On the Arithmetic and Geometry of Quaternion Algebras: a spectral correspondence for Maass waveforms

by

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Abstract

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Let $\mathcal{A}$ be an indefinite rational division quaternion algebra with discriminant d equal to pq where p and q are primes such that $p, q > 2$ and let $\mathcal{O}_{pq}$ be a maximal order in $\mathcal{A}$. Further, let $\mathcal{O}_{pq,p^{2r}q^{2s}}, r, s \geq 1$ be an order of index $p^{2r}q^{2s}$ in $\mathcal{O}_{pq}$ with Eichler invariant equal to negative one at p and at q . Finally, let $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ be the cocompact Fuchsian group given as the group of units of norm one in $\mathcal{O}_{pq,p^{2r}q^{2s}}$. Using the classical Selberg trace formula, we show that the positive Laplace eigenvalues, including multiplicities, for Maaß forms on $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ coincide with the Laplace spectrum for Maaß newforms defined on the Hecke congruence group $\Gamma_0(M)$ where, M, the level of the congruence group, is equal to $p^{2r+1}q^{2s+1}$, i.e., the discriminant of $\mathcal{O}_{pq,p^{2r}q^{2s}}$.

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CAARMS[Conference for African American Researchers in the Mathematical Sciences] conference played a vital role in motivating and inspiring my determination to complete this project. In this regard I want to reiterate my thanks to Emma Previato whose tireless efforts has sustained the AFRAMATH endeavor for almost ten years. I also want to thank William Massey of Princeton University and Don King of Northeastern University for their efforts in guiding and sustaining the CAARMS conference and for their personal support and guidance of my efforts directed towards the completion of this project.

I have taught continuously at Medgar Evers College, in some capacity, since the fall of 1993. In some sense, I have grown up here and I worked to make mathematics meaningful in this context. Let me first say a sincere thank you to the Department of Mathematics at Medgar Evers College for giving me the opportunity to serve the Central Brooklyn community in this arena. In particular, I want to thank the many students and faculty members who have supported me personally and professionally and have affirmed the importance of my efforts in this arena within the context of this community and it's aspirations over many many years. In this regard I make special mention of Edison Jackson, Darius Movasseghi and Frank Ragland. Their examples of generosity, patience, perseverance and sustained commitment to community stand for me as a beacon to be followed.

Walterine Matthews was my first mathematics teacher. She lifted the veil on some of my early mathematical mysteries, encouraged my boyhood curiosity, and above all taught me a reverence for mathematics which I have carried with me to this day. For her efforts on my behalf I will always be grateful. I am deeply indebted to my parents Terrence and Joan Blackman, my sister Nichelle Davis, my brother Horace Blackman and the rest of my family, in particular my aunts, Jean, Jane, Thelma, Desiree, Grethel, and uncles Calvin and Wesley Phillips for their steadfast support of my long held dream to complete this project. Finally, I want to thank my wife Anna, my son Madiba, my daughter Zuri Alexandra and my former wife Marie, for their unfailing love, support and encouragement over the years it took me to complete this project. Daddy has finished. We begin anew.

New York, April 08, 2011, T.R.B.

... But I have promises to keep,

And miles to go before I sleep,

And miles to go before I sleep.

-Robert Frost

This is a promise kept to my, now, forever late, grandmothers Grathel Phillips and Elaine Hope.

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Introduction

0.1 General Remarks

The spectral theory of automorphic Laplacians is traditionally engaged either from the adelic-representation theoretic point of view or from the classical perspective of the upper half-plane[49]. If we adopt the classical view, seen through the lens of geometry, it is natural for us to start with a consideration of the spectral resolution of the Laplacian on $L^2(X)$ where X is a closed compact surface endowed with a Riemannian metric of constant negative curvature. From an arithmetic perspective, it is natural for us to begin with the example of the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its automorphic Laplacian. In this instance, the corresponding modular surface is non-compact and the spectral resolution has an absolutely continuous part in addition to the discrete one that characterizes the compact case. In some circumstances it is possible to relate the spectral resolutions of the automorphic Laplacians in the compact and non-compact cases. To do so it is required that the compact surface come from a cocompact arithmetic Fuchsian group and in so doing we realize a spectral correspondence between spaces of automorphic forms for cocompact and non-cocompact Fuchsian groups which preserves Laplace eigenvalues[6].

These correspondences are well-known. They are covered by the Jacquet-Langlands correspondence[24], which establishes, among other things, that to any nonconstant eigenfunction of the Laplacian on a cocompact Fuchsian group there corresponds a nontrivial cuspform with the same eigenvalue on some non-cocompact Fuchsian group[19]. This fact was first discovered, independently, by M. Eichler and A. Selberg in the 1950's. It was first proved, using the language of representation theory, by Jacquet and Langlands [24] in 1970 and reproved by Hideo Shimizu[42] in 1972 using the language of adelic trace formulas.

Within the context of Maaß waveforms(square-integrable eigenfunctions of the Laplace-Beltrami operator on certain Riemann surfaces with constant negative curvature and finite area) and trace formulas it is desirable to formulate spectral corespondences in classical language so as to make them more explicit. In 1983, Hejhal[19] illustrated how a part of this correspondence could be established using completely classical techniques. He showed that a certain integral transform, Θ, mapped Maaß waveforms on a Fuchsian group derived from a quaternion algebra to Maaß forms of equal eigenvalue on a related congruence subgroup $\Gamma_0(d)$. This approach to establishing a correspondence between spaces of automorphic forms for cocompact and non-cocompact Fuchsian groups had its roots in unpublished work of A. Selberg from the 1950's[40]. Hejhal's work in this direction was extended by Bolte and Johansson in 1996[6] and in 1999 [7]. In [6] they worked out the details of a classical construction of the spectral correspondence when the cocompact arithmetic Fuchsian group is given by a unit group in an order in an arbitrary indefinite rational quaternion algebra in so doing they showed that Hejhal's constructions could be extended to arbitrary orders. They also improved the result concerning the level of the congruence group by illustrating the natural correspondence between the discriminant of the order and the level of the congruence group. To be precise, let $\mathcal{O}$ be an order in an indefinite rational quaternion algebra so that its group $\mathcal{O}^1$ of units of norm one can be considered as a cocompact Fuchsian group. Bolte and Johansson showed that Maaß waveforms for $\mathcal{O}^1$, i.e. eigenfunctions of the automorphic Laplacian associated with $\mathcal{O}^1$, can be lifted to Maaß cusp forms for the Hecke congruence group $\Gamma_0(d)$, where d is the (reduced) discriminant of the order $\mathcal{O}$ [6] thus establishing that theta-lifts preserve eigenvalues of the hyperbolic Laplacian. Left unaddressed however, was the question as to whether or not theta-lifts provided isomorphisms between Laplace eigenspaces in $L^2(\mathcal{O}^1\backslash\mathcal{H})$ and $L^2(\Gamma_0(d)\backslash\mathcal{H})$. Bolte and Johansson, in[7], continued to address this question by concentrating on maximal orders $\mathcal{O}$ in indefinite rational division quaternion algebras. By exploiting several versions of the (classical) Selberg trace formula [39] they showed that the Laplace eigenvalues and their multiplicities for the cocompact group $\mathcal{O}^1$ and those for the newforms of level d coincide. This, however, still did not imply that theta-lifts provide isomorphisms between Laplace eigenspaces in $L^2(\mathcal{O}^1\backslash\mathcal{H})$ and $L^2(\Gamma_0(d)\backslash\mathcal{H})$. Strömbergsson[47], in his doctoral thesis, studied this question independently and proved that Θ was indeed a bijection between the respective eigenspaces.

It should be emphasized that in the work referenced above, explicit correspondences are established between cocompact groups, $\mathcal{O}^1$, where $\mathcal{O}^1$ is a unit group in a maximal order $\mathcal{O}$ in an indefinite rational quaternion algebra $\mathcal{A}$ with reduced discriminant d and Hecke congruence groups, $\Gamma_0(d)$. We note that in this case d is necessarily the product of an even number of different primes, and that any such number may be realized in this way.

0.2 Risager's Problem

We denote by Δ_Γ the automorphic Laplacian related to Γ and by $N_\Gamma(\lambda)$ the corresponding spectral counting function which, we recall, is defined as follows:

$$N_\Gamma(\lambda) = \#\{\lambda_n \leq \lambda : \lambda_n \in dSpec(\Delta_\Gamma)\},$$

where $dSpec(\Delta_\Gamma)$ denotes the discrete spectrum of Δ_Γ. Since $\mathcal{O}^1$ is cocompact $N_{\mathcal{O}^1}(\lambda)$ has an asymptotic expansion of the form:

$$N_{\mathcal{O}^1}(\lambda) = \frac{Vol(\mathcal{O}^1\backslash\mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda}).$$

For congruence subgroups $\Gamma_0(d)$, $N_{\Gamma_0(d)}(\lambda)$ has an asymptotic expansion of the form:

$$N_{\Gamma_0(d)}(\lambda) = \frac{Vol(\Gamma_0(d)\backslash\mathcal{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log\lambda).$$

Risager notes the difference in the error terms between the cocompact and noncocompact case and defines a counting function, $N^{new}_{\Gamma_0(d)}(\lambda)$, which counts only the newforms when d is the product of an even number of different primes. He finds that

$$N^{new}_{\Gamma_0(d)}(\lambda) = C_d \frac{Vol(\Gamma_0(d)\backslash\mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda})$$

for some constant $0 < C_d \leq 1$. I.e., the asymptotic expansion characteristic of the cocompact case! How was this to be explained? One explains this generally by the Jacquet-Langlands correspondence[24] and explicitly by existence of an order $\mathcal{O}$ with discriminant $d_{\mathcal{O}} = d$ such that there is a correspondence between the λ-eigenspace of $\Delta_{\mathcal{O}^1}$ and the λ-eigenspace of $\Delta_{\Gamma_0(d)}$. Risager then goes a bit further. He defines counting functions such as this one to be of cocompact type, i.e., we will say that if $N^{new}_{\Gamma_0(M)}(\lambda)$ is of form $C_M\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda})$ then $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type, and asks: Are there values of M not equal to the product of an even number of different primes for which $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type? In [36] he answers this question and characterises those M's for which $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type by the theorem below.

Theorem 0.2.1 (Risager)**.** *Let $M \in \mathbb{N}$ and let $n, t \in \mathbb{N}$ be the integers defined uniquely by the requirements that n should be squarefree and $M = t^2 n$. Then $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type $\Longleftrightarrow$ n, t satisfies one of the following:*

1. *n contains at least two primes.*

2. *n is a prime and $4||M$.*

Evidently, there are a number cases where $N^{new}_{\Gamma_0(M)}(\lambda)$ was of cocompact type and M is not a product of an even number of different primes. The following question thus naturally arises: If $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type did this imply the existence of a cocompact group $\mathcal{O}^1$ such that $N_{\mathcal{O}^1}(\lambda)$ coincides with $N^{new}_{\Gamma_0(M)}(\lambda)$? I.e., were there spectral correspondences responsible for the remaining cases of Theorem 0.2.1? More specifically, he asks: Is there a spectral correspondence responsible for the fact that $N^{new}_{\Gamma_0(12)}(\lambda)$ was of cocompact type? In this paper we answer yes for some values of M not covered by the theorems of Johansson[6], Strömbergsson[47] and Risager[36]. As described above, our focus is on the correspondence between the eigenfunctions of the Laplace-Beltrami operator Δ on $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ and the discrete eigenfunctions of $L^2\left(\Gamma_0(d)\backslash\mathcal{H}\right)$. Consequently, we will discuss: indefinite rational division quaternion algebras, orders in quaternion algebras, arithmetic Fuchsian groups, spectral theory and spectral counting functions, elements of the theory of newforms and the theory of arithmetical functions and two versions of the Selberg trace formula which we will exploit in order to establish our main result. We will also introduce some ideas concerning quadratic field extensions of $\mathbb{Q}$ and $\mathbb{Q}_p$. In the concluding section we establish the claimed result.

Notations

$\mathbf{F}$	a field
$\mathrm{M}_2(\mathbf{F})$	the matrix algebra over $\mathbf{F}$
$\mathcal{A} = \left(\frac{\alpha,\beta}{\mathbf{F}}\right)$	the (generalized) quaternion algebra over $\mathbf{F}$
$\mathcal{A}_p$	a quaternion algebra over $\mathbb{Q}_p$
$d_{\mathcal{A}}$	the discriminant of the quaternion algebra $\mathcal{A}$
$\Gamma(N)$	the principal congruence subgroup of level N
$\Gamma_0(M)$	the Hecke congruence group of level M
$N_{\Gamma_0(M)}(\lambda)$	spectral counting function for Maaß forms on $\Gamma_0(M)$
$N^{new}_{\Gamma_0(M)}(\lambda)$	spectral counting function for Maaß newforms on $\Gamma_0(M)$
$\mathcal{O}$	an order in $\mathcal{A}$
$\mathcal{O}_p$	an order in $\mathcal{A}_p$
$J(\mathcal{O}_p)$	the Jacobson radical of $\mathcal{O}_p$
$d_{\mathcal{O}}$	the discriminant of the order $\mathcal{O}$.
$\mathcal{O}_{pq}$	a maximal order in an indefinite rational quaternion algebra with $d_{\mathcal{A}} = pq$

$e(\mathcal{O})_p$	the Eichler invariant of $\mathcal{O}$ at p
$\mathcal{O}_{pq,p^{2r}q^{2s}}$	an order of index $p^{2r}q^{2s}$ in $\mathcal{O}_{pq}$ with $e(\mathcal{O})_p = e(\mathcal{O})_q = -1$
$\mathcal{O}_{pq,p^{2(r-i)}q^{2(s-j)}}$	an order of index $p^{2(r-i)}q^{2(s-j)}$ in $\mathcal{O}_{pq}$ with $e(\mathcal{O})_p = e(\mathcal{O})_q = -1$
$\mathcal{O}^1$	a cocompact Fuchsian group derived from $\mathcal{O} \subset \mathcal{A}$
$\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$	a cocompact Fuchsian group derived from $\mathcal{O}_{pq,p^{2(r-i)}q^{2(s-j)}}$
$N_{\mathcal{O}^1}(\lambda)$,	spectral counting function for Maaß forms on a cocompact group, $\mathcal{O}^1$
$N^{new}_{\mathcal{O}^1}(\lambda)$,	spectral counting function for Maaß newforms on a cocompact group, $\mathcal{O}^1$
$\mathbb{Q}$	the rational field
$\mathbb{Q}_p$	the p-adic field where p is prime
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{H}$	the ring of real quaternions
$\mathcal{H}$	the Poincaré upper half plane
$\partial\mathcal{H}$	the boundary of $\mathcal{H}$
$\Im(z)$	the imaginary part of z
$\Re(z)$	the real part of z
$\mathbf{K}$	a ring
$\mathrm{M}_2(\mathbf{K})$	the ring of 2×2 matrices over $\mathbf{K}$
$\mathrm{GL}_2(\mathbf{K})$	the general linear group

$\mathrm{SL}_2(\mathbf{K})$	the special linear group
$\mathrm{PSL}_2(\mathbb{R})$	the group of Möbius transformations with real coefficients
$\mathrm{PSL}_2(\mathbb{Z})$	the modular group
X	closed compact surface with a Riemannian metric of constant negative curvature
$\Gamma_0(d)\backslash\mathcal{H}$	a non-compact Riemannian surface with finite volume
$\mathcal{O}^1\backslash\mathcal{H}$	a compact Riemannian surface
$L^2(X)$	the Hilbert space of square integrable functions on X
$L^2(\Gamma_0(d)\backslash\mathcal{H})$	the Hilbert space of square integrable functions on $\Gamma_0(d)\backslash\mathcal{H}$
$L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$	the Hilbert space of square integrable functions on $\mathcal{O}^1\backslash\mathcal{H}$
Δ_Γ	the Laplace-Beltrami operator(hyperbolic Laplacian) on $L^2(\Gamma\backslash\mathcal{H})$
Θ	an integral transform
$Spec(\Delta_\Gamma)$	the spectrum of Δ_Γ
$dSpec(\Delta_\Gamma)$	the discrete part of the spectrum of Δ_Γ
$cSpec(\Delta_\Gamma)$	the continuous component of the spectrum of Δ_Γ
$\mathcal{M}_\Gamma$	the space of Maaß forms on a Fuchsian group Γ
$\mathcal{M}_\Gamma^{new}$	the space of Maaß newforms on a Fuchsian group Γ
$\mathcal{M}_\Gamma^{old}$	the space of Maaß oldforms on a Fuchsian group Γ
$\mathcal{M}_{\mathcal{O}^1}$	the space of Maaß waveforms on $\mathcal{O}^1$

