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APPLICATIONS OF THE STATIONARY PHASE FORMULA TO SOLUTIONS OF THE HELMHOLTZ EQUATION IN EXTERIOR DOMAINS

by

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TABLE OF CONTENTS

§ 1. INTRODUCTION

This paper is concerned with certain aspects of the classical boundary problems for the reduced wave (or Helmholtz) equation, $\Delta u+k^2u=0$. This equation arises when sinusoidal solutions to the wave equation, $\psi_{tt} = c^2\Delta\psi$ are sought in the form $\psi(x,t) = u(x)e^{-ikt}$. Such complex-valued solutions are considered for several reasons; the most important of these is that many of the physically significant solutions must satisfy the Sommerfield radiation condition which can be nicely stated for such solutions as

$\frac{\partial u}{\partial r} - iku = o(r^{-1})$. ($r = |x|$, uniformly along rays).

In particular, we will exhibit the asymptotic expansions of certain integral operators in § 4 (as $k\to\infty$) associated with the classical boundary value problems for the reduced wave equation in exterior domains. In doing so, we will generalize in § 3 the method of stationary phase as expounded by Guillemin and Sternberg (8).

This will require some preliminary lemmas, the subject of § 2. In addition, we derive the eigenvalues associated with their "asymptotic operators" and show that, under certain assumptions, their eigenvalues are related as a certain formula relating those operators dictates they should be.

The method of stationary phase permits the evaluations of one-dimensional or multi-dimensional integrals of the type $\int a(y)e^{ik\phi(v)}\,dy$ for large k in powers of $\frac{1}{k}$. It was first explicitly stated by Lord Kelvin. A mathematical treatment of the method was given by Watson (17) However, Watson's discussion did not seem capable of producing the complete asymptotic expansion. Van der Corput (4) suggested the use of a so-called "neutralizing function" which Focke (7) used to produce a complete asymptotic expansion. However, Focke, as well as Guillemin and Sternberg, relied on changes of variable to transform the integral above. Hence, in order to obtain an explicit asymptotic expansion this entailed calculation of the Jacobian and its derivatives, a more complicated procedure. That approach is outlined in § 6. The approach of this paper is more straightforward and requires no additional calculations and immediately yields an asymptotic expansion to whatever power of $(\frac{1}{k})$ desired for an n-dimensional surface without boundary.

In the remaining sections we show additional properties of these operators. Completely independently of § 4, we show in § 5, that if our boundary surface is a 2-sphere and if we utilize the more usual technique of expanding eigenfunctions in a Fourier expansion of sums of products of Hankel functions and spherical harmonics, then the asymptotic operators defined on the 2-sphere agree with those on our general 2-manifold in the limit as our 2-manifold becomes a sphere. Also their associated eigenvalues agree.

NOTATION

A word about notation is necessary. If Ω is a subset of E^n and k is a positive integer, $C^k(\Omega)$ will denote the space of continuous functions possessing continuous derivatives up to order k on Ω. Also $C_o^k(\Omega)$ will denote the space of functions mentioned whose support is compact and lies in Ω. If we write $f(t) = O(g(t))$ as $t \to a$, where f(t) and g(t) are real-valued functions and a is a real number, this means, as usual, that

$$\limsup_{t \to a} \left|\frac{f(t)}{g(t)}\right| \leqq M$$

(M is a finite constant)

Writing $h^{(i)}(t)$ is equivalent to $\frac{\partial^i h(t)}{\partial t^i}$. Similarly, unless otherwise noted, $h_\rho(x) = \frac{\partial h(x)}{\partial \rho}$ where $x = \rho\xi$ and $\xi \varepsilon S^n$ (S^n is the n-dimensional unit sphere), and $h_{j_1 \ldots j_m}(x) =$

$\frac{\partial^m h(x)}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_m}}$ where $x = (x_1, x_2, \ldots, x_j, \ldots, x_n)$. Also h^o will denote evaluation at the origin. Occasionally, the usual summation convention is employed for an index occurring both as a superscript and subscript in a product. Indices under a summation sign when written in the form $n_{j_1} \neq n_{j_2} \neq \cdots \neq n_{j_m}$ has the meaning that no pair of indices are equal.

§ 2. LEMMAS

As stated earlier, in order to apply the method of stationary phase, some preliminary lemmas are necessary. Following are a collection of these lemmas.

Lemma 2.1 Suppose that $h(t) \in C_o^m(o,\infty)$ and $h^{(j)}(t) = O(t^{2m-j})$ as $t \to o$ for $o \leqq j \leqq m$. Then as $k \to +\infty$

$$\int_o^\infty e^{\pm ikt^2/2} h(t)\,dt = O(k^{-m})$$

Proof: The lemma is proved by integrating by parts m times. Rewrite the integral as,

$\int_o^\infty e^{ikt^2/2} \frac{h(t)}{t}\, t\,dt$. If we formally integrate by parts m times without justifying any steps for now, and collecting all the boundary terms together, its easy to see that we get an expression of the form,

$$\int_o^\infty e^{ikt^2/2} \frac{h(t)}{t}\, t\,dt$$

$$= \sum_{j=1}^{m} \left(\frac{-1}{ik}\right)^{j-1} \left\{\frac{1}{ik} e^{ikt^2/2} \left(\frac{1}{t}\frac{d}{dt}\right)^{j-1} \left(\frac{h(t)}{t}\right)\Big|_o^\infty\right\}$$

$$+ \left(\frac{-1}{ik}\right)^m \int_o^\infty e^{ikt^2/2} \frac{d}{dt}\left(\frac{1}{t}\frac{d}{dt}\right)^{m-1}\left(\frac{h(t)}{t}\right) dt$$

By an elementary induction argument, it can be shown that

$$\left(\frac{1}{t}\frac{d}{dt}\right)^{j-1}\left(\frac{h(t)}{t}\right) = \sum_{p=o}^{j-1} \frac{\alpha_p\, t^p h^{(p)}(t)}{t^{2j-1}} \qquad (1)$$

where $\{\alpha_p\}$ are constants not important in the analysis. Therefore, those boundary terms with $j \leq m$ vanish since $h^{(p)}(t) = O(t^{2j-p})$ for $0 \leq p \leq j-1 \leq m-1$.

The m integration by parts can be carried out if $\frac{d}{dt}(\frac{1}{t}\frac{d}{dt})^{j-1}(\frac{h(t)}{t})$ is bounded as $t \to o$ for $1 \leq j \leq m$. Using (1) again this expression can be written for each j as,

$$\frac{d}{dt}(\frac{1}{t}\frac{d}{dt})^{j-1}(\frac{h(t)}{t}) = \sum_{p=o}^{j} \frac{\beta_p \; t^p h^{(p)}(t)}{t^{2j}}$$

This result implies that for each j, where $1 \leq j \leq m$, the expression above is bounded if $h^{(p)}(t) = O(t^{2j-p})$ as $t \to o$ for $o \leq p \leq j \leq m$ which follow from the hypothesis.

A similar analysis follows with i replaced by -i. This Lemma also holds over the integration interval $(-\infty,\infty)$.

LEMMA 2.2

Assume that $h(t) = C_o^{2m}(-\infty,\infty)$ and $h^{(p)}(t) = o$ when $t = o$ for $1 \leq p \leq 2m$. Then as $k \to +\infty$

$$\int_{-\infty}^{\infty} e^{\pm ikt^2/2} h(t)\,dt = (\frac{2\pi}{k})^{\frac{1}{2}} e^{\pm \pi i/4}\, h(o) + O(k^{-m}) \qquad (1)$$

Proof: We can write $h(t) = h(o) + \{h(t)-h(o)\}$.

The integral (1) then splits into two parts; the first is

$$\int_{-\infty}^{\infty} e^{ikt^2/2}\, h(o)\,dt = 2h(o) \int_{o}^{\infty} e^{ikt^2/2}\,dt$$

by evenness of the integrand.

This can be evaluated by integrating the entire function e^{iz^2} around the closed contour Γ which extends rectilinearly from o to + R along the x axis, thence along $|z| = R$ to the point $z_1 = Re^{i\pi/4}$ and then rectilinearly back to o. Then we let $R\to\infty$.

By Cauchy's theorem $\int_\Gamma e^{iz^2}\,dz = 0$. Or, since $z = x$ (from $x = o$ to $x = R$); $z = Re^{i\theta}$ (from $\theta = o$ to $\theta = \frac{\pi}{4}$); and $z = re^{\pi i/4}$ (from $r = R$ to $r = o$) respectively along the three parts of Γ we also have

$$0 = \int_o^R e^{ix^2}\,dx + \int_o^{\pi/4} e^{iR^2 e^{2i\theta}}\, iRe^{i\theta}\,d\theta$$

$$+ \int_R^o e^{ir^2 e^{\pi i/2}}\, e^{\pi i/4}\,dr$$

that is,

$$\int_o^R e^{ix^2}\,dx = e^{\pi i/4} \int_o^R e^{-r^2}\,dr$$

$$- \int_o^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin 2\theta}\, iRe^{i\theta}\,d\theta \quad (2)$$

The limit of the first integral on the right side of (2) as $R\to\infty$ can be evaluated by converting into a Gamma function. The result is

$$e^{\pi i/4} \int_o^\infty e^{-r^2}\,dr = e^{\pi i/4}\,\frac{\sqrt{\pi}}{2}$$

The absolute value of the second integral on the right of (2) is,

$$\left| \int_o^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta}\, iRe^{i\theta}\,d\theta \right|$$

$$\leq \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\Theta$$

$$= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin \phi} \, d\phi$$

$$\leq \frac{R}{2} \int_0^{\pi/2} e^{-2R^2\phi/\pi} \, d\phi$$

$$= \frac{\pi}{4R} \left(1-e^{-R^2} \right),$$ where the transformation $\theta = \phi/2$

and the inequality $\sin\phi \geq \frac{2\phi}{\pi}$ for $0 \leq \phi \leq \pi/2$ has been used. This shows that as $R\to\infty$ the second integral approaches zero.

Then (2) becomes

$$\int_0^\infty e^{ix^2} dx = \frac{\sqrt{\pi}}{2} e^{\pi i/4} \tag{3}$$

If the entire function e^{-z^2} is integrated around the same contour as above, a similar analysis yields that

$$\int_0^\infty e^{-ix^2} dx = \frac{\sqrt{\pi}}{2} e^{-\pi i/4} \tag{4}$$

Hence results (3) and (4) together with the change of variables $t = \sqrt{\frac{2}{k}}\, x$ gives

$$\int_{-\infty}^\infty e^{\pm ikt^2/2} h(o)\, dt = \sqrt{\frac{2\pi}{k}}\; e^{\pm\pi i/4} h(o)$$

The second part into which the integral (1) splits is,

$$\int_{-\infty}^\infty e^{\pm ikt^2/2} \{ h(t)-h(o) \} dt.$$

$f(t) = h(t) - h(o)$ satisfies the hypothesis of Lemma 2.1 which gives the desired estimate for the remainder term. This proves the Lemma.

LEMMA 2.3

Suppose that h(t) satisfies the hypothesis of Lemma 2.2 and $h'(t)$ satisfies the hypothesis of Lemma 2.1. Then as $k \to +\infty$

$$\int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, h(t)\, t^{2p}\, dt$$

$$= \frac{(2p-1)!}{2^{p-1}(p-1)!} \left(\frac{1}{2\pi}\right)^p \left(\frac{2\pi}{k}\right)^{p+1/2} e^{\pm\pi i(1+2p)/4}\, h(o)$$

$$+ O(k^{-m-1})$$

where m is defined in Lemma 2.1. $(m \geq p)$

Proof: The proof is by integrating by parts p times but collecting aside all terms involving $h'(t)$ resulting from the integration. Since $h(t) \varepsilon C_o^1$ this is permissible and we get an expression of the form

$$\int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, h(t)\, t^{2p}\, dt$$

$$= \sum_{j=1}^{p} \frac{\alpha_j}{(\pm ik)^j} e^{\pm ikt^2/2}\, h(t)\, t^{2p-(2j-1)} \Big|_{-\infty}^{\infty}$$

$$+ (-1)^p \frac{(2p-1)\cdot(2p-3)\cdots(1)}{(\pm ik)^p} \int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, h(t)\, dt$$

$$- \sum_{j=1}^{p} \frac{\beta_j}{(\pm ik)^j} \int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, h'(t)\, t^{2p-(2j-1)}\, dt$$

The constants $\{\alpha_j\}_{j=1}^p$ and $\{\beta_j\}_{j=1}^p$ are not important in what follows. Note that each integral in the second summation, by the conditions on $h'(t)\cdot t^{2p-2j}$ is $O(k^{-m-1})$ for each j. Also each boundary term vanishes since h(t) is of compact support.

Since h(t) satisfies the hypothesis of Lemma 2.2 we can write

$$\int_{-\infty}^{\infty} e^{\pm ikt^2/2} \; h(t) \; t^{2p} \, dt$$

$$= (-1)^p \frac{(2p-1)\cdot(2p-3)\cdots(1)}{(\pm ik)^p} \{(\frac{2\pi}{k})^{1/2} \; e^{\pm \pi i/4} \; h(o)\}$$

$$+ \; O(k^{-m-1})$$

After some algebraic manipulations we get

$$\int_{-\infty}^{\infty} e^{\pm ikt^2/2} \; h(t) \; t^{2p} \, dt$$

$$= \frac{(2p-1)!}{2^{p-1}(p-1)!} \; (\frac{1}{2\pi})^p \; (\frac{2\pi}{k})^{p+1/2} \; e^{\pm \pi i(1+2p)/4} \quad h(o)$$

$$+ \; O(k^{-m-1})$$

which proves the Lemma.

LEMMA 2.4

Suppose that h'(t) satisfies the hypothesis of Lemma 2.1 then the following holds

$$\int_{0}^{\infty} e^{\pm ikt^2/2} \; h(t) \; t^{2p+1} \, dt$$
$$= \frac{2^p \; p!}{k^{p+1}} \; e^{\pm \pi i(p+1)/2} \quad h(o) + O(k^{-m-1})$$

where m is defined in lemma 2.1 $(m>p)$

Proof: The proof follows closely that of the previous lemma. Integrate by parts p times collecting aside all the results of these integrations involving $h'(t)$. Since $h\epsilon C_o^1$ this can be done and yields

$$\int_o^\infty e^{\pm ikt^2/2} h(t)\, t^{2p+1}\, dt = \frac{(2p)\cdots(2)}{(\mp ik)^p}\int_o^\infty e^{\pm ikt^2/2} h(t)\, t\, dt$$

$$+ \sum_{j=o}^{p-1} \frac{\alpha j}{(\pm ik)^{j+1}} e^{\pm ikt^2/2} h(t)\, t^{2p-2j} \Big|_o^\infty$$

$$+ \sum_{j=o}^{p-1} \frac{\beta j}{(\pm ik)^{j+1}} \int_o^\infty e^{\pm ikt^2/2} h'(t)\, t^{2p-2j}\, dt$$

The boundary terms obviously vanish and since $h'(t)\cdot t^{2p-2j}$ satisfies the hypothesis of Lemma 2.1 each integral in the summation is $O(k^{-m-1})$. These facts combined with one more integration by parts on the first integral on the right side yields the result.

These lemmas enable us to prove the following theorem which is of use in an alternate derivation of the stationary phase formula. This theorem gives the asymptotic development up to any desired remainder term of order k^{-n} of an integral of the basic type $\int_{-\infty}^\infty e^{ikt^2/2} h(t)dt$ for large k around the point $t=0$.

THEOREM 2.1

Let $h(t)\ \epsilon\ C_o^{2q+1}(-\infty,\infty)$ and $h(o)\neq o$, then

$$\int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, h(t)dt$$

$$= \left(\frac{2\pi}{k}\right)^{1/2} e^{\pm\pi i/4}\; h(o)\; \{1+ \sum_{j=1}^{q} \frac{h^{(2j)}(o)}{j!h(o)} \left(\frac{\pm i}{2k}\right)^{j}\} + O(k^{-q-1})$$

as $k \to +\infty$

Proof: Expand h(t) in a Taylor series about t=o and choose an even bump function g(t) which satisfies the hypothesis of Lemma 2.3 and is identically 1 in a sufficiently small neighborhood of t=o. Write the product h(t)·g(t) as

$$h(t)\cdot g(t) = \{\sum_{j=o}^{2q+1} \frac{h^{(j)}(o)}{j!} t^j\} g(t) + \{h(t) - \sum_{j=o}^{2q+1} \frac{h^{(j)}(o)}{j!} t^j\} g(t)$$

This splits our integral into two parts, the first of which is

$$\sum_{j=o}^{2q} \frac{h^{(j)}(o)}{j!} \int_{-\infty}^{\infty} e^{\pm ikt^2/2} g(t)t^j\, dt$$

Those integrals in the summation with odd powers of t vanish since g(t) is even. Those with even powers can be evaluated by the aid of Lemma 2.3 and the zeroth order term can be evaluated by Lemma 2.2 to get

$$\sum_{j=o}^{2q} \frac{h^{(j)}(o)}{j!} \int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, g(t)\, t^j\, dt$$

$$= \sum_{j=o}^{q} \frac{h^{(2j)}(o)}{(2j)!} \int_{-\infty}^{\infty} e^{\pm ikt^2/2}\, g(t)\, t^{2j}\, dt$$

$$= \left(\frac{2\pi}{k}\right)^{\frac{1}{2}} e^{\pm\pi i/4}\, h(o)$$

$$+\sum_{j=1}^{q} \left\{\frac{h^{(2j)}(o)}{(2j)!} \frac{(2j-1)!}{2^{j-1}(j-1)!} \left(\frac{1}{2\pi}\right)^{j} \left(\frac{2\pi}{k}\right)^{j+\frac{1}{2}} e^{\pm\pi i(1+2j)/4}\right\}$$

$$+ O(k^{-q-1})$$

Factoring the expression on the left side we have,

$$\left(\frac{2\pi}{k}\right)^{\frac{1}{2}} e^{\pm\pi i/4} h(o)\left\{1 + \sum_{j=1}^{q} \frac{h^{(2j)}(o)}{j!\,h(o)}\left(\frac{\pm i}{2k}\right)^{j}\right\} + O(k^{-q-1}),$$

which proves the theorem except for evaluation of the second part of the integral mentioned.

The second part of the integral is

$$\int_{-\infty}^{\infty} e^{\pm ik^2/2} g(t) \left\{h(t) - \sum_{j=o}^{2q+1} \frac{h^{(j)}(o)}{j!} t^j\right\} dt$$

which is $O(k^{-q-1})$ by Lemma 2.1.

Two additional useful Lemmas which are very similar to Lemmas 2.1 and 2.4 respectively are also presented here without proofs.

LEMMA 2.5

Suppose that $h(t) \in C_o^m(o,\infty)$ and $h^{(j)}(o) = o$ for $o \leq j \leq m-1$ then

$$\int^{\infty} e^{\pm ikt} h(t)dt = O(k^{-m}) \text{ as } k\to+\infty$$

LEMMA 2.6

Let $h'(t)$ satisfy the hypothesis of Lemma 2.5 and suppose $h(t)$ has compact support then

$$\int_0^\infty e^{\pm ikt}\, h(t)\, t^p\, dt = \frac{p!}{k^{p+1}}\, e^{\pm\pi i(p+1)/2}\, h(0) + O(k^{-m-1})$$

as $k\to+\infty$, where m is defined in Lemma 2.5. $(m>p)$

The last lemma is useful in the derivation of the stationary phase formula in order to obtain estimates on the remainder term of the Taylor series expansion.

LEMMA 2.7

Let U be open in E^n and let $f: U \to E^n$ be of class C^p. Let $x \in U$ and $y \in E^n$ be such that $(x + ty) \in U$ for $0 \le t \le 1$. Then in the Taylor Series expansion of $f(x + y)$ the remainder term can be estimated as,

$$|R_p| \le \frac{\sup\limits_{x \in U} \| D^p f(x) \| \cdot |y|^p}{p!}$$

Proof: The remainder term can be represented as

$$R_p = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\, D^p f(x + ty)\, y^{(p)}\, dt$$

where $y^{(p)}$ denotes the p-tuple $(y, y, \ldots, y)$

Hence,

$$|R_p| = \left| \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\, D^p f(x+ty)\, y^{(p)}\, dt \right|$$

$$\le \sup_{x \in U} \| D^p f(x) \| \cdot |y|^p \left| \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\, dt \right|$$

$$= \sup_{x \in U} \| D^p f(x) \| \cdot |y|^p\, \frac{1}{p!}$$

§ 3. THE METHOD OF STATIONARY PHASE

For these integral operators in which we are interested, the method of stationary phase suggests itself. This method permits the asymptotic evaluation of integrals of the type $\int_\Sigma a(y)\, e^{ik\phi(y)}\, dy$ for large k in a series expansion in powers of $\frac{1}{k}$. Here Σ is a two-dimensional surface, dy is the volume element on Σ, and a(y) is a function defined an Σ which is assumed not to have singularities in neighborhoods of the critical points of $\phi(y)$, that is the points where $d\phi(y)=0$.

