NILPOTENT Q[X]-POWERED GROUPS AND Z[X]-GROUPS

By

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Abstract

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by

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An *exponential group* or *A-group* (as defined by A. G. Myasnikov and V. N. Remeslennikov [12]) is a group, G, equipped with an action by an associative ring with unity, A, such that for all $g \in G$ and for all $\alpha \in A$, the element $g^\alpha \in G$ is uniquely defined and the following axioms hold:

1. $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in A$.

2. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and for all $\alpha \in A$.

3. If $g, h \in G$ satisfy the relation $[g, h] = 1$, then $(gh)^\mu = g^\mu h^\mu$ for all $\mu \in A$.

A particular example of an exponential group is a *nilpotent R-powered group*, where R is a binomial ring (that is, a commutative integral domain of characteristic zero with identity such that for any $r \in R$ and $k \in \mathbb{Z}^+$, $\binom{r}{k} \in R$, where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$). A nilpotent R-powered group (see P. Hall [5], [6] and R. B. Warfield, Jr. [14]) is a nilpotent group, G, equipped with an action by R such that, for all $g \in G$ and for all $\alpha \in R$, the element $g^\alpha \in G$ is uniquely defined and the following axioms hold:

1. $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in R$.

2. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and for all $\alpha \in R$.

3. $g_1^\alpha \cdots g_n^\alpha = \tau_1(\bar{g})^\alpha \tau_2(\bar{g})^{\binom{\alpha}{2}} \cdots \tau_{k-1}(\bar{g})^{\binom{\alpha}{k-1}} \tau_k(\bar{g})^{\binom{\alpha}{k}}$ for $g_i \in G$ and for every $\alpha \in R$, where $\tau_i(\bar{g}) = \tau_i(g_1, \ldots, g_n)$ and k is the class of $gp\,(g_1, \ldots g_n)$.. The $\tau_i(\bar{g})$'s are known as the *Hall-Petresco words*.

In this thesis I generalize the notion of a nilpotent group in two specific classes of exponential groups, namely the class of nilpotent $\mathbb{Q}[x]$-powered groups and the class of $\mathbb{Z}[x]$-groups. Many questions which arise in the study of ordinary nilpotent groups are explored in this thesis for these particular exponential groups. I also introduce several new concepts and results for nilpotent $\mathbb{Q}[x]$-powered groups and $\mathbb{Z}[x]$-groups in general, expanding on the existing theory. Some known results for nilpotent R-powered groups, where R is any binomial ring, are mentioned in the papers of P. Hall ([5], [6]) and in the book by R. B. Warfield, Jr. [14] (A. M. Duguid also has results for such groups, but I was unable to find any of these results in the literature). R. C. Lyndon [10] and A. G. Myasnikov and V. N. Remeslennikov [12] have studied free A-groups, where A is any associative ring with unity. In [12], some foundational work for A-groups is given as well.

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This is dedicated to my brother-in-law,

Tony Santoro

Introduction

One of the most interesting classes of groups is the class of nilpotent groups. These groups have been thoroughly studied and many fascinating results have emerged. For example, every finite p-group is nilpotent and every finite nilpotent group is the internal direct product of its Sylow p-subgroups. Finitely generated nilpotent groups are polycyclic and have solvable word problem, generalized word problem, conjugacy problem and isomorphism problem. They are also residually finite, which is an important result in combinatorial group theory. If a nilpotent group is finitely generated and torsion free, then it admits a Mal'cev basis which gives rise to a unique normal form for its elements. In particular, the set of $n \times n$ unitriangular matrices over $\mathbb{Z}$, often denoted as $UT_n(\mathbb{Z})$, is a finitely generated torsion free nilpotent group and has a Mal'cev basis. These are just a few of the results.

The purpose of this thesis is to discuss the notion of a nilpotent group in the class of nilpotent $\mathbb{Q}[x]$-powered groups and the class of $\mathbb{Z}[x]$-groups, as well as provide several new concepts and results for these groups. Some results for nilpotent R-powered groups, where R is any binomial ring (that is, a commutative integral domain of characteristic zero with identity such that for any $r \in R$ and $k \in \mathbb{Z}^+$, $\binom{r}{k} \in R$, where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$), are mentioned in the papers of P. Hall ([5], [6]) and in the book by R. B. Warfield, Jr. [14] (A. M. Duguid also has results for such groups, but I was unable to find any of these results in the literature). R. C. Lyndon [10] and A. G. Myasnikov and V. N. Remeslennikov [12] have studied free A-groups, where A is an associative ring with unity. Both nilpotent R-powered groups and A-groups lie in a broader class of groups known as *exponential groups* as defined in [12].

This thesis contains three chapters. In the first chapter I mention some classical results in the study of nilpotent groups, some of which will be proven for completeness. Among these results there are two theorems which play a crucial role in the development of the theory of nilpotent $\mathbb{Q}[x]$-powered groups, which I will now mention. Let G be a finitely generated torsion free nilpotent group. The first major theorem (see [7]) states that G has at a finite normal series

$$1 = G_0 \lhd \cdots \lhd G_n = G$$

such that the series is a central series and each factor group G_{i+1}/G_i is infinite cyclic for $i = 0, \ldots, n-1$. Put another way, G has a poly-infinite cyclic and central series. I will give the proof of this theorem by utilizing some of the results prior to it. Using the notations above, suppose $G_{i+1} = gp\,(u_{i+1}, G_i)$ for some $u_{i+1} \in G_{i+1}$ and for each

$i = 0, 1, \ldots, n-1$. Then every element $g \in G$ can be uniquely expressed in the normal form $g = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$. The set $(u_1, \ldots, u_n)$ of G associated with the series $\{G_i\}$ is called a *Mal'cev basis* for G. The second major theorem, due to Mal'cev and Hall (see [5] and [6]), states that if G has a Mal'cev basis $(u_1, \ldots, u_n)$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\bar{\beta} = (\beta_1, \ldots, \beta_n)$, then

$$(u_1^{\alpha_1} \cdots u_n^{\alpha_n})(u_1^{\beta_1} \cdots u_n^{\beta_n}) = u_1^{f_1(\bar{\alpha},\bar{\beta})} \cdots u_n^{f_n(\bar{\alpha},\bar{\beta})}$$

and

$$(u_1^{\alpha_1} \cdots u_n^{\alpha_n})^{\lambda} = u_1^{g_1(\bar{\alpha},\lambda)} \cdots u_n^{g_n(\bar{\alpha},\lambda)}$$

where each $f_i(\bar{\alpha}, \bar{\beta})$ for $\bar{\alpha},\ \bar{\beta} \in \mathbb{Z}^n$ is a polynomial with rational coefficients in $2n$ variables and each $g_i(\bar{\alpha}, \lambda)$ for $\lambda \in \mathbb{Z}$ is a polynomial with rational coefficients in $n+1$ variables. As Philip Hall points out in [5] and [6], if we consider the set $\{u_1^{\alpha_1} \cdots u_n^{\alpha_n} \mid \alpha_i \in R\}$ for some binomial ring R, then this set together with the binary operation defined by the polynomials for multiplication and exponentiation (called the *R-completion* of G with respect to $(u_1, \ldots, u_n)$) is a nilpotent group of the same class as G. For such groups, the exponents are now elements of R rather than of $\mathbb{Z}$ and it is clear that G embeds into G^R. As Hall states in [5], this idea of an R-completion suggests an axiomatic approach toward the development of a nilpotent R-powered group. Let G be a nilpotent group of class c and R any binomial ring. Suppose that G is equipped with an action by R such that, for all $g \in G$ and for all $\alpha \in R$, the element $g^{\alpha} \in G$ is uniquely defined. Then G is called a *nilpotent* $\mathbb{Q}[x]$*-powered group* if the following axioms hold:

1. $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in R$.

2. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and for all $\alpha \in R$.

3. $g_1^\alpha \cdots g_n^\alpha = \tau_1(\bar{g})^\alpha \tau_2(\bar{g})^{\binom{\alpha}{2}} \cdots \tau_{k-1}(\bar{g})^{\binom{\alpha}{k-1}} \tau_k(\bar{g})^{\binom{\alpha}{k}}$ for all $\alpha \in R$ and for all $(g_1, \ldots, g_n) = \bar{g} \in G^n$, where k is the class of $gp\,(g_1, \ldots g_n)$. The $\tau_i(\bar{g})$'s are the Hall-Petresco words.

Examples of such groups are given at the end of the chapter. These groups are sometimes referred to as *R-powered groups* [5] and *nilpotent R-groups* [14].

Chapter two is devoted to the study of nilpotent $\mathbb{Q}[x]$-powered groups. It begins with the definition of a nilpotent $\mathbb{Q}[x]$-powered group (which is a special case of the definition stated above) and is followed by some fundamental definitions and lemmas. I prove a theorem of R. B. Warfield, Jr. [14] which states that every finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group has a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series. This result is clearly of great importance for pursuing the study of finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered groups since it permits one to express an element of such a group in a normal form. I also prove that if G is a finitely generated torsion free nilpotent group with Mal'cev bases $\mathcal{B}_1$ and $\mathcal{B}_2$, then the $\mathbb{Q}[x]$-completions of G with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathbb{Q}[x]$-isomorphic to each other. The concept of a nilpotent $\mathbb{Q}[x]$-powered group of finite type and of a π-primary nilpotent

$\mathbb{Q}[x]$-powered group is introduced. The connection between the two is proven in the theorem which states that every nilpotent $\mathbb{Q}[x]$-powered group of finite type is the direct $\mathbb{Q}[x]$-product of its π-primary components. This resembles the situation for ordinary nilpotent groups in which we have that every finite nilpotent group is the direct product of its Sylow p-subgroups. I introduce the *Frattini* $\mathbb{Q}[x]$*-subgroup* of a nilpotent $\mathbb{Q}[x]$-powered group and prove results for such $\mathbb{Q}[x]$-subgroups. Some of Dehn's fundamental problems have positive solutions for finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered groups. In particular, if G is a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group, then G has solvable word problem, generalized word problem and conjugacy problem. Such groups are also Hopfian (as defined in the $\mathbb{Q}[x]$ sense) and residually finite dimensional over $\mathbb{Q}$. The chapter concludes with a discussion of the Mal'cev correspondence between $\mathbb{Q}[x]$-completions of finitely generated torsion free nilpotent groups and nilpotent Lie algebras over $\mathbb{Q}[x]$. The construction of this correspondence uses group rings over $\mathbb{Q}[x]$ and a straightening process mentioned by S. A. Jennings [7].

In the last chapter, I discuss $\mathbb{Z}[x]$-groups. The notion of an A-group was first introduced by R. C. Lyndon [10] while attempting to solve Tarski's problem, which asks whether or not the elementary theory of free groups is decidable. A new axiom was introduced to those of an A-group by A. G. Myasnikov and V. N. Remeslennikov

[12] which revised R. C. Lyndons' notion. The advantage of this new axiom is that it allows one to view an abelian A-group as an A-module. For completeness, I will state the definition of an A-group now. Let A be any associative ring with unity and G be an arbitrary group. Suppose that G is equipped with an action by A such that, for all $g \in G$ and for all $\alpha \in A$, the element $g^\alpha \in G$ is uniquely defined. Then G is called an *A-group* if the following axioms hold:

1. $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in A$.

2. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and for all $\alpha \in A$.

3. If $g, h \in G$ satisfy the relation $[g, h] = 1$, then $(gh)^\mu = g^\mu h^\mu$ for all $\mu \in A$.

I am interested in the case where $A = \mathbb{Z}[x]$. My goal in studying such groups is to develop the theory of what I call *$\mathcal{N}^*$-groups.* $\mathcal{N}^*$-groups are closely related to ordinary nilpotent groups. These are not to be confused with nilpotent R-powered groups, since $\mathbb{Z}[x]$ is not a binomial ring and the axioms for nilpotent R-powered groups are different from those of A-groups. To begin with, I define a $\mathbb{Z}[x]$-group and give other preliminary definitions and results as well. The notion of an ideal ([12]) is given and some basic results pertaining to ideals are mentioned. I define an $\mathcal{N}^*$-group and a special kind of series called an $\mathcal{S}^*$-series which plays a useful role in the study of $\mathcal{N}^*$-groups. Several results which hold for ordinary nilpotent groups also hold for $\mathcal{N}^*$-groups. For example, I prove that if G is a torsion free $\mathcal{N}^*$-group and if $g^\beta = h^\beta$ for some $g, h \in G$ and non-zero $\beta \in \mathbb{Z}[x]$, then $g = h$. I also develop the theory of $\mathcal{R}$-groups which are closely related to those mentioned by A. G. Kurosh [9]. The major relationship between $\mathcal{N}^*$-groups and nilpotent $\mathbb{Q}[x]$-powered groups will be discussed when studying a particular class of $\mathbb{Z}[x]$-groups which I refer to as $\mathbb{Z}[x]$-groups of type HP. The main example which exhibits this relationship deals with the $\mathbb{Z}[x]$-group consisting of all $n \times n$ unitriangular matrices over $\mathbb{Z}[x]$ and the nilpotent $\mathbb{Q}[x]$-powered group of $n \times n$ unitriangular matrices over $\mathbb{Q}[x]$. The chapter concludes with a look at the specific case of when $n = 3$. In this case, we obtain the *Heisenberg* $\mathbb{Z}[x]$*-group*, denoted by $\mathcal{H}^{\mathbb{Z}[x]}$.

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Notations

$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers
$\mathbb{Z}[x]$	the ring of polynomials over $\mathbb{Z}$
$\mathbb{Q}[x]$	the ring of polynomials over $\mathbb{Q}$
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$\mid G \mid$	the order of G
$H \cap G$	the intersection of H and G
$H \cup G$	the union of H and G
$H \subset G$	H is a subset of G
$Z(G)$	the center of G
$C_G(A)$	the centralizer of A in G
G/H	the factor group of G by H
$[g,h]$	$g^{-1}h^{-1}gh$
g^h	$h^{-1}gh$
$gp\,(X)$	the smallest subgroup of G containing the subset X
$[H,K]$	$gp\,([h,k] \mid h \in H, k \in K)$
$\gamma_2 G = \Gamma_2$	$[G,G]$
$\gamma_n G = \Gamma_n$	$[\gamma_{n-1}G, G]$, the n^{th} subgroup of the lower central series of G

$Ab(G)$	$G/[G,G]$, the abelianization of G
$\zeta_n G$	the n^{th} subgroup of the upper central series of G
$\tau(G)$	the torsion subgroup of G
$G \bigotimes H$	the tensor product of the modules G and H
$\Phi(G)$	the Frattini subgroup of G
RG	the group ring of G over the ring R
$UT_n(R)$	the $n \times n$ unitriangular matrices over R
E_{ij}	the $n \times n$ matrix with 1 in the $(i,j)^{th}$ place and $0's$ elsewhere
$t_{ij}(\alpha)$	$I + \alpha E_{ij}$ where I is the $n \times n$ identity matrix and $\alpha \in R$
$\tau_i(g_1, \ldots, g_k)$	the i^{th} Hall-Petresco word of G with entries $(g_1, \ldots, g_k)$
$\binom{\alpha}{i}$	$\frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}$ where $i \in \mathbb{Z}$ and $\alpha \in R$ for some binomial ring R
$\mathcal{H}$	the Heisenberg group
G^R	the R-completion of G with respect to a Mal'cev basis
$H \leq_R G$	H is an R-subgroup of G
$H \trianglelefteq_R G$	H is a normal R-subgroup of G
$gp_R(X)$	the smallest R-subgroup of G containing the subset X
$[H_1, H_2]_R$	$gp_R([h_1, h_2] \mid h_1 \in H_1, h_2 \in H_2)$
$G \cong_R H$	G is R-isomorphic to H
$\prod_{i \in I} G_i$	the direct product of $\{G_i \mid i \in I\}$
G_π	the π-primary component of G
$h \sim g$	h is conjugate to g
$\mathcal{N}_c$	the class of all nilpotent $\mathbb{Q}[x]$-powered groups of class at most c.
$\mathcal{A}$	the class of abelian $\mathbb{Q}[x]$-groups
log	the logarithm map
exp	the exponentiation map

$g * h$	$log(exp(g)exp(h))$
$\mathcal{L} = \mathcal{L}_{G^{\mathbb{Q}[x]}}$	the Lie algebra of $G^{\mathbb{Q}[x]}$
$\mathcal{N}^*$	a $\mathbb{Z}[x]$-group which possesses a central $\mathbb{Z}[x]$-series
$\Gamma_i^*(G)$	the i^{th} term of the $\mathcal{S}^*$-series of G
$\mathcal{R}$	the class of $\mathbb{Z}[x]$-groups in which α^{th} roots, if they exist, are unique
$I(S)$	the isolator of S in G
$\mathcal{Z}$	the class of $\mathbb{Z}[x]$-groups
$\mathbb{Z}[l_1, l_2, \cdots, l_k][x]$	the ring $\mathbb{Z}[x]$ with the elements $\{l_1, l_2, \cdots, l_k\}$ adjoined to it
$\mathcal{H}^{\mathbb{Z}[x]}$	the Heisenberg $\mathbb{Z}[x]$-group

Chapter 1

Preliminaries On Nilpotent Groups

I will begin with a review some classical results pertaining to nilpotent groups. The definition of a nilpotent group is given, as well as some theorems which are proven by utilizing the upper and lower central series. I will also recall some facts about basic commutators, free nilpotent groups and other topics which arise in the study of nilpotent groups. The main theorem of this chapter, due to Mal'cev and P. Hall, will allow us to construct an R-completion of a finitely generated torsion free nilpotent group, where R is a binomial ring. This, in turn, gives rise to one of the main objects of this paper.

1.1 The Basics

Definition 1.1.1. Let G be a group. Then the series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_s = G$$

of G is called a *finite normal series* if $G_i \trianglelefteq G$ for each $0 \leq i \leq s$.

Note. It easily follows from the definition that for each $0 \leq i \leq s-1$, we have $G_i \trianglelefteq G_{i+1}$.

Definition 1.1.2. Let G be a group. Then a finite normal series

$$1 = G_0 \trianglelefteq \cdots \trianglelefteq G_s = G$$

of G is called a *central series* if $G_{i+1}/G_i \leq Z(G/G_i)$ for each $0 \leq i \leq s$.

Definition 1.1.3. A group G is called a *nilpotent group* if it has at least one central series. The shortest length of all such series is called its *nilpotency class.*

Definition 1.1.4. A group G is *solvable* if G has a finite normal series

$$1 = G_0 \trianglelefteq \cdots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is an abelian group for each $0 \leq i \leq s-1$.

An alternative definition for a group to be nilpotent is

Definition 1.1.5. A group G is a *nilpotent group* if it is solvable, and the finite normal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_s = G$ can be chosen so that the action of G on each of the factors G_{i+1}/G_i is trivial for each $1 \leq i \leq s$.

A nilpotent group of class 0 is the trivial group, while nilpotent groups of class at most 1 are abelian. Note that, although every nilpotent group is a solvable group, the converse is false (for example, S_3 is solvable, but not nilpotent).

Note. If $a, b \in G$ for some group G, then I will denote the expression $b^{-1}ab$ by a^b.

Definition 1.1.6. Let G be a group and let $g_1 \in G$ and $g_2 \in G$. Then the *commutator* of g_1 and g_2 is $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 = g_1^{-1}g_1^{g_2}$.

More generally, we have:

Definition 1.1.7. A *simple commutator of weight* $n \geq 2$ is defined recursively as $[g_1, \ldots, g_n] = [[g_1, \ldots, g_{n-1}], g_n]$ where by convention $[g_j] = g_j$.

Next I will recall some computational identities for commutator calculus.

Lemma 1.1.1. *Let x, y and z be elements of any group. Then*

1. $[x, y] = [y, x]^{-1}$
2. $x^y = x[x, y]$
3. $[xy, z] = [x, z]^y[y, z]$
4. $[x, yz] = [x, z][x, y]^z$
5. $[x, y^{-1}] = ([x, y]^{y^{-1}})^{-1}$
6. $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$
7. $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ *(the Hall-Witt Identity).*

Proof. Follows by direct calculation. □

We can form commutators of subsets of G as well as elements. Let $\{X_1, X_2, \ldots\}$ be non-empty subsets of G.

Definition 1.1.8. The *commutator subgroup* of X_1 and X_2 is defined by $[X_1, X_2] = gp\,([x_1, x_2] \mid x_1 \in X_1,\ x_2 \in X_2)$.

Remark. If G is any group and $S \subseteq G$, then

$$gp\,(S) = \bigcap_{H_k \leq G} \{H_k\}$$

where $S \subseteq H_k$. Hence, $gp\,(S)$ is the smallest subgroup of G containing S.

More generally, let $[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n]$ where $n \geq 2$. Observe that $[X_i, X_j] = [X_j, X_i]$. An immediate lemma which follows from Definition 1.1.2 is

Lemma 1.1.2. *Let G be any group and let $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_s = G$ be a finite normal series for G. Then the series is central if and only if $[G_{i+1}, G] \leq G_i$ for $0 \leq i \leq s-1$.*

The definitions of the upper and lower central series of a group are stated next.

Definition 1.1.9. Let G be any group. Then the *upper central series* of G is $1 = \zeta_0 G \trianglelefteq \zeta_1 G \trianglelefteq \cdots$, where $\zeta_{i+1}G/\zeta_i G = Z(G/\zeta_i G)$ for $i = 0, 1, \ldots$.

Remark. By definition, $\zeta_1 G = Z(G)$ and each $\zeta_i G$ is characteristic in G, but not necessarily fully-invariant in G. Furthermore, the upper central series doesn't necessarily ascend to G. When there is no ambiguity, I will sometimes denote $\zeta_i G$ by Z_i and refer to these as the *upper central subgroups*.

Definition 1.1.10. Let G be any group. Then the *lower central series* of G is $G = \gamma_1 G \trianglerighteq \gamma_2 G \trianglerighteq \cdots$, where $\gamma_{i+1}G = [\gamma_i G, G]$ for $i = 1, 2, \ldots$.

Remark. Each $\gamma_i G$ is fully-invariant in G and the series need not descend to the identity element. $\gamma_2 G = [G, G] = G'$ is called the *derived subgroup* of G and $G/\gamma_2 G = Ab(G)$ the *abelianization* of G. When there is no ambiguity I will often denote $\gamma_i G$ by Γ_i and refer to these as the *lower central subgroups.*

Lemma 1.1.3. *Let G be a nilpotent group with a central series*

$$1 = G_0 \trianglelefteq \cdots \trianglelefteq G_r = G.$$

Then

1. $\zeta_i G \geq G_i$ *and*

2. $\gamma_{i+1} G \leq G_{r-i}$ *for all* $i = 0, 1, \ldots, r$.

Proof. By using Definition 1.1.9, Definition 1.1.10 and induction on i, the result follows. □

Remark. Clearly, the above lemma shows the following relationship between the terms of the upper and lower central series:

$$\gamma_{i+1} G \leq \zeta_{r-i} G \text{ for } i = 0, 1, \ldots, r.$$

By using induction and Lemma 1.1.1 repeatedly, one establishes the following lemma:

Lemma 1.1.4. *For any group G, we have $\gamma_n G = gp\left([g_1, \ldots, g_n] \mid g_i \in G\right)$.*

Putting together Lemmas 1.1.1, 1.1.3 and 1.1.4 with the definition of a nilpotent group we obtain:

Theorem 1.1.1. *Let G be a nilpotent group. Then the following are equivalent:*

1. $\gamma_{c+1} G = 1$
2. $\zeta_c G = G$
3. *G has nilpotency class at most c.*
4. *$[g_1, \ldots, g_{c+1}] = 1$ for all g_i in G.*

Theorem 1.1.2. *Let G be a nilpotent group of class c. Then*

1. *every subgroup of G is a nilpotent group of class $\leq c$.*
2. *every factor group of G is a nilpotent group of class $\leq c$.*
3. *if $G_1, \ldots, G_k$ are also nilpotent groups of class c, then their external direct product is a nilpotent group of class c.*

Proof. Follows by induction and the isomorphism theorems. See [11]. □

1.2 Results Obtained From The Lower and Upper Central Series

The results in this section are proven with the aid of the upper and lower central series. Most of these results will be stated with references.

Lemma 1.2.1. *If $H \trianglelefteq G$ and G is nilpotent of class c, then $\gamma_i(G/H) = ((\gamma_i G)H)/H$ for each $i = 1, \ldots, c+1$. Furthermore, if G has class c, then $G/\gamma_c G$ has class $c-1$.*

Lemma 1.2.2. *Let G be a nilpotent group of class $c \geq 2$. Then for any element $g \in G$, the subgroup $H = gp\,(g, \gamma_2 G)$ is nilpotent of class $< c$.*

Proof. Refer to [5] or [8]. □

Lemma 1.2.3. *Let G be a nilpotent group of class c and suppose that $H \leq G$ such that $gp\,(H, \gamma_2 G) = G$. Then $H = G$.*

Proof. See [8] or [14]. □

I will now discuss the torsion group and torsion elements of a group.

Definition 1.2.1. Let G be any group and let $\tau(G)$ denote the set of all elements of finite order, $\tau(G) = \{g \in G \mid g^n = 1 \text{ for some } n \neq 0 \in \mathbb{Z}\}$. An element $g \in \tau(G)$ is termed a *torsion element* of G and $\tau(G)$ the set of *torsion elements* of G.

Definition 1.2.2. A group G is called a *torsion group* or a *periodic group* if $\tau(G) = G$. If $m \in \mathbb{Z}^+$ is the smallest positive integer for which $G^m = 1$, then we say that G has *exponent* m.

Definition 1.2.3. If G is any group, then G is a *torsion free* group if it has no torsion elements other than the identity element (i.e. if $g \in G$ and $g^n = 1$ for some $n \in \mathbb{Z}$, then either $g = 1$ or $n = 0$).

Remark. It is obvious that every group contains at least one torsion element, namely the identity element. It is not always the case, however, that the set of torsion elements of a group form a subgroup. For any nilpotent group, though, it does.

Theorem 1.2.1. *Let G be a nilpotent group of class c. Then $\tau(G) \trianglelefteq G$.*

Proof. The proof is by induction on the class of G and uses Lemma 1.2.2. See [8]. □

A useful corollary which follows from Theorem 1.2.1 is

Corollary 1.2.2. *Let G be a nilpotent group of class c. Then $G/\tau(G)$ is a torsion free nilpotent group.*

Proof. By Theorem 1.2.1, $\tau(G) \trianglelefteq G$. Hence the factor group $G/\tau(G)$ can be formed. Let $g\tau(G) \in G/\tau(G)$ such that there exists $n \neq 0 \in \mathbb{Z}$ with $(g\tau(G))^n = \tau(G)$. Then $g^n \in \tau(G)$ and so $\exists\, m \in Z$ s.t. $(g^n)^m = g^{nm} = 1$. Therefore, $g \in \tau(G)$. □

The next theorem demonstrates how the abelianization of a group can give us information about the group itself. The proof I've given can be found in [15]. See [14] for a slightly different approach. The next couple of lemmas are needed in the proof.

Lemma 1.2.4 (The Three Subgroup Lemma: Kalužnin, P. Hall). *Let H, K and L be subgroups of a group G. If any two of the commutator subgroups $[H, K, L]$, $[K, L, H]$, $[L, H, K]$ are contained in a normal subgroup of G, the so is the third.*

Proof. The proof uses Lemma 1.1.1. See [15] for details. □

Lemma 1.2.5. *Let G be any group and let i and j be positive integers. Then $[\Gamma_i, \Gamma_j] \leq \Gamma_{i+j}$.*

Proof. The proof is by induction and uses Lemma 1.2.4. See D. J. S. Robinson [15]. □

Theorem 1.2.3. *Let G be any group. For each integer $n > 1$, the mapping $\varphi_n : \Gamma_{n-1}/\Gamma_n \times Ab(G) \to \Gamma_n/\Gamma_{n+1}$ defined by $\varphi_n(x\Gamma_n, y\Gamma_2) = [x, y]\Gamma_{n+1}$ is multiplicative in each variable (i.e. φ_n, when restricted to each component, is a homomorphism). Consequently there is a well-defined module epimorphism*

$$\psi_n : \Gamma_{n-1}/\Gamma_n \bigotimes Ab(G) \to \Gamma_n/\Gamma_{n+1}$$

$$(x\Gamma_n \otimes y\Gamma_2) \mapsto [x, y]\Gamma_{n+1}$$

between $\mathbb{Z}$-modules for every integer $n > 1$.

Proof. I will first check that φ_n is well-defined. Let $g \in G$, $g_{n-1} \in \Gamma_{n-1}$, $g_n \in \Gamma_n$ and $g_2 \in \Gamma_2$. By Lemma 1.2.5, $[g_{n-1}, g_2]$, $[g_n, g]$, $[g_n, g_2]$, $[g_{n-1}, g, g_n]$, $[g_{n-1}, g_2, g_n]$ and $[g_{n-1}, g, g_2]$ are all contained in Γ_{n+1}. Therefore,

$$\begin{aligned}
\varphi_n(g_{n-1}g_n\Gamma_n, gg_2\Gamma_2) &= [g_{n-1}g_n, gg_2]\Gamma_{n+1} \\
&= [g_{n-1}g_n, g_2][g_{n-1}g_n, g]^{g_2}\Gamma_{n+1} \\
&= [g_{n-1}, g_2]^{g_n}[g_n, g_2]([g_{n-1}, g]^{g_n}[g_n, g])^{g_2}\Gamma_{n+1} \\
&= [g_{n-1}, g_2][g_{n-1}, g_2, g_n][g_n, g_2]([g_{n-1}, g]^{g_n}[g_n, g])^{g_2}\Gamma_{n+1} \\
&= ([g_{n-1}, g][g_{n-1}, g, g_n][g_n, g])^{g_2}\Gamma_{n+1} \\
&= [g_{n-1}, g]^{g_2}\Gamma_{n+1} \\
&= [g_{n-1}, g][g_{n-1}, g, g_2]\Gamma_{n+1} \\
&= [g_{n-1}, g]\Gamma_{n+1} = \varphi_n(g_{n-1}\Gamma_n, g\Gamma_2).
\end{aligned}$$

Hence φ_n is well-defined.

Now I will verify that φ_n is multiplicative in each variable.

1. Let $a_1, a_2 \in \Gamma_{n-1}$. Then, since $[a_1, g, a_2] \in \Gamma_{n+1}$,

$$\begin{aligned}
\varphi_n(a_1a_2\Gamma_n, g\Gamma_2) &= [a_1a_2, g]\Gamma_{n+1} \\
&= [a_1, g][a_1, g, a_2][a_2, g]\Gamma_{n+1} \\
&= [a_1, g][a_2, g]\Gamma_{n+1} \\
&= \varphi_n(a_1\Gamma_n, g\Gamma_2)\varphi_n(a_2\Gamma_n, g\Gamma_2).
\end{aligned}$$

2. Let $b_1, b_2 \in G$. Then, since $[g_{n-1}, b_1, b_2] \in \Gamma_{n+1}$,

$$\begin{aligned}
\varphi_n(g_{n-1}\Gamma_n, b_1 b_2 \Gamma_2) &= [g_{n-1}, b_1 b_2]\Gamma_{n+1} \\
&= [g_{n-1}, b_2][g_{n-1}, b_1][g_{n-1}, b_1, b_2]\Gamma_{n+1} \\
&= [g_{n-1}, b_2][g_{n-1}, b_1]\Gamma_{n+1} \\
&= [g_{n-1}, b_1][g_{n-1}, b_2]\Gamma_{n+1} \\
&= \varphi_n(g_{n-1}\Gamma_n, b_1\Gamma_2)\varphi_n(g_{n-1}\Gamma_n, b_2\Gamma_2).
\end{aligned}$$

Therefore, φ_n is multiplicative in each variable. Clearly, φ_n is an epimorphism since $[\Gamma_{n-1}, G] = \Gamma_n$. Furthermore, by the fundamental mapping property of the tensor products (over $\mathbb{Z}$), there is an induced epimorphism

$$\psi_n : \Gamma_{n-1}/\Gamma_n \bigotimes Ab(G) \to \Gamma_n/\Gamma_{n+1}$$

defined by

$$(x\Gamma_n \otimes y\Gamma_2) \mapsto [x, y]\Gamma_{n+1}$$

This completes the proof. ☐

Corollary 1.2.4. *Let G be any group. For each integer $i > 0$, there is an epimorphism $\Psi_i : \underbrace{Ab(G) \bigotimes \cdots \bigotimes Ab(G)}_{i \text{ of these}} \to \Gamma_i/\Gamma_{i+1}$ between $\mathbb{Z}$-modules defined by $\Psi_i(g_1\Gamma_2 \otimes \cdots \otimes g_i\Gamma_2) = [g_1, \ldots, g_i]\Gamma_{i+1}$, where $g_k \in G$.*

Proof. By iterating the above result, the corollary follows. ☐

Some consequences of Corollary 1.2.4 are:

Corollary 1.2.5 (Dixmier). *If G is a nilpotent group of class c and $Ab(G)$ is a torsion group of exponent m, then G is also a torsion group with exponent dividing m^c.*

Corollary 1.2.6. *Let G be a finitely generated group with generating set $X = \{x_1, \ldots, x_n\}$. Then Γ_n/Γ_{n+1} is finitely generated by the set of all commutators, modulo Γ_{n+1}, of the form $[x_{i_1}, \ldots, x_{i_n}]$ where the x_{i_j}'s range over X.*

Proof. Observe that if G is a finitely generated group, then so is $Ab(G)$ by applying the natural epimorphism $\pi : G \to Ab(G)$. The rest follows by using Corollary 1.2.4 and Lemma 1.1.1 repeatedly. □

Theorem 1.2.7. *Every subgroup of a finitely generated nilpotent group G is also finitely generated.*

Proof. I will give the proof from [11]. Let $H \leq G$ and let G have lower central series of length s. By Corollary 1.2.6, each $\gamma_i G/\gamma_{i+1}G$ is finitely generated for $1 \leq i \leq s$. One can easily check that the series $H = H_1 \trianglerighteq H_2 \trianglerighteq \cdots H_s \trianglerighteq H_{s+1} = 1$, where $H_i = H \cap \gamma_i G$ for $1 \leq i \leq s+1$, is a central series for H. Furthermore,

$$\begin{aligned} H_i/H_{i+1} &= \{H \cap \gamma_i G\}/\{H \cap \gamma_i G \cap \gamma_{i+1} G\} \\ &\cong \gamma_{i+1} G\{H \cap \gamma_i G\}/\gamma_{i+1} G \end{aligned}$$

by the second isomorphism theorem. Therefore H_i/H_{i+1} is isomorphic to a subgroup of $\gamma_i G/\gamma_{i+1}G$. But $\gamma_i G/\gamma_{i+1}G$ is abelian and any subgroup of a finitely generated abelian group is finitely generated. Therefore, H_i/H_{i+1} is finitely generated for $1 \leq i \leq s$. In particular, $H_s = H_s/H_{s+1}$ is finitely generated. Hence, working our way up the central series for H inductively, it is easy to see that H is finitely generated. □

Theorem 1.2.8. *Let G be a nilpotent group and let $N \trianglelefteq G$ where $N \neq 1$. Then $N \cap Z(G) \neq 1$.*

Proof. See [14]. □

Corollary 1.2.9. *Let G be a nilpotent group. Then G is torsion free if and only if $Z(G)$ is torsion free.*

Proof. I will use the fact that the torsion subgroup is normal in G.

1. Suppose $Z(G)$ is torsion free. Then $\tau(G) \cap Z(G) = 1$. By Theorem 1.2.1, $\tau(G) \trianglelefteq G$. Therefore, by Theorem 1.2.8, $\tau(G) \cap Z(G) \neq 1$ unless $\tau(G) = 1$. This means that G is torsion free.

2. If G is torsion free, then $Z(G)$ is obviously torsion free as well.

□

Lemma 1.2.6. *Let G be a torsion free nilpotent group of class c. For any $x, y \in G$ and $n \in \mathbb{Z}^+$, if $x^n = y^n$, then $x = y$.*

Proof. The proof uses induction on the class of G along with Lemma 1.2.2. I will omit the proof. □

Lemma 1.2.7. *Let G be a torsion free nilpotent group. Then $G/Z(G)$ is a torsion free nilpotent group as well.*

Proof. Using the previous lemma and induction on the class of G, the proof follows. I will leave out the details. □

Theorem 1.2.10. *If G is a torsion free nilpotent group of class $c+1$, then $\zeta_{i+1}G/\zeta_i G$ is torsion free for each $0 \leq i \leq c$.*

Proof. The proof is by induction on i. It uses Corollary 1.2.9, Lemma 1.2.7 and the isomorphism theorems. Refer to either [5] or [7] for the proof. □

1.3 A Decomposition Theorem For Finite Nilpotent Groups

Definition 1.3.1. If p is a prime, then a finite group G is called a *p-group* if $|G| = p^n$ for some integer $n \geq 0$.

Theorem 1.3.1. *If G is a finite p-group, $G \neq 1$, then $Z(G) \neq 1$. Furthermore, G is nilpotent.*

Proof. The first part of the theorem follows from the class equation. The second part follows by induction using the upper central series. See [11]. □

Definition 1.3.2. Let G be an arbitrary finite group and let p^n be the highest power of p dividing $|G|$. Then H is a *Sylow p-subgroup* of G if $H \leq G$ and $|H| = p^n$.

Theorem 1.3.2 (Burnside, Wielandt). *If G is a finite nilpotent group, then G is the direct product of its Sylow p-subgroups.*

Proof. See [11] or [15]. □

1.4 The Frattini Subgroup

I will recall the definition of the Frattini subgroup of a group.

Definition 1.4.1. The *Frattini subgroup* of a group, G, is the intersection of all of the maximal subgroups of G. The Frattini subgroup is denoted by $\Phi(G)$. If G has no maximal subgroups, then $\Phi(G) = G$.

Definition 1.4.2. Let G be a group and suppose $g \in G$. Then g is called a *non-generator* of G if $G = gp\,(g, S)$ always implies that $G = gp\,(S)$ whenever $S \subset G$.

Lemma 1.4.1. *Let G be an arbitrary group. Then $\Phi(G)$ is the set of all non-generating elements of G.*

Proof. Refer to [11] or [15]. □

By Lemma 1.2.3, it follows that

Lemma 1.4.2. *If G is any nilpotent group, then* $[G, G] \leq \Phi(G)$.

Theorem 1.4.1 (Frattini). *The Frattini subgroup of a finite group is nilpotent.*

1.5 Group Rings And Jennings' Theorem

I will mention a theorem on residually nilpotent groups due to S. A. Jennings. The definitions and results in this section can be found in [2] and [7].

Definition 1.5.1. Let G be any group and R any ring with unity. Then the *group ring of G over R*, often denoted as RG, is defined as

$$RG = \left\{ \sum_{g \in G} \lambda_g g \text{ where all but finitely many of the coefficients } \lambda_g \in R \text{ are zero} \right\},$$

together with the addition and multiplication rules

$$\left(\sum_{g \in G} \lambda_g g \right) + \left(\sum_{g \in G} \bar{\lambda}_g g \right) = \sum_{g \in G} (\lambda_g + \bar{\lambda}_g) g$$

and

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{g \in G} \bar{\lambda}_g g \right) = \sum_{g \in G} \left(\sum_{hk = g} \lambda_h \bar{\lambda}_k \right) g$$

where λ_g, $\bar{\lambda}_g$, λ_h, $\bar{\lambda}_k \in R$.

Definition 1.5.2. Let $\Psi : RG \to R$ be the homomorphism defined by

$$\Psi \left(\sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} \lambda_g$$

Then $\ker \Psi$ is called the *augmentation ideal* of RG. I will denote this by A.

Remark. Clearly $\Psi(g-1) = 0$. Hence if $\tilde{g} \in A$, then $\tilde{g} = \sum_{g_i \in G} a_i g_i$ with $\sum a_i = 0$. Therefore $\sum_{g_i \in G} a_i(g_i - 1) = 0$ and so A is additively spanned by the elements $g - 1$ $(g \in G)$. In fact, Ψ is an epimorphism, so $\text{im}\Psi = R$ and consequently $RG/A \cong R$.

Definition 1.5.3. RG is called *residually nilpotent* if

$$\bigcap_{n=1}^{\infty} A^n = 0$$

where here A^n is the two-sided ideal of RG generated by all elements of the form $\{a_1 a_2 \cdots a_n \mid a_i \in A, 1 \leq i \leq n\}$.

Theorem 1.5.1 (S. A. Jennings). *Let G be a finitely generated torsion free nilpotent group and let F be a field. Then FG is residually nilpotent.*

Proof. See [2] or [7]. □

Remark. M. Lazard proved that the theorem holds if G is not finitely generated. E. Formanek [3] proved that if F is any associative ring, then the theorem holds.

1.6 Free Groups and Free Nilpotent Groups

In this section, I will give some results on free nilpotent groups. I'll begin by stating some definitions and theorems on free groups (see [15] for proofs).

Definition 1.6.1. Let F be a group, X a nonempty set and $\rho : X \to F$ a set map. Then (F, ρ), or simply F, is said to be *free on X* if to each function $\alpha : X \to G$, where G is any group, there corresponds a unique homomorphism $\beta : F \to G$ such that $\alpha = \beta \circ \rho$. A group which is free on some set is called a *free group.* If $X \subset F$ and ρ is the identity map, then F is *freely generated* by X. One usually writes $F(X)$ in this case.

As a consequence of the definition,

- $\rho : X \to F$ is one-to-one. Hence we may identify X with $\rho(X)$.
- The image of ρ generates F

Definition 1.6.2. Let F be a free group on a countably infinite set, $X = \{x_1, x_2, \ldots\}$ and let W be a nonempty subset of F. If $w = x_{i_1}^{k_1} \cdots x_{i_n}^{k_n} \in W$ and $\{g_1, \ldots, g_n\}$ are elements of a group G, then the *value of the word* w at $(g_1, \ldots, g_n)$ is $w(g_1, \ldots, g_n) = g_1^{k_1} \cdots g_n^{k_n}$. The subgroup of G which is generated by all values in G of words in W is called the *verbal subgroup* of G determined by W. Hence,

$$W(G) = gp\,(w(g_1, \ldots, g_i, \ldots) \mid g_i \in G,\ w \in W).$$

Definition 1.6.3. If W is a collection of words on a countably infinite set X, the class of all groups G such that $W(G) = 1$ is called a *variety* $\mathcal{B}(W)$ determined by W. W is called a *set of laws* for $\mathcal{B}(W)$.

Let's define for a countably infinite set, $X = \{x_1, x_2, \ldots\}$, the word $w = [x_{i_1}, \ldots, x_{i_k}]$ where the x_{i_j}'s are in X. If G is a nilpotent group of class c then, by Theorem 1.1.1, $w(g_1, \ldots, g_k) = [g_1, \ldots, g_k] = 1$ for any $\{g_1, \ldots, g_k\} \in G$ provided that $k \geq c$. Hence if $W = \{[x_{i_1}, \ldots, x_{i_k}]\}$ then $\mathcal{B}(W)$ is the variety of nilpotent groups of class less than $k + 1$.

Definition 1.6.4. Let $\mathcal{B}$ be a variety, F be a group in $\mathcal{B}$, X a nonempty set and $\rho : X \to F$ a set map. Then (F, ρ) is said to be *$\mathcal{B}$-free on X* if to each function $\alpha : X \to G$, where G is any group in $\mathcal{B}$, there corresponds a unique homomorphism $\beta : F \to G$ such that $\alpha = \beta \circ \rho$. A group which is $\mathcal{B}$-free on some set is called a *free $\mathcal{B}$-group*. As before, if $X \subset F$ and ρ is the identity map, then F is *freely generated* by X.

