1 \leq d \leq \min\{d_G, d_H, 3\}$. We can apply the second part of Theorem 3.1(a) and that completes the proof. ■

5.3 Corollary

Let $P_\sigma(G)$ be the σ−permutation graph of a graph G. Assume that there exists a fixed-point free involution $\varphi : G \longrightarrow G$. Assume further that $\sigma \circ \varphi = \varphi \circ \sigma$. Then

$$W(P_\sigma(G), m) = W\left(G^d, m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$.

If φ also satisfies property $(\mathbf{P}_{d_0})$ for some $d_0 \geq 1$, then

$$W\left(P_\sigma^d(G), m\right) = W\left(G^d, m\right) = B$$

for any $1 \leq d \leq \min\{d_0, 3\}$ and all integers $m \geq 1$.

5.4 Corollary

Let $C(n, \sigma)$ be a cycle permutation graph for some $\sigma \in \mathbf{S}_n$ with n a positive even integer. If

$$\sigma\left(i + \frac{n}{2}\right) = \sigma(i) + \frac{n}{2} (\operatorname{mod} n)$$

for all $0 \leq i \leq n-1$, then

$$W\left(C^d(n, \sigma), m\right) = B$$

for all integers $d \geq 1$ and all even integers $m \geq 2$ or for $1 \leq d \leq \min\left\{\frac{n}{2} - 1, 3\right\}$ and all integers $m \geq 1$.

Proof. Let the involution on the n−cycle G identify "diametrically opposed" vertices. More precisely, if $V(G) = \{0, 1, \ldots, n-1\}$ and

$$E(G) = \{i(i+1) \mid 0 \leq i \leq n-1\},$$

then we define the involution φ_G by

$$\varphi_G(i) = i + \frac{n}{2}$$

for all $0 \leq i \leq n-1$. Clearly, φ_G is an involution. Since

$$\sigma\left(i + \frac{n}{2}\right) = \sigma(i) + \frac{n}{2}$$

(mod n) for all $0 \leq i \leq n-1$, then $\sigma \circ \varphi = \varphi \circ \sigma$. Moreover, φ_G satisfies property ($\mathbf{P}_d$) for any $1 \leq d \leq \frac{n}{2}$. We can now apply Corollary 5.3. ■

As an example refer to the graph of $C(6, \sigma)$, where $\sigma = (012)(345)$, above. In that diagram $\sigma(i)$ is denoted by i'.

5.5 Generalized Petersen Graph

The **generalized Petersen graph** $P(n,k)$ is defined by the letting

$$V(P(n,k)) = \{u_i, v_i \mid 0 \leq i \leq n-1\}$$

be the set of vertices and

$$E(P(n,k)) = \{u_i u_{i+1},\ v_i v_{i+k},\ u_i v_i \mid 0 \leq i \leq n-1\}$$

be the set of edges where addition in the subscripts is modulo n and $1 \leq k < \frac{n}{2}$. (See page 45 of [8] for example.

In particular, $P(5,2) \cong P$, the Petersen graph shown in Figure 5.14.

Observe that any generalized Petersen graph $P(n,k)$ is isomorphic to $P_\sigma(G,H)$ for G an n–cycle and H a disjoint union of (one or more) cycles of equal length with

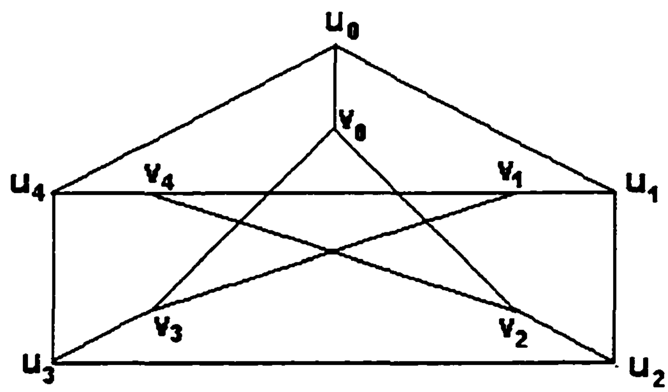


Figure 5.14.

$|V(H)| = |V(G)|$. More precisely, define G by setting $V(G) = \{u_0, \ldots, u_{n-1}\}$ and

$$E(G) = \{u_i u_{i+1} \mid 0 \le i \le n-1\}$$

$(u_n = u_0)$. the graph H will consist of one cycle (of length n) if $(k,n) = 1$ or of k cycles (of length n/k) if $(k,n) \neq 1$. Let $V(H) = \{v_0, \ldots, v_{n-1}\}$, with v_i and v_j belonging to the same cycle if and only if $|i-j| = 0 \pmod k$. Finally, define $\sigma : V(G) \longrightarrow V(H)$ by $\sigma(u_i) = v_i$, for $0 \le i \le n-1$.

5.6 Corollary

Let n be an even integer. Then

$$W\left(P^d(n,k), m\right) = B$$

for all integers $d \ge 1$ and even integers $m \ge 2$ or for $d = 1$ and all integers $m \ge 1$.

If $(k,n) = 1$, then $W\left(P^d(n,k), m\right) = B$ for all integers $1 \le d < \frac{n}{2}$ and all integers $m \ge 1$.

Proof. Let G and H and σ be defined as in the remarks preceding the corollary. Define φ_G and φ_H by

$$\varphi_G(u_i) = u_{i+\frac{n}{2}}$$

and

$$\varphi_H(v_i) = v_{i+\frac{n}{2}},$$

respectively, for all $0 \leq i \leq n-1$. Observe that since $k < \frac{n}{2}$, both φ_G and φ_H satisfy property $(\mathbf{P}_1)$. Moreover, if $(k,n) = 1$, both φ_G and φ_H satisfy property $(\mathbf{P}_\mathbf{d})$ for any integer $1 \leq d < \frac{n}{2}$.

Our results will now follow from the relevant parts of Corollary 4.8 once we check that the condition $\sigma \circ \varphi_G = \varphi_H \circ \sigma$ holds. However, for $0 \leq i \leq n-1$ we have

$$\begin{aligned}(\sigma \circ \varphi_G)(u_i) &= \sigma\left(u_{i+\frac{n}{2}}\right) = v_{i+\frac{n}{2}} = \varphi_H(v_i) \\ &= (\varphi_H \circ \sigma)(u_i), 0 \leq i \leq n-1.\end{aligned}$$

5.7 Note

Observe that $P(10,2)$ is isomorphic to the graph of the dodecahedron and the involution on $P(10,2)$ induces one on the dodecahedron which identifies antipodal points. See Figure 5.15. (This identification results in the Petersen graph as is pointed out on pages 280 and 318 of [8]).

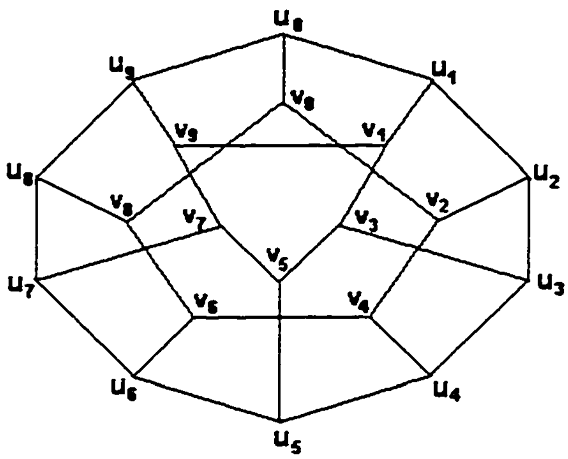


Figure 5.15.

Chapter 6
Complete K-Partite Graphs

Let $\mathbf{K}_{n_1,n_2,\ldots,n_k}$ be the complete k−partite graph with parts X_i of size n_i, $1 \leq i \leq k$. Let

$$\varepsilon = |\{n_i \mid n_i \text{ even}\}|$$

and

$$o = |\{n_i \mid n_i \text{ odd}\}|.$$

Observe that if a vertex of X_i is assigned the color c then in any subsequent move only the remaining vertices of X_i, if any, can be assigned the color c, that is in any subsequent move another vertex can be assigned the c if and only if that vertex belongs to X_i. Consequently, once a vertex of X_i is assigned a color all the vertices of X_i will be assigned colors by the end of the game. However, they need not all be assigned the same color.

Suppose that one of the players wanted to color to all the vertices of each X_i for which n_i is an odd number. By the previous remarks, he would simply have to assign a color to one vertex from each of these X_i's. To do so he would need o colors. However, the other player need not cooperate–he may color vertices of X_i's for which n_i is an even number. In this case, all the colors would have been assigned in m moves, where m is the number of colors, with player A having assigned $\lceil \frac{m}{2} \rceil$ colors and player B having assigned $\lfloor \frac{m}{2} \rfloor$ colors. Therefore, if we assume that $\lceil \frac{m}{2} \rceil \geq o$, then neither player can prevent the other from coloring all the vertices of all the X_i for which n_i is odd, in which case the player with the winning strategy is determined by the parity of o. Hence,

$$W(\mathbf{K}_{n_1,n_2,\ldots,n_k}, m) = \begin{cases} \text{A} & \text{for } o \text{ odd,} \\ \text{B} & \text{for } o \text{ even.} \end{cases}$$

6.1 Theorem

(a) If $o \geq m$, then

$$W(\mathbf{K}_{n_1,n_2,\ldots,n_k}, m) = \begin{cases} \text{A} & \text{for } m \text{ odd,} \\ \text{B} & \text{for } m \text{ even.} \end{cases}$$

(b) If $o < m$, then

$$W(\mathbf{K}_{n_1,n_2,\ldots,n_k}, m) = \begin{cases} \text{A} & \text{for } m, o \text{ odd,} \\ \text{B} & \text{for } m, o \text{ even.} \end{cases}$$

Proof. Choose an involution $\kappa : \mathbf{C} \longrightarrow \mathbf{C}$ that is fixed-point free for even m and with exactly one fixed point, say 1, for odd m. (Here $\mathbf{C}$ is the set of colors.)

We will prove parts (a) and (b) simultaneously. For odd m and either $o \geq m$ or o odd, player A starts off by assigning the color 1 to any vertex of any X_i with n_i odd, say X_1. From this point on, player A's strategy is identical to player B's strategy (for m even and either $o \geq m$ or o even) described below but with the roles of A and B reversed. So we will assume m is even and proceed to describe player B's winning strategy.

Whenever player A assigns the color c to a vertex $v \in X_i$ with n_i even, player B should respond by assigning the color $\kappa(c)$ to any other vertex of X_i. (Player B can always do this since n_i is even.)

Whenever player A assigns the color c to a vertex $v \in X_i$ with n_i odd and this is the first time a vertex of X_i has been assigned a color (and consequently the first time the color c has been assigned), player B should respond by assigning the color $\kappa(c)$ to any vertex of any X_j, $i \neq j$, with n_j odd and satisfying the condition that up to this point no vertex of X_j has been assigned any color. (Since either o is even or $o \geq m$, player B can always respond in this fashion.)

Finally, assume player A assigns a color c to a vertex $v \in X_i$ with n_i odd and this is *not* the first time a vertex of X_i has been assigned a color. Say the first vertex of X_i to have been assigned a color was assigned the color c_0.

If $c = c_0$, then a vertex of X_j, for some $j \neq i$ with n_j odd, must have been assigned the color $\kappa(c_0)$ in a previous move. In this case, player B responds by assigning the same color c_0 to any other vertex of X_i.

If $c \neq c_0$, then player B responds by assigning the color $\kappa(c)$ to any other vertex of X_i. (Since n_i is odd, player B can always respond in this fashion). ■

6.2 Corollary

If $k \geq m$, then

$$W(\mathbf{K}_k, m) = \begin{cases} A & \text{for } m \text{ odd,} \\ B & \text{for } m \text{ even.} \end{cases}$$

Proof. Let $n_i = 1$, for all $1 \leq i \leq k$. Then $o = k$. ■

6.3 Corollary

$$W\left(\mathbf{K}_{n_1,n_2,\ldots,n_k},1\right)=\begin{cases}A & \text{for } o\geq 1,\\ B & \text{for } o=0.\end{cases}$$

Chapter 7
The General Kneser Graphs

For integers $n > k > t > 0$, the **general Kneser graph** $K(n, k, t)$ is defined as the graph with the set of all k–subsets of $N = \{1, 2, \ldots, n\}$ as vertex set and two such sets X and Y and joined by an edge if and only if $|X \cap Y| < t$. (For example, see p.161 of [8] or p.296 of [9].)