$\mathcal{M}_{\Gamma_0(d)}$	the space of Maaß waveforms on $\Gamma_0(d)$
$\mathcal{M}^{new}_{\mathcal{O}^1}$	the space of Maaß newforms on $\mathcal{O}^1$
$\mathcal{M}^{new}_{\Gamma_0(d)}$	the space of Maaß newforms on $\Gamma_0(d)$
$\mathcal{M}_{\Gamma_0(d)}(\lambda)$	the subspace of $\mathcal{M}_{\Gamma_0(d)}$ with Laplace eigenvalue $\lambda > 0$
$\mathcal{M}_{pq,p^{2(r-i)}q^{2(s-j)}}$	the space of Maaß forms on $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$
$\mathcal{M}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$	the space of Maaß forms on $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$ with eigenvalue λ
$\mathcal{M}^{old}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$	the space of Maaß oldforms on $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$ with eigenvalue λ
$\mathcal{M}^{new}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$	the space of Maaß newforms on $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$ with eigenvalue λ
$\mathrm{Tr}(\gamma)$	the trace of $\gamma \in \Gamma$
$\mathrm{N}(\gamma)$	the norm of $\gamma \in \Gamma$
$E'(t, 1, \mathcal{O}^1)$	number of primitive conjugacy classes of $\gamma \in \mathcal{O}^1$ with $\mathrm{Tr}(\gamma) = t$ and $\mathrm{N}(\gamma) = 1$
$E'(t, n, \Gamma)$	number of conjugacy classes in Γ of primitive elements γ with $\mathrm{Tr}(\gamma) = t$ and $\mathrm{N}(\gamma) = n$
$\mathcal{F}_{\mathcal{O}^1}$	fundamental domain for $\mathcal{O}^1\backslash\mathcal{H}$
$\mathcal{F}_{\Gamma}$	fundamental domain for $\Gamma\backslash\mathcal{H}$
$A_{\mathcal{O}^1}$	the (hyperbolic) area of $\mathcal{F}_{\mathcal{O}^1}$
κ	the number of parabolic fixed points of a cofinite Fuchsian group Γ
κ_N	the number of inequivalent parabolic fixed points of the group $\Gamma_0(N)$

$\{\gamma\}$	the conjugacy class of $\gamma \in \mathrm{PSL}_2(\mathbb{R})$
$\phi(n)$	the Euler phi-function
$\omega(d)$	the number of prime divisors of d
$\tau(n)$	the number of positive divisors of n
$\delta(m,\lambda)$	the dimension of $\mathcal{M}_{\Gamma_0(m)}(\lambda)$
$\delta'(m,\lambda)$	the dimension of $\mathcal{M}^{new}_{\Gamma_0(m)}(\lambda)$
$\mu(n)$	the Möbius function
$\delta(p^{2r}q^{2s},\lambda)$	the dimension of the subspace of $\mathcal{M}_{pq,p^{2r}q^{2s}}(\lambda)$
$\delta'(p^{2r}q^{2s},\lambda)$	the dimension of the subspace of $\mathcal{M}^{new}_{pq,p^{2r}q^{2s}}(\lambda)$
$\mathcal{I}_{\mathcal{O}^1}$	the identity contribution on the geometric side of the Selberg trace formula for $\mathcal{O}^1$
$\mathcal{E}_{\mathcal{O}^1}$	the elliptic contribution on the geometric side of the Selberg trace formula for $\mathcal{O}^1$
$\mathcal{H}_{\mathcal{O}^1}$	the hyperbolic contribution on the geometric side of the Selberg trace formula for $\mathcal{O}^1$
$\mathcal{I}_{\Gamma_0(m)}$	the identity contribution on the geometric side of the Selberg trace formula for $\Gamma_0(m)$
$\mathcal{E}_{\Gamma_0(m)}$	the elliptic contribution on the geometric side of the Selberg trace formula for $\Gamma_0(m)$

$\mathcal{H}_{\Gamma_0(m)}$	the hyperbolic contribution on the geometric side of the Selberg trace formula for $\Gamma_0(m)$
$\mathcal{P}_{\Gamma_0(m)}$	the parabolic contribution on the geometric side of the Selberg trace formula for $\Gamma_0(m)$
$\mathbb{Q}\left(\sqrt{d}\right)$	real quadratic field
$\mathfrak{r}\left[f\right]$	an order of conductor f in a real quadratic field $\mathbb{Q}\left(\sqrt{d}\right)$
ϵ_d	the proper fundamental unit in $\mathbb{Q}\left(\sqrt{d}\right)$
$A^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$	the term that corresponds to the area of the fundamental domain in $N^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}(\lambda)$
$A^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$	the term that corresponds to the area of the fundamental domain in $N^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}(\lambda)$
$\mathcal{I}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$	the identity contribution of the newforms for the Hecke congruence subgroup $\Gamma_0(p^{2r+1}q^{2s+1})$
$\mathcal{E}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$	the elliptic contribution of the newforms for the Hecke congruence subgroup $\Gamma_0(p^{2r+1}q^{2s+1})$
$\mathcal{H}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$	the hyperbolic contribution of the newforms for the Hecke congruence subgroup $\Gamma_0(p^{2r+1}q^{2s+1})$
$\mathcal{P}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$	the parabolic contribution of the newforms for the Hecke congruence subgroup $\Gamma_0(p^{2r+1}q^{2s+1})$

$\mathcal{I}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$ the identity contribution of the newforms

for the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$

$\mathcal{E}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$ the elliptic contribution of the newforms

for the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$

$\mathcal{H}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$ the hyperbolic contribution of the newforms

for the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$

$E(\mathfrak{r}\,[f]\,,\Gamma)$ the number of optimal embeddings of $\mathfrak{r}\,[f]$ into an order $\mathcal{O} \subset \mathcal{A}$

up to conjugation by elements in Γ

$E(\mathfrak{r}\,[f]\,,\Gamma)_p$ the localization of $E(\mathfrak{r}\,[f]\,,\Gamma)$ at p

$E(\mathfrak{r}\,[f]\,,\Gamma)^{new}$ the newform part of number of $E(\mathfrak{r}\,[f]\,,\Gamma)$

$E(\mathfrak{r}\,[f]\,,\Gamma_0(p^{2r+1}))^{new}_p$ the newform part of $E(\mathfrak{r}\,[f]\,,\Gamma_0(p^{2r+1}))_p$

$E(\mathfrak{r}\,[f]\,,\mathcal{O}^1_{pq,p^{2r}})^{new}_p$ the newform part of $E(\mathfrak{r}\,[f]\,,\mathcal{O}^1_{pq,p^{2r}})_p$

Chapter 1

Preliminaries

We recall the basic facts and the motivating ideas.

1.1 The Poincaré upper half plane

Let $\mathcal{H}$ be the Poincaré upper half plane. I.e.,

$$\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\} \tag{1.1}$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{1.2}$$

This yields the area measure(the volume form derived from the metric ds^2)

$$d\mu = \frac{1}{y^2} dxdy \tag{1.3}$$

and the distance function

$$d(z, w) = log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}. \tag{1.4}$$

This metric is the hyperbolic metric, i.e., it is the metric of constant Gaussian curvature negative one.

1.2 Geometry in $\mathcal{H}$

The hyperbolic metric is conformal to the Euclidean metric, so angles are computed as in Euclidean geometry. The geodesics are arcs of generalized circles intersecting $\partial\mathcal{H} = \mathbb{R}\cup\infty$. orthogonally. The area of a hyperbolic triangle, i.e., a triangle with geodesic sides, with interior angles α, β and γ is equal $\pi - \alpha - \beta - \gamma$.[22, Gauss-Bonnet Thm., p.13]

1.3 Fuchsian Groups

Let $\mathbf{K}$ be a ring. $\mathrm{M}_2(\mathbf{K})$ is the ring of 2×2 matrices over $\mathbf{K}$. The general linear group, $\mathrm{GL}_2(\mathbf{K})$, is the group of invertible matrices in $\mathrm{M}_2(\mathbf{K})$ and the special linear group, $\mathrm{SL}_2(\mathbf{K})$, is the group of elements in $\mathrm{GL}_2(\mathbf{K})$ with determinant equal to one. $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R}) / \{\pm \mathrm{Id}\}$ is the group of Möbius transformations with real coefficients. I.e.,

$$\mathrm{PSL}_2(\mathbb{R}) = \left\{ z \longrightarrow \frac{az+b}{cz+d} \middle| a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}. \tag{1.5}$$

A Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ [27, p.26][22, p.39]. Such a group acts properly discontinuously and isometrically on the Poincaré upper half plane $\mathcal{H}$. If Γ is a Fuchsian group then the space of Γ-orbits,

$$\Gamma\backslash\mathcal{H} = \{\Gamma z | z \in \mathcal{H}\}, \tag{1.6}$$

can be given the analytical structure of a Riemann surface with marked points. The Klein-Poincaré uniformization theorem provides that any Riemann surface with constant negative curvature equal to negative one can be realized as some $\Gamma\backslash\mathcal{H}$. We visualize the Riemann surface by means of a fundamental domain $\mathcal{F}$ for Γstromberg. We recall that a closed set $\mathcal{F} \subseteq \mathcal{H}$ is called a fundamental domain for Γ if

1.
$$\bigcup_{\gamma\in\Gamma} \gamma(\mathcal{F}) = \mathcal{H} \tag{1.7}$$
and

2. if $\mathcal{F}^o$ denotes the interior of $\mathcal{F}$ then $\gamma_1(\mathcal{F}^o) \cap \gamma_2(\mathcal{F}^o) = \phi$ if $\gamma_1 \neq \gamma_2 \in \Gamma$[44, p.13].

A Fuchsian group Γ is cofinite if the orbit space $\Gamma\backslash\mathcal{H}$ is of finite volume. I.e., Γ is cofinite if

$$vol(\Gamma\backslash\mathcal{H}) = \int_{\mathcal{F}_\Gamma} d\mu\,(z) < \infty. \tag{1.8}$$

Here $\mathcal{F}_\Gamma$ is a suitable fundamental domain for $\Gamma\backslash\mathcal{H}$. If Γ is cofinite then Γ has a finite number, κ, of parabolic fixed points(cusps). Further, $\kappa = 0$ iff $\Gamma\backslash\mathcal{H}$ is compact. In this case we will refer to the Fuchsian group Γ as a cocompact group[27, p.84].

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element in $\mathrm{PSL}_2(\mathbb{R})$ different from $\pm\mathrm{Id}$. We denote its conjugacy class by

$$\{\gamma\} = \left\{\tau\gamma\tau^{-1}\middle|\tau \in \mathrm{PSL}_2(\mathbb{R})\right\}. \tag{1.9}$$

Conjugate motions act similarly. Note: the identity motion forms a class of itself. A geometric invariant of conjugation is the number of fixed points and we have three types of elements in the group action depending on their fixed points. γ is either:

(i) Hyperbolic: $|\mathrm{Tr}(\gamma)| > 2$. These transformations have two distinct fixed points in $\partial\mathcal{H} = \mathbb{R} \cup \infty$ and they are conjugate to $z \longmapsto e^l z$ where $l \in \mathbb{R}$;

(ii) Elliptic: $|\mathrm{Tr}(\gamma)| < 2$. These transformations have one fixed point in $\mathcal{H}$. The other fixed point is its complex conjugate. These transformations are rotations centered at the fixed point; or

(iii) Parabolic: $|\mathrm{Tr}(\gamma)| = 2$. These transformations have a unique fixed point which must lie in $\partial\mathcal{H} = \mathbb{R} \cup \infty$ and they are conjugate to transformations of form $z \longmapsto z + 1$.

1.4 Arithmetic Fuchsian Groups

There are two main types of arithmetic Fuchsian groups. These are subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ and groups of quaternion type[20]. Groups of quaternion type are cocompact and subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ are non-cocompact but cofinite[27, p.129]. We discuss below some examples of interest.

1.4.1 The Hecke congruence Group

The most common examples of subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ are the congruence subgroups. Let N be any positive integer. We define the principal congruence subgroup of level N,

$\Gamma(N) \subseteq \mathrm{PSL}_2(\mathbb{Z})$, by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{1.10}$$

This is a subgroup of finite index in $\mathrm{PSL}_2(\mathbb{Z})$. Any subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ containing some $\Gamma(N)$ is called a congruence subgroup. We will be concerned with the Hecke congruence group of level N. Denoted by $\Gamma_0(N)$, it is defined as follows:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \middle| c \equiv 0 \mod N \right\}. \tag{1.11}$$

Evidently $\Gamma(N) \subseteq \Gamma_0(N)$. This is a subgroup of index

$$[\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + \frac{1}{p}) < \infty \tag{1.12}$$

in the full modular group $\mathrm{PSL}_2(\mathbb{Z}) = \Gamma_0(1)$. Here the product extends over all prime divisors p of N. The standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z}) = \Gamma_0(1)$ is the subset

$$\mathcal{F} = \left\{ z \in \mathbb{C} \middle| \Im(z) > 0 \cap |\Re(z)| \leq \frac{1}{2} \cap |z| \geq 1 \right\}. \tag{1.13}$$

Evidently, $\mathrm{PSL}_2 \mathbb{Z})$ is cofinite and it follows that its subgroups of finite index $\Gamma_0(N)$ are also cofinite. The groups $\Gamma_0(N)$ have

$$\kappa_N = \sum_{m|N} \phi\left(\left(m, \frac{N}{m}\right)\right) \geq 1 \tag{1.14}$$

inequivalent parabolic fixed points, where $\phi(n)$ denotes the Euler phi-function[43, Prop. 1.43]. We recall the definition of the Euler phi-function

$$\phi(n) = \#\{d \in \mathbb{N} | 1 \leq d \leq n \wedge (d, n) = 1\}.$$

1.4.2 A formula for the hyperbolic area of the fundamental domain for noncocompact surface $\Gamma_0(d)\backslash\mathcal{H}$

Suppose d is a product of an even number of different primes. Then infinity and all rational numbers are cusps for $\Gamma_0(d)$. If $\omega(d)$ is the number of prime divisors of d, then the number of $\Gamma_0(d)$-equivalence classes of cusps is $2^{\omega(d)}$. As a set of representatives for these we choose

$$F(d) = \left\{ \frac{1}{v} : \ v|d, \ v > 0 \right\}. \tag{1.15}$$

A suitable fundamental domain $\mathcal{F}_d$ for $\Gamma_0(d)$ adopted to this choice then extends to $\mathbb{R}$ exactly in the points of $F(d)$. The hyperbolic area A_d of $\mathcal{F}_d$ satisfies[7]

$$A_d = \frac{\pi}{3} \prod_{p|d} (p+1). \tag{1.16}$$

Evidently if $d = pq$ then

$$A_{pq} = \frac{\pi}{3}(p+1)(q+1) \tag{1.17}$$

and it follows that

$$A_{p^r q^s} = \frac{\pi}{3} p^{r-1}(p+1) q^{s-1}(q+1). \tag{1.18}$$

1.4.3 Groups of quaternion type

An example of a group of quaternion type is:

$$G = \left\{ \left(\begin{array}{cc} a + b\sqrt{3} & c + d\sqrt{3} \\ (c - d\sqrt{3})5 & a - b\sqrt{3} \end{array} \right) \in \mathrm{PSL}_2\left(\mathbb{R}\right) \middle| a, b, c, d \in \mathbb{Z} \right\}. \tag{1.19}$$

One observes that the group elements are matrices with entries from an algebraic number field. Evidently, the precise definition of these groups will require some ideas from algebraic number theory. We present a sketch of those aspects of the theory which are relevant for our subsequent discussion. Broadly speaking, these groups are generalizations of the modular group. They are typically obtained from quaternion algebras over totally real number fields.

