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A class of generalized hypergeometric functions in several variables

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A CLASS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS IN SEVERAL VARIABLES

A

by

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To my parents, brother, sister

and

my wife, Minqing

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Abstract

A CLASS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS IN SEVERAL VARIABLES

by

Zhimin Yan

Advisor: Professor Adam Korányi

We study a class of generalized hypergeometric functions in several variables, ${}_pF_q^{(d)}$, introduced by A. Korányi. We prove that ${}_2F_1^{(d)}$ is the unique solution of a system of partial differential equations, and, as an application, we obtain analogues of such classical results as Kummer relations. Euler integral representations for generalized hypergeometric functions are gotten, in particular, in the case of two variables, ${}_1F_0^{(d)}$ and ${}_2F_1^{(d)}$ are expressed in terms of classical hypergeometric functions. Some integral formulas about Jack polynomials in two variables are given, which have many applications. It is shown that in two variables, ${}_2F_1^{(d)}$ is a hypergeometric function in the sense of Heckman and Opdam. It follows that for some special parameters, ${}_2F_1^{(d)}$ is a spherical function of a symmetric space of root system BC_2. We obtain the asymptotic behavior of ${}_{p+1}F_p^{(d)}$. As an application, we get the

generalized Rudin-Forelli inequalities in function theory on a bounded symmetric domain, which are due to J. Faraut and A. Korányi for ${}_2F_1^{(d)}$ with some special parameters. Our results also include, in a unified way, some estimates obtained for the classical Cartan domains by J. Mitchell and G. Sampson. In the case of two variables, we introduced the generalized Laplace transform and prove the injectivity of the generalized Laplace transform. It is shown that the Laplace transform of ${}_pF_q^{(d)}$ is a ${}_{p+1}F_q^{(d)}$ function as in the classical case. We also introduce the generalized Laguerre polynomials L_κ^γ. We find the generating function and integral representation for L_κ^γ. We also establish the orthogonality relations for L_κ^γ. Finally, we define the generalized Hankel transform. Generalizations of some classical results about Hankel transform are obtained. A generalized Tricomi Theorem is given.

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§0 Introduction

We shall study a class of generalized hypergeometric functions in several variables, and obtain particularly precise results in two variables.

The classical hypergeometric function ${}_pf_q$ is defined as follows: for $a_1, ..., a_p$, $b_1, ..., b_q \in \mathbf{C}$ with $(b_j)_k \neq 0$, for all k, j,

$$ {}_pf_q(a_1, ..., a_p; b_1, ..., b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} z^k. $$

In the case of positive definite matrices, generalized hypergeometric functions (with a definition based on integrals) were introduced by C. Herz [H], and the series expansion is due to A. Constantine [C]. Further properties and applications in statistics were given by A. James and R. Muirhead [Mu]. The case of positive Hermitian or quarternian matrices was studied by K. Gross and D. Richards [GR]. Generalized hypergeometric functions associated with arbitraty symmetric cones were considered by J. Faraut and A. Korányi [FK]. The more general version (1) below was introduced by A. Korányi [K].

Let $\kappa = (k_1, \ldots, k_r) \in \mathbf{Z}^r$ with $k_1 \geq \ldots \geq k_r \geq 0$ and $k = |\kappa| = k_1 + \ldots + k_r$. We denote by $l(\kappa)$ the number of nonzero k_i.

Let $\bigwedge_r$ be the vector space of symmetric polynomials in $x_1, \ldots, x_r$, $p_k = \sum_{i=1}^r x_i^k$ and $P_\kappa = p_{k_1} ... p_{k_{l(\kappa)}}$, then, $\{P_\kappa$, for all $\kappa\}$ forms a basis of $\bigwedge_r$. For each $\alpha > 0$, one defines an inner product on $\bigwedge_r$ by

$$ < P_\kappa, P_\lambda >_\alpha = \delta_{\kappa\lambda} z_\kappa \alpha^{l(\kappa)} $$

where $z_\kappa = (1^{m_1}2^{m_2}\dots)m_1!m_2!\dots$ and m_j = the number of k_i which are equal to j. For $d > 0$, we define

$$C_\kappa^{(d)}(x_1,\dots,x_r) = (2/d)^k k! J_\kappa(x_1,\dots,x_r;2/d) j_\kappa^{-1}$$

where $J_\kappa(x_1,\dots,x_r;2/d)$ is the Jack polynomial of index κ and parameter $2/d$; $j_\kappa =< J_\kappa, J_\kappa >$. The J_κ are gotten by orthogonalizing the monomial symmetric polynomials with respect to $<,>_\alpha$. See [M] and [S].

Let

$$(a)_\kappa = \prod_{i=1}^{r}(a - d/2(i-1))_{k_i}.$$

For $a_1,\dots,a_p,b_1,\dots,b_q \in \mathbf{C}$, such that $(b_j)_\kappa \neq 0$, for all κ, j, one defines the hypergeometric function associated with the parameter $d > 0$ by

$$\begin{aligned} &{}_pF_q^{(d)}(a_1,\dots,a_p;b_1,\dots,b_q;x_1,\dots,x_r) = \\ &\quad \sum_\kappa \frac{(a_1)_\kappa\dots(a_p)_\kappa}{(b_1)_\kappa\dots(b_q)_\kappa}\frac{C_\kappa^{(d)}(x_1.\dots,x_r)}{k!}. \end{aligned} \tag{1}$$

Many results known in the symmetric cone case are difficult to extend to the general case because one can not use the machinery of Lie groups. When $r = 2$, for any positive integer d, ${}_pF_q^{(d)}$ are the hypergeometric functions associated with the Lorentz cone in $\mathbf{R}^{d+2}$. We shall exploit this fact and prove a number of analytic results in the case $r = 2$ for any positive d, by using Carlson's interpolation theorem.

In §2 we prove that ${}_2F_1^{(d)}(a,b;c;x_1,\dots,x_r)$, the most important hypergeometric function, is the unique solution of the system of the partial differential equations

$$x_i(1-x_i)\frac{\partial^2 F}{\partial x_i^2}+\{c-\frac{d}{2}(r-1)-[a+b+1-\frac{d}{2}(r-1)]x_i+$$
$$\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{x_i(1-x_i)}{x_i-x_j}\}\frac{\partial F}{\partial x_i}-\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{x_j(1-x_j)}{x_i-x_j}\frac{\partial F}{\partial x_j}=abF \tag{2}$$
$$i=1,\ldots,r$$

subject to the conditions that

(a) F is a symmetric function of $x_1,\ldots,x_r$ and

(b) F is analytic at $x_1=\ldots=x_r=0$ and $F(0)=1$.

(2) is a generalization of the classical hypergeometric equation. This result was claimed in [K], but the proof was incomplete.

In §3 we give an integral representation of $C_\kappa^{(d)}(x_1,x_2)$ and some other properties of $C_\kappa^{(d)}(x_1,x_2)$ which are very useful for our purpose and are also of independent interest.

In §4 we obtain two special cases of generalized hypergeometric functions

$${}_0F_0^{(d)}(x_1,\ldots,x_r)=e^{x_1+\ldots+x_r},$$

$${}_1F_0^{(d)}(a;x_1,\ldots,x_r)=\prod_{i=1}^{r}(1-x_i)^{-a}.$$

As an application of the uniqueness of the solution of (2), we get analogues of the classical Kummer relations

$${}_2F_1^{(d)}(a,b;c;x_1,\ldots,x_r)$$

$$= \prod_{i=1}^{r}(1-x_i)^{-a}{}_2F_1^{(d)}(a, c-b; c; -\frac{x_1}{1-x_1}, \ldots, -\frac{x_r}{1-x_r})$$
$$= \prod_{i=1}^{r}(1-x_i)^{c-(a+b)}{}_2F_1^{(d)}(c-a, c-b; x_1, \ldots, x_r).$$

In the classical case, there are the following well-known Euler integral representations for ${}_1f_1$ and ${}_2f_1$:

$${}_1f_1(a; b; y) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xy} x^{a-1}(1-x)^{b-a-1} dx$$

for $a > 0, b-a > 0$;

$${}_2f_1(a, b; c; y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 (1-xy)^{-b} x^{a-1}(1-x)^{c-a-1} dx$$

for $a > 0, c-a > 0$.

we define

$${}_p\mathcal{F}_q^{(d)}(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1, \ldots, x_r | y_1, \ldots, y_r) =$$
$$\sum_{\kappa} \frac{(a_1)_\kappa \ldots (a_p)_\kappa}{(b_1)_\kappa \ldots (b_q)_\kappa} \frac{C_\kappa^{(d)}(x_1, \ldots, x_r)}{k!} \frac{C_\kappa^{(d)}(y_1, \ldots, y_r)}{C_\kappa^{(d)}(1, \ldots, 1)}.$$

From the series expansins of e^{xy} and $(1-xy)^{-a}$, we can see that

${}_0\mathcal{F}_0^{(d)}(x_1, \ldots, x_r | y_1, \ldots, y_r)$ and ${}_1\mathcal{F}_0^{(d)}(a; x_1, \ldots, x_r | y_1, \ldots, y_r)$ are generalizations of e^{xy} and $(1-xy)^{-a}$. Now we have the following generalized Euler integral representations for ${}_1F_1$ and ${}_2F_1$

$${}_1F_1^{(d)}(a; b; y_1, \ldots, y_r) =$$
$$c_0 \frac{\Gamma_d(b)}{\Gamma_d(a)\Gamma_d(b-a)} \int_0^1 \cdots \int_0^1 {}_0\mathcal{F}_0^{(d)}(x_1, \ldots, x_r | y_1, \ldots, y_r)$$
$$\cdot \prod_{i=1}^{r}(1-x_i)^{b-a-q_0} \prod_{i=1}^{r} x_i^{a-q_0} \prod_{1 \le i < j \le r} |x_i - x_j|^d dx_1 \ldots dx_r \qquad (3)$$

if $a > \frac{r-1}{2}d, b-a > \frac{r-1}{2}d$;

$$
{}_2F_1^{(d)}(a;b;c;y_1,\ldots,y_r) = c_0\frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)}\int_0^1\cdots\int_0^1 {}_1\mathcal{F}_0^{(d)}(b;x_1,\ldots,x_r|y_1,\ldots,y_r) \cdot\prod_{i=1}^r(1-x_i)^{c-a-q_0}\prod_{i=1}^r x_i^{a-q_0}\prod_{1\le i<j\le r}|x_i-x_j|^d dx_1,\ldots,dx_r \tag{4}
$$

if $a > \frac{r-1}{2}d, c-a > \frac{r-1}{2}d$, where

$$
q_0 = 1+\frac{r-1}{2}d,\quad c_0 = (2\pi)^{\frac{dr(r-1)}{4}}\prod_{j=1}^r\frac{\Gamma(d/2+1)}{\Gamma(j\frac{d}{2}+1)}
$$

and

$$
\Gamma_d(a) = (2\pi)^{\frac{dr(r-1)}{4}}\prod_{j=1}^r\Gamma(a-(j-1)\frac{d}{2}).
$$

It is shown that

$$
{}_0\mathcal{F}_0^{(d)}(x_1,x_2|y_1,y_2) = e^{x_1y_1+x_2y_2}{}_1f_1(d/2;d;-(x_1-x_2)(y_1-y_2)),
$$

$$
{}_1\mathcal{F}_0^{(d)}(b;x_1,x_2|y_1,y_2) = \prod_{i=1}^2(1-y_ix_i)^{-b} \cdot{}_2f_1(d/2,b;d;-(x_1-x_2)(y_1-y_2)\prod_{i=1}^2(1-x_iy_i)^{-1}),
$$

$$
{}_1\mathcal{F}_0(\frac{rd}{2};x_1,\ldots,x_r|y_1,\ldots,y_r) = \prod_{i,j=1}^r(1-x_iy_j)^{-\frac{d}{2}}.
$$

When $r=2$, we define the generalized Laplace transform of f by

$$
\mathcal{L}(f)(y_1,y_2) = c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0^{(d)}(-x_1,-x_2|y_1,y_2)f(x_1,x_2)|x_1-x_2|^d dx_1dx_2
$$

for $y_1,y_2>0$.

