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ROTHMAN, SHELDON NED

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THE VON NEUMANN KERNEL AND MINIMALLY ALMOST PERIODIC GROUPS

by

SHELDON ROTHMAN

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Abstract

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by

Sheldon Rothman

Advisor: Professor Martin Moskowitz

We first calculate the von Neumann kernel, $\mathfrak{n}(G)$, of an arbitrary connected Lie group. We conclude easily from this calculation that the closed normal subgroup $\mathfrak{n}(G)$ is also connected. Using our calculation of $\mathfrak{n}(G)$ we give various characterizations of minimally almost periodicity for a connected Lie group. Among the characterizations is the following: A connected Lie group G with radical R is minimally almost periodic (map) if and only if G/R is semisimple without compact factors and $G = \overline{[G,G]}$. We prove in the special case where R is also simply connected that $G = [G,G]$. This has the corollary that a simply connected radical of a connected map Lie group is nilpotent.

Using techniques established early in this paper together with a theorem of Tits [24] we prove that a connected map Lie group has no nontrivial automorphisms of bounded displacement. As a consequence we get a new proof via the results of F. Greenleaf, M. Moskowitz, and L. Rochschild [9] of the following theorem: If G is a map connected Lie

group, H is a closed subgroup of G such that G/H has finite volume, then $Z_G(H) = Z(G)$, and more generally if G and H are as above and α is an automorphism of G leaving H pointwise fixed, then α is trivial. With G and H again as above, we prove if $\mathrm{disp}_H\ \alpha$ is bounded, then α is trivial.

Using a decomposition due to Y. Matsushima [17], and projective limits of Lie groups, we extend most of our results on the characterization of map Lie groups to arbitrary locally compact topological groups, and get a relatively simple proof of the Freudenthal-Weil theorem.

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Introduction

In 1934 J. v. Neumann considered the set of all elements of a topological group G for which f(x) = f(1) for every almost periodic function, f, defined on G [22]. This subset, to be denoted $\mathfrak{N}(G)$, and referred to as the von Neumann kernel of G, was shown to be equal to the intersection of the kernels of all finite-dimensional continuous complex unitary representations of G. (Hereafter, called a representation of G). From this, $\mathfrak{N}(G)$ is evidently a closed normal subgroup of G. We need only consider irreducible representations in our intersection since a unitary representation is completely reducible. Von Neumann calls a topological group, G, minimally almost periodic, (map), if $\mathfrak{N}(G) = G$, implying G has no nontrivial representations (equivalently, no irreducible representations), and maximally almost periodic (MAP), if $\mathfrak{N}(G) = (1)$. The latter is equivalent to the statement that the representations (equivalently, the irreducible representations) separate the points. It is well known that both compact and locally compact abelian topological groups are MAP [11]. This implies that a map group has no nontrivial continuous homomorphisms into either a compact or a locally compact abelian topological group. Von Neumann calculates the irreducible representations of various groups. Referring to Van der Waarden it is shown that Sl(n,R) is map [21].

In 1936 Hans Frudenthal gave a necessary and sufficient condition for a connected topological group with a countable open basis to be MAP [4]. A. Weil's paper of 1941 enables one to remove the separability restriction giving the following (Freudenthal-Weil) theorem. A locally compact connected topological group G is MAP if and only if it is the direct product of a vector group and a compact group [25]. In particular, this shows that a locally compact connected topological group is MAP if and only if it has a continuous injective homomorphism into a compact topological group. Various generalizations of the Freudenthal-Weil theorem for locally compact topological groups which are almost connected, i.e., G/G_0 is compact where G_0 is the identity component of G, are provided by M. Kuranishi [15], S. Murakami [20], and S. Grosser and M. Moskowitz [10]. Recently, Ter-Jeng Huang gave a characterization of MAP groups in terms of transformation groups which implies several of the above-mentioned authors' results [12].

In 1940, J. v. Neumann and E. P. Wigner gave a sufficient condition for an element of a connected topological group G to be contained in $\mathfrak{N}(G)$ [22]. Using this criterion they calculated the von Neumann kernel of the "ax + b" group. and showed by another direct calculation that Sl(2,R) discretely topologized was map. From this together with the root space decomposition of a connected semisimple Lie group without compact factors it follows that a connected semisimple Lie group without compact factors is map. Another method of argument showed that the discrete subgroup Sl(2,Z) of Sl(2,R) was MAP. Some calculations of the von

Neumann kernel in a few particular cases were made in [10].

Very recently in [5] H. Furstenberg gave a generalization (with a new proof) of the Borel density theorem [2;23] and its consequences to map groups. This was further generalized by Moskowitz in [18]. One of the motivations for the present paper was to understand exactly the generality achieved by Furstenberg, at least in the case of connected groups.

In the present paper all groups will be assumed to be topological groups, and as mentioned previously all representations are finite-dimensional continuous complex unitary representations. For a group G and a subgroup H we establish the following notation. If G is locally compact, the radical of G is the largest compact connected solvable normal subgroup of G (see K. Iwasawa for the existence of the radical [5]). The derived of G is denoted [G,G], with $\overline{[G,G]}$ being its closure. Z(G) is the center of G and $Z_G(H)$ is the centralizer of H in G. G_0 will be the identity component of G. We say that an automorphism, α, of G, is of bounded displacement if for all $g \in G$, $\alpha(g)g^{-1}$ lies in some fixed compact set. The H-displacement of α is $\text{disp}_H\, \alpha = \{\alpha(h)h^{-1}\}_{h\epsilon H}$. For two groups A and B, $S = A \times_\eta B$ denotes the semidirect product of A and B where A is normal in S and η: $B \to \text{Aut}\ (A)$. is a homomorphism.

We now give a summary of results to be proved in this paper.

§1 is devoted to the calculation of the von Neumann kernel of an arbitrary connected Lie group. We conclude easily from this calculation that the closed normal subgroup $\mathfrak{N}(G)$ is also connected.

In §2 we give various characterizations of minimally almost periodicity for a connected Lie group. The results of the previous section enable us to restrict ourselves to the case where G has an abelian radical. Here, a structure theorem for connected abelian Lie groups and orbit considerations are the basis for the arguments. Among the characterizations is the following. A connected Lie group G with radical R is map if and only if G/R is semisimple without compact factors and $G = \overline{[G,G]}$. (A similar result has been stated by Guivarch'.) We prove in the special case where R is also simply connected that $G = [G,G]$. This has the corollary that a simply connected radical of a connected map Lie group is nilpotent.

In §3 we prove by using the techniques of the previous sections in this paper together with a theorem of Tits [24] that a connected map Lie group has no nontrivial automorphisms of bounded displacement. As a consequence we get a new proof via the results of F. Greenleaf, M. Moskowitz, and L. Rothschild [9] of the following theorem: If G is a map connected Lie group, H is a closed subgroup of G such that G/H has finite volume then $Z_G(H) = Z(G)$ [5], and more generally if G and H are as above and α is an automorphism of G leaving H pointwise fixed then α is trivial. We conclude §3 with the following result. For G and H as above, if $\mathrm{disp}_H\ \alpha$ is bounded then α is trivial. This type of theorem will be dealt with in the case of an arbitrary connected Lie group in [7].

In §4 making use of a decomposition due to Y. Matsushima [17], and projective limits of Lie groups, we extend most of the results of §2 to arbitrary locally compact connected topological groups. We get as a corollary a relatively simple proof of the Freudenthal-Weil theorem.

§1. Characterization of $\mathfrak{N}(G)$ for a Connected Lie Group

Given a connected Lie group G, we let $G = R \cdot KS$ be its Levi decomposition where R is the radical and KS is the decomposition of the Levi factor into compact and noncompact parts. Let $\overline{[R,R]}$ denote the closure of the derived of R. Since $\overline{[R,R]}$ is a characteristic subgroup of G we may form the projection $\pi\colon G \to G/\overline{[R,R]}$. In the quotient $G/\overline{[R,R]}$ we have the connected abelian radical $R/\overline{[R,R]}$ which by well-known structure theorems can be written as $V \times T$ where V is a vector group and T is a toroidal group, the T being characteristic in $G/\overline{[R,R]}$. To see that T is actually fixed under the induced action of $\pi(KS)$ on $R/\overline{[R,R]}$, the Levi factor of $G/\overline{[R,R]}$, consider the continuous homomorphism from the Levi factor into the automorphism group of T, Aut(T), defined by

$g \to \alpha_g|_T$; $g \in \pi(KS)$. The image, $B = \{\alpha_g|_T\}_{g \in \pi(KS)}$ is a connected subgroup of the discrete group Aut(T), the automorphism group of T, and therefore B = (1). This implies that $\alpha_{\pi(ks)}|_T = I$, the identity automorphism, for all $\pi(ks) \in \pi(KS)$ and hence that T is centralized by $\pi(KS)$. So, T is $\pi(KS)$-fixed. The semisimple group $\pi(KS)$ acts by conjugation on $V \times T$ and since T is stable, Weyl's theorem will enable us to replace V by a vector group of the same dimension, hereafter also denoted V, which is $\pi(KS)$-stable. This implies since $R / \overline{[R,R]}$ is abelian that V is normal in $G/\overline{[R,R]}$. We let V_f denote the set of all elements of V which have finite $\pi(KS)$-orbit. V_f is evidently normal in $G/\overline{[R,R]}$ since it is also $\pi(KS)$-stable. By Weyl's theorem [11; Thm. 2.3, p. 125] it must have a $\pi(KS)$-stable complement written as $V_f^{\perp}$, and as above, normal in $G/\overline{[R,R]}$. Because $\pi(KS)\cdot v$ is a connected set for each $v \in V$, if $v \in V_f$, then $\pi(KS)\cdot v$ is a finite and connected set and hence must be a point. Thus every element of V_f is actually $\pi(KS)$-fixed.

Using the notations above our first theorem is

<u>Theorem 1.1</u>. If G is a connected Lie group, then
$\eta(G) = \pi^{-1}(V_f^{\perp} \cdot \pi(S))$.

In particular,

<u>Theorem 1.2</u>. If the radical of G is abelian, $\eta(G) = V_f^{\perp} \cdot S$, and also in this case $V_f^{\perp}$ is unique.

We begin with a lemma.