It can be shown {(4),(8)} that the major contribution to the integral for large k will come from a neighborhood of precisely those points where $d\phi=0$. For we can break a(y) up into a sum of small pieces by using a partition of unity. If $d\phi \neq 0$ on the support of a then integration by parts shows that the above integral is $O(k^{-N})$ for any N, if we assume a is a C^∞ function of y.

Those integrals in which we will be interested possess a singularity in a(y) at a minimum value of the function $\phi^x(y) = r = |x-y|$. In fact $\phi(y)$ will not even be differentiable at that point. This difficulty is overcome in theorem 3.2 of this section.

Guillemin and Sternberg(8), derive the first term in the expansion of the integral. We derive two additional terms in the expansion and show how to derive, by analogous methods, any desired number of additional terms in the expansion of the integral. This derivation is shown in theorem 3.1.

It should be noted that the theorems of this section permit the asymptotic evaluation of integrals of the type desired to be more precise and more general in the sense that not only more of the asymptotic series is obtained but integrals possessing singularities in a(y) at a point in whose neighborhood we wish to develop the integral asymptotically can be dealt with.

In theorem 3.1 below we use the following notation: (The functions below are defined on Σ in local coordinates.)

$$\lambda_j = \phi_{jj}(p) = \frac{\partial^2 \phi}{\partial x^2}(p) \text{ (p is a critical point of } \phi(y))$$

$h(y) = a(y)\cdot\sqrt{G}$ where G is the determinant of the first fundamental form.

H, the Hessian, is the matrix of second derivatives of a function and sgnH(y) denotes the signature of the Hessian of $\phi(y)$ evaluated at y.

det $H\phi$ denotes the determinant of the Hessian of $\phi(y)$ evaluated at p.

Occasionally, we will also use the notation $\text{sgn}\lambda_j$ meaning the sign of λ_j, not to be confused with the above concept.

THEOREM 3.1

Let Σ be a closed, compact, orientable, n-manifold imbedded in E^{n+1} of class C^8. Suppose that $\phi(y) \varepsilon C^6$ and that the Hessian of $\phi(y)$ is nonsingular at each critical point. Suppose also that $a(y) \varepsilon C^4$ in a neighborhood of each critical point of $\phi(y)$. Assume, without loss of generality, that $\phi_{jq}=0$ at a critical point if $j \neq q$.

Then the asymptotic development for large k of the following integral is as follows:

(1) $$\int_\Sigma e^{ik\phi(y)} a(y)dy = \left(\frac{2\pi}{k}\right)^{n/2} \sum_{p | d\phi(p)=0} e^{\pi i \operatorname{sgn} H(p)/4} \frac{e^{ik\phi(p)}}{|\det H\phi|}$$

$$\cdot \Big(a(p) + \left(\frac{i}{2k}\right)\Big\{\sum_j \frac{-\phi_{jjj}(p)a_j(p)}{\lambda_j^2} + \sum_{j\neq q} \frac{-\phi_{qqj}(p)a_j(p)}{\lambda_j \lambda_q}$$

$$+ \sum_j \frac{h_{jj}(p)}{\lambda_j} + \sum_{j<q} \frac{-\phi_{jjqq}(p)a(p)}{2\lambda_j\lambda_q} + \sum_j \frac{-\phi_{jjjj}(p)a(p)}{4\lambda_j^2}$$

$$+ \sum_{j\neq k\neq q} \frac{a(p)}{36\lambda_j\lambda_q\lambda_k} \{36\phi_{jkq}^2(p) + 9\phi_{jqq}(p) \quad \phi_{jkk}(p)\}$$

$$+ \sum_{j\neq q} \frac{a(p)}{12\lambda_j^2\lambda_q} \{9\phi_{jjq}^2(p) + 6\phi_{jjj}(p)\phi_{jqq}(p)\}$$

$$+ \sum_j \frac{5\phi^2_{jjj}(p)a(p)}{12\lambda_j^3} \}$$

$$+ \frac{4\pi^2}{k^2} \{ \sum_{j=1}^{12} F_j(\phi_{q_1q_2q_3}(p), \phi_{q_1q_2q_3q_4}(p), \phi_{q_1q_2q_3q_4q_5}(p),$$

$$\phi_{q_1q_2q_3q_4q_5q_6}(p),\ h(p),\ h_q(p),\ h_{q_1q_2}(p),\ h_{q_1q_2q_3}(p),$$

$$h_{q_1q_2q_3q_4}(p)\} + O(k^{-n/2-3})$$

where the $\{F_j\}$ are described below,

<u>Proof</u>: The method of proof involves expanding various functions involved in (1) in series expansions, splitting the resultant integrals into other simpler integrals, and then evaluating the asymptotic contribution to (1) from each of these simpler integrals by using the lemmas of the previous section. (cf. Lemmas 2.1, 2.2, 2.3, 2.7)

It was already observed that the major contribution to the integral (1) comes from a neighborhood of the critical points of ϕ. The contribution from each critical point is calculated separately.

Choose a critical point p and fix a local coordinate system about p with $p = (o,o,\ldots,o)$. Let the local coordinates be designated by $(x_1,x_2,\ldots,x_n)$. Also let N(p) be a sufficiently small neighborhood of p for which the coordinate system is valid and which contains no other critical points of $\phi(y)$. Such a neighborhood exists because the critical points of ϕ are nondegenerate hence isolated.

The contribution to (1) from the critical point p can be written as

$$\int_{N(p)} e^{ik\phi(x)} h(x)g(x)dx$$

where $dx = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$ and $h(x) = a(x) \cdot \sqrt{G}$. We have introduced a bump function $g(x) = \prod_{j=1}^{n} g_j(x_j)$ where each $g_j(x_j)$ is even in x_j and satisfies the hypothesis of Lemma 2.3. Also, each $g_j(x_j)$ is identically 1 in a neighborhood of x=o. This bump function conveniently restricts the integration to a neighborhood of p.

If we substitute the Taylor series expansions (with remainder, cf. Lemma 2.7) for h(x) and $\phi(x)$, and use the exponential series expansion for the exponential factors, the integral assumes the desired form. Notice that the first order terms in the Taylor series for ϕ do not appear since p is a critical point and the mixed second order terms are zero by assumption. (cf. Lemma 2.7 for the form of the remainders) The integral then becomes,

$$(2) \qquad \int_{N(p)} e^{ik\phi(x)}\, h(x)g(x)dx$$

$$= e^{ik\phi(p)} \int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \{1 + \frac{ik}{6} \phi^o{}_{qrs}\, x^q x^r x^s$$

$$+ \frac{1}{2!}\left(\frac{ik}{6}\right)^2 (\phi^o{}_{qrs}\, x^q x^r x^s)^2 + \frac{1}{3!}\left(\frac{ik}{6}\right)^3 (\phi^o{}_{qrs}\, x^q x^r x^s)^3$$

$$+ \frac{1}{4!}\left(\frac{ik}{6}\right)^4 (\phi^o{}_{qrs} x^q x^r x^s)^4 + k^5 \cdot O(|x|^{15})\} \cdot$$

$$\{ 1 + \frac{ik}{24} \phi^o{}_{qrs} x^q x^r x^s x^v$$

$$+ \frac{1}{2!}\left(\frac{ik}{24}\right)^2 (\phi^o{}_{qrsv} x^q x^r x^s x^v)^2 + k^3 \cdot O(|x|^{12})\} \cdot$$

$$\{ 1 + \frac{ik}{120} \phi^o{}_{qrsvw} x^q x^r x^s x^v x^w + k^2 \cdot O(|x|^{10})\} \cdot$$

$$\{ 1 + \frac{ik}{720} \phi^o{}_{qrsvwu} x^q x^r x^s x^v x^w x^u + k^2 \cdot O(|x|^{12})\} \cdot$$

$$\{ 1 + k \cdot O(|x|^7)\} \cdot \{h^o + h^o_q\, x^q + \frac{1}{2!} h^o_{qr} x^q x^r$$

$$+ \frac{1}{3!} h^o_{qrs}\, x^q x^r x^s + \frac{1}{4!} h^o_{qrsv}\, x^q x^r x^s x^v + O(|x|^5)\} g(x)dx$$

This can be broken down into a sum of simpler integrals each of which is really a product of one-dimensional integrals. Note also that integrals involving odd powers of x vanish since g is even. Hence only products of one-dimensional integrals of even powers of x are considered. Any integral obtainable from (2) which can be shown by Lemma 2.3, or theorem 2.1, to be $O(k^{-n/2\ -3})$ are not analyzed. Hence the following integrals are examined:

(3) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \; h^o \; g(x)dx$

(4) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{ik}{6}\right) (\phi^o_{qrs} \; x^q x^r x^s)(h^o_v \; x^v) \; g(x)dx$

(5) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{1}{2}\right) (h^o_{qr} \; x^q x^r) \; g(x) \; dx$

(6) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{ik}{24}\right) (\phi^o_{qrsv} \; x^q x^r x^s x^v) h^o \; g(x)dx$

(7) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{1}{2!}\right)\left(\frac{ik}{6}\right)^2 (\phi^o_{qrs} \; x^q x^r x^s)^2 \; h^o g(x)dx$

(8) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{1}{3!}\right) \left(\frac{ik}{6}\right)^3 (\phi^o_{qrs} \; x^q x^r x^s)^3 (h_v x^v) g(x)dx$

(9) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{1}{4!}\right)\left(\frac{ik}{6}\right)^4 (\phi^o_{qrs} \; x^q x^r x^s)^4 \; h^o g(x)dx$

(10) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{1}{2!}\right) \left(\frac{ik}{24}\right)^2 (\phi^o_{qrsv} x^q x^r x^s x^v)^2 \; h^o g(x)dx$

(11) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{ik}{120}\right) (\phi^o_{qrsvw} x^q x^r x^s x^v x^w)(h^o_u x^u) g(x)dx$

(12) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{1}{4!}\right)(h^o_{qrsv} \; x^q x^r x^s x^v) \; g(x)dx$

(13) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{ik}{720}\right)(\phi^o_{qrsvwu} x^q x^r x^s x^w x^u x^v) h^o g(x)dx$

(14) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{ik}{6}\right)(\phi^o_{qrs} x^q x^r x^s)\left(\frac{ik}{120}\right) \cdot$

$$\cdot (\phi^o_{q'r's'v'w'} \; x^{q'} x^{r'} x^{v'} x^{w'} x^{s}) h^o g(x)dx$$

(15) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} \left(\frac{ik}{6}\right)(\phi^o_{qrs} \; x^q x^r x^s)\left(\frac{1}{3!}\right)(h^o_{vwu} x^v x^w x^u) g(x)dx$

(16) $\int_{N(p)} e^{ik\phi_j (x^j)^2/2} \left(\frac{1}{2!}\right)\left(\frac{ik}{6}\right)^2 (\phi^o_{qrs} \; x^q x^r x^s)^2 \left(\frac{ik}{24}\right)$

$$(\phi^o_{q'r's'v'} x^{q'} x^{r'} x^{s'} x^{v'}) \; h^o g(x)dx$$

(17) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} (\frac{ik}{24}) (\phi^o_{qrsv} x^q x^r x^s x^v) (\frac{1}{2!}) (h^o_{wu} x^w x^u) g(x) dx$

(18) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} (\frac{1}{2!}) (\frac{ik}{6})^2 (\phi^o_{qrs} x^q x^r x^s)^2 (\frac{1}{2!}) (h^o_{vw} x^v x^w) g(x) dx$

(19) $\int_{N(p)} e^{ik\lambda_j (x^j)^2/2} (\frac{ik}{6}) (\phi^o_{qrs} x^q x^r x^s) (\frac{ik}{24})$

$$(\phi^o_{q'r's'v'} x^{q'} x^{r'} x^{s'} x^{u'}) (h^o_v x^v) g(x) dx$$

Integral (3) is the dominant term in the asymptotic development of (1); the next four integrals (4) - (7) will be shown to be of order $k^{-n/2\ -1}$; integrals (8) - (19) are of order $k^{-n/2\ -2}$; The remaining integrals, which are not listed are of order $k^{-n/2\ -3}$. The analysis of the integrals is a straightforward application of the Lemmas.

Only integrals (3) and (4) are analyzed in detail. The others are similar to (4) so only calculations are indicated for (5) through (7). Results are indicated for the remaining integrals.

Written as a product of integrals (3) becomes,

$$\int e^{ik\lambda_j (x^j)^2/2}\ h^o\ g(x) dx = h^o \prod_{j=1}^{n} \{\int_{-\infty}^{\infty} e^{ik\lambda_j (x_j)^2} \quad g_j(x_j) dx_j\}$$

Application of Lemma 2.2 to each one-dimensional integral is justified since $g_j(x_j)$ for $1 \leq j \leq n$ satisfies the hypothesis of Lemma 2.2. The result is,

(3) $$h^o \prod_{j=1}^{n} \{ (\frac{2\pi}{|\lambda_j| k})^{\frac{1}{2}} e^{\pi i \operatorname{sgn} \lambda_j /4} + O(k^{-m}) \}$$

$$= h^o (\frac{2\pi}{k})^{n/2} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} + O(k^{-m-(n-1)/2})$$

To arrive at the last step we have used the fact that $\frac{1}{\lambda} = \frac{\text{sgn}\lambda}{|\lambda|}$ where recall that sgn λ denotes the sign of λ. Since we wish to obtain the asymptotic series of (1) up to a remainder of order $k^{-n/2-3}$ it is sufficient for our purpose to choose $g \varepsilon C^8$.

The next integral (4), when the vanishing of odd powers of x is taken into account becomes,

$$\int e^{ik\lambda_j (x^j)^2/2} (\tfrac{ik}{6})(\phi^o_{qpr}\ x^q x^p x^r)(h^o_s\ x^s) g(x)dx$$

$$= \sum_j \frac{ik}{6}\ \phi^o_{jjj} h^o_j \int_{N(p)} e^{ik\lambda_p (x^p)^2/2}\ x_j^4\ g(x)dx$$

$$+ \sum_{j \neq p} \frac{ik}{2}\ \phi^o_{jjp} h^o_p \int_{N(p)} e^{ik\lambda_q (x^q)^2/2}\ x_j^2\ x_p^2\ g(x)dx$$

$$= \sum_j \frac{ik}{6}\ \phi^o_{jjj} h^o_j \int_{-\infty}^{\infty} e^{ik\lambda_j (x_j)^2/2}\ x_j^4\ g_j(x_j)dx_j \cdot$$

$$\cdot \prod_{\substack{p=1 \\ p \neq j}}^{n} \{\int_{\infty}^{\infty} e^{ik\lambda_p (x_p)^2/2}\ g_p(x_p)dx_p\}$$

$$+ \sum_{j \neq p} \frac{ik}{2}\ \phi^o_{jjp} h^o_p \int_{-\infty}^{\infty} e^{ik\lambda_j (x_j)^2/2} x_j^2\ g_j(x_j)dx_j \cdot$$

$$\cdot \int_{\infty}^{\infty} e^{ik\lambda_p (x_p)^2/2}\ x_p^2\ g_p(x_p)dx_p \cdot \prod_{\substack{q=1 \\ q \neq j,p}}^{n} \{\int_{-\infty}^{\infty} e^{ik\lambda_q (x_q)^2/2} g_q(x_q)dx_q\}$$

In the second summation, as in other summations over more than one index, there is a combinatorial problem involved of determining the coefficients of the partial derivatives of ϕ. Coefficients are shown without analysis.

The one-dimensional integrals above are calculated by Lemmas 2.2 and 2.3. We then get

$$(4) \quad \sum_j \frac{ik}{6} \phi^o_{jjj} h^o_j \{3(\frac{1}{2\pi})^2 (\frac{2\pi}{|\lambda_j|k})^{5/2} e^{5\pi i \operatorname{sgn}\lambda_j/4}$$

$$+ O(k^{-m-1})\} \prod_{\substack{q=1\\ q\neq j}}^n \{(\frac{2\pi}{|\lambda_q|k})^{\frac{1}{2}} e^{\pi i \operatorname{sgn}\lambda_q/4} + O(k^{-m})\}$$

$$+ \sum_{j\neq p} \frac{ik}{2} \phi^o_{jjp} h^o_p \{\frac{1}{2\pi}(\frac{2\pi}{|\lambda_j|k})^{3/2} e^{3\pi i \operatorname{sgn}\lambda_j/4} + O(k^{-m-1})\}$$

$$\{\frac{1}{2\pi}(\frac{2\pi}{|\lambda_p|k})^{3/2} e^{\pi i \operatorname{sgn}\lambda_p/4} + O(k^{-m-1})\}$$

$$\prod_{\substack{q=1\\ q\neq j,p}}^n \{(\frac{2\pi}{|\lambda_q|k})^{\frac{1}{2}} e^{\pi i \operatorname{sgn}\lambda_q/4} + O(k^{-m})\}$$

$$= \sum_j \frac{-i}{4\pi\lambda_j^2} \phi^o_{jjj} h^o_j (\frac{2\pi}{k})^{n/2+1} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H_\phi|)^{\frac{1}{2}}}$$

$$+ \sum_{j\neq p} \frac{-i}{4\pi\lambda_j\lambda_p} \phi^o_{jjp} h^o_p (\frac{2\pi}{k})^{n/2+1} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} + O(k^{-m-(n+1)/2})$$

The next three integrals which are of order $k^{-n/2-1}$ when evaluated give

$$(5\ ) \quad \int_{N(p)} e^{ik\lambda_j(x^j)^2/2} \tfrac{1}{2}(h^o_{qp}\, x^q x^p)\, g(x)dx$$

$$= \sum_j \tfrac{1}{2} h^o_{jj} \{\frac{1}{2\pi} (\frac{2\pi}{|\lambda_j|k})^{3/2} e^{3\pi i \operatorname{sgn}\lambda_j/4} + O(k^{-m-1})\}\cdot$$

$$\cdot \prod_{\substack{q=1\\ q\neq j}}^n \{(\frac{2\pi}{|\lambda_q|k})^{\frac{1}{2}} e^{\pi i \operatorname{sgn}\lambda_q/4} + O(k^{-m})\}$$

$$= \sum_j \frac{ih^o_{jj}}{4\pi\lambda_j} (\frac{2\pi}{k})^{n/2+1} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} + O(k^{-m-(n+1)/2})$$

For integral (6) we have

$$(6)\quad \int_{N(p)} e^{ik\lambda_p (x^p)^2/2} \left(\frac{ik}{24}\right) (\phi^o_{qprs}\, x^q x^p x^r x^s)\, h^o\, g(x)dx$$

$$= \sum_j \frac{ik}{24} \phi^o_{jjjj} h^o \left\{3\left(\frac{1}{2\pi}\right)^2 \left(\frac{2\pi}{|\lambda_j|k}\right)^{5/2} e^{5\pi i\,\mathrm{sgn}\lambda_j/4} + O(k^{-m-1})\right\}\cdot$$

$$\cdot \prod_{\substack{q=1\\ q\neq j}}^{n} \left\{\left(\frac{2\pi}{|\lambda_q|k}\right)^{1/2} e^{\pi i\,\mathrm{sgn}\lambda_q/4} + O(k^{-m})\right\}$$

$$+ \sum_{j<p} \frac{ik}{4} \phi^o_{jjpp} h^o \left\{\frac{1}{2\pi} \left(\frac{2\pi}{|\lambda_j|k}\right)^{3/2} e^{3\pi i\,\mathrm{sgn}\lambda_j/4} + O(k^{-m-1})\right\}\cdot$$

$$\cdot\left\{\frac{1}{2\pi} \left(\frac{2\pi}{|\lambda_p|k}\right)^{3/2} e^{3\pi i\,\mathrm{sgn}\lambda_p/4}\ O(k^{-m-1})\right\}\cdot$$

$$\cdot \prod_{\substack{q=1\\ q\neq j,p}}^{n} \left\{\left(\frac{2\pi}{|\lambda_q|k}\right)^{1/2} e^{\pi i\,\mathrm{sgn}\lambda_q/4} + O(k^{-m})\right\}$$

$$= \sum_j \frac{-ih^o}{16\pi\lambda_j^2} \phi^o_{jjjj} \left(\frac{2\pi}{k}\right)^{n/2+1} \frac{e^{\pi i\,\mathrm{sgn}H(p)/4}}{(|\det H\phi|)^{1/2}}$$

$$+ \sum_{j<q} \frac{-ih^o}{8\pi\lambda_j\lambda_q} \phi^o_{jjqq} \left(\frac{2\pi}{k}\right)^{n/2+1} \frac{e^{\pi i\,\mathrm{sgn}H(p)/4}}{(|\det H\phi|)^{1/2}} + O(k^{-m-(n+1)/2})$$