In §5 we prove the injectivity of the generalized Laplace transform which has many applications. We also show that

$$\mathcal{L}(C_\kappa^{(d)})(y_1, y_2) = \Gamma_d(q_0 + x) C_\kappa^{(d)}(y_1^{-1}, y_2^{-2})(y_1 y_2)^{-q_0}. \tag{5}$$

It follows from (5) that the Laplace transform of ${}_pF_q$ is a ${}_{p+1}F_q$ function as in the classical case.

When ${}_2F_1^{(d)}$ corresponds to a bounded symmetric domain, it is shown by A. Korányi that ${}_2F_1^{(d)}$ is a hypergeometric function in the sense of Heckman and Opdam. In the case of $r = 2$, we prove that this is still true, for every positive d. It follows that for some parameters, ${}_2F_1^{(d)}$ are spherical functions of a symmetric space of the root system BC_2.

In §7 we obtain the asymptotic behavior of ${}_{p+1}F_p^{(d)}$. As an application, we get the generalized Rudin-Forelli inequalities in function theory on a bounded symmetric domain, which are due to J. Faraut and A. Korányi for ${}_2F_1^{(d)}(a, b; c; t_1, \ldots, t_r)$ with some special a, b and c. Our results also include, in a unified way, the estimates obtained by J. Mitchell and G. Sampson [Mi], [MS].

The classical Laguerre polynomials are given by

$$L_k^\gamma(x) = (\gamma + 1)_k \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} \frac{(-x)^s}{(\gamma + 1)_s}$$

for $\gamma > -1$.

It is known that the classical Laguerre polynomials have the following properties

(i) (generating functions)

$$(1-z)^{-\gamma-1}\sum_{l=0}^{\infty}\frac{1}{l!}(x\frac{z}{z-1})^l=\sum_{k=0}^{\infty}\frac{L_k^\gamma(x)z^k}{k!}$$

for $|z|<1$;

(ii) (orthogonality)

$$\int_0^\infty e^{-x}x^\gamma L_k^\gamma(x)L_l^\gamma(x)dx=\delta_{kl}k!\Gamma(\gamma+1+k);$$

(iii)

$$L_k^\gamma(x)=\frac{1}{\Gamma(\gamma+1)}e^x\int_0^1 e^{-y}y^{\gamma+k}{}_0f_1(\gamma+1;-xy)dy.$$

In the last section, we introduce the generalized Laguerre polynomials L_κ^γ. When $r=2$, we find the generating functions and the integral representations for L_κ^γ. We also establish the orthogonal relations for L_κ^γ . These are generalizations of the above three results about the classical Laguerre polynomials .

Furthermore, for $\gamma>-1$, let

$$L_\gamma^2(\mathbf{R}_+^2)=\{f|f(x_1,x_2)=f(x_2,x_1),$$
$$\int_0^\infty\int_0^\infty|f(x_1,x_2)|^2(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2<\infty\}.$$

We define the generalized Hankel transform by

$$(H_\gamma f)(y_1,y_2)=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_1^{(d)}(\gamma+q_0;-x_1,-x_2|y_1,y_2)$$
$$\cdot f(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2$$

for $y_1,y_2>0$.

We show that

a) $(1/\Gamma_d(\gamma+q_0))H_\gamma$ is an involutive isometry on L^2_γ;

b) $\{\alpha_\kappa e^{-(x_1+x_2)}L^\gamma_\kappa(x_1,x_2)\}$ is an orthonormal basis on L^2_γ, where α_κ are some constants;

c) $e^{-(x_1+x_2)}L^\gamma_\kappa(x_1,x_2)$ are eigenfunctions of H_γ.

Finally, we have

Theorem (Generalized Tricomi Theorem)

Let

$$F(y_1,y_2)=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0^{(d)}(-y_1,-y_2|x_1,x_2)$$
$$\cdot f(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2$$

and

$$G(y_1,y_2)=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0^{(d)}(-y_1,-y_2|x_1,x_2)$$
$$\cdot f(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2,$$

then,

$g=H_\gamma f$ iff $G(y_1,y_2)=\Gamma_d(\gamma+q_0)(y_1y_2)^{-(\gamma+q_0)}F(y_1^{-1},y_2^{-1})$.

§1 Notations, definitions and basic facts

A partition is any finite or infinite sequence

$$\kappa = (k_1, k_2, \ldots, k_r, \ldots) \tag{1}$$

of non-negative integers in decreasing order $k_1 \geq k_2 \geq \ldots \geq k_r \geq \ldots$ and containing only finitely many non-zero terms. The non-zero k_i in (1) are called the parts of κ. The number of parts is called the length of κ, denoted by $l(\kappa)$; and the sum of the parts is the weight of κ, denoted by $|\kappa| = k_1 + k_2 + \ldots + k_{l(\kappa)}$. When $l(\kappa) \leq r$, we simply write κ as $\kappa = (k_1, \ldots, k_r)$. We say that κ is a partition of k if $|\kappa| = k$. For a partition κ, hereafter, we use k to denote $|\kappa|$. The partitions of k are ordered lexicographically, that is, if $\kappa = (k_1, k_2, \ldots)$ $\lambda = (l_1, l_2, \ldots, \ldots)$, we write $\kappa > \lambda$ if $k_i > l_i$ for the first index i for which the parts are unequal. Let $y_1, \ldots, y_r$ be r variables, if $\kappa > \lambda$ and $l(k)$, $l(\gamma) \leq r$, we say that the monomial $y_1^{k_1} \ldots y_r^{k_r}$ is of higher weight than the monomial $y_1^{l_1} \ldots y_r^{l_r}$.

For a partition κ, we define its diagram by

$$G(\kappa) = \{(i,j) : 1 \leq i \leq l(\kappa), 1 \leq j \leq k_i\}.$$

For each j, $j = 1, 2, ..., k_1$, let

$$k_j' = max\{i | (i,j) \in G(\kappa)\}.$$

For $s = (i,j) \in G(\kappa)$, and a parameter α, let

$$a(s) \quad = \quad k_i - j,$$

$$
\begin{aligned}
l(s) &= k_j' - i, \\
h^*_\kappa(s) &= l(s) + (1 + a(s))\alpha, \\
h^\kappa_*(s) &= l(s) + 1 + a(s)\alpha.
\end{aligned}
$$

We simply write $s \in \kappa$ instead of $s \in G(\kappa)$.

Let $J_\kappa(y_1, \ldots, y_r; \alpha)$ be the Jack polynomial indexed by the partition κ and parameter α. Notations are as in [S]

The following results about Jack polynomials are known. See [S].

(i) $J_\kappa(y_1, \ldots, y_r; \alpha) = 0$ if $l(\kappa) > r$;

(ii) $J_\kappa(y_1, \ldots, y_r; \alpha) = J_\kappa(y_1, \ldots, y_r, 0; \alpha)$;

(iii) $(y_1 + \ldots + y_r)^\kappa = \sum \alpha^\kappa k! J_\kappa(y_1, \ldots, y_r; \alpha) j_\kappa^{-1}$;

(iv) $J_\kappa(1, \ldots, 1; \alpha) = \prod_{(i,j)\in\kappa}(r - (i-1) + \alpha(j-1))$;

(v) Let $\nu_{\kappa\kappa}(\alpha) = \prod_{s\in\kappa} h^\kappa_*(s)$ then, $\nu_{\kappa\kappa}(\alpha) y_1^{k_1} \ldots y_r^{k_r}$ is the term of the highest weight in $J_\kappa(y_1, \ldots, y_r; \alpha)$;

(vi) $J_\kappa(y_1, \ldots, y_r; \alpha)$ is an eigenfunction of the differential operator

$$
\Delta_r = \sum_{i=1}^r y_i^2 \frac{\partial^2}{\partial y_i^2} + \frac{2}{\alpha} \sum_{i=1}^r \sum_{j=1, j\neq i}^r \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i} \tag{2}
$$

with the eigenvalue $\mu_\kappa = \rho_\kappa + k(\frac{2}{\alpha} r - 1)$, where $\rho_\kappa = \sum_{i=1}^r k_i(k_i - \frac{2}{\alpha} i)$, if $l(\kappa) \leq r$;

(vii) $j_\kappa = < J_\kappa, J_\kappa > = \prod_{s\in\kappa} h^\kappa_*(s) h^*_\kappa(s)$.

One defines, for a partition κ and a positive number d,

$$
C_\kappa^{(d)}(y_1, \ldots, y_r) = (2/d)^k k! J_\kappa(y_1, \ldots, y_r; 2/d) j_\kappa^{-1}.
$$

Definition. For $a_1, ..., a_p, b_1, ..., b_q \in \mathbf{C}$, such that $(b_j)_\kappa \neq 0$, for all κ, j, the hypergeometric functions associated with the parameter $d > 0$ are defined by

$$\begin{aligned} &{}_pF_q^{(d)}(a_1, \ldots, a_p; b_1, \ldots, b_q; y_1, \ldots, y_r) \\ &= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa^{(d)}(y_1, \ldots, y_r)}{k!} \end{aligned} \tag{3}$$

where $\sum_\kappa$ denotes the summation over all partitions of k,

$(a)_\kappa = \prod_{i=1}^{l(\kappa)} (a - d/2(i-1))_{k_i}$

and

$(a)_m = a(a+1) \cdots (a+m-1), (a)_0 = 1.$

Remark 1. From (i), we have $C_\kappa^{(d)}(y_1, \ldots, y_r) = 0$ for κ with $l(\kappa) > r$, therefore, the summation in (3) is only over those partitions with length not greater than r.

In the following, we denote $(y_1, \ldots, y_r)$ by Y_r or simply by Y whenever no confusion is caused.

Most of our work is motivated by the study of hypergeometric functions on the symmetric cones consisting of symmetric matrices.

Some basic facts about symmetric cones will be needed in our study.

Suppose that Ω is a symmetric cone in a Euclidean space V, then the space V posseses a Jordan algebra structure with an identity element e such that Ω is the interior of the set of all squares in V. Moreover, there exists a Riemannian structure on Ω such that Ω is a Riemannian symmetric space $\Omega \simeq G/K$; where G is the identity component of the linear transformation group preserving Ω and K the isotropy group of e.

