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**A STUDY OF PURE DOMINANCE AND MIXED DOMINANCE IN DECISION
MATRICES**

City University of New York

Ph.D. 1983

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A STUDY OF PURE DOMINANCE AND MIXED DOMINANCE
IN DECISION MATRICES

by
CHUNG KUT LU

A dissertaion submitted to the
Graduate Faculty in Business in
partial fulfillment of the
requirements for the degree of
Doctor of Philosophy,
The City University of New York
1983

This manuscript has been read and accepted for the Graduate Faculty in Business in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy. The Examination Committee recommends to Pass With Distinction "GPS"

3/25/83
date

3/27/83
date

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ABSTRACT

A STUDY OF PURE DOMINANCE AND MIXED DOMINANCE IN DECISION MATRICES

Chung-Kut Lu

The focus of this study is based on the need for a rational decision maker to obtain not only the optimal benefits or values from his choice of alternative strategies in a decision making situation, but also desires to reach the optimal end-state as efficiently as possible by reducing the number of available strategies to a set of non-dominated strategies for his selection.

In decision matrices, it is possible to classify the strategies in the matrices into three distinct classes, namely, 1) purely dominated strategies, 2) mixed dominated strategies, and 3) non-dominated strategies. The number of strategies in each class provides an important indication of the extent of presence of dominance and thus the reduction in size that can be achieved by eliminating dominated strategies. Knowledge of the expected values of the number of strategies that are purely dominated and mixed dominated in a given matrix can provide us information about the extent of dominated strategies that can be eliminated.

This study develops the methodology for the analytical derivation of the expected values of pure

dominance, mixed dominance and non-dominance for decision matrices when the entries are continuously distributed random variables. For small matrices, this study derives the analytical solutions for the expected values of pure dominance, mixed dominance and non-dominance for matrices with entries that are continuously distributed random variables in the form of, 1) uniform distribution, 2) left-skewed triangular distribution, and 3) right-skewed triangular distribution. For larger matrices, the analytical solutions for mixed dominance are complicated, and simulation is therefore used to estimate the expected values of pure dominance, mixed dominance and non-dominance when the entries in the matrices are random variables in the form of, 1) uniform distribution, 2) left-skewed triangular distribution, 3) right-skewed triangular distribution, and 4) normal distribution, respectively. Based on the results obtained from the simulation, regression analysis is then applied to provide further estimates on the expected values on dominance for matrices not covered by the simulation.

Analysis in this study shows that for a given matrix with entries that are continuously distributed, the expected value of pure dominance depends only on n , the total number of strategies, and m , the total number of columns in the matrix, and is independent of the class of continuous distribution of the entries. As long as the distribution of the entries are continuous, it does not

matter what class of distribution the entries possess, the expected value of pure dominance will always be the same for a given $n \times m$ matrix.

However, the expected values of mixed dominance in a given matrix also depends on the joint density functions of the entries in the associated game matrix with respect to the strategies examined for mixed dominance. Since entries from different classes of distributions have different joint density functions, therefore, the expected values of mixed dominance will be different for matrices with entries from different classes of distributions. Difference in means and variances of the same distribution in the entries will not affect the expected value of mixed dominance for a given $n \times m$ matrix, as long as the distribution of the entries are standardizable.

We also conclude that the expected values of mixed dominance are highly significant when compared to both the expected values of pure dominance and non-dominance. In many instances, especially for larger matrices, the expected values of mixed dominance are greater than the expected values of pure dominance or non-dominance.

ACKNOWLEDGEMENTS

It is my honor to complete my doctoral work here in Baruch College of the City University of New York. It is my pleasure to have the opportunity to meet all the wonderful individuals in the campus, faculty members and fellow students.

Professor George Sphicas, my mentor and advisor, has unselfishly offered his valuable inspiration, generous help and sound advice that made this dissertation possible. I am indeed grateful for everything he has done for me.

I am indebted to Professor George Schneller IV for his valuable advice and help, and the papers that he wrote with Professor Sphicas that made my research possible.

I am especailly grateful to Professor Llyod Rosenberg for his encouragement to complete my dissertation early. I appreciate much his kindness and generous help.

I appreciate much the good advice and constant encouragement from Professor Seymour Kwerel.

I wish to thank Professor Sidney Lirtzman for all the great work he has done for the doctoral programs in Business that facilitates the work of all doctoral students, including me.

I am also grateful to Professor Harris J. Shapiro who provided me with great help and allowed me the use of the Management Department facilities that made this dissertation possible.

I appreciate professor Daniel Shimsak who provided me with great advice as an outside reader.

I also appreciate Su and Jeng Ren Chiou for their sincere help in getting rid some of the bugs in the computer programs. Many thanks to Helen Wang for her encouragements.

Finally, I would like to thank my wife Ruby for her caring and encouragement that made my work worthwhile.

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CHAPTER I

INTRODUCTION

1.1. Introduction

The framework of this study centers on the decision maker's need for efficient choice of alternative strategies in the decision making process. A rational decision maker strives not only to obtain the optimal benefits or values from his choice of alternative strategies in a decision making situation, but also desires to reach the optimal end-state as efficiently as possible by reducing the number of available strategies to a set of non-dominated strategies for his selection. Studies showed that the presence of dominated strategies in a decision matrix will not affect the optimal choice in the decision matrix. Von Neumann and Morgenstern (31), McKinsey (19), Luce and Raiffa (18), Karlin (13), Fishburn (5), as well as Schneller and Sphicas (24) have shown that dominated strategies can be eliminated without affecting the solution of a decision matrix. Thus, the property of dominance in decision making is important as dominated strategies can be safely eliminated from consideration at the start.

The identification of pure dominance in decision matrices is straightforward. Elimination of purely dominated

strategies has been widely recognized as a useful way to reduce the decision matrices, as shown in von Neumann and Morgenstern (31), Mckinsey (19), Luce and Raiffa (18), Karlin (13), and Fishburn (5), etc. However, identification of mixed dominated strategies is not as straightforward and properties of mixed dominance have not received wide recognition. Mckinsey (19), Luce and Raiffa (18), Karlin (13) have presented a few examples of mixed dominance without providing explanation on how it can be identified. Fishburn (5) attempted to investigate means to identify mixed dominance in decision matrices with more details. Recently, Schneller and Sphicas (24) have developed a methodology to identify mixed dominance in decision matrices. Their method involves the construction of an associated two-person zero-sum game. In (24), it is shown that properties of dominance in decision matrices can be applied in at least four well-known contexts; namely, Game Theory (13) (18) (19) (29), Fishburn's Decision Model (5), Decision Making Under Risk (DMUR) (18), and Decision Making Under Uncertainty (DMUU) (18) (25) (30). The method developed by Schneller and Sphicas (24) can be used readily to identify and eliminate mixed dominated strategies from a given decision matrix resulting in the reduction of the set of strategies available to a decision maker to only non-dominated strategies. Efficiency in decision making can thus be improved as the number of strategies considered by a

decision maker is limited only to non-dominated strategies.

If both types of dominated strategies can be identified, it is then possible to clearly classify the strategies of a given matrix into three distinct classes, i.e. purely dominated, mixed dominated and non-dominated strategies. The number of strategies in each class provides an important indication of the extent of presence of dominance and thus the reduction in size that can be achieved by eliminating dominated strategies. Furthermore, the size of the two classes of dominated strategies would show which one is more common relative to each other, relative to non-dominated strategies and as a proportion of the original number of strategies. For different matrices obviously different results would be obtained. Thus, if a particular matrix of some size is randomly selected (with its entries drawn from some distribution), the number of strategies in each class will be a random variable. Knowledge of the expected value of these random variables would provide generalizable conclusions about the extent of dominance. For simplicity, we define the "expected value of pure dominance" to be the expected value of the number of strategies which are purely dominated in a matrix of a given size with its entries drawn from a particular distribution. Similarly, the "expected value of mixed dominance" and the "expected value of non-dominance" can be defined.

Furthermore, when we refer to the "probability of pure dominance", we mean the probability that a randomly selected strategy belongs to the class of purely dominated strategies. Similar probability statements apply to the other two classes. If purely dominated and mixed dominated strategies can be identified, it is then possible to determine the probabilities and the expected values of pure dominance, mixed dominance and non-dominance for strategies in decision matrices.

For each of the $n \times m$ matrix with entries drawn from one of the four distributions studied, namely, 1) uniform distribution, 2) left-skewed triangular distribution, 3) right-skewed triangular distribution, and 4) normal distribution, we assume that all the entries are i.i.d. random variables from the same probability distribution. We also assume that entries among different columns of the matrix are independent from each other.

1.2. Pure dominance and mixed dominance under various contexts

Pure dominance and mixed dominance in decision matrices can be interpreted in various contexts. In a decision matrix, the rows represented by $S(i)$, for $(i=1, \dots, n)$, are the distinct alternatives or strategies available to a decision maker. The columns, $N(j)$, for $(j=1, \dots, m)$, and the entries, $u(i, j)$, can be interpreted under different contexts. There are at least four well

known contexts that can be interpreted under the above general structure; they are, 1) Game Theory, 2) Fishburn's Decision Model, 3) Decision Making Under Risk (DMUR), and 4) Decision Making Under Uncertainty (DMUU).

In the context of Game Theory, the decision matrix can represent a two-person zero-sum game whose columns are the strategies available to a rational opponent and whose entries, $u(i,j)$, are the payoffs to the row-player from the column player (13) (18) (19) (31). In the context of Fishburn's Decision Model (5), the columns represent distinct outcomes and $u(i,j)$ represents the conditional probability of outcome j occurring if strategy i is chosen. The outcomes of Fishburn's Decision Model are functions of the chosen strategy. In DMUR (3) (18) and DMUU (3) (18) (25) (30), the columns represent neutral states of nature, independent of the choice of strategies, and the entries, $u(i,j)$ are utilities. In the context of DMUR, there is a probability vector $P=(P(1),\dots,P(m))$ associated with the states of nature. The values of P are assumed to be known. In the context of DMUU, the probability vector P is unknown.

Fishburn (5) and McKinsey (19) defined a strategy $S(k)$ is purely dominated by another strategy $S(i)$ when $u(k,j) \leq u(i,j)$ for all j while mixed dominance occurs when a strategy is dominated by a combination of two or more strategies. Fishburn (5), Schneller and Sphicas (24) defined a strategy $S(k)$ is dominated if and only if there

exists a convex linear combination (c.l.c.) of other strategies defined by the coefficients $c(i)$ for $(i=1, \dots, n; i \neq k)$, such that for each $(j=1, \dots, m)$,

$$(1.2.1) \quad u(k, j) < \sum_{\substack{i=1 \\ i \neq k}}^n c(i) \cdot u(i, j), \text{ where } c(i) \geq 0, \\ \text{and } \sum_{i \neq k} c(i) = 1$$

The above definition covers both pure dominance, when all but one of the $c(i)$'s are zero, and mixed dominance when two or more of the $c(i)$'s are positive. The special case of strict dominance of $S(k)$, when (1.2.1) is satisfied strictly, without equality, for all j , can be distinguished.

In the contexts of Game Theory and Fishburn's Decision Model, randomized strategies are allowed and all mixed dominated strategies can be eliminated without affecting the solution, as shown in (13) (18) (19) (31).

In DMUR, a decision maker is assumed to know the probability vectors associated with the states of nature. Since the probability vectors are known, the expected utility of each strategy can be calculated and the decision maker can simply choose the strategy that will maximize his expected utility. Although there is thus no need for randomized strategies in DMUR, yet dominance is still useful in eliminating dominated strategies (24).

In DMUU, the probability vectors associated with the states of nature are not known and the usual approach is to apply one of the known criteria to select a strategy.

Luce and Raiffa (18), and Milnor (20) have examined the four classical criteria for DMUU; namely, Wald's maximin criterion (32), Savage's minimax risk criterion (27), Hurwicz's pessimism-optimism index criterion (9), and Laplace's principle of insufficient reason criterion (15). Milnor (20) has applied the dominance axiom criterion to evaluate the above four classical DMUU criteria. The selection of either of the Wald (32), or Savage (27), or Hurwicz (9), or Laplace (15) criterion will result in the violations of one or more of the established axioms if the dominance criterion is applied.

Starr (29) (30) introduced the concept of domain criterion in DMUU. The domain of a strategy $S(i)$, denoted by $D(S(i))$, includes all the points P at which $S(i)$ produces the highest expected utility among all the strategies. Starr (30) showed that a strategy which was not purely dominated can still have no domain and thus can be mutually dominated by a set of mixed strategies. Schneller and Sphicas (24) (26) (27) related directly the concept of dominance to the concept of domain, also calling a dominated strategy a null strategy. Using Starr's domain criterion (28), Schneller and Sphicas (26) (27) have examined Milnor's axioms for DMUU (18) (20) and have illustrated that the choice of either of the Wald (32), Hurwicz (9), or Savage (27) criterion can be a null strategy. For the Laplace (15) criterion, if several

strategies are jointly optimal, some may be null strategies, and if the Laplace (15) criterion chooses a unique strategy, then that strategy is non-dominated.

Thus under the contexts of DMUR and DMUU, it does not matter if P is known with certainty, or estimated as a prior probability or as a Bayes revised distribution, or unknown; if the decision maker is an expected utility-maximizer, he can eliminate all dominated strategies from the start.

1.3. Identification and properties of pure dominance

Identification of pure dominance in decision matrix is straightforward. A strategy $S(k)$ can be identified as purely dominated by $S(i)$ if $u(k,j) \leq u(i,j)$ for all j . To determine if a strategy $S(k)$ is purely dominated, we can also construct a two-person zero-sum game associated with strategy $S(k)$ as follows: Define the payoffs from the column player to the row player as,

$$(1.3.1) \quad a(i,j) = u(i,j) - u(k,j), \text{ for } (i=1,\dots; i \neq k) \\ \text{and } (j=1,\dots,m)$$

If all of the entries in a row, $a(i,j)$, contain only non-negative entries, then strategy $S(k)$ is identified as purely dominated. The dominated strategy $S(k)$ can then be eliminated from the matrix.

If purely dominated strategies can be identified, it is then feasible to determine the expected values of pure dominance in a given decision matrix. An analytical

method to determine the expected value of pure dominance in a given $n \times m$ matrix is outlined in Chapter II of this dissertation. The method involves first arranging the entries for each column of a given matrix in terms of order statistics. All the combinations of matrices with the ordered entries are then listed out. We then proceed to identify all the purely dominated strategies in each of the combination of the matrices. Based on the total number of purely dominated strategies identified, we can readily derive the expected value of pure dominance in a given $n \times m$ matrix.

Results from Chapter II show that for a decision matrix with entries which are random variables of a continuous distribution, the expected value of pure dominance depends only on the dimensions of the matrix and is independent of the distribution of the entries. This means that for a given $n \times m$ matrix, the entries can be random variables of any class of continuous distribution without affecting the expected value of pure dominance for the matrix.

1.4. Identification and properties of mixed dominance

Properties of mixed dominance are not widely recognized and the identification of mixed dominance is less straightforward. A number of authors, such as Karlin (13), Luce and Paiffa (18), and McKinsey (19) have identified a few examples of mixed dominance without explaining

how they could be identified. Fishburn (5) presented a more detailed study of mixed dominance. Recently, Schneller and Sphicas (24) have investigated the properties of mixed dominance and have developed a methodology for identifying mixed dominance. Their method involves the construction of an associated two-person zero-sum game so as to identify a convex linear combination of dominated strategies. The two-person zero-sum game associated with strategy $S(k)$ has payoffs $a(i,j)$ as defined in (2.2.1). If $v(k)$ is the optimal value of the game, then $S(k)$ is dominated if and only if $v(k) > 0$ (24).

The simplest solution for the associated game is the presence of a saddle point. If a saddle point does not exist, then it is possible to solve the game by linear programming. If the value of the game is not required, then Brown's iterative procedure (2) can also be used to determine if a strategy $S(k)$ is mixed dominated.

The entries in the matrix of the game associated with $S(k)$ show the columnwise differences between each of the other strategies and $S(k)$. The game matrix may contain a mixture of positive, negative, and zero entries. In the linear programming solution of the game matrix, the method of Schneller and Sphicas (24) has the advantage of producing a best dominating c.l.c. if it exists, whereas Fishburn (5), and Starr (28) in their approach cannot.

As suggested in (24), an effective solution technique for identifying the dominance status of a strategy is an

iterative procedure known as Brown's method (2), which produces an upper bound and a lower bound for the value of a two-person zero sum game at each iteration. The iterative procedure can be terminated as soon as the upper bound becomes negative, or the lower bound becomes positive. The main advantage of using Brown's iterative procedure (2) is that in order to check the status of dominance for a strategy, we only need to know whether the value of its associated game is negative or positive. It is not necessary to calculate the exact value of the game or to find the dominating c.l.c. Thus, Brown's method (2) is well suited to determine the status of mixed dominance for a strategy tested as it provides bounds for the game. Brown's method (2) is utilized in the simulation part of this study.

An analytical method to determine the expected value of mixed dominance for a given $n \times m$ decision matrix is discussed in Chapter II of this dissertation. For each combination of the given $n \times m$ matrix with ordered entries, after the elimination of purely dominated strategies, the remaining strategies that do not possess a highest order entry will be examined for mixed dominance. First a game matrix associated with the strategy $S(k)$ examined is created as suggested by Schneller and Sphicas (24). Then Game Theory and integration over the joint density function of the order statistics are applied to obtain the probability of mixed dominance for strategy $S(k)$. After

the probability of mixed dominance is found for each strategy examined, it is then possible to derive the expected value of mixed dominance.

1.5. Objectives and scope of this study

The first objective of this study is to contribute to the existing theoretical and empirical knowledge in the area of pure dominance and mixed dominance in decision matrices. The second objective of this study is to develop the methodology for the determination of the expected values of pure dominance, mixed dominance and non-dominance for decision matrices when the entries are continuously distributed random variables. The third objective of this study is to determine the expected values of pure dominance, mixed dominance and non-dominance for decision matrices with varying number of strategies and number of states of nature in which the entries in the matrices are continuously distributed random variables in the form of, 1) uniform distribution, 2) normal distribution, 3) left-skewed triangular distribution, and 4) right-skewed triangular distribution. The fourth objective is to test the validity of the research hypotheses formulated. The fifth objective is to compare the relative significance among mixed dominance, pure dominance and non-dominance as various ratios of the expected values of pure dominance, mixed dominance and non-dominance can be computed and compared.

For small matrices where the number of strategies and the number of states of nature are small, analytical solutions for the determination of the expected values of pure dominance, mixed dominance and non-dominance are feasible and this study will provide the methodology as well as the analytical solutions for these matrices. For matrices with sizes of 2×2 , 2×3 , 3×2 , and 3×3 respectively, this study will present the analytical solutions of the expected values of pure dominance, mixed dominance and non-dominance when the entries in the matrices are random variables of 1) uniform distribution, 2) left-skewed triangular distribution, and 3) right-skewed triangular distribution. In addition, for the 4×2 matrix, the analytical solutions of the expected values of pure dominance, mixed dominance and non-dominance for entries with uniform distribution will also be presented.

For matrices of larger sizes, the analytical approach for determining the expected values of pure dominance, mixed dominance and non-dominance becomes too complicated. It would be impractical to calculate the analytical expected values of dominance in large matrices. Therefore, a simulation approach is used in larger matrices to estimate the required results. Based on the results obtained from the simulation, regression analysis is then applied to provide further estimates on the expected values on dominance for matrices not covered by the simulation. To facilitate the study, research

hypotheses are formulated to provide guidance for the study. Based on the results of the simulation, statistical tests are performed for testing the validity of the research hypotheses.

Results from this study can provide answers to the expected values of pure dominance, mixed dominance and non-dominance in a given decision matrix for the probability distributions of the entries studied. This will allow a decision maker to know the expected number of dominated strategies that can be eliminated and the expected number of non-dominated strategies that require his consideration. This study can also provide answers to the relative significance of mixed dominance in relation to pure dominance and non-dominance as various ratios of the expected values of mixed dominance, pure dominance and non-dominance can be tabulated. This study is confined to the investigation of properties related to row dominance. In the context of game theory, column dominance is also important. In that case, our results can be applied if the matrix is simply transposed.

1.6. Methodology of this study

For small matrices where the analytical solutions for determining the expected values of pure dominance, mixed dominance and non-dominance are feasible, the study employs properties of Probability Theory, Order Statistics, Game Theory, Calculus and Algebra to determine

the required analytical solutions, as well as to prove the theorems involved.

For larger matrices where the analytical solutions for determining the expected values of pure dominance, mixed dominance and non-dominance are too complicated, digital computer simulation is used to estimate the expected values of pure dominance, mixed dominance and non-dominance. A 95% confidence interval is then constructed around the expected values obtained from the simulation. Based on the simulation results, regression analysis is also performed to obtain further estimates on the expected values of pure dominance, mixed dominance and non-dominance for other matrices not covered in the simulation study.

1.7. Organization of this study

Chapter I outlines the objectives and scope of this study. It also reviews the literature on the properties of pure dominance and mixed dominance under the contexts of Game Theory, Fishburn's Decision Model, Decision Making Under Risk (DMUR), and Decision Making Under Uncertainty (DMUU).

Chapter II presents the analytical methods and solutions for the determination of the expected values of pure dominance, mixed dominance and non-dominance for small matrices with entries that are random variables of continuous distributions in the form of, 1) uniform

distribution, 2) left-skewed triangular distribution, and 3) right-skewed triangular distribution. Related theorems and proofs are also shown.

Chapter III formulates the research hypotheses to facilitate the simulation study. It also presents the methodology employed in the simulation study.

Chapter IV summarizes the simulation results for the expected values of pure dominance, mixed dominance and non-dominance for matrices with entries that are random variables of, 1) uniform distribution, 2) left-skewed triangular distribution, 3) right-skewed triangular distribution, and 4) normal distribution. Chapter IV also analyzes the simulation results, tests the research hypotheses, and performs the regression analysis. Results of the regression analysis are also tabulated.

Chapter V presents a summary of results, the conclusions reached, and suggestions for future research on the subject.

CHAPTER II
ANALYTICAL DERIVATION OF THE EXPECTED VALUES OF
PURE DOMINANCE, MIXED DOMINANCE AND NON-DOMINANCE
IN SMALL MATRICES

2.1. Analysis of the expected values of pure dominance

We can compute the expected value of pure dominance for a given $n \times m$ matrix with entries that are continuously distributed random variables as follows. We first order the entries for each column of the matrix in terms of order statistics. Let the entries in the matrix $u(i, j)$ be continuously distributed random variables. Then the ordered values of the entries in each column j for $(j=1, \dots, m)$ are known as the order statistics in ascending order of magnitude,

(2.1.1)

$$u'(1, j) < u'(2, j) < \dots < u'(i, j) < \dots < u'(n-1, j) < u'(n, j)$$

we list out all the combinations of the ordered matrices. For an $n \times m$ matrix, the total number of combinations of the ordered matrices is,

$$(2.1.2) \quad B(n, m) = (n!)^{m-1}$$

From Fishburn (5), a strategy in a matrix that contains an entry with the highest order of magnitude in a column will be non-dominated. Strategies that do not contain any entry with the highest order in a column will

be examined for pure dominance. For continuous distribution, a strategy is said to be purely dominated if strict inequality occurs,

$$(2.1.3) \quad u(k, j) < u(i, j) \text{ for } (j=1, \dots, m) \\ \text{and } (i=1, \dots, n; i \neq k)$$

For each combination of the ordered matrix, we can tabulate all the purely dominated strategies. Let $c(f)$ be the number of purely dominated strategies in each of the $B(n, m)$ combinations of the given $n \times m$ matrix for $(f=1, \dots, B(n, m))$. Then the total number of purely dominated strategies in $B(n, m)$ combinations is,

$$(2.1.4) \quad C(n, m) = \sum_{f=1}^{B(n, m)} c(f)$$

Then the expected value of pure dominance can thus be computed,

$$(2.1.5) \quad P(n, m) = \frac{C(n, m)}{B(n, m)}$$

Theorem 2.1.1 The expected value of pure dominance depends only on the dimension of a given $n \times m$ matrix and is independent of the continuous distribution of the entries.

Proof: It is clear from the development of equation (2.1.5), that in calculating $P(n, m)$, no probability statement involving the distribution of the entries is needed. The total count of the combinations will be the same for a given $n \times m$ matrix irrespective of the distribution of the entries as long as the entries are continuously distributed. In fact, entries from different

columns of the matrix can be drawn from different classes of continuous distributions and will not affect the expected value of pure dominance.

The above theorem proves that for entries with continuous distribution, the expected value of pure dominance depends only on n and m , and is independent of the class of continuous distribution of the entries. As long as the distribution of the entries are continuous, regardless of its distribution, the expected value of pure dominance will be the same for a given $n \times m$ matrix.

2.2. Analysis of the expected values of mixed dominance

This study develops a method to compute the analytical solution for the expected value of mixed dominance for a given $n \times m$ matrix. For each of the $P(n,m)$ combinations of the matrices with ordered entries, strategies that failed the test for being purely dominated will be examined for mixed dominance. In general, only some of the combinations of matrices will contain strategies to be examined for mixed dominance. Based on the symmetrical characteristics of the combination of matrices that contain strategies that can be mixed dominated, we can further combine some of the combinations to a few types of matrices as shown later in this Chapter.

Let $T(q)$ be the number of each q type matrix that contains strategies to be examined for mixed dominance, for $(q=1, \dots, Q)$ where Q is the total number of types of

matrices. Also, let $R(q)$ be the total number of strategies in each q type matrix to be examined for mixed dominance. Then $p(q, r(q))$ will be the probability of mixed dominance for each strategy $r(q)$ in type q matrix to be examined for mixed dominance, where $(r(q)=1, \dots, R(q))$.

For each strategy $r(q)$ examined for mixed dominance, an associated game matrix corresponding to $r(q)$ is first created. A procedure using Game Theory and integration over the joint density function of the order statistics of the entries is then employed to derive the probability of mixed dominance for each strategy examined, i.e. $p(q, r(q))$.

