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CLASSES OF COMBINATORIAL EXTREMA

IN CERTAIN METRIC SPACES

by

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Section 1 . Preliminaries, Definitions, and Results.

Let $D = \{P_1,\ldots,P_k\}$ be a set of k points in R^n, real, n-dimensional coordinate space. We define a polygon π whose vertex set is D to be a set of k line segments, called edges, such that (1) each given point is an endpoint of exactly two edges, and (2) the ordering induced by the set is cyclic. The cyclic and symmetric notation $[P_{i_1},P_{i_2},\ldots,P_{i_k}] = \pi$ will denote such a polygon in the (covertex) class $\psi(D)$ of all polygons whose vertex set is D . Moreover, let (R^n,m) denote a metric space on the set R^n whose metric is "m" . Then " $\{\sum_{r=1}^{k-1} m(P_{i_r},P_{i_{r+1}})\} + m(P_{i_k},P_{i_1}) = m[\pi]$ " will denote the m-length of the polygon π . The problem considered in this paper is that of finding m-minimal and m-maximal polygons in $\psi(D)$.

Despite the importance of its application, the "euclidean case," $E^2 = (R^2,e)$, where "e" denotes the euclidean metric, does not yet have a satisfactory solution. But the problem has been solved here and in various other spaces for certain classes of vertex sets. In the euclidean plane, for example, Quintas and Supnick in [1], [2], and [3], have completely solved and discussed extensions of the case in which the points of the vertex set all lie on the boundary of their convex hull. (This is referred to as the convex case.) In this paper we solve further extensions of the convex case in various planar metric spaces. These extensions require the following definitions:

Let us call a set of points $D = \{P_1,\ldots,P_k\}$ m-cogeodesic in (R^n,m) if there is an ordering $(P_{i_1},\ldots,P_{i_k})$ of the points such that

$$m(P_{i_1},P_{i_k}) = \sum_{r=1}^{k-1} m(P_{i_r},P_{i_{r+1}}) .$$

By "2n-gon planes" we will mean normed linear spaces in R^2 (Minkowski planes") whose unit circles are geometric 2n-gons, $n \geqq 2$.*

We give special consideration to the space $S^2 = (R^2, s)$, whose unit circle is the square with vertices (1,1), (1,-1), (-1,1), (-1,-1), the "square metric space." This space is often cited as an example of a geodesic metric space whose topology is equivalent to that of E^2 , but which is not isometric to E^2 . The definition of s in (R^2, s) is given in n-dimensions (for $S^n = (R^n, s)$) by

$$s(P,Q) = \max_{1 \leq i \leq n} = |p_i - q_i| ,$$

where P and Q belong to R^n .

We begin by seeking criteria for s-extrema, that is, conditions which when satisfied enable us to find s-extrema in the class $\psi(D)$ without enumerating all $(k-1)!/2$ members of $\psi(D)$, D having k points. For example, a planar polygon in E^2 which is minimal in its covertex class cannot intersect itself; i.e., no closed edge will intersect an open edge in such a polygon. (cf.[1]) But in the space S^2 a polygon which is minimal in its covertex class may intersect itself, though certain well-defined types of self-intersection are precluded. (This is a situation similar to that encountered in Lorentz space, which is not a metric space, however. cf.[4] and [5].) In order to investigate these proscribed types in S^n and to extend the notion of self-intersection of a pölygon, we proceed as follows:

* These 2n-gons must be the boundaries of convex bodies with center, since they are boundaries of metric spheres in a Minkowski plane.

We define the between-set of two distinct points P and Q in (R^n,m) as the set of all T in R^n such that $m(P,T) + m(T,Q) = m(P,Q)$, and denote this set by "B(P,Q)" , where the metric will be understood. Unless we indicate the contrary, the symbol "PQ" will mean the directed line segment from P to Q , or, if P = Q , a point. Further, if $s(P,Q) = |p_i-q_i|$, we will say that PQ is i-like with either positive or negative i-orientation as $p_i-q_i \leqq 0$ or $p_i-q_i \geqq 0$, respectively. Denote by "insB(P,Q)" the set of all points T of B(P,Q) such that PT and QT are i-like only if PQ is i-like, union the set $\{P,Q\}$.

Now, suppose that PQ and RS are segments in S^n having distinct endpoints and such that whenever PQ and RS are both i-like, they have the same i-orientation. If, in this case, $\text{insB}(P,Q) \cap \text{insB}(R,S) \neq \emptyset$, we will say that the edges PQ and RS quasi-intersect.

Theorem 1 : Let D denote a finite set of points in S^n , and let $\pi \in \psi(D)$. Then, a necessary condition that π is s-minimal in the class $\psi(D)$ is that π have no quasi-intersecting edges. Further, there exist distributions for which this condition is also sufficient. (cf. §2, pp.10–23).

The existence of realizations of combinatorial extremal conditions is an immediate question. Indeed, a condition for maximal polygons which is realizable in the space E^2 and which also applies to the space S^2 is the following "dual necklace theorem" of D. Sanders: A polygon is longest in its covertex class if a (closed) sphere may be centered on each vertex such that each sphere intersects all others except the two spheres centered on the two adjacent vertices as given by the polygon. (cf.[5])

Theorem 2 : Realizations of vertex sets of k points satisfying the dual necklace theorem in S^n exist precisely for $3 \leqq k \leqq 2n + 3$, if $n \geqq 3$ and for $k \leqq 6$ if $n = 2$. (cf. pp.24-28)*

A distribution D of k points in (R^n,m) is said to satisfy the four point condition if the points of D can be labeled $P_1,P_2,\ldots,P_k$ so as to satisfy the following: For all sets of integers $\{a,b,c,d\}$ such that $1 \leqq a < b < c < d \leqq k$, we have

$$m(P_a,P_b) + m(P_c,P_d) \leqq m(P_a,P_c) + m(P_b,P_d) \leqq m(P_a,P_d) + m(P_b,P_c)$$

Lemma S: If a distribution D of k points in (R^n,m) satisfies the four point condition, then the polygons $[\ldots P_7P_5P_3P_1P_2P_4P_6\ldots]$ and $[\ldots P_{k-3}P_3P_{k-1}P_1P_kP_2P_{k-2}P_4P_{k-4}\ldots]$ are respectively m-minimal and m-maximal in their covertex class $\psi(D)$.

The proof of this lemma is identical with that of ([6],Th.III). The latter was proved in a Euclidean setting, but it is directly verified that this is not necessary.

Let U denote the unit circle of some 2n-gon plane M, and suppose that some n consecutive sides of U are labelled $1,2,\ldots,n$, respectively, in that order. Then, a segment PQ will be called i-like in M if the line through the origin which is parallel to PQ intersects the i^{th} side of U. (Notice that this agrees with the definition of "i-like" given in the space S^n.) A set of points, D, in a 2n-gon plane M (or in S^n) is cogeodesic if and only if there is an $i = 1,\ldots,n$ such that, for all P and Q in D, PQ is i-like.

*Added for final typing: realizations of the more general dual sphere cluster theorem of Sanders. cf. pg. 27.

We define a generalization of cogeodesic sets of points as follows:

We will call a set of points, D, in M <u>tracklike</u> if for some fixed $i = 1,\ldots,n$ and for each P_0 in D we have P_0Q is i-like for all Q in D, with the possible exception of a single point Q_0 depending on P_0. The set D will be called tracklike in S^n, if there exist a fixed i and j, $1 \leqq i, j \leqq n$, such that for each P_0 in D , P_0Q is i-like for all Q in D, with possible exception of a single point Q_0 depending on P_0. In that case, P_0Q_0 is j-like.

<u>Theorem 3</u>: Tracklike distributions in S^n and in 2n-gon planes satisfy the four point condition.

The characteristics of tracklike distribution are best seen in the space S^2 (cf.page 31 for methods of constructing tracklike distributions in S^2 and related remarks). It can be verified that the following set of ten points is tracklike in S^2 : (*) {(0,0),(3,7),(8,6),(12,5),(19,7),(20,5),(23,4),(28,12),(8,3),(15,9)} .

One can verify that no subset of six of these points is cogeodesic in S^2. Moreover, not all of the points lie on the boundary of their convex hull (where, we remark that all reference to convexity in this paper will mean ordinary euclidean convexity.) The relevance of this last remark stems from the fact that the "convex case," that is, the case in which $D = H(D)$, where "H(D)" denotes the points of D on the boundary of its convex hull, has been solved in E^2. The solutions are given by Quintas and Supnick in [1],[2], and [3], and are of the form "if π is e-extreme in its covertex class, then π must belong to the subclass C."

However, the above e-results are generally <u>false</u> for spaces (R^2,m)

and for S^2 in particular. For, if π is <u>any</u> polygon in $\psi(D)$ where D is an arbitrary finite point-set in R^2, then there exists a metric m on R^2 such that π is uniquely m-minimal (m-maximal) in $\psi(D)$. For example, call P and Q "adjacent" if and only if they are adjacent vertices of π; define m as follows:

$$m(P,Q) = \begin{cases} 1, \text{ if } P \neq Q \text{ and } P \text{ and } Q \text{ are adjacent} \\ 2, \text{ if } P \neq Q,\ P \text{ and } Q \text{ not adjacent} \\ 0, \text{ if } P = Q. \end{cases}$$

It is easy to check that (R^2,m) is a metric space, where π is uniquely minimal in $\psi(D)$, since every edge not in π is of m-length 2. For the maximal case, interchange the numbers 1 and 2 in the above.

Nevertheless, it is possible to give extensions of the above e-results which subsume the space S^2, as well as other classes of planar metric spaces, as follows:

Let m denote a planar metric. We will say that <u>m is of type I</u> if $e(P,Q) = e(P,T) + e(T,Q)$ always implies $m(P,Q) = m(P,T) + m(T,Q)$. We will say that <u>m is of type II</u> if, whenever ABCD is a convex quadrilateral in the plane with not all A,B,C, and D collinear,

$$m(A,C) + m(B,D) \geqq \max\{m(A,B) + m(C,D),\ m(A,D) + m(B,C)\}.$$

We will say that <u>m is of type III</u> if $\lim_n e(P_n,P_0) = 0$ and $\lim_n e(Q_n,Q_0) = 0$ imply $\lim_n m(P_n,Q_n) = m(P_0,Q_0)$, where $\{P_n\}$ and $\{Q_n\}$ are sequences of points in R^2; P_0,Q_0 are in R^2.

<u>Theorem 4</u>: Let D denote a set of non-collinear points in R^2, and let m denote a metric of type I. Then there exists an m-minimal polygon in the class $\psi(D)$ which does not self-intersect. Further, there exists an m-minimal polygon of $\psi(D)$ in $\psi_0(D)$, the subclass of poly-

gons of $\psi(D)$ whose vertices in $H(D)$ are in cyclic order. (cf.pg.33)

Theorem 5 : (a) Let $\{P_1,P_3,P_5,\ldots,P_{2p-1},P_2,P_4,\ldots,P_{2p-2}\} = D$ be distinct non-collinear points in R^2 which fall on the boundary B of their convex hull in the stated cyclic order, and let m be a metric of types II and III. Then $[P_1,P_2\ldots P_{2p-1}]$ is an m-maximal polygon in $\psi(D)$.

(b) Let D denote a set of 2n points in R^2 (non-collinear) which fall on the boundary B of their convex hull and let $P_1^1,P_1^2,\ldots,P_1^n$ denote any n points of D which are adjacent on B . Then, an m-maximal polygon in $\psi(D)$ is among the n polygons

$$[\ldots P^i_{2n-5}P^i_5P^i_{2n-3}P^i_3P^i_{2n-1}P^i_1P^i_{2n}P^i_2P^i_{2n-2}P^i_4\ldots]$$

where, for each i , starting with P_1^i and traversing B in a specified common direction the consecutive points of D are labeled

$$P^i_1,P^i_2,P^i_3,\ldots,P^i_n,P^i_{2n},P^i_{2n-1},\ldots,P^i_{n+1} .$$

(cf.pg. 34)

Observe that all planar norms are metrics of types I, II, and III. This fact and others concerning these metrics are discussed on page 39. Note that our extension of [1],viz., Th. 4, does not contain a corresponding statement about uniqueness in the convex minimal case. In S^2 , for example, this would be false. (cf. pg.35) However, a relation between non-unique m-minima (m-maxima) may be given as follows:

By <u>an arc-inversion on a polygon</u> we will mean:

the symbol	$[P_1\ldots P_{i-1}\ (P_i\ldots\ P_j)\quad P_{j+1}\ldots P_n]$
on a polygon	$[P_1\ldots P_{i-1}\ \ P_i\ \ldots P_j\quad P_{j+1}\ldots P_n]$
producing	$[P_1\ldots P_{i-1}P_jP_{j-1}\ldots P_{i+1}P_iP_{j+1}\ldots P_n]$

<u>Corollary 1</u> : Let D be a set of $n \geqq 3$ points in R^2 (where n is odd) such that $D = H(D)$. If π_1 and π_2 are two distinct m-minimal (m-maximal) polygons in $\psi(D)$, where m is a type I (type II and III) metric, then there is a finite sequence of m-minimal (m-maximal) polygons, beginning with π_1 and ending with π_2 , which are obtained by means of consecutive arc-inversions. (cf. page 36. Note that this corollary is <u>false</u> for the even maximal case.)