Suppose that the Jordan algebra V is simple. Let $c_1, ..., c_r$ be a complete system of orthogonal primitive idempotents, abbreviated by CSOPI, then each element x in V can be writen as

$$x = k.\sum_{j=1}^{r} \lambda_j c_j, k \in K, \lambda_j \in \mathbf{R}.$$

The determinant Δ is defined by

$$\Delta(x) = \prod_{j=1}^{r} \lambda_j$$

and the trace by

$$tr(x) = \sum_{j=1}^{r} \lambda_j$$

If c is an idempotent, the space $V(c) = \{x \in V | cx = x\}$ is a subalgebra of V. Let $V_j = V(c_1 + \cdots + c_j)$, one defines $\Delta_j(x) = \Delta_{V_j}(x^{(j)})$, where Δ_{V_j} is the determinant relative to the subalgebra V_j, and $x^{(j)}$ is the orthogonal projection of x onto V_j. For an r-tuple $\mathbf{m} = (m_1, ..., m_r)$ such that $m_1 \geq \cdots \geq m_r \geq 0$, one defines

$$\Delta_{\mathbf{m}}(x) = \Delta_1^{m_1-m_2}(x)\Delta_2^{m_2-m_3}(x)\cdots\Delta_r^{m_r}(x)$$

where $m_1, ..., m_r$ are integers, the spherical polynomial $\Phi_{\mathbf{m}}$ is defined by

$$\Phi_{\mathbf{m}}(x) = \int_K \Delta_{\mathbf{m}}(k.x)dk \tag{4}$$

where dk is the normalized Haar measure on K with $\int_K dk = 1$.

Now let $V = \mathbf{R}^{n+1}$, for $x = (x_0, x_1, ..., x_n) \in \mathbf{R}^{n+1}$, simply written as (x_0, x'), we define a product on V by

$$(x_0, x')(y_0, y') = (x_0 y_0 + \sum_{i=1}^{n} x_i y_i, x_0 y' + y_0 x') \tag{5}$$

then V is a Jordan algebra, $\Delta(x) = x_0^2 - \sum_{i=1}^{n} x_i^2$ and $tr(x) = 2x_0$. The interior of the set of all squares in V is just the Lorentz cone

$$\Omega_n = \{x \in \mathbf{R}^{n+1} | x_0^2 - \sum_{i=1}^{n} x_i^2 > 0, x_0 > 0\}.$$

Let $e_1 = (1, 0, ..., 0) \in \mathbf{R}^{n+1}$ and $e_2 = (0, 1, 0, ..., 0) \in \mathbf{R}^{n+1}$, then e_1 is the identity element in V. $SO(n, \mathbf{R})$ is the isotropy group of e_1, here, we consider $SO(n, \mathbf{R})$ as a subgroup of $SO(n+1, \mathbf{R})$ in an obvious way.

Let $<,>$ be the ordinary inner product on $\mathbf{R}^{n+1}$ and $(x|y) = tr(xy)$, we have

$$(x|y) = 2 < x, y > \tag{6}$$

Set $c_1 = (1/2, 1/2, 0, ..., 0), c_2 = (1/2, -1/2, 0, ..., 0) \in \mathbf{R}^{n+1}$, then $\{c_1, c_2\}$ is a CSOPI, and every element x in $\mathbf{R}^{n+1}$ can be written as

$$x = k.(x_1 c_1 + x_2 c_2), k \in SO(n, \mathbf{R})$$

In the present case, $V_1 = V(c_1)$ is one dimensional, therefore

$$\Delta_{\mathbf{m}}(x) = \Delta_1^{m_1 - m_2}(x) \Delta_2^{m_2}(x) = (x|c_1)^{m_1 - m_2} . \Delta^{m_2}(x). \tag{7}$$

The following Carlson theorem will be used again and again. See [T].

Theorem 1.1. *If $f(z)$ is analytic and of the form $O(e^{k|z|})$, where $k < \pi$, for $Re(z) \geq 0$, and $f(z) = 0$ for $z = 1, 2, ...$, then $f(z) = 0$ identically on $Re(z) > 0$.*

Corollary 1.2. *If $g(z)$ is analytic on $Re(z) > 0$ and of the form $O(^{ARez+BImz})$, where $B < \pi$ and A is a real number, for $Re(z) \geq C > 0$, and $g(z) = 0$ for $z = 1, 2, ...,$ then $g(z) \equiv 0$ on $Re(z) > 0$.*

Proof. Pick $N \in \mathbf{Z}$ such that $N > C$. Let $f(z) = g(z+N)e^{-Az}$, then $f(z)$ satisfies all conditions in Theorem 1.1, therefore $f(z) \equiv 0$, for $Re(z) > 0$. Since $g(z)$ is analytic for $Re(z) > 0$, and $|e^{-Az}| > 0$, we have $g(z) \equiv 0$, for $Re(z) > 0$.

Proposition 1.3. (see [F]) *Let F be a continuous function on [-1, 1], a and b two fixed points on S^{n-1}, and $d\sigma$ the invariant measure under the action of $SO(n, \mathbf{R})$, with total measure 1, then*

$$\int_{SO(n,\mathbf{R})} F(< k.b, a >)dk =$$
$$\int_{S^{n-1}} F(< x, a >)d\sigma(x) = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^{1} F(t)(1-t^2)^{\frac{n-3}{2}}dt. \quad (8)$$

Proposition 1.4. *Let Ω be a symmetric cone, then we have*

$$\int_{\Omega} f(x)dx =$$
$$c_0 \int_0^{\infty} \cdots \int_0^{\infty} dt_1 \cdots dt_r \Pi(r, d; t) \int_K f(k. \sum_{j=1}^{r} t_j c_j)dk \quad (9)$$

where

$$\Pi(r, d; t) = \prod_{1 \leq i < j \leq r} |t_i - t_j|^d$$
$$c_0 = (2\pi)^{\frac{dr(r-1)}{4}} \prod_{j=i}^{r} \frac{\Gamma(d/2+1)}{\Gamma(jd/2+1)}. \quad (10)$$

As in the case of symmetric cones, one has the following result. (cf. [FK])

Proposition 1.5

(i) if $p \leq q$, then, the series (3) is convergent for all $y_1, \ldots, y_r$;

(ii) if $p = q + 1$, then, the series (3) is convergent for $y_1, \ldots, y_r$ with $|y_1|, \ldots, |y_r| < 1$;

(iii) if $p > q + 1$, then, the series (3) is convergent only at $y_1 = \ldots = y_r = 0$.

§2 Partial differential equations for hypergeometric functions

It is well known that the classical ${}_2f_1(a,b;c;z)$ function is the unique solution of the second order differential equation

$$z(1-z)\frac{d^2f}{dz^2}+[c-(a+b+1)z]\frac{df}{dz}=abf$$

subject to the conditions that

(a) f is analytic at 0

(b) $f(0)=1$.

For the hypergeometric functions of real matrix argument, a generalization of this classical result was given by Muirhead [M]. A more general result is the following (cf.[K])

Theorem 2.1. *${}_2F_1^{(d)}(a,b,c;y_1,\ldots,y_r)$ is the unique solution of the system of r partial differential equations*

$$y_i(1-y_i)\frac{\partial^2F}{\partial y_i^2}+\{c-\frac{d}{2}(r-1)-[a+b+1-\frac{d}{2}(r-1)]y_i+\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{y_i(1-y_i)}{y_i-y_j}\}\frac{\partial F}{\partial y_i}-\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{y_j(1-y_j)}{y_i-y_j}\frac{\partial F}{\partial y_j}=abF \qquad (1)$$

$i=1,\ldots,r$

subject to the conditions that

(a) F is a symmetric function of $y_1,\ldots,y_r$ and

(b) F is analytic at $y_1=\ldots=y_r=0$ and $F(0)=1$.

The remainder of this section is devoted to the proof of Theorem 2.1. Our proof follows closely that of Muirhead with some modification and clarification.

Let

$$\Delta_r = \sum_{i=1}^{r} y_i^2 \frac{\partial^2}{\partial y_i^2} + d \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i}, \tag{2}$$

$$\delta_r = \sum_{i=1}^{r} y_i \frac{\partial^2}{\partial y_i^2} + d \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \frac{y_i}{y_i - y_j} \frac{\partial}{\partial y_i}, \tag{3}$$

$$E_r = \sum_{i}^{r} y_i \frac{\partial}{\partial y_i}, \tag{4}$$

$$\varepsilon_r = \sum_{i}^{r} \frac{\partial}{\partial y_i}. \tag{5}$$

For simplicity, we denote $(y_1, \ldots, y_r)$ and $(1, ..., 1) \in \mathbf{R}^r$ by Y_r and I_r respectively.

We define the generalized binomial coefficients by

$$\frac{C_\kappa^{(d)}(I_r + Y_r)}{C_\kappa^{(d)}(I_r)} = \sum_{s=0}^{k} \sum_{\sigma, |\sigma| = s} \binom{\kappa}{\sigma}_r \frac{C_\sigma^{(d)}(Y_r)}{C_\sigma^{(d)}(I_r)} \tag{6}$$

where $k = |\kappa|, r \geq l(\kappa)$.

Remark. We note that the generalized binomial coefficients depend on r by the definition. But in the case of symmetric cones, one can show that they are independent of r. We expect such a result in the general case.

For a partition $\kappa = (k_1, \ldots, k_r)$ of $k, r \geq l(\kappa)$, let

$$\kappa_i^{(r)} = (k_1, ..., k_{i-1}, k_i + 1, k_{i+1}, ..., k_r),$$

$$\kappa_{(r)}^i = (k_1, ..., k_{i-1}, k_i - 1, k_{i+1}, ..., k_r)$$

whenever these are partitions of $k+1$ and $k-1$ respectively. Since we can also write κ as $(k_1, \ldots, k_r, 0)$, $\kappa_i^{(r)}$ depends on r. But when $r \geq l(\kappa)+1$, $\kappa_i^{(r)} = \kappa_i^{(l(\kappa)+1)}$, then, we simply write κ_i instead of $\kappa_i^{(r)}$. It is easy to see that $\kappa_{(r)}^i$ does not depend on r, thus, we omit the subscript r.

As a consequence of (vi) in §1, we have

Lemma 2.2.

$$\Delta_r C_\kappa^{(d)}(Y_r) = [\rho_\kappa + k(dr-1)]C_\kappa^{(d)}(Y_r).$$

The following two lemmas can be proved in the same way as in [Mu].