1.4.4 Quaternion Algebras

A quaternion algebra over a field $\mathbf{F}$ is a central simple algebra of dimension four over $\mathbf{F}$ [34, p.199, Sec. 5.2][25, p.15]. It follows from Wedderburn's structure theorem on simple algebras[12] that every quaternion algebra over any field $\mathbf{F}$ is either isomorphic to $\mathrm{M}_2(\mathbf{F})$ or a division algebra with center $\mathbf{F}$[31]. Further, if the characteristic of $\mathbf{F} \neq 2$, then it is always possible to find a basis $\{1, i, j, k\}$ for $\mathcal{A}$ over $\mathbf{F}$ such that

$$i^2 = \alpha, j^2 = \beta, k = ij = -ji \tag{1.20}$$

where α and $\beta \in \mathbf{F}^* = \mathbf{F} - \{0\}$. We will refer to this algebra as

$$\mathcal{A} = \left(\frac{\alpha, \beta}{\mathbf{F}}\right). \tag{1.21}$$

Evidently $q \in \mathcal{A}$ is of the form

$$q = a + bi + cj + dk, \tag{1.22}$$

where $a,\ b,\ c,\ d \in \mathbf{F}$[51, p.2] . One notes that in this context Hamilton's quaternion algebra, $\mathbb{H}$, is isomorphic to $\left(\frac{-1,-1}{\mathbb{R}}\right)$.

Let $q \in \left(\frac{\alpha,\beta}{\mathbf{F}}\right)$, i.e., $q = a + bi + cj + dk$. The conjugate of q, denoted by $\bar{q}$, is equal to $a - bi - cj - dk$. For each $q \in \left(\frac{\alpha,\beta}{\mathbf{F}}\right)$ we define the (reduced) norm map

$$\mathrm{N} : \left(\frac{\alpha,\beta}{\mathbf{F}}\right) \to \mathbf{F}$$

by

$$\mathrm{N}(q) = \bar{q}q = a^2 - \alpha b^2 - \beta c^2 + \alpha\beta d^2 \tag{1.23}$$

and the (reduced) trace map

$$\mathrm{Tr} : \left(\frac{\alpha,\beta}{\mathbf{F}}\right) \to \mathbf{F}$$

by

$$\mathrm{Tr}(q) = \bar{q} + q = 2a. \tag{1.24}$$

We note that every element $q \in \mathbf{A}$ satisfies the quadratic equation

$$q^2 - \mathrm{Tr}(q)q + \mathrm{N}(q) = 0. \tag{1.25}$$

We now specialize to quaternion algebras over the rational field $\mathbb{Q}$, or over one of the completions, $\mathbb{Q}_p$ or $\mathbb{R}$. Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{Q}$ and let p be a prime. We define

$$\mathcal{A}_p = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}_p. \tag{1.26}$$

Either $\mathcal{A}_p \cong \mathrm{M}_2(\mathbb{Q}_p)$ or $\mathcal{A}_p$ is a division algebra over $\mathbb{Q}_p$. In this case we will say that $\mathcal{A}_p \cong \mathbb{H}_p$ and that $\mathcal{A}$ is ramified at p. When $\mathcal{A}_p \cong \mathrm{M}_2(\mathbb{Q}_p)$ we will say that $\mathcal{A}$ is unramified or split at p. If $\mathcal{A}$ is ramified at infinity it is called a definite rational quaternion algebra. I.e., if $\mathcal{A}$ is definite it means that

$$\mathcal{A}_\infty = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \tag{1.27}$$

and if $\mathcal{A}$ is split at infinity it is called an indefinite rational quaternion algebra. I.e., if $\mathcal{A}$ is indefinite then

$$\mathcal{A}_\infty = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R}) \, . \tag{1.28}$$

Let $d_{\mathcal{A}}$ be the product of all primes p that ramify $\mathcal{A}$. Evidently, $d_{\mathcal{A}}$ is square-free. $d_{\mathcal{A}}$ is the (reduced) discriminant of $\mathcal{A}$[51, p.58]. When $\mathcal{A}$ is indefinite $d_{\mathcal{A}}$ is the product of an even number of different primes[51, p. 74, Thm. 3.1]. For each square free number $d \in \mathbb{Z}^+$, up to isomorphism, there is exactly one quaternion algebra $\mathcal{A}$ over $\mathbb{Q}$ with $d_{\mathcal{A}} = d$. I.e., $d_{\mathcal{A}}$ determines the isomorphism class of $\mathcal{A}$ over $\mathbb{Q}$. We note also that $d_{\mathcal{A}} > 1$ iff $\mathcal{A}$ is a division algebra.

1.4.5 Orders in Quaternion Algebras

An element $q \in \mathcal{A}$ is integral if the $\mathrm{N}(q)$ and $\mathrm{Tr}(q) \in \mathbb{Z}$. An order $\mathcal{O} \subseteq \mathcal{A}$ is a $\mathbb{Z}$-algebra of elements of integral reduced trace and reduced norm such that $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathcal{A}$[46, pg. 2]. Observe that as an example of an order we always have $\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$. Let e_1, e_2, e_3, e_4 be any $\mathbb{Z}$-basis of $\mathcal{O}$ and let

$$d_{\mathcal{O}} = \sqrt{|det\left[\mathrm{Tr}\left(e_i e_j\right)\right]|}. \tag{1.29}$$

$d_{\mathcal{O}}$ is the (reduced) discriminant of the order $\mathcal{O}$. It is always an integer, it is independent of the choices of e_1, e_2, e_3, e_4, it is divisible by $d_{\mathcal{A}}$ and $d_{\mathcal{O}} = d_{\mathcal{A}}$ iff $\mathcal{O}$ is a maximal order in $\mathcal{A}$. We recall that a maximal order is one that cannot be properly contained in any other order. Let $\mathcal{O}_1$ be a any order in $\mathcal{A}$ with discriminant $d_{\mathcal{O}_1}$ and let $\mathcal{O}_2$ denote an order of index n in $\mathcal{O}_1$ then the discriminant of $\mathcal{O}_2$, i.e.,

$$d_{\mathcal{O}_2} = d_{\mathcal{O}_1} \cdot [\mathcal{O}_1 : \mathcal{O}_2] = d_{\mathcal{O}_1} n. \tag{1.30}$$

1.4.6 Eichler orders

Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{Q}$ with discriminant d and let $N \in \mathbb{Z}^+$. Now let $\mathcal{O} \subset \mathcal{A}$ be given as the intersection of two maximal orders so that its index in either of them is N. Such an order is called an Eichler order of level N. If we are given an Eichler order and a prime p, then $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is maximal if p divides d otherwise $\mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$.

1.4.7 The Eichler invariant of $\mathcal{O}$ at p

Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{Q}$ and let $\mathcal{O}$ be an order in $\mathcal{A}$. Eichler, in [13], introduced a local invariant of $\mathcal{O}$ called the Eichler invariant of $\mathcal{O}$ at p. We denote this invariant by $e(\mathcal{O})_p$ and we define it as follows: Let p be prime and let $\mathbb{Q}_p$ be the p-adic field with ring of integers $\mathbb{Z}_p$. Further, let $\mathcal{O}_p$ be an order in $\mathcal{A}_p$ a quaternion algebra over $\mathbb{Q}_p$, let $\mathbb{Z}/p\mathbb{Z}$ the residue class field of $\mathbb{Q}_p$ and let $J(\mathcal{O}_p)$ be the Jacobson radical of $\mathcal{O}_p$. If $\mathcal{O}_p \ncong \mathrm{M}_2(\mathbb{Z}_p)$, then the Eichler invariant $e(\mathcal{O})_p$ is defined as follows:

$$e(\mathcal{O})_p = \begin{cases} 1, & \text{if} \quad \mathcal{O}_p/J(\mathcal{O}_p) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \\ 0, & \text{if} \quad \mathcal{O}_p/J(\mathcal{O}_p) \cong \mathbb{Z}/p\mathbb{Z} \\ -1, & \text{if} \quad \mathcal{O}_p/J(\mathcal{O}_p) \text{ is a quadratic field extension of } \mathbb{Z}/p\mathbb{Z} \end{cases}.$$

Eichler also illustrated how to compute $e(\mathcal{O})_p$. For $\alpha \in \mathcal{A}$, we define the discriminant of α, $\Delta(\alpha) = Tr(\alpha)^2 - 4N(\alpha)$ and then determine Eichler invariant of $\mathcal{O}$ at p, $e(\mathcal{O})_p$, by using the following result:

1. If $e(\mathcal{O})_p = 0$, then $\left(\frac{\Delta(\alpha)}{p}\right) = 0 \ \forall \alpha \in \mathcal{O}$
2. If $e(\mathcal{O})_p = 1$, then $\left(\frac{\Delta(\alpha)}{p}\right) \neq -1 \ \forall \alpha \in \mathcal{O}$, and $\exists \alpha \in \mathcal{O} \ni \left(\frac{\Delta(\alpha)}{p}\right) = 1$
3. If $e(\mathcal{O})_p = -1$, then $\left(\frac{\Delta(\alpha)}{p}\right) \neq 1 \ \forall \alpha \in \mathcal{O}$, and $\exists \alpha \in \mathcal{O} \ni \left(\frac{\Delta(\alpha)}{p}\right) = -1$, where $\left(\frac{\Delta(\alpha)}{p}\right)$ is the Legendre symbol.

It follows from the above that

1. If $\exists \alpha \in \mathcal{O} \ni \left(\frac{\Delta(\alpha)}{p}\right) = 1$, then $e(\mathcal{O})_p = 1$
2. If $\exists \alpha \in \mathcal{O} \ni \left(\frac{\Delta(\alpha)}{p}\right) = -1$, then $e(\mathcal{O})_p = -1$

Orders in $\left(\frac{\alpha,\beta}{\mathbb{Q}_p}\right)$ with Eichler invariant -1 only occur in the maximal order with index p^{2n} and orders with Eichler invariant 1 only occur in $\mathrm{M}_2(\mathbb{Q}_p)$.

1.4.8 An arithmetic Fuchsian group of quaternion type

Let $\mathcal{O}$ be any order in an indefinite rational division quaternion algebra $\mathcal{A}$. We define

$$\mathcal{O}^1 = \{\alpha \in \mathcal{O} : N(\alpha) = 1\} \tag{1.31}$$

where $N(\alpha)$ is the norm from $\mathcal{A}$ to $\mathbb{Q}$. From the isomorphism $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$, it follows that $\mathcal{O}^1$ is (isomorphic to) a cocompact Fuchsian group[34, p.209, Thm 5.2.13]. A Fuchsian group Γ that is a subgroup of finite index in some $\mathcal{O}^1$ is called a Fuchsian group derived from the quaternion algebra $\mathcal{A}$. The phrases quaternion group or groups of quaternion type are also used to describe the group Γ. A Fuchsian group Γ that is commensurable with some $\mathcal{O}^1$ is called an arithmetic Fuchsian group[27, p.120][5, p.943].

1.4.9 A formula for the hyperbolic area of the fundamental domain of our compact surface $\mathcal{O}^1\backslash\mathcal{H}$

We can fix a suitable fundamental domain $\mathcal{F}_{\mathcal{O}^1} \subset \mathcal{H}$ for our compact surface $\mathcal{O}^1\backslash\mathcal{H}$. Denote by $A_{\mathcal{O}^1}$ the (hyperbolic) area of $\mathcal{F}_{\mathcal{O}^1}$. If $\mathcal{O}$ is a maximal order, then

$$A_{\mathcal{O}^1} = \frac{\pi}{3}\prod_{p|d_{\mathcal{O}}}(p-1) \tag{1.32}$$

[26, p.184]. We note here also that if $\Gamma_1 \subseteq \Gamma_2$ are two Fuchsian groups such that Γ_1 is of finite index in Γ_2 then,

$$A_{\Gamma_1} = A_{\Gamma_2} \cdot [\Gamma_2 : \Gamma_1]\,. \tag{1.33}$$

Let $\mathcal{O}_{pq}$ denote a maximal order with discriminant equal to pq then

$$A_{\mathcal{O}^1_{pq}} = \frac{\pi}{3}(p-1)(q-1), \tag{1.34}$$

where $A_{\mathcal{O}^1_{pq}}$ is the hyperbolic area of the fundamental domain $\mathcal{F}_{\mathcal{O}^1_{pq}}$ for $\mathcal{O}^1_{pq}$. Moreover, if $\mathcal{O}_{pq,p^{2r}q^{2s}} \subseteq \mathcal{O}_{pq} \ni \left[\mathcal{O}_{pq} : \mathcal{O}_{pq,p^{2r}q^{2s}}\right] = p^{2r}q^{2s}$ and $e\left(\mathcal{O}_{pq,p^{2r}q^{2s}}\right)_p = e\left(\mathcal{O}_{pq,p^{2r}q^{2s}}\right)_q = -1$, then

$$A_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}} = \frac{\pi}{3} p^{2r}(p-1)q^{2s}(q-1). \tag{1.35}$$

where $A_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$ is the hyperbolic area of the fundamental domain $\mathcal{F}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$ for $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$.

1.4.10 The fundamental correspondence

Let $\mathcal{O}^1$ be an arithmetic Fuchsian group derived from an order $\mathcal{O} \subset \mathcal{A}$ with discriminant $d_{\mathcal{O}}$. We will associate to each cocompact group $\mathcal{O}^1$ the Hecke congruence group $\Gamma_0(d_{\mathcal{O}})$. It is this link between the discriminant of the order and the level of the congruence group which provides us with the natural setting in which to examine the correspondence between the spaces of Maaß forms on $\mathcal{O}^1$ and spaces of Maaß forms on $\Gamma_0(d_{\mathcal{O}})$.

1.5 Spectral Theory

1.5.1 Maaß waveforms

The spectral theory of hyperbolic surfaces has its origins in the efforts by Atle Selberg to use the techniques of harmonic analysis in the study of automorphic forms[21]. It's development was influenced, in part, by the work of Hans Maaß[32] who studied nonanalytic automorphic functions. We outline below the elements of the theory that are necessary for our purposes. A comprehensive reference for what follows is [22].

Let $\mathcal{A}$ be an indefinite rational quaternion algebra and $\mathcal{O}$ be any order in $\mathcal{A}$ such that $d_{\mathcal{O}} = d$. Let $\mathcal{O}^1\backslash\mathcal{H}$ and $\Gamma_0(d)\backslash\mathcal{H}$ be the Riemann surfaces related to $\mathcal{O}^1$ and $\Gamma_0(d)$

respectively and let $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ and $L^2\left(\Gamma_0(d)\backslash\mathcal{H}\right)$ be the corresponding Hilbert spaces of square integrable functions on $\mathcal{O}^1\backslash\mathcal{H}$ and $\Gamma_0(d)\backslash\mathcal{H}$ respectively. Further, let

$$\Delta_\Gamma = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.36}$$

be the Laplace-Beltrami operator(hyperbolic Laplacian) on the respective Hilbert spaces.

If $f \in L^2\left(\Gamma\backslash\mathcal{H}\right)$ is a function such that:

(i) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$. I.e., f satisfies the automorphy condition relative to the cofinite discrete group Γ.

(ii) f vanishes at the cusps of Γ, and

(iii) $\Delta_\Gamma f = \lambda f$ for some $\lambda > 0$.

Then f is an eigenfunction of Δ_Γ. More precisely f is a real analytic eigenfunction of the hyperbolic Laplacian, i.e., f is a Maaß waveform and λ is an eigenvalue of Δ_Γ. It should be noted that solutions to the equation $\Delta_\Gamma f = \lambda f$ on hyperbolic surfaces have deep connections to physics. They are used to describe mathematical models of quantum chaos[4] and also play a role in the study of cosmology[44].

