Trees, Prisms and a Quillen Model Structure on Prismatic Sets

by

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Louis Thrall

Advisor: Professor Martin Bendersky

We define a category which we call the category of prisms, $\mathbb{P}$. This category interpolates between the simplex category, Δ and the box category $\square$. The way in which we interpolate these two categories is considering categories with a tensor functor and a cone functor. It turns out that the objects of the category $\mathbb{P}$ are in one to one correspondence with planer rooted trees. Furthermore the morphisms of this category may be defined as certain combinatorial decorations on certain trees. We then show that that the category $\mathbb{P}$ is a test category automatically giving a Quillen model structure on the presheaves on $\mathbb{P}$.

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Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

This project began with the problem of understanding the topological relationship between cubical sets and simplicial sets. It was already know that they both have Quillen model structures and that are both Quillen equivalent [10]. I was interested in the combinatorial relationship between the two. At a certain point, I found a mathoverflow question which had asked about the relative merits of cubes and simplices. The following is the answer that struck me

> *I think simplices are more convenient for constructing cones, while cubes are more convenient for constructing products. In fact, we can think of simplices as smallest collection of polyhedra which contains a point (0-simplex) and is closed under taking cones, while the cubes are the smallest collection which contain an interval (1-cube) and are closed under products. (Alternatively, contain a point and closed under taking product*

with an interval.) Personally, I think it is more convenient to do singular homology with the larger collection of polyhedra which is closed under both cones and products. (The n-dimensional polyhedra in this collection are indexed by rooted trees with n edges. The simplices correspond to maximal depth trees where the valence of a vertex is at most 2, while the cubes correspond to minimal depth (star-shaped) trees where the root vertex has valence n and all the other vertices have valence 1.) [1]

The idea here is that simplices are generated by a coning operation and cubes are generated by a tensoring operation. What this approach did was to combine the two operations. So we have the category of prisms [2] . The category of prisms is the category generated by coning and a tensor product. What was interesting about the objects is that they are in a one to one correspondence with rooted trees. So then the co-face and co-degeneracy maps, similar to those in cubical sets and simplicial sets, were worked out in terms of a graphical calculus on the trees. Motivated by [10], I was lead make the conjecture that the category of prismatic sets (the category of presheaves of prisms) is a Quillen model category which is Quillen equivalent to the category of simplicial sets with the standard Quillen model structure. Therefore the main theorem of this thesis is as follows.

Theorem 1.1.1. *The categories, $\widehat{\mathbb{P}}$, and $\widehat{\mathbb{P}}_{sym}$ are have Quillen model structures such that the cofibrations are monomorphisms and the weak equivalences*

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[1] This is the answer on mathoverflow by Kevin Walker. The address for this answer is http://mathoverflow.net/questions/3656/cubical-vs-simplicial-singular-homology/3670#3670

[2] Another author uses prisms in a slightly different way see [1]

are maps, $X \to Y$ such that $NX \to NY$ is a weak equivalence, where N denotes the simplicial nerve. Furthermore, the simplicial nerve functor is a left Quillen equivalence.

To prove this, we recalled some work of Cisinski. He took Grothendieck's notion of a test category and showed that the pre-sheaves of a test category have a Quillen model structure that is Quillen equivalent to the category of simplicial sets (Grothendieck had already shown that they have the same homotopy category). So our task was now to show that the category of prisms is a test category. It turned out that this was straightforward once we knew that any small contractible category with a cylinder functor is a test category.

1.2 Pre-sheaf Categories and Slice Categories

This section will be a general exposition of pre-sheaf categories. Throughout, $\mathcal{A}$ will be a small category, $\widehat{\mathcal{A}}$ will be the category such that the objects are functors of the form, $\mathcal{A}^{op} \to \mathbf{sets}$, and the morphisms are natural transformations. Furthermore, their is the Yoneda embedding, $\gamma : \mathcal{A} \to \widehat{\mathcal{A}}$ given by $\gamma(a)_b = hom_{\mathcal{A}}(b, a)$. Given such a small category, we have a functor, $i_{\mathcal{A}} : \widehat{\mathcal{A}} \to \mathbf{cat}$ given by

$$X \mapsto \mathcal{A} \downarrow X,$$

where $\mathcal{A} \downarrow X$ is the category whose objects are pairs, (a, f) such that $f : \gamma a \to X$, and the morphisms are commutative triangles of the form,

$$\begin{tikzcd}
\gamma a \arrow[rr, "\gamma(g:a\to a')"] \arrow[dr] & & \gamma a' \arrow[dl] \\
& X &
\end{tikzcd}$$

This functor has a right adjoint, $i^*_{\mathcal{A}} : \mathbf{cat} \to \widehat{\mathcal{A}}$ given by

$$i^*_{\mathcal{A}}(\mathcal{C})(a) = fun(\mathcal{A} \downarrow a, \mathcal{C}).$$

In order to simplify the notation, we will drop the subscripts, $\mathcal{A}$ when it is clear which category we are talking about. Of particular importance to us is the counit of this adjunction, $\varepsilon : ii^* \to 1_{\mathbf{cat}}$. If $\mathcal{C}$ is a category, then $ii^*(\mathcal{C})$ is also a category. We will now explicitly describe the category $ii^*(\mathcal{C})$. The objects of $ii^*(\mathcal{C})$ are functors,

$$\mathcal{A} \downarrow a \to \mathcal{C},$$

and the morphisms of $ii^*(\mathcal{C})$ are also pairs,

$$\begin{tikzcd}
a \arrow[rr, "f"] & & a' \\
\mathcal{A} \downarrow a \arrow[rr, "\mathcal{A}\downarrow f"] \arrow[dr, "F"'] & & \mathcal{A} \downarrow a' \arrow[dl, "G"] \\
& \mathcal{C} &
\end{tikzcd}$$

Example 1.2.1. *Let Δ be the simplex category X be any simplicial set. Then $\Delta \downarrow X$ is exactly the category of simplices of X. It turns out that the nerve of this category is the barycentric subdivision of X.*

1.3 Homotopical Algebra

Definition 1.3.1. *Let $\mathcal{C}$ be a category. Then a homotopy category on $\mathcal{C}$ is a subcategory $\mathcal{W}$. The category $\mathcal{W}^{-1}\mathcal{C}$, the localization of $\mathcal{C}$ at $\mathcal{W}$ is the homotopy category of $(\mathcal{C}, \mathcal{W})$, if it exists.*[3]

Some authors call such a pair from definition 1.3.1 a relative category. For more details, see [2].

For may instances, this is a difficult notion to work with, so we put some more structure on the category, and ask it to be a Quillen model category, defined as follows.

Definition 1.3.2. *A Quillen model category is a category, $\mathcal{C}$ along with three classes of maps called fibrations, cofibrations and weak equivalence that satisfies the following axioms.*

CM1*:The Category contains all small limits and co-limits.*

CM2*:Suppose that the following diagram commutes in $\mathcal{C}$ and that two out of the three morphisms are weak equivalences. Then so is the third.*

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\
& Z &
\end{tikzcd}$$

CM3*: If f is a retract of g and g is in one of the three classes of maps, then f is in the same class of maps.*

---

[3] The existence of the localization may or may not exist. If we choose the right kind of foundations, it will always exist. If not we might have to do some gymnastics to show that it exists.

CM4: *Suppose that their is a solid arrow diagram*

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
V \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where i is a cofibration, p is a fibration and one of the two maps is a weak equivalence. Then there is a dotted arrow making the diagram above commute.

CM5: *Given any map$f : X \to Y$, their is a diagram of the following form.*

$$\begin{tikzcd}
 & Z \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\
 & W \arrow[ur, "q"'] &
\end{tikzcd}$$

Where p and q are fibrations,and i and j are cofibrations. Furthermore q and i are weak equivalences. Maps that are both fibrations (cofibration) and weak equivalences are called trivial fibrations (cofibrations)

If $\mathcal{C}$ is a Quillen model category, $cof(\mathcal{C})$ will denote the class of cofibarations and $fib(\mathcal{C})$ will denote the class of fibrations.

Definition 1.3.3. *Let $\mathcal{C}, \mathcal{D}$, be a pair if Quillen model categories and let $L : \mathcal{C} \to \mathcal{D}$ be a functor that has a right adjoint, R. Then (L, R) is a Quillen pair if L preserves cofibrations and trivial cofibrations. The pair (L, R) is a Quillen Equivalence if for every pair of objects, $(c, d) \in cofib(\mathcal{C}) \times fib(\mathcal{D})$ a morphism, $c \to Rd$ is a weak equivalence if and only if the adjoint , $Lc \to d$ is a weak equivalence.*

Note that given any Quillen model category, we have a homotopy theory, given by considering the class of weak equivalences. We now state a some theorems due to Quillen.

Theorem 1.3.4. *Let $\mathcal{C}$ be a Quillen model category. Then the homotopy category associated to the class of weak equivalences, denoted $\boldsymbol{Ho}(\mathcal{C})$ exists.*

Proof. For he proof of this theorem, see [13] or more recently [9]. $\square$

Theorem 1.3.5. *Let (L, R) be a Quillen pair form $\mathcal{C}$ to $\mathcal{D}$. Then we have a commutative diagram of functors,*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "L"] \arrow[d] & \mathcal{D} \arrow[l, bend left, "R"] \arrow[d] & . \\
\boldsymbol{Ho}(\mathcal{C}) \arrow[r, bend left, "\boldsymbol{Ho}L"] & \boldsymbol{Ho}(\mathcal{D}) \arrow[l, bend left, "\boldsymbol{Ho}L"] &
\end{tikzcd}$$

Furthermore of (L, R) is a Quillen equivalence, then $\boldsymbol{Ho}L$, and $\boldsymbol{Ho}R$ are equivalences of categories.

Proof. For he proof of this theorem, see [13] or more recently [9]. $\square$

Theorem 1.3.6. *The geometric realization and singular simplices functor from simplicial sets to the category of compactly generated Haussdorf spaces is a Quillen equivalence.*

Proof. For he proof of this theorem, see [13] or more recently [9]. $\square$

Theorem 1.3.7. *Let **cat** be the category of small categories and $\mathcal{W}$ be the subcategory of **cat** such that the nerve of the morphisms of $\mathcal{W}$ are weak equivalences of simplicial sets. Then the homotopy category $(\boldsymbol{cat}, \mathcal{W})$ has an associated homotopy category.*

Proof. For he proof of this theorem, see [13] or more recently [9]. □

1.4 Test categories and Quillen model structures on them

In this section, we will define the notion of a test category. Using this idea, we will state the fact that that the pre-sheaves on a test category have a Quillen model structure and that this Quillen model structure is Quillen equivalent to simplicial sets, with the Kan model structure.

The problem that test categories solve is:

Question 1.4.1. *Let $\mathcal{A}$ be a small category. What conditions on $\mathcal{A}$ will give a Quillen model structure on $\widehat{\mathcal{A}}$ that is Quillen equivalent to the model structure on simplicial sets?*

The answer is what we define to be a test category in definition 1.4.2. We will put a equip **cat** with a homotopy theory by defining a functor to be a weak equivalence if the nerve of that functor is a weak equivalence of simplicial sets. We will also define a homotopy theory on $\widehat{\mathcal{A}}$ as a map, $f : X \to Y$, such that if is a weak equivalence of categories. If this homotopy theory on $\widehat{\mathcal{A}}$ has a homotopy category, then it is clear that the functor i is well defined on the associated homotopy categories. Therefore a necessary condition is that the co-unit of the adjunction, $\epsilon_{\mathcal{C}} : ii^*(\mathcal{C}) \to \mathcal{C}$ is a weak equivalence of categories. This partially motivates the following definition, first defined by Grothendieck in [8].

Definition 1.4.2. *Let $\mathcal{A}$ be a small category. Then $\mathcal{A}$ is a weak test category if $ii^*(\mathcal{C})$ is contractible for all categories $\mathcal{C}$ such that $\mathcal{C}$ has a terminal object. A category is a local test category if all of the categories $\mathcal{A} \downarrow a$ are weak test categories. A test category is a category that is a weak test category and a local test category.*

Theorem 1.4.3. *Let $\mathcal{A}$ be a weak test category. Then the functor, $\mathcal{A} \downarrow \vdots \widehat{\mathcal{A}} \rightarrow$ **cat** induces an equivalence of homotopy categories.*

Proof. This was proven in [8]. $\square$

Lemma 1.4.4. *The category, $\mathcal{A}$ is test category if and only if $\mathcal{A}$ is contractible and the categories, $i(i^*(\mathcal{C}) \times \gamma a)$ are contractible.*

Proof. See [11] and [5].

Theorem 1.4.5. *Let $\mathcal{A}$ be a test category. Then $\widehat{\mathcal{A}}$ has a Quillen Model structure such that the cofibrations are monomorphisms, and the weak equivalences are maps $f : X \to Y$ such that the nerve of the map $\mathcal{A} \downarrow f$ is a weak equivalence of simplicial sets. Furthermore,This Quillen model category is Quillen equivalent to the standard Quillen model structure on the category of simplicial sets.*

Proof. See [11] and [5].

1.5 Cylinder functors

In this section, we define the notion of a cylinder functor. We then show that if $\mathcal{A}$ is a contractible category with a cylinder functor, then $\mathcal{A}$ is a test category.

Definition 1.5.1. *Let $\mathcal{A}$ be any category. A cylinder functor on $\mathcal{A}$ is a functor, **Cyl** along with natural transformations,,*

1. $d^i : Id \to \boldsymbol{Cyl}, i = 0, 1$ *and a natural natural transformation* $p : \boldsymbol{Cyl} \to Id$ *such that* $pd^i = id$.
2. *diagrams of the form,*

$$\begin{tikzcd}
b \arrow[r] \arrow[d] & a \arrow[d, "d^0"] \\
a \arrow[r, "d^1"'] & \boldsymbol{Cyl}a
\end{tikzcd}$$

   do not exist.

Note that item 1 of definition 1.5.1 is equivalent to a diagram of the following form for each map $f : X \to Y$ in $\widehat{\mathcal{A}}$.

$$\begin{tikzcd}
a \arrow[r, "d^0"] \arrow[drrr] \arrow[ddr, "1_a"'] & \mathbf{Cyl}a \arrow[drrr, "\mathbf{Cyl}f"] \arrow[dd, "s"'] & a \arrow[l, "d^1"'] \arrow[drrr] \arrow[ddl, "1_a"] & & & \\
& & & a' \arrow[r, "d^0"'] \arrow[ddr, "1_{a'}"'] & \mathbf{Cyl}a' \arrow[dd, "s"'] & a' \arrow[l, "d^1"] \arrow[ddl, "1_{a'}"] \\
& a \arrow[drrr, "f"'] & & & & \\
& & & & a' &
\end{tikzcd}$$

Furthermore, any cylinder functor $\mathbf{Cyl} : \mathcal{A} \to \mathcal{A}$ may be extended to a cylinder functor, $\mathbf{Cyl} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$. The proof of the following lemma is an adaptation of the proof of lemma 3.11 in the paper [11].

Lemma 1.5.2. *Let $\mathcal{A}$ be a small category with a cylinder functor,* ***Cyl****. Then for every object $a \in \mathcal{A}$, the category $i(i^*\mathcal{C} \times a)$ is contractible whenever $\mathcal{C}$ has a terminal object.*

Proof. Let us begin by saying what the morphisms of the category $i(i^*\mathcal{C} \times a)$ are. They are pairs of commutative diagrams of the form

$$\begin{tikzcd}
\mathcal{A} \downarrow b \arrow[rr, "\mathcal{A}\downarrow f"] \arrow[dr, "F"'] & & \mathcal{A} \downarrow b' \arrow[dl, "G"] & b \arrow[rr, "f"] \arrow[dr] & & b' \arrow[dl] \\
& \mathcal{C} & & & a &
\end{tikzcd}$$

Now given any functor $F : \mathcal{A} \downarrow a \to \mathcal{C}$ such that $\mathcal{C}$ has a terminal object, t we may define a functor, $F_* : \mathcal{A} \downarrow \mathbf{Cyl}a \to \mathcal{C}$ on objects as follows

$$F_*(\sigma : b \to \mathbf{Cyl}a) = \begin{cases} F(\sigma_1) & \text{If } \sigma \text{ factors as } d^0\sigma_1 \\ t & \text{otherwise} \end{cases}$$

We must show that this is well defined on objects. To show that this is well defined, suppose that $\sigma : a \to \mathbf{Cyl}a$ factors as $d^0\sigma_1^i$ if $i = 0, 1$. Since d^0 is a monomorphism and the two factorings are the same, $\sigma_1^0 = \sigma_1^1$. This shows well definedness of F_* on objects. We will now show that this definition on objects induces a unique definition of F_* on morphisms. To this end note that we have a partition of the objects into two parts. The objects that factor through d^0 and those that do not. Now consider a morphism of the form,

$$\begin{tikzcd}
b \arrow[rr, "f"] \arrow[d, dashed, "\sigma_1"'] \arrow[dr, "\sigma"] & & b' \arrow[d, dashed, "\sigma'_1"] \arrow[dl, "\sigma'"'] \\
a \arrow[r, dashed, "d^0"'] & \mathbf{Cyl}a & a \arrow[l, dashed, "d^0"]
\end{tikzcd}$$

, where the dotted morphisms may or may not exist. We will write this map as $f : \sigma \to \sigma'$. We then define F_* on objects to be

$$F_*(f : \sigma \to \sigma') = \begin{cases} F(f : f : \sigma_1 \to \sigma_1') & \text{If } \sigma_1' \text{ exists} \\ F(\sigma_1) \to t & \text{If } \sigma_1 \text{ exists and } \sigma_1' \text{ does not exists} \\ t \to t & \text{otherwise} \end{cases}$$

We will now show that for any functor, we have a commutative digram of functors of the following form.

$$\begin{tikzcd}
\mathcal{A}\downarrow a \arrow[r, "\mathcal{A}{\downarrow}d^1"] \arrow[dr, "F"'] & \mathcal{A}\downarrow \mathbf{Cyl}a \arrow[d, "F_*"'] & \mathcal{A}\downarrow a \arrow[l, "\mathcal{A}{\downarrow}d^1"'] \arrow[dl, "t"] \\
& \mathcal{C} &
\end{tikzcd}$$