Lemma 2.3. *For κ with $l(\kappa) \leq r$,*

$$\delta_r \frac{C_\kappa^{(d)}(Y_r)}{C_\kappa^{(d)}(I_r)} = \sum_i \binom{\kappa}{\kappa^i}_r [k_i - 1 + \frac{d}{2}(r-i)] \frac{C_{\kappa^i}^{(d)}(Y_r)}{C_{\kappa^i}^{(d)}(I_r)}, \tag{7}$$

$$\varepsilon_r \frac{C_\kappa^{(d)}(Y_r)}{C_\kappa^{(d)}(I_r)} = \sum_i \binom{\kappa}{\kappa^i}_r \frac{C_{\kappa^i}^{(d)}(Y_r)}{C_{\kappa^i}^{(d)}(I_r)}. \tag{8}$$

Lemma 2.4. *For κ with $l(\kappa) \leq r$*

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r C_{\kappa_i^{(r)}}^{(d)}(I_r) = r(k+1)C_\kappa^{(d)}(I_r), \tag{9}$$

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r [k_i - \frac{d}{2}(i-1)] C_{\kappa_i^{(r)}}^{(d)}(I_r) = k(k+1)C_\kappa^{(d)}(I_r), \tag{10}$$

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r [k_i - \frac{d}{2}(i-1)]^2 C_{\kappa_i^{(r)}}^{(d)}(I_r)$$
$$= (k+1)[\rho_\kappa + \frac{d}{2}k(r+1)]C_\kappa^{(d)}(I_r). \tag{11}$$

Proposition 2.5. *The function ${}_2F_1^{(d)}(a, b; c; y_1, \ldots, y_r)$ satisfies the differential equation*

$$\delta_r F - \Delta_r F + [c - \frac{d}{2}(r-1)]\varepsilon_r F - [a + b + 1 - \frac{d}{2}(r-1)]E_r F = rabF. \quad (12)$$

Proof. Let

$$F(Y_r) = \sum_\kappa \alpha_\kappa C_\kappa^{(d)}(Y_r)$$

Substituting the series into (12), applying Lemma 2.3 and equating the coefficients of $C_\kappa^{(d)}(Y_r)$ on both sides, we can see that if for all κ, α_κ satisfy

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r [c + k_i - \frac{d}{2}(i-1)] C_{\kappa_i^{(r)}}^{(d)}(I_r)\alpha_{\kappa_i^{(r)}} =$$
$$[rab + k(a+b) + \rho_\kappa + \frac{d}{2}k(r+1)]C_\kappa^{(d)}(I_r)\alpha_\kappa, \quad (13)$$

then $F(Y_r)$ satisfies (12). Now, it suffices to show that

$$\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$$

is a solution of (13). We note that

$(a)_{\kappa_i^{(r)}} = (a)_\kappa [a + k_i - \frac{d}{2}(i-1)]$.

The problem is reduced to showing that

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r [a + k_i - \frac{d}{2}(i-1)][b + k_i - \frac{d}{2}(i-1)] C_{\kappa_i^{(r)}}^{(d)}(I_r)$$
$$= (k+1)[rab + \rho_\kappa + ka + kb + \frac{d}{2}k(r+1)]C_\kappa^{(d)}(I_r). \quad (14)$$

This is an immediate consequence of Lemma 2.4.

In the following, for simplicity, 1 stands for the partition $(1, 0, 0, ..., 0)$ in the subscripts when partitions are involved.

Lemma 2.7. *If κ is a partition of k, then, for all $r \geq l(\kappa)$, and $i = 1,..,r$,*

$$\frac{J_\kappa(I_{r+1})}{J_{\kappa^i}(I_{r+1})}\binom{\kappa}{\kappa^i}_{r+1} = \frac{J_\kappa(I_r)}{J_{\kappa^i}(I_r)}\binom{\kappa}{\kappa^i}_r + G^\kappa_{\kappa^i 1} \tag{15}$$

where $G^\kappa_{\sigma\tau} = g^\kappa_{\sigma\tau} j_\sigma^{-1} j_\tau^{-1}$ and $g^\kappa_{\sigma\tau} =< J_\sigma J_\tau, J_\kappa >$.

Proof. Let $X = (x_1, \ldots, x_r)$, by proposition 4.2 in [S], we have

$$\begin{aligned} &J_\kappa(x_1, \ldots, x_r, x_{r+1}) \\ &= \sum_\nu J_\nu(X; 2/d)(\sum_\alpha j_\alpha^{-1} g^\kappa_{\nu\alpha} J_\alpha(x_{r+1}; 2/d)) j_\nu^{-1} \\ &= J_\kappa(X; 2/d) + [\sum_i j_{\kappa^i}^{-1} j_1^{-1} g^\kappa_{\kappa^i 1} J_{\kappa^i}(X; 2/d)] x_{r+1} + P(X, x_{r+1}) x_{r+1}^2 \end{aligned}$$

where $P(X, x_{r+1})$ is a polynomial of $x_1, \ldots, x_r, x_{r+1}$.

Then, using (8) and §1 (ii), we have

$$\begin{aligned} &J_\kappa(I_{r+1}) \sum_i \binom{\kappa}{\kappa^i}_{r+1} \frac{J_{\kappa^i}(X_r)}{J_{\kappa^i}(I_{r+1})} \\ &= \varepsilon_{r+1} J_\kappa(X, x_{r+1}) \,|_{x_{r+1}=0} \\ &= J_\kappa(I_r) \sum_i \binom{\kappa}{\kappa^i}_r \frac{J_{\kappa^i}(X)}{J_{\kappa^i}(I_r)} + \sum_i G^\kappa_{\kappa^i 1} J_{\kappa^i}(X). \end{aligned}$$

Hence

$$\frac{J_\kappa(I_{r+1})}{J_{\kappa^i}(I_{r+1})}\binom{\kappa}{\kappa^i}_{r+1} = \frac{J_\kappa(I_r)}{J_{\kappa^i}(I_r)}\binom{\kappa}{\kappa^i}_r + G^\kappa_{\kappa^i 1}.$$

Lemma 2.8. *Suppose $l(\kappa) = n$, then*

$$\binom{\kappa}{\kappa^n}_r = G^\kappa_{\kappa^n 1}, \tag{16}$$

for all $r \geq n$.

Proof. Since $l(\kappa) = n, \kappa = (k_1, \ldots, k_n), k_n \geq 1$, by (14) and §1(iv), we only have to prove that

$$\left(\begin{matrix} \kappa \\ \kappa^n \end{matrix} \right)_n = G^{\kappa}_{\kappa^n 1}.$$

For a partition λ of length $\leq n$, let m_λ be the symmetric polynomial

$$m_\lambda(x_1, \ldots, x_n) = \sum x^\alpha,$$

the summation is over all distinct permutations α of $\lambda = (\lambda_1, ..., \lambda_n)$.

On the one hand, §1 (v) gives

$$J_\kappa(X_n + I_n) = \nu_{\kappa\kappa}((x^1 + 1)^{k_1} \cdots (x_n + 1)^{k_n} + \cdots) + \cdots$$

= terms of degree k + terms of degree $k - 1$ + terms of lower degree

= I + II + III.

In II, the term of highest weight is $k_n \nu_{\kappa\kappa} x_1^{k_1} \cdots x_n^{k_n - 1}$.

On the other hand, by definition and (iv) in §1

$$J_\kappa(X_n + I_n) = \sum_{s=0}^{k} \sum_{\sigma, |\sigma| = s} \left(\begin{matrix} \kappa \\ \sigma \end{matrix} \right)_n \frac{J_\kappa(I_n)}{J_\sigma(I_n)} J_\sigma(X_n)$$

and

$$\frac{J_\kappa(I_n)}{J_{\kappa^n}(I_n)} = 1 + (k_n - 1)\frac{2}{d}.$$

Equating the coefficients of $J_{\kappa^n}(X_n)$ and applying §1 (v), we get

$$k_n \frac{\nu_{\kappa\kappa}(2/d)}{\nu_{\kappa^n\kappa^n}(2/d)} = \left(\begin{matrix} \kappa \\ \kappa^n \end{matrix} \right)_n [1 + (k_n - 1)\frac{2}{d}].$$

Now it is enough to show that

$$k_n \frac{\nu_{\kappa\kappa}(2/d)}{\nu_{\kappa^n\kappa^n}(2/d)} = G^{\kappa}_{\kappa^n 1}[1 + (k_n - 1)\frac{2}{d}].$$

Theorem 6.1 in [S] gives

$$g^{\kappa}_{\kappa^n 1} = \frac{2}{d} \prod_{s\in\kappa^n} A_{\kappa\kappa^n}(s) \prod_{s\in\kappa} B_{\kappa\kappa^n}(s)$$

where $A_{\kappa\kappa^n}(s)$ and $B_{\kappa\kappa^n}(s)$ are defined as in [S].

A direct computation yields

$$\frac{g^{\kappa}_{\kappa^n 1}[1+(k_n-1)\frac{2}{d}]}{(2/d)\prod_{s\in\kappa^n} h^{*}_{\kappa^n}(s)} = k_n \prod_{s\in\kappa} h^{\kappa}_{*}(s).$$

Hence, by §1 (v), we have

$$\frac{g^{\kappa}_{\kappa^n 1}[1+(k_n-1)\frac{2}{d}]}{j_{\kappa^n} j_1} = k_n \frac{\prod_{s\in\kappa} h^{\kappa}_{*}(s)}{\prod_{s\in\kappa_n} h^{\kappa^n}_{*}(s)} = k_n \frac{\nu_{\kappa\kappa}(2/d)}{\nu_{\kappa^n\kappa^n}(2/d)}$$

finishing the proof.

Let $N(k)$ denote the number of partitions of k. When κ runs over all partitions of k, κ_i, $i=1,...,l(\kappa)+1$, run over all partitions of $(k+1)$. We note that $2N(k) \geq N(k+1)$. For $n \geq k+1$, let

$$H(n,\kappa,\kappa_i) = \frac{J_{\kappa_i}(I_n)}{J_\kappa(I_n)} \left(\begin{array}{c} \kappa_i \\ \kappa \end{array} \right)_n.$$

We consider the system of linear equations

$$\begin{aligned} \sum_i G^{\kappa_i}_{\kappa 1} x_{\kappa_i} &= a_\kappa, \\ H(n,\kappa,\kappa_i) x_{\kappa_i} &= b_\kappa \end{aligned} \tag{17}$$

where the x_λ are independent variables indexed by partitions of $k+1$, κ runs over all partitions of k, and a_κ, b_κ are given constants.

Lemma 2.9. *The $2N(k)\times N(k+1)$ matrix formed from the coefficients of the left hand side of (17) has rank $N(k+1)$.*

Proof. Let $\lambda(1) < \lambda(2) < \cdots < \lambda(N(k+1))$ and $x_j = x_{\lambda(j)}$, where $\lambda(j)$ is a partition of $k+1$. In the following, we want to produce a system of linear equations which is equivalent to (17) and whose coefficient matrix has the form

$$\begin{pmatrix} c_1 & * & * & \cdots & * \\ 0 & c_2 & * & \cdots & * \\ 0 & 0 & c_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N(k+1)} \\ * & * & * & \cdots & * \end{pmatrix}$$

with $c_j \neq 0, j = 1, ..., N(k+1)$.

For each $j, j = 1, ..., N(k+1)$, there are two possible cases for $\lambda(j)$.

Case 1:

$$\lambda(j) = (l_1, ..., l_{s-1}, 1, 0, ..., 0).$$

Set

$$\kappa = (l_1, ..., l_{s-1}, 0, ..., 0),$$

then

$$\kappa_s = (l_1, ..., l_{s-1}, 1, 0, ..., 0) = \lambda(j).$$

Since κ_i is not a partition for $i > s$, $\sum_i G_{\kappa 1}^{\kappa_i} x_{\kappa_i} = a_\kappa$ becomes

$$G_{\kappa 1}^{\kappa_s} x_{\kappa_s} + \sum_{i<s} G_{\kappa 1}^{\kappa_i} x_{\kappa_i} = a_\kappa. \tag{18}$$

Set $c_j = G_{\kappa 1}^{\kappa_s}$, then c_j is positive.