In particular, a *free nilpotent group of class c* is a group which satisfies the above in the variety of nilpotent groups of class $\leq$ c.

Theorem 1.6.1. *Let $X = \{x_1, x_2, \ldots\}$, $\tilde{F}(X)$ a free group on X and $\mathcal{B}$ a variety with a set of laws W. Then $F(X) = \tilde{F}(X)/W(\tilde{F}(X))$ is a free $\mathcal{B}$-group on X. Furthermore, every group which is a free $\mathcal{B}$-group is isomorphic to $F(X)$.*

In particular (see [5]),

Corollary 1.6.2. *If $\tilde{F}(X)$ is a free group freely generated by X and G is a free nilpotent group of class c, then $G \cong \tilde{F}(X)/gp\left([x_{i_1}, \ldots, x_{i_k}]\right)$ where $i_k \geq c$ and $x_{i_j} \in X$.*

Theorem 1.6.3. *If G is a free nilpotent group of class c, then the upper and lower series for G coincide.*

A consequence of Theorem 1.6.3 and Corollary 1.6.2 is

Corollary 1.6.4. *Suppose that G is a free nilpotent group of class c. Then $G/Z(G)$ is a free nilpotent group of class $c-1$.*

1.7 Basic Commutators

The definition of a sequence of basic commutators and its relationship to free groups is now given. This material can be found in [2], [4], [5] and [6].

Definition 1.7.1. Let $X = \{x_1, x_2, \dots x_p\}$ be any set. The *basic commutators in X* are defined recursively as follows:

1. The basic commutators of weight one are the elements $\{x_1, x_2, \dots x_p\}$ in this order.

2. After defining and ordering the basic commutators of weight less than n, one can obtain the basic commutators of weight n by taking the commutator of c_i and c_j, where c_i and c_j satisfy:

    - c_i and c_j are each basic commutators with $n = wt(c_i) + wt(c_j)$, where wt is the abbreviation for the weight of a commutator.

    - in the chosen order for the basic commutators of weight less than n, we have $c_j < c_i$ if $j < i$.

- if $c_i = [c_y, c_z]$ where c_y and c_z are basic commutators, then $c_z \leq c_i$ in the order that has been chosen for the basic commutators of weight $< n$.

3. the basic commutators of weight n follow all of the basic commutators of weight $< n$ in the order for the basic commutators of weight less than $n+1$, but the basic commutators of weight n may be arranged in any order.

The sequence of basic commutators $\{x_1, \ldots, x_p, c_{p+1}, \ldots\}$ constructed above is termed a *basic sequence in X.*

Note: In [2], the sequence above is constructed using a binary operation on groupoids called 'rep'. This proves to be an excellent way of developing such a sequence.

Theorem 1.7.1. *Let $G = gp(Y)$, where $Y = \{y_1, \ldots, y_q\}$, and let $c_1, c_2, \ldots$ be any basic sequence of basic commutators in Y. Then $\gamma_r G$ is generated, modulo $\gamma_{r+1}G$, by the basic commutators of weight r, where $r = 1, 2, \ldots$.*

The proof of this theorem uses Lie ring techniques, which will not be discussed here. Refer to G. Baumslag [2].

Theorem 1.7.2. *Let F be a free group, freely generated by $X = \{x_1, \ldots, x_q\}$. Let $b_1, b_2, \ldots$ be any basic sequence of basic commutators in X and let r be any positive integer. Then, modulo $\gamma_{r+1}F$, $\gamma_r F$ is a free abelian group, freely generated by the basic commutators of weight r.*

An immediate result of this theorem is:

Corollary 1.7.3. *Let F be a free nilpotent group of class c, freely generated by $X = \{x_1, \ldots, x_q\}$. Let $b_1, b_2, \ldots$ be any basic sequence of basic commutators in X and let $1 \leq r \leq c$. Then, modulo $\gamma_{r+1}F$, $\gamma_r F$ is a free abelian group, freely generated by the basic commutators of weight r.*

1.8 The Group $UT_n(\mathbb{Z})$

In this section I will discuss the group of $n \times n$ unitriangular matrices whose entries lie in $\mathbb{Z}$.

Definition 1.8.1. An $n \times n$ matrix is called *unitriangular over* $\mathbb{Z}$ if it has entries consisting of 0's under the main diagonal, 1's along the main diagonal and elements of $\mathbb{Z}$ above the main diagonal. The group consisting of all unitriangular matrices whose entries lie in $\mathbb{Z}$ will be denoted by $UT_n(\mathbb{Z})$.

Let $UT_n^m(\mathbb{Z})$ denote the subgroup of $UT_n(\mathbb{Z})$ consisting of those unitriangular matrices over $\mathbb{Z}$ whose $m-1$ super diagonals (those which are above the main diagonal) have 0's in their entries.

Remark. Notice that $UT_n^1(\mathbb{Z}) = UT_n(\mathbb{Z})$ and $UT_n^n(\mathbb{Z}) = I$, where I is the $n \times n$ identity matrix.

When doing computations with matrices, the objects which are most favorably used are transvections. I will now recall the definition of a transvection and some useful identities pertaining to them.

Let E_{ij} denote the $n \times n$ matrix with 1 in the $(i,j)^{th}$ place and 0's elsewhere. Then

$$E_{ij} \cdot E_{kl} = \begin{cases} E_{il} & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}$$

Moreover, if $A = (a_{ij})$ is an $n \times n$ matrix whose elements lie in $\mathbb{Z}$, then one can write

$$A = \sum_{i,j=1}^{n} a_{ij} E_{ij}.$$

Definition 1.8.2. Let $t_{ij}(\alpha) = I + \alpha E_{ij}$, where I is the $n \times n$ identity matrix and $\alpha \in \mathbb{Z}$. Then $t_{ij}(\alpha)$ is called a *transvection.*

For simplicity, one often abbreviates $t_{ij}(1) = t_{ij}$. The following identities can be verified by using the definition. Suppose $\alpha, \beta \in \mathbb{Z}$.

1. $t_{ij}(\alpha) \cdot t_{ij}(\beta) = t_{ij}(\alpha + \beta)$

2. $[t_{ij}(\alpha)]^{-1} = t_{ij}(-\alpha) = t_{ij}^{-\alpha}$

3. $t_{ij}(\alpha) = t_{ij}(1)^{\alpha} = t_{ij}^{\alpha}$

4. $[t_{ij}(\alpha), t_{kl}(\beta)] = \begin{cases} t_{il}(\alpha\beta) & \text{if } j = k, \\ t_{kj}(-\alpha\beta) & \text{if } j \neq k \text{ but } i = l, \\ 1 & \text{otherwise} \end{cases}$

By using the above identities, one can prove the following theorems. I will omit the proofs.

Theorem 1.8.1. *Each subgroup $UT_n^m(\mathbb{Z})$ of the group $UT_n(\mathbb{Z})$ is generated by transvections t_{ij} such that $j - i \geq m$.*

Theorem 1.8.2. $[UT_n^m(\mathbb{Z}), UT_n(\mathbb{Z})] = UT_n^{m+1}(\mathbb{Z})$. *Consequently, $UT_n(\mathbb{Z})$ is a nilpotent group of class $n-1$ with lower central series*

$$UT_n(\mathbb{Z}) \rhd UT_n^2(\mathbb{Z}) \rhd \cdots \rhd UT_n^{n-1}(\mathbb{Z}) \rhd 1.$$

Theorem 1.8.3. *The upper central series for $UT_n(\mathbb{Z})$ coincides with the lower central series.*

Theorem 1.8.4. *Suppose $UT_n^m(\mathbb{Z}) = gp\,(t_{ij} \mid j-i \geq m)$ and $UT_n^{m+1}(\mathbb{Z}) = gp\,(t_{kl} \mid l-k \geq m+1)$. Then*

$$UT_n^m(\mathbb{Z})/UT_n^{m+1}(\mathbb{Z}) = gp\,(t_{1m+1}, t_{2m+2}, \ldots, t_{n-mn}).$$

Theorem 1.8.5. *For each $1 \leq m \leq n$, the factor group*

$$UT_n^m(\mathbb{Z})/UT_n^{m+1}(\mathbb{Z})$$

is torsion free.

An example of a group of unitriangular matrices is the well-known *Heisenberg group*, $UT_3(\mathbb{Z})$. I will be studying an analogue of this group later in a more general setting. I will denote the Heisenberg group by $\mathcal{H}$. Hence, $\mathcal{H}$ is the collection of 3×3 upper triangular matrices

$$\left\{ \begin{pmatrix} 1 & g_1 & g_2 \\ 0 & 1 & g_3 \\ 0 & 0 & 1 \end{pmatrix} \middle| g_i \in \mathbb{Z} \right\}.$$

It is easy to verify by direct calculation that $\mathcal{H}$ is indeed a torsion free nilpotent group of class 2 and, in fact, a free nilpotent group. Later in this chapter, I will give a set of generators for this group which allow us to write an arbitrary element in a unique normal form. Such unique normal forms occur in the more generalized groups $UT_n(\mathbb{Z})$ as well.

1.9 The Hall-Petresco Words

Definition 1.9.1. Let $F(X)$ be a free group, freely generated by $X = \{x_1, x_2, \ldots\}$. The *Hall-Petresco words* $\tau_k(x_1, \ldots, x_n) = \tau_k(\bar{x})$ are defined inductively by

$$x_1^n \cdots x_m^n = \tau_1(\bar{x})^n \tau_2(\bar{x})^{\binom{n}{2}} \cdots \tau_{n-1}(\bar{x})^{\binom{n}{n-1}} \tau_n(\bar{x}),$$

where $n \in \mathbb{Z}^+$ and $\bar{x} = (x_1, \ldots, x_n)$.

The proof of the next theorem can be found in [2] and [5].

Theorem 1.9.1 (P. Hall). *Let F be a free group which is freely generated by the set $X = \{x_1, x_2, \ldots, x_n\}$. Then $\tau_k(\bar{x}) \in \gamma_k F$, where $\bar{x} = (x_1, \ldots, x_n)$. Furthermore, if G*

is a free group on $\{x_1, x_2, \ldots, x_n, y_1, \ldots, y_s\}$ *and* $\theta : G \to F$ *is the homomorphism of* G *to* F *such that* $\theta(x_i) = x_i$ *and* $\theta(y_i) = 1$, *then*

$$\theta(\tau_k(x_1, x_2, \ldots, x_n, y_1, \ldots, y_s)) = \tau_k(\bar{x})$$

.

Corollary 1.9.2. *Let* G *be an arbitrary group and let* $g_1, \ldots, g_n \in G$. *Then*

$$g_1^k \cdots g_n^k = \tau_1(\bar{g})^k \tau_2(\bar{g})^{\binom{k}{2}} \cdots \tau_{k-1}(\bar{g})^{\binom{k}{k-1}} \tau_k(\bar{g})$$

for all $k \in \mathbb{Z}^+$, *where* $\bar{g} = (g_1 \ldots, g_n)$.

Theorem 1.9.3. *For any group* G *and* $g_1, \ldots, g_n \in G$, $\tau_i(g_1 \ldots, g_n) \in \gamma_i G$.

Proof. See P. Hall [5]. □

1.10 Mal'cev Bases

In this section I will prove that every finitely generated torsion free nilpotent group G has a poly-infinite cyclic and central series. Hence any $g \in G$ has a unique normal form with respect to this series. A major theorem of Hall and Mal'cev will then be stated.

Definition 1.10.1. Let X be a property (or class) of groups. A finite normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_l = G$$

of G is called a *poly-X series for G* if $G_{i+1}/G_i \in X$ for each $i = 0, \ldots, l-1$. A group G is called *poly-X* if it has at least one poly-X series. The *length* of the series above is l.

Theorem 1.10.1. *Let G be a finitely generated torsion free nilpotent group. The G has at least one central series*

$$1 = G_0 \lhd \cdots \lhd G_n = G$$

such that G_{i+1}/G_i is infinite cyclic for each $i = 0, \ldots, n-1$. Put another way, G has a poly-infinite cyclic and central series.

Proof. I will give the proof from the paper by S. A. Jennings [7]. Let $\{\zeta_i G\}$ denote the upper central series of G as usual. By Theorem 1.2.10, each factor group $\zeta_{s+1}G/\zeta_s G$ is torsion free. Furthermore, since each $\zeta_{s+1}G$ is finitely generated by Theorem 1.2.7, each $\zeta_{s+1}G/\zeta_s G$ is finitely generated (the natural epimorphism $\pi : \zeta_{s+1}G \to \zeta_{s+1}G/\zeta_s G$ gives us this). Hence each $\zeta_{s+1}G/\zeta_s G$ is a finitely generated torsion free abelian group. Therefore each $\zeta_{s+1}G/\zeta_s G$ is a direct product of a finite number of infinite cyclic groups. As a result, the upper central series of G may be refined so that between any two consecutive terms $\zeta_s G$ and $\zeta_{s+1}G$ we have a finite chain of subgroups $\zeta_s G \rhd U_{s_1} \rhd U_{s_2} \rhd \cdots \rhd U_{s_r} = \zeta_{s-1}G$ so that each factor is infinite cyclic. Since $[\zeta_s G, G] \leq \zeta_{s-1}G$, we have $[U_{s_j}, G] \leq \zeta_{s-1}G \leq U_{s_{j+1}}$ and so the refinement forms part of a central series of G. Continuing down the chain, the result is established. □

Remark. From the proof it is clear that every finitely generated nilpotent group G is polycyclic. Hence, the additional hypothesis that G is torsion free is special.

Definition 1.10.2. The length n of any such series above is an invariant of the group, called the *Hirsch length* (or the *rank*) of G.

Remark. More generally, the number of infinite cyclic factors in any polycyclic group G is an invariant and is called the *Hirsch length* or the *torsion free rank* of G.

Let's choose $u_{i+1} \in G_{i+1}$ in such a way that $G_{i+1} = gp(u_{i+1}, G_i)$ for each $i = 0, 1, \ldots, n-1$. Then every element $g \in G$ can be uniquely expressed in the normal form $g = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$.

Definition 1.10.3. The set $\bar{u} = (u_1, \ldots, u_n)$ of G associated with the poly-infinite cyclic and central series $\{G_i\}$ is called a *Mal'cev basis* for G. The element g has *coordinates* $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and I will sometimes write $g = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ as $g = \bar{u}^{\bar{\alpha}}$.

Note. In [5], Philip Hall calls $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ the *canonical parameters* with respect to the *canonical basis* $\bar{u} = (u_1, \ldots, u_n)$.

The proof of the next theorem, which utilizes the collection process, can be found in [5] and [8]. I will utilize the collection process in the proof of a theorem in the next chapter.

Theorem 1.10.2 (Mal'cev and Hall). *Let G be a finitely generated torsion free nilpotent group and let $\bar{u} = (u_1, \ldots, u_n)$ be a Mal'cev basis of G with respect to some* *poly-infinite cyclic and central series. If* $x = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ *and* $y = u_1^{\beta_1} \cdots u_n^{\beta_n}$, *then*

1. $xy = u_1^{f_1(\bar{\alpha},\bar{\beta})} \cdots u_n^{f_n(\bar{\alpha},\bar{\beta})}$ *with* $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *and* $\bar{\beta} = (\beta_1, \ldots, \beta_n)$

2. $x^\lambda = u_1^{g_1(\bar{\alpha},\lambda)} \cdots u_n^{g_n(\bar{\alpha},\lambda)}$

where each $f_i(\bar{\alpha}, \bar{\beta})$ *for* $\bar{\alpha},\ \bar{\beta} \in \mathbb{Z}^n$ *is a polynomial with rational coefficients in* $2n$ *variables and each* $g_i(\bar{\alpha}, \lambda)$ *for* $\lambda \in \mathbb{Z}$ *is a polynomial with rational coefficients in* $n+1$ *variables.*

Definition 1.10.4. I will call $(f_1(\bar{\alpha}, \bar{\beta}), \cdots, f_n(\bar{\alpha}, \bar{\beta}))$ the *multiplication polynomials* and $(g_1(\bar{\alpha}, \lambda), \cdots, g_n(\bar{\alpha}, \lambda))$ the *exponentiation polynomials* for G with respect to the Mal'cev basis $\bar{u} = (u_1, \ldots, u_n)$. I will sometimes abbreviate $(f_1(\bar{\alpha}, \bar{\beta}), \cdots, f_n(\bar{\alpha}, \bar{\beta})) = \bar{f}(\bar{\alpha}, \bar{\beta})$ and $g_1(\bar{\alpha}, \lambda), \cdots, g_n(\bar{\alpha}, \lambda)) = \bar{g}(\bar{\alpha}, \lambda)$.

For example, let's consider the Heisenberg group $\mathcal{H}$. Let $a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [a, b]$. Then $\mathcal{H} = gp\,(a, b)$ and so any $g \in \mathcal{H}$ can be represented as a word in $\{a, b\}$. Observe that $c \in Z(\mathcal{H})$. One can verify that the set $\{a, b, [b, a] = c^{-1}\}$ is a basic sequence of basic commutators for the free nilpotent group $\mathcal{H}$ and is therefore a Mal'cev basis for $\mathcal{H}$. Hence every $g \in \mathcal{H}$ can be uniquely expressed in the normal form $a^\alpha b^\beta c^\gamma$ where $\alpha, \beta, \gamma \in \mathbb{Z}$. By using the collection process, one can obtain the following identities:

1. $(a^{\alpha_1}b^{\alpha_2}c^{\alpha_3})(a^{\beta_1}b^{\beta_2}c^{\beta_3}) = a^{\alpha_1+\beta_1}b^{\alpha_2+\beta_2}c^{\alpha_3+\beta_3-\beta_1\alpha_2}$

2. $(a^{\alpha_1}b^{\alpha_2}c^{\alpha_3})^{\lambda} = a^{\alpha_1\lambda}b^{\alpha_2\lambda}c^{\alpha_3\lambda-\binom{\lambda}{2}\alpha_1\alpha_2}$

where α_1, α_2, α_3, β_1, β_2, β_3, $\lambda \in \mathbb{Z}$. Therefore the multiplication and exponentiation polynomials for $\mathcal{H}$ are

- $f_1(\bar{\alpha}, \bar{\beta}) = \alpha_1 + \beta_1$
- $f_2(\bar{\alpha}, \bar{\beta}) = \alpha_2 + \beta_2$
- $f_3(\bar{\alpha}, \bar{\beta}) = \alpha_3 + \beta_3 - \beta_1\alpha_2$
- $g_1(\bar{\alpha}, \lambda) = \alpha_1\lambda$
- $g_2(\bar{\alpha}, \lambda) = \alpha_2\lambda$
- $g_3(\bar{\alpha}, \lambda) = \alpha_3\lambda - \binom{\lambda}{2}\alpha_1\alpha_2$

with respect to this Mal'cev basis.

1.11 The R-Completion Of A Finitely Generated Torsion Free Nilpotent Group

For any finitely generated torsion free nilpotent group with a specified Mal'cev basis, the coordinates (which lie in $\mathbb{Z}^n$ for some $n > 0$) may be replaced by elements which lie in a binomial ring and, by using the multiplication and exponentiation polynomials, we obtain a group again. I will now look at this situation.

Definition 1.11.1. A *binomial ring*, R, is a commutative integral domain of characteristic zero with identity such that for any $r \in R$ and $k \in \mathbb{Z}^+$, we have $\binom{r}{k} \in R$, where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$.

Let G be a finitely generated torsion free nilpotent group, $\bar{u} = (u_1, \dots, u_n)$ a Mal'cev basis for G and R any binomial ring. Suppose that $\bar{f}(\bar{\alpha}, \bar{\beta}) = (f_1(\bar{\alpha}, \bar{\beta}), \cdots, f_n(\bar{\alpha}, \bar{\beta}))$ are the multiplication polynomials for G and $\bar{g}(\bar{\alpha}, \lambda) = (g_1(\bar{\alpha}, \lambda), \cdots, g_n(\bar{\alpha}, \lambda))$ are the exponentiation polynomials for G with respect to the Mal'cev basis $\bar{u}$. Consider the set of formal products

$$G^R = \{u_1^{\alpha_1} \cdots u_n^{\alpha_n} \mid \alpha_i \in R\}$$

and let multiplication and exponentiation be defined in G^R by means of the polynomials $f_i(\bar{\alpha}, \bar{\beta})$ and $g_i(\bar{\alpha}, \lambda)$ where each of the arguments for each polynomial is an element of R. More precisely,

1. $(u_1^{\alpha_1} \cdots u_n^{\alpha_n})(u_1^{\beta_1} \cdots u_n^{\beta_n}) = u_1^{f_1(\bar{\alpha}, \bar{\beta})} \cdots u_n^{f_n(\bar{\alpha}, \bar{\beta})}$

2. $(u_1^{\alpha_1} \cdots u_n^{\alpha_n})^{\lambda} = u_1^{g_1(\bar{\alpha}, \lambda)} \cdots u_n^{g_n(\bar{\alpha}, \lambda)}$

where the α_j's, β_j's and λ all lie in the ring R. The set G^R with multiplication and R-exponentiation defined in this way becomes a group (I will prove this in the next theorem).

Definition 1.11.2. The set G^R defined above, together with the multiplication and R-exponentiation polynomials, is called the *Mal'cev completion* of G with respect to the Mal'cev basis $\bar{u} = (u_1, \ldots, u_n)$.

Theorem 1.11.1. *The set G^R described above is a group under the prescribed polynomials for multiplication and R-exponentiation.*

Proof. In order to prove the theorem, the following lemma is needed:

Lemma 1.11.1. *Suppose that F is a field of characteristic 0 and let $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$. If $f(k_1, \ldots, k_n) = 0$ for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, then $f(x_1, \ldots, x_n) \equiv 0$.*

Proof of theorem: Let G have Mal'cev basis $\bar{u} = (u_1, \ldots, u_n)$ as above. I will verify that the group axioms are satisfied.

- Associativity: I want to show that

$$(\bar{u}^{\bar{\alpha}}\bar{u}^{\bar{\beta}})\bar{u}^{\bar{\gamma}} = \bar{u}^{\bar{\alpha}}(\bar{u}^{\bar{\beta}}\bar{u}^{\bar{\gamma}})$$

for any $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in R^n$. Well,

$$(\bar{u}^{\bar{\alpha}}\bar{u}^{\bar{\beta}})\bar{u}^{\bar{\gamma}} = \bar{u}^{\bar{f}(\bar{\alpha},\bar{\beta})}\bar{u}^{\bar{\gamma}} = \bar{u}^{\bar{f}(\bar{f}(\bar{\alpha},\bar{\beta}),\bar{\gamma})}$$

and

$$\bar{u}^{\bar{\alpha}}(\bar{u}^{\bar{\beta}}\bar{u}^{\bar{\gamma}}) = \bar{u}^{\bar{\alpha}}\bar{u}^{\bar{f}(\bar{\beta},\bar{\gamma})} = \bar{u}^{\bar{f}(\bar{\alpha},\bar{f}(\bar{\beta},\bar{\gamma}))}.$$

I claim that

$$\bar{f}(\bar{f}(\bar{\alpha},\bar{\beta}),\bar{\gamma}) = \bar{f}(\bar{\alpha},\bar{f}(\bar{\beta},\bar{\gamma}))$$

for all $\bar{\alpha},\bar{\beta},\bar{\gamma} \in R^n$ i.e. for all $i = 1,\ldots,n$, I want

$$f_i(\bar{f}(\bar{\alpha},\bar{\beta}),\bar{\gamma}) = f_i(\bar{\alpha},\bar{f}(\bar{\beta},\bar{\gamma})).$$

Consider the function $h_i(\bar{x},\bar{y},\bar{z}) = f_i(\bar{f}(\bar{x},\bar{y}),\bar{z}) - f_i(\bar{x},\bar{f}(\bar{y},\bar{z}))$, where $\bar{x},\bar{y},\bar{z} \in R^n$. Clearly, $h_i(\bar{x},\bar{y},\bar{z}) = 0$ for all $\bar{x},\bar{y},\bar{z} \in \mathbb{Z}^n$, since this relation holds in G. Hence $h_i(\bar{x},\bar{y},\bar{z}) \equiv 0$ by Lemma 1.11.1. Therefore

$$f_i(\bar{f}(\bar{\alpha},\bar{\beta}),\bar{\gamma}) = f_i(\bar{\alpha},\bar{f}(\bar{\beta},\bar{\gamma}))$$

for all $\bar{\alpha},\bar{\beta},\bar{\gamma} \in R^n$.

- Identity: I claim that $u_1^0 \cdots u_n^0$ is the identity element in G^R (abbreviated as $\bar{u}^{\bar{0}}$). Well, $\bar{u}^{\bar{0}}$ is the identity in G^R if and only if $\bar{u}^{\bar{\alpha}}\bar{u}^{\bar{0}} = \bar{u}^{\bar{\alpha}}$ for any $\bar{u}^{\bar{\alpha}} \in G^R$. But $\bar{u}^{\bar{\alpha}}\bar{u}^{\bar{0}} = \bar{u}^{\bar{f}(\bar{\alpha},\bar{0})}$ yields $\bar{u}^{\bar{f}(\bar{\alpha},\bar{0})} = \bar{u}^{\bar{\alpha}}$. Hence I need only show that $\bar{f}(\bar{\alpha},\bar{0}) = \bar{\alpha}$ for all $\bar{\alpha} \in R^n$. Consider the function $h(\bar{x}) = \bar{f}(\bar{x},\bar{0}) - \bar{x}$ for $\bar{x},\bar{0} \in R^n$. Then $h(\bar{x}) = \bar{0}$ for all $\bar{x} \in \mathbb{Z}^n$, since this relation holds in G. Hence $h(\bar{x}) \equiv 0$ by Lemma 1.11.1. Therefore $\bar{f}(\bar{\alpha},\bar{0}) = \bar{\alpha}$ for all $\bar{\alpha} \in R^n$.

- Inverses: If $\bar{u}^{\bar{\alpha}} \in G^R$, I claim that there exists $(\bar{u}^{\bar{\alpha}})^{-1} \in G^R$ such that $(\bar{u}^{\bar{\alpha}})((\bar{u}^{\bar{\alpha}})^{-1}) = \bar{u}^{\bar{0}}$. Observe that $(\bar{u}^{\bar{\alpha}})((\bar{u}^{\bar{\alpha}})^{-1}) = \bar{u}^{\bar{0}}$ yields $\bar{u}^{\bar{\alpha}}\bar{u}^{\bar{g}(\bar{\alpha},-1)} = \bar{u}^{\bar{0}}$. This gives us $\bar{u}^{\bar{f}(\bar{\alpha},\bar{g}(\bar{\alpha},-1))} = \bar{u}^{\bar{0}}$. Hence, I need only show that $\bar{f}(\bar{\alpha},\bar{g}(\bar{\alpha},-1)) = \bar{0}$

for all $\bar{\alpha} \in R^n$. But this equality holds for all $\bar{\alpha} \in \mathbb{Z}^n$. Hence, once again by Lemma 1.11.1, $\bar{f}(\bar{\alpha}, \bar{g}(\bar{\alpha}, -1)) = \bar{0}$ for all $\bar{\alpha} \in R^n$.

□

Theorem 1.11.2. *The group G^R described above is a nilpotent group of the same class as G, namely c.*

Proof. As before, let $G^R = \{u_1^{\alpha_1} \cdots u_n^{\alpha_n} \mid \alpha_i \in R\}$. Suppose that $g_i \in G^R$ for $i = 1, \ldots, c+1$ and let's consider the commutator $[g_1, \cdots, g_{c+1}]$. Well,

$$\begin{aligned} [g_1, \ldots, g_{c+1}] &= [u_1^{\alpha_{1,1}} \cdots u_n^{\alpha_{1,n}}, \ldots, u_1^{\alpha_{c+1,1}} \cdots u_n^{\alpha_{c+1,n}}] \\ &= u_1^{P_1(\bar{\alpha})} \cdots u_n^{P_n(\bar{\alpha})} \end{aligned}$$

where $P_i(\bar{\alpha}) = P_i(\alpha_{1,1}, \ldots, \alpha_{1,n}, \ldots, \alpha_{c+1,1}, \ldots, \alpha_{c+1,n})$ for some polynomials P_i and $\alpha_{j,k} \in R$. Each $P_i(\bar{\alpha}) = 0$ for each $\alpha_{j,k} \in \mathbb{Z}$, since G is of class c and $[g_1, \ldots, g_{c+1}] = 1$ for $g_i \in G$. Hence, $P_i(\bar{\alpha}) \equiv 0$ for all $\alpha_{j,k} \in R$ and for each i. Therefore,

$$[g_1, \ldots, g_{c+1}] = u_1^0 \cdots u_n^0 = 1$$

for all $g_i \in G^R$. This means that G is nilpotent of class is $\leq c$. To obtain equality, observe that G embeds into G^R and so the nilpotency class of G^R is $\geq c$. This shows that the nilpotency class of G^R equals c. □

Note. If G is a finitely generated torsion free nilpotent group of class c, we see from the above that G^R, with respect to a given Mal'cev basis for G, satisfies the following:

- For all $g \in G^R$ and for all $r \in R$, $g^r \in G^R$ is a uniquely determined element of G^R. In other words, *R-exponentiation* is well-defined in G^R. This R-action can be written as

$$G^R \times R \to G^R$$

$$(g, r) \mapsto g^r.$$

- By using the multiplication and R-exponentiation polynomials together with Lemma 1.11.1, one can check that the following hold:

  1. $g^1 = g$, $g^r g^s = g^{r+s}$, $(g^r)^s = g^{rs}$ for all $g \in G^R$ and for all $r, s \in R$.
  2. $(h^{-1}gh)^r = h^{-1}g^r h$ for all $g, h \in G^R$ and for all $r \in R$.
  3. $g_1^r \cdots g_n^r = \tau_1(\bar{g})^r \tau_2(\bar{g})^{\binom{r}{2}} \cdots \tau_{k-1}(\bar{g})^{\binom{r}{k-1}} \tau_k(\bar{g})^{\binom{r}{k}}$ for all $r \in R$ and $(g_1, \ldots, g_n) = \bar{g} \in (G^R)^n$, where k is the class of $gp\,(g_1, \ldots, g_n)$.

1.12 Nilpotent R-Powered Groups

In the previous section, it was shown that the R-completion (R is a binomial ring) of a finitely generated torsion free nilpotent group G can be formed with respect to some Mal'cev basis. The multiplication and exponentiation polynomials for G

are used to define multiplication and R-exponentiation in the R-completion. The identities which are satisfied by this R-completion can be used in a more general setting. In this section, I will give the definition of a nilpotent R-powered group and some examples of such groups. P. Hall [5] and R. B. Warfield, Jr. [14] give an excellent survey of nilpotent R-powered groups. I would like to mention that A. M. Duguid also did some work with such groups, but I was not able to find it in the literature. His name is mentioned in [5] and [7].

1.12.1 What Is A Nilpotent R-Powered Group?

Definition 1.12.1. Let G be a nilpotent group of class c and R a binomial ring. Suppose that G is equipped with an action by R,

$$G \times R \to G \text{ defined by } (g, \alpha) \mapsto g^{\alpha},$$

such that for all $g \in G$ and for all $\alpha \in R$, the element $g^{\alpha} \in G$ is uniquely determined. Then G is called a *nilpotent R-powered group* if the following axioms hold:

1. $g^1 = g$, $g^{\alpha}g^{\beta} = g^{\alpha+\beta}$, $(g^{\alpha})^{\beta} = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in R$.

2. $(h^{-1}gh)^{\alpha} = h^{-1}g^{\alpha}h$ for all $g, h \in G$ and for all $\alpha \in R$.

3. $g_1^{\alpha} \cdots g_n^{\alpha} = \tau_1(\bar{g})^{\alpha}\tau_2(\bar{g})^{\binom{\alpha}{2}} \cdots \tau_{k-1}(\bar{g})^{\binom{\alpha}{k-1}}\tau_k(\bar{g})^{\binom{\alpha}{k}}$ for all $\alpha \in R$ and for all $(g_1, \ldots, g_n) = \bar{g} \in G^n$, where k is the class of $gp\,(g_1, \ldots g_n)$. The $\tau_i(\bar{g})$'s are the Hall-Petresco words.

The last axiom will be referred to as the *Hall-Petresco axiom.*

Remark. A nilpotent R-powered group is sometimes referred to as an *R-powered group* (see [5]) or a *nilpotent R-group* (see [14]).

From the axioms above, it is easy to see that

1. $(g^{\mu})^{-1} = g^{-\mu}$ for all $\mu \in R$

2. $g^0 = 1$ for all $g \in G$

3. $1^{\alpha} = 1$ for all $\alpha \in R$

1.12.2 Examples Of Nilpotent R-Powered Groups

I will now give some examples of nilpotent R-powered groups which can be found in P. Hall [5].

1. If G is any finitely generated torsion free nilpotent group, then G^R is a nilpotent R-powered group.

2. Let G be an abelian R-powered group. Then $(gh)^{\alpha} = g^{\alpha}h^{\alpha}$ for all $g, h \in G$ and for all $\alpha \in R$. This can be seen by using the Hall-Petresco axiom and observing that, since G is abelian and $\tau_i(\bar{g}) \in \gamma_i G$, we have $\tau_i(\bar{g}) = 1$ for $i \geq 2$. Therefore G can be viewed as an R-module if we interpret the group multiplication

and R-exponentiation operations of G as the R-module operations of addition and scalar multiplication, respectively. Let $g, h \in G$ and α, $\beta \in R$. Then the interpretation is as follows:

- $g^1 = g \quad \rightsquigarrow \quad 1 \cdot g = g$
- $(gh)^\alpha = g^\alpha h^\alpha \quad \rightsquigarrow \quad \alpha \cdot (g + h) = \alpha \cdot g + \alpha \cdot h$
- $g^{\alpha+\beta} = g^\alpha g^\beta \quad \rightsquigarrow \quad (\alpha + \beta) \cdot g = \alpha \cdot g + \beta \cdot g$
- $g^{\alpha\beta} = (g^\beta)^\alpha \quad \rightsquigarrow \quad (\alpha\beta) \cdot g = (\alpha \cdot (\beta \cdot g))$

3. Let $\tilde{R}$ be any associative ring with unity which contains R, a binomial ring, in its center. Let ρ be a nilpotent ideal of $\tilde{R}$, say $\rho^{n+1} = 0$. Then $G = \{1 + a \mid a \in \rho\}$ is a subgroup of the group of units of $\tilde{R}$. In fact, $G = 1 + \rho$ is nilpotent of class $\leq n$. For $\lambda \in \tilde{R}$ and $g \in \rho$, define

$$(1 + g)^\lambda = 1 + \lambda g + \binom{\lambda}{2} g^2 + \cdots + \binom{\lambda}{n} g^n \in G.$$

Then this definition makes G into a nilpotent R-powered group.

4. The group $UT_n(R)$ of all unitriangular $n \times n$ matrices with entries in R is a nilpotent R-powered group as a consequence of the previous example with the defined R-action.

5. Suppose R is a binomial ring and let RG be the group ring of a torsion free nilpotent group, G, over R. Consider the quotient ring RG/A^n, where A is the

augmentation ideal of RG. By Theorem 1.5.1,

$$\bigcap_{n=1}^{\infty} A^n = 0.$$

If $g \neq 1$, there exists $k \in \mathbb{Z}^+$ such that $g - 1 \notin A^k$, since $g - 1 \in A^k$ for all $k \in \mathbb{Z}^+$ yields $g - 1 = 0$ by Theorem 1.5.1, a contradiction. In fact, there exists a $k \in \mathbb{Z}^+$ such that

$$G \cap (1 + A^k) = 1.$$

This gives rise to an embedding

$$\Omega : G \to RG/A^k \quad \text{given by}$$

$$g \mapsto 1 + ((g-1) + A^k).$$

Hence $\Omega(G) \cong G$. Thus G embeds into $1 + (A/A^k)$ by Ω. Notice that $g - 1 \in A$ for each $g \in G$ and so $(g-1)^k \in A^k$. Hence, in $1 + (A/A^k)$, one can define the $R-$action

$$\bar{g}^{\lambda} = (1 + (\bar{g} - 1))^{\lambda} = 1 + \lambda(\bar{g} - 1) + \binom{\lambda}{2}(\bar{g} - 1)^2 + \cdots + \binom{\lambda}{k-1}(\bar{g} - 1)^{k-1}$$

where $\bar{g} = 1 + ((g-1) + A^k) \in 1 + (A/A^k)$ and $\lambda \in R$. This R-action turns $1 + (A/A^k)$ into a nilpotent R-powered group by Example 3 above. Hence, we obtain a way of embedding a torsion free nilpotent group into a nilpotent R-powered group of the same class.

Chapter 2

Nilpotent $\mathbb{Q}[x]$-Powered Groups

In the first chapter, I mentioned the definition of a nilpotent R-powered group where R is a binomial ring. I will now focus on the case where $R = \mathbb{Q}[x]$. Clearly, $\mathbb{Q}[x]$ is a binomial ring, since all of the ring requirements are satisfied and the binomial coefficients of elements in $\mathbb{Q}[x]$ lie within $\mathbb{Q}[x]$ (i.e. if $\mu \in \mathbb{Q}[x]$ and $k \in \mathbb{Z}^+$, then $\binom{\mu}{k} \in \mathbb{Q}[x]$).

In this chapter, I will begin by recalling the definition of a nilpotent R-powered group for the case where $R = \mathbb{Q}[x]$ and give some basic definitions and results. Some of these results can be generalized for arbitrary nilpotent R-powered groups. I will discuss some particular classes of nilpotent $\mathbb{Q}[x]$-powered groups such as π-primary nilpotent $\mathbb{Q}[x]$-powered groups, nilpotent $\mathbb{Q}[x]$-powered groups of finite type, the Frattini $\mathbb{Q}[x]$-subgroup and free nilpotent $\mathbb{Q}[x]$-powered groups. I will then discuss some of Dehns' fundamental problems and a residual property of nilpotent $\mathbb{Q}[x]$-powered groups. The chapter concludes with a look at the Mal'cev correspondence between nilpotent Lie algebras over $\mathbb{Q}[x]$ and a certain class of nilpotent $\mathbb{Q}[x]$-powered groups.

I would like to mention that some of this material can be found in P. Hall ([5], [6]) and R. B. Warfield, Jr. [14].

2.1 Basic Definitions and Basic Results

Definition 2.1.1. Let G be a nilpotent group of class c. Suppose that G is equipped with an action by $\mathbb{Q}[x]$,

$$G \times \mathbb{Q}[x] \to G \text{ defined by } (g, \alpha) \mapsto g^\alpha,$$

such that for all $g \in G$ and for all $\alpha \in \mathbb{Q}[x]$, the element $g^\alpha \in G$ is uniquely determined. Then G is called a *nilpotent* $\mathbb{Q}[x]$*-powered group* if the following axioms hold:

1. $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in \mathbb{Q}[x]$.
2. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and for all $\alpha \in \mathbb{Q}[x]$.
3. $g_1^\alpha \cdots g_n^\alpha = \tau_1(\bar{g})^\alpha \tau_2(\bar{g})^{\binom{\alpha}{2}} \cdots \tau_{k-1}(\bar{g})^{\binom{\alpha}{k-1}} \tau_k(\bar{g})^{\binom{\alpha}{k}}$ for all $g_i \in G$ and for every $\alpha \in \mathbb{Q}[x]$, where $\tau_i(\bar{g}) = \tau_i(g_1, \ldots, g_n)$ and k is the class of $gp\,(g_1, \ldots g_n)$. The $\tau_i(\bar{g})$'s are the *Hall-Petresco words.*

As mentioned in chapter 1, the last axiom will be referred to as the *Hall-Petresco axiom.*

From the axioms above, we see that

1. $(g^{\mu})^{-1} = g^{-\mu}$ for all $\mu \in \mathbb{Q}[x]$.

2. $g^0 = 1$ for all $g \in G$.

3. $1^{\alpha} = 1$ for all $\alpha \in \mathbb{Q}[x]$.

I will be referring to subrings of $\mathbb{Q}[x]$. From this point on, all such subrings will be with unity.

Definition 2.1.2. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and K a subring of $\mathbb{Q}[x]$. Then H is called a *K-subgroup* of G if $H \leq G$ and $g^{\alpha} \in H$ for all $g \in H$ and $\alpha \in K$.

I will denote "H is a K-subgroup of G" by $H \leq_K G$.

Definition 2.1.3. K is a *binomial subring* of $\mathbb{Q}[x]$ if K is a subring of $\mathbb{Q}[x]$ and is a binomial ring.

Note. It is clear that if G is a nilpotent $\mathbb{Q}[x]$-powered group, K is a binomial subring of $\mathbb{Q}[x]$ and $H \leq_K G$, then H is a nilpotent K-powered group.

Definition 2.1.4. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and K a subring of $\mathbb{Q}[x]$. Then N is a *normal K-subgroup* of a nilpotent $\mathbb{Q}[x]$-powered group G if $N \leq_{\mathbb{Q}[x]} G$ and $N \trianglelefteq G$.

I will denote "N is a normal K-subgroup of G" by $N \trianglelefteq_K G$.

Lemma 2.1.1. *Let K be a subring of $\mathbb{Q}[x]$ and let $\{G_1, \ldots, G_k\}$ be a collection of K-subgroups of a nilpotent $\mathbb{Q}[x]$-powered group, G. Then $\bar{G} = \bigcap_{i=1}^{k} G_i$ is a K-subgroup of G.*

Proof. Observe that $\bar{G} \leq_K G_i$ for each $i = 1, \ldots, k$. It is now routine to verify that $\bar{G}$ is a K-subgroup of G. □

Let K be a subring of $\mathbb{Q}[x]$. A K-subgroup of a nilpotent $\mathbb{Q}[x]$-powered group G which is K-generated by a subset S is defined as follows:

Definition 2.1.5. Let G be a nilpotent $\mathbb{Q}[x]$-powered group. If $S = \{x_1, \ldots, x_j\}$ is a subset of G, then

$$H = \bigcap_{S \subset H_i \leq_K G} \{H_i\} = gp_K(x_1, \ldots, x_j)$$

is called the *K-subgroup of G which is K-generated by* $\{x_1, \ldots, x_j\}$. We call S a set of *K-generators* for H.

Remark. Put another way, H is the smallest K-subgroup of G containing the set S. In particular, $gp_{\mathbb{Z}}(S)$ is the usual subgroup generated by S.

Let K be a subring of $\mathbb{Q}[x]$. The commutator of K-subgroups of a nilpotent $\mathbb{Q}[x]$-powered group is defined in the following way:

Definition 2.1.6. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and let $H_1, H_2 \leq_K G$.

Then

$$[H_1, H_2]_K = gp_K([h_1, h_2] \mid h_1 \in H_1, h_2 \in H_2)$$

is the *commutator* K*-subgroup of* H_1 *and* H_2. If $H_1, \ldots, H_i \leq_K G$ then we recursively define, for $i > 2$,

$$[H_1, \ldots, H_i]_K = [[H_1, \ldots, H_{i-1}]_K, H_i]_K.$$

In particular, $[G, G]_K = gp_K([g_1, g_2] \mid g_1, g_2 \in G)$ is the *derived* K*-subgroup.*

The next lemma can be found in [5] and [14].