The left hand triangle exists by construction. We only need to show that the right hand triangle exists. This occurs if for any $g \in \mathcal{A} \downarrow a$, $F_*(d^1 g) = t$. But since diagrams of the form

$$\begin{tikzcd}
b \arrow[r, "g"'] \arrow[d] & a \arrow[d, "d^0"] \\
a \arrow[r, "d^1"'] & \mathbf{Cyl}a
\end{tikzcd}$$

do not exist, this is true. We will now show that if we have a morphisms of categories, $\mathcal{A} \downarrow f \mathcal{A} \downarrow a \to \mathcal{A} \downarrow a'$, then the diagram

$$\begin{tikzcd}
\mathcal{A}\downarrow a \arrow[rrrrrr, "f"] \arrow[dr, "\mathcal{A}{\downarrow}d^1"] \arrow[ddddrrr, bend right, "t"'] & & & & & & \mathcal{A}\downarrow a \arrow[dl, "\mathcal{A}{\downarrow}d^1"'] \arrow[ddddlll, bend left, "t"] \\
& \mathcal{A}\downarrow \mathbf{Cyl}a \arrow[rrrr, "\substack{\mathbf{I} \\ \mathbf{cyl}f}"] \arrow[dddrr, bend right, "F_*"'] & & & & \mathcal{A}\downarrow \mathbf{Cyl}a' \arrow[dddll, bend left, "G_*"] & \\
& \mathbf{VI} & \mathcal{A}\downarrow a \arrow[ul, "\mathcal{A}{\downarrow}d^0"'] \arrow[rr, "\substack{\mathbf{II} \\ f}"] \arrow[ddr, "F"] & & \mathcal{A}\downarrow a' \arrow[ur, "\mathcal{A}{\downarrow}d^0"] \arrow[ddl, "G"'] & \mathbf{VII} & \\
& & \mathbf{IV} & \mathbf{III} & \mathbf{V} & & \\
& & & \mathcal{C} & & &
\end{tikzcd}$$

commutes. This is east to see, if we apply a general pasting lemma. This says that if we paste together two commutative diagrams, then the new commutative diagram also commutes. But this diagram is obtained by pasting together subdiagrams **I-VII**, all of which commute by construction. Recall that if we have a natural transformation between two functors, $F_i : \mathcal{D}_1 \to \mathcal{D}_2$ then the nerve of then their is a homotopy between the nerves of the functors. We remark here that the relation of two maps being homotopic is not usually an equivalence relation. The two maps are weakly equivalent however. With this in mind, we are in a position to show that the category $i(i^*(\mathcal{C}) \times a)$ is contractible by forming an endo-functor, on $i(i^*(\mathcal{C}) \times a)$, and constructing a zig-zag of natural transformations between two the functor (t, id) and (id, id), with roof H, where t is the functor that sends every functor, $F : \mathcal{A} \downarrow a \mapsto t : \mathcal{A} \downarrow a$. To each morphism, as in diagram , we get a commutatve square,

$$\begin{tikzcd}
\mathcal{A} \downarrow \mathbf{Cyl} b \arrow[rr, "\mathcal{A}\downarrow\mathbf{Cyl} f"] \arrow[ddr, "F_*"] & & \mathcal{A} \downarrow \mathbf{Cyl} b' \arrow[ddl, "G_*"'] \\
\mathcal{A} \downarrow b \arrow[u, "\mathcal{A}\downarrow d^0"] \arrow[rr, "\mathcal{A}\downarrow f"] \arrow[dr, "F"'] & & \mathcal{A} \downarrow b' \arrow[u, "\mathcal{A}\downarrow d^0"'] \arrow[dl, "G"] \\
& \mathcal{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Cyl} b \arrow[rr, "\mathbf{Cyl} f"] \arrow[ddr] & & \mathbf{Cyl} b' \arrow[ddl] \\
b \arrow[u, "d^0"] \arrow[rr, "f"] \arrow[dr] & & b' \arrow[u, "d^0"'] \arrow[dl] \\
& a &
\end{tikzcd}$$

This defines a natural transformation between $id \rightsquigarrow H$. Similarly, for each morphisms, we have a commutative square

$$\begin{tikzcd}
\mathcal{A} \downarrow \mathbf{Cyl} b \arrow[rr, "\mathcal{A}\downarrow \mathbf{Cyl} f"] \arrow[rdd, "t"] & & \mathcal{A} \downarrow \mathbf{Cyl} b' \arrow[ldd, "t"'] \\
\mathcal{A} \downarrow b \arrow[u, "\mathcal{A}\downarrow d^1"] \arrow[rr, "\mathcal{A}\downarrow f"] \arrow[rd, "t"'] & & \mathcal{A} \downarrow b' \arrow[u, "\mathcal{A}\downarrow d^1"'] \arrow[ld, "t"] \\
& \mathcal{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Cyl} b \arrow[rr, "\mathbf{Cyl} f"] \arrow[rdd] & & \mathbf{Cyl} b' \arrow[ldd] \\
b \arrow[u, "d^1"] \arrow[rr, "f"] \arrow[rd] & & b' \arrow[u, "d^1"'] \arrow[ld] \\
& a &
\end{tikzcd}$$

This provides a natural transformation between $t \rightsquigarrow H$. This shows that $i(i^*(\mathcal{C}) \times a)$ is weakly equivalent to $\mathcal{A}$, which is contractible.

Corollary 1.5.3. *Let $\mathcal{A}$ be a small category with a terminal object and a cylinder functor. Then $\widehat{\mathcal{A}}$ has a Quillen model structure such that the monomorphisms are the cofibrations and the weak equivalences are all maps, $f : X \to Y$ such that the nerve of the functor, $\mathcal{A} \downarrow f$ is a weak equivalence of simplicial sets. Furthermore $\widehat{\mathcal{A}}$ is Quillen equivalent to the standard Quillen model structure on simplicial sets.*

Proof. This follows directly from theorem 1.4.5 and lemma 1.5.2. □

Chapter 2

Trees and Prisms

2.1 Rooted trees

This section, we define rooted trees and planar rooted trees. We then define certain operations on the planar rooted trees that will be useful in the sequel.

Definition 2.1.1.

1. *A rooted tree, T is a finite poset such that for every pair of elements $a, b \in T$ the closed interval, $[a, b]$ is a total order, that has a minimal element called the root.* [1]

2. *An edge in T is a closed interval, $[v, v'] \neq \emptyset$ such that such that the open interval, (v, v') is empty.*

3. *An outer vertex of a rooted tree is a maximal element.*

---

[1] We may make the set of trees into a category by asking that the maps be poset maps.

4. *A leg is an edge of the tree, T that contains an outer vertex.*

5. *A planar rooted tree is a rooted tree, T such that for every vertex, v, the set of edges of the form, $[v, v,]$ is a total order (that is unrelated to the poset structure that defines the rooted tree), which we will call the horizontal order at a vertex, v. Unless otherwise specified, all trees we consider will be planar rooted trees.*

Unless otherwise specified, all trees we consider will be planar rooted trees.

One thing to note about this definition is that the empty set is vacuously a planar rooted tree. Given any (planar) rooted tree, T, $[v, \infty]$ is the subtree of T consisting of the vertices, $\{w \in T : v \leq w\}$. Likewise the open interval, (v, ∞) will be the subtree of T whose vertex set is $\{w \in T : v < w\}$.

A main feature of the class of rooted trees is that we may make larger rooted trees by building them from smaller rooted trees. We therefore define two operations on the set of rooted trees three operations on rooted trees.

Definition 2.1.2. .

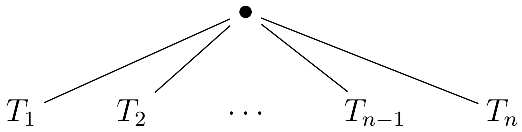
1. *Let T be a rooted tree. Then the whisker of T, denoted WT is the unique tree obtained by adding a new root.*

2. *If T, and T' are rooted trees, then the grafting of T, and T', denoted $T \vee T'$, is the unique rooted tree obtained by identifying the two roots of the trees. In terms of simplicial complexes, this is simply the join.*

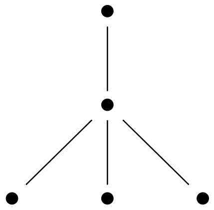
3. *We have a third operation that is a combination of the grafting, and whiskering. Given any ordered n-tuple of trees, $(T_1, T_2, \cdots T_n)$, we have*

multi-whisker, which is obtained by joining the (ordered) disjoint union with an initial point.

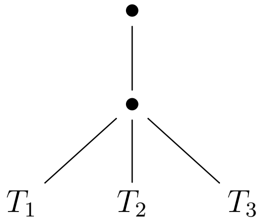
We will now give a way of pictorially denoting the multi-whisker. Given an ordered n-tuple of non-empty trees, $(T_1, T_2 \cdots T_n)$, we will draw the multi-whisker as follows:

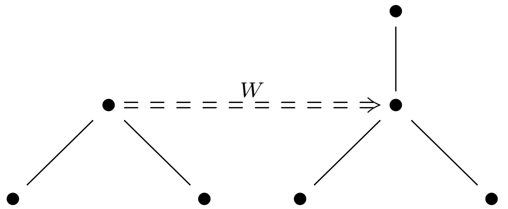


More generally, given any rooted tree, T, and any n-tuple $(T_1, T_2, \cdots T_N)$ where N is the number of maximal elements of the tree T, we may form the tree $T(T_1, T_2, \cdots T_N)$ which is obtained by identifying the i-th maximal element with the root of T_i. For instance given the tree,

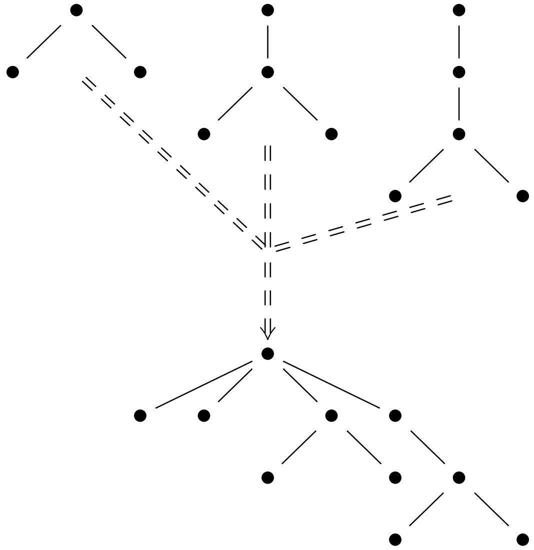


and an ordered triple of trees, (T_1, T_2, T_3), we will draw $T(T_1, T_2, T_3)$ as

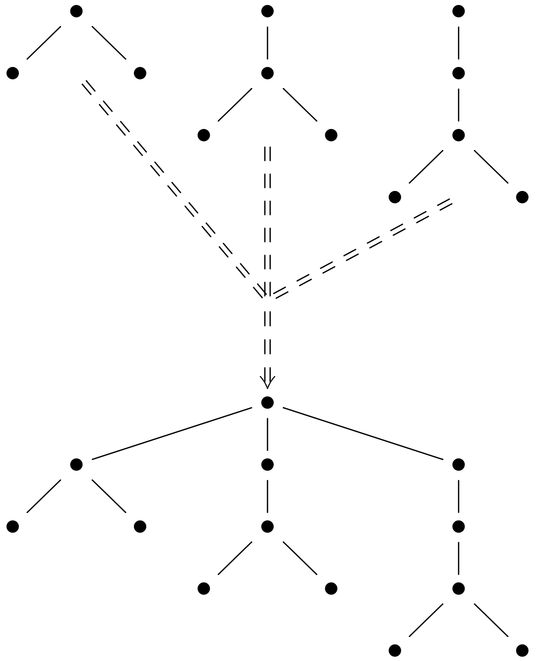




This diagram is an example of grafting.



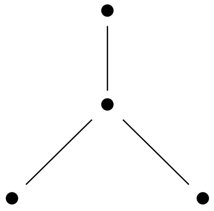
This diagram is an example of the multi-wisker.



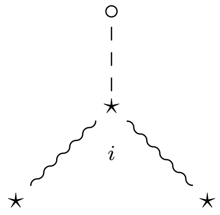
We will now describe some combinatorics on planar rooted trees that will be of use to us when we define the category of prisms. We will begin with the following definition.

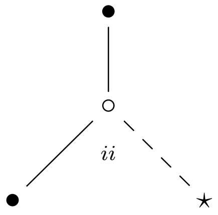
Definition 2.1.3. *Let T be a rooted tree, and v be any vertex. Then a tree under v as a pair, (v, S) such that S is a maximal subtree of the poset (v, ∞).*

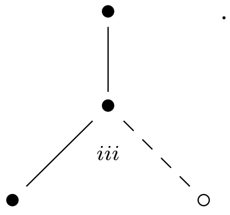
Given the rooted tree,

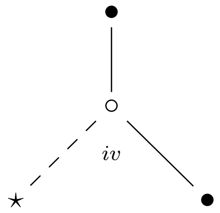


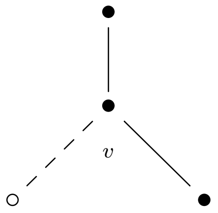
We have five pairs (v, S) such that S is a tree under v given below.





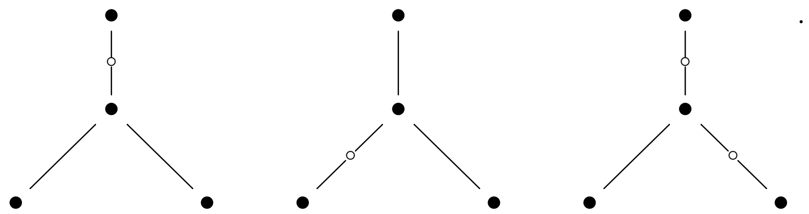




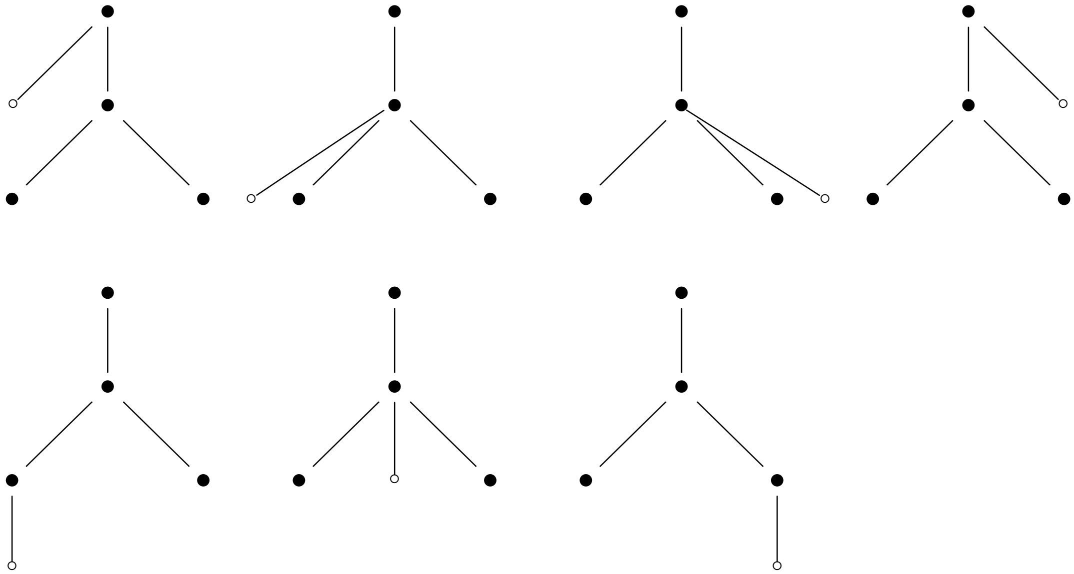


Where the the vertex is labeled by an open vertex, and the tree under the vertex is labeled by stars and squiggles. Note that we have two types of pairs, the type where S is the empty tree and the case where the tree, S is non-empty. Trees i, ii, iv are the correspond to the pairs where S is non-empty. Trees iii and v correspond to the case where S is empty Note that in each case, we have an associated edge. In the case that the vertex is an inner vertex, the associated edge is the edge $[v, v']$ such that v' is the root of the tree under v. In the case that the associated edge is the unique edge of the form, $[v', v]$. Note that we know that this edge is unique since the poset, $[r, v]$ is a total order, where r is the root. We have denote the associated edge as a dashed edge as we have done in the above diagram. Now given the associated edge, we have a canonical map of rooted trees given by $T \to T/e$.

The next construction that we may apply to rooted trees will be to subdivide an edge. Let T be a rooted tree, and $[w, w']$ be an edge. Then we will define a new rooted tree, $T^{[w,w']}$ to have vertex set, $V(T)\coprod\{[w, w']\}$. To define the poset structure on this rooted tree, it suffices to say where we place the new vertex, $[w, w']$. We will insist that $w < [w, w'] < w'$. Given the tree above, we have three such subdivisions, which are shown below



We must also define leg growth. Given any vertex, $v \in T$ we have the horizontal total order one the set of edges of the form, $[v, v']$. We will denote this total order on the set of edges emanating out of v by E_v. Now given any patrician of $E_v = L \coprod R$ such that $\forall (e_1, e_2) \in L \times R \implies e_1 < e_2$ in E_v, we add a new vertex, $v^{L,R}$ and a new edge $[v, v^{L,R}]$ such that $\forall (e_1, e_2) \in L \times R \implies e_1 < [v, v^{L,R}] < e_2$, if L and R are both non-empty. If L is empty we add the edge, $[v, v^{L,R}]$ to the total order E_v by making this new edge the less than all of the elements of E_v. Likewise if R is empty, we declare that the new edge is greater than all of the edges of E_v. We have seven different types of leg growths given the tree above.



It is facially clear that contacting along an edge associated to a vertex decrease the number of edges by one, and that a subdivisions and a leg growth increase the number of edges by one. We may imagine that a leg growth is a type of subdivision of an invisible edge, such that when ones subdivides the invisible edge, the upper part of the subdivision becomes a real edge.

For the following definition we use the fact that we may define a small category by defining a set of objects, as set of morphisms, and some relations between the morphisms. For details see [12].

Definition 2.1.4. *We define the category $\mathbb{T}$ to be the category such that the objects are isomorphism classes of rooted trees, and the morphisms are generated by the following*

1. *for each pair (v, S), we have a map of the form, $d_{S,v} : T \to T/e$, where v is a vertex of the tree T, S is a tree under v, and e is the edge*

associated to the pair, (S, v). These maps will be called co-faces.