Note that $K(n, 2, 1) = \overline{L(\mathbf{K}_n)}$, the complement of the line graph of the complete graph $\mathbf{K}_n$. If $n = 5$ then we obtain the Petersen graph $P \cong \overline{L(\mathbf{K}_5)}$ shown in Figure 7.16

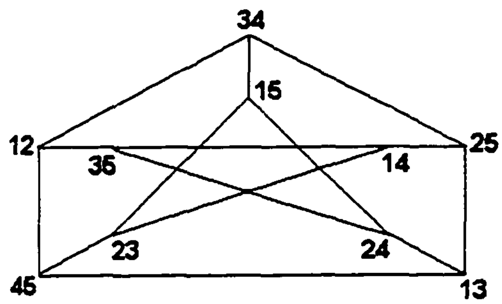


Figure 7.16.

The following two theorems are corollaries of Theorem 1.2.

7.1 Theorem

If n is even and k is odd, then $W(K(n,k,t),m) = B$ for any even integer $m \geq 2$. Moreover, $W_S(K(n,k,t),m) = B$ for any integer $m \geq 1$ and $W_P(K(n,k,t)) = B$.

Proof. Let $N = \{1,2,\ldots,n\}$. Define a bijection $\theta : N \longrightarrow N$ by $\theta(i) = n - i + 1$, $i \in N$. Since n is assumed even, θ has no fixed points.

The map θ induces a bijection $\overline{\theta} : V(K(n,k,t)) \longrightarrow V(K(n,k,t))$ defined by $\overline{\theta}(X) = \{\theta(i) \mid i \in X\}$, $X \in V(K(n,k,t))$. Clearly $\overline{\theta}$ preserves intersections, that is $\overline{\theta}(X_1 \cap X_2) = \overline{\theta}(X_1) \cap \overline{\theta}(X_2)$ for all $X_1, X_2 \in V(K(n,k,t))$. Since $X_1X_2 \in E(K(n,k,t))$ if and only if $|X_1 \cap X_2| \leq t-1$, this implies that $\overline{\theta}$ induces a graph automorphism of $K(n,k,t)$. Moreover, $\overline{\theta}^2 = id$ because $\theta^2 = id$. Consequently, $\overline{\theta}$ is an involution.

We now intend to apply the first part of Theorem 1.2(a). In order to do so we must show that $\overline{\theta}$ is fixed-point free.

Assume that $\overline{\theta}$ has a fixed point $X_0 \in V(K(n,k,t))$. Then

$$X_0 = \bigcup_{i \in X_0} \{i, \theta(i)\}.$$

Since θ has no fixed points, $|\{i,\theta(i)\}| = 2$ for all $i \in X_0$. But this implies that $k = |X_0|$ must also be even, which is a contradiction. Therefore, $\overline{\theta}$ must be fixed-point free and Theorem 1.2(a) applies. ■

7.2 Theorem

Let $j, k \geq 2i + 1$. Then

$$W\left(K\left(k+j, k, k-i\right), 2\right) = B.$$

Proof. Suppose player A assigns color 1, say, to the k-subset of X of N. Without loss of generality, we may assume $X = \{1, 2, \ldots, k\}$. Let the bijection $\theta : N \longrightarrow N$ be defined as in the last proof. As noted there θ induces an involution

$$\overline{\theta} : K\left(n, k, t\right) \longrightarrow K\left(n, k, t\right),$$

where $n = k + j$ and $t = k - i$. Now unless n is even and k is odd, $\overline{\theta}$ will not be fixed-point free. However, if player B responds to player A's move by assigning color 2 to $Y = \theta(X) = \{n - k + 1, n - k + 2, \ldots, n\}$, then all the fixed points of $\overline{\theta}$ will be rendered uncolorable and we will be able to once again apply the first part of Theorem 1.2(a).

If $Z \in V\left(K\left(n, k, t\right)\right)$ is a fixed point of $\overline{\theta}$, we will show that Z is a neighbor of both X and Y, and is therefore uncolorable. Consequently, the rest of the game is played out on the vertices of the graph of $K\left(n, k, t\right) \setminus W$, where $W \subset V\left(K\left(n, k, t\right)\right)$ is the set of fixed points of $\overline{\theta}$. But by restriction $\overline{\theta} : K\left(n, k, t\right) \longrightarrow K\left(n, k, t\right)$ induces a fixed-point free involution $\overline{\theta} : K\left(n, k, t\right) \setminus W \longrightarrow K\left(n, k, t\right) \setminus W$. We can then apply Theorem 1.2(a).

Observe that if $n - k + 1 \leq k$, that is if $n \leq 2k - 1$, then $|X \cap Y| = 2k - n$. On the other hand, if $n \geq 2k$, $X \cap Y = \emptyset$. Moreover since $\overline{\theta}$ preserves intersections then for any fixed point Z of $\overline{\theta}$,

$$|Z \cap X| = |\theta(Z \cap X)| = |\theta(Z) \cap \theta(X)| = |Z \cap Y|.$$

If $n \geq 2k$, that is if $X \cap Y = \emptyset$, then

$$|Z \cap X| = |Z \cap Y| \leq \frac{k}{2} < t.$$

Consequently, ZX and ZY are edges of $K(n,k,t)$. In other words, Z is a neighbor of both X and Y.

Finally, assume that $n \leq 2k-1$. In this case $|X \cap Y| = 2k-n$ and $Z \subset X \cup Y = N$. Therefore,

$$\begin{aligned} k &= |Z| = |Z \cap X| + |Z \cap Y| - |Z \cap X \cap Y| = 2|Z \cap X| - |Z \cap X \cap Y| \\ &\geq 2|Z \cap X| - 2k + n > 2|Z \cap X| - k + 2i. \end{aligned}$$

(The last inequality was obtained by substituting $k+j$ for n and using the fact that $j \geq 2i+1$.) Once again we get

$$|Z \cap X| = |Z \cap Y| < k - i = t,$$

that is that Z is a neighbor of both X and Y. ■

7.3 Note

If $j \leq i$, then $K(k+j,k,k-i)$ is a discrete graph since for any two k–subsets X and Y we have

$$|X \cap Y| = |X| + |Y| - |X \cup Y| \geq 2k - (k+j) \geq k - j \geq k - i.$$

Consequently $W(K(k+j,k,k-i),2)$ is determined by the parity of $|V(K(k+j,k,k-i))|$. So in light of Theorem 7.2 we have determined $W(K(k+j,k,k-i),2)$ for $k \geq 2i+1$ and all values of $j>0$ except for those in the range $i+1 \leq j \leq 2i$. For $i=2$ and $k \geq 5$, for example, that leaves $j=3$ and $j=4$, that is $W(K(k+3,k,k-2),2)$ and $W(K(k+4,k,k-2),2)$ to determine. The following lemma tells us that these are equal to $W(K(k+3,3,1),2)$ and $W(K(k+4,4,2),2)$, respectively.

7.4 Lemma

Let $i,j \geq 1$ and $j>i$. There is a graph isomorphism

$$K(k+j,k,k-i) \cong K(k+j,j,j-i).$$

Proof. We define a map

$$\Psi : K(k+j,k,k-i) \longrightarrow K(k+j,j,j-i)$$

by complementation, that is we let

$$\Psi(X) = X' = \{1,2,\ldots,n\} \setminus Xf$$

or any $k-$subset X. Suppose that X and Y are $k-$subsets with $|X \cap Y| < k-i$. Then

$$\begin{aligned}
|X' \cap Y'| &= |(X \cup Y)'| = k+j-|X \cup Y| \\
&= k+j-(|X|+|Y|-|X \cap Y|) \\
&= k+j-2k+|X \cap Y|
\end{aligned}$$

$$< \; k + j - 2k + k - i$$
$$= \; j - i.$$

Since $X'' = X$, this shows that Ψ is an isomorphism of graphs. ■

7.5 Discussion

Using r−MIC's (maximal intersection colorings with r colors), Harary and Tuza [6] computed $W(P, 2) = W(K(5,2,1), m)$, where $P \cong K(5,2,1) = \overline{L(\mathbf{K}_5)}$ is the Petersen graph and $m = 1,2,3$. We extend their results to $K(n,2,1) = \overline{L(\mathbf{K}_n)}$. In fact, by the lemma, we will actually be computing $W(K(k+2,k,k-1), m)$ for $k \geq 1$ and $m = 1,2,3$.

Recall that in a proper coloring of the vertices of a graph each color class forms an *independent* set of vertices, that is no two vertices in the set are joined by an edge. Observe that a set of distinct k−subsets $\{X_i\}$ of $\{1,2,\ldots,n\}$ forms an independent set of vertices in $K(n,k,k-1)$ if and only if $|X_{i_1} \cap X_{i_2}| = k-1$ for any two k−subsets X_{i_1} and X_{i_2} in the set.

The game of Achievement played with r colors on the vertices of a finite graph G ends with a proper r−coloring of a subset of vertices of G such that each color class C_i, $1 \leq i \leq r$, so produced is maximally independent with respect to uncolored vertices. By this we mean that if for any color class C_{i_0} there exists an independent set D with $C_{i_0} \subseteq D$ and

$$(D \setminus C_{i_0}) \cap \bigcup_{j \neq i_0} C_j = \emptyset,$$

then $D = C_{i_0}$.

A collection of r such maximally independent sets is referred to as an $r-$MIC by F. Harary and Z. Tuza.

Observe that a $1-$MIC is nothing but a maximal independent set of vertices. More generally, an $r-$MIC is a maximal $r-$partite graph.

Let S be a $(k-1)-$subset of $\{1, 2, \ldots, n\}$ and

$$T \subseteq \{1, 2, \ldots, n\} \setminus S$$

an $m-$subset. (Hence $1 \leq m \leq n-k+1$.) The set $\{S \cup \{t\} \mid t \in T\}$ is an independent set of vertices in $K(n, k, k-1)$ and will be referred to as an $m-$**star**. We denote $m-$stars by S_m.

A subset of a star is a star. More precisely, if S_m is an $m-$star and $X \subseteq S_m$, then $X = S_{m'}$ for some $1 \leq m' \leq m$. (In fact, $m' = |X|$.)

Let R be a $(k+1)-$subset of $\{1, 2, \ldots, n\}$. The set of $k-$subsets of R is also an independent set of vertices in $K(n, k, k-1)$ and will be referred to as a $k-$**simplex**. We denote $k-$simplices by Δ_k or $\overline{\Delta}_k$. We note that $|\Delta_k| = k+1$.

If $X \subseteq \Delta_k$, then X is an independent set with $1 \leq |X| \leq k+1$. Denote X by Δ_k^i, where $i = |X|$. Note that $\Delta_k^{k+1} = \Delta_k$ and $S_i = \Delta_k^i$ for $i = 1, 2$.

7.6 Lemma

Let K be a $1-$MIC in $K(n, k, k-1)$. If $n = k+1$, then $K = \Delta_k$. If $n \geq k+2$, then $K = \Delta_k$ or $K = S_{n-k+1}$.

Proof. If $n = k + 1$, then clearly

$$K = V(K(n, k, k - 1)) = \Delta_k.$$

So assume that $n \geq k + 2$. Let $X, Y \in K$ be distinct k–subsets in a 1–MIC K. Then $|X \cap Y| = k - 1$. Without lose of generality, assume that $X = \{1, 2, \ldots, k\}$ and $Y = \{2, 3, \ldots, k + 1\}$. Let $Z \in K \setminus \{X, Y\}$. Then either

$$Z \subset X \cup Y = \{1, 2, \ldots, k + 1\}$$

or $Z \not\subseteq X \cup Y$. In the first case we claim that

$$K = \Delta_k = \{k - \text{subsets of } \{1, \ldots, k + 1\}\};$$

in the second case that

$$K = S_{n-k+1} = \{\{2, \ldots, k\} \cup \{t\} \mid t = 1 \text{ or } t \geq k + 1\}.$$

Suppose that $Z \subset X \cup Y$. Then $|X \cap Y \cap Z| = k - 2$. Given $Z' \in K \setminus \{X, Y, Z\}$ we need to show that $Z' \subset X \cup Y$. If not then there exists $z' \in Z' \setminus (X \cup Y)$ with

$$Z' \setminus \{z'\} = Z' \cap Z = Z' \cap X = Z' \cap Y$$

(since all these sets must have cardinality $k - 1$.) But this implies that $|X \cap Y \cap Z| = k - 1$, a contradiction. Therefore, $Z' \subset X \cup Y$ for any $Z' \in K \setminus \{X, Y, Z\}$, which is to say $K = \Delta_k$.