Integral (7) gives

$$(7)\quad \int_{N(p)} e^{ik\ _j (x^j)^2/2} \left(\frac{1}{2!}\right)\left(\frac{ik}{6}\right)^2 (\phi^o_{qpr}\, x^q x^p x^r)^2\, h^o\, g(x)dx$$

$$= \sum_j \frac{-k^2}{72} h^o (\phi^o_{jjj})^2 \left\{15 \left(\frac{1}{2\pi}\right)^3 \left(\frac{2\pi}{|\lambda_j|k}\right)^{7/2} e^{7\pi i\,\mathrm{sgn}\lambda_j/4}\right.$$

$$\left. + O(k^{-m-1})\right\} \prod_{\substack{q=1\\ q\neq j}}^{n} \left\{\left(\frac{2\pi}{|\lambda q|k}\right)^{1/2} e^{\pi i\,\mathrm{sgn}\lambda n/4} + O(k^{-m})\right\}$$

$$+ \sum_{j\neq p} \frac{-k^2}{72} h^o \{9(\phi^o_{jjp})^2 + 6\phi^o_{jjj}\phi^o_{jpp}\}\left\{3\left(\frac{1}{2\pi}\right)^2 \left(\frac{2\pi}{|\lambda_j|k}\right)^{5/2}\cdot\right.$$

$$\cdot\, e^{5\pi i\,\mathrm{sgn}\lambda_j/4}$$

$$+ \quad O(k^{-m-1})\}\{\frac{1}{2\pi}(\frac{2\pi}{|\lambda_p|k})^{3/2} e^{3\pi i \operatorname{sgn}\lambda_p/4} + O(k^{-m-1})\}\cdot$$

$$\cdot \prod_{\substack{q=1 \\ q\neq j,p}}^{n} \{(\frac{2\pi}{|\lambda_q|k})^{\frac{1}{2}} e^{\pi i \operatorname{sgn}\lambda_q/4} + O(k^{-m})\}$$

$$+ \sum_{j\neq q\neq p} \frac{-k^2}{72} h^o\{36(\phi^o_{jqp})^2 + 9\phi^o_{jqq}\phi^o_{jpp}\}\{\frac{1}{2\pi}(\frac{2\pi}{|\lambda_j|k})^{3/2} e^{3\pi i \operatorname{sgn}\lambda_j/4}$$

$$+ \quad O(k^{-m-1})\}\{\frac{1}{2\pi}(\frac{2\pi}{|\lambda_q|k})^{3/2} e^{3\pi i \operatorname{sgn}\lambda_q/4} + O(k^{-m-1})\}\cdot$$

$$\cdot\{\frac{1}{2\pi}(\frac{2\pi}{|\lambda_p|k})^{3/2} e^{3\pi i \operatorname{sgn}\lambda_j/4} + O(k^{-m-1})\}\cdot$$

$$\cdot \prod_{\substack{r=1 \\ r\neq j,q,p}}^{n} \{(\frac{2\pi}{|\lambda_r|k})^{\frac{1}{2}} e^{\pi i \operatorname{sgn}\lambda_r/4} + O(k^{-m})\}$$

$$= \sum_j \frac{5ih^o}{48\pi\lambda_j^3}(\phi^o_{jjj})^2 (\frac{2\pi}{k})^{n/2+1} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}}$$

$$+ \sum_{j\neq q} \frac{ih^o}{48\pi\lambda_j^2\lambda_q}\{9(\phi^o_{jjq})^2 + 6\phi^o_{jjj}\phi^o_{jqq}\}(\frac{2\pi}{k})^{n/2+1} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}}$$

$$+ \sum_{j\neq q\neq p} \frac{ih^o}{144\pi\lambda_j\lambda_q\lambda_p}\{36(\phi^o_{jqp})^2 + 9\phi^o_{jqq}\phi^o_{jpp}\}\cdot$$

$$\cdot(\frac{2\pi}{k})^{n/2+1} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|^{\frac{1}{2}})} + O(k^{-m-(n+1)/2})$$

The results for the remaining integrals are noted below. The $\{F_j\}$ described in the statement of the theorem are just the factors of

$$(\frac{2\pi}{k})^{n/2+2} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} \quad \text{given below.}$$

$$(8)\quad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{\pi i\,\mathrm{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} \Big\{ \sum_j \frac{35}{192\pi^2\lambda_j^5} \{(\phi^o_{jjj})^3\, h^o_j\}$$

$$+ \sum_{j\neq q} \frac{35}{1728\pi^2\lambda_j^4\lambda q} \{9(\phi^o_{jjj})^2\,\phi^o_{jjq} h^o_q$$

$$+ 9(\phi^o_{jjj})^2\,\phi^o_{jqq} h^o_j + 27(\phi^o_{jjq})^2\,\phi^o_{jjj} h^o_j\}$$

$$+ \sum_{j\neq q} \frac{5}{576\pi^2\lambda_j^3\lambda_q^2} \{3(\phi^o_{jjj})^2\,\phi^o_{qqq} h^o_q + 54\phi^o_{jjj}\,\phi^o_{jqq}\phi^o_{jjq} h^o_q$$

$$+ 27(\phi^o_{jjq})^3\, h^o_q + 18\,\phi^o_{qqq}\phi^o_{qjj}\,\phi^o_{jjj} h^o_j$$

$$+ 27(\phi^o_{qqj})^2\,\phi^o_{jjj} h^o_j + 81\,(\phi^o_{jjq})^2\,\phi^o_{jqq} h^o_j\}$$

$$+ \sum_{j\neq p\neq q\neq r\neq s} \frac{1}{5184\pi^2\lambda_j\lambda_p\lambda_q\lambda_r\lambda_s} \{27\,\phi^o_{jjp}\,\phi^o_{pqq}\,\phi^o_{rrs} h^o_s$$

$$+ 324\,(\phi^o_{jpq})^2\,\phi^o_{rrs} h^o_s + 324\,\phi^o_{jqq}\,\phi^o_{rpj}\,\phi^o_{rps} h^o_s$$

$$+ 162\,\phi^o_{pjj}\,\phi^o_{rqq}\,\phi^o_{rps} h^o_s + 216\,\phi^o_{jpq}\,\phi^o_{jrq}\,\phi^o_{prs} h^o_s\}$$

$$+ \sum_{j\neq p\neq q} \frac{5}{1728\pi^2\lambda_j^3\lambda_p\lambda_q} \{54\,\phi^o_{jjj}\,\phi^o_{qjj}\,\phi^o_{ppj} h^o_q + 9\,(\phi^o_{jjj})^2\,\phi^o_{qpp} h^o_q$$

$$+ 108\,\phi^o_{jjj}\,\phi^o_{jjp}\,\phi^o_{qpj} h^o_q + 27(\phi^o_{jjp})^2\,\phi^o_{jjq} h^o_q + 27\,\phi^o_{jjj}\,\phi^o_{ppj}\,\phi^o_{qqj} h^o_j$$

$$+ 81\,(\phi^o_{pjj})^2\,\phi^o_{qqj} h^o_j + 162\,\phi^o_{pjj}\,\phi^o_{pqj}\,\phi^o_{qjj} h^o_j + 108(\phi^o_{pqj})^2\phi^o_{jjj} h^o_j$$

$$+ 54\,\phi^o_{jjj}\,\phi^o_{qjj}\,\phi^o_{qpp}\, h^o_j\} + \sum_{j\neq q\neq r\neq p} \frac{1}{1728\pi^2\lambda_j^2\,\lambda_q\lambda_r\lambda_p}$$

$$\cdot\{108(\phi^o_{pqr})^2\,\phi^o_{jjj} h^o_j + 27\,\phi^o_{jjj}\,\phi^o_{rqq}\,\phi^o_{ppr} h^o_j$$

$$+ \quad 648\, \phi^o_{pqr}\, \phi^o_{jqr}\, \phi^o_{jjp} h^o_j + 27\, \phi^o_{jpp}\, \phi^o_{jqq}\, \phi^o_{jrr} h^o_j$$

$$+ \quad 304 (\phi^o_{jpq})^2\, \phi^o_{jrr} h^o_j + 216\, \phi^o_{jpq}\, \phi^o_{jpr}\, \phi^o_{jqr} h^o_j$$

$$+ \quad 54 \phi^o_{jjj}\, \phi^o_{jqq}\, \phi^o_{prr} h^o_p + 108 \phi^o_{jjj}\, \phi^o_{qqr}\, \phi^o_{pjr} h^o_p$$

$$+ \quad 216\, \phi^o_{jjj}\, \phi^o_{jqr}\, \phi^o_{pqr} h^o_p + 81\, \phi^o_{jqq}\, \phi^o_{jrr}\, \phi^o_{pjj} h^o_p$$

$$+ \quad 162\, \phi^o_{jjq}\, \phi^o_{jjr}\, \phi^o_{pqr} h^o_p + 81\, (\phi^o_{jjq})^2 \phi^o_{pqq} h^o_p$$

$$+ \quad 81\, \phi^o_{jjq}\, \phi^o_{qrr}\, \phi^o_{pjj} h^o_p + 324\, (\phi^o_{jqr})^2\, \phi^o_{pjj} h^o_p$$

$$+ \quad 324\, \phi^o_{jjr}\, \phi^o_{jqq}\, \phi^o_{pjr} h^o_p + 324\, \phi^o_{jjq}\, \phi^o_{jrq}\, \phi^o_{pjr} h^o_p \}$$

$$+ \sum_{j \neq p \neq r} \frac{1}{576 \pi^2 \lambda_j^2 \lambda_p^2 \lambda_r^2} \{ 81 (\phi^o_{jpp})^2\, \phi^o_{rjj} h^o_r$$

$$+ \quad 54\, \phi^o_{ppp}\, \phi^o_{jjp}\, \phi^o_{rjj} h^o_r + 18 \phi^o_{jjj}\, \phi^o_{ppp}\, \phi^o_{rjp} h^o_r$$

$$+ \quad 162\, \phi^o_{jpp}\, \phi^o_{jjp}\, \phi^o_{rjp} h^o_r + 324 (\phi^o_{jrp})^2\, \phi^o_{jpp} h^o_j \}$$

$$+ \quad 9\, \phi^o_{jjj}\, \phi^o_{rrp}\, \phi^o_{ppp} h^o_j + 27 \phi^o_{jjj} (\phi^o_{rpp})^2\, h^o_j$$

$$+ \quad 108\, \phi^o_{jjr}\, \phi^o_{jrp}\, \phi^o_{ppp} h^o_j + 81\, \phi^o_{jjr}\, \phi^o_{jpp}\, \phi^o_{rpp} h^o_j$$

$$+ \quad 162\, \phi^o_{jjp}\, \phi^o_{rpp}\, \phi^o_{rjp} h^o_j + 54\, \phi^o_{jjp}\, \phi^o_{jrr}\, \phi^o_{ppp} h^o_j$$

$$+ \ 162\, \phi^{o}_{jjp}\, \phi^{o}_{jpp}\, \phi^{o}_{rrp} h^{o}_{j} \ + \ 81\, \phi^{o}_{jrr} (\phi^{o}_{jpp})^{2}\ h^{o}_{j} \}$$

$$(9) \quad (\frac{2\pi}{k})^{n/2\ +\ 2} \ \frac{e^{\pi i \,\mathrm{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} \{ \sum_{j} \frac{-385h}{4608\pi^{2} \lambda_{j}^{6}} \{ (\phi^{o}_{jjj})^{4} \}$$

$$+ \sum_{j\neq p} \frac{-35h^{o}}{4608\pi^{2} \lambda_{j}^{5} \lambda_{p}} \{ 12(\phi^{o}_{jjj})^{3}\ \phi^{o}_{jpp} + 54(\phi^{o}_{jjp})^{2}\ (\phi^{o}_{jjj})^{2} \}$$

$$+ \sum_{j\neq p} \frac{-35h^{o}}{13{,}824\pi^{2} \lambda_{j}^{4} \lambda_{p}^{2}} \{ 36(\phi^{o}_{jjj})^{2}\ \phi^{o}_{jjp}\ \phi^{o}_{ppp} + 54(\phi^{o}_{jjj})^{2}\ (\phi^{o}_{jpp})^{2}$$

$$+ \ 324\, \phi^{o}_{jjj} (\phi^{o}_{jjp})^{2}\ \phi^{o}_{ppj} \ + \ 81 (\phi^{o}_{jjp})^{4} \}$$

$$+ \sum_{j\neq q\neq p} \frac{-35h^{o}}{41{,}472\pi^{2} \lambda_{j}^{4}\ \lambda_{p} \lambda_{q}} \{ 108\, (\phi^{o}_{jjj})^{2}\ \phi^{o}_{jjp}\ \phi^{o}_{pqq}$$

$$+ \ 36(\phi^{o}_{jjj})^{2}\ \phi^{o}_{jqq}\ \phi^{o}_{jpp} + 216(\phi^{o}_{jjj})^{2}\ (\phi^{o}_{jqp})^{2}$$

$$+ \ 324(\phi^{o}_{jjp})^{2} \phi^{o}_{jjj}\ \phi^{o}_{jqq} + 648\, \phi^{o}_{jjj}\ \phi^{o}_{jjp}\ \phi^{o}_{jjq}\ \phi^{o}_{jqp}$$

$$+ \ 486(\phi^{o}_{jjq})^{2}\ (\phi^{o}_{jjp})^{2} \} + \sum_{j\neq p} \frac{-25h^{o}}{13{,}824\pi^{2} \lambda_{j}^{3}\ \lambda_{p}^{3}} \{ (\phi^{o}_{jjj})^{2}\ (\phi^{o}_{ppp})^{2}$$

$$+ \ 108\, \phi^{o}_{jjj}\ \phi^{o}_{ppp}\ \phi^{o}_{jjp}\ \phi^{o}_{ppj} + 162(\phi^{o}_{jjp})^{2}\ (\phi^{o}_{ppj})^{2} + 108(\phi^{o}_{jpp})^{3}\ \phi^{o}_{jjj} \}$$

$$+ \sum_{j\neq p\neq q} \frac{-5h^{o}}{13{,}824\pi^{2} \lambda_{j}^{3} \lambda_{p}^{2} \lambda_{q}} \{ 16(\phi^{o}_{jjj})^{2}\ \phi^{o}_{qqp}\ \phi^{o}_{ppp} + 54\, (\phi^{o}_{jjj})^{2}\, (\phi^{o}_{qpp})^{2}$$

$$+ \ 432\, \phi^{o}_{jjj}\ \phi^{o}_{jjq}\ \phi^{o}_{jqp}\ \phi^{o}_{ppp} + 648\, \phi^{o}_{jjj}\ \phi^{o}_{jjq}\ \phi^{o}_{jpp}\ \phi^{o}_{qpp}$$

$$+ \quad 216\,\phi^{o}_{jjj}\;\phi^{o}_{jjp}\;\phi^{o}_{jqq}\;\phi^{o}_{ppp} + 648\,\phi^{o}_{jjj}\;\phi^{o}_{jjp}\;\phi^{o}_{jpp}\;\phi^{o}_{qqp}$$

$$+ \quad 1296\,\phi^{o}_{jjj}\;\phi^{o}_{jjp}\;\phi^{o}_{jqp}\;\phi^{o}_{qpp} + 324\,\phi^{o}_{jjj}\;(\phi^{o}_{jpp})^{2}\,\phi^{o}_{jqq}$$

$$+ \quad 1296\;(\phi^{o}_{jpq})^{2}\;\phi^{o}_{jjj}\;\phi^{o}_{jpp} + 324\,(\phi^{o}_{jjp})^{3}\;\phi^{o}_{qqp}$$

$$+ \quad 324\,(\phi^{o}_{jjq})^{2}\;\phi^{o}_{jjp}\;\phi^{o}_{ppp} + 972\,(\phi^{o}_{jjp})^{2}\,\phi^{o}_{jjq}\;\phi^{o}_{qpp}$$

$$+ \quad 324\;(\phi^{o}_{jjp})^{2}\;\phi^{o}_{jpp}\;\phi^{o}_{jqq} + 486\,(\phi^{o}_{jjq})^{2}\;(\phi^{o}_{jpp})^{2}$$

$$+ \quad 1944\;(\phi^{o}_{jjp})^{2}\;(\phi^{o}_{jqp})^{2} + 1944\,\phi^{o}_{jjq}\;\phi^{o}_{jjp}\;\phi^{o}_{jqp}\;\phi^{o}_{jpp}\}$$

$$+ \quad \sum_{j\neq p\neq q\neq r} \frac{-5h^{o}}{41472\pi^{2}\lambda_{j}^{3}\lambda_{p}\lambda_{q}\lambda_{r}}\{216\,(\phi^{o}_{jjj})^{2}\;(\phi^{o}_{pqr})^{2}$$

$$+ \quad 36\;(\phi^{o}_{jjj})^{2}\;\phi^{o}_{ppq}\;\phi^{o}_{qrr} + 1296\,\phi^{o}_{jjj}\;\phi^{o}_{jjp}\;\phi^{o}_{jpq}\;\phi^{o}_{qrr}$$

$$+ \quad 648\;\phi^{o}_{jjj}\;\phi^{o}_{jjp}\;\phi^{o}_{jqq}\;\phi^{o}_{prr} + 2592\,\phi^{o}_{jjj}\;\phi^{o}_{jjp}\;\phi^{o}_{jrq}\;\phi^{o}_{prq}$$

$$+ \quad 108\;\phi^{o}_{jjj}\;\phi^{o}_{jpp}\;\phi^{o}_{jqq}\;\phi^{o}_{jrr} + 1296\;\phi^{o}_{jjj}\;(\phi^{o}_{jpq})^{2}\;\phi^{o}_{jrr}$$

$$+ \quad 864\;\phi^{o}_{jjj}\;\phi^{o}_{jpq}\;\phi^{o}_{jpr}\;\phi^{o}_{jqr} + 972\;(\phi^{o}_{jjp})^{2}\;\phi^{o}_{jjq}\;\phi^{o}_{qrr}$$

$$+ \quad 648\;\phi^{o}_{jjp}\;\phi^{o}_{jjq}\;\phi^{o}_{jjr}\;\phi^{o}_{pqr} + 324\;(\phi^{o}_{jjp})^{2}\;\phi^{o}_{jqq}\;\phi^{o}_{jrr}$$

$$+ \; 972\,(\phi^{o}_{jjp})^{2}\,(\phi^{o}_{jqr})^{2} + 3888\,\phi^{o}_{jjq}\,\phi^{o}_{jjr}\,\phi^{o}_{jpq}\,\phi^{o}_{jpr}$$

$$+ \; 3888\,\phi^{o}_{jjq}\,\phi^{o}_{jjr}\,\phi^{o}_{jrq}\,\phi^{o}_{jpp}\} + \sum_{j\neq p\neq q} \frac{-h^{o}}{4608\pi^{2}\lambda_{j}^{2}\lambda_{p}^{2}\lambda_{q}^{2}} \cdot$$

$$\cdot\{24\,\phi^{o}_{jjj}\,\phi^{o}_{ppp}\,\phi^{o}_{qqq}\,\phi^{o}_{jpq} + 108\,\phi^{o}_{jjj}\,\phi^{o}_{ppp}\,\phi^{o}_{qqp}\,\phi^{o}_{qqj}$$

$$+ \; 648\,\phi^{o}_{jjj}\,\phi^{o}_{qqp}\,\phi^{o}_{ppq}\,\phi^{o}_{jpq} + 324\,(\phi^{o}_{ppq})^{2}\,\phi^{o}_{jjj}\,\phi^{o}_{qqj}$$