Lemma 2.1.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c and let $N \trianglelefteq_{\mathbb{Q}[x]} G$. Then*

$$gN = hN \text{ implies } g^{\beta}N = h^{\beta}N$$

for any $g, h \in G$ and any $\beta \in \mathbb{Q}[x]$.

Proof. Suppose $gN = hN$ as in the hypothesis. Then there exists $n \in N$ such that $gn = h$. For every $\beta \in \mathbb{Q}[x]$, we have $(gn)^{\beta} = h^{\beta}$. By axiom 3, we obtain

$$g^{\beta}n^{\beta} = (gn)^{\beta}\tau_2(g,n)^{\binom{\beta}{2}} \cdots \tau_{k-1}(g,n)^{\binom{\beta}{k-1}}\tau_k(g,n)^{\binom{\beta}{k}},$$

where k is the class of $gp\,(g,n)$. Clearly, each of the $\tau_i(g,n)$'s necessarily lie in N since $N \trianglelefteq_{\mathbb{Q}[x]} G$. Hence, $g^{\beta}n^{\beta} = (gn)^{\beta}\bar{n}$ for some $\bar{n} \in N$. Therefore $(gn)^{\beta} = g^{\beta}\tilde{n}$ for some $\tilde{n} \in N$. Thus $(gn)^{\beta}N = (g^{\beta}\tilde{n})N = g^{\beta}N$. Since $(gn)^{\beta} = h^{\beta}$, it follows that $g^{\beta}N = h^{\beta}N$. ☐

Consequently, we have

Lemma 2.1.3. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group and $N \trianglelefteq_{\mathbb{Q}[x]} G$. Then the $\mathbb{Q}[x]$-action on G induces a $\mathbb{Q}[x]$-action on G/N,*

$$(gN)^{\mu} = g^{\mu}N \text{ for all } g \in G \text{ and } \mu \in \mathbb{Q}[x],$$

which turns G/N into a nilpotent $\mathbb{Q}[x]$-powered group.

Proof. Let $gN, hN \in G/N$. By the previous lemma, $(gN)^{\mu} = (hN)^{\mu}$ implies $g^{\mu}N = h^{\mu}N$ for all $\mu \in \mathbb{Q}[x]$. Therefore this $\mathbb{Q}[x]$-action is well-defined. Verifying that G/N equipped with this $\mathbb{Q}[x]$-action is a nilpotent $\mathbb{Q}[x]$-powered group is straightforward. □

Theorem 2.1.1. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group. Then the subgroups of the upper and lower central series of G are $\mathbb{Q}[x]$-subgroups of G.*

Proof. If $z \in Z(G)$ and $g \in G$, then $g^{-1}zg = z$ yields $(g^{-1}zg)^{\alpha} = z^{\alpha}$ for any $\alpha \in \mathbb{Q}[x]$. Hence, by axiom 2, we have $g^{-1}z^{\alpha}g = z^{\alpha}$ and so $z^{\alpha} \in Z(G)$. Since $Z(G) \trianglelefteq G$, we have that $Z(G) \trianglelefteq_{\mathbb{Q}[x]} G$ and $G/Z(G)$ is a nilpotent $\mathbb{Q}[x]$-powered group by the above lemma. By induction, it follows that $\zeta_i G \trianglelefteq_{\mathbb{Q}[x]} G$ for each i. The proof that $\gamma_i G \trianglelefteq_{\mathbb{Q}[x]} G$ for each i can be found in [14]. □

Corollary 2.1.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group. Then*

$$\gamma_n G = gp\left([g_1, \ldots, g_n] \mid g_i \in G\right).$$

The following terms will be used later.

Definition 2.1.7. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and suppose that G has a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G.$$

Then the series is called a

1. $\mathbb{Q}[x]$*-series* if $G_i \leq_{\mathbb{Q}[x]} G$ for $i = 0, \ldots, n$
2. *normal* $\mathbb{Q}[x]$*-series* if $G_i \trianglelefteq_{\mathbb{Q}[x]} G$ for each $i = 0, 1, \ldots, n$
3. *central* $\mathbb{Q}[x]$*-series* if it is a normal $\mathbb{Q}[x]$-series which is central

The product of two $\mathbb{Q}[x]$-subgroups of a nilpotent $\mathbb{Q}[x]$-powered group is defined in the obvious way.

Lemma 2.1.4. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group and let $N \trianglelefteq_{\mathbb{Q}[x]} G$ and $H \leq_{\mathbb{Q}[x]} G$. Then $HN \leq_{\mathbb{Q}[x]} G$ and $HN = gp_{\mathbb{Q}[x]}(H, N)$. Furthermore, if $H \trianglelefteq_{\mathbb{Q}[x]} G$ as well, then $HN \trianglelefteq_{\mathbb{Q}[x]} G$.*

Proof. Let $hn \in HN$ and $\beta \in \mathbb{Q}[x]$. I claim that $(hn)^\beta \in HN$. Observe that

$$h^\beta n^\beta = (hn)^\beta \tau_2(h,n)^{\binom{\beta}{2}} \cdots \tau_{c-1}(h,n)^{\binom{\beta}{c-1}} \tau_c(h,n)^{\binom{\beta}{c}},$$

where c is the class of $gp\,(h,n)$. Each of the $\tau_i(h,n)$'s lie in N since $N \trianglelefteq_{\mathbb{Q}[x]} G$. Hence, $h^\beta n^\beta = (hn)^\beta \bar{n}$ for some $\bar{n} \in N$. Therefore $(hn)^\beta = h^\beta \tilde{n} \in HN$ for some $\tilde{n} \in N$. The rest of the proof is obvious. □

2.2 $\mathbb{Q}[x]$-Mappings

The definitions of $\mathbb{Q}[x]$-mappings and the $\mathbb{Q}[x]$-isomorphism theorems are analogous to the ordinary group case.

Definition 2.2.1. Let $\phi : G \to \bar{G}$ be a mapping between two nilpotent $\mathbb{Q}[x]$-powered groups. Then ϕ is termed a $\mathbb{Q}[x]$-*homomorphism* if $\phi(gh) = \phi(g)\phi(h)$ and $\phi(g^\lambda) = [\phi(g)]^\lambda$ for all $g, h \in G$ and $\lambda \in \mathbb{Q}[x]$. $Hom_{\mathbb{Q}[x]}(G, \bar{G})$ will represent the collection of all $\mathbb{Q}[x]$-homomorphisms from G to $\bar{G}$.

The terms $\mathbb{Q}[x]$-*monomorphism*, $\mathbb{Q}[x]$-*epimorphism*, $\mathbb{Q}[x]$-*isomorphism* and $\mathbb{Q}[x]$-*automorphism* are defined in the obvious way. $Aut_{\mathbb{Q}[x]}(G)$ will represent the collection of all $\mathbb{Q}[x]$-automorphisms of G.

As usual, $\phi(G)$ is the *image of* ϕ and will be denoted by *im* ϕ. The *kernel of* ϕ is $\{g \in G| \ \phi(g) = 1\}$ and will be denoted by *ker* ϕ.

Lemma 2.2.1. *Let $\phi : G \to \bar{G}$ be a $\mathbb{Q}[x]$-homomorphism between two nilpotent $\mathbb{Q}[x]$-powered groups. Then ker ϕ and im ϕ are $\mathbb{Q}[x]$-subgroups of G and $\bar{G}$, respectively. Moreover, ker ϕ is a normal $\mathbb{Q}[x]$-subgroup of G.*

Proof. If $g \in \ker \phi$, then for any $\beta \in \mathbb{Q}[x]$, $\phi(g^\beta) = [\phi(g)]^\beta = 1^\beta = 1$. Hence, $g^\beta \in \ker \phi$. If $\bar{g} \in \operatorname{im} \phi$, then there exists $g \in G$ such that $\phi(g) = \bar{g}$. But then $\phi(g^\alpha) = [\phi(g)]^\alpha = \bar{g}^\alpha$ for any $\alpha \in \mathbb{Q}[x]$. Hence, $\bar{g}^\alpha \in \operatorname{im} \phi$. The rest of the proof is straightforward. □

Theorem 2.2.1 (First $\mathbb{Q}[x]$-Isomorphism Theorem). *Let $\phi : G \to \bar{G}$ be a $\mathbb{Q}[x]$-homomorphism between two nilpotent $\mathbb{Q}[x]$-powered groups. Then*

$$G/ker\phi \cong_{\mathbb{Q}[x]} \phi(G).$$

Theorem 2.2.2 (Second $\mathbb{Q}[x]$-Isomorphism Theorem). *Let G be a nilpotent $\mathbb{Q}[x]$-powered group, $H \leq_{\mathbb{Q}[x]} G$ and $N \trianglelefteq_{\mathbb{Q}[x]} G$. Then*

$$HN/N \cong_{\mathbb{Q}[x]} H/H \cap N.$$

Theorem 2.2.3 (Third $\mathbb{Q}[x]$-Isomorphism Theorem). *Let G be a nilpotent $\mathbb{Q}[x]$-powered group and let $H \trianglelefteq_{\mathbb{Q}[x]} G$ and $K \trianglelefteq_{\mathbb{Q}[x]} G$ with $K \leq_{\mathbb{Q}[x]} H$. Then*

$$G/H \cong_{\mathbb{Q}[x]} \frac{G/K}{H/K}.$$

2.3 Abelian $\mathbb{Q}[x]$-groups

An *abelian $\mathbb{Q}[x]$-group* is simply a nilpotent $\mathbb{Q}[x]$-powered groups of class at most 1. As it was pointed out at the end chapter 1, if G is an abelian $\mathbb{Q}[x]$-group and $g, h \in G$, then $[g, h] = 1$ implies that $(gh)^\lambda = g^\lambda h^\lambda$ for any $\lambda \in \mathbb{Q}[x]$. Consequently, abelian $\mathbb{Q}[x]$-groups can be interpreted as $\mathbb{Q}[x]$-modules and their $\mathbb{Q}[x]$-subgroups as submodules. Much of the theory of modules can therefore be applied to abelian $\mathbb{Q}[x]$-groups.

Definition 2.3.1. A nilpotent $\mathbb{Q}[x]$-powered group, G, is a *cyclic $\mathbb{Q}[x]$-group* if there exists $g \in G$ such that $G = gp_{\mathbb{Q}[x]}(g)$.

Theorem 2.3.1. *Let G be a cyclic $\mathbb{Q}[x]$-group. Then every $\mathbb{Q}[x]$-subgroup of G is also a cyclic $\mathbb{Q}[x]$-group.*

Proof. The key to proving this theorem is the fact that the division algorithm for $\mathbb{Q}[x]$ holds, since $\mathbb{Q}$ is a field. The proof follows the same way as in the ordinary group case. $\square$

Lemma 2.3.1. *If G is a nilpotent $\mathbb{Q}[x]$-powered group then Γ_2 is the smallest normal $\mathbb{Q}[x]$-subgroup for which G/Γ_2 is an abelian $\mathbb{Q}[x]$-group.*

Proof. Let $N \trianglelefteq_{\mathbb{Q}[x]} G$ be any normal $\mathbb{Q}[x]$-subgroup of G such that G/N is an abelian $\mathbb{Q}[x]$-group. Then $gNhN = hNgN$ for arbitrary $g, h \in G$. Therefore, $g^{-1}h^{-1}gh \in N$ and so $[g, h] \in N$. This implies that $N \geq_{\mathbb{Q}[x]} \Gamma_2$ since $[g, h] \in [G, G] = \Gamma_2$. The result now follows. $\square$

Lemma 2.3.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group and suppose that $G/Z(G)$ is a cyclic $\mathbb{Q}[x]$-group. Then G is an abelian $\mathbb{Q}[x]$-group.*

Proof. Let G and $G/Z(G)$ be as above. If $G = Z(G)$ then we are done. Suppose that $G \neq Z(G)$. Then there exists $g \in G$ such that $g \notin Z(G)$ and $G/Z(G) = gp_{\mathbb{Q}[x]}(gZ(G))$. Hence every $h \in G$ has the form $h = g^{\lambda}z$, where $\lambda \in \mathbb{Q}[x]$ and $z \in Z(G)$. Then, for any $z_1, z_2 \in Z(G)$, we have

$$g^{\lambda_1}z_1g^{\lambda_2}z_2 = g^{\lambda_2}z_2g^{\lambda_1}z_1.$$

Therefore, G is an abelian $\mathbb{Q}[x]$-group. $\square$

Theorem 2.3.2. *Let G be a finitely $\mathbb{Q}[x]$-generated abelian $\mathbb{Q}[x]$-group and suppose $H \leq_{\mathbb{Q}[x]} G$. Then H is also a finitely $\mathbb{Q}[x]$-generated abelian $\mathbb{Q}[x]$-group.*

Proof. Let G be a finitely $\mathbb{Q}[x]$-generated abelian $\mathbb{Q}[x]$-group with $H \leq_{\mathbb{Q}[x]} G$. Then G can be viewed as a finitely generated $\mathbb{Q}[x]$-module. Since $\mathbb{Q}[x]$ is a noetherian ring, every submodule of a finitely generated $\mathbb{Q}[x]$-module is itself finitely generated. In particular, H (viewed as a submodule of G) is finitely generated. Therefore H is a finitely $\mathbb{Q}[x]$-generated abelian $\mathbb{Q}[x]$-subgroup of G. □

2.4 Direct Products

Let $\{G_i | \ i \in I\}$ be a family of nilpotent $\mathbb{Q}[x]$-powered groups indexed by a non-empty set I. Define the set $\bar{G} = \{f : I \to \bigcup_{i \in I} G_i \mid f(i) \in G_i \text{ for all } i \in I\}$. Suppose that there is a bound on the set of classes of the family of groups $G_i \, (i \in I)$. Then $\bar{G}$ becomes a nilpotent $\mathbb{Q}[x]$-powered group on defining multiplication and $\mathbb{Q}[x]$-exponentiation as follows:

1. $(ff')(i) = f(i)f'(i)$, where $f, f' \in \bar{G}$ and $i \in I$

2. $f^{\lambda}(i) = (f(i))^{\lambda}$ for all $\lambda \in \mathbb{Q}[x]$

An element $g \in \bar{G}$ can be viewed as a "vector" $g = (g_1, \ldots, g_i, \ldots)$ whose i^{th} coordinate is $g_i = f(i) \in G_i$ for all $i \in I$. By viewing the elements of $\bar{G}$ in this way, the group operations become

1. $(g_1, \ldots, g_i, \ldots)(h_1, \ldots, h_i, \ldots) = (g_1h_1, \ldots, g_ih_i, \ldots)$ for all $g_i, h_i \in G_i$

2. $(g_1, \ldots, g_i, \ldots)^\lambda = (g_1^\lambda, \ldots, g_i^\lambda, \ldots)$ for all $g_i \in G_i$ and for all $\lambda \in \mathbb{Q}[x]$

Definition 2.4.1. The nilpotent $\mathbb{Q}[x]$-powered group $\bar{G}$ described above is called the *unrestricted direct product* of the $\{G_i\}$'s, denoted by

$$\bar{G} = \overline{\prod_{i \in I}} G_i.$$

Definition 2.4.2. Let $\{G_i |\ i \in I\}$ be a family of nilpotent $\mathbb{Q}[x]$-powered groups indexed by a non-empty set I. Then a group G is termed a *direct product* of the G_i's, which will be denoted by

$$G = \prod_{i \in I} G_i,$$

if there exists $\mathbb{Q}[x]$-monomorphisms $\varphi_i : G_i \to G$ such that

1. $\varphi_i(G_i) \trianglelefteq_{\mathbb{Q}[x]} G$

2. $G = gp_{\mathbb{Q}[x]}(\bigcup_{i \in I} \varphi_i(G_i))$

3. $\varphi_i(G_i) \cap gp_{\mathbb{Q}[x]}(\bigcup_{j \neq i} \varphi_j(G_j)) = 1$ for $i, j \in I$ and $1 \in G$ denoting the identity element.

As usual, if I is a finite index set $I = \{1, 2, \ldots, n\}$, then the direct product of the $\{G_i\}$'s is written as $\prod_{i \in I} G_i = G_1 \times \cdots \times G_n$.

Note. Given a family of nilpotent $\mathbb{Q}[x]$-powered groups $\{G_i | \ i \in I\}$ for which there is a bound on the set of classes of this family, if we set

$$G = \left\{ f \in \overline{\prod_{i \in I}} G_i \mid f(i) = 1 \text{ except for finitely many } i \in I \right\},$$

then $G = \prod_{i \in I} G_i$ and G is a nilpotent $\mathbb{Q}[x]$-powered group.

Theorem 2.4.1. *Every finitely $\mathbb{Q}[x]$-generated abelian $\mathbb{Q}[x]$-group is a direct product of cyclic $\mathbb{Q}[x]$-groups.*

Proof. Let G be a finitely $\mathbb{Q}[x]$-generated abelian $\mathbb{Q}[x]$-group. Then G can be interpreted as a finitely generated $\mathbb{Q}[x]$-module. Since $\mathbb{Q}[x]$ is a principal ideal domain, G is a direct sum of cyclic $\mathbb{Q}[x]$-modules. However, the notion of the direct sum of cyclic $\mathbb{Q}[x]$-modules corresponds to the notion of the direct product of cyclic $\mathbb{Q}[x]$-groups in the obvious way. The result now follows. □

2.5 Results Obtained From The Upper And Lower Central Series

In this section I will prove some results which are obtained with the help of the upper and lower central series.

Lemma 2.5.1 (The Three $\mathbb{Q}[x]$-Subgroup Lemma). *Let H, K and L be $\mathbb{Q}[x]$-subgroups of a nilpotent $\mathbb{Q}[x]$-powered group G. If any two of the $\mathbb{Q}[x]$-subgroups*

$[H,K,L]_{\mathbb{Q}[x]}$, $[K,L,H]_{\mathbb{Q}[x]}$, $[L,H,K]_{\mathbb{Q}[x]}$ *are contained in a normal* $\mathbb{Q}[x]$*-subgroup of* G*, then so is the third.*

Proof. Just like in Lemma 1.2.4, we may apply the Hall-Witt identity (Lemma 1.1.1) to obtain the result. □

Lemma 2.5.2. *Let* G *be a nilpotent* $\mathbb{Q}[x]$*-powered group. Then* $[\Gamma_i, \Gamma_j]_{\mathbb{Q}[x]} \leq \Gamma_{i+j}$.

Proof. Since each Γ_k is a normal $\mathbb{Q}[x]$-subgroup of G, the result follows from Lemma 2.5.1 and induction. □

The abelianization of a nilpotent $\mathbb{Q}[x]$-powered group gives us information about the group itself.

Theorem 2.5.1. *Let* G *be a nilpotent* $\mathbb{Q}[x]$*-powered group. Then for every integer* $n > 1$*, the mapping*

$$\Omega_n : \Gamma_{n-1}/\Gamma_n \times Ab(G) \to \Gamma_n/\Gamma_{n+1}$$

defined by

$$\Omega_n(g\Gamma_n, h\Gamma_2) = [g,h]\Gamma_{n+1}$$

is multiplicative and $\mathbb{Q}[x]$*-exponential in each variable (hence* Ω_n*, when restricted to each component, is a* $\mathbb{Q}[x]$*-homomorphism). Consequently, there is a well-defined module epimorphism*

$$\Psi_n : \Gamma_{n-1}/\Gamma_n \bigotimes Ab(G) \to \Gamma_n/\Gamma_{n+1}$$

$$\Psi_n(g\Gamma_n \otimes h\Gamma_2) = [g,h]\Gamma_{n+1}$$

between $\mathbb{Q}[x]$*-modules for every integer* $n > 1$.

Proof. We know, by recalling the proof of Theorem 1.2.3 and using Lemma 2.5.2, that each map Ω_n is well-defined and is multiplicative in each variable. We need to prove that Ω_n is $\mathbb{Q}[x]$-exponential in each variable.

1. Let $g \in \Gamma_{n-1}$ and $h \in G$. I claim that $[g, h^\alpha] \equiv [g,h]^\alpha \mod \Gamma_{n+1}$ for all $\alpha \in \mathbb{Q}[x]$.

   By the Hall-Petresco axiom, we know that

   $$[g,h^\alpha] = [g,h]^\alpha \bar{\tau}_2^{\binom{\alpha}{2}} \cdots \bar{\tau}_k^{\binom{\alpha}{k}},$$

   where $\bar{\tau}_j = \tau_j(g^{-1}h^{-1}g, h)$ and k is the class of $gp\,(g^{-1}h^{-1}g, h)$. I claim that $\bar{\tau}_i \in \Gamma_{n+1}$ for each $i = 2, \ldots, k$. The proof is by induction on i :

   If $i = 2$, we have:

   $$\begin{aligned} [g,h^2] &= [g,h]^2\bar{\tau}_2 \Rightarrow \\ \bar{\tau}_2 &= [g,h]^{-2}[g,h^2] \\ &= [g,h]^{-2}[g,h][g,h]^h \\ &= [g,h]^{-2}[g,h][g,h][g,h,h] \\ &= [g,h,h] \in \Gamma_{n+1}, \text{ since } g \in \Gamma_{n-1} \text{ and } h \in G. \end{aligned}$$

   Suppose $\bar{\tau}_{j-1} \in \Gamma_{n+1}$ for $j = 3, 4, \ldots, i$. Observe that

   $$\bar{\tau}_i = \bar{\tau}_{i-1}^{-\binom{i}{i-1}} \bar{\tau}_{i-2}^{-\binom{i}{i-2}} \cdots \bar{\tau}_2^{-\binom{i}{2}} [g,h]^{-i}[g,h^i].$$

It is easy to verify (using Lemma 1.1.1 and induction) that $[g, h^i] = [g, h]^i d$ for all $i \geq 2$, where $d \in \Gamma_{n+1}$. Therefore, by the induction hypothesis,

$$\begin{aligned} \bar{\tau}_i &= \bar{\tau}_{i-1}^{-\binom{i}{i-1}} \bar{\tau}_{i-2}^{-\binom{i}{i-2}} \cdots \bar{\tau}_2^{-\binom{i}{2}} [g, h]^{-i} [g, h]^i d \\ &= \bar{\tau}_{i-1}^{-\binom{i}{i-1}} \bar{\tau}_{i-2}^{-\binom{i}{i-2}} \cdots \bar{\tau}_2^{-\binom{i}{2}} d \in \Gamma_{n+1}. \end{aligned}$$

Hence, for all $\alpha \in \mathbb{Q}[x]$, we have

$$[g, h^\alpha] = [g, h]^\alpha \bar{\tau}_2^{\binom{\alpha}{2}} \cdots \bar{\tau}_c^{\binom{\alpha}{c}} \equiv [g, h]^\alpha \mod \Gamma_{n+1}.$$

2. Verifying that $[g^\alpha, h] \equiv [g, h]^\alpha \mod \Gamma_{n+1}$ is the same.

Now, since Γ_{n-1}/Γ_n and $Ab(G)$ are both abelian $\mathbb{Q}[x]$-groups, we can view them as $\mathbb{Q}[x]$-modules. Furthermore, since Ω_n is multiplicative and $\mathbb{Q}[x]$-exponential in each variable and is clearly a $\mathbb{Q}[x]$-epimorphism, it induces an epimorphism between modules:

$$\Psi_n : \Gamma_{n-1}/\Gamma_n \bigotimes Ab(G) \to \Gamma_n/\Gamma_{n+1}$$

defined by $\Psi_n(g\Gamma_n \otimes h\Gamma_2) = [g, h]\Gamma_{n+1}$. This completes the proof. □

Corollary 2.5.2. *Let G be any nilpotent $\mathbb{Q}[x]$-powered group. For each integer $i > 0$, there is a $\mathbb{Q}[x]$-epimorphism*

$$\Psi_i : \underbrace{Ab(G) \bigotimes \cdots \bigotimes Ab(G)}_{i \text{ of these}} \to \Gamma_i/\Gamma_{i+1}$$

between $\mathbb{Q}[x]$-modules defined by

$$\Psi_n(\underbrace{g_1\Gamma_2 \otimes \cdots \otimes g_i\Gamma_2}_{i \text{ of these}}) = [g_1, \ldots, g_i]\Gamma_{i+1}.$$

Proof. As in Corollary 1.2.4, we may iterate the above result to obtain our result. □

Corollary 2.5.3. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group with $\mathbb{Q}[x]$-generators $X = \{x_1, \ldots, x_p\}$. Then $\gamma_n G/\gamma_{n+1}G$ is finitely $\mathbb{Q}[x]$-generated by the set $\{[x_{i_1}, \ldots, x_{i_n}]\gamma_{n+1}G\}$ where the x_{i_j}'s range over X.*

Proof. Since G is a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group, so is $Ab(G)$. By Corollary 2.5.2, we see that Γ_n/Γ_{n+1} is $\mathbb{Q}[x]$-generated by elements of the form $[y_1, \ldots, y_n]\Gamma_{n+1}$, where $y_i \in G$. But each y_i is expressible as a product of $\mathbb{Q}[x]$-powers in the elements of $\{x_1, \ldots, x_p\}$. By using the Hall-Petresco axiom and Lemmas 1.1.1 and 2.5.2, we obtain the result. □

Corollary 2.5.4. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c and let $N < G$ such that $G = gp_{\mathbb{Q}[x]}(N)$. Then N also has class c.*

Proof. I will supply the proof given by R. B. Warfield, Jr. [14], where he proves it for an arbitrary binomial ring. As usual, let Γ_i denote the subgroups of the lower central series of G. Let $\pi : G \to Ab(G)$ be the natural $\mathbb{Q}[x]$-epimorphism. Observe that, since N is a set of $\mathbb{Q}[x]$-generators for G, $\pi(N)$ is a set of $\mathbb{Q}[x]$-generators for

$Ab(G)$. By applying the $\mathbb{Q}[x]$-epimorphism

$$\Psi_i : \underbrace{Ab(G) \bigotimes \cdots \bigotimes Ab(G)}_{i \text{ of these}} \to \Gamma_i/\Gamma_{i+1},$$

we see that the images of n-fold commutators of elements of N will $\mathbb{Q}[x]$-generate Γ_n/Γ_{n+1}. Applying this result to $\Gamma_c = \Gamma_c/\Gamma_{c+1}$, we see that N has nontrivial c-fold commutators, so N has class c. □

Theorem 2.5.5. *Every $\mathbb{Q}[x]$-subgroup of a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group is finitely $\mathbb{Q}[x]$-generated.*

Proof. The case for nilpotency class 1 was proven in Theorem 2.3.2. The rest of the proof resembles that of Theorem 1.2.7 using Theorem 2.2.2 and Corollary 2.5.3. □

In R. B. Warfield, Jr. [14] it is stated that if R is a binomial ring then a finitely R-generated nilpotent R-powered group has a series of a special kind. I will prove this result next for the particular case of $R = \mathbb{Q}[x]$. First,

Definition 2.5.1. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and let P be a property of such groups. Then a $\mathbb{Q}[x]$-series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_{n+1} = G$$

for G is a *poly-$\mathbb{Q}[x]$ P series* if

1. $G_i \trianglelefteq_{\mathbb{Q}[x]} G_{i+1}$ for each $0 \leq i \leq n$ and

2. G_{i+1}/G_i is a P $\mathbb{Q}[x]$-group for each $0 \leq i \leq n$.

Theorem 2.5.6. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group. Then G is finitely $\mathbb{Q}[x]$-generated if and only if it has a central $\mathbb{Q}[x]$-series*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n+1} = G$$

such that G_{i+1}/G_i is a cyclic $\mathbb{Q}[x]$-group for each $0 \leq i \leq n$. In other words, G has a central $\mathbb{Q}[x]$-series which is poly-$\mathbb{Q}[x]$ cyclic.

Proof. Let G be finitely $\mathbb{Q}[x]$-generated. We know that each of the $\mathbb{Q}[x]$-subgroups of the upper central series, $\zeta_s G$, is finitely $\mathbb{Q}[x]$-generated by Theorem 2.5.5. Thus, we have that each factor $\mathbb{Q}[x]$-group $\zeta_{s+1}G/\zeta_s G$ is finitely $\mathbb{Q}[x]$-generated. Since each factor $\mathbb{Q}[x]$-group $\zeta_{s+1}G/\zeta_s G$ is abelian, we can express each as the direct product of a finite number of cyclic $\mathbb{Q}[x]$-groups by Theorem 2.4.1. Hence, refining the upper central series yields the result. The converse is trivial. □

Definition 2.5.2. Let R be any binomial ring and let G be a finitely $R-$generated nilpotent R-powered group. Then the *Hirsch R-length* of G is the minimal length of all poly-R cyclic and central R-series for G.

Corollary 2.5.7. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group and let $H \leq_{\mathbb{Q}[x]} G$. Then H has a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series.*

Proof. Follows from Theorem 2.5.6 and Theorem 2.5.5. □

2.6 Uniqueness Of $\mathbb{Q}[x]$-Completions

Given a finitely generated torsion free nilpotent group, G, there are many different Mal'cev bases one can choose to represent an element in normal form. Therefore there are many different ways to form the $\mathbb{Q}[x]$-completion of G. I will show in this section that all such $\mathbb{Q}[x]$-completions are $\mathbb{Q}[x]$-isomorphic to each other.

Theorem 2.6.1. *Suppose G and $\bar{G}$ are finitely generated torsion free nilpotent groups of class c and that $\varphi : G \to \bar{G}$ is a homomorphism. Suppose that $\bar{G}$ is contained in some nilpotent $\mathbb{Q}[x]$-powered group of class c, say D. Then for every choice of a poly-infinite cyclic and central series for G there exists a $\mathbb{Q}[x]$-homomorphism $\Phi : G^{\mathbb{Q}[x]} \to D$ which extends φ, where $G^{\mathbb{Q}[x]}$ is the $\mathbb{Q}[x]$-completion of G with respect to the given series.*

Proof. Let

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

be any poly-infinite cyclic and central series for G (such a series exists by Theorem 1.10.1). Let $G_{i+1} = gp(u_{i+1}, G_i)$ for $i = 0, \ldots, n-1$. Then the collection $\bar{u} = (u_1, \ldots, u_n)$ is a Mal'cev basis for G with respect to the given series. Hence, any $u \in G$ has the unique form $u = u_1^{\alpha_1} \cdots u_n^{\alpha_n} = \bar{u}^{\bar{\alpha}}$. Let $(f_1(\bar{\alpha}, \bar{\beta}), \ldots, f_n(\bar{\alpha}, \bar{\beta}))$ and

$(g_1(\bar{\alpha}, \lambda), \ldots, g_n(\bar{\alpha}, \lambda))$ be the multiplication and exponentiation polynomials, respectively. Hence, if $u = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ and $v = u_1^{\beta_1} \cdots u_n^{\beta_n}$, then we have the following:

$$\begin{aligned} uv &= (u_1^{\alpha_1} \cdots u_n^{\alpha_n})(u_1^{\beta_1} \cdots u_n^{\beta_n}) \\ &= u_1^{f_1(\bar{\alpha},\bar{\beta})} \cdots u_n^{f_n(\bar{\alpha},\bar{\beta})} \text{ and} \\ u^\lambda &= (u_1^{\alpha_1} \cdots u_n^{\alpha_n})^\lambda \\ &= u_1^{g_1(\bar{\alpha},\lambda)} \cdots u_n^{g_n(\bar{\alpha},\lambda)}. \end{aligned}$$

Now let's consider the homomorphism $\varphi : G \to \bar{G}$. Put $w_i = \varphi(u_i)$. If $u = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ and $v = u_1^{\beta_1} \cdots u_n^{\beta_n}$ as above, then

$$\begin{aligned} \varphi(u) &= w_1^{\alpha_1} \cdots w_n^{\alpha_n} \text{ and} \\ \varphi(v) &= w_1^{\beta_1} \cdots w_n^{\beta_n}. \end{aligned}$$

Observe that

$$\varphi(u)\varphi(v) = (w_1^{\alpha_1} \cdots w_n^{\alpha_n})(w_1^{\beta_1} \cdots w_n^{\beta_n})$$

and

$$\varphi(uv) = w_1^{f_1(\bar{\alpha},\bar{\beta})} \cdots w_n^{f_n(\bar{\alpha},\bar{\beta})}.$$

Hence, since φ is a homomorphism, we have

$$(w_1^{\alpha_1} \cdots w_n^{\alpha_n})(w_1^{\beta_1} \cdots w_n^{\beta_n}) = w_1^{f_1(\bar{\alpha},\bar{\beta})} \cdots w_n^{f_n(\bar{\alpha},\bar{\beta})}.$$

Moreover,

$$[\varphi(u)]^{\lambda} = [w_1^{\alpha_1} \cdots w_n^{\alpha_n}]^{\lambda}$$

and

$$\varphi(u^{\lambda}) = w_1^{g_1(\bar{\alpha},\lambda)} \cdots w_n^{g_n(\bar{\alpha},\lambda)}$$

yields the equality

$$[w_1^{\alpha_1} \cdots w_n^{\alpha_n}]^{\lambda} = w_1^{g_1(\bar{\alpha},\lambda)} \cdots w_n^{g_n(\bar{\alpha},\lambda)}.$$

From the above observation, it follows that in $\varphi(G)$ (the image of φ), elements multiply and exponentiate according to the polynomials for G. Now, since G has a Mal'cev basis $\bar{u} = \{u_1, \ldots, u_n\}$ with respect to the given poly-infinite cyclic and central series, the elements of $G^{\mathbb{Q}[x]}$ are of the form $u_1^{\alpha_1(x)} \cdots u_n^{\alpha_n(x)}$.

Consider the map

$$\Phi : G^{\mathbb{Q}[x]} \to D$$

defined by

$$\Phi(u_1^{\alpha_1(x)} \cdots u_n^{\alpha_n(x)}) = w_1^{\alpha_1(x)} \cdots w_n^{\alpha_n(x)}.$$

I claim that Φ is a $\mathbb{Q}[x]$-homomorphism extending φ. Suppose $u = u_1^{\alpha_1(x)} \cdots u_n^{\alpha_n(x)}$ and $v = u_1^{\beta_1(x)} \cdots u_n^{\beta_n(x)}$ are elements of $G^{\mathbb{Q}[x]}$. I will prove the following identities:

$$(w_1^{\alpha_1(x)} \cdots w_n^{\alpha_n(x)})(w_1^{\beta_1(x)} \cdots w_n^{\beta_n(x)}) = w_1^{f_1(\bar{\alpha}(x),\bar{\beta}(x))} \cdots w_n^{f_n(\bar{\alpha}(x),\bar{\beta}(x))}$$

and

$$(w_1^{\alpha_1(x)} \cdots w_n^{\alpha_n(x)})^{\lambda(x)} = w_1^{g_1(\bar{\alpha}(x),\lambda(x))} \cdots w_n^{g_n(\bar{\alpha}(x),\lambda(x))}.$$

Once the above are proven, we will have

$$
\begin{aligned}
\Phi(uv) &= \Phi[(u_1^{\alpha_1(x)} \cdots u_n^{\alpha_n(x)})(u_1^{\beta_1(x)} \cdots u_n^{\beta_n(x)})] \\
&= \Phi(u_1^{f_1(\bar{\alpha}(x),\bar{\beta}(x))} \cdots u_n^{f_n(\bar{\alpha}(x),\bar{\beta}(x))}) \\
&= w_1^{f_1(\bar{\alpha}(x),\bar{\beta}(x))} \cdots w_n^{f_n(\bar{\alpha}(x),\bar{\beta}(x))} \\
&= (w_1^{\alpha_1(x)} \cdots w_n^{\alpha_n(x)})(w_1^{\beta_1(x)} \cdots w_n^{\beta_n(x)}) \\
&= \Phi(u)\Phi(v) \text{ and} \\
\Phi(u^{\lambda(x)}) &= \Phi[(u_1^{\alpha_1(x)} \cdots u_n^{\alpha_n(x)})^{\lambda(x)}] \\
&= \Phi(u_1^{g_1(\bar{\alpha}(x),\lambda(x))} \cdots u_n^{g_n(\bar{\alpha}(x),\lambda(x))}) \\
&= w_1^{g_1(\bar{\alpha}(x),\lambda(x))} \cdots w_n^{g_n(\bar{\alpha}(x),\lambda(x))} \\
&= [w_1^{\alpha_1(x)} \cdots w_n^{\alpha_n(x)}]^{\lambda(x)} \\
&= [\Phi(u)]^{\lambda(x)}
\end{aligned}
$$

and the proof is complete.

The proof is by induction on the Hirsch $\mathbb{Q}[x]$-length of $G^{\mathbb{Q}[x]}$.

1. Suppose the Hirsch $\mathbb{Q}[x]$-length of $G^{\mathbb{Q}[x]}$ is $n = 1$. Then $G^{\mathbb{Q}[x]}$ is a cyclic $\mathbb{Q}[x]$-group and the result is obvious.

2. Suppose now that Φ is a $\mathbb{Q}[x]$-homomorphism for those $G^{\mathbb{Q}[x]}$ such that the Hirsch $\mathbb{Q}[x]$-length $1 \leq i < n$. I will prove it for n. For simplicity, let's abbreviate $\alpha_i(x) = \alpha_i$, $\beta_i(x) = \beta_i$, $\mu_i(x) = \mu_i$ and $\lambda(x) = \lambda$.

(a) I will first prove that

$$(w_1^{\alpha_1}\cdots w_n^{\alpha_n})(w_1^{\beta_1}\cdots w_n^{\beta_n}) = w_1^{f_1(\bar{\alpha},\bar{\beta})}\cdots w_n^{f_n(\bar{\alpha},\bar{\beta})}.$$

Observe that

$$(w_1^{\alpha_1}\cdots w_n^{\alpha_n})(w_1^{\beta_1}\cdots w_n^{\beta_n}) = w_1^{\alpha_1+\beta_1}\prod_{i=2}^{n}(w_1^{-\beta_1}w_i^{-1}w_1^{\beta_1})^{-\alpha_i}w_2^{\beta_2}\cdots w_n^{\beta_n}$$

(note that $(w_1^{-\beta_1}w_i^{-1}w_1^{\beta_1})^{-\alpha_i} = w_1^{-\beta_1}w_i^{\alpha_i}w_1^{\beta_1}$ by axiom 2). Now,

$$w_1^{-\beta_1}w_i^{-1}w_1^{\beta_1} = w_1^{-\beta_1}(w_i^{-1}w_1w_i)^{\beta_1}w_i^{-1},$$

again by axiom 2. Furthermore,

$$w_i^{-1}w_1w_i = w_1w_{i+1}^{r_{i,1}}\cdots w_n^{r_{i,n-i}}$$

for some $r_{i,j}$'s in $\mathbb{Z}$, since $u_i^{-1}u_1u_i = u_1u_{i+1}^{r_{i,1}}\cdots u_n^{r_{i,n-i}}$ in G, and so $\varphi(u_i^{-1}u_1u_i) = \varphi(u_1u_{i+1}^{r_{i,1}}\cdots u_n^{r_{i,n-i}})$, giving us

$$\varphi(u_i)^{-1}\varphi(u_1)\varphi(u_i) = \varphi(u_1)\varphi(u_{i+1})^{r_{i,1}}\cdots\varphi(u_n)^{r_{i,n-i}}.$$

Observe that the subgroup $H_i = gp\,(u_1, u_{i+1}, \ldots, u_n)$, $i > 1$, has a Mal'cev basis $\{u_1, u_{i+1}, \ldots, u_n\}$, hence the Hirsch $\mathbb{Q}[x]$-length of $H_i^{\mathbb{Q}[x]}$ is less than

n for each $1 \leq i < n$. Now,

$$\begin{aligned}
(w_i^{-1} w_1 w_i)^{\beta_1} &= (w_1 w_{i+1}^{r_{i,1}} \cdots w_n^{r_{i,n-i}})^{\beta_1} \\
&= [\Phi(u_1 u_{i+1}^{r_{i,1}} \cdots u_n^{r_{i,n-i}})]^{\beta_1} \\
&= \Phi[(u_1 u_{i+1}^{r_{i,1}} \cdots u_n^{r_{i,n-i}})^{\beta_1}] \quad \text{(by induction)} \\
&= \Phi(u_1^{\beta_1} u_{i+1}^{g_{i+1}(r_{i,j},\beta_1)} \cdots u_n^{g_n(r_{i,j},\beta_1)}) \\
&= w_1^{\beta_1} w_{i+1}^{\tilde{g}_{i+1}} \cdots w_n^{\tilde{g}_n}
\end{aligned}$$

where the expressions $\{g_1, \ldots, g_n\}$ are obtained using the polynomials for G and $\tilde{g}_k = g_k(r_{i,j}, \beta_1)$.

Hence,

$$\begin{aligned}
w_1^{-\beta_1} w_i^{-1} w_1^{\beta_1} &= w_1^{-\beta_1} (w_i^{-1} w_1 w_i)^{\beta_1} w_i^{-1} \\
&= w_1^{-\beta_1} (w_1^{\beta_1} w_{i+1}^{\tilde{g}_{i+1}} \cdots w_n^{\tilde{g}_n}) w_i^{-1} \\
&= w_{i+1}^{\tilde{g}_{i+1}} \cdots w_n^{\tilde{g}_n} w_i^{-1} \\
&= \Phi(u_{i+1}^{\tilde{g}_{i+1}} \cdots u_n^{\tilde{g}_n}) \Phi(u_i^{-1}) \\
&= \Phi(u_{i+1}^{\tilde{g}_{i+1}} \cdots u_n^{\tilde{g}_n} u_i^{-1}) \quad \text{(by induction)} \\
&= \Phi(u_i^{-1} u_{i+1}^{\tilde{f}_{i+1}} \cdots u_n^{\tilde{f}_n}) \\
&= w_i^{-1} w_{i+1}^{\tilde{f}_{i+1}} \cdots w_n^{\tilde{f}_n}
\end{aligned}$$

where the expressions $\{\tilde{f}_1, \ldots, \tilde{f}_n\}$ are obtained using the polynomials for $G^{\mathbb{Q}[x]}$.

Therefore,

$$\begin{aligned}
(w_1^{-\beta_1} w_i^{-1} w_1^{\beta_1})^{-\alpha_i} &= (w_i^{-1} w_{i+1}^{\tilde{f}_{i+1}} \cdots w_n^{\tilde{f}_n})^{-\alpha_i} \\
&= [\Phi(u_i^{-1} u_{i+1}^{\tilde{f}_{i+1}} \cdots u_n^{\tilde{f}_n})]^{-\alpha_i} \\
&= \Phi[(u_i^{-1} u_{i+1}^{\tilde{f}_{i+1}} \cdots u_n^{\tilde{f}_n})^{-\alpha_i}] \quad \text{(by induction)} \\
&= \Phi(u_i^{\alpha_i} u_{i+1}^{\hat{g}_{i+1}} \cdots u_n^{\hat{g}_n}) \\
&= w_i^{\alpha_i} w_{i+1}^{\hat{g}_{i+1}} \cdots w_n^{\hat{g}_n},
\end{aligned}$$

where the expressions $\{\hat{g}_1, \ldots, \hat{g}_n\}$ are again obtained by the polynomials for $G^{\mathbb{Q}[x]}$. Note that $\hat{g}_k = g_k(\tilde{f}, \alpha_i)$. As a result, we obtain

$$w_1^{\alpha_1} \cdots w_n^{\alpha_n} w_1^{\beta_1} \cdots w_n^{\beta_n} = w_1^{\alpha_1+\beta_1} \prod_{i=2}^{n} (w_i^{\alpha_i} w_{i+1}^{\hat{g}_{i+1}} \cdots w_n^{\hat{g}_n}) w_2^{\beta_2} \cdots w_n^{\beta_n}.$$

Now, if we continue this procedure of collecting to the left and use the prescribed polynomials, we obtain a parallel method to that of the non-$\mathbb{Q}[x]$-case using the same polynomials. Hence

$$(w_1^{\alpha_1} \cdots w_n^{\alpha_n})(w_1^{\beta_1} \cdots w_n^{\beta_n}) = w_1^{f_1(\bar{\alpha}, \bar{\beta})} \cdots w_n^{f_n(\bar{\alpha}, \bar{\beta})}.$$

(b) I will now prove that $(w_1^{\alpha_1} \cdots w_n^{\alpha_n})^{\lambda} = w_1^{g_1(\bar{\alpha}, \lambda)} \cdots w_n^{g_n(\bar{\alpha}, \lambda)}$.