Lemma 1.3. If G is a connected Lie group with Levi decomposition, $G = R \cdot KS$, then both S and $\overline{[R,R]}$ are contained in $\mathfrak{N}(G)$. If R is abelian, $V_f^{\perp}$ is also contained in $\mathfrak{N}(G)$.

Proof of Lemma 1.3. Let ρ be an irreducible representation of G. The restriction of ρ to S is a representation of S, and since S is map must be trivial on S. Thus, $S \subseteq \mathfrak{N}(G)$.

By Clifford's theorem [3; Theorem 1, p. 534] applied to the normal subgroup R of G,

$$\rho|_R = \nu(\chi \oplus \text{conjugates of } \chi) ,$$

where ν is a natural number, and χ is an irreducible representation of R (clearly, ρ being continuous and unitary implies the same for χ and its conjugates). By Lie's theorem [11; Theorem 1.3, p. 119], χ and its conjugates, χ^g, being irreducible, implies $\chi^g([R,R]) = (1)$, and so by continuity $\chi^g(\overline{[R,R]}) = (1)$. Thus $\rho(\overline{[R,R]}) = (1)$ and hence $\overline{[R,R]} \subseteq \mathfrak{N}(G)$.

The standard identification of V with $\hat{V}$ is given by the map $w \to \chi_w$, where $\chi_w(v) = \exp(i\langle w,v\rangle)$ (where $\langle\ ,\ \rangle$ is the inner product on V). If R is abelian, then $V_f^{\perp}$ is a normal subgroup of G, and then by Clifford's theorem,

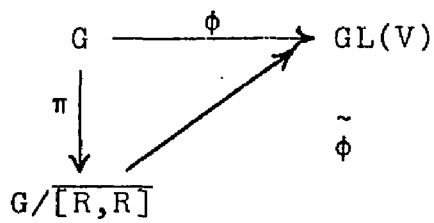
$$\rho|_{V_f^{\perp}} = \nu_0(\chi_w \oplus \text{conjugates of } \chi_w), \text{ for some } w \in V_f^{\perp} ,$$

where ν_0 is a natural number, and up to equivalence all conjugates must occur. Here, equivalence means equality since the irreducible representations of V are one-dimensional, and $\chi_{w_1} = \chi_{w_2}$ if and only if $w_1 = w_2$. Since the KS-orbit of χ_w is infinite for all $0 \neq w \in V_f^{\perp}$, and ρ is finite-dimensional

$$\rho|_{V_f^{\perp}} = \nu_0 \chi_0 = I_{d_\rho} \; .$$

Thus, $V_f^{\perp} \subseteq \mathcal{N}(G)$. This completes the proof of Lemma (1.3).

Proof of Theorem 1.1. We begin the proof by the reduction to the case of an abelian radical. We show that $\pi^{-1}(\mathcal{N}(G/\overline{[R,R]})) = \mathcal{N}(G)$. Let ϕ be an irreducible unitary representation of G. By (1.3), $\phi(\overline{[R,R]}) = (1)$, and therefore ϕ induces a continuous representation $\tilde{\phi}$ such that the diagram below is commutative.



Let $y \in \pi^{-1}(\mathcal{N}(G/\overline{[R,R]}))$. From the diagram $\phi(y) = \tilde{\phi}(\pi(y))$. Since $\pi(y) \in \mathcal{N}(G/\overline{[R,R]})$, $\phi(y) = 1$ and $y \in \mathcal{N}(G)$. Thus, $\pi^{-1}(\mathcal{N}(G/\overline{[R,R]})) \subseteq \mathcal{N}(G)$. Let $x \in \mathcal{N}(G)$ and let $\tilde{\rho}$ be an irreducible representation of $G/\overline{[R,R]}$. Then, $\tilde{\rho} \circ \pi$ is a representation of G, and therefore $(\tilde{\rho} \circ \pi)(x) = 1$. This implies $\tilde{\rho}(\pi(x)) = 1$ and since $\tilde{\rho}$ was arbitrary, $\pi(x) \in \mathcal{N}(G/\overline{[R,R]})$. Then, $x \in \pi^{-1}(\mathcal{N}(G/\overline{[R,R]}))$, and $\mathcal{N}(G) \subseteq \pi^{-1}(\mathcal{N}(G/\overline{[R,R]}))$. The two inclusions imply that $\mathcal{N}(G) = \pi^{-1}(\mathcal{N}(G/\overline{[R,R]}))$.

To complete the proof of (1.1) it remains to prove for $G = (V \times T) \cdot \pi(S)$ that $\mathcal{N}(G/\overline{[R,R]}) = V_f^{\perp} \cdot \pi(S)$. This will follow immediately from (1.2). However, we require some preliminary results.

Definition 1.4. A group G is said to be a Z-group if G/Z(G) is compact.

A proof is outlined in [11; Ex. 2, p. 194] that a connected group is MAP if and only if it is a Z-group.

Lemma 1.5. Assume $G = Z(G) \cdot K$ is a connected Lie group. Then G is MAP.

Proof of Lemma 1.5. From the Second Isomorphism Theorem, $G/Z(G) \simeq (Z(G) \cdot K)/Z(G) \simeq K/Z(G) \cap K$, which is compact. Hence, G is a connected Z-group and therefore MAP.

Proposition 1.6. Let G be a connected Lie group. Then any Levi factor of G is closed.

Proof. Let $G = R \cdot KS$ be a Levi decomposition. Since any Levi factor of G is conjugate to KS, it is sufficient to prove that KS is closed. In fact, since K is compact, if we show that S is closed (in G), it will follow that KS is closed [11; Lemma 2.2, p. 5]. By Greenleaf and Moskowitz [7], Z(S) is a discrete central subgroup of G. Therefore, $\pi: G \to G/Z(S)$ is a covering map. In the connected group G/Z(S) the Levi factor is clearly S/Z(S). Since S/Z(S) is isomorphic to the linear semisimple Lie group Ad(S), it is closed in G/Z(S) [6; Lemma 4, p. 115 and Lemma 5, p. 116]. Therefore, $L = \pi^{-1}(S/Z(S))$ is closed by continuity and is thus a Lie subgroup of G containing S. And $\pi|_L: L \to S/Z(S)$

is a covering map since it is surjective and has the discrete central kernel, $\ker\pi|_L = L \cap Z(S)$. Therefore, dim L = = dim S/Z(S) = dim S. So, S is an open subgroup of L and therefore closed in L. Then, S is closed in G because L is closed in G.

Lemma 1.7. (a) Assume that H is a closed, normal, connected subgroup in G and that $H \subseteq \eta(G)$. If G/H is MAP, then $H = \eta(G)$. In particular,

(b) Let G be a connected reductive Lie group with Levi decomposition $G = Z(G)\cdot KS$. Then $\eta(G) = S$.

In particular,

(c) If G is a connected, reductive Lie group which is map, then G = S.

Proof. (a) Suppose that $x_0 \in G - H$ and consider the element $Hx_0 \in G/H$. Since G/H is MAP, and $Hx_0 \neq 1$, there exists a representation $\tilde{D}$ of G/H such that $\tilde{D}(Hx_0) \neq 1$. Define the representation D of G by $D(x) = \tilde{D}(Hx)$ for $x \in G$. We have $D(x_0) = \tilde{D}(Hx_0) \neq 1$. Therefore, $x_0 \notin \eta(G)$ and $H = \eta(G)$.

(b) Since S is normal in KS and commutes with Z(G), S is a normal subgroup of G. From (1.6), S is closed in G, and hence we may apply the Second Isomorphism Theorem to obtain

$$(1) \qquad G/S \simeq (Z(G)\cdot K)\cdot S/S \simeq Z(G)\cdot K/(Z(G)\cdot K) \cap S$$

The connected group $Z(G)\cdot K$ is MAP, (1.5), and hence a Z-group. Because $Z(G)\cdot K/(Z(G)\cdot K) \cap S$ is the homomorphic image (under the canonical projection), of this Z-group, it is also a Z-group. This implies, using (1), that G/S is MAP and by (1.3), $S \subseteq \mathfrak{N}(G)$. We conclude by part (a) that $\mathfrak{N}(G) = S$.

(c) Since G is map, $\mathfrak{N}(G) = G$. But by part (b) above, $\mathfrak{N}(G) = S$. Thus, $G = S$.

This completes the proof of the lemma.

<u>Proof of Theorem 1.2</u>. We begin the proof in the case when G is of the form $G = (V \times T) \times_\eta KS$. Here it will be shown that G is isomorphic to $T \oplus (V \times_\eta KS)$. Once this has been established, since $\mathfrak{N}(T \oplus (V \times_\eta KS)) = \mathfrak{N}(T) \oplus \mathfrak{N}(V \times_\eta KS) =$ $= (1) \oplus \mathfrak{N}(V \times_\eta KS)$, because T is compact and hence MAP [11; Theorem 1.4, p. 18], we may assume T is trivial. To see the isomorphism, define the function

$$\phi : (V \times T) \times_\eta KS \to T \oplus (V \times_\eta KS)$$

by

$$\phi\{((v,t),ks)\} = (t,(v,ks));\ v \in V,\ t \in T,\ ks \in KS\ .$$

Let $x = ((v,t),ks)$ and $y = ((v',t'),k's')$ be arbitrary elements of $(V \times T) \times_\eta KS$. If $x = y$, then $v = v'$, $t = t'$, $ks = k's'$; and hence $(t,(v,ks)) = (t',(v',k's'))$, whereby $\phi(x) = \phi(y)$, and so ϕ is well defined. It is clear that ϕ is a continuous, open, and surjective map. We show that ϕ is a homomorphism. Calculating, for arbitrary x and y above,

$$\phi(xy) = \phi\{((v,t),ks)((v',t'),k's')\}$$

$$= \phi\{((v,t)(\eta(ks)(v',t')),ksk's')\}$$

$$= \Phi\{((v,t)(\eta(ks)(v'),\eta(ks)(t'),ksk's')\}$$

$= \phi\{((v,t)(\eta(ks)(v'),t'),ksk's')\}$ (because $t' \in T$ is KS-fixed)

$= \phi\{(v\eta(ks)(v'),tt'),ksk's')\}$

$= (tt',(v\eta(ks)(v'),ksk's'))$

$= (t,(v,ks))(t',(v',k's'))$

$= \phi(x)\phi(y)$.