$$+ \; 324\,\phi^{o}_{jjp}\,\phi^{o}_{jjq}\,\phi^{o}_{ppq}\,\phi^{o}_{qqp} + 1944\,\phi^{o}_{jjp}\,\phi^{o}_{ppq}\,\phi^{o}_{qqj}\,\phi^{o}_{jpq}$$

$$+ \; 486\,(\phi^{o}_{jjq})^{2}\,(\phi^{o}_{ppq})^{2} + 3888(\phi^{o}_{jpq})^{2}\,\phi^{o}_{jjq}\,\phi^{o}_{ppq} + 1296(\phi^{o}_{jpq})^{4}\}$$

$$+ \sum_{j\neq p\neq q\neq r} \frac{-h^{o}}{2304\pi^{2}\lambda_{j}^{2}\lambda_{p}^{2}\lambda_{q}\lambda_{r}} \{54\,\phi^{o}_{jjj}\,\phi^{o}_{ppp}\,\phi^{o}_{qqj}\,\phi^{o}_{rrp}$$

$$+ \; 216\,\phi^{o}_{jjj}\,\phi^{o}_{ppp}\,\phi^{o}_{jrq}\,\phi^{o}_{prq} + 432\,\phi^{o}_{jjj}\,\phi^{o}_{ppp}\,\phi^{o}_{qqr}\,\phi^{o}_{jpr}$$

$$+ \; 648\,\phi^{o}_{jjj}\,\phi^{o}_{ppr}\,\phi^{o}_{ppj}\,\phi^{o}_{qqr} + 324\,\phi^{o}_{jjj}\,\phi^{o}_{ppj}\,\phi^{o}_{pqq}\,\phi^{o}_{prr}$$

$$+ \; 1296\,\phi^{o}_{jjj}\,\phi^{o}_{ppj}\,(\phi^{o}_{pqr})^{2} + 324\,(\phi^{o}_{ppq})^{2}\,\phi^{o}_{jjj}\,\phi^{o}_{jrr}$$

$$+ \; 648\,\phi^{o}_{jjj}\,\phi^{o}_{ppq}\,\phi^{o}_{ppr}\,\phi^{o}_{jqr} + 648\,\phi^{o}_{jjj}\,\phi^{o}_{ppq}\,\phi^{o}_{jpq}\,\phi^{o}_{prr}$$

$$+ \; 1296\,\phi^{o}_{jjj}\,\phi^{o}_{ppq}\,\phi^{o}_{jpr}\,\phi^{o}_{pqr} + 81\,(\phi^{o}_{jjq})^{2}\,(\phi^{o}_{ppr})^{2}$$

$$+ \; 81\,\phi^{o}_{jjq}\,\phi^{o}_{jjr}\,\phi^{o}_{ppq}\,\phi^{o}_{ppr} + 972\,(\phi^{o}_{jjp})^{2}\,\phi^{o}_{ppq}\,\phi^{o}_{qrr}$$

$$+ \ 81 \, \phi^{o}_{jjp} \, \phi^{o}_{jjr} \, \phi^{o}_{ppr} \, \phi^{o}_{pqq} + 162 \, \phi^{o}_{jjp} \, \phi^{o}_{jjr} \, \phi^{o}_{ppq} \, \phi^{o}_{pqr}$$

$$+ \ 81 \, (\phi^{o}_{jjp})^2 \, \phi^{o}_{pqq} \, \phi^{o}_{prr} + 324 \, (\phi^{o}_{jqr})^2 \, \phi^{o}_{jjq} \, \phi^{o}_{ppq}$$

$$+ \ 324 \, \phi^{o}_{jjr} \, \phi^{o}_{jpr} \, \phi^{o}_{jpq} \, \phi^{o}_{ppq} + 162 \, \phi^{o}_{jjp} \, \phi^{o}_{jpq} \, \phi^{o}_{jrr} \, \phi^{o}_{ppq}$$

$$+ \ 324 \, (\phi^{o}_{jjp})^2 \, (\phi^{o}_{pqr})^2 + 324 \, \phi^{o}_{jjq} \, \phi^{o}_{jpp} \, \phi^{o}_{jpr} \, \phi^{o}_{pqr}$$

$$+ \ 81 \, \phi^{o}_{jjp} \, \phi^{o}_{jpp} \, \phi^{o}_{jrr} \, \phi^{o}_{pqq} + 1296 (\phi^{o}_{jpq})^2 \, (\phi^{o}_{jpr})^2$$

$$+ \ 324 \, (\phi^{o}_{jpr})^2 \, \phi^{o}_{jjp} \, \phi^{o}_{pqq} + 648 \, \phi^{o}_{jjp} \, \phi^{o}_{jpq} \, \phi^{o}_{pqr} \, \phi^{o}_{jpr}$$

$$+ \ 324 \, \phi^{o}_{jjp} \, \phi^{o}_{jpp} \, \phi^{o}_{jqr} \, \phi^{o}_{pqr} + 162 \, \phi^{o}_{jjp} \, \phi^{o}_{jpp} \, \phi^{o}_{jpq} \, \phi^{o}_{qrr} \}$$

$$+ \sum_{j \neq p \neq q \neq r \neq s} \frac{-h^{o}}{41472 \pi^2 \lambda_j^2 \lambda_p \lambda_q \lambda_r \lambda_s} \{ \ 27 \, \phi^{o}_{jjj} \, \phi^{o}_{jpp} \, \phi^{o}_{qqr} \, \phi^{o}_{rss}$$

$$+ \ 108 \, (\phi^{o}_{qrs})^2 \, \phi^{o}_{jjj} \, \phi^{o}_{jpp} + 54 \, \phi^{o}_{jjj} \, \phi^{o}_{jpq} \, \phi^{o}_{rrp} \, \phi^{o}_{ssq}$$

$$+ \ 216 \, \phi^{o}_{jjj} \, \phi^{o}_{jpq} \, \phi^{o}_{rsp} \, \phi^{o}_{rsq} + 108 \, \phi^{o}_{jjj} \, \phi^{o}_{jpq} \, \phi^{o}_{pqr} \, \phi^{o}_{rss}$$

$$+ \ 81 \, (\phi^{o}_{jjp})^2 \, \phi^{o}_{qqr} \, \phi^{o}_{rss} + 324 \, (\phi^{o}_{jjp})^2 \, (\phi^{o}_{qrs})^2$$

$$+ \ 81 \, \phi^{o}_{jjp} \, \phi^{o}_{jjq} \, \phi^{o}_{prr} \, \phi^{o}_{qss} + 324 \, \phi^{o}_{jjp} \, \phi^{o}_{jjq} \, \phi^{o}_{prs} \, \phi^{o}_{qrs}$$

$$+ \; 648\, \phi^{o}_{jjp}\, \phi^{o}_{jjq}\, \phi^{o}_{pqr}\, \phi^{o}_{rss} + 81\, \phi^{o}_{jjq}\, \phi^{o}_{jrr}\, \phi^{o}_{jpp}\, \phi^{o}_{ssq}$$

$$+ \; 1944\, (\phi^{o}_{jqs})^{2}\, \phi^{o}_{jjp}\, \phi^{o}_{rrp} + 1944\, \phi^{o}_{jjq}\, \phi^{o}_{jps}\, \phi^{o}_{rrp}\, \phi^{o}_{jqs}$$

$$+ \; 648\, \phi^{o}_{jjq}\, \phi^{o}_{jpq}\, \phi^{o}_{jss}\, \phi^{o}_{rrp} + 648\, \phi^{o}_{jjp}\, \phi^{o}_{jjq}\, \phi^{o}_{pqr}\, \phi^{o}_{rss}$$

$$+ \; 2592\, \phi^{o}_{jjp}\, \phi^{o}_{jrq}\, \phi^{o}_{jrs}\, \phi^{o}_{pqs} + 2592\, \phi^{o}_{jjr}\, \phi^{o}_{jpr}\, \phi^{o}_{jqs}\, \phi^{o}_{pqs}$$

$$+ \; 1944\, \phi^{o}_{jjs}\, \phi^{o}_{jpq}\, \phi^{o}_{jrr}\, \phi^{o}_{spq} + 81\, \phi^{o}_{jpp}\, \phi^{o}_{jqq}\, \phi^{o}_{jrr}\, \phi^{o}_{jss}$$

$$+ \; 1944\, \phi^{o}_{jpp}\, \phi^{o}_{jqq}\, (\phi^{o}_{jrs})^{2} + 2592\, \phi^{o}_{jpp}\, \phi^{o}_{jqr}\, \phi^{o}_{jrs}\, \phi^{o}_{jqs}$$

$$+ \; 1296\, (\phi^{o}_{jpq})^{2}\, (\phi^{o}_{jrs})^{2} + 1296\, \phi^{o}_{jpq}\, \phi^{o}_{jpr}\, \phi^{o}_{jsq}\, \phi^{o}_{jsr} \}$$

$$+ \sum_{j \neq m \neq q \neq p \neq r \neq s} \frac{-\, h^{o}}{124416\pi^{2} \lambda_{j} \lambda_{m} \lambda_{q} \lambda_{p} \lambda_{r} \lambda_{s}} \{ 81\, \phi^{o}_{jjp}\, \phi^{o}_{mmp}\, \phi^{o}_{rrs}\, \phi^{o}_{qqs}$$

$$+ \; 648\, \phi^{o}_{jjp}\, \phi^{o}_{mms}\, \phi^{o}_{rrq}\, \phi^{o}_{pqs} + 1944\, \phi^{o}_{jjs}\, \phi^{o}_{mms} (\phi^{o}_{pqr})^{2}$$

$$+ \; 1944\, \phi^{o}_{jjs}\, \phi^{o}_{mmr}\, \phi^{o}_{pqr}\, \phi^{o}_{spq} + 2592\, \phi^{o}_{jjm}\, \phi^{o}_{mpq}\, \phi^{o}_{spr}\, \phi^{o}_{srq}$$

$$+ \; 1296\, (\phi^{o}_{jpq})^{2}\, (\phi^{o}_{mrs})^{2} + 1296\, \phi^{o}_{jpq}\, \phi^{o}_{jpr}\, \phi^{o}_{msq}\, \phi^{o}_{msr}$$

$$+ \; 1296\, \phi^{o}_{jpq}\, \phi^{o}_{jsr}\, \phi^{o}_{mpr}\, \phi^{o}_{msq} \} \}$$

(10) $$\left(\frac{2}{k}\right)^{n/2\,+\,2} \frac{e^{\pi i\,\mathrm{sgn} H(p)/4}}{(|\det H\phi|)^{1/2}} \Big\{ \sum_{j} \frac{-35h^{o}}{1536\pi^{2}\lambda_j^{4}} \{(\mathring{\phi}_{jjjj})^{2}\}$$

$$+ \sum_{j\neq p} \frac{-5h^{o}}{1536\pi^{2}\lambda_j^{3}\lambda_p} \{16(\mathring{\phi}_{jjjp})^{2} + 12\,\mathring{\phi}_{jjjj}\,\mathring{\phi}_{jjpp}\}$$

$$+ \sum_{j\neq p} \frac{-h^{o}}{512\pi^{2}\lambda_j^{2}\lambda_p^{2}} \{\mathring{\phi}_{jjjj}\,\mathring{\phi}_{pppp} + 16\,\mathring{\phi}_{jjjp}\,\mathring{\phi}_{pppj} + 36(\mathring{\phi}_{jjpp})^{2}\}$$

$$+ \sum_{j\neq p\neq q} \frac{-h^{o}}{1536\pi^{2}\lambda_j^{2}\lambda_p\lambda_q} \{12\,\mathring{\phi}_{jjjj}\,\mathring{\phi}_{ppqq} + 96\,\mathring{\phi}_{jjjp}\,\mathring{\phi}_{jpqq}$$

$$+ 36\,\mathring{\phi}_{jjpp}\,\mathring{\phi}_{jjqq} + 144(\mathring{\phi}_{jjpq})^{2}\} + \sum_{j\neq r\neq p\neq q} \frac{-h^{o}}{4608\pi^{2}\lambda_j\lambda_r\lambda_p\lambda_q}$$

$$\{36\,\mathring{\phi}_{jjrr}\,\mathring{\phi}_{ppqq} + 144\,\mathring{\phi}_{jjrp}\,\mathring{\phi}_{qqrp} + 576(\mathring{\phi}_{jrpq})^{2}\}\Big\}$$

(11) $$\left(\frac{2\pi}{k}\right)^{n/2\,+\,2} \frac{e^{\pi i\,\mathrm{sgn} H(p)/4}}{(|\det H\phi|)^{1/2}} \Big\{ \sum_{j} \frac{1}{32\pi^{2}\lambda_j^{3}} \{\mathring{\phi}_{jjjjj}\,\mathring{h}_j\}$$

$$+ \sum_{j\neq p} \frac{1}{160\pi^{2}\lambda_j^{2}\lambda_p} \{5\,\mathring{\phi}_{jjjjp}\,\mathring{h}_p + 10\,\mathring{\phi}_{jjjpp}\,\mathring{h}_j\}$$

$$+ \sum_{j\neq p\neq q} \frac{1}{480\pi^{2}\lambda_j\lambda_p\lambda_q} \{30\,\mathring{\phi}_{jjppq}\,\mathring{h}_q\}\Big\}$$

(12) $$\left(\frac{2\pi}{k}\right)^{n/2\,+\,2} \frac{e^{\pi i\,\mathrm{sgn} H(p)/4}}{(|\det H\phi|)^{1/2}} \Big\{ \sum_{j} \frac{1}{32\pi^{2}\lambda_j^{2}} \{\mathring{h}_{jjjj}\}$$

$$+ \sum_{j<p} \frac{-1}{96\pi^{2}\lambda_j\lambda_p} \{6\,\mathring{h}_{jjpp}\}\Big\}$$

$$(13)\qquad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{i\,\mathrm{sgn}H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}}\Big\{\sum_{j} \frac{h^{\circ}}{192\pi^{2}\lambda_j^{3}} \{\mathring{\phi}_{jjjjjj}\}$$

$$+ \sum_{j\neq p} \frac{h^{\circ}}{960\pi^{2}\lambda_j^{2}\lambda_p} \{15\ \mathring{\phi}_{jjjjpp}\}$$

$$+ \sum_{j\neq p\neq q} \frac{h^{\circ}}{2880\pi^{2}\lambda_j\lambda_p\lambda_q} \{90\ \mathring{\phi}_{jjppqq}\}\Big\}$$

$$(14)\qquad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{\pi i\,\mathrm{sgn}H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}}\Big\{\sum_{j} \frac{-7h^{\circ}}{192\pi^{2}\lambda_j^{4}} \{\mathring{\phi}_{jjjjj}\ \mathring{\phi}_{jjj}\}$$

$$+ \sum_{j\neq p} \frac{-h^{\circ}}{192\pi^{2}\lambda_j^{3}\lambda_p} \{3\ \mathring{\phi}_{jjjjj}\ \mathring{\phi}_{jpp} + 10\ \mathring{\phi}_{ppjjj}\ \mathring{\phi}_{jjj}$$

$$+ 15\ \mathring{\phi}_{pjjjj}\ \mathring{\phi}_{pjj}\} + \sum_{j\neq p} \frac{-h^{\circ}}{320\pi^{2}\lambda_j^{2}\lambda_p^{2}} \{5\ \mathring{\phi}_{jjjjp}\ \mathring{\phi}_{ppp}$$

$$+ 30\ \mathring{\phi}_{jjjpp}\ \mathring{\phi}_{jpp}\} + \sum_{j\neq p\neq q} \frac{-h^{\circ}}{2880\pi^{2}\lambda_j^{2}\lambda_p\lambda_q} \{15\ \mathring{\phi}_{jjjjp}\ \mathring{\phi}_{pqq}$$

$$+ 30\ \mathring{\phi}_{jjjpp}\ \mathring{\phi}_{jqq} + 120\ \mathring{\phi}_{jjjpq}\ \mathring{\phi}_{jpq} + 90\ \mathring{\phi}_{jjppq}\ \mathring{\phi}_{qjj}$$

$$+ 30\ \mathring{\phi}_{jppqq}\ \mathring{\phi}_{jjj}\} + \sum_{j\neq p\neq q\neq r} \frac{-h^{\circ}}{2880\pi^{2}\lambda_j\lambda_p\lambda_q\lambda_r}\ \cdot$$

$$\cdot\{90\ \mathring{\phi}_{jjppq}\ \mathring{\phi}_{rrq} + 360\ \mathring{\phi}_{jjpqr}\ \mathring{\phi}_{pqr}\}\Big\}$$

$$(15)\qquad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{i\,\mathrm{sgn}H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}}\Big\{\sum_{j} \frac{5}{48\pi^{2}\lambda_j^{3}} \{\mathring{\phi}_{jjj}\ h_{jjj}\}$$

$$+ \sum_{j\neq p} \frac{1}{144\pi^2 \lambda_j^2 \lambda_p} \{3\, \mathring{\phi}_{jjj}\, \mathring{h}_{jpp} + 3\, \mathring{\phi}_{ppj}\, \mathring{h}_{jjj} + 9\, \mathring{\phi}_{pjj}\, \mathring{h}_{pjj}\}$$

$$+ \sum_{j\neq p\neq q} \frac{1}{144\pi^2 \lambda_j \lambda_p \lambda_q} \{9\, \mathring{\phi}_{jjp}\, \mathring{h}_{pqq} + 36\, \mathring{\phi}_{jqp}\, \mathring{h}_{jqp}\}$$

$$(16) \quad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{1/2}} \{\sum_{j} \frac{35\mathring{h}}{256\pi^2 \lambda^5} \{(\mathring{\phi}_{jjj})^2\, \mathring{\phi}_{jjjj}\}$$

$$+ \sum_{j\neq p} \frac{35\mathring{h}}{2304\pi^2 \lambda_j^4 \lambda_p} \{9(\mathring{\phi}_{jjp})^2\, \mathring{\phi}_{jjjj} + 24\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{jjp}\, \mathring{\phi}_{jjjp}$$

$$+ 6\, \mathring{\phi}_{jpp}\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{jjjj} + 6\, (\mathring{\phi}_{jjj})^2\, \mathring{\phi}_{jjpp}\}$$

$$+ \sum_{j\neq p} \frac{5\mathring{h}}{768\pi^2 \lambda_j^3 \lambda_p^2} \{(\mathring{\phi}_{jjj})^2\, \mathring{\phi}_{pppp} + 6\, \mathring{\phi}_{jjp}\, \mathring{\phi}_{ppp}\, \mathring{\phi}_{jjjj}$$

$$+ 9\, (\mathring{\phi}_{jpp})^2\, \mathring{\phi}_{jjjj} + 24\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{jjp}\, \mathring{\phi}_{jppp} + 54\, (\mathring{\phi}_{jjp})^2\, \mathring{\phi}_{jjpp}$$

$$+ 36\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{jpp}\, \mathring{\phi}_{jjpp} + 4\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{ppp}\, \mathring{\phi}_{jjjp}$$

$$+ 36\, \mathring{\phi}_{jjp}\, \mathring{\phi}_{jpp}\, \mathring{\phi}_{jjjp}\} + \sum_{j\neq p\neq q} \frac{5\mathring{h}}{2304\pi^2 \lambda_j^3 \lambda_p \lambda_q} \cdot$$

$$\cdot \{6\, (\mathring{\phi}_{jjj})^2\, \mathring{\phi}_{ppqq} + 72\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{jjp}\, \mathring{\phi}_{qqpj}$$

$$+ 109\, \mathring{\phi}_{jjp}\, \mathring{\phi}_{jjq}\, \mathring{\phi}_{jjpq} + 144\, \mathring{\phi}_{pqj}\, \mathring{\phi}_{jjj}\, \mathring{\phi}_{jjpq} + 54\, (\mathring{\phi}_{qjj})^2\, \mathring{\phi}_{ppjj}$$

$$+ \; 36\,\mathring{\phi}_{jjj}\,\mathring{\phi}_{qqj}\,\mathring{\phi}_{ppjj} + 24\,\mathring{\phi}_{jjj}\,\mathring{\phi}_{pqq}\,\mathring{\phi}_{pjjj}$$

$$+ \; 144\,\mathring{\phi}_{qjj}\,\mathring{\phi}_{pqj}\,\mathring{\phi}_{jjjp} + 36\,\mathring{\phi}_{qqj}\,\mathring{\phi}_{jjp}\,\mathring{\phi}_{jjjp}$$

