# An Unstable Adams Spectral Sequence based on a Generalized Homology Theory

by

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# Introduction

An Adams spectral sequence is, roughly speaking, a sequence converging, for a specific homology theory, from homology to homotopy of a certain spectrum or space. The classical stable Adams spectral sequence uses ordinary mod-$p$ homology theory and converges to the $p$-component in homotopy for a spectrum $X$, i.e. $H_*(X; Z/p) \Rightarrow \pi_*(X) \otimes Z_{(p)}$ — $Z_{(p)}$ are the integers localized at $p$. Two restrictions make this particular spectral sequence simpler than all its descendants. It is a stable spectral sequence using homology of spectra instead of spaces and computes each $p$-component at a time, i.e. everything is localized at a prime $p$. Later, other homology theories replaced mod-$p$ homology, best known example being the stable Adams-Novikov spectral sequence based on complex bordism $MU_*(X)$ or its localized version using Brown-Peterson homology $BP_*(X)$ and converging again to the $p$-component in homotopy. Localizaton makes the latter especially useful for computations and a large amount of work has been done in understanding and using it [**15**].

Parallel to the development of these stable sequences unstable Adams spectral sequences where also constructed and studied, although on a comparatively smaller scale. First, the unstable equivalent of the classical Adams spectral sequence with coefficients in a suitable ring [**7**] and later an unstable version of the Adams-Novikov spectral sequence [**3**] were introduced. Only recently have been attempts made to set up an unstable spectral sequence based on a non-connective homology theory, like periodic complex $K$-theory [4].

Since this work has to be seen in the context of unstable homotopy theory, I would like to point out the peculiarities of unstable sequences versus stable sequences, which are in general more familiar.

In both cases one starts with a commutative, associative ring spectrum $E$ and its associated generalized homology theory $E_*$ to construct a certain tower of fibrations. In the stable case this are simply fibrations generated by the unit $\eta : S \longrightarrow E$ of the ring spectrum $E$. Applying homotopy to this tower generates an exact couple which determines, as usual, a spectral sequence. In the stable setting the $E_2$-term of this spectral sequence can be described as an $Ext$-term in the category of $E_*E$-comodules, where $E_*E$ is the Hopf algebroid of stable cooperations similar to the dual of the Steenrod Algebra in the classical case. The description requires the category of $E_*E$-comodules to be abelian, which is the case when $E_*E$ is flat as a $E_*$-module. Most interesting homology theories meet this requirement.

In the unstable case, on the other hand, no such abelian category exists and one has to settle for a description as an unstable $Ext$-term in a non-abelian category. Identifying this $Ext$-term requires a certain amount of non-abelian homological algebra.

Convergence considerations of such an unstable spectral sequence are even more subtle. In certain cases—like $BP$-theory—convergence has been proved directly [3]. for non-connective homology theories the only currently known way is by computing the $E_2$-term and detecting a vanishing line. This vanishing line together with results by Bousfield and Kan in [6] determines convergence and the target of the spectral sequence.

The actual computation of the $E_2$-term as an unstable $Ext$-term poses the most difficult problem. The non-abelian category $\mathcal{M}(G)$ of unstable $G$-coalgebras in which this $Ext$-term is defined is not well understood. especially for non-connective homology theories. In the connected case— like $BP$-theory in [3] —the computation for spaces with nice homology can be carried out in a related abelian category $\mathcal{A}(\Gamma)$ of unstable comodules using a composite functor spectral sequence. For non-connective homolgy theories the use of this spectral sequence is more difficult and. so far. the transfer of the computation to $\mathcal{A}(\Gamma)$ has been only proved directly for the homology of certain spaces .

The main result of this work concerns the computation of the $E_2$-term in this abelian category $\mathcal{A}(\Gamma)$. I will present an unstable version of a change of rings theorem for certain non-connective spectra. $E(n)$-spectra. which plays a role in future computations using the associated homology theory. This has been carried out for $E(1)$. a summand of periodic complex $K$-theory. in [4].

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# Contents

CHAPTER 1

# An Unstable Adams Spectral Sequence based on a generalized homology theory

## 1. Background and Definintions

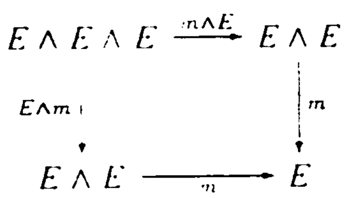
In [3] M. Bendersky, E. Curtis and H. Miller generalized the Boustield-Kan construction [7] of an unstable Adams spectral sequence to a spectral sequence based on generalized homology theories associated to certain connective spectra. With $BP$-theory as the prime example in mind, the authors did not consider non-connective spectra, although large parts of their approach generalize in a straightforward way. This chapter discusses the details of this more general setting.

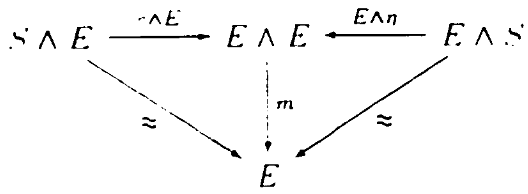
DEFINITION 1.1. *A spectrum $E$ is a sequence of spaces $\mathbf{E_n}$ together with structure maps $\epsilon_n : \Sigma\mathbf{E_n} \longrightarrow \mathbf{E_{n+1}}$. If each adjoint $\tilde{\epsilon}_n : \mathbf{E_n} \longrightarrow \Omega\mathbf{E_{n+1}}$ is a weak homotopy equivalence, then $E$ is called an $\Omega$-spectrum. Any spectrum $E$ is homotopy equivalent to an $\Omega$-spectrum $F$, which can be defined by*

$$\mathbf{F_n} = \varinjlim \Omega^k \mathbf{E_{n+k}}.$$

A special example is the *suspension spectrum* $\Sigma^\infty X$ of a space $X$ with $(\Sigma^\infty X)_n = \Sigma^n X$.

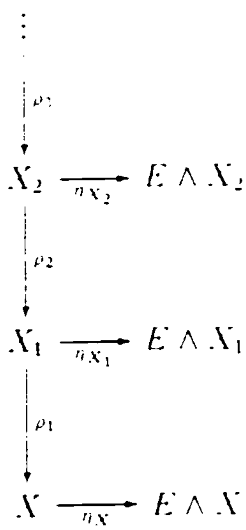
DEFINITION 1.2. *A ring spectrum* $E$ *is a spectrum with a multiplicative structure* $m : E \wedge E \longrightarrow E$ *and a unit* $\eta : S \longrightarrow E$ *(*$S$ *is the suspension spectrum of the sphere) such that the following diagrams commute up to homotopy:*




A ring spectrum determines in the usual way [1] a multiplicative homology theory with coefficients in the ring $E_* = \pi_*(E)$, where $\pi_r(E) = \varinjlim \pi_{n+r}(\mathbf{E_n})$ is the homotopy of the spectrum $E$.

Given a ring spectrum $E$, the unit induces a natural transformation $\eta_X = \eta \wedge X : X = S \wedge X \longrightarrow E \wedge X$ for spectra $X$. In the stable situation a tower of fibrations is used to define a spectral sequence. Specifically, the homotopy spectral sequence of the tower



of fibrations determines the stable Adams spectral sequence based on $E$-theory.

## 2. The triple $(E, \mu, \eta)$ and its associated homotopy spectral sequence

To destabilize the construction in the previous section a space $E(X)$, to replace $E \wedge X$ in the tower above, has to be introduced.

DEFINITION 1.3. *For a space $X$ let $E(X)$ be the $0^{th}$ space of the $\Omega$-spectrum associated to the spectrum $E \wedge \Sigma^\infty X$. Using notation of [2]:*

$$E(X) = \Omega^\infty(E \wedge \Sigma^\infty X)$$

This defines an endofunctor on the category of spaces which will be denoted by $E$. The context should make it clear if $E$ is this endofunctor or the spectrum.

Observe that for $n \geq 0$,

$$\pi_n(E(X)) \approx E_n(X),$$

where $E_n(X)$ denotes reduced $E$-homology. Compare this to the stable statement

$$\pi_n(E \wedge X) \approx E_n(X),$$

for a spectrum $X$.

The unit $\eta : S \longrightarrow E$ induces the Hurewicz map

$$\eta_X : X \longrightarrow \Omega^\infty \Sigma^\infty X \xrightarrow{\Omega^\infty(\eta \wedge \Sigma^\infty X)} \Omega^\infty(E \wedge \Sigma^\infty X) = E(X)$$

The multiplicative structure $m : E \wedge E \longrightarrow E$ also induces a structure map for the functor $E$ considered now as a functor on the homotopy category of spaces. Following §4 of [**3**] pairings $\mathbf{E_n} \wedge \mathbf{E_m} \longrightarrow \mathbf{E_{n+m}}$ induce a natural transformation $\mu_X : E(E(X)) \longrightarrow E(X)$ in the homotopy category of spaces.

The defining properties of the ring spectrum $E$ make the following diagrams in the homotopy category of spaces commutative:

$$\begin{array}{ccc} E^3(X) & \xrightarrow{E\mu_X} & E^2(X) \\ \downarrow{\scriptstyle \mu_X E} & & \downarrow{\scriptstyle \mu_X} \\ E^2(X) & \xrightarrow[\mu_X]{} & E(X) \end{array} \qquad \begin{array}{ccccc} E(X) & \xrightarrow{\eta_{E(X)}} & E^2(X) & \xleftarrow{E\eta_X} & E(X) \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu_X} & \swarrow{\scriptstyle =} & \\ & & E(X) & & \end{array}$$

The system $(E, \mu, \eta)$ thus forms a triple on the homotopy category $\mathcal{C}$ of spaces (see 2.5 for the definition of a triple).