1.5.2 The spectrum of Δ_Γ

This set of eigenvalues is called the spectrum of Δ_Γ. We will denote it by $Spec\left(\Delta_\Gamma\right)$. The spectrum of Δ_Γ decomposes into discrete and continuous parts. We denote these components by $dSpec\,(\Delta_\Gamma)$ and $cSpec\,(\Delta_\Gamma)$ respectively . I.e.,

$$Spec\,(\Delta_\Gamma) = dSpec\,(\Delta_\Gamma) \cup cSpec\,(\Delta_\Gamma)\,. \tag{1.37}$$

If Γ is a cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$, say $\mathcal{O}^1$, then the Laplace operator $\Delta_{\mathcal{O}^1}$ of $\mathcal{O}^1\backslash\mathcal{H}$ acting on the Hilbert space $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ has only the discrete spectrum, i.e., $cSpec\,(\Delta_{\mathcal{O}^1})$ is empty and $dSpec\,(\Delta_{\mathcal{O}^1})$ consists of 0 and a discrete subset of the non-negative real numbers. I.e., the spectrum of the hyperbolic Laplacian $\Delta_{\mathcal{O}^1}$ on $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ is discrete, and is comprised of the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\ , \lambda_n \longrightarrow \infty.$$

Each of the eigenvalues is known to occur with finite multiplicity. It is also known that $dSpec\,(\Delta_{\mathcal{O}^1})$ has infinitely many eigenvalues and that its counting function satisfies Weyl's asymptotic law.

If Γ is a non-cocompact but cofinite subgroup of $\mathrm{PSL}_2(\mathbb{R})$, say $\Gamma_0(d)$, then the spectral resolution of the Laplace operator $\Delta_{\Gamma_0(d)}$ acting on $L^2\left(\Gamma_0(d)\backslash\mathcal{H}\right)$ has both a continuous spectrum $\left[\frac{1}{4}, \infty\right)$ and a discrete spectrum contained in $[0, \infty)$[40]. In a manner similar to that of the compact case the discrete eigenvalues satisfy

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots\ , \mu_n \longrightarrow \infty.$$

The spectral correspondence that we focus on is the correspondence between $dSpec\left(\Delta_{\Gamma_0(d)}\right)$ and $dSpec\,(\Delta_{\mathcal{O}^1})$. I.e, the correspondence between the λ-eigenspace of $\Delta_{\mathcal{O}^1}$, and the μ-

eigenspace of $\Delta_{\Gamma_0(d)}$ or between the eigenfunctions of the Laplace-Beltrami operator Δ on $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ and the discrete eigenfunctions of $L^2\left(\Gamma_0(d)\backslash\mathcal{H}\right)$.

1.5.3 Spectral counting functions

To make explicit this correspondence we first define the spectral counting function $N_\Gamma(\lambda)$:

$$N_\Gamma(\lambda) = \#\{\lambda_n \leq \lambda : \lambda_n \in dSpec(\Delta_\Gamma)\}. \tag{1.38}$$

1.5.4 Cocompact case

Since $\mathcal{O}^1$ has no cusps then $\Delta_{\mathcal{O}^1}$ has infinitely many eigenfunctions and $N_{\mathcal{O}^1}(\lambda)$ has an asymptotic expansion of the form:

$$N_{\mathcal{O}^1}(\lambda) = \frac{Vol(\mathcal{O}^1\backslash\mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda}). \tag{1.39}$$

1.5.5 Non-cocompact case

For congruence subgroups Selberg[41] showed that $N_{\Gamma_0(d)}(\lambda)$ has an asymptotic expansion of the form:

$$N_{\Gamma_0(d)}(\lambda) = \frac{Vol(\Gamma_0(d)\backslash\mathcal{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log\lambda). \tag{1.40}$$

We note here the difference in the error terms between the cocompact and noncocompact case.

1.6 Newforms and Oldforms

1.6.1 Newforms on Hecke congruence groups

The theory of newforms was originally developed by Atkin and Lehner for holomorphic forms[2]. This theory has been translated into a similar theory for Maaß forms by various people and details may be found in Strömbergsson[46] and also in Strömberg[44, p.28]. We need one result from this theory and as such we present below only what is necessary in order for this result to make sense. What follows is essentially a restatement of the discussion in [6] which leads to the desired result.

Let $a, m, d \in \mathbb{N}$ such that $m < d$ and $am|d$. Further, let $\Gamma_0(m)$ and $\Gamma_0(d)$ be the Hecke congruence groups of level m and d respectively. Denote by $\mathcal{M}_{\Gamma_0(d)}$ the space of Maaß forms on $\Gamma_0(d)$ and by $\mathcal{M}_{\Gamma_0(m)}$ the space of Maaß forms on $\Gamma_0(m)$. Evidently, if $m|d$ then $\Gamma_0(d) \subset \Gamma_0(m)$ which implies that $\mathcal{M}_{\Gamma_0(m)} \subset \mathcal{M}_{\Gamma_0(d)}$. Moreover, if $f(z)$ is a Maaß form on $\Gamma_0(m)$ then $f(az)$ is a Maaß form on $\Gamma_0(d)$ for all $a|\frac{d}{m}$. Such functions are called oldforms on $\Gamma_0(d)$ and it is natural to avoid such functions in a search for Maass forms on the subgroup, for they naturally belong on the larger group. The Maass forms which naturally live on $\Gamma_0(d)$ are called newforms. Alternatively, if a group Γ has Maass waveforms, then these descend to all subgroups of Γ. As a Maass waveform on the subgroup we will call it an oldform. The linear span of all forms $f(az) \in \mathcal{M}_{\Gamma_0(d)}$ that derive from all possible a, m is called the oldspace $\mathcal{M}^{old}_{\Gamma_0(d)}$. Its orthogonal complement within $\mathcal{M}_{\Gamma_0(d)}$ is the newspace $\mathcal{M}^{new}_{\Gamma_0(d)}$, so that $\mathcal{M}_{\Gamma_0(d)} = \mathcal{M}^{old}_{\Gamma_0(d)} \oplus \mathcal{M}^{new}_{\Gamma_0(d)}$ and functions in $\mathcal{M}^{new}_{\Gamma_0(d)}$ are called newforms. It is trivial to check that all $f(az)$ corresponding to a fixed $f(z) \in \mathcal{M}_{\Gamma_0(m)}$ have the same Laplace eigenvalue. If $f(z) \in \mathcal{M}^{new}_{\Gamma_0(m)}$, then there are $\tau(\frac{d}{m})$ forms $f(az)$ in $\mathcal{M}_{\Gamma_0(d)}$ corresponding to $f(z)$, where $\tau(n)$ is the number of positive divisors of n.

Let $\delta(m,\lambda) = dim(\mathcal{M}_{\Gamma_0(m)}(\lambda))$, i.e., the dimension of the subspace of $\mathcal{M}_{\Gamma_0(m)}$ with Laplace eigenvalue $\lambda > 0$, and let $\delta'(m,\lambda) = dim(\mathcal{M}^{new}_{\Gamma_0(m)}(\lambda))$ be the dimension of the corresponding subspace $\mathcal{M}^{new}_{\Gamma_0(m)}(\lambda)$. Since $\mathcal{M}_{\Gamma_0(d)} = \mathcal{M}^{old}_{\Gamma_0(d)} \oplus \mathcal{M}^{new}_{\Gamma_0(d)}$, these satisfy

$$\delta(d,\lambda) = \sum_{m|d} \tau\left(\tfrac{d}{m}\right) \delta'(m,\lambda).$$

Inverting this formula, one gets [2, (6.7)]

$$\delta'(d,\lambda) = \sum_{m|d} \beta\left(\tfrac{d}{m}\right) \delta(m,\lambda), \tag{1.41}$$

where

$$\beta(n) = \sum_{k|n} \mu(k)\mu(\tfrac{n}{k}), \tag{1.42}$$

where $\mu(n)$ is the Möbius function. I.e.,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \\ 0 & \text{otherwise.} \end{cases} \tag{1.43}$$

It follows that if n is a product of r distinct primes, then

$$\beta(n) = (-2)^r \tag{1.44}$$

and in particular, $\beta(1) = 1$, $\beta(p) = -2$, $\beta(p^2) = 1$, and $\beta(p^k) = 0, k > 2$.

1.6.2 Newforms on Cocompact Groups Derived from Quaternion Algebras

In the proof of our main theorem we will show that the linear combination, described above, of $\Gamma_0(m)$-trace formulas has the property of canceling all cuspidal terms, and making the area term, the elliptic terms and the hyperbolic terms agree with the corresponding terms for a special linear combination of $\mathcal{O}^1_{pq}$-trace formulas. We provide a similar language for this linear combination of $\mathcal{O}^1_{pq}$-trace formulas by defining, below, a notion of newforms on the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$.

Let $\mathcal{A}$ be an indefinite rational division quaternion algebra with discriminant $d = pq$ where p and q are primes such that $p, q > 2$. Let $\mathcal{O}_{pq}$ be a maximal order in $\mathcal{A}$. Now, let $r, s \in \mathbb{N}$ and let $\mathcal{O}_{pq,p^{2r}q^{2s}}, r, s \geq 1$ be an order of index $p^{2r}q^{2s}$ in $\mathcal{O}_{pq}$ with Eichler invariant equal to negative one at p and at q.

Now, let $\mathcal{O}_{pq,p^{2(r-i)}q^{2(s-j)}}$ be an order of index $p^{2(r-i)}q^{2(s-j)}$ in $\mathcal{O}_{pq}$, where $i \in \{0, 1\}$ and $j \in \{0, 1\}$ and $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$ the respective cocompact Fuchsian groups given as the group of units of norm one in $\mathcal{O}_{pq,p^{2(r-i)}q^{2(s-j)}}$.

Denote by $\mathcal{M}_{pq,p^{2(r-i)}q^{2(s-j)}}$, the spaces of Maaß forms on the respective groups. The subspace $\mathcal{M}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$, of Maaß forms with eigenvalue λ, can be decomposed into two subspaces

$$\mathcal{M}^{old}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda) \oplus \mathcal{M}^{new}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$$

where the space $\mathcal{M}^{old}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$ is the linear span of all forms with the same eigenvalue λ coming from all overgroups $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$. The orthogonal complement of $\mathcal{M}^{old}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$ is defined to be $\mathcal{M}^{new}_{pq,p^{2(r-i)}q^{2(s-j)}}(\lambda)$. Evidently,

- $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}^1_{pq} \Rightarrow \mathcal{M}_{pq} \subset \mathcal{M}_{pq,p^{2r}q^{2s}}$
- $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}^1_{pq,p^{2r-2}q^{2s}} \Rightarrow \mathcal{M}_{pq,p^{2r-2}q^{2s}} \subset \mathcal{M}_{pq,p^{2r}q^{2s}}$
- $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}^1_{pq,p^{2r}q^{2s-2}} \Rightarrow \mathcal{M}_{pq,p^{2r}q^{2s-2}} \subset \mathcal{M}_{pq,p^{2r}q^{2s}}$
- $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}^1_{pq,p^{2r-2}q^{2s-2}} \Rightarrow \mathcal{M}_{pq,p^{2r-2}q^{2s-2}} \subset \mathcal{M}_{pq,p^{2r}q^{2s}}$
- $\mathcal{O}^1_{pq,p^{2r-2}q^{2s}} \subset \mathcal{O}^1_{pq} \Rightarrow \mathcal{M}_{pq} \subset \mathcal{M}_{pq,p^{2r-2}q^{2s}}$
- $\mathcal{O}^1_{pq,p^{2r}q^{2s-2}} \subset \mathcal{O}^1_{pq} \Rightarrow \mathcal{M}_{pq} \subset \mathcal{M}_{pq,p^{2r}q^{2s-2}}$
- $\mathcal{O}^1_{pq,p^{2r-2}q^{2s-2}} \subset \mathcal{O}^1_{pq} \Rightarrow \mathcal{M}_{pq} \subset \mathcal{M}_{pq,p^{2r-2}q^{2s-2}}$

Let $\delta(p^{2r}q^{2s},\lambda)$ be the dimension of the subspace of $\mathcal{M}_{pq,p^{2r}q^{2s}}(\lambda)$ and let $\delta'(p^{2r}q^{2s},\lambda)$ be the dimension of the corresponding subspace of $\mathcal{M}^{new}_{pq,p^{2r}q^{2s}}(\lambda)$. These satisfy

$$\delta'(p^{2r}q^{2s},\lambda) = \sum_{k|p^rq^s} \mu(\frac{p^rq^s}{k})\delta(k^2,\lambda) \tag{1.45}$$

where $\mu(n)$ is the Möbius function. I.e.,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \\ 0 & \text{otherwise.} \end{cases}$$

1.7 The Selberg Trace Formula

In this section we introduce the Selberg trace formula. It establishes a quantitative connection between the spectrum and the geometry of the Riemann surface $\Gamma\backslash\mathcal{H}$. See [22], [18] and [19] for a comprehensive classical introduction to the trace formula. It is a general identity connecting geometrical and spectral terms, i.e., an identity of form:

$$\sum \text{spectral terms} = \sum \text{geometric terms}. \tag{1.46}$$

The spectral terms come from the discrete and continuous spectra of the automorphic hyperbolic Laplacian Δ_Γ for a cofinite Fuchsian group Γ and the geometrical terms are integral operators depending on the conjugacy classes of Γ. One can calculate these integrals explicitly and achieve the final form of the Selberg trace formula. We will need two versions of it; For cocompact groups $\mathcal{O}^1$ and for Hecke congruence groups $\Gamma_0(m)$. The trace formula for both types of Fuchsian groups under consideration are well-known. The result for $\mathcal{O}^1$ can be found in [18, ch.V,Thm.8.1] and for $\Gamma_0(m)$ in [17]. We recall the known results: In what follows $h : \mathbb{C} \to \mathbb{C}$ always denotes a function satisfying,

1. $h(r) = h(-r)$

2. $h(r)$ is holomorphic in the strip $|\Im(r)| \leq \frac{1}{2} + \varepsilon$, for some $\varepsilon > 0$

3. $|h(r)| \leq C\,(1 + \Re(r))^{-2-\delta}$ for some $C > 0$ and $\delta > 0$.

The Fourier transform of h will then be written as

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\, e^{-iru}\; dr\ .$$

1.7.1 The Selberg Trace Formula for cocompact groups $\mathcal{O}^1$

Since the unit group $\mathcal{O}^1$ is a cocompact Fuchsian group, the Selberg trace formula reads as follows:

Proposition 1.7.1. *Let $\lambda_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(\mathcal{O}^1\backslash\mathcal{H})$, counted with multiplicities. Then*

$$\begin{aligned}\sum_{k=0}^{\infty} h(r_k) =& \frac{A_{\mathcal{O}^1}}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\; dr \\ &+ \sum_{t\in\{0,1\}} \frac{E'(t,1,\mathcal{O}^1)}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\; dr \\ &+ \sum_{t=3}^{\infty} E'(t,1,\mathcal{O}^1)\, \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)}\ .\end{aligned} \tag{1.47}$$

Proof. This is [7, Proposition 4.1]. The sums over representatives of the primitive elliptic and hyperbolic conjugacy classes of elements in $\mathcal{O}^1$ are rewritten as sums over the traces $t \in \mathbb{N}_0$ of the representatives. The number of primitive conjugacy classes with trace t is $E'(t, 1, \mathcal{O}^1)$. In the elliptic case, where $t \in \{0, 1\}, m_t$ denotes the order of the primitive element with trace t. □

We recall that an element $\gamma \in \Gamma$, Γ a Fuchsian group, is called primitive in Γ, if there is no element $\alpha \in \Gamma$ such that $\gamma = \alpha^r$, with $r \geq 2$ and note that if $\gamma \neq 1$ is hyperbolic then $\gamma = \gamma_0^k$ for some primitive γ_0 with $k \geq 1$. We define $E'(t, n, \Gamma)$ to be the number of conjugacy classes in Γ of primitive elements with trace t and norm equal to n.

We will view equation 1.47 as having form

$$\sum_{k=0}^{\infty} h(r_k) = \mathcal{I} + \mathcal{E} + \mathcal{H} \tag{1.48}$$

where

$$\mathcal{I}_{\mathcal{O}^1} = \frac{A_{\mathcal{O}^1}}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\; dr \tag{1.49}$$

will denote the identity contribution,

$$\mathcal{E}_{\mathcal{O}^1} = \sum_{t \in \{0,1\}} \frac{E'(t, 1, \mathcal{O}^1)}{2m_t} \sum_{k=1}^{m_t - 1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1 + e^{-2\pi r}}\; dr \tag{1.50}$$

the elliptic contribution and

$$\mathcal{H}_{\mathcal{O}^1} = \sum_{t=3}^{\infty} E'(t, 1, \mathcal{O}^1)\; \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)} \tag{1.51}$$

the hyperbolic contribution. We recall that in the case of cocompact groups there is no continuous spectrum and no parabolic element.