$$+ \; 9\,\mathring{\phi}_{ppj}\,\mathring{\phi}_{qqj}\,\mathring{\phi}_{jjjj} + 9\,\mathring{\phi}_{jjp}\,\mathring{\phi}_{qqp}\,\mathring{\phi}_{jjjj} + 36\,(\mathring{\phi}_{jpq})^2\,\mathring{\phi}_{jjjj}\}$$

$$+ \sum_{j\neq p\neq q\neq r} \frac{h^{o}}{2304\pi^2\lambda_j^2\lambda_p\lambda_q\lambda_r}\,\{\,36\,(\mathring{\phi}_{pqr})^2\,\mathring{\phi}_{jjjj}$$

$$+ \; 432\,\mathring{\phi}_{jqq}\,\mathring{\phi}_{pjr}\,\mathring{\phi}_{jjrp} + 36\,\mathring{\phi}_{prr}\,\mathring{\phi}_{jqq}\,\mathring{\phi}_{jjjp}$$

$$+ \; 144\,\mathring{\phi}_{pqr}\,\mathring{\phi}_{jqr}\,\mathring{\phi}_{jjjp} + 144\,\mathring{\phi}_{qrr}\,\mathring{\phi}_{jpq}\,\mathring{\phi}_{jjjp}$$

$$+ \; 54\,\mathring{\phi}_{rrq}\,\mathring{\phi}_{jjq}\,\mathring{\phi}_{jjpp} + 54\,\mathring{\phi}_{jqq}\,\mathring{\phi}_{jrr}\,\mathring{\phi}_{jjpp}$$

$$+ \; 216\,(\mathring{\phi}_{jqr})^2\,\mathring{\phi}_{jjpp} + 432\,\mathring{\phi}_{pqr}\,\mathring{\phi}_{jjr}\,\mathring{\phi}_{jjpq}$$

$$+ \; 108\,\mathring{\phi}_{rrq}\,\mathring{\phi}_{jjp}\,\mathring{\phi}_{jjpq} + 108\,\mathring{\phi}_{jrr}\,\mathring{\phi}_{jjq}\,\mathring{\phi}_{ppjq}$$

$$+ \; 432\,\mathring{\phi}_{jjr}\,\mathring{\phi}_{jqr}\,\mathring{\phi}_{ppjq} + 72\,\mathring{\phi}_{jjj}\,\mathring{\phi}_{rrq}\,\mathring{\phi}_{ppjq}$$

$$+ \; 864\,\mathring{\phi}_{jqr}\,\mathring{\phi}_{jjp}\,\mathring{\phi}_{jpqr} + 288\,\mathring{\phi}_{jjj}\,\mathring{\phi}_{pqr}\,\mathring{\phi}_{jpqr}$$

$$+ \; 144\,\mathring{\phi}_{jjj}\,\mathring{\phi}_{jpq}\,\mathring{\phi}_{rrpq} + 36\,\mathring{\phi}_{jjj}\,\mathring{\phi}_{ppj}\,\mathring{\phi}_{qqrr}$$

$$+ \; 54\,(\mathring{\phi}_{jjp})^2\,\mathring{\phi}_{qqrr} + 432\,\mathring{\phi}_{jqr}\,\mathring{\phi}_{jqp}\,\mathring{\phi}_{jjrp}$$

$$+ \quad 108\, \overset{o}{\phi}_{jjp}\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{rrpq} + \sum_{j\neq p\neq q} \frac{h^{o}}{768\pi^{2}\lambda_j^{2}\lambda_p^{2}\lambda_q}$$

$$\{\, 6\overset{o}{\phi}_{jjjj}\, \overset{o}{\phi}_{ppp}\, \overset{o}{\phi}_{pqq} + 9\, (\overset{o}{\phi}_{ppq})^{2}\, \overset{o}{\phi}_{jjjj} + 24\, \overset{o}{\phi}_{jjjp}\, \overset{o}{\phi}_{jqq}\, \overset{o}{\phi}_{ppp}$$

$$+ \quad 36\, \overset{o}{\phi}_{jjjp}\, \overset{o}{\phi}_{jpp}\, \overset{o}{\phi}_{pqq} + 144\, \overset{o}{\phi}_{jjjp}\, \overset{o}{\phi}_{jpq}\, \overset{o}{\phi}_{ppq}$$

$$+ \quad 48\, \overset{o}{\phi}_{ppp}\, \overset{o}{\phi}_{jqp}\, \overset{o}{\phi}_{jjjq} + 36\, \overset{o}{\phi}_{ppq}\, \overset{o}{\phi}_{ppj}\, \overset{o}{\phi}_{jjjq}$$

$$+ \quad 36\, \overset{o}{\phi}_{ppp}\, \overset{o}{\phi}_{jjp}\, \overset{o}{\phi}_{jjqq} + 54\, \overset{o}{\phi}_{jjp}\, \overset{o}{\phi}_{qqp}\, \overset{o}{\phi}_{jjpp} + 54\, (\overset{o}{\phi}_{jpp})^{2}\, \overset{o}{\phi}_{jjqq}$$

$$+ \quad 54\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{ppq}\, \overset{o}{\phi}_{jjpp} + 216\, (\overset{o}{\phi}_{jpq})^{2}\, \overset{o}{\phi}_{jjpp}$$

$$+ \quad 72\, \overset{o}{\phi}_{ppp}\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{jjpq} + 108\, \overset{o}{\phi}_{jjp}\, \overset{o}{\phi}_{ppq}\, \overset{o}{\phi}_{jjpq}$$

$$+ \quad 432\, \overset{o}{\phi}_{ppj}\, \overset{o}{\phi}_{jpq}\, \overset{o}{\phi}_{jjpq} + 12\, \overset{o}{\phi}_{ppp}\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{qqjp}$$

$$+ \quad 108\, \overset{o}{\phi}_{jjp}\, \overset{o}{\phi}_{ppj}\, \overset{o}{\phi}_{qqjp}\,\} + \sum_{j\neq p\neq q\neq r\neq s} \frac{h^{o}}{6912\pi^{2}\lambda_j\lambda_p\lambda_q\lambda_r\lambda_s}$$

$$\{54\, \overset{o}{\phi}_{qqr}\, \overset{o}{\phi}_{ssr}\, \overset{o}{\phi}_{jjpp} + 216\, (\overset{o}{\phi}_{qrs})^{2}\, \overset{o}{\phi}_{jjpp}$$

$$+ \quad 108\, \overset{o}{\phi}_{rrp}\, \overset{o}{\phi}_{ssq}\, \overset{o}{\phi}_{jjpq} + 432\, \overset{o}{\phi}_{rrs}\, \overset{o}{\phi}_{qps}\, \overset{o}{\phi}_{jjpq}$$

$$+ \quad 432\, \overset{\circ}{\phi}_{rsp}\, \overset{\circ}{\phi}_{rsq}\, \overset{\circ}{\phi}_{jjpq} + 864\, \overset{\circ}{\phi}_{pqr}\, \overset{\circ}{\phi}_{ssj}\, \overset{\circ}{\phi}_{jpqr}$$

$$+ \quad 864\, \overset{\circ}{\phi}_{jps}\, \overset{\circ}{\phi}_{qrs}\, \overset{\circ}{\phi}_{jpqr} \}$$

$$(17) \qquad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} \sum_{j} \frac{5}{64\pi^2 \lambda_j^3} \{ \overset{\circ}{\phi}_{jjjj}\, \overset{\circ}{h}_{jj} \}$$

$$+ \sum_{j \neq p} \frac{1}{64\pi^2 \lambda_j^2 \lambda_p} \{ \overset{\circ}{\phi}_{jjjj}\, \overset{\circ}{h}_{pp} + 6\, \overset{\circ}{\phi}_{jjpp}\, \overset{\circ}{h}_{jj} + 8\, \overset{\circ}{\phi}_{jjjp}\, \overset{\circ}{h}_{jp} \}$$

$$+ \sum_{j \neq p \neq q} \frac{1}{192\pi^2 \lambda_j \lambda_p \lambda_q} \{ 6\, \overset{\circ}{\phi}_{jjpp}\, \overset{\circ}{h}_{qq} + 24\, \overset{\circ}{\phi}_{jjqp}\, \overset{\circ}{h}_{qp} \}$$

$$(18) \qquad \left(\frac{2\pi}{k}\right)^{n/2+2} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(|\det H\phi|)^{\frac{1}{2}}} \sum_{j} \frac{-35}{192\pi^2 \lambda_j^4} \{ (\overset{\circ}{\phi}_{jjj})^2\, \overset{\circ}{h}_{jj} \}$$

$$+ \sum_{j \neq p} \frac{-5}{192\pi^2 \lambda_j^3 \lambda_p} \{ (\overset{\circ}{\phi}_{jjj})^2\, \overset{\circ}{h}_{pp} + 6\, \overset{\circ}{\phi}_{jjj}\, \overset{\circ}{\phi}_{ppj}\, \overset{\circ}{h}_{jj}$$

$$+ \quad 12\, \overset{\circ}{\phi}_{jjj}\, \overset{\circ}{\phi}_{jjp}\, \overset{\circ}{h}_{jp} + 9\, (\overset{\circ}{\phi}_{jjp})^2\, \overset{\circ}{h}_{jj} \}$$

$$+ \sum_{j \neq p} \frac{-1}{64\pi^2 \lambda_j^2 \lambda_p^2} \{ 6\, \overset{\circ}{\phi}_{ppp}\, \overset{\circ}{\phi}_{jjp}\, \overset{\circ}{h}_{jj} + 9\, (\overset{\circ}{\phi}_{jpp})^2\, \overset{\circ}{h}_{jj}$$

$$+ \quad 18\, \overset{o}{\phi}_{ppj}\, \overset{o}{\phi}_{jjp}\, \overset{o}{h}_{jp} + 2\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{ppp}\, \overset{o}{h}_{pj} \}$$

$$+ \sum_{j\neq p\neq q} \frac{-1}{192\pi^2 \lambda_j^2 \lambda_p \lambda_q} \{ 6\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{ppj}\, \overset{o}{h}_{qq} + 9\, (\overset{o}{\phi}_{jjp})^2\, h_{qq}$$

$$+ \quad 24\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{jpq}\, \overset{o}{h}_{pq} + 18\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{jjp}\, \overset{o}{h}_{pq} + 12\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{ppq}\, \overset{o}{h}_{jq}$$

$$+ \quad 18\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{ppj}\, \overset{o}{h}_{jq} + 72\, \overset{o}{\phi}_{jpq}\, \overset{o}{\phi}_{jjp}\, \overset{o}{h}_{jq} + 9\, \overset{o}{\phi}_{ppq}\, \overset{o}{\phi}_{jjq}\, \overset{o}{h}_{jj}$$

$$+ \quad 9\, \overset{o}{\phi}_{qqj}\, \overset{o}{\phi}_{ppj}\, \overset{o}{h}_{jj} + 36\, (\overset{o}{\phi}_{jqp})^2\, \overset{o}{h}_{jj} \}$$

$$+ \sum_{j\neq p\neq q\neq r} \frac{-1}{576\pi^2 \lambda_j \lambda_p \lambda_q \lambda_r} \{ 9\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{rrq}\, \overset{o}{h}_{pp} + 36\, (\overset{o}{\phi}_{jqr})^2\, \overset{o}{h}_{pp}$$

$$+ \quad 72\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{pqr}\, \overset{o}{h}_{pr} + 18\, \overset{o}{\phi}_{jjp}\, \overset{o}{\phi}_{qqr}\, \overset{o}{h}_{pr} + 72\, \overset{o}{\phi}_{jpq}\, \overset{o}{\phi}_{jqr}\, \overset{o}{h}_{pr} \}$$

$$(19) \qquad \left(\frac{2\pi}{k}\right)^{n/2\, +\, 2} \frac{e^{\pi i \operatorname{sgn} H(p)/4}}{(\,|\det H\phi|\,)^{\frac{1}{2}}} \{ \sum_{j} \frac{-35}{192\pi^2 \lambda_j^4} \{ \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{jjjj}\, \overset{o}{h}_{j} \}$$

$$+ \sum_{j\neq q} \frac{-5}{192\pi^2 \lambda_j^3 \lambda_q} \{ 3\, \overset{o}{\phi}_{jqq}\, \overset{o}{\phi}_{jjjj}\, \overset{o}{h}_{j} + 6\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{jjqq}\, \overset{o}{h}_{j}$$

$$+ \quad 3\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{jjjj}\, \overset{o}{h}_{q} + 4\, \overset{o}{\phi}_{jjj}\, \overset{o}{\phi}_{jjjq}\, \overset{o}{h}_{q} + 12\, \overset{o}{\phi}_{jjq}\, \overset{o}{\phi}_{jjjq}\, \overset{o}{h}_{j} \}$$

$$+ \sum_{j \neq q} \frac{-1}{64\pi^2 \lambda_j^2 \lambda_q^2} \{ \mathring{\phi}_{qqq} \mathring{\phi}_{jjjj} \mathring{h}_q + 4 \mathring{\phi}_{qqq} \mathring{\phi}_{jjjq} \mathring{h}_j$$

$$+ 12 \mathring{\phi}_{qqj} \mathring{\phi}_{jjjq} \mathring{h}_q + 18 \mathring{\phi}_{jjq} \mathring{\phi}_{jjqq} \mathring{h}_q + 18 \mathring{\phi}_{jqq} \mathring{\phi}_{jjqq} \mathring{h}_j \}$$

$$+ \sum_{j \neq p \neq q} \frac{-1}{192\pi^2 \lambda_j^2 \lambda_p \lambda_q} \{ 3 \mathring{\phi}_{ppq} \mathring{\phi}_{jjjj} \mathring{h}_q + 12 \mathring{\phi}_{ppq} \mathring{\phi}_{jjjq} \mathring{h}_j$$

$$+ 12 \mathring{\phi}_{jpp} \mathring{\phi}_{jjjq} \mathring{h}_q + 24 \mathring{\phi}_{jqp} \mathring{\phi}_{jjjq} \mathring{h}_p + 18 \mathring{\phi}_{jjp} \mathring{\phi}_{jjqq} \mathring{h}_p$$

$$+ 36 \mathring{\phi}_{jjp} \mathring{\phi}_{jjpq} \mathring{h}_q + 12 \mathring{\phi}_{jjj} \mathring{\phi}_{jpqq} \mathring{h}_p + 6 \mathring{\phi}_{jjj} \mathring{\phi}_{qqpp} \mathring{h}_j$$

$$+ 36 \mathring{\phi}_{jjp} \mathring{\phi}_{jpqq} \mathring{h}_j \}$$

$$+ \sum_{j \neq p \neq q \neq r} \frac{-1}{576\pi^2 \lambda_j \lambda_p \lambda_q \lambda_r} \{ 18 \mathring{\phi}_{pqq} \mathring{\phi}_{jjrr} \mathring{h}_p$$

$$+ 36 \mathring{\phi}_{rqq} \mathring{\phi}_{jjrp} \mathring{h}_p + 72 \mathring{\phi}_{pqr} \mathring{\phi}_{jjrp} \mathring{h}_q + 144 \mathring{\phi}_{rpq} \mathring{\phi}_{jrpq} \mathring{h}_j \}$$

The next theorem treats the case where $\int_\Sigma e^{ik\phi(y)} a(y)dy$ has a singularity in a(y) at a value around which we wish to expand the integral. The point x denotes the singularity and $r = |x-y|$.

THEOREM 3.2

Let Σ be a closed, compact, orientable, n-manifold of class C^3 imbedded in E^{n+1}. Suppose that $J(x,y) = \frac{A(x,y)}{|x-y|^{n-1}}$ is a C^2 kernel that is A(x,y) is C^2 as a function of y in a sufficiently small neighborhood of x denoted by $N(x) \subseteq \Sigma$. Suppose also that the manifold is explicitly represented in N(x) by $y_{n+1} = f(y_1, \ldots, y_n)$ where, without loss of generality, assume that $f_j^o = 0$ (for all j) and $f_{jq}^o = 0$ if $j \neq q$. Then as $k \to +\infty$

$$\int_{N(x)} e^{ikr} J(x,y)\, dy = \frac{C_{n-1}\, A(x,x)}{(-ik)} + \frac{B_n (\Delta A)(x)}{(-ik)^3}$$

$$+ \quad \frac{B_n A(x,x)}{4(-ik)^3} \{ n^2 . H^2 - 2n(n-1)K_2 \} + O(k^{-4})$$

where C_n is the area of the unit n-sphere, B_n is a function of n defined below, $\Delta A(x) = \Sigma \frac{\partial^2 A}{\partial^2 y_j{}^2}(x)$, H is the mean curvature $(\frac{\Sigma f_{jj}^o}{n})$, and K_2 is the "Gaussian" curvature defined by $\binom{n}{2} K_2 = \sum_{j<q} f_{jj}^o \; f_{qq}^o$.

Proof: Let x be the origin of the Euclidean coordinate system and the y_{n+1} axis have the same orientation as the unit outward normal to Σ to x. By compactness and smoothness of the surface and the fact that $A \epsilon C^2(\Sigma)$ it will be clear that all terms included in $O(k^{-4})$ can be made uniform in x.

We can write the integral as

$$\int_{S(O)} \int_o^\infty e^{ikr} \frac{A(x,y)}{|x-y|^{n-1}} g(\rho) \cdot \sqrt{G} \; \rho^{n-1} \, d\rho \, dS^{n-1} + O(k^{-4})$$

where the volume element in N(x) has been expressed in polar coordinates in the tangent plane to x, dS^{n-1} is surface measure on the n-1 dimensional sphere, S(O) is the unit sphere in E^n, and G denotes the determinant of the first fundamental form. A cut-off function $g(\rho)$ has been inserted which satisfies the hypothesis of Lemma 2.6 and $g(\rho) = 1$ near x.

If we expand the functions of ρ above in a similar manner to the previous theorem we obtain

(1) $$\int_{S(0)} \int_o^\infty e^{ik\rho} \{1 + ik(\frac{f^2}{2\rho}) + k^2 O(\rho^6)\} \{1 + k O(\rho^5)\} \cdot$$

$$\cdot \{h^o + h^o_\rho(\rho) + \tfrac{1}{2} h^o_{\rho\rho}(\rho^2) + O(\rho^3)\}\{1 - \frac{n-1}{2} \frac{f^2}{\rho^2} + O(\rho^4)\} \cdot$$

$$\cdot g(\rho) d\rho dS^{n-1} + O(k^{-4}),$$ where $h(y) = A(x,y) \cdot \sqrt{G}$

The five integrals from (1) which give contributions are listed below. The others by Lemma 2.6 can be shown to be $O(k^{-4})$.

(2) $$\int_{S(0)} \int_0^\infty e^{ik\rho}\, h^o\, g(\rho)\, d\rho\, dS^{n-1}$$

(3) $$\int_{S(0)} \int_0^\infty e^{ik\rho}\, h^o_\rho\, g(\rho)\, \rho\, d\rho\, dS^{n-1}$$

(4) $$\int_{S(0)} \int_0^\infty e^{ik\rho}\, \tfrac{1}{2} h^o_{\rho\rho}\, g(\rho)\, \rho^2\, d\rho\, dS^{n-1}$$

(5) $$\frac{-(n-1)}{2} \int_{S(0)} \int_0^\infty e^{ik\rho}\, h^o\, g(\rho)\, \left(\frac{f^2}{\rho^2}\right) d\rho\, dS^{n-1}$$

(6) $$\frac{ik}{2} \int_{S(0)} \int_0^\infty e^{ik\rho}\, h^o\, g(\rho)\, \left(\frac{f^2}{\rho}\right) d\rho\, dS^{n-1}$$

In each case integration is performed with respect to ρ first using Lemma 2.6. Since $A \varepsilon C^2$ and $f \varepsilon C^3$ any integration by parts is justified as are any Taylor series expansions up to the second order.

Integrating (2) by parts we have by Lemma 2.6

(2) $$\int_{S(0)} \int_0^\infty e^{ik\rho}\, h^o\, g(\rho)\, d\rho\, dS^{n-1}$$

$$= \int_{S(0)} h^o\, dS^{n-1} \left\{ \frac{1}{-ik} + O(k^{-m-1}) \right\}$$

$$= C_{n-1}\, h^o \left\{ \frac{1}{-ik} + O(k^{-m-1}) \right\}$$

where m is defined in Lemma 2.6. For the purposes of this theorem it is sufficient to choose $g \varepsilon C^4$, that is m=4.