REMARK 1.4. *This triple is defined only on the homotopy category of spaces, since the diagrams that define the ring spectrum E commute only up to homotopy. In* [**8**] *a category of spectra with concrete point-set topological models are studied, where strictly commutative diagrams make sense. In the context of this category an S-algebra is a spectrum with strictly associative, unital multiplication. If E is an S-algebra, the triple* $(E, \mu, \eta)$ *can be considered a triple in* *a category of spaces. Some ring spectra, like MU, BP and K, have been shown to be representable as S-algebras. It is conjectured the same thing holds for E(n), the Johnson-Wilson spectra. Everything said in section 3 has to be seen modulo a proof of this conjecture. (see also [9])*

Using the unit $\eta : I \longrightarrow E$ a sequence of endofunctors $D_i : \mathcal{C} \longrightarrow \mathcal{C}$ and natural transformations $\rho_i : D_{i+1} \longrightarrow D_i$ are constructed in the following way. Let

$$\begin{array}{ccc} D_1(X) & & \\ \downarrow{\scriptstyle \rho_1} & & \\ X & \xrightarrow[\eta_X]{} & E(X) \end{array}$$

be the pullback of the path-space fibration over $E(X)$ via $\eta_X$, and in general

$$\begin{array}{ccc} D_{i+1}(X) & & \\ \downarrow{\scriptstyle \rho_{i+1}} & & \\ D_i(X) & \xrightarrow[D_i\eta_X]{} & D_i(E(X)) \end{array}$$

be the pullback of the path-space fibration over $D_i(E(X))$ via $D_i\eta_X$. This construction gives a sequence of fibrations, which is called the tower of the triple $(E, \mu, \eta)$ over the space $X$ :

$$\begin{array}{ccc}
\vdots & & \\
\downarrow{\scriptstyle p_3} & & \\
D_2(X) & \xrightarrow[D_2\eta_X]{} & D_2(E(X)) \\
\downarrow{\scriptstyle p_2} & & \\
D_1(X) & \xrightarrow[D_1\eta_X]{} & D_1(E(X)) \\
\downarrow{\scriptstyle p_1} & & \\
X & \xrightarrow[\eta_X]{} & E(X)
\end{array}$$

DEFINITION 1.5. *The homotopy spectral sequence of the triple $(E, \eta, \mu)$ at $X$ is the spectral sequence arising from the homotopy exact couple of this tower of fibrations, i.e. from*

$$\cdots \xrightarrow{\delta} \pi_*D_*(X) \xrightarrow{p_*} \pi_*D_*(X) \xrightarrow{D_*\eta_X} \pi_*D_*(E(X)) \xrightarrow{i} \pi_*D_*(X) \xrightarrow{\delta_*} \cdots$$

*where $\delta$ is the boundary map of the long exact sequence in homotopy of the above fibrations. The $E_1$-term of this spectral sequence is given by*

$$E_1^{s,t} = \begin{cases} \pi_{t-s}D_s(E(X)), & \textit{for } t > s \geq 0 \\ 0, & \textit{otherwise.} \end{cases}$$

REMARK 1.6. 1. *The definition works for an arbitrary triple on the homotopy category of spaces.*

2. *Since $\pi_n(E(X)) \approx E_n(X)$, the $E_1$-term of this spectral sequence depends only on $E_*(X)$. (compare this to 1.2 in [7])*

## 3. Some Remarks about Convergence

In §3 of [**3**] Bendersky, Curtis and Miller prove that for a connective ring spectrum $E$ that allows a Thom map $E \longrightarrow H$, where $H$ is the integral Eilenberg-Mac Lane spectrum, the spectral sequence defined in the previous section converges for a simply connected space $X$ to the homotopy groups of $X$. In their work they point at similar arguments using a $p$-local Thom map to show that the spectral sequence converges for $BP$-theory to the $BP$-localization of $X$.

DEFINITION 1.7. *The $E$-localization of a space $X$ is a space $X_E$ with a localization map $\eta_E : X \longrightarrow X_E$ which induces an isomorphism in $E$-homology and is terminal under such maps, i.e. suppose there is another map $f : X \longrightarrow Y$ which induces an $E_*$-isomorphism, then there exists a unique map $r : Y \longrightarrow X_E$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle \eta_E}\searrow & \swarrow {\scriptstyle r} \\ & X_E & \end{array}$$

If $E$ is a connective spectrum and $X$ a simply connected space, then $X_E$ is just the localization with respect to some primes. If we fix a prime $p$, for example, then the spectral sequence based on $BP$-theory converges to the homotopy groups of $X$ localized at $p$, i.e. to $\pi_*(X) \otimes Z_{(p)} \approx \pi_*(X_{BP})$. If $E$ fails to be connective, $X_E$ is far more mysterious. As an example let $E_*$ be $p$-local complex $K$-theory, then it is shown in [12] that $(S^{2n+1})_E$ is not $2n$-connected, in fact the results of [4] imply that $\pi_{2n-1}((S^{2n+1})_E) = Q/Z_{(p)}$.

To study convergence for non-connected homology theories requires a different approach outlined in [4]. Localization, the appropriate target in the connected case has to be replaced by the more general notion of $E$-completion following [6]. In this book Bousfield and Kan study thoroughly the notion of completion of a space with respect to a ring $R$, using mostly simplicial techniques. As one application they relate certain spectral sequences converging to the homotopy of a completion of a space $X$ to the unstable Adams spectral sequence based on homology with coefficients in the ring $R$ defined in [7] by the same authors. The following arguments are basically the same.

REMARK 1.8. *In many cases $E$-localizations and $E$-completions are closely related, in fact the example above concerning p-local complex $K$-theory was proved for $E$-completions.*

In order to define the $E$-completion $E^{\wedge}X$ of a space $X$ one has to work simplicially and define:

- the cosimplicial resolution of a space with respect to a triple
- the total space of a cosimplicial space

DEFINITION 1.9. *A cosimplicial space $\check{X}$ is a cosimplicial object over a category $\mathcal{C}$ of spaces, i.e. $\check{X}$ consists of*

1. a space $\tilde{X}_n$ for every $n \geq 0$ and
2. for $0 \leq i \leq n$ coface and codegeneracy maps

$$d^i : \tilde{X}_{n-1} \longrightarrow \tilde{X}_n \qquad \text{and}$$

$$s^i : \tilde{X}_{n+1} \longrightarrow \tilde{X}_n$$

satisfying the cosimplicial identities dual to the simplicial identities as in [**13**].

EXAMPLE 1.10. *Any triple $(E, \mu, \eta)$ on the category $\mathcal{C}$ of spaces determines a functor $E^\bullet$ from $\mathcal{C}$ to the category of cosimplicial objects over $\mathcal{C}$:*

$$(E^\bullet(X))^n = E^{n+1}(X)$$

$$d^i = E^i\eta_{E^{n-i}X} : E^n(X) \longrightarrow E^{n+1}(X), \qquad 0 \leq i \leq n$$

$$s^i = E^i\mu_{E^{n-i}(X)} : E^{n+2}(X) \longrightarrow E^{n+1}(X), \qquad 0 \leq i \leq n$$

EXAMPLE 1.11. *The cosimplicial* standard simplex $\tilde{\Delta}$ *consists in each dimension $n$ of the standard $n$-simplex $\Delta[n]$ with the usual coface and codegeneracy maps.*

DEFINITION 1.12. *The* total space $Tot_\infty \tilde{X}$ *of a cosimplicial space is the function space $hom(\tilde{\Delta}, \tilde{X})$.*

**Definition 1.13.** *For a triple $(E, \mu, \eta)$ in the category of spaces the $E$-completion of a space $X$ is defined by*

$$E^\wedge X = Tot_\infty E^\bullet(X)$$

As in [6] using the simplicial $s$-skeleton $\Delta^{[s]}$ of $\Delta$ and defining

$$Tot_s X = hom(\Delta^{[s]}, X)$$

$E^\wedge X$ can be described as the homotopy inverse limit of a tower of fibrations consisting of the $Tot_s E^\bullet(X)$

$$E^\wedge X = \varprojlim Tot_s E^\bullet(X)$$

Given a tower under a space $X$

$$\begin{array}{ccc}
 & & X \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
F_2 & \longrightarrow & X_2 \\
 & & \downarrow \\
F_1 & \longrightarrow & X_1 \\
 & & \downarrow \\
F_0 & \xrightarrow{=} & X_0
\end{array}$$

it is also possible to define a homotopy spectral sequence, where $E_1^{s,t} = \pi_{t-s} F_s$.

If $X$ is the homotopy inverse limit of this tower the following result is proved in chapter IX of [6]:

PROPOSITION 1.14 (Bousfield, Kan). *Suppose that under the situation above for $r \geq 1$*

$$\varprojlim{}^1 E_r^{s,s+t} = 0 = \varprojlim{}^1 E_r^{s,s+t+1}$$

*for all $s \geq 0$, then the homotopy spectral sequence of the tower above converges completely to $\pi_*(X)$.*

REMARK 1.15. *The proposition above proves convergence if the spectral sequence has a horizontal vanishing line, i.e. if for some $r$ there is an $N$ such that $E_r^{s,t} = 0$ for $s \geq N$.*

Recall that our spectral sequence is defined by a tower over $X$, but the spectral sequence above is determined by a tower under $E^\wedge X$. In order to make $E^\wedge X$ the target of our spectral sequence it is necessary to study the relationship of such towers. Every tower under $X$ determines a tower over $X$. Let

$$\begin{array}{ccc}
 & & X \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
F_2 & \longrightarrow & X_2 \\
 & & \downarrow \\
F_1 & \longrightarrow & X_1 \\
 & & \downarrow \\
F_0 & \xrightarrow{\simeq} & X_0
\end{array}$$

be a tower under $X$. Define $X^{s+1}$ as the homotopy fiber of $X \longrightarrow X_s$ and let $D_s = F_s$, then

$$\begin{array}{ccc}
\vdots & & \\
\uparrow & & \\
X^2 & \longrightarrow & D^2 \\
\uparrow & & \\
X^1 & \longrightarrow & D^1 \\
\uparrow & & \\
X & \longrightarrow & D^0
\end{array}$$

is a tower over $X$. The following diagram shows that this two towers induce the same homotopy spectral sequence.

$$\begin{array}{ccccc}
X^{s+1} & \longrightarrow & X^s & \longrightarrow & F_s = D^s \\
\Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow \\
X^{s+1} & \longrightarrow & X & \longrightarrow & X_s \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
* & \longrightarrow & X_{s-1} & \xrightarrow{=} & X_s
\end{array}$$

Following the same argument as in [6] chapter X with respect to [7] shows that the tower under $E^*X$ induces the tower of fibration defining our spectral sequence.

This shows that if the spectral sequence has a horizontal vanishing line, then it converges for a simply connected space $X$ to the homotopy of the $E$-completion of $X$.

CHAPTER 2

# The $E_2$-term of an unstable Adams spectral sequence based on a generalized homology theory

## 1. A cosimplicial description

In this chapter we start with the triple $(E, \mu, \eta)$ in the category $\mathcal{C}$ of spaces and the description in Definition 1.5 of its associated homotopy spectral sequence. The first section will lead to a cosimplicial description of its $E_2$-term.

Recall Example 1.10 which defines a functor $E^\bullet$ which assigns to a space $X$ a cosimplicial space $E^\bullet(X)$. The map $\eta_X : X \longrightarrow E(X)$ provides a coaugmentation of this cosimplicial space in the sense that $d^0 \circ \eta = d^1 \circ \eta$. Applying homotopy gives the cosimplicial abelian group $\pi_*(E^\bullet X)$, which determines a cochain complex $ch(\pi_*(E^\bullet X))$ by defining the differentials $\delta^k$ as

$$\delta^0 = \pi_*(\eta)$$

$$\delta^k = \sum_{i=0}^{k} (-1)^i \pi_*(d^i), \qquad \text{for } k \geq 1$$

DEFINITION 2.1. *The homology $H^* ch(\pi_*(E^\bullet X))$ of the cochain complex defined above is called the cohomotopy of the cosimplicial abelian group $\pi_*(E^\bullet X)$ and is denoted by $\pi^*\pi_* E^\bullet X$.*

THEOREM 2.2. *The $E_2$-term of the homotopy spectral sequence of the triple $(E, \mu, \eta)$ at $X$ is given by:*

$$E_2^{s,t}(X) = \begin{cases} \pi^s \pi_t E^\bullet(X), & \text{for } t > s \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The proof of this Theorem relies on a double cochain complex argument which uses the fact that the homology of the total complex can be computed in two different ways. The following two Lemmas provide the necessary collapsing arguments.