Moreover, we obtain the following:

<u>Corollary 2</u> : Let D be a finite set of non-collinear points in R^2 , and let m be a type I metric. Then $m[\pi_0]$ is a tight lower bound on the m-lengths of the polygons in $\psi(D)$, where "π_0" denotes the unique polygon in the class $\gamma(H(D))$. (cf. page 44)

The utility of the last corollary can be seen by the following example: Let $D = \{P_1, P_2, \ldots, P_{12}\}$ where $P_1(0,0), P_2(3,7), P_3(8,6)$, $P_4(12,5), P_5(19,7), P_6(20,5), P_7(23,4), P_8(28,12), P_9(8,3), P_{10}(15,9), P_{11}(10,23)$, $P_{12}(8,24)$. Then $H(D) = \{P_1, P_2, P_8, P_7\}$. If $\pi_0 = [P_1, P_2, P_8, P_7]$ and $\pi = [P_1P_2P_3P_{10}P_5P_{11}P_8P_{12}P_7P_6P_4P_9]$, then $s[\pi] = 20 = s[\pi_0]$. Thus, π is s-minimal in $\psi(D)$; it is unnecessary to compare π with the other $(12-1)!/(4-1)!-1$ members of $\gamma(D)$. (cf. page 41)

We discuss the identification of vertex sets D in the spaces (R^2, m) for which the m-minimum in $\psi(D)$ has the same m-length as the m-minimum in $\gamma(H(D))$ on pages 44-50 . These vertex sets will be called m-chain distributions. A chain of edges $P_1P_2, P_2P_3, \ldots, P_{k-1}P_k$ will be called an m-geodesic chain if $m(P_1, P_k) = \sum_{r=1}^{k-1} \{m(P_r, P_{r+1})\}$.

We obtain the following characterization in 2n-gon planes:

<u>Theorem 6</u> : Let M denote a 2n-gon plane and let $D \subsetneq M$ be a finite set of points. Then D is an m-chain distribution if and only if there exists a polygon in $\psi(D)$ which can be expressed as no more than 2n consecutive m-geodesic chains which are joined at points of H(D) in cyclic order (where "m" denotes the M metric). (cf. pg. 45)

Indeed, the following corollary shows that the tracklike distributions discussed earlier (page 5) are m-chain distributions, though the converse is false:

<u>Corollary 3</u> : Let D be a finite set of points in M and suppose that $D = X \cup Y$, where X and Y are both i-like distributions for some $i = 1, \ldots, n$ in M and $X \cap Y = \emptyset$. Then D is an m-chain distribution. (Note: A <u>distribution</u> is i-like if every segment determined by a pair of its points is i-like in M . cf. pg. 49)

Section 2. Properties of S^n

In the following paragraphs we develop certain properties of S^n metric betweenness which will enable us to prove theorem 1 as well as the theorems concerning 2n-gon planes as they apply to the space S^2. Results obtained for the space S^2 will then be extended in later sections for the 2n-gon planes.

If $P(p_1,\ldots,p_n)$ belongs to S^n, then let us define:

$$C_i^+(P) = \{(x_1,\ldots,x_n) \text{ in } S^n : x_i - p_i \geqq |x_j - p_j|,\ j=1,\ldots,n\}$$
$$C_i^-(P) = \{(x_1,\ldots,x_n) \text{ in } S^n : p_i - x_i \geqq |x_j - p_j|,\ j=1,\ldots,n\}$$
$$C_i(P) = C_i^+(P) \cup C_i^-(P)$$

(2.1) If P_1P_2 is i-like with $p_i^{(1)} < p_i^{(2)}, P_j = (p_1^{(j)},\ldots,p_n^{(j)})$, then $B(P_1,P_2) = C_i^+(P_1) \cap C_i^-(P_2)$.

Proof: If Q is in $C_i^+(P_1) \cap C_i^-(P_2)$, then $s(P_2,Q) = p_i^{(2)} - q_i$; and $s(P_1,Q) = q_i - p_i^{(1)}$. $s(P_1,P_2) = p_i^{(2)} - p_i^{(1)}$ by assumption. Hence, $s(P_1,Q) + s(P_2,Q) = s(P_1,P_2)$; and so $C_i^+(P_1) \cap C_i^-(P_2) \subseteq B(P_1,P_2)$.

If $s(P_1,Q) + s(P_2,Q) = s(P_1,P_2)$, we claim that Q belongs to ("$\in$") $C_i(P_1) \cap C_i(P_2)$. For, suppose that $Q \notin C_i(P_1)$. Then there exists an $i \neq j$ such that $|q_j-p_j^{(1)}| > |q_i-p_i^{(1)}|$, where $s(Q,P_1) = |q_j-p_j^{(1)}|$. We also have $\max\limits_{k=1,\ldots,n} |q_k-p_k^{(2)}| \geqq |q_i-p_i^{(2)}|$. By hypothesis

$|p_i^{(2)} - p_i^{(1)}| = \max\limits_k |p_k^{(1)} - q_k| + \max\limits_k |p_k^{(2)} - q_k|$. But this yields

$s(P_1,P_2) = |p_i^{(2)} - p_i^{(1)}| \leqq |p_i^{(1)} - q_i| + |p_i^{(2)} - q_i| < |p_j^{(1)} - q_j| + |p_i^{(2)} - q_i| \leqq s(P_1,Q) + s(P_2,Q)$, which is a contradiction. Similarly, $Q \in C_i(P_2)$.

Suppose that $Q \notin C_i^+(P_1)$. Then clearly $Q \notin C_i^+(P_2)$ and $0 < p_i^{(1)} - q_i < p_i^{(2)} - q_i$. Then, $p_i^{(2)} - p_i^{(1)} = (p_i^{(2)} - q_i) + (p_i^{(1)} - q_i) = p_i^{(1)} + p_i^{(2)} - 2q_i$. So $p_i^{(1)} = q_i$ and $P_1 = Q$. But, $Q = P_1 \in C_i^+(P_1)$, which is a contradiction. Similarly, $Q \in C_i^-(P_2)$. qed

The interpretation of the above in S^2 is that $B(P,Q)$ is a rectangular region, in general, with sides whose slopes are 1, having P and Q as opposite vertices of the rectangular boundary. It also follows that $B(P,Q)$ is a compact (closed and bounded) convex set. The compactness, in S^2 , is clear by definition, while the convexity follows from the fact that $C_i^+(P_1)$ and $C_i^-(P_2)$ are both intersections of a finite number of half-spaces in R^n .

(2.2) If $P_1 \in C_i^{\pm}(P_2)$ and $P_2 \in C_i^{\pm}(P_3)$, then $P_1 \in C^{\pm}(P_3)$.

<u>Proof</u>: $P_1 \in C_i^-(P_2)$ implies that $p_i^{(2)} - p_i^{(1)} \geq \max_j |p_j^{(2)} - p_j^{(1)}|$; and $P_2 \in C_i^-(P_3)$ implies that $p_i^{(3)} - p_i^{(2)} \geq \max_j |p_j^{(3)} - p_j^{(2)}|$. Hence, $p_i^{(3)} - p_i^{(1)} = (p_i^{(3)} - p_i^{(2)}) + (p_i^{(2)} - p_i^{(1)}) \geq \max_k |p_k^{(3)} - p_k^{(2)}| + \max_j |p_j^{(2)} - p_j^{(1)}| \geq \max_{1 \leq m \leq n} \{ |p_m^{(3)} - p_m^{(2)}| + |p_m^{(2)} - p_m^{(1)}| \} \geq \max_m \{ |(p_m^{(3)} - p_m^{(2)}) + (p_m^{(2)} - p_m^{(1)})| \} = \max_m |p_m^{(3)} - p_1^{(1)}|$. Hence, $P_3 \in C_i^-(P_1)$. The case $P_3 \in C_i^+(P_1)$ is similar.

(2.3) It is clear that: $P \in C_i^{\pm}(Q)$ if and only if $Q \in C_i^{\mp}(P)$ if and only if PQ is i-like. Also, if $P \in B(R,S)$ and RS is i-like, then RP and PS are i-like and they have the same i-orientation as

RS . However, recall the definition of insB(R,S) , and observe that if RP and PS are i-like, then RS need not be i-like. For example, consider the points R(0,0), P(1,1), and S(2,0); then $P \in B(R,S)$, PR and PS are both 2-like, but RS is not 2-like.

(2.4) If $P \neq Q$, PQ is i-like, and if R and S belong to the line determined by P and Q , then RS is i-like. This is a consequence of the fact that the determination of the direction numbers of a line does not depend upon which pair of distinct points on the line one uses. Hence, $\text{insB}(P,Q) \supseteq PQ$.

(2.5) Let $D = \{P_1, \ldots, P_m\}$ be a set of m distinct points in S^n , S^n, $n \geqq 2$. Suppose that there exist i,j, $i \neq j$ such that for all P,Q in D , PQ is either i-like or j-like. Then the perpendicular projection of D into the i-j coordinate plane is 1-1 and S^n distance preserving.

<u>Proof</u>: Suppose that the image of P_r is P_r' . Then P_r and P_r' agree in their i^{th} and j^{th} coordinates; all the other coordinates of P_r' are zero. Suppose that $P_k' = P_\ell'$. Then,

$$0 = s(P_k', P_\ell') = \max_r |p_r^{(k)} - p_r^{(\ell)}| = s(P_k, P_\ell) .$$

Hence, $P_k = P_\ell$, and the map is 1-1. The chain of equalities, without the 0 , shows the rest. qed

We will call such distributions "i-j distributions," and our later work in the space S^2 will apply to them in the natural way. Observe that i-j distributions in S^n, $n > 2$, need not be coplanar; also, coplanar distributions in S^n are not necessarily i-j distributions, if $n > 2$. For example, the points (0,0,1), (1,0,0), (-1,0,0), (0,1,0) and (0,-1,0) are a non-coplanar 1-2 distribution. On the other hand, the points (-3,6,-2), (3,-1,-1), (6,-3,-2), (-1,3,-1), (-2,-3,6), and (-1,+3,-1) all belong to the plane in R^3 whose equation is $x_1 + x_2 + x_3 = 1$. But the first two points are only 2-like, the next two are only 1-like, and the last two are only 3-like in S^3. Hence, this can be neither a 1-3, 2-3, or 1-2 distribution, though it is coplanar.

(2.6) An important special case of (2.5) is the following: Let $D = \{P_1, \ldots, P_m\}$, where all the points of D are distinct. Suppose that there exists an i such that for all $P,Q \in D$, PQ is i-like. Then the projection of D into the i^{th} coordinate axis is 1-1 and S^n distance preserving. The proof is the same as that of (2.5). Observe that these point sets are s-cogeodesic (cf. pg.). Hence, co-geodesic distributions in S^n need not be collinear. By the direction number argument given in (2.4), however, collinear distributions are s-cogeodesic. In fact, (2.6) actually characterizes s-cogeodesic distributions:

(2.7) Three distinct points $P,Q,R \in S^n$ are such that

a) $s(P,Q) + s(Q,R) = s(P,R)$, if and only if ("iff")

b) $C_i^+(P) \supseteq C_i^+(Q) \supseteq C_i^+(R)$ or $C_i^-(P) \supseteq C_i^-(Q) \supseteq C_i^-(R)$, some i, iff

c) PQ, QR, and PR are all i-like with the same i-orientation, iff

d) P, Q, and R are s-cogeodesic and $s(P,R) > s(Q,P), s(Q,R)$.

Proof: a) iff b) follows by (2.1); b) iff c) follows by the definitions of "i-like," "i-orientation," and $C_i^{\pm}(P)$; c) implies d) by (2.6); and d) implies a) by definition of "s-cogeodesic."

(2.8) It follows quite easily (by induction) that k points $\{P_j\}$ are s-cogeodesic iff we can write $C_i^+(P_1) \supseteq C_i^+(P_2) \supseteq \ldots \supseteq C_i^+(P_k)$ or $C_i^-(P_1) \supseteq C_i^-(P_2) \supseteq \ldots \supseteq C_i^-(P_k)$ for some $i = 1,\ldots,n$. Also, these k points actually lie on a geodesic in S^n, viz., the union of the chain of straight line segments $P_2P_3, P_3P_4, \ldots, P_{k-2}P_{k-1}$ and the rays $\overrightarrow{P_2P_1}$ and $\overrightarrow{P_{k-1}P_k}$. This is clear since straight lines are examples of geodesics in this space.

We shall prove Theorems 1 and 3, using the properties we have developed. First, however, we shall prove a lemma which gives us somewhat more than we need for this immediate purpose but which will be useful later.