Finally suppose that $Z \not\subseteq X \cup Y$. Then there exists $z \in Z \setminus (X \cup Y)$ with

$$Z \setminus \{z\} = Z \cap X = Z \cap Y = \{2, 3, \ldots, k\}.$$

Thus $Z = \{2, 3, \ldots, k\} \cup \{z\}$ and $\{X, Y, Z\}$ is a 3–star. If $Z' \in K \setminus \{X, Y, Z\}$, then $Z' \not\subseteq X \cup Y$. Else we would have $Z' \cap Z = Z \setminus \{z\} = \{2, 3, \ldots, k\}$. But this would imply that $Z' = X$ or $Z' = Y$, a contradiction. Consequently, there exists $z' \in Z' \setminus (X \cap Y)$ with $Z' \setminus \{z'\} = Z' \cap X = Z' \cap Y$. In other words, $Z' = \{2, 3, \ldots, k\} \cup \{z'\}$. This shows that $K = S_{n-k+1}$. ■

7.7 Note

As a corollary of the proof of the above lemma, we have that for $n \geq k+2$ the winner of the Achievement game is determined as soon as player A makes his second move. (If $n = k + 1$, then the winner is determined from the very beginning.)

7.8 Theorem

$$W(K(n, k, k-1), 1) = \begin{cases} \text{B} & \text{if } n \text{ is even and } k \text{ is odd,} \\ \text{A} & \text{otherwise.} \end{cases}$$

In particular,

$$W\left(\overline{L(\mathbf{K}_n)}, 1\right) = W(K(n, 2, 1), 1) = A$$

for any $n \geq 3$.

Proof. If $n = k + 1$ then every 1–MIC is a k–simplex. Thus player A wins if k is even while player B wins if k is odd. If $n \geq k + 2$, player A determines in his second move whether the 1–MIC is going to be a k–simplex or an $(n - k + 1)$–star. For player A to lose (that is for player B to win), $n-k+1$ and $k+1$ must both be even. But this occurs precisely when n is even and k is odd. ■

7.9 Theorem

If $n = k+1$, then

$$W'(K(n,k,k-1),1) = \begin{cases} \text{A} & \text{if } k \text{ is odd,} \\ \text{B} & \text{if } k \text{ is even.} \end{cases}$$

If $n \geq k+2$, then

$$W'(K(n,k,k-1),1) = \begin{cases} \text{B} & \text{if } n \text{ and } k \text{ are both even,} \\ \text{A} & \text{otherwise.} \end{cases}$$

Proof. Since player A determines whether the 1–MIC is going to be Δ_{k+1} or S_{n-k+1}, if either $k+1$ or $n-k+1$ is even, then $W'(K(n,k,k-1),1) = A$. Equivalently, $W'(K(n,k,k-1),1) = B$ if and only if both $k+1$ and $n-k+1$ are odd, that is if both n and k are even. ■

7.10 Lemma

A 2–MIC $K \cup L$ in $K(n,2,1)$ takes one of the following forms:

(1) $\Delta_2 \cup S_{n-1}$ $(n \geq 4)$

(2) $\Delta_2 \dot{\cup} \overline{\Delta}_2$ $(n \geq 5)$

(3) $S_{n-1} \cup S_{n-2}$ $(n \geq 5)$

(4) $\Delta_2 \cup S_{n-3}$ $(n \geq 6)$

(5) $S_m \cup S_{m'}$ $\quad (m, m' \geq 3,\ m + m' = n - 1 \geq 6).$

Note In the isomorphism $K(n,2,1) \cong \overline{L(\mathbf{K}_n)}$, vertices of $K(n,2,1)$ correspond to edges of $\mathbf{K}_n$ and two vertices of $K(n,2,1)$ are nonadjacent if and only if the corresponding edges of $\mathbf{K}_n$ have a common vertex. Therefore, a 2−simplex Δ_2 in $K(n,2,1)$ corresponds to a triangle in $\mathbf{K}_n$, while an m−star in $K(n,2,1)$ corresponds to a set of m edges in $\mathbf{K}_n$ having exactly one vertex in common.

Proof. Let $K \cup L$ be a 2−MIC in $K(n,2,1)$ where $n \geq 4$. Then $K = \Delta_2$ or $K = S_m$ for some $3 \leq m \leq n-1$, and similarly for L. This follows from the fact K and L are independent sets and therefore contained in maximally independent sets. But the latter, the 1−MIC's, are of the forms Δ_2 or S_{n-1} by Lemma 7.6. Since $n \geq 4$, we must have $m \geq 3$.

Let $K = \Delta_2$, the 2−subsets of some 3−subset R of N. Then either L is $\overline{\Delta}_2$, the 2−subsets of some (different) 3−subset $\overline{R}$ of N, or L is a star. We show that both these possibilities do arise and the result is either case (1), (2) or (4).

If $n \geq 5$, then $|N \setminus R| = n - |R| = n - 3 \geq 2$. This allows us to choose $\overline{R}$ such that $\left|R \cap \overline{R}\right| \leq 1$, in which case $\Delta_2 \cap \overline{\Delta}_2 = \emptyset$. Thus $L = \overline{\Delta}_2$ is a possibility. Conversely if we have $L = \overline{\Delta}_2$ then, since

$$\Delta_2 \cap \overline{\Delta}_2 = \emptyset, \quad \left|R \cap \overline{R}\right| \leq 1.$$

So $L = \overline{\Delta}_2$ can only occur if

$$n \geq \left|R \cup \overline{R}\right| = |R| + \left|\overline{R}\right| - \left|R \cap \overline{R}\right| \geq 3 + 3 - 1 = 5.$$

This gives us case (2).

If L is an m–star, then

$$L = \{\{s_0, t\} \mid t \in T\}$$

for some $s_0 \in N$ and $T \subseteq N \setminus \{s_0\}$ with $|T| = m$. We need to determine the values of m (if any) for which this can occur. Either $s_0 \in R$ or $s_0 \notin R$.

If $s_0 \in R$, then for any $t \in R \setminus S_0$ we have $\{s_0, t\} \in \Delta_2$ because $\{s_0, t\} \subset R$. Moreover, $\{s_0, t\} \notin L$ since we must have $L \cap \Delta_2 = \emptyset$. Therefore, $T \subseteq N \setminus R$. Since L is required to be maximally independent with respect to uncolored vertices and $\{s_0, t\} \notin \Delta_2$ for all $t \in N \setminus R$, we must have $T = N \setminus R$.

Consequently we could set $L = \{\{s_0, t\} \mid t \in T\}$ as long as $|T| \geq 3$. Since $|T| = n - 3$, $|T| \geq 3$ if and only if $n \geq 6$. Hence, for $n \geq 6$, we may have $K \cup L = \Delta_2 \cup S_{n-3}$. This is case (4).

If $s_0 \notin R$, then $\{s_0, t\} \not\subseteq R$ for all $t \in T \triangleq N \setminus \{s_0\}$. As in the last case, we could set $L = \{\{s_0, t\} \mid t \in T\}$ as long as $|T| \geq 3$. Since $|T| = n - 1$, $|T| \geq 3$ if and only if $n \geq 4$. Hence, for $n \geq 4$, we may have $K \cup L = \Delta_2 \cup S_{n-1}$. This is case (1).

We now only need to consider the cases where both K and L are stars of cardinality at least three. We need to show that there are exactly two of these, namely (3) and (5). Let K be an m–star, $m \geq 3$. In the above language, $K = \{\{s_0, t\} \mid t \in T\}$, where T is a subset of N with $|T| = m$. Either $m = n - 1 = |N \setminus \{s_0\}|$, in which case $T = N \setminus \{s_0\}$ (and so $K = S_{n-1}$), or $m \leq n - 1$, in which case $T \subsetneqq N \setminus \{s_0\}$.

Assume first that $m \leq n-1$. Since K is maximally independent with respect to uncolored vertices, then $L = \{\{s_0, t\} \mid t \in T'\}$, where $T' = N \setminus (T \cup \{s_0\})$. Furthermore, since we are assuming that both $|T| \geq 3$ and $|T'| \geq 3$, then

$$n \geq \left|\{s_0\} \cup T \cup T'\right| \geq 1 + 3 + 3 = 7.$$

So for $n \geq 7$ we have the 2−MIC $K \cup L = S_m \cup S_{m'}$, with $m + m' = n - 1$. This is case (5).

Suppose that $m = n - 1$, in which case $n \geq 4$ since we're requiring that $m \geq 3$. Then

$$L = \left\{\{\overline{s_0}\} \cup \{t\} \mid t \in \overline{T}\right\},$$

where $\overline{s_0} \in N$ such that $\overline{s_0} \neq s_0$ and $\overline{T} \subseteq N \setminus \{\overline{s_0}\}$. Since $K \cap L = \emptyset$, $\overline{T} = N \setminus \{s_0, \overline{s_0}\}$. Note that $|\overline{T}| \geq 3$ if and only if $n \geq 5$. Hence for $n \geq 5$ we have the 2−MIC $K \cup L = S_{n-1} \cup S_{n-2}$. This takes care of the remaining case and so we are done. ■

7.11 Theorem

For $n \geq 3$,

$$W\left(\overline{L(\mathbf{K}_n)}, 2\right) = W(K(n,2,1), 2) = \begin{cases} \text{A} & n \text{ odd}, \\ \text{B} & n \text{ even}. \end{cases}$$

In particular, $W(P, 2) = A$ for the Petersen graph $P \cong \overline{L(\mathbf{K}_n)}$.

Proof. It follows from Theorem 7.8 that for n even player B has a winning strategy for Achievement if and only if he can force at least one monochromatic Δ_2. He does this as follows.

Suppose that player A assigns color 1, say, to the vertex $\{1,2\}$. Since we are assuming that $n \geq 4$, then there exists at least one vertex $\{3,4\}$ which is nonadjacent to $\{1,2\}$. Let player B assign color 2 to $\{3,4\}$. Without loss of generality we suppose that player A then assigns color 1 to vertex $\{1,i\}$, for some $i \in N \setminus \{1\}$. Player B wins by assigning color 1 to $\{2,i\}$. (This results in the monochromatic $\Delta_2 = \{\{1,2\},\{1,i\},\{2,i\}\}$.)

Assume that n is odd. By Lemma 7.10, player A has a winning strategy for Achievement if he can force at least one monochromatic S_{n-1}. (Here we are assuming $n \geq 5$. However, player A can win any game if $n = 3$ since $K(3,2,1)$ is the discrete graph on 3 vertices.) He does this as follows. Let player A assign the color 1 to the vertex $\{1,2\}$.

If player B assigns the color 1 to an adjacent vertex, say $\{1,3\}$, then player A should assign the color 2 to $\{2,3\}$. The unique 2−simplex containing $\{1,2\}$ and $\{2,3\}$, namely $\Delta_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$, is now bichromatic. Consequently, the 1−MIC induced by color 1 will be star. We claim that is the star $S_{n-1} = \{\{1,i\} \mid i \in N \setminus \{1\}\}$. The vertices $\{1,2\}$ and $\{2,3\}$ have already been assigned the color 1, while for $i \in N \setminus \{2,3\}$, $\{1,i\}$ is adjacent to $\{2,3\}$ and so it can not be assigned the color 2. Consequently all the vertices of $S_{n-1} = \{\{1,i\} \mid i \in N \setminus \{1\}\}$ must be assigned the color 1.

If player B assigns the color 2 to a nonadjacent vertex, say $\{3,4\}$, then player A should assign the color 1 to $\{1,3\}$. Player B must then assign the color 2 to $\{1,4\}$ in order to prevent a monochromatic $S_{n-1} = \{\{1,i\} \mid i \in N \setminus \{1\}\}$. But then player A can assign the color 2 to $\{2,4\}$ and this results in a monochromatic $S_{n-1} = \{\{4,i\} \mid i \in N \setminus \{4\}\}$. ■

7.12 Corollary

Since player A can force at least one Δ_2, then $W'(K(n,2,1),2) = A$ for even n. Since player B can force at least one S_{n-1}, then $W'(K(n,2,1),2) = B$ for odd n.

Proof. The result follows at once from the fact that

$$K(k+2,k,k-1) \cong K(k+2,2,1).$$

7.13 Corollary

$$W(K(k+2,k,k-1),2) = \begin{cases} \text{A} & k \text{ odd,} \\ \text{B} & k \text{ even.} \end{cases}$$

As an added bonus, we would like to determine $W(K(n,2,1),3) = W\left(\overline{L(\mathbf{K}_n)},3\right)$ for any $n \geq 3$, the case $n = 5$ having already been settled in [1].