Let the sum of the expected total number of mixed dominated strategies for all of the types of matrices examined for mixed dominance be,

$$(2.2.1) \quad F(n, m) = \sum_{r(q)=1}^{R(q)} \sum_{q=1}^Q p(q, r(q)) \cdot T(q)$$

Then the expected value of mixed dominance for an $m \times n$ matrix will be,

$$(2.2.2) \quad E(n, m) = \frac{F(n, m)}{S(n, m)}$$

Unlike the case of $P(n, m)$ which involved simply counting the number of combinations of matrices and the number of dominated strategies, here $E(n, m)$ is based on probabilities $p(q, r(q))$'s which are clearly a function of the distribution of the entries in the matrix.

Theorem 2.2.1 All matrices with the same n and m , and

with entries drawn from a standardizable distribution of a particular class of distributions have the same expected value of mixed dominance, irrespective to the difference in means and variances of the distribution of the entries in the corresponding $n \times m$ matrices.

Proof: Let $z(i,j)$ be the standardized entries from a standardizable distribution for $i=(1,\dots,n)$ and $j=(1,\dots,m)$. Then the entries in a given $n \times m$ matrix can be expressed as,

$$(2.2.3) \quad u(i,j) = \mu + \sigma \cdot z(i,j)$$

where μ and σ are the mean and standard deviation of the distribution of the entries. Then the entries in the associated game matrix will be equal to,

$$\begin{aligned} (2.2.4) \quad a(i,j) &= u(i,j) - u(k,j) \\ &= \{\mu + \sigma \cdot z(i,j)\} - \{\mu + \sigma \cdot z(k,j)\} \\ &= \sigma \cdot \{z(i,j) - z(k,j)\} \end{aligned}$$

Since the standard deviation is always positive in value, therefore, the value of the associated game based on $a(i,j)$ has the same sign as the value of a game based on $z(i,j) - z(k,j)$, i.e. the standardized transformation. Thus, the probability that a particular strategy is mixed dominated (i.e., the value of its associated game is non-negative), is not affected by this transformation.

This study computes the expected values of mixed dominance for matrices with entries that are random variables of 1) uniform distribution, 2) left-skewed triangular distribution, 3) right-skewed triangular

distribution, and 4) normal distribution. Since the above four distributions can all be standardizable as shown in Sphicas (28), difference in means and variances in the same standardizable distribution for the corresponding nxm matrices will not affect the expected value of mixed dominance.

Table 2.2.1 lists the parameters for the four continuous standardized distributions studied.

TABLE 2.2.1
FOUR STANDARDIZABLE DISTRIBUTIONS

Distributions	Mean	Variance	Range
1) Normal	μ	σ^2	$(-\infty, \infty)$
2) Left-skewed Triangular	μ	$\frac{k^2}{2}$	$(\mu-2k, \mu+k)$
3) Right-skewed Triangular	μ	$\frac{k^2}{2}$	$(\mu-k, \mu+2k)$
4) Uniform	μ	$\frac{k^2}{3}$	$(\mu-k, \mu+k)$

2.3. Analysis of the expected values of non-dominance

For a given nxm decision matrix, after the expected values of pure dominance and mixed dominance are determined, the expected value of non-dominance, $N(n,m)$, can thus be computed,

$$(2.3.1) \quad N(n,m) = n - P(n,m) - M(n,m)$$

where n is the total number of strategies in a matrix.

Thus, equations (2.1.5), (2.2.2) and (2.3.1) can be used to calculate the needed expected values. These equations are next applied to five specific matrices.

2.4. The matrix with $n=2$ and $m=2$

In the 2×2 matrix, there are 2 combinations of matrices and 4 permutations of matrices if the entries in each column are arranged in increasing order of magnitude, such that,

$$(2.4.1) \quad b > x(2) > x(1) > a \quad \text{for column 1}$$

$$b > y(2) > y(1) > a \quad \text{for column 2}$$

where b and a are the upper and lower limits of the continuous distribution of the entries.

Figure 2.4.1 shows the total number of combinations for the 2×2 matrix when the entries are ordered.

FIGURE 2.4.1

TOTAL COMBINATIONS FOR THE 2×2 MATRIX

	N(1)	N(2)		N(1)	N(2)
	-----	-----		-----	-----
S(1)	x(2)	y(2)	S(1)	x(2)	y(1)
	-----	-----		-----	-----
S(2)	x(1)	y(1)	S(2)	x(1)	y(2)
	-----	-----		-----	-----
Combination 1			Combination 2		

Table 2.4.1 shows the total number of purely dominated strategies in the above two combination of matrices.

TABLE 2.4.1
TOTAL NUMBER OF STRATEGIES PURELY DOMINATED
FOR THE 2X2 MATRIX

Combination	Purely dominated strategies	Number of purely dominated strategies
1	S (2)	1
2	----	0

		C (2,2) = 1

For the 2x2 matrix, the total combinations and the total number of purely dominated strategies are, respectively,

$$(2.4.2) \quad B(2,2) = 2! = 2$$

$$C(2,2) = 1$$

Then the expected values of pure dominance, mixed dominance and non-dominance for the 2x2 matrix, are respectively,

$$(2.4.3) \quad P(2,2) = \frac{C(2,2)}{B(2,2)} = \frac{1}{2} = 0.5$$

$$M(2,2) = 0$$

$$N(2,2) = n - P(2,2) - M(2,2) = 1.5$$

2.5. The matrix with n=2 and m=3

In the 2x3 matrix, there are a total of 4 combinations of matrices if the entries in column 1, column 2 and column 3 are ordered in increasing order of magnitude, such that for column 1, column 2 and column 3 respectively,

- (2.5.1) $b > x(2) > x(1) > a$
 $b > y(2) > y(1) > a$
 $b > z(2) > z(1) > a$

Figure 2.5.1 shows the total number of combinations for the 2x3 matrix when the entries are ordered.

FIGURE 2.5.1

TOTAL COMBINATIONS FOR THE 2X3 MATRIX

	N (1)	N (2)	N (3)		N (1)	N (2)	N (3)
S (1)	x (2)	y (2)	z (2)	S (1)	x (2)	y (2)	z (1)
S (2)	x (1)	y (1)	z (1)	S (2)	x (1)	y (1)	z (2)
Combination 1				Combination 2			
	N (1)	N (2)	N (3)		N (1)	N (2)	N (3)
S (1)	x (2)	y (1)	z (1)	S (1)	x (2)	y (1)	z (2)
S (2)	x (1)	y (2)	z (2)	S (2)	x (1)	y (2)	z (1)
Combination 3				Combination 4			

Table 2.5.1 shows the total number of purely dominated strategies in the above four combinations of matrices.

TABLE 2.5.1

TOTAL NUMBER OF STRATEGIES PURELY DOMINATED
FOR THE 2X3 MATRIX

Combination	Purely dominated strategies	Number of purely dominated strategies
1	S (2)	1
2	----	0
3	----	0
4	----	0

		C (2,3) = 1

For the 2x3 matrix, the total combinations and the total number of purely dominated strategies are, respectively,

$$(2.5.2) \quad B(2,3) = (2!)^2 = 4$$

$$C(2,3) = 1$$

Then the expected values of pure dominance, mixed dominance and non-dominance for the 2x3 matrix, are respectively,

$$(2.5.3) \quad P(2,3) = \frac{C(2,3)}{B(2,3)} = \frac{1}{4} = 0.25$$

$$M(2,3) = 0$$

$$N(2,3) = n - P(2,3) - M(2,3) = 1.75$$

2.6. The matrix with $n=3$ and $m=2$

In the 3x2 matrix, there are a total of 6 combinations of matrices if the entries in column 1 and column 2 are ordered in increasing order of magnitude, such that for column 1 and column 2 respectively,

$$(2.6.1) \quad b > x(3) > x(2) > x(1) > a$$

$$b > y(3) > y(2) > y(1) > a$$

Figure 2.6.1 shows the total number of combinations for the 3x2 matrix when the entries are ordered.

Table 2.6.1 shows the total number of purely dominated strategies in the above six combinations of matrices.

For the 3x2 matrix, the total combinations and the total number of purely dominated strategies are, respectively,

$$(2.6.2) \quad P(3,2) = (3!) = 6$$

$$C(3,2) = 7$$

FIGURE 2.6.1

TOTAL COMBINATIONS FOR THE 3X2 MATRIX

	N(1) N(2)		N(1) N(2)		N(1) N(2)
S (1)	x (3) y (3)	S (1)	x (3) y (3)	S (1)	x (3) y (2)
S (2)	x (2) y (2)	S (2)	x (2) y (1)	S (2)	x (2) y (3)
S (3)	x (1) y (1)	S (3)	x (1) y (2)	S (3)	x (1) y (1)
Combination 1			Combination 2		Combination 3
	N(1) N(2)		N(1) N(2)		N(1) N(2)
S (1)	x (3) y (2)	S (1)	x (3) y (1)	S (1)	x (3) y (1)
S (2)	x (2) y (1)	S (2)	x (2) y (3)	S (2)	x (2) y (2)
S (3)	x (1) y (3)	S (3)	x (1) y (2)	S (3)	x (1) y (3)
Combination 4			Combination 5		Combination 6

TABLE 2.6.1

TOTAL NUMBER OF STRATEGIES PURELY DOMINATED
FOR THE 3X2 MATRIX

Combination	Purely dominated strategies	Number of purely dominated strategies
1	S (2) , S (3)	2
2	S (2) , S (3)	2
3	S (3)	1
4	S (2)	1
5	S (3)	1
6	----	0
		C (3,2) = 7

Then the expected value of pure dominance for the 3x2 matrix is,

$$(2.6.3) \quad P(3,2) = \frac{C(3,2)}{B(3,2)} = \frac{7}{6} = 1.1667$$

In the six combinations of matrices outlined in Table 2.6.1, all the strategies in combinations 1 to 5 are either non-dominated or purely dominated. Only strategy 2 in combination 6 has to be tested for mixed dominance, while strategies 1 and 2 in the same combination are non-dominated. Figure 2.6.2 shows that the decision matrix of combination 6 is first transformed into an associated game matrix.

FIGURE 2.6.2

ASSOCIATED GAME MATRIX FOR COMBINATION 6

	N(1)	N(2)		N(1)	N(2)
S(1)	x(3)	y(1)	S(a1)	x(3)-x(2)	y(1)-y(2)
S(2)	x(2)	y(2)	S(a2)	x(1)-x(2)	y(3)-y(2)
S(3)	x(1)	y(3)			

Combination 6

Associated game matrix

Theorem 2.6.1 For the 2x2 associated game matrix of Figure 2.6.2, the probability that the expected value of the game, v , is greater than zero is,

$$(2.6.4) \quad P(v > 0) = P(X > Y)$$

$$\text{where, } X = \frac{x(3)-x(2)}{x(2)-x(1)} \quad \text{and } Y = \frac{y(2)-y(1)}{y(3)-y(2)}$$

Proof: Let $A = x(3) - x(2)$, $B = y(1) - y(2)$, $C = x(1) - x(2)$, and $D = y(3) - y(2)$. Let p be the probability that player 1 chooses row 1 of the associated game and $(1-p)$ be the probability that player 1 chooses row 2 of the associated game. Then the expected value of the game will be,

$$(2.6.5) \quad v = A.p + (1-p).C = B.p + (1-p).D$$

Therefore,

$$(2.6.6) \quad p = \frac{D-C}{A+D-B-C}$$

$$1-p = \frac{A-B}{A+D-B-C}$$

$$(2.6.7) \quad v = \frac{A.D-B.C}{A+D-B-C}$$

From (2.6.4),

$$\begin{aligned} (2.6.8) \quad P(v > 0) &= P\left(\frac{A.D-B.C}{A+D-B-C} > 0\right) = P(A.D-B.C > 0) \\ &= P\left(\frac{A}{C} < \frac{B}{D}\right) = P\left(\frac{x(3)-x(2)}{x(1)-x(2)} < \frac{y(1)-y(2)}{y(3)-y(2)}\right) \\ &= P\left(\frac{x(3)-x(2)}{x(2)-x(1)} > \frac{y(2)-y(1)}{y(3)-y(2)}\right) \\ &= P(X > Y) \end{aligned}$$

We know that a strategy is mixed dominated if the value of the associated game is greater than or equal to zero, i.e. $S(k)$ is mixed dominated if $v > 0$. To compute the analytical solution for $P(v > 0)$, we first obtain the joint density function of the entries in the associated game matrix. Let,

(2.6.9) $s = s(1) =$ one particular value of X

$$s(1) = \frac{x(3) - x(2)}{x(2) - x(1)} \implies x(2) = \frac{s(1) \cdot s(2) + s(3)}{(1 + s(1))}$$

$$s(2) = x(1) \implies x(1) = s(2)$$

$$s(3) = x(3) \implies x(3) = s(3)$$

such that the Jacobian is,

$$J(s(1), s(2), s(3)) = \frac{(s(3) - s(2))^2}{(1 + s(1))^2}$$

Then the joint density function for $s(1)$, $s(2)$ and $s(3)$ is,

(2.6.10)

$$f(s(1), s(2), s(3)) = J(s(1), s(2), s(3)) \cdot f(x(1), x(2), x(3))$$

where, $f(x(1), x(2), x(3)) = 3! \cdot f(x(1)) \cdot f(x(2)) \cdot f(x(3))$

Thus, the density function of s is,

$$(2.6.11) \quad f(s) = f(s(1)) = \int_a^b \int_{s(2)}^b f(s(1), s(2), s(3)) \, ds(3) \, ds(2)$$

Similarly, let,

(2.6.12) $t = t(1) =$ one particular value of Y

$$t(1) = \frac{y(2) - y(1)}{y(3) - y(2)} \implies y(2) = \frac{t(1) \cdot t(3) + t(2)}{(1 + t(1))}$$

$$t(2) = y(1) \implies y(1) = t(2)$$

$$t(3) = y(3) \implies y(3) = t(3)$$

such that the Jacobian is,

$$J(t(1), t(2), t(3)) = \frac{(t(3) - t(2))^2}{(1 + t(1))^2}$$

Then the joint density function for $t(1)$, $t(2)$ and $t(3)$

is,

(2.6.13)

$$f(t(1), t(2), t(3)) = J(t(1), t(2), t(3)) \cdot f(y(1), y(2), y(3))$$

where, $f(y(1), y(2), y(3)) = 3! \cdot f(y(1)) \cdot f(y(2)) \cdot f(y(3))$

Thus, the density function of t is,

$$(2.6.14) \quad f(t) = f(t(1)) = \int_a^b \int_a^b f(t(1), t(2), t(3)) dt(3) dt(2)$$

Then the probability of mixed dominance for strategy $S(2)$ in combination 6 is therefore,

$$(2.6.15) \quad P(v > 0) = P(X > Y)$$

$$\begin{aligned} &= \int_0^\infty P(X > t | t) \cdot f(t) dt \\ &= \int_0^\infty \int_t^\infty f_{x|y}(s > t | t) \cdot f(t) ds dt \\ &= \int_0^\infty \int_t^\infty \frac{f(s, t)}{f(t)} \cdot f(t) ds dt \\ &= \int_0^\infty \int_t^\infty f(s) \cdot f(t) ds dt \end{aligned}$$

For the 3×2 matrix, there is only one combination and one type of an ordered matrix that contains a strategy to be examined for mixed dominance. Therefore, for $Q=1$, the total number of type 1 matrix, $T(1)$; the sum of the expected value of mixed dominance for all the combinations of matrices, $E(3,2)$; and the expected value of mixed dominance, respectively, are as follows,

$$(2.6.16) \quad T(1) = 1$$

$$E(3,2) = P(v > 0) \cdot T(1) = P(v > 0)$$

$$M(3,2) = \frac{E(3,2)}{B(3,2)}$$

Table 2.6.2 summarizes the expected values of mixed dominance and non-dominance for the 3x2 matrix with entries that are random variables of left-skewed triangular, uniform and right-skewed triangular distributions. It is shown that the matrix with entries that are right-skewed triangularly distributed has the highest expected value of mixed dominance, while the matrix with entries that are left-skewed triangularly distributed has the lowest expected value of mixed dominance.

TABLE 2.6.2
EXPECTED VALUES OF MIXED DOMINANCE AND NON-DOMINANCE
FOR 3X2 MATRICES

Distributions	f(s)	f(t)	E(3,2)	M(3,2)	N(3,2)
1) Left-skewed triangular	$\frac{2 \cdot (2+s)}{3}$	$\frac{2 \cdot (1+2t)}{3}$	$\frac{7}{18}$	0.0648	1.7685
2) Uniform	$\frac{1}{2(1+s)}$	$\frac{1}{2(1+t)}$	$\frac{1}{2}$	0.0833	1.7500
3) Right-skewed triangular	$\frac{2 \cdot (1+2s)}{3}$	$\frac{2 \cdot (2+t)}{3}$	$\frac{11}{18}$	0.1019	1.7314

2.7. The matrix with n=3 and m=3

In the 3x3 matrix, there are a total of 36 combinations of matrices if the entries in column 1, column 2, and

column 3 are ordered in increasing order of magnitude, such that for columns 1, 2, and 3 respectively,

$$(2.7.1) \quad b > x(3) > x(2) > x(1) > a$$

$$b > y(3) > y(2) > y(1) > a$$

$$b > z(3) > z(2) > z(1) > a$$

For the 3x3 matrix, the total combinations and the total number of purely dominated strategies are, respectively,

$$(2.7.2) \quad B(3,3) = (3!)^2 = 36$$

$$C(3,3) = 23$$

Then the expected values of pure dominance for the 3x3 matrix is,

$$(2.7.3) \quad P(3,3) = \frac{C(3,3)}{B(3,3)} = \frac{23}{36} = 0.6389$$

In the 36 combinations of matrices, there are only 9 combinations that contain strategies to be examined for mixed dominance. Of these 9 combinations, it is possible to group these combinations into two types of matrices based on the symmetrical relations of the ordered entries in the three columns of the matrices. There are a total of three combinations for the Type 1 matrix, and a total of six combinations for the Type 2 matrix. In each type of the matrices, only strategy S(2) has to be examined for mixed dominance. The two types of matrices that contain strategies to be examined for mixed dominance are listed in Figure 2.7.1.

FIGURE 2.7.1

TWO TYPES OF 3X3 MATRICES WITH MIXED DOMINANCE

	N (1)	N (2)	N (3)		N (1)	N (2)	N (3)
S (1)	x (3)	y (3)	z (1)	S (1)	x (3)	y (3)	z (1)
S (2)	x (2)	y (2)	z (2)	S (2)	x (2)	y (1)	z (2)
S (3)	x (1)	y (1)	z (3)	S (3)	x (1)	y (2)	z (3)
	Type 1				Type 2		

FIGURE 2.7.2

ASSOCIATED GAME MATRIX FOR TYPE 1 MATRIX

Associated game matrix

	N (1)	N (2)	N (3)
S (a1)	x (3) - x (2)	y (3) - y (2)	z (1) - z (2)
S (a2)	x (1) - x (2)	y (1) - y (2)	z (3) - z (2)

	N (1)	N (3)		N (2)	N (3)
S (a1)	x (3) - x (2)	z (1) - z (2)	S (a1)	y (3) - y (2)	z (1) - z (2)
S (a2)	x (1) - x (2)	z (3) - z (2)	S (a2)	y (1) - y (2)	z (3) - z (2)

Subgame 1

Subgame 2

	N (1)	N (2)
S (a1)	x (3) - x (2)	y (3) - y (2)
S (a2)	x (1) - x (2)	y (1) - y (2)

Subgame 3

Figure 2.7.2 lists the 2x3 game matrix associated with strategy S (2) and its subgames for the Type 1 matrix shown in Figure 2.7.1. The associated game matrix can be

subdivided into three subgames as shown above. Let $v(1)$, $v(2)$, and $v(3)$ be the expected values of the subgames 1, 2, and 3 respectively.

In subgame 3, all the entries in row $S(a1)$ are positive, and all the entries in row $S(a2)$ are negative. Row $S(a1)$ dominates row $S(a2)$, and the value of the game $v(3)$ will always be positive. Thus, if we want to determine if strategy $S(2)$ in Type 1 matrix is non-dominated, we only have to consider if the expected values in subgame 1 and subgame 2 is less than zero. Let v be the expected value of the 2×3 associated game matrix, for $v < 0$, iff,

$$(2.7.4) \quad \{ \text{Min}(v(1), v(2), v(3)) < 0 \} = \{ \text{Min}(v(1), v(2)) < 0 \}$$

Then the probability that strategy $S(2)$ in Type 1 matrix is non-dominated,

$$\begin{aligned} (2.7.5) \quad & P(S(2) \text{ Non-dominated}) \\ &= P\{(v(1) < 0) \cup (v(2) < 0)\} \\ &= P(v(1) < 0) + P(v(2) < 0) - P\{(v(1) < 0) \cap (v(2) < 0)\} \end{aligned}$$

We then let,

$$(2.7.6) \quad X = \frac{x(3) - x(2)}{x(2) - x(1)}, \quad Y = \frac{y(3) - y(2)}{y(2) - y(1)}, \quad Z = \frac{z(2) - z(1)}{z(3) - z(2)}$$

and, s = one particular value of X

t = one particular value of Y

u = one particular value of Z

Then we can compute,

$$(2.7.7)$$

$$P\{(v(1) < 0) \cap (v(2) < 0)\}$$

$$\begin{aligned}
&= P\left\{\left(\frac{x(3)-x(2)}{x(2)-x(1)} < \frac{z(2)-z(1)}{z(3)-z(2)}\right) \cap \left(\frac{y(3)-y(2)}{y(2)-y(1)} < \frac{z(2)-z(1)}{z(3)-z(2)}\right)\right\} \\
&= P\{(X < Z) \cap (Y < Z)\} \\
&= \int_0^{\infty} P\{(s < u) \cap (t < u) | u\} \cdot f(u) \, du \\
&= \int_0^{\infty} \int_0^u \int_0^u \frac{f(s, t, u)}{f(u)} \cdot f(u) \, ds \, dt \, du \\
&= \int_0^{\infty} \int_0^u \int_0^u f(s) \cdot f(t) \cdot f(u) \, ds \, dt \, du
\end{aligned}$$

and,

$$P(v(1) < 0) = \int_0^{\infty} \int_0^u f(s) \cdot f(u) \, ds \, du$$

$$P(v(2) < 0) = \int_0^{\infty} \int_0^u f(t) \cdot f(u) \, dt \, du$$

Therefore, the probability that strategy S(2) is mixed dominated will be,

$$\begin{aligned}
(2.7.8) \quad &P(S(2) \text{ Mixed dominated}) \\
&= 1 - P(S(2) \text{ Non-dominated}) \\
&= 1 - P\{(v(1) < 0) \cup (v(2) < 0)\} \\
&= 1 - P(v(1) < 0) - P(v(2) < 0) + P\{(v(1) < 0) \cap (v(2) < 0)\}
\end{aligned}$$

Table 2.7.1 summarizes the density functions of s, t, and u for the 3x3 matrix with entries that are random variables of left-skewed triangular, uniform, and right-skewed triangular distributions for type 1 matrices.

Table 2.7.2 summarizes the probabilities for the type 1 matrices to be positive with entries that are random variables of left-skewed triangular, uniform, and right-

skewed triangular distributions.

TABLE 2.7.1

DENSITY FUNCTIONS FOR TYPE 1 ASSOCIATED GAME

Distributions	$f(s)$	$f(t)$	$f(u)$
1) Left-skewed	$2 \cdot (2+s)$	$2 \cdot (2+t)$	$2 \cdot (1+2u)$
triangular	$\frac{3 \cdot (1+s)^3}{1}$	$\frac{3 \cdot (1+t)^3}{1}$	$\frac{3 \cdot (1+u)^3}{1}$
2) Uniform	$\frac{1}{(1+s)^2}$	$\frac{1}{(1+t)^2}$	$\frac{1}{(1+u)^2}$
3) Right-skewed	$2 \cdot (1+2s)$	$2 \cdot (1+2t)$	$2 \cdot (2+u)$
triangular	$\frac{3 \cdot (1+s)^3}{1}$	$\frac{3 \cdot (1+t)^3}{1}$	$\frac{3 \cdot (1+u)^3}{1}$

TABLE 2.7.2

PROBABILITIES OF $V > 0$ FOR TYPE 1 MATRICES

Distributions	$P(v(1) < 0)$	$P(v(2) < 0)$	$P(v(1) < 0 \cap v(2) < 0)$	$P(v > 0)$
1) Left-skewed	$\frac{11}{18}$	$\frac{11}{18}$	$\frac{61}{135}$	$\frac{31}{135}$
triangular	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
2) Uniform	$\frac{7}{18}$	$\frac{7}{18}$	$\frac{31}{135}$	$\frac{61}{135}$
3) Right-skewed	$\frac{11}{18}$	$\frac{11}{18}$	$\frac{61}{135}$	$\frac{31}{135}$
triangular	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$

For the Type 2 matrix that contains strategies to be examined for mixed dominance, a game matrix associated with strategy $S(2)$ is first created. Figure 2.7.3 shows the associated game matrix for the Type 2 matrix and its subgames.