Lemma 1: Let AB and CD be directed line segments in S^n with $n \geqq 2$, A, B, C, and D all distinct, and suppose that $\text{insB}(A,B) \cap \text{insB}(C,D) \neq \emptyset$. If (a) when AB and CD are both i-like they have the same i-orientation, then, $s(A,B) + s(C,D) > s(A,C) + s(B,D)$. (b) If, however, there exists $i = 1,\ldots,n$ such that AB and CD are both i-like but have different i-orientations, then $s(A,B) + s(C,D) = s(A,C) + s(B,D)$.

Proof: (a) Suppose that there exists a $P \in \text{insB}(A,B) \cap \text{insB}(C,D)$ such that $P \notin \{A,B,C,D\}$. By definition, $P \in B(A,B) \cap B(C,D)$. Hence,

$$s(A,P) + s(P,B) = s(A,B) \quad \text{and} \quad s(C,P) + s(P,D) = s(C,D)$$

We claim that $s(A,C) < s(A,P) + s(P,C)$. This is clear if A, P, and C are not s-cogeodesic. Therefore, let us suppose that A, P, and C are s-cogeodesic. Hence, there exists an $i=1,\dots,n$, such that AC, CP, and AP are i-like, and we may write, therefore,

$$s(A,C) = |a_i - c_i| \ , \ s(C,P) = |c_i - p_i| \ , \ s(A,P) = |a_i - p_i| \ .$$

If $s(A,C) = s(A,P) + s(C,P)$, then by (2.7) we have either $a_i < p_i < c_i$ or $c_i < p_i < a_i$. But this implies that, by (2.1), AP and PC have the same i-orientation; so, AP and CP have different i-orientations. Now, AP and AB , and, CP and CD have the same i-orientations, resp., by (2.3). Thus, AB and CD have different i-orientations which contradicts the hypothesis in (a). Hence, $s(A,C) < s(A,P) + s(C,P)$, as claimed. The triangle inequality yields $s(B,D) \leqq s(B,P) + s(P,D)$. So, $s(A,B) + s(C,D) =$ $(s(A,P) + s(P,B)) + (s(C,P) + s(P,D)) = \dots = (s(A,P) + s(P,C)) +$ $(s(B,P) + s(P,D)) > s(A,C) + s(B,D)$.

Next suppose that $insB(A,B) \cap insB(C,D) \subseteq \{A,B,C,D\}$. It will suffice to consider the cases in which C or $D \in insB(A,B) \cap insB(C,D)$.

In the first case, if AB and CD are not both i-like for any $i=1,\dots n$, then neither are CD and CB , by (2.3); thus, C, D, and B are not s-cogeodesic, by (2.7). And so, $s(C,D) > s(B,D) - s(C,B)$. Also, $C \in insB(A,B) \cap insB(C,D)$ implies that $s(A,B) + s(C,D) = \dots$ $\dots = (s(A,C) + s(C,B)) + s(C,D) > s(A,C) + (s(C,B) - s(C,B)) + s(B,D) =$ $s(A,C) + s(B,D)$.

If AB and CD <u>are</u> both i-like, they have the same i-orientation.

Hence, CD and CB have the same i-orientation. If C, D, and B are not s-cogeodesic, the last paragraph holds. Otherwise, we have one of the following cases:

$c_i < b_i < d_i$, $d_i < b_i < c_i$, $c_i < d_i < b_i$, or $b_i < d_i < c_i$. The first two of these yield $s(C,D) > s(B,D)$. The third and fourth, $s(C,B) > s(D,B)$. Thus, $s(A,B) + s(C,D) = (s(A,C) + s(C,B)) + s(C,D) > s(A,C) + s(D,B) + s(C,D) > \,>. > s(A,C) + s(B,D)$ results in the first case, or, $s(A,B) + s(C,D) = \,>. = (s(A,C) + s(C,B)) + (s(C,B) - s(D,B)) = s(A,C) + 2s(C,B) - s(D,B) > \,>. > s(A,C) + s(B,D)$ results in the second.

If $\{D\} = insB(A,B) \cap insB(C,D)$, and AB and CD are not both i-like for any $i=1,\ldots n$, then neither are AD and CD . As above, we obtain

$$s(A,C) < s(A,D) + s(C,D) \text{ , or, } s(A,D) > s(A,C) - s(C,D) .$$

Since $D \in B(A,B) \cap B(C,D)$, we have $s(A,B) + s(C,D) = \ldots$

$\ldots = s(A,D) + s(D,B) + s(C,D) > s(A,C) - s(C,D) + s(B,D) + s(C,D) = s(AC) + s(B,D)$.

If, AB and CD are i-like for some i , then A, D, and C are s-cogeodesic; arguing as above we obtain: $s(A,D) > s(A,C)$; or $s(C,D) = s(A,C) + s(D,C)$, so that $s(A,C) < s(C,D)$. This will again yield the result $s(A,B) + s(C,D) > s(A,C) + s(B,D)$. The remaining cases, with A and B , follow by symmetry.

(b) Suppose that AB and CD are both i-like, but that they have opposite i-orientations, for some $i=1,\ldots,n$. As in part (a), we first suppose that there is a point $P \in insB(A,B) \cap insB(C,D)$, $P \notin \{A,B,C,D\}$. We claim that $s(A,P) + s(C,P) = s(A,C)$ and also

$s(B,P) + s(P,D) = s(B,D)$. For the first equality, observe that AP and PC are both i-like with opposite i-orientations, by (2.3). Now, $a_i \neq c_i$, for otherwise we would have $a_i - p_i = c_i - p_i$, by substitution; hence, AP and CP would have the same i-orientation, and so would AB and CD. Moreover, $a_i \neq p_i \neq c_i$, since $A \neq P \neq C$. We claim that AC is i-like. For, by our orientation hypothesis either $a_i < p_i < c_i$ or $a_i > p_i > c_i$ holds. Then, in the first case, by (2.7), $P \in C_i^+(A)$ and $C \in C_i^+(P)$. Thus, $C \in C_i^+(A)$, and AC is i-like. The other case is analogous. By (2.7), $s(A,P) + s(P,C) = s(A,C)$. The other equality is proved the same way.

Hence, $P \in \text{insB}(A,B) \cap \text{insB}(C,D) \subseteq B(A,B) \cap B(C,D)$ yields:
$s(A,B) + s(C,D) = (s(A,P) + s(P,B)) + (s(C,P) + s(P,D)) = \ldots$
$\ldots = (s(A,P) + s(P,C)) + (s(P,D) + s(B,D)) = s(A,C) + s(B,D)$.

Next, if $\text{insB}(A,B) \cap \text{insB}(C,D) \subseteq \{A,B,C,D\}$ we first suppose that $C \in \text{insB}(A,B) \cap \text{insB}(C,D)$. Then, $s(A,B) + s(C,B) = s(A,B)$. Now, BC and CD have the same i-orientation. So, $C_i^+(B) \supseteq C_i^+(C) \supseteq C_i^+(D)$, or, $C_i^+(B) \subseteq C_i^+(C) \subseteq C_i^+(D)$. Whence, $s(B,C) + s(C,D) = s(D,B)$. Thus, $s(A,B) + s(C,D) = (s(A,C) + s(C,B)) + (s(B,D) - s(B,C)) = s(A,C) + s(B,D)$.

If $D \in \text{insB}(A,B) \cap \text{insB}(C,D)$, then $s(A,D) + s(D,B) = s(A,B)$, and $C_i^+(A) \supseteq C_i^+(D) \supseteq C_i^+(C)$, or, $C_i^+(A) \subseteq C_i^+(D) \subseteq C_i^+(C)$; this implies $s(A,C) = s(A,D) + s(D,C)$. Hence,
$s(A,B) + s(C,D) = (s(A,D) + s(D,B)) + (s(A,C) - s(A,D)) = s(A,C) + s(B,D)$.
As in the first part of the Lemma "(a)" the other cases are analogous.

qed

<u>Proof of Theorem 1 (necessity only)</u>: Let $\pi = [\ldots AB \ldots CD \ldots]$ be

a polygon in S^n with distinct vertices A,B,C, and D , and edges AB and CD which quasi-intersect. Then the arc-inversion [...A(B...C)D...] yields the polygon $\pi^1 = [\ldots AC \ldots BD \ldots]$ which differs from π only in having the edges AC and BD instead of the edges AB and CD. By Lemma 1 (a), $s(A,C) + s(B,D) < s(A,B) + s(C,D)$. Hence, $s[\pi] > s[\pi^1]$. So π could not have been s-minimal in its covertex class, and the condition is necessary.

<u>Remark:</u> Observe that if $insB(A,B) \cap insB(C,D) \neq \emptyset$ but that AB and CD had different i-orientations, as in the second part of Lemma 1, then π might in fact be minimal in its covertex class. For example, let $P_1(0,0), P_2(1,0), P_3(2,2.5)$, and $P_4(3,0.75)$. Then $\pi = [P_1P_2P_3P_4]$ is s-minimal in its covertex class, but the edges P_2P_3 and P_4P_1 intersect; so, $insB(P_2,P_3) \cap insB(P_4,P_1) \neq \emptyset$. This possibility is suggested by part (b) of the lemma, which shows that the conditions in part (a) are not too stringent.

<u>Remark:</u> If in the definition of "quasi-intersect" we merely require that $B(A,B) \cap B(C,D) \neq \emptyset$ and keep the "orientation requirements" intact, then Theorem 1 becomes false. For a counter-example, let A(0,0), B(1,0), C(1,1), D(0,1) . Then [ABCD] is s-minimal in its covertex class although $B(A,B) \cap B(C,D) = (0.5,0.5)$; both AB and CD are only 1-like and have the same 1-orientation.

<u>Remark:</u> Theorem 1 is not generally sufficient for s-minimality. If $P_1(4,0), P_2(0,1), P_3(0,4), P_4(2,3), P_5(5,9), P_6(8,9), P_7(10,7)$, then with respect to the polygon $\pi = [P_1P_2P_3P_4P_5P_6P_7]$, for any pair of edges PQ and RS with distinct vertices as endpoints we have $\emptyset =$ $B(P,Q) \cap B(R,S) \supseteq insB(P,Q) \cap insB(R,S)$, which can be checked by draw-

ing the sets $B(P,Q)$ etc. following (2.1) . But π is not s-minimal, since if we let $\pi^1 = [P_1P_2P_4P_3P_5P_6P_7]$, then

$$s[\pi] = 28 \text{ , and } s[\pi^1] = 25 \text{ .}$$

Proof of the sufficiency assertion of Theorem 1: For the following class of points in S^n the necessity hypothesis of Theorem 1 is sufficient as well: Let i,j, $1 \leqq i$, $j \leqq n$ be fixed. Let D be a finite set of points in S^n such that for each $P_0 \in D$ we have that $Q \in$ interior of $C_i(P_0)$ for all $Q \in D$, with perhaps the exception of a point Q_0 depending on P_0 . In that case, $Q_0 \in$ interior of $C_j(P_0)$.

First we prove the following:

Lemma 2: Let A, A', B, and B' be four distinct points in S^2 such that AA' and BB' are only 2-like, but AB, AB', $A'B$, and $A'B'$ are all 1-like. Then A, A', B, and B' are the vertices of a (non-degenerate) convex quadrilateral.

Proof: Without loss of generality we may assume that $a_2 > a_2'$, $b_2 > b_2'$, where $A = (a_1, a_2)$, etc. Observe that either max $\{a_1, a_2\} <$ min $\{b_1, b_2\}$ or max $\{b_1, b_2\} <$ min $\{a_1, a_2\}$. For, if $a_1 < b_1 < a_1'$, then AB and BA' being 1-like imply that $B \in C_1^+(A)$ and $A' \in C_1^+(B)$; so $A' \in C_1^+(A)$ and AA' is 1-like -- which is a contradiction. The other cases are quite similar. So, we assume without loss of generality that max $\{a_1, a_1'\} <$ min $\{b_1, b_1'\}$.

Now, if our conclusion were false, then one of the points which we will denote by P , would lie interior to a triangular region whose vertices are the other three points. In that case, it would be necessary that at least one of these vertices lies in each of the sets $C_1^+(P)$ and $C_1^-(P)$. For, otherwise all three vertices would lie on the same side of a line through P with slope 1 or -1. Hence, suppose that $P = A$. But then B and $B^{\downarrow} \in C_1^+(A)$; and, since $AA^{\downarrow}$ is only 2-like, $A \notin C_1^-(A^{\downarrow})$ -- which contradicts the last fact. Similarly, letting $P = A^{\downarrow}$, B, or $B^{\downarrow}$ we obtain a contradiction. This proves the lemma.

Now let us proceed with the main proof; i.e., that of the theorem. First observe that by (2.5) we may consider D to be lying in the i-j coordinate plane, or equivalently in the space S^2 with i=1, j=2.

Our plan is to first characterize all the minima in $\psi(D)$ using the necessary condition of the theorem. Then we will show that only the polygons in $\psi(D)$ fitting our characterization satisfy this condition.