Let $\tau_i = \tau_i(w_1^{\mu_1}, \ldots, w_n^{\mu_n})$ be the Hall-Petresco words with arguments $\{w_1^{\mu_1}, \ldots, w_n^{\mu_n}\}$. It is clear that each τ_i can be expressed as a product of the $w_i^{\mu_i}$'s. Therefore, by the result above, we have

$$\begin{aligned}
\tau_i(w_1^{\mu_1}, \ldots, w_n^{\mu_n}) &= \Phi(\tau_i(u_1^{\mu_1}, \ldots, u_n^{\mu_n})) \\
&= \Phi(u_1^{\sigma_{i,1}} \cdots u_n^{\sigma_{i,n}}) \\
&= w_1^{\sigma_{i,1}} \cdots w_n^{\sigma_{i,n}},
\end{aligned}$$

where the $\sigma_{i,n}$'s are obtained using the polynomials as before. Furthermore, since the class of the nilpotent $\mathbb{Q}[x]$-powered group, D equals c, we have $\Gamma_{c+1}(D) = 1$. Hence,

$$\tau_{c+1} = \tau_{c+2} = \cdots = 1$$

since $\tau_i \in \Gamma_i(D)$ for all i. But an easy calculation shows that $\tau_1 = w_1^{\mu_1} \cdots w_n^{\mu_n}$ and so the Hall-Petresco axiom gives

$$(w_1^{\mu_1} \cdots w_n^{\mu_n})^\lambda = w_1^{\lambda\mu_1} \cdots w_n^{\lambda\mu_n} \tau_c^{-\binom{\lambda}{c}} \tau_{c-1}^{-\binom{\lambda}{c-1}} \cdots \tau_2^{-\binom{\lambda}{2}}.$$

Now, since all of the τ_k lie in $\Gamma_2(D)$ for $k \geq 2$, we have

$$\begin{aligned}
\tau_k^{-\binom{\lambda}{k}} &= (w_1^{\sigma_{k,1}} \cdots w_n^{\sigma_{k,n}})^{-\binom{\lambda}{k}} \\
&= [\Phi(u_1^{\sigma_{k,1}} \cdots u_n^{\sigma_{k,n}})]^{-\binom{\lambda}{k}} \\
&= [\Phi(u_2^{\xi_{k,2}} \cdots u_n^{\xi_{k,n}})]^{-\binom{\lambda}{k}} \\
&= \Phi[(u_2^{\xi_{k,2}} \cdots u_n^{\xi_{k,n}})^{-\binom{\lambda}{k}}] \text{ by induction} \\
&= \Phi(u_2^{\gamma_{k,2}} \cdots u_n^{\gamma_{k,n}}) \\
&= w_2^{\gamma_{k,2}} \cdots w_n^{\gamma_{k,n}}
\end{aligned}$$

where each $\xi_{k,j}$ comes about from the fact that every element $(u_1^{\sigma_{k,1}} \cdots u_n^{\sigma_{k,n}}) = \tau_i(u_1^{\mu_1}, \ldots, u_n^{\mu_n})$ in $\Gamma_2(G^{\mathbb{Q}[x]})$ can be expressed in the basis $\{u_2, \ldots, u_n\}$ and each $\gamma_{k,j} = g_i(\xi_{k,2}, \xi_{k,3}, \ldots, \xi_{k,n}, -\binom{\lambda}{k})$. Now substituting these $\tau_k^{-\binom{\lambda}{k}}$'s into the expression

$$(w_1^{\mu_1} \cdots w_n^{\mu_n})^\lambda = w_1^{\lambda\mu_1} \cdots w_n^{\lambda\mu_n} \tau_c^{-\binom{\lambda}{c}} \tau_{c-1}^{-\binom{\lambda}{c-1}} \cdots \tau_2^{-\binom{\lambda}{2}}$$

and using the multiplication polynomials, by induction we obtain $(w_1^{\alpha_1} \cdots w_n^{\alpha_n})^\lambda = w_1^{g_1(\bar{\alpha},\lambda)} \cdots w_n^{g_n(\bar{\alpha},\lambda)}$ in exactly the same way we obtain the result for G itself. This completes the proof.

□

For example, let G be a finitely generated torsion free nilpotent group of class c and let $\bar{u} = (u_1, \ldots, u_k)$ be any Mal'cev basis for G. It is known that there exists a faithful embedding $\psi : G \to UT_n(\mathbb{Z})$ for some $n > 0$. If $UT_n(\mathbb{Q}[x])$ is the group of unitriangular matrices with entries in $\mathbb{Q}[x]$, then we have a natural embedding $\sigma : UT_n(\mathbb{Z}) \hookrightarrow UT_n(\mathbb{Q}[x])$. This gives rise to an extension of ψ to the map $\Psi = \sigma \circ \psi$ such that $\Psi : G \to UT_n(\mathbb{Q}[x])$. Note that Ψ is clearly an embedding. As we have seen in the first chapter, $UT_n(\mathbb{Q}[x])$ can be turned into a nilpotent $\mathbb{Q}[x]$-powered group by defining the $\mathbb{Q}[x]$-action on $g = I + N \in UT_n(\mathbb{Q}[x])$ as

$$(I + N)^\alpha = 1 + \alpha N + \cdots + \binom{\alpha}{n-1} N^{n-1},$$

where $\alpha \in \mathbb{Q}[x]$. Therefore, by Theorem 2.6.1, $\Psi : G \to UT_n(\mathbb{Q}[x])$ can be extended to a $\mathbb{Q}[x]$-homomorphism $\Phi : G^{\mathbb{Q}[x]} \to UT_n(\mathbb{Q}[x])$ with respect to the Mal'cev basis u of G and, in fact, Φ is a $\mathbb{Q}[x]$-monomorphism.

A particular case of the above example is when $G = UT_n(\mathbb{Z})$. Let's choose the collection $\{t_{12}, \ldots, t_{1n}, \ldots, t_{n-11}, \ldots, t_{n-1n}\}$ of transvections as our Mal'cev basis. Then the $\mathbb{Q}[x]$-completion of $UT_n(\mathbb{Z})$ is the collection of elements

$$[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]} = \{t_{12}^{\alpha_{12}(x)} \cdots t_{n-1n}^{\alpha_{n-1n}(x)} \mid \alpha_{ij}(x) \in \mathbb{Q}[x]\}$$

and the polynomials for multiplication and $\mathbb{Q}[x]$-exponentiation are the same as those of $UT_n(\mathbb{Z})$. I claim that

$$[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} UT_n(\mathbb{Q}[x]),$$

where the $\mathbb{Q}[x]$-action on $UT_n(\mathbb{Q}[x])$ is defined above. Well, let $\varphi : UT_n(\mathbb{Z}) \to UT_n(\mathbb{Z})$ be the identity map $\varphi(t_{12}^{\alpha_{12}} \cdots t_{n-1n}^{\alpha_{n-1n}}) = t_{12}^{\alpha_{12}} \cdots t_{n-1n}^{\alpha_{n-1n}}$ where $\alpha_{ij} \in \mathbb{Z}$. Then φ extends to an embedding $\phi : UT_n(\mathbb{Z}) \to UT_n(\mathbb{Q}[x])$. By the above theorem, ϕ can be extended to a $\mathbb{Q}[x]$-homomorphism $\Phi : [UT_n(\mathbb{Z})]^{\mathbb{Q}[x]} \to UT_n(\mathbb{Q}[x])$ given by

$$\begin{aligned} \Phi(t_{12}^{\alpha_{12}(x)} \cdots t_{n-1n}^{\alpha_{n-1n}(x)}) &= \varphi(t_{12})^{\alpha_{12}(x)} \cdots \varphi(t_{n-1n})^{\alpha_{n-1n}(x)} \\ &= t_{12}^{\alpha_{12}(x)} \cdots t_{n-1n}^{\alpha_{n-1n}(x)} \end{aligned}$$

where the $\alpha_{ij}(x)$'s are in $\mathbb{Q}[x]$.

- Suppose $t_{12}^{\mu_{12}(x)} \cdots t_{n-1n}^{\mu_{n-1n}(x)} \in \ker \Phi$ with $\mu_{ij}(x) \in \mathbb{Q}[x]$. Then

$$\Phi(t_{12}^{\mu_{12}(x)} \cdots t_{n-1n}^{\mu_{n-1n}(x)}) = t_{12}^{\mu_{12}(x)} \cdots t_{n-1n}^{\mu_{n-1n}(x)} = 1.$$

Using the properties of transvections, it follows that this can only happen if each $\mu_{ij}(x) \equiv 0$. Hence the kernel of Φ is trivial and so Φ is a $\mathbb{Q}[x]$-monomorphism.

- Let $g \in UT_n(\mathbb{Q}[x])$. Then g can be written in the form $g = t_{12}^{\beta_{12}(x)} \cdots t_{n-1n}^{\beta_{n-1n}(x)}$ for certain $\beta_{ij}(x) \in \mathbb{Q}[x]$ where $t_{ij}^{\beta_{ij}(x)} = I + \beta_{ij}(x)E_{ij}$ as in the usual sense. Hence $g = \Phi(t_{12}^{\beta_{12}(x)} \cdots t_{n-1n}^{\beta_{n-1n}(x)})$ and so Φ is a $\mathbb{Q}[x]$-epimorphism and the claim is proven.

Corollary 2.6.2. *Let $G_1 = (G, \mathcal{B}_1)$ and $G_2 = (G, \mathcal{B}_2)$ denote the finitely generated torsion free nilpotent group, G, with distinct Mal'cev bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Let $G_1^{\mathbb{Q}[x]}$ and $G_2^{\mathbb{Q}[x]}$ be their $\mathbb{Q}[x]$-completions, respectively. Then $G_1^{\mathbb{Q}[x]}$ and $G_2^{\mathbb{Q}[x]}$ are $\mathbb{Q}[x]$-isomorphic.*

Proof. The identity map $\iota_1 : G_1 \to G_2 \leq G_2^{\mathbb{Q}[x]}$ gives rise to a $\mathbb{Q}[x]$-epimorphism

$$\Sigma_1 : G_1^{\mathbb{Q}[x]} \to G_2^{\mathbb{Q}[x]}$$

by Theorem 2.6.1. Similarly, $\iota_2 : G_2 \to G_1 \leq G_1^{\mathbb{Q}[x]}$ gives rise to a $\mathbb{Q}[x]$-epimorphism

$$\Sigma_2 : G_2^{\mathbb{Q}[x]} \to G_1^{\mathbb{Q}[x]}.$$

It is easy to see that Σ_1 and Σ_2 are each $\mathbb{Q}[x]$-monomorphic mappings. Therefore, $G_1^{\mathbb{Q}[x]}$ and $G_2^{\mathbb{Q}[x]}$ are $\mathbb{Q}[x]$-isomorphic. □

Lemma 2.6.1. *Suppose that G is a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group. Then there exists a finitely generated subgroup H of G such that $G \cong_{\mathbb{Q}[x]} H^{\mathbb{Q}[x]}$.*

Note: H is not a $\mathbb{Q}[x]$-subgroup of G here.

Proof. Let $G = gp_{\mathbb{Q}[x]}(a_1, \ldots, a_n)$ and set $H = gp\,(a_1, \ldots, a_n)$. I claim that $G \cong_{\mathbb{Q}[x]} H^{\mathbb{Q}[x]}$. Observe that H is an ordinary group with exponents in $\mathbb{Z}$, hence is torsion free as a subgroup of G (if $g^\alpha = 1$ for some $\alpha \in \mathbb{Z}$ with $g \in H$, then $(g^\alpha)^{1/\alpha} = 1^{1/\alpha} = 1$ and so $g = 1$ in H). Since H is a finitely generated torsion free nilpotent group, we can form its $\mathbb{Q}[x]$-completion with respect to any Mal'cev basis. By Corollary 2.6.2, all such $\mathbb{Q}[x]$-completions are $\mathbb{Q}[x]$-isomorphic. Thus we can write $H^{\mathbb{Q}[x]}$ as the $\mathbb{Q}[x]$-completion of H without any ambiguity as to which Mal'cev basis we are considering. It is clear that $G = gp_{\mathbb{Q}[x]}(H)$ since the generators for H are precisely the $\mathbb{Q}[x]$-generators for G. Moreover, $H^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} gp_{\mathbb{Q}[x]}(H)$. Therefore $H^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} G$. □

2.7 Torsion Nilpotent $\mathbb{Q}[x]$-Powered Groups

In this section I will give the definition of a torsion and torsion free nilpotent $\mathbb{Q}[x]$-powered group. The main result is that the set of torsion elements of a nilpotent $\mathbb{Q}[x]$-powered group form a normal $\mathbb{Q}[x]$-subgroup.

Definition 2.7.1. Let G be a nilpotent $\mathbb{Q}[x]$-powered group.

1. An element $g \in G$ is called a *torsion element* if there exists an $\alpha \in \mathbb{Q}[x]$, $\alpha \neq 0$, for which $g^\alpha = 1$. The set of torsion elements of G will be denoted as $\tau(G)$.

2. If $\tau(G) = G$ then G is called a *torsion nilpotent* $\mathbb{Q}[x]$*-powered group.*

3. G is called *torsion free* if $g^\alpha = 1$ for some $\alpha \in \mathbb{Q}[x]$ implies $g = 1$ or $\alpha = 0$.

Some lemmas are required before proving the main result of this section.

Lemma 2.7.1. *If G is a nilpotent $\mathbb{Q}[x]$-powered group and $H \leq_{\mathbb{Q}[x]} G$ such that $[G,G] \leq H$, then $H \trianglelefteq_{\mathbb{Q}[x]} G$.*

Lemma 2.7.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class $c \geq 2$ and let $g \in G$. Then $H = gp_{\mathbb{Q}[x]}(g, [G,G])$ is of class $< c$.*

Proof. Since $[G,G] \leq (\zeta_{c-1}G) \cap H \leq_{\mathbb{Q}[x]} \zeta_{c-1}H$, it follows by Theorem 2.3.1 that $H/\zeta_{c-1}H$ is a cyclic $\mathbb{Q}[x]$-group. By Lemma 2.3.2, we must have that $\zeta_{c-1}H = H$, and so H is of class $< c$ as desired. □

The next theorem is stated in R. B. Warfield, Jr. [14] for arbitrary nilpotent R-powered groups.

Theorem 2.7.1. *Let G be an arbitrary nilpotent $\mathbb{Q}[x]$-powered group of class c. Then $\tau(G) \trianglelefteq_{\mathbb{Q}[x]} G$.*

Proof. Let G be a nilpotent $\mathbb{Q}[x]$-powered group, of class c and suppose $a, b \in \tau(G)$ are non-identity elements. There exists $\alpha, \beta \in \mathbb{Q}[x]$, both non-zero, such that $a^\alpha = 1$ and $b^\beta = 1$. The proof is by induction on the class.

- If $c = 1$, then G is an abelian $\mathbb{Q}[x]$-group. We need to show that $ab \in \tau(G)$.

Well, observe that

$$
\begin{aligned}
(ab)^{\alpha\beta} &= a^{\alpha\beta}b^{\alpha\beta} \text{ (since G is abelian)} \\
&= (a^{\alpha})^{\beta}(b^{\beta})^{\alpha} = 1.
\end{aligned}
$$

Hence $ab \in \tau(G)$. Clearly, $a \in \tau(G)$ implies a^{-1} and $a^{\mu} \in \tau(G)$ for any $\mu \in \mathbb{Q}[x]$. Furthermore, since G is abelian, $\tau(G)$ is a normal $\mathbb{Q}[x]$-subgroup of G. Therefore, it is proven for $c = 1$.

- Suppose the theorem holds for class $< c$. Consider the $\mathbb{Q}[x]$-subgroups $A = gp_{\mathbb{Q}[x]}(a, [G,G])$ and $B = gp_{\mathbb{Q}[x]}(b, [G,G])$ of G. By Lemma 2.7.2, we know that A and B each have class $\leq c-1$. Hence, by induction, we have that $\tau(A) \trianglelefteq_{\mathbb{Q}[x]} A$ and $\tau(B) \trianglelefteq_{\mathbb{Q}[x]} B$. Now observe that

  1. $A \trianglelefteq_{\mathbb{Q}[x]} G$ and $B \trianglelefteq_{\mathbb{Q}[x]} G$. This follows from Lemma 2.7.1, since both A and B contain $[G,G]$.

  2. $\tau(A) \trianglelefteq_{\mathbb{Q}[x]} G$ and $\tau(B) \trianglelefteq_{\mathbb{Q}[x]} G$. To verify this, observe that if $\psi \in Aut_{\mathbb{Q}[x]}(A)$ and $\tilde{a} \in \tau(A)$ with $\tilde{a}^{\lambda} = 1$ for some $\lambda \in \mathbb{Q}[x]$, $\lambda \neq 0$, then $(\psi(\tilde{a}))^{\lambda} = 1$ implies $\psi(\tilde{a}) \in \tau(A)$. Hence $\tau(A)$ is invariant under $\mathbb{Q}[x]$-automorphisms of A. In particular, $\tau(A)$ is invariant under conjugation by $g \in G$. Put another way, for any $g \in G$, we have

  $$g^{-1}Ag = A \Longrightarrow g^{-1}\tau(A)g = \tau(A),$$

  and so $\tau(A) \trianglelefteq_{\mathbb{Q}[x]} G$. The same argument shows that $\tau(B) \trianglelefteq_{\mathbb{Q}[x]} G$.

Define

$$K = gp_{\mathbb{Q}[x]}(\tau(A), \tau(B)) = \tau(A)\tau(B) \trianglelefteq_{\mathbb{Q}[x]} G.$$

For any $\tilde{a} \in \tau(A)$ and $\tilde{b} \in \tau(B)$, we have

$$(\tilde{a}\tilde{b})^{\mu} = \tilde{a}^{\mu}\hat{b}$$

where $\mu \in \mathbb{Q}[x]$ and $\hat{b} \in \tau(B)$ (this follows from normality and the Hall-Petresco axiom). In particular,

$$(ab)^{\alpha} = a^{\alpha}b_0 = b_0 \in \tau(B)$$

for some $b_0 \in B$, since $a^{\alpha} = 1$ by hypothesis. Hence, there exists a non-zero $\gamma \in \mathbb{Q}[x]$ such that

$$(ab)^{\alpha\gamma} = b_0^{\gamma} = 1$$

and so $ab \in \tau(G)$. The rest of the proof is obvious.

□

From now on, $\tau(G)$ will be called the *torsion* $\mathbb{Q}[x]$*-subgroup* of G.

Corollary 2.7.2. *If G is a nilpotent $\mathbb{Q}[x]$-powered group, then $G/\tau(G)$ is torsion-free.*

Proof. Follows the same as in Corollary 1.2.2. □

Theorem 2.7.3. *Let G be a torsion free nilpotent $\mathbb{Q}[x]$-powered group and suppose $g, h \in G$. If $g^{\alpha} = h^{\alpha}$ for some non-zero $\alpha \in \mathbb{Q}[x]$ then $g = h$.*

Proof. The proof is by induction on the nilpotency class of G. Let $g, h \in G$ and $\alpha \in \mathbb{Q}[x]$.

- If $c = 1$, then G is an abelian $\mathbb{Q}[x]$-group. Let $g, h \in G$ such that $g^\alpha = h^\alpha$ for some non-zero $\alpha \in \mathbb{Q}[x]$. Then $g^\alpha h^{-\alpha} = 1$ and a simple application of the Hall-Petresco axiom yields $(gh^{-1})^\alpha = 1$. Since G is torsion free, we have $gh^{-1} = 1$ and so $g = h$.

- Suppose that $c \geq 2$ and that the result holds for every torsion free nilpotent $\mathbb{Q}[x]$-powered group of class less than c. Let $H = gp_{\mathbb{Q}[x]}(g, [G, G])$, which is a nilpotent $\mathbb{Q}[x]$-powered group of class less than c by Lemma 2.7.2. Notice that $h^{-1}gh \in H$ since $h^{-1}gh = g[g, h] \in gp_{\mathbb{Q}[x]}(g, [G, G])$. Now for every non-zero $\alpha \in \mathbb{Q}[x]$,

$$\begin{aligned} g^\alpha = h^\alpha &\implies g^\alpha = h^{-1}h^\alpha h \\ &\implies g^\alpha = h^{-1}g^\alpha h \\ &\implies g^\alpha = (h^{-1}gh)^\alpha. \end{aligned}$$

By induction, we have $g = h^{-1}gh \Longrightarrow [g, h] = 1$. Therefore,

$$
\begin{aligned}
g^\alpha = h^\alpha \quad &\Longrightarrow \quad g^\alpha h^{-\alpha} = 1 \\
&\Longrightarrow \quad (gh^{-1})^\alpha = 1 \\
&\Longrightarrow \quad gh^{-1} = 1 \\
&\Longrightarrow \quad g = h.
\end{aligned}
$$

□

Lemma 2.7.3. *G is a torsion nilpotent $\mathbb{Q}[x]$-powered group if and only if $Ab(G)$ is a torsion nilpotent $\mathbb{Q}[x]$-powered group.*

Proof. Clearly, if G is a torsion nilpotent $\mathbb{Q}[x]$-powered group, then so is $Ab(G)$. For the converse, we apply Corollary 2.5.2. □

I will end this section with an embedding theorem which can be found in [14].

Theorem 2.7.4. *Let G be a finitely $\mathbb{Q}[x]$-generated torsion free nilpotent $\mathbb{Q}[x]$-powered group. Then there exists a $\mathbb{Q}[x]$-monomorphism of G into $UT_n(\mathbb{Q}[x])$ for some integer $n > 0$.*

2.8 Nilpotent $\mathbb{Q}[x]$-Powered Groups Of Finite Type

In this section we will focus our attention on a particular class of nilpotent $\mathbb{Q}[x]$-powered groups which I will call the nilpotent $\mathbb{Q}[x]$-powered groups of finite type. The interesting thing about this class is that such groups have a nice decomposition property which is similar to the finite case of ordinary nilpotent groups. This will be examined in the next section.

Definition 2.8.1. Let G be a nilpotent $\mathbb{Q}[x]$-powered group. Then G is of *finite type* if it is a finitely $\mathbb{Q}[x]$-generated torsion $\mathbb{Q}[x]$-group.

Clearly, the nilpotent $\mathbb{Q}[x]$-powered group $G = 1$ is of finite type. Moreover, since every $\mathbb{Q}[x]$-subgroup of a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group is finitely $\mathbb{Q}[x]$-generated, we immediately have

Lemma 2.8.1. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group. Then $\tau(G)$ is of finite type.*

The center of a nilpotent $\mathbb{Q}[x]$-powered group , G, determines whether or not G itself is of finite type, as we shall now see.

Lemma 2.8.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group. If $H \trianglelefteq_{\mathbb{Q}[x]} G$ is of finite type and G/H is of finite type, then G is also of finite type.*

Proof. Let $g \in G$ and let $\{g_1H, \ldots, g_nH\}$ be a set of $\mathbb{Q}[x]$-generators for G/H. Then we have that $gH \in gp_{\mathbb{Q}[x]}(g_1H, \ldots, g_nH)$. But H is finitely $\mathbb{Q}[x]$-generated, so if $\{h_1, \ldots, h_m\}$ is a set of $\mathbb{Q}[x]$-generators for H, then $g \in gp_{\mathbb{Q}[x]}(g_1, \ldots, g_n, h_1, \ldots, h_m)$. Hence, G is finitely $\mathbb{Q}[x]$-generated. Now, since $gH \in G/H$ and G/H is of finite type, there exists $\alpha \in \mathbb{Q}[x]$ such that $(gH)^\alpha = H$. This yields $g^\alpha \in H$. Since H is also of finite type, there exists $\beta \in \mathbb{Q}[x]$ such that $(g^\alpha)^\beta = 1$. Hence $g^{\alpha\beta} = 1$ and so g is a torsion element of G. Therefore, G is a torsion $\mathbb{Q}[x]$-group. □

Theorem 2.8.1. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group of class c. If $Z(G)$ is of finite type, then G is also of finite type.*

Proof. The proof is by induction on the class.

- If $c = 1$, then $Z(G) = G$ and there is nothing to prove.

- Let $c > 1$ and suppose that the result holds for finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered groups of class $< c$. Let $Z(G)$ be of finite type. If $g \in G$ and $a \in \zeta_2 G$, then for some $\alpha \in \mathbb{Q}[x]$, we have

  $$[g, a^\alpha] = [g, a]^\alpha = 1,$$

  since $[g, a] \in [G, \zeta_2 G]_{\mathbb{Q}[x]} \leq_{\mathbb{Q}[x]} Z(G)$ and every element of $Z(G)$ is a torsion element. Therefore, $[g, a^\alpha] = 1$ for all $g \in G$ and so $a^\alpha \in Z(G)$. This shows that

  $$(aZ(G))^\alpha = a^\alpha Z(G) = Z(G)$$

for $a \in \zeta_2 G$. As a result, $\zeta_2 G/Z(G)$ is a torsion $\mathbb{Q}[x]$-group. Furthermore, the fact that

$$\zeta_2 G/Z(G) = Z(G/Z(G)) \trianglelefteq_{\mathbb{Q}[x]} G/Z(G)$$

and $G/Z(G)$ is a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group implies that $\zeta_2 G/Z(G)$ is also finitely $\mathbb{Q}[x]$-generated (since it is a $\mathbb{Q}[x]$-subgroup of a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group). Therefore, $\zeta_2 G/Z(G)$ is of finite type. By induction, $G/Z(G)$ is also of finite type. Hence, G is of finite type by Lemma 2.8.2.

□

2.9 π-Primary Nilpotent $\mathbb{Q}[x]$-Powered Groups

In this section I will prove that a nilpotent $\mathbb{Q}[x]$-powered group of finite type can be decomposed into a direct product of $\mathbb{Q}[x]$-subgroups of a special kind. This is somewhat analogous to the corresponding theorem for finite nilpotent groups.

Definition 2.9.1. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and let $\pi \in \mathbb{Q}[x]$ be a prime in $\mathbb{Q}[x]$. Then the *π-primary component* of G is the set

$$G_\pi = \{g \in G \mid g^{\pi^k} = 1 \text{ for some } k \in \mathbb{Z}^+\}.$$

Definition 2.9.2. Let G be a nilpotent $\mathbb{Q}[x]$-powered group. We call G a *π-primary $\mathbb{Q}[x]$-group* if, for every $g \in G$, there exists $k \in \mathbb{Z}^+$ such that $g^{\pi^k} = 1$.

Note. It is clear that every finitely $\mathbb{Q}[x]$-generated π-primary $\mathbb{Q}[x]$-group is of finite type.

Theorem 2.9.1. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c and let $\pi \in \mathbb{Q}[x]$ be prime. Then G_π is a normal $\mathbb{Q}[x]$-subgroup of G.*

Proof. The proof is by induction on the class.

- Suppose the class of G is $c = 1$ and let $g, h \in G_\pi$. Then $g^{\pi^m} = 1$ and $h^{\pi^n} = 1$ for some $m, n \in \mathbb{Z}^+$. Hence,

$$(gh)^{\pi^{m+n}} = (g^{\pi^m})^{\pi^n}(h^{\pi^n})^{\pi^m} = 1$$

and so $gh \in G_\pi$. Furthermore, $(g^\alpha)^{\pi^m} = (g^{\pi^m})^\alpha = 1$ for all $\alpha \in \mathbb{Q}[x]$ and so $g^\alpha \in G_\pi$.

- Suppose that the result holds for all nilpotent $\mathbb{Q}[x]$-powered groups of class less than c and let G_π be the π-primary component of G. Observe that if $H = gp_{\mathbb{Q}[x]}(G_\pi) \leq_{\mathbb{Q}[x]} G$, then proving that $G_\pi \leq_{\mathbb{Q}[x]} G$ is equivalent to proving that $g \in G_\pi$ for every $g \in H$. Hence we may as well assume that $G = gp_{\mathbb{Q}[x]}(G_\pi)$. I claim that for every $g \in G$, we have $g \in G_\pi$. The equality $G = G_\pi$ then follows.

  For any $g \in G$ there is an $m \in \mathbb{Z}^+$ such that $(g\Gamma_2)^{\pi^m} = \Gamma_2$ since $Ab(G) = G/\Gamma_2$ is abelian. This is equivalent to $g^{\pi^m} \in \Gamma_2$. Now if $g_1, \ldots, g_r \in G$ and

$$g_1\Gamma_2 \otimes \cdots \otimes g_r\Gamma_2 \in \bigotimes^r Ab(G) = \underbrace{Ab(G) \bigotimes \cdots \bigotimes Ab(G)}_{r \text{ of these}},$$

then there exists $m_1, \ldots, m_r \in \mathbb{Z}^+$ such that $g_i^{\pi^{m_i}} \in \Gamma_2$ and so

$$\begin{aligned}(g_1\Gamma_2 \otimes \cdots \otimes g_r\Gamma_2)^{\pi^{m_1+\cdots+m_r}} &= (g_1\Gamma_2 \otimes \cdots \otimes g_r\Gamma_2)^{\pi^{m_1}\cdots\pi^{m_r}} \\ &= g_1^{\pi^{m_1}}\Gamma_2 \otimes \cdots \otimes g_r^{\pi^{m_r}}\Gamma_2 \\ &= \underbrace{\Gamma_2 \otimes \cdots \otimes \Gamma_2}_{r\ of\ these}\end{aligned}$$

Now, by Corollary 2.5.2, there exists a $\mathbb{Q}[x]$-epimorphism

$$\varphi : \bigotimes^{r} Ab(G) \to \Gamma_r/\Gamma_{r+1}$$

given by

$$\varphi(g_1\Gamma_2 \otimes \cdots \otimes g_r\Gamma_2) = [g_1, \ldots, g_r]\Gamma_{r+1}.$$

Thus, we have

$$\begin{aligned}\varphi\Big\{(g_1\Gamma_2 \otimes \cdots \otimes g_r\Gamma_2)^{\pi^{m_1+\cdots+m_r}}\Big\} &= \varphi\Big\{(g_1\Gamma_2)^{\pi^{m_1}} \otimes \cdots \otimes (g_r\Gamma_2)^{\pi^{m_r}}\Big\} \\ &= [g_1^{\pi^{m_1}}, \ldots, g_r^{\pi^{m_r}}]\Gamma_{r+1} = \Gamma_{r+1}.\end{aligned}$$

Hence, if we let $\bar{g}\Gamma_{r+1} = [g_1, \ldots, g_r]\Gamma_{r+1} \in \Gamma_r/\Gamma_{r+1}$, then there is an $l \in \mathbb{Z}^+$ such that $\bar{g}^{\pi^l}\Gamma_{r+1} = \Gamma_{r+1}$. In particular, observe that if $\bar{g} \in \Gamma_c = \Gamma_c/\Gamma_{c+1}$, then there exists $l \in \mathbb{Z}^+$ such that $\bar{g}^{\pi^l} = 1$.

Now, by induction (since G/Γ_c is of class $c-1$), we may assume that for any $k \in G$, there is a $p \in \mathbb{Z}^+$ such that $(k\Gamma_c)^{\pi^p} = \Gamma_c$. This implies that there exists $\hat{g} \in \Gamma_c$ such that $k^{\pi^p} = \hat{g}$. But, since $\hat{g} \in \Gamma_c$, there is a $q \in \mathbb{Z}^+$ for which $\hat{g}^{\pi^q} = 1$ by the observation made above. Therefore $k^{\pi^p} = \hat{g}$ implies that

$k^{\pi^{p+q}} = 1$. Hence, $k \in G_\pi$ for any $k \in G$ and, consequently, $G = G_\pi$ as claimed.

Now suppose that G is any nilpotent $\mathbb{Q}[x]$-powered group and let $k \in G_\pi$ for some prime $\pi \in \mathbb{Q}[x]$. Then there is an $n \in \mathbb{Z}^+$ for which $k^{\pi^n} = 1$. Hence, for any $g \in G$, we have

$$(g^{-1}kg)^{\pi^n} = g^{-1}k^{\pi^n}g = 1.$$

Thus, $g^{-1}kg \in G_\pi$ and so $G_\pi \trianglelefteq_{\mathbb{Q}[x]} G$. This completes the proof.

□

Definition 2.9.3. Let G be a nilpotent $\mathbb{Q}[x]$-powered group. If $g \in G$, then the *annihilator* of g is

$$ann(g) = \{\beta \in \mathbb{Q}[x] \mid g^\beta = 1\}.$$

Lemma 2.9.1. *If G is a nilpotent $\mathbb{Q}[x]$-powered group and $g \in G$, then $ann(g)$ is an ideal in $\mathbb{Q}[x]$.*

Proof. Clearly $0 \in ann(g)$, since $g^0 = 1$ for any $g \in G$. If $\beta_1, \beta_2 \in \mathbb{Q}[x]$, then $g^{\beta_1} = g^{\beta_2} = 1$. Hence, $g^{\beta_1}g^{\beta_2} = 1$, and so $g^{\beta_1+\beta_2} = 1$ implies $\beta_1 + \beta_2 \in ann(g)$. If $a \in \mathbb{Q}[x]$ and $\beta \in ann(g)$, then $g^\beta = 1$ implies $(g^\beta)^a = 1^a = 1$. Therefore, $g^{\beta a} = 1$ and so $\beta a \in ann(g)$. □

Theorem 2.9.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c of finite type. If I is some index set and $\{\pi_i \mid i \in I\}$ is the set of all primes in $\mathbb{Q}[x]$, then G is the direct product of its π_i-primary components, $\prod_{i\in I} G_{\pi_i}$.*

Proof. Let $\{G_{\pi_i} \mid i \in I\}$ be the set of all π_i-primary components of G. By Theorem 2.9.1, we know that $G_{\pi_i} \trianglelefteq_{\mathbb{Q}[x]} G$ for each $i \in I$. I claim that $G = gp_{\mathbb{Q}[x]}(G_{\pi_1}, G_{\pi_2}, \cdots)$ and $gp_{\mathbb{Q}[x]}(G_{\pi_1}, \cdots, \widehat{G_{\pi_i}}, \cdots) \cap G_{\pi_i} = 1$ for any $i \in I$.

1. Let $g \in G$ such that $g \neq 1$. Since G is of finite type, $ann(g) =< d > \neq \{0\}$ for some $d \in \mathbb{Q}[x]$ $(d \neq 1)$, where $< d >$ is the ideal of $\mathbb{Q}[x]$ generated by d. Now, since $\mathbb{Q}[x]$ is a unique factorization domain, there exists irreducible elements $\pi_1, \ldots, \pi_n$ of $\mathbb{Q}[x]$, no two of which are associates, and $m_1, \ldots, m_n$ of $\mathbb{Z}^+$ such that $d = \pi_1^{m_1} \cdots \pi_n^{m_n}$. One can show that $P_i =< \pi_i >$ is a prime ideal of $\mathbb{Q}[x]$ for each i. Now, define $r_i = d/\pi_i^{m_i}$, so that $\pi_i^{m_i} r_i = d$ in $\mathbb{Q}[x]$. It follows that $(g^{r_i})^{\pi_i^{m_i}} = g^d = 1$, so that $g^{r_i} \in G_{\pi_i}$ for each i. But the greatest common divisor of the elements $\{r_1, \ldots, r_n\}$ is 1 and so there are elements $\{s_1, \ldots, s_n\} \in \mathbb{Q}[x]$ such that
$$\sum_{i=1}^{n} r_i s_i = 1.$$
Therefore $g = g^{r_1 s_1 + \cdots + r_n s_n} = g^{r_1 s_1} \cdots g^{r_n s_n} \in G_{\pi_1} G_{\pi_2} \cdots G_{\pi_n}$. This follows since each $g^{r_i} \in G_{\pi_i}$ implies that $g^{r_i s_i} \in G_{\pi_i}$.

2. Let $K_i = gp_{\mathbb{Q}[x]}(G_{\pi_1}, \cdots, \widehat{G_{\pi_i}}, \cdots G_{\pi_n}) = G_{\pi_1} \cdots \hat{G_{\pi_i}} \cdots G_{\pi_n}$. It suffices to prove that if $g \in G_{\pi_i} \cap K_i$, then $g = 1$. Well, since $g \in G_{\pi_i}$, there exists $q \in \mathbb{Z}^+$ such that $g^{\pi_i^q} = 1$. But $g \in K_i$ implies that $g^t = 1$, where $t = \pi_1^{t_1} \cdots \hat{\pi_i^{t_i}} \cdots \pi_n^{t_n}$ for some suitable $t_1, \ldots, t_n$ lying in $\mathbb{Z}^+$. Furthermore, since π_i^q and t are relatively

prime, there exists $s_1, s_2 \in \mathbb{Q}[x]$ with

$$s_1\pi_i^q + s_2t = 1.$$

Therefore $g = g^{s_1\pi_i^q+s_2t} = (g^{\pi_i^q})^{s_1}(g^t)^{s_2} = 1$. This completes the proof.

□

2.10 The Frattini $\mathbb{Q}[x]$-Subgroup

In this section the Frattini $\mathbb{Q}[x]$-subgroup of a nilpotent $\mathbb{Q}[x]$-powered group is introduced. Several basic results are given which are analogous to the usual group case.

Definition 2.10.1. Let G be a nilpotent $\mathbb{Q}[x]$-powered group and let H be a proper $\mathbb{Q}[x]$-subgroup of G. Then H is a *maximal* $\mathbb{Q}[x]$*-subgroup* of G if it is not contained in any other proper $\mathbb{Q}[x]$-subgroup.

Theorem 2.10.1. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c. Then if G has a maximal $\mathbb{Q}[x]$-subgroup, it is a normal $\mathbb{Q}[x]$-subgroup of G.*

Proof. Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c and suppose H is a proper maximal $\mathbb{Q}[x]$-subgroup of G. It is easy to verify that the collection $\{H\zeta_iG\}$ satisfies the condition $H\zeta_jG \leq_{\mathbb{Q}[x]} H\zeta_{j+1}G$. Hence H is subnormal in G (in the $\mathbb{Q}[x]$ sense). Furthermore, it can be proven in the same way as for ordinary nilpotent groups that

$N_G(H) >_{\mathbb{Q}[x]} H$, where $N_G(H)$ is the normalizer of H in G. Since H is maximal, we have that $N_G(H) = H$. Therefore $H \trianglelefteq_{\mathbb{Q}[x]} G$. □

Definition 2.10.2. The *Frattini* $\mathbb{Q}[x]$*-subgroup* of a nilpotent $\mathbb{Q}[x]$-powered group, G, is the intersection of all maximal $\mathbb{Q}[x]$-subgroups of G. It will be denoted by $\Phi(G)$. If G has no maximal $\mathbb{Q}[x]$-subgroups, we set $\Phi(G) = G$.

Remark. It is easy to check that $\Phi(G) \trianglelefteq_{\mathbb{Q}[x]} G$. In fact, $\Phi(G)$ is a characteristic $\mathbb{Q}[x]$-subgroup of G (if $\Psi \in \mathrm{Aut}_{\mathbb{Q}[x]}(G)$, then $\Psi(\Phi(G)) \leq_{\mathbb{Q}[x]} \Phi(G)$).

Theorem 2.10.2. *The Frattini* $\mathbb{Q}[x]$*-subgroup of a non-trivial nilpotent* $\mathbb{Q}[x]$*-powered group is precisely the set of non-*$\mathbb{Q}[x]$*-generators of* G.

The following lemma is needed in the proof of the theorem:

Lemma 2.10.1. *Let* G *be a nilpotent* $\mathbb{Q}[x]$*-powered group and suppose that* $H <_{\mathbb{Q}[x]} G$ *with* $g \in G \setminus H$*. Then there exists a* $\mathbb{Q}[x]$*-subgroup* $K <_{\mathbb{Q}[x]} G$ *which is maximal in* G *with respect to the properties* $H <_{\mathbb{Q}[x]} K$ *and* $g \notin K$*.*

Proof. Let $R = \{J <_{\mathbb{Q}[x]} G \mid H \subseteq J \text{ and } g \notin J\}$. Then $R \neq \varnothing$ since $H \in R$. Furthermore, R is partially ordered by inclusion and the union of any chain in R is again in R. By Zorn's lemma, R has a maximal element. □

I will now prove the theorem:

Proof. Let $g \in \Phi(G)$ and suppose that $G = gp_{\mathbb{Q}[x]}(g, X)$ but $G \neq gp_{\mathbb{Q}[x]}(X)$ (hence g is a $\mathbb{Q}[x]$-generator of G). Then $g \notin gp_{\mathbb{Q}[x]}(X)$. By the lemma, there exists a $\mathbb{Q}[x]$-subgroup M of G which is maximal in G with respect to the properties $gp_{\mathbb{Q}[x]}(X) <_{\mathbb{Q}[x]} M$ and $g \notin M$. Now if $M <_{\mathbb{Q}[x]} H <_{\mathbb{Q}[x]} G$ for some $H \subset G$, then $g \in H$ and, consequently, $H = G$. Hence M is maximal in G. But $g \in \Phi(G) \leq_{\mathbb{Q}[x]} M$ and so $G = gp_{\mathbb{Q}[x]}(g, X) = M$ which is a contradiction. Hence g is a non-$\mathbb{Q}[x]$-generator of G. Conversely, suppose that g is a non-$\mathbb{Q}[x]$-generator of G and suppose $g \notin \Phi(G)$. Then there exists a maximal $\mathbb{Q}[x]$-subgroup, M, of G for which $g \notin M$. Then $M \neq gp_{\mathbb{Q}[x]}(g, M)$ and so $G = gp_{\mathbb{Q}[x]}(g, M)$ by the maximality of M in G. Since g is a non-$\mathbb{Q}[x]$-generator of G, this yields $G = M$, a contradiction. Hence $g \in \Phi(G)$. □

I will now prove a variety of lemmas. The first one is analogous to Lemma 1.2.3. I will omit the proof.