LEMMA 2.3. *Let $Y = E(X)$ for $E$ and $X$ as above. Then*

$$E_2^{s,t}(Y) = 0, \qquad \text{for } s > 0$$

$$E_2^{0,t}(Y) = \pi_t(Y), \qquad \text{for } t > 0$$

PROOF. Because $(E, \mu, \eta)$ is a triple and the $D_t$ are functors $D_t\mu_X$ is left inverse to $D_t\eta_{EX}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} D_t(E(X)) & \xrightarrow{D_t\eta_{EX}} & D_t(E^2(X)) \\ & \overset{=}{\searrow} & \downarrow D_t\mu_X \\ & & D_t(E(X)) \end{array}$$

Therefore the long exact sequence in homotopy of the fibrations

$$\begin{array}{ccc} D_{i+1}(E(X)) & & \\ \downarrow & & \\ D_i(E(X)) & \xrightarrow[D_i(\eta_{E(X)})]{} & D_i(E^2(X)) \end{array}$$

splits into short exact sequences of the form:

$$0 \longrightarrow \pi_t D_i(E(X)) \xrightarrow{\pi_t D_i(\eta_{E(X)})} \pi_t D_i(E^2(X)) \xrightarrow{\partial} \pi_{t-1} D_{i+1}(E(X)) \longrightarrow 0$$

Splicing the short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{t+i} E(X) & \longrightarrow & \pi_{t+i} E^2(X) & \longrightarrow & \pi_{t+i-1} D_1(E(X)) & \longrightarrow & 0 \\
0 & \longrightarrow & \pi_{t+i-1} D_1(E(X)) & \longrightarrow & \pi_{t+i-1} D_1(E^2(X)) & \longrightarrow & \pi_{t+i-2} D_2(E(X)) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \pi_{t+1} D_{i-1}(E(X)) & \longrightarrow & \pi_{t+1} D_{i-1}(E^2(X)) & \longrightarrow & \pi_t D_i(E(X)) & \longrightarrow & 0 \\
0 & \longrightarrow & \pi_t D_i(E(X)) & \longrightarrow & \pi_t D_i(E^2(X)) & \longrightarrow & \pi_{t-1} D_{i+1}(E(X)) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

together gives the following long exact sequence with $d_1$ the first differential in the spectral sequence:

$$0 \longrightarrow \pi_{t+i} E(X) \longrightarrow \pi_{t+i} E^2(X) \xrightarrow{d_1} \pi_{t+i-1} D_1(E^2(X)) \xrightarrow{d_1} \cdots$$
$$\cdots \xrightarrow{d_1} \pi_{t+1} D_{i-1}(E^2(X)) \xrightarrow{d_1} \pi_t D_i(E^2(X)) \xrightarrow{d_1} \cdots$$

Looking at the definition of the $E_1$-term this is the sequence:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{t+s}E(X) & \longrightarrow & E_1^{0,t+s} & \xrightarrow{d_1} & E_1^{1,t+s-1} & \xrightarrow{d_1} & \dots \\
 & & \dots & \xrightarrow{d_1} & E_1^{s-1,t+1} & \xrightarrow{d_1} & E_1^{s,t} & \xrightarrow{d_1} & \dots
\end{array}
$$

Taking homology proves the statement. $\square$

LEMMA 2.4. *Let $F$ be a functor from our category $\mathcal{C}$ of spaces to the category of abelian groups. Then*

$$
\pi^s FE(E^\bullet X) = \begin{cases} FE(X), & \text{if } s = 0 \\ 0, & \text{otherwise.} \end{cases}
$$

PROOF. $\mu : E^2 \longrightarrow E$ is a natural transformation i.e. the following diagram commutes for every $f : C \longrightarrow D$.

$$
\begin{array}{ccc}
E^2(C) & \xrightarrow{\mu_C} & E(C) \\
\big\downarrow{\scriptstyle E^2 f} & & \big\downarrow{\scriptstyle Ef} \\
E^2(D) & \xrightarrow[\mu_D]{} & E(D)
\end{array}
$$

Now let $f = d^{i-1} : E^n(X) \longrightarrow E^{n+1}(X)$ for $1 \le i \le n$, then the diagram above becomes:

$$
\begin{array}{ccc}
E^2(E^n(X)) & \xrightarrow{\mu_{E^n(X)}} & E(E^n(X)) \\
\big\downarrow{\scriptstyle E^2 d^{i-1}} & & \big\downarrow{\scriptstyle Ed^{i-1}} \\
E^2(E^{n+1}(X)) & \xrightarrow[\mu_{E^{n+1}(X)}]{} & E(E^{n+1}(X))
\end{array}
$$

If I define

$$\mu_n = \mu_{(E\bullet X)^n} = \mu_{E^{n+1}(X)}$$

$$\mu_{-1} = \mu_X$$

and observe that $E^2 d^{i-1} = E^2 E^{i-1} \eta_{E^{n-i+1}(X)} = E E^i \eta_{E^{n-i+1}(X)} = Ed^i$ . then

$$Ed^{i-1} \circ \mu_{n-1} = \mu_n \circ Ed^i .$$

Looking at the following diagram

$$FE(X) \underset{F\mu_{-1}}{\overset{F(E\eta_X)}{\rightleftarrows}} FE(E(X)) \underset{F\mu_0}{\overset{\delta^1}{\rightleftarrows}} FE(E^2(X)) \underset{F\mu_1}{\overset{\delta^2}{\rightleftarrows}} FE(E^3(X)) \underset{F\mu_2}{\overset{\delta^3}{\rightleftarrows}} \cdots$$

where $\delta^k = \sum_{i=0}^{k} (-1)^i F(Ed^i)$. I claim that the $F\mu_k$ form a contracting homotopy of this cochain complex. It is clear that

$$F\mu_{-1} \circ FE\eta_X = F(\mu_X \circ E\eta_X) = id_{F(EX)}$$

Furthermore

$$
\begin{aligned}
F\mu_n \circ \delta^{n+1} + \delta^n \circ F\mu_{n-1}
&= F\mu_n \circ \sum_{i=0}^{n+1} (-1)^i F(Ed^i) + \sum_{i=0}^{n} (-1)^i F(Ed^i) \circ F\mu_{n-1} \\
&= F(\mu_n \circ Ed^0) + \sum_{i=1}^{n+1} (-1)^i F(\mu_n \circ Ed^i) + \sum_{i=0}^{n} (-1)^i F(Ed^i \circ \mu_{n-1}) \\
&= F(\mu_n \circ Ed^0) + \sum_{i=1}^{n+1} (-1)^i F(Ed^{i-1} \circ \mu_{n-1}) + \sum_{i=1}^{n+1} (-1)^{i-1} F(Ed^{i-1} \circ \mu_{n-1}) \\
&= F(\mu_n \circ Ed^0) \\
&= F(\mu_{(E^\bullet X)^n} \circ E\eta_{(E^\bullet X)^n}) \\
&= F(id_{(E^\bullet X)^n}) \\
&= id_{F(E^\bullet X)^n}
\end{aligned}
$$

Therefore, $\pi^s FE(E^\bullet X) = 0$ if $s > 0$ and $\pi^0 FE(E^\bullet X) = ker\,\delta^1 = FE(X)$. □

Proof of Theorem 2.2. Define for every $k \geq 1$ the following doule chain complex:

$$
C^{m,n} = \begin{cases} \pi_{k-m} D_m E(E^\bullet X)^n, & \text{for } k \geq m \geq 0,\ n \geq 0 \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \partial_3} & & \uparrow & & \uparrow{\scriptstyle \partial_3} & & \\
0 & \longrightarrow & \pi_k E(E^3(X)) & \xrightarrow{d_1} & \pi_{k-1} D_1 E(E^3(X)) & \xrightarrow{d_1} & \pi_{k-2} D_2 E(E^3(X)) & \xrightarrow{d_1} & \cdots \\
 & & \uparrow{\scriptstyle \partial_2} & & \uparrow{\scriptstyle \partial_2} & & \uparrow{\scriptstyle \partial_2} & & \\
0 & \longrightarrow & \pi_k E(E^2(X)) & \xrightarrow{d_1} & \pi_{k-1} D_1 E(E^2(X)) & \xrightarrow{d_1} & \pi_{k-2} D_2 E(E^2(X)) & \xrightarrow{d_1} & \cdots \\
 & & \uparrow{\scriptstyle \partial_1} & & \uparrow{\scriptstyle \partial_1} & & \uparrow{\scriptstyle \partial_1} & & \\
0 & \longrightarrow & \pi_k E(E(X)) & \xrightarrow{d_1} & \pi_{k-1} D_1 E(E(X)) & \xrightarrow{d_1} & \pi_{k-2} D_2 E(E(X)) & \xrightarrow{d_1} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The horizontal differentials are the differentials $d_1$ of the spectral sequence, whereas the vertical differentials are the same as in Lemma 2.4 for $F = \pi_{k-j} \circ D_j$. Because of Lemma 2.4 taking homology vertically gives:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & & & \\
\pi_k E(X) & \xrightarrow{d_1} & \pi_{k-1} D_1 E(X) & \xrightarrow{d_1} & \pi_{k-2} D_2 E(X) & \xrightarrow{d_1} & \cdots
\end{array}
$$

The horizontal homology of this gives then:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
 & & & \\
0 & 0 & 0 & 0 \\
 & & & \\
E_2^{0,k}(X) & E_2^{1,k}(X) & E_2^{2,k}(X) & \cdots
\end{array}
$$

Alternately, because of Lemma1 taking homology first horizontally gives:

$$
\begin{array}{ccc}
\vdots & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
\pi_k E^3(X) & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
\pi_k E^2(X) & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
\pi_k E(X) & 0 & 0
\end{array}
$$

And the vertical homology of this is:

$$
\begin{array}{ccc}
\vdots & 0 & 0 \\
 & & \\
\pi^2 \pi_k E^{\bullet} X & 0 & 0 \\
 & & \\
\pi^1 \pi_k E^{\bullet} X & 0 & 0 \\
 & & \\
\pi^0 \pi_k E^{\bullet} X & 0 & 0
\end{array}
$$

Both of this diagrams must coincide with the homology of the total cochain complex which means that

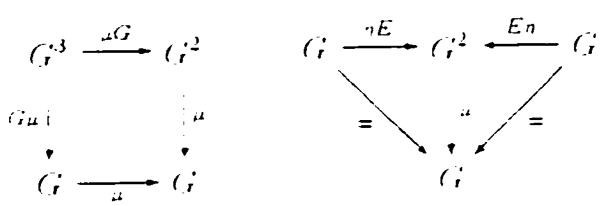
$$E_2^{s,t} = \pi^s \pi_t E^\bullet X$$

□

## 2. Triples, Cotriples and Derived Functors

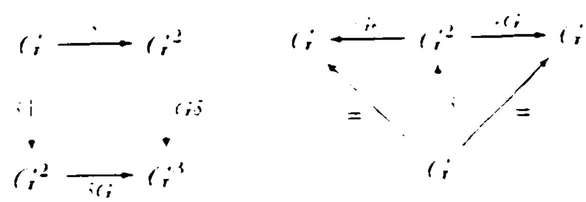
This section provides most of the theoretical foundation that helps to identify the $E_2$-term in a different way. The constructions are kept very general and will be applied to concrete functors and categories in the next section.