2. *For each edge, $e \in T$, an inclusion, $s_e : T \to T_e$, where T_e is the tree obtained by the subdividing the edge e*

3. *For each vertex, $v \in T$ and each patrician (L, R) of the horizontal total order, E_v, we get the inclusion of $T \to T_{[v, v^{L,R}]}$, where $T_{[v, v^{L,R}]}$ is the tree obtained by applying the leg growth associated to (L, R).*

The maps of the form 2 or 3 will be called co-degeneracies. Furthermore, the relations are given by the following rules,

(a) *Given any two distinct pairs (v, S), and (v', S') such that the associated edges are distinct, we have a commutative diagram*

$$\begin{tikzcd}
 & T/e \arrow[dr] & \\
T \arrow[ur] \arrow[dr] & & T/\{e, e'\} \\
 & T/e' \arrow[ur] &
\end{tikzcd}$$

(b) *Any composite of subdivisions or leg growths that give rise to the same map in the category of trees will be identified.*

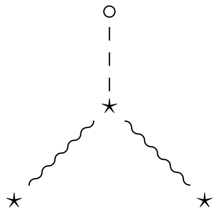
(c) *Given a contraction (v, S) and a subdivision, the order that the two are applied does not matter as long as the new edges made and the edge associated to the contraction are distinct.*

(d) *Given a contraction (v, S) and a leg growth, the order that the two are applied does not matter as long as the new edges made and the edge* *associated to the contraction are distinct.*

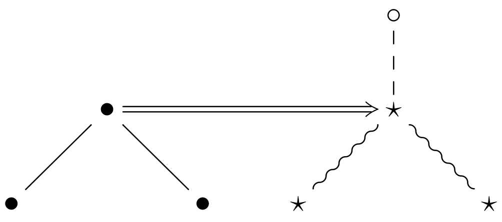
(e) A subdivision followed by one of the two ways of contacting the graph which reverses the subdivisions is the identity.

(f) A leg growth followed by one of the two ways of contacting the graph which reverses the subdivisions is the identity.

We will now set up a graphical calculus for the various types of maps that occur in $\mathbb{T}^{op}$. Now given a morphism in $\mathbb{T}$ that arises associated to a pair (v, S), where v is a vertex of T, and S is a non-empty tree under v, we graphically denote this map by the following diagram, where the open vertex is the vertex v , the subtree S marked by the stars and squiggles is the and the dashed edge is the edge associated to to the pair, (S, v).

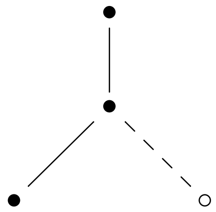


If we are in the category $\mathbb{T}^{op}$, then the dual of the map associated to (v, S) we will abbreviate this denote as follows.

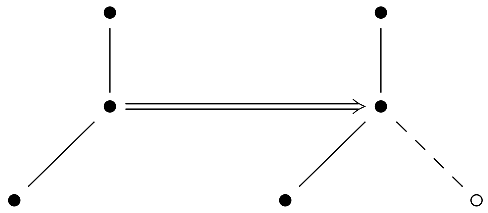


This map will be called the dual contraction associated to the pair, (S, v).

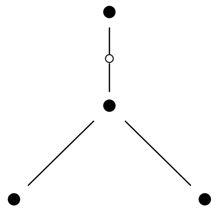
If we have a pair (S, v) such that the tree S under v is the empty tree, this means that the vertex, v is an outer vertex. We will denote the associated map in $\mathbb{T}$ by



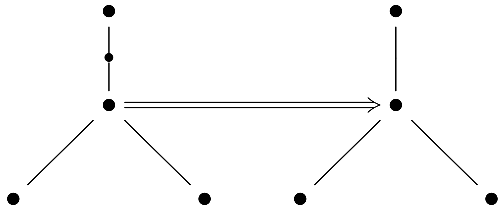
If we are in the category $\mathbb{T}^{op}$, then the dual of the map associated to (v, S) we will abbreviate this denote as follows.



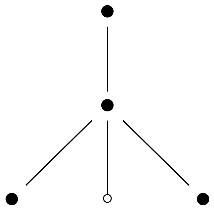
The following diagram is a map in $\mathbb{T}$ that arises as the subdivision of an edge, we put a small vertex in the middle of the edge to mark the edge that has been subdivided. The following diagram is an instance of this.



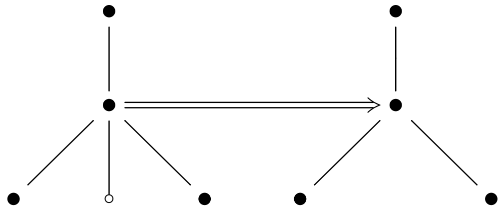
If we are in the category $\mathbb{T}^{op}$, then we will abbreviate this map as follows:



We will call this typeof map in $\mathbb{T}^{op}$ a dual subdivision. Similarly, if we have a leg growth, then we will graphically represent the map in $\mathbb{T}$ by the following decorated tree.



The dual of this map in $\mathbb{T}^{op}$ will be abbreviated as follows,



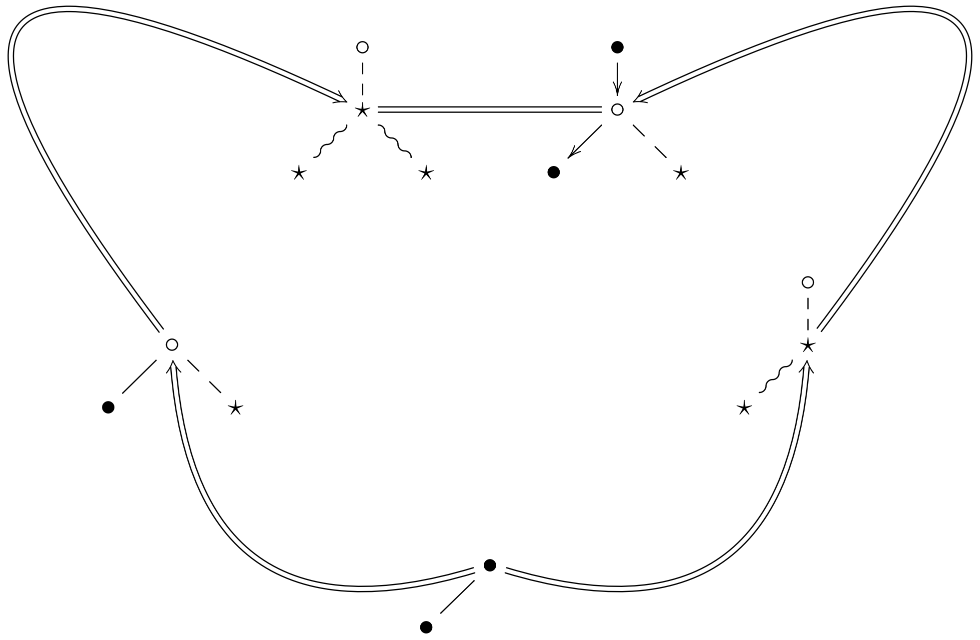
.

We will call this type of map in $\mathbb{T}^{op}$ a dual subdivision.

2.1.1 The relations for $\mathbf{T}^{op}$

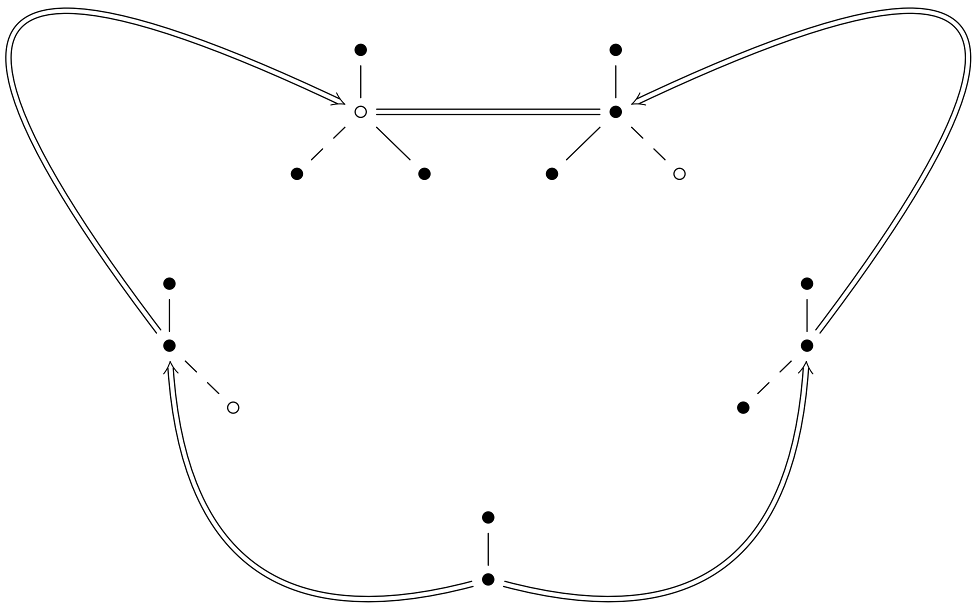
The purpose of this section is to illustrate the different types of relations that occur, and to help understand the graphical calculus for $\mathbb{T}^{op}$ that we set earlier in this chapter. The first four diagrams will illustrate the different ways that relations of type a in definition 2.1.4 occur.

The first case is the case where both maps maps correspond to a pair of the form, (v, S), where both vertices are inner vertices and remain inner contracting after contracting one of the associated edges. The diagram that we get for this case is a follows. In all four of the diagrams, we are performing the dual of contracting the dashed edges.



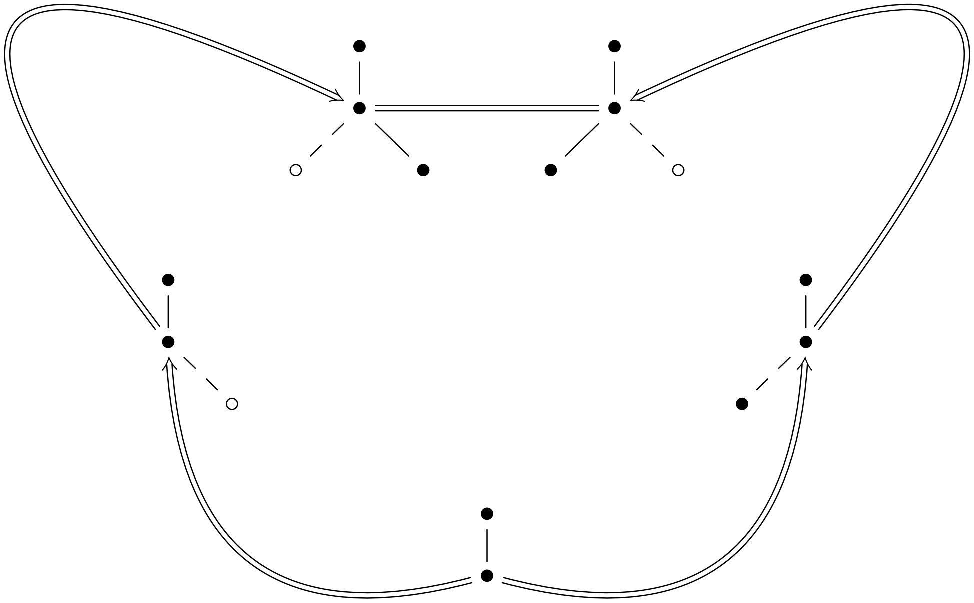
As we move around this left side of this diagram, we are performing the dual of contracting the right hand dashed leg followed by the dual of contacting the upper dashed edge. As we move around the right side of the diagram, we perform the dual of contracting the upper dashed edge, followed by contacting the right hand dashed leg. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The next diagram is expressing what happens when one of the vertices is an outer vertex and the other is and the other is an inner vertex.



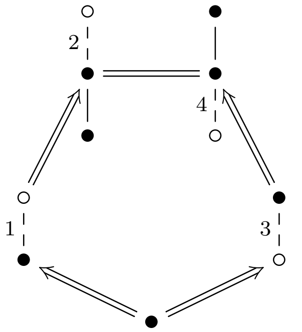
As we move around this left side of this diagram, we are performing the dual of contracting the right hand dashed leg followed by the dual of contacting the left dashed leg. As we move around the right side of the diagram, we perform the dual of contracting the left dashed leg, followed by contacting the right hand dashed leg. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The next diagram is expressing the case where both vertices are outer vertices.



The explanation of this diagram is almost identical to the last, since the edges that are contracted in the dual diagram in the category $\mathbb{T}$ are identical in both diagrams. However, the difference is that in the previous diagram, only one of the vertices was an outer vertex, whereas in this diagram both of the vertices are outer vertices.

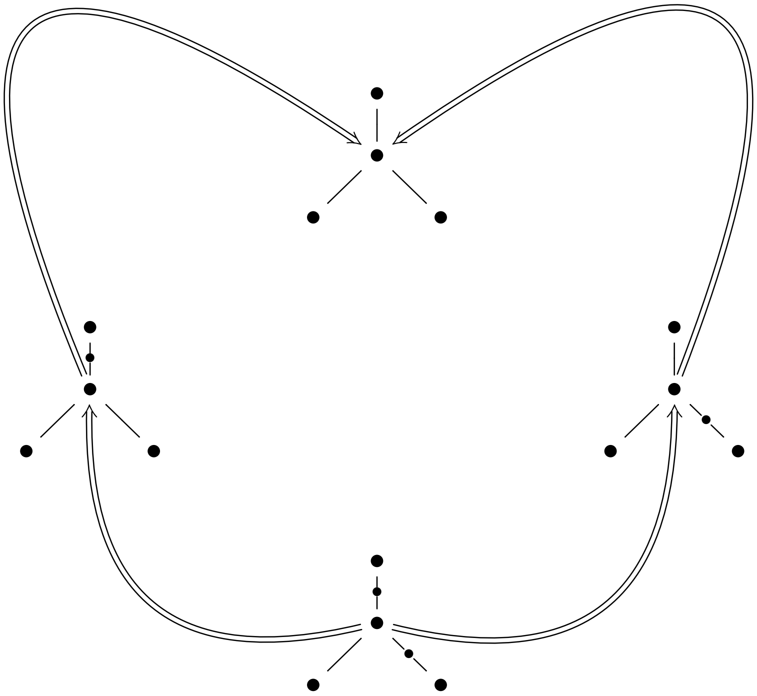
The way that relation one in the definition 2.1.4 may occur is if contracting one of the edges leaves the other edge exposed as a leg. We will show one of three ways that this can happen. We will also note that these three ways (only one of which is shown) are related to the three maps, $\Delta^0 \to D^2$ in the category of simplices.



When we view this diagram in the dual category $\mathbb{T}$, it is telling us that contracting edge 2 followed by contracting edge one is the same map as contracting edge 4 followed by contracting edge three. Note that when we contract edge 4 we are leaving edge three exposed as a leg.

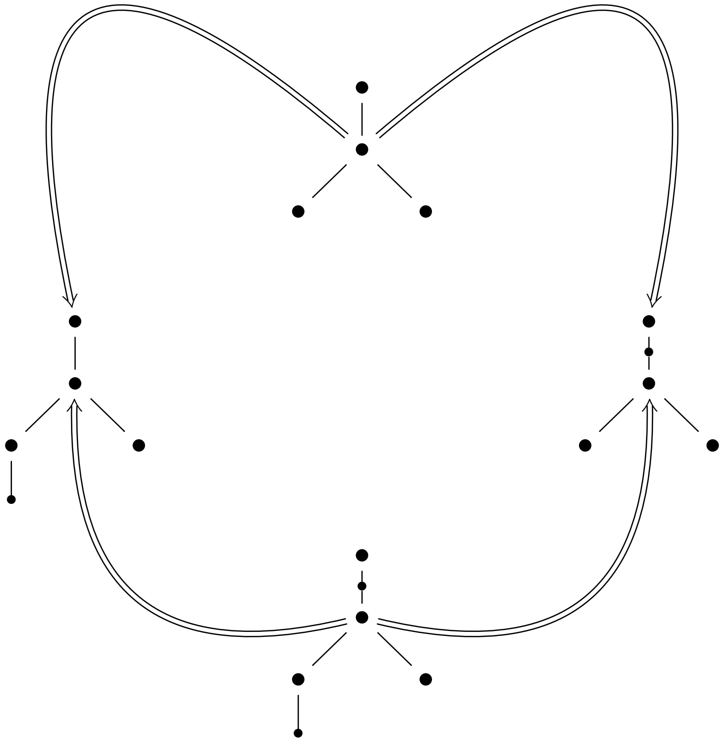
Let us now recall relations of type b from definition 2.1.4. This tells us that any two composites subdivisions or leg growths in $\mathbb{T}$ are equal if the corresponding maps are equal in the category of posets. The next four diagrams will illustrate the various types of relations of type b that can occur.

The first case of this is when we perform the dual of subdivision on two distinct edges.



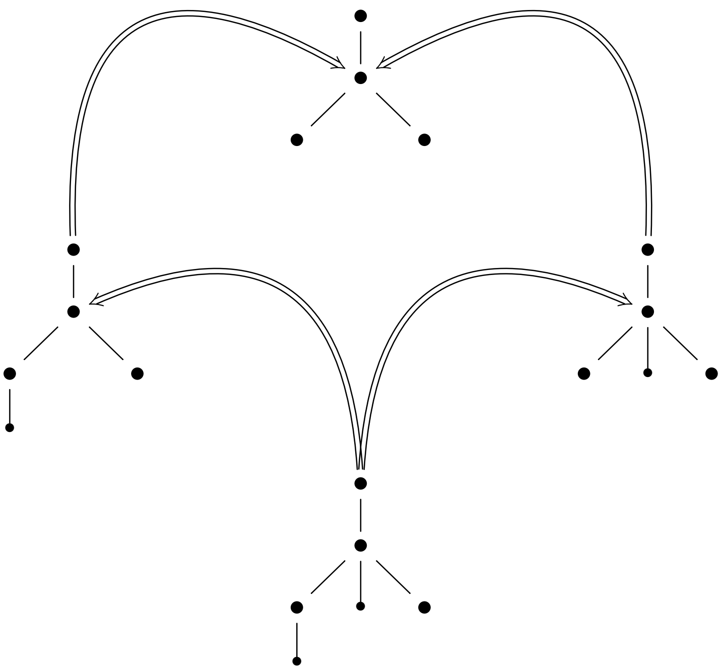
As we move around the left side of the diagram, we are performing the dual of subdividing right hand leg followed by the dual of subdividing the upper edge. Going around the right side of the diagram, we are performing the dual of subdividing the upper edge followed by subdividing the right hand leg. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The next diagram illustrates when relation of type b of definition 2.1.4 when the map in $\mathbb{T}$ is the composite consists of a leg growth and a subdivision.