1.7.2 The Selberg Trace Formula For Hecke congruence groups $\Gamma_0(m)$

The trace formula for $\Gamma_0(m)$ is well-known. We recall [17, Thm.9.9] together with [17, (10.2),(10.4)] and use the same notation as in Proposition 1.7.1:

Proposition 1.7.2. *Let $\mu_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(\Gamma_0(m)\backslash\mathcal{H})$, counted with multiplicities. Then*

$$\begin{aligned}\sum_{k=0}^{\infty} h(r_k) = &\frac{A_m}{4\pi}\int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\; dr\\ &+ \sum_{t\in\{0,1\}} \frac{E'(t,1,\Gamma_0(m))}{2m_t}\sum_{k=1}^{m_t-1}\frac{1}{\sin(\frac{k\pi}{m_t})}\int_{-\infty}^{+\infty} h(r)\,\frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\; dr\\ &+ \sum_{t=3}^{\infty} E'(t,1,\Gamma_0(m))\; \operatorname{arccosh}(\tfrac{t}{2})\sum_{k=1}^{\infty}\frac{\hat{h}\left(2k\operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k\operatorname{arccosh}(\frac{t}{2})\right)}\\ &+ 2^{\omega(m)}\Bigg\{\hat{h}(0)\log(\tfrac{\pi}{2}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\,\left[\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\right]\; dr\\ &\qquad + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\,\hat{h}(2\log n) - \sum_{\substack{p\mid m\\ p\ prime}}\sum_{k=0}^{\infty}\frac{\log p}{p^k}\,\hat{h}(2k\log p)\Bigg\}\,.\end{aligned}$$

Proof. This is [7, Proposition 4.3]. □

As above, we will view this equation as having form

$$\sum_{k=0}^{\infty} h(r_k) = \mathcal{I} + \mathcal{E} + \mathcal{H} + \mathcal{P} \tag{1.52}$$

where

$$\mathcal{I}_{\Gamma_0(m)} = \frac{A_m}{4\pi}\int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\; dr \tag{1.53}$$

will denote the identity contribution,

$$\mathcal{E}_{\Gamma_0(m)} = \sum_{t\in\{0,1\}} \frac{E'(t,1,\Gamma_0(m))}{2m_t}\sum_{k=1}^{m_t-1}\frac{1}{\sin(\frac{k\pi}{m_t})}\int_{-\infty}^{+\infty} h(r)\,\frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\; dr \tag{1.54}$$

the elliptic contribution,

$$\mathcal{H}_{\Gamma_0(m)} = \sum_{t=3}^{\infty} E'(t,1,\Gamma_0(m))\; \operatorname{arccosh}(\tfrac{t}{2})\sum_{k=1}^{\infty}\frac{\hat{h}\left(2k\operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k\operatorname{arccosh}(\frac{t}{2})\right)} \tag{1.55}$$

the hyperbolic contribution and

$$\begin{aligned}\mathcal{P}_{\Gamma_0(m)} =& 2^{\omega(m)}\Big\{\hat{h}(0)\log(\tfrac{\pi}{2}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\left[\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\right] dr \\ & + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\hat{h}(2\log n) - \sum_{\substack{p|m \\ p \text{ prime}}}\sum_{k=0}^{\infty}\frac{\log p}{p^k}\hat{h}(2k\log p)\Big\}\end{aligned} \tag{1.56}$$

the parabolic contribution[35, p.8].

1.8 Real Quadratic Fields

It is known that conjugacy classes of hyperbolic elements of $\mathrm{SL}_2(\mathbb{Z})$ are represented by units ϵ_d in orders $\mathfrak{r}[f]$ in real quadratic fields $\mathbb{Q}\left(\sqrt{d}\right)$, with multiplicity h_d= the narrow class numbers of the order [48, pg 276] and in particular, the hyperbolic sums in the trace formula for $\mathcal{O}^1_{pq}$ and $\Gamma_0(p^{2r+1}q^{2s+1})$ can be enumerated in terms of class numbers and fundamental units of quadratic field extensions of $\mathbb{Q}$ [46]. In [37] there is a description of this problem in the language of quadratic forms which goes all the way back to Gauss. We recall for the convenience of the reader some ideas associated with real quadratic fields which will prove crucial to establishing the main theorem. Our exposition is based on [46, Section 3].

The non isomorphic quadratic extensions of $\mathbb{Q}$ are given by

$$\mathbb{Q}\left(\sqrt{d}\right) = x + y\sqrt{d} \tag{1.57}$$

where x and y are in $\mathbb{Q}$ and d runs through all of the fundamental discriminants. I.e., the non zero integers d such that

1. $d \equiv 1 \mod 4$, d is square free, and $d \neq 1$ or

2. $d \equiv 0 \mod 4$, $\frac{d}{4} \not\equiv 0 \mod 4$ and $\frac{d}{4}$ is square free. Observe that this is equivalent to

   $d \equiv 8, 12 \mod 16$, and $\frac{d}{4}$ is square free

The norm and the trace of $\mathbb{Q}\left(\sqrt{d}\right)$ over $\mathbb{Q}$ are given by

$$\mathrm{N}\left(x + y\sqrt{d}\right) = x^2 - dy^2 \tag{1.58}$$

and

$$\mathrm{Tr}\left(x + y\sqrt{d}\right) = 2x \tag{1.59}$$

for $x, y \in \mathbb{Q}$.

Let's now recall that for an element $\alpha \in \mathcal{A}$,

$$P(X) = X^2 - \mathrm{Tr}(\alpha)X + \mathrm{N}(\alpha), \tag{1.60}$$

is called the principal polynomial of α. Evidently, $P(\alpha) = 0$ and if $\alpha \notin \mathbb{Q}$ then $P(\alpha)$ is the minimal polynomial of α over $\mathbb{Q}$. If $P(X)$ is irreducible over $\mathbb{Q}$, then $\mathbb{Q}(\alpha)$ is a quadratic extension of $\mathbb{Q}$ and the restriction of the trace and norm maps of the algebra to the quadratic extension $\mathbb{Q}(\alpha)$ coincides with the trace and norm of the field $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$[34, pg 200].

$P(X)$ is irreducible over $\mathbb{Q}$ if and only if $\sqrt{t^2 - 4n} \notin \mathbb{Q}$. When this happens, there are unique numbers l and d such that

$$t^2 - 4n = l^2 d,$$

where $l \in \mathbb{Z}^+$ and d is a fundamental discriminant. It follows that the splitting field of $P(X)$ is

$$\mathbb{Q}\left(\sqrt{t^2-4n}\right) = \mathbb{Q}\left(\sqrt{d}\right) \tag{1.61}$$

and that the zeros of $P(X)$ are

$$x_1 = \frac{t}{2} + \frac{l\sqrt{d}}{2} \tag{1.62}$$

and

$$x_2 = \frac{t}{2} - \frac{l\sqrt{d}}{2}. \tag{1.63}$$

To reiterate: given a quaternion algebra $\mathcal{A}$ over $\mathbb{Q}$, we can naturally associate to a set of elements in $\mathcal{A}$ of fixed trace t and fixed norm n, a polynomial,

$$P(X) = X^2 - tX + n, \tag{1.64}$$

and a quadratic extension, $\mathbb{Q}\left(\sqrt{d}\right)$, which arises as the splitting field of this polynomial.

1.8.1 Orders in $\mathbb{Q}\left(\sqrt{d}\right)$

Given any d in the set of fundamental discriminants, we define for each $f \in \mathbb{Z}^+$,

$$\mathfrak{r}[f] = \mathbb{Z} + f\omega\mathbb{Z}, \tag{1.65}$$

where

$$\omega = \frac{d+\sqrt{d}}{2}. \tag{1.66}$$

$\mathfrak{r}[f]$ is called an order of conductor f in $\mathbb{Q}\left(\sqrt{d}\right)$. $\mathfrak{r}[1]$, $\mathfrak{r}[2]$, $\mathfrak{r}[3]$, ..., are all distinct orders in $\mathbb{Q}\left(\sqrt{d}\right)$. In particular, $\mathfrak{r}[1]$ is the unique maximal order in $\mathbb{Q}\left(\sqrt{d}\right)$. We note that

1. $f_2|f_1 \Longleftrightarrow \mathfrak{r}[f_1] \subset \mathfrak{r}[f_2]$.

2. The roots of $P(X)$, i.e., x_1 and x_2 satisfy

$$\mathbb{Z} + \mathbb{Z}x_1 = \mathbb{Z} + \mathbb{Z}x_2 = \mathfrak{r}[l],$$

   for $P(X)$, d, and l as above.

Fundamental units in $\mathfrak{r}$

For any order $\mathfrak{r} \in \mathbb{Q}\left(\sqrt{d}\right)$, we define

$$\mathfrak{r}^1 = \{\alpha \in \mathfrak{r} : N(\alpha) = 1\}. \tag{1.67}$$

Evidently, $\mathfrak{r}^1$ is a group under multiplication. We refer to $\mathfrak{r}^1$ as the group of units in $\mathfrak{r}$. If d is negative then $\mathfrak{r}^1$ is of finite cardinality[34, Remark 6.7.1, pg 259]. We will only consider values of $d > 0$. If d is positive then $\mathfrak{r}[f]^1$ is an infinite cyclic group of finite index in $\mathfrak{r}[1]^1$. The generator of the infinite cyclic group is called the fundamental unit of the real quadratic field[48]. We define the proper fundamental unit, ϵ_d, in $\mathbb{Q}\left(\sqrt{d}\right)$ by

$$\epsilon_d = \frac{x + y\sqrt{d}}{2}, \tag{1.68}$$

where $\langle x, y\rangle$ is the positive integer solution to $x^2 - dy^2 = 4$ for which y is minimal. See [37, Section 1] for a related discussion in the language of quadratic forms. This ϵ_d always exists. It is a member of $\mathfrak{r}[1]^1$ and the groups $\mathfrak{r}[f]^1$ are given by

$$\mathfrak{r}[f]^1 = \left\{\pm\left(\epsilon_d^A\right)^k | k \in \mathbb{Z}\right\}, \tag{1.69}$$

where

$$A = \left[\mathfrak{r}[1]^1 : \mathfrak{r}[f]^1\right]. \tag{1.70}$$

See[11, Ch. V1, Secs. 4 and 5].

Finally, we also recall here some general definitions. Let A be any finite dimensional algebra over $\mathbb{Q}$. For each place v of $\mathbb{Q}$, i.e., for $v = p$, p- a prime, and $v = \infty$, we define

$$A_v = A \otimes_{\mathbb{Q}} \mathbb{Q}_v. \tag{1.71}$$

A_v is an algebra over $\mathbb{Q}_v$ and the dimension of A_v over $\mathbb{Q}_v$ is the same as the dimension of A over $\mathbb{Q}$. For each $v = p$ we define an order in A_p as follows: A subset $\mathcal{O}_p \subset A_p$ is called an order in A_p if the following three conditions are satisfied:

1. $\mathcal{O}_p$ is a finitely generated $\mathbb{Z}_p$-module.

2. $\mathcal{O}_p$ contains a basis for A_p over $\mathbb{Z}_p$

3. $\mathcal{O}_p$ is a subring of A_p, and $1_{A_p} \in \mathcal{O}_p$

If $\mathcal{O}$ is an order in A, taking the closure of $\mathcal{O}$ in A_p results in the order $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in A_p[34, Lemma 5.11].

Chapter 2

A spectral Correspondence

We now establish the claimed spectral correspondence between Maaß forms on a cocompact Fuchsian group derived from an indefinite rational division quaternion algebra and Maaß newforms on a related non-cocompact but cofinite Hecke congruence group. Strictly speaking, we establish a correspondence between newforms on the cocompact group and newforms on the related Hecke congruence group.

2.1 Risager's Problem Revisited

For the convenience of the reader we begin by recalling the key elements of the narrative outlined in the introduction. Let d be a product of an even number of different primes and let $\Gamma_0(d)$ be the Hecke congruence group of level d. The spectral counting function, $N_{\Gamma_0(d)}(\lambda)$, has an asymptotic expansion of form

$$N_{\Gamma_0(d)}(\lambda) = \frac{Vol(\Gamma_0(d)\backslash\mathcal{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log\lambda)$$

while the spectral counting function, $N^{new}_{\Gamma_0(d)}(\lambda)$, which counts only the new forms on $\Gamma_0(d)$, has an expansion of form

$$N^{new}_{\Gamma_0(d)}(\lambda) = C_d \frac{Vol(\Gamma_0(d)\backslash\mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda}).$$

This is precisely the form of the expansion characteristic of the cocompact group $\mathcal{O}^1$! We recall that

$$N_{\mathcal{O}^1}(\lambda) = \frac{Vol(\mathcal{O}^1\backslash\mathcal{H})}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda}).$$

We will set $Vol(\mathcal{O}^1\backslash\mathcal{H}) = \mathcal{A}_{\mathcal{O}^1}$ and $Vol(\Gamma_0(d)\backslash\mathcal{H}) = \mathcal{A}_d$ and $\mathcal{A}^{new}_d = C_d\mathcal{A}_d$. Spectral counting functions for newforms on $\Gamma_0(d)$ which have asymptotic expansions of form

$$N^{new}_{\Gamma_0(d)}(\lambda) = \frac{\mathcal{A}^{new}_d}{4\pi}\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda})$$

are defined to be of cocompact type. I.e., if

$$N^{new}_{\Gamma_0(M)}(\lambda) = C_M\lambda + O(\frac{\sqrt{\lambda}}{\log\lambda})$$

for some constant C_M then $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type. Risager asks, are there values of M not equal to the product of an even number of different primes for which $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type? In Theorem 0.2.1 he answers this question in the affirmative and characterises those M's for which $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type as follows: Suppose M, n and $t \in \mathbb{N}$ are integers defined uniquely by the requirement that n should be squarefree and $M = nt^2$ then $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type iff n, t satisfies one of the following conditions: either n contains at least two primes or n is a prime and $4||M$.

Evidently, there are a number of cases where $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type and M is not a product of an even number of different primes. Now, let's suppose that $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type. Does this imply the existence of a group $\mathcal{O}^1$ such that $N_{\mathcal{O}^1}(\lambda)$ coincides with $N^{new}_{\Gamma_0(M)}(\lambda)$? I.e., are there spectral correspondences responsible for the remaining cases of Theorem 0.2.1? We answer yes for some values of M not covered by the theorems of Bolte and Johansson [6], Strömbergsson [47] and Risager [36]. In particular we show that if $\mathcal{O}_{pq} \subset \mathcal{A}$ is a maximal order in an indefinite rational division quaternion algebra with discriminant $d = pq$ where p and q are primes such that $p, q > 2$ and $\mathcal{O}_{pq,p^{2r}q^{2s}}$ is an order of index $p^{2r}q^{2s}$, $r, s \geq 1$, in $\mathcal{O}_{pq}$ with Eichler invariant at p and at q equal to negative one then the main term, i.e., the term that corresponds to the area of the fundamental domain, of $N^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}(\lambda)$ coincides with the main term of $N^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}(\lambda)$. We also show that the positive Laplace eigenvalues, including multiplicities for Maaß newforms on $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ coincides with the Laplace spectrum on Maaß newforms for the Hecke congruence group $\Gamma_0(p^{2r+1}q^{2s+1})$. We situate our contribution to the identification of correspondences predicted in Part I of Theorem 0.2.1 by placing it within the context of the correspondences covered by the theorems of Bolte and Johansson [6], Strömbergsson [47] and Risager [36].

2.1.1 Strömbergsson: $n = \prod_{i=1}^{2n} p_i,\ t^2 = 1.$

Let $M = nt^2$, where n is the product of an even number of different primes and $t^2 = 1$. By Theorem 0.2.1, $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type.