For (3) we have $h_\rho^o = h_j^o \xi^j$ where $h = h(\rho\xi^1, \ldots, \rho\xi^n)$ and $\xi = (\xi, \ldots, \xi^n) \varepsilon S(0)$.

$$(3) \qquad \int_{S(0)} \int_o^\infty e^{ik\rho} h_\rho^o \, g(\rho)\rho d\rho \, dS^{n-1}$$

$$= \int_{S(0)} h_\rho^o \, dS^{n-1}\{ \frac{1}{(-ik)^2} + O(k^{-m-1})\}$$

$$= h_j^o \int_{S(0)} \xi^j \, dS^{n-1}\{ \frac{1}{(-ik)^2} + O(k^{-m-1}) \}$$

$= 0$ by symmetry of the surface integral.

For (4) we have,

$$(4) \qquad \tfrac{1}{2} \int_{S(0)} \int_o^\infty e^{ik\rho} h_{\rho\rho}^o \, g(\rho)\rho^2 \, d\rho dS^{n-1}$$

$$= \tfrac{1}{2} h_{jq}^o \int_{S(0)} \xi^j \xi^q \, dS^{n-1} \{\frac{2}{(-ik)^3} + O(k^{-m-1})\}$$

By symmetry, the surface integral vanishes unless j=q and has the same value for all j if j=q. Since $dS^{n-1} = \sin^{n-2}\alpha \, d\alpha \, dS^{n-2}$ and $\xi^n = \cos\alpha$, the surface integral is calculated to be

$$\int_{S(0)} (\xi^n)^2 \, dS^{n-1} = \int_{S(0)} \int_o^\pi \cos^2\alpha \sin^{n-2}\alpha d\alpha dS^{n-2} = B_n$$

where

$$B_n = \begin{cases} \left\{\dfrac{C_{n-2}\,(n-2)!\,\pi}{n2^{n-2}\left(\left(\frac{n-2}{2}\right)!\right)^2}\right\} & n \text{ even} \\[2ex] \left\{\dfrac{C_{n-2}\,2^{n-2}\left(\left(\frac{n-3}{2}\right)!\right)^2}{n(n-2)!}\right\} & n \text{ odd} \end{cases}$$

Recalling that $h(y) = A(x,y)\cdot\sqrt{G}$ so that $\Delta h(x) = \Delta A(x) + A(x,x)\sum_j (f^o_{jj})^2$ since $f^o_j = 0$ for all j and combining with the above result yields

$$\tfrac{1}{2}\int_{S(0)}\int_0^\infty e^{ik\rho}\, h^o_{\rho\rho}\, g(\rho)\rho^2\, d\rho\, dS^{n-1}$$

$$= \frac{B_n\{\Delta A(x) + A(x)\cdot\Sigma(f^o_{jj})^2\}}{(-ik)^3} + O(k^{-m-1})$$

Integrals (5) and (6) are evaluated in a similar manner hence (6) only is analyzed in detail. For (5) expand f in a Taylor series in ρ to get

$$f^2 = \tfrac{1}{4}(f^o_{\rho\rho})^2\,\rho^4 + O(\rho^5)$$

Then (5) becomes

$$\frac{-(n-1)}{2}\int_{S(0)}\int_0^\infty e^{ik\rho}\, h^o\, g(\rho)\left(\frac{f^2}{\rho^2}\right) d\rho\, dS^{n-1}$$

$$= \frac{-(n-1)h^o}{8}\int_{S(0)} (f^o_{\rho\rho})^2\, dS^{n-1}\left\{\frac{2}{(-ik)^3} + O(k^{-4})\right\}$$

$$= \frac{-(n-1)h^o}{4(-ik)^3}\Big\{\sum_j (f^o_{jj})^2\int_{S(0)} (\xi^j)^4\, dS^{n-1}$$

$$+ 2\sum_{j<q} f^o_{jj}\, f^o_{qq}\int_{S(0)} (\xi^j)^2(\xi^q)^2\, dS^{n-1} + O(k^{-4})$$

By symmetry the first integral is independent of j and in a similar manner to (4) is evaluated to be $\frac{3Bn}{n+2}$. The second integral is also independent of j and q where $j \neq q$ and this is analyzed as follows:

$$\int_{S(0)} (\xi^n)^2 (\xi^{n-1})^2 \, dS^{n-1}$$

$$= \int_{S(0)} \{1 - \sum_{j \neq n} (\xi^j)^2\} (\xi^{n-1}) \quad dS^{n-1}$$

$$= \int_{S(0)} (\xi^{n-1})^2 \, dS^{n-1} - \int_{S(0)} (\xi^{n-1})^4 \, dS^{n-1}$$

$$- \sum_{j \neq n, n-1} \int_{S(0)} (\xi^j)^2 (\xi^{n-1})^2 \, dS^{n-1}$$

$$= \int_{S(0)} (\xi^{n-1})^2 \, dS^{n-1} - \int_{S(0)} (\xi^{n-1})^4 \, dS^{n-1}$$

$$- (n-2) \int_{S(0)} (\xi^n) \quad (\xi^{n-1})^2 \quad dS^{n-1}$$

So

$$(n-1) \int_{S(0)} (\xi^n)^2 (\xi^{n-1})^2 \, dS^{n-1}$$

$$= \int_{S(0)} (\xi^{n-1})^2 \, dS^{n-1} - \int_{S(0)} (\xi^{n-1})^4 \, dS^{n-1}$$

or

$$\int_{S(0)} (\xi^n)^2 (\xi^{n-1})^2 \, dS^{n-1} = \frac{1}{n-1} \{B_n - \frac{3B_n}{n+2}\} = \frac{B_n}{n+2}$$

Hence (5) gives

$$\frac{-(n-1)h^{o}B_n}{4(-ik)^3}\left\{\frac{3}{n+2}\sum_j (f^{o}_{jj})^2 + \frac{2}{n+2}\sum_{j<q} f^{o}_{jj}\, f^{o}_{qq}\right\}$$

$$+ \quad O(k^{-4})$$

A similar analysis for (6) yields

$$\frac{ik}{2}\int_{S(0)}\int_{o}^{\infty} e^{ik\rho}\, h^{o}\, g(\rho)\left(\frac{f^2}{\rho}\right)d\rho\, dS^{n-1}$$

$$= \frac{-3h^{o}\,B_n}{4(-ik)^3}\left\{\frac{3}{n+2}\sum_j (f^{o}_{jj})^2 + \frac{2}{n+2}\sum_{j<q} f^{o}_{jj}\, f^{o}_{qq}\right\} + O(k^{-4})$$

Combining (2), (3), (4), (5), and (6) and using the definitions of H and K_2 proves the theorem.

It is possible to derive several recursion formulas for B_n which is defined in theorem 3.2. The first two and last recursion formulas can be obtained by evaluating a particular surface integral in two independent ways. The third can easily be deduced from the first two. These recursion formulas are:

(1) $$\frac{B_n}{n+2} = \frac{B_{n-1}\quad B_{n+2}}{C_n}$$

(2) $$\frac{B_n}{(n+4)\ (n+2)} = \frac{B_{n-2}\quad B_{n+1}\quad B_{n+4}}{C_{n-1}\quad C_{n+2}}$$

(3) $$\frac{B_n}{C_{n+1}} = \frac{B_{n+1}}{C_{n+2}}$$

(4) $$B_n = \frac{C_{n-1}}{n}$$

For example (1) can be obtained by evaluating $\int_{S(0)} (\xi^n)^2 (\xi^{n-1})^2 \, dS^{n-1}$ first in the manner performed in theorem 3.2 and second by letting $\xi^n = \cos\alpha$ and $\xi^{n-1} = \sin\alpha \cos\beta$ and evaluating $\int_{S(0)} \cos^2\alpha \sin^2\alpha \cos^2\beta \, dS^{n-1}$ where $dS^{n-1} = \sin^{n-2}\alpha \sin^{n-3}\beta d\alpha d\beta dS^{n-3}$

In a similar manner to (1), (2) and (4) can be derived by evaluating respectively the following integrals in the two ways mentioned above.

$$\int_{S(0)} (\xi^n)^2 (\xi^{n-1})^2 (\xi^{n-1})^2 \, dS^{n-1}$$

$$\int_{S(0)} (\xi^n)^2 \, dS^{n-1}$$

§ 4 ASYMPTOTIC PROPERTIES OF RELATED OPERATORS

We wish to derive the asymptotic development of certain integral operators related to the solution of the classical boundary value problem for the reduced wave equation.

Let Σ, except in Lemma 4.1, be a closed, compact, 2-manifold of class C^4 imbedded in E^3.

For y on Σ and $x \neq y$ in E^3, define the following kernels for the integral operators we are interested in:

$$Q(x,y) = \frac{e^{ikr}}{4\pi r} \quad \text{where } r = |x-y|;$$

and $K(x,y) = \frac{\partial}{\partial N_y} Q(x,y)$ where N_y is the unit outward normal vector to Σ at y.

Then these kernels define compact integral operators, as follows, where we will let the same letter denote both the kernel and the operator

$$Qg(x) = \int_\Sigma \frac{e^{ikr}}{4\pi r}\, g(y)\, dy, \quad \text{and}$$

$$Kh(x) = \int_\Sigma \frac{\partial}{\partial N_y} \left(\frac{e^{ikr}}{4\pi r}\right) h(y)dy, \text{ where}$$

dy is the volume element on Σ and g(y) and h(y) are suitably defined functions. For further properties of these and related operators consult Sacksteder's paper (13)

The method of stationary phase comprised of theorems 3.1 and 3.2 enables us to evaluate these integral operators asymptotically for large k. Applications of these theorems is immediate in the case of the operator Q. The extension in E^{n+1} of theorem 3.2 to the integral operator K is contained in the next Lemma.

First a brief comment on "fundamental solutions" of the reduced wave equation $L(u) = \Delta u + k^2 u = 0$. With $k^2 \neq 0$, in complex notation, we find that for odd $n > 1$ the singular ("fundamental") solutions are

$$u = \frac{U}{r^{n-2}} + f(r)$$

and for even n

$$u = \frac{U}{r^{n-2}} + W \log r + f(r)$$

where f(r) is real analytic and where U and W are real analytic solutions of $L(U) = L(W) = 0$.

In Lemma 4.1 we analyze the contribution due to the singularity at x from the "principal part" of the "fundamental" solution (that is, the highest power of $\frac{1}{r}$). The remaining terms are easy to analyze.

LEMMA 4.1

Let Σ be a closed, compact, orientable, n-manifold of class C^4 imbedded in E^{n+1}. Let $h(y)\varepsilon\ C^2(\Sigma)$ and belong to the domain of the operator K. Suppose also that in a sufficiently small neighborhood N(x) of x the surface is explicitly represented by $y_{n+1} = f(y_1,\ldots,y_n)$ where, without loss of generality, $f^o_{jq} = 0$ if $j \neq q$. Then as $k \to +\infty$

$$\int_{N(x)} \frac{\partial}{\partial N_y}\left(\frac{e^{ikr}}{4\pi r^{n-1}}\right) h(y)\, dy = \frac{n^2 B_n h(x) H}{8\pi(-ik)}$$

$$- \frac{B_n h(x)}{32\pi(-ik)^3}\{15\, n^3 H^3 - 36n\, H\binom{n}{2}K_2 + 24\binom{n}{3} K_3\}$$

$$+ \frac{B_n}{4\pi(-ik)^3}\{ H\Delta h(x) + (\vec{\Delta} f \cdot \vec{\Delta} h)(x)\} + \frac{nB_n}{2\pi(-ik)^3}\{(\nabla h \cdot \nabla H)(x)\}$$

$$+ \frac{3nB_n h(x)\Delta H(x)}{16\pi(-ik)^3} + O(k^{-4})$$

where B_n is the same function of n defined in theorem 3.2,

$H = \frac{1}{n}\sum_j f^o_{jj}$ (mean curvature), $\binom{n}{j}K_j = \sum_{q_1<q_2<\ldots<q_j} (f_{q_1q_1})\ldots(f_{q_jq_j})$ (Gaussian curvature), $\vec{\Delta} = \left(\frac{\partial^2}{\partial y^2},\ldots,\frac{\partial^2}{\partial y_n^2}\right)$, and $\Delta = \sum_j \frac{\partial^2}{\partial y_j^2}$.

<u>Proof</u>: The set-up of the coordinate system and opening remarks in this proof are the same as in theorem 3.2. The same notation also holds.

Hence the integral can be written as

$$\frac{1}{4\pi} \int_{N(x)} \frac{e^{ikr}}{r^{n-2}} \quad \frac{\cos(rN)}{r}\, h(y) \,\{\frac{n-1}{r} - ik\}\, dy$$

and noting that

$$\frac{\cos(rN)}{r} = (\tfrac{1}{2}\ (\xi^j)^2\ f^o_{jj} + \tfrac{1}{3}\ (\xi^j \xi^q \xi^p)\ f^o_{jqp}\ \rho$$

$$+ \quad \tfrac{1}{8}\ (\xi^j \xi^q \xi^p \xi^m)\ f^o_{jqpm}\ \rho^2)\frac{\{1 - \frac{f^2}{\rho^2} + O(\rho^4)\}}{(1 + \Sigma\, f_j^{\,2})^{\frac{1}{2}}} + O(\rho^3)$$

we obtain after the other functions of ρ are expanded

$$\frac{1}{4\pi} \int_{S(O)} \int_o^\infty \ e^{ik\rho}\{1 + ik\ (\frac{f^2}{2\rho}) + k^2\cdot O(\rho^6)\}$$

$$\{1 + k\cdot O(\rho^5)\ \}\{\tfrac{1}{2}(\xi^j)^2\ f^o_{jj} + \tfrac{1}{3}\ (\xi^j\ \xi^q\ \xi^r)\ f^o_{jqr}\ \rho$$

$$+ \quad \tfrac{1}{8}\ (\xi^j\ \xi^q\ \xi^r\ \xi^m)\ f^o_{jqrm}\ \rho^2\}$$

$$\{1 - \frac{f^2}{\rho^2} + O(\rho^4)\}\{\ 1 + O(\rho^3)\}\{\ h^o + h^o_\rho\, \rho + \tfrac{1}{2}\, h^o_{\rho\rho}\ \rho^2$$

$$+ \quad O(\rho^3)\}\ \{1 - \frac{n-2}{2}\ \frac{f^2}{\rho^2} + O(\rho^4)\}\ \{\frac{n-1}{\rho} - \frac{n-1}{2}\ \frac{f^2}{\rho^3} + O(\rho^3) - ik\}\cdot$$

$$\cdot g(\rho)\ \rho d\rho dS^{n-1} + O(k^{-4})$$

where $g(\rho)$ satisfies the hypothesis of Lemma 2.6.

. The non-vanishing integrals which contribute are listed below. The others are $O(k^{-4})$ by Lemma 2.6.

(1) $\frac{h^o(n-1)}{8\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; g(\rho) \; d\rho dS^{n-1}$

(2) $\frac{(-ik)h^o}{8\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; \dot{g}(\rho) \; \rho d\rho dS^{n-1}$

(3) $\frac{-(n-1)h^o}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; (\frac{f^2}{\rho^2}) \; g(\rho) \; d\rho dS^{n-1}$

(4) $\frac{(ik)(n-1)h^o}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\frac{f^2}{\rho}) \; (\xi^j)^2 \; f^o_{jj} \; g(\rho) \; d\rho dS^{n-1}$

(5) $\frac{-(ik)^2 \; h^o}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (f^2) \; (\xi^j)^2 \; f^o_{jj} \; g(\rho) \; d\rho dS^{n-1}$

(6) $\frac{-(n-1)h^o}{8\pi} \int_{S(0)} \int_o^\infty e^{ik} (\xi^j)^2 \; f^o_{jj} \; (\frac{f^2}{\rho^2}) \; g(\rho) \; d\rho dS^{n-1}$

(7) $\frac{(ik)h^o}{8\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; (\frac{f^2}{\rho^2}) \; g(\rho) \; d\rho dS^{n-1}$

(8) $\frac{-(n-1)(n-2)h^o}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; (\frac{f^2}{\rho^2}) \; g(\rho) \; d\rho dS^{n-1}$

(9) $\frac{(ik)(n-2)h^o}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; (\frac{f^2}{\rho}) g(\rho) \; d\rho dS^{n-1}$

(10) $\frac{(n-1)}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; h^o_{\rho\rho} \; g(\rho) \rho^2 \; d\rho dS^{n-1}$

(11) $\frac{(-ik)}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 \; f^o_{jj} \; h^o_{\rho\rho} \; g(\rho) \rho^3 \; d\rho dS^{n-1}$

(12) $\frac{(n-1)}{12\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j \; \xi^q \; \xi^m) \; f^o_{jqm} \; h^o_\rho \; g(\rho)\rho^2 \; d\rho dS^{n-1}$

(13) $\frac{-ik}{12\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j \; \xi^q \; \xi^m) \; f^o_{jqm} \; h^o_\rho \; \rho^3 \; d\rho dS^{n-1}$

(14) $\frac{(n-1)h^{o}}{32\pi} \int_{S(0)} \int_{o}^{\infty} e^{ik} (\xi^{j}\ \xi^{q}\ \xi^{m}\ \xi^{n})\ f^{o}_{jqmn}\ g(\rho)\rho^{2}\, d\rho\, dS^{n-1}$

(15) $\frac{(-ik)h^{o}}{32\pi} \int_{S(0)} \int_{o}^{\infty} e^{ik\rho} (\xi^{j}\ \xi^{q}\ \xi^{m}\ \xi^{r})\ f^{o}_{jqmr}\ g(\rho)\rho^{3}\, d\rho\, dS^{n-1}$

Several representative integrals from (1) - (15) will be analyzed in detail.

Integral (3) gives the following after expanding f in a Taylor series in ρ to get $f^{2} = \frac{1}{4}(f^{o}_{\rho\rho})^{2}\ \rho^{4} + O(\rho^{5})$ and noting that $(f^{o}_{\rho\rho})^{2} = (f^{o}_{jj})^{2}\ (\xi^{j})^{4} + 2\ f^{o}_{jj}\ f^{o}_{qq}\ (\xi^{j}\ \xi^{q})^{2}$, $j\neq q$, where $f = f\ (\rho\xi^{1},\ldots,\ \rho\xi^{n})$. (Note that integrals involving odd powers of ξ vanish by symmetry and $j\neq q\neq m$).

$$\frac{-(n-1)h^{o}}{16\pi} \int_{S(0)} \int_{o}^{\infty} e^{ik\rho} (\xi^{j})^{2}\ f^{o}_{jj}\ \left(\frac{f^{2}}{\rho^{2}}\right)\ g(\rho)\ d\rho\, dS^{n-1}$$

$$= \frac{-(n-1)h^{o}}{64\pi} \{ (f^{o}_{jj})^{3} \int_{S(0)} (\xi^{j})^{6}\ dS^{n-1}$$

$$+ \ 3(f^{o}_{jj})^{2}\ f^{o}_{qq} \int_{S(0)} (\xi^{j})^{4}\ (\xi^{q})^{2}\ dS^{n-1}$$

$$+ \ 2\ f^{o}_{jj}\ f^{o}_{qq}\ f^{o}_{mm} \int_{S(0)} (\xi^{j})^{2}\ (\xi^{q})^{2}\ (\xi^{m})^{2}\ dS^{n-1} \}$$

$$\int_{o}^{\infty} e^{ik\rho}\ \rho^{2}\ g(\rho)\, d\rho$$

$$= \frac{-(n-1)h^{o}}{64\pi} \{ \frac{15\ B_{n}}{(n+4)(n+2)} \sum_{j} (f^{o}_{jj})^{3} + \frac{9\ B_{n}}{(n+4)\ (n+2)} \sum_{j\neq q} (f^{o}_{jj})^{2}\ f^{o}_{qq}$$

$$+ \quad \frac{6 \; B_n}{(n+4)\;(n+2)} \sum_{j<q<m} f^o_{jj} \; f^o_{qq} \; f^o_{mm} \} \{ \frac{2}{(-ik)^3} + O(k^{-4})$$

$$= \quad \frac{-(n-1)h^o B_n}{32\pi\;(n+4)\;(n+2)} \{15 \sum_j (f^o_{jj})^2 + 9 \sum_{j \neq q} (f^o_{jj})^2 \; f^o_{qq} \}$$

$$+ \quad 6 \sum_{j<q<m} f^o_{jj} \; f^o_{qq} \; f^o_{rr} \} \; + O(k^{-4})$$

$$= \quad \frac{-(n-1)h^o B_n}{32\pi(n+4)\;(n+2)(-ik)^3} \{15 \; n^3 H^3 - 36_n \; H\binom{n}{2} K_2 + 24\binom{n}{3} K_3 \}$$

$$+ \; O(k^{-4}).$$

where B_n, H, and K_j (j = 2, 3)) are defined in the hypothesis.