7.14 Lemma

A 3–MIC $K \cup L \cup M$ in $K(n,2,1)$ takes one of the following forms:

(1) $\Delta_2 \cup \Delta_2 \cup \Delta_2$

(2) $\Delta_2 \cup \Delta_2 \cup S_{n-1}$

(3) $\Delta_2 \cup \Delta_2 \cup S_{n-3}$

(4) $\Delta_2 \cup \Delta_2 \cup S_{n-5}$

(5) $\Delta_2 \cup S_{n-1} \cup S_{n-2}$

(6) $\Delta_2 \cup S_{n-1} \cup S_{n-4}$

(7) $\Delta_2 \cup S_{n-2} \cup S_{n-3}$

(8) $\Delta_2 \cup S_{n-3} \cup S_{n-3}$

(9) $\Delta_2 \cup S_m \cup S_{n-m-1}$ $(m, n-m-3 \geq 1)$

(10) $\Delta_2 \cup S_m \cup S_{n-m-3}$ $(m, n-m-3 \geq 3)$

(11) $S_{n-1} \cup S_{n-2} \cup S_{n-3}$

(12) $S_{n-2} \cup S_{n-2} \cup S_{n-2}$

(13) $S_{n-2} \cup S_m \cup S_{n-m-1}$ $(m, n-m-3 \geq 1,\ m, n-m-3 \neq 1)$

(14) $S_{n-1} \cup S_m \cup S_{n-m-2}$ $(m, n-m-3 \geq 3)$

(15) $S_m \cup S_{m'} \cup S_{m''}$ $(m, m'm'' \geq 3,\ m+m'+m'' = n-1)$

Proof. Each color class is an independent set and as such is contained in a maximal independent set. But the latter, the 1−MIC's, for 2−simplices and stars according to Lemma 7.6 (with $n = k+1$). Hence, each color class forms either a 2−simplex or a star. Case (1) is the situation where all three are simplices. Assume next that we have precisely two 2–simplices (and one star). In $K(n,2,1)$, a 2–simplex consists of all the 2–subsets of some 3–subset of $N = \{1,\ldots,n\}$ and the set 2–subsets can be construed as edges of a triangle in $\mathbf{K}_n$. Denotes these two 3–subsets by R and R'. Then $R \cap R'$ is either empty or contains exactly one point (this point thought of as a vertex in $\mathbf{K}_n$.) The star consists of edges sharing a single vertex, the basepoint. If this vertex is in $R \cap R' \neq \emptyset$, then the star has cardinality $n-5$ (case (4)). If the vertex is an element of $R \setminus R'$ or $R' \setminus R$, then the star has cardinality $n-3$ (case (3)). Finally, if the vertex is an element of $N \setminus (R \cup R')$, then the star has cardinality $n-1$ (case (2)).

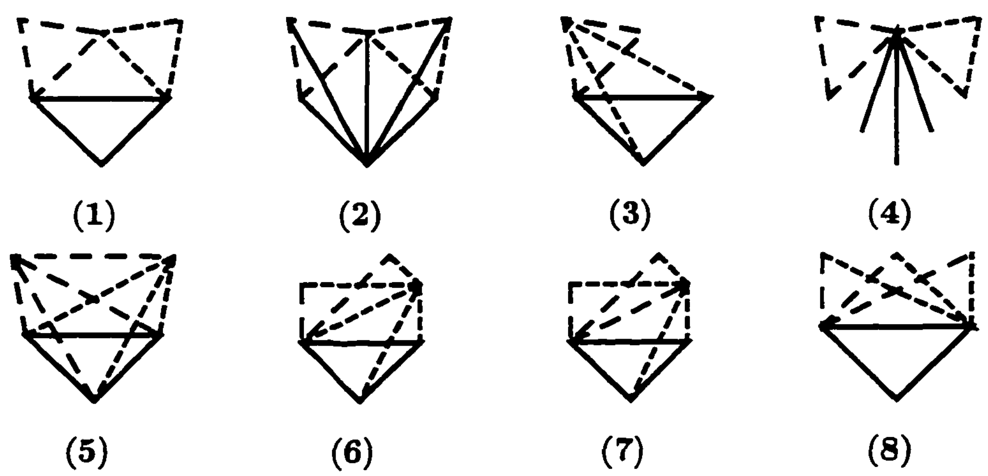
Assume next that there are two stars and one simplex. Suppose that the simplex consists of the 2–subsets of the 3–subset R of N. If the two stars share the same basepoint and this basepoint is an element of R, then the sum of the cardinality of the two stars must be $n-3$ (case (10)). If they share the same basepoint and this basepoint is not an element of R, then the sum of the cardinality of the two stars must be $n-1$ (case (9)). If the basepoints are distinct and neither one is an element of R, then one star must have cardinality $n-1$ and the other star cardinality $n-2$ (case (5)). (The star with cardinality $n-1$ is the one that contains the edge connecting the two basepoints.) If the basepoints are distinct and precisely one of them is an element of R, then the star with basepoint in R has cardinality $n-3$ (if it contains the edge between basepoints) or $n-4$ (if it doesn't). This forces the other star to have cardinality $n-2$ and $n-1$, respectively. That takes care of cases (6) and (7).

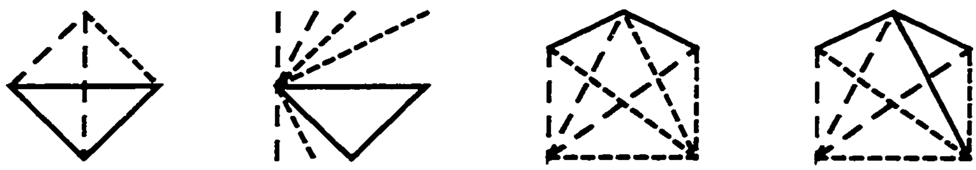
Last but not least, if the basepoints are distinct and both are elements of R, then both stars have cardinality $n-3$ (case (8)).

Now suppose that the 3–MIC consists of three stars. We need to show that this results in cases (11) through (15). If the three basepoints are distinct, then the edges between pairs of basepoints form a triangle. This triangle will either be bichromatic or trichromatic, that is its three edges will be colored with exactly two or three colors. (It cannot be monochromatic since the 3–MIC consists of three stars.) A bichromatic triangle leads to case (11), a trichromatic triangle to case (12).

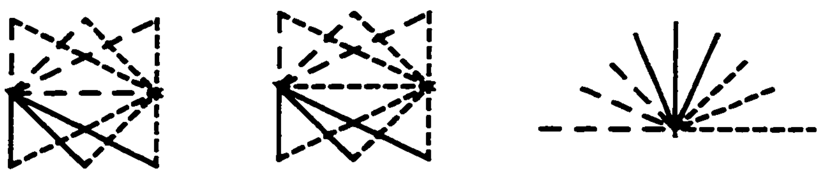
Suppose next that exactly two of the stars share the same basepoint. Thus there are two distinct basepoints. The edge between these two basepoints is either contained in one of the two stars sharing a basepoint or in the third star. That leads to cases (13) and (14), respectively. Finally, we have the case where all three stars share the same basepoint. This is case (15). ■

The diagrams below illustrate the fifteen cases of the lemma. (Red edges are denoted by dotted line, blue edges by dashed lines and black edges by solid lines.)





(9) (10) (11) (12)



(13) (14) (15)

7.15 Theorem

$$W\left(\overline{L(\mathbf{K}_n)}, 3\right) = W(K(n, 2, 1), 3) = B$$

for all $n \geq 4$. In particular, $W(P, 3) = B$ for the Petersen graph $P \cong \overline{L(\mathbf{K}_5)}$.

Proof. We should point out first that not all cases in Lemma 7.14 can occur unless $n \geq 10$. We ignored such restrictions in the lemma because we won't be needing them here. Though obviously some of player A's moves considered below will require that n be large enough, player B's response in each of these cases will not require that n be any larger as long as $n \geq 5$. Hence player B's strategy as outlined below will be valid for all values of $n \geq 5$. Player B has a forced win for $n = 4$ since in this case all the vertices of $K(4, 2, 1)$ must be colored. (In fact, $K(4, 2, 1) == P_2 \cup P_2 \cup P_2$, the disjoint union of three paths of length one.)

First we may assume that $n \geq 4$ is even. Then the fifteen cases from Lemma 7.14 divide into two groups.

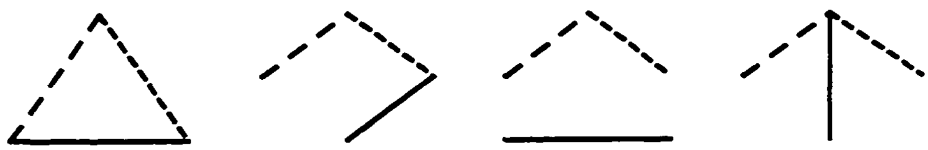
(1) $\Delta_2 \cup \Delta_2 \cup \Delta_2$
(2) $\Delta_2 \cup \Delta_2 \cup S_{n-1}$
(3) $\Delta_2 \cup \Delta_2 \cup S_{n-3}$
(4) $\Delta_2 \cup \Delta_2 \cup S_{n-5}$
(8) $\Delta_2 \cup S_{n-3} \cup S_{n-3}$
(13) $S_{n-2} \cup S_m \cup S_{n-m-1}$
(14) $S_{n-1} \cup S_m \cup S_{n-m-2}$
(15) $S_m \cup S_{m'} \cup S_{m''}$

(5) $\Delta_2 \cup S_{n-1} \cup S_{n-2}$
(6) $\Delta_2 \cup S_{n-1} \cup S_{n-4}$
(7) $\Delta_2 \cup S_{n-2} \cup S_{n-3}$
(9) $\Delta_2 \cup S_m \cup S_{n-m-1}$
(10) $\Delta_2 \cup S_m \cup S_{n-m-3}$
(11) $S_{n-1} \cup S_{n-2} \cup S_{n-3}$
(12) $S_{n-2} \cup S_{n-2} \cup S_{n-2}$

The cases in the first column are wins for player A, the ones in the second column wins for player B. In order to win, player B must force either three stars with three distinct basepoints or one simplex and two stars. Furthermore, the two stars must either share a basepoint on the simplex or have distinct basepoints at least one of which is not on the simplex. He achieves this goal as follows.

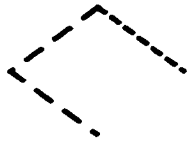
Whichever edge player A colors, player B responds by coloring an adjacent edge with a different color.

Player A now has seven essentially different ways to respond to player B's move.



(1) (2) (3) (4)

(5)

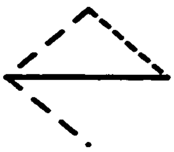


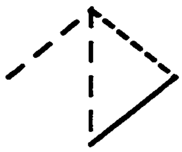
(6)

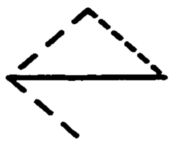
(7)

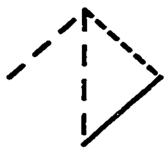
In both case (3) and (5), player B responds to player A's move by forming the following configuration.

In cases (1), (2), (6) and (7), player B responds in such a way as to construct the following equivalent configurations, respectively.

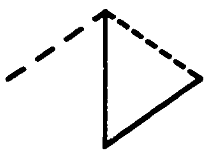






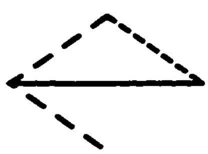


For case (4), player B should respond as follows.

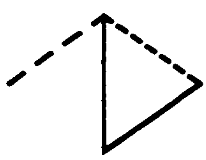


Hence, it now suffices to consider the following three configurations.

(a)

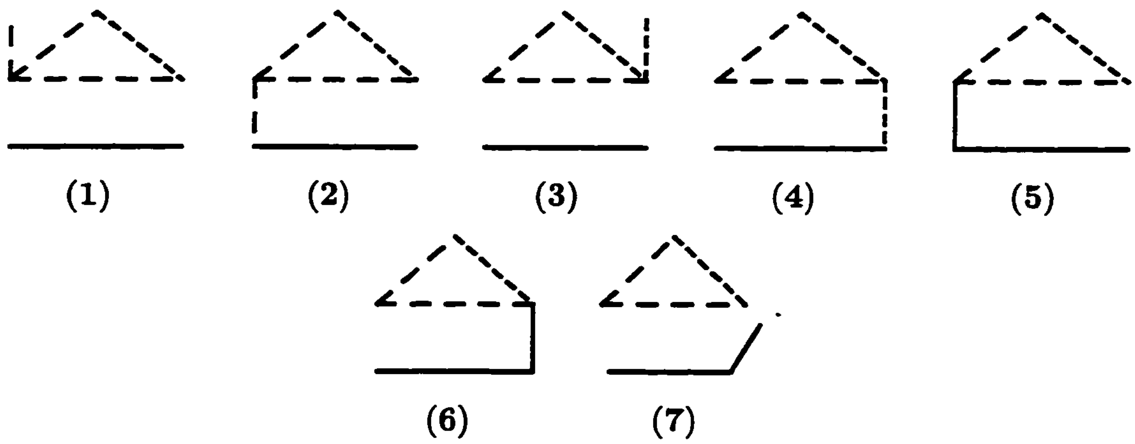


(b)

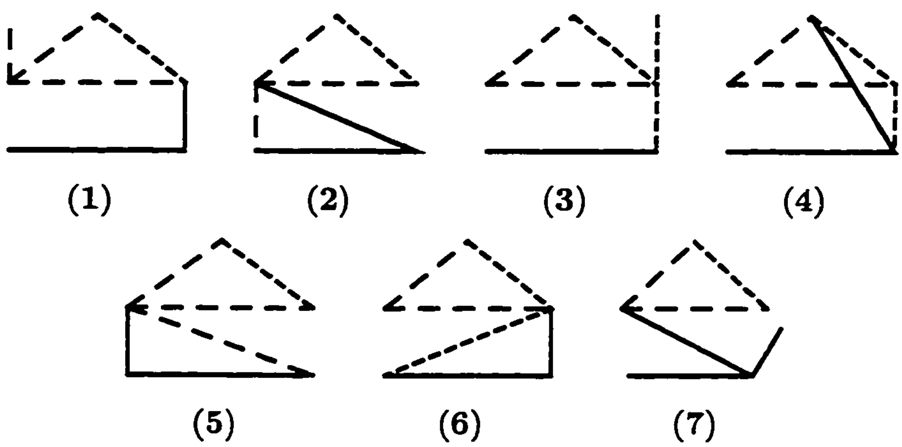


(c)

Given case (a), player A must make one of the following seven moves.