Call the set $\{A,A^{\downarrow}\}$, where $A,A^{\downarrow} \in D$, a "v-pair"(vertical pair) if $A \in C_2(A^{\downarrow})$. Suppose that $A = (a_1,a_2)$ and $A^{\downarrow} = (a_1^{\downarrow},a_2^{\downarrow})$ where we may assume that $a_2 < a_2^{\downarrow}$ and $a_1 \leqq a_1^{\downarrow}$. Then there does not exist a $B \in D$ such that $B = (b_1,b_2)$ and $a_1 \leqq b_1 \leqq a_1^{\downarrow}$, where $A \neq B \neq A^{\downarrow}$. For, if there were, then by hypothesis, $B \in C_1^+(A)$ and $A \in C_1^+(B)$. But then we would have that $A^{\downarrow} \in C_1^+(A)$, which contradicts the fact that we have a v-pair.

Now, if $\{A,A^{\downarrow}\}$ is a v-pair, we will call it an "e-pair" (end-pair) if either $\max\{a_1,a_1^{\downarrow}\} \geqq \max_{(b_1,b_2)\in D} b_1$ or $\min\{a_1,a_1^{\downarrow}\} \leqq \min_{(b_1,b_2)\in D} b_1$ holds.

Let E denote an edge of an s-minimal polygon π in $\psi(D)$.

Then we assert that E is <u>not</u> 1-like if and only if the end points of E belong to the same e-pair. For, suppose that the endpoints of E do not belong to the same e-pair, but that E is not 1-like. Let $E = MN$. Then there exists an edge AB of π such that AB is 1-like and $M \in \text{insB}(A,B)$. For, there must exist points to the right and to left of both M and N in D connected by a strictly 1-like edge, by previous considerations. But $M \in \text{insB}(M,N)$; and since π is s-minimal in $\psi(D)$, we have contradicted our necessity hypothesis. Conversely, suppose that A and B are points of an e-pair but that no edge of π joins them. Then we must have in π directed 1-like edges AA', $A''A$, BB', and $B''B$, where the points A, B, A'', B'', A', B'' are distinct. Two of these must be similarly oriented; suppose AA' and BB' are such. We claim that $\text{insB}(A,A') \cap \text{insB}(B,B') \neq \emptyset$. For, without loss of generality we may assume that AA' has a positive 1-orientation. Now, observe that the open segment $PQ \leq \text{insB}(P,Q)$ if $P \neq Q$. Moreover, if A' and B' did not belong to the same v-pair, then as in the first part of this argument, we arrive at a contradiction. If they do, however, then A,A',B,B' are the vertices of a convex quadrilateral by Lemma 1, and, clearly, no three are collinear. If AA' and BB' are the diagonals of this quadrilateral, they intersect, and we contradict the necessity hypothesis.

On the other hand, if AA' and BB' are not the diagonals, then AB' and $A'B$ must be the diagonals; and then $AB' \cap A'B \neq \emptyset$. Hence,

$$\emptyset \neq (\text{int}C_1^+(B) \cap \text{int } C_1^-(A')) \cap (\text{int } C_1^+(A) \cap \text{int}C_1^-(B'))$$

$$= (\text{int}C_1^+(A) \cap \text{int } C_1^-(A')) \cap (\text{int } C_1^+(B) \cap \text{int}C_1^+(B'))$$

$$= \text{insB}(A,A') \cap \text{insB}(B,B') , \qquad \text{by (1.9)}.$$

But this is impossible by our necessary condition, since π is s-minimal in the covertex class $\psi(D)$. Thus, our assertion concerning the edge E is proved in full.

Next, if D has two distinct e-pairs $\{A,B\}$ and $\{C,D\}$, AB and CD the corr. edges of an s-minimal polygon (which we have been implicitly considering to be directed), then all the s-minimal polygons of the class $\psi(D)$ may be characterized thus: A chain of 1-like edges from B to C ; followed by the edge CD ; followed by a chain of 1-like chain of edges from D to A ; followed by the edge AB . In fact, it is clear that the s-length of all the minimal polygons must be: $s(A,B) + s(C,D) + s(A,C) + s(B,D) = .=. = s(A,B) + s(C,D) + s(A,D) + s(CB)$, since the points A,B,C, and D are the vertices of a convex quadrilateral, and the preceeding argument showed that the sum of the "horizontal sides" is equal to the sum of the diagonals. Also, the 1-like chains must be monotone; that is, the endpoints of the edges of a chain must have their first coordinates in natural order, or else we would obtain a larger number than the s-length given above.

If there is only one or no e-pairs (there cannot be more than two) in D , then we may consider the e-pairs to be the leftmost or rightmost points of D , whichever is appropriate, where the two components of a pair are identical. That is, A = B and/or C = D . In any case, from now on we consider the e-pairs to be single units V_1 and V_2 joined by chains of 1-like edges in any polygon in $\psi(D)$ satisfying the necessity hypothesis. In such a polygon π we have shown that the components of an e-pair, if they are distinct, must be joined by an edge. We now consider the two chains of π joining V_1 and V_2 and show that

that, under the assumption that one of them is not monotone, (and so π is not s-minimal in $\psi(D)$) we contradict the necessity hypothesis.

Let X denote a non-monotone 1-like chain from V_1 to V_2 , and consider the maximally monotone subchain of X beginning at V_1 and ending at some point B . Let AB and BC belong to X ; then AB and BC have opposite 1-orientations. We may assume without loss of generality that they are positively and negatively 1-oriented respectively.

If B does not belong to a v-pair, or if $\{B,B'\}$ is a v-pair but B' is not an endpoint of an edge in X , then there exists an edge RS in the chain X which is positively 1-oriented and such that $B \in insB(R,S)$. Hence, $insB(A,B) \cap insB(R,S) \neq \emptyset$, which contradicts the hypothesis that π satisfies the necessity condition.

If $\{B,B'\}$ is a v-pair with $B \neq B'$ and B' is used as an endpoint of an edge in X , then there is an edge RS in the <u>other</u> 1-like chain such that $B \in insB(R,S)$ and RS is negatively 1-oriented. This also contradicts the necessity hypothesis, and we again have a contradiction. Thus, a polygon is s-minimal in the class $\psi(D)$ if and only if it contains no quasi-intersections. qed

<u>Remark</u>: The distributions given above are slightly specialized examples of the tracklike distributions (cf. pg. 5) which we shall discuss at further length in Section 4.

Section 3. Necklace Theorems

Proof of Theorem 2: We shall show that there does not exist any such collection of rectangular parallelopipedon regions of more than the specified number of members in n-dimensional coordinate space. Suppose that there were such a collection, $\mathcal{R}$, whose members form a dual necklace.

Let P belong to $\mathcal{R}$ and let P_1 and Q_1 be the two members of $\mathcal{R}$ which do not intersect P . Then P_1 intersects Q_1 , or else we would be forced to join the centers (the intersections of their diagonals) of P_0, P_1, Q_1 to obtain a triangle. This is impossible since $n \geqq 2$. Moreover, there are additional members P_2 and Q_2 of $\mathcal{R}$ such that $P_2 \cap Q_1 = \emptyset$, $Q_2 \cap P_1 = \emptyset$, but $P_2 \cap P_1 \neq \emptyset$, $Q_2 \cap Q_1 \neq \emptyset$ and $P_2 \cap Q_2 \neq \emptyset$. If this were not the case, that is, if there were a member S in $\mathcal{R}$ which intersected P_0 but neither P_1 nor Q_1 we must join the centers of P_0, P_1, S, Q_1, and P_0 in that order, obtaining a quadrilateral. Similarly, P_2 and Q_2 must intersect or we obtain a pentagon. Finally, each of the remaining members of $\mathcal{R}$ must intersect P_0, P_1, and Q_1 .

Let us continue by defining P_0 to be a particular member of $\mathcal{R}$. We will denote the perpendicular projection of a member R of $\mathcal{R}$ on the i^{th} coordinate axis by $R^i, i=1,\dots,n$. Let P_0 be any member of $\mathcal{R}$ such that the righthand endpoint of P_0^1 is minimal among all righthand endpoints of the intervals R^1 , for all R in $\mathcal{R}$. Observe that two members P and Q in $\mathcal{R}$ intersect if and only if we have $P^i \cap Q^i \neq \emptyset$ for all $i=1,\dots,n$. But $P_2 \cap P_0 \neq \emptyset$ and $Q_2 \cap P_0 \neq \emptyset$ implies $Q_2^1 \cap P_0^1 \neq \emptyset$ and $P_2^1 \cap P_0^1 \neq \emptyset$. Hence, we may assume, by symmetry, that $P_2^1 \cap P_1^1 \neq \emptyset$ and $P_2^1 \cap Q_1^1 \neq \emptyset$. But then $P_2 \cap Q_1 = \emptyset$ implies that $P_2^i \cap Q_1^i = \emptyset$ for some $i > 1$; and, by symmetry, we may assume that $i = 2$.

Moreover, $P_2^2 \cap Q_2^2 \neq \emptyset$ and $Q_2^2 \cap Q_1^2 \neq \emptyset$. But P_2^2 and Q_1^2 are separated by an (open) interval I_1. Hence, Q_2^2 contains I_1, since Q_2^2 is connected. In fact, if R belongs to $\mathcal{R}$, $R^2 \cap P_2^2 \neq \emptyset$ and $R^2 \cap Q_1^2 \neq \emptyset$ then R contains I_1; so, $R^2 \cap Q_2^2 \neq \emptyset$. Further, if $R^1 \cap P_0^1 \neq \emptyset$, then $R^1 \cap Q_2^1 \neq 0$, since both R^1 and Q_2^1 contain the righthand endpoint of P_0^1.

Thus, we have shown that, if we are in the plane, any R in $\mathcal{R}$ intersecting P_0, P_2 and Q_1 must intersect Q_2 as well, contradicting the requirement that exactly two rectangles of $\mathcal{R}$ do not intersect Q_2.

In $n > 2$ dimensions, we suppose that $\mathcal{R}$ has at least $2n + 4$ members, the first five of which are P_0, P_1, Q_1, P_2, and Q_2 as above. We then iterate the above argument, obtaining a list of pairs of members of $\mathcal{R}$: P_3, Q_3; P_4, Q_4; ... ;P_{n+1}, Q_{n+1}, such that each P and Q of a pair intersect each other and all the preceeding members of the list except the preceeding Q or P respectively. Thus, Q_{n+1} intersects all others on the list except P_n. Further, we may assume, by symmetry, that for each $i=2,\ldots,n-1$, P_{i+1}^{i+1} and Q_i^{i+1} are separated on the $(i+1)^{th}$ coordinate axis by an (open) interval I_i. Hence, any member Q_0 of $\mathcal{R}$ not already accounted for on the preceeding list must intersect P_{n+1}. For, P_{n+1} contains the intervals I_i by virtue of P_{n+1} intersecting P_i and Q_{i-1}, $2 \leq i \leq n$; and Q_0 also has this property. Moreover, both P_{n+1}^1 and Q_0^1 contain the righthand endpoint of P_0^1. But this contradicts the requirement that P_{n+1} intersect all but two members of $\mathcal{R}$, completing the proof for parallelepipeds.

In order to show the existence of dual necklaces as claimed in Theorem 2, we proceed as follows: A realization of $2n + 3$ (hypercubic) spheres in S^n , $n \geqq 3$; $P_0, P_1, Q_1, \ldots, P_{n+1}, Q_{n+1}$ is obtained by following the pattern discussed in the previous proof, except that we require P_{n+1} and Q_{n+1} to be disjoint. We need only give their projections on the n coordinate axes. (See Table I, pg. 26a.)

The symbol "*" on an axis means that the projections of spheres not explicitly accounted for on that axis take position "*" . Observe that on the x_1-axis P_0 does not intersect P_1 nor Q_1 ; P_2 does not intersect Q_1 . On the x_n-axis, P_{n+1} does not intersect Q_{n+1} nor Q_n ; P_{n-1} does not intersect Q_n . Finally, on the remaining x_i-axes, P_{i+1} does not intersect Q_{i+2} , or Q_i ; P_{i-1} does not intersect Q_i . However, all other intersections do occur, thus yielding the required realization of $2n + 3$ members.

Similarly, Table II (pg. 26a) gives a realization of $2n + 2$ spheres in the space S^n, $n \geqq 3$.

It is clear that a realization in one space gives realizations in all higher dimensional S^n . Thus, we will be finished when we give realizations of size six and seven in S^2 and S^3 , respectively. For the latter, we take spheres with centers at (-1,8,8), (10,2,8), (3,5,3), (6,8,1), (12,6,8), (6,12,10), and (6,8,12) . For the former, take spheres with centers at (2,9), (5,12), (8,14), (15,8), (12,5), (8,2) . In each case let the spheres have radii equal to four. q.e.d.