Lemma 2.10.2. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group of class c and suppose that $H \leq_{\mathbb{Q}[x]} G$ such that $gp_{\mathbb{Q}[x]}(H, \gamma_2 G) = G$. Then $H = G$ and, consequently, $\gamma_2 G \subseteq \Phi(G)$.*

Lemma 2.10.3. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group and suppose $H \leq_{\mathbb{Q}[x]} G$. If $gp_{\mathbb{Q}[x]}(H, \Phi(G)) = G$, then $H = G$.*

Proof. Suppose $G = gp_{\mathbb{Q}[x]}(g_1, \ldots, g_n)$. Since $gp_{\mathbb{Q}[x]}(H, \Phi(G)) = G$ we have $g_i = h_i u_i$ for some $h_i \in H$, $u_i \in \Phi(G)$ and $1 \leq i \leq n$. Therefore $G =$

$gp_{\mathbb{Q}[x]}(h_1, \ldots, h_n, u_1, \ldots, u_n)$. But each $u_i \in \Phi(G)$ and so each u_i is a non-$\mathbb{Q}[x]$-generator of G by Theorem 2.10.2. Hence $G = gp_{\mathbb{Q}[x]}(h_1, \ldots, h_n) \subseteq H$. Therefore $G = H$. □

Lemma 2.10.4. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group and suppose that $\gamma_2 G$ is finitely $\mathbb{Q}[x]$-generated. If $H \leq_{\mathbb{Q}[x]} G$ such that $gp_{\mathbb{Q}[x]}(H, \gamma_2 G) = G$, then $H = G$.*

Proof. By Lemma 2.10.2, $\gamma_2 G \subseteq \Phi(G)$. If $\gamma_2 G = gp_{\mathbb{Q}[x]}(g_1, \ldots, g_n)$, then we have

$$gp_{\mathbb{Q}[x]}(H, \gamma_2 G) = gp_{\mathbb{Q}[x]}(H, g_1, \ldots, g_n) = gp_{\mathbb{Q}[x]}(H) = H,$$

since each $g_i \in \Phi(G)$. Hence $G = H$. □

Lemma 2.10.5. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group whose Frattini $\mathbb{Q}[x]$-subgroup is finitely $\mathbb{Q}[x]$-generated. Then the only $\mathbb{Q}[x]$-subgroup H of G which satisfies $gp_{\mathbb{Q}[x]}(H, \Phi(G)) = G$ is $H = G$.*

Proof. Let $gp_{\mathbb{Q}[x]}(H, \Phi(G)) = G$ and suppose $\Phi(G) = gp_{\mathbb{Q}[x]}(g_1, \ldots, g_n)$. Then we can write $gp_{\mathbb{Q}[x]}(H, g_1, \ldots, g_n) = G$. Since each g_i is a non-$\mathbb{Q}[x]$-generator for G by Theorem 2.10.2, we have $G = gp_{\mathbb{Q}[x]}(H) = H$. □

Lemma 2.10.6. *Let G be a nilpotent $\mathbb{Q}[x]$-powered group whose Frattini $\mathbb{Q}[x]$-subgroup, $\Phi(G)$, is finitely $\mathbb{Q}[x]$-generated. If $G/\Phi(G)$ can be $\mathbb{Q}[x]$-generated by p elements and no fewer, then so can G.*

Proof. Let $G/\Phi(G) = gp_{\mathbb{Q}[x]}(g_1\Phi(G), \ldots, g_p\Phi(G))$. If $H = gp_{\mathbb{Q}[x]}(g_1, \ldots, g_p)$, then we obtain $G = gp_{\mathbb{Q}[x]}(H, \Phi(G))$. By Lemma 2.10.5, we have $H = G$. Hence G is $\mathbb{Q}[x]$-generated by p elements. Moreover, if $g_1, \ldots, g_m$ is a set of $\mathbb{Q}[x]$-generators for G, then the set $g_1\Phi(G), \ldots, g_m\Phi(G)$ is a set of $\mathbb{Q}[x]$-generators for $G/\Phi(G)$. Since p was minimal for such a set, we deduce that G has no set of $\mathbb{Q}[x]$-generators consisting of fewer than p elements. □

2.11 A Residual Property

I will prove that finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered groups are residually finite dimensional over $\mathbb{Q}$.

Definition 2.11.1. Let R be an arbitrary binomial ring and let K be a binomial subring of R which is also a field. A finitely R-generated nilpotent R-powered group, G, is *finite dimensional over* K if G has a central R-series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_q = G$ satisfying, for each $i = 0, \ldots, q-1$,

1. $G_{i+1}/G_i = gp_R(g_{i+1}, G_i)$ is a cyclic R-group and
2. each G_{i+1}/G_i, viewed as a vector space over K, is finite dimensional.

Lemma 2.11.1. *Let $f(x)$ be a non-zero polynomial in $\mathbb{Q}[x]$. Then there exists a homomorphism $\varphi : \mathbb{Q}[x] \to \mathbb{Q}$ with $\varphi(f(x)) \neq 0$.*

Proof. Let $a \in \mathbb{Q}$ such that $f(a) \neq 0$. Such an $a \in \mathbb{Q}$ clearly exists since $f(x)$ can only have finitely many zeros, being of finite degree. Let φ_a be the evaluation homomorphism at $x = a$:

$$\varphi_a : \mathbb{Q}[x] \to \mathbb{Q} \text{ is defined by } \varphi_a(f(x)) = f(a).$$

Then φ_a is the required homomorphism, since $\varphi_a(f(x)) = f(a) \neq 0$. □

Note. The above lemma also holds with $\mathbb{Q}[x]$ replaced by $\mathbb{Z}[x]$ and with $\mathbb{Q}$ replaced by $\mathbb{Z}$.

Lemma 2.11.2. *Let $\{f_1(x), \ldots, f_k(x)\}$ be a finite collection of polynomials in $\mathbb{Q}[x]$, some of which may equal the zero polynomial, and let $deg(f_i(x)) = n_i$ for those $i = 1, \ldots, k$ in which $f_i(x)$ is not the zero polynomial. Then there is an $\tilde{x} \in \mathbb{Q}$ such that $f_i(\tilde{x}) \neq 0$ for each $i = 1, \ldots, k$ in which $f_i(x)$ is not the zero polynomial.*

Proof. Since $deg(f_i(x)) = n_i$ for each $i = 1, \ldots, k$ in which $f_i(x)$ is not the zero polynomial, $f_i(x)$ can have at most n_i distinct zeros. Hence there exists a set $X_i = \{x_{i_1}, x_{i_2}, \ldots\}$ satisfying $f_i(x_{i_j}) \neq 0$ for each $i = 1, \ldots, k$ in which $f_i(x)$ which is not the zero polynomial. As a result, there is an $\tilde{x} \in \mathbb{Q}$ such that

$$\tilde{x} \in \bigcap_{i=1}^{k} X_i$$

and $f_i(\tilde{x}) \neq 0$ for each $i = 1, \ldots, k$ in which $f_i(x)$ is not the zero polynomial. □

Note. Once again, this lemma also holds with $\mathbb{Q}[x]$ replaced by $\mathbb{Z}[x]$ and with $\mathbb{Q}$ replaced by $\mathbb{Z}$.

Theorem 2.11.1. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group of class c. Then G is residually finite dimensional over $\mathbb{Q}$.*

Proof. Let $G = gp_{\mathbb{Q}[x]}(g_1, \ldots, g_n)$ and suppose $g \in G$, $g \neq 1$. I claim that there exists a $\mathbb{Q}[x]$-homomorphism $\varphi : G \to B$, where B is some finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group of finite dimension over $\mathbb{Q}$ and $\varphi(g) \neq 1$ in B. Suppose that $B = gp_{\mathbb{Q}}(g_1, \ldots, g_n) <_{\mathbb{Q}} G$. If the indeterminate variable $x \in \mathbb{Q}[x]$ acts as the identity on elements of B, then B can be viewed as a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group.

Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$ be a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series for G (such a series exists by Theorem 2.5.6). Let $G_{i+1}/G_i = gp_{\mathbb{Q}[x]}(a_{i+1}, G_i)$. Then $g \in G$ can be expressed, with respect to this series, as $g = a_1^{\alpha_1(x)} \cdots a_r^{\alpha_r(x)}$, where the $\alpha_i(x) \in \mathbb{Q}[x]$. By Lemma 2.11.2, there exists an $\tilde{x} \in \mathbb{Q}$ such that $\alpha_i(\tilde{x}) \neq 0$ for every $i = 1, \ldots, r$ in which $\alpha_i(x)$ is a non-zero polynomial (note that at least one of the $\alpha_i(x)$'s is non-zero).

By Lemma 2.11.1, choose $\mu_{\tilde{x}} : \mathbb{Q}[x] \to \mathbb{Q}$ so that $\mu_{\tilde{x}}(f(x)) = f(\tilde{x})$. Then $\mu_{\tilde{x}}(\alpha_i(x)) = \alpha_i(\tilde{x}) \neq 0$ for those $i = 1, \ldots, r$ in which $\alpha_i(x)$ is a non-zero polynomial. Therefore, if $\varphi_{\tilde{x}} : G \to B$ is the $\mathbb{Q}[x]$-mapping defined by

$$\begin{aligned} \varphi_{\tilde{x}}(g) &= \varphi_{\tilde{x}}(a_1^{\alpha_1(x)} \cdots a_r^{\alpha_r(x)}) \\ &= a_1^{\alpha_1(\tilde{x})} \cdots a_r^{\alpha_r(\tilde{x})} \\ &= a_1^{\mu_{\tilde{x}}(\alpha_1(x))} \cdots a_r^{\mu_{\tilde{x}}(\alpha_r(x))} \end{aligned}$$

then it is easy to check that $\varphi_{\bar{x}}$ is a $\mathbb{Q}[x]$-homomorphism and $\varphi_{\bar{x}}(g) \neq 1$. □

2.12 Dehns' Fundamental Problems

In this section I will discuss the problems proposed by Dehn. I will begin by proving that the conjugacy problem for finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered groups is solvable. Afterwards the word problem and the the generalized word problem are examined. The section closes with a result on Hopfian nilpotent $\mathbb{Q}[x]$-powered groups. The $\mathbb{Q}[x]$-isomorphism problem has not been studied yet.

Theorem 2.12.1. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group of class c. Then G has a solvable conjugacy problem. Put another way, if $g, h \in G$, then there is an algorithm which decides whether or not there exists $f \in G$ such that $h^f = g$.*

Proof. Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ be a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series for G of Hirsch $\mathbb{Q}[x]$-length n (such a series exists by Theorem 2.5.6). The proof is by induction on n.

1. If $n = 1$ the $\mathbb{Q}[x]$-series for G is $1 = G_0 \lhd G_1 = G$, so G is a cyclic $\mathbb{Q}[x]$-group. The conjugacy problem is certainly solvable in this case.

2. Suppose $n > 1$ and assume the conjugacy problem is solvable for every nilpotent $\mathbb{Q}[x]$-powered group of Hirsch $\mathbb{Q}[x]$-length less than n. Then, in particular,

G/G_1 has a solvable conjugacy problem (note that $G_1 \leq_{\mathbb{Q}[x]} Z(G)$). Suppose $g, h \in G$. We want to decide whether or not $h \sim g$, where the symbol "$\sim$" means "conjugate to". By induction, we can decide whether or not $hG_1 \sim gG_1$. If $hG_1 \nsim gG_1$, then $h \nsim g$ in G and the conjugacy problem is solved. Suppose that $hG_1 \sim gG_1$. Then there exists $f \in G$ such that $h^f = ga$ for some $a \in G_1$. It suffices to check whether or not h^f and g are conjugate in G. We can assume that $a \neq 1$ (otherwise, $h^f = g$ and the we're done). Hence, deciding whether or not $h^f \sim g$ in G is the same as deciding whether or not $ga \sim g$ in G. Put another way, does there exist $c \in G$ such that $g = (ga)^c$? If such a $c \in G$ exists, then we have (since $a \in G_1$)

$$(ga)^c = c^{-1}gac = c^{-1}gca = g^c a.$$

Since $g = (ga)^c$, we obtain $g = g^c a$ which is equivalent to $[c, g] = a$. Let $H = \{h \in G \mid [h, g] \in G_1\}$. By Lemma 2.12.1 following this proof, we have $H \leq_{\mathbb{Q}[x]} G$. Now, consider the set $[H, g] = \{[h, g] \mid h \in H\}$. By Lemma 2.12.2 following this proof, we have

$$[H, g] \leq_{\mathbb{Q}[x]} G_1.$$

Clearly, $[H, g]$ is finitely $\mathbb{Q}[x]$-generated. Therefore $[H, g]$ is a finitely generated submodule of the finitely generated $\mathbb{Q}[x]$-module G_1. We only need to check whether or not the element $a \in G_1$ satisfies the containment $a \in [H, g]$. By [1] (Theorem 2.7), there is an algorithm which decides this problem.

□

Lemma 2.12.1. *Let $H = \{h \in G \mid [h, g] \in G_1\}$ as above. Then $H \leq_{\mathbb{Q}[x]} G$.*

Proof. Let $h_1, h_2 \in H$. Then

$$\begin{aligned} [h_1 h_2, g] &= [h_1, g]^{h_2}[h_2, g] \\ &= [h_1, g][h_2, g] \text{ since } [h_1, g] \in G_1 \leq_{\mathbb{Q}[x]} Z(G). \end{aligned}$$

Hence, $h_1 h_2 \in H$. Furthermore, for any $\alpha \in \mathbb{Q}[x]$, we have

$$\begin{aligned} [h_1^\alpha, g] &= h_1^{-\alpha} g^{-1} h_1^\alpha g \\ &= (h_1^{-1})^\alpha (g^{-1} h_1 g)^\alpha \\ &= (h_1^{-1} g^{-1} h_1 g)^\alpha \tau_2(h_1^{-1}, g^{-1} h_1 g)^{\binom{\alpha}{2}} \ldots \\ &= (h_1^{-1} g^{-1} h_1 g)^\alpha, \text{ since each } \tau_i = 1 \text{ for } i \geq 2 \\ &= [h_1, g]^\alpha. \end{aligned}$$

Hence, $h_1^\alpha \in H$. □

Lemma 2.12.2. *Let $[H, g] = \{[h, g] \mid h \in H\}$ be as above. Then we have $[H, g] \leq_{\mathbb{Q}[x]} G_1$.*

Proof. Let $[h_1, g], [h_2, g] \in [H, g]$. Then, since $h_1, h_2 \in H$, we have by Lemma 2.12.1 that $h_1 h_2 \in H$. Therefore, $[h_1 h_2, g] \in [H, g]$. But

$$[h_1 h_2, g] = [h_1, g][h_2, g]$$

by Lemma 2.12.1, and so $[h_1, g][h_2, g] \in [H, g]$. Proving that $[h_1, g]^\alpha \in [H, g]$ for any $\alpha \in \mathbb{Q}[x]$ is just as easy. □

Theorem 2.12.2. *The word problem for any finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group is solvable.*

Proof. Suppose that G is finitely $\mathbb{Q}[x]$-generated by $X = \{x_1, \ldots, x_k\}$. By Theorem 2.5.6, G has a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G.$$

Therefore there exists a collection of elements $u_1, \ldots, u_r$ in G such that $G_{i+1} = gp_{\mathbb{Q}[x]}(u_{i+1}, G_i)$. Then every $g \in G$ has the form $g = u_1^{\alpha_1} \cdots u_r^{\alpha_r}$, where $\alpha_i \in \mathbb{Q}[x]$. Clearly, any relation which holds in G can be written in the form $u_1^{\gamma_1} \cdots u_r^{\gamma_r} = 1$ for suitable $\gamma_i \in \mathbb{Q}[x]$.

Let $w = w_{\mathbb{Q}[x]}(x_1, \ldots, x_r)$ be any word in $X = \{x_1, \ldots, x_k\}$ which comes from taking products of $\mathbb{Q}[x]$-powers of elements in W. Utilizing P. Halls' collection process and the Hall-Petresco axiom provides us with an algorithm which allows us to rewrite w in the form $w = u_1^{\eta_1} \cdots u_r^{\eta_r}$ for suitable $\eta_i \in \mathbb{Q}[x]$. Once in this form, we can determine whether or not $w = 1$ in G. □

Theorem 2.12.3. *Let G be any finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group and let $H \leq_{\mathbb{Q}[x]} G$. Then G has a solvable generalized word problem. Put another way, if $g \in G$, then there is an algorithm which decides whether or not $g \in H$.*

Proof. Let G and H be as in the hypothesis. Suppose that

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_p = G$$

is a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series for G. Then $G_i = gp_{\mathbb{Q}[x]}(a_i, G_{i-1})$ for $i = 1, \ldots, p$ and for some $a_1, \ldots, a_p$ in G. By Corollary 2.5.7, we know that H also has such a $\mathbb{Q}[x]$-series. Let's construct this $\mathbb{Q}[x]$-series of H.

1. First, let's consider the $\mathbb{Q}[x]$-subgroup $HG_{p-1}/G_{p-1} \trianglelefteq_{\mathbb{Q}[x]} G/G_{p-1}$. Then HG_{p-1}/G_{p-1} is a $\mathbb{Q}[x]$-subgroup of a cyclic $\mathbb{Q}[x]$-group, hence is itself a cyclic $\mathbb{Q}[x]$-group. Suppose that $HG_{p-1}/G_{p-1} = gp_{\mathbb{Q}[x]}(b_p, G_{p-1})$ with $b_p \in H$. Then $b_p = a_p^{\lambda_p} a_{p-1}^{\lambda_{p-1}} \cdots a_1^{\lambda_1}$ for some $\lambda_i \in \mathbb{Q}[x]$. I claim that $H = gp_{\mathbb{Q}[x]}(b_p, H \cap G_{p-1})$. To see this, let $h \in H$ and let $hG_{p-1} \in HG_{p-1}/G_{p-1}$. Notice that $hG_{p-1} = b_p^{\gamma_p} G_{p-1}$ for some $\gamma_p \in \mathbb{Q}[x]$. Hence $h = b_p^{\gamma_p} s_p$ for some $s_p \in H \cap G_{p-1}$ and the claim is verified.

2. Now consider the $\mathbb{Q}[x]$-subgroup $HG_{p-2}/G_{p-2} \trianglelefteq_{\mathbb{Q}[x]} G_{p-1}/G_{p-2}$, which is again a cyclic $\mathbb{Q}[x]$-group. Then we can write $HG_{p-2}/G_{p-2} = gp_{\mathbb{Q}[x]}(b_{p-1}, G_{p-2})$ with $b_{p-1} \in H \cap G_{p-1}$. I claim that $H \cap G_{p-1} = gp_{\mathbb{Q}[x]}(b_{p-1}, H \cap G_{p-2})$. The verification is identical to the previous case. Let $h \in H \cap G_{p-1}$. Then $hG_{p-2} \in HG_{p-2}/G_{p-2}$, and so $hG_{p-2} = b_{p-1}^{\gamma_{p-1}} G_{p-2}$ for some $\gamma_{p-1} \in \mathbb{Q}[x]$. Therefore $h = b_{p-1}^{\gamma_{p-1}} s_{p-1}$ for some $s_{p-1} \in H \cap G_{p-2}$ and the claim is proven.

3. Putting the above observations together, we inductively obtain

$$\begin{aligned} H &= gp_{\mathbb{Q}[x]}(b_p, H \cap G_{p-1}) \\ &= gp_{\mathbb{Q}[x]}(b_p, b_{p-1}, H \cap G_{p-2}) \\ &= gp_{\mathbb{Q}[x]}(b_p, b_{p-1}, \ldots, b_1) \end{aligned}$$

and thus we have a poly-$\mathbb{Q}[x]$ cyclic and central $\mathbb{Q}[x]$-series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_p = H$$

such that $H_i = gp_{\mathbb{Q}[x]}(b_i, H_{i-1})$ for $i = 1, \ldots, p$.

Let's now return to the generalized word problem for G. Suppose that $g \in G$. We want to determine whether or not there is an algorithm which decides whether or not $g \in H$. The proof is by induction on the length of the $\mathbb{Q}[x]$-series for G. Well, notice that $G_1 = gp_{\mathbb{Q}[x]}(a_1)$ is a cyclic $\mathbb{Q}[x]$-group. Hence if $g = a_1^{\alpha}$ for some $\alpha \in \mathbb{Q}[x]$, we want to know whether or not we can write a_1^{α} in the form $b_p^{\alpha_p} \cdots b_1^{\alpha_1}$ for some $\alpha_i \in \mathbb{Q}[x]$. This is easy to determine since each b_i has the form $b_i = a_1^{\beta_1} \cdots a_p^{\beta_p}$ for some $\beta_i \in \mathbb{Q}[x]$ and we can effectively express a_1^{α} in terms of the a_i's. Now let's examine G/G_{p-1}. Let $gG_{p-1} \in G/G_{p-1}$. If it were the case that $g \in H$, then we would have $gG_{p-1} \in HG_{p-1}/G_{p-1}$. This means that $g = b_p^{\mu_p} f_p$ for some $\mu_p \in \mathbb{Q}[x]$ and $f_p \in G_{p-1}$. Now observe that if $g = b_p^{\mu_p} f_p$, then $g \in H$ if and only if $f_p \in H$. But $f_p \in G_{p-1}$ and, by induction, we can decide whether or not $f_p \in H$. This completes the proof. $\square$

Recall that a group G is called *hopfian* if whenever $\varphi : G \to G$ is a surjective homomorphism, then φ is an isomorphism. It is known (see [2]) that finitely generated nilpotent groups are hopfian. A similar result holds for nilpotent $\mathbb{Q}[x]$-powered groups.

Definition 2.12.1. Let G be a nilpotent $\mathbb{Q}[x]$-powered group. Then G is said to satisfy the *ascending chain condition on* $\mathbb{Q}[x]$*-subgroups* if there does not exist an infinite properly ascending chain of $\mathbb{Q}[x]$-subgroups of G.

As in the ordinary group case, it can be proven that a nilpotent $\mathbb{Q}[x]$-powered group G satisfies the ascending chain condition on $\mathbb{Q}[x]$-subgroups if and only if each $\mathbb{Q}[x]$-subgroup of G is finitely $\mathbb{Q}[x]$-generated.

Definition 2.12.2. A nilpotent $\mathbb{Q}[x]$-powered group G is called *hopfian* if, whenever $\Phi : G \to G$ is a $\mathbb{Q}[x]$-epimorphism, then Φ is a $\mathbb{Q}[x]$-isomorphism.

Proposition 2.12.4. *If G is a nilpotent $\mathbb{Q}[x]$-powered group and G satisfies the ascending chain condition on $\mathbb{Q}[x]$-subgroups, then G is hopfian.*

Proof. Let $\Phi : G \to G$ be a $\mathbb{Q}[x]$-epimorphism and suppose that ker $\Phi \neq 1$. The proof is by contradiction. For $j \geq 1$, let $\Phi^{\circ j} = \underbrace{\Phi \circ \cdots \circ \Phi}_{j \text{ of these}}$ denote the composition of Φ with itself j times. Clearly $\Phi^{\circ j} : G \to G$ is a $\mathbb{Q}[x]$-homomorphism for each $j \geq 1$. Let $K_j = \ker(\Phi^{\circ j})$. Observe that $\Phi^{\circ j}(K_j) = 1$ and $\Phi^{\circ j}(K_{j+1}) = \ker \Phi$. Since ker $\Phi \neq 1$ and $\Phi^{\circ j}$ is a $\mathbb{Q}[x]$-epimorphism, we have $K_j \subset K_{j+1}$. It follows that $K_1 \subset K_2 \subset \cdots$ is an infinite properly ascending chain of $\mathbb{Q}[x]$-subgroups of G. This contradicts the fact that G satisfies the ascending chain condition on $\mathbb{Q}[x]$-subgroups. Hence ker Φ is trivial and so Φ is a $\mathbb{Q}[x]$-monomorphism. Consequently, Φ is a $\mathbb{Q}[x]$-isomorphism and so G is hopfian. $\square$

Corollary 2.12.5. *Let G be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group. Then G is hopfian.*

Proof. Let G be as in the hypothesis. Then by Theorem 2.5.5, every $\mathbb{Q}[x]$-subgroup of G is finitely $\mathbb{Q}[x]$-generated. Hence the above proposition applies. $\square$

2.13 Free Nilpotent $\mathbb{Q}[x]$-Powered Groups

Denote by $\mathcal{N}_c$ be the class of all nilpotent $\mathbb{Q}[x]$-powered groups of class at most c. Since every nilpotent $\mathbb{Q}[x]$-powered group G of class c is defined by a set of axioms and satisfies the relation

$$[g_1, \ldots, g_j] = 1 \text{ for all } j \geq c+1,$$

$\mathcal{N}_c$ is a variety. Therefore, $\mathcal{N}_c$ contains free objects. I will now give the definition of freeness in this variety.

Definition 2.13.1. Let $\mathcal{N}_c$ be the class of all nilpotent $\mathbb{Q}[x]$-powered groups of class at most c. Then a group $H \in \mathcal{N}_c$ is *free in $\mathcal{N}_c$* if it comes equipped with a set X and a map $\mu : X \to H$ such that for every nilpotent $\mathbb{Q}[x]$-powered group $K \in \mathcal{N}_c$ and every map $\theta : X \to K$, there exists a unique $\mathbb{Q}[x]$-homomorphism $\varphi : H \to K$ such that $\varphi \circ \mu = \theta$. We say that H *is free on* X. If $X \subset H$ and μ is the identity map, we say that H *is freely* $\mathbb{Q}[x]$*-generated by* X.

The next theorem can be found in P. Hall [6].

Theorem 2.13.1. *Let G be a free nilpotent group of class c, freely generated by $X = \{x_1, \ldots, x_n\}$. Then the $\mathbb{Q}[x]$-completion of G with respect to some Mal'cev basis is a free nilpotent $\mathbb{Q}[x]$-powered group of class c, freely $\mathbb{Q}[x]$-generated by X. Moreover, if H is a free nilpotent $\mathbb{Q}[x]$-powered group of class c, freely $\mathbb{Q}[x]$-generated by X, then $H \cong G^{\mathbb{Q}[x]}$, where G is a free nilpotent group of class c, freely generated by X and $G^{\mathbb{Q}[x]}$ is its $\mathbb{Q}[x]$-completion with respect to some Mal'cev basis.*

Theorem 2.13.2. *Let $G \in \mathcal{N}_c$, freely $\mathbb{Q}[x]$-generated by $X = \{x_1, \ldots, x_k\}$. Then $G/Z(G) \in \mathcal{N}_{c-1}$ is of one less class than G and is freely $\mathbb{Q}[x]$-generated by the set $\bar{X} = \{x_1Z(G), \ldots, x_kZ(G)\}$.*

Proof. Suppose G be a free nilpotent $\mathbb{Q}[x]$-powered group of class c, freely $\mathbb{Q}[x]$-generated by $X = \{x_1, \ldots, x_k\}$. Then there exists a free nilpotent group, H, of class c, freely generated by X, such that $H^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} G$. Now, it is known that $H/Z(H)$ is free nilpotent of class $c - 1$, freely generated by $\bar{X} = \{x_1Z(H), \ldots, x_kZ(H)\}$. By taking the $\mathbb{Q}[x]$-completion of $H/Z(H)$, choosing as the Mal'cev basis for $H/Z(H)$ the basic sequence of basic commutators of the set $\bar{X} = \{x_1Z(H), \ldots, x_kZ(H)\}$, we obtain a free nilpotent $\mathbb{Q}[x]$-powered group of class c, freely $\mathbb{Q}[x]$-generated by $X = \{x_1Z(H), \ldots, x_kZ(H)\}$, namely $(H/Z(H))^{\mathbb{Q}[x]}$. Since $H^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} G$, it is easy to see that

$$(H/Z(H))^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} G/Z(G).$$

Hence, $G/Z(G)$ is a free nilpotent $\mathbb{Q}[x]$-powered group of class $c-1$, freely $\mathbb{Q}[x]$-generated by $\bar{X} = \{x_1Z(G), \ldots, x_kZ(G)\}$. □

The next theorem can be proven the same way as for ordinary groups.

Theorem 2.13.3. *If G is a free nilpotent $\mathbb{Q}[x]$-powered group of class c and $A < G$ is an abelian subgroup of G, then A is a free abelian group.*

Let $\mathcal{A}$ denote the class of abelian $\mathbb{Q}[x]$-groups. The following are proven exactly the same way as for $\mathbb{Q}[x]$-modules.

Theorem 2.13.4. *Let $G \in \mathcal{A}$ be an abelian $\mathbb{Q}[x]$-group with $\mathbb{Q}[x]$-generators $X = \{x_j | \; j \in J\}$. Then G is free in $\mathcal{A}$ if and only if $G = \prod_{i \in I} G_i$ where each $G_i = gp_{\mathbb{Q}[x]}(x_i | \; i \in I)$ is a torsion free cyclic $\mathbb{Q}[x]$-group and I is some index set.*

Corollary 2.13.5. *Every $\mathbb{Q}[x]$-subgroup of a free abelian $\mathbb{Q}[x]$-group is a free abelian $\mathbb{Q}[x]$-group.*

2.14 The Mal'cev Correspondence

I will end the chapter with a discussion of the Mal'cev correspondence between nilpotent Lie algebras over $\mathbb{Q}[x]$ and $\mathbb{Q}[x]$-completions of finitely generated torsion free nilpotent groups. Similar results are mentioned in R. B. Warfield, Jr. [14].

Let G be a finitely generated torsion free nilpotent group and let $u = (u_1, \ldots, u_n)$ be a Mal'cev basis for G. Denote by $\mathbb{Q}[x]G$ the group ring of G over $\mathbb{Q}[x]$. E. Formaneks' proof [3] of Jennings' theorem shows that $\mathbb{Q}[x]G$ is residually nilpotent. Put another way, if A denotes the augmentation ideal of $\mathbb{Q}[x]G$ then

$$\bigcap_{m=1}^{\infty} A^m = 0.$$

As usual, A is $\mathbb{Q}[x]$-generated by elements of the form $\{g-1 \mid g \in G\}$. One can deduce from Jennings' theorem that there exists an integer $k > 0$ such that G embeds in a natural way into the quotient ring $\mathbb{Q}[x]G/A^k$. More precisely, there exists an embedding $\psi : G \to 1+A/A^k$ given by $\psi(g) = 1+(g-1)+A^k$. If we uniquely express g in terms of the Mal'cev bases u, then $g = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$. By applying the straightening process described by S. A. Jennings [7], we have that ψ maps g to

$$\sum_{0 \le i_1 + \cdots + i_n \le k-1} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_n}{i_n} (u_1 - 1)^{i_1} \cdots (u_n - 1)^{i_n} + A^k.$$

As mentioned at the end of chapter one, $1 + A/A^k$ is a nilpotent $\mathbb{Q}[x]$-powered group with $\mathbb{Q}[x]$-exponentiation defined as

$$(1 + (g-1) + A^k)^\lambda = \sum_{i=0}^{k-1} \binom{\lambda}{i} (g-1)^i + A^k$$

for all $\lambda \in \mathbb{Q}[x]$ and $(g-1) + A^k \in A/A^k$. Therefore, $\psi : G \to 1 + A/A^k$ is a homomorphic mapping from a finitely generated torsion free nilpotent group into a nilpotent $\mathbb{Q}[x]$-powered group. By Theorem 2.6.1 ψ can be extended to a $\mathbb{Q}[x]$-homomorphism (in fact, an embedding) $\bar{\psi} : G^{\mathbb{Q}[x]} \to 1 + A/A^k$ defined as

$$\bar{\psi}(u_1^{\alpha_1} \cdots u_n^{\alpha_n}) = \sum_{0 \le i_1 + \cdots + i_n \le k-1} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_n}{i_n} (u_1 - 1)^{i_1} \cdots (u_n - 1)^{i_n} + A^k$$

where $\alpha \in \mathbb{Q}[x]$ and $G^{\mathbb{Q}[x]}$ denotes the $\mathbb{Q}[x]$-completion of G with respect to $u = (u_1, \ldots, u_n)$.

Note. ψ and $\bar{\psi}$ agree with each other whenever $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}$. Moreover, the Hall polynomials which hold in G with respect to the basis $u = (u_1, \ldots, u_n)$ also hold in $\psi(G)$. Therefore, as the proof of Theorem 2.6.1 demonstrates, the same polynomials (which hold in $G^{\mathbb{Q}[x]}$ as well as in G) hold in $\bar{\psi}(G^{\mathbb{Q}[x]})$. This shows that elements of $\bar{\psi}(G^{\mathbb{Q}[x]})$ multiply and $\mathbb{Q}[x]$-exponentiate in the same way as those of $G^{\mathbb{Q}[x]}$.

From now on, we will identify $G^{\mathbb{Q}[x]}$ with its embedded image $\bar{\psi}(G^{\mathbb{Q}[x]})$ in $1 + A/A^k$. That is to say, $u_1^{\alpha_1} \cdots u_n^{\alpha_n} \in G^{\mathbb{Q}[x]}$ is identified with

$$\sum_{0 \le i_1 + \cdots + i_n \le k-1} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_n}{i_n} (u_1 - 1)^{i_1} \cdots (u_n - 1)^{i_n} + A^k \in 1 + A/A^k.$$

I will now define the log and exp maps for our setting.

1. $\log : 1 + A/A^k \to A/A^k$ is defined as

$$\log(1 + (a + A^k)) = \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i} a^i + A^k.$$

2. $\exp : A/A^k \to 1 + A/A^k$ is defined as

$$\exp(a + A^k) = \sum_{i=0}^{k-1} \frac{1}{i!} a^i + A^k.$$

As usual the log and exp maps are inverses of each other:

1. $\log(\exp(a + A^k)) = a + A^k$ and

2. $\exp(\log(1 + (a + A^k)) = 1 + (a + A^k)$.

We will focus our attention on the restriction of the log and exp maps to particular subrings:

1. $\log : G^{\mathbb{Q}[x]} \to A/A^k$ is defined as

$$\log(u_1^{\alpha_1} \cdots u_n^{\alpha_n}) = \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i} (u_1^{\alpha_1} \cdots u_n^{\alpha_n} - 1)^i + A^k$$

2. Let $\mathcal{M} =$ im $\log(G^{\mathbb{Q}[x]})$ in A/A^k. Then $\exp : \mathcal{M} \to G^{\mathbb{Q}[x]}$ is defined as

$$\exp(m) = \sum_{i=0}^{k-1} \frac{1}{i!} m^i + A^k,$$

where $m = \log(u_1^{\gamma_1} \cdots u_n^{\gamma_n})$ for some $\gamma_1, \ldots, \gamma_n \in \mathbb{Q}[x]$. Observe that, by the inverse property above, we can simply write $\exp(m) = u_1^{\gamma_1} \cdots u_n^{\gamma_n}$.

The next theorem will be used in the discussion of the Mal'cev correspondence. The notations above will be the same as in what follows. Many of these results are analogues of the work done in S. A. Jennings [7], G. Baumslag [2] and R. B. Warfield, Jr. [14].

Theorem 2.14.1. *The submodule $\mathcal{L}$ of $\mathbb{Q}[x]G/A^k$ spanned by $\log(G^{\mathbb{Q}[x]})$ is a $\mathbb{Q}[x]$-module of dimension n and is a nilpotent Lie subalgebra over $\mathbb{Q}[x]$ of the commutation Lie algebra on $\mathbb{Q}[x]G/A^k$.*

Proof. Let $G^{\mathbb{Q}[x]} = \{u_1^{\alpha_1} \cdots u_n^{\alpha_n} \mid \alpha_i \in \mathbb{Q}[x]\}$ be a nilpotent $\mathbb{Q}[x]$-powered group of class c. I claim that if w is a positive integer satisfying $w > c$ and $g_i = \log(h_i)$ for $i = 1, \ldots, w$ where each $h_i \in G^{\mathbb{Q}[x]}$, then $(g_1, g_2, \ldots, g_w) \equiv 0 \mod A^k$ (as usual, we define $(x_1, x_2) = x_1x_2 - x_2x_1$ and, inductively, $(x_1, \ldots, x_{n-1}, x_n) = ((x_1, \ldots, x_{n-1}), x_n)$). Well, note that we can write $h_i = \exp(g_i)$ for each $i = 1, \ldots, w$. Now, by applying the Baker-Campbell-Hausdorff formula (see [2] or [7]), we obtain

$$\begin{aligned}
[h_1, h_2] &= h_1^{-1}h_2^{-1}h_1h_2 \\
&= \exp(-g_1)\exp(-g_2)\exp(g_1)\exp(g_2) \\
&= \exp(g_1, g_2) + \cdots
\end{aligned}$$

where the rest of the sum consists of commutator terms in g_1 and g_2, each of whose commutator weight exceeds 2. By induction, we have $[h_1, h_2, \ldots, h_w] =$

$\exp(g_1, g_2, \ldots, g_w) + \cdots$, where the where the rest of the sum consists of commutator terms in the g_i's, each of whose commutator weight exceeds w. Since $G^{\mathbb{Q}[x]}$ has class c and $w > c$, we have that $[h_1, h_2, \ldots, h_w] = 1$. Therefore $(g_1, g_2, \ldots, g_w) + \sum p_j \equiv 0 \mod A^k$ where each p_j is a linear combination of commutators of the form $(g_{s_1}, \ldots, g_{s_j})$ where $j = w+1, w+2, \ldots$. Rewriting the equation, we obtain $(g_1, g_2, \ldots, g_w) + A^k = -\sum p_j + A^k$. By applying the same technique described in [7], $(g_1, g_2, \ldots, g_w) + A^k \in (A/A^k)^{w+N}$ for all $N \geq 0$. For large enough N, we have

$$\bigcap_{N=0}^{\infty} (A/A^k)^N = A^k.$$

Therefore $(g_1, g_2, \ldots, g_w) + A^k = A^k$ and the claim is proven. This shows that $(\log(g_{s_1}), \ldots, \log(g_{s_n})) \equiv 0 \mod A^k$ where each $s_j \in \{1, \ldots, w\}$. Hence $\mathcal{L}$, the span of $\log(G^{\mathbb{Q}[x]})$, is a nilpotent Lie subalgebra of the commutation Lie algebra of $\mathbb{Q}[x]G/A^k$. It can be proven just as in [7] that if $g \in \mathcal{L}$, then there exists $\beta_1, \ldots, \beta_n \in \mathbb{Q}[x]$ such that $g = \beta_1 \log(u_1) + \cdots + \beta_n \log(u_n)$. Hence the elements $\log(u_1), \ldots, \log(u_n)$ form a basis for $\mathcal{L}$. This completes the proof. □

Definition 2.14.1. $\mathcal{L} = \mathcal{L}_{G^{\mathbb{Q}[x]}}$ is called the *Lie algebra* of $G^{\mathbb{Q}[x]}$.

I will now discuss the relationship between the $\mathbb{Q}[x]$-automorphisms of $G^{\mathbb{Q}[x]}$ and the automorphisms of its Lie algebra, $\mathcal{L}$ (see [2] for a similar result). Suppose that $\rho \in \mathrm{Aut}_{\mathbb{Q}[x]}(G^{\mathbb{Q}[x]})$. Then ρ induces an automorphism $\bar{\rho} : \mathbb{Q}[x]G/A^k \to \mathbb{Q}[x]G/A^k$

defined by

$$\bar{\rho}\left(\sum_{finite} \gamma_i g_i + A^k\right) = \sum_{finite} \gamma_i \rho(g_i) + A^k$$

where $\gamma_i \in \mathbb{Q}[x]$ and $g_i \in G$. Note that $\bar{\rho}$ is indeed an automorphism, since for any $g_1, g_2 \in \mathbb{Q}[x]G$ and $\beta_1,\ \beta_2 \in \mathbb{Q}[x]$, we have

$$\bar{\rho}(\beta_1 g_1 + \beta_2 g_2 + A^k) = \beta_1 \bar{\rho}(g_1 + A^k) + \beta_2 \bar{\rho}(g_2 + A^k).$$

For any $g \in G^{\mathbb{Q}[x]}$ we have

$$\begin{aligned}
\bar{\rho}(\log(g)) &= \bar{\rho}\left(\sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i}(g-1)^i + A^k\right) \\
&= \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i}\left(\bar{\rho}[(g-1)^i]) + A^k\right) \\
&= \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i}\left(\bar{\rho}\left[\sum_{j=0}^{i} (-1)^j \binom{i}{j} g^{i-j}\right]\right) + A^k \\
&= \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i}\left(\sum_{j=0}^{i} (-1)^j \binom{i}{j} [\rho(g)]^{i-j}\right) + A^k \\
&= \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{i}(\rho(g)-1)^i + A^k \\
&= \log(\rho(g)).
\end{aligned}$$

The above computation shows that every $\rho \in \mathrm{Aut}_{\mathbb{Q}[x]}(G^{\mathbb{Q}[x]})$ gives rise to an automorphism of the Lie algebra $\mathcal{L}$ which maps $\log(G^{\mathbb{Q}[x]})$ onto itself. Now suppose that we are given $\rho \in \mathrm{Aut}(\mathcal{L})$ which maps $\log(G^{\mathbb{Q}[x]})$ onto itself. Define the map $\eta : G^{\mathbb{Q}[x]} \to G^{\mathbb{Q}[x]}$ by $\eta(g) = h$ if $\rho(\log(g)) = \log(h)$. I claim that $\eta \in \mathrm{Aut}_{\mathbb{Q}[x]}(G^{\mathbb{Q}[x]})$. Let g_1, g_2 and $h \in G^{\mathbb{Q}[x]}$.

1. Suppose $\eta(g_1) = \eta(g_2) = h$. Then $\rho(\log(g_1)) = \rho(\log(g_2)) = \log(h)$. Since ρ is injective, we obtain $\log(g_1) = \log(g_2)$. Therefore, since log is an injective map, we have $g_1 = g_2$ and so η is a $\mathbb{Q}[x]$-monomorphism.

2. Let's suppose we are given $h \in G^{\mathbb{Q}[x]}$. Since ρ is surjective there exists $g \in G^{\mathbb{Q}[x]}$ such that $\rho(\log(g)) = \log(h)$. Hence $\eta(g) = h$ and so η is a $\mathbb{Q}[x]$-epimorphism.

3. Let $\eta(g_1 g_2) = h$. Then we know that $\rho(\log(g_1 g_2)) = \log(h)$ by definition of η. We also know that $\log(g_1 g_2) = \log(g_1) * \log(g_2)$ by the Baker-Campbell-Hausdorff formula (recall that $\exp x \exp y = \exp(x*y)$). Hence $\log(g_1 g_2) = \log(g_1) * \log(g_2)$ yields

$$\begin{aligned} \rho(\log(g_1 g_2)) &= \rho(\log(g_1) * \log(g_2)) \\ &= \rho(\log(g_1)) * \rho(\log(g_2)) \\ &= \log(h). \end{aligned}$$

   Therefore, if $\rho(\log(g_1)) = \log(h_1)$ and $\rho(\log(g_2)) = \log(h_2)$, then

$$\log(h) = \log(h_1) * \log(h_2) = \log(h_1 h_2).$$

   Since log is injective, $h = h_1 h_2$ and so $\eta(g_1 g_2) = \eta(g_1)\eta(g_2)$.

In conclusion, $\mathrm{Aut}_{\mathbb{Q}[x]}(G^{\mathbb{Q}[x]})$ is isomorphic to the group of those automorphisms of the finite dimensional $\mathbb{Q}[x]$-module $\mathcal{L} =$ the span of $\log(G^{\mathbb{Q}[x]})$ which maps the subset $\log(G^{\mathbb{Q}[x]})$ onto itself.