Proposition 2.1.1. *If $\mathcal{O}^1 \subset \mathcal{O} \subset \mathcal{A}$ is the group of units in a maximal order $\mathcal{O}$ in an indefinite rational division quaternion algebra $\mathcal{A}$ with discriminant $d = n$ then $N^{new}_{\mathcal{O}^1}(\lambda)$ coincides with $N^{new}_{\Gamma_0(M)}(\lambda)$.*

Proof. This correspondence is described classically in [47] where the following is proven:

> Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion division algebra over $\mathbb{Q}$, and let $d = d(\mathcal{O})$ be its (reduced) discriminant. This is always a *squarefree integer with an even number of prime factors*. The norm one unit group $\mathcal{O}^1$ can be viewed as a Fuchsian group *which is cocompact*. Then:
>
> *The eigenvalues of the Laplacian on $\mathcal{O}^1 \setminus \mathcal{H}$ are exactly the same (with multiplicities) as the eigenvalues corresponding to the newspace on $\Gamma_0(d) \setminus \mathcal{H}$.*

That $N^{new}_{\mathcal{O}^1}(\lambda)$ coincides with $N^{new}_{\Gamma_0(M)}(\lambda)$ follows immediately from this result since $N^{new}_{\mathcal{O}^1}(\lambda) = N_{\mathcal{O}^1}(\lambda)$. This is also a result of Bolte and Johansson [7]. See also [35, p.10]. □

2.1.2 Risager: $n = \prod_{i=1}^{2n} p_i,\ t^2 \neq 1,\ (n, t^2) = 1$

Let $M = nt^2$, where n is the product of an even number of different primes and $t^2 \neq 1$, with $(n, t^2) = 1$. Then, by Theorem 0.2.1, $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type.

Proposition 2.1.2. *If $\mathcal{O}^1_{n,t^2} \subset \mathcal{O}_{n,t^2} \subset \mathcal{O}_n \subset \mathcal{A}$ is the group of units in an Eichler order $\mathcal{O}_{n,t^2}$ of index t^2 in a maximal order $\mathcal{O}_n$ in an indefinite rational division quaternion algebra $\mathcal{A}$ with discriminant $d = n$ then $N_{\mathcal{O}^1}(\lambda)$ coincides with $N^{new}_{\Gamma_0(M)}(\lambda)$.*

Proof. This is Theorem B of [36]. □

Note, in this case, it is enough that at least two primes divide n which do not divide t. We take an even number of these primes for the discriminant d of the quaternion algebra and we take the Eichler order of of index $\frac{M}{d}$ in a maximal order $\mathcal{O}$. We note that this is an order of Eichler invariant 1.

2.1.3 Blackman: $n = pq,\ t^2 = (p^r q^s)^2,\ (n, t^2) \neq 1$

Let $M = nt^2$ where n is the product of two primes p and q such that $p, q > 2, p \neq q$ and $t^2 = (p^r q^s)^2$ where $r, s \geq 1$. I.e.,

$$M = p^{2r+1} q^{2s+1} \tag{2.1}$$

is a product of two different odd primes raised to odd powers. By Theorem 0.2.1, $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type.

Proposition 2.1.3 (Blackman). *Let $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}_{pq,p^{2r}q^{2s}} \subset \mathcal{O}_{pq} \subset \mathcal{A}$ be the group of units in an order of index $p^{2r}q^{2s}$ in a maximal order $\mathcal{O}_{pq}$ in an indefinite rational division quaternion algebra $\mathcal{A}$ with discriminant $d = pq$. Further, let $e(\mathcal{O}_{pq,p^{2r}q^{2s}})_p = e(\mathcal{O}_{pq,p^{2r}q^{2s}})_q = -1$. Then, the main term, i.e., the term that corresponds to the area of the fundamental domain, of $N^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}(\lambda)$ coincides with the main term of $N^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}(\lambda)$. I.e.,*

$$A^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = A^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}. \tag{2.2}$$

Proof. We prove this result by comparing the Selberg trace formula for the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ with the trace formula for the related Hecke congruence group $\Gamma_0(p^{2r+1}q^{2s+1})$. We effect the comparison by evaluating the identity term on the geometric side of the trace formula for the newspace of $\Gamma_0(p^{2r+1}q^{2s+1})$ and showing that this is equal to the identity term of geometric side of the trace formula for the newspace of $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$.

Evaluation of $A^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$

We begin with the identity term on the right-hand side of the trace formula in Proposition 1.7.2. The formula (1.41) implies that

$$\sum_{r_j \in Spec\left(\Delta^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}\right)} h(r_j) = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) \sum_{r_j \in Spec\left(\Delta_{\Gamma_0(k)}\right)} h(r_j). \tag{2.3}$$

As such,

$$A^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) A_k \tag{2.4}$$

where

$$A_{p^r q^s} = p^{r-1}\,(p+1)\,q^{s-1}\,(q+1)\,. \tag{2.5}$$

We recall that if n is a product of r distinct primes, then $\beta(n) = (-2)^r$ and in particular,

$\beta(1) = 1, \beta(p) = -2, \beta(p^2) = 1$, and $\beta(p^k) = 0, k > 2$. Consequently

$$\begin{aligned}\sum_{k|p^{2r+1}q^{2s+1}} \beta\Big(\frac{p^{2r+1}q^{2s+1}}{k}\Big)\,A_k =& \beta(1)\,A_{p^{2r+1}q^{2s+1}} + \beta(p)\,A_{\frac{p^{2r+1}q^{2s+1}}{p}} + \beta(q)\,A_{\frac{p^{2r+1}q^{2s+1}}{q}}\\ &+ \beta(pq)\,A_{\frac{p^{2r+1}q^{2s+1}}{pq}} + \beta\big(p^2\big)\,A_{\frac{p^{2r+1}q^{2s+1}}{p^2}} + \beta\big(q^2\big)\,A_{\frac{p^{2r+1}q^{2s+1}}{q^2}}\\ &+ \beta\big(p^2q\big)\,A_{\frac{p^{2r+1}q^{2s+1}}{p^2q}} + \beta\big(pq^2\big)\,A_{\frac{p^{2r+1}q^{2s+1}}{pq^2}} + \beta\big(p^2q^2\big)\,A_{\frac{p^{2r+1}q^{2s+1}}{p^2q^2}}\,.\end{aligned} \tag{2.6}$$

It follows that

$$\begin{aligned}\sum_{k|p^{2r+1}q^{2s+1}} \beta\Big(\frac{p^{2r+1}q^{2s+1}}{k}\Big)\,A_k =& p^{2\,r}\,(p+1)\,q^{2\,s}\,(q+1)\\ &- 2\,p^{2\,r-1}\,(p+1)\,q^{2\,s}\,(q+1)\\ &- 2\,p^{2\,r}\,(p+1)\,q^{2\,s-1}\,(q+1)\\ &+ 4\,p^{2\,r-1}\,(p+1)\,q^{2\,s-1}\,(q+1)\\ &+ p^{2\,r-2}\,(p+1)\,q^{2\,s}\,(q+1)\\ &+ p^{2\,r}\,(p+1)\,q^{2\,s-2}\,(q+1)\\ &- 2\,p^{2\,r-2}\,(p+1)\,q^{2\,s-1}\,(q+1)\\ &- 2\,p^{2\,r-1}\,(p+1)\,q^{2\,s-2}\,(q+1)\\ &+ p^{2\,r-2}\,(p+1)\,q^{2\,s-2}\,(q+1)\,,\end{aligned} \tag{2.7}$$

which yields

$$\sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) A_k = (q+1)\,(p+1)\,(q-1)^2\, q^{2\,s-2}\,(p-1)^2\, p^{2\,r-2}. \tag{2.8}$$

I.e.,

$$A^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = (q+1)\,(p+1)\,(q-1)^2\, q^{2\,s-2}\,(p-1)^2\, p^{2\,r-2}. \tag{2.9}$$

Evaluation of $A^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$

As above, we start with the right hand side of the trace formula in Proposition 1.7.1. The formula (1.45) implies that

$$\sum_{r_j \in Spec\left(\Delta^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}\right)} h(r_j) = \sum_{k|p^r q^s} \mu\left(\frac{p^r q^s}{k}\right) \sum_{r_j \in Spec\left(\Delta_{\mathcal{O}^1_{pq,k^2}}\right)} h(r_j), \tag{2.10}$$

where μ is the Möbius function, i.e.,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathcal{O}^1_{pq,k^2}$ is a unit group in the order $\mathcal{O}_{pq,k^2}$. I.e., the order of index k^2 in the maximal order $\mathcal{O}_{pq}$. As such

$$A^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}} = \sum_{k|p^r q^s} \mu(\frac{p^r q^s}{k}) A_{\mathcal{O}^1_{pq,k^2}} \tag{2.11}$$

where

$$A_{\mathcal{O}^1_{pq,k^2}} = k^2 (p-1)(q-1). \tag{2.12}$$

I.e., $A_{\mathcal{O}^1_{pq,k^2}}$ is the area of the fundamental domain of $\mathcal{O}^1_{pq,k^2} \backslash \mathcal{H}$.

We observe that

$$\sum_{k|p^r q^s} \mu(\frac{p^r q^s}{k}) A_{\mathcal{O}^1_{pq,k^2}} = (p-1)\,(q-1)\left(p^{2\,r}q^{2\,s} - p^{2\,r-2}q^{2\,s} - p^{2\,r}q^{2\,s-2} + p^{2\,r-2}q^{2\,s-2}\right)$$

Evidently,

$$A^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = A^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}. \tag{2.13}$$

We have thus established the equality of the terms corresponding to the area of the fundamental domain on the right hand side of the trace formula for the newspace of the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ and that of right hand side of the trace formula for the newspace of the non cocompact group $\Gamma_0(p^{2r+1}q^{2s+1})$ and consequently the coincidence of the main terms of the counting functions $N^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}(\lambda)$ and $N^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}(\lambda)$. Note: we can allow for an additional factor N so that $M = p^{2r+1}q^{2s+1}N$. It is enough that N be such that all of its prime divisors occur at least to the second power. □

2.2 Main Theorem

Finally, our main theorem:

Theorem 2.2.1 (Blackman). *Let $\mathcal{O}^1_{pq,p^{2r}q^{2s}} \subset \mathcal{O}_{pq,p^{2r}q^{2s}} \subset \mathcal{O}_{pq} \subset \mathcal{A}$ be the group of units in an order of index $p^{2r}q^{2s}$ in a maximal order $\mathcal{O}_{pq}$ in an indefinite rational division quaternion algebra $\mathcal{A}$ with $d_{\mathcal{A}} = pq$. Further, let $e(\mathcal{O}_{pq,p^{2r}q^{2s}})_p = e(\mathcal{O}_{pq,p^{2r}q^{2s}})_q = -1$. Then the positive Laplace eigenvalues, including multiplicities for Maaß newforms on $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$ coincide with the Laplace spectrum on Maaß newforms for the Hecke congruence group $\Gamma_0(p^{2r+1}q^{2s+1})$ where $r, s \geq 1$.*

Proof. It suffices to show that

$$\sum_{r_k \in Spec\left(\Delta^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}\right)} h(r_k) = \sum_{r_k \in Spec\left(\Delta^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}\right)} h(r_k) \tag{2.14}$$

for an arbitrary test function h.

In order to prove (2.14), we will once more use the right-hand sides of the trace formulae in Propositions 1.7.1 and 1.7.2. To be able to compute with the trace formula on the right-hand side of (2.14), we again observe that the formula (1.41) implies

$$\sum_{r_k \in Spec\left(\Delta^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}\right)} h(r_k) = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) \sum_{r_k \in Spec\left(\Delta_{\Gamma_0(k)}\right)} h(r_k). \tag{2.15}$$

.

To compute with the trace formula on the left-hand side of (2.14), we observe that (1.45) implies

$$\sum_{r_k \in Spec\left(\Delta^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}\right)} h(r_k) = \sum_{k|p^r q^s} \mu\left(\frac{p^r q^s}{k}\right) \sum_{r_k \in Spec\left(\Delta_{\mathcal{O}^1_{pq,k^2}}\right)} h(r_k). \tag{2.16}$$

.

Using Propositions 1.7.1 and 1.7.2, we therefore have to show that

$$\sum_{k|p^r q^s} \mu\left(\frac{p^r q^s}{k}\right) \sum_{r_k \in Spec\left(\Delta_{\mathcal{O}^1_{pq,k^2}}\right)} h(r_k) = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) \sum_{r_k \in Spec\left(\Delta_{\Gamma_0(k)}\right)} h(r_k)$$

.

We will execute this by way of four lemmas:

Lemma 2.2.2 (Equivalence of Identity contributions)**.**

$$\mathcal{I}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = \mathcal{I}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}. \tag{2.18}$$

Proof. For the terms corresponding to the identity on the right-hand sides of the trace formulae we need to show that

$$\sum_{k|p^r q^s} \mu\left(\frac{p^r q^s}{k}\right) A_{\mathcal{O}^1_{pq,k^2}} = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) A_k.$$

This is the content of Proposition 2.1.3. □

2.2.1 The elliptic contribution

Lemma 2.2.3 (Equivalence of Elliptic contributions)**.**

$$\mathcal{E}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = \mathcal{E}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}. \tag{2.19}$$

Proof. Recall that $\mathcal{E}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}$ is the elliptic term in the newspace for the non-cocompact group $\Gamma_0(p^{2r+1}q^{2s+1})$ and $\mathcal{E}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}$ is the elliptic term in the newspace for the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$. In the elliptic term of the trace formula, only the numbers $E'(t,1,\Gamma)$ depend on the group Γ. That these terms are identical is equivalent to

$$\sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{r+1}q^{s+1}}{k}\right) E'\big(t,1,\Gamma_0(k)\big) = \sum_{k|p^rq^s} \mu\left(\frac{p^rq^s}{k}\right) E'(t,1,\mathcal{O}^1_{pq,k^2}) \tag{2.20}$$

for all traces $t \in \{0,1\}$. Recall that $E'\big(t,1,\Gamma\big)$ is the number of conjugacy classes of elements in Γ with trace t and norm =1.

We denote by

$$E'(0,1,\Gamma_0(p^{2r+1}q^{2s+1}))^{new} = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(0,1,\Gamma_0(k)\big). \tag{2.21}$$

When $t=0$ we have

$$E'(0,1,\Gamma_0(p^{2r+1}q^{2s+1}))^{new} = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(0,1,\Gamma_0(k)\big).$$

To count the number of conjugacy classes of elements in $\Gamma_0(p^{2r+1}q^{2s+1})$ with trace $t=0$ and norm equal to one we recall that if $\gamma \in \Gamma$ is of trace zero then γ has one fixed point in $\mathcal{H}$ and γ is an order 2 elliptic transformation. Hence what is required is the number of inequivalent elliptic fixed points of order two on $\Gamma_0(p^{2r+1}q^{2s+1})\backslash\mathcal{H}$.

It follows from [43, Prop 1.43], see also [23], that

$$E'(0,1,\Gamma_0(p^k)) = \begin{cases} 2 & \text{if } p \equiv 1 \bmod 4, \\ 1 & \text{if } p = 2 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and that

$$E'(0,1,\Gamma_0(p^k q^l)) = E'(0,1,\Gamma_0(p^k))E'(0,1,\Gamma_0(q^l))$$

so consequently

$$E'(0,1,\Gamma_0(p^k q^l) = \begin{cases} 4 & \text{if } p \equiv 1 \bmod 4 \text{ and } q \equiv 1 \bmod 4, \\ 2 & \text{if } p = 2,\ k = 1 \text{ and } q \equiv 1 \bmod 4 \text{ or } q = 2,\ l = 1 \text{ and } p \equiv 1 \bmod 4\ , \\ 0 & \text{otherwise.} \end{cases}$$

Note: if $p = 2$, $k = 1$ and $q = 2$, $l = 1$ then $pq = 4$ and $\Gamma_0(4)$ has no elliptic points of order two.