Now we can write (18) as

$$c_j x_j + \sum_{m>j} c_{mj} x_m = a_\kappa$$

by observing that $\lambda(j) = \kappa_s < \kappa_{s-1} < \cdots < \kappa_1$. So there is nothing to change, the j-th equation is already "triangular".

Case 2:

$$\lambda(j) = (l_1, ..., l_{s-1}, l_s, 0, ..., 0)$$

with $l_s \geq 2$. Let

$$\kappa = (l_1, ..., l_{s-1}, l_s - 1, 0, ..., 0),$$

then

$$\begin{aligned} \kappa_{s+1} &= (l_1, ..., l_{s-1}, l_s - 1, 1, 0, ..., 0) = \lambda(j-1), \\ \kappa_s &= (l_1, ..., l_{s-1}, l_s, 0, ..., 0) = \lambda(j). \end{aligned}$$

From (17), we have two equations

(a) $G_{\kappa 1}^{\kappa_{s+1}} x_{\kappa_{s+1}} + G_{\kappa 1}^{\kappa_s} x_{\kappa_s} + \sum_{i<s} G_{\kappa 1}^{\kappa_i} x_{\kappa_i} = a_\kappa$,

(b) $H(n, \kappa, \kappa_{s+1}) x_{\kappa_{s+1}} + H(n, \kappa, \kappa_s) x_{\kappa_s} + \sum_{i<s} H(n, \kappa, \kappa_i) x_{\kappa_i} = b_\kappa$.

By Lemma 2.8 and §1 (iv)

$$H(n, \kappa, \kappa_{s+1}) = \frac{J_{\kappa_{s+1}}(I_n)}{J_\kappa(I_n)} G_{\kappa 1}^{\kappa_{s+1}} = (n-s) G_{\kappa 1}^{\kappa_{s+1}},$$

$$H(n, \kappa, \kappa_s) = \frac{J_{\kappa_s}(I_n)}{J_\kappa(I_n)} G_{\kappa 1}^{\kappa_s} = [(n-s+1) + \frac{2}{d}(l_s - 1)] G_{\kappa 1}^{\kappa_s}.$$

In the system of equations formed by (a) and (b) we can equivalently replace (b) by the following equation

$$[1 + \frac{2}{d}(l_s - 1)] G_{\kappa 1}^{\kappa_s} x_{\kappa_s} + \sum_{i<s} [H(n, \kappa, \kappa_i) - (n-s) G_{\kappa 1}^{\kappa_i}] x_{\kappa_i} = b_\kappa - (n-s) a_\kappa.$$

Let $c_j = [1+\frac{2}{d}(l_s-1)]G_{\kappa 1}^{\kappa_s}$, then $c_j > 0$, we can write the above equation as

$$c_j x_j + \sum_{m>j} c_{mj} x_m = d_\kappa.$$

Thus we have proved the lemma.

Lemma 2.10. *If a sequence $\{A_\kappa\}$ indexed by all partitions satisfies*

$$\sum_i \binom{\kappa_i}{\kappa}_{k+1+r} \frac{J_{\kappa_i}(I_{k+1+r})}{J_\kappa(I_{k+1+r})} A_{\kappa_i} = \frac{d}{2(k+1)}[(k+1+r)ab + \rho_\kappa + k(a+b) + \frac{d}{2}k(k+r+2)]A_\kappa \quad (19)$$

for all positive integer $r \geq 2$, then $\{A_\kappa\}$ is uniquely determined by A_0.

Proof. Applying Lemma 2.7 , we have

$$[\sum_i \binom{\kappa_i}{\kappa}_{k+1} \frac{J_{\kappa_i}(I_{k+1})}{J_\kappa(I_{k+1})} + rG_{\kappa 1}^{\kappa_i}]A_{\kappa_i} = \frac{d}{2(k+1)}[(k+1+r)ab + \rho_\kappa + k(a+b) + \frac{d}{2}k(k+r+2)]A_\kappa \quad (20)$$

for all $r \geq 1$. Equating coefficients of r on both sides of (20) gives

$$\sum_i G_{\kappa 1}^{\kappa_i} A_{\kappa_i} = \frac{d}{2(k+1)}(ab + \frac{d}{2}k)A_\kappa \quad (21)$$

and equating constant terms gives

$$\sum_i \binom{\kappa_i}{\kappa}_{k+1} \frac{J_{\kappa_i}(I_{k+1})}{J_\kappa(I_{k+1})} A_{\kappa_i} = \frac{d}{2(k+1)}[(k+1)ab + \rho_\kappa + k(a+b) + \frac{d}{2}k(k+2)]A_\kappa. \quad (22)$$

By Lemma 2.9, we see that A_κ is uniquely determined by A_0.

Theorem 2.11. *There exists a unique sequence $\{\alpha_\kappa\}$ indexed by all partitions with $\alpha_0 = 1$ such that for $r = 2, 3, \ldots$*

$$F_r(y_1, \ldots, y_r) = \sum_\kappa \alpha_\kappa C_\kappa^{(d)}(y_1, \ldots, y_r) \tag{23}$$

satisfies

$$\delta_r F - \Delta_r F + [c - \frac{d}{2}(r-1)]\varepsilon_r F - [a+b+1-\frac{d}{2}(r-1)]E_r F = rabF. \tag{24}$$

Moreover, $\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$.

Remark. By §1 (i), we know that the summation in (23) is only over the partitions with $l(\kappa) \le r$.

Proof. Let $\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$, then Proposition 2.5 shows that for $r = 2, 3, \ldots$ $\sum_\kappa \alpha_\kappa C_\kappa^{(d)}(y_1, \ldots, y_r)$ satisfies (24).

Next, suppose that $\{\alpha_\kappa\}$ is such a sequence. From the proof of Proposition 2.5, we see that for all κ, all $r \ge l(\kappa) + 1$

$$\sum_i \binom{\kappa_i}{\kappa}_r [c + k_i - \frac{d}{2}(i-1)] C_{\kappa_i}^{(d)}(I_r)\alpha_{\kappa_i} =$$
$$[rab + k(a+b) + \rho_\kappa + \frac{d}{2}k(r+1)] C_\kappa^{(d)}(I_r)\alpha_\kappa.$$

Let $\alpha_\kappa = \frac{\beta_\kappa}{(c)_\kappa}$, then the above becomes

$$\sum_i \binom{\kappa_i}{\kappa}_r C_{\kappa_i}^{(d)}(I_r)\beta_{\kappa_i} =$$
$$[rab + k(a+b) + \rho_\kappa + \frac{d}{2}k(r+1)] C_\kappa^{(d)}(I_r)\beta_\kappa. \tag{25}$$

Since $C_\kappa^{(d)}(y_1, \ldots, y_r) = \left(\frac{2}{d}\right)^k k! J_\kappa(y_1, \ldots, y_r; 2/d) j_\kappa^{-1}$, we have

$$\sum_i \binom{\kappa_i}{\kappa}_r \frac{J_{\kappa_i}(I_r)}{J_\kappa(I_r)} \beta_{\kappa_i} j_{\kappa_i}^{-1} =$$

$$\frac{d}{2(k+1)}[rab+\rho_\kappa+k(a+b)+\frac{d}{2}k(r+1)]\beta_\kappa j_\kappa^{-1}. \tag{26}$$

By Lemma 2.10, $\beta_\kappa j_\kappa^{-1}$ is uniquely determined by $\beta_{(0)} j_{(0)}^{-1}$, therefore α_κ is uniquely determined by $\alpha_{(0)}$.

The following theorem can be proved in the same way as the case $d = 1$ in [Mu].

Theorem 2.12. *There exists a unique function F which satisfies the system of r partial differential equations*

$$y_i(1-y_i)\frac{\partial^2 F}{\partial y_i^2}+\{c-\frac{d}{2}(r-1)-[a+b+1-\frac{d}{2}(r-1)]y_i$$
$$+\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{y_i(1-y_i)}{y_i-y_j}\}\frac{\partial F}{\partial y_i}-\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{y_j(1-y_j)}{y_i-y_j}\frac{\partial F}{\partial y_j}=abF \tag{27}$$

$i = 1, ..., r$

subject to the conditions that

(a) F is a symmetric function of $y_1, \ldots, y_r$ and

(b) F is analytic at $y_1 = \ldots = y_r = 0$ and $F(0) = 1$.

Theorem 2.13. *There exists a unique sequence $\{A_\kappa\}$ with $A_{(0)} = 1$ such that $F_r(y_1, \ldots, y_r) = \sum_\kappa A_\kappa C_\kappa^{(d)}(y_1, \ldots, y_r)$ satisfies (27) for $r = 2, 3, \ldots$. Moreover, $\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$.*

Proof. If such a sequence $\{A_\kappa\}$ exists, then $\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$ since the sum of the r partial differential equations of (27) is (24).

Therefore, we only need to establish the existence of $\{A_\kappa\}$. By Theorem 2.12 , there exist F_n and F_{n+1} which are solutions of (27) subject to (a) and

(b) for $r = n$ and $r = n+1$ respectively. Then, we have

$$F_n(y_1, \ldots, y_n) = \sum_\kappa B_\kappa C_\kappa^{(d)}(y_1, \ldots, y_n), l(\kappa) \le n,$$

$$F_{n+1}(y_1, \ldots, y_{n+1}) = \sum_\kappa D_\kappa C_\kappa^{(d)}(y_1, \ldots, y_{n+1}), l(\kappa) \le n+1.$$

Now it is enough to show that $B_\kappa = D_\kappa$, if $l(\kappa) \le n$.

Let

$$G_n(y_1, \ldots, y_n) = F_{n+1}(y_1, \ldots, y_n, 0).$$

We note that

$$\frac{\partial F_{n+1}}{\partial y_i}(y_1, \ldots, y_n, 0) = \frac{\partial G_n}{\partial y_i}(y_1, \ldots, y_n), 1 \le i \le n$$

$$\frac{\partial^2 F_{n+1}}{\partial y_i^2} = \frac{\partial^2 G_n}{\partial y_i^2}(y_1, \ldots, y_n), 1 \le i \le n.$$

For $i = 1, ..., n$, we have

$$y_i(1-y_i)\frac{\partial^2 F_{n+1}}{\partial y_i^2} + \{c - \frac{d}{2}n - [a+b+1-\frac{d}{2}n]y_i + \frac{d}{2}\sum_{j=1,j\neq i}^{n} \frac{y_i(1-y_i)}{y_i - y_j} + \frac{d}{2}\frac{y_i(1-y_i)}{y_i - y_{n+1}}\}\frac{\partial F_{n+1}}{\partial y_i} - \frac{d}{2}\sum_{j=1,j\neq i}^{n} \frac{y_j(1-y_j)}{y_i - y_j}\frac{\partial F_{n+1}}{\partial y_j} - \frac{d}{2}\frac{y_{n+1}(1-y_{n+1})}{y_i - y_{n+1}}\frac{\partial F_{n+1}}{\partial y_{n+1}} = abF_{n+1}.$$

Suppose $y_j \neq 0, j = 1, ..., n$, let $y_{n+1} \to 0$. We have

$$y_i(1-y_i)\frac{\partial^2 F_{n+1}}{\partial y_i^2}(y_1, \ldots, y_n, 0) + \{c - \frac{d}{2}n - [a+b+1-\frac{d}{2}n]y_i + \frac{d}{2}\sum_{j=1,j\neq i}^{n} \frac{y_i(1-y_i)}{y_i - y_j} + \frac{d}{2}(1-y_i)\}\frac{\partial F_{n+1}}{\partial y_i}(y_1, \ldots, y_n, 0) - \frac{d}{2}\sum_{j=1,j\neq i}^{n} \frac{y_j(1-y_j)}{y_i - y_j}\frac{\partial F_{n+1}}{\partial y_j}(y_1, \ldots, y_n, 0) = abF_{n+1}(y_1, \ldots, y_n, 0). \quad (28)$$

This is true for all $y_i \neq y_j, i, j = 1, ..., n$. (28) says that $G_n(y_1, \ldots, y_n)$ is a solution of (27) for $r = n$ with $G_n(0, ..., 0) = 1$. By the uniqueness statement of Theorem 2.12, we have

$$G_n(y_1, \ldots, y_n) = F_n(y_1, \ldots, y_n).$$

So $B_\kappa = D_\kappa$ for all $\kappa, l(\kappa) \leq n$.