Therefore, ϕ is a homomorphism. We next show that ϕ is injective. Suppose $\phi(x) = \phi(y)$. This implies $(t,(v,ks)) = (t',(v',k's'))$, and so $t = t'$, $v = v'$, and $ks = k's'$. Therefore, $x = ((v,t),ks) = ((v',t'),k's') = y$ and ϕ is injective. This completes the argument that ϕ is an isomorphism.

Assuming T is trivial, we have $G = V \times_\eta KS$. From (1.3) both S and $V_f^\perp$ are contained in $\mathcal{N}(G)$ and so $V_f^\perp \times_\eta S$, the group generated by $V_f^\perp$ and S is contained in $\mathcal{N}(G)$. Using the notation of (1.3) for the standard identification of V with $\hat{V}$, from the Mackey theory [15; Theorem A, p. 42] representations of the form $\chi_w \otimes \gamma$, $w \in V$, γ an irreducible representation of KS, defined by $(\chi_w \otimes \gamma)(v,ks) = \chi_w(v)\gamma(ks)$, for $(v,ks) \in G$, is a representation of G (not necessarily finite-dimensional), and is finite-dimensional if $w \in V_f$. (Actually, this collection exhausts the (finite-dimensional) irreducible representations of G, although we will not have need of this fact.)

Let $(v,ks) \in G$, where $ks \notin S$. We show that

$(v,ks) \notin \eta(G)$. Since $ks \notin S$ and $S = \eta(KS)$ (1.7 (c)), there exists an irreducible representation γ_{ks} of S such that $\gamma_{ks}(ks) \neq 1$. But then $(\chi_0 \otimes \gamma_{ks})(v,ks) = \chi_0(v)\chi_{ks}(ks) = = 1\cdot\gamma_{ks}(ks) \neq 1$, and $(v,ks) \notin \eta(G)$. Consider an element $(v,ks) \in G$, with $v \in V_f$. Since $V_f^{\perp}$ is a subspace of V, $V/V_f^{\perp}$ is a vector group and hence has a separating family of irreducible representations (indeed it has a faithful one). Thus, for any element $v \in V_f \subseteq V - V_f^{\perp}$ we can find an irreducible representation χ_w of V such that $\chi_w(v) \neq 1$. Calculating, $(\chi_w \otimes I)(v,ks) = \chi_w(v)\cdot 1 \neq 1$, and so $(v,ks) \notin \eta(G)$. We have proven that $V_f^{\perp} \times_{\eta} S \subseteq \eta(G)$ and that an element $(v,ks) \in G - (V_f^{\perp} \times_{\eta} S)$ is not in $\eta(G)$. Therefore, $\eta(G) = V_f^{\perp} \times_{\eta} S$, and since $\eta(G)$ is defined independently of V_f, $V_f^{\perp}$ must be unique. This completes the proof of (1.2) in the case where G is of the form $G = R \times_{\eta} KS$. For the general case of (1.2) we require another lemma.

<u>Lemma 1.8</u>. Assume G is as in (1.2). Then $V_f^{\perp} \cdot S$ is closed in G.

<u>Proof</u>. In view of Hochschild [11; Ex. 2, p. 194], it is sufficient to prove that for every compact subgroup M of G, $(V_f^{\perp} \cdot S) \cap M$ is closed in M.

Let L be an arbitrary compact subgroup of G. The Iwasawa Decomposition Theorem [13; Theorem 2, p. 515] tells us that any two maximal compact subgroups of G are conjugate by an element of G and since V has no compact subgroups it also tells us that a maximal compact subgroup of T·KS is a maximal compact subgroup of G as well. These assertions allows us to choose an element $x \in G$ such that

(1) $$L^x = xLx^{-1} \subseteq T \cdot KS \ .$$

The group $V \cap T\cdot KS$ is a normal subgroup of G. Its Lie algebra, $\mathcal{V} \cap \mathfrak{t}\cdot\mathfrak{ks}$ is contained in the Lie algebra $(\mathcal{V}+\mathfrak{t}) \cap \mathfrak{ks}$. And, $(\mathcal{V}+\mathfrak{t}) \cap \mathfrak{ks} = (0)$ since $\mathcal{V}+\mathfrak{t}$ is the radical of $\mathfrak{g}$, the Lie algebra of G, and $\mathfrak{ks}$ is the Levi factor of $\mathfrak{g}$. Hence, the subgroup $V \cap T\cdot KS$ is both normal and discrete and therefore central in G. So,

(2) $$V \cap T\cdot KS \subseteq V_f$$

and because $V_f^{\perp} \cap T\cdot KS \subseteq V \cap T\cdot KS$, we have using (2) that

(3) $$V_f^{\perp} \cap T\cdot KS \subseteq V_f$$

Clearly, $V_f^{\perp} \cap T\cdot KS \subseteq V_f^{\perp}$ and hence combined with (3),

(4) $$V_f^{\perp} \cap T\cdot KS \ , \ \ V_f \cap V_f^{\perp} = (1) \ ,$$

implying

(5) $$V_f^{\perp} \cap T\cdot KS = (1) \ .$$

Obviously, $(V_f^{\perp}\cdot S) \cap L^x \supseteq S \cap L^x$. Suppose $v_f^{\perp}\cdot s = \ell$ is an element of $(V_f^{\perp}\cdot S) \cap L^x$. Then, $v_f^{\perp} = \ell s^{-1} \in L^x S \subseteq T\cdot KS\cdot S$ by (1). But, $T\cdot KS\cdot S = T\cdot KS$ and so $v_f^{\perp} \in T\cdot KS$, and therefore by (5), $V_f^{\perp} \cap T\cdot KS = (1)$. Thus, $v_f^{\perp} = 1$ and hence

(6) $$(V_f^{\perp}\cdot S) \cap L^x = S \cap L^x$$

which is closed in L^x since S is closed in G (1.6). So, $(V_f^{\perp} \cdot S) \cap L^x$ is closed in L^x.

We construct the group $G' = R \times_\eta KS$ where the action η is conjugation in the group G. Consider the map $\phi: G' \to G$ defined by

$$\phi((r,ks)) = (r,ks) \ .$$

It is a continuous open homomorphism of G' onto G mapping the subgroup $V_f^{\perp} \times_\eta S$ of G' onto the subgroup $V_f^{\perp} \cdot S$ of G. By (1.2), $V_f^{\perp} \times_\eta S = \mathcal{N}(G')$ and so it is a (closed) normal subgroup of G'. Therefore, $V_f^{\perp} \cdot S$ is a normal subgroup of G. Conjugation by the element x^{-1} is a homeomorphism of G which then maps the closed set $(V_f^{\perp} \cdot S) \cap L^x$, in L^x, onto the set $(V_f^{\perp} \cdot S) \cap L$, in L. This implies that $(V_f^{\perp} \cdot S) \cap L$ is closed in L, and thus $V_f^{\perp} \cdot S$ is closed in G. This completes the proof of the lemma.

To complete the proof of (1.2), let G, G', and ϕ be as above. We have seen that $V_f^{\perp} \cdot S$ is normal in G, and by (1.3) that it is closed in G as well. We can therefore form $G/V_f^{\perp} \cdot S$ and we let $\tilde{\phi}: G'/V_f^{\perp} \times_\eta S \longrightarrow G/V_f^{\perp} \cdot S$ be the homomorphism induced from ϕ. Earlier in the proof we showed that $\mathcal{N}(G') = \mathcal{N}(V \times_\eta KS) = V_f^{\perp} \times_\eta S$, and thus $G'/V \times_\eta KS$ is a connected MAP group, and hence a Z-group. The homomorphic image, $G/V_f^{\perp} \cdot S$, of $G'/V_f^{\perp} \times_\eta S$, under $\tilde{\phi}$, must consequently be a connected Z-group as well. Hence, $G/V_f^{\perp} \cdot S$ is MAP, and by (1.3), both S and $V_f^{\perp}$ are contained in $\mathcal{N}(G)$. Thus,

$V_f^{\perp} \cdot S$ is contained in $\mathcal{N}(G)$ and so $\mathcal{N}(G) = V_f^{\perp} \cdot S$ (1.5(a)). This completes the proof of (1.2).

Proof of Theorem 1.1. We have shown that $\pi^{-1}(\mathcal{N}(G/\overline{[R,R]})) = \mathcal{N}(G)$. By (1.2), $\mathcal{N}(G/\overline{[R,R]}) = V_f^{\perp} \cdot \pi(S)$. Therefore, $\mathcal{N}(G) = \pi^{-1}(V_f^{\perp} \cdot \pi(S))$, and (1.1) is proven.

Corollary 1.9. If G is a connected Lie group, then $\mathcal{N}(G)$ is a closed, normal, and connected subgroup of G.

Proof. We have seen earlier from the definition of $\mathcal{N}(G)$ that it is a closed normal subgroup of G. From (1.1) and (1.2), $\mathcal{N}(G)/\overline{[R,R]} = \mathcal{N}(G/\overline{[R,R]}) = V_f^{\perp} \cdot \pi(S)$ which is clearly connected. Therefore, $\mathcal{N}(G)/\overline{[R,R]}$ is connected, and the characteristic subgroup $\overline{[R,R]}$ is connected since R is connected. Thus, $\mathcal{N}(G)$ is connected.

Theorem 1.2 allows us to calculate the Von Neumann kernel readily as the following example shows.

Example 1.10. Let $G = R^3 \times_\eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$, where η is the usual action. Then $\mathcal{N}(G)$ = subspace generated by $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. To see this let $K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \right\}_{\theta \in R}$, and let $V = R^3$. We have that $\eta(K)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, i.e., $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is K-fixed. This implies that the compact group K leaves the line $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $a \in R$, fixed. It is clear that all other elements of R^3 have infinite K-orbit, and so

$V_f^{\perp}$ = subspace generated $\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. By (1.2), with

$S = (1)$, $\mathfrak{N}(G) = V_f^{\perp}$ = subspace generated $\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

If G is of the form, $G = R^n \times_\eta K$, where K is a compact connected Lie group and R^n is the radical of G, then $\mathfrak{N}(G)$ is a subspace of R^n (1.2). Thus it makes sense to speak of the dimension of $\mathfrak{N}(G)$ as a vector space, denoted dim $\mathfrak{N}(G)$. We assert the following about dim $\mathfrak{N}(G)$.

Corollary 1.11. If $G = R^n \times_\eta K$, where K is a compact connected Lie group and R^n is the radical of G, then $\dim \mathfrak{N}(G) \neq 1$.