The surface integrals above were evaluated as follows. By symmetry the first surface integral is independent of j. Hence letting $\xi^n = \cos \alpha$, the first integral is $\int_{S(0)} \cos^6\alpha \; \sin^{n-2}\alpha \; d\alpha \; dS^{n-2}$ which is easily evaluated to be $\frac{15 \; B_n}{(n+4)\;(n+2)}$.

By symmetry considerations and the fact that for fixed m, $(\xi^m)^2 = 1 - \sum_{j \neq m} (\xi^j)^2$, the second surface integral can be written as

$$\int_{S(0)} (\xi^n)^4 \; (\xi^{n-1})^2 \quad dS^{n-1}$$

$$= \int_{S(0)} (\xi^n)^4 \{1 - \sum_{j \neq n-1} (\xi^j)^2 \} \; dS^{n-1}$$

$$= \int_{S(0)} (\xi^n)^4 \, dS^{n-1} - \int_{S(0)} (\xi^n)^6 \, dS^{n-1}$$

$$- \sum_{j \neq n, n-1} \int_{S(0)} (\xi^n)^4 (\xi^j)^2 \, dS^{n-1}$$

$$= \frac{3 B_n}{n+2} - \frac{15 B_n}{(n+4)(n+2)} - (n-2) \int_{S(0)} (\xi^n)^4 (\xi^{n-1})^2 \, dS^{n-1}$$

This implies

$$(n-1) \int_{S(0)} (\xi^n)^4 (\xi^{n-1})^2 \, dS^{n-1} = \frac{3 B_n}{n+2} - \frac{15 B_n}{(n+4)(n+2)}$$

which gives our result

$$\int_{S(0)} (\xi^n)^4 (\xi^{n-1})^2 \, dS^{n-1} = \frac{3 B_n}{(n+4)(n+2)} .$$

In a similar manner, it is shown that

$$\int_{S(0)} (\xi^j)^2 (\xi^q)^2 (\xi^m)^2 \, dS^{m-1} = \frac{B_n}{(n+4)(n+2)} \text{ for } j \neq q \neq m.$$

For integral (10) we have, since $h = h(\rho\xi^1, \ldots, \rho\xi^n)$ implies

$$h^o_{\rho\rho} = h^o_{jq} \, \xi^j \, \xi^q ,$$

$$\frac{(n-1)}{16\pi} \int_{S(0)} \int_o^\infty e^{ik\rho} (\xi^j)^2 f^o_{jj} \, h^o_{\rho\rho} \, g(\rho) \, \rho^2 \, d\rho \, dS^{n-1}$$

$$= \frac{(n-1)}{16} \{ f^o_{jj} \, h^o_{jj} \int_{S(0)} (\xi^j)^4 \, dS^{n-1}$$

$$+ f^o_{jj} \, h^o_{qq} \int_{S(0)} (\xi^j)^2 (\xi^q)^2 \, dS^{n-1} \} \int_o^\infty e^{ik\rho} \, g(\rho) \, \rho^2 \, d\rho$$

$$= \frac{(n-1)}{16\pi} \cdot \{\frac{3\ B_n}{n+2} \sum_j f^o_{jj}\ h^o_{jj} + \frac{B_n}{n+2} \sum_{j\neq q} f^o_{jj}\ h^o_{qq}\{\frac{2}{(-ik)^3} + O(k^{-4})\}$$

$$= \frac{(n-1)\ B_n}{8\pi(n+2)\ (-ik)^3}\{3 \sum_j f^o_{jj}\ h^o_{jj} + \sum_{j\neq q} f^o_{jj}\ h^o_{qq}\} + O(k^{-4})$$

$$= \frac{(n-1)\ B_n}{4\pi(n+2)\ (-ik)^3}\ \{H\Delta h(x) + (\vec{\Delta}f \cdot \vec{\Delta}h)(x)\} + O(k^{-4})$$

where $\vec{\Delta} = (\frac{\partial^2}{\partial y^2}, \ldots, \frac{\partial^2}{\partial y_n{}^2})$ and the surface integrals are evaluated as in (3).

For integral (12), note that $h^o_\rho = h^o_j \xi^j$

so that $\frac{n-1}{12\pi} \int_{S(0)} \int_o^\infty e^{ik\rho}\ (\xi^j\ \xi^q\ \xi^m)\ f^o_{jqm}\ h^o_\rho\ g(\rho)\,\rho^2\ d\rho\ dS^{n-1}$

$$= \frac{n-1}{12\pi}\{ f^o_{jjj}\ h^o_j \int_{S(0)} (\xi^j)^4\ dS^{n-1}$$

$$+ 3\ f^o_{jjq}\ h^o_q \int_{S(0)} (\xi^j)^2\ (\xi^j)^2\ dS^{n-1}\} \int_o^\infty e^{ik\rho}\rho g(\rho)\ d\rho$$

$$= \frac{n-1}{12\pi}\{\frac{3\ B_n}{n+2} \sum_j f^o_{jjj}\ h^o_j + \frac{3\ B_n}{n+2} \sum_{j\neq q} f^o_{jjq}\ h^o_q\}\{\frac{2}{(-ik)^3} + O(k^{-4})\}$$

$$= \frac{(n-1)\ B_n}{2\pi(n+2)\ (-ik)^3}\ \{\nabla h \cdot \nabla(\Delta f)\}\ (x) + O(k^{-4})$$

Finally, for integral (14) we have

$$\frac{n-1}{32\pi} \int_{S(0)} \int_o^\infty e^{ik\rho}\ (\xi^j\ \xi^q\ \xi^m\ \xi^r)\ f^o_{jqmr}\ g(\rho)\ \rho^2\ d\rho\ dS^{n-1}$$

$$= \frac{(n-1)h^o}{32\pi}\{ f^o_{jjjj} \int_{S(0)} (\xi^j)\ dS^{n-1}$$

$$+ \quad 6 \, f^{o}_{jjqq} \int_{S(0)} (\xi^{j})^{2} \, (\xi^{q})^{2} \, dS^{n-1} \} \quad \int_{o}^{\infty} e^{ik\rho} \rho^{2} g(\rho) \, d\rho$$

$$= \quad \frac{(n-1)h^{o}}{32\pi} \left\{ \frac{3 \, B_n}{n+2} \sum_{j} f^{o}_{jjjj} + \frac{6 \, B_n}{n+2} \sum_{j<q} f^{o}_{jjqq} \right\} \left\{ \frac{2}{(-ik)^{3}} + O(k^{-4}) \right\}$$

$$= \quad \frac{3n(n-1)h^{o} \, B_n \, \Delta H}{16\pi(n+2) \, (-ik)^{3}} + O(k^{-4}).$$

Results for all the integrals are noted below (modulo $O(k^{-4})$).

(1) $$\frac{n(n-1) \, h(x) \, B_n \, H}{8\pi(-ik)}$$

(2) $$\frac{n \, h(x) \, B_n \, H}{8\pi(-ik)}$$

(3) $$\frac{-(n-1) \, h(x) \, B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 36nH\tbinom{n}{2}K_2 + 24\tbinom{n}{3}K_3 \}$$

(4) $$\frac{-3(n-1) \, h(x) \, B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 36nH\tbinom{n}{2}K_2 + 24\tbinom{n}{3}K_3 \}$$

(5) $$\frac{-12 \, h(x) \, B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 36nH\tbinom{n}{2}K_2 + 24\tbinom{n}{2}K_3 \}$$

(6) $$\frac{-2(n-1)h(x)B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 36nH\tbinom{n}{2}K_2 + 24\tbinom{n}{3}K_3 \}$$

(7) $$\frac{-6h(x) \, B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 36nH\tbinom{n}{2}K_2 + 24\tbinom{n}{3}K_3 \}$$

(8) $$\frac{-(n-2)(n-1)h(x)B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 35nH\tbinom{n}{2}K_2 + 24\tbinom{n}{3}K_3 \}$$

(9) $$\frac{-3(n-2)h(x)B_n}{32\pi(n+4)(n+2)(-ik)^{3}} \{ 15 \, n^{3}H^{3} - 36nH\tbinom{n}{2}K_2 + 24\tbinom{n}{3}K_3 \}$$

(10) $\frac{(n-1)\ B_n}{4\pi(n+2)(-ik)^3}\{\ H\Delta h(x) + (\vec{\Delta} f\cdot\vec{\Delta} h)(x)\}$

(11) $\frac{3B_n}{4\ (n+2)(-ik)^3}\{\ H\Delta h(x) + (\vec{\Delta} f\cdot\vec{\Delta} h)(x)\}$

(12) $\frac{(n-1)B_n}{4\pi(n+2)(-ik)^3}\ (\{\nabla h\cdot\ \nabla(\Delta f)\}(x))$

(13) $\frac{3B_n}{2\pi(n+2)(-ik)^3}\ (\{\nabla h\cdot\nabla(\Delta f)\}(x))$

(14) $\frac{3n(n-1)h(x)B_n\Delta H}{16\pi(n+2)(-ik)^3}$

(15) $\frac{9n\ h(x)\ B_n\ \Delta H}{16\pi(n+2)(-ik)^3}$

Combining (1) through (15) proves the Lemma.

ASYMPTOTIC DEVELOPMENT OF THE OPERATORS K AND Q

We are now in a position using theorems 3.1, 3.2, and Lemma 4.1 to obtain the asymptotic development of the operators. This is obtained first for the case of a general two-dimensional surface and secondly, we specialize to the case of the two-dimensional sphere.

If we set $\phi^x(y) = |x-y| = r$ and $\Psi^x(y) = \frac{1}{2}\{\phi^x(y)\}^2$ then the Hessians satisfy $H\phi = \frac{1}{\Psi} H\Psi$ at critical points. In our case Σ is a two-dimensional surface so that we have,

$$\frac{1}{(|\det H\phi|)^{\frac{1}{2}}} = \frac{|x-y|}{(|\det H\Psi|)^{\frac{1}{2}}} = \frac{|x-y|}{|(1-\mu_1 r)(1-\mu_2 r)|^{\frac{1}{2}}}$$

where μ_1 and μ_2 are the principal curvatures and $r = \phi$.

For each $x\varepsilon\,\Sigma$ and critical point y we set d equal to the distance between x and y. The signature of H is 2-2s where s is the index of the quadratic form $d^2\phi$ (or equivalently the number of focal points between x and y counted with multiplicity).

For the sake of brevity, the asymptotic expansions of Q and K for a general two-dimensional surface are given only up to order k^{-2}. There follows first a reference list of formulas for the values of the partial derivatives of $h(y) = a(y)\cdot(1+\Sigma f_j^{\,2})^{\frac{1}{2}}$ and $\phi^x(y) = |x-y| = (y_1^{\,2} + y_2^{\,2} + (d + f)^2)^{\frac{1}{2}}$ on a 2-manifold M and their corresponding values on the unit 2-sphere S_2. We are assuming $f^o = f^o_j = f^o_{jq} = 0$ if $j \neq q$ and also $f^o_{jpq} = f^o_{jpqrs} = 0$ for all j,p,q,r,s where evaluation is at a critical point (denoted by -x on S_2).

($j \neq s \neq q \neq r \neq p$ unless otherwise indicated).

<u>M</u>	<u>S_2</u>
$h^o = a^o$	$a(x)$
$h^o_j = a^o_j$	$a_j(-x)$
$h^o_{jj} = a^o_{jj} + a^o(f^o_{jj})^2$	$a_{jj}(-x) + a(-x)$
$h^o_{js} = a^o_{js}$	$a_{js}(-x)$
$h^o_{jjj} = a^o_{jjj} + 3\ a^o_j(f^o_{jj})^2$	$a_{jjj}(-x) + 3\ a_j(-x)$
$h^o_{jjs} = a^o_{jjs} + a^o_s(f^o_{jj})^2$	$a_{jjs}(-x) + a_s(-x)$
$h^o_{jsq} = a^o_{jsq}$	$a_{jsq}(-x)$
$h^o_{jjjj} = a^o_{jjjj} + 6\ a^o_{jj}(f^o_{jj})^2$ $+ 4\ a^o\ f^o_{jj}\ f^o_{jjjj} - 3\ a^o(f^o_{jj})^4$	$a_{jjjj}(-x) + 6\ a_{jj}(-x)$ $+ 9\ a(-x)$
$h^o_{jjss} = a^o_{jjss} + a^o_{jj}(f^o_{ss})^2 + a^o_{ss}(f^o_{jj})^2$ $+ 2\ a^o\ f^o_{jj}\ f^o_{jjss} + 2\ a^o_{ss}\ f^o_{jjss}$ $- a(f^o_{jj})^2\ (f^o_{ss})^2$	$a_{jjss}(-x) + a_{jj}(-x)$ $+ a_{ss}(-x) + 3\ a(x)$

M	S_2
$\phi^o_{jj} = \frac{1 + d\, f^o_{jj}}{d}$	$-\frac{1}{2}$
$\phi^o_{js} = f^o_{js}$	0
$\phi^o_{jsq} = f^o_{jsq}$ (for all j,s,q)	0
$\phi^o_{jjjj} = \frac{3}{d}(f^o_{jj})^2 + f^o_{jjjj} - \frac{3}{d^3}(1+df^o_{jj})^2$	$-\frac{15}{8}$
$\phi^o_{jjjs} = f^o_{jjjs}$	0
$\phi^o_{jjss} = \frac{1}{d} f^o_{jj} f^o_{ss} + f^o_{jjss} - \frac{1}{d^3}(1+d\, f^o_{jj})(1+df^o_{ss})$	$-\frac{5}{8}$
$\phi^o_{jjsq} = f^o_{jjsq}$	0
$\phi^o_{jsqr} = f^o_{jsqr}$	0
$\phi^o_{jsqrp} = f^o_{jsqrp}$ (for all j,l,q,r,p)	0
$\phi^o_{jjjjss} = \frac{1}{d}\{f^o_{jjjj} f^o_{ss} + 6 f^o_{jj} f^o_{jjss} + d\, f^o_{jjjjss}\} - \frac{1}{d^3}\{1 + df^o_{ss}\}\{3(f^o_{jj})^2 + df^o_{jjjj}\} + \frac{9}{d^5}\{1 + df^o_{jj}\}^2 \{1+df^o_{ss}\}^2 - \frac{6}{d^3}\{1 + df^o_{jj}\}\{f^o_{jj} f^o_{ss} + df^o_{jjss}\}$	$-\frac{189}{32}$

M S

$$ {}^{o}_{jjjjjj} = \frac{1}{d} \{15\ f^{o}_{jj}\ f^{o}_{jjjj} + d\ f^{o}_{jjjjjj}\} - \frac{945}{32} $$

$$ - \frac{15}{d^3} \{3(f^{o}_{jj})^2 + df^{o}_{jjjj}\} \{1 + df^{o}_{jj}\} $$

$$ + \frac{45}{d^5} \{1 + df^{o}_{jj}\}^3 $$

VALUES OF PARTIAL DERIVATIVES OF f ON S_2

$$ f^{o} = 0 $$

$$ f^{o}_{j} = 0 $$

$$ f^{o}_{js} = -\delta_{js} $$

$$ f^{o}_{jsq} = 0 \text{ (for all } j,s,q) $$

$$ f^{o}_{jjjj} = -3 $$

$$ f^{o}_{jjss} = -1 $$

$$ f^{o}_{jsqp} = 0 \quad (\text{for } j \neq s \neq q) $$

$$ f^{o}_{jsqpr} = 0 \text{ (for all } j,s,q,p,r) $$

$$ f^{o}_{jjjjjj} = -45 $$

$$ f^{o}_{jjjjss} = -9 $$

GENERAL TWO-DIMENSIONAL SURFACE

ASYMPTOTIC DEVELOPMENT OF Q

Asymptotic development due to contribution from non-singular points.

$$Qg(x) = \int_{\Sigma} e^{ikr} \frac{g(x)}{4\pi r}\, dy = \sum_{p|dr(p)=0} \frac{-(-i)^{s+1} e^{ikd}}{2|(1-\mu_1 d)(1-\mu_2 d)|^{\frac{1}{2}} k}$$

$$\{g(p) + (\frac{i}{2k})\{\sum_j \frac{-d^2 f_{jjj}(p)\; g_j(p)}{(1 + df_{jj}(p))^2}$$

$$+ \sum_{j\neq q} \frac{-d^2 f_{qqj}(p)\; g_j(p)}{(1+df_{jj}(p)\; (1+df_{qq}(p))}$$

$$+ \sum_j \frac{d}{1+df_{jj}(p)} \{g_{jj}(p) - \frac{g(p)}{d^2}\{1+df_{jj}(p)\} + g(p) f^2_{jj}(p)\}$$

$$+ \sum_{j\neq q} \frac{-g(p)\{d^3 f_{jjqq}(p) - d(f_{jj}(p)) - 1\}}{2d(1+df_{jj}(p))(1+df_{qq}(p))}$$

$$+ \sum_j \frac{-g(p)\{d^3 f_{jjjj}(p) - 6df_{jj}(p) - 3\}}{4d(1+df_{jj}(p))^2}$$

$$+ \sum_{j\neq q} \frac{d^3 g(p)\{9\; f^2_{jjq}(p) + \; 6f_{jjj}(p) f_{jqq}(p)\}}{12(1+\; df_{jj}(p))^2\; (1 + df_{qq}(p))}$$

$$+ \sum_j \frac{5d^3 g(p) f^2_{jjj}(p)}{12(1 + df_{jj}(p))^3}\} \quad + O(k^{-3}) \text{ as } k\to+\infty$$

The contribution due to the singularity at x is from theorem 3.2.

$$\int_{N(x)} e^{ikr} \frac{g(x)}{4\pi r}\, dy = \frac{ig(x)}{2k} - \frac{i\Delta g(x)}{4k^3} - \frac{(ig(x)}{4k^3} (H^2-K)$$

$$+ O(k^{-4}) \text{ as } k\to+\infty$$

ASYMPTOTIC DEVELOPMENT OF K

Asymptotic development due to contribution from non-singular points.

$$Kg(x) = \int_\Sigma \frac{\partial}{\partial N_y}\left(\frac{e^{ikr}}{4\pi r}\right) g(y)\,dy = \sum_{p\,|\,dr(p)=0} \frac{-(-i)^{s+1}\, e^{ikd}}{2\,|(1-\mu_1 d)(1-\mu_2 d)|^{\frac{1}{2}} k}$$

$$\Big\{-f(p)\left(\frac{1}{d} - ik\right) + \left(\frac{i}{2k}\right)\Big\{\sum_j \frac{-d^2 f_{jjj}(p) g_j(p)}{(1+df_{jj}(p))^2}\left(ik - \frac{1}{d}\right)$$

$$+ \sum_{j\neq q} \frac{-d^2 f_{qqj}(p) g_j(p)}{(1+df_{jj}(p))(1+df_{qq}(p))}\left(ik - \frac{1}{d}\right)$$

$$+ \sum_j \frac{d}{1+df_{jj}(p)}\left\{\left(\frac{1}{d} - ik\right)\left(-g_{jj}(p) + \frac{g(p) f_{jj}(p)}{d}\right.\right.$$

$$+ \frac{2g(p)(1+df_{jj}(p)}{d^2}) + \left(\frac{g(p)(1+df_{jj}(p)}{d^3}\right)$$

$$+ \sum_{j\neq q} \frac{-g(p)\left(ik - \frac{1}{d}\right)\{d^3 f_{jjqq}(p) - d(f_{jj}(p) + f_{qq}(p)) - 1\}}{2d(1+df_{jj}(p))(1+df_{qq}(p)}$$

$$+ \sum_j \frac{-g(p)\left(ik - \frac{1}{d}\right)\{d^3 f_{jjjj}(p) - 6\,df_{jj}(p) - 3\}}{4d(1+df_{jj}(p)^2}$$

$$+ \sum_{j\neq q} \frac{d^3 g(p)\left(ik - \frac{1}{d}\right)\{9\,f^2_{jjq}(p) + 6f_{jjj}(p) f_{jqq}(p)\}}{12(1+df_{jj}(p))^2\,(1+df_{qq}(p))}$$

$$+ \sum_j \frac{5d^3 f(p)\left(ik - \frac{1}{d}\right) f^2_{jjj}(p)}{12(1+df_{jj}(p))^3}\Big\} + O(k^{-3}) \text{ as } k\to+\infty$$