DEFINITION 2.5. *A triple $(G, \mu, \eta)$ on a category $\mathcal{C}$ consists of a functor $G : \mathcal{C} \longrightarrow \mathcal{C}$ and natural transformations $\mu : G^2 \longrightarrow G$ and $\eta : I \longrightarrow G$ such that the following diagrams commute:*

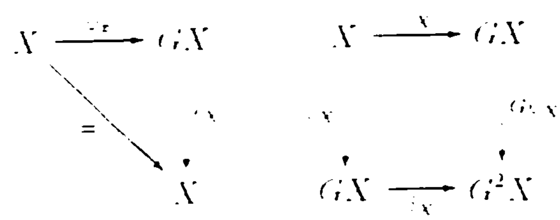



DEFINITION 2.6. *A cotriple $(G, \delta, \epsilon)$ on a category $\mathcal{C}$ consists of a functor $G : \mathcal{C} \longrightarrow \mathcal{C}$ and natural transformations $\delta : G \longrightarrow G^2$ and $\epsilon : G \longrightarrow I$ such*

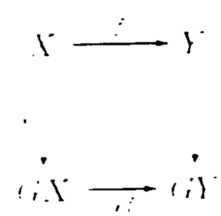
*that the following diagrams commute:*



DEFINITION 2.7. *Given a cotriple $(G, \delta, \epsilon)$ on a category $\mathcal{C}$ a $G$-coalgebra is an object $X$ in $\mathcal{C}$ together with a structure map $\upsilon_X : X \longrightarrow GX$ such that the following diagrams commute:*



*A map $f : (X, \upsilon_X) \longrightarrow (Y, \upsilon_Y)$ between $G$-coalgebras is a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ such that*



*commutes. Let $\mathcal{C}(G)$ denote the category of $G$-coalgebras.*

Observe that $G$ defines in the definition above a triple on the category $\mathcal{C}(G)$ with $\mu_X = G\epsilon_X$ and $\eta_X = \upsilon_X$.

REMARK 2.8. *$G$ can also be considered a functor from $\mathcal{C}$ to $\mathcal{C}(G)$, where the structure map of $GX$ is given by $\delta_X$. The forgetful functor from $\mathcal{C}(G)$ to $\mathcal{C}$*

*is then left adjoint to $G$, i.e.*

$$Hom_{\mathcal{C}}(X, Y) \approx Hom_{\mathcal{C}(G)}(X, GY)$$

*for $X$ in $\mathcal{C}(G)$ and $Y$ in $\mathcal{C}$. This is readily proved by checking all definitions.*

Let us now consider a pair of adjoint functors between categories $\mathcal{B}$ and $\mathcal{C}$. Let

$$\mathcal{B} \underset{H}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

such that $F$ is left adjoint to $H$. Let $\eta : I \longrightarrow HF$ and $\epsilon : FH \longrightarrow I$ be the unit and counit of this adjunction. This defines a triple $(E, \mu, \eta)$ on $\mathcal{B}$ with

$$E = HF \quad \text{and} \quad \mu = H\epsilon F$$

and a cotriple $(G, \delta, \epsilon)$ on $\mathcal{C}$ with

$$G = FH \quad \text{and} \quad \delta = F\eta H$$

Observe that for any object $B$ in $\mathcal{B}$, $FB$ is a $G$-coalgebra with structure map

$$F\eta_B : F(B) \longrightarrow FE(B) = FHF(B) = GF(B)$$

$F$ can therefore be considered a functor from $\mathcal{B}$ to $\mathcal{C}(G)$, with $G$ as the functor of a triple on $\mathcal{C}(G)$. The triple $(E, \mu, \eta)$ on $\mathcal{B}$ and the newly defined triple $(G, \bar{\mu}, \bar{\eta})$ on $\mathcal{C}(G)$ are compatible via $F$ in the sense that for all objects $B$ in $\mathcal{B}$

$$FE(B) \approx GF(B)$$

and the following diagrams commute

$$
\begin{array}{ccc}
F(B) & \xrightarrow{F\eta} & FE(B) \\
 & \searrow^{\eta_{F(B)}} & \downarrow \approx \\
 & & GF(B)
\end{array}
\qquad
\begin{array}{ccc}
FE^2(B) & \xrightarrow{F\mu_B} & FE(B) \\
\downarrow \approx & & \downarrow \approx \\
G^2F(B) & \xrightarrow[\mu_{F(B)}]{} & GF(B)
\end{array}
$$

The proof only consists of checking definitions.

**DEFINITION 2.9.** *Let $(G, \mu, \eta)$ be a triple on the category $\mathcal{C}$. The standard cosimplicial $G$-resolution of an object $X$ is the cosimplicial object $G^\bullet X$ over $X$ defined as in 1.10 by*

$$(G^\bullet X)^n = G^{n+1}(X)$$

$$d^i = G^i \eta_{G^{n-i}X} : G^n(X) \longrightarrow G^{n+1}(X), \qquad 0 \le i \le n$$

$$s^i = G^i \mu_{G^{n-i}(X)} : G^{n+2}(X) \longrightarrow G^{n+1}(X), \qquad 0 \le i \le n$$

Together with the previous considerations the following proposition is proved.

**PROPOSITION 2.10.** *If $\mathcal{B}$ and $\mathcal{C}$ are categories with functors*

$$\mathcal{B} \underset{H}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

*$F$ left adjoint to $H$, then for $F$ considered as a functor from $\mathcal{B}$ to $\mathcal{C}(G)$ with triples $(E, \mu, \eta)$ and $(G, \bar{\mu}, \bar{\eta})$ as defined above*

$$FE^\bullet B \approx G^\bullet FB$$

*for all objects $B$ in $\mathcal{B}$.*

DEFINITION 2.11. *Let $(G, \mu, \eta)$ be a triple on a category $\mathcal{C}$ and let $T : \mathcal{C} \longrightarrow \mathcal{A}$ be a functor into an abelian category. The right derived functor $R^*_G T : \mathcal{C} \longrightarrow \mathcal{A}$ of $T$ with respect to the triple $(G, \mu, \eta)$ is defined by*

$$R^*_G T(C) = H^*(chT(G^\bullet C))$$

*where $ch\dot{X}$ is the cochain complex associated to the cosimplicial object $\dot{X}$ with differentials defined as in the previous section.*

The description of the $E_2$-term in the previous section is just as a derived functor of $\pi_*$ with respect to the triple $(E, \mu, \eta)$. In the next section Proposition 2.10 will be used to transfer the description to a certain category of $G$-coalgebras.

## 3. The category $\mathcal{M}(G)$ of $G$-coalgebras

From now on it is assumed that the ring spectrum $E$ which is represented by an $\Omega$-spectrum $\{\mathbf{E_n}\}$ satisfies the following properties:

- $E$ is a homotopy commutative, flat $CW$ ring spectrum
- For each $n \geq 0$, $E_*(\mathbf{E_n})$ is a free $E_*$-module
- The primitives $PE_*(\mathbf{E_n})$ in the coalgebra $E_*(\mathbf{E_n})$ form a free $E_*$-module and the composite

$$PE_*(\mathbf{E_n}) \longrightarrow E_*(\mathbf{E_n}) \xrightarrow{\sigma_*} E_*E$$

is injective, where $(E_*, E_*E)$ is the Hopf algebroid of stable cooperations—like the dual of the Steenrod algebra in ordinary homology mod $p$—and $\sigma_*$ is the homology suspension.

REMARK 2.12. *The Johnson-Wilson spectra $E(n)$ meet the requirements above (see* [10] *together with results from* [17]*).*

For this section only the first two requirements are necessary, the third one is used in the next section.

Given a spectrum as above, let $\mathcal{M}$ be the category of free graded $E_*$-modules and let $\mathcal{H}$ be the homotopy category of spaces $X$ with $E_*(X)$ free as an $E_*$-module. $E_*$-homology is therefore a functor

$$\mathcal{H} \xrightarrow{E_*} \mathcal{M}$$

In order to construct a right adjoint to this functor some definitions are necessary:

**Definition 2.13.** *A module spectrum $F$ over a ring spectrum $E$ is one equipped with a map $\phi : E \wedge F \longrightarrow F$ such that the following diagrams commute up to homotopy:*

$$\begin{array}{ccc} E \wedge E \wedge F & \xrightarrow{\mu \wedge F} & E \wedge F \\ \Big\downarrow{\scriptstyle E\wedge\phi} & & \Big\downarrow{\scriptstyle \phi} \\ E \wedge F & \longrightarrow & F \end{array} \qquad\qquad \begin{array}{ccc} S \wedge F & \xrightarrow{\iota \wedge F} & E \wedge F \\ & \searrow{\scriptstyle \approx} & \Big\downarrow{\scriptstyle \phi} \\ & & F \end{array}$$

*$F$ is a free $E$-module spectrum, if $\pi_*(F)$ is a free $E_*$-module.*

Given a free $E_*$-module $M$ with generators $\{m_\alpha\}$ one can describe $F$ with $\pi_*(F) = M$ as a wedge of suspensions of $E$, i.e. $F = \bigvee \Sigma^{|m_\alpha|} E$.
The $E_*$-homology of $\Omega^\infty F$ for such a spectrum $F$ is again a free $E_*$-module, it is a sum of $E_*(\mathbf{E_n})$ which are free by one of the assumptions above.

**Definition 2.14.** *The functor $K_E : \mathcal{M} \longrightarrow \mathcal{H}$ is defined by*

$$K_E(M) = \Omega^\infty F$$

*where $F$ is defined as above as a wedge of suspensions of $E$ with $\pi_*(F) = M$.*

**Remark 2.15.** *This is just a generalization of Eilenberg-Mac Lane spaces in ordinary homology theory.*

PROPOSITION 2.16. *The functor $K_E$ is right adjoint to $E_*$, i.e.*

$$[X, K_E(M)] \approx Hom_{\mathcal{M}}(E_*(X), M)$$

*where $X$ is in $\mathcal{H}$ and $M$ in $\mathcal{M}$.*

LEMMA 2.17. *Let $\mathcal{F}$ be the homotopy category of $E$-module spectra with appropriately defined morphisms, then for any $E$-module spectrum $Y$ and spectrum $X$*

$$[X, Y] \approx Hom_{\mathcal{F}}(E \wedge X, Y)$$

PROOF. If $f : X \longrightarrow Y$, then

$$E \wedge X \xrightarrow{E \wedge f} E \wedge Y \longrightarrow Y$$

is a morphism in $\mathcal{F}$.