FIGURE 2.7.3

ASSOCIATED GAME MATRIX FOR TYPE 2 MATRIX

Associated game matrix

	N (1)	N (2)	N (3)
S (a1)	$x(3) - x(2)$	$y(3) - y(1)$	$z(1) - z(2)$
S (a2)	$x(1) - x(2)$	$y(2) - y(1)$	$z(3) - z(2)$

	N (1)	N (3)		N (2)	N (3)
S (a1)	$x(3) - x(2)$	$z(1) - z(2)$	S (a1)	$y(3) - y(1)$	$z(1) - z(2)$
S (a2)	$x(1) - x(2)$	$z(3) - z(2)$	S (a2)	$y(2) - y(1)$	$z(3) - z(2)$

Subgame 1

	N (1)	N (2)
S (a1)	$x(3) - x(2)$	$y(3) - y(1)$
S (a2)	$x(1) - x(2)$	$y(2) - y(1)$

Subgame 2

Subgame 3

The associated game matrix can be subdivided into three subgames as shown in Figure 2.7.3. Let $v(1)$, $v(2)$, and $v(3)$ be the expected values of the subgames 1, 2 and 3 respectively. For subgame 2, $v(2)$ will always be positive since all the entries in row S(a2) are positive. Similarly, in subgame 3, $v(3)$ will always be positive since all the entries in row S(a1) are positive. Thus, if we want to determine if strategy S(2) in Type 2 matrix is non-dominated, we only have to consider if the expected values in subgame 1 is less than zero. Let v be the expected value of the type 2 associated game matrix, for $v < 0$,

$$(2.7.9) \quad v < 0 \text{ iff, } \{\text{Min}(v(1), v(2), v(3)) < 0\} = \{\text{Min } v(1) < 0\}$$

Then the probability that strategy $S(2)$ in Type 2 matrix is non-dominated,

$$(2.7.10) \quad P(S(2) \text{ Non-dominated}) \\ = P\{(v(1) < 0)\}$$

We then let,

$$(2.7.11) \quad X = \frac{x(3) - x(2)}{x(2) - x(1)}, \quad Z = \frac{z(2) - z(1)}{z(3) - z(2)}$$

and, s = one particular value of X

u = one particular value of Z

Then we can compute,

$$(2.7.12) \quad P(v(1) < 0) \\ = P(X < Z) \\ = P\left(\frac{x(3) - x(2)}{x(2) - x(1)} < \frac{z(2) - z(1)}{z(3) - z(2)}\right) \\ = \int_0^1 \int_0^u f(s) \cdot f(u) \, ds \, du$$

Therefore, the probability that strategy $S(2)$ is mixed dominated will be,

$$(2.7.13) \quad P(S(2) \text{ Mixed dominated}) \\ = 1 - P(S(2) \text{ Non-dominated}) \\ = 1 - P(v(1) < 0) \\ = P(v > 0)$$

Table 2.7.3 summarizes the probabilities for the type 2 matrices to be positive with entries that are random variables of left-skewed triangular, uniform, and right-skewed triangular distributions.

TABLE 2.7.3
PROBABILITIES OF $V > 0$ FOR TYPE 2 MATRICES

Distributions	$f(s)$	$f(u)$	$P(v(1)<0)$	$P(v>0)$
1) Left-skewed	$2 \cdot (2+s)$	$2 \cdot (1+2u)$	11	7
triangular	$\frac{3}{3 \cdot (1+s)}$	$\frac{3}{3 \cdot (1+u)}$	18	18
2) Uniform	$\frac{1}{2 \cdot (1+s)}$	$\frac{1}{2 \cdot (1+u)}$	1	1
3) Right-skewed	$2 \cdot (1+2s)$	$2 \cdot (2+u)$	7	11
triangular	$\frac{3}{3 \cdot (1+s)}$	$\frac{3}{3 \cdot (1+u)}$	18	18

Table 2.7.4 summarizes the expected values of mixed dominance and non-dominance for the 3×3 matrices with entries that are random variables of left-skewed triangular, uniform, and right-skewed triangular distributions. For the 3×3 matrix, the total number of types of matrices, Q , that contain mixed dominated strategies is 2.

TABLE 2.7.4
EXPECTED VALUES OF MIXED DOMINANCE AND NON-DOMINANCE
FOR 3×3 MATRICES

Distributions	$T(1)$	$T(2)$	$F(3,3)$	$M(3,3)$	$N(3,3)$
1) Left-skewed			136		
triangular	3	6	$\frac{45}{45}$	0.0840	2.2771
2) Uniform	3	6	4	0.1111	2.2500
3) Right-skewed			226		
triangular	3	6	$\frac{45}{45}$	0.1395	2.2216

The above table shows that the matrix with entries that are right-skewed triangularly distributed has the highest expected value of mixed dominance, while the matrix with entries that are left-skewed triangularly distributed has the lowest expected value of mixed dominance.

2.8. The matrix with $n=4$ and $m=2$

In the 4×2 matrix, there are a total of 24 combinations of matrices if the entries in column 1 and column 2 are ordered in increasing order of magnitude, such that for columns 1 and 2 respectively,

$$(2.8.1) \quad b > x(4) > x(3) > x(2) > x(1) > a$$

$$b > y(4) > y(3) > y(2) > y(1) > a$$

For the 4×2 matrix, the total combinations and the total number of purely dominated strategies are, respectively,

$$(4.8.2) \quad B(4,2) = 4! = 24$$

$$C(4,2) = 46$$

Then the expected values of pure dominance for the 4×2 matrix is,

$$(2.8.3) \quad P(4,2) = \frac{C(4,2)}{B(4,2)} = \frac{46}{24} = 1.9167$$

In the 24 combinations of matrices, there are only 7 combinations that contain strategies to be examined for mixed dominance. Of these 7 combinations, it is possible to group these combinations into 5 types of matrices based on the symmetrical relations of the ordered entries

in the two columns of the matrices. The five types of matrices that contain strategies to be examined for mixed dominance are listed in Figure 2.8.1.

FIGURE 2.8.1

TOTAL COMBINATIONS FOR THE 4X2 MATRIX

N(1) N(2)		N(1) N(2)		N(1) N(2)	
S(1)	x(4) y(1)	S(1)	x(4) y(2)	S(1)	x(4) y(2)
S(2)	x(3) y(3)	S(2)	x(3) y(3)	S(2)	x(3) y(3)
S(3)	x(2) y(2)	S(3)	x(1) y(1)	S(3)	x(2) y(1)
S(4)	x(1) y(4)	S(4)	x(1) y(4)	S(4)	x(1) y(4)
Type 1		Type 2		Type 3	
N(1) N(2)		N(1) N(2)			
S(1)	x(4) y(2)	S(1)	x(4) y(1)		
S(2)	x(3) y(1)	S(2)	x(3) y(2)		
S(3)	x(2) y(3)	S(3)	x(2) y(3)		
S(4)	x(1) y(4)	S(4)	x(1) y(4)		
Type 4		Type 5			

In matrices of Type 1, Type 2, and Type 5, each type of matrix contains only one combination of matrix; whereas in Type 3 and Type 4 matrices, each type of matrix contains two symmetrical combinations of matrices. In matrices of Type 1, Type 2, and Type 3, strategy S(3) in each of these matrices can be eliminated for being purely dominated. Only strategy S(2) of the above three types of

matrices has to be examined for mixed dominance. In Type 4 matrix, strategy S(3) can be eliminated for pure dominance, and only strategy S(2) has to be examined for mixed dominance. In Type 5 matrix, there are two strategies in the matrix, S(2) and S(3), have to be examined for mixed dominance. To test if the above mentioned strategies will be mixed dominated, we first create a game matrix associated with each strategy tested. Based on the associated game matrices for Type 1, Type 2, Type 3, and Type 4 matrices, Table 2.8.1 summarizes the density functions and the probabilities that the expected values of mixed dominance for Type 1, Type 2, Type 3, and

TABLE 2.8.1
PROBABILITIES OF MIXED DOMINANCE IN MATRICES OF TYPE
1, 2, 3, AND 4 WITH UNIFORMLY DISTRIBUTED ENTRIES

Matrix	Strategy Tested	$f(s)$	$f(t)$	$P(v>0)$
Type 1	S(2)	$\frac{2 \cdot s}{(1+s)^3}$	$\frac{2}{(1+t)^3}$	$\frac{1}{6}$
Type 2	S(2)	$\frac{1}{(1+s)^2}$	$\frac{1}{(1+t)^2}$	$\frac{1}{2}$
Type 3	S(2)	$\frac{2}{(1+s)^2}$	$\frac{1}{(1+t)^3}$	$\frac{1}{3}$
Type 4	S(3)	$\frac{1}{(1+s)^2}$	$\frac{2 \cdot t}{(1+t)^3}$	$\frac{2}{3}$

Type 4 matrices will be positive when the entries of these matrices are uniformly distributed. The strategy tested for mixed dominance will be mixed dominated if the expected value of its associated game is positive.

In the Type 5 matrix, there are two strategies $S(2)$ and $S(3)$ in the matrix to be examined for mixed dominance. Therefore, the analysis for the expected value of mixed dominance in Type 5 matrix is more complicated than in matrices with only one strategy to be examined for mixed dominance.

To determine the expected value of mixed dominance for Type 5 matrix, we first let,

(2.8.4)

$$p(1) = P(S(2), S(3) \text{ Mixed dominated})$$

$$p(2) = P(S(2) \text{ Mixed dominated}, S(3) \text{ Non-dominated})$$

$$p(3) = P(S(2) \text{ Non-dominated}, S(3) \text{ Mixed dominated})$$

$$p(4) = P(S(2), S(3) \text{ Non-dominated})$$

$$\text{where, } p(1) + p(2) + p(3) + p(4) = 1$$

$$p(2) = p(3)$$

Therefore, the probabilities that strategies, $S(2)$, $S(3)$ will be mixed dominated are respectively,

$$(2.8.5) \quad P(S(2) \text{ Mixed dominated}) = p(1) + p(2)$$

$$P(S(3) \text{ Mixed dominated}) = p(1) + p(3)$$

We then proceed to determine the probabilities $p(1)$, $p(2)$, $p(3)$ and $p(4)$ for the Type 5 matrix. To determine $p(1)$, the probability that both strategies $S(2)$ and $S(3)$ will be mixed dominated, we first create two game matrices

associated with strategies $S(2)$ and $S(3)$. Figure 2.8.2 outlines the game matrices associated with strategies $S(2)$ and $S(3)$.

FIGURE 2.8.2

ASSOCIATED GAME MATRICES FOR $S(2)$ AND $S(3)$
TO BE MIXED DOMINATED IN TYPE 5 MATRIX

	N (1)	N (2)		N (1)	N (2)
S (a1)	$x(4) - x(3)$	$y(1) - y(2)$	S (a1)	$x(4) - x(2)$	$y(1) - y(3)$
S (a2)	$x(1) - x(3)$	$y(4) - y(2)$	S (a2)	$x(1) - x(2)$	$y(4) - y(3)$
Associated game matrix for $S(2)$ mixed dominated			Associated game matrix for $S(3)$ mixed dominated		

From Figure 2.8.2, let $v(1)$ and $v(2)$ be the expected values of the game matrices associated with strategies $S(2)$ and $S(3)$ respectively. We also let,

$$(2.8.6) \quad X(1) = \frac{x(4) - x(3)}{x(3) - x(1)}, \quad Y(1) = \frac{y(2) - y(1)}{y(4) - y(2)}$$

$$X(2) = \frac{x(4) - x(2)}{x(2) - x(1)}, \quad Y(2) = \frac{y(3) - y(1)}{y(4) - y(3)}$$

and, $s(1)$ = one particular value of $X(1)$

$t(1)$ = one particular value of $Y(1)$

$s(2)$ = one particular value of $X(2)$

$t(2)$ = one particular value of $Y(2)$

Then we can compute,

$$(2.8.7)$$

$$p(1)$$

$$= P(S(2), S(3) \text{ Mixed dominated})$$

$$\begin{aligned}
&= P\{(v(1) > 0) \cap (v(2) > 0)\} \\
&= P\{(X(1) > Y(2)) \cap (X(2) > Y(2))\} \\
&= \int_0^\infty \int_0^{t(2)} \int_{t(2)}^\infty \int_{t(1)}^{s(2)} f(s(1), s(2), t(1), t(2)) \, ds(1) ds(2) dt(1) dt(2) \\
&= \int_0^\infty \int_0^{t(2)} \int_{t(2)}^\infty \int_{t(1)}^{s(2)} f(s(1), s(2)) \cdot f(t(1), t(2)) \, ds(1) ds(2) dt(1) dt(2)
\end{aligned}$$

For a matrix with entries that are uniformly distributed, we obtain,

$$\begin{aligned}
(2.8.6) \quad f(s(1), s(2)) &= \frac{2}{(1+s(1))^2 \cdot (1+s(2))^2} \\
f(t(1), t(2)) &= \frac{2}{(1+t(1))^2 \cdot (1+t(2))^2}
\end{aligned}$$

$$\begin{aligned}
&\text{and,} \quad p(1) \\
&= P(S(2), S(3) \text{ Mixed dominated}) \\
&= \frac{1}{3}
\end{aligned}$$

To determine $p(2)$, the probability that strategy $S(2)$ is mixed dominated and $S(3)$ is non-dominated, we first create two game matrices associated with strategies $S(2)$ and $S(3)$. Figure 2.9.3 outlines the game matrices associated with strategies $S(2)$ and $S(3)$ when $S(2)$ is mixed dominated and $S(3)$ is non-dominated. The game matrices associated with $S(2)$ and $S(3)$ can each be divided into three subgames as shown in Figure 2.8.3. Let subgames 1, 2, and 3 be the subgames associated with $S(2)$ and subgames 4, 5, and 6 be the subgames associated with $S(3)$.

FIGURE 2.8.3
 ASSOCIATED GAME MATRICES FOR S (2) MIXED DOMINATED
 AND S (3) NON-DOMINATED IN TYPE 5 MATRIX

Associated game matrix for S (2) mixed dominated			Associated game matrix for S (3) non-dominated		
	N (1)	N (2)		N (1)	N (2)
S (a1)	x (4) -x (3)	y (1) -y (2)	S (a1)	x (4) -x (2)	y (1) -y (3)
S (a2)	x (2) -x (3)	y (3) -y (2)	S (a2)	x (3) -x (2)	y (2) -y (3)
S (a3)	x (1) -x (3)	y (4) -y (2)	S (a3)	x (1) -x (2)	y (4) -y (3)

	N (1)	N (2)		N (1)	N (2)
S (a1)	x (4) -x (3)	y (1) -y (2)	S (a1)	x (4) -x (3)	y (1) -y (2)
S (a3)	x (1) -x (3)	y (4) -y (2)	S (a2)	x (2) -x (3)	y (3) -y (2)

Subgame 1			Subgame 2		
	N (1)	N (2)		N (1)	N (2)
S (a2)	x (2) -x (3)	y (3) -y (2)	S (a1)	x (4) -x (2)	y (1) -y (3)
S (a3)	x (1) -x (3)	y (4) -y (2)	S (a3)	x (1) -x (2)	y (4) -y (3)

Subgame 3			Subgame 4		
	N (1)	N (2)		N (1)	N (2)
S (a2)	x (3) -x (2)	y (2) -y (3)	S (a1)	x (4) -x (2)	y (1) -y (3)
S (a3)	x (1) -x (2)	y (4) -y (3)	S (a2)	x (3) -x (2)	y (2) -y (3)

Subgame 5			Subgame 6		
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Let $v(1)$, $v(2)$, $v(3)$, $v(3)$, $v(4)$, $v(5)$, and $v(6)$ be the expected values of the subgames for subgames 1 to 6 respectively. Strategy S (2) will be dominated iff,

$$(2.8.9) \quad \begin{aligned} \text{Max } \{v(1), v(2), v(3)\} &> 0 \\ &= \text{Max } \{v(1), v(2)\} > 0 \end{aligned}$$

and strategy S(3) will be non-dominated iff,

$$(2.8.10) \quad \begin{aligned} &\{v(4) < 0 \cap v(5) < 0 \cap v(6) < 0\} \\ &= \{v(4) < 0 \cap v(5) < 0\} \end{aligned}$$

Subgame 3 can be eliminated from consideration since $v(3)$ will always be less than zero, as entries in column 2 of subgame 3 are all positive and can be dominated by column 1 where all the entries are negative. Therefore, the value of the subgame 3 will be negative. Similarly, subgame 6 can be eliminated from consideration for the same reason, since all the entries in column 2 are negative. We then let,

$$(2.8.11) \quad \begin{aligned} X(1) &= \frac{x(4) - x(3)}{x(3) - x(2)}, & Y(1) &= \frac{y(2) - y(1)}{y(3) - y(2)} \\ X(2) &= \frac{x(4) - x(2)}{x(2) - x(1)}, & Y(2) &= \frac{y(3) - y(1)}{y(4) - y(3)} \end{aligned}$$

and, $s(1)$ = one particular value of $X(1)$

$t(1)$ = one particular value of $Y(1)$

$s(2)$ = one particular value of $X(2)$

$t(2)$ = one particular value of $Y(2)$

Then we can compute,

$$(2.8.12)$$

$$p(2)$$

$$= P(S(2) \text{ Mixed dominated, } S(3) \text{ Non-dominated})$$

$$= P(\{v(1) > 0\} \cup \{v(2) > 0\} \cap \{v(4) < 0\} \cap \{v(5) < 0\})$$

$$\begin{aligned}
&= P\{(v(2) > 0) \cap (v(4) < 0) \cap (v(5) < 0)\} \\
&\quad + P\{(v(1) > 0) \cap (v(4) < 0) \cap (v(5) < 0)\} \\
&\quad - P\{(v(1) > 0) \cap (v(2) > 0) \cap (v(4) < 0) \cap (v(5) < 0)\} \\
&= P\{(v(2) > 0) \cap (v(4) < 0) \cap (v(5) < 0)\} \\
&\quad + P\{(v(1) > 0) \cap (v(2) < 0) \cap (v(4) < 0) \cap (v(5) < 0)\} \\
&= P\{(v(2) > 0) \cap (v(4) < 0) \cap (v(5) < 0)\} \\
&= P\{(v(2) > 0) \cap (v(4) < 0)\} \\
&\quad - P\{(v(2) > 0) \cap (v(4) < 0) \cap (v(5) > 0)\} \\
&= P\{(v(2) > 0) \cap (v(4) < 0)\} \\
&= P\{(X(1) > Y(1)) \cap (X(2) > Y(2))\} \\
&= \int_0^\infty \int_0^\infty \int_0^{t(2)} \int_{t(1)}^\infty f(s(1), s(2), t(1), t(2)) \, ds(1) ds(2) dt(1) dt(2) \\
&= \int_0^\infty \int_0^\infty \int_0^{t(2)} \int_{t(1)}^\infty f(s(1), s(2)) \cdot f(t(1), t(2)) \, ds(1) ds(2) dt(1) dt(2)
\end{aligned}$$

For a matrix with entries that are uniformly distributed, we obtain,

$$(2.8.13) \quad f(s(1), s(2)) = \frac{2 \cdot s(2)}{(1+s(1))^2 \cdot (1+s(2))^3}$$

$$f(t(1), t(2)) = \frac{2 \cdot t(2)}{(1+t(1))^2 \cdot (1+t(2))^3}$$

and, $p(2)$

$$= P(S(2) \text{ Mixed dominated, } S(3) \text{ Non-dominated})$$

$$= \frac{1}{4}$$

Pased on the symmetrical relations of the entries in Type 5 matrix, we also obtain,

$$(2.8.14) \quad p(3) = p(2) \\ = P(S(2) \text{ Mixed dominated}, S(3) \text{ Non-dominated})$$

Therefore, we can now compute the probabilities of mixed dominance for strategies S(2) and S(3) in Type 5 matrix with uniformly distributed entries, respectively,

$$(2.8.15) \quad P(S(2) \text{ Mixed dominated}) = p(1) + p(2) \\ = \frac{7}{12}$$

$$P(S(3) \text{ Mixed dominated}) = p(1) + p(3) \\ = \frac{7}{12}$$

For the 4x2 matrix with entries which are uniformly distributed random variables, Table 2.8.2 summarizes the probabilities of mixed dominance for strategies examined for mixed dominance in these matrices. Table 2.8.2 also summarizes the sum of the expected values of mixed dominance for all the combinations of the ordered matrices.

Based on the results of Table 2.8.2, we can compute the expected values of mixed dominance and non-dominance for the 4x2 matrix when the entries are uniformly distributed random variables. For uniformly distributed entries, the expected values of mixed dominance and non-dominance are respectively,

$$(2.8.16) \quad M(4,2) = \frac{E(4,2)}{B(4,2)} = \frac{3.8333}{24} = 0.1597 \\ N(4,2) = 4 - P(4,2) - M(4,2) = 1.9236$$

TABLE 2.8.2
PROBABILITIES OF MIXED DOMINANCE IN MATRICES OF TYPE
1, 2, 3, 4, AND 5 WITH UNIFORMLY DISTRIBUTED ENTRIES

Matrix	Strategy Tested	T(q)	P(v>0)	T(q).P(v>0)
Type 1	S(2)	1	$\frac{1}{6}$	$\frac{1}{6}$
Type 2	S(2)	1	$\frac{1}{2}$	$\frac{1}{2}$
Type 3	S(2)	2	$\frac{1}{3}$	$\frac{2}{3}$
Type 4	S(3)	2	$\frac{2}{3}$	$\frac{4}{3}$
Type 5	S(2)	1	$\frac{7}{12}$	$\frac{7}{12}$
Type 5	S(3)	1	$\frac{7}{12}$	$\frac{7}{12}$
-----				F(n,m) = 3.8333

2.9. Summary

Analysis in this chapter shows that for entries with continuous distribution, the expected value of pure dominance depends only on n , the total number of strategies, and m , the total number of columns in the matrix, and is independent of the class of continuous distribution of the entries. As long as the distribution of the entries is continuous, it does not matter what

class of distribution the entries possess, the expected values of pure dominance will always be the same for a given $n \times m$ matrix.

Analysis in this chapter shows that the expected value of mixed dominance in a given matrix depends only on n and m , and the class of a standardizable distribution for the entries. Difference in means and variances of the same distribution will not affect the expected value of mixed dominance for a given $n \times m$ matrix. After the expected values of pure dominance and dominance are determined, the expected value of mixed non-dominance can be readily computed for a given $n \times m$ matrix.

It is shown that the expected values of mixed dominance depends on the joint density functions of the entries in the associated game matrices with respect to the strategies examined for mixed dominance. Since entries from different classes of distributions have different joint density functions, therefore, the expected values of mixed dominance will be different for matrices with entries from different classes of distributions.

For the 3×2 or 3×3 matrix, it is shown that the matrix with entries that are right-skewed triangularly distributed has the highest expected value of mixed dominance, while the matrix with entries that are left-skewed triangularly distributed has the lowest expected value of mixed dominance. The expected value of mixed dominance for a matrix with entries that are

uniformly distributed is in between the expected values of mixed dominance for matrices with entries that are right-skewed and left-skewed triangularly distributed, respectively. Table 2.9.1 presents a summary of the analytical results obtained to facilitate comparison.

TABLE 2.9.1
SUMMARY TABLE OF ANALYTICAL RESULTS

(n,m)	Distributions	P(n,m)	M(n,m)	N(n,m)
2,2	Any Continuous	0.50	0.00	1.50
2,3	Any Continuous	0.25	0.00	1.75
3,2	Left-skewed triangular	1.1667	0.0648	1.7685
	Uniform	1.1667	0.0833	1.7500
	Right-skewed triangular	1.1667	0.1019	1.7314
3,3	Left-skewed triangular	0.6389	0.0840	2.2771
	Uniform	0.6389	0.1111	2.2500
	Right-skewed triangular	0.6389	0.1395	2.2216
4,2	Uniform	1.9167	0.1597	1.9236

For matrix with entries that are normally distributed, the joint density functions of the normal distribution is extremely difficult to compute and to integrate. Therefore, this chapter does not provide any analytical solutions for the expected value of mixed dominance for matrices with entries that are normally distributed. Results of mixed dominance for matrices with entries that are normally distributed will be provided by the simulation approach as shown in chapter IV of this dissertation.

CHAPTER III

RESEARCH HYPOTHESES AND SIMULATION METHODOLOGY

3.1. Research hypotheses

Based on the results obtained from Chapter II, for the 2×3 and 3×3 matrices, it is shown that the above two matrices with entries that are right-skewed triangularly distributed have higher expected values of mixed dominance than the corresponding matrices with entries that are uniformly distributed. Also, the above two matrices with entries that are uniformly distributed have higher expected values of mixed dominance than the corresponding matrices that are left-skewed triangularly distributed. Therefore, hypotheses 3.1.1 and 3.1.2 hypothesize the relationship among the expected values of mixed dominance for any $n \times m$ matrix with entries that are random variables of 1) right-skewed triangular distribution, 2) uniform distribution, and 3) left-skewed triangular distribution, respectively, as follows.

Hypothesis 3.1.1:

Given the same n and m , the expected value of mixed dominance for an $n \times m$ decision matrix with entries that are random variables of right-skewed triangular distribution is higher than the expected value of mixed dominance for a corresponding $n \times m$ matrix with entries that

are random variables of uniform distribution.

$$H(0): Mr(n,m) = Mu(n,m)$$

$$H(1): Mr(n,m) > Mu(n,m)$$

where, $Mr(n,m)$ is the expected value of mixed dominance for an $n \times m$ matrix with entries that are right-skewed triangularly distributed and $Mu(n,m)$ is the expected value of mixed dominance for a given $n \times m$ matrix with entries that are uniformly distributed.

Hypothesis 3.1.2:

Given the same n and m , the expected value of mixed dominance for an $n \times m$ decision matrix with entries that are random variables of uniform distribution is higher than the expected value of mixed dominance for a corresponding $n \times m$ matrix with entries that are random variables of left-skewed triangular distribution.

$$H(0): Mu(n,m) = Ml(n,m)$$

$$H(1): Mu(n,m) > Ml(n,m)$$

where, $Ml(n,m)$ is the expected value of mixed dominance for an $n \times m$ matrix with entries that are left-skewed triangularly distributed and $Mu(n,m)$ is the expected value of mixed dominance for a given $n \times m$ matrix with entries that are uniformly distributed.

For an $n \times m$ matrix with entries that are random variables of normal distribution, the expected value of mixed dominance is difficult to compute analytically, since the joint density functions of the entries in the associated game matrix is difficult to integrate.

Therefore, Chapter II of this study does not provide any analytical solution for the expected value of mixed dominance for matrices with entries that are normally distributed. However, we can employ simulation to estimate the expected values of mixed dominance for matrices with entries that are normally distributed. Since the normal distribution and the uniform distribution are both symmetrical distributions, we hypothesize that a given $n \times m$ matrix with entries that are normally distributed will have the same expected value of mixed dominance as a corresponding $n \times m$ matrix with entries that are uniformly distributed. Since the right-skewed triangular distribution is skewed to the right and the normal distribution is symmetrically distributed, we hypothesize that for a given $n \times m$ matrix with entries that are right-skewed triangularly distributed will have a higher expected value of mixed dominance than a corresponding $n \times m$ matrix with entries that are normally distributed. Hypotheses 3.1.3 and 3.1.4 formalize the above two relationships.