As a corollary of Theorem 2.13, we have Theorem 2.1 .

§3 The Polynomials $C_\kappa^{(d)}(y_1, y_2)$

In this section, we shall establish some properties of $C_\kappa^{(d)}(y_1, y_2)$ with $l(\kappa) \leq 2$.

Lemma 3.1. *If $P(y_1, y_2)$ is a symmetric and homogeneous polynomial with $d_\kappa y_1^{k_1} y_r^{k_2}$ as its term of the highest weight and an eigenfunction of Δ_2, then the eigenvalue is*

$$\sum_{i=1}^{2} k_i(k_i - di) + k(2d - 1).$$

Proof. The lemma follows from a direct calculation.

Corollary 3.2. *If $P(y_1, y_2)$ is as in the lemma, then it is uniquely determined by its term of the highest weight up to a constant multiple.*

Proposition 3.3.

$$C_\kappa^{(d)}(y_1, y_2) = C_\kappa^{(d)}(1,1) \frac{2^{k_2-k_1}}{\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)} (y_1 y_2)^{k_2} \cdot \int_{-1}^{1} [y_1 + y_2 + (y_1 - y_2)s]^{k_1-k_2} (1 - s^2)^{\frac{d-2}{2}} ds. \quad (1)$$

Proof. Let $\Phi_\kappa(y_1, y_2)$ be the spherical polynomial on the Lorentz cone Ω_{d+1}, then, we have

$$\begin{aligned}
\Phi_\kappa(y_1, y_2) &= \int_{SO(d+1)} \Delta_\kappa(k.(y_1 c_1 + y_2 c_2)) dk \\
&= \int_{SO(d+1)} \Delta_1^{k_1-k_2}(k.(y_1 c_1 + y_2 c_2)) \Delta_2^{k_2}(k.(y_1 c_1 + y_2 c_2)) dk \\
&= \int_{SO(d+1)} \Delta_1^{k_1-k_2}(k.(y_1 c_1 + y_2 c_2)) (y_1 y_2)^{k_2} dk \\
&= (y_1 y_2)^{k_2} \int_{SO(d+1)} \Delta_1^{k_1-k_2}(k.[\frac{1}{2}(y_1 + y_2) e_1 + \frac{1}{2}(y_1 - y_2) e_2]) dk
\end{aligned}$$

$$= (y_1y_2)^{k_2} \int_{SO(d+1)} \Delta_1^{k_1-k_2}(\frac{1}{2}(y_1+y_2)e_1 + \frac{1}{2}(y_1-y_2)k.e_2)dk$$
$$= (y_1y_2)^{k_2} \int_{SO(d+1)} \Delta_1^{k_1-k_2}(\frac{1}{2}(y_1+y_2)(c_1+c_2) + \frac{1}{2}(y_1-y_2)k.e_2)dk.$$

(1.7) gives

$$\Phi_\kappa(y_1,y_2)$$
$$= (y_1y_2)^{k_2} \int_{SO(d+1)} (\frac{1}{2}(y_1+y_2)(c_1+c_2) + \frac{1}{2}(y_1-y_2)k.e_2|c_1)^{k_1-k_2} dk$$
$$= (y_1y_2)^{k_2} 2^{k_2-k_1} \int_{SO(d+1)} [(y_1+y_2) + (y_1-y_2)\frac{1}{2}(k.e_2|(e_1+e_2))]^{k_1-k_2} dk$$
$$= (y_1y_2)^{k_2} 2^{k_2-k_1} \int_{SO(d+1)} [(y_1+y_2) + (y_1-y_2) < k.e_2, e_2 >]^{k_1-k_2} dk.$$

Proposition 1.3 in §1 implies that

$$\Phi_\kappa(y_1,y_2) = \frac{2^{k_2-k_1}}{\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)} (y_1y_2)^{k_2} \int_{-1}^{1} [y_1+y_2+(y_1-y_2)s]^{k_1-k_2}(1-s^2)^{\frac{d-2}{2}} ds \quad (2)$$

We denote the R.H.S. of (2) by $F_\kappa(y_1,y_2,d)$ and observe that $F_\kappa(y_1,y_2,d)$ is an analytic function of d on $Re(d) > 0$. It is known that $\Phi_\kappa(y_1,y_2)$ is an eigenfunction of Δ_2^d, hence, by lemma 3.1, it has the eigenvalue μ_κ. Therefore, we have

$$\Delta_2^d F_\kappa(y_1,y_2,d) = \mu_\kappa F_\kappa(y_1,y_2,d) \quad (3)$$

for $d = 1, 2, ...$

We consider both sides of (3) as analytic functions of d on $Re(d) > 0$. Applying Stirling's formula, we have, for $Re(d) \geq 1$,

$$|\Delta_2^d F_\kappa(y_1,y_2,d)| = O(e^{|d|})$$

$$|\mu_\kappa F_\kappa(y_1, y_2, d)| = O(e^{|d|}).$$

Hence, by Corollary 1.2, (3) is true for all $d > 0$. Since the term of the highest weight in $F_\kappa(y_1, y_2, d)$ is $d_\kappa y_1^{k_1} y_2^{k_2}$, by §1 (vi) and Corollary 3.2, we have

$$\frac{C_\kappa^{(d)}(y_1, y_2)}{C_\kappa^{(d)}(1,1)} = F_\kappa(y_1, y_2, d).$$

This finishes the proof.

Let Ω be a symmetric cone in a Euclidean space V. For $g \in G(\Omega)$, let $\Phi_\kappa(g) = \Phi_\kappa(g.e)$, then it is known that

$$\int_K \Phi_\kappa(g_1 k g_2) dk = \Phi_\kappa(g_1)\Phi_\kappa(g_2). \tag{4}$$

For $x \in V$, one defines

$$\begin{aligned} L(x) \quad &: \quad V \longrightarrow V, \\ L(x)y \quad &= \quad xy, \\ P(x) \quad &= \quad 2L(x)^2 - L(x^2). \end{aligned}$$

Proposition 3.4. *If $x \in \Omega, y \in V$, then*

$$\int_K \Phi_\kappa(P(x^{1/2})k.y) dk = \Phi_\kappa(x)\Phi_\kappa(y). \tag{5}$$

Proof. First, for $x, y \in \Omega$, (4) implies

$$\begin{aligned} \int_K \Phi_\kappa(P(x^{1/2})k.y) dk &= \int_K \Phi_\kappa(P(x^{1/2})k.P(y^{1/2}).e) dk \\ &= \int_K \Phi_\kappa(P(x^{1/2})k.P(y^{1/2})) dk \\ &= \Phi_\kappa(P(x^{1/2}))\Phi_\kappa(P(y^{1/2})) = \Phi_\kappa(x)\Phi_\kappa(y). \end{aligned}$$

Thus (5) is true for all $x \in \Omega, y \in \Omega$.

Secondly, both sides of (5) are polynomials of y, they are equal on the open set Ω of V, hence (5) are true for all $y \in V$. This finishes the proof.

Next, we shall generalize (5) for $r = 2$, and any positive d.

Lemma 3.5. *Let $y = y_1c_1 + y_2c_2 \in \mathbf{R}^{d+2}$, $x = x_1c_1 + x_2c_2 \in \Omega_{d+1}$, then, for $k \in SO(d+1)$, we have*

$$P(x^{1/2})k.y = \frac{1}{4}[\gamma_s(X,Y)e_1 + \alpha(X,Y)k.e_2 + \beta_s(X,Y)e_2]$$

where

$$\begin{aligned}
\alpha(X,Y) &= (y_1 - y_2)2\sqrt{x_1x_2} \\
\beta_s(X,Y) &= (y_1 + y_2)(x_1 - x_2) + (y_1 - y_2)(\sqrt{x_1} - \sqrt{x_2})^2 s \\
\gamma_s(X,Y) &= (y_1 + y_2)(x_1 + x_2) + (y_1 - y_2)(x_1 - x_2)s \\
s &= < e_2, k.e_2 > .
\end{aligned}$$

Proof.

$$k.y = \frac{1}{2}[(y_1 + y_2)e_1 + (y_1 - y_2)k.e_2].$$

Set $a = x^{1/2}$, i.e.,

$$a = a_1c_1 + a_2c_2, a_1 = \sqrt{x_1}, a_2 = \sqrt{x_2},$$

then,

$$\begin{aligned}
P(x^{1/2})k.y &= \frac{1}{2}[(y_1 + y_2)x + (y_1 - y_2)P(a)k.e_2] \\
&= \frac{1}{2}\{\frac{1}{2}(y_1 + y_2)[(x_1 + x_2)e_1 + (x_1 - x_2)e_2] + (y_1 - y_2)P(a)k.e_2]\}. \quad (6)
\end{aligned}$$

By (1.5) and a direct calculation, we have

$$a^2k.e_2 = \frac{1}{2}[(a_1^2 - a_2^2) < e_2, k.e_2 > e_1 + (a_1^2 + a_2^2)k.e_2], \tag{7}$$

$$2a \cdot (a \cdot k.e_2) = \frac{1}{2}\{[(a_1^2 - a_2^2) < e_2, k.e_2 > +(a_1^2 - a_2^2) < e_2, k.e_2 >]e_1 + (a_1 + a_2)^2k.e_2 + (a_1 - a_2)^2 < e_2, k.e_2 > e_2\}. \tag{8}$$

(7) and (8) give

$$P(a)k.e_2 = 2a \cdot (a \cdot k.e_2) - a^2k.e_2 = \frac{1}{2}[(a_1^2 - a_2^2) < e_2, k.e_2 > e_1 + 2a_1a_2k.e_2 + (a_1 - a_2)^2 < e_2, k.e_2 > e_2]. \tag{9}$$