If $g : E \wedge X \longrightarrow Y$ is a morphism in $\mathcal{F}$ then

$$X \xrightarrow{\wedge X} E \wedge X \longrightarrow Y$$

defines a morphism in the homotopy category of spectra. Using the definitions of ring spectra and module spectra shows that this relationship is a natural bijection. □

LEMMA 2.18. *If $X$ and $Y$ are free $E$-module spectra, then*

$$Hom_{\mathcal{F}}(X, Y) \approx Hom_{\mathcal{M}}(\pi_* X, \pi_* Y)$$

PROOF. $X$ is a wedge of suspensions of $E$

$$X = \bigvee \Sigma^{|p|} E = E \wedge \bigvee S^{|p|}$$

where $\bigvee S^{|p|}$ is a bouquet of spheres.

$$\begin{aligned}
Hom_{\mathcal{F}}(X, Y) &= Hom_{\mathcal{F}}(E \wedge \bigvee S^{|p|}, Y) \\
&\approx [\bigvee S^{|p|}, Y] \qquad \text{by Lemma 2.17} \\
&\approx \prod_{i} \pi_* Y \\
&\approx Hom(\bigoplus_{i} Z, \pi_* Y) \\
&\approx Hom_{\mathcal{M}}(E_* \bigoplus_{i} Z, \pi_* Y) \\
&\approx Hom_{\mathcal{M}}(\pi_*(E \wedge \bigvee S^{|p|}), \pi_* Y) \\
&\approx Hom_{\mathcal{M}}(\pi_* X, \pi_* Y)
\end{aligned}$$

□

PROOF OF PROPOSITION 2.16. For $X$ in $\mathcal{H}$ and $M$ in $\mathcal{M}$

$$[X, K_E(M)] = [X, \Omega^\infty F]$$

$$\approx [\Sigma^\infty X, F] \qquad \text{since } \Sigma^\infty \text{ is left adjoint to } \Omega^\infty, \text{ see [2]}$$

$$\approx Hom_{\mathcal{F}}(E \wedge \Sigma^\infty X, F) \qquad \text{using Lemma 2.17}$$

$$\approx Hom_{\mathcal{M}}(\pi_*(E \wedge \Sigma^\infty X), \pi_* F) \qquad \text{using Lemma 2.18}$$

$$\approx Hom_{\mathcal{M}}(E_* X, M)$$

Observe that $E \wedge \Sigma^\infty X$ is a free $E$-module spectrum, since $\pi_*(E \wedge \Sigma^\infty X) = E_* X$ is $E_*$-free. □

REMARK 2.19. *The functor $E$ defined in 1.3 is exactly the composite $K_E \circ E_*$.*

To complete the definitions let $G$ be the composite $E_* \circ K_E$, then the previous section shows that this is a functor of a cotriple on $\mathcal{M}$ giving rise to the category $\mathcal{M}(G)$ of $G$-coalgebras. The triple $(E, \mu, \eta)$ on $\mathcal{H}$ and the triple $(G, \mu, \eta)$ are now compatible via $E_*$, i.e. $E_*(E(X)) \approx G(E_*(X))$.

THEOREM 2.20. *Suppose the spectrum $E$ satisfies the assumption above, let $X$ be a simply connected space with $E_*(X)$ free as an $E_*$-module, then*

$$E_2^{s,t}(X) = Ext^s_{\mathcal{M}(G)}(E_*(S^t), E_*(X)) \qquad \textit{for } t > s \geq 0$$

*where the expression on the right is the derived functor of* $Hom_{\mathcal{M}(G)}(E_*(S^t), \_\_)$ *with respect to the triple* $(G, \mu, \eta)$.

PROOF. It follows from

$$\begin{aligned}
\pi_t(E(X)) &= [S^t, E(X)] \\
&\approx [S^t, K_E(E_*(X))] \\
&\approx Hom_{\mathcal{M}}(E_*(S^t), E_*(X)) \qquad \text{by adjointness} \\
&\approx Hom_{\mathcal{M}(G)}(E_*(S^t), G(E_*(X))) \qquad \text{again by adjointness} \\
&\approx Hom_{\mathcal{M}(G)}(E_*(S^t), E_*(E(X))) \qquad \text{since the triples are compatible}
\end{aligned}$$

using Propositon 2.10 that the following cosimplicial abelian groups are isomorphic:

$$\begin{aligned}
\pi_t(E^\bullet X) &\approx Hom_{\mathcal{M}(G)}(E_*(S^t), E_*(E^\bullet X)) \\
&\approx Hom_{\mathcal{M}(G)}(E_*(S^t), G^\bullet E_*(X))
\end{aligned}$$

Applying cohomotopy gives

$$\begin{aligned}
E_2^{s,t}(X) = \pi^s\pi_t E^\bullet X &\approx \pi^s Hom_{\mathcal{M}(G)}(E_*(S^t), G^\bullet E_*(X)) \\
&\approx Ext^s_{\mathcal{M}(G)}(E_*(S^t), E_*(X))
\end{aligned}$$

for $t > s \geq 0$. □

## 4. The category $\mathcal{A}(\Gamma)$ of unstable comodules

The identification of the $E_2$-term as an unstable $Ext$ in the category $\mathcal{M}(G)$ of $G$-coalgebras leads closer to an actual computation. The category $\mathcal{M}(G)$ is still somewhat mysterious and needs to be studied further. One way to understand it is to study functors from $\mathcal{M}(G)$ into certain abelian categories. The category $\mathcal{A}(\Gamma)$ of unstable $\Gamma$-comodules is such a category. $\Gamma$ stands in this section for $E_*E$ the Hopf algebroid of stable cooperations. This category was defined in [**3**] and the following arguments repeat the approach outlined there.

REMARK 2.21. *For the following discussion it is necessary to consider the full subcategory of homology coalgebras in $\mathcal{M}(G)$. The definition of the primitive functor below requires an augmented coalgebra. Therefore, it is from now on assumed that all spaces involved come with a special component containing a basepoint. This restriction does not affect the description of the $E_2$-term, since all constructions where made in this subcategory. This subcategory will be again denoted by $\mathcal{M}(G)$.*

The functor from $\mathcal{M}(G)$ to $\mathcal{A}(\Gamma)$ will be the functor $P$ which assigns to a $G$-coalgebra the $E_*$-module of its primitives. Before defining this functor let us observe that an object in $\mathcal{M}(G)$ is a coalgebra over $E_*$ using the fact that

$G(M)$ is the $E_*$-homology of a space, just define

$$\Delta_M : M \longrightarrow G(M) = E_*(\Omega^\infty F)$$

$$\longrightarrow E_*(\Omega^\infty F) \otimes_{E_*} E_*(\Omega^\infty F) = G(M) \otimes_{E_*} G(M)$$

$$\longrightarrow M \otimes_{E_*} M$$

using the counit $\epsilon_M$ of $G$ for the last map. Let $P(M)$ denote the module of primitives with respect to this coalgebra structure, i.e. let $M$ be the unit coideal

$$E_* \longrightarrow M \longrightarrow M \longrightarrow 0$$

and $\Delta_M$ the induced coproduct on $M$, then

$$P(M) = ker(\Delta_M : M \longrightarrow M \otimes_{E_*} M)$$

$P$ can be considered a functor from $\mathcal{M}(G)$ to the category of graded $E_*$-modules. Any $G$-coalgebra $M$ is also a $\Gamma$-comodule with structure map

$$\Psi_M : M \longrightarrow G(M) = E_*(\Omega^\infty F) \xrightarrow{\sigma_*} E_*(F) \approx \bigoplus_I E_*E \approx \Gamma \otimes_{E_*} M$$

since $\Gamma$ and $M$ are free $E_*$-modules and $F$ is a wedge of suspensions of $E$. Since

$$PE_*(\mathbf{E_n}) \longrightarrow E_*(\mathbf{E_n}) \longrightarrow E_*E$$

is injective by assumption and $G(M)$ is a direct sum of $E_*(\mathbf{E_n})$

$$PG(M) \longrightarrow G(M) \longrightarrow \Gamma \otimes_{E_*} M$$

is also injective, in particular $PG(M)$ is again $E_*$-free. This leads to the following definition:

DEFINITION 2.22. *The functor* $U : \mathcal{M} \longrightarrow \mathcal{M}$ *is defined by* $U = PG$.

This functor is the functor of a cotriple $(U, \delta^U, \epsilon^U)$ defined in the following way:

For $M$ in $\mathcal{M}(G)$ with structure map $\psi : M \longrightarrow G(M)$ the following pull-back square of injections

$$\begin{array}{ccc} UP(M) & \longrightarrow & \Gamma \otimes_{E_*} P(M) \\ \downarrow & & \downarrow \\ U(M) & \longrightarrow & \Gamma \otimes_{E_*} M \end{array}$$

together with the commutative square

$$\begin{array}{ccc} P(M) & \xrightarrow{\psi_M} & \Gamma \otimes_{E_*} P(M) \\ \downarrow{\scriptstyle P\psi} & & \downarrow \\ U(M) & \longrightarrow & \Gamma \otimes_{E_*} M \end{array}$$

defines a unique map $P(M) \longrightarrow UP(M)$ which defines an unstable comodule structure on $P(M)$. For $M = G(N)$ this defines $\delta^U_N : U(N) \longrightarrow U^2(N)$. $\epsilon^U_N$ is just the composite $U(N) \longrightarrow G(N) \longrightarrow N$ using again the counit of $G$.

This triple $(\Gamma, \delta^{\Gamma}, \epsilon^{\Gamma})$ is then extended to the category $\mathcal{A}$ of all $E_*$-modules in the following way: Let $M$ be in $\mathcal{A}$ and

$$F_1 \xrightarrow{f} F_0 \longrightarrow M \longrightarrow 0$$

a resolution by free $E_*$-modules, then

$$\Gamma(M) = coker(\Gamma(f) : \Gamma(F_1) \longrightarrow \Gamma(F_0))$$

This automatically makes $\Gamma$ an exact functor on the category $\mathcal{A}$.

DEFINITION 2.23. *$\mathcal{A}(\Gamma)$ is the category of unstable $\Gamma$-comodules, i.e. $E_*$-modules $M$ with structure maps $M \longrightarrow \Gamma(M)$ compatible with the cotriple $(\Gamma, \delta^{\Gamma}, \epsilon^{\Gamma})$.*

Since $\Gamma$ is exact on $\mathcal{A}$ this category is abelian. The functor $P$ can now be considered a functor

$$P : \mathcal{M}(G) \longrightarrow \mathcal{A}(\Gamma)$$

and used to study homological properties in $\mathcal{M}(G)$.