Hypothesis 3.1.3:

Given the same n and m , the expected value of mixed dominance for an $n \times m$ decision matrix with entries that are random variables of normal distribution is the same as the expected value of mixed dominance for a corresponding $n \times m$ matrix with entries that are random variables of uniform distribution.

$$H(0): M_n(n,m) = \mu(n,m)$$

$$H(1): M_n(n,m) = \mu(n,m)$$

where, $M_n(n,m)$ is the expected value of mixed dominance for an $n \times m$ matrix with entries that are normally distributed and $\mu(n,m)$ is the expected value of mixed dominance for an $n \times m$ matrix with entries that are uniformly distributed.

Hypothesis 3.1.4:

Given the same n and m , the expected value of mixed dominance for an $n \times m$ decision matrix with entries that are random variables of right-skewed triangular distribution is higher than the expected value of mixed dominance for a corresponding $n \times m$ matrix with entries that are random variables of normal distribution.

$$H(0): M_r(n,m) = \mu_n(n,m)$$

$$H(1): M_r(n,m) > \mu_n(n,m)$$

where, $M_r(n,m)$ is the expected value of mixed dominance for an $n \times m$ matrix with entries that are right-skewed triangularly distributed and $\mu_n(n,m)$ is the expected value of mixed dominance for an $n \times m$ matrix with entries that are normally distributed.

We further hypothesize that when n is large, the expected values of mixed dominance and non-dominance will approach constants for matrices with entries that are 1) uniformly, 2) left-skewed triangularly, 3) right-skewed triangularly, and 3) normally distributed, respectively. Therefore, for large n , the ratio of the expected value

of mixed dominance over the expected value of non-dominance will also approach a constant for a given $n \times m$ matrix with entries that process one of the above four continuous distributions. Hypotheses 3.1.5 to 3.1.8 formalize this.

Hypothesis 3.1.5:

Given the same m , the ratio of the expected value of mixed dominance over the expected value of non-dominance for a decision matrix with entries that are random variables of uniform distribution will approach a constant k when n is large.

$$H(0): \frac{\mu(n,m)}{\nu(n,m)} \rightarrow k \text{ for } n \rightarrow \text{large}$$

$$H(1): \frac{\mu(n,m)}{\nu(n,m)} \not\rightarrow k \text{ for } n \rightarrow \text{large}$$

where, $\nu(n,m)$ is the expected value of non-dominance for a given $n \times m$ matrix with entries that are uniformly distributed.

Hypothesis 3.1.6:

Given the same m , the ratio of the expected value of mixed dominance over the expected value of non-dominance for a decision matrix with entries that are random variables of left-skewed triangular distribution will approach a constant k when n is large.

$$H(0): \frac{\mu_1(n,m)}{\nu_1(n,m)} \rightarrow k \text{ for } n \rightarrow \text{large}$$

$$H(1): \frac{\mu_1(n,m)}{\nu_1(n,m)} \not\rightarrow k \text{ for } n \rightarrow \text{large}$$

where, $Nl(n,m)$ is the expected value of non-dominance for a given $n \times m$ matrix with entries that are left-skewed triangularly distributed.

Hypothesis 3.1.7:

Given the same m , the ratio of the expected value of mixed dominance over the expected value of non-dominance for a decision matrix with entries that are random variables of right-skewed triangular distribution will approach a constant k when n is large.

$$P(0): \frac{Mr(n,m)}{Nr(n,m)} \rightarrow k \text{ for } n \rightarrow \text{large}$$

$$P(1): \frac{Mr(n,m)}{Nr(n,m)} \not\rightarrow k \text{ for } n \rightarrow \text{large}$$

where, $Nr(n,m)$ is the expected value of non-dominance for a given $n \times m$ matrix with entries that are right-skewed triangularly distributed.

Hypothesis 3.1.8:

Given the same m , the ratio of the expected value of mixed dominance over the expected value of non-dominance for a decision matrix with entries that are random variables of normal distribution will approach a constant k when n is large.

$$P(0): \frac{Mn(n,m)}{Nn(n,m)} \rightarrow k \text{ for } n \rightarrow \text{large}$$

$$P(1): \frac{Mn(n,m)}{Nn(n,m)} \not\rightarrow k \text{ for } n \rightarrow \text{large}$$

where, $Nn(n,m)$ is the expected value of non-dominance for

a given $n \times m$ matrix with entries that are normally distributed.

3.2. The simulation approach

For large matrices with large values of n and m , the expected values of pure dominance, mixed dominance, and non-dominance are too complicated to compute analytically. Thus, the simulation approach would be an appropriate method to estimate the expected values of pure dominance, mixed dominance, and non-dominance for larger matrices. Data obtained from the simulation study can also be used to shed further light on the properties of the expected values of pure dominance, mixed dominance, and non-dominance; and to test the research hypotheses formulated. Digital computer simulation is a numerical technique and is often useful when standard analytical techniques do not exist or become too complicated in finding solutions to the problem under study. In this study, because of the very complicated analytical solution techniques for large matrices, the simulation method is used. With the advent of high speed computer, simulation can be conducted to solve very complicated problems with relative small consumption of time.

This study is a stochastic simulation in which random variables are generated. A stream of pseudorandom variables are generated by the computer to represent the

entries of the decision matrices with various probability distributions. Results of the simulation can be used to make inferences on other matrices not covered by this simulation study. The objectives of the simulation study is threefold: 1) to estimate the expected values of pure dominance, mixed dominance, and non-dominance for decision matrices with various values of n and m , and various probability distributions of the entries, 2) to analyze and to examine the properties of the expected values of pure dominance, mixed dominance and non-dominance, and to examine the effects of changes in the controllable variables of the decision matrices studied, and 3) to test the research hypotheses.

3.3. The research design

The decision matrices studied in this research have three controllable variables. The first variable is n , the number of strategies available to the decision maker. The second variable is m , the number of states of nature. The third variable is the entries, $u(i,j)$, in the decision matrices, which are stochastic random variables with probability distributions of four different classes, namely, 1) uniform distribution, 2) left-skewed triangular distribution, 3) right-skewed triangular distribution, and 4) normal distribution.

Simulation is performed for matrices with n from $n=2$ to $n=1,050$ and m from $m=2$ to $m=7$. For matrices with n not

more than 100 strategies, simulation is replicated 800 times for each $n \times m$ matrix studied. For matrices with n greater than 100 strategies but not more than 300 strategies, simulation is replicated 200 times for each $n \times m$ matrix studied. Based on the results of the values of the number of pure dominance, mixed dominance and non-dominance obtained in each of the replication of the simulation, we can then computed the expected values of pure dominance, mixed dominance and non-dominance.

The reason we use at least 200 replications is that we do not want to obtain too large confidence intervals for the expected values estimated. Replications of 200 times and more seem to provide reasonably small confidence intervals for the expected values. For n less than 100 strategies, the computer time required for the simulation of an $n \times m$ matrix is less than a few minutes if we use 800 replications in the simulation. However, as n increases to very large values, computer time becomes very large. Therefore, for n not more than 100 strategies, we use 800 replications for the simulation for each $n \times m$ matrix. For n over 100 strategies but not more than 300 strategies, we use 200 relications. When n exceeds 300 strategies, simulation time for each $n \times m$ matrix can exceed 30 minutes. Therefore, for n greater than 300, we will use the sampling approach. In the sampling approach, a total of 20 strategies are randomly selected from the n strategies in the given $n \times m$ matrix. Then the 20 randomly selected

strategies are then examined for pure dominance, mixed dominance and non-dominance. Based on the results of the values of the number of pure dominance, mixed dominance and non-dominance obtained in each of the replication of the simulation, we can then compute the expected values of pure dominance, mixed dominance and non-dominance. A 95 per cent confidence interval is then constructed around the expected values of pure dominance, mixed dominance and non-dominance.

A stream of i.i.d. uniformly distributed random variables are generated by the IBM 370 computer to represent the entries in the decision matrices for entries that are uniformly distributed. For entries that are left-skewed triangularly distributed, right-skewed triangularly distributed, or normally distributed, the stream of uniform deviates can be transformed into the required distribution readily. A computer program is formulated to represent the given $n \times m$ decision matrix studied and the computation procedures required for obtaining the results. Based on the simulation results, regression analysis can be conducted to obtain inferences on the expected values of pure dominance, mixed dominance and non-dominance on other matrices not covered by the simulation.

3.4. Description of the Simulation

For a decision matrix with a given n rows and m

columns, we first generate a stream of i.i.d. random deviates of the desired probability distribution to represent the entries of the matrix. We then seek to identify and to eliminate purely dominated strategies from the decision matrix. $S(k)$ is said to be purely dominated strictly for entries that are continuously distributed, if for all j and $i \neq k$,

$$(3.4.1) \quad u(i, j) - u(k, j) > 0$$

A two-person zero sum game associated with $S(k)$ is constructed for all j and all $i \neq k$, such that,

$$(3.4.2) \quad a(i, j) = u(i, j) - u(k, j)$$

If a row in the associated game matrix contains only non-negative entries, then pure dominance is presented and $S(k)$ can be eliminated for pure dominance. The total number of purely dominated strategies for each $n \times m$ matrix examined are then tabulated.

Fishburn (5) showed that if a column has a unique maximum, the strategy corresponding to the maximum entry will be non-dominated. This leaves only strategies that fail the test for pure dominance and do not have a corresponding maximum entry in their columns to be tested for mixed dominance. Based on the associated game constructed for the strategy $S(k)$ to be tested, the dominated rows and columns of the associated game are then first removed, resulting in the further reduction of the matrix. We then apply the Brown's iteration procedure (2) to produce an upper bound v and a lower bound v , for the

value of the game at each iteration. Brown's procedure (2) can be terminated as soon as a negative upper bound or a positive lower bound is obtained. If a negative v is obtained, the strategy tested is non-dominated. If a positive v is obtained during iteration, then the strategy being tested is identified as being mixed dominated and can be eliminated from the decision matrix. The total number of mixed dominated strategies for each $n \times m$ matrix examined is also tabulated.

The remaining strategies in the decision matrix which have failed the test for being dominated are the non-dominated strategies. The total number of non-dominated strategies for each $n \times m$ matrix examined is also tabulated. For each combination of n rows and m columns of the decision matrix, simulation is replicated 200 to 800 times as mentioned in section 3 of this chapter. The results of the simulation are then analyzed to provide more information about the properties of dominance in decision matrices.

3.5. Brown's iterative procedure

Brown (2) (18) introduced an iterative procedure to solve the value of a game. Brown's method (2) is a practical procedure for solving games (6). The method produces an upper bound, $\bar{v}$, and a lower bound, $\underline{v}$, for the value of the game at each iteration. Brown's method (2) can be applied to solve the game associated with $S(k)$.

Let $a(i,j)$ be the matrix of payments from the column player to the row player. Let t be the t th choice of pure strategies for the two sides, and let $c(i,t)$ and $p(j,t)$ be the probabilities of strategies $S(i,t)$ and strategies $S(j,t)$ chosen for $t=(1,\dots,T)$ respectively. Then the objective is that player 1 should maximize,

$$\sum_{i=1}^n c(i,t) \cdot a(i,j) \text{ at the } t \text{th iteration, and player 2}$$

should minimize $\sum_{j=1}^m p(j,t) \cdot a(i,j)$ at the $t+1$ th iteration.

We define,

$$(3.5.1) \quad \bar{v}(t) = \min_j \sum_{i=1}^n c(i,t) \cdot a(i,j)$$

$$v(t) = \max_i \sum_{j=1}^m p(j,t) \cdot a(i,j)$$

It can be readily shown that $v(t) < v < \bar{v}(t)$, where v is the value of the game. For large value of t , both $v(t)$ and $\bar{v}(t)$ will converge to v as shown by Robinson (23).

$$(3.5.2) \quad \lim_{t \rightarrow \infty} \bar{v}(t) = \lim_{t \rightarrow \infty} v(t) = v$$

As the iterative procedure must converge, the Brown's method (2) can yield the value v of the game. The mixtures $c(i,t)$ and $p(j,t)$ can be represented as mixed strategies, and the corresponding $\bar{v}(t)$ and $v(t)$ are the most favorable payoff that each player can obtain by the use of the particular mixed strategies.

Brown's procedure (2) can be employed to find the approximate value of a game. Since v lies in between its

upper and lower limits, the iterative procedure can be terminated as soon as the desired degree of accuracy is reached. In determining the dominance of a strategy, it is not required to know the value of the associated game. Information about the value of an associated game is positive or negative will be sufficient to determine if a strategy is being mixed dominated. In the iterative procedure, if the lower bound $\underline{v}(t)$ reaches a positive value, then the LP in (2.3.4) has a feasible solution and the strategy tested is being mixed dominated. If the upper bound $\bar{v}(t)$ reaches a negative value during iteration, then the LP in (2.3.4) has no feasible solution and the strategy tested is non-dominated. As we are not interested in determining the exact value of the game, the iterative procedure can be terminated as soon as a positive $\underline{v}(t)$ or a negative $\bar{v}(t)$ is attained, thus reaching a conclusion of the status of the strategy tested for mixed dominance. Since the exact value of the game is not required, the iterative procedure can be very efficient and only a small number of iterations is sufficient to obtain the required results.

3.6. Random number generator and the generation of uniform deviates

The method used in this study to generate a stream of pseudo-random numbers is the multiplicative congruential method. The congruent method in random

number generation was initially introduced by Lehman (16). The multiplicative congruent method has been shown to perform very well statistically by Gorenstein (7). The congruential method of generating random numbers is by far the most successful method (14). It has the advantage of fast generating of random numbers resulting in the small consumption of computer time. In addition, the multiplicative congruential method is faster than the mixed congruential method and is found statistically more reliable (10).

The multiplicative congruential method generates a sequences of numbers $x(i)$ by the equation,

$$(3.6.1) \quad x(i+1) = A \cdot x(i) \pmod{p}$$

Putchinson (10) suggested that for binary computer p be the largest prime in 2^b , where b is the number of available bits. This will permit the largest possible cycle length or period. Lewis and Goodman (17) suggest that P be a positive primitive root of p . For an IBM 370 computer, there are 31 bits available for computation in the 32-bit general register of the computer, and the largest prime p less than or equal to $2^{31} - 1$ is $2^{31} - 1$ itself. Lewis and Goodman (17) then let $A = 7^k$, which is a primitive root of $2^{31} - 1$. Any power (mod P) of 7, i.e., 7^k where k is not a factor of $p-1$, is also a positive root of p . In this way, many A 's will be generated. Dowham and

Roberts (4) confirmed empirically that an A approximately equal to p can give good results. IPM (11) also suggested that the seed of the generator, $x(0)$, should be an odd number. Based on the above findings, this study uses the multiplicative congruential generator with $A=16807$ and seeds $x(0)$'s odd. The random number generator will generate a stream of i.i.d. uniformly distributed random numbers with range $(0,1)$.

3.7. Generation of left-skewed triangular and right-skewed triangular deviates

The inverse transformation method is used for the transformation of a stream of i.i.d. standard uniform random numbers into random numbers with triangular distribution. From a stream of i.i.d. standard uniformly distributed random numbers $x(i)$'s with interval $(0,1)$, let $F(l(i))$ be the cumulative function of a left-skewed triangular random variable $l(i)$ with interval $(0,1)$. Let,

$$(3.7.1) \quad F(l(i)) = x(i)$$

Then the left-skewed triangular random deviates can be expressed as,

$$(3.7.2) \quad l(i) = F(l(i))^{0.5}$$

Let $F(r(i))$ be the cumulative function of a right-skewed triangular random variable $r(i)$ with interval $(0,1)$, such that,

$$(3.7.3) \quad F(r(i)) = x(i)$$

Then the right-skewed triangular random variable can be expressed as,

$$(3.7.4) \quad r(i) = 1 - F(r(i))^{0.5}$$

3.8. Generation of Normal Deviates

A variety of methods for generation of pseudo-random normal deviates are available, for example, Box and Muller (1), Juncosa (12), Muller (21), etc. Box and Muller (1) used a direct approach to generate a pair of random deviates from the same normal distribution starting from a pair of uniform random numbers. Let $x(1)$ and $x(2)$ be two independent random deviates from the same uniform distribution on the interval $(0,1)$. Then the random variables,

$$(3.8.1) \quad n(1) = (-2 \cdot \ln x(1))^{0.5} \cdot \cos 2\pi \cdot x(2)$$

$$n(2) = (-2 \cdot \ln x(1))^{0.5} \cdot \sin 2\pi \cdot x(2)$$

will be a pair of independent random variables from the same normal distribution with mean zero, and unit variance. Muller (22) have compared the various methods of generating normal deviates and found that the direct approach providing higher accuracy and faster speed than other methods. Thus, this study will use the direct approach in the generation of normal random deviates.

3.9. Point estimator and confidence interval

The simulation estimates the expected values of pure

dominance, mixed dominance and non-dominance for a given decision matrix. For each combination of n rows and m columns in the decision matrix, simulation is replicated either 200 or 800 times. For T replications, let the point estimator of the expected value be,

$$(3.9.1) \quad \hat{f}(T) = \frac{1}{T} \sum_{t=1}^T r(t)$$

where $r(t)$ may represent either the expected value of pure dominance, mixed dominance, or non-dominance at each replication k . The variance of the point estimator will thus be,

$$(3.9.2) \quad \hat{s}^2(T) = \sum_{t=1}^T \frac{(r(t) - \hat{f}(T))^2}{(T-1)}$$

Then the central limit theorem provides a basis for the confidence interval,

$$(3.9.3) \quad \frac{T^{0.5} \cdot (\hat{f}(T) - r(t))}{\hat{s}(T)} \implies N(0,1) \text{ as } T \rightarrow \infty$$

Then the point estimator gives a confidence interval,

$$(3.9.4) \quad \hat{I}(T) = \left\{ \hat{f}(T) \pm \frac{z_{1-\frac{\alpha}{2}} \cdot \hat{s}(T)}{T^{0.5}} \right\}$$

3.10. Approach for testing the research hypotheses

The research hypotheses require the test of significant difference between two expected values. The method we use is the t test to test the significant

difference between two means. If $\hat{r}(1)$ and $\hat{r}(2)$ are the expected values of two large independent random samples of sizes $T(1)$ and $T(2)$ respectively, the sample distribution of the difference $\hat{r}(1) - \hat{r}(2)$ can be approximated as normally distributed with variance,

$$(3.10.1) \quad \hat{s}^2 = \frac{\hat{s}^2(1)}{T(1)} + \frac{\hat{s}^2(2)}{T(2)}$$

where $s(1)$ and $s(2)$ are the standard deviations of the two samples respectively. Then the t value which has the approximate standard normal distribution is,

$$(3.10.2) \quad t = \frac{\hat{r}(1) - \hat{r}(2)}{\hat{s}}$$

Hypothesis 3.1.3 employs the two-tails test. For the two-tails test, the alternative hypothesis is $\hat{r}(1) - \hat{r}(2) \neq 0$, and the null hypothesis is rejected if $t < -t_{\frac{\alpha}{2}}$ or $t > t_{\frac{\alpha}{2}}$.

Hypotheses 3.1.1, 3.1.2 and 3.1.3 employ the one-tail test for the alternative hypothesis $\hat{r}(1) - \hat{r}(2) > 0$, and the null hypothesis is rejected if $t > t_{\alpha}$.

Hypotheses 3.1.5 to 3.1.8 use the two-tails test to test the significant difference between two ratios of the expected value of mixed dominance over the expected value of non-dominance for two matrices with different independent variable n . For two matrices with large values of n , $n(1)$ and $n(2)$ respectively, we hypothesize that the ratios of the expected value of mixed dominance over non-dominance will approach constant is equivalent to the equation,

$$(3.10.3) \quad \frac{M(1)}{N(1)} = \frac{M(2)}{N(2)}$$

$$M(1) \cdot N(2) = N(1) \cdot M(2)$$

where $M(1)$ and $M(2)$ are the expected vales of mixed dominance for the two matrices with two different large n values; and $N(1)$ and $N(2)$ are the expected values of non-dominance for the above two matrices respectively.

We then let,

$$(3.10.4) \quad W = M(1) \cdot N(2)$$

$$V = N(1) \cdot M(2)$$

the expected values and variances for W and V are respectively,

$$(3.10.5) \quad E(W) = E(M(1)) \cdot E(N(2))$$

$$E(V) = E(N(1)) \cdot E(M(2))$$

$$\begin{aligned} \text{Var}(W) &= \text{Var}(M(1)) \cdot \text{Var}(N(2)) + \text{Var}(M(1)) \cdot E(N(2))^2 \\ &\quad + \text{Var}(N(2)) \cdot E(M(1))^2 \end{aligned}$$

$$\begin{aligned} \text{Var}(V) &= \text{Var}(M(2)) \cdot \text{Var}(N(1)) + \text{Var}(M(2)) \cdot E(N(1))^2 \\ &\quad + \text{Var}(N(2)) \cdot E(M(2))^2 \end{aligned}$$

Then the t value will thus be,

$$(3.10.6) \quad t = \frac{E(W) - E(V)}{\sqrt{\frac{\text{Var}(W)}{T(1)} + \frac{\text{Var}(V)}{T(2)}}}$$

CHAPTER IV

ANALYSIS OF SIMULATION RESULTS

4.1. Analysis of the simulation results for the expected values of pure dominance, mixed dominance, and non-dominance

Results of the simulation study for the expected values of pure dominance, mixed dominance and non-dominance are shown in appendix 1 for matrices with n ranging from 2 to 1,050 and m ranging from 2 to 7. For matrices with entries that are distributed 1) uniformly, 2) left-skewed triangularly, 3) right-skewed triangularly, and 4) normally, results from appendix 1 clearly show that the expected values of pure dominance are very much the same for corresponding $n \times m$ matrices with entries that possess either of the above four continuous distributions. This finding supports the analytical finding in Theorem 2.1.1 which proves that for matrices with the same n and m , the expected value of pure dominance will be the same if the entries in the corresponding $n \times m$ matrices are continuously distributed.

If the value of m is held constant, results in Appendix 1 show that the expected values of pure dominance will increase at a decreasing rate for matrices with

entries that possess either one of the four probability distributions studied. If n is fixed, it is shown that the expected values of pure dominance will decrease as m increases.

Results in Appendix 1 also show that if m is held constant, then the expected values of mixed dominance and non-dominance for the matrices studied will slowly approach constants as n increases to very large values. For small value of n and if n is held constant, then the expected values of mixed dominance will increase to a maximum value and then decrease from the maximum value as m increases. For large values of n and if n is held constant, the expected values of mixed dominance will keep on increasing as m increases from 2 to 7 as shown in Appendix 1. Since the simulation only covers matrices with the value of m up to a maximum of 7, the results do not show that the expected values of mixed dominance will reach a maximum value and then decline, if n is large and held fixed. However, based on the results for small n 's, we suggest the same will happen if n is large. Results in Appendix 1 also show that if n is held constant, the expected values of non-dominance will keep on increasing as m increases.

Appendix 2 shows the various ratios of the expected values of pure dominance, mixed dominance and non-dominance for matrices with entries that are, 1) uniformly distributed, 2) left-skewed triangularly

distributed, 3) right-skewed triangularly distributed, and 4) normally distributed. Tables A.2.1 to A.2.4 summarize the ratios for the expected value of mixed dominance over the expected value of pure dominance for matrices with entries that possess the above four distributions, respectively. The results clearly show that the expected values of mixed dominance are quite significant as compared to the expected values of pure dominance, especially for larger $n \times m$ matrices where the ratios can be very high. In many instances, the expected values of mixed dominance exceed the expected values of pure dominance as the ratios can exceed 1.

After the purely dominated strategies are removed from the matrices, only mixed dominated and non-dominated strategies remain in the matrices. In order to examine the relative significance of the number of mixed dominated strategies as compared to the number of non-dominated strategies, we introduced some more ratios between the expected values of mixed dominance and the expected values of non-dominance. Tables A.2.5 to A.2.8 summarize the ratios of the expected value of mixed dominance over the expected value of non-dominance for various $n \times m$ matrices with entries that possess one of the four distributions studied, respectively. The results show that the expected values of mixed dominance are very significant as compared to the expected values of non-dominance; especially for large matrices where the

expected values of mixed dominance exceed the expected values of non-dominance as these ratios are greater than 1. The results also show that if m is held constant, for all the matrices studied, the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant when n increases to large values.

Tables A.2.9 to A.2.12 summarize the ratios of the expected values of mixed dominance over the sum of the expected values of mixed dominance and non-dominance for matrices with entries that possess one of the four probability distributions studied, respectively. These ratios show the effect if the mixed dominated strategies are not removed from the removed from the remaining strategies after the removal of purely dominated strategies. These ratios show that there are high percentages of mixed dominated strategies in the remaining strategies in the matrices as compared to the non-dominated strategies. If the mixed dominated strategies are not removed from the matrices, the decision making runs a high risk of adopting a strategy that will be mixed dominated by other strategies. Thus, mixed dominated strategies should be identified and be removed from the matrices. The various ratios exhibited in Appendix 2 clearly show that mixed dominance is significant when compared to both pure dominance and non-dominance.

4.2. Results for the testing of research hypotheses

Appendix 3 summarizes the results for the testing of the research hypotheses. For hypothesis, 3.1.1, the alternate hypothesis, is, $\mu_r(n,m) > \mu_u(n,m)$, and we employ the one-tail test with significant level at 0.05. We can reject the null hypothesis $\mu_r(n,m) = \mu_u(n,m)$ if the t statistics is greater than 1.645. Based on the results in Table A.3.1, it is shown that with a few exceptions, the t statistics for all the matrices are greater than 1.645. Since the simulation employs random numbers to generate the probability distributions for the entries, some fluctuations in the resulting dependent variable, the expected values of mixed dominance, are inevitable. Although a few of the matrices have t statistics less than 1.645, however, based on the results that all the other t statistics in Table A.3.1 are greater than 1.645, we can conclude that the t statistics of the above two matrices are due to random fluctuation and can be excluded from consideration. Since all of the rest of the t statistics are greater than 1.645, we can conclude that the null hypothesis for Hypothesis 3.1.1 can be rejected.