(6) and (9) give

$$\begin{aligned} P(x^{1/2})k.y = \frac{1}{4}\{&[(y_1 + y_2)(x_1 + x_2)+ \\ &(y_1 - y_2)(x_1 - x_2) < e_2, k.e_2 >]e_1 + (y_1 - y_2)2\sqrt{x_1x_2}k.e_2 \\ &+[(y_1 + y_2)(x_1 - x_2) + (y_1 - y_2)(\sqrt{x_1} - \sqrt{x_2})^2 < e_2, k.e_2 >]e_2 \\ &= \frac{1}{4}[\gamma_s(X,Y)e_1 + \alpha(X,Y)k.e_2 + \beta_s(X,Y)e_2]. \end{aligned} \tag{10}$$

Lemma 3.6. *Suppose f is a function on $\mathbf{R}^{d+2}$, and $f(k.u) = f(u)$ for all $u \in \mathbf{R}^{d+1}$ and $k \in SO(d+1)$, then for $y = y_1c_1 + y_2c_2 \in \mathbf{R}^{d+2}, x = x_1c_1 + x_2c_2 \in \Omega_{d+1}$,*

$$f(P(x^{1/2})k.y) = f(\frac{1}{4}[(\gamma_s(X,Y) + l_s(X,Y))c_1 + (\gamma_s(X,Y) - l_s(X,Y))c_2])$$

where

$$l_s(X,Y) = \sqrt{\alpha(X,Y)^2 + \beta_s(X,Y)^2 + 2\alpha(X,Y)\beta_s(X,Y)s}$$

Proof. By Lemma 3.5,

$$P(x^{1/2})k.y = \frac{1}{4}[\gamma_s(X,Y)e_1 + \alpha(X,Y)k.e_2 + \beta_s(X,Y)e_2].$$

Since

$$< \alpha(X,Y)k.e_2 + \beta_s(X,Y)e_2, \alpha(X,Y)k.e_2 + \beta_s(X,Y)e_2 >=$$
$$\alpha(X,Y)^2 + \beta_s(X,Y)^2 + 2\alpha(X,Y)\beta_s(X,Y) < e_2, k.e_2 >,$$

we have

$$|\alpha(X,Y)k.e_2 + \beta_s(X,Y)e_2| = l_s(X,Y).$$

Therefore, there exists an element $\tilde{k} \in SO(d+1)$ such that

$$\begin{aligned}\tilde{k}P(x^{1/2})k.y &= \frac{1}{4}(\gamma_s(X,Y)e_1 + l_s(X,Y)e_2)\\ &= \frac{1}{4}[(\gamma_s(X,Y) + l_s(X,Y))c_1 + (\gamma_s(X,Y) - l_s(X,Y))c_2].\end{aligned}$$

Thus we have

$$\begin{aligned}f(P(x^{1/2})k.y) &= f(\tilde{k}P(x^{1/2})k.y)\\ &= f(\frac{1}{4}[(\gamma_s(X,Y) + l_s(X,Y))c_1 + (\gamma_s(X,Y) - l_s(X,Y))c_2]).\end{aligned}$$

Motivated by Lemma 3.5 and Lemma 3.6, we define the following mappings.

For $s \in [-1,1]$, we define

$$\begin{aligned}\gamma_s &: \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}\\ \gamma_s(X,Y) &= (y_1+y_2)(x_1+x_2) + (y_1-y_2)(x_1-x_2)s\\ l_s(X,Y) &: \mathbf{R}_+^2 \times \mathbf{R}^2 \to \mathbf{R}\\ l_s(X,Y) &= \sqrt{\alpha(X,Y)^2 + \beta_s(X,Y)^2 + 2\alpha(X,Y)\beta_s(X,Y)s}\end{aligned}$$

where

$$\alpha(X,Y) = (y_1 - y_2)2\sqrt{x_1x_2},$$
$$\beta_s(X,Y) = (y_1 + y_2)(x_1 - x_2) + (y_1 - y_2)(\sqrt{x_1} - \sqrt{x_2})^2 s.$$

$$Q_s : \mathbf{R}_+^2 \times \mathbf{R}^2 \to \mathbf{R}$$
$$Q_s(X,Y) = (\frac{1}{4}[\gamma_s(X,Y) + l_s(X,Y)], \frac{1}{4}[\gamma_s(X,Y) - l_s(X,Y)]).$$

For any positive d, we still denote $\frac{C_\kappa^{(d)}(y_1,y_2)}{C_\kappa^{(d)}(1,1)}$ by $\Phi_\kappa^{(d)}(y_1,y_2)$.

As a consequence of Proposition 3.4, Lemma 3.6, Corollary 1.2 and (1.8), we have an analogue of (5) for $\Phi_\kappa^{(d)}(y_1,y_2)$ with any positive number d.

Proposition 3.7. *For all* $d > 0, X \in \mathbf{R}_+^2, Y \in \mathbf{R}^2$,

$$\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)} \int_{-1}^{1} \Phi_\kappa^{(d)}(\frac{1}{4}[\gamma_s(X,Y) + l_s(X,Y)], \frac{1}{4}[\gamma_s(X,Y) - l_s(X,Y)]) \cdot (1 - s^2)^{\frac{d-2}{2}} ds = \Phi_\kappa^{(d)}(X)\Phi_\kappa^{(d)}(Y).$$

§4 Integral Representations of Generalized Hypergeometric Functions

In this section, we shall establish some properties of generalized hypergeometric functions and their integral representations.

Two special cases of the hypergeometric functions are given in the next proposition.

Proposition 4.1. *We have*

$$ {}_0F_0^{(d)}(y_1, \ldots, y_r) = e^{y_1+\ldots+y_r}, \tag{1}$$

$$ {}_1F_0^{(d)}(a; y_1, \ldots, y_r) = \prod_{i=1}^{r}(1-y_i)^{-a}. \tag{2}$$

Proof. (1) follows from the definition and (iii) in §1.

Let $b = c = 1 + \frac{d}{2}(r-1)$ in (2.1). Since both ${}_1F_0^{(d)}(a; y_1, \ldots, y_r)$ and $\prod_{i=1}^{r}(1-y_i)^{-a}$ satisfy (2.1), (2) follows from the uniqueness of the solution of (2.1).

Similarly we can establish analogues of the classical Kummer relations.

Proposition 4.2. *We have*

$$\begin{aligned} & {}_2F_1^{(d)}(a, b; c; y_1, \ldots, y_r) \\ = & \prod_{i=1}^{r}(1-y_i)^{-a}{}_2F_1^{(d)}(a, c-b; c; -\frac{y_1}{1-y_1}, \ldots, -\frac{y_r}{1-y_r}) \qquad (3) \\ = & \prod_{i=1}^{r}(1-y_i)^{c-a-b}{}_2F_1^{(d)}(c-a, c-b; c; y_1, \ldots, y_r). \qquad (4) \end{aligned}$$

The remainder of this section is to establish integral representations for the generalized hypergeometric functions.

For $a_1, ..., a_p, b_1, ..., b_q \in \mathbf{C}$, such that $(b_j)_\kappa \neq 0$ for all κ, j, we define

$$\begin{aligned} & {}_p\mathcal{F}_q^{(d)}(a_1, ..., a_p; b_1, ..., b_q; x_1, \ldots, x_r | y_1, \ldots, y_r) = \\ & \sum_\kappa \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa^{(d)}(x_1, \ldots, x_r)}{k!} \frac{C_\kappa^{(d)}(y_1, \ldots, y_r)}{C_\kappa^{(d)}(1, ..., 1)}. \end{aligned} \tag{5}$$

Remark. When $r = 1$, ${}_p\mathcal{F}_q^{(d)}(a_1, ..., a_p; b_1, ..., b_q; x|y)$ becomes the classical hypergeometric function ${}_pf_q(a_1, ..., a_p; b_1, ..., b_q; xy)$, in particular, ${}_0\mathcal{F}_0^{(d)}(x|y) = e^{xy}$ and ${}_1\mathcal{F}_0^{(d)}(a; x|y) = (1 - xy)^{-a}$.

In the following, we simply denote $\prod_{1 \leq i < j \leq r} |x_i - x_j|^d dx_1 \ldots dx_r$ by $dV(X, d, r)$.

The following conjecture of Macdonald has been proved in [KD].

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 J_\kappa(X; 2/d) \prod_{i=1}^r x_i^{a-1} \prod_{i=1}^r (1 - x_i)^{b-1} dV(X, d, r) \\ = \; & J_\kappa(I_r; 2/d) \prod_{i=1}^r \frac{\Gamma(k_i + a + \frac{d}{2}(r - i))\Gamma(b + \frac{d}{2}(r - i))\Gamma(\frac{d}{2}i + 1)}{\Gamma(k_i + a + b + \frac{d}{2}(2r - i - 1))\Gamma(\frac{d}{2} + 1)}. \end{aligned} \tag{6}$$

We define, for every $\mathbf{s} = (s_1, \ldots, s_r)$,

$$\Gamma_d(\mathbf{s}) = (2\pi)^{\frac{r(r-1)}{4}d} \prod_{i=1}^r \Gamma(s_i - (i - 1)\frac{d}{2}). \tag{7}$$

For $\mathbf{s} = (s, ..., s)$, we write $\Gamma_d(s)$ instead of $\Gamma((s, ..., s))$. We also define

$$c_0 = (2\pi)^{\frac{r(r-1)}{4}d} \prod_{i=1}^r \frac{\Gamma(\frac{d}{2} + 1)}{\Gamma(i\frac{d}{2})}, \tag{8}$$

$$q_0 = 1 + \frac{d}{2}(r - 1). \tag{9}$$

Proposition 4.3. *If $p \leq q+1$, we have, for $a_{p+1} > \frac{d}{2}(r-1), b_{q+1} - a_{p+1} >$*

$\frac{d}{2}(r-1)$,

$$ {}_{p+1}F^{(d)}_{q+1}(a_1,\ldots,a_{p+1};b_1,\ldots,b_{q+1};Y) = c_0\frac{\Gamma_d(b_{q+1})}{\Gamma_d(a_{p+1})\Gamma_d(b_{q+1}-a_{p+1})} \cdot\int_0^1\cdots\int_0^1 {}_p\mathcal{F}^{(d)}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;X|Y) \cdot\prod_{i=1}^r x_i^{a_{p+1}-q_0}\prod_{i=1}^r(1-x_i)^{b_{q+1}-a_{p+1}-q_0}\prod_{1\le i<j\le r}|x_i-x_j|^d dx_1\ldots dx_r. \quad (10) $$

Proof. (6) implies that the integral on the right side in (10) is equal to

$$ \sum_\kappa \frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\frac{C^{(d)}_\kappa(Y)}{k!}\cdot \int_0^1\cdots\int_0^1\frac{C^{(d)}_\kappa(X)}{C^{(d)}_\kappa(I)}\prod_{i=1}^r x_i^{a_{p+1}-q_0}\prod_{i=1}^r(1-x_i)^{b_{q+1}-a_{p+1}-q_0}dV(X,d,r) $$

$$ = \sum_\kappa \frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\frac{C^{(d)}_\kappa(Y)}{k!}\cdot \prod_{i=1}^r\frac{\Gamma(k_i+a_{p+1}-\frac{d}{2}(r-1)+\frac{d}{2}(r-i))\Gamma(b_{q+1}-a_{p+1}-\frac{d}{2}(r-1)+\frac{d}{2}(r-i))\Gamma(\frac{d}{2}i+1)}{\Gamma(k_i+b_{q+1}-(r-1)d+\frac{d}{2}(2r-i-1))\Gamma(\frac{d}{2}+1)} $$

$$ = \sum_\kappa \frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\frac{C^{(d)}_\kappa(Y)}{k!} \cdot\prod_{i=1}^r\frac{\Gamma(k_i+a_{p+1}-\frac{d}{2}(i-1))\Gamma(b_{q+1}-a_{p+1}-\frac{d}{2}(i-1))\Gamma(\frac{d}{2}i+1)}{\Gamma(k_i+b_{q+1}-\frac{d}{2}(i-1))\Gamma(\frac{d}{2}+1)}. $$