For a module $M$ in $\mathcal{A}(\Gamma)$ we can again construct the cosimplicial resolution $\Gamma^{\bullet}M$, applying $Hom_{\mathcal{A}(\Gamma)}(E_*(S^t), \_\_)$ to it gives

$$C^{s,t}(M) = Hom_{\mathcal{A}(\Gamma)}(E_*(S^t), (\Gamma^{\bullet}M)^s),$$

the *unstable cobar complex* for $M$. Using adjointness

$$Hom_{\mathcal{A}(\Gamma)}(E_*(S^t), \Gamma(M)) \approx Hom_{\mathcal{A}}(E_*(S^t), M)$$

shows that

$$C^{s,t}(M) = \Gamma^{s}(M)_t.$$

If $M$ is a free $E_*$-module this complex is a sub-chain complex of the stable cobar complex over $\Gamma$ and its homology can be computed by desuspending elements of the stable complex. (see [3] and [4])

The abelian group valued functor $Hom_{\mathcal{M}(G)}(E_*(S^t), \_\_)$ that is used to define the $E_2$-term now factors over $\mathcal{A}(\Gamma)$ in the following way

$$\begin{array}{ccc} \mathcal{M}(G) & \xrightarrow{P} & \mathcal{A}(\Gamma) \\ & \searrow^{Hom_{\mathcal{M}(G)}(E_*(S^t),\_\_)} & \downarrow^{Hom_{\mathcal{A}(\Gamma)}(E_*(S^t),\_\_)} \\ & & Ab \end{array}$$

This is used in [3] by M. Bendersky, E. Curtis and H. Miller to set up a composite functor spectral sequence

$$E_2^{r,s} = Ext^r_{\mathcal{A}(\Gamma)}(E_*(S^t), R^s_{\mathcal{M}(G)}P(M)) \Longrightarrow Ext^{r-s}_{\mathcal{M}(G)}(E_*(S^t), M)$$

converging to the $E_2$-term.

Defining $\mathcal{M}(S)$ to be the category of free $E_*$-coalgebras—connected in [3]—the authors were able to show that the derived functor $R^*_{\mathcal{M}(G)}P$ coincides with $R^*_{\mathcal{M}(S)}P$ in the connected case. The latter has been investigated by Bousfield in [5]. In particular, the cofree coalgebra $E_*(\mathbf{E_n})$ in $\mathcal{M}(G)$ turns out to be cofree in $\mathcal{M}(S)$ too. This makes the composite functor spectral sequence collapse immediately and allows to compute $E_2(E_*(S^{2n+1}))$ as the homology of the unstable cobar complex.

In the non-connective case these arguments do not work. $E_*(\mathbf{E_n})$ does not seem to be cofree in $\mathcal{M}(S)$, moreover the whole notion of cofree is not so clear in the non-connective case. The problem is that in [5] Bousfield works strictly with the homology of connected spaces and connected coalgebras, there the cofree functor and cofree objects are well-understood.

In the non-connective case the category of $E_*$-coalgebras does not admit a straightforward description of cofree objects. The smaller category of "irreducible" coalgebras used by Bousfield is too small to contain the homology of non-connected spaces which appear in spectra that are of interest (e.g. $BU \times Z$ in K-theory). A larger category of "completed" coalgebras might finally solve some problems and more work has to be done in this direction.

For now the fact that the $E_2$-term for certain $E_*(X)$ can be computed as the homology of the unstable cobar complex has to be proved directly. In [4] this has been done for $E(1)_*(S^{2n+1})$.

After this the discussion the focus shifts in the next chapter to the category $\mathcal{A}(U)$. The $Ext$-terms based on $E(n)$-theory will be related to those in $BP$-theory for a certain class of modules. In [4] this is the starting point for the computations.

CHAPTER 3

# An unstable change of rings isomorphism

## 1. Introduction and Background

The main theorem of this chapter is an unstable analogue of the change of rings ismorphism presented in [14] by H. R. Miller and D. C. Ravenel. There, the authors identify for certain comodules $M$ the $E_2$-term $Ext_{BP_*BP}(BP_*, M)$ of the stable Adams-Novikov spectral sequence, defined in the category of $BP_*BP$-comodules, with an $Ext$-term over a smaller Hopf algebroid. The theorem is also a generalization of the special case for $M = v_1^{-1}BP_*(S^{2n+1})/p$ and $E(1)$-theory presented in [4] by M. Bendersky and R. Thompson.

$BP$ stands for the Brown-Peterson spectrum, $BP_* = Z_{(p)}[v_1, v_2, v_3, \ldots]$, with $|v_i| = 2(p^i - 1)$ is the coefficient ring of the associated homology theory, $BP_*BP = BP_*[h_1, h_2, h_3, \ldots]$ the Hopf algebroid of cooperations, i.e. the analogue of the dual of the Steenrod Algebra. Recall that $BP_*BP$ is close to an Hopf algebra, but its product structure comes with two units $\eta_L, \eta_R : BP_* \longrightarrow BP_*BP$. These units arise by taking homotopy of $BP \longrightarrow BP \wedge BP$, where $BP$ is mapped into the left or right factor. This provides $BP_*BP$ with the structure of a $BP_*$-bimodule with two different actions. Given a spectrum $X$, $BP_*(X)$ is a comodule over this Hopf algebroid.

A $BP_*$-module $N$ is called $v_n$-*local* if $v_n$ acts bijectively on $N$. The prime example is just the localization of a $BP_*$-module $M$, i.e. the direct limit of

$$M \xrightarrow{v_n} M \xrightarrow{v_n} M \xrightarrow{v_n} \ldots v_n^{-1}M$$

If $M$ is a $BP_*BP$-comodule killed as a $BP_*$-module by the ideal $I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset BP_*$, then multiplication by $v_n$ is in fact a comodule map. This can be seen by observing that $\eta_R(v_n) \equiv v_n \mod I_n$ which allows $v_n$ to be passed through the tensor of $BP_*BP \otimes_{BP_*} M$ so that

$$\begin{array}{ccccc}
M & \xrightarrow{v_n} & M & \xrightarrow{v_n} & \ldots v_n^{-1}M \\
\downarrow & & \downarrow & & \downarrow \\
BP_*BP \otimes_{BP_*} M & \xrightarrow{v_n} & BP_*BP \otimes_{BP_*} M & \xrightarrow{v_n} & \ldots BP_*BP \otimes_{BP_*} v_n^{-1}M \\
 & & & & \| \\
 & & & & v_n^{-1}BP_*BP \otimes_{BP_*} M
\end{array}$$

commutes. This allows to carry out this localization in the category of $BP_*BP$-comodules. A similar argument will be used in Lemma 3.4 in the unstable case.

The Johnson-Wilson spectra $E(n)$ determine non-connective homology theories with coefficient ring $E(n)_* = Z_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$. These homology theories are Landweber exact, i.e. $E(n)_*(X) = E(n)_* \otimes_{BP_*} BP_*(X)$ [**11**], and the Hopf algebroid $E(n)_*E(n)$ can be described as

$$E(n)_*E(n) = E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*$$

where $E(n)_*$ is a $BP_*$-module via the obvious ring homomorphism $BP_* \longrightarrow E(n)_*$. Tensoring $BP_*BP$ on both sides with $E(n)_*$ introduces relations in $BP_*BP$—e.g. $\eta_R(v_i) = 0$ for $i > n$—and makes it smaller but more complicated.

In the previous mentioned paper by Miller amd Ravenel the following statement is proven:

THEOREM 3.1. *If $M$ is a $v_n$-local $BP_*BP$-comodule annihilated by $I_n$, then*

$$Ext_{BP_*BP}(BP_*, M) \approx Ext_{E(n)_*E(n)}(E(n)_*, E(n)_* \otimes_{BP_*} M)$$

The following notations and definitions will be used throughout this chapter:

- $\Gamma = BP_*BP$
- $\Sigma = E(n)_*E(n) = E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*$
- For a free $BP_*$-module $M$, $U_\Gamma(M) = PG(M)$, where $G(M)$ is the cofree $G$-coalgebra for $BP$-theory and $P$ is the primitive functor from the category of $G$-coalgebras to the category of unstable $\Gamma$-comodules. $U_\Gamma(M)$ for general $M$ is defined as in the previous chapter.
- For a free $E(n)_*$-module $N$, $U_\Sigma(N) = PG(M)$, where $G(M)$ is the cofree $G$-coalgebra for $E(n)$-theory and $P$ is the primitive functor from the category of $G$-coalgebras to the category of unstable $\Sigma$-comodules. $U_\Sigma(M)$ for general $M$ is defined as in the previous chapter.

The unstable change of rings isomorphism uses the fact that $E(n)$-theory is Landweber exact. In the stable situation this allows a description of $E(n)_*E(n)$ in terms of $BP_*BP$. Unstably, one has to deal with the $E(n)$-homology of the spaces in the $E(n)$-spectrum and it will be necessary to express those in terms of the $BP$-homology of the spaces in the $BP$-spectrum. Studying the modules $BP_*(\mathbf{BP_n})$ for various $n$ is best done using the notion of a Hopf ring [16].

A *Hopf ring* is a graded ring object in the category of graded coalgebras over a certain ring, which means it is a bigraded object with two different product structures, a coproduct and a list of properties relating those structures to each other. In the case of the Hopf ring $BP_*(\mathbf{BP_*}) = \{BP_*(\mathbf{BP_k})\}_{k\in Z}$ each $BP_*(\mathbf{BP_k})$ is a Hopf algebra with the usual coalgebra structure and a $*$-product derived from the $H$-space structure of each space $\mathbf{BP_k}$. These Hopf algebras are connected by a product

$$\circ : BP_*(\mathbf{BP_i}) \otimes_{BP_*} BP_*(\mathbf{BP_j}) \longrightarrow BP_*(\mathbf{BP_{i+j}})$$

using the multiplicative structure of the spectrum $BP$ and a Künneth isomorphism

$$BP_*(\mathbf{BP_i} \wedge \mathbf{BP_j}) \approx BP_*(\mathbf{BP_i}) \otimes_{BP_*} BP_*(\mathbf{BP_j})$$

which holds since all $BP_*(\mathbf{BP_k})$ are torsion free.