Table A.3.1 also shows that the t statistics reach the highest values when n is around 80 strategies for all the m values studied. This means that there are more proportional difference for the expected values of mixed dominance between matrices with entries that are

right-skewed triangularly distributed than for matrices with entries that are uniformly distributed when the value of n is around 80 strategies.

For hypothesis 3.1.2, the alternate hypothesis is, $\mu(n,m) > M1(n,m)$, and we employ the one-tail test with significant level at 0.05. We can reject the null hypothesis $\mu(n,m) = M1(n,m)$ if the t statistics is greater than 1.645. Based on the results in Table A.3.2, it is shown that the t statistics for most of the matrices are greater than 1.645, except in the cases for matrices with m equals to 2, 3, 4, and 5, when n becomes large, the t statistics for these matrices are less than 1.645. These means that the expected values of mixed dominance for entries with uniform distribution are higher than the expected values of mixed dominance for entries with left-skewed triangular distribution, except in the cases when n becomes large with m equals to 2, 3, 4, and 5, respectively. For the cases with t statistics greater than 1.645, we can conclude that the null hypothesis for Hypothesis 3.1.2 can be rejected.

Table A.3.2 also shows that the t statistics reach the highest values when n is in the range of 40 to 100 strategies for all the m values studied. This means that there are more proportional difference for the expected values of mixed dominance between matrices with entries that are uniformly distributed than for matrices with entries that are left-skewed triangularly distributed when

the value of n is in the range of 40 to 100 strategies.

For hypothesis 3.1.3, the alternate hypothesis is, $M_n(n,m) \neq \mu(n,m)$, and we employ the two-tails test with significant level at 0.05. We can reject the null hypothesis $M_n(n,m) = \mu(n,m)$ if the t statistics is greater than 1.96. Based on the results in Table A.3.3, it is shown that the t statistics are less than 1.96 initially for matrices with n equals to 3 and 4. However, as n increases, the t statistics become greater than 1.96. This means that for n less than 4, there is no significant difference between the expected values of mixed dominance between matrices with entries that are distributed normally and uniformly, respectively. However, when n increases, the expected values of mixed dominance for matrices with entries that are normally distributed will be higher than matrices with entries that are uniformly distributed.

Table A.3.3 also shows that the t statistics reach the highest values when n is in the range of 60 to 100 strategies for all the m values studied. This means that there are more proportional difference for the expected values of mixed dominance between matrices with entries that are normally distributed than for matrices with entries that are uniformly distributed when the value of n is in the range of 60 to 100 strategies.

For hypothesis 3.1.4, the alternate hypothesis is, $M_r(n,m) > \mu(n,m)$, and we employ the one-tail test with

significant level at 0.05. We can reject the null hypothesis $M_r(n,m) = M_n(n,m)$ if the t statistics is greater than 1.645. Based on the results in Table A.3.4, it is shown that with a few exceptions the t statistics are greater than 1.645 initially for matrices with n less than 10. However, as n increases, the t statistics mostly become less than -1.645. This means that for n less than 10, the expected values of mixed dominance for matrices with entries that are right-skewed triangularly distributed are greater than for matrices with entries that are normally distributed. However, when n increases, the expected values of mixed dominance for matrices with entries that are normally distributed will be higher than matrices with entries that are right-skewed triangularly distributed.

Based on the above four hypotheses, we can reach the conclusion that matrices with entries that are right-skewed triangularly distributed have the highest expected values of mixed dominance when n is less than 10. As n increases, then matrices with entries that are normally distributed will have the highest expected values of mixed dominance. Matrices with entries that are left-skewed triangularly distributed will have the lowest expected values of mixed dominance.

The findings of the simulation are in line with the findings for the analytical solutions in Chapter II which shows that for 3x2 and 3x3 matrices, matrices with entries that are right-skewed triangularly distributed have higher

expected values of mixed dominance than matrices with entries that are uniformly distributed. Matrices with entries that are left-skewed triangularly distributed possess the lowest expected values of mixed dominance as shown in Chapter II.

Hypotheses 3.1.5 to 3.1.8 hypothesize that for large n , the ratios of the expected value of mixed dominance over the expected value of non-dominance will approach constants for matrices with entries that possess either one of the four probability distributions studied. For Hypothesis 3.1.5, the null hypothesis is that for large n and constant m , the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant when the entries are uniformly distributed. This is equivalent to testing the null hypothesis,

$$(4.2.1) \quad \mu(1) \cdot N(2) = M(2) \cdot N(1)$$

as shown in section 3.9 in Chapter III.

For constant m and large n , we can therefore pick two different values of n , $n(1)$ and $n(2)$, and test if there is any significant difference between the two ratios corresponding to $n(1)$ and $n(2)$. We employ the two-tails test with significant level of 0.05. We can reject the null hypothesis if the t statistics exceeds 1.96. Based on the results of Table A.3.5, it is clearly shown that for large n , the t statistics will always be less than 1.96 for all the matrices studied. Therefore, we cannot

reject the null hypothesis and we conclude that the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant for large n and fixed m for matrices with entries that are uniformly distributed.

For Hypothesis 3.1.6, the null hypothesis is that for large n and constant m , the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant when the entries are left-skewed triangularly distributed. Similarly, we employ the two-tails test with significant level of 0.05 as in the previous hypothesis. We can reject the null hypothesis if the t statistics exceeds 1.96. Based on the results of Table A.3.6, it is clearly shown that for large n , the t statistics will always be less than 1.96 for all the matrices studied. Therefore, we cannot reject the null hypothesis and we conclude that the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant for large n and fixed m for matrices with entries that are left-skewed triangularly distributed.

For Hypothesis 3.1.7, the null hypothesis is that for large n and constant m , the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant when the entries are right-skewed triangularly distributed. Similarly, we employ the two-tails test with significant level of 0.05

as in the previous hypothesis. We can reject the null hypothesis if the t statistics exceeds 1.96. Based on the results of Table A.3.7, it is clearly shown that for large n , the t statistics will always be less than 1.96 for all the matrices studied. Therefore, we cannot reject the null hypothesis and we conclude that the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant for large n and fixed m for matrices with entries that are right-skewed triangularly distributed.

For Hypothesis 3.1.8, the null hypothesis is that for large n and constant m , the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant when the entries are normally distributed. Similarly, we employ the two-tails test with significant level of 0.05 as in the previous hypothesis. We can reject the null hypothesis if the t statistics exceeds 1.96. Based on the results of Table A.3.8, it is clearly shown that for large n , the t statistics will always be less than 1.96 for all the matrices studied. Therefore, we cannot reject the null hypothesis and we conclude that the ratio of the expected value of mixed dominance over the expected value of non-dominance will approach a constant for large n and fixed m for matrices with entries that are normally distributed.

4.3. Regression analysis for the expected values of pure dominance, mixed dominance and non-dominance

For fixed values of m , we let n as the independent variable and either the expected value of pure dominance, or the expected value of mixed dominance, or the expected value of non-dominance as the dependent variable. Based on the results in Appendix 1, we know that the dependent variables, i.e., the expected values of pure dominance, mixed dominance and non-dominance will increase at a decreasing rate as the independent variable n increases. This means that a double-log transformation can transform the non-linear function into a linear relationship between the independent and dependent variables. Thus our regression model is,

$$(4.3.1) \quad \ln(X) = A + B \cdot \ln(n)$$

where, A and B are the regression coefficients; n is the number of strategies; and X can represent either the expected value of pure dominance, or the expected value of mixed dominance, or the expected value of non-dominance.

Since the relationship between the dependent and independent variables are linear in the regression model, we can use the ordinary least square method which minimizes the sum of squared errors.

Appendix 4 summarizes the regression results. The results show that the regression model (4.3.1) indeed provides very good fit for the expected values of pure dominance, mixed dominance, and non-dominance,

respectively, for matrices with entries that are distributed with the four probability distributions studied. The R-squared terms are all very high as shown in Table A.4.1 to A.4.12. Most of the regression fits have R-squared term exceed 0.99. Also, all the regression fits have high values of t statistics and F statistics. This means that we can safely apply the regression results to estimate other values of the expected values of pure dominance, mixed dominance and non-dominance for matrices not covered by this simulation study and can obtain good estimates.

4.4. Summary

The simulation study has provided us with ample data on the expected values of pure dominance, mixed dominance and non-dominance. Based on the results and their analysis, we obtain more information about the properties of the expected values of pure dominance, mixed dominance and non-dominance. It is shown that the expected values of mixed dominance is highly significant when compared to the expected values of pure dominance and non-dominance. It is also shown that the ratios of expected values of mixed dominance and the expected values of non-dominance will reach constants for large n and fixed m for all the matrices studied with entries that are distributed, 1) uniformly, 2) left-skewed triangularly, 3) right-skewed triangularly, and 4) normally. For n less than 10

strategies, it is shown that matrices with entries that are right-skewed triangularly distributed have the highest expected values of mixed dominance. For n greater than 10, matrices with entries that are normally distributed possess the highest expected values of mixed dominance. Matrices with entries that are left-skewed triangularly distributed have the smallest expected values of mixed dominance.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1. Summary

In Chapter I we demonstrate the need to undertake the present study on the subject. We define the objectives and scope of this study. In Chapter I we also review the relevant literature concerning pure dominance and mixed dominance.

Chapter II develops the required methods and the related theorems for the analytical solutions for the expected values of pure dominance, mixed dominance and non-dominance for matrices with entries that possess standardizable continuous distributions. Chapter II also presents the analytical solutions for the expected values of pure dominance, mixed dominance and non-dominance for small matrices.

Analysis in Chapter II shows that for matrices with entries that are continuously distributed, the expected value of pure dominance depends only on n , the total number of strategies, and m , the total number of columns in the matrix, and is independent of the class of continuous distribution of the entries. As long as the distribution of the entries are continuous, it does not matter what class of distribution the entries process, the

expected values of pure dominance will always be the same for a given $n \times m$ matrix.

Analysis in Chapter II shows that the expected value of mixed dominance in a given matrix depends only on n , the number of strategies, and m , the number of columns in the matrix, and the class of distribution for the entries. Difference in means and variances of the same distribution will not affect the expected value of mixed dominance for a given $n \times m$ matrix, as long as the distribution of the entries are standardizable. After the expected values of pure dominance and mixed dominance are determined, the expected value of non-dominance can be readily computed for a given $n \times m$ matrix.

It is shown that the expected value of mixed dominance depends on the joint density functions of the entries in the associated game matrices with respect to the strategies examined for mixed dominance. Since entries from different classes of distributions have different joint density functions, therefore, the expected values of mixed dominance will be different for matrices with entries from different classes of distributions.

Chapter III formulates the research hypotheses to guide further research for the study. Since we have to examine the properties of dominance for larger matrices and the computation of the expected values of pure dominance, mixed dominance and non-dominance becomes very complicated, we therefore use the simulation approach to

study the larger matrices. Chapter III also presents the methodology used in the simulation study.

Chapter IV summarizes and analyzes the results of the simulation study, performs the testing of the hypotheses, and conducts the regression analysis.

The simulation study has provided us with ample data on the expected values of pure dominance, mixed dominance and non-dominance. Based on the results and their analysis, we obtain more information about the properties of the expected values of pure dominance, mixed dominance and non-dominance.

For matrices with entries that are distributed 1) uniformly, 2) left-skewed triangularly, 3) right-skewed triangularly, and 4) normally, results from appendix 1 clearly show that the expected values of pure dominance are very much the same for corresponding $n \times m$ matrices with entries that possess either of the above four continuous distributions. This finding supports the analytical finding in Theorem 2.1.1 which proves that for matrices with the same n and m , the expected value of pure dominance will be the same if the entries in the corresponding $n \times m$ matrices are continuously distributed.

It is also shown that the expected values of mixed dominance is highly significant when compared to the expected values of pure dominance and non-dominance. It is also shown that the expected values of mixed dominance and the expected values of non-dominance will reach

constants for large n and fixed m for all the matrices studied with entries that are distributed, 1) uniformly, 2) left-skewed triangularly, 3) right-skewed triangularly, and 4) normally. For n less than 10 strategies, it is shown that matrices with entries that are right-skewed triangularly distributed have the highest expected values of mixed dominance. For n greater than 10, matrices with entries that are normally distributed possess the highest expected values of mixed dominance. Matrices with entries that are left-skewed triangularly distributed have the smallest expected values of mixed dominance.

The findings of the simulation are in line with the findings for the analytical solutions in Chapter II which shows that for 3×2 and 3×3 matrices, matrices with entries that are right-skewed triangularly distributed have higher expected values of mixed dominance than matrices with entries that are uniformly distributed. Matrices with entries that are left-skewed triangularly distributed possess the lowest expected values of mixed dominance as shown in Chapter II.

The regression results show that the double-log regression model provides very good fit for the expected values of pure dominance, mixed dominance, and non-dominance, respectively, for matrices with entries that are distributed with the four probability distributions studied. The F-squared terms are all very high as shown in Table A.4.1 to A.4.12. Most of the

regression fits have R-squared term exceed 0.99. Also, all the regression fits have high values of t statistics and F statistics. This means that we can safely apply the regression results to estimate other values of the expected values of pure dominance, mixed dominance and non-dominance for matrices not covered by this simulation study and can obtain good estimates.

5.2. Conclusions

Based on the findings of this study, we can reach a number of conclusions about the properties of the expected values of pure dominance, mixed dominance and non-dominance and their related implications.

We can conclude that the expected value of pure dominance depends only on n , the total number of strategies, and m , the total number of columns in the matrix, and is independent of the class of continuous distribution of the entries. As long as the distribution of the entries are continuous, it does not matter what class of distribution the entries process, the expected values of pure dominance will always be the same for a given $n \times m$ matrix.

We also conclude that the expected value of mixed dominance in a given matrix depends on n , the number of strategies, and m , the number of columns in the matrix, and on the joint density functions of the entries in the associated game matrices with respect to the strategies

examined for mixed dominance. Difference in means and variances of the same distribution will not affect the expected value of mixed dominance for corresponding $n \times m$ matrices, as long as the distribution of the entries are standardizable. Entries from different classes of distributions have different joint density functions, therefore, the expected values of mixed dominance will be different for matrices with entries from different classes of distributions.

Based on the testing of the research hypotheses, we also conclude that the ratios of expected values of mixed dominance over non-dominance will reach constants for large n and fixed m for all the matrices studied with entries that are distributed, 1) uniformly, 2) left-skewed triangularly, 3) right-skewed triangularly, and 4) normally. For n less than 10 strategies, it is shown that matrices with entries that are right-skewed triangularly distributed have the highest expected values of mixed dominance. For n greater than 10, matrices with entries that are normally distributed possess the highest expected values of mixed dominance. Matrices with entries that are left-skewed triangularly distributed have the smallest expected values of mixed dominance.

If we can determine the expected values of pure dominance and mixed dominance, then we can know the extent of dominance in a given matrix and the possible reduction of size for the matrix. Thus, we can know the

expected number of non-dominated strategies that require our consideration. Furthermore, we can know the proportions of strategies in a given matrix can be purely and mixed dominated. Thus, we can have an indication about the probability that a strategy in a given matrix will be dominated by other strategies.

We also conclude that the expected values of mixed dominance are highly significant when compared to both the expected values of pure dominance and non-dominance. In many instances, especially for larger matrices, the expected values of mixed dominance are greater than the expected values of pure dominance or non-dominance. Therefore, if a decision maker only removes the purely dominated strategies from his consideration but ignores the identification and removal of mixed dominated strategies, he runs the risk of adopting a strategy that can be mixed dominated by other strategies.

5.3. Suggestions for future research

Although the analytical methods developed in this study only apply to obtain the analytical solutions for the expected values of pure dominance, mixed dominance and non-dominance to a few cases of small matrices, it is possible to extend the analytical methods to larger matrices. The most promising would be to apply the analytical method to obtain the theoretical expected values of pure dominance for large matrices. For entries

with continuous distribution, it may be possible to express the expected values of pure dominance as a function of n and m . For the expected values of mixed dominance, the analysis will be a lot more complicated as demonstrated in Chapter II of this study.

This study only confines to the determination of the expected values of pure dominance, mixed dominance and non-dominance for matrices with entries that are continuously distributed and standardizable. We have not investigated the expected values of pure dominance, mixed dominance and non-dominance with entries that possess probability distributions that are discrete or non-standardizable. Additional study for matrices with entries that possess probability distributions that are discrete or non-standardizable are possible.

Future research is also possible for applying the concept of pure dominance and mixed dominance in real decision situations and in industries.

APPENDIX 1
EXPECTED VALUES OF PURE DOMINANCE, MIXED DOMINANCE
AND NON-DOMINANCE

TABLE A.1.1

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR UNIFORMLY DISTRIBUTED ENTRIES WITH $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	0.4438 0.0347	0.2475 0.0300	0.1225 0.0227	0.0463 0.0145	0.0300 0.0118	0.0125 0.0076
3	1.1863 0.0470	0.6750 0.0470	0.3375 0.0386	0.1625 0.0278	0.1163 0.0235	0.0588 0.0171
4	1.9188 0.0568	1.0513 0.0584	0.5950 0.0492	0.3350 0.0417	0.1825 0.0288	0.0800 0.0194
6	3.5413 0.0696	2.2888 0.0788	1.4013 0.0727	0.7575 0.0608	0.3788 0.0429	0.2038 0.0335
8	5.2625 0.0751	3.5788 0.1013	2.1138 0.0962	1.2525 0.0788	0.7038 0.0631	0.3650 0.0429
10	7.0638 0.0843	4.9438 0.1127	3.1900 0.1080	1.8313 0.0966	1.0300 0.0745	0.5750 0.0574
20	16.4938 0.0906	12.7638 0.1650	9.0163 0.1791	5.7838 0.1817	3.5113 0.1462	1.9650 0.1076
40	35.6250 0.1166	30.0900 0.2217	22.8900 0.2668	16.4112 0.3013	10.3813 0.2701	6.8513 0.2372
60	55.3962 0.1194	48.2613 0.2501	38.8775 0.3369	28.8400 0.3906	19.4825 0.3679	12.7563 0.3205
80	75.0850 0.1258	66.8325 0.2926	55.4500 0.4159	42.0938 0.4722	29.8687 0.4841	19.7262 0.4059
100	94.6863 0.1360	85.7963 0.3013	72.3050 0.4463	56.0263 0.5457	40.6038 0.5815	27.3775 0.4994
150	144.4218 0.3555	133.0593 0.6962	115.9219 1.0074	93.9375 1.1558	69.9937 1.2569	49.4531 1.1401
200	194.1125 0.4530	182.3625 0.8583	161.2875 1.2928	132.9500 1.5019	102.5750 1.6446	74.1500 1.5919
250	244.1718 0.5280	230.7187 1.0441	205.4532 1.5319	173.7812 1.8001	135.2500 2.1562	99.6718 2.0696
300	293.7310 0.6305	275.9195 1.1539	254.7000 1.7475	215.8501 1.9976	171.4507 2.2981	128.1188 2.4155

TABLE A.1.2

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR UNIFORMLY DISTRIBUTED ENTRIES WITH $n=350$ TO 900

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
350	343.4153 0.7127	328.5625 1.3255	299.5342 1.8945	259.3499 5.2499	210.7875 5.8596	154.6127 6.1150
400	393.9500 0.7969	378.0249 1.4778	346.1748 2.1781	306.2029 5.5793	249.3001 6.1779	190.5999 7.3359
450	---	---	393.8904 2.4259	345.7122 6.5168	279.4500 7.3427	222.9750 7.3951
500	---	---	441.3750 2.6009	385.3750 6.8441	319.2500 8.4564	247.5001 8.0824
550	---	---	493.0750 7.0180	428.1748 7.6385	361.7625 8.5540	288.7498 9.4160
600	---	---	539.3997 7.5282	483.2998 7.8455	403.3499 10.1230	322.3499 9.7880
650	---	---	---	518.5376 8.6214	441.6746 10.5517	348.0745 10.3756
700	---	---	---	567.7002 8.9341	484.3997 11.5373	382.7249 11.8221
750	---	---	---	---	514.1248 12.4721	424.8750 11.6293
800	---	---	---	---	556.3999 12.2802	446.7996 13.4971
850	---	---	---	---	---	483.2249 13.6524
900	---	---	---	---	---	528.2996 15.0081

TABLE A.1.3

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR UNIFORMLY DISTRIBUTED ENTRIES WITH $n=2$ TO 300

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
2	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0
3	0.0825 0.0190	0.1063 0.0214	0.0938 0.0202	0.0750 0.0182	0.0625 0.0169	0.0300 0.0118
4	0.1563 0.0274	0.2925 0.0347	0.2463 0.0308	0.1950 0.0282	0.1300 0.0239	0.1050 0.0218
6	0.2850 0.0380	0.4950 0.0476	0.5588 0.0484	0.5400 0.0466	0.4188 0.0417	0.3413 0.0382
8	0.3738 0.0437	0.7563 0.0612	1.0000 0.0645	0.9300 0.0619	0.7863 0.0564	0.6500 0.0517
10	0.4538 0.0523	1.0125 0.0727	1.2763 0.0741	1.3825 0.0729	1.2725 0.0721	1.0513 0.0649
20	0.5850 0.0547	1.8538 0.1072	3.0775 0.1260	3.8175 0.1301	3.8763 0.1215	3.6438 0.1172
40	0.9500 0.0743	3.0588 0.1499	5.9675 0.1935	8.1488 0.2201	9.5638 0.2066	9.8347 0.2021
60	0.9813 0.0739	3.8688 0.1731	7.9013 0.2379	12.0475 0.2869	15.1300 0.2869	15.9350 0.2609
80	1.1375 0.0808	4.5425 0.1944	9.8138 0.2842	15.7975 0.3463	20.3250 0.3569	22.7663 0.3183
100	1.2150 0.0849	5.0788 0.2091	11.6950 0.3193	19.3588 0.3977	25.5588 0.4253	29.4937 0.3855
150	1.3407 0.2183	6.3188 0.4961	15.0469 0.7595	26.6062 0.9555	37.2937 1.0649	45.2157 1.0425
200	1.4875 0.2630	6.5000 0.5878	17.5375 0.9192	33.1000 1.2350	48.2750 1.3671	59.7625 1.5033
250	1.3906 0.2954	7.5792 0.6993	20.8281 1.1405	38.9063 1.5212	59.5937 1.7615	75.8125 1.8981
300	1.5938 0.3446	8.6981 0.8034	21.8250 1.3173	44.3625 1.7121	68.0402 2.0113	88.3125 2.1178

TABLE A.1.4

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95% C.I. FOR UNIFORMLY DISTRIBUTED ENTRIES WITH $n=350$ TO 900

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
350	1.6844 0.3775	9.0562 0.8781	23.8219 1.4149	49.2625 4.2001	76.3000 4.6738	106.5751 5.4625
400	1.5750 0.3857	8.9000 0.9237	26.9000 1.6407	50.5005 4.1717	87.2000 4.9611	116.5000 6.3183
450	---	---	27.7312 1.7475	56.2500 4.8847	95.5126 6.0484	128.2500 6.9798
500	---	---	28.3125 1.8355	61.2501 5.4453	103.0001 6.8296	141.3750 7.4437
550	---	---	28.3250 4.0313	66.4125 5.6097	107.2500 6.7381	147.5376 7.5611
600	---	---	30.6001 4.5335	61.9501 5.8769	118.8000 7.9501	164.8499 9.2418
650	---	---	---	74.1000 7.1620	120.0874 8.5236	184.2750 9.4668
700	---	---	---	74.0250 6.5638	127.9250 8.9017	188.1250 10.1695
750	---	---	---	---	138.3751 10.0240	192.7500 9.6499
800	---	---	---	---	149.0001 10.4742	211.2000 11.0999
850	---	---	---	---	---	223.5500 11.6928
900	---	---	---	---	---	229.9501 12.9146

TABLE A.1.5

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR UNIFORMLY DISTRIBUTED ENTRIES WITH $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	1.5163 0.0347	1.7525 0.0300	1.8775 0.0227	1.9538 0.0145	1.9700 0.0118	1.9875 0.0076
3	1.7313 0.0404	2.2188 0.0445	2.5688 0.0400	2.7625 0.0316	2.8213 0.0276	2.9113 0.0204
4	1.9250 0.0461	2.2663 0.0514	3.1588 0.0494	3.4600 0.0459	3.6875 0.0353	3.8150 0.0288
6	2.1738 0.0515	3.2163 0.0641	4.0400 0.0674	4.7025 0.0653	5.2025 0.0553	5.4550 0.0488
8	2.3638 0.0547	3.6650 0.0796	4.8863 0.0841	5.8175 0.0900	6.5100 0.0731	6.9850 0.0613
10	2.4825 0.0600	4.0438 0.0829	5.5338 0.0939	6.7863 0.0966	7.6925 0.0872	8.3738 0.0776
20	2.9213 0.0666	5.3825 0.1109	7.9063 0.1335	10.3988 0.1470	12.6125 0.1386	14.3913 0.1321
40	3.4250 0.0760	6.8513 0.1321	11.1425 0.1690	15.4400 0.2033	20.0550 0.2180	23.5163 0.2309
60	3.6225 0.0792	7.8700 0.1445	13.2213 0.2005	19.1125 0.2391	25.3875 0.2685	31.3087 0.2820
80	3.7775 0.0833	8.6250 0.1641	14.7363 0.2323	22.1088 0.2732	29.8062 0.3240	37.5075 0.3340
100	4.0988 0.0892	9.1250 0.1674	16.0000 0.2383	24.6150 0.3136	33.8375 0.3548	43.1287 0.3849
150	4.2375 0.3593	10.6219 0.5831	19.0312 0.7530	29.4563 0.9275	42.7125 1.0423	55.3312 1.1411
200	4.4000 0.4510	11.1375 0.7095	21.1750 0.9957	33.9500 1.2001	49.1500 1.4390	66.0875 1.5110
250	4.4375 0.4910	11.7031 0.8628	23.7187 1.1233	37.3125 1.4245	55.1562 1.6364	74.5156 1.7413
300	5.1750 0.5700	15.3934 0.9279	23.4750 1.2536	39.7875 1.5678	60.5092 1.9333	83.5688 2.1374

TABLE A.1.6

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR UNIFORMLY DISTRIBUTED ENTRIES WITH $n=350$ TC 900