I will now discuss the Mal'cev correspondence between $\mathbb{Q}[x]$-completions of finitely generated torsion free nilpotent groups and nilpotent Lie algebras over $\mathbb{Q}[x]$. Let G be a finitely generated torsion free nilpotent groups with some Mal'cev basis $u = (u_1, \ldots, u_n)$. By Theorem 2.14.1, the submodule of $\mathbb{Q}[x]G/A^k$ spanned by $\log(G^{\mathbb{Q}[x]})$, namely $\mathcal{L}$, is a nilpotent Lie subalgebra over $\mathbb{Q}[x]$ of the commutation Lie algebra on $\mathbb{Q}[x]G/A^k$. In [2]), G. Baumslag proves that if Λ is any residually nilpotent Lie algebra over a field of characteristic 0, then $(\Lambda, *)$ is a group, where the binary operation, $*$, is defined by the Baker-Campbell-Hausdorff formula

$$x * y = \log(\exp(x)\exp(y)) = x + y + \frac{1}{2}(x,y) + \frac{1}{12}(y,x,x) + \cdots$$

for any $x, y \in \Lambda$. Using methods similar to those in [2], it can be seen that $(\mathcal{L}, *)$ is a group. In fact, $(\mathcal{L}, *) = \mathcal{L}^*$ is a nilpotent $\mathbb{Q}[x]$-powered group with multiplication $*$ and $\mathbb{Q}[x]$-exponentiation defined by $g^{*\alpha} = \alpha g$ for any $g \in \mathcal{L}^*$ and $\alpha \in \mathbb{Q}[x]$. Notice, in particular, that if $\log(h) \in \mathcal{L}^*$ and $\beta \in \mathbb{Q}[x]$, then $\log(h^\beta) = \beta \log(h) = [\log(h)]^{*\beta}$. Define the mapping $\Phi : G^{\mathbb{Q}[x]} \to \mathcal{L}^*$ by $\Phi(g) = \log(g)$. Then Φ is a $\mathbb{Q}[x]$-isomorphism:

1. Let $g_1, g_2 \in G^{\mathbb{Q}[x]}$ such that $\Phi(g_1) = \Phi(g_2)$. Then $\log(g_1) = \log(g_2)$ and so $g_1 = g_2$. Hence Φ is a $\mathbb{Q}[x]$-monomorphism.

2. Let $h + A^k \in \mathcal{L}^*$. Then $\exp(h + A^k) = g$ for some $g \in G^{\mathbb{Q}[x]}$. Hence $h + A^k = \log(g) = \Phi(g)$ and so Φ is a $\mathbb{Q}[x]$-epimorphism.

3. Φ is a $\mathbb{Q}[x]$-homomorphism since

(a) $\Phi(g_1g_2) = \log(g_1g_2) = \log(g_1) * \log(g_2) = \Phi(g_1) * \Phi(g_2)$

(b) $\Phi(g^\alpha) = \log(g^\alpha) = \alpha \log(g) = \alpha\Phi(g) = [\Phi(g)]^{*\alpha}$ for any $\alpha \in \mathbb{Q}[x]$.

The inverse map, $\Phi^{-1} : \mathcal{L}^* \to G^{\mathbb{Q}[x]}$, is given by $\Phi^{-1}(l) = \exp(l)$. Observe that if $\sigma_1 \log(g_1) * \cdots * \sigma_r \log(g_r) \in \mathcal{L}^*$ where $\sigma_i \in \mathbb{Q}[x]$ and $g_i \in G^{\mathbb{Q}[x]}$, then we have

$$\begin{aligned} \Phi^{-1}(\sigma_1 \log(g_1) * \cdots * \sigma_r \log(g_r)) &= \exp(\sigma_1 \log(g_1) * \cdots * \sigma_r \log(g_r)) \\ &= \exp(\sigma_1 \log(g_1)) \cdots \exp(\sigma_r \log(g_r)) \\ &= g_1^{\sigma_1} \cdots g_r^{\sigma_r} \end{aligned}$$

by repeatedly using the Baker-Campbell-Hausdorff formula. Now, suppose that G is as above with Mal'cev basis $u = (u_1, \ldots, u_n)$ and let H be another finitely generated torsion free nilpotent group with some Mal'cev basis $v = (v_1, \ldots, v_m)$. Let $\tau : G^{\mathbb{Q}[x]} \to \mathcal{L}_{G^{\mathbb{Q}[x]}}$ be defined as $\tau(u_1^{\alpha_1} \cdots u_n^{\alpha_n}) = \log(u_1^{\alpha_1} \cdots u_n^{\alpha_n})$ and $\tilde{\tau} : H^{\mathbb{Q}[x]} \to \mathcal{L}_{H^{\mathbb{Q}[x]}}$ as $\tilde{\tau}(v_1^{\mu_1} \cdots v_m^{\mu_m}) = \log(v_1^{\mu_1} \cdots v_m^{\mu_m})$. Then the $\mathbb{Q}[x]$-homomorphism $\phi : G^{\mathbb{Q}[x]} \to H^{\mathbb{Q}[x]}$ defined by $\phi(u_1^{\alpha_1} \cdots u_n^{\alpha_n}) = v_1^{\mu_1} \cdots v_m^{\mu_m}$ induces a homomorphism $\bar{\phi} : \mathcal{L}_{G^{\mathbb{Q}[x]}} \to \mathcal{L}_{H^{\mathbb{Q}[x]}}$ between their respective Lie algebras. This mapping is defined as $\bar{\phi}(\log(u_1^{\alpha_1} \cdots u_n^{\alpha_n})) = \log(v_1^{\mu_1} \cdots v_m^{\mu_m})$. Observe that $\bar{\phi} = \tilde{\tau} \circ \phi \circ \tau^{-1}$ and so the maps commute. Moreover, suppose $\Omega : \mathcal{L} \to \mathcal{M}$ is any homomorphism between 2 nilpotent Lie algebras over $\mathbb{Q}[x]$. Let $\iota_1 : \mathcal{L} \to \mathcal{L}^*$ and $\iota_2 : \mathcal{M} \to \mathcal{M}^*$ be the identity mappings of the Lie algebras into their respective groups with operation $*$. Then Ω induces a $\mathbb{Q}[x]$-homomorphism, $\bar{\Omega} : \mathcal{L}^* \to \mathcal{M}^*$ in the obvious way. In fact, $\bar{\Omega}$ is a $\mathbb{Q}[x]$-isomorphism if and only if Ω is a Lie algebra isomorphism (see P. F. Pickel [13] for the similar result of when the Lie algebras are over a field of characteristic 0). In conclusion, the correspondence between nilpotent Lie algebras over $\mathbb{Q}[x]$ and $\mathbb{Q}[x]$-completions of finitely generated torsion free nilpotent groups is categorical.

Chapter 3

$\mathbb{Z}[x]$-Groups

In this chapter, I will discuss a specific kind of exponential group known as a $\mathbb{Z}[x]$-group. I'll begin by defining an A-group and focus on the case where $A = \mathbb{Z}[x]$. The notion of an *A-group* was first introduced by R. C. Lyndon [10]. An additional axiom was incorporated into Lyndons' list by A. G. Myasnikov and V. N. Remeslennikov in [12]. This axiom allows one to view an abelian A-group as an A-module. It is the latter definition that I will be using from now on.

I will develop the theory and terminology of several different types of classes of $\mathbb{Z}[x]$-groups. Such classes include $\mathcal{N}^*$-groups, $\mathcal{I}$-groups, torsion $\mathbb{Z}[x]$-groups, $\mathcal{R}$-groups and $\mathbb{Z}[x]$-groups of type HP. The chapter concludes with a discussion of unitriangular $\mathbb{Z}[x]$-groups.

3.1 Basic Notions and Results

Definition 3.1.1. Let A be any associative ring with unity and G be an arbitrary group. Suppose that G is equipped with an action by A,

$$G \times A \to G \text{ defined by } (g, \alpha) \mapsto g^{\alpha},$$

such that for all $g \in G$ and for all $\alpha \in A$, the element $g^{\alpha} \in G$ is uniquely determined. Then G is called an *A-group* or an *exponential group* if the following axioms hold:

1. $g^1 = g$, $g^{\alpha} g^{\beta} = g^{\alpha+\beta}$, $(g^{\alpha})^{\beta} = g^{\alpha\beta}$ for all $g \in G$ and for all $\alpha, \beta \in A$.

2. $(h^{-1}gh)^{\alpha} = h^{-1}g^{\alpha}h$ for all $g, h \in G$ and for all $\alpha \in A$.

3. If $g, h \in G$ satisfy the relation $[g, h] = 1$, then $(gh)^{\mu} = g^{\mu}h^{\mu}$ for all $\mu \in A$.

Note. If R is a binomial ring and G is a nilpotent R-powered group, then axiom 3 above holds by the Hall-Petresco axiom. Therefore every nilpotent R-powered group is an R-group.

We focus our attention on the specific case where $A = \mathbb{Z}[x]$. Some consequences of the definition are

Lemma 3.1.1. *Let G be a $\mathbb{Z}[x]$-group. Then*

1. $(g^{\mu})^{-1} = g^{-\mu}$ *for all* $\mu \in \mathbb{Z}[x]$.

2. $g^0 = 1$ *for all* $g \in G$.

3. $1^\alpha = 1$ *for all* $\alpha \in \mathbb{Z}[x]$.

4. *if* $[g, h] = 1$ *where* $g, h \in G$, *then* $[g^\alpha, h^\beta] = 1$ *for all* $\alpha, \beta \in \mathbb{Z}[x]$.

Proof. The first two statements are clear. As for 3, notice that if $g \in G$ then $1^\alpha g^\alpha = (1g)^\alpha$ since $[1, g] = 1$. Thus $1^\alpha g^\alpha = g^\alpha$ and so $1^\alpha = 1$. For 4, observe that if $[g, h] = 1$ where $g, h \in G$, then $[g^\alpha, h] = 1$ for all $\alpha \in \mathbb{Z}[x]$ since $h^{-1}g^{-\alpha}h = (h^{-1}gh)^{-\alpha} = g^{-\alpha}$. Hence, $g^{-\alpha}h^{-\beta}g^\alpha = (g^{-\alpha}h^{-1}g^\alpha)^\beta = (h^{-1})^\beta$ and the result follows. □

Next I will give some preliminary definitions which will be used throughout this chapter. From this point on, all subrings of $\mathbb{Z}[x]$ will be with unity.

Definition 3.1.2. Let G be a $\mathbb{Z}[x]$-group and let K be a subring of $\mathbb{Z}[x]$. Then H is called a *K-subgroup* of G if $H \leq G$ and $g^\alpha \in H$ for all $g \in H$ and $\alpha \in K$.

I will denote "H is a K-subgroup of G" by $H \leq_K G$.

Definition 3.1.3. Let G be a $\mathbb{Z}[x]$-group and K a subring of $\mathbb{Z}[x]$. Then N is a *normal K-subgroup* of G if $N \leq_{\mathbb{Z}[x]} G$ and $N \trianglelefteq G$.

I will denote "N is a normal K-subgroup of G" by $N \trianglelefteq_K G$.

Lemma 3.1.2. *Let* $\{G_1, \ldots, G_n\}$ *be any (possibly infinite) collection of* $\mathbb{Z}[x]$*-groups. Then* $\bigcap_{i=1}^n G_i$ *is a* $\mathbb{Z}[x]$*-group.*

Proof. Verifying that the axioms hold in $\bigcap_{i=1}^n G_i$ is straightforward. □

Definition 3.1.4. Let G be a $\mathbb{Z}[x]$-group and let K be a subring of $\mathbb{Z}[x]$. If $S = \{g_1, \ldots, g_k\}$ is a subset of G, then

$$H = \bigcap_{S \subset H_i \leq_K G} \{H_i\} = gp_K(g_1, \ldots, g_k)$$

is called the *K-subgroup of G which is K-generated by* $\{g_1, \ldots, g_k\}$. We call S a set of *K-generators* for H.

In particular, suppose that $K = \mathbb{Z}[x]$. One can describe $gp_{\mathbb{Z}[x]}(S)$ in the following way: let $S_0 = gp(S)$, $S_1 = gp(g_0^{\alpha_0} \mid g_0 \in S_0, \alpha_0 \in \mathbb{Z}[x])$ and define inductively $S_{n+1} = gp(g_n^{\alpha_n} \mid g_n \in S_n, \alpha_n \in \mathbb{Z}[x])$. Then

$$gp_{\mathbb{Z}[x]}(S) = \bigcup_{n=0}^{\infty} S_n.$$

An element $g \in gp_{\mathbb{Z}[x]}(S)$ is called a *$\mathbb{Z}[x]$-word in S.*

The next lemma can be found in [12].

Lemma 3.1.3. *Let G be a $\mathbb{Z}[x]$-group. If $N \trianglelefteq G$, then $gp_{\mathbb{Z}[x]}(N) \trianglelefteq G$.*

Definition 3.1.5. Let G be any $\mathbb{Z}[x]$-group and let $H_1, H_2 \leq_{\mathbb{Z}[x]} G$. Then

$$[H_1, H_2]_{\mathbb{Z}[x]} = gp_{\mathbb{Z}[x]}(\,[h_1, h_2] \mid h_1 \in H_1, h_2 \in H_2)$$

is the *commutator $\mathbb{Z}[x]$-subgroup of H_1 and H_2*. In particular, $[G, G]_{\mathbb{Z}[x]}$ is the *derived $\mathbb{Z}[x]$-subgroup* of G. If $H_1, \ldots, H_i \leq_{\mathbb{Z}[x]} G$ then we recursively define, for $i > 2$,

$$[H_1, \ldots, H_i]_{\mathbb{Z}[x]} = [[H_1, \ldots, H_{i-1}]_{\mathbb{Z}[x]}, H_i]_{\mathbb{Z}[x]}.$$

Remark. By the previous definition and lemma, it follows that if G is any $\mathbb{Z}[x]$-group, then $[G, G]_{\mathbb{Z}[x]} \trianglelefteq G$.

Lemma 3.1.4. *Let G be a $\mathbb{Z}[x]$-group with $H \leq_{\mathbb{Z}[x]} G$ and $N \leq_{\mathbb{Z}[x]} G$. If $H \trianglelefteq G$ and $N \trianglelefteq G$, then $[H, N]_{\mathbb{Z}[x]} \trianglelefteq G$.*

Proof. If $H \trianglelefteq G$ and $N \trianglelefteq G$, then $[H, N] \trianglelefteq G$. However, $gp_{\mathbb{Z}[x]}([H, K]) \trianglelefteq G$ by Lemma 3.1.3. Since $gp_{\mathbb{Z}[x]}([H, K]) = [H, K]_{\mathbb{Z}[x]}$, the proof is complete. □

The product of two $\mathbb{Z}[x]$-subgroups of a $\mathbb{Z}[x]$-group is defined in the obvious way.

Lemma 3.1.5. *Suppose G is a $\mathbb{Z}[x]$-group with normal $\mathbb{Z}[x]$-subgroups H, K and L. Then*

$$[HK, L]_{\mathbb{Z}[x]} = gp_{\mathbb{Z}[x]}([H, L],\ [K, L])$$

and, similarly,

$$[H, KL]_{\mathbb{Z}[x]} = gp_{\mathbb{Z}[x]}([H, K],\ [H, L])$$

Proof. I will prove the first of the claims. The other works the same way. Suppose $x \in H$, $y \in K$ and $z \in L$. Then we have $[xy, z] \in [HK, L]$. Now, $[xy, z] = [x, z]^y[y, z]$ and $[x, z]^y \in [H, L]$ since $[H, L] \trianglelefteq G$. Hence

$$[xy, z] \in gp_{\mathbb{Z}[x]}([H, L],\ [K, L]).$$

If we now take $\mathbb{Z}[x]$-words in elements of $[HK, L]_{\mathbb{Z}[x]}$, we can repeatedly use the above identity to rewrite our $\mathbb{Z}[x]$-word in terms of elements of both $[H, L]$ and $[K, L]$. Hence

$$[HK, L]_{\mathbb{Z}[x]} \subset gp_{\mathbb{Z}[x]}([H, L], [K, L]).$$

Clearly,

$$gp_{\mathbb{Z}[x]}([H, L], [K, L]) \subset [HK, L]_{\mathbb{Z}[x]}.$$

This completes the proof. □

3.2 Ideals

Given a normal $\mathbb{Z}[x]$-subgroup N of a $\mathbb{Z}[x]$-group G, the quotient group G/N need not be a $\mathbb{Z}[x]$-group. In this section I will discuss a special class of normal $\mathbb{Z}[x]$-subgroups for which the $\mathbb{Z}[x]$-action of G induces a $\mathbb{Z}[x]$-action on G/N. Such subgroups are called *ideals* (see [12]). I will give the definition of an ideal for an arbitrary A-group.

Definition 3.2.1. Let G be an A-group and let N be a normal A-subgroup of G. Then N is called an *ideal* of G if

1. $g^{-\alpha}(gn)^{\alpha} \in N$ for any $g \in G$, $n \in N$, $\alpha \in A$ and
2. if $[g, h] \in N$ then $h^{-\alpha}g^{-\alpha}(gh)^{\alpha} \in N$ for any $g, h \in G, \alpha \in A$.

Lemma 3.2.1. *Let G be a $\mathbb{Z}[x]$-group and let N be an ideal of G. Then*

$$gN = hN \textit{ implies } g^{\beta}N = h^{\beta}N$$

for any $g, h \in G$ and any $\beta \in \mathbb{Z}[x]$.

Proof. Suppose $gN = hN$. Then there exists $n \in N$ such that $gn = h$. For every $\beta \in \mathbb{Z}[x]$, we have $(gn)^\beta = h^\beta$. Since N is an ideal of G, $g^{-\beta}(gn)^\beta = \bar{n}$ for some $\bar{n} \in N$. Therefore $(gn)^\beta = g^\beta \bar{n} = h^\beta$. Thus $g^\beta \bar{n} N = h^\beta N$ and so $g^\beta N = h^\beta N$. □

Theorem 3.2.1. *Let G be a $\mathbb{Z}[x]$-group and let N be an ideal of G. Then the $\mathbb{Z}[x]$-action on G induces a $\mathbb{Z}[x]$-action on G/N,*

$$(gN)^\beta = g^\beta N \text{ for all } g \in G \text{ and } \beta \in \mathbb{Z}[x],$$

which turns G/N is a $\mathbb{Z}[x]$-group.

Proof. Suppose $gN = hN$ for some $g, h \in G$. Then for every $\beta \in \mathbb{Z}[x]$, we have $(gN)^\beta = (hN)^\beta$. By the previous lemma, this becomes $g^\alpha N = h^\alpha N$. Hence this $\mathbb{Z}[x]$-action is well-defined. I will verify the $\mathbb{Z}[x]$-axioms for G/N:

1. $(gN)^1 = g^1 N = gN$.

2. $(gN)^\alpha (gN)^\beta = (g^\alpha N)(g^\beta N) = (g^\alpha g^\beta)N = g^{\alpha+\beta}N = (gN)^{\alpha+\beta}$.

3. $[(gN)^\alpha]^\beta = (g^\alpha N)^\beta = [(g^\alpha)^\beta]N = g^{\alpha\beta}N = (gN)^{\alpha\beta}$.

4. $(gN)^{-1}(hN)^\alpha(gN) = (g^{-1}N)(h^\alpha N)(gN) = (g^{-1}h^\alpha g)N = [(g^{-1}hg)^\alpha]N = [(g^{-1}hg)N]^\alpha = [(gN)^{-1}(hN)(gN)]^\alpha$.

5. $(gN)(hN) = (hN)(gN)$ yields $[g, h] \in N$. Hence $h^{-\alpha}g^{-\alpha}(gh)^\alpha \in N$ and, consequently, $(hN)^{-\alpha}(gN)^{-\alpha}(gNhN)^\alpha \in G/N$.

□

It is clear that if G is a $\mathbb{Z}[x]$-group, then G and $\{1\}$ are ideals of G. However, it may not be the case that $Z(G)$ and $[G,G]_{\mathbb{Z}[x]}$ are ideals of G. It is worth noting that $Z(G)$ does satisfy the first condition of the definition, for if G is a $\mathbb{Z}[x]$-group with $g \in G$, then $[g,z] = 1$ for any $z \in Z(G)$ and $\lambda \in \mathbb{Z}[x]$. Consequently, $(gz)^\lambda = g^\lambda z^\lambda$ and so $g^{-\lambda}(gz)^\lambda \in Z(G)$.

Lemma 3.2.2. *Let G be a $\mathbb{Z}[x]$-group and suppose $\{N_1, \ldots, N_k\}$ be a collection of ideals of G. Then $\cap_{i=1}^k N_i$ is an ideal of G.*

Proof. It suffices to show that $N_1 \cap N_2$ is an ideal of G. The result will follow by induction. Let $g \in G$ and suppose that $n \in N_1 \cap N_2$. Then $n \in N_1$ and $n \in N_2$ implies that $g^{-\alpha}(gn)^\alpha \in N_1$ and $g^{-\alpha}(gn)^\alpha \in N_2$ for all $\alpha \in \mathbb{Z}[x]$. Therefore $g^{-\alpha}(gn)^\alpha \in N_1 \cap N_2$. Now suppose that $[g,h] \in N_1 \cap N_2$ for some $g, h \in G$. Then $[g,h] \in N_1$ and $[g,h] \in N_2$. Hence $h^{-\alpha}g^{-\alpha}(gh)^\alpha \in N_1$ and $h^{-\alpha}g^{-\alpha}(gh)^\alpha \in N_2$ for every $\alpha \in \mathbb{Z}[x]$. Therefore $h^{-\alpha}g^{-\alpha}(gh)^\alpha \in N_1 \cap N_2$. This completes the proof. □

Lemma 3.2.3. *Let G be a $\mathbb{Z}[x]$-group. If N is an ideal of G and $H \leq_{\mathbb{Z}[x]} G$, then $HN \leq_{\mathbb{Z}[x]} G$ and $gp_{\mathbb{Z}[x]}(H,N) = HN$.*

Proof. Let $h \in H, n \in N$ and $\lambda \in \mathbb{Z}[x]$. Then $h^{-\lambda}(hn)^\lambda \in N$ since N is an ideal of G. Therefore there exists $\bar{n} \in N$ such that $(hn)^\lambda = h^\lambda \bar{n}$. Hence $(hn)^\lambda \in HN$. The rest of the proof is straightforward. □

Lemma 3.2.4. *Let G be a $\mathbb{Z}[x]$-group and suppose $N \leq_{\mathbb{Z}[x]} K$ and $K \leq_{\mathbb{Z}[x]} G$. If N is an ideal of G, then N is an ideal of K.*

Proof. Let N be an ideal of G. Clearly, $N \trianglelefteq_{\mathbb{Z}[x]} K$. If $n \in N$ then $g^{-\alpha}(gn)^{\alpha} \in N$ for all $g \in G$ and $\alpha \in \mathbb{Z}[x]$. Since $K \leq_{\mathbb{Z}[x]} G$, the above holds, in particular, for all $g \in K$. Verifying the other part of the definition is just as easy. $\square$

Lemma 3.2.5. *Let G be a $\mathbb{Z}[x]$-group. Suppose that $H \leq_{\mathbb{Z}[x]} G$ and $N \trianglelefteq_{\mathbb{Z}[x]} G$. If N is an ideal of G, then $H \cap N$ is an ideal of H.*

Proof. It is clear that $H \cap N \trianglelefteq_{\mathbb{Z}[x]} H$. I claim that $H \cap N$ is an ideal of H. Let $n \in H \cap N$. Since N is an ideal of G, we have $h^{-\alpha}(hn)^{\alpha} \in N$ for all $h \in H$ and $\alpha \in \mathbb{Z}[x]$. However, $n \in H$ yields $h^{-\alpha}(hn)^{\alpha} \in H$. Therefore, $h^{-\alpha}(hn)^{\alpha} \in H \cap N$. Suppose now that $h_1, h_2 \in H$ satisfy the condition $[h_1, h_2] \in H \cap N$. Then for arbitrary $\alpha \in \mathbb{Z}[x]$, we have $h_2^{-\alpha} h_1^{-\alpha}(h_1 h_2)^{\alpha} \in N$ since N is an ideal of G. Clearly $h_2^{-\alpha} h_1^{-\alpha}(h_1 h_2)^{\alpha} \in H$. Therefore $h_2^{-\alpha} h_1^{-\alpha}(h_1 h_2)^{\alpha} \in H \cap N$ and so $H \cap N$ is an ideal of H. $\square$

3.3 $\mathbb{Z}[x]$-Mappings

Definition 3.3.1. Let $\phi : G \to \bar{G}$ be a mapping between two $\mathbb{Z}[x]$-groups. Then ϕ is a $\mathbb{Z}[x]$-*homomorphism* if $\phi(gh) = \phi(g)\phi(h)$ and $\phi(g^{\lambda}) = [\phi(g)]^{\lambda}$ for all $g, h \in G$ and $\lambda \in \mathbb{Z}[x]$. $Hom_{\mathbb{Z}[x]}(G, \bar{G})$ will represent the collection of all $\mathbb{Z}[x]$-homomorphisms from G to $\bar{G}$.

The terms $\mathbb{Z}[x]$-*monomorphism*, $\mathbb{Z}[x]$-*epimorphism*, $\mathbb{Z}[x]$-*isomorphism* and $\mathbb{Z}[x]$-*automorphism* are defined in the obvious way. $Aut_{\mathbb{Z}[x]}(G)$ will represent the collection of all $\mathbb{Z}[x]$-automorphisms of G.

The *image of* ϕ is $\phi(G)$ and will be denoted by *im* ϕ. The *kernel of* ϕ, denoted by *ker* ϕ, is $\{g \in G \mid \phi(g) = 1\}$.

Lemma 3.3.1. *Let* $\phi : G \to \bar{G}$ *be an* $\mathbb{Z}[x]$*-homomorphism between two* $\mathbb{Z}[x]$*-groups. Then im* ϕ *is a* $\mathbb{Z}[x]$*-subgroup of* $\bar{G}$ *and ker* ϕ *is an ideal of* G*.*

Proof. It is easy to show that im ϕ is a $\mathbb{Z}[x]$-subgroup of $\bar{G}$ and that ker ϕ is a normal $\mathbb{Z}[x]$-subgroup of G. I will prove that ker ϕ is an ideal of G. Let $\phi : G \to \bar{G}$ be as above.

- Suppose $k \in \ker \phi$. Then for every $g \in G$ and $\beta \in \mathbb{Z}[x]$, we have

  $$\phi(g^{-\beta}(gk)^{\beta}) = [\phi(g)]^{-\beta}[\phi(g)\phi(k)]^{\beta} = 1$$

  since $\phi(k) = 1$. Therefore $g^{-\beta}(gk)^{\beta} \in \ker \phi$.

- Now suppose $g, h \in G$ satisfy $[g, h] \in \ker \phi$. Then $\phi(g^{-1}h^{-1}gh) = 1$ gives us $\phi(g)\phi(h) = \phi(h)\phi(g)$. Since $\phi(G)$ is a $\mathbb{Z}[x]$-subgroup of $\bar{G}$, by one of our $\mathbb{Z}[x]$-group axioms, we have $(\phi(g)\phi(h))^{\alpha} = \phi(g)^{\alpha}\phi(h)^{\alpha}$. Therefore,

  $$\phi(h^{-\alpha}g^{-\alpha}(gh)^{\alpha}) = \phi(h)^{-\alpha}\phi(g)^{-\alpha}(\phi(g)\phi(h))^{\alpha} = 1.$$

  Hence, $h^{-\alpha}g^{-\alpha}(gh)^{\alpha} \in \ker \phi$.

□

Corollary 3.3.1. *Let $\phi : G \to \bar{G}$ be an $\mathbb{Z}[x]$-homomorphism between two $\mathbb{Z}[x]$-groups. Then $G/ker\,\phi$ is a $\mathbb{Z}[x]$-group.*

Theorem 3.3.2 (First $\mathbb{Z}[x]$-Isomorphism Theorem). *Let $\phi : G \to \bar{G}$ be an $\mathbb{Z}[x]$-homomorphism between two $\mathbb{Z}[x]$-groups. Then*

$$G/ker\phi \ \cong_{\mathbb{Z}[x]} \ \phi(G).$$

Theorem 3.3.3 (Second $\mathbb{Z}[x]$-Isomorphism Theorem). *Let G be a $\mathbb{Z}[x]$-group, $H \leq_{\mathbb{Z}[x]} G$ and N an ideal of G. Then*

$$HN/N \ \cong_{\mathbb{Z}[x]} \ H/H \cap N.$$

Theorem 3.3.4 (Third $\mathbb{Z}[x]$-Isomorphism Theorem). *Let G be a $\mathbb{Z}[x]$-group and let H, K be ideals of G with $K \leq_{\mathbb{Z}[x]} H$. Then*

$$G/H \ \cong_{\mathbb{Z}[x]} \ \frac{G/K}{H/K}.$$

Lemma 3.3.2. *Let G and H be $\mathbb{Z}[x]$-groups. Suppose that $\phi : G \to H$ be a $\mathbb{Z}[x]$-epimorphism and N is an ideal of G containing $ker\ \phi$. Then $\phi(N)$ is an ideal of H.*

Proof. It is easy to verify that $\phi(N) \trianglelefteq_{\mathbb{Z}[x]} H$. I claim that $\phi(N)$ is an ideal of H. Since ϕ is a $\mathbb{Z}[x]$-epimorphism, if $a \in H$ then $a = \phi(\bar{a})$ for some $\bar{a} \in G$. If $\phi(g) \in \phi(N)$ for some $g \in G$ and $\alpha \in \mathbb{Z}[x]$, then

$$\begin{aligned}(a\phi(g))^{\alpha} &= (\phi(\bar{a})\phi(g))^{\alpha} = (\phi(\bar{a}g))^{\alpha} \\ &= \phi((\bar{a}g)^{\alpha}) = \phi(\bar{a}^{\alpha}\bar{g}) \text{ for some } \bar{g} \in N \\ &= (\phi(\bar{a}))^{\alpha}\phi(\bar{g}) = a^{\alpha}\phi(\bar{g}).\end{aligned}$$

Therefore, $a^{-\alpha}(a\phi(g))^{\alpha} \in \phi(N)$.

If $a = \phi(\bar{a}),\ b = \phi(\bar{b}) \in H,\ [a, b] \in \phi(N)$ and $\alpha \in \mathbb{Z}[x]$, then

$$\begin{aligned}[\phi(\bar{a}), \phi(\bar{b})] \in \phi(N) &\implies \phi([\bar{a}, \bar{b}]) \in \phi(N) \\ &\implies [\bar{a}, \bar{b}] \in N \\ &\implies \bar{b}^{-\alpha}\bar{a}^{-\alpha}(\bar{a}\bar{b})^{\alpha} \in N \\ &\implies \phi(\bar{b}^{-\alpha}\bar{a}^{-\alpha}(\bar{a}\bar{b})^{\alpha}) \in \phi(N) \\ &\implies \phi(\bar{b})^{-\alpha}\phi(\bar{a})^{-\alpha}(\phi(\bar{a})\phi(\bar{b})^{\alpha}) \in \phi(N).\end{aligned}$$

Hence $\phi(N)$ is an ideal of H. □

Lemma 3.3.3. *Let G and H be $\mathbb{Z}[x]$-groups. Suppose that $\phi : G \to H$ is a $\mathbb{Z}[x]$-epimorphism and N is an ideal of H. Then $\phi^{-1}(N)$ is an ideal of G, where*

$$\phi^{-1}(N) = \{g \in G \,|\, \phi(g) \in N\}.$$

Proof. Clearly, $\phi^{-1}(N) \trianglelefteq_{\mathbb{Z}[x]} G$. I will show that $\phi^{-1}(N)$ is an ideal of G. Let $g \in G, \alpha \in \mathbb{Z}[x]$ and $n \in \phi^{-1}(N)$. Then $\phi(n) \in N$ implies

$$\begin{aligned} \phi((gn)^\alpha) &= (\phi(gn))^\alpha = (\phi(g)\phi(n))^\alpha \\ &= (\phi(g))^\alpha m \text{ for some } m \in N. \end{aligned}$$

Consequently, we have that

$$\begin{aligned} \phi((gn))^\alpha &= (\phi(g))^\alpha m \Longrightarrow \\ \phi((gn)^\alpha) &= \phi(g^\alpha)m \Longrightarrow \\ g^{-\alpha}(gn)^\alpha &\in \phi^{-1}(N). \end{aligned}$$

If $g, h \in G$ such that $[g, h] \in \phi^{-1}(N)$, then it easy to verify that $h^{-\alpha}g^{-\alpha}(gh)^\alpha \in \phi^{-1}(N)$. □

Lemma 3.3.4. *Let G be a $\mathbb{Z}[x]$-group. Suppose $M \leq_{\mathbb{Z}[x]} G$ and N is an ideal of G. If M/N is an ideal of G/N, then M is an ideal of G.*

Proof. Let $gN \in G/N$ and $mN \in M/N$. Then for every $\alpha \in \mathbb{Z}[x]$, we have $(gN)^{-\alpha}(gNmN)^\alpha \in M/N$. This condition is equivalent to $g^{-\alpha}(gm)^\alpha N \in M/N$. This implies that $g^{-\alpha}(gm)^\alpha \in M$. Now suppose $g, h \in G$ satisfy $[g, h] \in M$. Then by applying the natural $\mathbb{Z}[x]$-epimorphism $\pi : G \to G/N$ we obtain, for any $\alpha \in \mathbb{Z}[x]$,

$$\begin{aligned}
\pi([g,h]) \in \pi(M) &\implies [gN, hN] \in M/N \\
&\implies (hN)^{-\alpha}(gN)^{-\alpha}(gNhN)^{\alpha} \in M/N \\
&\implies h^{-\alpha}g^{-\alpha}(gh)^{\alpha}N \in M/N \\
&\implies h^{-\alpha}g^{-\alpha}(gh)^{\alpha} \in M.
\end{aligned}$$

□

3.4 Direct Products

Let $\{G_i | \ i \in I\}$ be a family of $\mathbb{Z}[x]$-groups indexed by a non-empty set I. Define the set $\bar{G} = \{f : I \to \bigcup_{i\in I} G_i \mid f(i) \in G_i \text{ for all } i \in I\}$. Then $\bar{G}$ becomes a $\mathbb{Z}[x]$-group on defining multiplication and $\mathbb{Z}[x]$-exponentiation as follows:

1. $(ff')(i) = f(i)f'(i)$, where $f, f' \in \bar{G}$ and $i \in I$

2. $f^{\lambda}(i) = (f(i))^{\lambda}$ for all $\lambda \in \mathbb{Z}[x]$

An element $g \in \bar{G}$ can be viewed as a "vector" $g = (g_1, \ldots, g_i, \ldots)$ whose i^{th} coordinate is $g_i = f(i) \in G_i$ for all $i \in I$. By viewing the elements of $\bar{G}$ in this way, the group operations become

1. $(g_1, \ldots, g_i, \ldots)(h_1, \ldots, h_i, \ldots) = (g_1h_1, \ldots, g_ih_i, \ldots)$ for all $g_i, h_i \in G_i$

2. $(g_1, \ldots, g_i, \ldots)^{\lambda} = (g_1^{\lambda}, \ldots, g_i^{\lambda}, \ldots)$ for all $g_i \in G_i$ and for all $\lambda \in \mathbb{Z}[x]$

Definition 3.4.1. The $\mathbb{Z}[x]$-group $\bar{G}$ described above is called the *unrestricted direct product* of the $\{G_i\}$'s, denoted by

$$\bar{G} = \overline{\prod_{i \in I}} G_i.$$

Definition 3.4.2. Let $\{G_i |\ i \in I\}$ be a family of $\mathbb{Z}[x]$-groups indexed by a non-empty set I. Then a group G is termed a *direct product* of the G_i's, which will be denoted by

$$G = \prod_{i \in I} G_i,$$

if there exists $\mathbb{Z}[x]$-monomorphisms $\varphi_i : G_i \to G$ such that

1. $\varphi_i(G_i) \trianglelefteq_{\mathbb{Z}[x]} G$

2. $G = gp_{\mathbb{Z}[x]}(\bigcup_{i \in I} \varphi_i(G_i))$

3. $\varphi_i(G_i) \cap gp_{\mathbb{Z}[x]}(\bigcup_{j \neq i} \varphi_j(G_j)) = 1$ for $i, j \in I$ and $1 \in G$ denoting the identity element.

As usual, if I is a finite index set $I = \{1, 2, \ldots, n\}$, then the direct product of the $\{G_i\}$'s is written as $\prod_{i \in I} G_i = G_1 \times \cdots \times G_n$.

Note. Given a family of $\mathbb{Z}[x]$-groups $\{G_i |\ i \in I\}$, if we set

$$G = \left\{ f \in \overline{\prod_{i \in I}} G_i \mid f(i) = 1 \text{ except for finitely many } i \in I \right\},$$

then $G = \prod_{i \in I} G_i$. By the above observation, $G = \prod_{i \in I} G_i$ is a $\mathbb{Z}[x]$-group.

3.5 Abelian $\mathbb{Z}[x]$-Groups

It was mentioned in the introduction of this chapter that an abelian $\mathbb{Z}[x]$-group can be viewed as a $\mathbb{Z}[x]$-module by interpreting group multiplication and $\mathbb{Z}[x]$-exponentiation as module addition and scalar multiplication, respectively. If $g \in G$ and $\alpha \in \mathbb{Z}[x]$, one simply writes the group element g^α as αg and all of the module axioms are satisfied. Notice that $\mathbb{Z}[x]$-subgroups of an abelian $\mathbb{Z}[x]$-group are just submodules of the $\mathbb{Z}[x]$-module.

Lemma 3.5.1. *Every $\mathbb{Z}[x]$-subgroup of an abelian $\mathbb{Z}[x]$-group G is an ideal of G.*

Proof. Let $N \leq_{\mathbb{Z}[x]} G$. It is obvious that N is normal in G. If $g \in G$ and $n \in N$, then $g^{-\beta}(gn)^\beta = n^\beta \in N$ for any $\beta \in \mathbb{Z}[x]$ since $[g, n] = 1$. Moreover, $[g_1, g_2] = 1 \in N$ for all $g_1, g_2 \in G$ implies that $g_2^{-\beta} g_1^{-\beta}(g_1 g_2)^\beta = 1 \in N$. Hence N is an ideal of G. □

Definition 3.5.1. G is a *cyclic $\mathbb{Z}[x]$-group* if $G = gp_{\mathbb{Z}[x]}(g)$ for some $g \in G$.

Note. One can easily see that a $\mathbb{Z}[x]$-subgroup of a cyclic $\mathbb{Z}[x]$-group is not necessarily cyclic. For example, if $G = gp_{\mathbb{Z}[x]}(g)$ then the $\mathbb{Z}[x]$-subgroup $H = gp_{\mathbb{Z}[x]}(g^2, g^x)$ cannot be $\mathbb{Z}[x]$-generated by a single element of the form g^β for some $\beta \in \mathbb{Z}[x]$. This is due to the fact that $< 2, x >$ is not a principal ideal in $\mathbb{Z}[x]$.

Theorem 3.5.1. *Every $\mathbb{Z}[x]$-subgroup of a finitely $\mathbb{Z}[x]$-generated abelian $\mathbb{Z}[x]$-group is finitely $\mathbb{Z}[x]$-generated.*

Proof. Let G be a finitely $\mathbb{Z}[x]$-generated abelian $\mathbb{Z}[x]$-group with $H \leq_{\mathbb{Z}[x]} G$. Then G can be viewed as a $\mathbb{Z}[x]$-module and H a submodule of G. Since $\mathbb{Z}[x]$ is a noetherian ring every submodule of G is finitely generated. In particular, H (viewed as a submodule of G) is a finitely generated submodule of G. Therefore H is finitely $\mathbb{Z}[x]$-generated as an abelian $\mathbb{Z}[x]$-group. □

Lemma 3.5.2. *Let $\varphi : G \to \bar{G}$ be a $\mathbb{Z}[x]$-homomorphism between two $\mathbb{Z}[x]$-groups and suppose that $\bar{G}$ is an abelian $\mathbb{Z}[x]$-group. Then im φ is an ideal of $\bar{G}$.*

Proof. Let im $\varphi = I$. Clearly, $I \trianglelefteq_{\mathbb{Z}[x]} \bar{G}$ since I is a $\mathbb{Z}[x]$-subgroup of an abelian $\mathbb{Z}[x]$-group. Suppose $\varphi(g) \in I$, $\bar{g} \in \bar{G}$, and $\alpha \in \mathbb{Z}[x]$. Then

$$(\bar{g}\varphi(g))^{\alpha} = \bar{g}^{\alpha}(\varphi(g))^{\alpha}$$

because $\bar{G}$ is an abelian $\mathbb{Z}[x]$-group. Since $I \leq_{\mathbb{Z}[x]} \bar{G}$, we have $(\varphi(g))^{\alpha} \in I$. Hence, $\bar{g}^{-\alpha}(\bar{g}\varphi(g))^{\alpha} \in I$. If $\bar{h} \in \bar{G}$ satisfies $[\bar{g}, \bar{h}] \in I$ then $\bar{h}^{-\alpha}\bar{g}^{-\alpha}(\bar{g}\bar{h})^{\alpha} \in I$ since both conditions become vacuous. □

3.6 $\mathcal{I}$-Groups

Definition 3.6.1. G is termed an $\mathcal{I}$*-group* if every normal $\mathbb{Z}[x]$-subgroup of G is an ideal of G.

Remark. It follows immediately that if G is an $\mathcal{I}$-group, then $Z(G)$ and $[G, G]_{\mathbb{Z}[x]}$ are ideals of G.

Lemma 3.6.1. *Every abelian $\mathbb{Z}[x]$-group is an $\mathcal{I}$-group.*

Proof. We know that every $\mathbb{Z}[x]$-subgroup of an abelian $\mathbb{Z}[x]$-subgroup is a normal $\mathbb{Z}[x]$-subgroup. Lemma 3.5.1 now applies. □

Theorem 3.6.1. *Suppose G is an $\mathcal{I}$-group and $N \trianglelefteq_{\mathbb{Z}[x]} G$. Then G/N is an $\mathcal{I}$-group.*

Proof. Let $H/N \trianglelefteq_{\mathbb{Z}[x]} G/N$. Observe that $H \trianglelefteq_{\mathbb{Z}[x]} G$ (clearly the usual correspondence theorem holds for $\mathbb{Z}[x]$-groups). Hence both N and H are ideals in G. I claim that H/N is an ideal of G/N.

1. Let $gN \in G/N$, $hN \in H/N$ and $\alpha \in \mathbb{Z}[x]$. Then

$$(gN)^{-\alpha}(gNhN)^{\alpha} = g^{-\alpha}(gh)^{\alpha}N = g^{-\alpha}g^{\alpha}\bar{h}N \in H/N,$$

   where $\bar{h} \in H$.

2. Let $g_1N, g_2N \in G/N$ such that $[g_1N, g_2N] \in H/N$. Then $[g_1N, g_2N] = [g_1, g_2]N \in H/N$ implies that $[g_1, g_2] \in H$. Now, since H is an ideal of G, we have that $g_2^{-\alpha}g_1^{-\alpha}(g_1g_2)^{\alpha} \in H$ for any $\alpha \in \mathbb{Z}[x]$. Hence, $(g_2N)^{-\alpha}(g_1N)^{-\alpha}(g_1Ng_2N)^{\alpha} \in H/N$.

□

Corollary 3.6.2. *Let G be an $\mathcal{I}$-group and $N \trianglelefteq_{\mathbb{Z}[x]} G$. Then $Z(G/N)$ is an ideal of G/N.*

Proof. By Theorem 3.6.1 we know that G/N is an $\mathcal{I}$-group. Since $Z(G/N) \trianglelefteq_{\mathbb{Z}[x]} G/N$, the result immediately follows. □

3.7 $\mathbb{Z}[x]$-Series

Definition 3.7.1. Let G be a $\mathbb{Z}[x]$-group and suppose that G has a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G.$$

Then the series is termed a

1. $\mathbb{Z}[x]$*-series* if $G_i \leq_{\mathbb{Z}[x]} G$ for $i = 0, \ldots, n$
2. *normal* $\mathbb{Z}[x]$*-series* if $G_i \trianglelefteq_{\mathbb{Z}[x]} G$ for each $i = 0, 1, \ldots, n$
3. *ideal* $\mathbb{Z}[x]$*-series* if G_i is an ideal of G for each $i = 0, 1, \ldots, n$
4. *central* $\mathbb{Z}[x]$*-series* if it is an ideal $\mathbb{Z}[x]$-series which is central. In other words, each G_i is an ideal of G and $G_{i+1}/G_i \leq Z(G/G_i)$ for each $i = 0, 1, \ldots, n-1$.