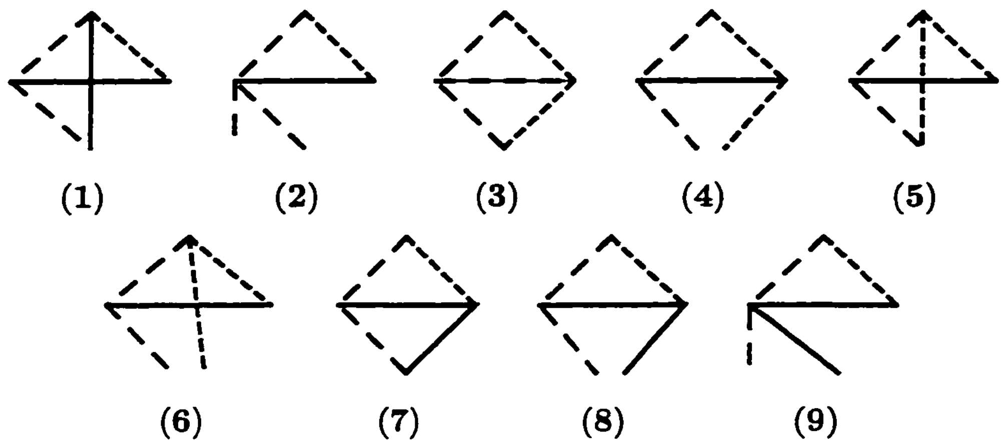


Player B responds, respectively, as follows:

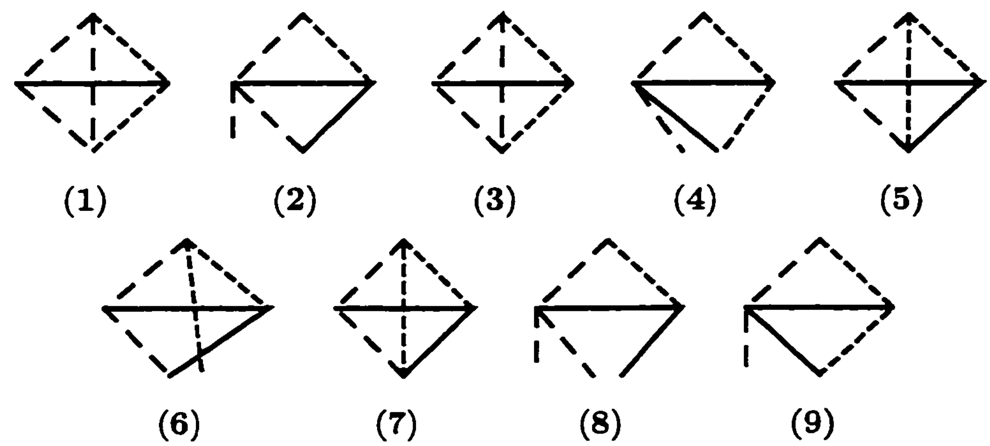


In each case it is clear that if there are going to be three stars then they must all have different basepoints. Moreover, in all but the first case it is also clear that at most one simplex will be constructed and if so at least one of the two star's basepoints won't be on it. In the first case, there is still the possibility of two simplices forming. However, player B can easily prevent this in his next move (if player A doesn't do this first) by forming a bichromatic simplex. (In a bichromatic simplex, the intersection of the two edges having the same color will turn out to be the basepoint of a star in that color.)

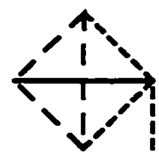
Player A can respond to case (b) in one of the following nine ways.



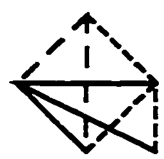
Player B's response in each of these cases is as follows.



Cases (1) and (3) which are identical, will lead to a blue simplex and red star (since there is a bichromatic simplex with two red edges). since the red star's basepoint won't be on the blue simplex, all player B needs to do to win is force a black star. Suppose player A colors an edge black. Since n is even and consequently $n \geq 6$, there exists an edge which shares a vertex with the two black edges. Player B can color this edge black insuring that a black star results. Suppose instead player A colored an edge red.



Then player B could again force a black star by making the following move.



Cases (2) and (8) are identical and lead to black and blue stars with different basepoints. If we had a red simplex to boot, then player B would win since the blue star's basepoint would not be on it. The only way that player A could win would be by forcing a red star having the same basepoint as the black one.

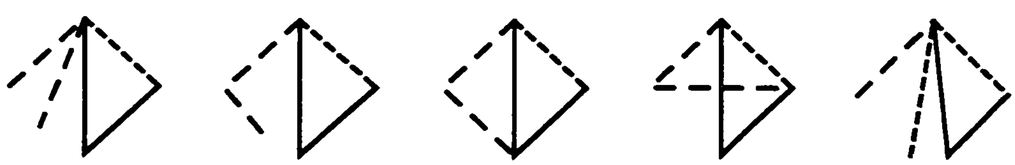
If player A colored an edge black or blue, then player B would respond by forming a bichromatic simplex consisting of two red edges and one black edge. This would result in a red star with basepoint distinct from the black star's So to prevent this from happening, player A must assign the color red to an edge that intersects the red and black edges. Since this cannot result in a bichromatic simplex with two red edges, player B can complete a red simplex and thus wins the game.

Cases (4) and (9) are identical as well. They are also a bit tricky. The bichromatic simplex with the two black edges guarantees that there will be a black star.

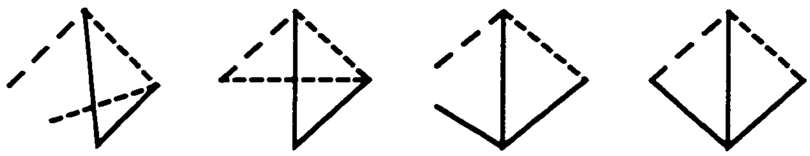
If player A colored an edge black and a black star is not completed, then player B would also color an edge black. Observe that since n is even, player B could not possibly have completed the star. If player A colored an edge black and that completed the star, then it would also force a blue simplex. At this point, player B can force a red star and since this star's base would not be on the blue simplex he would win. If short of the completion of the black star, player A colored an edge blue or red forming a star (respectively a simplex), then player B would respond by coloring an edge red or blue, respectively, and forming a simplex (respectively a star). Since the basepoint of the star that is formed is not on the simplex, player B will win the game.

The identical cases (5) and (7) lead to $S_{n-2} \cup S_{n-2} \cup S_{n-2}$, which is a win for player B. That only leaves case (6) to consider. Player A cannot force both a blue and red simplex. (If player A forces a simplex in blue, say, in his next move, then player B will force a star in red.) So we will end up with either three stars with distinct bases or one simplex and two stars with at least one of the stars' bases off the simplex. Again player B wins.

It only remains to consider case (c). In this case, player A must make one of the following nine moves.

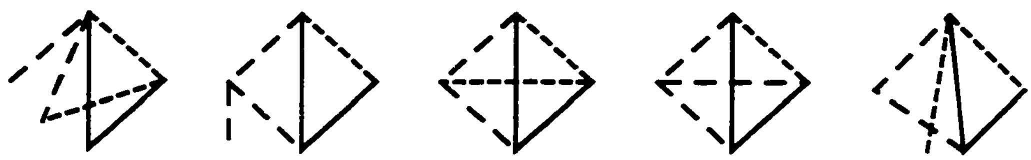


(1) (2) (3) (4) (5)

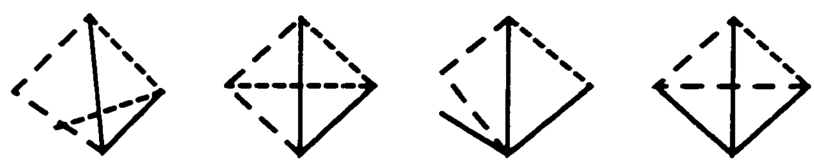


(6) (7) (8) (9)

Player B's response to these nine cases, respectively, is as follows.



(1) (2) (3) (4) (5)



(6) (7) (8) (9)

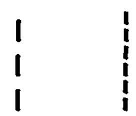
Cases (3) and (7) are identical and both give rise to $S_{n-2} \cup S_{n-2} \cup S_{n-2}$. The remaining cases each gives rise either to three stars with three distinct bases or a simplex and two stars, both whose basepoints are distinct and "off the simplex" ($\Delta_2 \cup S_{n-1} \cup S_{n-2}$). that takes care of the first half of the theorem.

Last but not least, we assume that n is odd with $n \geq 5$. The fifteen cases then split up as follows.

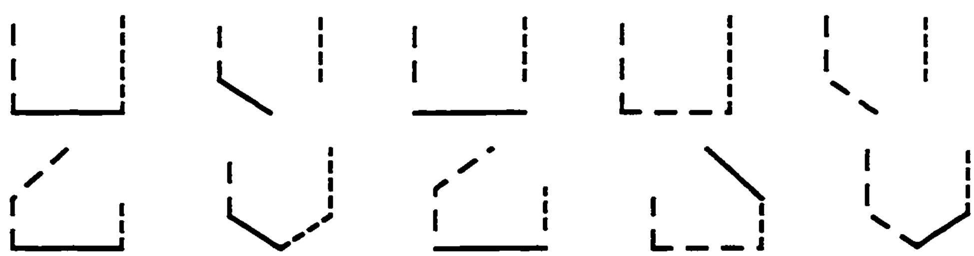
(1) $\Delta_2 \cup \Delta_2 \cup \Delta_2$	(2) $\Delta_2 \cup \Delta_2 \cup S_{n-1}$
(8) $\Delta_2 \cup S_{n-3} \cup S_{n-3}$	(3) $\Delta_2 \cup \Delta_2 \cup S_{n-3}$
(9) $\Delta_2 \cup S_m \cup S_{n-m-1}$	(4) $\Delta_2 \cup \Delta_2 \cup S_{n-5}$
(10) $\Delta_2 \cup S_m \cup S_{n-m-3}$	(5) $\Delta_2 \cup S_{n-1} \cup S_{n-2}$
(11) $S_{n-1} \cup S_{n-2} \cup S_{n-3}$	(6) $\Delta_2 \cup S_{n-1} \cup S_{n-4}$
(12) $S_{n-2} \cup S_{n-2} \cup S_{n-2}$	(7) $\Delta_2 \cup S_{n-2} \cup S_{n-3}$
(13) $S_{n-2} \cup S_m \cup S_{n-m-1}$	(15) $S_m \cup S_{m'} \cup S_{m''}$
(14) $S_{n-1} \cup S_m \cup S_{n-m-2}$	

Again, the cases in the first column are wins for player A, the ones in the second column wins for player B. In order to win, player B must force three stars with a common basepoint, one simplex and two stars with distinct basepoints not both of whose basepoints are on the simplex, or precisely two simplices. He achieves this goal as follows.

Whichever edge player A colors player B responds by coloring a nonadjacent edge (with a different color).



Player A now has five essentially different ways to respond to player B's move. The first row of diagrams below illustrate player A's possible responses; the second row how player B counters each of these moves.