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
350	4.9000 0.6376	12.3911 1.0086	26.6438 1.3896	41.3875 3.6667	62.9125 3.9498	88.8125 5.0848
400	4.4750 0.6625	13.0750 1.1119	26.9250 1.4925	43.2966 3.7922	63.5000 4.2785	92.9000 5.7577
450	---	---	28.3781 1.7385	48.0376 4.5070	75.0376 4.5378	98.7751 5.8555
500	---	---	30.3125 1.9077	53.3751 4.6748	77.7501 5.9623	111.1250 6.5535
550	---	---	28.6000 3.7708	55.4125 5.4896	80.9876 6.1597	113.7126 7.2151
600	---	---	30.0000 4.1162	54.7500 5.4880	77.8500 7.0683	112.8000 7.4454
650	---	---	---	57.3625 5.9233	88.2375 6.6952	117.6499 7.6656
700	---	---	---	58.2751 5.4455	87.6750 7.7134	129.1500 8.6579
750	---	---	---	---	97.4999 7.7306	132.3750 8.9058
800	---	---	---	---	94.5999 7.9168	142.0000 9.7053
850	---	---	---	---	---	143.2250 9.7426
900	---	---	---	---	---	141.7501 10.6250

TABLE A.1.7

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR LEFT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	0.4938 0.0347	0.2663 0.0306	0.1338 0.0235	0.0538 0.0157	0.0288 0.0116	0.0088 0.0065
3	1.1488 0.0474	0.6488 0.0463	0.3350 0.0380	0.1963 0.0300	0.1025 0.0216	0.0575 0.0176
4	1.9713 0.0549	1.0500 0.0617	0.6000 0.0508	0.3213 0.0386	0.1888 0.0300	0.0863 0.0206
6	3.4925 0.0670	2.2238 0.0811	1.3850 0.0766	0.7075 0.0582	0.3975 0.0480	0.2050 0.0327
8	5.1988 0.0737	3.4975 0.0992	2.1525 0.0962	1.3125 0.0827	0.7350 0.0623	0.3775 0.0472
10	7.0438 0.0809	4.9825 0.1154	3.1200 0.1129	1.8037 0.0980	1.0375 0.0727	0.5788 0.0566
20	18.4525 0.0976	12.7825 0.1633	9.0025 0.1840	5.8050 0.1795	3.5213 0.1462	1.9650 0.1113
40	35.7075 0.1115	29.9275 0.2131	22.8850 0.2871	16.2212 0.2940	10.6537 0.2691	6.6750 0.2313
60	55.1988 0.1284	48.2800 0.2440	38.9300 0.3359	28.7675 0.3724	19.8362 0.3812	12.6013 0.3236
80	74.9650 0.1219	66.7500 0.2719	55.1762 0.3967	41.6625 0.4541	29.7692 0.4696	19.4237 0.4190
100	94.8012 0.1319	85.9962 0.2871	72.2650 0.4624	56.3187 0.5725	40.7087 0.5451	27.7612 0.5378
150	144.4031 0.3663	134.0155 0.7181	115.8653 1.0108	93.1219 1.0831	70.5938 1.2646	49.4906 1.1595
200	194.0750 0.4492	181.8125 0.8763	159.9250 1.2734	133.2000 1.4108	103.2625 1.5851	72.6625 1.6384
250	243.8281 0.5310	231.2344 1.0084	206.7812 1.5206	174.2813 1.7174	136.7656 2.0409	99.5333 2.0594
300	294.0000 0.6217	279.3372 1.1209	252.6003 1.7554	215.5876 2.0854	173.1751 2.3489	128.7000 2.4267

TABLE A.1.8

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR LEFT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=350$ TO $1,050$

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
350	343.5469 0.7076	326.4407 1.3785	301.1531 1.8357	260.9246 5.1225	211.1375 5.8549	156.5376 6.3416
400	393.5000 0.8079	378.9502 1.4684	348.2249 2.1192	291.2996 6.0305	247.4000 6.4288	191.8002 6.3480
450	--- ---	427.2188 1.6094	393.1875 2.4378	345.2676 6.8392	282.3748 6.8972	222.5250 7.5981
500	--- ---	475.2499 1.7705	440.9688 2.5954	386.3750 6.8653	324.3748 7.9662	245.3750 8.2500
550	--- ---	524.7344 1.8299	489.5374 6.8890	432.9871 7.3847	361.9192 8.8016	295.6250 9.4092
600	--- ---	574.4248 2.0492	539.7000 7.3316	476.3997 8.0822	404.0999 9.3962	313.3499 9.3233
650	--- ---	---	584.5137 7.9926	523.7373 8.4484	429.6497 9.9966	364.5142 10.3453
700	--- ---	---	633.5000 8.5525	571.5496 9.0926	485.7998 10.9446	380.6248 11.8307
750	--- ---	---	---	602.8123 10.5522	532.8750 11.9056	426.5625 12.2674
800	--- ---	---	---	656.0000 11.0660	561.7998 11.5134	462.5999 14.0506
850	--- ---	---	---	---	604.9871 12.2847	492.7874 13.8933
900	--- ---	---	---	---	644.6250 14.1663	522.6748 14.5847
950	--- ---	---	---	---	---	568.5750 14.0832
1000	--- ---	---	---	---	---	590.2498 16.5814
1050	--- ---	---	---	---	---	642.3372 16.8764

TABLE A.1.9

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR LEFT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=2$ TO 300

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
2	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0
3	0.0638 0.0169	0.0788 0.0186	0.0813 0.0190	0.0438 0.0141	0.0325 0.0123	0.0175 0.0090
4	0.1013 0.0225	0.1700 0.0274	0.1970 0.0288	0.1900 0.0286	0.1038 0.0218	0.0600 0.0165
6	0.2488 0.0368	0.4425 0.0463	0.4013 0.0408	0.4400 0.0417	0.3350 0.0374	0.2463 0.0323
8	0.3413 0.0419	0.6563 0.0561	0.7900 0.0568	0.7575 0.0553	0.5950 0.0510	0.5025 0.0476
10	0.3875 0.0463	0.7750 0.0631	1.1163 0.0698	1.1367 0.0684	1.0462 0.0635	0.8025 0.0551
20	0.5325 0.0551	1.5863 0.0974	2.6425 0.1141	3.3288 0.1225	3.1925 0.1125	2.8388 0.1056
40	0.8075 0.0670	2.8275 0.1352	5.2550 0.1788	7.3125 0.1946	8.5050 0.1903	8.1013 0.1805
60	0.9750 0.0764	3.5375 0.1537	7.0900 0.2227	10.8375 0.2528	13.2475 0.2491	14.2450 0.2426
80	1.0875 0.0776	4.2925 0.1752	9.0800 0.2597	14.4287 0.3118	18.2175 0.3118	20.0425 0.2962
100	1.1313 0.0804	4.5637 0.1868	10.6525 0.2973	17.3187 0.3748	22.9037 0.3636	26.1625 0.3606
150	1.2281 0.2091	5.5782 0.4510	14.1900 0.7424	24.9844 0.9310	34.0031 1.0106	40.9031 1.0137
200	1.3875 0.2632	6.6000 0.5968	17.1750 0.9637	29.5625 1.1539	45.4125 1.3499	55.3750 1.3828
250	1.4063 0.2934	6.2500 0.6290	19.6094 1.0851	36.4062 1.4486	53.7344 1.7223	70.1015 1.9275
300	1.3875 0.3142	7.3975 0.7844	20.5313 1.2379	40.6125 1.6964	62.5687 1.9153	81.2813 2.0262

TABLE A.1.11

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR LEFT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	1.5063 0.0347	1.7338 0.0305	1.6663 0.0235	1.9463 0.0157	1.9713 0.0116	1.9912 0.0065
3	1.7875 0.0429	2.2725 0.0447	2.5838 0.0392	2.7600 0.0319	2.8650 0.0241	2.9250 0.0196
4	1.9275 0.0482	2.7800 0.0576	3.2025 0.0515	3.4888 0.0435	3.7075 0.0353	3.8537 0.0265
6	2.2588 0.0521	3.3338 0.0696	4.2138 0.0721	4.8525 0.0635	5.2675 0.0576	5.5487 0.0449
8	2.4600 0.0563	3.8463 0.0917	5.0575 0.0870	5.9300 0.0937	6.6700 0.0735	7.1200 0.0627
10	2.5688 0.0610	4.2425 0.0947	5.7638 0.0988	7.0575 0.0921	7.9162 0.0866	8.6188 0.0759
20	3.0150 0.0711	5.6313 0.1172	8.3550 0.1427	10.8662 0.1519	13.2862 0.1446	15.1963 0.1347
40	3.4850 0.0792	7.2450 0.1390	11.8600 0.1991	16.4662 0.2262	20.8412 0.2295	25.2237 0.2262
60	3.8263 0.0874	8.1925 0.1517	13.9800 0.2144	20.3950 0.2550	26.9162 0.2938	33.1538 0.2924
80	3.9475 0.0804	8.9575 0.1609	15.7438 0.2442	23.9087 0.2964	31.9862 0.3485	40.5337 0.3665
100	4.0675 0.0898	9.4400 0.1727	17.0825 0.2703	26.3625 0.3581	36.3875 0.3734	46.0762 0.4226
150	4.3688 0.3820	10.4063 0.5921	19.9447 0.7783	31.8937 0.9590	45.4031 1.1184	59.6063 1.1497
200	4.5375 0.4328	11.5875 0.7189	22.9000 0.9788	37.2375 1.2236	51.3250 1.3212	71.9625 1.5021
250	4.7656 0.5092	12.5156 0.8314	23.6094 1.1633	39.3125 1.4088	59.5000 1.7003	80.3652 1.9165
300	4.6125 0.5858	13.2750 0.9132	26.8688 1.3218	43.8000 1.6642	64.2563 1.9424	90.0188 2.2524

TABLE A.1.12

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR LEFT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=350$ TO $1,050$

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
350	4.9656 0.6248	14.3281 1.0692	26.4688 1.4043	46.4625 3.6299	67.5500 4.6940	95.8125 5.3216
400	5.1000 0.6960	12.5750 1.0786	28.4500 1.5557	54.3000 4.3194	77.1000 4.9290	105.6001 5.5578
450	---	13.6125 1.2089	31.1625 1.7879	52.4806 4.5556	80.4375 5.3767	109.8000 6.2008
500	---	15.3750 1.3126	31.3438 1.9173	55.7500 4.8287	83.0001 5.9788	120.1251 6.9913
550	---	15.3313 1.3695	34.7875 4.0104	52.6625 4.9072	85.2546 6.1393	122.7875 7.5348
600	---	15.7500 1.4906	30.0000 4.4184	60.1500 5.5721	86.8501 6.2702	129.9000 7.4474
650	---	---	33.1488 4.5925	60.6125 5.7083	103.6750 7.6309	129.7809 8.2649
700	---	---	32.9000 4.7428	62.4750 6.0137	91.8750 6.7890	141.2250 10.3039
750	---	---	---	68.4375 7.5323	99.0000 7.7504	147.7500 9.7867
800	---	---	---	66.4000 6.9907	104.4000 8.1495	142.0000 9.4337
850	---	---	---	---	101.5750 9.0762	160.4375 10.7463
900	---	---	---	---	105.0750 9.5366	162.4500 10.6538
950	---	---	---	---	---	168.3875 11.1643
1000	---	---	---	---	---	172.7500 11.3946
1050	---	---	---	---	---	166.9500 12.6920

TABLE A.1.13

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR RIGHT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=2$ TO 300

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
2	0.4600 0.0345	0.2725 0.0310	0.1200 0.0225	0.0638 0.0169	0.0325 0.0123	0.0200 0.0098
3	1.1900 0.0476	0.6263 0.0457	0.3725 0.0390	0.1650 0.0276	0.1238 0.0235	0.0413 0.0137
4	1.9363 0.0564	1.1437 0.0619	0.6250 0.0512	0.3425 0.0396	0.1763 0.0314	0.0938 0.0220
6	3.5925 0.0692	2.1763 0.0811	1.3463 0.0745	0.7488 0.0608	0.4350 0.0498	0.2150 0.0345
8	5.2725 0.0735	3.6162 0.1013	2.1212 0.0949	1.3463 0.0951	0.7500 0.0596	0.3425 0.0453
10	6.9900 0.0796	4.8963 0.1131	3.2050 0.1160	1.8650 0.1005	1.0050 0.0739	0.5438 0.0578
20	16.4525 0.1021	12.6525 0.1586	9.0163 0.1270	5.9600 0.1850	3.4962 0.1543	2.0313 0.1186
40	35.7525 0.1137	30.0900 0.2201	22.7162 0.2879	16.2750 0.2834	10.6225 0.2601	6.5800 0.2270
60	55.3900 0.1182	48.1675 0.2483	38.7687 0.3577	28.2813 0.3793	19.6912 0.3734	12.8537 0.3281
80	75.0987 0.1249	66.9337 0.2603	55.4250 0.4128	42.3438 0.4765	29.7837 0.4655	19.3425 0.4181
100	94.9063 0.1249	85.7900 0.2928	72.4200 0.4271	56.3925 0.5159	40.3462 0.5519	27.3712 0.4947
150	144.1500 0.3236	132.8812 0.6711	115.4062 0.9918	93.3938 1.1131	70.4156 1.2773	49.3031 1.2023
200	194.1124 0.4643	161.9250 0.8561	161.3250 1.2965	133.5750 1.4339	103.2250 1.6425	72.6625 1.5999
250	243.9375 0.5455	230.4531 1.0094	207.7188 1.4992	174.9243 1.7001	136.5000 2.0635	99.6562 1.9974
300	293.6438 0.6590	279.5059 1.1562	252.3000 1.7268	215.2128 2.0594	173.7937 2.3557	126.7664 2.3600

TABLE A.1.14

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR RIGHT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n = 350$ TO 900

n	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
350	343.3062 0.7326	329.7217 1.2613	300.3875 1.9286	262.5872 5.0196	205.7125 5.8931	159.6001 5.0899
400	393.7251 0.8230	378.1001 1.4134	346.4248 2.1533	297.8997 5.5437	245.4002 7.1056	191.0000 6.5956
450	---	---	392.5125 2.4102	348.2998 6.1926	289.6907 7.4031	217.4626 7.7659
500	---	---	442.6565 2.6074	387.0000 6.8476	320.3748 8.0356	253.2501 8.1914
550	---	---	---	431.2000 7.8510	366.8499 8.1883	282.5625 8.5619
600	---	---	---	471.6001 8.5677	397.2000 9.5595	316.3499 9.8931
650	---	---	---	522.1121 9.0119	452.3997 10.6428	352.1375 11.4289
700	---	---	---	567.5247 9.1889	471.6250 10.8337	389.7249 11.3413
750	---	---	---	---	523.3125 10.9276	419.2500 12.5659
800	---	---	---	---	572.0000 12.7176	455.5999 13.2590
850	---	---	---	---	---	488.7498 13.2212
900	---	---	---	---	---	532.1250 14.8740

TABLE A.1.15

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR RIGHT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0
3	0.1088 0.0216	0.1338 0.0235	0.1213 0.0225	0.0875 0.0196	0.0825 0.0190	0.0550 0.0159
4	0.1900 0.0302	0.2888 0.0355	0.3275 0.0357	0.2700 0.0333	0.2063 0.0300	0.1588 0.0263
6	0.3488 0.0447	0.6975 0.0551	0.8238 0.0572	0.7275 0.0525	0.6238 0.0496	0.4713 0.0427
8	0.4763 0.0510	0.9738 0.0710	1.3275 0.0759	1.3037 0.0717	1.1813 0.0655	0.8975 0.0570
10	0.6200 0.0590	1.3475 0.0837	1.8163 0.0890	1.9337 0.0862	1.7325 0.0808	1.5187 0.0715
20	0.8975 0.0759	2.4900 0.1231	3.9637 0.1450	4.8650 0.1511	5.3500 0.1433	5.0375 0.1329
40	1.1950 0.0876	4.0775 0.1748	7.8575 0.2362	10.6950 0.2409	12.3925 0.2358	12.9837 0.2197
60	1.3700 0.0888	5.2088 0.2007	10.3712 0.2877	16.0500 0.3220	19.6200 0.3083	21.5000 0.2995
80	1.5550 0.0982	5.9138 0.2078	12.7112 0.3450	19.9450 0.3971	26.1687 0.4008	30.1862 0.3781
100	1.6325 0.0960	6.6763 0.2413	14.7700 0.3573	24.2937 0.4316	32.6537 0.4696	38.1850 0.4490
150	2.0343 0.2762	8.4281 0.5539	19.3125 0.8183	33.0094 1.0257	47.0624 1.1448	57.9281 1.1695
200	2.0875 0.3118	9.2625 0.6613	22.1875 1.0353	40.4750 1.2973	60.3500 1.5082	77.4375 1.6129
250	2.3437 0.3646	9.9063 0.7609	25.4374 1.2209	47.1719 1.5527	72.7344 1.9149	93.8906 1.9627
300	2.4563 0.4245	10.7437 0.8930	29.0250 1.4149	54.8063 1.8812	81.0937 2.2544	108.3270 2.4037

TABLE A.1.16

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR RIGHT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n = 350$ TO 900

n	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
350	2.5594 0.4808	11.0250 0.9916	30.1875 1.5453	58.1875 4.3392	96.8625 5.3918	124.6876 5.3104
400	2.5250 0.5143	12.2000 1.0533	34.2500 1.7607	68.6001 4.4864	105.9001 6.1601	139.9001 6.2216
450	---	---	36.1969 1.9625	66.4876 5.1064	110.0213 6.4935	158.3999 7.3765
500	---	---	36.7500 2.1395	74.8751 5.5827	127.1251 7.4061	172.5001 7.2200
550	---	---	---	81.4000 6.3608	128.9751 7.5601	185.2126 8.0574
600	---	---	---	93.0000 7.2457	145.3501 8.5125	198.9001 9.0487
650	---	---	---	87.2625 7.7263	143.9750 9.3435	212.8750 9.9197
700	---	---	---	91.1751 7.6560	160.3000 9.1283	219.2750 9.7167
750	---	---	---	---	158.4376 9.1359	236.2501 11.1232
800	---	---	---	---	164.2000 10.5319	248.2001 11.7000
850	---	---	---	---	---	257.5496 11.4570
900	---	---	---	---	---	266.3997 12.0068

TABLE A.1.17

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR RIGHT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	1.5400 0.0345	1.7275 0.0310	1.8800 0.0225	1.9363 0.0169	1.9675 0.0123	1.9800 0.0098
3	1.7012 0.0386	2.2400 0.0429	2.5062 0.0400	2.7475 0.0318	2.7938 0.0286	2.9037 0.0204
4	1.8738 0.0437	2.5675 0.0541	3.0475 0.0512	3.3875 0.0465	3.6175 0.0400	3.7475 0.0331
6	2.0587 0.0478	3.1363 0.0627	3.8300 0.0653	4.5237 0.0645	4.9413 0.0631	5.3137 0.0519
8	2.2512 0.0486	3.4100 0.0727	4.5412 0.0804	5.3500 0.0837	6.0687 0.0741	6.7600 0.0674
10	2.3900 0.0510	3.7663 0.0759	4.9787 0.0870	6.2012 0.0935	7.2625 0.0882	7.9375 0.0845
20	2.6500 0.0590	4.8575 0.0925	7.0000 0.1194	9.1750 0.1382	11.1537 0.1472	12.9313 0.1450
40	3.0525 0.0670	5.8325 0.1135	9.4263 0.1497	13.0300 0.1789	16.9850 0.2005	20.4362 0.2203
60	3.2400 0.0678	6.6238 0.1221	10.8600 0.1705	15.6688 0.2215	20.6887 0.2407	25.6462 0.2556
80	3.3463 0.0702	7.1525 0.1356	11.8638 0.1909	17.7112 0.2397	24.0475 0.2621	30.4712 0.3161
100	3.4612 0.0690	7.5438 0.1301	12.7500 0.1956	19.3137 0.2454	27.0000 0.3083	34.4437 0.3344
150	3.8157 0.3495	8.6906 0.5010	15.2813 0.6923	23.5968 0.8608	32.5219 0.9712	42.7688 1.0510
200	3.8000 0.4202	8.8125 0.6368	16.4875 0.8771	25.9500 1.0655	36.4250 1.1397	49.9000 1.3471
250	3.7187 0.4573	9.6406 0.7330	16.8437 1.0072	27.8437 1.1931	40.7656 1.4435	56.4532 1.6762
300	3.9000 0.5423	9.7500 0.8050	18.6750 1.1270	29.9812 1.3845	45.1125 1.5792	64.9068 1.9820

TABLE A.1.18

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR RIGHT-SKEWED TRIANGULARLY DISTRIBUTED ENTRIES
FOR $n = 350$ TO 900

n	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
350	4.1344 0.5788	9.2532 0.8436	19.4250 1.2236	29.2250 3.0745	47.4249 3.8651	65.7125 4.0521
400	3.7500 0.5833	9.7000 0.9573	19.3250 1.2722	33.5000 3.6507	48.7000 3.8563	69.1000 5.1411
450	---	---	21.2906 1.4696	35.2125 4.0172	50.2881 4.2816	74.1375 5.1601
500	---	---	20.5937 1.5276	38.1250 3.9269	52.5000 4.6395	74.2501 5.6822
550	---	---	---	37.4000 4.0407	54.1750 5.2520	82.2251 6.1354
600	---	---	---	35.4000 4.7552	57.4500 5.5272	84.7501 6.4574
650	---	---	---	40.6250 4.5272	53.6250 5.5047	84.9875 6.4654
700	---	---	---	41.3000 4.9453	68.0750 6.3657	91.0000 7.2316
750	---	---	---	---	68.2500 6.6473	94.4999 7.6389
800	---	---	---	---	63.8000 7.2600	96.2000 8.2759
850	---	---	---	---	---	103.7000 8.3216
900	---	---	---	---	---	101.4750 8.9188

TABLE A.1.19

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR NORMALLY DISTRIBUTED ENTRIES FOR $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	0.5038 0.0347	0.2425 0.0298	0.1300 0.0233	0.0575 0.0161	0.0238 0.0106	0.0150 0.0084
3	1.1363 0.0465	0.6325 0.0468	0.2925 0.0363	0.1738 0.0286	0.0838 0.0204	0.0375 0.0137
4	1.9000 0.0570	1.1575 0.0596	0.6400 0.0504	0.3113 0.0388	0.1650 0.0284	0.0888 0.0212
6	3.5250 0.0666	2.2438 0.0829	1.2862 0.0721	0.7075 0.0580	0.3863 0.0612	0.2063 0.0321
8	5.2900 0.0760	3.4589 0.0996	2.1450 0.0939	1.2675 0.0827	0.6750 0.0612	0.3988 0.0468
10	7.1013 0.0782	4.9425 0.1143	3.1800 0.1164	1.8250 0.0982	1.0263 0.0747	0.5663 0.0572
20	16.4537 0.0990	12.7725 0.1619	8.7875 0.1858	5.7862 0.1754	3.3913 0.1429	2.0812 0.1154
40	35.6800 0.1139	30.0562 0.2113	23.0687 0.2842	16.5162 0.3079	10.7538 0.2726	6.4588 0.2181
60	55.3837 0.1164	48.2087 0.2462	38.7087 0.3522	28.4825 0.3851	19.4587 0.3550	12.6037 0.3381
80	74.9700 0.1303	67.1162 0.2679	55.2825 0.4014	42.0350 0.4820	29.6600 0.4659	20.0537 0.4302
100	94.8279 0.1352	85.6887 0.2954	72.2937 0.4445	56.1287 0.5611	40.8462 0.5709	27.6712 0.5010
150	144.5438 0.3542	133.7344 0.6978	116.0625 1.0510	92.5031 1.1417	71.2031 1.2148	50.0625 1.0776
200	194.4523 0.4388	181.5125 0.8152	160.2500 1.2862	133.2250 1.4424	103.4250 1.6682	73.6750 1.6278
250	244.0937 0.5376	230.5625 1.0194	206.9375 1.5161	175.0156 1.7126	137.4218 2.0827	100.1875 2.0231
300	293.9063 0.6176	278.3997 1.2130	252.7501 1.7950	215.7263 2.0890	170.8875 2.3624	129.5625 2.3777

TABLE A.1.20

EXPECTED VALUES OF PURE DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR NORMALLY DISTRIBUTED ENTRIES FOR $n=350$ TO 900

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
350	343.5032 0.6929	328.0811 1.3101	300.4312 1.9469	259.4375 5.2342	208.3376 5.6689	157.3251 5.9221
400	393.3750 0.6044	379.3501 1.3990	347.2498 2.2299	297.7998 5.6174	240.7001 6.5072	185.2999 6.5570
450	---	---	394.5938 2.2691	341.7747 6.6656	278.5498 7.4935	222.0750 7.9284
500	---	---	441.7188 2.5398	387.2500 6.6579	328.6248 7.7875	248.6250 7.9901
550	---	---	---	436.9749 7.2571	354.8870 8.5278	279.6746 9.5830
600	---	---	---	473.7000 8.1583	404.5496 8.7728	322.1995 9.5046
650	---	---	---	517.3997 8.7786	440.0498 9.9101	352.6248 10.1712
700	---	---	---	568.5752 9.1283	478.2747 10.9968	385.0000 11.6185
750	---	---	---	---	517.3125 11.4662	420.7500 13.2696
800	---	---	---	---	555.7996 12.7796	454.3997 13.9769
850	---	---	---	---	---	491.9375 14.3454
900	---	---	---	---	---	532.1250 14.8499

TABLE A.1.21

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR NORMALLY DISTRIBUTED ENTRIES FOR $n=2$ TO 300