Proof. Let $V = R^n$. From (1.2), $\mathfrak{N}(G) = V_f^{\perp}$ and hence $\dim \mathfrak{N}(G) = \dim V_f^{\perp}$. To show that $\dim \mathfrak{N}(G) \neq 1$ it is therefore sufficient to show $\dim V_f^{\perp} \neq 1$. Suppose to the contrary that $\dim V_f^{\perp} = 1$. Then there exists a vector $0 \neq v_0 \in V_f^{\perp}$ such that v_0 is a basis of $V_f^{\perp}$. For such $k \in K$, $\eta(k)$ is nonsingular and hence $\eta(k)(v_0) \neq 0$. Therefore, there is a continuous homomorphism $\phi: K \to R^x$ (the nonzero real numbers), such that

$$(1) \quad \eta(k)(v_0) = \phi(k)v_0; \ k \in K.$$

So, $\phi(K)$ is a compact connected subgroup of R^x, and therefore, $\phi(K) = (1)$. This implies, from (1), that $v_0 \in V_f$. This is a contradiction and so $\dim V_f^{\perp} \neq 1$.

§2. Minimally Almost Periodic Groups

In this section we give various characterizations of minimally almost periodic groups, (map), i.e., those where $\mathfrak{N}(G) = G$. We continue with the notation established in §1.

Lemma 2.1. Let G be a connected Lie group. Then G is map if and only if $G/\overline{[R,R]}$ is map.

Proof. We have seen in the proof of (1.1) that $\mathfrak{N}(G) = \pi^{-1}(\mathfrak{N}(G/\overline{[R,R]}))$. Therefore, $\mathfrak{N}(G)/\overline{[R,R]} = \mathfrak{N}(G/\overline{[R,R]})$. If $G/\overline{[R,R]}$ is map, then $\mathfrak{N}(G/\overline{[R,R]}) = G/\overline{[R,R]}$ and $\mathfrak{N}(G)/\overline{[R,R]} = G/\overline{[R,R]}$. Hence, $\overline{[R,R]}\,\mathfrak{N}(G) = G$, $\overline{[R,R]}$ being the kernel of π. Consequently, $\mathfrak{N}(G) = G$ since $\overline{[R,R]} \subseteq \mathfrak{N}(G)$, (1.3), and thus G is map.

In the other direction, since $G/\overline{[R,R]}$ is a quotient group of G, the minimally almost periodicity of G implies the same for $G/\overline{[R,R]}$.

Since Lemma 2.1 enables us to consider groups with abelian radicals we now come to

Theorem 2.2. Let G be a connected Lie group with Levi decomposition $G = R \cdot KS$. Then the following are equivalent.

(i) G is map

(ii) $G/\overline{[R,R]}$ can be written in the form $G/\overline{[R,R]} = V \cdot \pi(S)$ with all nonzero elements of V of unbounded $\pi(S)$-orbit.

(iii) G/R has no compact factors and every nonzero element of $R/\overline{[R,R]}$ is of unbounded $\pi(S)$-orbit.

(iv) G/R has no compact factors and $\overline{[R,R]} = \{r \in R: [G,r] \subseteq \overline{[R,R]}\}$.

(v) G/R has no compact factors and $G = \overline{[G,G]}$.

<u>Proof</u> (i $\Longleftrightarrow$ ii). If G is map, then $G/\overline{[R,R]}$ is map and so the semisimple group $G/\overline{[R,R]} \big/ R/\overline{[R,R]}$ is map and hence by (1.7(c)) has no compact factors. Therefore, $G/\overline{[R,R]}$ has Levi decomposition $G/\overline{[R,R]} = R/\overline{[R,R]} \cdot \pi(S)$, and $\eta(G/\overline{[R,R]}) = V_f^{\perp} \cdot \pi(S)$ (see 1.2). Thus since $G/\overline{[R,R]}$ is map, $R/\overline{[R,R]} \cdot \pi(S) = V_f^{\perp} \cdot \pi(S)$, which implies that $T = (1)$ and $R/\overline{[R,R]} = V_f^{\perp} = V$. So, $G/\overline{[R,R]} = V \cdot \pi(S)$ with each nonzero element of $V = (V_f^{\perp})$ of infinite $\pi(S)$-orbit. Since $\pi(S)$ is a connected semisimple Lie group without compact factors each nonzero element of V has infinite $\pi(S)$-orbit if and only if it has unbounded $\pi(S)$-orbit [8; Prop. 8.4, p. 238]. This proves (ii).

If (ii) holds, by (1.2), $\eta(G/\overline{[R,R]} = V_f^{\perp} \cdot \pi(S) = G/\overline{[R,R]}$ and so $G/\overline{[R,R]}$ is map. By (2.1), G is map.

(i $\Longleftrightarrow$ iii). Assume that G is map. Then G/R is map, and since it is also semisimple must have no compact factors (1.7(c)). That every nonzero element of $R/\overline{[R,R]}$ is of unbounded $\pi(S)$-orbit follows immediately from (ii).

If (iii) holds, $K = (1)$ and $T = (1)$, and hence $R/\overline{[R,R]} = V$ and $G/\overline{[R,R]}$ has Levi decomposition, $G/\overline{[R,R]} = V \cdot \pi(S)$ with all nonzero elements of V $(= R/\overline{[R,R]})$ of unbounded $\pi(S)$-orbit. By (ii), G is map.

(i $\Longleftrightarrow$ iv). Suppose G is map. By (iii), G/R has no compact factors. Set $A = \{r \in R: [G,r]\} \leq \overline{[R,R]}$. Clearly, $[R,R] \leq A$ because $\overline{[R,R]}$ is contained in R and is a characteristic subgroup of G. Let $x \in A$. Then, $[G,x] \leq \overline{[R,R]}$. In particular,

$$(1)\quad [S,x] \leq \overline{[R,R]}$$

From (1) we obtain

$$(2)\quad [\pi(S),\pi(x)] = \pi([S,x]) \leq \pi(\overline{[R,R]}) = (1) .$$

This says

$$(3)\ \pi(s)\pi(x)\pi(s)^{-1}\pi(x)^{-1} = 1 \text{ for all } s \in S,$$

and hence,

$$(4)\ \pi(s)\cdot\pi(X) = \pi(s)\pi(x)\pi(s)^{-1} = \pi(x) \text{ for all } s \in S.$$

Thus $\pi(x) \in R/\overline{[R,R]}$ is $\pi(S)$-fixed. By (iii), the only element of $R/\overline{[R,R]}$ that is $\pi(S)$-fixed is 1. So $\pi(x) = 1$ and $x \in \ker \pi = \overline{[R,R]}$. Therefore, $A \leq \overline{[R,R]}$ and with the other inclusion, $A = \overline{[R,R]}$. We have actually proven

$$(5)\quad \overline{[R,R]} = \{r \in R: [S,r] \leq \overline{[R,R]}\} = \{r \in R: [G,r] \leq \overline{[R,R]}\}.$$

Conversely, assume (iv) holds. Then $K = (1)$ and $G/\overline{[R,R]}$ has Levi decomposition, $G/\overline{[R,R]} = R/\overline{[R,R]} \cdot \pi(S)$. Suppose a nonzero element $\overline{[R,R]}x \in R/\overline{[R,R]}$ has finite (and hence fixed) $\pi(S)$-orbit. We can express this as

$$(6)\quad \pi(s)\cdot\pi(x) = \pi(s)\pi(x)\pi(s)^{-1} = \pi(x) \text{ for all } s \in S.$$

Then

(7) $\pi([s,x]) = \pi(s)\pi(x)\pi(s)^{-1}\pi(x)^{-1} = 1$ for all $s \in S$,

and

(8) $\pi([S,x]) = (1)$.

From (8) we have that $[S,x] \subseteq \ker \pi = \overline{[R,R]}$. By (5), $x \in \overline{[R,R]}$. However, this is a contradction since we assumed $\overline{[R,R]}x$ to be nonzero in $R/\overline{[R,R]}$. Therefore, every nonzero element of $R/\overline{[R,R]}$ has infinite and hence unbounded $\pi(S)$-orbit. By (iii), G is map.

(i $\Longleftrightarrow$ v). Assume (i) holds. By (iv), G/R has no compact factors. Suppose now that $G \neq \overline{[G,G]}$ and consider the canonical projection π': $G \to G/\overline{[G,G]}$. It is a continuous homomorphism of G into the nontrivial abelian group, $G/\overline{[G,G]}$. Since G is map this cannot happen and so we must have $G = \overline{[G,G]}$.

Conversely, if (v) holds then $G = R \cdot S$ is the Levi decomposition of G. Let ρ be an irreducible representation of G. By Clifford's theorem, the restriction $\rho|_R$ can be written in the form

$$\rho|_R = \nu(\chi \oplus \text{conjugates of } \chi) ,$$

where ν is a natural number, χ is an irreducible representation of R; and up to equivalence all conjugates of χ must occur. Since the irreducible representations of R are one-dimensional, equivalence means equality. Because $\rho|_R$ is finite-dimensional, χ must have a finite number of conjugates, and since $G\cdot\chi$ is connected, χ must actually be fixed under the action of G. Thus, $\rho|_R = \nu\chi$. Therefore,

$$(1) \quad \rho(r) = \nu\chi(r)I; \; r \in R.$$

Let $\phi\colon G \to R$ be the continuous homomorphism defined by

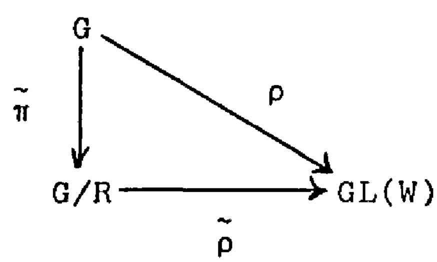
$$(2) \quad \phi(g) = \det \rho(g).$$

Since R is abelian, $\phi(\overline{[G,G]}) = (1)$. However, by assumption $G = \overline{[G,G]}$ and so $\phi(G) = (1)$. In particular, upon restricting ϕ to R we have

$$(3) \quad \phi(r) = \det \rho(r) = 1, \text{ for all } r \in R.$$

This implies that $\rho(r) \in SU(n,\mathbb{C})$ for all $r \in R$. From (1), for every $r \in R$ $\rho(r)$ is a scalar matrix and therefore $\rho(r) \in Z(SU(n,\mathbb{C}))$. Hence $\rho(R)$ is a connected subgroup of the finite group $Z(SU(n,\mathbb{C}))$, and so $\rho(R) = I$.