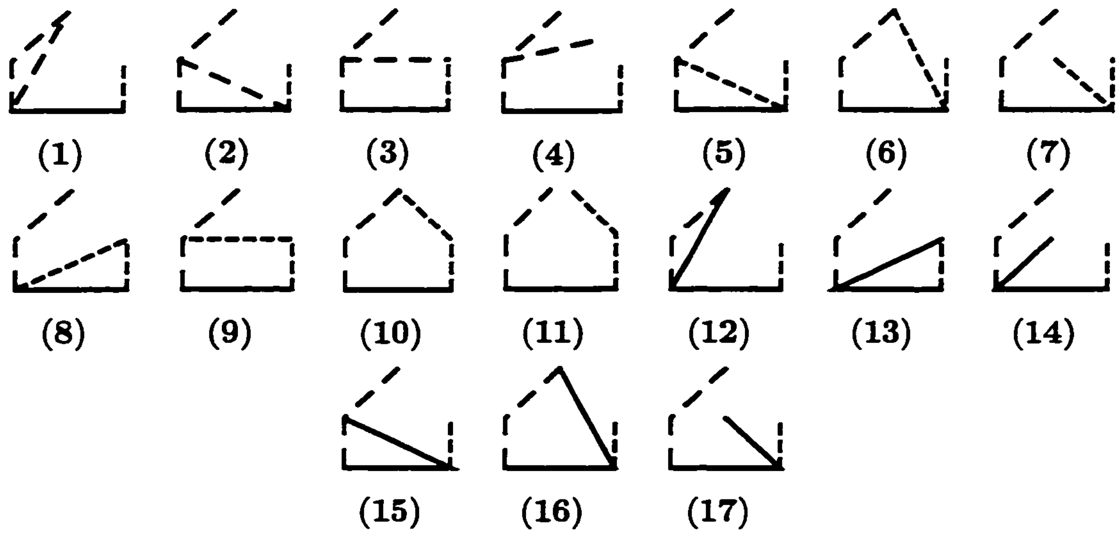


(1) (2) (3) (4) (5)

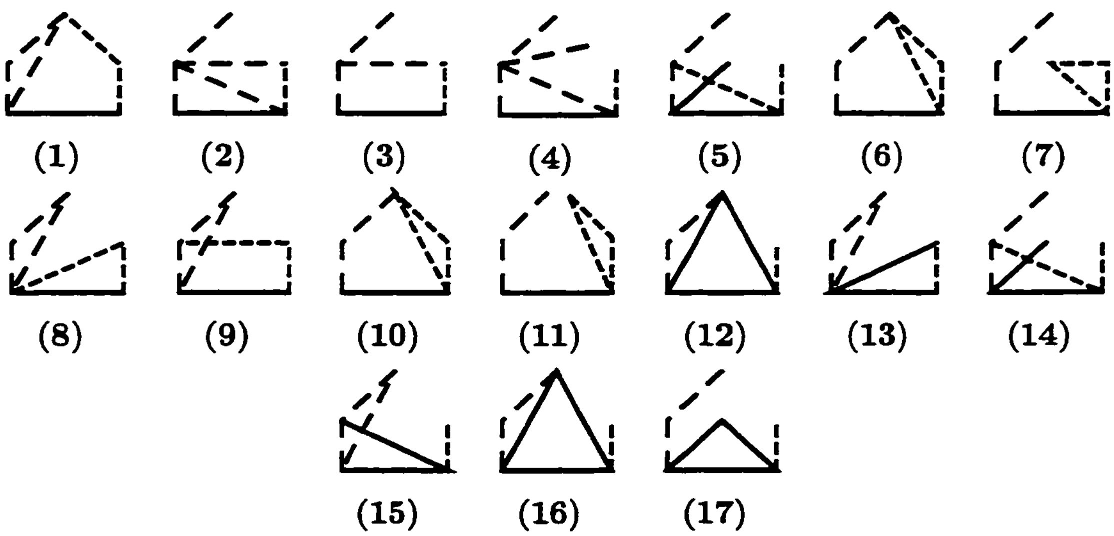
Since cases (1), (2), (4) and (5) are equivalent, there are in fact only two essentially different cases to deal with.

(a) (b)

Player A may respond to case (a) in any one of 17 different ways:



Player B responds to each of these as follows.



Cases (1), (6) through (13), (15), (16) and (17) already have one colored simplex. In his next move, player A can force at most one more. So all that player B needs to do is force a star and this he can do. (He may be required to do this in one move by forming a bichromatic simplex.) In all these cases except perhaps (15) it is clear that if exactly one simplex is formed, the two stars will not share a basepoint and least one of their base points will not be on the simplex. In case (15), player B must take care that the game doesn't end with a black and red star sharing a basepoint ($\Delta_2 \cup S_m \cup S_{n-m-1}$). The black edges will form a star; however player A cannot force a red star having the same basepoint. (If he starts to do so, player B would force a red simplex and this would lead to $\Delta_2 \cup \Delta_2 \cup S_{n-m-1}$ which is a win for player B.)

Cases (5) and (14) are identical. In his next move, player A can force at most two stars and at most one simplex. (He can simultaneously force a blue and red star by forming a triangle with one blue side and two red sides.) If player A forces a star, player B forces simplex and vice versa. This results in one or two simplices. In the event that there is only one simplex at least one of the two stars' basepoint won't be on it. Moreover, the two stars will have distinct basepoints.

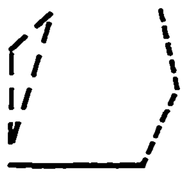
Cases (2) and (3) are identical and are treated as follows. If player A colors an edge blue, then player B colors an edge blue. Since n is assumed odd, player A will lose this "waiting game", that is player B will color the last blue edge if player A persist in coloring edges blue. So eventually, player A will either choose or be forced to color an edge red or black. Player A can force a red or black star in his next move by forming a bichromatic simplex in black and red.

If player A forms such a bichromatic simplex, then player B responds by coloring another edge with the same color, the color of the star thus formed. For argument's sake let us say it is red. Again, since n is odd, player A cannot win the "waiting game" on red either, so player A must eventually color an edge black. However, he cannot form another bichromatic simplex involving black. So whatever edge he colors black, player B responds by completing the black simplex. Since the blue star's basepoint (which is distinct from the red star's) does not lie on the simplex, the resulting configuration is a win for player B.

If player A instead colored an edge red or black but doesn't form a bichromatic simplex (in red and black), then player B can follow suit with the same color forming a simplex in that color. Once again this leads to a win for player B.

This same strategy can be applied to case (4). However, this time it is possible to form two bichromatic simplices, one consisting of one blue and two red edges and the other consisting of one red and two black edges. This would result in three stars which player B does not want. But player A cannot force both of these at the same time–they involve coloring different edges and each of these may be assigned more than one color. More precisely, if player A formed a red and black simplex, then player B can form a simplex consisting of one and two blue edges by coloring the remaining edge blue (assuming it hasn't already been colored blue). This prevents player A from forcing a red star. If instead player A formed a red and blue simplex, then player B can form a simplex consisting of one black and two red sides by coloring the remaining edge red (assuming it hasn't already been colored red). this prevents player A from forcing a black star.

It only remains to consider case (b). If player A makes a blue simplex, player B will respond as follows.



Similarly, if player A forms a red and black path of length three, then player B would form a blue simplex. All player B has to do now is force a red or black star (or both which would necessarily have different basepoints.) It is easy to see that he can achieve this.

If instead player A formed a blue star, then player B would respond by coloring another edge blue as well. If the edge colored blue by player A is nonadjacent to the red (respectively black) edge, then player B can also choose an edge nonadjacent to the red (respectively black) edge to color blue. If the edge colored blue by player A is adjacent to the red (respectively black) edge, then player B can also choose an edge adjacent to the red (respectively black) edge to color blue. This prevents player A from forming either a bichromatic simplex consisting of one blue and two red edges or a bichromatic simplex consisting of one blue and two black edges. If player A continued to color edges blue, then so would player B by following the above prescription. Since n is odd, player B can always respond in this fashion. So eventually player A must assign the color red or black to an edge at which point player B can complete a simplex in that color. The blue star's basepoint won't be one it and so this is a win for player B.

Finally, suppose player A assigns the color red (respectively black) to an edge and this edge is not adjacent to a black (respectively red) edge. Then player B may assign the color black (respectively red) to an edge which is not adjacent to a red (respectively black) edge. Depending on the configuration at this point, player A may force at most two stars. If he does so, then player B can force a simplex in a remaining color. Conversely, if player A forms a simplex in one color, player B can force a star in another. The usual arguments convince us that at this point player B will win. ■

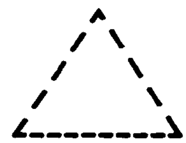
7.16 Theorem

$$W'(K(n,2,1),3) = W'\left(\overline{L(\mathbf{K}_n)},3\right) = A$$

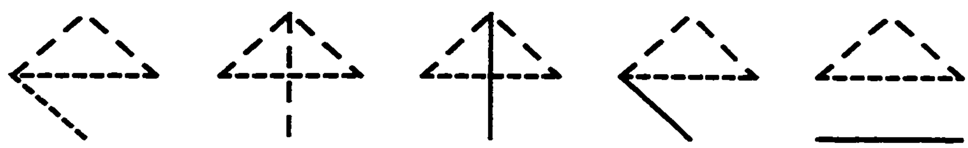
for all $n \geq 4$. In particular, $W'(P,3) = A$ for the Petersen graph $P \cong \overline{L(\mathbf{K}_5)}$.

Proof. Assume first that n is even. As noted in the proof of theorem 44, the case $n = 4$ is a forces win for player B in Achievement (and hence a forced win for player A in Avoidance). So we may as well assume $n \geq 6$. We also observed in the proof of theorem 44 that player B wins Achievement if he can force either three stars with three distinct basepoints or one simplex and two stars, at least one whose basepoints is not on the simplex. We show that for even n player A can do the same and hence win Avoidance.

Player A begins by coloring an edge blue. If player B then colors an adjacent edge blue (respectively red), then player a forms a bichromatic triangle–coloring its third edge red (respectively blue).

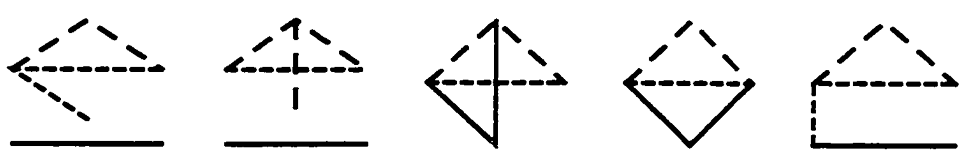


At this point, player B has five possible moves.



(a) (b) (c) (d) (e)

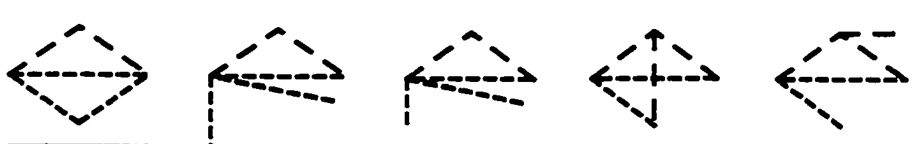
Player A responds to each of these moves as follows.



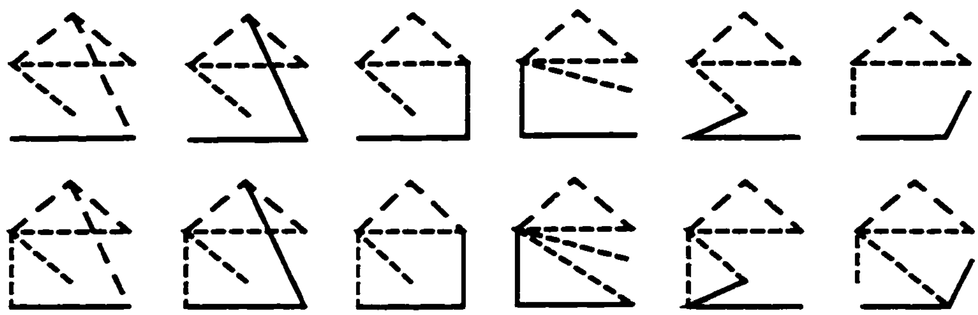
(a′) (b′) (c′) (d′) (e′)

In cases (c′) and (d′), it is clear that at most one triangle will be formed and that if precisely one triangle is formed then the two stars' basepoints will be distinct and at least one won't lie on the triangle. So it remains to consider cases (a′), (b′) and (e′).

Player B can respond to case (a′) in one of eleven ways. We list them below together with player A's countermoves.