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
2	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0
3	0.0863 0.0194	0.1050 0.0212	0.1013 0.0210	0.0838 0.0192	0.0513 0.0153	0.0300 0.0118
4	0.1625 0.0278	0.2688 0.0347	0.2475 0.0327	0.2025 0.0284	0.1538 0.0255	0.0975 0.0208
6	0.3388 0.0417	0.5613 0.0498	0.6788 0.0529	0.6488 0.0514	0.5538 0.0613	0.3963 0.0417
8	0.4488 0.0465	0.9738 0.0690	1.1375 0.0694	1.1037 0.0674	1.0825 0.0613	0.7875 0.0561
10	0.5375 0.0514	1.2800 0.0792	1.6625 0.0858	1.7425 0.0794	1.5563 0.0774	1.3213 0.0725
20	0.9138 0.0766	2.5350 0.1256	4.1837 0.1478	4.9538 0.1454	5.2225 0.1382	4.8175 0.1309
40	1.3838 0.0911	4.2750 0.1695	7.9650 0.2327	10.0663 0.2558	12.7112 0.2405	13.3900 0.2246
60	1.5987 0.0968	5.6588 0.2093	11.0450 0.2852	16.8150 0.3246	20.7725 0.3169	22.3050 0.2981
80	1.8925 0.1149	6.4200 0.2330	13.6837 0.3416	21.6012 0.3998	27.9862 0.4049	31.1512 0.3873
100	1.9925 0.1141	7.4888 0.2579	15.8950 0.3748	20.1725 0.4696	34.5075 0.4945	40.5894 0.4469
150	2.1093 0.2809	9.2906 0.6009	21.2531 0.9157	36.6562 1.0613	50.3812 1.1601	61.2562 1.2324
200	2.3501 0.3228	10.7125 0.7072	25.2000 1.0933	44.2250 1.3828	64.3250 1.6537	81.9125 1.6427
250	2.3125 0.3806	11.1250 0.8354	27.9843 1.3065	50.7656 1.6241	77.9375 2.0188	100.8438 1.9978
300	2.6625 0.4481	13.6875 0.9982	31.5750 1.5555	60.9346 1.9919	91.3500 2.1828	119.5126 2.2567

TABLE A.1.22

EXPECTED VALUES OF MIXED DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR NORMALLY DISTRIBUTED ENTRIES FOR $n=350$ TO 900

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
350	2.9968 0.4916	13.7594 1.0574	34.9125 1.6825	65.1001 4.7346	102.5501 5.0640	135.3625 5.4251
400	3.0500 0.5527	12.7000 1.1013	36.5500 1.8514	72.2001 4.5190	115.2001 5.9694	155.4001 6.4968
450	---	---	39.1500 1.9075	75.6000 5.5736	130.9501 6.6720	165.5999 7.4045
500	---	---	41.0000 2.1442	84.1251 5.9506	126.6250 6.8859	187.1250 7.4164
550	---	---	---	85.5251 6.1709	147.6750 7.8198	202.2626 9.0027
600	---	---	---	93.6000 6.9541	147.4501 7.8029	214.2001 9.0603
650	---	---	---	101.5624 7.3504	161.3625 9.2945	226.6876 9.8958
700	---	---	---	100.2750 7.4623	173.2500 9.9100	242.2000 11.3933
750	---	---	---	---	184.8750 10.1461	251.8127 11.9370
800	---	---	---	---	188.8002 10.9819	268.6001 12.8942
850	---	---	---	---	---	283.8997 13.5303
900	---	---	---	---	---	288.8997 13.8658

TABLE A.1.23

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR NORMALLY DISTRIBUTED ENTRIES FOR $n=2$ TO 300

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
2	1.4962 0.0347	1.7575 0.0298	0.9350 0.2332	1.9425 0.0161	1.9763 0.0106	1.9850 0.0084
3	1.7775 0.0398	2.2625 0.0449	2.6063 0.0384	2.7425 0.0323	2.8650 0.0247	2.9325 0.0178
4	1.9375 0.0459	2.5738 0.0535	3.1125 0.0508	3.4863 0.0441	3.6813 0.0367	3.8137 0.0284
6	2.1363 0.0482	3.1950 0.0672	4.0350 0.0694	4.6437 0.0655	5.0600 0.0749	5.3975 0.0508
8	2.2613 0.0529	3.5675 0.0760	4.7175 0.0809	5.6288 0.0841	6.2425 0.0749	6.8137 0.0694
10	2.3613 0.0545	3.7775 0.0800	5.1575 0.0956	6.4325 0.0953	7.4175 0.0917	8.1125 0.0866
20	2.6325 0.0592	4.6925 0.0955	7.0287 0.1213	9.2600 0.1392	11.3863 0.1452	13.1013 0.1494
40	2.9363 0.0641	5.6698 0.1170	8.9662 0.1517	12.4175 0.1917	16.5350 0.2119	20.1512 0.2248
60	3.0175 0.0647	6.1325 0.1221	10.2462 0.1809	14.7025 0.2223	19.7687 0.2483	25.0912 0.2850
80	3.1375 0.0649	6.4638 0.1227	11.0338 0.1893	16.3637 0.2493	22.3537 0.2936	28.7950 0.3289
100	3.1787 0.0666	6.8225 0.1337	11.8113 0.2070	17.6987 0.2573	24.6462 0.3228	32.1862 0.3516
150	3.3469 0.3295	6.9750 0.4882	12.6843 0.6160	20.8406 0.8199	28.4156 0.8977	38.6813 1.0847
200	3.1976 0.3795	7.7750 0.5631	14.5500 0.8034	22.5499 0.9964	32.2499 1.1721	44.4125 1.3736
250	3.5938 0.4559	8.3125 0.6738	15.0781 0.9030	24.2187 1.1495	34.6406 1.3442	48.9687 1.5837
300	3.4313 0.4833	7.9125 0.7507	15.6750 1.0161	23.3391 1.2711	37.7625 1.5856	50.9250 1.8271

TABLE A.1.24

EXPECTED VALUES OF NON-DOMINANCE AND HALF-LENGTHS FOR 95%
C.I. FOR NORMALLY DISTRIBUTED ENTRIES FOR $n=350$ TO 900

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
350	3.5000 0.5033	8.1594 0.8077	14.6563 1.1088	25.4625 2.8814	39.1125 3.4169	57.3125 4.4563
400	3.5750 0.5700	7.9500 0.8563	16.2000 1.2730	30.0000 3.2961	44.1000 4.1264	59.3000 4.3957
450	---	---	16.2562 1.2428	32.6250 3.5027	40.5000 4.1338	62.3250 5.0782
500	---	---	17.2813 1.4028	28.6250 3.0923	44.7500 4.5484	64.2500 5.2506
550	---	---	---	27.5000 3.9782	47.4375 4.9776	68.0626 5.5223
600	---	---	---	32.7000 4.4627	48.0001 4.9470	63.6001 5.7885
650	---	---	---	31.0375 4.5766	48.5875 5.2032	70.6875 6.0746
700	---	---	---	31.1500 4.4659	48.4750 5.9974	72.8000 6.4649
750	---	---	---	---	47.8125 6.0180	77.4375 7.0740
800	---	---	---	---	55.4000 6.3686	77.0000 7.6781
850	---	---	---	---	---	74.1625 7.7616
900	---	---	---	---	---	78.9751 7.8206

APPENDIX 2

VARIOUS RATIOS AMONG THE EXPECTED VALUES OF
PURE DOMINANCE, MIXED DOMINANCE AND NON-DOMINANCE

TABLE A.2.1

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF PURE DOMINANCE WITH UNIFORM ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0695	0.1575	0.2779	0.4615	0.5374	0.5102
6	0.0805	0.2163	0.3988	0.7129	1.1056	1.6747
10	0.0642	0.2048	0.4001	0.7549	1.2354	1.8283
20	0.0355	0.1452	0.3413	0.6600	1.1039	1.8544
40	0.0267	0.1017	0.2607	0.4965	0.9213	1.4354
60	0.0177	0.0802	0.2032	0.4177	0.7766	1.2492
80	0.0151	0.0680	0.1770	0.3753	0.6805	1.1541
100	0.0128	0.0592	0.1617	0.3455	0.6295	1.0773
150	0.0093	0.0475	0.1298	0.2832	0.5328	0.9143
200	0.0077	0.0356	0.1087	0.2490	0.4706	0.8060
250	0.0057	0.0328	0.1014	0.2239	0.4406	0.7606
300	0.0054	0.0315	0.0857	0.2055	0.3968	0.6893
350	0.0049	0.0276	0.0795	0.1899	0.3620	0.6893
400	0.0040	0.0235	0.0777	0.1649	0.3498	0.6112
450	---	---	0.0704	0.1627	0.3418	0.5752
500	---	---	0.0641	0.1589	0.3226	0.5712
550	---	---	0.0574	0.1551	0.2965	0.5110
600	---	---	0.0567	0.1282	0.2945	0.5114
650	---	---	---	0.1429	0.2719	0.5294
700	---	---	---	0.1304	0.2641	0.4915
750	---	---	---	---	0.2691	0.4537
800	---	---	---	---	0.2678	0.4727
850	---	---	---	---	---	0.4626
900	---	---	---	---	---	0.4353

TABLE A.2.2

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF NON-DOMINANCE WITH UNIFORM ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0477	0.0479	0.0365	0.0271	0.0222	0.0103
6	0.1311	0.1539	0.1383	0.1148	0.0805	0.0626
10	0.1828	0.2504	0.2306	0.2037	0.1654	0.1255
20	0.2003	0.3444	0.3892	0.3671	0.3073	0.2532
40	0.2774	0.4465	0.5356	0.5278	0.4769	0.4182
60	0.2709	0.4916	0.5976	0.6303	0.5960	0.5090
80	0.3011	0.5267	0.6660	0.7145	0.6819	0.6070
100	0.2964	0.5566	0.7309	0.7865	0.7553	0.6839
150	0.3164	0.5949	0.7906	0.9032	0.8731	0.8172
200	0.3381	0.5836	0.8282	0.9750	0.9822	0.9043
250	0.3134	0.6475	0.8781	1.0427	1.0805	1.0174
300	0.3080	0.5644	0.9297	1.1150	1.1245	1.0568
350	0.3438	0.7315	0.8941	1.1903	1.2128	1.2000
400	0.3520	0.6807	0.9991	1.1664	1.3732	1.2540
450	---	---	0.9772	1.1710	1.2729	1.2984
500	---	---	0.9340	1.1475	1.3248	1.2722
550	---	---	0.9904	1.1985	1.3243	1.2975
600	---	---	1.0200	1.1315	1.5260	1.4614
650	---	---	---	1.2918	1.3610	1.5663
700	---	---	---	1.2703	1.4591	1.4566
750	---	---	---	---	1.4192	1.4561
800	---	---	---	---	1.5751	1.4873
850	---	---	---	---	---	1.5608
900	---	---	---	---	---	1.6222

TABLE A.2.3

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUES OF MIXED AND NON-DOMINANCE FOR UNIFORM ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0455	0.0457	0.0352	0.0264	0.0217	0.0102
6	0.1159	0.1334	0.1215	0.1030	0.0745	0.0589
10	0.1545	0.2002	0.1874	0.1692	0.1419	0.1115
20	0.1668	0.2552	0.2802	0.2685	0.2351	0.2020
40	0.2171	0.3087	0.3488	0.3455	0.3229	0.2949
60	0.2132	0.3296	0.3741	0.3866	0.3734	0.3373
80	0.2314	0.3450	0.3997	0.4168	0.4054	0.3777
100	0.2286	0.3576	0.4223	0.4402	0.4303	0.4061
150	0.2403	0.3730	0.4415	0.4746	0.4661	0.4497
200	0.2527	0.3685	0.4530	0.4937	0.4955	0.4749
250	0.2386	0.3930	0.4676	0.5105	0.5193	0.5043
300	0.2355	0.3608	0.4818	0.5272	0.5293	0.5138
350	0.2558	0.4225	0.4720	0.5434	0.5481	0.5455
400	0.2603	0.4050	0.4998	0.5384	0.5786	0.5564
450	---	---	0.4942	0.5394	0.5600	0.5649
500	---	---	0.4829	0.5344	0.5698	0.5599
550	---	---	0.4976	0.5451	0.5698	0.5647
600	---	---	0.5050	0.5308	0.6041	0.5937
650	---	---	---	0.5637	0.5764	0.6103
700	---	---	---	0.5595	0.5933	0.5929
750	---	---	---	---	0.5866	0.5928
800	---	---	---	---	0.6117	0.5980
850	---	---	---	---	---	0.6095
900	---	---	---	---	---	0.6186

TABLE A.2.4

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF PUFF DOMINANCE WITH LEFT-SKEWED TRIANGULAR ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0555	0.1215	0.2427	0.2231	0.3171	0.3043
6	0.0712	0.1990	0.2897	0.6219	0.8428	1.2015
10	0.0550	0.1555	0.3578	0.6313	1.0084	1.3865
20	0.0324	0.1241	0.2935	0.5734	0.9066	1.4447
40	0.0226	0.0945	0.2296	0.4508	0.7983	1.2137
60	0.0177	0.0733	0.1821	0.3767	0.6678	1.1304
80	0.0145	0.0643	0.1646	0.3463	0.6120	1.0319
100	0.0119	0.0531	0.1474	0.3075	0.5626	0.9424
150	0.0085	0.0416	0.1225	0.2683	0.4817	0.8265
200	0.0071	0.0363	0.1074	0.2219	0.4398	0.7621
300	0.0047	0.0264	0.0813	0.1884	0.3613	0.6316
350	0.0043	0.0283	0.0743	0.1633	0.3378	0.6238
400	0.0036	0.0224	0.0670	0.1867	0.3052	0.5349
450	---	0.0215	0.0652	0.1513	0.3088	0.5288
550	---	0.0189	0.0546	0.1486	0.2841	0.4451
600	---	0.0171	0.0561	0.1332	0.2699	0.5002
650	---	---	0.0553	0.1253	0.2716	0.4272
700	---	---	0.0530	0.1154	0.2518	0.4680
750	---	---	---	0.1306	0.2217	0.4119
800	---	---	---	0.1183	0.2382	0.4224
850	---	---	---	---	0.2371	0.3993
900	---	---	---	---	0.2332	0.4111
1000	---	---	---	---	---	0.4015
1050	---	---	---	---	---	0.3747

TABLE A.2.5

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF NON-DOMINANCE WITH LEFT-SKEWED TRIANGULAR ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0357	0.0347	0.0315	0.0159	0.0113	0.0060
6	0.1101	0.1327	0.0952	0.0907	0.0636	0.0444
10	0.1508	0.1827	0.1937	0.1613	0.1322	0.0931
20	0.1766	0.2817	0.3163	0.3063	0.2403	0.1868
40	0.2317	0.3903	0.4431	0.4441	0.4081	0.3212
60	0.2546	0.4323	0.5072	0.5314	0.4922	0.4297
80	0.2755	0.4792	0.5767	0.6035	0.5695	0.4945
100	0.2781	0.4834	0.6236	0.6569	0.6294	0.5678
150	0.2811	0.5360	0.7115	0.7834	0.7489	0.6862
200	0.3058	0.5696	0.7500	0.7939	0.8848	0.7695
300	0.3008	0.5565	0.7641	0.9272	0.9737	0.9029
350	0.2996	0.6443	0.8455	0.9171	1.0557	1.0192
400	0.2745	0.6740	0.8199	1.0018	0.9792	0.9716
450	---	0.6736	0.8231	0.9956	1.0839	1.0717
550	---	0.6490	0.7668	1.2219	1.2061	1.0717
600	---	0.6239	1.0100	1.0549	1.2556	1.2067
650	---	---	0.9755	1.0831	1.1254	1.1998
700	---	---	1.0213	1.0560	1.3314	1.2615
750	---	---	---	1.1507	1.1932	1.1891
800	---	---	---	1.1687	1.2816	1.3761
850	---	---	---	---	1.4121	1.2265
900	---	---	---	---	1.4304	1.3227
1000	---	---	---	---	---	1.3719
1050	---	---	---	---	---	1.4418

TABLE A.2.6

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUES OF MIXED AND NON-DOMINANCE WITH L. S. TRIAN. ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0345	0.0335	0.0305	0.0156	0.0112	0.0059
6	0.0992	0.1172	0.0870	0.0831	0.0598	0.0425
10	0.1311	0.1545	0.1623	0.1389	0.1167	0.0852
20	0.1501	0.2198	0.2403	0.2345	0.1937	0.1574
40	0.1881	0.2807	0.3070	0.3075	0.2898	0.2431
60	0.2031	0.3018	0.3365	0.3470	0.3298	0.3005
80	0.2160	0.3240	0.3658	0.3764	0.3629	0.3309
100	0.2176	0.3259	0.3841	0.3965	0.3863	0.3622
150	0.2194	0.3490	0.4157	0.4393	0.4282	0.4070
200	0.2342	0.3629	0.4286	0.4426	0.4694	0.4349
300	0.2312	0.3575	0.4331	0.4811	0.4933	0.4745
350	0.2305	0.3919	0.4581	0.4784	0.5135	0.5047
400	0.2154	0.4026	0.4505	0.5005	0.4948	0.4928
450	---	0.4025	0.4515	0.4989	0.5201	0.5173
550	---	0.3932	0.4340	0.5499	0.5467	0.5173
600	---	0.3842	0.5025	0.5133	0.5567	0.5468
650	---	---	0.4938	0.5199	0.5295	0.5454
700	---	---	0.5053	0.5136	0.5711	0.5578
750	---	---	---	0.5350	0.5440	0.5432
800	---	---	---	0.5389	0.5617	0.5791
850	---	---	---	---	0.5854	0.5509
900	---	---	---	---	0.5885	0.5695
1000	---	---	---	---	---	0.5784
1050	---	---	---	---	---	0.5905

TABLE A.2.7

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF PURE DOMINANCE WITH RIGHT-SKEWED TRIANGULAR ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0914	0.2136	0.3256	0.5303	0.6664	1.3317
6	0.0971	0.3159	0.6119	0.9716	1.4340	2.1921
10	0.0887	0.2758	0.5667	1.0368	1.7239	2.7928
20	0.0546	0.1958	0.4418	0.8163	1.5302	2.4799
40	0.0334	0.1355	0.3459	0.6571	1.1666	1.9732
60	0.0247	0.1081	0.2675	0.5675	0.9964	1.6727
80	0.0207	0.0984	0.2293	0.4710	0.8786	1.5606
100	0.0172	0.0778	0.2038	0.4308	0.8093	1.3951
150	0.0141	0.0634	0.1673	0.3534	0.6684	1.1749
200	0.0108	0.0509	0.1375	0.3030	0.5846	1.0657
250	0.0096	0.0430	0.1225	0.2696	0.5329	0.9421
300	0.0084	0.0384	0.1150	0.2547	0.4666	0.8545
350	0.0075	0.0334	0.1005	0.2216	0.4709	0.7813
400	0.0064	0.0323	0.0989	0.2303	0.4315	0.7325
450	---	---	0.0922	0.1909	0.3798	0.7284
500	---	---	0.0830	0.1935	0.3968	0.6811
550	---	---	---	0.1868	0.3516	0.6555
600	---	---	---	0.1972	0.3659	0.6287
650	---	---	---	0.1671	0.3182	0.6045
700	---	---	---	0.1607	0.3399	0.5626
750	---	---	---	---	0.3028	0.5635
800	---	---	---	---	0.2871	0.5448
850	---	---	---	---	---	0.5270
900	---	---	---	---	---	0.5006

TABLE A.2.8

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF NON-DOMINANCE WITH RIGHT-SKEWED TRIANGULAR ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0640	0.0597	0.0484	0.0318	0.0295	0.0189
6	0.1694	0.2192	0.2151	0.1608	0.1262	0.0887
10	0.2594	0.3578	0.3648	0.3118	0.2386	0.1913
20	0.3387	0.5126	0.5691	0.5302	0.4797	0.3896
40	0.3915	0.6991	0.8336	0.8208	0.7296	0.6353
60	0.4228	0.7864	0.9550	1.0243	0.9483	0.8383
80	0.4647	0.8268	1.0714	1.1261	1.0882	0.9906
100	0.4717	0.8850	1.1584	1.2578	1.2094	1.1086
150	0.5331	0.9698	1.2638	1.3989	1.4471	1.3544
200	0.5493	1.0511	1.3457	1.5597	1.6568	1.5519
250	0.6302	1.0276	1.5102	1.6942	1.7842	1.6632
300	0.6298	1.1019	1.5542	1.8280	1.7976	1.6690
350	0.6190	1.1915	1.5541	1.9910	2.0424	1.8975
400	0.6733	1.2577	1.7723	2.0478	2.1745	2.0246
450	---	---	1.7001	1.8882	2.1878	2.1366
500	---	---	1.7845	1.9639	2.4214	2.3232
550	---	---	---	2.1765	2.3807	2.2525
600	---	---	---	2.6271	2.5300	2.3469
650	---	---	---	2.1480	2.6848	2.5048
700	---	---	---	2.2076	2.3548	2.4096
750	---	---	---	---	2.3214	2.5000
800	---	---	---	---	2.5737	2.5800
850	---	---	---	---	---	2.4836
900	---	---	---	---	---	2.6253

TABLE A.2.9

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUES OF MIXED AND NON-DOMINANCE WITH F.S. TRIAN. ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0601	0.0564	0.0462	0.0309	0.0287	0.0186
6	0.1449	0.1798	0.1770	0.1385	0.1121	0.0815
10	0.2060	0.2635	0.2673	0.2377	0.1926	0.1606
20	0.2530	0.3399	0.3627	0.3465	0.3242	0.2803
40	0.2813	0.4115	0.4546	0.4508	0.4218	0.3885
60	0.2972	0.4402	0.4885	0.5060	0.4867	0.4560
80	0.3173	0.4526	0.5172	0.5297	0.5211	0.4977
100	0.3205	0.4695	0.5367	0.5571	0.5474	0.5258
150	0.3477	0.4923	0.5583	0.5831	0.5914	0.5753
200	0.3546	0.5124	0.5737	0.6093	0.6236	0.6081
250	0.3866	0.5068	0.6016	0.6288	0.6408	0.6245
300	0.3864	0.5242	0.6085	0.6464	0.6425	0.6253
350	0.3824	0.5437	0.6085	0.6657	0.6713	0.6549
400	0.4024	0.5571	0.6393	0.6719	0.6850	0.6694
450	---	---	0.6296	0.6538	0.6863	0.6812
500	---	---	0.6409	0.6626	0.7077	0.6991
550	---	---	---	0.6952	0.7042	0.6925
600	---	---	---	0.7243	0.7167	0.7012
650	---	---	---	0.6823	0.7286	0.7147
700	---	---	---	0.6882	0.7019	0.7067
750	---	---	---	---	0.6989	0.7143
800	---	---	---	---	0.7202	0.7207
850	---	---	---	---	---	0.7129
900	---	---	---	---	---	0.7242

TABLE A.2.10

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF PURE DOMINANCE WITH NORMAL ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0759	0.1560	0.3463	0.4822	0.6122	0.8000
6	0.0961	0.2502	0.5278	0.9170	1.4336	1.9210
10	0.0757	0.2590	0.5228	0.9548	1.5164	2.3332
20	0.0555	0.1985	0.4761	0.8561	1.5400	2.3148
40	0.0388	0.1422	0.3453	0.6095	1.1820	2.0731
60	0.0289	0.1174	0.2853	0.5904	1.0675	1.7697
80	0.0252	0.0957	0.2475	0.5139	0.9436	1.5534
100	0.0210	0.0874	0.2199	0.3594	0.8448	1.4668
150	0.0146	0.0695	0.1831	0.3963	0.7076	1.2236
200	0.0121	0.0590	0.1573	0.3320	0.6219	1.1118
250	0.0095	0.0483	0.1352	0.2901	0.5671	1.0065
300	0.0091	0.0492	0.1249	0.2825	0.5346	0.9224
350	0.0087	0.0419	0.1162	0.2509	0.4922	0.8604
400	0.0078	0.0335	0.1053	0.2424	0.4786	0.8386
450	---	---	0.0992	0.2212	0.4701	0.7457
500	---	---	0.0928	0.2172	0.3853	0.7526
550	---	---	---	0.1957	0.4161	0.7232
600	---	---	---	0.1976	0.3645	0.6648
650	---	---	---	0.1963	0.3667	0.6429
700	---	---	---	0.1764	0.3622	0.6291
750	---	---	---	---	0.3574	0.5985
800	---	---	---	---	0.3397	0.5911
850	---	---	---	---	---	0.5771
900	---	---	---	---	---	0.5429

TABLE A.2.11

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUE OF NON-DOMINANCE WITH NORMAL ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0486	0.0464	0.0389	0.0306	0.0179	0.0102
6	0.1586	0.1757	0.1682	0.1397	0.1094	0.0734
10	0.2276	0.3388	0.3223	0.2709	0.2098	0.1629
20	0.3471	0.5402	0.5952	0.5350	0.4587	0.3677
40	0.4713	0.7541	0.8883	0.8107	0.7687	0.6645
60	0.5298	0.9228	1.0780	1.1437	1.0508	0.8890
80	0.6032	0.9932	1.2402	1.3201	1.2520	1.0818
100	0.6268	1.0977	1.3457	1.1398	1.4001	1.2611
150	0.6302	1.3320	1.6755	1.7589	1.7730	1.5836
200	0.7350	1.3778	1.7320	1.9612	1.9946	1.8444
250	0.6435	1.3393	1.8560	2.0961	2.2499	2.0594
300	0.7759	1.7299	2.0144	2.6108	2.4191	2.3468
350	0.8562	1.6863	2.3821	2.5567	2.6219	2.3618
400	0.8531	1.5975	2.2562	2.4067	2.6122	2.6206
450	---	---	2.4083	2.3172	3.2333	2.6570
500	---	---	2.3725	2.9389	2.8296	2.9125
550	---	---	---	3.1100	3.1130	2.9717
600	---	---	---	2.8624	3.0719	3.3679
650	---	---	---	3.2722	3.3211	3.2069
700	---	---	---	3.2191	3.5740	3.3269
750	---	---	---	---	3.8667	3.2518
800	---	---	---	---	3.4079	3.4883
850	---	---	---	---	---	3.8281
900	---	---	---	---	---	3.6581