Consider the following commutative diagram:



where $\tilde{\pi}$ is the canonical projection of G onto G/R and $\tilde{\rho}$ is the representation of G/R induced from ρ. Evidently, $\tilde{\rho}$ must be trivial since G/R is semisimple without compact factors and so map. Hence, $\rho = \tilde{\rho} \circ \tilde{\pi}$ is trivial and since ρ was arbitrary, G is map. This completes the proof of (2.2).

In the case of simply connected radical we can say somewhat more.

Theorem 2.3. Assume G is a connected Lie group with Levi decomposition, $G = R \cdot KS$.

(a) If G is map and R is simply connected, then $G = [G,G]$. In particular,

(b) If G is map and R is abelian or G is simply connected, then $G = [G,G]$.

Proof. We will first prove the result for the special case when R is a vector group V. By (2.2 (ii)),

(1) $G = V \cdot S$ with all nonzero elements of V of unbounded S-orbit. Calculating for $v \in V$, $s \in S$,

(2) $[v,s] = [(v,1),(1,s)] = (s \cdot v - v, 1)$.

Put $L = \text{linear span } \{s \cdot v - v\}_{\substack{s \in S \\ v \in V}}$. L can be seen to be a submodule of V as follows. For $s,t \in S$, $v \in V$,

$$\begin{aligned} t.(s \cdot v - v) &= t \cdot (s \cdot v) - t \cdot v \\ &= t \cdot (s \cdot v) - s \cdot v + s \cdot v - v + v - t \cdot v \\ &= [t \cdot (s \cdot v) - s \cdot v] + [s \cdot v - v] - [t \cdot v - v] . \end{aligned}$$

All these expressions in brackets in the last line above are in L and so $t \cdot (s \cdot v - v)$ is also in L. This shows that L is a submodule of V. Observe that $L \neq (0)$ if $V \neq (0)$ since S acts on nonzero elements of V by unbounded orbits. Thus, if V is irreducible, L = V. By (1), the linear span $\{[v,s]\}_{\substack{v\in V\\ s\in S}}$ = linear span $\{s\cdot v - v\}_{\substack{s\in S\\ v\in V}}$, and so,

$$(2)\quad [V,S] = \text{linear span } \{[v,s]\}_{\substack{v\in V\\ s\in S}} = \text{linear span } \{s\cdot v - v\}_{\substack{v\in V\\ s\in S}} = L\ .$$

Thus, if V is irreducible, [V,S] = L = V.

In the case where R = V is not necessarily irreducible, since S is semisimple Weyl's theorem enables us to write $V = W_1 \oplus \cdots \oplus W_k$, where each W_i is an irreducible submodule of V. We have shown above that for each i, L_i = linear span $\{s\cdot w_i - w_i\}_{\substack{s\in S\\ w_i\in W_i}}$ is a submodule, respectively, of W_i, with $L_i \neq (0)$ if $W_i \neq (0)$ since S acts by unbounded orbits on all nonzero elements of W_i. As above, $L_i = W_i$, and $[W_i,S] = W_i$, for $i = 1,2,\ldots,k$. Hence

$$(3)\quad [V,S] = [W_1 \oplus \cdots \oplus W_k,S] = \sum_{i=1}^{k} [W_i,S] = \sum_{i=1}^{k} W_i = V.$$

We know [S,S] = S because S is semisimple. From this fact and (3), we see that [G,G] contains both V and S, and so must contain V·S, the group generated by V and S. Since V·S = G, by (1), we conclude that [G,G] contains G and so G = [G,G].

In the general case, (ii) of (2.2) implies that

$R/\overline{[R,R]}$ is a vector group. By (2.1), $G/\overline{[R,R]}$ is map and so by the special case proven above for a map group whose radical is a vector group, $G/\overline{[R,R]} = [G/\overline{[R,R]}, G/\overline{[R,R]}]$. This implies $\pi([G,G]) = [\pi(G),\pi(G)] = [G/\overline{[R,R]}, G/\overline{[R,R]}] = G/\overline{[R,R]}$. Therefore,

(4) $\ker\pi \cdot [G,G] = G$.

But $\ker \pi = \overline{[R,R]}$ and $\overline{[R,R]} = [R,R]$ since R is simply connected [11; Theorem 1.2, p. 135]. Thus $\ker \pi = [R,R] \subseteq [G,G]$ and (4) becomes $[G,G] = G$. This completes the proof of (2.3) (a).

To prove (2.3) (b) we note that if G is simply connected, then R is simply connected and so $G = [G,G]$ by (a). We saw in the proof of (2.2) that $R/\overline{[R,R]}$ is simply connected. If R is abelian, this says R is simply connected and so $G = [G,G]$ follows again by (a).

Corollary 2.4. If G is a connected map Lie group with simply connected radical R, then R is nilpotent.

Proof. By (2.3), $G = [G,G]$, and therefore $R = R \cap G = R \cap [G,G]$. And, $R \cap [G,G]$ is nilpotent because it is a connected Lie group and its Lie algebra, $\mathfrak{s} = \mathfrak{r} \cap [\mathfrak{g},\mathfrak{g}]$ is nilpotent [11; Theorem 3.2, p. 128]. Hence, R is nilpotent.

We close with an example. We will construct a connected map Lie group G where $[G,G] \neq G$. This will show that we may not drop the assumption that the radical of G is simply connected in (2.3).

Example 2.5. Let $H = \{(x,y,z,t): x,y,z,t \in R\}$ with the manifold structure of R^4 and with its multiplication defined by

$$(1) \quad (x_1,y_1,z_1,t_1)(x_2,y_2,z_2,t_2) = (x_1+x_2+z_1t_2,\ y_1+y_2+\alpha z_1t_2,\ z_1+z_2,\ t_1+t_2) ,$$

where α is a fixed irrational real number. Let D be the discrete central subgroup of H consisting of the elements $(p,q,0,0)$ with arbitrary integers p and q. Form H/D. We observe that H/D is not simply connected [11; Ex. 2, p. 140]. Let $\ell = D(x_1,y_1,z_1,t_1)$ and $m = D(x_2,y_2,z_2,t_2)$ be elements of H/D. We have $\ell^{-1} = D(-x_1+z_1t_1,-y_1+\alpha z_1t_1,-z_1,-t_1)$ and $m^{-1} = D(-x_2+z_2t_2,-y_2+\alpha z_2t_2,-z_2,-t_2)$. Thus, using (1),

$$\begin{aligned}[\ell,m] &= \ell m\ell^{-1}m^{-1} \\ &= D(x_1,y_1,z_1,t_1)D(x_2,y_2,z_2,t_2)D(-x_1+z_1t_1,-y_1+\alpha z_1t_1-z_1-t_1) \\ &\qquad D(-x_2+z_2t_2,-y_2+\alpha z_2t_2,-z_2,-t_2) \\ &= D(x_1+x_2+z_1t_2,y_1+y_2+\alpha z_1t_2,z_1+z_2,t_1+t_2) \\ &\qquad D(-x_1-x_2+z_1t_1+z_2t_2+z_1t_2,-y_1-y_2+\alpha(z_1t_1+z_2t_2+z_1t_2), \\ &\qquad\qquad -z_1-z_2,-t_1-t_2) \\ &= D(z_1t_2+z_1t_1+z_1t_2+(z_1+z_2)(-t_1-t_2), \\ &\qquad \alpha[z_1t_2+z_1t_1+z_1t_2+(z_1+z_2)(-t_1-t_2)],0,0)\end{aligned}$$

Simplifying we get

$$(2) \quad [\ell,m] = D(z_1t_2 - z_2t_1,\ \alpha(z_1t_2 - z_2t_1),0,0) .$$

Then, $[H/D, H/D]$ = group generated $\{[\ell,m]: \ell,m \in H/D\}$, and (2) implies since α is irrational that

(3) $\overline{[H/D,\ H/D]} = \{D(x,y,0,0): x,y \in R\}$.

Consider the action of $Sl(2,R)$ on H defined for $s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in Sl(2,R)$, and $(x,y,z,t) \in H$ by

(4) $s\cdot(x,y,z,t) = (x,y,s_{11}z+s_{12}t,s_{21}z+s_{22}t)$.

We have

(5) $s\cdot(p,q,0,0) = (p,q,0,0)$ for all $p,q \in R$, $s \in Sl(2,R)$.

In particular, (5) holds for all $p,q \in Z$, $s \in Sl(2,R)$. Thus, $Sl(2,R)$ acts trivially on D and we have an induced action of $Sl(2,R)$ on H/D defined by

(6) $s.Dh = D(s\cdot h)$, for $s \in Sl(2,R)$, $Dh \in H/D$,

where $s\cdot h$ is the action of $Sl(2,R)$ on H defined by (4). Using (6) define $G = H/D \cdot Sl(2,R)$.

Claim. (i) G is a connected map Lie group

(ii) $[G,G] \neq G$.

<u>Proof</u>. (i) It is clear that G is a connected Lie group. The radical of G is H/D and $G/H/D$ has no compact factors. since $Sl(2,R)$ has none. Since $\overline{[H/D,\ H/D]}$ is a characteristic subgroup of G we can consider the induced action of $Sl(2,R)$ on $H/D\big/\overline{[H/D,H/D]}$. For $a = \overline{[H/D,\ H/D]}D(x,y,z,t) \in H/D\big/\overline{[H/D,\ H/D]}$ and $s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in Sl(2,R)$,

(7) $s\cdot a = \overline{[H/D,H/D]}D(x,y,s_{11}z+s_{12}t,s_{21}z+s_{22}t)$

Observe that the usual action of $Sl(2,R)$ on R^2,

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}\begin{pmatrix} z \\ t \end{pmatrix} = (s_{11}z+s_{12}t,s_{21}z+s_{22}t)$$

is such that every nonzero element $\begin{pmatrix} z \\ t \end{pmatrix} \varepsilon R^2$ has an infinite, and hence unbounded $Sl(2,R)$-orbit. Then, from (7), the element a above with z and t not both zero has unbounded $Sl(2,R)$-orbit. Such elements a are the nonzero elements of $H/D\big/\overline{[H/D,H/D]}$ (see (3)), and so we have shown that $Sl(2,R)$ acts by unbounded orbits on every nonzero element of $H/D\big/\overline{[H/D,H/D]}$. Hence, G is map ((2.2) (iii)).