Certain elements of this Hopf ring can be constructed in the following way:

Let $v \in BP_*$ be a homogeneous element represented by a map $v$ : *point* $\longrightarrow \mathbf{BP_k}$. Taking $BP$-homology one identifies elements $[v] \in BP_0(\mathbf{BP_*})$ as the image of a selected generator in $BP_*$. These elements form a sub-Hopf ring of $BP_*(\mathbf{BP_*})$ with

- $[v] * [w] = [v + w]$
- $[v] \circ [w] = [vw]$
- $\psi[v] = [v] \otimes [v]$

This Hopf ring is for integral homology in fact just the zero-dimensional part of $H_*(\mathbf{BP_*})$, i.e. $H_0(\mathbf{BP_*})$. For $BP$-theory this sub-Hopf ring of $BP_*(\mathbf{BP_*})$ will denoted by $BP_*[BP^*]$. This notation is justified since the above description already suggests that it is the group ring of all homogenous elements of $BP^*$ over $BP_*$.

In [16] $BP_*(\mathbf{BP_*})$ is shown to be a free Hopf ring over $BP_*[BP^*]$ generated by elements $b_{(m)} \in BP_{2p^m}(\mathbf{BP_2})$. This description is then used to give an explicit $BP_*$-basis for $PBP_*(\mathbf{BP_k})$ in terms of elements of the form

$$[v^I] \circ b^J$$

where

(1) $$v^I = v_1^{i_1} v_2^{i_2} v_3^{i_3} \ldots$$

(2) $$b^J = b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ b_{(2)}^{\circ j_2} \ldots$$

and $I = (i_1, i_2, i_3, \ldots)$ and $J = (j_0, j_1, j_2, \ldots)$ are certain sequences of non-negative integers.

There is a relationship between the o-product and the product in $BP_*BP$, more specifically, between the "right" action of $BP_*[BP^{-*}]$ on $BP_*(\mathbf{BP}_*)$ using the o-product and the right action of $BP_*$ on $BP_*BP$. Under the stabilization map $\sigma_* : BP_*(\mathbf{BP_k}) \longrightarrow BP_*BP$ the o-multiplication by $[v]$ turns into multiplication by $\eta_R(v)$. Using the monomorphism

$$PBP_*(\mathbf{BP_k}) \longrightarrow BP_*(\mathbf{BP_k}) \longrightarrow BP_*BP$$

allows to describe $PBP_*(\mathbf{BP_k})$ as a $BP_*$-submodule of $BP_*BP$. This was done in [**3**] using the Ravenel-Wilson basis.

For sequences $I = (i_1, i_2, i_3, \ldots)$ and $J = (j_0, j_1, j_2, \ldots)$ of non-negative integers the pair $(J, I)$ is considered *allowable with respect to the dimension k* if

1) Let

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} + p^3\Delta_{k_3} + \cdots + p^n\Delta_{k_n} + J'$$

where $k_1 \leq k_2 \leq k_3 \leq \ldots$ and $J'$ is a sequence of non-negative integers, then $i_n = 0$. ($\Delta_l$ is the sequence with 1 in the $l^{th}$ position and 0 elsewhere.)

2) $2\sum j_i = |v^I| + k$

Denoting the right action in $BP_*BP$ still by $\circ$, a $BP_*$-basis of $PBP_*(\mathbf{BP_k})$ is given by elements of the form $b^J \circ v^I$ where $(J, I)$ is allowable with respect to the dimension of $k$.

In general it is possible to define a Hopf ring for any multiplicative $\Omega$-spectrum $E$ for which all $E_*(\mathbf{E_k})$ are torsion free. In the case of $E(n)$-theory this is a consequence of [**10**].

## 2. Some properties of the functor $U_\Gamma$

The following three statements are familiar in the stable setting. They all rely on the fact that $\eta_R(v_n) \equiv v_n \mod I_n$ in the Hopf algebroid $BP_*BP$.

LEMMA 3.2. *Let $M$ be a $BP_*$-module, then $U_\Gamma(v_n M) \equiv v_n U_\Gamma(M) \mod I_n$.*

PROOF. If

$$F_1 \xrightarrow{f} F_0 \longrightarrow M$$

is a resolution by free $BP_*$-modules, then

$$v_n F_1 \longrightarrow v_n F_0 \longrightarrow v_n M$$

is also a free resolution. Applying $U_\Gamma$ defines $U_\Gamma(v_n M)$ by

$$U_\Gamma(v_n F_1) \xrightarrow{U_\Gamma(v_n f)} U_\Gamma(v_n F_0) \longrightarrow U_\Gamma(v_n M) \longrightarrow 0.$$

where $U_\Gamma(v_n f)$ is the restriction of

$$\Gamma \otimes v_n f : \Gamma \otimes v_n F_1 \longrightarrow \Gamma \otimes v_n F_0$$

to the unstable part. Since $\eta_R(v_n) \equiv v_n \mod I_n$ for the right unit $\eta_R : BP_* \longrightarrow BP_*BP$, it follows that

$$\Gamma \otimes v_n f \equiv v_n(\Gamma \otimes f) : v_n\Gamma \otimes F_1 \longrightarrow v_n\Gamma \otimes F_0 \mod I_n .$$

and therefore $U_\Gamma(v_n f) \equiv v_n U_\Gamma(f) \mod I_n$, which implies

$$U_\Gamma(v_n M) \equiv v_n U_\Gamma(M) \mod I_n$$

□

LEMMA 3.3. *Let $M$ be a $BP_*$-module annihilated by $I_n$, i.e. $I_n M = 0$, then $U_\Gamma(M)$ is also annihilated by $I_n$.*

PROOF. The proof is by induction on $n$. For $n = 0$ the statement is obvious, since by Lemma 3.2:

$$pU_\Gamma(M) = U_\Gamma(pM) = 0$$

Let us now assume that M is killed by $I_{n+1}$, then it is also killed by $I_n$, which implies that $I_n U_\Gamma(M) = 0$. Using Lemma 3.2 gives

$$0 = U_\Gamma(v_n M) \equiv v_n U_\Gamma(M) \mod I_n.$$

which shows that $v_n$ also kills $U_\Gamma(M)$. □

LEMMA 3.4. *Let $M$ be a $BP_*$-module annihilated by $I_n$ which is also $v_n$-local, i.e. $v_n$ acts bijectively on $M$, then $U_\Gamma(M)$ is also $v_n$-local.*

PROOF. For an arbitrary $BP_*$-module $M$ we define $v_n^{-1}M$ by a direct limit of

$$M \xrightarrow{v_n} M \xrightarrow{v_n} M \xrightarrow{v_n} \cdots \quad v_n^{-1}M$$

Applying $U_\Gamma$ to this directed system we get

$$U_\Gamma(M) \xrightarrow{U_\Gamma(v_n)} U_\Gamma(M) \xrightarrow{U_\Gamma(v_n)} U_\Gamma(M) \xrightarrow{U_\Gamma(v_n)} \cdots \quad U_\Gamma(v_n^{-1}M)$$

as a direct limit, since $U_\Gamma$ is an exact functor and direct limits can be defined as cokernels. If $M$ is annihilated by $I_n$ this directed system is equal to

$$U_\Gamma(M) \xrightarrow{v_n} U_\Gamma(M) \xrightarrow{v_n} U_\Gamma(M) \xrightarrow{v_n} \cdots$$

whose direct limit is $v_n^{-1}U_\Gamma(M)$. This implies that $U_\Gamma(v_n^{-1}M) = v_n^{-1}U_\Gamma(M)$. If $M$ is additionally $v_n$-local, i.e. $M = v_n^{-1}M$, then $U_\Gamma(M) = v_n^{-1}U_\Gamma(M)$ and $U_\Gamma(M)$ is $v_n$-local. □

Combining Lemma 3.3 and Lemma 3.4 proves the following statement:

COROLLARY 3.5. *Let $M$ be a $v_n$-local $BP_*$-module annihilated by $I_n$, then $U_\Gamma(M)$ is also $v_n$-local and annihilated by $I_n$.*

## 3. Description of $U_\Sigma$ in terms of $U_\Gamma$

Proposition 3.8 serves as the key to relate the functors $U_\Gamma$ and $U_\Sigma$ for $E(n)_*$-modules. Observe that any $E(n)_*$-module $N$ is also a $BP_*$-module via the obvious ring homomorphism $BP_* \longrightarrow E(n)_*$ and $U_\Gamma(N)$ is well-defined.

Proposition 3.8 is based on the fact that $E(n)$-theory is Landweber exact and allows a convenient description of the Hopf ring $E(n)_*(\mathbf{E(n)}_*)$ in terms of $BP_*(\mathbf{BP}_*)$. The stable analogue is completely obvious once we know that $\Sigma = E(n)_* \otimes_{BP_*} \Gamma \otimes_{BP_*} E(n)_*$, since for any $E(n)_*$-module $X$

$$\Sigma \otimes_{E(n)_*} X = E(n)_* \otimes_{BP_*} \Gamma \otimes_{BP_*} E(n)_* \otimes_{E(n)_*} X$$

$$= E(n)_* \otimes_{BP_*} \Gamma \otimes_{BP_*} X$$

For a free $E(n)_*$-module $X$ on one generator in dimension $k$, $U_\Sigma(X)$ is defined by $U_\Sigma(X) = PE(n)_*(\mathbf{E(n)}_k)$. The following Proposition helps to describe this module.

**Proposition 3.6** (Hopkins, Hunton). *Let $E_*$ represent a Landweber exact multiplicative homology theory localized at a prime $p$. If the coefficients $E_*$ are concentrated in even dimensions and are a free $R$-module of countable rank for some subring $R$ of the rationals, then*

$$E_*(\mathbf{E}_*) = E_* \otimes_{BP_*} BP_*(\mathbf{BP}_*) \otimes_{BP_*[BP^*]} BP_*[E^*]$$

**Proof.** Similar to the proof in [10]. □

**Remark 3.7.** *$E(n)$-theory fulfills all requirements of Proposition 3.6 since $E(n)_* = Z_{(p)}[v_1, v_2, v_3, \ldots, v_n, v_n^{-1}]$ and $|v_i| = 2(p^i - 1)$.*

**Proposition 3.8.** *For any $E(n)_*$-module $X$*

$$U_\Sigma(X) \approx E(n)_* \otimes_{BP_*} U_\Gamma(X)$$

*as* $E(n)_*$*-modules.*

PROOF. The proof consists of four parts. First the Proposition is proven for a free $E(n)_*$-module on one generator, then for arbitrary finitely generated free $E(n)_*$-modules and finally, after a short discussion about naturality, for arbitrary finitely presented $E(n)_*$-modules.