We can now compute $E'(0,1,\Gamma_0(p^{2r+1}q^{2s+1}))^{new}$

$$\begin{aligned} \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(0,1,\Gamma_0(k)\big) =& \beta(1)\, E'\big(0,1,\Gamma_0(p^{2r+1}q^{2s+1})\big) + \beta(p)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p})\big) \\ &+ \beta(q)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{q})\big) + \beta(pq)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{pq})\big) \\ &+ \beta\big(p^2\big)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p^2})\big) + \beta\big(q^2\big)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{q^2})\big) \\ &+ \beta\big(p^2q\big)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p^2q})\big) + \beta\big(pq^2\big)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{pq^2})\big) \\ &+ \beta\big(p^2q^2\big)\, E'\big(0,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p^2q^2})\big). \end{aligned} \tag{2.22}$$

It follows that

$$\begin{aligned}\sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(0,1,\Gamma_0(k)\big) =& E'\big(0,1,\Gamma_0(p^{2r+1}q^{2s+1})\big) - 2E'\big(0,1,\Gamma_0(p^{2r}q^{2s+1})\big) \\ & - 2E'\big(0,1,\Gamma_0(p^{2r+1}q^{2s})\big) + 4E'\big(0,1,\Gamma_0(p^{2r}q^{2s})\big) \\ & + E'\big(0,1,\Gamma_0(p^{2r-1}q^{2s+1})\big) + E'\big(0,1,\Gamma_0(p^{2r+1}q^{2s-1})\big) \\ & - 2E'\big(0,1,\Gamma_0(p^{2r-1}q^{2s})\big) - 2E'\big(0,1,\Gamma_0(p^{2r}q^{2s-1})\big) \\ & + E'\big(0,1,\Gamma_0(p^{2r-1}q^{2s-1})\big).\end{aligned} \tag{2.23}$$

If we assume that r and s are greater than or equal to one and that p and q are greater than two then there are only two possibilities:

1. All $E'\left(0,1,\Gamma_0(p^kq^l)\right) = 0 \Longrightarrow E'\left(0,1,\Gamma_0(p^{2r+1}q^{2s+1})\right)^{new} = 0$ or

2. All $E'\left(0,1,\Gamma_0(p^kq^l)\right) = 4 \Longrightarrow E'\left(0,1,\Gamma_0(p^{2r+1}q^{2s+1})\right)^{new} = 0$.

The elliptic contribution to the non-cocompact newspace with trace=0 is zero!

In a similar manner, when $t = 1$ we have

$$E'(1,1,\Gamma_0(p^{2r+1}q^{2s+1}))^{new} = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(1,1,\Gamma_0(k)\big).$$

To count the number of conjugacy classes of elements in $\Gamma_0(p^{2r+1}q^{2s+1})$ with trace $t = 1$ and norm equal one we once again recall that if $\gamma \in \Gamma$ is of trace one then γ has fixed point in $\mathcal{H}$ and γ is an order 3 elliptic transformation. Hence what is also required is the number of inequivalent elliptic fixed points of order three on $\Gamma_0(p^{2r+1}q^{2s+1})\backslash\mathcal{H}$. We once again appeal to [43, Prop 1.43] which shows that

$$E'(1,1,\Gamma_0(p^k)) = \begin{cases} 2 & \text{if } p \equiv 1 \bmod 3\ , \\ 1 & \text{if } p = 3 \text{ and } k = 1, \\ 0 & \text{if } p \equiv 2 \bmod 3, \end{cases}$$

and that

$$E'\left(1,1,\Gamma_0(p^k q^l)\right) = E'\left(1,1,\Gamma_0(p^k)\right) E'\left(1,1,\Gamma_0(q^l)\right)$$

so consequently

$$E'(1,1,\Gamma_0(p^k q^l)) = \begin{cases} 4 & \text{if } p \equiv 1 \bmod 3 \text{ and } q \equiv 1 \bmod 3, \\ 2 & \text{if } p = 3,\ k = 1 \text{ and } q \equiv 1 \bmod 3 \text{ or } q = 3,\ l = 1 \text{ and } p \equiv 1 \bmod 3\ , \\ 0 & \text{otherwise.} \end{cases}$$

Note: if $p = 3$, $k = 1$ and $q = 3$, $l = 1$ then $pq = 9$ and $\Gamma_0(9)$ has no elliptic points of order three.

We can now compute $E'(1,1,\Gamma_0(p^{2r+1}q^{2s+1})^{new}$

$$\begin{aligned} \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(1,1,\Gamma_0(k)\big) =& \beta(1)\, E'\big(1,1,\Gamma_0(p^{2r+1}q^{2s+1})\big) + \beta(p)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p})\big) \\ &+ \beta(q)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{q})\big) + \beta(pq)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{pq})\big) \\ &+ \beta\big(p^2\big)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p^2})\big) + \beta\big(q^2\big)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{q^2})\big) \\ &+ \beta\big(p^2q\big)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p^2q})\big) + \beta\big(pq^2\big)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{pq^2})\big) \\ &+ \beta\big(p^2q^2\big)\, E'\big(1,1,\Gamma_0(\frac{p^{2r+1}q^{2s+1}}{p^2q^2})\big). \end{aligned} \tag{2.24}$$

It follows that

$$\begin{aligned}\sum_{k\mid p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(1,1,\Gamma_0(k)\big) =& E'\big(1,1,\Gamma_0(p^{2r+1}q^{2s+1})\big) - 2E'\big(1,1,\Gamma_0(p^{2r}q^{2s+1})\big)\\ &- 2E'\big(1,1,\Gamma_0(p^{2r+1}q^{2s})\big) + 4E'\big(1,1,\Gamma_0(p^{2r}q^{2s})\big)\\ &+ E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s+1})\big) + E'\big(1,1,\Gamma_0(p^{2r+1}q^{2s-1})\big)\\ &- 2E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s})\big) - 2E'\big(1,1,\Gamma_0(p^{2r}q^{2s-1})\big)\\ &+ E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s-1})\big).\end{aligned} \tag{2.25}$$

If we assume that r and s are greater than or equal to one and that p and q are greater than two this time there are three possible outcomes:

1. All $E'\left(1,1,\Gamma_0(p^k q^l)\right) = 0 \Longrightarrow E'\left(1,1,\Gamma_0(p^{2r+1}q^{2s+1})\right)^{new} = 0$.

2. All $E'\left(1,1,\Gamma_0(p^k q^l)\right) = 4 \Longrightarrow E'\left(1,1,\Gamma_0(p^{2r+1}q^{2s+1})\right)^{new} = 0$.

3. There are some cases in which $E'(1,1,\Gamma_0(p^k q^l) = 2$. These will occur when $p = 3$, $r = 1$ and $q \equiv 1 \bmod 3$. However, in this case we see that all of the terms in the sum above are zero except $E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s+1})\big)$, $E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s})\big)$, and $E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s-1})\big)$ which are all two. Evidently

$$E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s+1})\big) - 2E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s})\big) + E'\big(1,1,\Gamma_0(p^{2r-1}q^{2s-1})\big) = 0.$$

   The same applies if $q = 3$, $s = 1$ and $p \equiv 1 \bmod 3$.

Thus, the elliptic contribution to the non-cocompact newspace when $t = 1$ is zero and consequently, the contribution to the non-cocompact newspace of the elliptic conjugacy classes of $\Gamma_0\left(p^{2r+1}q^{2s+1}\right)$ is zero! I.e.,

$$E'\left(t,1,\Gamma_0(p^{2r+1}q^{2s+1})\right)^{new}=0. \tag{2.26}$$

Let's now consider the contribution of the elliptic conjugacy classes to the cocompact newspace. That is,

$$E'(t,1,\mathcal{O}^1_{pq,p^{2r}q^{2s}})^{new}=\sum_{k|p^rq^s}\mu\left(\frac{p^rq^s}{k}\right)E'(t,1,\mathcal{O}^1_{pq,k^2})$$

for traces $t\in\{0,1\}$.

As above we start with the case $t=0$. Recall that $E'\left(0,1,\mathcal{O}^1\right)$ is the number of conjugacy classes of elements in $\mathcal{O}^1$ with $t=0$ and norm $=1$. We recall that if $\gamma\in\mathcal{O}^1$ is of trace zero then γ has one fixed point in $\mathcal{H}$ and γ is an order two elliptic transformation. Hence what we require is the number of inequivalent elliptic fixed points of order two on $\mathcal{O}^1\backslash\mathcal{H}$. Specifically, we are required to compute

$$E'(0,1,\mathcal{O}^1_{pq,p^{2r}q^{2s}})^{new}=\sum_{k|p^rq^s}\mu\left(\frac{p^rq^s}{k}\right)E'(0,1,\mathcal{O}^1_{pq,k^2}).$$

Now

$$\begin{aligned}\sum_{k|p^rq^s}\mu(\frac{p^rq^s}{k})E'(0,1,\mathcal{O}^1_{pq,k^2})=&\mu(1)E'(0,1,\mathcal{O}^1_{pq,p^{2r}q^{2s}})\\&+\mu(p)E'(0,1,\mathcal{O}^1_{pq,p^{2r-2}q^{2s}})\\&+\mu(q)E'(0,1,\mathcal{O}^1_{pq,p^{2r}q^{2s-2}})\\&+\mu(pq)E'(0,1,\mathcal{O}^1_{pq,p^{2r-2}q^{2s-2}}).\end{aligned}\tag{2.27}$$

It follows that

$$\begin{aligned}\sum_{k|p^r q^s} \mu(\frac{p^r q^s}{k}) E'(0,1,\mathcal{O}^1_{pq,k^2}) =& E'(0,1,\mathcal{O}^1_{pq,p^{2r}q^{2s}}) \\ &- E'(0,1,\mathcal{O}^1_{pq,p^{2r-2}q^{2s}}) \\ &- E'(0,1,\mathcal{O}^1_{pq,p^{2r}q^{2s-2}}) \\ &+ E'(0,1,\mathcal{O}^1_{pq,p^{2r-2}q^{2s-2}}).\end{aligned} \tag{2.28}$$

From [26, Prop 5.9,Table 1], it follows that all terms on the right hand side of the equation above are either simultaneously four which occurs when both p and q are congruent to 3 mod 4 or zero otherwise. In either case

$$\sum_{k|p^r q^s} \mu(\frac{p^r q^s}{k}) E'(0,1,\mathcal{O}^1_{pq,k^2}) = 0.$$

A similar argument works for the case $t = 1$. Again, recall that $E'\big(1,1,\mathcal{O}^1)\big)$ is the number of conjugacy classes of elements in $\mathcal{O}^1$ with $t = 1$ and norm $= 1$. If $\gamma \in \mathcal{O}^1$ is of trace one then γ has fixed point in $\mathcal{H}$ and γ is an order three elliptic transformation. Thus what is required are the number of inequivalent elliptic fixed points of order three on $\mathcal{O}^1\backslash\mathcal{H}$. Specifically, we are required to compute

$$E'(1,1,\mathcal{O}^1_{pq,p^2q^2})^{new} = \sum_{k|p^r q^s} \mu\left(\frac{p^r q^s}{k}\right) E'(1,1,\mathcal{O}^1_{pq,k^2}).$$

As above, we have:

$$\begin{aligned}\sum_{k|p^rq^s}\mu\left(\frac{p^rq^s}{k}\right)E'\left(1,1,\mathcal{O}^1_{pq,k^2}\right)=&E'(1,1,\mathcal{O}^1_{pq,p^{2r}q^{2s}})\\&-E'(1,1,\mathcal{O}^1_{pq,p^{2r-2}q^{2s}})\\&-E'(1,1,\mathcal{O}^1_{pq,p^{2r}q^{2s-2}})\\&+E'(1,1,\mathcal{O}^1_{pq,p^{2r-2}q^{2s-2}}).\end{aligned}\tag{2.29}$$

and as before, an appeal to [26, Prop 5.9 and Table 1], shows that all terms on the right hand side of the equation above are either

1. Simultaneously all four if both p and q are congruent to 2 mod 3,
2. Simultaneously all two if either p or q is exactly three and the other congruent to 2 mod 3.
3. Simultaneously all zero otherwise.

In each case

$$\sum_{k|p^rq^s}\mu(\frac{p^rq^s}{k})E'(1,1,\mathcal{O}^1_{pq,k^2})=0.$$

Thus what we have shown is that

$$\sum_{k|p^rq^s}\mu(\frac{p^rq^s}{k})E'(t,1,\mathcal{O}^1_{pq,k^2})=\sum_{k|p^{2r+1}q^{2s+1}}\beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right)E'\big(t,1,\Gamma_0(k)\big)$$

for $t\in\{0,1\}$. I.e., the identity of the elliptic contributions on

$$\sum_{r_k\in Spec\left(\Delta^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}}\right)}h(r_k)$$

and on

$$\sum_{r_k \in Spec\left(\Delta^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}\right)} h(r_k).$$

It remains now for us to show that

$$\sum_{k|p^r q^s} \mu(\frac{p^r q^s}{k}) E'(t,1,\mathcal{O}^1_{pq,k^2}) = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(t,1,\Gamma_0(k)\big)$$

for $t \in (2,\infty)$, i.e., the hyperbolic contributions are the same and that

$$\sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(2,1,\Gamma_0(k)\big) = 0.$$

I.e., the parabolic terms vanish.

□

2.2.2 The parabolic contribution

Lemma 2.2.4 (The Parabolic contribution)**.**

$$\mathcal{P}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})} = 0. \tag{2.30}$$

Proof. We show that the parabolic terms vanish. The number of inequivalent cusps for $\Gamma_0\left(p^{r+1}q^{s+1}\right)$ is given by

$$\sum_{d|p^{2r+1}q^{2s+1}} \varphi\left((d, \frac{p^{2r+1}q^{2s+1}}{d})\right)$$

where $\varphi\left((d, \frac{p^{2r+1}q^{2s+1}}{d})\right)$ is the Euler φ-function [43, Prop 1.43]. Evidently

$$\sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) \sum_{d|p^{2r+1}q^{2s+1}} \varphi\left((d, \frac{p^{2r+1}q^{2s+1}}{d})\right) = 0.$$

□

2.2.3 The hyperbolic contribution

Lemma 2.2.5 (Equivalence of Hyperbolic contributions)**.**

$$\mathcal{H}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}} = \mathcal{H}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}. \tag{2.31}$$

To establish this lemma we need to show that

$$\sum_{k|p^r q^s} \mu(\frac{p^r q^s}{k}) E'(t,1,\mathcal{O}^1_{pq,k^2}) = \sum_{k|p^{2r+1}q^{2s+1}} \beta\left(\frac{p^{2r+1}q^{2s+1}}{k}\right) E'\big(t,1,\Gamma_0(k)\big)$$

for all $t \geq 3$. To do so we will employ a variation of the strategy used in [7]. For $t, n \in \mathbb{Z}$, we define $E(t,n,\Gamma)$ to be the number of conjugacy classes $\{\gamma\}_\Gamma$ in Γ with $\mathrm{Tr}(\gamma) = t$ and $\mathrm{N}(\gamma) = n$ and we recall that the number of primitive conjugacy classes of elements in Γ with trace t and norm n is denoted by $E'(t,n,\Gamma)$.