(ii). We calculate $\overline{[G,G]}$. Let $a = D((x_1,y_1,z_1,t_1),s)$ and $b = D((x_2,y_2,z_2,t_2),p)$ be elements of G where $s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ and $p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$. We have $a^{-1} = D((-x_1-z_1[s_{21}(s_{12}t_1-s_{22}z_1)+$
$+s_{22}(s_{21}z_1-s_{11}t_1)],-y_1-\alpha\{z_1[s_{21}(s_{12}t_1-s_{22}z_1)+s_{22}(s_{21}z_1-$
$-s_{11}t_1)]\},s_{12}t_1-s_{22}z_1,s_{21}z_1-s_{11}t_1),s^{-1})$, and
$b^{-1} = D((-x_2-z_2[p_{21}(p_{12}t_2-p_{22}z_2)+p_{22}(p_{21}z_2-p_{11}t_2)],$
$-y_2-\alpha\{z_2[p_{21}(p_{12}t_2-p_{22}z_2)+p_{22}(p_{21}z_2-p_{11}t_2)]\},p_{12}t_2-$
$p_{22}z_2,p_{21}z_2-p_{11}t_2,p^{-1})$. We have

(8) $ab = D((x_1+x_2+z_1c,\ y_1+y_2+\ \alpha z_1c,*,**),sp)$

and

$$(9) \quad a^{-1}b^{-1} = D((-x_1-x_2-d,-y_1-y_2-\alpha d,*',(**)'),s^{-1}p^{-1})$$

and $[a,b] = aba^{-1}b^{-1} = D((x_1+x_2+z_1c,y_1+y_2+\alpha z_1c,*,**),sp)$
$D((-x_1-x_2-d,-y_1-y_2-\alpha d,*',(**)'),s^{-1}p^{-1})$ where
$c = s_{21}z_2+s_{22}t_2$, and $d = z_1[s_{21}(s_{12}t_1-s_{22}z_1)+s_{22}(s_{21}z_1-s_{11}t_1)] +$
$+ z_2[p_{21}(p_{12}t_2 - p_{22}z_2) + p_{22}(p_{21}z_2 - p_{11}t_2)] +$
$+ [s_{12}t_1 - s_{22}z_1][s_{21}(p_{12}t_2 - p_{22}z_2) + s_{22}(p_{21}z_2 - p_{11}t_2)]$.
Continuing, $[a,b] = D((z_1c - d + [*](s_{21}(*') + s_{22}(**)'),$
$\alpha(z_1c-d+ [*](s_{21}(*')+s_{22}(**)'),*'',(**)''),sps^{-1}p^{-1})$.
Letting $\varepsilon = z_1c-d+[*]\ (s_{21}(*')+s_{22}(**)')$, we get

$$(10) \quad [a,b] = D((\varepsilon,\alpha\varepsilon,*'',(**)''),sps^{-1}p^{-1}) \ .$$

From (8), we see that any product of elements $[a,b]$ of the form in (10) will still have that form. Thus $[G,G]$ = group generated $\{[a,b]: a,b \in G\}$ cannot be equal to all of G. This completes the proof of the claim.

§3. Automorphisms of Bounded Displacement and Homogeneous Spaces of Finite Volume for Minimally Almost Periodic Groups

Recall that an automorphism, α, of G, has bounded displacement if for all $g \in G$, $\alpha(g)g^{-1}$ lies in some fixed compact set and that $\mathrm{disp}_H\alpha = \{\alpha(h)h^{-1}\}_{h\in H}$ is the H-displacement of α. $B(G)$ consists of those $g \in G$ whose conjugacy class has compact closure. Clearly, an inner automorphism

α_g of G is of bounded displacement if and only if $g \in B(G)$. In this section we prove

<u>Theorem 3.1</u>. A connected map Lie group G has no nontrivial automorphisms of bounded displacement.

<u>Proof</u>. We first prove the theorem for inner automorphisms of bounded displacement, i.e., we show $B(G) = Z(G)$. It is clear that $Z(G) \subseteq B(G)$. Let G be a subgroup of $Gl(V)$. We define the continuous homomorphism $\rho: G \to Gl(\text{End } V)$ by

$$g \to \rho_g, \text{ where } \rho_g(T) = g \cdot T = gTg^{-1};\ g \in G,\ T \in \text{End } V.$$

For each $T \in \text{End } V$, we define the continuous 1-cocycle.

$$\phi_T: G \to \text{End } V \text{ by } \phi_T(g) = T - g \cdot T\ .$$

If $T \in B(G) \subseteq \text{End } V$, then ϕ_T is bounded, and since G is map, $\phi_T \equiv 0$ [18; Prop. 1.6]. That is, $T - g \cdot T = 0$ for every $g \in G$. Hence, $gTg^{-1} = T$ for every $g \in G$, and so $T \in Z(G)$. Thus, $B(G) \subseteq Z(G)$, and the two inclusions imply $B(G) = Z(G)$.

For the general case, let $g_0 \in B(G)$. Then the closure of the G-orbit of g_0 under conjugation, $\overline{\mathcal{O}_G(g_0)}$ is compact and so $\text{Ad}(\overline{\mathcal{O}_G(g_0)})$ is compact. Therefore

$$\begin{aligned} \text{Ad}(\overline{\mathcal{O}_G(g_0)}) \supseteq \text{Ad}(\mathcal{O}_G(g_0)) &= \text{Ad}\{gg_0g^{-1}\}_{g \in G} \\ &= \{(\text{Ad } g)(\text{Ad } g_0(\text{Ad } g)^{-1}\}_{g \in G} \\ &= \mathcal{O}_{\text{Ad } G}(\text{Ad } g_0)\ . \end{aligned}$$

Then, $\overline{\mathcal{O}_{\mathrm{Ad}\ G}(\mathrm{Ad}\ g_0)}$ is a closed subset of a compact group and so compact. Hence, Ad $g_0 \in B(\mathrm{Ad}\ G)$. Since Ad G is a connected map linear group we know from the case proven above that $B(\mathrm{Ad}\ G) = Z(\mathrm{Ad}\ G)$. Then, Ad $g_0 \in B(\mathrm{Ad}\ G)$ implies that Ad $g_0 \in Z(\mathrm{Ad}\ G)$. Thus, $\mathrm{Ad}g_0\mathrm{Ad}g = \mathrm{Ad}g\mathrm{Ad}g_0$ for all $g \in G$, and so $\mathrm{Ad}[g_0,g] = 1$ for all $g \in G$. This says $[g_0,g] \in \ker \mathrm{Ad}$, and since G is connected, $\ker \mathrm{Ad} = Z(G)$, and $[g_0,g] \in Z(G)$. Consider the map from G to Z(G) defined for each $g \in G$ by $g \to [g_0,g] = \alpha_{g_0}(g)g^{-1}$. This map is a continuous homomorphism of G into the abelian group Z(G), and because G is map must be trivial. We conclude from this that $[g_0,g] = 1$ for all $g \in G$ and hence $g_0 \in Z(G)$. Thus, $B(G) \subseteq Z(G)$ and so $B(G) = Z(G)$.

To complete the proof of the theorem we will need the following result of Tits [24; Theorem 3, p. 99].

If G is a connected Lie group having no nondiscrete normal semisimple compact subgroups, T' the largest compact connected subgroup of Z(G), then given any automorphism of bounded displacement, α, of G, there exists an element $g \in B(G)$ and a homomorphism without fixed points, except 1, $\phi: G \to T'$ such that α is defined by the relation $\alpha(x) = gxg^{-1}\phi(x^{-1})$.

Continuing with the proof of the theorem, suppose N is a normal semisimple compact connected subgroup of G. Then $\pi(N)$ is a semisimple compact normal subgroup of $G/\overline{[R,R]}$. There is a conjugate of $\pi(N)$ by an element of g, denoted $\pi(N)^g$, such that $\pi(N)^g \subseteq \pi(S)$, the Levi factor of $G/\overline{[R,R]}$

(see 2.2 (ii)). Since $\pi(S)$ has no compact factors the group $\pi(N)^{g}$ is trivial. Therefore, also $\pi(N) = (1)$, and $N \subseteq \ker \pi = \overline{[R,R]}$. But $\overline{[R,R]}$ is solvable and N is semisimple, connected, and normal, and hence $N = (1)$ and we can apply [24].

If α is an automorphism of bounded displacement of G, there exists an element $g \in B(G)$ and a homomorphism $\phi: G \to T'$ without fixed points, except 1, such that $\alpha(x) = gxg^{-1}\phi(x^{-1})$. Since $B(G) = Z(G)$, $g \in Z(G)$ and $\alpha(x) = x\phi(x^{-1})$. Since the map group G has no nontrivial homomorphisms into a compact group, ϕ, and therefore α, is trivial. Hence,

<u>Corollary 3.2</u>. Let G be a map connected Lie group and H a closed subgroup of G such that G/H has finite volume. If $\alpha \in \mathrm{Aut}\,(G)$ leaves H pointwise fixed then α is trivial.

<u>Proof</u>. By [18; Theorem 3.1] α has bounded displacement. Then, by (3.1) α is trivial.

In particular,

<u>Corollary 3.3</u> (Furstenberg [5; Cor. 3, p. 211]). If G and H are as in (3.2), then $Z_G(H) = Z(G)$.

Finally we have the following generalization of (3.2).

<u>Theorem 3.4</u>. Let G be a map connected Lie group and H a closed subgroup of G such that G/H has finite volume. If for $\alpha \in \mathrm{Aut}\,(G)$ the $\mathrm{disp}_H\, \alpha = \{\alpha(h)h^{-1}\}_{h \in H}$ is bounded, then α is trivial.