**I.** Suppose $M$ is a free $BP_*$-module on one generator of dimension $k$ and $N = E(n)_* \otimes_{BP_*} M$. Remember that

$$\Gamma_P(M) = PBP_*(\mathbf{BP_k}) \qquad \text{and}$$

$$\Gamma_E(N) = PE(n)_*(\mathbf{E(n)_k})$$

Using Proposition 3.6 the graded algebra $PE(n)_*(\mathbf{E(n)_*})$ can be described as

$$PE(n)_*(\mathbf{E(n)_*}) = P(E(n)_* \otimes_{BP_*} BP_*(\mathbf{BP_*}) \otimes_{BP_*[BP^*]} BP_*[E(n)^*])$$

$$= E(n)_* \otimes_{BP_*} P(BP_*(\mathbf{BP_*}) \otimes_{BP_*[BP^*]} BP_*[E(n)^*])$$

since $E(n)_*$ is flat as a $BP_*$-module and the primitives are defined by a kernel. $BP_*(\mathbf{BP_*}) \otimes_{BP_*[BP^*]} BP_*[E(n)^*]$ is the tensor product of two Hopf rings considered as algebras over $BP_*[BP^*]$ using the $\circ$-product. In the Appendix it is shown that

$$P(BP_*(\mathbf{BP_*}) \otimes_{BP_*[BP^*]} BP_*[E(n)^*])$$

$$= (PBP_*(\mathbf{BP_*}) \otimes_{BP_*[BP^*]} 1) + (1 \otimes_{BP_*[BP^*]} PBP_*[E(n)^*])$$

Observe that the second term is trivial, since for all $v_i \in BP_*[E(n)^*]$ the codiagonal is $\Delta[v_i] = [v_i] \otimes [v_i]$, so there are no primitives. (Compare this to the fact that there are no primitives in $H_0(X)$ even for a space $X$ which is not connected. This leaves us with $PBP_*(\mathbf{BP_*}) \otimes_{BP_*[BP_*]} 1$ which using the Ravenel-Wilson basis [**16**] has a $BP_*$-basis of elements of the form $b^J \circ v^I$ where $(J, I)$ are allowable. The tensor has an additional effect on $I$, it kills all $v_i$'s for $i > n$ and allows the exponent of $v_n$ to be negative. This is the same as tensoring $PBP_*(\mathbf{BP_*})$ on the right with $E(n)_*$. Therefore, restricting ourselves just to dimension $k$, we get

$$PE(n)_*(\mathbf{E(n)_k}) = E(n)_* \otimes_{BP_*} U_\Gamma(E(n)_* \otimes_{BP_*} M) \qquad \text{or}$$

$$U_\Gamma(X) = E(n)_* \otimes_{BP_*} U_\Gamma(X)$$

**II.** This quickly generalizes to free $E(n)_*$-modules, since $U_\Gamma$ is an additive functor and certainly commutes with direct sums of $E(n)_*$-modules.

**III.** The ring homomorphism $BP_* \longrightarrow E(n)_*$ induces a map between Hopf algebroids $\Gamma \longrightarrow \Sigma$. For a $BP_*$-module $M$ this gives a map

$$\Gamma \otimes_{BP_*} M \longrightarrow \Sigma \otimes_{BP_*} M$$

whose $E(n)_*$-linear extension is

$$E(n)_* \otimes_{BP_*} \Gamma \otimes_{BP_*} M \longrightarrow \Sigma \otimes_{BP_*} M.$$

If $X$ is an $E(n)_*$-module this map becomes the obvious isomorphism.

$$\begin{array}{ccc} E(n)_* \otimes_{BP_*} \Gamma \otimes_{BP_*} N & \longrightarrow & \Sigma \otimes_{BP_*} N \\ \vdots\wr & & \vdots \\ E(n)_* \otimes_{BP_*} \Gamma \otimes_{BP_*} E(n)_* \otimes_{E(n)_*} N & \longrightarrow & \Sigma \otimes_{E(n)_*} N \end{array}$$

For a free $BP_*$-module $M$, $U_\Gamma(M) \subset \Gamma \otimes_{BP_*} M$. Killing $v_i$'s for $i > n$ and inverting $v_n$ on both sides from the right we can consider $U_\Gamma(N)$ still as a submodule of $\Gamma \otimes_{BP_*} N$ for $N = E(n)_* \otimes_{BP_*} M$. The map $E(n)_* \otimes_{BP_*} U_\Gamma(N) \longrightarrow U_\Sigma(N)$ is just the restriction of the isomorphism above.

**IV.** Given an arbitrary finitely presented $E(n)_*$-module $N$, let

$$F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

be a free resolution by $E(n)_*$-modules.

The diagram

$$\begin{array}{ccccccc} U_\Sigma(F_1) & \longrightarrow & U_\Sigma(F_0) & \longrightarrow & U_\Sigma(N) & \longrightarrow & 0 \\ \vdots & & \vdots & & = & & \\ E(n)_* \otimes_{BP_*} U_\Gamma(F_1) & \longrightarrow & E(n)_* \otimes_{BP_*} U_\Gamma(F_0) & \longrightarrow & E(n)_* \otimes_{BP_*} U_\Gamma(N) & \longrightarrow & 0 \end{array}$$

shows that $U_\Sigma(N) \approx E(n)_* \otimes_{BP_*} U_\Gamma(N)$.

$\square$

## 4. Theorem and Proof

PROPOSITION 3.9. *If $M$ is a $BP_*$-module killed by $I_n$ and $N = E(n)_* \otimes_{BP_*} M$, then $U_\Gamma(N)$ is a sum $\bigoplus U_\Sigma(N)$ as $E(n)_*$-modules.*

PROOF. $Z_{(p)}[v_{n+1}, v_{n+2}, \dots] \otimes_{Z_{(p)}} U_\Sigma(N)$ is a sum $\bigoplus U_\Sigma(N)$ indexed by monomials in the $v_i$'s for $i > n$. Using Proposition 3.8

$$Z_{(p)}[v_{n+1}, v_{n+2}, \dots] \otimes_{Z_{(p)}} U_\Sigma(N) = Z_{(p)}[v_{n+1}, v_{n+2}, \dots] \otimes_{Z_{(p)}} E(n)_* \otimes_{BP_*} U_\Gamma(N)$$

$$= Z_{(p)}[v_{n+1}, v_{n+2}, \dots] \otimes_{Z_{(p)}} Z_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}] \otimes_{BP_*} U_\Gamma(N)$$

$$= v_n^{-1} Z_{(p)}[v_1, v_2, \dots] \otimes_{BP_*} U_\Gamma(N)$$

$$= v^{-1} BP_* \otimes_{BP_*} U_\Gamma(N)$$

$$= v_n^{-1} U_\Gamma(N) = U_\Gamma(N)$$

by Corollary 3.5, since $N$ is $v_n$-local and killed by $I_n$. $\square$

THEOREM 3.10. *If $M$ is an unstable $\Gamma$-comodule, which is $v_n$-local and killed by $I_n$ (e.g. $M = v_n^{-1} BP_*/I_n$), then*

$$Ext_{U_\Gamma}(BP_*, M) \approx Ext_{U_\Sigma}(E(n)_*, E(n)_* \otimes_{BP_*} M)$$

PROOF. Let $N = E(n)_* \otimes_{BP_*} M$. $N$ turns out to be an unstable $\Sigma$-comodule whose coaction is induced by the coaction of $M$. Specifically, given the coaction $M \longrightarrow U_\Gamma(M)$ using the statements of Section 2—$U_\Gamma(N)$ is again $v_n$-local— one gets an unstable coaction $N \longrightarrow U_\Gamma(N)$, which reduces to

$$\psi_N : N = E(n)_* \otimes_{BP_*} N \longrightarrow E(n)_* \otimes_{BP_*} U_\Gamma(N) \approx U_\Sigma(N).$$

Let

$$N \longrightarrow U_\Sigma(N) \longrightarrow U_\Sigma^2(N) \longrightarrow U_\Sigma^3(N) \longrightarrow \dots$$

be the unstable cobar resolution of $N$ in the category of unstable $U_{\Sigma}$-comodules. Taking a sum of this resolution over all monomials in the $v_i$'s for $i > n$ gives an exact sequence

$$\bigoplus N \longrightarrow \bigoplus U_{\Sigma}(N) \longrightarrow \bigoplus U_{\Sigma}^{2}(N) \longrightarrow \bigoplus U_{\Sigma}^{3}(N) \longrightarrow \cdots$$

which, considering Propositon 3.9 and the fact that $M = \bigoplus N$, gives rise to

$$M \longrightarrow U_{\Gamma}(N) \longrightarrow U_{\Gamma}(U_{\Sigma}(N)) \longrightarrow U_{\Gamma}(U_{\Sigma}^{2}(N)) \longrightarrow \cdots ,$$

a resolution of M in the category of unstable Γ-comodules. Applying $Hom_{U\Gamma}(BP_*, -)$ to this resolution gives exactly the cobar complex associated to the resolution of $N$, since

$$Hom_{U\Gamma}(BP_*, U_{\Gamma}(U_{\Sigma}^{(s)}(N))) = Hom_{BP_*}(BP_*, U_{\Sigma}^{(s)}(N)) = U_{\Sigma}^{(s)}(N).$$

Taking homology proves the theorem.

∎

REMARK 3.11. *This proof of an unstable change of rings isomorphism can be easily adapted to provide a different proof in the stable case.*

## Appendix

LEMMA. Let $C$ and $D$ be coalgebras with unit over a ring $R$. Let $P$ be the primitive functor from the category of such coalgebras to the category of $R$-modules, then

$$P(C \otimes D) = (PC \otimes 1) \oplus (1 \otimes PD)$$

where the tensors are over the ring $R$. □

PROOF. Denote the codiagonals of $C, D$ and $C \otimes D$ by $\Delta_C, \Delta_D$ and $\Delta$, let $\eta_C$ and $\eta_R$ be the units of $C$ and $D$ respectively. Taking the cokernels of

$$R \xrightarrow{\eta_C} C \longrightarrow C \longrightarrow 0$$

$$R \xrightarrow{\eta_D} D \longrightarrow D \longrightarrow 0$$

defines the unit coideals $C$ and $D$.

- **claim:** $(PC \otimes 1) \oplus (1 \otimes PD) \subset P(C \otimes D)$

Let $a \in C$ and $a \in C$ such that $\Delta_C a = 0$, i.e $\Delta_C a = \sum a' \otimes a''$ where either $a' \in R$ or $a'' \in R$. $\Delta(a \otimes 1) = \sum a' \otimes 1 \otimes a'' \otimes 1$ and $a' \otimes 1 \in R \otimes R$ or $a'' \otimes 1 \in R \otimes R$. Therefore, $a \otimes 1 \in P(C \otimes D)$. Similarly, is $1 \otimes PD$ a submodule of $P(C \otimes D)$ and the claim follows.

- **<u>claim:</u>** $P(C \otimes D) \subset (PC \otimes 1) + (1 \otimes PD)$

Let $a \otimes b \in C \otimes D$ with $a \notin R$ and $b \notin R$. If $\Delta a = \sum a' \otimes a''$ and $\Delta b = \sum b' \otimes b''$, then $\Delta(a \otimes b) = \sum a' \otimes b' \otimes a'' \otimes b''$ and there have to be mixed terms in this expression where $a' \otimes b' \notin R \otimes R$ and $a'' \otimes b'' \notin R \otimes R$. (Compare this argument to the dual setting, where a product of indecomposables is, of course, not indecomposable.) Therefore, $a \otimes b \notin P(C \otimes D)$.

$\square$

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