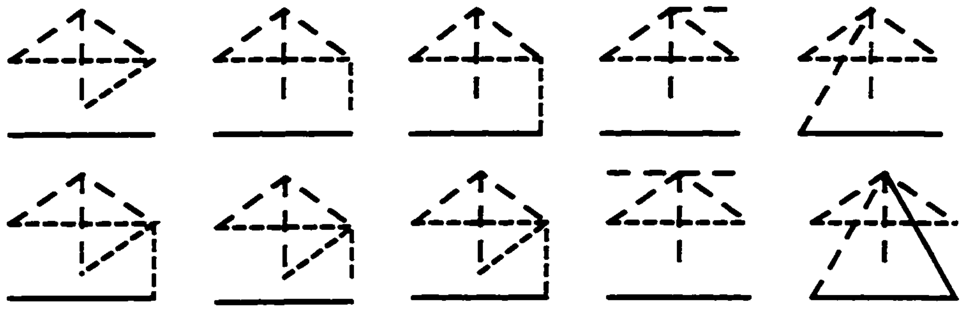
(1) (2) (3) (4) (5)



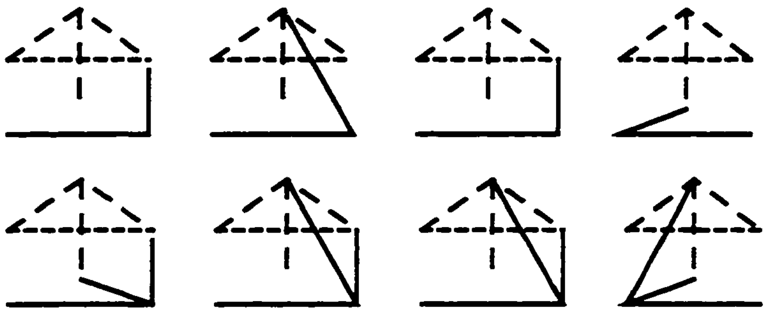
(6) (7) (8) (9) (10) (11)

We consider case (1) first. If any subsequent move player B colors an edge nonadjacent to the black one blue, then player A colors any other edge nonadjacent to the black one blue. The fact that n is even guarantees that player A can always do this. Eventually, player B must either color an edge adjacent to the black one blue or color an edge (adjacent to the black one) black. In the first case, player A forms a triangle with two blacks sides and one blue side. This forces a black star. In the second case, player A can also force a black star by coloring an edge black. The configuration that results is $\Delta_2 \cup S_{n-1} \cup S_{n-2}$, a win for player A. Cases (2) and (9) will result in three stars with three distinct basepoints, again a win for player A. In the remaining cases, three stars would also be a win for player A since they must automatically have distinct basepoints. However, in these cases there is the additional possibility that precisely one (black) triangle will be formed. But since the edge connecting the basepoints of the red and blue stars is already colored, both of these basepoints cannot lie on the triangle.

Next we consider player B's responses to case (b′) above. There are nine of them. We list them below together with player A's next move.



(1) (2) (3) (4) (5)

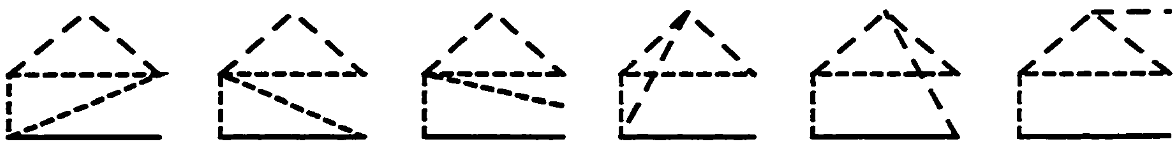


(6) (7) (8) (9)

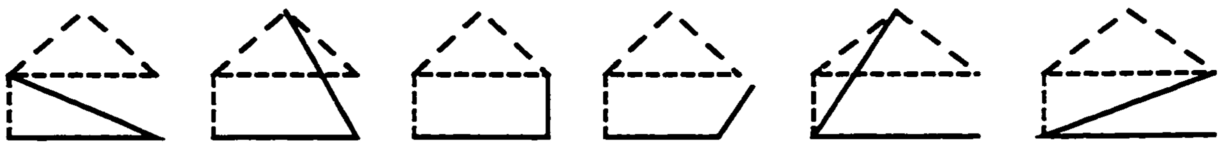
In all cases it is clear that for there to be three stars there must be three distinct basepoints. In all cases except (4) it is also evident that (at least) two stars will result and that they will have distinct basepoints. (In each case the edge connecting these basepoints is already colored and so if a triangle is also formed, at least one of these basepoints won't lie on it). It remains to consider case (4).

If in any subsequent move player B colors an edge nonadjacent to the black one blue, then player A colors any other edge nonadjacent to the black one blue. The fact that n is even guarantees that player A can always do this. Eventually, player B must either color an edge adjacent to the black one blue or color an edge red or black. In the first case, player A forms a triangle with two black sides and one blue side. This forces a black star. On the other hand, if player B colors an edge red (respectively black) then player A forces a red (respectively black) star by coloring another edge red (respectively black). The analysis now proceeds as in the other cases. That takes care of case (b′). Finally, we consider case (e′)

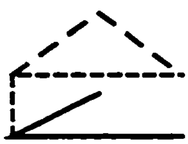
Player B must respond to (e′) in one of thirteen ways.



(1) (2) (3) (4) (5) (6)



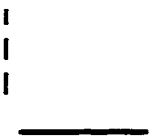
(7) (8) (9) (10) (11) (12)



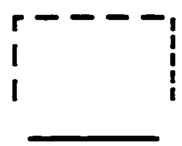
(13)

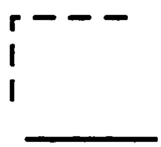
In all cases it is clear that for there to be three stars they must all have distinct basepoints. By forming a star in his next move if he has to, player A can prevent more than one triangle from forming. Moreover, if precisely one triangle results, then the two stars' basepoints won't both lie on it.

Now we consider the case where player B responds to player A's initial move by coloring a nonadjacent edge (with a different color of course.) Since we are assuming $n \geq 6$, there exists a third edge nonadjacent to the first. Let player A color this edge (with the third color).



Player B must respond in one of essentially two ways.





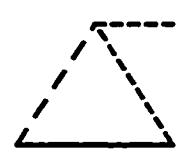
In the first case, player A can force the configuration in case (a′) above. In the second case, player A responds with

I will now show that starting with this configuration, player A can force three stars with three distinct basepoints. If player B colors an edge blue in any subsequent move, then player A responds by coloring another edge blue. Player A can always respond tin this fashion because n is assumed to be even. Eventually, player B must color an edge a different color, say red. Player A then forces a red star. If in any subsequent move player B colors an edge red or blue, then player A colors an edge red or blue. The players cannot continue coloring edges red and blue indefinitely. In fact they must stop once the red and blue stars are complete. But their total cardinality is $2n - 1$. So player B cannot force player A to color a second black edge: player B must do that himself and at that point player A forces the third (black) star. Since the three stars will have distinct basepoints, player A wins once again.

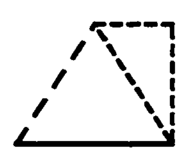
We now assume that n is an odd number. In order to win Avoidance, player A must force three stars with a common basepoint, one simplex and two stars with distinct basepoints not both of whose basepoints are on the simplex, or precisely two simplices.

If player B responds to player A's initial move by coloring an adjacent edge a different color, then player A colors the mutually edge adjacent to both with the third color.

Player B has essentially only one possible response.



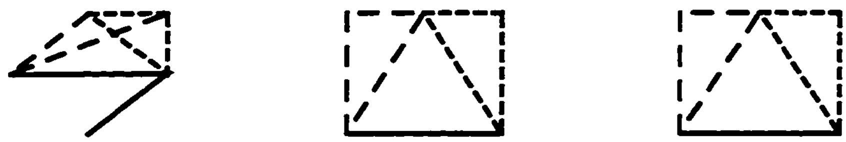
Player A then completes the red triangle.



(♣)

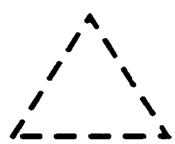
Player B must then respond in one of essentially three ways.

In each case player A then proceeds as follows.



The first configuration leads to either $\Delta_2 \cup \Delta_2 \cup S_{n-3}$ or $\Delta_2 \cup S_{n-2} \cup S_{n-3}$, both of which are wins for player A. The second and third configurations are identical and Player B responds to them by coloring an edge black. Player A then forces a black star to win the game.

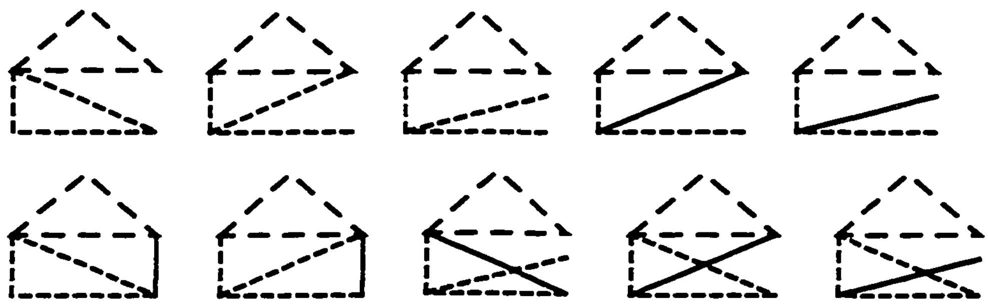
If instead player B responds to player A's initial move by coloring an adjacent edge the same color, say blue, then player A completes the triangle in that color.



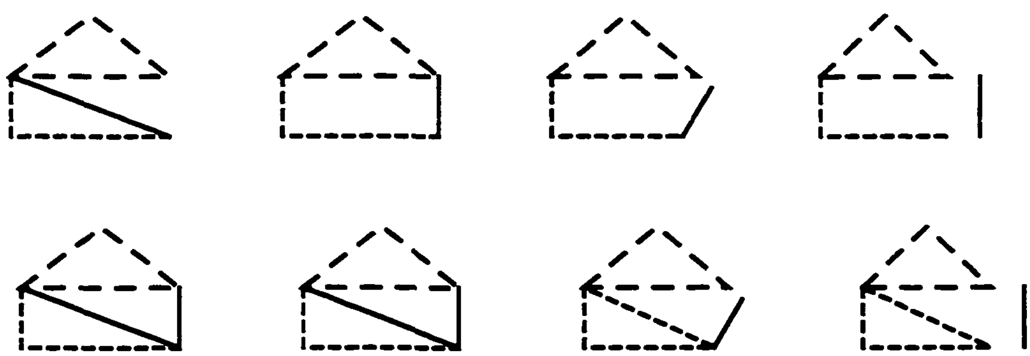
Player B now has essentially two possible moves.

Player A responds to each as follows.

The first configurations is equivalent to (♣) above and we have already shown that player A has a winning strategy with this configuration. Thus we only need to consider the second configuration. Player B must respond in one of essentially nine ways. We list them below together with player A's countermove.



(1) (2) (3) (4) (5)

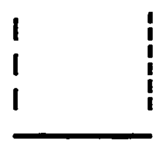


(6) (7) (8) (9)

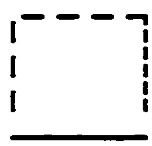
Cases (6) and (7) are identical and lead to $\Delta_2 \cup S_{n-1} \cup S_{n-2}$ which is a win for player A. In cases (1), (4), (5), (8) and (9), player B must color an edge black in his next move. Player A then forces a black star. If player B colors an edge red in response to case (2) or (3), then player A can color a black edge so as to form a triangle with two black sides and one blue one. This results in a black star. If instead player B colors a black edge then player A can still force a black star.

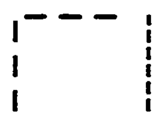
It only remains to consider the scenario where player B starts off by coloring an edge nonadjacent to the one player A colored.

Assume first that $n \geq 7$. Then player A responds as follows.

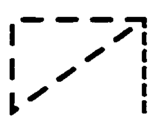


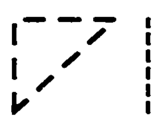
Player B must make essentially one of two moves.





In either case, player A completes the blue triangle.

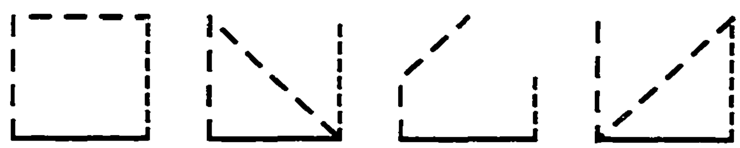




In either case, whatever color player B next assigns, player A forces a star in that color. This insures that three triangles won't arise. Moreover, if exactly one (blue) simplex results, then at least one of the two (red and black) stars' basepoints won't lie on it.

It only remains to describe player A's strategy when $n = 5$ for the case where player B responds to player A's initial move by coloring a nonadjacent edge. In this case, player A moves as follows.