Proof. Since Z(G) is a characteristic subgroup of G, $\alpha(Z(G)) \subseteq Z(G)$ and so α induces an automorphism, $\tilde{\alpha}$, of $G/Z(G) \simeq Ad(G)$. Consider the diagram below

$$\begin{array}{ccc} G & \xrightarrow{\ \pi\ } & Ad\ (G) \\ \uparrow\!\cup & & \uparrow\!\cup \\ H & \xrightarrow[\ \pi|_H\ ]{} & \pi(H) \end{array} \quad , \ \pi: G \longrightarrow G/Z(G) \simeq Ad(G) \ .$$

Since G/H has finite volume, Ad $(G)/\overline{\pi(H)}$ has finite volume [8; Cor. 5, p. 227] and the $disp_H\ \alpha$ being bounded implies that $disp_{\pi(H)}\ \tilde{\alpha}$ is bounded, and so by continuity the $disp_{\overline{\pi(H)}}\ \tilde{\alpha}$ is bounded. Since Ad(G) is a linear map group, $\tilde{\alpha}$ must have bounded displacement [7]. Then, by (3.1), $\tilde{\alpha}$ is trivial. This means that for every $g \in G$, $\alpha(g)g^{-1}$ is in Z(G). Define the continuous map $\phi: G \to Z(G)$ by $\phi(g) = \alpha(g)g^{-1}$, $g \in G$. Since ϕ takes values in Z(G) it is a homomorphism. But G has no continuous homomorphisms into an abelian group. Therefore, α is trivial.

§4. Extensions of Results on Minimally Almost Periodic Groups to Locally Compact Connected Groups and Some Results on Maximally Almost Periodic Groups

Given a locally compact connected group G, then G can be written in the form G = R·KS where R is the radical of G, i.e., the largest (closed) normal connected solvable subgroup of G. (See Iwasawa for the existence of R [13;

Theorem 1.5, p. 552]), K is a compact connected subgroup with trivial radical and S is a semisimple Lie subgroup without compact factors. We have $[K,S] = (1)$ and $K \cap S$ is a finite group (see Matsushima for this decomposition [17; Theorem 1, p. 266]. As in the case of Lie groups, we let $\overline{[R,R]}$ denote the closure of the derived of R, and since $\overline{[R,R]}$ is a characteristic subgroup of G we again form the projection $\pi: G \to G/\overline{[R,R]}$. The connected abelian radical, $R/\overline{[R,R]}$, of $G/\overline{[R,R]}$, can be written as $V \times L$, where V is a vector group and L is a compact group, the L being characteristic in $G/\overline{[R,R]}$ [14]. Since L is a compact abelian group, the automorphism group of L, Aut (L), is totally disconnected [13; Cor., p. 514]. Arguing now exactly as in the introduction of §1, we see that L is actually fixed under the induced action of $\pi(KS) \subseteq G/\overline{[R,R]}$.

<u>Lemma 4.1</u>. If $G = R.KS$ is the decomposition above and $K = (1)$ (or $S = (1)$), then we can replace V by a vector group of the same dimension, also to be denoted as V, which is either $\pi(S)$-stable (or $\pi(K)$-stable).

<u>Proof</u>. If $K = (1)$, then $\pi(S)$ is a semisimple Lie group and since L is $\pi(S)$-stable, an application of Weyl's Theorem completes the argument. If $S = (1)$, then $\pi(K)$ is a compact subgroup of Aut $(V \times L)$ leaving L fixed. The conclusion follows from [10; Theorem 1.1, p. 4].

<u>Definition 4.2</u>. Let G be a locally compact connected group. It is well known that $G = \varprojlim G_\alpha$ (projective limit) of con-

nected Lie groups G_α. If every G_α is semisimple we call G_α a <u>prosemisimple group</u>.

It has been shown in [19; Cor. 2.5, p. 404] that if $G = \varprojlim G_\alpha$ and is prosemisimple, then G_α is semisimple.

For $G = \varprojlim G_\alpha$, $\pi_\alpha: G \to G_\alpha$ will be the corresponding projections, and $G_\alpha = R_\alpha \cdot K_\alpha S_\alpha$ will be the Levi decomposition of G_α.

<u>Definition 4.3</u>. If $G = R\cdot KS$ is the decomposition above and if $K = (1)$ (or $S = (1)$), we let V_f denote the set of all elements of V which either have finite $\pi(S)$-orbit (or finite $\pi(K)$-orbit). As in §1, V_f is a normal subgroup of $G/\overline{[R,R]}$, and the connectedness of $\pi(K)$ (or $\pi(S)$), implies that each element of V_f is actually $\pi(K)$-fixed (or $\pi(S)$-fixed). If $K = (1)$, $\pi(S)$ is a semisimple Lie group and we may apply Weyl's Theorem to obtain a $\pi(S)$-stable complement to V_f, written as $V_f^\perp$, also as in §1, that is normal in $G/\overline{[R,R]}$. If $S = (1)$, let $\pi(K) = \varprojlim K_\alpha$. Since the radical of $\pi(K)$ is (1), we conclude by [19; Theorem 2.11, p. 405] that $\pi(K)$ is prosemisimple and therefore each connected Lie group H_α is semisimple. Let $\rho: \pi(K) \to Gl(V)$ be the continuous homomorphism induced by the action of $\pi(K)$ on $R/\overline{[R,R]}$ restricted to V. (By (4.2), V is $\pi(K)$-stable). By [19, Lemma 2.2, p. 403] there exists an α and a compact subgroup C_α such that $K_\alpha = \pi(K)/C_\alpha$ and $\rho(C_\alpha) = (1)$, i.e., ρ induces a continuous homomorphism $\tilde{\rho}$ of K_α such that the following diagram commutes.

$$\begin{array}{ccc} \pi(K) & \xrightarrow{\rho} & GL(V) \\ \downarrow \pi_\alpha & \nearrow \tilde{\rho} & \\ \pi(K)/C_\alpha = K_\alpha & & \end{array}$$

Thus, we may consider ρ as a homomorphism from the semisimple Lie group K_α into $Gl(V)$. (Clearly, $\tilde{\rho}$ must be unitary since ρ is unitary). Since V_f is $\pi(K)$-stable, it is therefore K_α-stable and now Weyl's Theorem enables us to find a K_α-stable, and therefore $\pi(K)$-stable, complement written as $V_f^\perp$, which as before is normal in $G/\overline{[R,R]}$.

For the remainder of this section any reference to V_f or $V_f^\perp$ will, of course, only be in the case when either $K = (1)$ (or $S = (1)$), i.e., $G = R \cdot S$ (or $G = R \cdot K$).

<u>Lemma 4.4</u>. If $G = \varprojlim G_\alpha$ is a locally compact connected group, then G is map if and only if each G_α is map.

<u>Proof</u>. Suppose first that each G_α is map and let ρ be an irreducible representation of G. Again by [19, Lemma 2.2, p. 403] for some α, ρ induces a representation $\tilde{\rho}$, of G_α, such that $\tilde{\rho} \circ \pi_\alpha = \rho$. Since G_α is map, $\tilde{\rho}$ is trivial and hence ρ is trivial. Therefore, G is map.

Conversely, assume that G is map and let $\tilde{\phi}$ be a representation of G_α, for arbitrary α. Then, $\tilde{\phi} \circ \pi_\alpha$ is a representation of G, and since G is map, $\tilde{\phi} \circ \pi_\alpha$ is trivial. Thus $\tilde{\phi}(\pi_\alpha(G)) = \tilde{\phi}(G_\alpha) = (1)$ and therefore $\tilde{\phi}$ is trivial. It follows that G_α is map, and since α was arbitrary, that every G_α is map.

<u>Lemma 4.5</u>. If G is a locally compact connected group, then G is map if and only if $G/\overline{[R,R]}$ is map.

<u>Proof</u>. This is proven exactly as in the Lie group case by first showing $\mathcal{N}(G) = \pi^{-1}(\mathcal{N}(G/\overline{[R,R]}))$, as in the proof of (1.1), and then applying the argument of (2.1).

The next theorem generalizes (2.2) to locally compact connected groups.

<u>Theorem 4.5</u>. Let $G = \varprojlim G_\alpha$ be a locally compact connected group. Then the following are equivalent:

(i) G is map

(ii) $G/\overline{[R,R]}$ can be written in the form $G/\overline{[R,R]} =$ $= V\cdot\pi(S)$ with all nonzero elements of V of unbounded $\pi(S)$-orbit.

(iii) G/R is a semisimple Lie group without compact factors and every nonzero element of $R/\overline{[R,R]}$ is of unbounded $\pi(S)$-orbit.

(iv) G/R is a semisimple Lie group without compact factors and $\overline{[R,R]} = \{r \in R: [G,r] \subseteq \overline{[R,R]}\}$.

(v) G/R is a semisimple Lie group without compact factors and $G = \overline{[G,G]}$.

<u>Proof</u>. Let $G/R = \varprojlim H_\alpha$ with corresponding projections $\pi_\alpha: G/R \to H_\alpha$. (i $\iff$ ii). Suppose G is map. Then, G/R is map and therefore each H_α is also map (4.3), Thus, every connected Lie group H_α is map and semisimple, and therefore has no compact factors (1.7(c)). Let L be a compact

connected normal subgroup of G/R. For each α, $\pi_\alpha(L)$ is a compact connected normal subgroup of H_α and so $\pi_\alpha(L) = (1)$, and we have $L \subseteq \bigcap_\alpha \ker \pi_\alpha = (1)$. Hence, $L = (1)$ and so G/R has no nontrivial compact connected normal subgroups. Therefore, $K = (1)$, and $G = R\cdot S$. This implies that G/R is a semisimple Lie group without compact factors. The group $G/\overline{[R,R]}$ is map since G is map, and we can write its connected abelian radical $R/\overline{[R,R]}$ as $R/\overline{[R,R]} = V \times L$. We have seen that the compact connected group L is fixed under the action of $\pi(KS)$. Since L is also abelian and commutes with V it must be contained in $Z(G/\overline{[R,R]})_0$, the identity component of the center of $G/\overline{[R,R]}$. We see that $G/\overline{[R,R]} = \varprojlim G_\alpha/\overline{[R_\alpha,R_\alpha]}$ (ϕ_α: $G/\overline{[R,R]} \to G_\alpha/\overline{[R_\alpha,R_\alpha]}$ are the corresponding projections), and each $G_\alpha/\overline{[R_\alpha,R_\alpha]}$ is a connected map Lie group (4.1), with abelian radical $R_\alpha/\overline{[R_\alpha,R_\alpha]}$. Hence, by (1.2), for each α,

$$(1) \quad G_\alpha/\overline{[R_\alpha,R_\alpha]} = \eta(G/\overline{[R_\alpha,R_\alpha]}) = (V_\alpha)_f^{\perp}\tilde{\pi}_\alpha(S_\alpha) ,$$

where $G_\alpha/\overline{[R_\alpha,R_\alpha]} = R_\alpha/\overline{[R_\alpha,R_\alpha]} \cdot \tilde{\pi}_\alpha(S_\alpha)$ is the Levi decomposition of $G_\alpha/\overline{[R_\alpha,R_\alpha]}$ ($\tilde{\pi}_\alpha$: $G \to G_\alpha/\overline{[R_\alpha,R_\alpha]}$ is the canonical projection), and $R_\alpha/\overline{[R_\alpha,R_\alpha]} = V_\alpha \times T_\alpha$, V_α a vector group, and T_α a toroidal group. From (1), $(V_\alpha)_f = (1)$ and since $Z(G_\alpha/\overline{[R_\alpha,R_\alpha]})_0 \subseteq (V_\alpha)_f = (1)$, we have $Z(G_\alpha/\overline{[R_\alpha,R_\alpha]})_0 = (1)$ for all α. Therefore, since $\phi_\alpha(Z(G/\overline{[R,R]})_0) \subseteq Z(G_\alpha/\overline{[R_\alpha,R_\alpha]})_0 = (1)$, for each α, $Z(G/\overline{[R,R]})_0 \subseteq \bigcap_\alpha \ker \phi_\alpha = (1)$. This shows that $L = (1)$, and so $R/\overline{[R,R]} = V$ is a vector group.