TABLE I

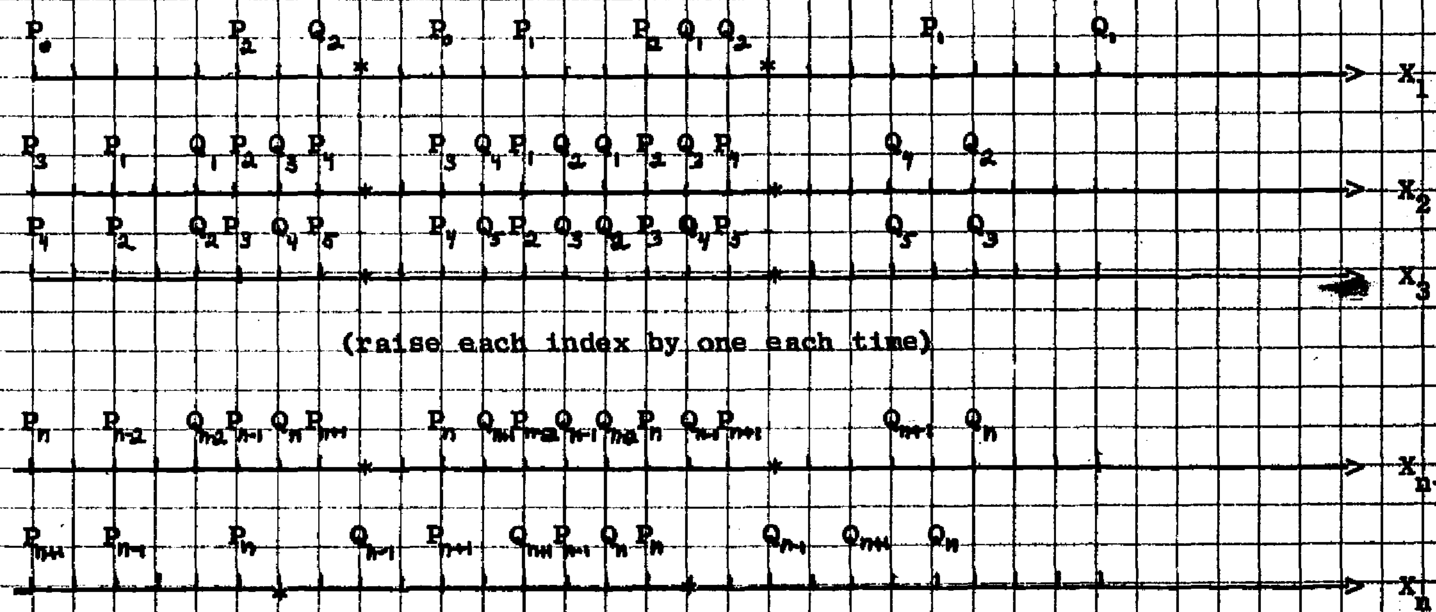
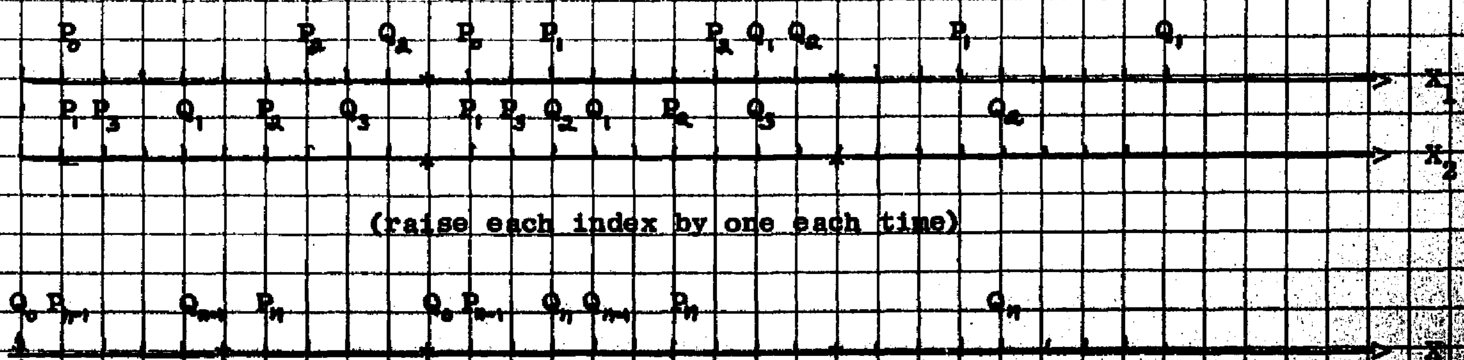


TABLE II



<u>Remark</u>: The dual necklace theorem is, a priori, a special case of the following "cluster" theorem (Th. 0.3,[6]) of Sanders: "Given a collection of spheres in [a geodesic metric space] with at least one circuit on the sphere centers satisfying (i) the interiors of two spheres do not intersect if there is an edge of the circuit joining their centers and (ii) two spheres do meet if there is no edge of the circuit joining their centers, then all circuits satisfying (i) and (ii) and only those are maximal on the sphere centers." Notice that here one is not led directly to the maxima on the sphere center by intersections and non-intersections as with the necklace theorem. This will become clear if we consider the following <u>realizations of the "dual cluster theorem," above, of k spheres in S^2, for all $k \geq 3$</u> :

For even $k \geqq 4$, let the sphere centers be at the points $P_1(0,0)$, $Q_1(2,1)$, $Q_2(2,-1)$, $Q_3(4,1)$, $Q_4(4,-1)$, ... , $Q_{k-2}(k-2,-1)$, $P_2(k,0)$, except that we substitute the points $(\frac{k}{2} - \frac{1}{4}, -\frac{3}{4})$ and $(\frac{k}{2} + \frac{1}{4}, \frac{3}{4})$ for the points $(\frac{k}{2}, -1)$ and $(\frac{k}{2}, 1)$ whenever the latter should occur. Take the sphere radii so that the spheres are all tangent to the line $x = \frac{k}{22}$. For odd k , delete the sphere at center P_2 and substitute $(\frac{k}{2} + \frac{1}{4}, -\frac{3}{4})$ for $(\frac{k}{2} - \frac{1}{4}, -\frac{3}{4})$. One can find the required circuit (polygon) by observing that these distributions are tracklike. (cf. pg. 5) Observe that none of the spheres in each cluster has radius equal to zero, something which the cluster theorem actually requires, although it does not explicitly say so.

Remark: The argument given above works equally well for the spheres of a 2n-gon plane, that is, dual necklaces of parallel, centrally symmetric 2n-gons. Two of these spheres will intersect if and only if the connected regions formed by extending both pairs of corresponding parallel sides intersect. Therefore, we need only set up a pencil of n axes in the plane, that is, lines which are perpendicular to the sides of the 2n-gon spheres of the space. Then, in the above proof, use the intersections of the strip-like regions with the relevant perpendicular axis, instead of the projections on these n-dimensional axes in R^n, and proceed as before.

Thus we have shown that dual necklaces of k parallel, centrally symmetric 2n-gons do not exist for $k > 2n + 3$. However, Warren Becker has shown that there exist realizations of the dual necklace theorem in E^2 for all $k \geqq 3$ [8].

Remark: Realizations can be found in S^2 for all $n = 4,5,6,\ldots$ for the following "necklace theorem" of D. Sanders: A polygon is shortest of its covertex class if a (closed) sphere may be centered on each vertex such that each sphere touches precisely two others, namely, those centered on the two adjacent vertices as given by the polygon. (cf. [7]). If n is even, $n \geqq 6$, the collection of S^2 spheres with center at $(0,\pm 1)$, $(1,\pm 2),\ldots,(n/2-1,\pm 2)$ and $(n/2,\pm 1)$ with radius 1/2 provides such a realization. If n is odd, $n \geqq 5$, substitute one sphere with center at $(0,0)$ and radius 1 for those at $(0,\pm 1)$.

Section 4. S^n and the Four Point Condition

Proof of Theorem 3: Let D denote of tracklike distribution of m points, $m \geqq 4$, in S^2; that is, we assume that for P_0 in D, we have that P_0Q is 1-like for all Q in D, with the possible exception of a single point Q_0 in D depending on P_0. Let us label the points of D as follows: Let $P(p_1, p_2)$ and $Q(q_1, q_2)$ belong to D. If $p_1 < q_1$, then P has a lower index than Q; if $p_1 = q_1$, but $p_2 < q_2$, then P has a lower index than Q. It is clear that this will suffice to label all the points of D, unambiguously, $D = \{P_1, P_2, \ldots, P_m\}$.

Let $P_{i_1}, P_{i_2}, P_{i_3}$, and P_{i_4} denote four distinct points of D, labeled so that $i_1 < i_2 < i_3 < i_4$; for convenience, we shall let $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, and $i_4 = 4$. Now, consider all possible segments determined by these four points, the segments $P_{i_j}P_{i_k}$. We first assert that only the segments P_1P_2, P_2P_3, P_3P_4, out of all these may not be 1-like; that is, all the others must be 1-like. For, if P_1P_3 is not 1-like, since D is a tracklike distribution in S^2, it follows that P_1P_2 and P_2P_3 must be 1-like. Then, by our labeling procedure and (2.7), $P_1 \in C_1^+(P_1) \supseteq C_1^+(P_2) \supseteq C_1^+(P_3)$. Hence, $P_1 \in C_1^+(P_3)$ and P_1P_3 is 1-like -- contradiction. Similarly, neither P_1P_4 nor P_2P_4 is not 1-like.

Therefore, we only need consider the following three cases:

Case I: Each of the segments P_1P_2, P_2P_3, P_3P_4 are 1-like. Then by our labeling procedure, $C_1^-(P_1) \subseteq C_1^-(P_2) \subseteq C_1^-(P_3) \subseteq C_1^-(P_4)$. Hence, the points P_1, P_2, P_3, and P_4 are s-cogeodesic and are labeled in their geodesic order. Such points, thus labeled, obviously satisfy the four point condition since collinear points in E^2 labeled in their linear order do so.

<u>Case II</u>: Exactly one of (i) P_1P_2, (ii) P_2P_3, (iii) P_3P_4 not 1-like.

(i) In this case each of the segments P_1P_3 , P_3P_4 , P_1P_4 , P_2P_3 , P_2P_4 are 1-like. Our labeling procedure implies $C_1^-(P_1) \leq C_1^-(P_2) \leq C_1^-(P_4)$ and $C_1^-(P_2) \leq C_1^-(P_3) \leq C_1^-(P_4)$. Hence, by (2.7),

$s(P_1,P_3) + s(P_3,P_4) = s(P_1,P_4)$ and $s(P_2,P_3) + s(P_3,P_4) = s(P_2,P_4)$.

By the triangle inequality in S^2 , $s(P_1,P_2) \leqq s(P_1,P_3) + s(P_3,P_2)$.

Hence, $s(P_1,P_2) + s(P_3,P_4) = s(P_1,P_2) + (s(P_2,P_4) - s(P_2,P_3)) = \ldots$

$\ldots = (s(P_1,P_2) - s(P_2,P_3)) + s(P_2,P_4) \leqq s(P_1,P_3) + s(P_2,P_4) = \ldots$

$\ldots = (s(P_1,P_2) + s(P_3,P_4)) + (s(P_2,P_4) - s(P_3,P_4)) = s(P_1,P_4) + s(P_2,P_3)$,

as required.

(ii) If only P_2P_3 is not 1-like, we again obtain $C_1^-(P_1) \leq C_1^-(P_3) \leq C_1^-(P_4)$, and $C_1^-(P_1) \leq C_1^-(P_2) \leq C_1^-(P_4)$. Hence, by (2.7),

$s(P_1,P_3) + s(P_3,P_4) = s(P_1,P_4)$ and $s(P_1,P_2) + s(P_2,P_4) = s(P_1,P_4)$.

Moreover, if $P_i(p_1^{(i)},p_2^{(i)})$, our labeling yields $p_1^{(1)} < p_1^{(2)} \leqq p_1^{(3)} < p_1^{(4)}$.

Hence, $s(P_1,P_2) \leqq s(P_1,P_3)$ and $s(P_2,P_4) \geqq s(P_3,P_4)$. So,

$$s(P_1,P_2) + s(P_3,P_4) \leqq s(P_1,P_3) + s(P_2,P_4) \leqq \ldots$$

$\ldots \leqq s(P_1,P_3) + (s(P_2,P_3) + s(P_3,P_4)) = (s(P_1,P_3) + s(P_3,P_4)) + s(P_2,P_3) = \ldots$

$\ldots = s(P_1,P_4) + s(P_2,P_3)$.