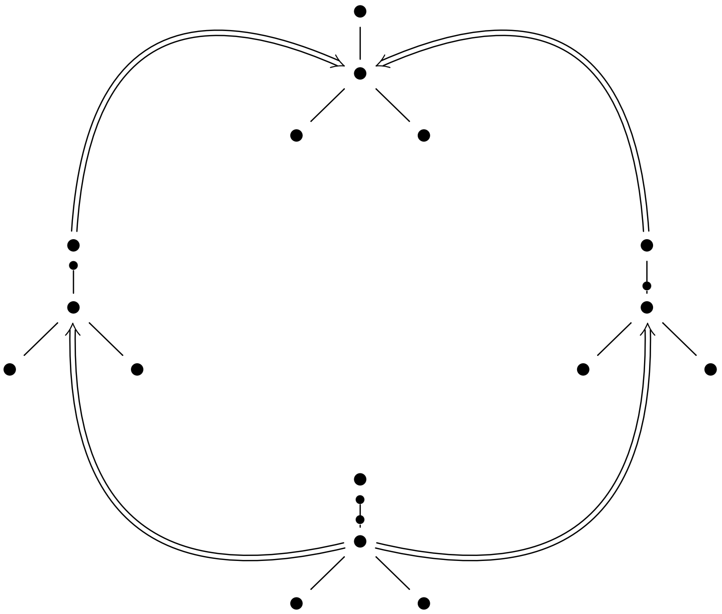
As we move around the left side of the diagram, we are performing the dual of the subdivision of the upper edge, followed by a dual of a leg growth coming out of the leftmost outer edge . As we move around the right side of the diagram, we perform the dual of the leg growth out of the leftmost outer edge followed by the dual of the subdivision of subdividing the upper edge. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The next diagram illustrates when relation b of definition 2.1.4 when the map in $\mathbb{T}$ is the composite consists of two leg growths.

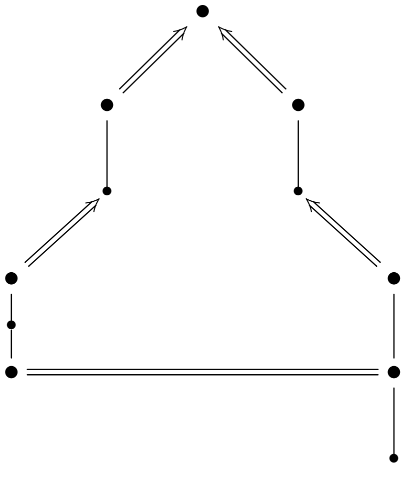


As we move around the left side of the diagram we are performing the dual of a leg growth out of the middle vertex followed by the dual of a leg growth out of the leftmost outer edge. As we move around the right side of the diagram, we are performing the dual of the leg growth out of the leftmost outer edge followed by the dual of the leg growth out of the middle vertex. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

When we subdivide an edge, we create two new edges, an upper edge and a lower edge. From that point, we may choose to subdivide the upper edge or we may choose to subdivide the lower edge. The following diagram is an illustration of this fact in the dual category $\mathbb{T}^{op}$.

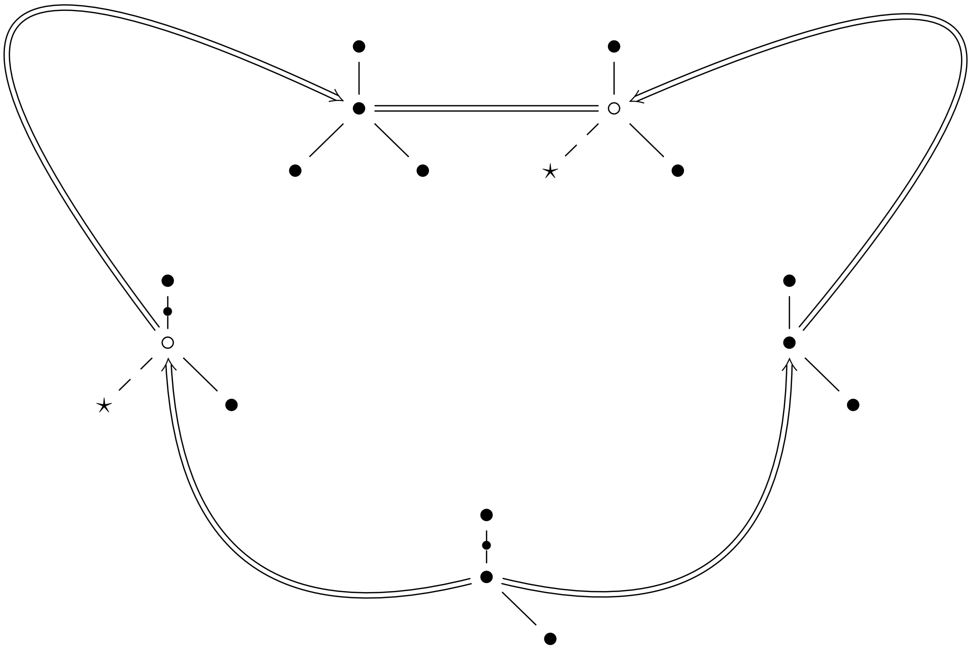


The last relation of type two that we will illustrate that when we perform the dual to a subdivision followed by the dual of the leg growth, we get the same answer if we perform the dual of a leg growth followed by the dual of a leg growth. This situation is shown on the following diagram.



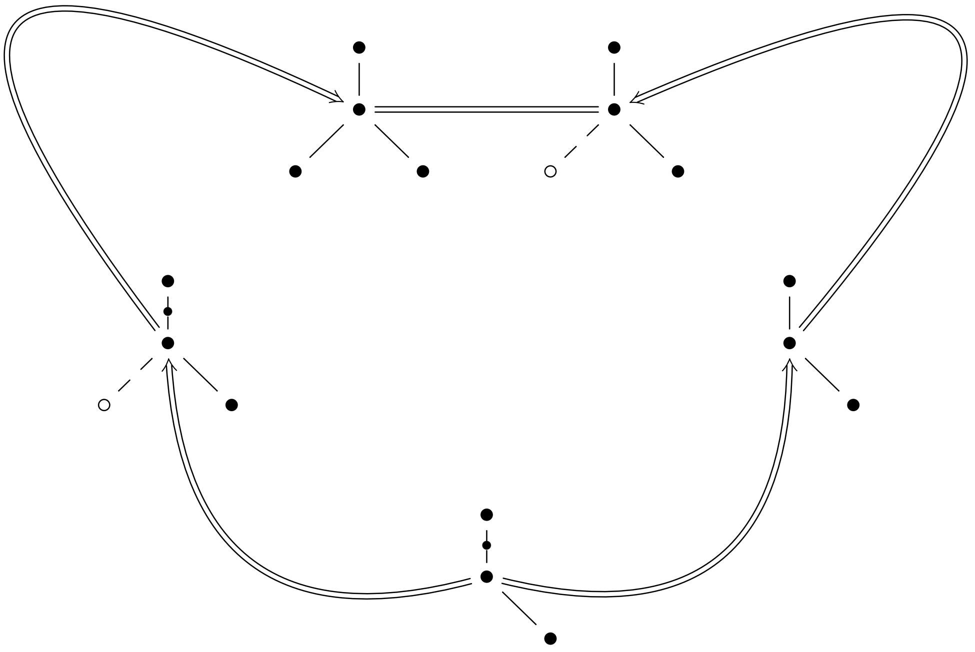
In particular as we move around the left side of the diagram, we start by performing the dual of a subdivision. The result in this case is that the edge that was subdivided is now a leg. We then perform the dual of a leg growth on the newly exposed leg. As we move around the right side of the diagram, we perform the dual of a leg growth. After the dual of this leg growth has been performed, the edge above the leg that was there is now a leg., We then perform the dual of a leg growth on that leg. This diagram is expressing the fact that

The following two diagrams will illustrate relations of type c from definition 2.1.4. Recall that this type of relation tells us that if we subdivide one edge and perform a contraction of the form (S, v) such that the associated edge and the subdivide edge are distinct, then it does dot matter which order we perform the two operations. The following diagram illustrates this relation when the vertex, v, marked by the open vertex, is an inner vertex.



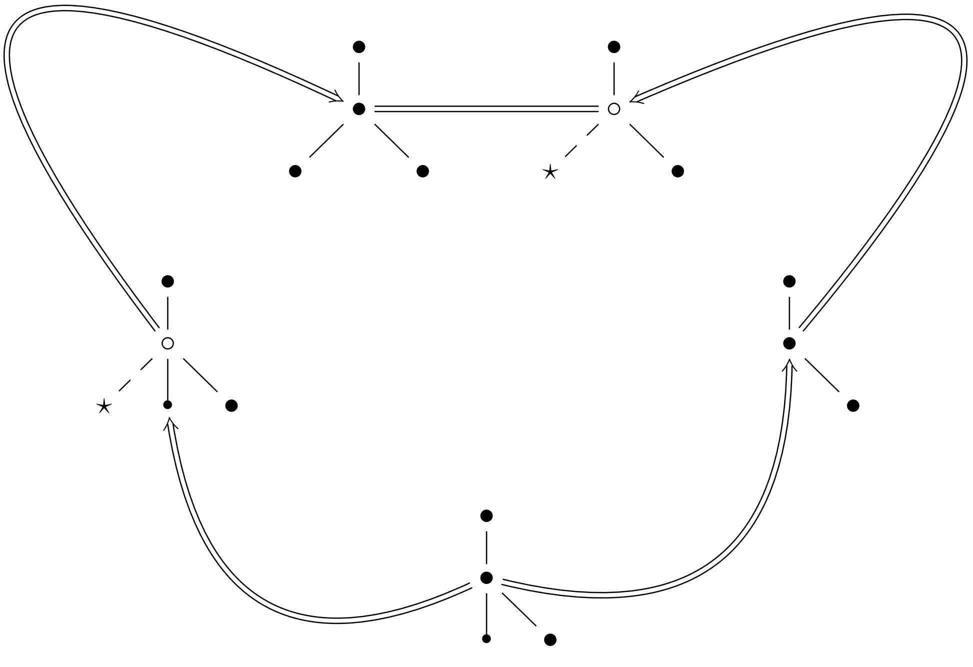
As we move around the left side of the diagram, we perform the dual contraction that is marked by the dashed arrow, followed by the dual of the subdivision of the upper edge. As we move around the right side of the diagram, we perform the dual of the subdivision of the upper edge, followed by the dual of the contraction of the leftmost leg. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The following diagram illustrates this relation when the vertex, v, marked by the open vertex, is an outer vertex.



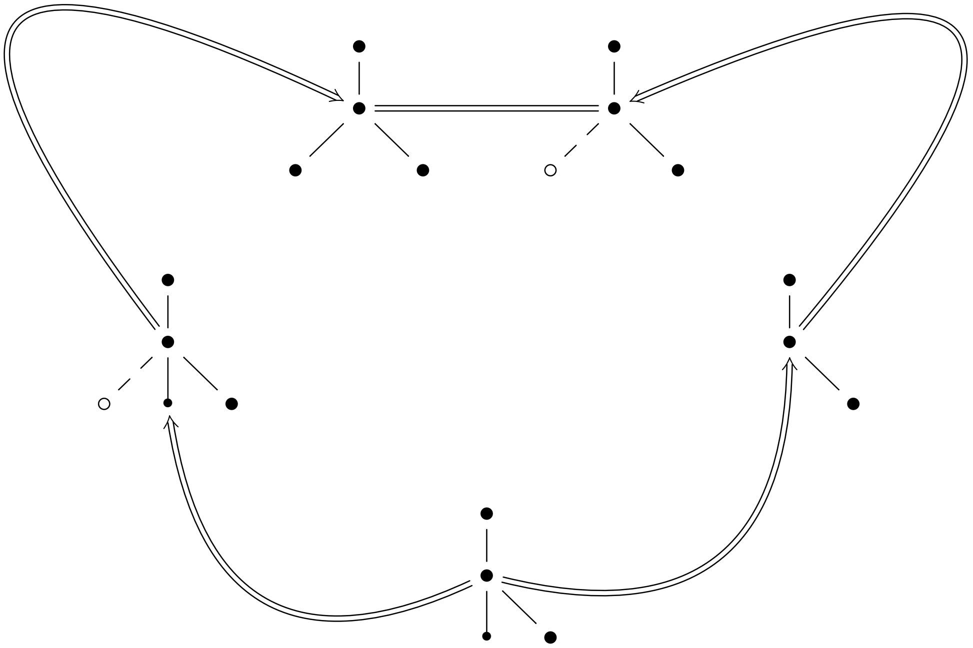
As we move around the left side of the diagram, we perform the dual contraction that is marked by the dashed arrow, followed by the dual of the subdivision of the upper edge. As we move around the right side of the diagram, we perform the dual of the subdivision of the upper edge, followed by the dual of the contraction of the leftmost leg. Note again that the previous two diagrams are remarkably similar. The difference here is that the vertex v is an outer vertex whereas in the previous digram the vertex was an inner vertex. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The following two diagrams will illustrate relations of type d from definition 2.1.4. Recall that this type of relation tells us that if we perform a leg growth on one edge and perform a contraction of the form (S, v) such that the associated edge and the newly grown edge are distinct, then it does not matter which order we perform the two operations. The following diagram illustrates this relation when the vertex, v, marked by the open vertex, is an inner vertex.



As we move around the left side of the diagram, we perform the dual contraction that is marked by the dashed arrow, followed by the dual of the leg growth of the upper edge. As we move around the right side of the diagram, we perform the dual of the leg growth of the upper edge, followed by the dual of the contraction of the leftmost leg.

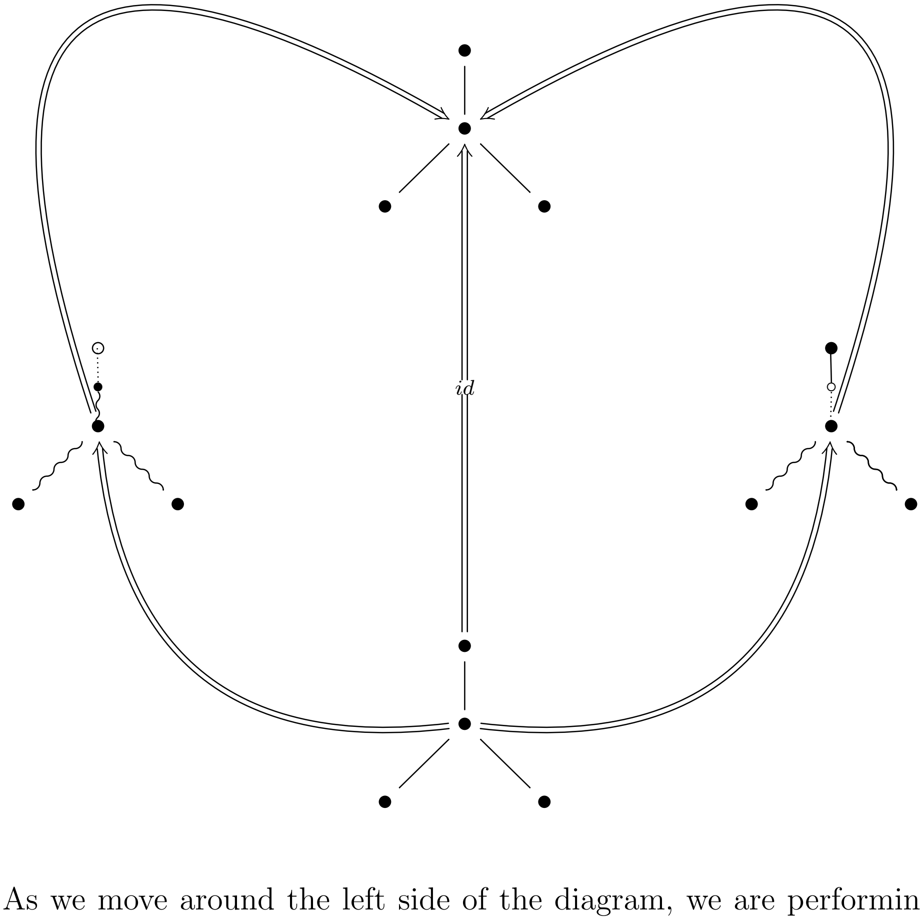
The following diagram illustrates this relation when the vertex, v, marked by the open vertex, is an outer vertex.



As we move around the left side of the diagram, we perform the dual contraction that is marked by the dashed arrow, followed by the dual of the leg grwoth of the upper edge. As we move around the right side of the diagram, we perform the dual of the leg growth of the upper edge, followed by the dual of the contraction of the leftmost leg. Note again that the previous two diagrams are remarkably similar. The difference here is that the vertex v is an outer vertex whereas in the previous digram the vertex was an inner vertex. This diagram is expressing the fact that the result of moving around the left side of the diagram is equal to the result of moving around the right side of the diagram.

The next two relations are relations that express when either a the dual of a leg growth or a the dual of a subdivision have a section.

The following diagram is an illustration the type e of relation in definition 2.1.4. Recall that when we subdivide an edge, we end up with two new edges. If we then perform a contraction of one of the two edges then the result is the identity map. Note that in these two diagrams, we are dotting the edges associated to pairs, (S, v) rather than denoting them by dashes, purely for typographical reasons.



As we move around the left side of the diagram, we are performing the dual to the contraction associated to the pair, (v, S) where the v is the rooted and the tree, S is the tree marked by the squiggles followed by the dual to the followed by the dual of the subdivision of the upper edge. As we move around the right side of the diagram, we are performing the dual to the contraction associated to the pair, (v', S') where the v' is the root and the tree, S' is the subtree of the right hand tree marked by the squiggles followed by the dual to the followed by the dual of the subdivision of the upper edge. Type f relations are similar to type e relations excett instead of undoing a subdivision, we are undoing a leg growth.

2.2 Prismatic categories and Prisms

In this section, we will defined what a prismatic category and prismatic functor. A prismatic category is a category that has a notion of a cone functor and a tensor product that have some relations We will then define that category of prisms to be the initial category in the category of prismatic categories (which must be shown to exist). Prismatic sets then will be the category of presheaves of prisms. We actually have two different notions of prisms, depending on whether we make our tensor product commutative or not.

Definition 2.2.1. *A prismatic category is a category $\mathcal{C}$ such that*

1. *Their is a initial object,* 0
2. *Their is an endofunctior,* $\boldsymbol{T} : \mathcal{C} \to \mathcal{C}$
3. $X \otimes 0 = 0 \otimes X = 0$

4. *$\mathcal{C}$ has a (non)-commutative associative bifunctor, $\otimes$ such that with unit $\boldsymbol{T}0$.*

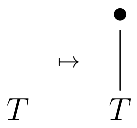
5. *Their is a natural transformation, $\mu : \boldsymbol{T}^2 \to \boldsymbol{T}$ and $\epsilon : Id_{\mathcal{C}} \to \boldsymbol{T}$ making $\boldsymbol{T}$ a monad*

A prismatic functor is a functor between two prismatic categories such that the structure is all preserved. The category of augmented prisms is the initial object in the category of prismatic categories. If we ask that the tensor is commutative, then we get what we call commutative augmented prisms. We will denote the category of prisms by $\mathbb{P}_+$. The prism category is the largest full subcategory of the augmented prism category not containing the initial object.