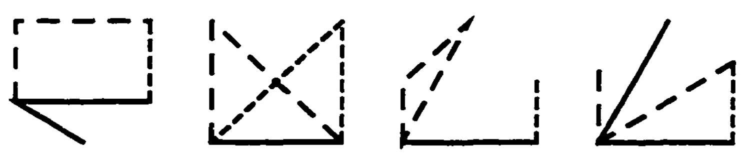
Player B must then make one of essentially seven moves.



(1) (2) (3) (4)

(5) (6) (7)

Player A then responds to each of these possibilities as follows.



(1′) (2′) (3′) (4′)

(5′) (6′) (7′)

Three simplices can only arise when $n \geq 6$. So player A only needs to insure that at least one simplex arises and if precisely one does then the two stars have distinct basepoints at least one of which is not on the simplex. In cases (3′), (5′) and

(7′), a simplex has already formed while in case (1′), (2′), (4′) and (6′) it is clear that at least one simplex is forced due to the fact that $n = 5$. (This is the reason this strategy is not valid for $n \geq 6$.) Moreover, player A can insure in one move, in the cases involving two stars, that at least one of the two star's distinct basepoints won't lie on the simplex. Thus we've show that player A has a winning strategy for $n = 5$ and that completes the proof of the theorem. ■

7.17 Note

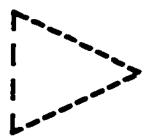
The $m-$color SC-VCG played on the vertices of $K(n,2,1) = \overline{L(\mathbf{K}_n)}$ corresponds to an $m-$color coloring game played on the edges of $\mathbf{K}_n$: players A and B take turns assigning any one of m colors to the edges of $\mathbf{K}_n$ with the proviso that nonadjacent edges must be assigned the same colors. (Equivalently, the proviso states that every colored edge just intersect all the edges that have been assigned a different color.)

7.18 Theorem

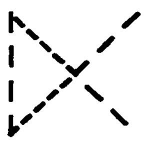
$$W_S(K(n,2,1), m) = W_S\left(\overline{L(\mathbf{K}_n)}, m\right) = A$$

if and only if $n \equiv 3 \pmod 4$, where m and n are integers with $m \geq 2$ and $n \geq 4$. In particular, for all $m \geq 2$, $W_S(P, m) = B$ for the Petersen graph $P \cong \overline{L(\mathbf{K}_5)}$.

Proof. Assume firs that m is odd. We will show that player A has the winning strategy as long as $n = 4k+3$, for some positive integer k. Suppose player A assigns the color blue to some edge and player B responds by assigning either the same (blue) color or a different color, say red, to an adjacent edge. In either case, player A can form a bichromatic simplex (in red and blue).



Since there are no edge adjacent to both the red and blue edge, no edge can be assigned a third color. Moreover, the only edges that can be colored blue are those adjacent to the red edges. Since n is odd, there exists an even number of these. If player B colors one of them blue, then player A can color another blue as well.



Observe that at this point neither player can color an edge red since there are no uncolored edges adjacent to all the blue edges Consequently, the players must continue coloring edges blue and since there are an even number of edges that can be so colored player A wins the game.

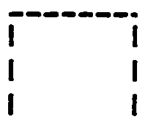
Suppose instead that player B added a red edge to the bichromatic simplex.

Since n is odd, player A can respond as follows

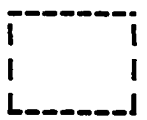
No uncolored edge is adjacent to all the red edges, so no uncolored edge can be colored blue. However all edges adjacent to the one blue edge can and will be colored by the end of the game. This results in two red $(n-2)$−stars (with basepoints the endpoints of the blue edge.) All together there will be an odd number of colored edges and that is a winning situation for player A.

Consequently, the only feasible response player B has to player A's initial move is to color a nonadjacent edge (with the same color).

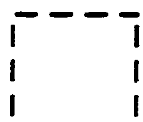
Player A can either respond with the same color or with a different color. If player a chooses to assign a different color, say red

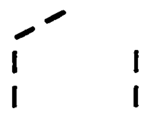


then player B should assign red as well

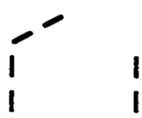


Player A must then color one of the two "diagonal" edges; player B wins by coloring the other. Thus player A's only good move is to assign the same color to another edge. There are basically two ways of doing this.

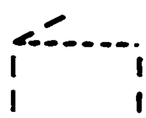




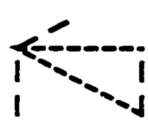
If $n = 4k + 1$, then $\binom{n}{2}$ is an even number. In this case, player B can assign the color blue to one more edge of $\mathbf{K}_n$ insuring that all the edge be colored blue. This is a win for player B. However if $n = 4k + 3$, then $\binom{n}{2}$ is an odd number. In this case, player A should form the configuration



Player B must then assign a different color, say red.



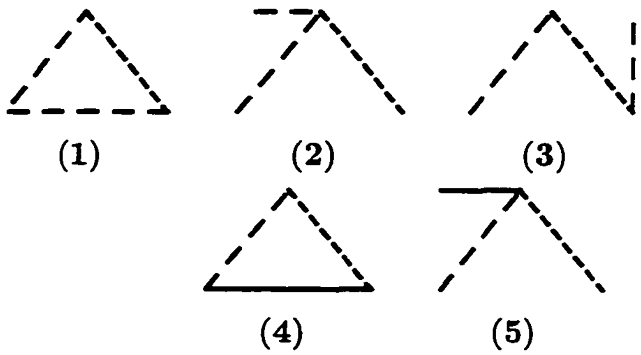
Else, a monochromatic $\mathbf{K}_n$ would result, a win for player A for $n = 4k + 3$.) Player A responds as follows.



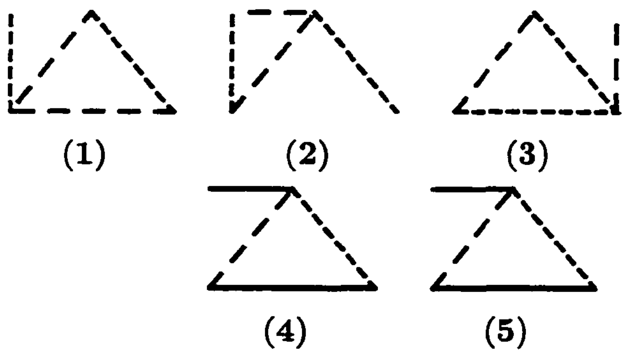
Since no edge is adjacent to all the blue edges only the color blue can be assigned from this point on. Since the only edges that can be so colored are those adjacent to the two red ones, this will result in a blue $(n-3)$–star. Together with the bichromatic simplex, that gives n colored edges, a win for player A.

Finally, assume n is even. We wish to show that player B has the winning strategy. He responds to player A's initial move by assigning an adjacent edge a different color.

Player A has essentially five responses.



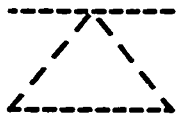
Player B responds to each of these as follows:



Here we have essentially two different configurations.

In the first case, player A can either color an edge blue

or color an edge red

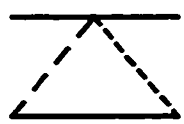


If player A colors an edge blue, then player B colors the remaining “diagonal” edge blue and wins

If player A colors an edge red, then that leads to a red $(n-3)$–star which, together with the bichromatic simplex, yields a total of n colored edges. Since n is assumed to be an even number, player B wins once again.

Finally, we need to consider the second case above.

Player A has essentially two responses.



In the first case, player B wins the game with his very next move

In the second case, player B must assign the color black to any one of the edges adjacent to both the red and blue one. In fact, from this point on both players must take turns coloring these edges black until the edges are exhausted. In the end there will be a total of n colored edges and since n is assumed to be even that means player B will once again win. ■

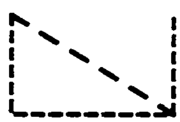
7.19 Theorem

$$W_S'\left(K(n,2,1),m\right) = W_S'\left(\overline{L(\mathbf{K}_n)},m\right) = B$$

if and only if $n \equiv 3 \pmod 4$, where m and n are integers with $m \geq 2$ and $n \geq 3$. In particular, for all $m \geq 2$, $W_S'(P,m) = A$ for the Petersen graph $P \cong \overline{L(\mathbf{K}_5)}$.

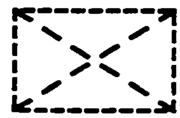
Proof. For any $n \geq 3$, if player B follows player A's first move by coloring an adjacent edge, then player A wins by forming a bichromatic triangle.

More precisely, player B can respond in essentially one of two ways.

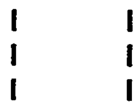


and in both cases player A can force the configuration

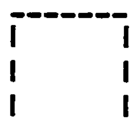
Player B loses since he then has to make the last move of the game



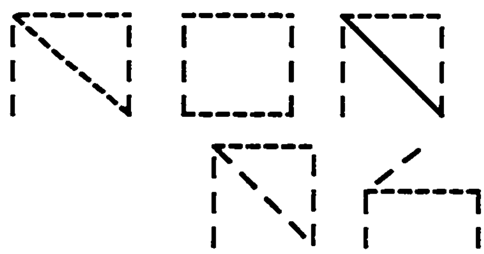
So regardless of whether n is odd or even player B should respond to player A's initial move by coloring an adjacent edge (with the same color).



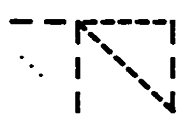
Now assume that n is even. If player A colors an edge red



then player B can respond in one of essentially five ways.

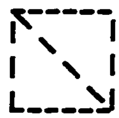


In the first and last cases, player A can force the following configuration



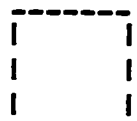
which consists of a bichromatic simplex and a star with basepoint on it. Altogether there are n colored edges and since n is assumed even this is a win for player A.

In the second and fourth cases, player A can respond as follows.

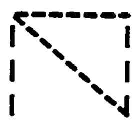


As before, player B would then make the last move of the game and lose. The third case is treated similarly. So for n even we have shown that player A has the winning strategy.

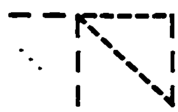
If, on the other hand, n is odd and player A colors an edge red



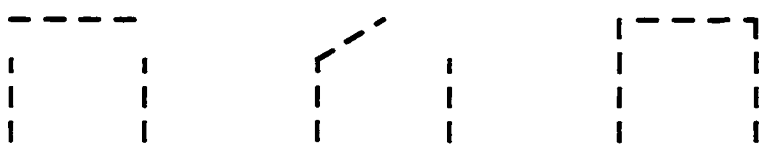
Then player B wins the game by forming the following configuration



since this leads to

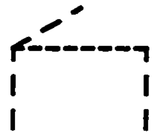


So for n odd player A avoids losing only if he colors an edge blue. There are three possibilities.



In the first case all the edge of $\mathbf{K}_n$ can and must be colored blue. Since $\binom{n}{2}$ is odd precisely when $n \equiv 3 \pmod 4$, player A wins for values of $n \equiv 1 \pmod 4$ while player B wins for values of $n \equiv 3 \pmod 4$.

In the second case, player B can color an edge red

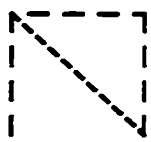


This leads either to

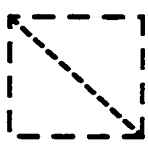
which is a win for player B since n is odd, or to

which is a win for player B for any n.

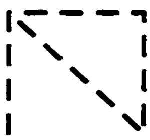
It only remains to consider the third case. If player B colored an edge red

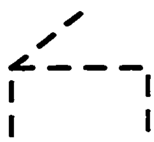


then player A would win the game by completing the "blue square".



Similar events would ensue if player B formed a "blue square" instead. So player B has essentially only two ways out:

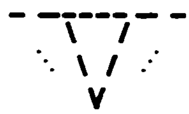




In the first case all the edge of K_n can and must be colored blue. Since $\binom{n}{2}$ is odd precisely when $n \equiv 3 \pmod{4}$, player A wins for values of $n \equiv 1 \pmod{4}$ while player B wins for values of $n \equiv 3 \pmod{4}$.

To prevent all the edges from being colored blue in the second case, player A would have to color an edge red:

But this would lead to



which is a win for player B for any n. Consequently, player A must color an edge blue. Then all the edges must be colored (blue) and, since n is odd, player B wins if and only if $n \equiv 3 \pmod{4}$. ■

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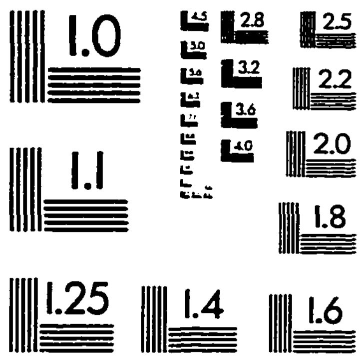
IMAGE EVALUATION TEST TARGET (QA-3)



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