(iii) If only P_3P_4 is not 1-like, then $P_1P_2, P_2P_4, P_1P_4, P_2P_3$, and P_1P_3, are 1-like. Thus, $C_1^-(P_1) \leq C_1^-(P_2) \leq C_1^-(P_4)$ and $C_1^-(P_1) \leq C_1^-(P_2) \leq C_1^-(P_3)$. Hence, $s(P_1,P_2) + s(P_2,P_4) = s(P_1,P_4)$ and $s(P_1,P_2) + s(P_2,P_3) = s(P_1,P_3)$. By the triangle inequality, $s(P_2,P_4) \leqq s(P_2,P_3) + s(P_3,P_4)$. So,

$$s(P_1,P_2) + s(P_3,P_4) = (s(P_1,P_3) - s(P_2,P_4)) + s(P_3,P_4) = \dots$$
$$\dots = s(P_1,P_3) + (s(P_3,P_4) - s(P_2,P_3)) \leqq s(P_1,P_3) + s(P_2,P_4) = \dots$$
$$\dots = (s(P_1,P_3) - s(P_1,P_2)) + (s(P_2,P_4) + s(P_1,P_2)) = s(P_1,P_4) + s(P_2,P_3) \ .$$

<u>Case III</u>: Finally, assume that P_1P_2 and P_3P_4 are not 1-like, but that all the rest are. We first claim that P_1,P_2,P_3,P_4 are the vertices of a convex quadrilateral. This follows directly from Lemma 2. Moreover, it is clear that P_1P_2 and P_3P_4 are sides of this quadrilateral, while either P_1P_3 and P_2P_4 , or, P_1P_4 and P_2P_3 are its diagonals, which, of course, intersect and must have opposite 1-orientations. Thus, by Lemma 1 (b), there is an arc inversion which interchanges these pairs of segments but does <u>not</u> change the polygonal s-length. This just means:

$$s(P_1,P_2) + s(P_3,P_4) \leqq s(P_1,P_3) + s(P_2,P_4) = s(P_1,P_4) + s(P_2,P_3) \ .$$

Therefore, we have proved Theorem 3 in the special case of S^2 . We shall use this result to prove the Theorem in its full generality in Section 8, though (2.5) extends the above to a proof for S^n, $n \geqq 2$.

<u>Remark</u>: In 1964 Lerman obtained the result (unpublished) that if a set of k non-collinear points in the euclidean plane satisfies the four point condition, then $k \leqq 8$. The collinear case in E^2 has its counterpart in s-cogeodesic sets in S^2 , which includes collinear point sets as a proper subclass. However, we can give a procedure for the construction of the still more general tracklike distributions, as follows:

We effect this construction by specifying "pairs" of points in $S^2, \{P,Q\}$, such that $p_2 - q_2 \geqq |p_1 - q_1|$. Now, given such a pair

$R_1 = \{P_1, Q_1\}$, we choose a second pair R_2 in the set $C_1^+(P_1) \cap C_1^+(Q_1)$, which may be described as that region of the plane containing the positive unbounded x-axis, bounded by the half-lines L_Q and L_P with slopes +1 and -1 , respectively, through the points Q and P , respectively. Then choose a third pair $R_3 = \{P_3, Q_3\}$ in $C_1^+(P_2) \cap C_1^+(Q_2)$. After k pairs have been thus selected, choose any pair $R_{k+1} = \{P_{k+1}, Q_{k+1}\}$ in $C_1^+(P_k) \cap C_1^+(Q_k)$. By induction on k , it follows that $R_{k+1} \subseteq \bigcap_{i=1}^{k} (C_1^+(P_i) \cap C_1^+(Q_2))$; and so the distribution is tracklike in S^2 . If both points of every pair are identical, then we have an s-cogeodesic distribution. An example (*) of a distribution constructed in this manner was given on page 5. Observe that this construction also contains a way to determine whether a particular distribution is tracklike in S^2 ; namely, order the points with respect to one of the coordinates and check the direction numbers determined by consecutive pairs of points -- something which must be done, at any event, in order to calculate s-distances between the pairs of points.

Section 5. Proof of Theorem 4

Let m denote a type I metric and let π denote a polygon that intersects itself; that is, there is a closed edge $Cl(P_1P_2)$ of π and an open edge P_3P_4 of π that intersect each other, where $\pi = [P_1P_2 \ldots P_3P_4 \ldots]$.

There are the following cases:

1. The intersection of P_3P_4 and $Cl(P_1P_2)$ is a point.

2. The intersection of P_3P_4 and $Cl(P_1P_2)$ is a line segment, and P_1P_2 and P_3P_4 induce the same orientation of this segment.

3. The same as "2", but P_1P_2 and P_3P_4 induce opposite orientations of the line segment of intersection.

In cases 1 and 2 the arc-inversion $[P_1(P_2 \ldots P_3)P_4 \ldots]$ yields a polygon π' strictly e-shorter than π . But we can only say that $m[\pi'] \leqq m[\pi]$ in case 1, since, as noted below, type I metrics are also of type II. We shall merely observe here that case 3 can be reduced (cf. [1]) to either case 1 or case 2.

The preceding shows that a polygon which intersects itself cannot be e-minimal in its covertex class (although we have already seen that such a polygon (in the case of S^2) can be m-minimal).

Next, it is shown in [1] that if a polygon π on the vertex set D does not belong to the subclass $\Psi(D)$ of $\psi(D)$, then π must intersect itself. Hence, there is an arc-inversion of the type described above yielding a polygon π' in $\psi(D)$ such that $e[\pi'] < e[\pi]$ and $m[\pi'] \leqq m[\pi]$.

Now, some polygon π_0 must be m-minimal in $\psi(D)$, since D is a finite set. If π_0 belongs to $\Psi(D)$, we are done. If not, by the above, there is an arc-inversion producing π_1 in $\psi(D)$ such that $e[\pi_0] > e[\pi_1]$ and $m[\pi_0] \geqq m[\pi_1]$. If π_1 belongs to $\Psi(D)$, we are done. If not, we continue in this way, and after a finite number of steps, we obtain two sequences

$$e[\pi_0] > e[\pi_1] >\ldots > e[\pi_k] \quad \text{and}$$

$$m[\pi_0] \geqq m[\pi_1] \geqq\ldots \geqq m[\pi_k]$$

such that π_k belongs to $\Psi(D)$, since there are only finitely many polygons in $\psi(D)$. But π_k is m-minimal, since $m[\pi_k] \leqq m[\pi_0]$, and π_0 was assumed to be m-minimal in $\psi(D)$. qed.

<u>Proof of Theorem 5</u>: Let m denote a metric of types II and III. Just as the preceding proof made use of the way that the euclidean case was handled in [1], the present argument will depend upon the proofs given in [2] and [3], where the following two cases were and shall be considered:

<u>Case 1</u>. Suppose that no three points of D are collinear, and, further, that π is a polygon in $\psi(D)$ which is not a member of the class specified in (a) or (b), as the number of points of D is odd or even, respectively. It is shown in [2] and [3] (with considerable effort) in this case that there exist edges P_1P_2 and P_3P_4 of π, where all the P_i are distinct, such that $P_1P_2P_3P_4$ is a (non-degenerate) convex quadrilateral. We may write

$$\pi = [P_1P_2\ldots P_3P_4\ldots] .$$

The arc-inversion $[P_1(P_2\ldots P_3)P_4\ldots]$ produces a polygon π' in $\psi(D)$ such that $e[\pi] < e[\pi']$ and $m[\pi] \leqq m[\pi']$, since m is a type II metric.

If we begin with an m-maximal polygon π_0 in $\psi(D)$ such that π_0 does not belong to the appropriate class given in the theorem, we derive, as in the last proof, two sequences:

$$e[\pi_0] < e[\pi_1] < \ldots < e[\pi_k] \qquad \text{and}$$

$$m[\pi_0] \leqq m[\pi_1] \leqq \ldots \leqq m[\pi_k]$$

by means of the arc-inversions described above, where π_k is a member of the correct class (but not necessarily e-maximal). Since π_0 was assumed to be m-maximal to begin with, we have $m[\pi_0] = m[\pi_k]$. This completes case 1.

Case 2.: B has support lines passing through at least three points of D . Let X be a point in the interior of the convex hull of D and $B(t)$ $(0 \leqq t \leqq 1)$ be a family of strongly convex curves circumscribing B and converging to B as t approaches 1 . Let $P_i(t)$ be the intersection of $B(t)$ with the ray emanating from X and passing through P_i in D , and let $D(t) = \{P_i(t)\}$. Then, for each t $(0 \leqq t \leqq 1)$ case 1 implies that there exists an m-maximal polygon of $\psi(D(t))$ among those of the appropriate class. Now, for t sufficiently close to 1, a polygon on the vertex set $D(t)$ is arbitrarily close to the corresponding polygon on D , since m is assumed to be of type III. Thus, there always exists an m-maximal polygon of $\psi(D)$ in the appropriate subclass of polygons. qed.

Remark: In the convex maximal odd case, e-maximal polygons in $\psi(D)$ are uniquely so, if no three points of D are collinear. That this is not true for general metrics of types II and III is shown for the following set of five points in S^2 : Let $\{P_1(-1,2)$, $P_2(1,2)$, P_3([illegible]),

$P_4(0,-1)$, $P_5(-1,0)\} = D$. Then both $[P_1P_4P_2P_5P_3]$ and $[P_1P_4P_2P_3P_5]$ are s-maximal in $\psi(D)$. In the convex maximal even case this uniqueness condition is not so; if the points of D are evenly distributed on a circle, then all the n polygons specified by part (b) of the theorem have the same e-length, since the metric e is rotation invariant. On the other hand, in case 2 of the convex odd maximal case, the single polygon specified in (a) is not necessarily the only e-longest polygon, even if the points of D are not all collinear. For example, if $D = \{P_1(0,0), P_2(1,0), P_3(2,0), P_4(3,0), P_5(4,0), P_6(5,0), P_7(0,1)\}$, then both $[P_1P_5P_2P_6P_3P_7P_4]$ and $[P_1P_6P_3P_7P_4P_2P_5]$ are e-maximal in $\psi(D)$.

Proof of Corollary 1: For the m-minimality and, if no three points of D are collinear, the m-maximality assertion the proof is almost immediate. For, if in these cases π_0 is an appropriate m-minimal or m-maximal polygon in $\psi(D)$ specified by the relevant theorem, there exist the required types of sequences

$$\pi_1, \pi_1^1, \pi_1^2, \ldots, \pi_1^{n_1}, \pi_0 \qquad \text{and}$$

$$\pi_2, \pi_2^1, \pi_2^2, \ldots, \pi_2^{n_2}, \pi_0$$

of m-minimal or m-maximal polygons of $\psi(D)$ derived by consecutive arc-inversions, as in the theorems. Thus,

$$\pi_1, \pi_1^1, \pi_1^2, \ldots, \pi_1^{n_1}, \pi_0, \pi_2^{n_2}, \ldots, \pi_2^2, \pi_2^1, \pi_2$$

is the required sequence from π_1 to π_2 , since each of the arc-inversions used can obviously be reversed.

Suppose now that at least three points of D are collinear. Let the set D' in case 2 of the proof of Theorem 5 (lying on a strongly convex curve) approximate D so closely that the difference in the m-lengths between the polygons in $\psi(D)$ and the corresponding polygons in $\psi(D')$ is no more than $r/8$, where r is the (non-negative) difference between the m-lengths of the m-maximal and next-to m-maximal polygons in $\psi(D)$. If $r = 0$, then the proof given above applies, since in that case all the polygons in $\psi(D)$ have the same m-length. Hence, suppose that $r > 0$. Let π_0^1 and π_1^1 be the polygons of $\psi(D')$ corresponding to the polygons π_0 and π_1 of $\psi(D)$ as in the last paragraph. Since D' has no three points collinear, there exists the right kind of sequence

$$\pi_1^1, \pi_2^1, \ldots, \pi_k^1, \pi_0^1 \qquad (1)$$

of polygons in $\psi(D')$ from π_1^1 to π_0^1. Let the sequence

$$\pi_1, \pi_2, \ldots, \pi_k, \pi_0 \qquad (2)$$

be the term-wise corresponding polygons in $\psi(D)$. Since π_1 and π_0 are both m-maximal in $\psi(D)$ by assumption, π_1^1 and π_0^1 cannot differ in m-length by more than $r/4$. But, by assumption, each of the pairs π_1^1 and π_1, and, π_2^1 and π_2, cannot differ by more than $r/8$. Hence, π_2 and π_1 cannot differ by more than $r/4$, which forces $m[\pi_2] = m[\pi_1]$, by the definition of "r". Repeating this argument k times if necessary, we find that sequence (2) is of the required type; that is, all the polygons in (2) are m-maximal in $\psi(D)$. We may now apply the argument in the first paragraph to complete the proof. qed.

Remark: Corollary 1 is false if D has an even number of points. For, let A, B, C, E, F, and G be the consecutive vertices of a regular hexagon in the plane. As noted above, every polygon on this vertex set specified by Theorem 5(b) is e-maximal, and these are the only ones. In particular, $\pi = [ACGEBF]$ is e-maximal, but any arc-inversion on π yields a polygon of $\psi(D)$ with an edge joining adjacent vertices of the regular hexagon. No such polygon is e-maximal in $\psi(D)$.

Section 6. Remarks on Metric Properties.

It is clear that a metric is of type III if the topology it induces on the plane is no finer than that induced by the euclidean metric. Moreover, while an easy application of the triangle inequality shows that type I metrics are also of type II, the converse is false.