Note. If G is an $\mathcal{I}$-group, then every normal $\mathbb{Z}[x]$-series is an ideal $\mathbb{Z}[x]$-series. Furthermore, if G has a central $\mathbb{Z}[x]$-series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$, then each factor group, G_{i+1}/G_i, is a $\mathbb{Z}[x]$-group.

The next lemma is proven in the usual way.

Lemma 3.7.1. *If G is a $\mathbb{Z}[x]$-group, then the $\mathbb{Z}[x]$-series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ is a central $\mathbb{Z}[x]$-series if and only if $[G_{i+1}, G]_{\mathbb{Z}[x]} \leq G_i$ for each $i = 0, \ldots, n-1$.*

As we already know, not every $\mathbb{Z}[x]$-group necessarily has a center or derived $\mathbb{Z}[x]$-subgroup which is an ideal. Consequently, the subgroups of the upper and lower central series need not be ideals. This means that the factor groups of these series may not be $\mathbb{Z}[x]$-groups. For the class of $\mathcal{I}$-groups, though, all such subgroups are ideals.

Lemma 3.7.2. *Let G be an $\mathcal{I}$-group. Then each of the subgroups of the upper and lower central series is an ideal of G.*

Proof. I will prove the result for the subgroups of the upper central series. The proof is by induction on i^{th} term of the series.

- When $i = 0$, then we have $\zeta_1 G/\zeta_0 G = Z(G/\zeta_0 G)$ which is equivalent to $\zeta_1 G = Z(G)$. Since G is an $\mathcal{I}$-group, $Z(G)$ is an ideal.
- Suppose $\zeta_j G$ is an ideal of G for all $j < i + 1$. I claim that $\zeta_{i+1} G$ is also an ideal of G. We know that $\zeta_i G$ is an ideal of G by induction. Therefore $G/\zeta_i G$ is an $\mathcal{I}$-group by Lemma 3.6.1. Hence $Z(G/\zeta_i G)$ is an ideal of $G/\zeta_i G$. However, $Z(G/\zeta_i G) = \zeta_{i+1} G/\zeta_i G$ and so $\zeta_{i+1} G$ is an ideal of G by Lemma 3.3.4.

The proof for the subgroups of the lower central series is easy. □

I will now define a special kind of $\mathbb{Z}[x]$-series for a $\mathbb{Z}[x]$-group.

Definition 3.7.2. Let G be any $\mathbb{Z}[x]$-group and let

$$G = \Gamma_1^*(G) \leq \Gamma_2^*(G) \leq \ldots \leq \Gamma_i^*(G) \leq \ldots$$

be a $\mathbb{Z}[x]$-series for which the $\Gamma_i^*(G)$'s are defined as follows:

$$G = \Gamma_1^*(G) \quad \text{and}$$

$$\Gamma_{j+1}^*(G) = \bigcap_{[\Gamma_j^*(G),G]_{\mathbb{Z}[x]} \leq H_k} \{H_k\},$$

where the $\{H_k\}$'s are ideals of G indexed by some set K. Then the $\mathbb{Z}[x]$-series of $\{\Gamma_i^*(G)\}$'s is called an *$\mathcal{S}^*$-series.*

Remark. It is clear by Lemma 3.2.2 that each $\Gamma_i^*(G)$ is an ideal of G.

In particular,

$$\Gamma_2^*(G) = \bigcap_{[G,G]_{\mathbb{Z}[x]} \leq H_k} \{H_k\}$$

where the H_k's are ideals of G indexed by some set K. It is clear from the definition that $[\Gamma_i^*(G), G]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G)$. Hence the $\mathcal{S}^*$-series is a central $\mathbb{Z}[x]$-series if $\Gamma_j^*(G) = 1$ for some $j \in \mathbb{Z}^+$. Whenever there is no ambiguity, I will abbreviate $\Gamma_i^*(G) = \Gamma_i^*$.

Note that if G is an $\mathcal{I}$-group, then $\Gamma_i^*(G) = \Gamma_i(G)$ for each i.

3.8 $\mathcal{N}^*$-Groups

In this section I will introduce the notion of an $\mathcal{N}^*$-group. The $\mathcal{S}^*$-series of an $\mathcal{N}^*$-group plays an important role in proving the results of this section.

Definition 3.8.1. A $\mathbb{Z}[x]$-group G which possesses a central $\mathbb{Z}[x]$-series is called an *$\mathcal{N}^*$-group.* The minimal length of all possible central $\mathbb{Z}[x]$-series for G is called the *$\mathcal{N}^*$ class* of G.

Theorem 3.8.1. *Let G be an $\mathcal{N}^*$-group and suppose that*

$$G = G_1 \trianglerighteq G_2 \trianglerighteq \ldots \trianglerighteq G_{m+1} = 1$$

is any central $\mathbb{Z}[x]$-series for G. Then $\Gamma^*_{i+1} \leq G_{i+1}$ for $i = 0, 1, \ldots, m$. Furthermore, if G is an $\mathcal{I}$-group as well, then $\zeta_i G \geq G_{m-i+1}$.

Proof. Let G be an $\mathcal{N}^*$-group with the series given above. The second part of the theorem follows from Lemma 3.7.2 and standard arguments. I will prove the first part by induction on i :

- For $i = 0$, we have $\Gamma^*_1 = G = G_1$.

- Suppose that the result holds for all $j < i$ and assume $\Gamma^*_j \leq G_j$. Then

  $$[\Gamma^*_j, G]_{\mathbb{Z}[x]} \leq [G_j, G]_{\mathbb{Z}[x]} \leq G_{j+1}.$$

  Now each G_p is an ideal of G for $p = 1, 2, \ldots, m+1$. In particular, we have that $G_{j+1} \in \{H_k\}$, where

  $$\Gamma^*_{j+1} = \bigcap_{[\Gamma^*_j, G]_{\mathbb{Z}[x]} \leq H_k} \{H_k\}$$

  and the H_k's are ideals of G indexed by some set K. Hence, $\Gamma^*_{j+1} \leq G_{j+1}$. This completes the proof.

□

Corollary 3.8.2. *Suppose G is an $\mathcal{N}^*$-group. Then $\Gamma^*_{c+1} = 1$ if and only if G is of $\mathcal{N}^*$ class c.*

Corollary 3.8.3. *Suppose G is an $\mathcal{N}^*$-group and an $\mathcal{I}$-group as well. Then the following are equivalent:*

1. $\zeta_c G = G$

2. $\Gamma^*_{c+1} = \Gamma_{c+1} = 1$

3. *the $\mathcal{N}^*$ class of G is c*

Lemma 3.8.1. *Suppose that G is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c. Then $\Gamma^*_c \leq Z(G)$.*

Proof. Suppose G is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c. Then

$$\Gamma^*_{c+1} = \bigcap_{[\Gamma^*_c, G]_{\mathbb{Z}[x]} \leq H_k} \{H_k\} = 1,$$

where the $\{H_k\}$'s are ideals of G indexed by some set K. Therefore $[\Gamma^*_c, G]_{\mathbb{Z}[x]} \leq 1$ and so $\Gamma^*_c \leq Z(G)$. □

Theorem 3.8.4. *Let G be an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c.*

1. *If $H \leq_{\mathbb{Z}[x]} G$, then H is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c or less.*

2. *If N is an ideal of G, then G/N is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c or less.*

Proof. By Corollary 3.8.2, the $\mathcal{S}^*$-series for G is

$$G = \Gamma^*_1(G) \rhd \Gamma^*_2(G) \rhd \ldots \rhd \Gamma^*_{c+1}(G) = 1.$$

1. Consider the $\mathbb{Z}[x]$-series

$$H = H_1 \geq H_2 \geq \ldots \geq H_{c+1} = 1,$$

where $H_i = H \cap \Gamma_i^*(G)$ for $1 \leq i \leq c+1$. I claim that each H_i is an ideal of H.

   (a) By the usual argument, we have that $H_i \trianglelefteq_{\mathbb{Z}[x]} H$.

   (b) Let $h \in H$ and $k \in H \cap \Gamma_i^*(G)$. Then $h^{-\alpha}(hk)^\alpha \in \Gamma_i^*(G)$ for arbitrary $\alpha \in \mathbb{Z}[x]$ since $\Gamma_i^*(G)$ is an ideal of G. Clearly, $h^{-\alpha}(hk)^\alpha \in K$. Therefore,

$$h^{-\alpha}(hk)^\alpha \in H \cap \Gamma_i^*(G).$$

   (c) Let $h_1, h_2 \in H$ satisfy $[h_1, h_2] \in H \cap \Gamma_i^*(G)$. As before, it is clear that $h_2^{-\alpha}h_1^{-\alpha}(h_1h_2)^\alpha \in \Gamma_i^*(G)$ since $\Gamma_i^*(G)$ is an ideal of G. Obviously, $h_2^{-\alpha}h_1^{-\alpha}(h_1h_2)^\alpha \in H$ as well. Hence $h_2^{-\alpha}h_1^{-\alpha}(h_1h_2)^\alpha \in H \cap \Gamma_i^*(G)$.

The above shows that H_i is an ideal of H. I will now verify that $[H_i, H]_{\mathbb{Z}[x]} \leq H_{i+1}$ for $i = 1, \ldots, c$. Observe that $H \leq_{\mathbb{Z}[x]} G$ and $H_i \leq_{\mathbb{Z}[x]} \Gamma_i^*(G)$ imply that $[H_i, H]_{\mathbb{Z}[x]} \leq [\Gamma_i^*(G), G]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G)$. Clearly, $[H_i, H]_{\mathbb{Z}[x]} \leq H$ and it now follows that $[H_i, H]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G) \cap H = H_{i+1}$. Hence, H is an $\mathcal{N}^*$-group.

2. Consider the following $\mathbb{Z}[x]$-series for G/N:

$$G/N = \Gamma_1^*(G)N/N \geq \Gamma_2^*(G)N/N \geq \ldots \geq \Gamma_{c+1}^*(G)N/N = 1.$$

I claim that each $\Gamma_i^*(G)N/N$ is an ideal of G/N for $1 \leq i \leq c+1$.

(a) It is straightforward to verify that $\Gamma_i^*(G)N/N \trianglelefteq_{\mathbb{Z}[x]} G/N$.

(b) Let $gN \in G/N$ and $g_iN \in \Gamma_i^*(G)N/N$, where $g \in G$ and $g_i \in \Gamma_i^*(G)$. Then for any $\alpha \in \mathbb{Z}[x]$, we have

$$\begin{aligned}(gN)^{-\alpha}(gNg_iN)^{\alpha} &= (gN)^{-\alpha}(gg_iN)^{\alpha} = g^{-\alpha}(gg_i)^{\alpha}N \\ &= g^{-\alpha}g^{\alpha}\bar{g}_iN \in \Gamma_i^*(G)N/N,\end{aligned}$$

where $\bar{g}_i \in \Gamma_i^*(G)$.

(c) Let $g_1N, g_2N \in G/N$ such that $[g_1N, g_2N] \in \Gamma_i^*(G)N/N$. Then $[g_1, g_2]N \in \Gamma_i^*(G)N/N$ implies $[g_1, g_2] \in \Gamma_i^*(G)$. Since $\Gamma_i^*(G)$ is an ideal of G, we have $g_2^{-\alpha}g_1^{-\alpha}(g_1g_2)^{\alpha} \in \Gamma_i^*(G)$ for every $\alpha \in \mathbb{Z}[x]$. Hence $(g_2N)^{-\alpha}(g_1N)^{-\alpha}(g_1Ng_2N)^{\alpha} \in \Gamma_i^*(G)N/N$.

Consequently, each $\Gamma_i^*(G)N/N$ is an ideal of G/N. Now the $\mathbb{Z}[x]$-series above is a central $\mathbb{Z}[x]$-series since $[\Gamma_i^*(G)N/N, G/N]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G)N/N$. This follows directly from the fact that $[\Gamma_i^*(G), G]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G)$. This completes the proof.

□

Corollary 3.8.5. *Let G be an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c. Then $G/\Gamma_c^*(G)$ is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class $c-1$.*

Proof. Follows directly from the previous theorem with $N = \Gamma_c^*(G)$. □

Lemma 3.8.2. *Let G be a $\mathbb{Z}[x]$-group. Then G/Γ_2^* is an abelian $\mathbb{Z}[x]$-group. Moreover, if A is an ideal of G such that G/A is an abelian $\mathbb{Z}[x]$-group, then $\Gamma_2^* \leq_{\mathbb{Z}[x]} A$. Hence, Γ_2^* is the smallest ideal of G for which G/Γ_2^* is an abelian $\mathbb{Z}[x]$-group.*

Proof. Suppose $x\Gamma_2^*, y\Gamma_2^* \in G/\Gamma_2^*$. Then

$$x\Gamma_2^* y\Gamma_2^* = xy\Gamma_2^* = xyy^{-1}x^{-1}yx\Gamma_2^* = yx\Gamma_2^* = y\Gamma_2^* x\Gamma_2^*.$$

Therefore G/Γ_2^* is an abelian $\mathbb{Z}[x]$-group. Now let $A \trianglelefteq_{\mathbb{Z}[x]} G$ be an ideal of G such that G/A is an abelian $\mathbb{Z}[x]$-group. Then $xAyA = yAxA$ for arbitrary $xA, yA \in G/A$. Hence, $x^{-1}y^{-1}xy \in A$ and so $[x,y] \in A$. This implies that $[G,G]_{\mathbb{Z}[x]} \leq A$ since $[x,y] \in [G,G]_{\mathbb{Z}[x]}$. Therefore $\Gamma_2^* \leq A$ as well, since Γ_2^* is the intersection of all ideals containing $[G,G]_{\mathbb{Z}[x]}$, A being one of them. □

Theorem 3.8.6. *Let G be an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c and let H be an ideal of G. Then $H\Gamma_2^* = G$ implies $H = G$, where $\Gamma_2^* = \Gamma_2^*(G)$.*

Proof. The proof is by induction on the $\mathcal{N}^*$ class of G.

1. If $c = 1$, then we have that $[G,G] = 1$ and so $\Gamma_2^* = 1$. The result follows.

2. Suppose the result holds for $\mathcal{N}^*$ class $c > 1$. We consider the factor group G/Γ_c^* which is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class $c-1$ by Corollary 3.8.5. Let $H\Gamma_2^* = G$. Under the natural $\mathbb{Z}[x]$-epimorphism $\theta : G \to G/\Gamma_c^*$ we then have $\theta(H)\theta(\Gamma_2^*) = \theta(G)$ which yields $\theta(H) = \theta(G) = G/\Gamma_c^*$ by induction. This is equivalent to $H\Gamma_c^* = G$.

Now, since $H \trianglelefteq_{\mathbb{Z}[x]} G$ and $\Gamma_c^* \trianglelefteq_{\mathbb{Z}[x]} G$, we may apply Lemma 3.1.5 and Lemma 3.8.1 to obtain

$$\begin{aligned}
[G, G]_{\mathbb{Z}[x]} &= [H\Gamma_c^*, H\Gamma_c^*]_{\mathbb{Z}[x]} \\
&= gp_{\mathbb{Z}[x]}([H, H\Gamma_c^*], [\Gamma_c^*, H\Gamma_c^*]) \\
&= [H, H\Gamma_c^*]_{\mathbb{Z}[x]} \\
&= gp_{\mathbb{Z}[x]}([H, H], [H, H\Gamma_c^*]) \\
&= [H, H]_{\mathbb{Z}[x]} \leq H
\end{aligned}$$

Hence $[G, G]_{\mathbb{Z}[x]} \leq H$. Since H is an ideal of G, we have $\Gamma_2^* \leq H$.

□

Remark. The above theorem shows that if $S = \{x_1, \ldots, x_r\}$ is a subset of G for which $H = gp_{\mathbb{Z}[x]}(S)$ is an ideal of G and $H \cong_{\mathbb{Z}[x]} G/\Gamma_2^*$, then $H = G$. As a result, Γ_2^* is a non-$\mathbb{Z}[x]$-generating set for G.

Corollary 3.8.7. *Suppose that G is an $\mathcal{N}^*$-group and let $X \subseteq G$. If $\varphi : G \to G/\Gamma_2^*$ is the natural $\mathbb{Z}[x]$-epimorphism and G/Γ_2^* is $\mathbb{Z}[x]$-isomorphic to an ideal of G, then $G = gp_{\mathbb{Z}[x]}(X)$ if and only if $G/\Gamma_2^* = gp_{\mathbb{Z}[x]}(\varphi(X))$.*

Proof. Clearly, if $G = gp_{\mathbb{Z}[x]}(X)$, then $G/\Gamma_2^* = gp_{\mathbb{Z}[x]}(\varphi(X))$. For the converse, we apply the previous theorem. □

Theorem 3.8.8. *Let G be an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class $c \geq 2$. Then for any $g \in G$, the $\mathbb{Z}[x]$-subgroup $\bar{G} = gp_{\mathbb{Z}[x]}(g, \Gamma_2^*(G))$ is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class less than c.*

Proof. I will construct the $\mathcal{S}^*$-series for $\bar{G}$ and compare it to that of G.

1. $\Gamma_1^*(\bar{G}) = \bar{G}$

2. By definition, we have that

$$\Gamma_2^*(\bar{G}) = \bigcap_{[\bar{G},\bar{G}]_{\mathbb{Z}[x]} \leq H_k} \{H_k\},$$

where the $\{H_k\}$'s are ideals of G indexed by some set K. I claim that $\Gamma_2^*(\bar{G}) \leq_{\mathbb{Z}[x]} \Gamma_3^*(G)$. It is enough to prove that $[\bar{G}, \bar{G}]_{\mathbb{Z}[x]} \leq \Gamma_3^*(G)$. This will be sufficient since $\Gamma_3^*(G)$ is an ideal of G and $\Gamma_3^*(G) \trianglelefteq \Gamma_2^*(G) < \bar{G} < G$. Now Lemma 3.2.4 applies and gives us $\Gamma_3^*(G)$ is an ideal of $\bar{G}$. If it is true that $[\bar{G}, \bar{G}]_{\mathbb{Z}[x]} \leq \Gamma_3^*(G)$, then $\Gamma_3^*(G) \in \{H_k\}$ and so the claim will be proven.

Well,

$$\begin{aligned}
[\bar{G}, \bar{G}]_{\mathbb{Z}[x]} &= [gp_{\mathbb{Z}[x]}(g, \Gamma_2^*(G)), gp_{\mathbb{Z}[x]}(g, \Gamma_2^*(G))]_{\mathbb{Z}[x]} \\
&= gp_{\mathbb{Z}[x]}([g^{\alpha_i} p_i, g^{\beta_i} q_i])
\end{aligned}$$

for all $\alpha_i, \beta_i \in \mathbb{Z}[x]$ and $p_i, q_i \in \Gamma_2^*(G)$. Observe that for any $\alpha, \beta \in \mathbb{Z}[x]$ and $p, q \in \Gamma_2^*(G)$, we have

$$\begin{aligned}
[g^{\alpha} p, g^{\beta} q] &= [g^{\alpha}, g^{\beta} q]^p [p, g^{\beta} q] \\
&= ([g^{\alpha}, q][g^{\alpha}, g^{\beta}]^q)^p [p, g^{\beta} q].
\end{aligned}$$

But $[g^\alpha, g^\beta] = 1$ and both $[g^\alpha, q]$ and $[p, g^\beta q]$ lie in $[G, \Gamma_2^*(G)]_{\mathbb{Z}[x]} \leq \Gamma_3^*(G)$. Hence $[g^\alpha p, g^\beta q] \in \Gamma_3^*(G)$ and the claim is proven.

3. Assume that $\Gamma_{i-1}^*(\bar{G}) \leq_{\mathbb{Z}[x]} \Gamma_i^*(G)$ for all $i = 3, 4, \ldots c$. The claim is that $\Gamma_i^*(\bar{G}) \leq_{\mathbb{Z}[x]} \Gamma_{i+1}^*(G)$. Just as in the previous case where $i = 3$, it suffices to show that $[\Gamma_{i-1}^*(\bar{G}), \bar{G}]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G)$. Well, observe that

$$[\Gamma_{i-1}^*(\bar{G}), \bar{G}]_{\mathbb{Z}[x]} \leq [\Gamma_i^*(G), \bar{G}]_{\mathbb{Z}[x]} \leq [\Gamma_i^*(G), G]_{\mathbb{Z}[x]} \leq \Gamma_{i+1}^*(G).$$

In particular, $\Gamma_c^*(\bar{G}) \leq \Gamma_{c+1}^*(G) = 1$ and so $\bar{G}$ has $\mathcal{N}^*$ class less than c.

□

Lemma 3.8.3. *Let G be an $\mathcal{N}^*$-group with a central $\mathbb{Z}[x]$-series*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G.$$

Then each factor group G_{i+1}/G_i may be viewed as a $\mathbb{Z}[x]$-module.

Proof. $G_{i+1}/G_i \trianglelefteq_{\mathbb{Z}[x]} Z(G/G_i)$ and $Z(G/G_i)$ is an abelian $\mathbb{Z}[x]$-group. Hence, G_{i+1}/G_i is an abelian $\mathbb{Z}[x]$-group as well. Therefore we can view $Z(G/G_i)$ as a $\mathbb{Z}[x]$-module and so G_{i+1}/G_i is a submodule of $Z(G/G_i)$. □

Lemma 3.8.4. *Let G be an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class 2. Then we have $[G, G]_{\mathbb{Z}[x]} \leq Z(G)$.*

Proof. Since $\Gamma_3^* = 1$, we have that $\Gamma_2^* \leq Z(G)$ by Lemma 3.8.1. However, we know that $[G, G]_{\mathbb{Z}[x]} \leq \Gamma_2^*$. □

Lemma 3.8.5. *For any $\mathcal{N}^*$-group of $\mathcal{N}^*$ class 2, we have $[g_1^\alpha, g_2^\beta] = [g_1, g_2]^{\alpha\beta}$ where $g_1, g_2 \in G$ and $\alpha, \beta \in \mathbb{Z}[x]$.*

Proof. By the previous lemma, we have $[G, G]_{\mathbb{Z}[x]} \leq Z(G)$.

1. I claim that $[g_1^{-\alpha} g_2^{-1} g_1^\alpha, g_2] = 1$. Well,

$$\begin{aligned} g_1^{-\alpha} g_2^{-1} g_1^\alpha g_2 &= g_2^{-1}(g_2 g_1^{-\alpha} g_2^{-1} g_1^\alpha) g_2 \\ &= g_2^{-1}[g_2^{-1}, g_1^\alpha] g_2 \\ &= g_2 g_1^{-\alpha} g_2^{-1} g_1^\alpha \end{aligned}$$

2. The second claim is that $[g_1^{-1}, g_2^{-1} g_1 g_2] = 1$. The computation is similar to the above:

$$\begin{aligned} g_1^{-1} g_2^{-1} g_1 g_2 &= g_1^{-1}(g_2^{-1} g_1 g_2 g_1^{-1}) g_1 \\ &= g_1^{-1}[g_2, g_1^{-1}] g_1 \\ &= g_2^{-1} g_1 g_2 g_1^{-1} \end{aligned}$$

Putting the above together, we have

$$\begin{aligned} [g_1^\alpha, g_2^\beta] &= g_1^{-\alpha} g_2^{-\beta} g_1^\alpha g_2^\beta \quad \text{and} \\ g_1^{-\alpha} g_2^{-\beta} g_1^\alpha g_2^\beta &= (g_1^{-\alpha} g_2^{-1} g_1^\alpha)^\beta g_2^\beta \\ &= (g_1^{-\alpha} g_2^{-1} g_1^\alpha g_2)^\beta \\ &= [g_1^{-\alpha} (g_2^{-1} g_1 g_2)^\alpha]^\beta \\ &= [(g_1^{-1} g_2^{-1} g_1 g_2)^\alpha]^\beta \\ &= [g_1, g_2]^{\alpha\beta}. \end{aligned}$$

□

3.9 Torsion $\mathbb{Z}[x]$-Groups

The definitions of torsion and torsion free $\mathbb{Z}[x]$-groups are now given.

Definition 3.9.1. Let G be an arbitrary $\mathbb{Z}[x]$-group.

1. An element $g \in G$ is called a *torsion element* if there exists $\alpha \in \mathbb{Z}[x]$ with $\alpha \neq 0$ for which $g^\alpha = 1$. The set of torsion elements of G will be written as $\tau(G)$.
2. If $\tau(G) = G$, then G is called a *torsion $\mathbb{Z}[x]$-group.*
3. G is called *torsion free* if $g^\alpha = 1$ implies either $g = 1$ or $\alpha = 0$ for $\alpha \in \mathbb{Z}[x]$.
4. If G a torsion $\mathbb{Z}[x]$-group and if there should exist a non-zero $\beta \in \mathbb{Z}[x]$ such that

for all $g \in G$ we have $g^\beta = 1$ where β is of minimal degree and has minimal positive leading coefficient, then G has *exponent* β.

Theorem 3.9.1. *Let G be a $\mathbb{Z}[x]$-group and let $1 = G_1 \leq \ldots \leq G_n = G$ be an ideal $\mathbb{Z}[x]$-series of G. If each factor group G_{i+1}/G_i has exponent $\mu_i \in \mathbb{Z}[x]$, then G has exponent dividing $\mu_1\mu_2\cdots\mu_{n-1}$.*

Proof. Let G have the series above and suppose that $gG_{n-1} \in G/G_{n-1}$. Then $g^{\mu_{n-1}} \in G_{n-1}$ by hypothesis. Now $g^{\mu_{n-1}}G_{n-2} \in G_{n-1}/G_{n-2}$ implies that $g^{\mu_{n-1}\mu_{n-2}} \in G_{n-2}$ by hypothesis. Continuing in this way, we obtain $g^{\mu_{n-1}\mu_{n-2}\cdots\mu_1} \in G_1$. Therefore, $g^{\mu_{n-1}\mu_{n-2}\cdots\mu_1} = 1$ and so G has exponent dividing $\mu_{n-1}\mu_{n-2}\cdots\mu_1$. $\square$

Theorem 3.9.2. *Let G is a torsion free $\mathcal{N}^*$-group of $\mathcal{N}^*$ class c. If $g, h \in G$ and $g^\beta = h^\beta$ for some non-zero $\beta \in \mathbb{Z}[x]$, then $g = h$.*

Proof. The proof is by induction on the $\mathcal{N}^*$ class of G. Let $g, h \in G$ and $\beta \in \mathbb{Z}[x]$.

- If $c = 1$, then G is an abelian $\mathbb{Z}[x]$-group. Let $g, h \in G$ such that $g^\beta = h^\beta$ for some non-zero $\beta \in \mathbb{Z}[x]$. Then

$$g^\beta h^{-\beta} = 1 \Longrightarrow (gh^{-1})^\beta = 1,$$

  since $[g^{-1}, h] = 1$. Since G is torsion free, this gives us $gh^{-1} = 1$. Hence $g = h$.

- Suppose that $c \geq 2$ and suppose that the result holds for torsion free $\mathcal{N}^*$-groups of $\mathcal{N}^*$ class less than c. Consider the $\mathbb{Z}[x]$-subgroup $A = gp_{\mathbb{Z}[x]}(g, \Gamma_2^*(G))$ of G, which is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class less than c by Theorem 3.8.8. Observe that $h^{-1}gh \in A$ because $h^{-1}gh = g[g,h]$ lies in $gp_{\mathbb{Z}[x]}(g, \Gamma_2^*)$. Hence

$$\begin{aligned} g^\beta = h^\beta &\implies g^\beta = h^{-1}h^\beta h \\ &\implies g^\beta = h^{-1}g^\beta h \\ &\implies g^\beta = (h^{-1}gh)^\beta. \end{aligned}$$

By induction, we have $g = h^{-1}gh \implies [g,h] = 1$. Therefore,

$$\begin{aligned} g^\beta = h^\beta &\implies g^\beta h^{-\beta} = 1 \\ &\implies (gh^{-1})^\beta = 1 \\ &\implies gh^{-1} = 1 \text{ , since G is torsion free} \\ &\implies g = h. \end{aligned}$$

□

Lemma 3.9.1. *Let G be a torsion free $\mathcal{N}^*$-group whose center is an ideal. Then $G/Z(G)$ is a torsion free $\mathcal{N}^*$-group.*

Proof. We know that $G/Z(G)$ is an $\mathcal{N}^*$-group. To show that $G/Z(G)$ is torsion free, we need to verify that if $g \in G$ such that $g^\mu \in Z(G)$, $\mu \in \mathbb{Z}[x]$, $\mu \neq 0$, then $g \in Z(G)$. Suppose $g \in G$ such that $g^\mu \in Z(G)$ and $\mu \neq 0$. Then for all $h \in G$, $[g^\mu, h] = 1$

implies $h^{-1}g^{\mu}h = g^{\mu}$. This is equivalent to $(h^{-1}gh)^{\mu} = g^{\mu}$ and so $h^{-1}gh = g$ since G is torsion free. Therefore $g \in Z(G)$. □

3.10 $\mathcal{R}$-groups

In this section I will discuss $\mathcal{R}$-groups. These are $\mathbb{Z}[x]$-groups which admit unique root extraction whenever such an extraction exists. Similar groups are discussed in A. G. Kurosh [9].

Definition 3.10.1. Let G be a $\mathbb{Z}[x]$-group. Then G is an *$\mathcal{R}$-group* if for every $f \in G$ and every $\alpha \in \mathbb{Z}[x]$, the equation $g^{\alpha} = f$ has at most one solution $g \in G$. Equivalently, for every pair of elements $g, h \in G$ and every $\alpha \in \mathbb{Z}[x]$, we have

$$g^{\alpha} = h^{\alpha} \Longrightarrow g = h.$$

Note. If $f \in G$ and $\alpha \in \mathbb{Z}[x]$, then the equation $g^{\alpha} = f$ may not have a solution $g \in G$. If it does, though, it is unique.

Lemma 3.10.1. *Every $\mathcal{R}$-group is torsion free and every torsion free abelian $\mathbb{Z}[x]$-group is an $\mathcal{R}$-group.*

Proof. Let G be an $\mathcal{R}$-group. Clearly, if $g \in G$ and $g^{\alpha} = 1$ for any non-zero $\alpha \in \mathbb{Z}[x]$, then $g = 1$ and so G is torsion free. Now let G be a torsion free abelian $\mathbb{Z}[x]$-group. If $g, h \in G$ and $g^{\alpha} = h^{\alpha}$ for some $\alpha \in \mathbb{Z}[x]$, then we have that $g^{\alpha}h^{-\alpha} = 1$ implies

$(gh^{-1})^\alpha = 1$ since G is abelian. Since G is torsion free, we have $gh^{-1} = 1$. Therefore $g = h$. □

Definition 3.10.2. Let G be an $\mathcal{R}$-group. Then $H \leq_{\mathbb{Z}[x]} G$ is called an *isolated* $\mathbb{Z}[x]$*-subgroup* of G if the following condition holds:

$$\text{if } g \in G \text{ and } g^\alpha \in H \text{ for some } \alpha \in \mathbb{Z}[x], \text{ then } g \in H.$$

In other words, the solution to the equation $g^\alpha = h$, if it exists in G, belongs to H.

Remark. Since G is a $\mathbb{Z}[x]$-group, we have that G is an isolated $\mathbb{Z}[x]$-subgroup of itself. Moreover, the identity subgroup of G is an isolated $\mathbb{Z}[x]$-subgroup of G, since $1^\alpha = 1$ for all $\alpha \in \mathbb{Z}[x]$.

Theorem 3.10.1. *Let G be an $\mathcal{R}$-group and suppose that $\{G_1, \ldots, G_n\}$ is a finite collection of isolated $\mathbb{Z}[x]$-subgroups of G. Then $\cap_{i=1}^n G_i$ is also an isolated $\mathbb{Z}[x]$-subgroup of G.*

Proof. Let $\{G_1, \ldots, G_n\}$ be a finite collection of isolated $\mathbb{Z}[x]$-subgroups of G and let $\bar{G} = \cap_{i=1}^n G_i$. We know that $\bar{G}$ is a $\mathbb{Z}[x]$-subgroup of G, since it is the intersection of $\mathbb{Z}[x]$-subgroups of G.

We want to show that $\bar{G}$ is an isolated $\mathbb{Z}[x]$-subgroup of G. Let $h \in G$ and $\alpha \in \mathbb{Z}[x]$ such that $h^\alpha \in \bar{G}$. Then $h^\alpha \in G_i$ for each $i = 1, \ldots, n$. Since each G_i is an isolated subgroup of G and extraction of roots is unique in G, we have that $h \in G_i$ for each $i = 1, \ldots, n$. We conclude that $h \in \cap_{i=1}^n G_i = \bar{G}$. □

The above theorem suggests that there exists a unique minimal isolated subgroup of an $\mathcal{R}$-group, G, containing a given collection of isolated $\mathbb{Z}[x]$-subgroups. More precisely, we have

Definition 3.10.3. Let G be an $\mathcal{R}$-group and let S be an arbitrary set of elements of G. The *isolator* of S in G, denoted by $I(S)$, is the $\mathbb{Z}[x]$-subgroup

$$I(S) = \bigcap_{S \subseteq G_i} G_i$$

where each G_i is an isolated $\mathbb{Z}[x]$-subgroup of G.

Lemma 3.10.2. *Let G be an $\mathcal{R}$-group and let $H \trianglelefteq_{\mathbb{Z}[x]} G$ be an ideal of G. Then H is an isolated $\mathbb{Z}[x]$-subgroup of G if and only if G/H is a torsion free $\mathbb{Z}[x]$-group.*

Proof. Let $H \leq G$ be an ideal of G and suppose that H is an isolated $\mathbb{Z}[x]$-subgroup of G. We know that G/H is a $\mathbb{Z}[x]$-group. Let $g \in G$ such that $(gH)^\beta = H$ for some non-zero $\beta \in \mathbb{Z}[x]$. I claim that $gH = H$. Obviously, $(gH)^\beta = H$ implies that $g^\beta \in H$ and so $g \in H$ since H is isolated. Therefore $gH = H$ and so G/H is torsion free. Conversely, let $H \leq G$ be an ideal of G and let G/H be a torsion free $\mathbb{Z}[x]$-group. Then, for every $g \in G$ and $\beta \in \mathbb{Z}[x]$, we have that $(gH)^\beta = H$ implies $gH = H$. Put another way, $g^\beta \in H$ implies $g \in H$. Hence, H is an isolated $\mathbb{Z}[x]$-subgroup of G. $\square$

Definition 3.10.4. Let G be an $\mathcal{R}$-group and suppose $H \leq G$ is an ideal and is also an isolated $\mathbb{Z}[x]$-subgroup of G. Then H is termed an *isolated ideal* of G.

I will now prove a theorem for $\mathbb{Z}[x]$-groups which is a generalization of the correspondence theorem between $\mathbb{Z}[x]$-subgroups of a $\mathbb{Z}[x]$-group and its factor $\mathbb{Z}[x]$-group.

Theorem 3.10.2. *Let G be an $\mathcal{R}$-group and let H be an isolated ideal of G. Suppose that G/H is an $\mathcal{R}$-group. If $K \leq_{\mathbb{Z}[x]} G$ and H is an ideal of K, then K is an isolated $\mathbb{Z}[x]$-subgroup of G if and only if K/H is an isolated $\mathbb{Z}[x]$-subgroup of G/H.*

Proof. Let K be an isolated $\mathbb{Z}[x]$-subgroup of G such that H is an ideal of K. I claim that if $(gH)^\alpha \in K/H$, then $gH \in K/H$. Well, if $(gH)^\alpha = kH$, where $g \in G, k \in K$ and $\alpha \in \mathbb{Z}[x]$, then $g^\alpha H = kH$ gives us $g^\alpha H \in K/H$. From this, we see that there exists $h \in H$ such that $g^\alpha = kh \in K$. Since K is an isolated $\mathbb{Z}[x]$-subgroup of G, $g^\alpha \in K$ implies that $g \in K$. This yields $gH \in K/H$. Hence, K/H is an isolated $\mathbb{Z}[x]$-subgroup of G/H. Now let K/H be an isolated $\mathbb{Z}[x]$-subgroup of G/H. If $g^\alpha = k$ for some $g \in G, k \in K$ and $\alpha \in \mathbb{Z}[x]$, then $g^\alpha H = kH$ in K/H. This means that $(gH)^\alpha = kH$ and so $gH \in K/H$ because K/H is an isolated $\mathbb{Z}[x]$-subgroup of G/H. Therefore, $g \in K$ and so $g^\alpha \in K$ yields $g \in K$. Hence, K is an isolated $\mathbb{Z}[x]$-subgroup of G. ☐

Theorem 3.10.3. *Let G be an $\mathcal{R}$-group and let A be an arbitrary set of elements of G. Then $C_G(A)$, the centralizer of A in G, is an isolated $\mathbb{Z}[x]$-subgroup of G.*

Proof. It is easy to verify that $C_G(A)$ is a $\mathbb{Z}[x]$-subgroup of G. Let $A = \{a_1, a_2, \ldots\}$ be the set of elements and let $g \in G$ be such that $g^\alpha \in C_G(A)$ for some $\alpha \in \mathbb{Z}[x]$.

Then for every $a_i \in A$, we have

$$\begin{aligned} a_i^{-1} g^\alpha a_i &= g^\alpha \Longrightarrow \\ (a_i^{-1} g a_i)^\alpha &= g^\alpha \Longrightarrow \\ a_i^{-1} g a_i &= g. \end{aligned}$$

Therefore, $g \in C_G(A)$. □

In particular,

Lemma 3.10.3. *The center of any $\mathcal{R}$-group is an isolated $\mathbb{Z}[x]$-subgroup.*

Lemma 3.10.4. *Let G be any $\mathcal{R}$-group. Then the equation $a^\alpha b^\beta = b^\beta a^\alpha$ implies that $ab = ba$, where $a, b \in G$ and $\alpha, \beta \in \mathbb{Z}[x]$.*

Theorem 3.10.4. *Let G be a torsion free $\mathcal{I}$-group. Then G is an $\mathcal{R}$-group if and only if $G/Z(G)$ is an $\mathcal{R}$-group.*

Proof. Suppose G is a torsion free $\mathcal{R}$-group. I claim that if $g, h \in G$ and $(gZ(G))^\alpha = (hZ(G))^\alpha$ for some $\alpha \in \mathbb{Z}[x]$, then $gZ(G) = hZ(G)$.

Let $(gZ(G))^\alpha = (hZ(G))^\alpha$ in $G/Z(G)$. Then $g^\alpha Z(G) = h^\alpha Z(G)$ gives us $g^\alpha = h^\alpha z$ for some $z \in Z(G)$. Observe that $g^\alpha h^\alpha = h^\alpha z h^\alpha = h^\alpha h^\alpha z = h^\alpha g^\alpha$. Therefore, $g^\alpha h^\alpha = h^\alpha g^\alpha$. By Lemma 3.10.4, this gives us $gh = hg$ and so $[g, h^{-1}] = 1$. By axiom 3, $[g, h^{-1}] = 1$ implies $(gh^{-1})^\alpha = g^\alpha h^{-\alpha}$. Hence, $g^\alpha = h^\alpha z$ implies $(gh^{-1})^\alpha = z$ and so $(gh^{-1})^\alpha \in Z(G)$. But $Z(G)$ is an isolated $\mathbb{Z}[x]$-subgroup of G, so that $gh^{-1} \in Z(G)$.

Therefore, $gZ(G) = hZ(G)$.

Conversely, let $G/Z(G)$ be an $\mathcal{R}$-group, where G is a torsion free $\mathbb{Z}[x]$-group. I claim that G is an $\mathcal{R}$-group. Let $g^\alpha = h^\alpha$ for some $g, h \in G$ and $\alpha \in \mathbb{Z}[x]$. Then $(gZ(G))^\alpha = (hZ(G))^\alpha$ in $G/Z(G)$. Since $G/Z(G)$ is an $\mathcal{R}$-group, we have

$$\begin{aligned} g^\alpha Z(G) &= h^\alpha Z(G) \Longrightarrow \\ gZ(G) &= hZ(G) \Longrightarrow \\ g &= hz \text{ for some } z \in Z(G). \end{aligned}$$

We therefore have that $g^\alpha = (hz)^\alpha$ implies $g^\alpha = h^\alpha z^\alpha$ and so $z^\alpha = 1$. Since G is torsion free, $z^\alpha = 1$ yields $z = 1$ and so $g = h$. □

Theorem 3.10.5. *Let G be both an $\mathcal{I}$-group and an $\mathcal{R}$-group and let*

$$1 = \zeta_0 G \trianglelefteq \zeta_1 G \trianglelefteq \ldots \trianglelefteq \zeta_i G \trianglelefteq \ldots$$

be the upper central series for G. Suppose that each factor $\mathbb{Z}[x]$-group $G/\zeta_j G$ is an $\mathcal{I}$-group. Then

1. *each $\zeta_i G$ is an isolated normal $\mathbb{Z}[x]$-subgroup of G*

2. *each factor $\mathbb{Z}[x]$-group $\zeta_i G/\zeta_{i-1} G$ is a torsion free abelian $\mathbb{Z}[x]$-group*

3. *each factor $\mathbb{Z}[x]$-group $G/\zeta_j G$ is an $\mathcal{R}$-group.*

Proof. The proof is by induction on i :

- When $i = 1$, $\zeta_1 G = Z(G)$ is an isolated $\mathbb{Z}[x]$-subgroup of G by Lemma 3.10.3. Furthermore, $Z(G) = \zeta_1 G/\zeta_0 G$ is a torsion free abelian $\mathbb{Z}[x]$-group since it is a $\mathbb{Z}[x]$-subgroup of an $\mathcal{R}$-group. Finally, by Theorem 3.10.4, $G/Z(G)$ is an $\mathcal{R}$-group.

- Suppose the theorem holds for all indices $i < j$:

  1. If $j-1$ exists, then by assumption, $G/\zeta_{j-1}G$ is an $\mathcal{R}$-group. Therefore, $Z(G/\zeta_{j-1}G) = \zeta_j G/\zeta_{j-1}G$ is a torsion free abelian $\mathbb{Z}[x]$-group. The factor group
  $$\frac{G/\zeta_{j-1}G}{Z(G/\zeta_{j-1}G)} = \frac{G/\zeta_{j-1}G}{\zeta_j G/\zeta_{j-1}G} \cong_{\mathbb{Z}[x]} G/\zeta_j G$$
  is an $\mathcal{R}$-group as well by Theorem 3.10.4. Hence, since $Z(G/\zeta_{j-1}G) = \zeta_j G/\zeta_{j-1}G$ is an isolated $\mathbb{Z}[x]$-subgroup of $G/\zeta_{j-1}G$ (by Lemma 3.10.3), we have that $\zeta_j G$ is an isolated $\mathbb{Z}[x]$-subgroup of G by Theorem 3.10.2.