Thus, $G/\overline{[R,R]} = V\cdot\pi(S)$ is a Lie group and it is map since G is map. We apply the argument of (2.2) (i$\Longleftrightarrow$ii) to complete the proof of (ii).

Conversely, assume (ii) holds. We see then that $G/\overline{[R,R]}$ is a Lie group and by (2.2) (i $\Longleftrightarrow$ ii), it is map. By (4.4), G is also map.

(i$\Longleftrightarrow$iii). Assume that(i) holds. By (ii) above, G/R is a semisimple Lie group without compact factors. The remainder of (iii) is shown as in (2.2) (i$\Longrightarrow$iii). Conversely, if (iii) holds, then K = (1), and L = (1) since L is $\pi(KS)$-fixed. Therefore, $G/\overline{[R,R]} = V\cdot\pi(S)$ with all nonzero elements of V of unbounded $\pi(S)$-orbit. By (ii), G is map.

(i $\Longleftrightarrow$iv). Assume that G is map. From (iii), G/R is a semisimple Lie group without compact factors. Exactly as in (2.2) (i)$\Longleftrightarrow$(iv) we show that $\overline{[R,R]} = \{r \in R: [G,r] \subseteq \overline{[R,R]}\}$. Conversely, if (iv) holds, we apply the argument of (2.2) (iv $\Longrightarrow$ i).

(i$\Longleftrightarrow$v). If (i) holds, by (iv), G/R is a semisimple Lie group without compact factors. To see that $G = \overline{[G,G]}$, argue as in (2.2) (i $\Longrightarrow$ v). In the other direction, see (2.2) (v $\Longrightarrow$ i).

This completes the proof of (4.5).

As a consequence of our work in this section we can give an alternate, and much simpler proof of the following well known result.

Corollary 4.6 (Freudenthal-Weil [4; p. 129]). A locally compact connected group G is MAP if and only if G is the direct product of a vector group and a compact group.

We require the following lemma.

Lemma 4.7. If G is a locally compact connected group with decomposition $G = R\cdot KS$, then both S and $\overline{[R,R]}$ are contained in $\mathfrak{N}(G)$. If R is abelian, $S = (1)$ (or $K = (1)$) then $V_f^{\perp}$ is also contained in $\mathfrak{N}(G)$.

Proof. Let $G = \varprojlim G_\alpha$ with $\pi_\alpha\colon G \to G_\alpha$ the corresponding projections, and let ρ be an irreducible representation of G. Again from [19] there is an α for which ρ induces a representation $\tilde\rho$ of G_α, where $\tilde\rho \circ \pi_\alpha = \rho$. We have $\rho([R,R]) = \tilde\rho(\pi_\alpha([R,R])) \subseteq \tilde\rho([R_\alpha,R_\alpha]) = (1)$ (by (1.3)). Therefore, $\rho([R,R]) = (1)$, and by continuity $\rho(\overline{[R,R]}) = (1)$. Hence, $\overline{[R,R]} \subseteq \mathfrak{N}(G)$. The remainder of (4.7) is proven exactly as in (1.3). This completes the proof of (4.7).

Proof of Corollary 4.6. Let G be MAP with decomposition $G = R\cdot KS$. Here, $\mathfrak{N}(G) = (1)$, and by (4.7) both S and $\overline{[R,R]}$ are contained in $\mathfrak{N}(G)$, whence $\overline{[R,R]} = S = (1)$. Since $\overline{[R,R]} = (1)$, R is abelian, and $S = (1)$ from above, and so again by (4.7), $V_f^{\perp} \subseteq \mathfrak{N}(G)$. Thus, $V_f^{\perp} = (1)$, implying that $V = V_f$, and $G = (V \times L)\cdot K$. The subgroup LK is a compact normal subgroup of G, $LK \cap V = (1)$, and $[V,LK] = (1)$ since V and L commute and $V = V_f$. By the Iwasawa splitting theorem [13; Lemma 3.8, p. 521], $G = V \times LK = V \times K'$, where

$K' = LK$.. Thus, G is the direct product of the vector group V and the compact group K'.

Conversely, if $G = V \times M$, V a vector group, M a compact group, then $\mathfrak{N}(G) = \mathfrak{N}(V) \times \mathfrak{N}(M) = (1)$ since both V and M are MAP. Hence, $\mathfrak{N}(G) = (1)$ and G is MAP.

For the reader's information we now record a fact which gives a necessary condition for a discrete group G to be MAP, i.e., $\mathfrak{N}(G) = (1)$. It is essentially a reproduction of the method used by von Neumann and Wigner in [22; p. 748] to show that $Sl(2,Z)$ is MAP.

Proposition 4.8. If $G \subseteq Gl(n,Z)$ is a Lie group, then $G_Z = G \cap Gl(n,Z)$ is MAP.

In particular, $Gl(n,Z)$, $Sl(n,Z)$, and $Sp(2n,Z)$ are all MAP.

Proof. The second statement follows immediately from the first. To prove the first, define for each integer n

$$G_n = \{x \in G_Z : x \equiv I \pmod{n}\}, \text{ (see [1; p. 49])} .$$

Each G_n is a normal subgroup of G_Z with finite index. Moreover, $\bigcap_n G_n = (1)$, and since $\mathfrak{N}(G_Z) \subseteq G_1$, the intersection of all normal subgroups of G_Z with finite index [22; p. 745], we have $\mathfrak{N}(G_Z) \subseteq G_1 \subseteq \bigcap_n G_n = (1)$. Hence, G_Z is MAP.

BIBLIOGRAPHY

1. Borel, A. <u>Introduction aux Groupes Arithmétiques</u> (Hermann; Paris, 1969).

2. ____________. Density properties for certain subgroups of semisimple groups without compact components. Ann. of Math. 72 (1960), 179-188.

3. Clifford, A. H. Representations induced in an invariant subgroup. Ann. of Math. 38 (1937), 533-550.

4. Freudenthal, H. Topologische Gruppen mit genugend vielen fastperiodischen Funktionen. Ann. of Math. 37 (1936), 57-77.

5. Furstenberg, H. A note on Borel's density theorem. Proc. Amer. Math. Soc. 55, No. 1 (1976), 209-212.

6. Gotô, M. Faithful representations of Lie groups I. Mathematica Japonicae, 1, no. 3 (1948), 107-119.

7. Greenleaf, F. P. and Moskowitz, M. Orbits and homogeneous spaces of finite volume. Monatschefte fur Mathematik. (To appear.)

8. Greenleaf, F. P., Moskowitz, M. and Rothschild, L. P. Unbounded conjugacy classes in Lie groups and location of central measures. Acta Mathematica, 132 (1974), 225-243.

9. ____________. Automorphisms, orbits, and homogeneous spaces of nonconnected Lie groups. Math. Annalen, 212 (1974/75), 145-155.

10. Grosser, S. and Moskowitz, M. Compactness conditions in topological groups (I and II). Jour. fur Reine und Agnew. Math. 246 (1971), 1-40.

11. Hochschild, G. The Structure of Lie Groups (Holden Day; San Francisco, 1965).

12. Huang, T.-J. A characterization of maximally almost periodic groups. (To appear.)

13. Iwasawa, K. On some types of topological groups. Ann. of Math., 50 (1949), 507-558.

14. Kampen, E. R. van. Locally bicompact Abelian groups and their character groups. Ann. of Math., 36 (1935), 448-463.

15. Kuranishi, M. On non-connected maximally almost periodic groups. Tokohu Math. Jour., 2 (1950), 40-46.

16. Mackey, G. W. Induced Representations of Groups and Quantum Mechanics (W. A. Benjamin, Inc., and Editore Boringhieri; New York, Amsterdam, and Torino, 1968).

17. Matsushima, Y. On the decomposition of an (L)-group. Jour. Math. Soc. of Japan, 1 (1950), 264-274.

18. Moskowitz, M. Some remarks on automorphisms of bounded displacement and bounded cocycles. Arkiv för Matematik. (To appear.)

19. ____________. On proreductive groups. Proc. Camb. Phil. Soc. 76 (1974), 401-406.

20. Murakami, S. Remarks on the structure of maximally almost periodic groups. Osaka Math. Jour., 1 (1950), 119-129.

21. Neumann, J. von. Almost periodic functions in a group. Trans. Amer. Math. Soc. 36 (1934), 445-492.

22. Neumann, J. von and Wigner, E. P. Minimally almost periodic groups. Ann. of Math. 41, No. 4 (1940), 746-750.

23. Raghunathan, M. S. Discrete Subgroups of Lie Groups (Springer-Verlag; New York, Heidelberg, and Berlin, 1972).

24. Tits, J. Automorphismes à déplacement borné des groupes de Lie. Topology, v. 3 (Suppl. 1)(1964), 97-107.

25. Weil, A. L'integration dans les groupes topologiques et ses applications, deuxième édition (Hermann; Paris, 1965).