Example: Let m denote any type I metric (for instance "e"), and let "t" denote the trivial metric. If we define $d = m + t$, then d is also a metric, since any finite sum of metrics on the same set is a metric. Let P, Q and S be three distinct points in R^2 such that $m(P,Q) + m(Q,S) = m(P,S)$. Then,

$d(P,Q) + d(Q,S) = m(P,S) + 2 > m(P,S) + 1 = d(P,S)$. Hence, d is not of type I. However, d is of type II. For, if A, B, C, and D are any four distinct points in R^2 such that

$$m(A,C) + m(B,D) \geqq m(A,B) + m(C,D) \text{ , then}$$

$$d(A,C) + d(B,D) = m(A,C) + m(B,D) + 2 \geqq m(A,B) + m(C,D) + 2 \geqslant d(A,B) + d(C,D) .$$

In any case, all norms on R^2 are metrics of types I, II, and III. It is well known that all norms on R^2 induce the same topology, and so are of type III Let "$\|\ \|$" be a norm on R^2 and let $A,B \in R^2$. Then, for all t in R we have, of course, $\|t(A + B)\| = |t| \cdot \|A + B\|$. Now, all points C in R^2 such that $e(A,C) + e(C,B) = e(A,B)$ must have the form $C = tA + (1-t)B$ where $0 \leqq t \leqq 1$. That is, they must lie on a straight line segment whose endpoints are A and B . Thus, letting $d(A,B) = \|A - B\|$, we may write

$$d(A,C) + d(C,B) = \|A - [(1-t)B + tA]\| + \|B - [tA - (1-t)B]\| = \ldots$$
$$= |1-t| \cdot \|A-B\| + |t| \cdot \|A-B\| = (1-t[(1-t)B + t \cdot \|A-B\| = \|A-B\| = d(A,B) .$$

Thus, all norms are also metrics of type II. The next example shows that a type I metric does not have to induce the euclidean topology on R^2, and therefore does not have to be a norm; nor does it have to be of type III.

<u>Example</u>: Let D_1 denote the union of the open upper half-plane and the x-axis, and let D_2 denote the lower open half-plane. We define $m : R^2 \times R^2 \longrightarrow R$ by $m(P,Q) = e(P,Q)$ if P and Q are both in D_i, $i = 1,2$; $m(P,Q) = e(P,Q) + 1$ if $P \in D_i$, $Q \in D_j$, $i \neq j$.

Clearly, $m(P,Q) = m(Q,P)$, and $m(P,Q) = 0$ if and only if $P = Q$. For the triangle inequality, suppose that we have $e(P,Q) + e(Q,S) \geqq e(P,S)$. We wish to derive the corresponding statement for m . But we can add at most 1 to $e(P,S)$ to obtain the value of $m(P,S)$, and we do this only if $P \in D_i$; $S \in D_j$, $i \neq j$. But Q belongs to exactly one of the D_i . So, in other words, if we add 1 to the righthand side of the e-inequality, then we add exactly one to the lefthand side, to obtain the expression for m. Since both the regions D_i are convex, this not only shows that the triangle inequality is satisfied for m , but also that m is a type I metric.

Moreover, since both D_1 and D_2 are open sets in this space, the topology induced by m is disconnected, and therefore not the euclidean topology: in fact, it is clearly a finer topology.

Finally, we observe that type III metrics are not necessarily of type I, but <u>if a metric is of types II and III, then it is of type I.</u> For, let "m" denote the metric $m(P,Q) = \min\{e(P,Q),1\}$. The topology induced by this metric is known to be the same as that induced by e . However, if P,Q, and S are any points in R^2 such that $e(P,Q) > 1$,

$e(Q,S) > 1$, and $e(P,S) > 1$, then $m(P,Q) = m(Q,S) = m(P,S) = 1$ and $m(P,Q) + m(Q,S) = 2 > m(P,S)$.

Suppose that m is both a type II and type III metric. Let A, B, and C denote distinct collinear points in R^2 on some line L , with B between A and C . Now, given an arbitrary $\epsilon > 0$, there exists a point $X \in R^2$ such that $m(X,B) \leq \epsilon$ and X does not belong to L . Then AXCB is a non-degenerate convex quadrilateral. Hence, $2(m(A,C) + \epsilon/2) \geqq (m(A,X) + m(C,B)) + (m(A,B) + m(X,C))$. By the triangle inequality for m we have:

$$m(A,X) + m(X,C) \geqq m(A,C) \quad \text{and} \quad m(A,B) + m(B,C) \geqq m(A,C) .$$

Thus, $2(m(A,C) + \epsilon/2) \geqq [m(A,B) + m(B,C)] + [m(A,X) + m(X,C)] \geqq 2m(A,C)$.

Since, for all X , chosen as above each of the bracketed expressions is $\geqq m(A,C)$; and since ϵ is arbitrarily small, we must have $m(A,B) + m(B,C) = m(A,C)$, as required.

<u>Remark</u>: We next consider the following question: How large is the class $\gamma(D)$ of polygons on a vertex set D such that the vertices on H(D) are connected in cyclic order? We claim that <u>if there are $m + n + 3$ $(m,n = 0, 1,\ldots)$ points in D and $n+3$ points in H(D)</u> , <u>then $\gamma(D)$ has $(m + n + 2)!/(n + 2)!$ members</u>.

<u>Proof:</u> Let the points on H(D) be $P_1, P_2, \ldots, P_{n+2}, P_{n+3}$, occurring in the given cyclic order (we suppose that the points of D are not collinear), and let $Q_1, Q_2, \ldots, Q_{m-1}, Q_m$ be the remaining points of D , in some arbitrary order. Now, the Q_i can be permuted in m! ways; let $Q_{i_1}, Q_{i_2}, \ldots, Q_{i_m}$ be one such permutation. Consider the following scheme:

(*) $$P_1 \text{———} Q_{i_1} \text{———} Q_{i_2} \text{———} \cdots \text{———} Q_{i_m} \text{———}$$

To obtain a polygon in the class $\gamma(D)$ we must insert the remaining P_i in some of the $m + 1$ spaces provided in (*), in such a way that their cyclic order is preserved. We may do this in a number of steps, as follows: First choose k arbitrary spaces out of the $m + 1$ possible ones. This may be done in a total of $\binom{m+1}{k}$ ways for each $k = 1,2,\ldots m + 1$. Further, for each k in these mutually exclusive and exhaustive cases, we may partition the sequence of the P_i, keeping their given order fixed, by placing $k - 1$ partition "slashes" in the $n + 1$ possible positions, as shown below, to obtain k groups of P_i as above:

$$P_2 \quad \Big| P_3 \quad \Big| P_4 \quad \Big| \cdots \quad \Big| P_{n+2} \ \Big| P_{n+3}$$

$n + 1$ positions

For each k, we can do this in $\binom{n+1}{k-1}$ different ways. Hence, each of the $m + 1$ mutually exclusive cases corresponds to $\binom{n+1}{k-1} \cdot \binom{m+1}{k}$. Letting $n_1 = n + 2$ and employing the reduction formula $$\sum_{k=1}^{r_2} \binom{r_2+1}{k} \cdot \binom{r_1-1}{k-1} = \sum_{k=1}^{r_1} \binom{r_2+1}{k} \cdot \binom{r_1-1}{k-1} = \binom{r_1+r_2}{r_1},$$

(cf. [9]) we obtain

$$m! \left[\sum_{k=1}^{m+1} \binom{m+1}{k} \cdot \binom{n_1-1}{k-1} \right] = m! \binom{m+n_1}{n_1} = (m + n + 2)!/(n+2)! \ .$$

qed.

We observe that, although we have discovered this result independently, Fred Supnick was the first to prove it (c. 1966 -- unpublished) by a somewhat different method, in connection with Theorem II of [1].

Section 7. m-Geodesic Distributions.

<u>Proof of Corollary 2</u>: By Theorem 4, the distinction between the unique polygon π_0 in the class $\gamma(H(D))$ and any minimal polygon π of the subclass $\gamma(D)$ of $\psi(D)$ may be described thus: vertices joined by an edge of π_0 are joined by a chain of edges in π. By the triangle inequality, the m-length of an edge in π_0 is less than or equal to the sum of the m-lengths of the edges in the corresponding chain in π. Repeating this for each edge of π_0, we are done. qed.

<u>Remarks</u>: It is clear that the m-lengths of polygons in $\psi(D)$ will attain the lower bound of Corollary 2 if and only if there exists a polygon in $\psi(D)$ which can be represented as a sequence of m-geodesic chains joining consecutive vertices of the set $H(D)$ in their cyclic order (where we assume henceforth that D is not a collinear distribution). If m and d denote two different type I metrics, then a vertex set D may be an m but not a d-chain distribution, as is seen by the example on page 8, letting $m = s$ and $d = e$. Now, D is an m-chain distribution only if every point of $D-H(D)$ is m-between two adjacent points of $H(D)$. For, $m(P_1,P_n) = \sum_{i=1}^{n-1} m(P_i,P_{i+1})$ implies that for $i = 2,\ldots,n$ we have $m(P_1,P_n) = m(P_1,P_i) + m(P_i,P_n)$, by the triangle inequality. But this condition is not sufficient:

<u>Example</u>: Let $D = \{P_1,P_2,P_3,Q_1,Q_2\}$ where $P_1(0,10),P_2(0,0),P_3(10,0)$, $Q_1(1,5),Q_2(2,5)$. Then $H(D) = \{P_1,P_2,P_3\}$ and if $\pi = [P_1P_2P_3]$, then $s[\pi] = 30$. But if $\pi' = [P_1Q_2Q_1P_2P_3]$, then it can be verified that π' is s-minimal in $\psi(D)$ and $s[\pi'] = 31$. But, [illegible] $P_3)$.

Nevertheless, we can represent the sets $B(P,Q)$ in some spaces in a simple graphic manner, as in S^2 , and therefore, check in some cases, whether our tight lower bound, which is easily calculated, is attained.

To give sufficient conditions in this vein, let us confine our remarks for the moment to the metric s . If we are given two points P and Q which are adjacent in H(D) , where D is some vertex set, we will call the set of points of $B(P,Q) \cap (D-H(D))$ the between-cell of D determined by P and Q . Now, observe that a pair of distinct points M and N belonging to the same between-cell determined by P and Q can belong to the same s-geodesic chain connecting P to Q only if PQ and MN are both i-like, for $i = 1,2$. In this case, we will say that MN is consistent with B(P,Q) , and simply inconsistent if it is inconsistent with every between-cell. In fact, inconsistency with respect to one between-cell implies inconsistency with respect to every between-cell, by Lemma 1. Hence, a sufficient (but not necessary) condition that a vertex set D has an s-chain decomposition is that (1) every point of D-H(D) belong to some between-cell of D , and, (2) there are no inconsistent segments. For, we may then partition the points of D-H(D) so that two points belong to the same class only if they belong to the same between-cell. The second condition enables us to order the points in each class into s-geodesic chains. The non-necessity of this condition is clear by the example on page 8 .

Proof of Theorem 6: Again we shall prove the theorem here in the special case of S^2 , extending the proof later.

Sufficiency. Any polygon of $\psi(D)$ which can be expressed as any number of s-geodesic chains connecting points of H(D) in cyclic order attains the lower bound of Corollary 2, since the vertices of $H(D)$ cannot be thus connected by shorter chains of edges.

Necessity. If D is collinear the theorem is trivial. If not, then there exist four distinct support lines of D, touching D, such that two have slopes 1 and two have slopes -1. For, we may rotate D through a positive angle of 45°, pass lines through the points of D with maximum and minimum x and y coordinates, respectively, and then rotate the entire plane through a negative angle of 45°. These four lines enclose D in a rectangle $ABCE$, whose vertices are their points of intersection. For definiteness, let us suppose that the "left-most" vertex of $ABCE$ is A and that the others are taken clockwise.

By properties of s-betweenness (2.9), we may assume that, for this part of the proof, $D = H(D)$. Further, let us denote the unique polygon in the class $\gamma(D)$ by π. We will show that π has the required representation.

Now, there is a point of D on each side of the rectangle $ABCE$. Let us suppose that $P_1, P_2 \in D$, $P_1 \neq P_2$, $P_1 \in AB$, and $P_2 \in BC$. Then, π consists of two connected chains of edges, one of which lies above the line L through P_1 and P_2; denote this chain by X. (By "above" we mean "entirely in the positive closed half-plane determined by L.") But X also lies below the support lines corresponding to AB and BC. If either $P_1 = B$ or $P_2 = B$, then all of the edges of X lie along the segment P_1P_2 and are therefore 1-like. But if neither equals B, then all of the edges of X lie in the closure of the region bounded by the triangle P_1BP_2. The extension of any such edges which are not

1-like must intersect the open segment $P_1P_2 - \{P_1,P_2\}$. However, all of the edges of X , being edges of π , connect adjacent vertices of D-H(D) . Further, the extensions of these edges are support lines of D , so that they cannot separate the points P_1 and P_2 as above. Moreover, if π is given an arbitrary orientation, then all of the edges of X must have the same 1-orientation. Hence, X is an s-geodesic chain of 1-like edges. In the same way, if $P_3, P_4 \in D$, $P_3 \in CE$, $P_4 \in EA$, $P_1 \neq P_2 \neq P_3 \neq P_4 \neq P_1$, then one can show, as above, that the chain in π connecting P_i to P_{i+1} and lying in the positive closed half-plane determined by P_iP_{i+1} is an s-geodesic chain. Hence, by choosing the least number of points of D lying on ABCE such that each side of ABCE will contain exactly one point, we divide up π into four or fewer s-geodesic chains, as required. This completes the proof for S^2 .