From Lemma 4.1 the contribution due to the singularity at x is

$$\int_{N(x)} \frac{\partial}{\partial N y} \frac{(e^{ikr})}{4\pi r} g(y)\,dy = \frac{Hg(x)}{2(-ik)} - \frac{3g(x)}{4(-ik)^3}(5H^3 - 3HK)$$

$$+ \frac{\{H\Delta g(x) + (\vec{\Delta} f \cdot \vec{\Delta} g)(x)\}}{4(-ik)^3} + \frac{\nabla g \cdot \nabla H}{(-ik)^3} + \frac{3g(x)\Delta H(x)}{8(-ik)^3} + O(k^{-4})$$

as $k \to +\infty$

THE UNIT TWO-DIMENSIONAL SPHERE

We let x denote the singularity and -x its antipodal point on the sphere of radius one. The asymptotic expansions of Q and K are then obtained to be:

$$Qg(x) = \frac{i}{2k} g(x) - \frac{i}{4k^3} \Delta g(x) - \frac{ie^{2ik}}{2k} g(-x)$$

$$- \frac{e^{2ik}}{2k^2} \Delta g(-x) - \frac{25ie^{2ik}}{16k^3} \Delta g(x) + \frac{ie^{2ik}}{8k^3} \vec{\Delta}^2 g(-x)$$

$$+ \frac{ie^{2ik}}{8k^3} \Delta^2 g(-x) - \frac{63ie^{2ik}}{64k^3} g(-x) + O(k^{-4})$$

$$Kg(x) = \frac{-i}{2k} g(x) + \frac{e^{2ik}}{2} g(-x) + \frac{ie^{2ik}}{2k} g(-x)$$

$$- \frac{ie^{2ik}}{2k} \Delta g(-x) + 9\frac{e^{2ik}}{4k^2} \Delta g(-x) - \frac{e^{2ik}}{8k^2} \vec{\Delta}^2 g(-x)$$

$$- \frac{e^{2ik}}{8k^2} \Delta^2 g(-x) + \frac{3e^{2ik}}{4k^2} g(-x) + O(k^{-3})$$

as $k \to +\infty$ where $\Delta^2 = \Delta(\Delta)$ and $\vec{\Delta}^2 = \vec{\Delta} \cdot \vec{\Delta}$

§ 5 ASYMPTOTIC RELATIONSHIPS ON S_2

For the Neumann problem in an exterior domain Ω_+ we can define the compact operator $Z_k(g) = ikh$ where $h = h_+$ is the solution corresponding to the given normal derivative $g = g_+$ in Σ (cf. R. Sacksteder's paper (13)). If u represents a point on the unit sphere and $x = ru$ ($r = |x|$) represents a point in 3-space then using the same notation as above we can write $Z_k g(u) = \lambda g(u) = ikh(u)$. This implies $\lambda = \frac{ikh(u)}{g(u)}$.

Its know (cf. 9) that because of the completeness in $L^2(S_2)$ of the spherical harmonics that every solution V(x) of the Helmholtz wave equation $\Delta V + V = 0$ which satisfies the Sommerfeld radiation condition has an expansion of the form

$$V(ru) \sim \left(\frac{\pi}{2r}\right) \sum_{n=0}^{\infty} \sum_{j=1}^{N} i^n \{a_{nj} H_\mu^{(1)}(r)\} K_{nj}(u).$$

where $H_\mu^{(1)}$ is a Hankel function of the first kind of order $\mu = n + 1/2$ and $K_{nj}(u)$, $j=1,\ldots,N$ are a orthonormalized basis of spherical harmonics (here $N = N(n) = 2n+1$).

Expanding h(u) and g(u) from above in such a series yields for the eigenvalues of Z_k that

$$\lambda_\mu = \frac{ikh(u)}{g(u)} = \frac{ik(kr)^{-1/2} H_\mu^{(1)}(kr)}{\{(kr)^{-1/2} H_\mu^{(1)}(kr)\}'}$$

where the multiplicity of the eigenvalue is $N = 2n+1$. We can investigate the behavior of λ_μ as $r \to \infty$ (or equivalently as $k \to \infty$) and also the behavior of λ_μ as $\mu \to \infty$ (that is $n \to \infty$). By using the function $h_n^{(1)}(r) = (\frac{\pi}{2r})^{1/2} H^{(1)}_{n+1/2}(r)$ and standard asymptotic and recurrence relations for $h_n^{(1)}(r)$ (cf. 1) it is possible to derive expressions for $\lambda_n(r)$ for large r (or k) or large n.

(1) CALCULATION OF λn FOR LARGE k

We have $\lambda_n = \dfrac{ikh_n^{(1)}(r)}{\frac{\partial}{\partial r} h^{(1)}(kr)}$, or using

an identity for $h_n^{(1)}(kr)$ (cf. 1) this becomes

$$\lambda_n = \frac{ih_n^{(1)}(kr)}{(\frac{n}{2n+1})h_{n-1}^{(1)}(kr) - (\frac{n+1}{2n+1})h_{n+1}^{(1)}(kr)}$$

Inverting and using the functions $h_\nu^{(1)}(kr) = \dfrac{krH_{\nu-1}^{(1)}(kr)}{H_\nu^{(1)}(kr)}$

and $\tilde{h}_\nu^{(1)}(kr) = \dfrac{krH_{\nu+1}^{(1)}(kr)}{H_\nu^{(1)}(kr)}$ (cf. 11) gives

$$\frac{1}{\lambda_n} = (\frac{-in}{2n+1})(\frac{1}{kr})\{h_n^{(1)}(kr) - \tilde{h}_n^{(1)}(kr)\} + (\frac{1i}{2n+1})(\frac{1}{kr})\tilde{h}_n^{(1)}(kr)$$

Since for large k, $\tilde{h}_\nu^{(1)}(kr) \sim ikr$ and $h_\nu^{(1)}(kr) \sim -ikr$ (cf.9) this yields $\lambda_n \to 1$ as $k \to \infty$ for all n.

(2) AN ALTERNATE CALCULATION OF λ_n FOR LARGE k

We have the following formula for large k (cf. 1).

$$h_n^{(1)}(kr) = i^{n-1}(kr)^{-1} \; e^{ikr} \sum_{j=0}^{n} (n+\tfrac{1}{2},j)(-2ikr)^{-j}$$

$$\text{where } (\nu,m) = \frac{\Gamma(\tfrac{1}{2}+\nu+m)}{m!\,\Gamma(\tfrac{1}{2}+\nu-m)}$$

$$= \frac{(4\nu^2-1^2)(4\nu^2-3^2)\ldots(4\nu^2-(2n-1)^2)}{2^{2m}.m!}$$

By inserting into the expression for $\lambda_n(kr)$ in 1 we find that (with r = 1)

$$\lambda_n(k) = 1 + \frac{1}{ik} + \frac{n^2+n-2}{2k^2} + O(\frac{1}{k^3})$$

as $k \to \infty$

(3) SPECIAL CASES OF λ_n (n=0,1,2) FOR LARGE k

It is possible to get closed explicit asymptotic expressions for λ_n as $k\to\infty$ for specific values of n since the formulas for $h_r^{(1)}(kr)$ are much more explicit themselves. We set r = 1 here. The results are:

a) $$\lambda_o(k) = \frac{1}{1-\frac{1}{ik}} = \sum_p \left(\frac{1}{ik}\right)^p$$

b) $$\lambda_1(k) = \frac{(1-\frac{1}{ik})}{(1-\frac{2}{ik}-\frac{2}{k^2})} = 1-\frac{i}{k} + 0(\frac{1}{k^3})$$

c) $$\lambda_2(k) = \frac{(1-\frac{3}{ik}-\frac{3}{k^2})}{(1-\frac{4}{ik}-\frac{9}{k^2}-\frac{9}{ik^3})} = 1-\frac{i}{k} + \frac{2}{k^2} + 0(\frac{1}{k^3})$$

We see that as $k\to\infty$ the real part of λ_n approaches 1 and the imaginary part approaches $-i$ (for n=0,1,2) which is additional confirmation of the first two results.

(4) ASYMPTOTIC EXPRESSION FOR λ_n AS $n\to\infty$

Here the relevant formula to be used is (cf. 1):

$$h_n^{(1)}(r) \sim \frac{1}{((2n+1)r)^{1/2}}\left\{\frac{1}{(2)^{1/2}}\left(\frac{er}{2n+1}\right)^{n+1/2} - i\sqrt{2}\left(\frac{er}{2n+1}\right)^{-n-1/2}\right\}$$

as $n\to+\infty$ (r fixed). After some manipulations the result for large n using the expression for λ_n in(1) is

$$\lambda_n(r) = r\left\{\frac{1}{n} + \frac{r^2}{4}\cdot\frac{1}{n^3} - \frac{r^2}{2}\bullet\frac{1}{n^4} + \frac{15r^2+r^4}{16}\cdot\frac{1}{n^5} + 0(n^{-6})\right\}$$

AN ASYMPTOTIC RELATION BETWEEN THE OPERATORS

These results above for the "eigenvalues" of Z_k can be used in relating our results for Q and K. For, we defined operators (cf. 13) which we can write (1) $\frac{1}{k}(K-\frac{1}{2}I)Z_k(g)=Q(g)$.

Then the asymptotic expansions for Q and K and the result for Z_k (g) can be used in the expression above. We assume that for large k, the eigenfunction g as well as λ is of the form below (at a critical point x)

$$g \sim g_o(x) + \frac{g_1(x)}{k} + \frac{g_2(x)}{k^3} + 0(\frac{1}{k^3})$$

$$\lambda^j \sim \sum_p \lambda_p^j \ k^{-p}$$

Z_k (g) can now be written

$$Z_k \ (g) = \lambda_o \ g_o(x) + \frac{\lambda_o}{k} \ g_1(x) + \frac{\lambda_1}{k} \ g_o(x) + 0(\frac{1}{k^2})$$

Using (1) and our results for a general two-dimensional surface for Q and K and Z_k above yields after equating the $\frac{1}{k}$ terms

$$\frac{\lambda_o \ Me^{ikd}}{k} \ g_o(-x) + \frac{i\lambda_o}{2k} \ g_o(x) = \frac{Me^{ikd}}{k} \ g_o(-x) + \frac{i}{2k} \ g_o(x)$$

Where $M = \dfrac{-(-i)^{s} + 1}{2 \ |(1 - \mu_1 d)(1 - \mu_2 d)|^{1/2}}$

and we assume x and -x denote points critical with respect to each other.

This implies that $\lambda_0 = 1$. (or the right hand side is zero).

Similarly, equating $\frac{1}{k^2}$ terms yields,

$$\frac{M\lambda_1 e^{ikd}}{k^2} g_o(-x) + \frac{iMe^{ikd}}{dk^2} g_o(-x)$$

$$- \frac{iMe^{ikd}}{2k^2} \{ \sum_j \frac{2}{d} g_o(-x) + \sum_j \frac{f^o_{jj}}{1 + df^o_{jj}} g_o(-x) \}$$

$$+ \frac{(i\lambda_1 - 1)}{2k^2} g_o(x)$$

$$= \frac{iMe^{ikd}}{2k^2} \{ \sum_j - \frac{1}{d} g_o(-x) + \sum_j \frac{df^{o\,2}_{jj}}{1 + df^o_{jj}} g_o(-x) \}$$

After simplification, this yields (assuming $\lambda_o = 1$ and $\cos(x \wedge -x) = -1$ where '$\wedge$' denotes angle between the critical points).

$$\lambda_1 = \frac{-i + 2MM'ie^{2ikd}(H+H')}{(4MM'e^{2ikd}+1)}$$

$$\pm \frac{2e^{ikd}\{MM'(H+H'+HH'+1 - MM'e^{2ikd}(H-H')^2)\}^{1/2}}{(4MM'e^{2ikd}+1)}$$

where H and H' (M and M') are mean curvatures (quantities) defined at x and $-x$ respectively. We also assumed here that x and $-x$ are points most distant from each other. On the sphere this last expression gives $\lambda_1 = -i$. These two results agree with the first two terms of λ^n (written λ_n above) for Z_k.

It is also possible to obtain independent confirmation of the results for Q and K on the sphere. That is, on the sphere $g(x) = 1$ so

$$Q1 = \int_{S(0)} \frac{e^{ikr}}{4\pi r} \, dy, \quad \text{and}$$

$$K1 = \int_{S(0)} \frac{e^{ikr}}{4\pi r} \cos(rN) \left\{\frac{1}{r} - ik\right\} dy$$

Let ϕ be the polar angle between the north pole (x) and an arbitrary point y on the sphere, hence $r = 2\cos\phi/2$ is the distance between the critical point -x (the south pole) and the point y.

Then

$$Q1 = \int_0^\pi \frac{e^{2ik\cos\phi/2}}{4\cos\phi/2} \sin\phi \, d\phi$$

$$= \frac{i}{2k} - \frac{ie^{2ik}}{2k}$$

and

$$K1 = \int_0^\pi \frac{e^{2ik\cos\phi/2}}{4\cos\phi/2} \left\{ik - \frac{1}{2\cos\phi/2}\right\} \cos\phi/2 \sin\phi \, d\phi$$

$$= \frac{e^{2ik}}{2} - \frac{i}{2k} + \frac{ie^{2ik}}{2k}$$

These results agree with the previous asymptotic expansions obtained for Q and K on S^2 up to order $\frac{1}{k^2}$ and $\frac{1}{k}$ respectively. Why this method doesn't yield higher order terms is a potential subject of further investigation.

§ 6. ALTERNATE APPROACHES TO THE STATIONARY PHASE FORMULA

Points of departure from this thesis include investigating the reason for the absence of higher order terms in the classical method of obtaining asympototic expansions for Q and K on S_2 versus their presence when the stationary phase formula is applied as well as analyzing the asymptotic expansions on other different surfaces such as spheroids (oblate and prolate), ellipsoids, and hyperboloids. Here some of the prime difficulties would arise from the fact that the normal derivative to Σ is no longer in a radial direction and the harmonics involved are more complex. Also, another interesting area of research would be to apply the stationary phase formula to other types of integral operators such as $Pg(x) = \frac{\partial}{\partial N_x} \int \frac{\partial}{\partial N_y} \left(\frac{e^{ikr}}{4\pi r} \right) g(y)dy$ (cf 13). Here the difficulty is that x is not fixed.

A totally different approach to this paper would be to obtain a generalization of the stationary phase formula by a more indirect route involving calculation of the Jacobian of the map appearing in Morse's lemma and its derivatives. That method should yield the same results obtained in this paper. This approach is outlined here (cf. 8)

The basic integral we are expanding in an asymptotic series is of the form $\int_\Sigma a(x)\, e^{ikf(x)} dx$ (cf. §3). By Morse's lemma, there is a change of coordinates $\phi(x)$ ($\phi:U\to V$, a diffeomorphism) valid in some neighborhood U of the origin so that f is quadratic in those coordinates i.e. $f o\phi= d^2 f$ where $df(0)=0$ and $d^2 f(0)=Q$.

This changes the integral above to a sum of products of integrals of the basic type (1) $\int_{N(p)} e^{ikQ(z)/2} h(z)dz$ in a neighbourhood of a critical point, N(p), where $h(z)=((ao\phi)\cdot |\det\frac{\partial\phi}{\partial z}|\cdot g\cdot\sqrt{G})$ and g is a conveniently defined "bump" function. Then by lemmas 2.1, 2.2, 2.3 (or directly by lemma 2.4) and lemma 2.7 it is possible to obtain, as in §3, the asymptotic expansion of (1) provided the Jacobian and its derivatives can be calculated. A sketch of this procedure is also shown (cf.8 for more details).

Define $f^t(x)=Q(x) + t(f(x) - Q(x))$ where $f^1 = f$ and $f^o = Q$. Hence a one parameter family of diffeomorphisms, ϕ^t, is sought such that $f^t o\phi^t=Q$ and clearly ϕ^1 works. We let ζ^t be a vector field tangent to ϕ^t i.e. (2) $\zeta^t(\phi^t(x)) = \frac{d\phi^t}{dt}(x)$.

After some manipulations it is possible to obtain (3) $B_x^t(u,\zeta^t) = C_x(u,x)$ where ζ^t is the unique solution $\forall$ $u\varepsilon V$ and where ζ^t and where both B_x^t and C_x are quadratic forms defined by

$$B_x^t(u,v) = \int_o^1 d^2f^t_{sx}(u,v)ds, \text{ and}$$

$$C_x(u,v) = \int_o^1\int_o^1 d^2w_{rsx}(su,u)\, drds$$

$$(w = f^o - f^1, \quad B^t = B^o + t(B^1 - B^o))$$

From (3) we have $\zeta_i^t = \Sigma_{j,k}\, b^{-1}_{ij}c_{jk}\, x_k$ where the notation is obvious.

Now define the matrices

$$E(\phi,t) = (\frac{\partial \zeta^j}{\partial \phi^m}) \quad \text{and,}$$

$$\Phi(x,t) = (\frac{\partial \phi^m}{\partial x^k}).$$

From (2) we get

$$\frac{\partial \Phi}{\partial t}(x,t) = E(\phi,t) \cdot \Phi(x,t)$$

By Liouville's formula this yields

$$J(x) = \det \Phi(x,1) = \exp\left(\int_0^1 \text{Trace } E(\phi(x,t),t)dt\right)$$

Taking derivatives of this expression and of (3) and solving the resultant differential equations the quantities sought can be found.

BIBLIOGRAPHY

(1) Abramowitz, M. and Stegun, I.A., eds., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series, 55, U.S. Department of Commerce, 1970. pp.435-479.

(2) Bruijn, N.G. de, Asymptotic Methods in Analysis, Dover Publications, Inc., New York, 1958.

(3) Colton, D. and Kress, R., Integral Equation Methods in Scattering theory, John Wiley & Sons, Inc., 1983.

(4) Corput, J.G. van der, "On the Method of Critical Points", Indag. Math., 10 (1948), p.201.

(5) Courant, R. and Hilbert D., Methods of Mathematical Physics, Vol. 2, Interscience Publishers, 1962.

(6) Erdelyi, A., Asymptotic Expansions, Dover Publications, Inc., New York, 1956.

(7) Focke, J., "Asymptotische Entwicklungen mittels der Methode der Stationären Phase," Ber. Sächs. Ges. (Akad.) Wiss, 101 (1954), Heft 3 pp.26-48.

(8) Guillemin, V. and Sternberg, S., Geometric Asymptotics, AMS, Mathematical Surveys, no.14, 1977 pp. 1-14.

(9) Hartman, P., "On the solutions of $\Delta V+V=0$ in an exterior region," Mathematische Zeitschrift. 71 (1959), pp. 251-257.

(10) Hartman, P. and Wilcox, C., "On the solutions of the Helmholtz equation in an exterior domain," Mathematische Zeitschrift, 75 (1961), pp.228-255.

(11) Kolmogorov, A.N. and Fomin, S.V., Introductory Real Analysis, Dover Publications, Inc., New York, 1975.

(12) Onoe, M., Tables of Modified Quotients of Bessel Functions of the First Kind For Real and Imaginary Arguments, Columbia University Press, New York, 1958, pp.1-10.

(13) Sacksteder, R., "On the Helmholtz Equation", Contributions to Analysis and Geometry, supplement to American Journal of Mathematics, John Hopkins University Press, 1981 pp.293-305.

(14) Schauder, J., "Potentialtheoretische Untersuchungen", Mathematische Zietschrift, 33 (1931), pp.602-640.

(15) Spivak, M., A Comprehensive Introduction to Differential Geometry, Vol. 2 Publish or Perish Inc., Berkeley, 1979, pp.86-205.

(16) Thomson, W., "Waves Produced by a Single Impulse," Proc. Roy Soc. A 42 (1887), pp.80-83.

(17) Watson, G.N., "The limits of Applicability of the Principle of Stationary Phase," Proc. Camb. Phil. Soc., 19 (1918), pp.49-53

(18) ________, A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge University Press, 1962, pp.194-230.