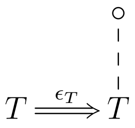
This definition contains the following implicit proposition.

Proposition 2.2.2. *The Augmented prism category exists, and is isomorphic to $\mathbb{T}^{op}$.*

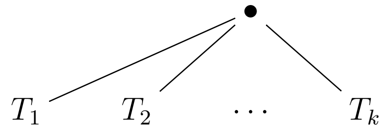
Proof. We begin this proof by putting a prismatic structure on T^{op}. The endofunctor, $\mathbf{C}$ is defined as the following diagram.



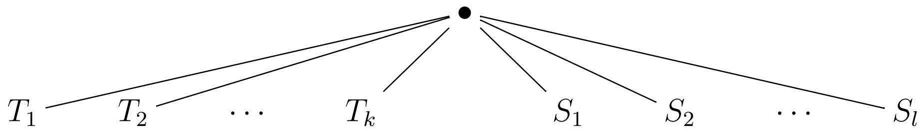
Furthermore, the following diagram also defines the natural transformation $\epsilon : Id \to \mathbf{C}$, as follows.



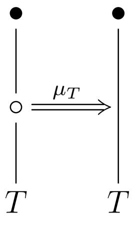
We now define the tensor functor, $\otimes$. Recall that ant non-empty tree may be written uniquely as follows.



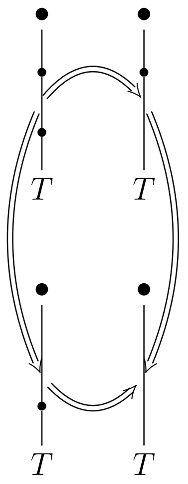
If S, T are two nonempty trees, then we define $T \otimes S$ to be the following tree.

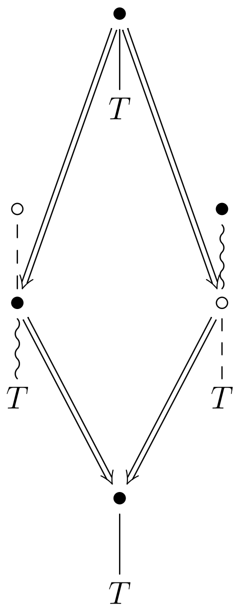


If on the other hand, one of the two trees are empty, we define the tensor product to also be empty. Note that this is associative, and the unit is the tree with no edges (which is the whisker on the empty tree). The last thing that we need to define is the natural transformation, $\mu : \mathbf{C}^2 \to \mathbf{C}$. Given a tree, T, we define μ_T as ,



We must now show that the triple, $(\mathbf{C}, \epsilon, \mu)$ is a monad. To the end, we note that the relaltions imply that the following diagrams commute.

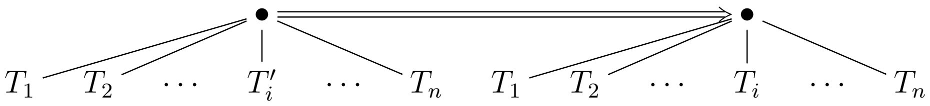




We now show that the category this category , $\mathbb{T}^{op}$ is the initial prismatic category. To this end, we let $(\mathcal{C}, \mathbf{T}, \epsilon, \mu, 0)$ be a prismatic category. Then we will define a prismatic functor $F : \mathbb{T}^{op} \to \mathcal{C}$. Note that F must preserve the initial object or in symbols, $F0 = 0$. This pins down the target of every other object. We can see this by noting that any non-initial object T may be uniquely represented as an object of the form, $\mathbf{C}T_1 \otimes \mathbf{C}T_2 \otimes \cdots \mathbf{C}T_n$. We suppose by induction on the number of edges that the functor has been defined on the T_i. Thus we must define

$$F(\mathbf{C}T_1 \otimes \mathbf{C}T_2 \otimes \cdots \mathbf{C}T_n) = \mathbf{T}FT_1 \otimes \mathbf{T}FT_2 \cdots \mathbf{T}FT_n$$

We must now say how the fact that F is a prismatic functor will force the definition on the morphisms. To this end, we will start with the co-face maps. Any co-face map arises in one of two ways. Given a map co-face of the form, $T_i' \to T_i$, we may obtain a co-face map, as shown in the following diagram.



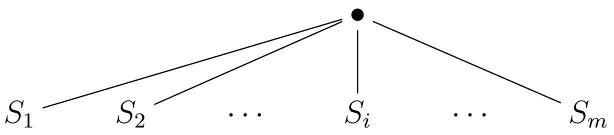
In symbols, this is a map of the form,

$$\mathbf{T}T_1 \otimes \cdots \mathbf{T}\phi \otimes \cdots \mathbf{T}T_n : \mathbf{T}T_1 \otimes \cdots \mathbf{T}T_i' \otimes \cdots \mathbf{T}T_n \to \mathbf{T}T_1 \otimes \cdots \mathbf{T}T_i \otimes \cdots \mathbf{T}T_n,$$

where $\phi : T_i' \to T_i$ is a co-face map. In this case

$$F(\mathbf{T}T_1 \otimes \cdots \mathbf{T}\phi \otimes \cdots \mathbf{T}T_n) = \mathbf{C}FT_1 \otimes \cdots \mathbf{C}F\phi \otimes \cdots \mathbf{C}FT_n.$$

Now let T_i be of the following form.



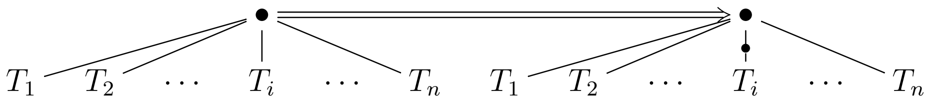
Then the other way that a co-face map map may arise is as a map of the form,

$$\mathbf{C}T_1 \otimes \cdots \epsilon_{T_i} \otimes \cdots \mathbf{C}T_n : \mathbf{C}T_1 \otimes \cdots T_i \otimes \cdots T_n \to \mathbf{C}T_1 \otimes \cdots \mathbf{C}T_i \otimes \cdots T_n.$$

In this case

$$F(\mathbf{C}T_1 \otimes \cdots \epsilon_{T_i} \otimes \cdots \mathbf{C}T_n) = \mathbf{T}FT_1 \otimes \cdots \epsilon_{FT_i} \otimes \cdots F\mathbf{T}T_n.$$

We will now move onto the co-degeneracies. Co-degeneracies may arise in one of two ways.The first way is the same as the co-face of type 1, except that the map ϕ is a co-degeneracy. The condition that F is prismatic forces the definition of F on these types of co-degeneracies the same way it does so for the corresponding co-faces. The second way that A co-degeneracy may occur is shown in the following diagram.



In symbols this map is of the form,

$$\mathbf{C}T_1 \otimes \cdots \epsilon_{C_i} \otimes \cdots \mathbf{T}C_n : \mathbf{C}T_1 \otimes \cdots \mathbf{C}^2 T_i \otimes \cdots \mathbf{C}T_n \to \mathbf{C}T_1 \otimes \cdots \mathbf{C}T_i \otimes \cdots \mathbf{C}T_n.$$

In this case we must define F to be

$$F(\mathbf{C}T_1 \otimes \cdots \epsilon_{T_i} \otimes \cdots \mathbf{C}T_n) = \mathbf{T}T_1 \otimes \cdots \epsilon_{\mathbf{T}T_i} \otimes \cdots \mathbf{T}T_n.$$

Note that by setting $T_i = 0$, this also covers leg growths out of the root. The other leg growths are taken subsumed by the first type of co-degeneracy. From this, the theorem has been shown. $\square$

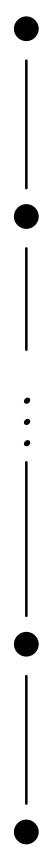
We will note that Joyal has defined cellular sets as as the category of pre-sheaves on a category we will call Θ. This category is remarkably similar (but not the same) to our category $\mathbb{P}$ in that it is dual to a category of trees. The paper, [3] is a good introduction. Furthermore [6] is remarkably similar to our main theorem.

Now that we have the definition of $\mathbb{P}$ in place, we should prove the following proposition.

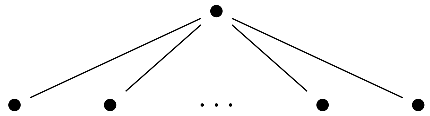
Proposition 2.2.3. *Both the simplex category, Δ and the box category $\Box$ embed into $\mathbb{P}$ as subcategories.*

Sketch of Proof

The objects of Δ correspond to the rooted trees that correspond to total orders. More precisely, the finite ordinal N maps to the linear rooted tree with N edges. Pictorially, these are trees of the following form.



The n-box, $\Box^n$ maps to the rooted tree such that all of the non-root vertices are outer vertices. Pictorially, these are trees of the following form.



One must then show the the co-face and co-degeneracy maps of Δ *and* $\Box$ *correspond the the co-face and the co-degeneracy maps of* $\mathbb{P}$.

■

Lemma 2.2.4. *All degeneracy maps are epimorphisms.*

Proof. This is since all co-degeneracies have sections.. □

Definition 2.2.5. *Any composite of co-face maps in* $\mathbb{P}_{aug}$ *will be referred to as a cell. Any composite of co-degeneracy maps in* $\mathbb{P}_{aug}$ *will be referred to as a co-cell.*

Proposition 2.2.6. *Any map in* $\mathbb{P}_{aug}$ *may uniquely written* $\mathcal{S}\mathcal{D}$*, where* $\mathcal{S}$ *is a c-cell and* $\mathcal{D}$ *is a cell. We will call such a factorization of a map a cell-cocell factorization.*

Proof. Let us first note that an empty string of faces or degeneracies corresponds to an identity map. Likewise any co-face or co-degeneracy is such a composition. Now let us suppose that any legnth n string may be written as a sequence of co-degeneracies followed by a sequence of co-face maps. Let $f_1 f_2 \cdots = f_n f_{n+1}$ be an length $n+1$ string of maps. Then $f_1 f_2 \cdots = f_n = s_1 s_2 \cdots s_i d_1 d_2 \cdots d_j$ where the s_i are co-degeneracies and the the d_i are co-faces. If f_{n+1} is a co-face we are done. If f_{n+1} is a co-degeneracy then we have two cases. In the first case, $d_j f_{n+1}$ is the identity in which case we are done. The other case is if $d_j f_{n+1}$ is not the identity. In this case, the exchange relations give us that $d_j f_{n+1} = s' d'_j$. In this case, $f_1 f_2 \cdots = f_n f_{n+1} = f_1 f_2 \cdots =$

$f_{n-1}s'd'_j$. By induction, $f_1 f_2 \cdots = f_{n-1}s'$ may be written as a composition of co-degeneracies followed by a composition of co-faces. We will now show that any two cocell-cell factorizations of the same map are equal. Suppose that their are two ways of writing some map as follows,

$$\begin{tikzcd}
& Y \arrow[dr, "\mathcal{D}"] & \\
X_1 \arrow[ur, "\mathcal{S}"] \arrow[dr, "\mathcal{S}'"'] & & Z \\
& Y' \arrow[ur, "\mathcal{D}'"'] &
\end{tikzcd}$$

We know that $\mathcal{S}$ and $\mathcal{S}'$ both have sections. Hence both are epimophisms, we have a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, bend right, "\iota"'] \arrow[dd, bend right, "\mathcal{S}'\iota"'] \\
X_1 \arrow[ur, "\mathcal{S}"] \arrow[dr, "\mathcal{S}'"'] & \\
& Y' \arrow[ul, bend left, "\iota'"] \arrow[uu, bend right, "\mathcal{S}\iota'"']
\end{tikzcd}$$

It is a general fact that when we have a commutative triangle such that the two arrows emanating out the initial point of the diagram are epimorphisms, so is the third. Thus the two maps, $\mathcal{S}'\iota$ and $\mathcal{S}\iota'$ are both split epimorphisms. We must show that $\mathcal{S}'\iota\mathcal{S}\iota' = id$.

To show this consider any commutative diagram of the form

$$\begin{tikzcd}
& Y \arrow[dd, bend right, "\pi"'] \\
X_1 \arrow[ur, two heads, "p"] \arrow[dr, two heads, "p'"'] & \\
& Y' \arrow[uu, bend right, "\pi'"']
\end{tikzcd}$$

Note that by the previous argument that every morphism both $\pi = \mathcal{S}'\iota$ and $\pi' = \mathcal{S}\iota'$ are epimorphisms. It is easy to verify that $\pi'\pi p = \pi' p' = p = id_Y p$. But since p is an epimorphism, $\pi'\pi = id_Y$. This implies that π, π' are isomorphisms. Since $\mathbb{P}_{aug}$ is skeletal, this implies that the decomposition is unique. ☐

Lemma 2.2.7. *The object $T0 = *$ is the terminal object of $\mathbb{P}_{aug}$.*

Proof. We will show that for each object X, their is exactly one morphism $X \to *$. Of $X = 0$ or $X = *$ we are done. We will now show this fact to be true for all objects by induction on the dimension of the objects. We know from proposition 2.2.6 that any maps of the form $\phi : X \to *$ is a composite of downward arrows. Suppose we have two different maps, $f_1^- f_2^- \cdots f_n^-$ and $f_1^{'-} f_2^{'-} \cdots f_n^{'-}$ Then we have solid morphisms of the diagram.

$$\begin{tikzcd}
 & Y_1 \arrow[dr, dashed, "f_1^- f_2^- \cdots f_{n-1}^-"] \arrow[drr, bend left] & & \\
X \arrow[ur, "f_n^-"] \arrow[dr, "f_n^{'-}"'] & & Z \arrow[r] & * \\
 & Y_2 \arrow[ur, dashed, "f_1^{'-} f_2^{'-} \cdots f_{n-1}^{'-}"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

If we can show that their are the dotted arrows as shown, then we are done since their is only one morphism from Y_2 to $*$. We may show this from the relations. ☐

Proposition 2.2.8. *Any two isomorphic objects in $\mathbb{P}$ are equal.*

Proof. Suppose $\mathcal{C}$ is a prismatic category and their is a non-identity isomorphism $\phi : X \to X'$. Then we will build a pair of functors, $i : \mathbb{C} - X' \to \mathbb{C}$, and $\mathbb{C} \to \mathbb{C} - X'$, where $\mathbb{C} - X'$ is the full subcategory of $\mathbb{C}$ generated by all objects not equal to X'. The map i is simply the inclusion. We must now define r. We will define r to be the identity on the subcategory $\mathbb{C} - X'$. We then define $r(\phi) = 1_X$. If we have a map of the form, $g : X' \to Y$, we define $r(g) = g\phi$, and if we have a map of the form, $f : Y' \to X'$, we define $r(f) = \phi^{-1} f$. Note that if we have a map in $\mathbb{C} - X'$ that factors through X' r of that map will be the identity making r a functor. The functor i is essentially surjective full and faithful, as is r. We will now define the prismatic structure on $\mathbb{C} - X'$.We define

1. $T' = rTi$
2. $\otimes' = r \otimes (i \times i)$
3. $\epsilon' = r\epsilon i$
4. $\mu' = r\mu i$

It is an easy verification that $(\mathcal{C} - X', T', \otimes', 0, \epsilon', \mu')$ is prismatic. Now suppose that $\mathbb{P}_{aug}$, has two distinct objects that are isomorphic. Then the above argument shows that we have a sub-category of $\mathbb{P}_{aug}$ that is also a prismatic subcategory. But this contradicts the fact that $\mathbb{P}_{aug}$ is the initial object in the category of prismatic categories. □

Lemma 2.2.9. *1. The functor* $T\mathbb{P}_{aug} \to \mathbb{P}_{aug}$ *is faithful.*

2. *The following functors are faithful*
   - *(a)* $\Phi \mapsto X \otimes \Phi$
   - *(b)* $\Phi \mapsto \Phi \otimes X$

Proof. This follows from the tree pictures. □

Lemma 2.2.10. *Let $f = \mathcal{D}\mathcal{S}$ such that $\mathcal{S}$ is a cocell and $\mathcal{D}$ is a cell. If f is injective, then $\mathcal{S}$ is the identity map.*

Proof. To show this it suffices to find two different maps f_1, f_2 such that $\mathcal{D}\mathcal{S}f_1 = \mathcal{D}\mathcal{S}f_2$. But this is easy since every co-degeneracy has two sections. □

Recall from definition 2.2.5 that a cell is a map in $\mathbf{P}_{aug}$ such that is is writable as a composition of co-face maps. At this point, we will enumerate the cells of a prism of the form,

$$T_1 \otimes T_2 \otimes \cdots T_k,$$

and

$$\mathbf{C}(T_1 \otimes T_2 \otimes \cdots T_k).$$

Lemma 2.2.11. *Let $(T_1, T_2 \cdots T_n)$ be an n-tuple of prisms. Then the set of cells, $cell(T_1 \otimes T_2 \otimes \cdots T_n)$ is isomorphic to $cells(T_1) \times cells(T_2) \times \cdots cells(T_n)$.*

Proof. Note that it suffices to show the lemma for two factors. It is clear that we have a surjective map, $cell(T_1) \times cell(T_2) \to cell(T_1 \otimes T_2)$. Since $\otimes$ is a functor, we get a natural transformation $\iota : cell(-) \times cell(-) \to cell(- \otimes -)$. Suppose that we have two pairs, (ϕ_1, ϕ_2), and (θ_1, θ_2) such that $\phi_1 \otimes \theta_1 = \phi_2 \otimes \theta_2$ such that $\phi_i : X_i \to X$, and $\theta_i : Y_i \to Y$. Then we have a diagram of the form,

$$\begin{tikzcd}
X_1 \arrow[r, "\phi_1"] & X & X_2 \arrow[l, "\phi_2"'] \\
X_1 \otimes Y_1 \arrow[u, "\pi_{X_1}"] \arrow[ur, "f_1"] \arrow[r, "\phi_1\otimes\theta_1"] \arrow[d, "\pi_{Y_1}"'] \arrow[dr, "f_3"] & X \otimes Y \arrow[u, "\pi_X"] \arrow[d, "\pi_Y"] & X_2 \otimes Y_2 \arrow[l] \arrow[ul, "f_2"'] \arrow[u, "\pi_{X_2}"'] \arrow[d, "\pi_{Y_2}"] \arrow[dl, "f_4"'] \\
Y_1 \arrow[r, "\theta_1"'] & Y & Y_2 \arrow[l, "\theta_2"]
\end{tikzcd}$$