<u>Corollary 4</u>: If D = H(D) and π is the unique polygon in $\Psi(D)$, then π contains no more than two maximal i-like geodesic chains, i = 1,2 .

<u>Proof</u>: Choose the points P_1,P_2,P_3,P_4 in the last proof so that they are the closest point on AB , BC , CE , or EA , to A or C , respectively.

<u>Corollary 5</u>: A sufficient condition that a vertex set D have an s-chain decomposition is that there exist a polygon in $\Psi(D)$ which can be expressed as no more than three consecutive s-geodesic chains.

<u>Proof</u>: If π denotes such a polygon in $\Psi(D)$, then π is clearly s-minimal in $\Psi(D)$, since the two or three vertices in question must be joined with some chains in, essentially, only one cyclic and symmetric

permutation. These are the shortest possible chains joining them.

Next, suppose that we have a polygon consisting of two s-geodesic chains, say, joining a point A to a point B. We claim that A and B belong to H(D). For, if AB is strictly i-like, i = 1 or 2, then every edge in either chain must lie entirely inside the set $C_i^+(A) \cap C_i^-(B)$, or the set $C_i^-(A) \cap C_i^+(B)$: equivalently, all the endpoints of these edges must lie in the first set, or they all lie in the second. Hence, both pairs of lines through A and B with slopes equal to ± 1 are support lines of D. If AB is both 1-like and 2-like, then A and B lie on a line with slope equal to 1 or −1 ; moreover, all of the endpoints lie on the segment AB. This proves our claim and finishes this case.

Suppose now that our polygon consists of three such chains: one from A to B, another from B to C, and a third from C back to A. If these are all i-like for some i = 1,2, we can reduce this case to the first by joining two of these chains to make one new one. Therefore, suppose that we cannot coelesce our chains at any of the three endpoints. The, for example, the path from A to B lies entirely within $C_i^+(A)$ or $C_i^-(A)$; the chain from C to A lies entirely within the set $C_j^+(A)$ or $C_j^-(A)$, where $i \neq j$. Hence, all the edges in these two paths lie in one of the half-planes determined by a line L through A with slope +1 or −1. But all the edges in the chain from B to C lie in this half-plane; that is, all points between B and C lie in this half-plane. Hence, the line L is a support line of D, and $A \in H(D)$. Similarly, B and C belong to H(D). The rest of the argument holds as in Theorem 6.

qed.

Remark: It is not, however, sufficient that $\psi(D)$ merely have a polygon which can be expressed as four consecutive s-geodesic paths. For example, given $A(0,1)$, $B(1,0)$, $C(0,-1)$, $E(-1,0)$, and $F(0,0)$, and letting $D = \{A,B,C,E,F\}$ then the polygon $[AFCBE] = \pi$ can be expressed as the consecutive s-geodesic chains AF, FC ; CB ; BE; EA . But F is not s-between any two adjacent points of $H(D)$. Hence, D does not have an s-chain decomposition.

Proof of Corollary 3: It follows by (2.7) that there is one and only one geodesic chain through the points of X ; likewise for the points of Y . If either of X or Y is empty, then this corollary follows directly from Corollary 4. Then, suppose that neither is empty and, without loss of generality, that both are 1-like. If either of X or Y has exactly one point, Corollary 4 applies -- suppose not. Then the geodesic paths through the points of X and Y have right and left endpoints X_1, Y_1 and X_2, Y_2 respectively. If either of X_1Y_1 or X_2Y_2 is 1-like, it is clear that we have a polygon in $\psi(D)$ which can be expressed as two or three consecutive s-geodesic paths, by (2.7) .

Hence, suppose that neither X_2Y_2 nor X_1Y_1 is 1-like. We claim that X_i and Y_i are adjacent points on $H(D)$, $i = 1, 2$. For, observe that there can be no points of D on the line L through X_i and Y_i other than X_i and Y_i . Because, such a point Q on L would have to belong to exactly one of X or Y , and neither X_iQ nor Y_iQ could be 1-like, by (2.7) . Also, if L were not a support line of D , then there would exist a point P of D in the open left half-plane, if $i = 2$ say, determined by L . Observing that $C_1^+(X_2) \cup C_1^+(Y_2)$ belongs to the right half-plane determined by L , we see that this is impossible since, in this case, we must have at least one of PX_2 or PY_2 1-like. But they must have a positive 1-orientation in either case, which contradicts the fact that X_2 and Y_2 are leftmost points. So, our claim holds.

Now, if the geodesic chains corresponding to X and Y are C_1 and C_2 , and the four chains C_1, X_1Y_1 , C_2 , and Y_2X_2 join X_1 , Y_1 , Y_2 , and X_2 in their cyclic order on H(D) , we are done. If not, an application of Lemma 1(b) shows that, nevertheless, we obtain the same polygonal s-length as if the points were joined cyclically.

qed.

Remark: This corollary is clearly not necessary, but the requirement that both X and Y be i-like is not too restrictive. For, in the example on page , if $X = \{A,F,C\}$ and $Y = \{E,B\}$, then X is only 1-like and Y is only 2-like. As noted, we do not have an s-chain decomposition in this case.

The fact that tracklike distributions in S^2 have s-chain decompositions is seen by letting, in each "pair" (cf. page 31), one member belong to X and the other, if the points of the pair are different, belong to Y . The example on page 8 shows that s-chain decomposed vertex sets are not necessarily tracklike.

Section 8. Extensions to 2n-gon Planes.

As noted in Section 1, the space S^2 is an example of a structure known as a Minkowski plane, which may be defined, following Benson [10], thus: "Let U be the boundary of a two dimensional convex body (in R^2) with center at the origin 0. If $X,Y \in R^2$, let $L_0(X,Y)$ be the line through 0 parallel to $L(X,Y)$, (the line through X and Y), and let P be a point in $L_0(X,Y) \cap U$. Define $m(X,Y) = e(X,Y)/e(0,P)$." Then (R^2,m) is a metric space, in fact, a normed linear space, and U is its unit circle. (R^2,m) is defined to be a Minkowski plane. (cf. [10], 29.3) The converse is also true; any two dimensional normed linear space $(R^2, \| \|)$ is a Minkowski plane. In fact, "Let M_1 and M_2 be two Minkowski planes with unit circles U_1 and U_2 and metrics m_1 and m_2, respectively. M_1 is isometric to M_2 if and only if U_1 is the image of U_2 under a linear transformation of R^2 onto itself." (cf. [10], 29.5) Thus, the most general Minkowski planes which are isometric to S^2 are those whose unit circle (up to translation) is the image of a square; that is, those whose unit circle is a parallelogram. Hence, all the theorems on 2n-gon planes cited in the first Section hold for $n = 2$. For, let us also recall that non-singular affine transformations preserve convex sets (since straight line segments go into straight line segments) interiors, boundaries, and the order of points occurring on simple curves (since they are homeomorphisms.)

Before developing the properties of 2n-gon planes which will enable us to extend the proofs of the relevant theorems for $n > 2$, let us note that not all 2n-gon planes are isometric for $n > 2$, even if we restrict the angles of their unit circles to be less than π. For, let $\underset{\circ}{M}$ denote any space whose unit circle is a regular hexagon. Consider the 6-gon plane whose unit circle is the hexagon $P_1P_2P_3P_4P_5P_6$, where the vertices are given as $P_1(0,4)$, $P_2(3,3)$, $P_3(2,-2)$, $P_4(0,-4)$, $P_5(-3,-3)$, $P_6(-2,2)$. There is no linear transformation mapping the hexagon $P_1P_2P_3P_4P_5P_6$ onto a regular hexagon. For, any proper diagonal of a regular hexagon is parallel to a pair of opposite sides. But the diagonal P_1P_4 of hexagon $P_1P_2P_3P_4P_5P_6$ is not parallel to any pair of sides, and linear transformations take parallel lines into parallel lines.

1) In a 4-gon plane, the points $P_1, P_2, \ldots, P_r$ will be cogeodesic if and only if for all $k \neq j$ the lines $L_0(P_k, P_j)$ intersect the same pair of opposite sides of the 4-gon unit circle (a parallelogram). For, this is what corresponds exactly to the statement that all of the edges P_kP_j are i-like in S^2 (under a suitable linear transformation), which, by (2.7) implies the preceding statement.

2) Recall the definition of "an edge PQ in a 2n-gon plane is i-like" and suppose that we have the points $P_1, P_2, \ldots, P_k$ in some 2n-gon plane M which are i-like for some $i = 1, \ldots, n$. Then these points are cogeodesic in M. For, if we extend the pair of sides corresponding to i, and also another pair of sides of the unit circle U, then the two pairs of sides when extended will meet in a parallelogram with center at 0. Hence, the points $P_1, P_2, \ldots P_k$ may be considered as lying in some 4-gon plane, and we merely apply (1). (On the other hand, if two segments are not both i-like for <u>any</u> $i = 1, \ldots, n$, then the points P_i are not metrically cogeodesic, by a similar argument.)

3) Non-singular affinities of a 2n-gon plane M preserve cogeodesic sets of points. In particular, we may transform M so that 0 remains fixed, but any side of U, the unit circle, is mapped onto the vertical side of the S^2 (square) circle. But then, the i-like segments of M correspond exactly to what we have called 1-like segments with respect to S^2. For, $T(L_0(X,Y)) \cap T(s_i) \neq \emptyset$ if and only if $L_0(X,Y) \cap s_i \neq \emptyset$, where "T" denotes the required transformation and "s_i" denotes the i^{th} side of U.

<u>Proof of Theorem 6 (continued)</u>: Suppose that M denotes a 2n-gon plane and that D is a vertex set in R^2. The sufficiency of this theorem is again clear, by Theorem 4. We shall now consider the necessity. First observe that <u>the necessity condition is not artificially strong</u>, that is, that the number "2n" is not too large. For, suppose that the unit circle, U, of M is a regular 2n-gon, $n > 2$, and let D consist of its 2n vertices, rotated about 0 through an angle of 90°. We claim that the unique polygon π in the class $\gamma(D)$ can be represented as <u>no fewer</u> than 2n geodesic chains in M. To see this, let us denote by P_i' the point in D corresponding to the vertex P_i of U under the above rotation. Then each edge $A'B'$ of π corresponding to the side AB of U is parallel to the perpendicular bisector of AB in the triangle AOB. Thus, no three consecutive vertices of π are cogeodesic in M; so, each edge in the polygon π is a maximal geodesic chain in M. Hence, π can be represented as no fewer than 2n chains.

More generally, let M be any 2n-gon plane and let D be any vertex set, where we can again assume without loss of generality that

$D = H(D)$. We will show that the polygon π in $\gamma(D)$ has the required representation. First notice that each edge of π is contained in a maximal i-like geodesic chain, that is, one that cannot be extended by the addition of adjacent i-like edges. We claim that for any $i = 1,\ldots,n$ there do not exist more than two maximal i-like geodesic chains in π . For, as in (3) we may transform M by means of a non-singular linear transformation T , so that for a particular i , the i-like segments of M become the 1-like segments of S^2 . Now we merely apply Corollary 4, which tells us that there are no more than two maximal 1-like chains in each of the image polygons $T(\pi)$ of π . qed.

Proofs of Theorem 3 and Corollary 2 (continued): To complete these proofs, it will suffice, given any 2n-gon plane M , to transform M by means of a non-singular linear transformation so that the i-like segments of M and the 1-like segments of S^2 coincide, as in the last proof. Then, it is immediately verified that the proofs given for the case $M = S^2$ may be repeated here without change, by (3). qed.

We will conclude by proving the following: *If an arbitrary point-set in a 2n-gon plane has enough members, it will have an m-cogeodesic subset of k points, where k is an arbitrary preassigned positive integer.* Indeed, this follows immediately from (2) and a well-known theorem on chromatic graphs: If the edges of a complete graph on m vertices, G , are colored with r distinct colors, then there exists a positive integer $N_k(r)$ such that $m \geqq N_k(r)$ implies that G contains a complete monochromatic subgraph on k vertices. Letting $r = n$, we are done.

What we have just shown seems to indicate that the spaces we have been considering are, in the sense of "cogeodesicness," intermediate in linearity between the plane and the line. For, in the plane there are strongly convex curves on which any three points are not e-cogeodesic.

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AUTOBIOGRAPHICAL STATEMENT

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Mrs. Kenneth Kalmanson was born Judith Rothenberg. Judi and Ken just recently acquired a new addition - Andrew David.

Judi, Ken, Andrew and "Bird", (a pet parakeet) are presently living in Brooklyn.