  2. If j is a limit number, then $\zeta_j G$ is the union of an ascending sequence of isolated $\mathbb{Z}[x]$-subgroups and is therefore an isolated $\mathbb{Z}[x]$-subgroup as well. Furthermore, if $(g\zeta_j G)^\mu = (h\zeta_j G)^\mu$, where $g, h \in G$ and $\mu \in \mathbb{Z}[x]$, then $g^\mu = h^\mu z$ for some $z \in \zeta_j G$. Hence, $z \in \zeta_k G$ for all $k < j$. This means that $(g\zeta_k G)^\mu = (h\zeta_k G)^\mu$ and so we have $g\zeta_k G = h\zeta_k G$. Consequently, $g\zeta_j G = h\zeta_j G$ and so $G/\zeta_j G$ is an $\mathcal{R}$-group.

□

Theorem 3.10.6. *Every torsion free $\mathcal{N}^*$-group is an $\mathcal{R}$-group.*

Proof. This follows from Theorem 3.9.2. □

3.11 $\mathbb{Z}[x]$-Groups of Type HP

If G is a $\mathbb{Z}[x]$-group and $g, h \in G$ with $\beta \in \mathbb{Z}[x]$, one can't always expand the product $(gh)^\beta$ in terms of g and h by applying only the axioms. In this section we focus our attention on specific $\mathbb{Z}[x]$-groups which admit such expansions.

Definition 3.11.1. Let G be a $\mathbb{Z}[x]$-group. Then G is of *type HP* if there exists a nilpotent $\mathbb{Q}[x]$-powered group, $\widetilde{G}$, such that $G <_{\mathbb{Z}[x]} \widetilde{G}$.

Theorem 3.11.1. *Let G be a $\mathbb{Z}[x]$-group of type HP and let $H \trianglelefteq_{\mathbb{Z}[x]} G$. Then H is an ideal of G. Consequently, every $\mathbb{Z}[x]$-group of type HP is an $\mathcal{I}$-group.*

Proof. Let G and H be as in the hypothesis and let $g \in G$. Suppose that $\widetilde{G}$ is a nilpotent $\mathbb{Q}[x]$-powered group such that $G <_{\mathbb{Z}[x]} \widetilde{G}$. I claim that H is an ideal of G. Let $\alpha \in \mathbb{Z}[x]$.

1. Suppose $h \in H$. Then the Hall-Petresco axiom give us

$$g^{-\alpha}(gh)^\alpha = h^\alpha \tau_k(g,h)^{-\binom{\alpha}{k}} \cdots \tau_2(g,h)^{-\binom{\alpha}{2}},$$

where k is the $\mathcal{N}^*$ class of $gp\,(g,h)$. For each $i = 2, \ldots, k$, we know that $\tau_i(g,h) \in H$ since $H \trianglelefteq_{\mathbb{Z}[x]} G$. Hence $\tau_i(g,h)^{-\binom{\alpha}{i}} \in gp_{\mathbb{Q}[x]}(H)$ in $\widetilde{G}$. Therefore

$g^{-\alpha}(gh)^{\alpha} \in G$ and $h^{\alpha}\tau_k(g,h)^{-\binom{\alpha}{k}} \cdots \tau_2(g,h)^{-\binom{\alpha}{2}} \in gp_{\mathbb{Q}[x]}(H)$. The above shows that $g^{-\alpha}(gh)^{\alpha} \in G \cap gp_{\mathbb{Q}[x]}(H) \leq_{\mathbb{Z}[x]} H$.

2. Let $g_1, g_2 \in G$ satisfy $[g_1, g_2] \in H$. Again, by applying the Hall-Petresco axiom, we obtain

$$g_2^{-\alpha}g_1^{-\alpha}(g_1g_2)^{\alpha} = \tau_k(g_1,g_2)^{-\binom{\alpha}{k}} \cdots \tau_2(g_1,g_2)^{-\binom{\alpha}{2}},$$

where k is the class of $gp\,(g_1, g_2)$. Since $[g_1, g_2] \in H$ and $H \trianglelefteq_{\mathbb{Z}[x]} G$, we have that each $\tau_i(g_1, g_2) \in H$. Therefore

$$g_2^{-\alpha}g_1^{-\alpha}(g_1g_2)^{\alpha} \in G \cap gp_{\mathbb{Q}[x]}(H) \leq_{\mathbb{Z}[x]} H.$$

Hence H is an ideal of G.

□

Corollary 3.11.2. *If G is a $\mathbb{Z}[x]$-group of type HP, then $\Gamma_i^*(G) = \Gamma_i(G)$.*

Corollary 3.11.3. *If G is $\mathbb{Z}[x]$-group of type HP, then G is an $\mathcal{N}^*$-group.*

Proof. Let $\widetilde{G}$ be a nilpotent $\mathbb{Q}[x]$-powered group of class c with lower central series $\widetilde{G} = \Gamma_1(\widetilde{G}) \triangleright \Gamma_2(\widetilde{G}) \triangleright \cdots \triangleright \Gamma_{c+1}(\widetilde{G}) = 1$. Suppose G is a $\mathbb{Z}[x]$-group of type HP such that $G <_{\mathbb{Z}[x]} \widetilde{G}$ and define $G_i = G \cap \Gamma_i(\widetilde{G}) <_{\mathbb{Z}[x]} \Gamma_i(\widetilde{G})$. Then G has a $\mathbb{Z}[x]$-series $G = G_1 > G_2 > \cdots > G_{c+1} = 1$. I claim that this is a central $\mathbb{Z}[x]$-series. It can be easily verified that $G_i \triangleleft_{\mathbb{Z}[x]} G$ for each $i = 1, \ldots, c+1$. Hence, by the previous theorem, each G_i is an ideal of G. By applying the same argument as in the proof of Theorem 3.8.4, we have that $[G_i, G]_{\mathbb{Z}[x]} \leq G_{i+1}$ for each $i = 1, \ldots, c$. Therefore the $\mathbb{Z}[x]$-series for G is central and so G is an $\mathcal{N}^*$-group. □

Corollary 3.11.4. *Let G be a finitely generated torsion free nilpotent group of class c and suppose $G^{\mathbb{Q}[x]}$ is the $\mathbb{Q}[x]$-completion of G with respect to some Mal'cev basis $u = (u_1, \ldots, u_k)$ for G. Then $\bar{G} = gp_{\mathbb{Z}[x]}(u_1, \ldots, u_k)$ is a finitely $\mathbb{Z}[x]$-generated $\mathcal{N}^*$-group of $\mathcal{N}^*$ class $\leq c$.*

Proof. It is clear that that $\bar{G}$ is a $\mathbb{Z}[x]$-group which is finitely $\mathbb{Z}[x]$-generated by the set $\{u_1, \ldots, u_k\}$. Moreover, $\bar{G}$ is of type HP since it is a $\mathbb{Z}[x]$-subgroup of $G^{\mathbb{Q}[x]}$. By Corollary 3.11.3, $\bar{G}$ is an $\mathcal{N}^*$-group. Clearly, the $\mathcal{N}^*$ class of $\bar{G}$ can't exceed c. □

Corollary 3.11.5. *Let G be a $\mathbb{Z}[x]$-group is of type HP and suppose $N \trianglelefteq_{\mathbb{Z}[x]} G$. Then G/N is a $\mathbb{Z}[x]$-group.*

Proof. Follows directly from Theorem 3.2.1 and Theorem 3.11.1. □

Theorem 3.11.6. *Let G be a $\mathbb{Z}[x]$-group of type HP and let $\tau(G)$ denote the set of torsion elements of G as usual. Then $\tau(G) \trianglelefteq_{\mathbb{Z}[x]} G$ and $G/\tau(G)$ is a torsion free $\mathbb{Z}[x]$-group.*

Proof. Let G be a $\mathbb{Z}[x]$-group of type HP such that $G <_{\mathbb{Z}[x]} \widetilde{G}$ for some nilpotent $\mathbb{Q}[x]$-powered group, $\widetilde{G}$. By Theorem 2.7.1, $\tau(\widetilde{G}) \trianglelefteq_{\mathbb{Q}[x]} \widetilde{G}$. I will first show that $\tau(\widetilde{G}) \cap G = \tau(G)$.

1. Let $g \in \tau(G)$. Then there exists a non-zero $\alpha \in \mathbb{Z}[x]$ such that $g^\alpha = 1$. Since $\alpha \in \mathbb{Q}[x]$ as well, we have that $g \in \tau(\widetilde{G})$. Therefore $g \in \tau(\widetilde{G}) \cap G$ and so $\tau(G) \subseteq \tau(\widetilde{G}) \cap G$.

2. Let $g \in \tau(\widetilde{G}) \cap G$. Since $g \in \tau(\widetilde{G})$, there exists a non-zero $\alpha \in \mathbb{Q}[x]$ such that $g^\alpha = 1$. Suppose $\alpha = \frac{a_0}{b_0} + \frac{a_1}{b_1}x + \cdots + \frac{a_n}{b_n}x^n$ where $a_i, b_i \in \mathbb{Z}$ and $b_i \neq 0$. If $d = l.c.m.(b_0, \ldots, b_n)$, then $\alpha d \in \mathbb{Z}[x]$. Hence, $(g^\alpha)^d = g^{\alpha d} = 1$ and so $g \in \tau(G)$. Therefore $\tau(\widetilde{G}) \cap G \subseteq \tau(G)$.

The above shows that $\tau(\widetilde{G}) \cap G = \tau(G)$. It can be easily verified that $\tau(\widetilde{G}) \cap G \trianglelefteq_{\mathbb{Z}[x]} G$. Hence $\tau(G) \trianglelefteq_{\mathbb{Z}[x]} G$. Moreover, $\tau(G)$ is an ideal of G by Theorem 3.11.1. The proof that $G/\tau(G)$ is torsion-free follows as in the ordinary group case. □

Theorem 3.11.7. *Let G be a $\mathbb{Z}[x]$-group of type HP such that $G <_{\mathbb{Z}[x]} \widetilde{G}$ for some nilpotent $\mathbb{Q}[x]$-powered group $\widetilde{G}$. If $\widetilde{G}$ is torsion free, then G is an $\mathcal{R}$-group.*

Proof. Let $\widetilde{G}$ and G be as in the hypothesis. Suppose that $g^\alpha = h^\alpha$ for some $g, h \in G$ and $\alpha \in \mathbb{Z}[x]$, $\alpha \neq 0$. Then this equality also holds in $\widetilde{G}$. By Theorem 2.7.3 we have $g = h$ in $\widetilde{G}$. Hence, $g = h$ in G. □

Theorem 3.11.8. *Let $\widetilde{G}$ be a finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group and suppose G is a $\mathbb{Z}[x]$-group of type HP such that $G <_{\mathbb{Z}[x]} \widetilde{G}$. Then G has a solvable word problem.*

Proof. Recall that every finitely $\mathbb{Q}[x]$-generated nilpotent $\mathbb{Q}[x]$-powered group has a solvable word problem. Hence the result immediately follows. $\square$

3.12 The $\mathbb{Z}[x]$-Group $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$

In section 2.7, we saw that $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} UT_n(\mathbb{Q}[x])$, where $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]}$ is the $\mathbb{Q}[x]$-completion of $UT_n(\mathbb{Z})$ with respect to any Mal'cev basis for $UT_n(\mathbb{Z})$ and $UT_n(\mathbb{Q}[x])$ is the nilpotent $\mathbb{Q}[x]$-powered group of unitriangular matrices with entries in $\mathbb{Q}[x]$ and with the $\mathbb{Q}[x]$-action defined by the binomial expansion. Suppose that $UT_n(\mathbb{Z})$ has Mal'cev basis $\bar{u} = (u_1, \ldots, u_n)$ and $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]}$ is its $\mathbb{Q}[x]$-completion. Then the $\mathbb{Z}[x]$-subgroup $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ of $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]}$ can be described in the following way, where we are using the polynomials for multiplication and $\mathbb{Q}[x]$-exponentiation as our group operations for $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ with respect to the basis $\bar{u}$:

- Let $K_0 = UT_n(\mathbb{Z})$, $K_1 = gp\,(g_{i0}^{\alpha_0} \mid g_{i_0} \in K_0, \alpha_0 \in \mathbb{Z}[x])$ and, inductively, $K_{j+1} = gp\,(g_{i_j}^{\alpha_j} \mid g_{i_j} \in K_j, \alpha_j \in \mathbb{Z}[x])$. Then

$$[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]} = \bigcup_{j=0}^{\infty} K_j = \{u_1^{\beta_1} \cdots u_n^{\beta_n} \mid \beta_i \in \mathbb{Q}[x]\}.$$

A particular case is when the Mal'cev basis is the collection of transvections of $UT_n(\mathbb{Z})$. In this case, every element $g \in [UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ has the normal form $g = t_{12}^{\beta_{12}} \cdots t_{n-1n}^{\beta_{n-1n}}$ where $\beta_{ij} \in \mathbb{Q}[x]$.

On the other hand, since $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]} \cong_{\mathbb{Q}[x]} UT_n(\mathbb{Q}[x])$, we know that $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$, viewed as a $\mathbb{Z}[x]$-subgroup of $UT_n(\mathbb{Q}[x])$ without making reference to any Mal'cev basis, inherits the $\mathbb{Q}[x]$-action which is defined on $UT_n(\mathbb{Q}[x])$, namely the binomial expansion. Hence, if $g = I + N \in UT_n(\mathbb{Z})$ where I is the $n \times n$ identity matrix, then

$$g^{\beta} = 1 + \alpha N + \cdots + \binom{\beta}{n-1} N^{n-1} \in [UT_n(\mathbb{Z})]^{\mathbb{Z}[x]},$$

where $\beta \in \mathbb{Z}[x]$ and, by constructing the $\{K_j\}$'s in the same way as above (now using the binomial expansion as the $\mathbb{Z}[x]$-action rather than the polynomials for multiplication and $\mathbb{Q}[x]$-exponentiation) we obtain $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ once again. We can therefore interpret $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ in either way.

Definition 3.12.1. The group $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is called the $n \times n$ *unitriangular* $\mathbb{Z}[x]$-*group.*

By Corollary 3.11.3 we immediately have

Lemma 3.12.1. *$[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is a $\mathbb{Z}[x]$-group of type HP and, consequently, is an $\mathcal{N}^*$-group.*

The $\mathbb{Z}[x]$-action defined on the collection $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is inherited from $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]}$. It turns out that, although $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is a $\mathbb{Z}[x]$-group, the matrix entries of its elements may not be in $\mathbb{Z}[x]$. For example, if $g = I + N \in UT_3(\mathbb{Z})$, then

$$g^{\beta} = (I + N)^{\beta} = I + \beta N + \binom{\beta}{2} N^2$$

where $\beta \in \mathbb{Z}[x]$ and $N = \begin{pmatrix} 0 & h_1 & h_2 \\ 0 & 0 & h_3 \\ 0 & 0 & 0 \end{pmatrix}$ with $h_1, h_2, h_3 \in \mathbb{Z}$. Notice that

$$N^2 = \begin{pmatrix} 0 & 0 & h_1 h_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $k \in \mathbb{Z}$. Hence we have

$$g^\beta = \begin{pmatrix} 1 & \beta h_1 & h_2 + \binom{\beta}{2} h_1 h_3 \\ 0 & 1 & \beta h_3 \\ 0 & 0 & 1 \end{pmatrix} \in [UT_3(\mathbb{Z})]^{\mathbb{Z}[x]}.$$

Clearly, $\binom{\beta}{2}$ is not always an element of $\mathbb{Z}[x]$. The natural question which arises is whether or not the entries of the matrices in $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ take a special form. It turns out that they do. Let $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{p}\right][x]$ denote the ring $\mathbb{Z}[x]$ with the elements $\left\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{p}\right\}$ adjoined to it.

Theorem 3.12.1. *Let $g \in [UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ and let a_{ij} denote the entry of g in the i^{th} row and j^{th} column. Then*

$$a_{ij} \in \begin{cases} \mathbb{Z}[x] & \text{if } j - i = 1, \\ \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{j-i}\right][x] & \text{if } j - i > 1. \end{cases}$$

Proof. Suppose $g = I + N \in UT_n(\mathbb{Z})$. Then

$$g^\beta = I + \beta N + \cdots + \binom{\beta}{n-1} N^{n-1}$$

for all $\beta \in \mathbb{Z}[x]$. An easy computation shows that $N^i \in UT_n^i(\mathbb{Z})$ for each $i = 1, \ldots, n$ and so the entries we obtain after multiplying N^i by $\binom{\beta}{i} \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{i}\right][x]$ and summing terms up appear as in the claim. It follows, by taking products of such matrices, that every element of $K_1 = gp\,(g_{i_0}^{\alpha_0} \mid g_{i_0} \in K_0, \alpha_0 \in \mathbb{Z}[x])$ has entries of the required form. If we now act on elements of K_1 in the same way as $UT_n(\mathbb{Z})$ and take products of such matrices, then $K_2 = gp\,(g_{i_1}^{\alpha_1} \mid g_{i_1} \in K_1, \alpha_1 \in \mathbb{Z}[x])$ again have entries of the required form. Continuing this we see, inductively, that every element of $K_{j+1} = gp\,(g_{i_j}^{\alpha_j} \mid g_{i_j} \in K_j, \alpha_j \in \mathbb{Z}[x])$ has the form claimed. $\square$

For example, any element of $[UT_5(\mathbb{Z})]^{\mathbb{Z}[x]}$ is contained in the collection of matrices of the form

$$\begin{pmatrix} 1 & \mathbb{Z}[x] & \mathbb{Z}\left[\frac{1}{2}\right][x] & \mathbb{Z}\left[\frac{1}{2},\frac{1}{3}\right][x] & \mathbb{Z}\left[\frac{1}{2},\frac{1}{3},\frac{1}{4}\right][x] \\ 0 & 1 & \mathbb{Z}[x] & \mathbb{Z}\left[\frac{1}{2}\right][x] & \mathbb{Z}\left[\frac{1}{2},\frac{1}{3}\right][x] \\ 0 & 0 & 1 & \mathbb{Z}[x] & \mathbb{Z}\left[\frac{1}{2}\right][x] \\ 0 & 0 & 0 & 1 & \mathbb{Z}[x] \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's view $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ as the $\mathbb{Z}[x]$-subgroup of $[UT_n(\mathbb{Z})]^{\mathbb{Q}[x]}$. We know that every $g \in [UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ has the unique normal form $g = u_1^{\beta_1(x)} \cdots u_n^{\beta_n(x)}$ where each $\beta_i(x) \in \mathbb{Q}[x]$. The Hall polynomials which give us such a normal form yield $\beta_i(x)$'s which are rational polynomials in x consisting of sums and/or differences of products of binomial coefficients over $\mathbb{Z}[x]$ by Theorem 1.10.2. It is known that if $p \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$ then $\binom{p}{k} \in \mathbb{Z}$. Therefore, if we substitute integer values into the $\beta_i(x)$'s of the element g, we will obtain an element in $UT_n(\mathbb{Z})$.

Theorem 3.12.2. *Suppose that $UT_n(\mathbb{Z})$ has Mal'cev basis $u = (u_1, \ldots, u_n)$. Then*

$[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is residually $UT_n(\mathbb{Z})$, where $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is viewed as the $\mathbb{Z}[x]$-subgroup of the $\mathbb{Q}[x]$-completion of $UT_n(\mathbb{Z})$ with respect to u.

Remark. $UT_n(\mathbb{Z})$ is a $\mathbb{Z}[x]$-subgroup of $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ in which the indeterminate variable, x, acts as the identity on elements of $UT_n(\mathbb{Z})$.

Proof. Let $UT_n(\mathbb{Z})$ have Mal'cev basis $\bar{u} = (u_1, \ldots, u_n)$ and let $g \in [UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ be any non-identity matrix. Then $g = u_1^{\beta_1(x)} \cdots u_n^{\beta_n(x)}$ where each $\beta_i(x) \in \mathbb{Q}[x]$. We know that at least one of the $\beta_i(x)$'s, say $\beta_t(x)$, is non-zero. Suppose the degree of $\beta_t(x)$ is m. Then $\beta_t(x)$ has at most m zeros. Moreover, as discussed above, $\beta_t(x)$ is a polynomial in x consisting of combinations of products of binomial coefficients over $\mathbb{Z}[x]$. Hence, there exists $q \in \mathbb{Z}$ such that $\beta_t(q) \neq 0$ and $\beta_i(q) \in \mathbb{Z}$ for all $i = 1, \ldots, n$. There exists a $\mathbb{Z}[x]$-homomorphism $\varphi : [UT_n(\mathbb{Z})]^{\mathbb{Z}[x]} \to UT_n(\mathbb{Z})$ which maps $u_1^{\beta_1(x)} \cdots u_n^{\beta_n(x)} \mapsto u_1^{\beta_1(q)} \cdots u_n^{\beta_n(q)}$ (see Theorem 2.11.1 for similar result). But $u_1^{\beta_1(q)} \cdots u_n^{\beta_n(q)} \neq 1$ since $\beta_t(q) \neq 0$. This completes the proof. □

Remark. This theorem could also have been proven by directly applying the remarks following Lemma 2.11.1 and Lemma 2.11.2, along with Theorem 1.10.2.

3.12.1 The Heisenberg $\mathbb{Z}[x]$-Group

I will now discuss the results of the previous section for the group $[UT_3(\mathbb{Z})]^{\mathbb{Z}[x]}$.

In this section, I will denote 3×3 identity matrix.

Let $\mathcal{H}$ be the Heisenberg group

$$\mathcal{H} = \left\{ \begin{pmatrix} 1 & g_1 & g_2 \\ 0 & 1 & g_3 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, g_i \in \mathbb{Z} \right\}.$$

There is a $\mathbb{Z}[x]$-action on $\mathcal{H}$ which is induced by $UT_3(\mathbb{Q}[x])$. Recall that if $g = I + N \in UT_3(\mathbb{Q}[x])$, where $N \in \left\{ \begin{pmatrix} 0 & g_1 & g_2 \\ 0 & 0 & g_3 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, g_i \in \mathbb{Q}[x] \right\}$, then the $\mathbb{Q}[x]$-action is given by

$$g^{\beta} = (I+N)^{\beta} = I + \beta N + \cdots + \binom{\beta}{n-1} N^{n-1}$$

for all $\beta \in \mathbb{Q}[x]$. Clearly this $\mathbb{Q}[x]$-action induces a $\mathbb{Z}[x]$-action on $\mathcal{H}$ defined by

$$(I+N)^{\alpha} = I + \alpha N + \binom{\alpha}{2} N^2 = I + \alpha N + \frac{\alpha(\alpha-1)}{2} N^2$$

where $N \in \left\{ \begin{pmatrix} 0 & g_1 & g_2 \\ 0 & 0 & g_3 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, g_i \in \mathbb{Z} \right\}$ and $\alpha \in \mathbb{Z}[x]$. If we define the sets $\{K_j\}$ as $K_0 = \mathcal{H}$, $K_1 = gp\,(g_{i_0}^{\alpha_0} \mid g_{i_0} \in K_0, \alpha_0 \in \mathbb{Z}[x])$ and, inductively, $K_{j+1} = gp\,(g_{i_j}^{\alpha_j} \mid g_{i_j} \in K_j, \alpha_j \in \mathbb{Z}[x])$, then

$$[UT_3(\mathbb{Z})]^{\mathbb{Z}[x]} = \bigcup_{j=0}^{\infty} K_j$$

equipped with this induced $\mathbb{Z}[x]$-action becomes a $\mathbb{Z}[x]$-group.

Definition 3.12.2. The $\mathbb{Z}[x]$-group $[UT_3(\mathbb{Z})]^{\mathbb{Z}[x]}$ described above is called the *Heisenberg $\mathbb{Z}[x]$-group.* It will be denoted by $\mathcal{H}^{\mathbb{Z}[x]}$.

Remark. As mentioned in the previous section, the $\mathbb{Z}[x]$-group $\mathcal{H}^{\mathbb{Z}[x]}$ can be viewed as a $\mathbb{Z}[x]$-subgroup of $\mathcal{H}^{\mathbb{Q}[x]}$, the $\mathbb{Q}[x]$-completion of $\mathcal{H}$ with respect to some Mal'cev basis.

Hence, by using the polynomials for multiplication and $\mathbb{Q}[x]$-exponentiation as the group operations for $\mathcal{H}^{\mathbb{Z}[x]}$ with respect to the Mal'cev basis, we obtain $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]} = \bigcup_{j=0}^{\infty} K_j$ as described earlier. This gives us an alternative way of constructing $\mathcal{H}^{\mathbb{Z}[x]}$.

In the last section we observed that the entries of a matrix of $\mathcal{H}^{\mathbb{Z}[x]}$ do not necessarily lie in $\mathbb{Z}[x]$. Let $\mathbb{Z}\left[\frac{1}{2}\right][x] = \{\frac{\beta}{2^n} \mid \beta \in \mathbb{Z}[x],\ n = 0, 1, \ldots\}$ be the set obtained by adjoining $\{\frac{1}{2}\}$ to the ring $\mathbb{Z}[x]$.

Lemma 3.12.2. *If* $g = \begin{pmatrix} 1 & m_1 & m_2 \\ 0 & 1 & m_3 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}^{\mathbb{Z}[x]}$, *then* $m_1, m_3 \in \mathbb{Z}[x]$ *and either* $m_2 \in \mathbb{Z}[x]$ *or* $m_2 \in \mathbb{Z}\left[\frac{1}{2}\right][x]$.

Proof. Let's construct $\mathcal{H}^{\mathbb{Z}[x]}$ in stages by examining the sets $\{K_j\}$ as defined above.

1. Suppose $g = \begin{pmatrix} 1 & g_1 & g_2 \\ 0 & 1 & g_3 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}$. Then

$$g^\alpha = (I + N)^\alpha = I + \alpha N + \binom{\alpha}{2} N^2$$

for any $\alpha \in \mathbb{Z}[x]$. Observe that $N^2 = \begin{pmatrix} 0 & 0 & g_1 g_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. An easy computation shows that

$$g^\alpha = \begin{pmatrix} 1 & \alpha g_1 & g_2 + \binom{\alpha}{2} g_1 g_3 \\ 0 & 1 & \alpha g_3 \\ 0 & 0 & 1 \end{pmatrix}$$

and the upper right entry contains a binomial coefficient $\binom{\alpha}{2}$. If $\alpha = 2\mu$ or $\alpha = 2\mu - 1$ for some $\mu \in \mathbb{Z}[x]$, then $\binom{\alpha}{2} \in \mathbb{Z}[x]$. Otherwise, $\binom{\alpha}{2} \in \mathbb{Z}\left[\frac{1}{2}\right][x]$. The other 2 entries above the diagonal lie in $\mathbb{Z}[x]$.

2. Suppose $g_1, g_2 \in \mathcal{H}^{\mathbb{Z}[x]}$ and each of these is obtained as described above. Let

$$g_1 = I + N_1 = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } g_2 = I + N_2 = \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It was shown above that $a_i, c_i \in \mathbb{Z}[x]$ and either $b_i \in \mathbb{Z}[x]$ or $b_i \in \mathbb{Z}\left[\frac{1}{2}\right][x]$ for $i = 1, 2$. Therefore

$$g_1 g_2 = \begin{pmatrix} 1 & a_1 + a_2 & b_1 + b_2 + a_1 c_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that $a_1 + a_2,\ c_1 + c_2 \in \mathbb{Z}[x]$ whereas $b_1 + b_2 + a_1 c_2$ may lie in either $\mathbb{Z}[x]$ or $\mathbb{Z}\left[\frac{1}{2}\right][x]$. We conclude that every $g \in K_1 = gp\,(g_{io}^{\alpha_o} \mid g_{i_0} \in \mathcal{H},\ \alpha_0 \in \mathbb{Z}[x]\}$ has entries of the form claimed.

3. Let g_1, g_2 be as above in 2. and let's calculate $(g_1g_2)^\gamma$ where $\gamma \in \mathbb{Z}[x]$. We obtain

$$\begin{aligned}(g_1g_2)^\gamma &= I+\gamma\begin{pmatrix}0 & a_1+a_2 & b_1+b_2+a_1c_2\\ 0 & 0 & c_1+c_2\\ 0 & 0 & 0\end{pmatrix}+\binom{\gamma}{2}\begin{pmatrix}0 & a_1+a_2 & b_1+b_2+a_1c_2\\ 0 & 0 & c_1+c_2\\ 0 & 0 & 0\end{pmatrix}^2\\ &= \begin{pmatrix}1 & \gamma(a_1+a_2) & \gamma(b_1+b_2+a_1c_2)+\binom{\gamma}{2}(a_1+a_2)(c_1+c_2)\\ 0 & 1 & \gamma(c_1+c_2)\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

Observe that $\gamma(a_1 + a_2), \gamma(c_1 + c_2) \in \mathbb{Z}[x]$ whereas $\gamma(b_1+b_2+a_1c_2)+\binom{\gamma}{2}(a_1+a_2)(c_1+c_2) \in \mathbb{Z}[x]$ or $\mathbb{Z}\left[\frac{1}{2}\right][x]$. Moreover, an easy computation shows that taking products of such matrices again yields a matrix whose entries satisfy the form claimed. Therefore, every $g \in K_2 = gp\,(g_{i_1}^{\alpha_1} \mid g_{i_1} \in K_1,\, \alpha_1 \in \mathbb{Z}[x]\}$ has entries of the form claimed.

If we now iterate the process above to form the collection

$$\bigcup_{j=0}^{\infty} K_j = \mathcal{H}^{\mathbb{Z}[x]}$$

we conclude that every $g \in \mathcal{H}^{\mathbb{Z}[x]}$ has entries of the form claimed. □

Theorem 3.12.3. *$\mathcal{H}^{\mathbb{Z}[x]}$ is an $\mathcal{N}^*$-group of $\mathcal{N}^*$ class 2.*

Proof. By Lemma 3.12.1, we know that $\mathcal{H}^{\mathbb{Z}[x]}$ is an $\mathcal{N}^*$-group and is an $\mathcal{I}$-group since it is of type HP. It is easy to verify that $Z(\mathcal{H}^{\mathbb{Z}[x]}) = [\mathcal{H}^{\mathbb{Z}[x]}, \mathcal{H}^{\mathbb{Z}[x]}]_{\mathbb{Z}[x]}$. Therefore, $\mathcal{H}^{\mathbb{Z}[x]}$ is of $\mathcal{N}^*$ class 2. □

It was mentioned earlier that if we choose a Mal'cev basis for $\mathcal{H}$ we can view $\mathcal{H}^{\mathbb{Z}[x]}$ as a $\mathbb{Z}[x]$-subgroup of the $\mathbb{Q}[x]$-completion of $\mathcal{H}$ with respect to this basis. Let $\{t_{12}, t_{13}, t_{23}\}$ be the set of transvections which is a Mal'cev basis for $\mathcal{H}$. Clearly, $\{t_{12}, t_{13}\}$ is a set of generators for $\mathcal{H}$. Therefore it is also a set of $\mathbb{Q}[x]$-generators for $\mathcal{H}^{\mathbb{Q}[x]}$, the $\mathbb{Q}[x]$-completion of $\mathcal{H}$ with respect to $\{t_{12}, t_{13}, t_{23}\}$. The next lemma immediately follows:

Lemma 3.12.3. $\mathcal{H}^{\mathbb{Z}[x]} = gp_{\mathbb{Z}[x]}(t_{12}, t_{23})$.

Lemma 3.12.4. *Let* $\mathcal{H}^{\mathbb{Z}[x]} = gp_{\mathbb{Z}[x]}(t_{12}, t_{23})$ *suppose that* α, α_i, β, β_i, $\mu \in \mathbb{Z}[x]$ *and either* γ, $\gamma_i \in \mathbb{Z}[x]$ *or* γ, $\gamma_i \in \mathbb{Z}\left[\frac{1}{2}\right][x]$. *Then*

1. $(t_{12}^{\alpha_1} t_{23}^{\beta_1} t_{13}^{\gamma_1})(t_{12}^{\alpha_2} t_{23}^{\beta_2} t_{13}^{\gamma_2}) = t_{12}^{\alpha_1+\alpha_2} t_{23}^{\beta_1+\beta_2} t_{13}^{\gamma_1+\gamma_2-\alpha_2\beta_1}$

2. $(t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma})^{\mu} = t_{12}^{\alpha\mu} t_{23}^{\beta\mu} t_{13}^{\gamma\mu-\binom{\mu}{2}\alpha\beta}$

Proof. These are exactly the Hall polynomials which hold in $\mathcal{H}$ with respect to the basis $\{t_{12}, t_{23}, t_{13}\}$. Hence they hold in the $\mathbb{Q}[x]$-completion of $\mathcal{H}$ and, therefore, to the restriction $\mathcal{H}^{\mathbb{Z}[x]}$. □

Lemma 3.12.5. $\mathcal{H}^{\mathbb{Z}[x]}$ *is torsion free.*

Proof. Let $g = t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma}$ be in normal form. Suppose that $g^{\mu} = 1$ for some non-zero $\mu \in \mathbb{Z}[x]$. Then

$$(t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma})^{\mu} = t_{12}^{\alpha\mu} t_{23}^{\beta\mu} t_{13}^{\gamma\mu-\binom{\mu}{2}\alpha\beta} = 1.$$

This implies that $\alpha = \beta = \gamma = 0$. Therefore $g = 1$. □

Lemma 3.12.6. *Let $g = t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma} \in \mathcal{H}^{\mathbb{Z}[x]}$ be in normal form, where both α, $\beta \in \mathbb{Z}[x]$ and either $\gamma \in \mathbb{Z}[x]$ or $\gamma \in \mathbb{Z}\left[\frac{1}{2}\right][x]$. Then $g = 1$ if and only if $\alpha = \beta = \gamma = 0$.*

$\mathcal{H}^{\mathbb{Z}[x]}$ has the property that it is free in the class of all $\mathcal{N}^*$-groups of $\mathcal{N}^*$ class 2 or less.

Definition 3.12.3. Let $\mathcal{N}_2$ be the class of all $\mathcal{N}^*$-groups of $\mathcal{N}^*$ class 2 or less. Then a group $F \in \mathcal{N}_2$ is *free in* $\mathcal{N}_2$ if it comes equipped with a set S and a map $\mu : S \to F$ such that for every $\mathcal{N}^*$-group $H \in \mathcal{N}_2$ and every map $\theta : S \to H$, there exists a unique $\mathbb{Z}[x]$-homomorphism $\varphi : F \to H$ such that $\varphi \circ \mu = \theta$. We say that F *is free on* S. If $S \subset F$ and μ is the identity map, we say that F *is freely* $\mathbb{Z}[x]$*-generated by* S.

Theorem 3.12.4. *$\mathcal{H}^{\mathbb{Z}[x]}$ is free in $\mathcal{N}_2$.*

Proof. Let $S = \{t_{12}, t_{23}\}$ be the set of transvections which $\mathbb{Z}[x]$-generate $\mathcal{H}^{\mathbb{Z}[x]}$ and let $G \in \mathcal{N}_2$ with $\mathbb{Z}[x]$-generating set $\{g_i \mid i \in I\}$ for some index set I. Suppose that $\theta : S \to G$ is a set map from S to G defined by $\theta(t_{12}) = g_1$, $\theta(t_{23}) = g_2$. I claim that θ can be uniquely extended to a $\mathbb{Z}[x]$-homomorphism from $\mathcal{H}^{\mathbb{Z}[x]}$ to G. Consider the mapping $\hat{\theta} : \mathcal{H}^{\mathbb{Z}[x]} \to G$ defined by

$$\hat{\theta}(t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma}) = g_1^{\alpha} g_2^{\beta} [g_1, g_2]^{\gamma},$$

where $\alpha, \beta, \gamma \in \mathbb{Z}[x]$. Then $\hat{\theta}$ is a $\mathbb{Z}[x]$-homomorphism:

$$\begin{aligned}
\hat{\theta}[(t_{12}^{\alpha_1} t_{23}^{\beta_1} t_{13}^{\gamma_1})(t_{12}^{\alpha_2} t_{23}^{\beta_2} t_{13}^{\gamma_2})] &= \hat{\theta}(t_{12}^{\alpha_1+\alpha_2} t_{23}^{\beta_1+\beta_2} t_{13}^{\gamma_1+\gamma_2-\alpha_2\beta_1}) \\
&= g_1^{\alpha_1+\alpha_2} g_2^{\beta_1+\beta_2} [g_1, g_2]^{\gamma_1+\gamma_2-\alpha_2\beta_1} \\
&= (g_1^{\alpha_1} g_2^{\beta_1} [g_1, g_2]^{\gamma_1})(g_1^{\alpha_2} g_2^{\beta_2} [g_1, g_2]^{\gamma_2}) \\
&= \hat{\theta}(t_{12}^{\alpha_1} t_{23}^{\beta_1} t_{13}^{\gamma_1}) \hat{\theta}(t_{12}^{\alpha_2} t_{23}^{\beta_2} t_{13}^{\gamma_2}). \\
\hat{\theta}[(t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma})^{\mu}] &= \hat{\theta}(t_{12}^{\alpha\mu} t_{23}^{\beta\mu} t_{13}^{\gamma\mu - \binom{\mu}{2}\alpha\beta}) \\
&= g_1^{\alpha\mu} g_2^{\beta\mu} [g_1, g_2]^{\gamma\mu - \binom{\mu}{2}\alpha\beta} \\
&= (g_1^{\alpha} g_2^{\beta} [g_1, g_2]^{\gamma})^{\mu} \\
&= [\hat{\theta}(t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma})]^{\mu}.
\end{aligned}$$

Suppose that w is a $\mathbb{Z}[x]$-word in $\mathcal{H}^{\mathbb{Z}[x]} = gp_{\mathbb{Z}[x]}(t_{12}, t_{23})$. We can represent w in the normal form $w = t_{12}^{\alpha} t_{23}^{\beta} t_{13}^{\gamma}$ for some $\alpha, \beta \in \mathbb{Z}[x]$ and either $\gamma \in \mathbb{Z}[x]$ or $\gamma \in \mathbb{Z}\left[\frac{1}{2}\right][x]$. If $w = 1$ in $\mathcal{H}^{\mathbb{Z}[x]}$, then $\alpha = \beta = \gamma = 0$. This implies that

$$\hat{\theta}(w) = \hat{\theta}(t_{12}^0 t_{23}^0 t_{13}^0) = g_1^0 g_2^0 [g_1, g_2]^0 = 1.$$

Therefore, $\mathbb{Z}[x]$-words in $\mathcal{H}^{\mathbb{Z}[x]}$ which reduce to 1 in $\mathcal{H}^{\mathbb{Z}[x]}$ are mapped to 1 in G. Thus $\hat{\theta}$ is a $\mathbb{Z}[x]$-homomorphism. □

We saw earlier that $[UT_n(\mathbb{Z})]^{\mathbb{Z}[x]}$ is residually $UT_n(\mathbb{Z})$. I will repeat the proof for $\mathcal{H}^{\mathbb{Z}[x]}$.

Theorem 3.12.5. *$\mathcal{H}^{\mathbb{Z}[x]}$ is residually $\mathcal{H}$.*

Remark. $\mathcal{H}$ can be viewed as a $\mathbb{Z}[x]$-group where the indeterminate variable, x, acts identically on $\mathcal{H}$.

Proof. Let $g \in \mathcal{H}^{\mathbb{Z}[x]}$ be a non-identity matrix. Suppose that $\{t_{12}, t_{23}, t_{13}\}$ is the chosen Mal'cev basis for $\mathcal{H}$. Then $g = t_{12}^{\alpha_1(x)} t_{23}^{\alpha_2(x)} t_{13}^{\alpha_3(x)}$ for some $\alpha_1(x), \alpha_2(x) \in \mathbb{Z}[x]$ and either $\alpha_3(x) \in \mathbb{Z}[x]$ or $\alpha_3(x) \in \mathbb{Z}\left[\frac{1}{2}\right][x]$. Now suppose that $\alpha_i(x)$ is not the zero polynomial for at least one $i \in \{1, 2, 3\}$ (clearly such an i exists or else we would have $g = 1$). Then $\alpha_i(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree p_i, say, which is a sum and/or difference of products of binomial coefficients of the form $\binom{\beta(x)}{k}$ where $\beta(x) \in \mathbb{Z}[x]$ and $k \in \mathbb{Z}^+$. Hence there exists $q \in \mathbb{Z}$ such that $\alpha_i(q) \neq 0$ and $\alpha_i(q) \in \mathbb{Z}$. Let $\varphi_q : \mathbb{Z}[x] \to \mathbb{Z}$ be the evaluation $\mathbb{Z}[x]$-homomorphism at q defined by $\varphi_q(f) = f(q)$. Then $\varphi_q(\alpha_i(x)) = \alpha_i(q) \neq 0$ and so, if we define $\psi : \mathcal{H}^{\mathbb{Z}[x]} \to \mathcal{H}$ by

$$\begin{aligned}
\psi(g) &= \psi(t_{12}^{\alpha_1(x)} t_{23}^{\alpha_2(x)} t_{13}^{\alpha_3(x)}) \\
&= t_{12}^{\varphi_q(\alpha_1(x))} t_{23}^{\varphi_q(\alpha_2(x))} t_{13}^{\varphi_q(\alpha_3(x))} \\
&= t_{12}^{\alpha_1(q)} t_{23}^{\alpha_2(q)} t_{13}^{\alpha_3(q)},
\end{aligned}$$

it is not hard to see that ψ is a $\mathbb{Z}[x]$-homomorphism and $\psi(g) \neq 1$. □

Bibliography

[1] Baumslag, Cannonito, and Miller, *Computable Algebra and Group Embeddings*, J. Algebra **69** (1981), 186–212, Theorem 2.7.

[2] Gilbert Baumslag, *Lecture Notes on Nilpotent Groups*, C.B.M.S. Regional Conference Series no. 2, 1971.

[3] Edward Formanek, *A Short Proof of a Theorem of Jennings*, Proc. Amer. Math. Soc **26** (1970), 405–407.

[4] M. Hall, *The Theory of Groups*, Macimillan, New York, 1959.

[5] Philip Hall, *Nilpotent Groups*, 1957, Canad. Math. Congress, Summer Seminar, University of Alberta.

[6] ______, *Some Word Problems*, J. London Math. Soc. **33** (1958), 482–496.

[7] S. A. Jennings, *The Group Ring of a Class of Infinite Nilpotent Groups*, Canad. J. Math. **7** (1955), 169–187.

[8] M.I. Kargapolov and Ju. I. Merzljakov, *Fundamentals of the Theory of Groups*, Springer-Verlag, 1979, Translated by Robert G. Burns.

[9] A.G. Kurosh, *Theory of Groups*, vol. II (translated from Russian), Chelsea Publishing Company, New York, N.Y., 1960.

[10] Roger C. Lyndon, *Groups With Parametric Exponents*, Trans. of the Amer. Math. Soc. **96** (1960), 518–533.

[11] Ian Macdonald, *The Theory of Groups*, Oxford, Clarendon P., 1968.

[12] A.G. Myasnikov and V.N. Remeslennikov, *Exponential Groups I: Fundamentals of the Theory and Tensor Completions*, Siberian Math. Journal **35** (1994), Number 5.

[13] P. F. Pickel, *Finitely Generated Nilpotent Groups with Isomorphic Finite Quotients*, Trans. Amer. Math. Soc. **160** (1971), 327–341.

[14] Jr. Robert B. Warfield, *Nilpotent Groups*, Springer-Verlag, 1976.

[15] Derek J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, 1996.