By assumption, $f_1 = f_2$, and $f_3 = f_4$. This means that $\phi_1\pi_{X_1} = \phi_2\pi_{X_2}$. Both of these are factorizations of the type proven to uniquely exist in lemma 2.2.6. Therefore, $\phi_1 = \phi_2$. Likewise $\theta_1 = \theta_2$. □

Lemma 2.2.12. *Let $\phi : X \to Y$ be a map in $\mathbb{P}_{aug}$. The following are equivalent:*

1. *ϕ is a monmorphism*

2. *If $\phi\iota_1 = \phi\iota_2 \implies \iota_1 = \iota_2$ for all pair $(\iota_1, \iota_2) \in cell(X) \times cell(X)$.*

Proof. Item 1 implies item 2 is immediate. Let us first note that by lemma 2.2.10 any monomrphism must be a cell. Now suppose item 2 and that we have a diagram of the form,

$$\begin{tikzcd}
W \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & X \arrow[r, "\phi"] & Y
\end{tikzcd}$$

such that the two composites, are equal. By lemma 2.2.6 we have the following diagram.

$$\begin{tikzcd}
& W_1 \arrow[rd] \arrow[rrd] & & \\
W \arrow[ru] \arrow[rd] & & X \arrow[r, "\phi"] & Y \\
& W_2 \arrow[ru] \arrow[rru] & &
\end{tikzcd}$$

Since the two composites are equal and are co-cell-cell factorizations from lemma 2.2.6, they are equal, and therefore $W_1 = W_2$. Therefore we have the following diagram.

$$\begin{tikzcd}
W_1 = W_2 \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_2"'] & X \arrow[r, "\phi"] & Y
\end{tikzcd}$$

Note that the two composites are equal. Therefore $f_1 = f_2$ is equal if and only if $g_1 = g_2$ are equal. ☐

Lemma 2.2.13. *If $\phi : X \to Y$ is a monomorphism in $\mathbb{P}_{aug}$, then so is $Z \otimes \phi$, and $\phi \otimes Z$.*

Proof. This is a direct consequence of lemma 2.2.11 and lemma 2.2.12. ☐

Lemma 2.2.14. *Any map in $\mathbb{P}_{aug}$ of the form, $\epsilon_T : T \to \boldsymbol{C}T$ is a monmorphism.*

Proof. Suppose that we have two morphisms

$$\begin{tikzcd}
Z \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_1"'] & T \arrow[r, "\epsilon_T"] & \mathbf{C}T
\end{tikzcd}$$

such that the two compositions are equal. Then the following two compositions are equal.

$$\begin{tikzcd}
\mathbf{C}Z \arrow[r, bend left, "\mathbf{C}f_2"] \arrow[r, bend right, "\mathbf{C}f_1"'] & \mathbf{C}T \arrow[r, "\epsilon_T"] & \mathbf{C}^2T .
\end{tikzcd}$$

But then the map, $\mathbf{C}T \to \mathbf{C}^2T$ is injective since it has a section. This implies that $\mathbf{C}f_1 = \mathbf{C}f_2$. Since $\mathbf{C}$ is faithfull, this implies that $f_1 = f_2$. $\square$

We will now describe the functor $T \mapsto hom_{\mathbb{P}_{aug}}(*, T)$. Since $*$ is the unique zero dimensional object, by Lemma 2.2.11, we have an isomorphism

$$hom_{\mathbb{P}_{aug}}(*, T_1 \otimes T_2) \sim hom_{\mathbb{P}_{aug}}(*, T_1) \times hom_{\mathbb{P}_{aug}}(*, T_2)$$

. We also know that $hom_{\mathbb{P}_{aug}}(*, 0)$ is the empty set. Now we must describe $hom_{\mathbb{P}_{aug}}(*, \mathbf{C}T)$ in terms of $hom_{\mathbb{P}_{aug}}(*, T)$. To this end, we always have a map,

$$hom_{\mathbb{P}_{aug}}(*, T) \to hom_{\mathbb{P}_{aug}}(*, \mathbf{C}T)$$

given by composition. Since ϵ_T is a monomorphism, so is the $hom_{\mathbb{P}_{aug}}(*, \epsilon_T)$. Therefore we only need to describe those maps, $* \to \mathbf{C}T$ that do not factor through T. To describe such maps, we prove the following lemma:

Lemma 2.2.15. *Their is a unique map $* \to \mathbf{C}T$ that does not factor through T in $\mathbb{P}$.*

Proof. If we work in the slightly larger category of $\mathbb{P}_{aug}$, we certainly have the map, $\mathbf{C}(0 \to T)$. We will first show that this map does not factor through T. If this map did factor through T, we would have a diagram of the form,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] \arrow[ur] & \mathbf{C}T \arrow[r] & \mathbf{C}*
\end{tikzcd}$$

We must now show that this is the only such map. Suppose that we have two maps,

$$\begin{tikzcd}
* \arrow[r, bend left] \arrow[r, bend right] & \mathbf{C}T
\end{tikzcd}$$

such that neither map factors through T. Then we may write each as a composition of the form,

$$\begin{tikzcd}
 & \mathbf{C}W_{n-1} \arrow[dr, "\mathbf{C}\phi_n"] & \\
* \arrow[ur] \arrow[dr] & & \mathbf{C}T \\
 & \mathbf{C}W_2 \arrow[ur, "\mathbf{C}\phi'_n"'] &
\end{tikzcd} .$$

The maps emanating out of W_n, and W'_n must be of the form ,$\mathbf{C}\phi, \mathbf{C}\phi'$ since otherwise the maps emanating out of $*$ into $\mathbf{C}$ would factor through T. By iteratively applying the same reasoning we see that we must get a diagram of the form

$$\begin{tikzcd}
 & \mathbf{C}W_1 \arrow[r, "\mathbf{C}\phi_2"] & \mathbf{C}W_2 \arrow[r, "\mathbf{C}\phi_3"] & \cdots & \mathbf{C}W_{n-1} \arrow[dr, "\mathbf{C}\phi_n"] & \\
* \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] & & & & & \mathbf{C}T \\
 & \mathbf{C}W'_1 \arrow[r, "\mathbf{C}\phi'_2"'] & \mathbf{C}W'_2 \arrow[r, "\mathbf{C}\phi'_3"'] & \cdots & \mathbf{C}W'_{n-1} \arrow[ur, "\mathbf{C}\phi'_n"'] &
\end{tikzcd} .$$

Where the two maps are factored maximally factored. Since$*$ is zero dimensional and $\mathbf{C}W_1, \mathbf{C}W'_1$ are one dimensional, then they are equal to $\mathbf{C}*$. But then both maps ϕ, ϕ' must be of the form $\mathbf{C}(0 \to *)$ since their are only two face maps from $*$ to $\mathbf{C}*$ and the other face map is not of the form $\mathbf{C}(0 \to T)$. We may therefore see that both maps are of the form $\mathbf{C}(0 \to T)$, which mean that the two maps are equal since $\mathbf{C}$ is faithful by lemma 2.2.9. $\square$

Lemma 2.2.16. *Suppose that $\phi : X \to \boldsymbol{C}Y$ is a cell map in $\mathbb{P}$. Then exactly one of the following hold*

1. $\phi = \epsilon_Y \theta$
2. $\phi = \boldsymbol{C}\phi'$ *for some face map* ϕ'

Proof. We will begin by showing that at least one of the two items holds. We will prove this by induction on the codimension of the map. If the dodimension of the map is one, then we are done. Now suppose that the that at least one of the two items holds when the codimension is n. Now let $Y \to \mathbf{C}X$ be a codimension $n+1$ cell. Then we may decompose this cell into a composition of cells, $Y \to Z \to \mathbf{C}X$, where the map $Z \to \mathbf{C}X$ is a codimension one cell. If the map $Z \to \mathbf{C}X$ is equal to ϵ_X, then we have item 1. If the map $Z \to \mathbf{C}X$ is of the form, $\mathbf{C}\phi'$, then by induction we have two cases. Case one is the case where the map $Y \to Z$ is of the form $\mathbf{C}\phi'$. In this case item 1 holds. The other case is when $Y \to Z$ is of the form given in item 2. In this case we have a diagram of the form,

$$\begin{tikzcd}
 & & \mathbf{C}T' \arrow[dr, "\mathbf{C}\phi"] & \\
T'' \arrow[r, "\phi'"'] & T' \arrow[ur, "\epsilon_{T'}"] \arrow[dr, "\phi"'] & & {} \\
 & & T \arrow[ur, "\epsilon_T"'] &
\end{tikzcd}$$

which gives shows that item 2 holds. Suppose that we have a cell map that satisfies both item 1 and item 2. Then we have a diagram of the form

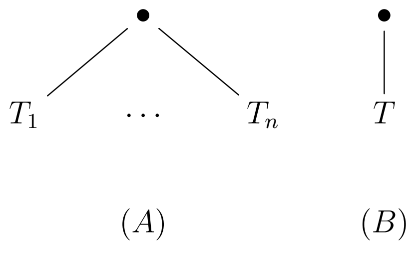
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & T \arrow[d] \\
* \arrow[r] & \mathbf{C}T' \arrow[r] \arrow[ur] & \mathbf{C}T
\end{tikzcd}$$

which contradicts lemma 2.2.15 □

Before we prove the next proposition let us recall the notion of a cell in the category $\mathbb{P}_{aug}$. From definition 2.2.5, a cell is a map that may be writable as a composition of co-face maps.

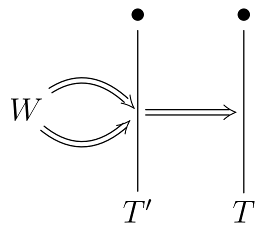
Proposition 2.2.17. *All face maps in $\mathbb{P}_{aug}$ are monomorphisms. Therefore all cells are monomorphisms and any map may be uniquly decomposed as an epimorphism followed by a monomorphism.*

Proof. We will show this by induction on the target of the face maps. Let us consider the zero dimensional prism. There is a unique sub-cell, given by $0 \to \mathbf{C}0$, and this is a monomorphism. Now let us assume that any face map $X \to Y$ such that Y is n-dimensional is a monomorphism. We will show that any face map of the form $X \to Y$ is a monomorphism if Y has dimension $n+1$. To this end Y has one of two forms shown in the following diagram.

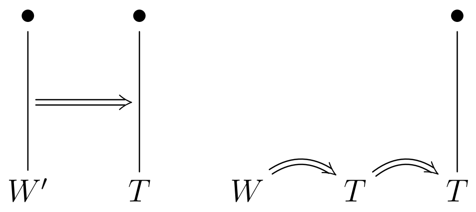


If our map is of form A, then this is lemma 2.2.13. Now suppose that Y is of form B. If $X \to Y$ is a face map, then it is of one of the two forms in the following diagram.

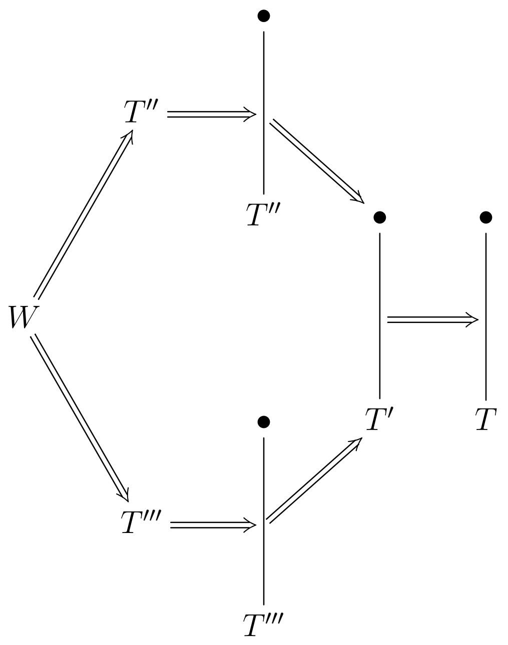
If the face map is of the first form, the face map is a monomorphism by lemma 2.2.14. If the map is of the second form, then we have some work to do. Suppose that we have a diagram of the form



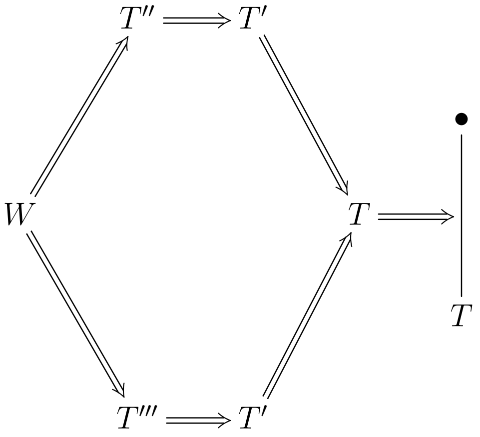
such that the two composites are equal. By lemma 2.2.12 we may assume that the two maps are cells. By Lemma 2.2.16 is of one of the two forms



If the left hand part of this diagram is the case we are done by since the functor $\mathbf{C}$ is faithful and the two parallel arrows would have to be in the image of $\mathbf{C}$. Now suppose that the right hand part of the above diagram holds. Then by lemma 2.2.16 both maps from W to $\mathbf{C}T'$ factorize as follows



By the relations the following diagram is equivalent.



But since ϵ_T is a monomorphism, the two morphisms are equal. □

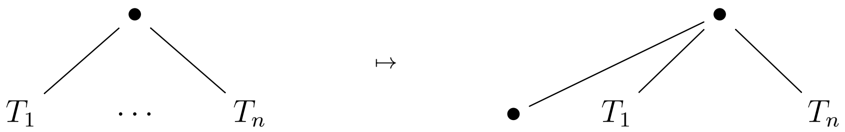
Chapter 3

The Proof of the Main Theorem

We will now prove the main theorem this thesis. Let us now recall the main theorem.

Theorem. *The categories,* $\widehat{\mathbb{P}}$*, and* $\widehat{\mathbb{P}}_{sym}$ *are have Quillen model structures such that the cofibrations are monomorphisms and the weak equivalences are maps,* $X \to Y$ *such that* $NX \to NY$ *is a weak equivalence, where* N *denotes the simplicial nerve. Furthermore, the simplicial nerve functor is a left Quillen equivalence.*

Proof. The category $\mathbb{P}$ is contractible since $\mathbb{P}$ has a terminal object. By corollary 1.5.3 it now suffices to show that $\mathbb{P}$ has a cylinder functor making $\mathbb{P}$ into a test category. We claim that the functor, $f \mapsto T^2 0 \otimes f$ provides us with such a functor. This functor may be pictured with trees as follows.



Recalling the definition of a item 1 of the definition of a cylinder functor (see definition 1.5.1), we must have natural transformations,

$$d^i : Id_{\mathbb{P}} \to T^2 0 \otimes -, i = 01,$$

, and a natural transformation,

$$p : T^2 0 \otimes - \to Id_{\mathbb{P}}$$

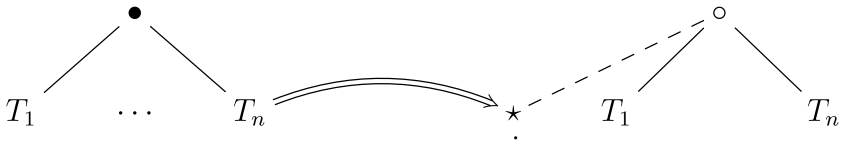
satisfying the relations,

$$pd^i = Id, i = 0, 1.$$

The natural transformation, d^0 is given by,

$$\epsilon_{\mathbf{C}0} \otimes Id : T \sim \mathbf{C}0 \otimes T \to \mathbf{C}^2 0 \otimes T.$$

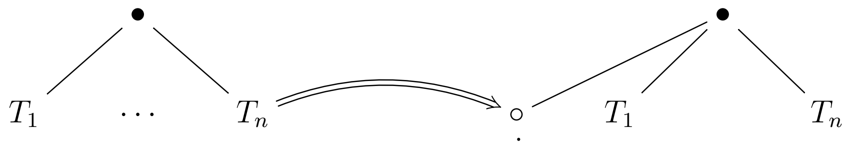
We may pictorially represent this natural transformation as follows.



The natural transformation, d^1 is given by

$$\mathbf{C}(0 \to *) \otimes Id : T \sim \mathbf{C}0 \otimes T \to \mathbf{C}^2 0 \otimes T.$$

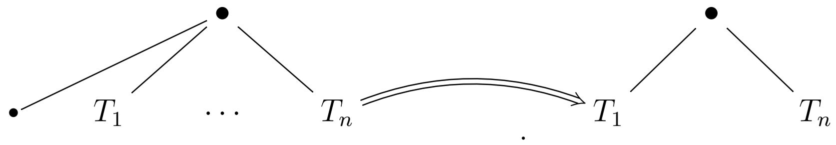
We may pictorially represent this natural transformation as follows.



The natural transformation p is given by,

$$\mu_0 \otimes Id : \mathbf{C}^2 0 \otimes T \to \mathbf{C}0 \otimes T = T.$$

We may represent this pictorially as follows.



Note that the relations required to make this functor a cylinder functor hold. Lastly, it suffices to show that the two natural transformations are disjoint. That is item 2 of definition 1.5.1 must also hold. Suppose that these two functors are not disjoint. Then we have a diagram of the form,

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & T \arrow[d, "d^1"] \arrow[dr] & \\
T \arrow[r, "d^0"'] \arrow[dr] & \mathbf{C}^2 0 \otimes T \arrow[dr] & * \arrow[d, "d^1"] \\
& * \arrow[r, "d^0"'] & \mathbf{C}^2 0
\end{tikzcd}$$

But we know that the only object that intersects in the two ends of the interval in $\mathbb{P}_{aug}$ is the object 0 which is not present in $\mathbb{P}$. The proof is similar for $\mathbb{P}_{sym}$. □

Appendix A

Regular cell complexes

In this sections we will briefly define a category of regular cell complexes which we will denote **Reg**. We will then define a functor to posets, which we will denote as N, and referred to it as the cell functor. We then show that this nerve functor reflects isomorphisms. This will allow us to define a functor from the image of the cell functor back to the category of regular cell complexes. We refer to the image category of the cell functor as **CWposet**.We will then define cone functors and product functors on each category. We will lastly make a conjecture that two functors defined above are equivalences of infinity categories [1]. We should note at this point we cannot hope to have a effective description of the category **CWposet**, since we have the word problem for finitely presented groups. We believe that the category **CWposet** should have some kind of combinatorial description (if we assume the word problem away by means of an oracle). To get such a

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[1] Although, we will not phrase the conjecture in these terms. For the reader interested in the subject, see these references. This will be the subject of a later paper.

definition would be the same as a sphere recognition problem.