Now, let $\mathcal{A}$ be an indefinite rational division quaternion algebra and $\mathbb{Q}\left(\sqrt{d}\right)$ a quadratic field extension of $\mathbb{Q}$. Further, let $\mathcal{O}$ and $\mathfrak{r}[f]$ be orders in $\mathcal{A}$ and $\mathbb{Q}\left(\sqrt{d}\right)$ respectively. Assume that there is an embedding $\varphi : \mathbb{Q}\left(\sqrt{d}\right) \longrightarrow \mathcal{A}$. The order $\mathfrak{r}[f] \subset \mathbb{Q}\left(\sqrt{d}\right)$ is said to be optimally embedded into an order $\mathcal{O} \subset \mathcal{A}$ with respect to φ if $\varphi(\mathfrak{r}[f]) = \mathcal{O} \cap \varphi(\mathbb{Q}\left(\sqrt{d}\right))$[8, p.177] [3, Chapter 4]. Two optimal embeddings, φ_1 and φ_2, are conjugate by γ if $\varphi_1(\mathfrak{r}[f]) = \gamma \cdot \varphi_2(\mathfrak{r}[f]) \cdot \gamma^{-1}$. We denote the number of optimal embeddings of $\mathfrak{r}[f]$ up to conjugation by elements in Γ by $E(\mathfrak{r}[f], \Gamma)$. It is known [51, p.96] that

$$E(t,n,\Gamma) = \sum_{\mathfrak{r}[f] \supseteq \mathbb{Z}[\gamma]} E(\mathfrak{r}[f], \Gamma), \tag{2.32}$$

where the sum is taken over all orders that contain an element γ with trace t and norm n. So, instead of comparing the numbers $E(t,n,\Gamma)$ for different groups Γ we can compare the number of optimal embeddings $E(\mathfrak{r}[f],\Gamma)$ of orders $\mathfrak{r}[f]$ in the quadratic field in K. This idea can be extended to the newforms part, $E(t,n,\Gamma)^{new}$, or to any linear combination of groups since

$$\begin{aligned} E(t,n,\Gamma)^{new} &= \sum_i k_i E(t,n,\Gamma_i) = \sum_i k_i \sum_{\mathfrak{r}[f]\supseteq\mathbb{Z}[\gamma]} E(\mathfrak{r}[f],\Gamma_i) \\ &= \sum_{\mathfrak{r}[f]\supseteq\mathbb{Z}[\gamma]} \sum_i k_i E(\mathfrak{r}[f],\Gamma_i) = \sum_{\mathfrak{r}[f]\supseteq\mathbb{Z}[\gamma]} E(\mathfrak{r}[f],\Gamma)^{new} \end{aligned}$$

where the k_i and Γ_i are the coefficients and groups that occur in the formula for the newforms part. Thus we can establish our result by computing and comparing the numbers $E(\mathfrak{r}[f],\Gamma)^{new}$ for the groups associated with $\Gamma=\Gamma_0(p^{2r+1}q^{2s+1})$ and for those associated with the cocompact group $\mathcal{O}^1_{pq,p^{2r}q^{2s}}$, i.e., $\mathcal{O}^1_{pq,p^{2(r-i)}q^{2(s-j)}}$ with $i,j\in\{0,1\}$.

We recall that f is multiplicative if $f(mn)=f(m)f(n)$ whenever $(m,n)=1$. Suppose f and g are multiplicative functions and

$$h(d)=\sum_{k|d} f(k)g\left(\frac{d}{k}\right),$$

then also h is multiplicative. Evidently, both μ and β are multiplicative functions and since $E(\mathfrak{r}[f],\Gamma)$ is also multiplicative it follows that $E(\mathfrak{r}[f],\Gamma)^{new}$ is also multiplicative. I.e., the numbers $E(\mathfrak{r}[f],\Gamma)^{new}$ are multiplicative in the sense that they can be computed as the product of the newforms part for each prime, as such:

$$E(\mathfrak{r}[f],\Gamma)^{new} = E(\mathfrak{r}[f],\Gamma)_p^{new} E(\mathfrak{r}[f],\Gamma)_q^{new}.$$

Thus what we need to compute and compare are the local embedding numbers

$$E(\mathfrak{r}[f],\Gamma_0(p^{2r+1}))_p^{new} = \sum_{i=0}^{2r+1} \beta(p^i) E(\mathfrak{r}[f],\Gamma_0(p^{2r+1-i}))_p \tag{2.33}$$

and

$$E(\mathfrak{r}[f],\mathcal{O}^1_{pq,p^{2r}})_p^{new} = \sum_{i=0}^{r} \mu(p^i) E(\mathfrak{r}[f],\mathcal{O}^1_{pq,p^{2(r-i)}})_p. \tag{2.34}$$

Let p be an odd prime and let $\mathbb{Q}_p$ be the field of the p-adic numbers. Further, let L be a quadratic extension of $\mathbb{Q}_p$. There are three different types of quadratic extensions of $\mathbb{Q}_p$. We characterize these types below: let d be a fundamental discriminant and p an odd prime. If $\left(\frac{d}{p}\right) = \begin{cases} 1 & \text{then } L \cong \mathbb{Q}_p \oplus \mathbb{Q}_p \\ 0 & \text{then } L \text{ a fully ramified extension of } \mathbb{Q}_p \\ -1 & \text{then } L \text{ an unramified extension of } \mathbb{Q}_p \end{cases}$

Now, let $\mathfrak{r}[f]_0$ be the maximal order in L. It is well known that any other order in L is of form $\mathfrak{r}[f]_i = \mathbb{Z}_p + p^i \mathfrak{r}[f]_0$ with $i \geq 0$[46, p.23]. The three propositions below give the local embedding numbers for the orders $\mathfrak{r}[f]_i$ in each of these three cases.

Proposition 2.2.6. *Let L be a ramified quadratic extension of $\mathbb{Q}_p$ where p is an odd prime, $\mathfrak{r}[f] = \mathfrak{r}[f]_0$ the maximal order in L and $\mathfrak{r}[f]_i = \mathbb{Z}_p + p^i \mathfrak{r}[f]_0$ with $i \geq 0$ a general* *order in L. Then the number of optimal embeddings are given by*

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = \begin{cases} 0, & i < k \\ p^k, & i = k \\ 2p^k, & i > k, \end{cases}$$

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = \begin{cases} 0, & i < k \\ p^{k-1}(p+1), & i \geq k, \end{cases}$$

$$E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = \begin{cases} p^k, & i = k \\ 0, & i \neq k. \end{cases}$$

Proposition 2.2.7. *Let L be an unramified quadratic extension of $\mathbb{Q}_p$ where p is an odd prime, $\mathfrak{r}\,[f] = \mathfrak{r}\,[f]_0$ the maximal order in L and $\mathfrak{r}\,[f]_i = \mathbb{Z}_p + p^i\mathfrak{r}\,[f]_0$ with $i \geq 0$ a general order in L. Then the number of optimal embeddings are given by*

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = \begin{cases} 0, & i \leq k \\ 2p^k, & i > k, \end{cases}$$

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = \begin{cases} 0, & i < k \\ p^k, & i = k, \\ p^{k-1}(p+1), & i > k, \end{cases}$$

$$E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = \begin{cases} 2, & i = 0 \\ 2p^{i-1}(p-1), & 1 \leq i \leq k \\ 0, & i > k. \end{cases}$$

Proposition 2.2.8. *Let L be a split quadratic extension of $\mathbb{Q}_p$ where p is an odd prime, $\mathfrak{r}\,[f] = \mathfrak{r}\,[f]_0$ the maximal order in L and $\mathfrak{r}\,[f]_i = \mathbb{Z}_p + p^i\mathfrak{r}\,[f]_0$ with $i \geq 0$ a general order* *in L. Then the number of optimal embeddings are given by*

$$E(\mathfrak{r}[f]_i, \Gamma_0(p^{2k+1}))_p = \begin{cases} 2, & i = 0 \\ 2p^{i-1}(p+1), & 0 < i \leq k, \\ 2p^k, & i > k, \end{cases}$$

$$E(\mathfrak{r}[f]_i, \Gamma_0(p^{2k}))_p = \begin{cases} 2, & i = 0 \\ 2p^{i-1}(p+1), & 0 < i < k, \\ p^{k-1}(p+2), & i = k \\ p^{k-1}(p+1), & i > k, \end{cases}$$

$$E(\mathfrak{r}[f]_i, \mathcal{O}^1_{pq,p^{2k}})_p = 0$$

Proofs of these formulas can be found in the preprint of Lemurell[30]. They can also be deduced from recursive formulas already extant in the literature. For $\mathcal{O}^1_{pq,p^{2k}}$ these formulas can be found in the paper by Brzezinski[8, Section 3] and for $\Gamma_0(p^{2k+1})$ in Vigneras[51].

We now compute and compare the new form part of the local embedding numbers in each of these three cases. I.e. we compute and compare

$$E(\mathfrak{r}[f], \Gamma_0(p^{2r+1}))_p^{new} \text{ and } E(\mathfrak{r}[f], \mathcal{O}^1_{pq,p^{2r}})_p^{new}.$$

Proposition 2.2.9. *Let L be a ramified quadratic extension of $\mathbb{Q}_p$ where p is an odd prime, $\mathfrak{r}[f] = \mathfrak{r}[f]_0$ the maximal order in L and $\mathfrak{r}[f]_i = \mathbb{Z}_p + p^i\mathfrak{r}[f]_0$ with $i \geq 0$ a general order in L. Then the new part of the number of optimal embeddings are given by*

$$E(\mathfrak{r}[f]_i, \Gamma_0(p^{2k+1}))_p^{new} = \begin{cases} p^i, & i = k-1 \\ -p^i, & i = k, \\ 0, & \textit{otherwise}, \end{cases}$$

$$E(\mathfrak{r}[f]_i, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = \begin{cases} -p^i, & i = k-1 \\ p^i, & i = k, \\ 0, & \textit{otherwise}. \end{cases}$$

In particular $E(\mathfrak{r}[f]_i, \Gamma_0(p^{2k+1}))_p^{new} = -E(\mathfrak{r}[f]_i, \mathcal{O}^1_{pq,p^{2k}})_p^{new}$ for all $i \geq 0$ and $k \geq 1$.

Proof. We use Proposition 2.2.6, formula (2.33) and the values of β to evaluate $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new}$ for all $i \geq 0$ and $k \geq 1$. We first observe that

$$\begin{aligned} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} &= \sum_{j=0}^{2k+1} \beta(p^j) E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1-j}))_p \\ &= E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p - 2E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p + E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p. \end{aligned}$$

If $i < k-1$, then

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 0$$

so $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 0$ in this case.

If $i = k-1$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = 0 \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = p^{k-1} = p^i \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = p^i$ in this case.

If $i = k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = p^i \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = p^{i-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{i-1} \end{cases}$$

so

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = p^i - 2p^{i-1}(p+1) + 2p^{i-1} = -p^i$$

in this case.

If $i > k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = 2p^k \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = p^{k-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{k-1} \end{cases}$$

so

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 2p^k - 2p^{k-1}(p+1) + 2p^{k-1} = 0$$

in this case and the proof for $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new}$ is complete.

We now use the formulas in Proposition 2.2.6, the formula (2.34) and the values of μ to evaluate $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new}$ for all $i \geq 0$ and $k \geq 1$. First we get

$$\begin{aligned} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} &= \sum_{j=0}^{k} \mu(p^j) E(S_i, \mathcal{O}^1_{pq,p^{2(k-j)}})_p \\ &= E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p - E(S_i, \mathcal{O}^1_{pq,p^{2(k-1)}})_p. \end{aligned}$$

If $i < k-1$ or $i > k$, then

$$E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = E(S_i, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = 0$$

so $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = 0$ in these cases.

If $i = k-1$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = 0 \\ E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = p^{k-1} = p^i \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = -p^i$ in this case.

If $i = k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = p^i \\ E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = 0 \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})^{new}_p = p^i$ in this case. This establishes the claim for $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})^{new}_p$ and as such the last statement in the proposition is now evident. □

Proposition 2.2.10. *Let L be an unramified quadratic extension of $\mathbb{Q}_p$ where p is an odd prime, $\mathfrak{r}\,[f] = \mathfrak{r}\,[f]_0$ the maximal order in L and $\mathfrak{r}\,[f]_i = \mathbb{Z}_p + p^i\mathfrak{r}\,[f]_0$ with $i \geq 0$ a general order in L. Then the new part of the number of optimal embeddings are given by*

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))^{new}_p = \begin{cases} 2p^{i-1}(1-p), & i = k, \\ 0, & \textit{otherwise.} \end{cases}$$

$$E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})^{new}_p = \begin{cases} 2p^{i-1}(p-1), & i = k, \\ 0, & \textit{otherwise.} \end{cases}$$

In particular $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))^{new}_p = -E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})^{new}_p$ for all $i \geq 0$ and $k \geq 1$.

Proof. As above we start use Proposition 2.2.7, formula (2.33) and the values of β to evaluate $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))^{new}_p$ for all $i \geq 0$ and $k \geq 1$. We again observe that

$$\begin{aligned} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))^{new}_p &= \sum_{j=0}^{2k+1} \beta(p^j) E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1-j}))_p \\ &= E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p - 2E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p + E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p. \end{aligned}$$

If $i < k$, then

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 0$$

so $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))^{new}_p = 0$ in this case.

If $i = k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = 0 \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = p^k = p^i \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{i-1} \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 2p^{i-1}(1-p)$ in this case.

If $i > k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = 2p^k \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = p^{k-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{k-1} \end{cases}$$

so

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 2p^k - 2p^{k-1}(p+1) + 2p^{k-1} = 0$$

in this case and the proof for $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new}$ is complete.

We now use the formulas in Proposition 2.2.7, the formula (2.34) and the values of μ to evaluate $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new}$ for all $i \geq 0$ and $k \geq 1$. First we get

$$\begin{aligned} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} &= \sum_{j=0}^{k} \mu(p^j) E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-j)}})_p \\ &= E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p - E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p. \end{aligned}$$

If $i < k$ then

$$E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = 2p^{i-1}(p-1)$$

so $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = 0$ in this case.

If $i = k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = 2p^{i-1}(p-1) \\ E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = 0 \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = 2p^{i-1}(p-1)$ in this case.

If $i > k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = 0 \\ E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = 0 \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = 0$ in this case. This establishes the claim for $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new}$ □

Proposition 2.2.11. *Let L be a split quadratic extension of $\mathbb{Q}_p$ where p is an odd prime, $\mathfrak{r}\,[f] = \mathfrak{r}\,[f]_0$ the maximal order in L and $\mathfrak{r}\,[f]_i = \mathbb{Z}_p + p^i\mathfrak{r}\,[f]_0$ with $i \geq 0$ a general order in L. Then the new part of the number of optimal embeddings are given by*

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} = 0$$

for all $i \geq 0$ and $k \geq 1$.

Proof. As above we once again begin with Proposition 2.2.8, formula (2.33) and the values of β to evaluate $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new}$ for all $i \geq 0$ and $k \geq 1$. We again observe that

$$\begin{aligned} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} &= \sum_{j=0}^{2k+1} \beta(p^j) E(S_i, \Gamma_0(p^{2k+1-j}))_p \\ &= E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p - 2E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p + E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p. \end{aligned}$$

If $i < k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = 2p^{i-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = 2p^{i-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{i-1}(p+1) \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 0$ in this case.

If $i = k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = 2p^{i-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = 2p^{i-1}(p+2) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{k-1} \end{cases}$$

so $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 0$ in this case.

If $i > k$, then

$$\begin{cases} E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p = 2p^{k} \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k}))_p = p^{k-1}(p+1) \\ E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k-1}))_p = 2p^{k-1} \end{cases}$$

so

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = 2p^k - 2p^{k-1}(p+1) + 2p^{k-1} = 0$$

in this case and the proof for $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new}$ is complete.

We now use the formulas in Proposition 2.2.8, the formula (2.34) and the values of μ to evaluate $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new}$ for all $i \geq 0$ and $k \geq 1$. First we get

$$\begin{aligned} E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new} &= \sum_{j=0}^{k} \mu(p^j) E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-j)}})_p \\ &= E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p - E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p. \end{aligned}$$

If $i < k, i = k, \text{or}, i > k$ then

$$E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p = E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2(k-1)}})_p = 0$$

□

Corollary 2.2.12. *For all quadratic orders* $\mathfrak{r}\,[f]_i$ *and all* $k \geq 1$

$$E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))_p^{new} = -E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})_p^{new}.$$

Proof. This follows from the last three propositions. □

Evidently we have a proof of Lemma 2.2.5. I.e.,

$$\mathcal{H}^{new}_{\mathcal{O}^1_{pq,p^{2r}q^{2s}}} = \mathcal{H}^{new}_{\Gamma_0(p^{2r+1}q^{2s+1})}.$$

We first observe that $E(\mathfrak{r}\,[f]_i\,, \Gamma_0(p^{2k+1}))^{new}_p$ and $E(\mathfrak{r}\,[f]_i\,, \mathcal{O}^1_{pq,p^{2k}})^{new}_p$ are not the same but that they are the negatives of each other and it is because we have product of the two primes p and q that these sign differences cancel forcing the equality of the contributions. One also observes that this corollary not only establishes the result for the hyperbolic conjugacy classes but it also does so for the elliptic and parabolic conjugacy classes as well.

2.2.4 Concluding Remarks

It is instructive to note that our research has not answered the specific question posed by Risager: Is there a spectral correspondence responsible for the fact that $N^{new}_{\Gamma_0(12)}(\lambda)$ is of cocompact type? We will address this question in subsequent work.

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