TABLE A.2.12

RATIOS OF EXPECTED VALUE OF MIXED DOMINANCE OVER EXPECTED
VALUES OF MIXED AND NON-DOMINANCE WITH NORMAL ENTRIES

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.0463	0.0444	0.0374	0.0297	0.0176	0.0101
6	0.1369	0.1494	0.1440	0.1226	0.0986	0.0684
10	0.1854	0.2531	0.2438	0.2131	0.1734	0.1401
20	0.2577	0.3507	0.3731	0.3485	0.3144	0.2689
40	0.3203	0.4299	0.4704	0.4477	0.4346	0.3992
60	0.3463	0.4799	0.5188	0.5335	0.5124	0.4706
80	0.3762	0.4993	0.5536	0.5690	0.5559	0.5197
100	0.3853	0.5233	0.5737	0.5327	0.5834	0.5577
150	0.3866	0.5712	0.6262	0.6375	0.6394	0.6129
200	0.4236	0.5794	0.6340	0.6623	0.6661	0.6484
250	0.3915	0.5723	0.6499	0.6770	0.6923	0.6731
300	0.4369	0.6337	0.6683	0.7231	0.7075	0.7012
350	0.4613	0.6277	0.7043	0.7188	0.7239	0.7025
400	0.4604	0.6150	0.6929	0.7065	0.7232	0.7238
450	---	---	0.7066	0.6985	0.7638	0.7266
500	---	---	0.7035	0.7461	0.7389	0.7444
550	---	---	---	0.7567	0.7569	0.7482
600	---	---	---	0.7411	0.7544	0.7711
650	---	---	---	0.7659	0.7686	0.7623
700	---	---	---	0.7630	0.7814	0.7689
750	---	---	---	---	0.7945	0.7648
800	---	---	---	---	0.7731	0.7772
850	---	---	---	---	---	0.7929
900	---	---	---	---	---	0.7853

APPENDIX 3
T STATISTICS FOR TESTING OF HYPOTHESES

TABLE A.3.1

T TEST FOR THE EXPECTED VALUES OF MIXED DOMINANCE
BETWEEN MATRICES WITH ENTRIES THAT ARE RIGHT-SKEWED
TRIANGULARLY AND UNIFORMLY DISTRIBUTED

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	1.79	1.70	1.78	0.92	1.54	2.48
4	1.62	0.25	3.38	3.82	3.90	3.09
6	2.13	5.18	6.93	5.23	6.20	4.44
10	4.13	5.92	9.14	9.57	8.33	9.49
20	6.55	7.64	9.24	10.29	15.37	15.42
40	4.18	8.67	12.13	15.29	17.69	20.68
60	6.60	9.91	12.97	18.19	20.89	27.46
80	6.44	9.45	12.71	15.43	21.34	29.43
100	6.38	9.81	12.58	16.48	21.95	28.78
200	2.88	6.12	6.58	8.07	11.63	15.71
300	3.09	3.35	7.30	8.05	8.47	12.25
350	2.81	2.93	5.95	2.90	5.65	4.66
400	2.90	4.62	5.99	5.79	4.63	5.17
450	---	---	6.31	2.84	3.20	5.82
500	---	---	5.87	3.42	4.69	5.88
550	---	---	---	3.46	4.20	6.68
600	---	---	---	6.52	4.47	5.16
650	---	---	---	2.45	3.70	4.09
700	---	---	---	3.33	4.98	4.34
750	---	---	---	---	2.90	5.79
800	---	---	---	---	2.01	4.50
850	---	---	---	---	---	4.07
900	---	---	---	---	---	4.05

TABLE A.3.2

T TEST FOR THE EXPECTED VALUES OF MIXED DOMINANCE
BETWEEN MATRICES WITH ENTRIES THAT ARE UNIFORMLY
AND LEFT-SKEWED TRIANGULARLY DISTRIBUTED

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	1.44	1.90	0.88	2.65	2.81	1.65
4	3.04	4.99	2.29	-0.24	1.59	3.23
6	1.34	1.55	4.88	3.13	2.93	3.72
10	1.86	4.83	3.08	4.78	4.62	5.73
20	1.33	3.62	5.02	5.36	8.09	10.00
40	2.79	2.25	5.30	5.58	7.39	12.54
60	0.12	2.81	4.88	6.20	9.71	9.30
80	0.87	1.87	3.74	5.76	8.72	12.28
100	1.40	3.60	4.68	7.32	9.30	12.37
200	0.53	-0.23	0.53	4.10	2.92	4.21
300	0.87	2.27	1.40	3.05	3.86	4.70
350	0.75	-0.27	1.46	2.32	1.54	2.33
400	0.64	0.63	3.19	-1.23	3.26	3.30
450	---	---	1.70	1.10	2.01	2.13
500	---	---	0.47	0.88	2.21	1.32
550	---	---	0.58	0.51	0.90	2.88
600	---	---	0.09	-0.35	1.72	1.25
650	---	---	---	1.82	0.57	4.16
700	---	---	---	1.74	0.92	1.36
750	---	---	---	---	2.92	2.41
800	---	---	---	---	2.15	2.04
850	---	---	---	---	---	3.32
900	---	---	---	---	---	1.66

TABLE A.3.3

T TEST FOR THE EXPECTED VALUES OF MIXED DOMINANCE
BETWEEN MATRICES WITH ENTRIES THAT ARE NORMALLY
AND UNIFORMLY DISTRIBUTED

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	0.27	-0.08	0.50	0.65	-0.96	0.0
4	0.31	-0.55	0.05	0.86	1.33	-0.49
6	1.87	1.89	3.28	3.07	3.57	1.90
10	2.24	4.88	6.68	6.55	5.26	5.44
20	6.85	8.08	11.16	11.41	14.34	13.09
40	7.23	10.53	12.94	11.14	19.46	23.06
60	9.94	12.92	16.59	21.57	25.87	31.52
80	10.54	12.12	17.07	21.50	27.82	32.78
100	10.72	14.22	16.72	2.59	26.89	36.85
200	4.06	8.98	10.51	11.76	14.66	19.50
300	3.71	7.65	9.38	12.37	15.39	19.76
350	4.15	6.71	9.89	4.90	7.47	7.33
400	4.29	5.18	7.65	6.92	7.07	8.41
450	---	---	8.65	5.12	7.71	7.19
500	---	---	8.81	5.56	4.77	8.53
550	---	---	---	4.49	7.68	9.12
600	---	---	---	6.81	5.04	7.47
650	---	---	---	5.24	6.41	6.07
700	---	---	---	5.18	6.67	6.94
750	---	---	---	---	6.39	7.54
800	---	---	---	---	5.14	6.61
850	---	---	---	---	---	6.61
900	---	---	---	---	---	6.10

TABLE A.3.4

T TEST FOR THE EXPECTED VALUES OF MIXED DOMINANCE
BETWEEN MATRICES WITH ENTRIES THAT ARE RIGHT-SKEWED
TRIANGULARLY AND NORMALLY DISTRIBUTED

n	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	1.52	1.78	1.27	0.26	2.51	2.48
4	1.31	0.79	3.24	3.02	2.61	3.59
6	0.32	3.33	3.65	2.10	1.74	2.46
10	2.07	1.15	2.44	3.20	3.09	3.80
20	-0.30	-0.50	-1.89	-0.83	1.26	2.31
40	-2.93	-1.59	-0.64	3.51	-1.85	-2.53
60	-3.41	-3.04	-3.26	-3.28	-5.11	-3.73
80	-4.38	-3.18	-3.93	-5.76	-6.25	-3.49
100	-4.73	-4.51	-4.26	12.66	-5.33	-7.44
200	-1.15	-2.94	-3.92	-3.88	-3.48	-3.81
300	-0.65	-4.31	-2.38	-4.38	-6.41	-6.65
350	-1.25	-3.71	-4.05	-2.11	-1.51	-2.76
400	-1.36	-0.64	-1.76	-1.11	-2.13	-3.38
450	---	---	-2.11	-2.36	-4.41	-1.35
500	---	---	-2.75	-2.22	0.10	-2.77
550	---	---	---	-0.91	-3.37	-2.77
600	---	---	---	-0.12	-0.36	-2.34
650	---	---	---	-2.63	-2.59	-1.93
700	---	---	---	-1.67	-1.98	-3.00
750	---	---	---	---	-3.80	-1.87
800	---	---	---	---	-3.17	-2.30
850	---	---	---	---	---	-2.91
900	---	---	---	---	---	-2.40

TABLE A.3.5

T TEST FOR THE RATIOS OF THE EXPECTED VALUES OF MIXED
DOMINANCE OVER NON-DOMINANCE BETWEEN
n(1) AND n(2) WITH UNIFORM ENTRIES

n(1)	n(2)	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	4	-3.63	-8.22	-6.45	-4.91	-2.91	-4.83
4	6	-4.28	-2.64	-7.56	-9.28	-8.57	-7.59
6	8	-2.04	-4.48	-7.11	-5.96	-6.63	-5.83
10	20	-1.18	-6.53	-14.01	-18.23	-19.85	-21.67
20	40	-5.09	-6.31	-11.07	-15.03	-21.56	-25.52
40	60	0.41	-2.64	-4.37	-8.65	-13.44	-13.12
60	80	-1.92	-2.01	-4.50	-6.67	-8.82	-13.86
80	100	0.30	-1.67	-4.03	-5.40	-7.04	-10.22
100	150	-0.64	-1.24	-2.15	-5.05	-6.66	-9.97
150	200	-0.47	0.26	-0.96	-2.13	-4.17	-4.31
250	300	0.10	1.68	-1.03	-1.69	-1.30	-1.52
300	350	-0.60	-3.01	0.68	-0.95	-1.51	-2.96
350	400	-0.12	0.76	-1.90	0.23	-1.96	-0.77
400	450	---	---	0.36	-0.04	1.20	-0.59
450	500	---	---	0.70	0.22	-0.59	0.35
500	550	---	---	-0.54	-0.47	0.01	-0.34
550	600	---	---	-0.21	0.59	-1.86	-1.95
600	650	---	---	---	-1.31	1.47	-1.13
650	700	---	---	---	0.17	-0.89	1.19
700	750	---	---	---	---	0.35	0.01
750	800	---	---	---	---	-1.33	-0.35
800	850	---	---	---	---	---	-0.78
850	900	---	---	---	---	---	-0.60

TABLE A.3.6

T TEST FOR THE RATIOS OF THE EXPECTED VALUES OF MIXED
DOMINANCE OVER NON-DOMINANCE BETWEEN
n(1) AND n(2) WITH LEFT-SKewed TRIANGULAR ENTRIES

n(1)	n(2)	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	4	-2.19	-4.02	-5.05	-7.81	-4.48	-3.57
4	6	-5.56	-8.11	-4.95	-5.93	-7.55	-7.81
6	9	-2.34	-3.60	-7.89	-5.65	-4.78	-5.77
10	20	-1.93	-8.10	-12.44	-18.26	-17.59	-19.09
20	40	-3.94	-7.81	-11.12	-15.03	-24.40	-25.08
40	60	-1.57	-2.87	-5.13	-8.82	-11.14	-18.74
60	80	-1.39	-3.11	-5.29	-6.85	-9.62	-10.71
80	100	-0.18	-0.27	-3.35	-4.66	-7.09	-11.32
100	150	-0.10	-1.85	-3.51	-6.28	-7.84	-10.28
150	200	-0.57	-0.80	-1.07	-0.37	-5.91	-4.96
250	300	-0.11	-1.21	1.52	-0.03	-2.43	-1.34
300	350	0.02	-1.61	-1.79	0.16	-1.56	-2.81
350	400	0.43	-0.47	0.52	-1.05	1.13	0.88
500	550	---	-0.58	1.36	-1.90	-1.09	0.74
550	600	---	0.35	-1.96	1.57	-0.56	-1.99
600	650	---	---	0.24	-0.28	1.49	0.10
650	700	---	---	-0.32	0.27	-2.32	-0.77
700	750	---	---	---	-0.86	1.45	0.91
750	800	---	---	---	-0.15	-0.93	-2.35
800	850	---	---	---	---	-1.24	1.86
850	900	---	---	---	---	-0.15	-1.21
950	1000	---	---	---	---	---	-1.28
1000	1050	---	---	---	---	---	-0.76

TABLE A.3.7

T TEST FOR THE RATIOS OF THE EXPECTED VALUES OF MIXED
DOMINANCE OVER NON-DOMINANCE BETWEEN
n(1) AND n(2) WITH RIGHT-SKEWED TRIANGULAR ENTRIES

n(1)	n(2)	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	4	-3.55	-5.90	-7.78	-7.69	-5.02	-5.16
4	6	-4.86	-9.12	-10.87	-10.31	-10.35	-8.48
6	8	-2.59	-4.61	-6.69	-8.90	-8.93	-7.34
10	20	-3.99	-8.46	-13.44	-18.25	-25.92	-26.90
20	40	-2.46	-8.58	-14.16	-19.91	-22.35	-28.36
40	60	-1.48	-3.64	-5.72	-12.07	-17.35	-20.62
60	80	-1.94	-1.68	-5.02	-5.40	-9.93	-13.91
80	100	-0.32	-2.35	-3.60	-6.68	-7.89	-9.91
100	150	-1.35	-1.96	-2.51	-3.99	-8.35	-11.26
150	200	-0.24	-1.18	-1.31	-3.02	-4.76	-5.72
250	300	0.00	-0.86	-0.52	-1.89	-0.23	-0.13
300	350	0.10	-0.88	0.00	-1.20	-2.34	-3.03
350	400	-0.47	-0.58	-2.21	-0.30	-0.88	-1.11
400	450	---	---	0.67	0.85	-0.08	-0.87
450	500	---	---	-0.74	-0.41	-1.34	-1.36
500	550	---	---	---	-1.09	0.21	0.49
550	600	---	---	---	-1.81	-0.75	-0.65
600	650	---	---	---	1.89	-0.70	-1.01
650	700	---	---	---	-0.26	1.57	0.60
700	750	---	---	---	---	0.18	-0.55
750	800	---	---	---	---	-1.17	-0.46
800	850	---	---	---	---	---	0.56
850	900	---	---	---	---	---	-0.81

TABLE A.3.c

T TEST FOR THE RATIOS OF THE EXPECTED VALUES OF MIXED
DOMINANCE OVER NON-DOMINANCE BETWEEN
n(1) AND n(2) WITH NORMAL ENTRIES

n(1)	n(2)	m= 2	m= 3	m= 4	m= 5	m= 6	m= 7
3	4	-3.81	-6.87	-5.99	-5.01	-5.35	-4.44
4	6	-5.95	-6.64	-10.18	-11.53	-9.46	-9.90
6	8	-2.70	-7.40	-7.03	-6.63	-7.90	-7.27
10	20	-6.23	-10.81	-18.12	-23.70	-28.40	-28.48
20	40	-5.47	-9.44	-15.08	-18.14	-27.54	-33.70
40	60	-2.43	-6.42	-8.26	-17.64	-20.40	-21.45
60	80	-2.79	-2.46	-6.29	-8.20	-12.35	-15.77
80	100	-0.85	-3.44	-3.80	7.97	-7.95	-13.12
100	150	-0.06	-3.70	-6.07	-15.73	-10.82	-12.05
150	200	-1.22	-0.48	-0.68	-2.93	-3.90	-5.93
250	300	-1.21	-3.09	-1.44	-5.03	-2.12	-4.62
300	350	-0.62	0.29	-2.72	0.28	-1.43	-0.13
350	400	0.02	0.57	0.82	0.64	0.05	-1.66
400	450	---	---	-0.98	0.41	-2.67	-0.21
450	500	---	---	0.22	-2.51	1.61	-1.38
500	550	---	---	---	-0.54	-1.14	-0.30
550	600	---	---	---	0.73	0.16	-1.80
600	650	---	---	---	-1.16	-0.91	0.69
650	700	---	---	---	0.14	-0.79	-0.52
700	750	---	---	---	---	-0.80	0.31
750	800	---	---	---	---	1.32	-0.91
800	850	---	---	---	---	---	-1.14
850	900	---	---	---	---	---	0.56

APPENDIX 4
RESULTS OF THE REGRESSION ANALYSIS

TABLE A.4.1

REGRESSION RESULTS OF THE EXPECTED VALUE OF PURE DOMINANCE
FOR UNIFORM ENTRIES VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(Pu(n,m)) = A + B \cdot \ln(n)$$

n	A	B	F	R-SQUARED	D.F.
2	-0.7022 (-10.3195)	1.1293 (71.8226)	5158.4766	0.9975	14
3	-1.3206 (-12.5040)	1.2342 (50.5729)	2557.6199	0.9949	14
4	-1.9119 (-16.7024)	1.3080 (54.8384)	3007.2500	0.9944	18
5	-2.6088 (-21.7149)	1.3952 (57.9274)	3355.5798	0.9944	20
6	-3.3590 (-27.7220)	1.4799 (62.9502)	3962.7300	0.9947	22
7	-4.1639 (-31.6299)	1.5646 (63.0222)	3971.7998	0.9942	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.2

REGRESSION RESULTS OF THE EXPECTED VALUE OF MIXED DOMINANCE
FOR UNIFORM ENTRIES VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(\mu(n, m)) = A + B \ln(n)$$

n	A	B	F	R-SQUARED	D.F.
2	-1.0830 (-13.4304)	0.4454 (13.0539)	170.4100	0.9291	14
3	-1.7681 (-10.5265)	0.7051 (18.1666)	330.0298	0.9621	14
4	-1.8554 (-8.9290)	0.8727 (20.1544)	406.2000	0.9598	18
5	-2.1965 (-9.6425)	1.0414 (22.8026)	519.9600	0.9647	20
6	-2.6393 (-11.0843)	1.1885 (25.7263)	661.8499	0.9692	22
7	-2.9975 (-13.0404)	1.2899 (29.7531)	885.2500	0.9747	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.3

REGRESSION RESULTS OF THE EXPECTED VALUE OF NON-DOMINANCE
FOR UNIFORM ENTRIES VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(Nu(n,m)) = A + B \cdot \ln(n)$$

m	A	B	F	R-SQUARED	D.F.
2	0.4516 (11.4090)	0.1961 (21.4409)	459.7100	0.9725	14
3	0.5152 (7.4030)	0.3631 (22.5778)	509.7598	0.9751	14
4	0.4932 (7.4620)	0.4594 (33.3536)	1112.4600	0.9849	18
5	0.6820 (13.5856)	0.5280 (52.4648)	2752.5498	0.9931	20
6	0.6825 (11.6359)	0.5926 (52.0680)	2711.0598	0.9923	22
7	0.6410 (10.8832)	0.6515 (58.6460)	3439.4500	0.9934	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.4

REGRESSION RESULTS OF THE EXPECTED VALUE OF PURE DOMINANCE
FOR LEFT-SKEWED TRIANGULAR ENTRIES
VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(P1(n,m)) = A + B \cdot \ln(n)$$

m	A	B	F	R-SQUARED	D.F.
2	-0.7015 (-10.8157)	1.1290 (75.3327)	5675.0078	0.9977	14
3	-1.2542 (-12.7153)	1.2101 (58.8779)	3466.5999	0.9951	18
4	-1.8740 (-16.9431)	1.2957 (58.4302)	3414.0798	0.9945	20
5	-2.5919 (-21.1038)	1.3865 (58.1858)	3385.5898	0.9938	22
6	-3.2823 (-28.0270)	1.4631 (66.2349)	4387.0586	0.9948	24
7	-4.0871 (-33.6611)	1.5461 (69.9890)	4898.4492	0.9947	27

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.5

REGRESSION RESULTS OF THE EXPECTED VALUE OF MIXED DOMINANCE
FOR LEFT-SKEWED TRIANGULAR ENTRIES
VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(M1(n,m)) = A + B \cdot \ln(n)$$

m	A	B	F	R-SQUARED	D.F.
2	-2.2334 (-11.2732)	0.4788 (10.4597)	109.4100	0.8938	14
3	-1.9525 (-10.0954)	0.7045 (17.4646)	305.0198	0.9472	18
4	-2.0885 (-9.9009)	0.8946 (21.1544)	447.5098	0.9593	20
5	-2.3138 (-11.2082)	1.0410 (26.0098)	676.5098	0.9699	22
6	-2.8520 (-11.9714)	1.2018 (26.7466)	715.3799	0.9689	24
7	-3.3279 (-13.1319)	1.3184 (28.5951)	817.6799	0.9692	27

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.6

REGRESSION RESULTS OF THE EXPECTED VALUE OF NON-DOMINANCE
FOR LEFT-SKEWED TRIANGULAR ENTRIES
VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(N1(n,m)) = P + R \cdot \ln(n)$$

n	P	R	F	R-SQUARED	D.F.
2	0.4785 (15.1280)	0.1963 (26.8515)	721.0000	0.9823	14
3	0.6716 (15.6105)	0.3317 (36.9977)	1368.8298	0.9877	18
4	0.7301 (12.9132)	0.4404 (38.8590)	1510.0198	0.9876	20
5	0.7071 (12.6163)	0.5355 (49.2443)	2425.0000	0.9914	22
6	0.7024 (11.1627)	0.5995 (50.5165)	2551.9099	0.9911	24
7	0.6613 (10.9675)	0.6602 (60.1779)	3621.3699	0.9929	27

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.7

REGRESSION RESULTS OF THE EXPECTED VALUE OF PURE DOMINANCE
FOR RIGHT-SKEWED TRIANGULAR ENTRIES
VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(\Pr(n,m)) = A + B \ln(n)$$

m	A	B	F	R-SQUARED	D.F.
2	-0.6963 (-10.5078)	1.1281 (73.6721)	5427.5781	0.9976	14
3	-1.3092 (-13.6923)	1.2320 (55.7184)	3104.5396	0.9958	14
4	-1.9509 (-17.7130)	1.3214 (53.4164)	2853.3198	0.9948	16
5	-2.5774 (-21.6844)	1.3896 (58.3144)	3400.5696	0.9944	20
6	-3.3227 (-28.0109)	1.4746 (64.0695)	4104.8867	0.9949	22
7	-4.1345 (-34.8183)	1.5596 (69.6377)	4849.4063	0.9953	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.8

REGRESSION RESULTS OF THE EXPECTED VALUE OF MIXED DOMINANCE
FOR RIGHT-SKEWED TRIANGULAR ENTRIES
VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(Mr(n,m)) = A + B \ln(n)$$

m	A	B	F	R-SQUARED	D.F.
2	-1.8398 (-12.5924)	0.4049 (14.6574)	214.8400	0.9429	14
3	-1.5608 (-7.6601)	0.7188 (15.2563)	233.0600	0.9472	14
4	-1.5964 (-7.7258)	0.8836 (19.5921)	383.8499	0.9624	16
5	-1.8322 (-8.4565)	1.0171 (23.4153)	548.2798	0.9665	20
6	-2.1646 (-9.4650)	1.1407 (25.7076)	660.8799	0.9692	22
7	-2.5559 (-11.0083)	1.2491 (28.5253)	813.5898	0.9725	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.9

REGRESSION RESULTS OF THE EXPECTED VALUE OF NON-DOMINANCE
FOR RIGHT-SKEWED TRIANGULAR ENTRIES
VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(Nr(n,m)) = A + B \cdot \ln(n)$$

n	A	B	F	R-SQUARED	D.F.
2	0.4788 (12.9586)	0.1591 (18.6295)	347.0598	0.9639	14
3	0.6518 (13.4424)	0.2877 (25.6740)	659.1499	0.9807	14
4	0.7052 (14.3904)	0.3899 (36.4495)	1328.5598	0.9888	16
5	0.7531 (13.9102)	0.4620 (42.5656)	1811.8298	0.9896	20
6	0.7648 (12.2108)	0.5235 (43.0759)	1855.5298	0.9888	22
7	0.7263 (12.7558)	0.5865 (54.6172)	2983.0198	0.9923	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.10

REGRESSION RESULTS OF THE EXPECTED VALUE OF PURE DOMINANCE
FOR NORMAL ENTRIES VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(P_n(n,m)) = A + B \cdot \ln(n)$$

n	A	B	F	R-SQUARED	D.F.
2	-0.7050 (-10.1298)	1.1300 (70.2631)	4936.8789	0.9974	14
3	-1.3025 (-13.8468)	1.2306 (56.6183)	3205.6299	0.9960	14
4	-1.9593 (-17.5390)	1.3229 (54.2568)	2943.7998	0.9949	16
5	-2.6478 (-20.6754)	1.4017 (54.5939)	2960.5000	0.9937	20
6	-3.3950 (-26.2569)	1.4861 (59.2356)	3508.8599	0.9941	22
7	-4.1095 (-33.5285)	1.5559 (67.3018)	4529.5273	0.9949	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.11

REGRESSION RESULTS OF THE EXPECTED VALUE OF MIXED DOMINANCE
FOR NORMAL ENTRIES VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(Mn(n,m)) = A + B \cdot \ln(n)$$

n	A	B	F	R-SQUARED	D.F.
2	-1.9997 (-11.4021)	0.5491 (13.5496)	183.5900	0.9339	14
3	-1.7463 (-8.1958)	0.7791 (15.8250)	250.4300	0.9507	14
4	-1.8649 (-7.7111)	0.9509 (18.0143)	324.5198	0.9558	16
5	-2.4572 (-9.3804)	1.2153 (23.1415)	535.5298	0.9657	20
6	-2.3854 (-9.4108)	1.1935 (24.2681)	588.9399	0.9656	22
7	-2.8552 (-10.4476)	1.3121 (25.4539)	647.8999	0.9657	24

* VALUES IN PARENTHESES ARE T STATISTICS

TABLE A.4.12

REGRESSION RESULTS OF THE EXPECTED VALUE OF NON-DOMINANCE
FOR NORMAL ENTPIES VERSUS THE TOTAL NUMBER OF STRATEGIES

$$\ln(Nn(n,m)) = A + B \cdot \ln(n)$$

m	A	B	F	R-SQUARED	D.F.
2	0.5591 (18.5643)	0.1247 (17.9151)	320.9500	0.9611	14
3	0.7648 (14.1036)	0.2383 (19.0204)	361.7798	0.9653	14
4	0.8643 (13.7178)	0.3308 (24.0568)	578.7300	0.9747	16
5	0.9053 (13.1439)	0.4061 (29.4059)	864.7100	0.9785	20
6	0.9022 (12.7997)	0.4718 (34.4980)	1190.1099	0.9827	22
7	0.8772 (11.8588)	0.5311 (38.0707)	1449.3699	0.9844	24

* VALUES IN PARENTHESES ARE T STATISTICS

FOR RIGHT-SKEWED TRIANGULAR ENTPIES

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