Definition A.0.18. *A regular cell complex is a filtered topological space,*

$$\emptyset = X_{-1} \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots X_n \hookrightarrow \cdots$$

along with a pushout of the form ,

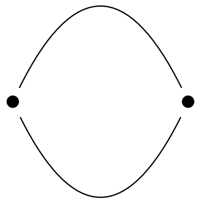
$$\begin{tikzcd}
\coprod_{i \in \mathcal{I}_n} S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in \mathcal{I}_n} D^n \arrow[r] & X_n
\end{tikzcd}$$

for each n. Such that the maps, $D^n \hookrightarrow \coprod_{i \in \mathcal{I}_n} D^n \to X_n$*must all be injective. These maps will be called will be called the n-cells of the regular cell complex. we will denote the regular cell complex by* X*. The number n will be called the dimension of the cell.*

Definition A.0.19. *Let* $X_0 \hookrightarrow X_1 \hookrightarrow \cdots X_n \hookrightarrow \cdots$ *be a regular cell complex,* X*. Then the functor* $X \mapsto colim X_i$ *to the category of compactly generated Haussdorf spaces, denotes* ***CGHauss*** *will be denoted* U *and will be referred to as the forgetful functor from the category of regular cell complexes to the category of compactly generated Haussdorf spaces. We will also refer to this the space associated to a regular cell complex.*

Note that this definition is essentially the the usual forgetful functor to the category of compactly generated haussdorf spaces. We will abuse the notion and denote the associated topological space and the regualar cell complex by the same symbol. If we were dealing with CW complexes, we would consider maps of the associated topological space that map the n-skeleta to n-skeleta.

However, this can be combinatorially complicated since a single cell in the domain may be mapped some kind of union of cells in the codomain. We will ask for more rigidity in the types of maps we would like to consider. Note that in the definition given, the set, $\mathcal{I}_n$ may be empty. This is an important special case, since a coproduct indexed by the empty set is the initial object of the category. Furthermore, if $\mathcal{I}_n$ is empty, so is $\mathcal{I}_{n+1}$. We will say that our regular cell complex is n-dimensional if $\mathcal{I}_{n+1}$ is first place time one of the indexing sets are empty. This tells us that X_n is also a regular cell complex by making $\mathcal{I}_{n+1}$ empty. Furthermore, we may add an number of disks to the same $n-1$ sphere. An a example of this would be the two sphere with the decomposition given as follows:



In this case, we have two zero cells, and two one cells, no cells above. The associated topological space is the circle. The zeroth space, X_0 is two disjoint points, which incidentally is the zero sphere, and X_n is the circle for all $n > 0$. The pushout that gives us X_1 form X_0, is as follows,

$$\begin{tikzcd}
S^0 \coprod S^0 \arrow[r, "Id \coprod Id"] \arrow[d] & X_0 \arrow[d] \\
D^1 \coprod D^1 \arrow[r, "u \coprod d"'] & X_1
\end{tikzcd}$$

All the other pushouts are obtained from pushouts indexed by the empty set.

Definition A.0.20. *Let X and Y be two filtered topological spaces. A regular map, $f : X \to Y$ will be a map of the associated topological spaces such that the image of of a cell in X is a cell in Y.*

The collection of regular cell complexes along with the regular maps, will be the category of regular cell complexes.

Definition A.0.21. *Let X be a regular cell complex. Then the nerve of X is a poset, such that the elements of the poset will be the cells of X. Given two cells, ι_1, ι_2, we define $\iota_1 \leq \iota_2$ if we have a commutative diagram of the form,*

$$\begin{tikzcd}
B^n \arrow[r] \arrow[d] & B^m \arrow[d] \\
X_n \arrow[r] & X_m
\end{tikzcd}$$

Given a regular cell complex X, we will denote the poset of cells of X by $\textbf{\textit{cells}}(X)$.

We want to now show that the assignment, $X \mapsto \mathbf{cells}(X)$ is a functor. Recall that in many categories, (the category of topological spaces being an example), that we have an epi-mono factorization. What this means is that if we have a map, $f : X \to Y$, then we have a factorization,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \\
& Im(f) \arrow[ur] &
\end{tikzcd}$$

Furthermore, any two such factorizations we differ by a an isomorphism between any two choices of $Im(f)$. If we have a map, $f : X \to Y$, and a cell $\iota : B^n \to X$, then we get a map $\iota f : B^n \to Y$. Since the category of topological spaces has epi-mono factorizations, we have a diagram of the form,

$$\begin{tikzcd}
B^n \arrow[rr, "\iota f"] \arrow[dr] & & Y . \\
& Im(\iota f) \arrow[ur, "\iota_f"'] &
\end{tikzcd}$$

Since we are in the category of regular cell complexes, this implies that ι_f is a cell of Y. Thus if we have a regular map $f : X \to Y$, then we have a set map,

$$\textbf{cells}(f) : \textbf{cells}(X) \to \textbf{cells}(Y)$$

given by

$$\textbf{cells}(f)(\iota) = \iota_f.$$

It is straightforward to show that this is in fact a map of posets. It follows from the following fact from category theory.

Lemma A.0.22. *Let $\mathcal{C}$ be a category with all pullbacks. Suppose further that every map , $f : X \to Y$ in $\mathcal{C}$ has an essentially unique epi-mono factorization. Then* ***sub*** $: \mathcal{C} \to$ ***posets****, the assignment that takes an object to the poset of sub-objects is a functor.*

Corollary A.0.23. *The functor* ***cells*** $:$ ***Reg*** $\to$ ***CWposets*** $\hookrightarrow$ ***Posets*** *is a sub-functor of* ***sub*** $\circ\, U$*, where* $U :$ ***Reg*** $\to$ ***CGHaus*** *is the functor which takes a regular cell complex to the associated topological space.*

Now note that **cells** is a functor that maps inclusions to inclusions. Now given the filtration,

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots X_n \hookrightarrow \cdots ,$$

we may form a filtration of posets

$$\mathbf{cells}(X_0) \hookrightarrow \mathbf{cells}(X_1) \hookrightarrow \cdots \mathbf{cells}(X_n) \hookrightarrow \cdots$$

of the poset, $\mathbf{cells}(X)$. We also have the augmented cells, where we add a single object of the poset, which will be less than every object. This will correspond to the empty cell. We will denote this poset as $\mathbf{cells}(X)_+$. We will denote this bottom element, 0. This device will allow us to define the dimension of an element of a CW poset. Recall that a poset may be regarded as a category. If $f : X \to Y$ is a map, then a factorization of f is a sequence of maps, $(f_1, f_2, \cdots f_n)$ such that $f = f_1 f_2 \cdots f_n$. Such a factorization is irreducible if whenever $f_i = g_1 g_2$, at least one of the g_i is an isomorphism. Note that the uniqueness of irreducible factorization is rare. If $f = f_1 f_2 \cdots f_n$ is irreducible, then n will be the length of the factorization. Note that in a poset, all isomorphisms are identities.

Lemma A.0.24. *Let X be a regular cell complex. Then for any $z \in \boldsymbol{cells}(X)_+$ the length of any irreducible factorization of $0 < z$ are all equal . One less than this length will be the dimension of the cell.*

Theorem A.0.25. *Let X and Y be two regular cell complexes and $f : \boldsymbol{cells}(X) \to \boldsymbol{cells}(Y)$ an isomorphism. Then X and Y are isomorphic regular cell complexes.*

We recall the proof of this theorem from [4].

Proof. First let us note that any CW poset has a natural filtration,

$$P_0 \hookrightarrow P_1 \hookrightarrow \cdots$$

, and that this filtration is isomorpphic to the filtration,

$$\mathbf{cells}(X_0) \hookrightarrow \mathbf{cells}(X_1) \hookrightarrow \cdots \mathbf{cells}(X_n) \hookrightarrow \cdots$$

. We will prove this by induction for any two finite dimensional cell complexes, then make a brief limit argument as to why the proposition must be true for infinite dimensional cell complexes. So the case where the cell complexes are zero dimensional, the proposition is trivial. Now suppose that the proposition is true for all cell complexes of dimension $k - 1$ We will show this proposition to be true for all cell complexes of dimension k. So Let $f : cells(X) \to cells(Y)$ be an isomorphism of posets, with X, Y, both k dimensional cell complexes. Then $f_{k-1} : cells(X_{k-1}) \to cells(Y_{k-1})$ is an isomorphism so by induction their is an isomorphism $g_{k-1} : X_{k-1} \to Y_{k-1}$ with $cells(g_{k-1}) = f_{k-1}$. We also know that the elements of the posets $cells(X)$, and $cells(Y)$ corresponding k-cells correspond to have grading k. For $c \in cells(X)$ of grading k we have the following commutative diagrams:

$$\begin{tikzcd}
\{x \in cells(X) : x < c\} \arrow[r, "\sim"] \arrow[d, "g"'] & \{f(x) \in cells(Y) : f(x) < f(c)\} \arrow[d] \\
\{x \in cells(X) : x \leq c\} \arrow[r, "\sim"] \arrow[d] & \{f(x) \in cells(Y) : f(x) \leq f(c)\} \arrow[d] \\
cells(X) \arrow[r, "\sim"] & cells(Y)
\end{tikzcd}$$

By induction, the isomorphism,

$$\{x \in cells(X) : x < c\} \to \{f(x) \in cells(Y) : f(x) < f(c)\}$$

(since c is of dimension k), are both posets $k - 1$ spheres with isomorphic cell decompositions. We no extend the two isomorphic cell decompositions of the two $k-1$-spheres to k-disks by adding a single k-cell. We do this for each c of grading k, thus showing that the two cell complexes, X and Y isomorphic. We have proven that any two finite dimensional cell complexes with isomorphic posets are isomorphic. We must now show this to be true for infinite dimensional cell complexes. But this is a simple limit argument. □

Definition A.0.26. 1. *The category P_n^k will then be the full subcategory of $\mathbb{N}^k$ consisting of objects, $(i_1, i_2, \cdots i_k)$ such that $i_1 + i_2 + \cdots i_k \leq n$.*

2. *Now given any pair of regular cell complexes, X and Y, we have a functor,*

$$F_{X,Y} : P_n^2 \to \boldsymbol{CGHauss}$$

given by

$$F^2_{X,Y,n}(i,j) = X_i \times Y_j.$$

3. *More generally we define*

$$F^n_{X^1,X^2,\cdots X^k}(i_1, i_2, \cdots i_k) = X^1_{i_1} \times X^2_{i_2} \times \cdots X^k_{i_k}$$

*from $P_n^k \to$ **CGHauss**.*

Definition A.0.27. *Let X and Y be two filtered topological spaces. Then $X \otimes Y$ is a filtered topological space where $(X \otimes Y)_n$ is $colim F_{X,Y,n}$.*

Lemma A.0.28. *Suppose that X, and Y are isomorphic filtered topological spaces and X is a regular cell complex. Then Y is an isomorphic regular cell complex in an essentially unique way.*

Proof. We will show that this is true by induction on the grading. If n then X_0 is isomorphic to Y_0, both of which are discrete sets of points.Now suppose that the lemma is true for all gradings less then n. Then we must show that Y_n may be obtained from Y_{n-1} in one way. Note that $Y_n - Y_{n-1}$ is isomorphic to $X_n - X_{n-1}$ which is a disjoint union of open balls. We then attach those balls to Y_{n-1} to obtain the regular cell complex structure on Y. □

What this lemma allow us to do is to regard regular cell complexes as filtered spaces with some special conditions.

Proposition A.0.29. *Let X and Y be regular cell complexes. Then $X \otimes Y$ is a regular cell complex.*

Proof. The needed pushouts are as follows:

$$\begin{tikzcd}
\coprod\limits_{(x,y)\in cell_i(X)\times cells_j(Y)} \partial(B^i \times B^j) \arrow[r, "\coprod\limits_{(x,y)\in cell_i(X)\times cells_j(Y)} \phi_i\times\phi_j"] \arrow[d] & (X \otimes Y)_{n-1} \arrow[d] \\
\coprod\limits_{(x,y)\in cell_i(X)\times cells_j(Y)} B^i \times B^j \arrow[r, "\underset{\text{i+j=n}}{\coprod\limits_{(x,y)\in cell_i(X)\times cells_j(Y)} \phi_i\times\phi_j}"'] & (X \otimes Y)_n
\end{tikzcd}$$

□

Lemma A.0.30. *Let X, and Y be two regular cell complexes. Then **cells**$(X \otimes Y)$ is naturally isomorphic to **cells**$(X) \times$ **cells**(Y), where the product is taken in the category of posets.*

Proof. This is readily seen by contemplating the pushout that is needed to make $X \otimes Y$ a regular cell complex. □

Lemma A.0.31. *Let X be a regular cell complex. Then $X \sim colim G_X$ where $G : \boldsymbol{cells}(X) \to \boldsymbol{Top}$ is a functor that sends the cell $B^n \to X$ to B^n, and an inclusion of cells, $B^m \hookrightarrow B^n \to X$ to the inclusion of balls, $B^m \to B^n$.*

Lemma A.0.32. *The functor $\otimes : \boldsymbol{Reg} \times \boldsymbol{Reg} \to \boldsymbol{Reg}$ is a tensor functor.*

Proof. It is clear that the one point space is a unit for $\otimes$. By lemma A.0.30,

$$\mathbf{Cells}((X \otimes Y) \otimes Z) \sim \mathbf{Cells}(X \otimes Y) \times \mathbf{Cells}(Z) \sim$$

$$(\mathbf{Cells}(X) \times \mathbf{Cells}(Y)) \times \mathbf{Cells}(Z) \sim \mathbf{Cells}(X) \times (\mathbf{Cells}(Y) \times \mathbf{Cells}(Z)) \sim$$

$$\mathbf{Cells}(X) \times \mathbf{Cells}(Y \otimes Z) \sim \mathbf{Cells}(X \otimes (Y \otimes Z)).$$

Furthermore the functors $G_{(X \otimes Y) \otimes Z}$ and $G_{X \otimes (Y \otimes Z)}$ defined in lemma are also isomorphic. The pentagonal axiom and hexagonal axioms are similar. ☐

Let B^n be a ball. Then $\partial C_T B^n$, where $C_T : \mathbf{Top} \to \mathbf{Top}$ is the functor that takes a space to it's cone, may be written as a pushout,

$$\begin{tikzcd}
\partial B^n \arrow[r] \arrow[d] & C_T \partial B^n \arrow[d] \\
B^n \arrow[r] & \partial C_T B^n
\end{tikzcd}$$

Definition A.0.33. *Let*

$$X_{-1} \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots X_n \hookrightarrow$$

be a regular cell complex. Then the cone on X denoted CX will be define as $CX_0 = X_0 \coprod \{*\}$ on the zeroth dimensional part of the filtrartion. We define the space CX_n to be the following pushout.

$$\begin{tikzcd}
\coprod_{x\in\mathcal{I}_n} \partial B^{n-1} \coprod \coprod_{x\in\mathcal{I}_{n-1}} \partial C_T B^{n-1} \arrow[r] \arrow[d] & CX_{n-1} \arrow[d] \\
\coprod_{x\in\mathcal{I}_n} B^{n-1} \coprod \coprod_{x\in\mathcal{I}_{n-1}} C_T B^{n-1} \arrow[r] & CX_n
\end{tikzcd}$$

Proposition A.0.34. *Let X be a regular cell complex. Then $\boldsymbol{cells}(CX)$ is isomorphic to the poset, whose objects are of the form, $\boldsymbol{cells}(X) \coprod \widetilde{\boldsymbol{cells}}(X) \coprod \{*\}$ where $\widetilde{\boldsymbol{cells}}(X)$ is isomorphic to $\boldsymbol{cells}(X)$ The poset structure is given by demanding that $\boldsymbol{cells}(X)$ is isomorphic to $\widetilde{\boldsymbol{cells}}(X)$., that the disjoint point will be less than all elements $\widetilde{\boldsymbol{cells}}(X)$. We also ask that each element in $\boldsymbol{cells}(X)$ is less than the corresponding element in $\widetilde{\boldsymbol{cells}}(X)$*

We will need the following lemmas in the next section.

Lemma A.0.35. *Let X and Y be regular cell complexes such that $CX \sim CY$. Then $X \sim Y$.*

Proof. By A.0.25, it suffices to show that if $\mathbf{cells}(CX) \sim \mathbf{cells}(CY)$, then $\mathbf{cells}(X) \sim \mathbf{cells}(Y)$. To this end, we will define a functor, $C_p : \mathbf{poset} \to \mathbf{poset}$. This takes a poset, P to a poset, whose objects are $P \coprod \tilde{P} \coprod \{*\}$, where $\tilde{P} \sim P$. We will now define $\leq$ for these elements. The poset structure restricted to $P \hookrightarrow P \coprod \tilde{P} \coprod \{*\}$ will be a poset map, which we will call the base inclusion. We will also ask that $\tilde{P}$be isomorphic to P. Furthermore each elemet of $\tilde{P}$ will be greater than it's corresponding element in P. Lastly the disjoint point will be less than all elements of $\tilde{P}$. Now let P be a poset. $x \in P$ is said to be a cone point if the map,

$$sup(-, x) : \{y \in P : y \ngeq x\} \to \{y \in P : y > x\}$$

is an isomorphism. It is easy to see that $C(\{y \in P : y \ngeq x\}) \sim P$. It is also easy to see that any map of posets sends cone points to cone points. When we take the cone on a poset, Q, we have a distinguished cone point. If the isomorphism $CP \to CP'$ maps the distinguished cone point to distinguished cone point, then we know that $P \sim P'$. Thus we must show that for any poset, P, that has more than one cone point, then we may permute any two cone points. If the isomorphism does not map the distinguished cone point to a distinguished cone point, we may post compose this map with on that permutes the cone points so that the new map does in fact take the distinguished cone point to the distinguished cone point. If P' has two cone points, x, and x', then we may patrician P into the following posets:

$$\{x\}, \{x'\}, \{sup(x, x')\}$$

$$\{y \in P : y > x, x'\}, \{y \in P : y > x, y \ngeq x'\}, \{y \in P : y \ngeq x, y > x'\}$$

$$\{y \in P : y \ngeq x, x'\}$$

Now if we send $x \mapsto x'$, this forces $\{sup(x, x')\}, \{y \in P : y > x, x'\}, \{y \in P : y \ngeq x, y > x'\}$to all be fixed. We must now show that make an isomorphism $\{y \in P : y > x, y \ngeq x'\} \to \{y \in P : y \ngeq x, y > x'\}$. To this end we have a diagram of the form,

$$\begin{tikzcd}
\{y \in P : y \ngeq x, y > x'\} \arrow[r, "sup(-{,}x)"] \arrow[d] & \{y \in P : y \ngeq x, y > x'\} \arrow[d] \\
\{y \in P : y \ngeq x'\} \arrow[r, "sup(-{,}x)"'] & \{y \in P : y > x'\}
\end{tikzcd}$$

which complete the proof. $\square$

Lemma A.0.36. *Suppose that $CX \sim Y \otimes Z$ is the category of regular cell complexes and is nonempty.Then either $X \sim \{*\}$ or $Y \sim \{*\}$.*

Proof. Suppose that Y and Z are both not isomorphic to $\{*\}$. Then since CX is connected, so is Y and Z. Now let t be a cone point in $Y \otimes Z$. Let t_Y and t_Z be the image of the cone point, t under the maps, $Y \otimes Z \to Y$, and $Y \otimes Z \to Z$, respectively. Now pick a zero cell $y \in Y - t_Y$, and a zero cell $z \in Z - t_Z$. Since t is a cone point, each set, $\{t, (y, t_Z)\}, \{t, (t_y, z)\}, \{t, (y, z)\}$. But the projection of the one cell containing $\{t, (y, t_Z)\}$ onto Y is a one cell in Y. Likewise the projection of the one cell containing $\{t, (t_Y, z)\}$ onto Z is a one cell in Z. Both of these cells are isomorphic to the interval, I. We therefore get a two cell of the form $I \otimes I \to X \times Y$ such that the image of one of the zero dimensional subcells of this square is the cone point, which is impossible. $\square$

Let us end this section with a conjecture. Recall the definition of the forgetful functor,

$$U : \mathbf{Reg} \to \mathbf{CGHAuss}$$

from definition A.0.19. The categories **Reg** and **CWposet** are both topological categories. Since the

$$U_{X,Y} : hom_{\mathbf{Reg}}(X, Y) \to hom_{\mathbf{Top}}(UX, UY)$$

is injective, we may topologize $hom_{\mathbf{Reg}}(X, Y)$ by regarding it as a subspace of $hom_{\mathbf{Top}}(UX, UY)$ which is given the compact open topology. We give the hom sets in **CWposet** the discrete topology. The conjecture then is that the maps, $U_{X,Y}$ are weak equivalences.

Appendix B

The geometric relaization of Prisms

Proposition B.0.37. *The geometric realization functor,* $|| : \mathbb{P}_{aug} \to$ ***Reg*** *is the functor such that*

1. $|0| = \emptyset$
2. $|X \otimes Y| = |X| \times |Y|$
3. $|b_X : X \to CX| = |X| \to C(|X|)$*, where the right hand map is the base inclusion, where* C *is the cone functor on the category of regular cell complexes (see definition A.0.33).*
4. $|p_X : C^2X \to CX|$ *is given by* $([0,1] \to \{pnt\}) * X$*, where* $*$ *is the join of topological spaces*

[1]

---

[1] These conditions do not really pin down the functor precisely. However if we pick a join functor and a product functor, then this does pin down the geometric realization.

Definition B.0.38. *A prod-simplicial complex will a regular cell complex that is in the image of the geometric realization functor,* $|| : \mathbb{P} \to \boldsymbol{Reg}$*. We will denote the full subcategory of* $\boldsymbol{Reg}$ *whose objects are the objects in the image of the geometric realization functor as* $\boldsymbol{PS}$

Definition B.0.39. *We will define the graphical category of rooted trees, to be the category such that the objects are the class of rooted trees, and the morphisms are injections that preserve the poset structure of the trees, such that the morphism maps edges to edges. We will denote this category by* $\boldsymbol{RT}_G$*.*

Recall from definition 2.1.2, that the class of rooted trees has a number of operations we may perform. The first is whiskering, which we will denote by W. The other operation is grafting where identify their respective roots. The grafting operation will be denoted by $\vee$.

Definition B.0.40. *Let* T *be the category,* RT_G*. Then the manifestation of a rooted tree will be a function,* $M : obj(\boldsymbol{RT}_G) \to obj(\boldsymbol{PS})$ *is defined by the following rules:*

1. $MT^0 = \{*\}$*, where* T^0 *is the (essentially) unique tree with one vertex.*
2. $M(WT) = CMT$*, where* C *is the cone functor on the category of regular cell complexes.*
3. $M(T_1 \vee T_2) = MT_1 \otimes MT_2$

.

Remark B.0.41. *With a little bit more work, we may upgrade* M *to a functor from* $\boldsymbol{RT}_G$ *to* $\boldsymbol{PS}$*. What we do is take a rooted tree, send it to the* *corresponding object in $\mathbb{P}$, then we find a way of contracting the larger tree onto the smaller tree by a series of edge contractions of the form, (v, S), where v is a vertex and S is a tree under v.*

Lemma B.0.42. *Let T be a rooted tree. Then $dim(MT) = E(T)$, where $E(T)$ is the number of edges.*

Sketch of Proof

Whiskering raises the number of edges by one, grafting two trees gives a tree with the number of edges equal to the sum of the number of edges of the two trees. Coning raises dimension of a complex by one and products are additive on dimension. The dimension on a point is zero as is the number of edges in the tree without edges.

■

Lemma B.0.43. *The geometric realization of all prisms have one top cell.*

Here we will define the notion of a unique factorization category, since it seems like an interesting notion and it is a nice way of expressing the propositions that follow.

Definition B.0.44. *Let $(\mathcal{C}, \otimes, I)$ be a (symmetric) tensor category. An non-unit object w of $\mathcal{C}$ is said to be prime if $w \sim x \otimes y$ then either x or y is isomorphic to I. The category is said to be a unique factorization category non-initial every object has a prime factorization and the prime factorization is unique up to reordering the factors, and the coherence isomorphisms that are part of the definition of a tensor category . A functor of tensor categories is said to preserve primes if the image of a prime is prime.*

Without too much trouble we have the following proposition:

Proposition B.0.45. *The tensor category,* $(RT_G, \vee, P^0)$*, where* P^0 *is the rooted tree with no edges is a unique factorization category.*

Sketch of Proof

We will say what the primes are and leave the rest to the reader. The primes in the category of rooted trees are those that have a root whose valance in one.

■

Lemma B.0.46. *Suppose that* PS *is a category with unique factorization. Then two trees that manifest themselves as the same cell complex are isomorphic.*

Proof. Assuming throughout, that the category of prod simplicial complexes is a UFC, we prove that a prod-simplicial complex determines a unique tree. This will be proved by induction on the number of edges. The number of trees with one edge is one. This corresponds to the interval. Now we suppose by induction that the lemma is true for all trees such that the number of edges is less than k. Now if we have two trees with k edges, then we may uniquely decompose the tree as $WT_1 \vee WT_2 \ldots WT_m$, and $WS_1 \vee WS_2 \ldots WS_m$. We have the following string of implications.

$$\begin{gathered}
M(WT_1 \vee WT_2 \ldots MWT_n) \sim MWS_1 \vee MWS_2 \ldots MWS_m \Longrightarrow \\
CMT_1 \vee CMT_2 \ldots CMT_n \sim CMS_1 \vee CMS_2 \ldots CMS_m \Longrightarrow \\
\exists \sigma \in AUT\{1 \ldots n\} \forall i \; MT_i \sim MS_{\sigma i} \Longrightarrow \\
\exists \sigma \in AUT\{1 \ldots n\} \forall i \; T_i \sim S_{\sigma i}, \text{ and } \exists \sigma \in \{1 \ldots n\} \forall i \; WT_i \sim WS_{\sigma i} \Longrightarrow \\
WT_1 \vee WT_2 \ldots WT_n \sim WS_1 \vee WS_2 \ldots WS_m
\end{gathered}$$

We will now prove the unique factorization proposition.

Proposition B.0.47. *Suppose that* $CX_1 \times CX_2 \times \dots CX_n \sim CY_1 \times CY_2 \times \dots CY_n$*. Then the lists (multi-sets),* $\{X_1, X_2 \dots X_n\} \sim \{Y_1, Y_2 \dots Y_n\}$ *of isomorphism classes of cell complexes are isomorphic.*

Proof. Suppose that

$$CX_1 \times CX_2 \times \dots CX_n \sim CY_1 \times CY_2 \times \dots CY_m,$$

and without loss of generality that $m \geq n$. Furthermore, for each $i \in \{1, 2, \cdots n\}$, we choose a map, $1_i : * \to CX_i$. Given these choices, for each projection,

$$CX_1 \times CX_2 \times \dots CX_n \to CX_{i_1} \times CX_{i_2} \times \dots CX_{i_k},$$

we get a section induced by the product of maps, $\phi_1 \times \phi_2 \cdots \phi_n$ such that

$$\phi_i = \begin{cases} id_{CX_i} & i \in \{i_1, i_2, \cdots i_k\} \\ 1_i & i \notin \{i_1, i_2, \cdots i_k\} \end{cases}$$

Furthermore, if we have a composition of projections,

$$CX_1 \times CX_2 \times \dots CX_n \to CX_{i_1} \times CX_{i_2} \times \dots CX_{i_k} \to CX_{j_1} \times CX_{j_2} \times \dots CX_{j_{k'}},$$

then we get a commuting triangle of sections,

$$\begin{tikzcd}
CX_{j_1} \times CX_{j_2} \times \dots CX_{j_{k'}} \arrow[rr] \arrow[dr] & & CX_{i_1} \times CX_{i_2} \times \dots CX_{i_k} \arrow[dl] \\
& CX_1 \times CX_2 \times \dots CX_n &
\end{tikzcd}$$

Note that what we did above may be done in any category such that for every object, X, the hom-set $hom(*, X)$ is non-empty (* is the terminal object). We will now consider the following diagram, for each $i \in \{1, 2 \cdots n\}$

$$\begin{tikzcd}
CX_1 \times CX_2 \times \dots CX_n \arrow[r, "\sim"] & CY_1 \times CY_2 \times \dots CY_m \\
CX_i \arrow[u] \arrow[ur, "\iota^i"'] &
\end{tikzcd}$$

Furthermore, $\iota^i = \iota^i_1 \times \iota^i_2 \times \cdots \iota^i_n$, where $\iota^i_j = p_{CY_j}\iota^i$. Thus $Im(\iota) = Im(\iota^i_1) \times Im(\iota^i_2) \times \cdots Im(\iota^i_n)$. Since $Im(\iota^i) \sim CX_i$, we have that $CX_i \sim Im(\iota^i_1) \times Im(\iota^i_2) \times \cdots Im(\iota^i_n)$. But by lemma A.0.36 CX_i There is a unique k such that $CX_i \sim Im(\iota^i_k)$, and $Im(\iota^i_{k'}) = \{*\}$ if $k' \neq k$. For each i we have a unique digram of the form

$$\begin{tikzcd}
CX_1 \times CX_2 \times \dots CX_n \arrow[r, "\sim"] \arrow[d] & CY_1 \times CY_2 \times \dots CY_m \arrow[d] \\
CX_i \arrow[u, bend left] \arrow[ur, "\iota^i"'] \arrow[r, "\iota^i_k"'] & CY_k \arrow[u, bend right]
\end{tikzcd}$$

This defines a function, $\psi : \{1, 2, \cdots n\} \to \{1, 2, \cdots m\}$ by sending $i \mapsto k$. Note that the bottom line of the last diagram may be rewritten as

$$\begin{tikzcd}
CX_i \arrow[r, "\iota^i_{\psi(i)}"] & CY_{\psi(i)}
\end{tikzcd} .$$

For each $k \in Im(\psi)$, we may form the following inclusion,

$$\begin{tikzcd}
\prod\limits_{l\in\psi^{-1}(k)} X_l \arrow[rr] \arrow[dr] & & \prod\limits_{0\leq l\leq n} CX_l \arrow[dl] \\
& \prod\limits_{0\leq l\leq m} CY_l &
\end{tikzcd}$$

This allows us to make the following diagrams, for each $l \in \psi^{-1}(k)$

$$\begin{tikzcd}
\prod\limits_{l\in\psi^{-1}(k)} CX_l \arrow[r, "\alpha^k"] \arrow[d] \arrow[dr, "\alpha^k_k"] & \prod\limits_{0\leq l\leq m} CY_l \arrow[d] \\
CX_l \arrow[u, bend left] \arrow[r] & CY_k \arrow[u, bend right]
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod\limits_{l\in\psi^{-1}(k)} CX_l \arrow[r, "\alpha^k"] \arrow[d] \arrow[dr, "\alpha^k_{k'}"] & \prod\limits_{0\leq l\leq m} CY_l \arrow[d] \\
CX_l \arrow[u, bend left] \arrow[r] \arrow[d] \arrow[dr, phantom, "k\neq k'"] & CY_{k'} \arrow[u, bend right] \\
\{*\} \arrow[ur, bend right] &
\end{tikzcd}$$

We therefore have that our map, $\alpha^k = \alpha_1^k \times a_2^k \times \cdots \times \alpha_m^k$. but by looking at the right hand diagram, we see that $\alpha_{k'}^k$ factors through$\{*\}$ if $k' \neq k$. Thus the injection α_k^k must be an injection. The rest of the argument is a cell counting argument. To this end, let us first assume that each factor of $\prod_{0 \leq l \leq m} CY_l$ has more than one cell (in other words, the factors, CY_i are not points, or the Y_i are not the empty cell complex). We will now show that the map, $\psi : \{1, 2, \cdots n\} \to \{1, 2, \cdots m\}$ is surjective. Suppose that ψ is not surjective. Denote $[X]$ as the number of cells in a cell complex, X. We know that $[X \times Y] = [X][Y]$. Up to reordering, $Im(\psi) = \{1, 2. \cdots p\}$. We also know that

$$\prod_{0 \leq l \leq n} [CX_l] = \prod_{0 \leq l \leq m} [CY_l],$$

as the two products are isomorphic. We also have just shown that we have an injection for each l,

$$\alpha_k^k : \prod_{l \in \psi^{-1}(k)} CX_l \hookrightarrow CY_k.$$

We also know that

$$\prod_{1 \leq l \leq n} CX_l$$

is isomorphic to

$$\prod_{k \in Im(\psi)} \prod_{l \in \psi^{-1}(k)} CX_l.$$

But then

$$[\prod_{1 \leq l \leq n} CX_l] = [\prod_{k \in Im(\psi)} \prod_{l \in \psi^{-1}(k)} CX_l]$$

$$\leq [\prod_{k \in Im(\psi)} CY_l] < [\prod_{k \in Im(\psi)} CY_l][\prod_{k \notin Im(\psi)} CY_l] =$$

$$[\prod_{0\leq l\leq m} CY_l] = [\prod_{1\leq l\leq n} CX_l].$$

This implies that ψ is surjective which implies that $m \geq n$. But we assumed at the outset that $n \leq m$. Thus $m = n$, and since we know that a surjective map between finite sets of the same cardinality is an isomorphism, ψ is an isomorphism. Thus for each i, $CX_i \hookrightarrow CY_{\psi(i)}$, and

$$[CX_i] \leq [CY_{\psi(i)}].$$

So suppose that this is not an isomorphism for some chosen i, without loss of generality set $i = 1$. This means that $[CX_1] < [CY_{\psi(1)}]$. This implies that

$$[\prod_{1\leq l\leq n} CX_l] = [CX_1][\prod_{2\leq l\leq n} CX_l] < [CY_{\psi(1)}][\prod_{2\leq l\leq n} CY_{\psi(l)}] = [\prod_{1\leq l\leq n} CY_{\psi(l)}]$$

which is a contradiction which means that $CX_i \sim CY_{\psi(i)}$. □

B.1 Some polynomial invariants of rooted trees and prisms

I came across a beautiful invariant of trees that and of prisms. It turned out that this invariant had already been defined in [7].

Definition B.1.1. *Let RT be the collection of rooted trees. We will define a class of polynomial invariants, which will be functions, $P : RT \to \mathbb{Z}[z]$ by the following inductive rules:*

1. *, The one point tree, $\Delta^0 \mapsto 1$*

2. If $X \mapsto P_\alpha(X : z)$, then $WX \mapsto (z+\alpha)P_\alpha(X :, z)$

3. If $X_i \mapsto P_\alpha(X_i :, z), i = 1, 2$, then $X_1 \times X_2 \mapsto P_\alpha(X_1 :, z)P_\alpha(X_2 :, z)$

Lemma B.1.2. *Let α be any number. The for every rooted tree T, the following two polynomials are equal, $P_0(T : z - \alpha) = P_\alpha(T : z)$.*

Sketch of Proof

This is certainly true for the one point tree. Now suppose that these two polynomials are the same for trees with $k-1$ edges. Then use rules (2) and (3) to show that the two polynomials are the same for trees with k edges.

■

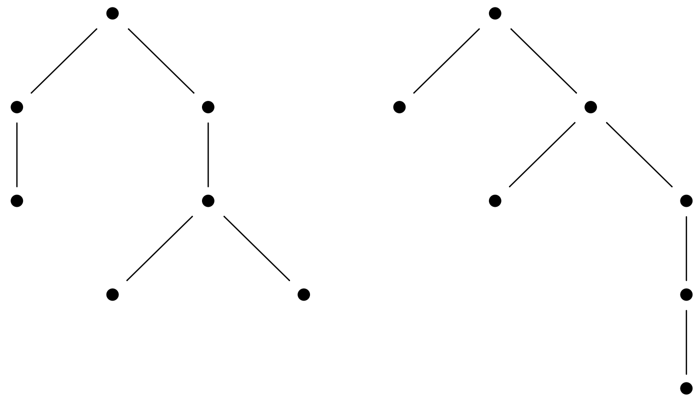
What is interesting about this polynomial is that if $\alpha = 0$, then the polynomial such that the i-th coefficient tells us how may rooted subtrees with i edges the tree has. When $\alpha = 1$ the i-th coefficient is the number of subcells of the dimension i of the manifestation of the tree.[2]

Question B.1.3. *Does this polynomial distinguish non-isomorphic trees ?*

The answer to this question is no. A counterexample, given in [7], is given by the following pair of trees.

---

[2] See http://mathoverflow.net/questions/107863/is-the-following-invariant-of-rooted-trees-a-complete-invariant for a proof



This is the first counterexample that occurs. The polynomial is

$$z^6 + 3z^5 + 4z^4 + 4z^3 + 3z^2 + 2z + 1$$

. This raises another question: is it possible to understand and explain when this invariant does not distinguish tree. This pair of trees may be used to cook up other counterexamples. If we whisker these two counterexamples, we get another pair of counterexamples. If we wedge these two with a a third common tree, we get another pair of example. The question then becomes, can we produce other counterexamples not obtained in this way.

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