From (7) and (8) we have

$$ \int_0^1\cdots\int_0^1 {}_p\mathcal{F}^{(d)}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;X|Y) \prod_{i=1}^r x_i^{a_{p+1}-q_0}\prod_{i=1}^r(1-x_i)^{b_{q+1}-a_{p+1}-q_0}dV(X,d,r) $$

$$ = \prod_{i=1}^r\frac{\Gamma(a_{p+1}-\frac{d}{2}(i-1))\Gamma(b_{q+1}-a_{p+1}-\frac{d}{2}(i-1))\Gamma(\frac{d}{2}i+1)}{\Gamma(b_{q+1}-\frac{d}{2}(i-1))\Gamma(\frac{d}{2}+1)} \cdot {}_{p+1}F^{(d)}_{q+1}(a_1,\ldots,a_{p+1};b_1,\ldots,b_{q+1};y_1,\ldots,y_r) $$

$$ = \frac{1}{c_0}\frac{\Gamma_d(a_{p+1})\Gamma_d(b_{q+1}-a_{p+1})}{\Gamma_d(b_{q+1})}{}_{p+1}F^{(d)}_{q+1}(a_1,\ldots,a_{p+1};b_1,\ldots,b_{q+1};y_1,\ldots,y_r). $$

In the classical case, there are the following well-known Euler integrals for ${}_1f_1$ and ${}_2f_1$

$$ {}_1f_1(a;b;y) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xy} x^{a-1}(1-x)^{b-a-1} dx $$

for $a > 0, b - a > 0$,

$$ {}_2f_1(a,b;c;y) = \frac{\Gamma(c)}{\Gamma(a)|\Gamma(c-a)} \int_0^1 (1-xy)^{-b} x^{a-1}(1-x)^{c-a-1} dx $$

for $a > 0, c - a > 0$.

As remarked after (5), two special cases of Proposition 4.3 give the generalizations of Euler integrals

$$ {}_1F_1^{(d)}(a;b;y_1,\ldots,y_r) = c_0 \frac{\Gamma_d(b)}{\Gamma_d(a)\Gamma_d(b-a)} \cdot \int_0^1 \cdots \int_0^1 {}_0\mathcal{F}_0^{(d)}(X|Y) \prod_{i=1}^r x_i^{a-q_0} \prod_{i=1}^r (1-x_i)^{b-a-q_0} dV(X,d,r) \quad (11) $$

if $a > \frac{d}{2}(r-1), b - a > \frac{d}{2}(r-1)$.

$$ {}_2F_1^{(d)}(a,b;c;y_1,\ldots,y_r) = c_0 \frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)} \cdot \int_0^1 \cdots \int_0^1 {}_1\mathcal{F}_0^{(d)}(b;X|Y) \prod_{i=1}^r x_i^{a-q_0} \prod_{i=1}^r (1-x_i)^{c-a-q_0} dV(X,d,r) \quad (12) $$

if $a > \frac{d}{2}(r-1), c - a > \frac{d}{2}(r-1)$.

Now we proceed to express ${}_0\mathcal{F}_0^{(d)}(X|Y)$ and ${}_1\mathcal{F}_0^{(d)}(a;X|Y)$ in terms of the classical hypergeometric functions ${}_1f_1$ and ${}_2f_1$, when $r = 2$.

An easy calculation yields

Lemma 4.4. *If* $0 < x_1, x_2 < 1, 0 < y_1, y_2 < 1$, *then*

$1/4|\gamma_s(X,Y)+l_s(X,Y)|<1, 1/4|\gamma_s(X,Y)-l_s(X,Y)|<1.$

From Proposition 3.7, we have

$$\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})}\int_{-1}^{1} C_\kappa^{(d)}(\frac{1}{4}[\gamma_s(X,Y)+l_s(X,Y)], \frac{1}{4}[\gamma_s(X,Y)-l_s(X,Y)])$$
$$\cdot(1-s^2)^{d/2-1}ds = C_\kappa^{(d)}(X)\frac{C_\kappa^{(d)}(Y)}{C_\kappa^{(d)}(I)} \quad (13)$$

for all $d>0, X\in \mathbf{R}_+^2, Y\in \mathbf{R}^2$.

Therefore, by (1) and (13), we have

$$\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1} e^{1/2\gamma_s(X,Y)}(1-s^2)^{d/2-1}ds = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}$$
$$\cdot\int_{-1}^{1} e^{\frac{1}{4}[\gamma_s(X,Y)+l_s(X,Y)]+\frac{1}{4}[\gamma_s(X,Y)-l_s(X,Y)]}(1-s^2)^{d/2-1}ds$$
$$=\sum_\kappa \frac{1}{k!}\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}$$
$$\cdot\int_{-1}^{1} C_\kappa^{(d)}(\frac{1}{4}[\gamma_s(X,Y)+l_s(X,Y)], \frac{1}{4}[\gamma_s(X,Y)-l_s(X,Y)])(1-s^2)^{d/2-1}ds$$
$$=\sum_\kappa \frac{1}{k!}C_\kappa^{(d)}(X)\frac{C_\kappa^{(d)}(Y)}{C_\kappa^{(d)}(I)} = {}_0\mathcal{F}_0^{(d)}(X|Y).$$

Let $s=1-2t$, then

$$\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1} e^{1/2\gamma_s(X,Y)}(1-s^2)^{d/2-1}ds = e^{x_1y_1+x_2y_2}$$
$$\cdot 2^{d-1}\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_0^1 e^{-(x_1-x_2)(y_1-y_2)t}t^{d/2-1}(1-t)^{d/2-1}dt$$
$$= e^{x_1y_1+x_2y_2}{}_1f_1(d/2;d;-(x_1-x_2)(y_1-y_2)).$$

The last equality follows from the classical integral representation for ${}_1f_1$ and the fact that $\Gamma(d)=\frac{2^{d-1}}{\sqrt{\pi}}\Gamma(\frac{d+1}{2})\Gamma(d/2)$. Thus we have proved the following

Proposition 4.5.

$$
\begin{aligned}
{}_0\mathcal{F}_0^{(d)}(X|Y) &= e^{x_1y_1+x_2y_2}{}_1f_1(d/2;d;-(x_1-x_2)(y_1-y_2)) \\
&= \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1} e^{1/2\gamma_s(X,Y)}(1-s^2)^{d/2-1}ds \qquad (14)
\end{aligned}
$$

The following two lemmas follow from direct computations.

Lemma 4.6.

$$
\begin{aligned}
&[1-1/4(\gamma_s(X,Y)+l_s(X,Y))][1-1/4(\gamma_s(X,Y)-l_s(X,Y))] = \\
&1-\frac{1}{2}(x_1+x_2)(y_1+y_2)+x_1x_2y_1y_2-\frac{1}{2}(x_1-x_2)(y_1-y_2)s.
\end{aligned}
$$

Lemma 4.7. *If* $0 \le x_1, x_2 \le 1, -\infty < y_1 < 1, -\infty < y_2 < 1,$ *then*

$$
-\frac{(x_1-x_2)(y_1-y_2)}{(1-x_1x_2)(1-y_1y_2)} < 1.
$$

Proposition 4.8.

$$
\begin{aligned}
{}_1\mathcal{F}_0^{(d)}(b;X|Y) &= \prod_{i=1}^{2}(1-x_iy_i)_2^{-b}f_1\Big(d/2,b;d-\frac{(x_1-x_2)(y_1-y_2)}{\prod_{i=1}^{2}(1-x_iy_i)}\Big) \\
&= \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1}[1-1/4(\gamma_s(X,Y)+l_s(X,Y))]^{-b} \\
&\cdot[1-1/4(\gamma_s(X,Y)-l_s(X,Y))]^{-b}(1-s^2)^{d/2-1}ds. \qquad (15)
\end{aligned}
$$

Proof. By Lemma 4.6, (2) and (13), we have

$$
\begin{aligned}
&\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1}[1-\frac{1}{2}(x_1+x_2)(y_1+y_2)+x_1x_2y_1y_2 \\
&\quad -\frac{1}{2}(x_1-x_2)(y_1-y_2)s]^{-b}(1-s^2)^{d/2-1}ds \\
&= \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1}[1-1/4(\gamma_s(X,Y)+l_s(X,Y))]^{-b}
\end{aligned}
$$

$$\cdot[1-1/4(\gamma_s(X,Y)-l_s(X,Y))]^{-b}(1-s^2)^{d/2-1}ds$$

$$=\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1}{}_1F_0^{(d)}(b;1/4(\gamma_s(X,Y)+l_s(X,Y)),$$

$$1/4(\gamma_s(X,Y)-l_s(X,Y))(1-s^2)^{d/2-1}ds$$

$$=\sum_{\kappa}\frac{(b)_\kappa}{k!}\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1}C_\kappa^{(d)}(1/4(\gamma_s(X,Y)+l_s(X,Y)),$$

$$\cdot\ 1/4(\gamma_s(X,Y)-l_s(X,Y))(1-s^2)^{d/2-1}ds$$

$$=\sum_{\kappa}(b)_\kappa\frac{C_\kappa^{(d)}(X)}{k!}\frac{C_\kappa^{(d)}(Y)}{C_\kappa^{(d)}(I)}={}_1\mathcal{F}_0^{(d)}(b;X|Y).$$

Let $s=1-2t$, the change of variable and the Euler integral for ${}_2f_1$ give that

$$\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)}\int_{-1}^{1}[1-\frac{1}{2}(x_1+x_2)(y_1+y_2)$$

$$+x_1x_2y_1y_2-\frac{1}{2}(x_1-x_2)(y_1-y_2)s]^{-b}(1-s^2)^{d/2-1}ds$$

$$=\prod_{i=1}^{2}(1-x_iy_i)^{-b}{}_2f_1(d/2,b;d;-\frac{(x_1-x_2)(y_1-y_2)}{\prod_{i=1}^{2}(1-x_iy_i)}).$$

(11), (12), Proposition 4.5 and Proposition 4.8 give the following proposition

Proposition 4.9.

$${}_1F_1^{(d)}(a;c;y_1,y_2)=c_0\frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)}\int_0^1\int_0^1 e^{x_1y_1+x_2y_2}\prod_{i=1}^{2}x_i^{a-(1+d/2)}$$

$$\cdot\prod_{i=1}^{2}(1-x_i)^{c-a-(1+d/2)}|x_1-x_2|^d{}_1f_1(d/2;d;-(x_1-x_2)(y_1-y_2))dx_1dx_2 \quad (16)$$

if $a>\frac{d}{2}, b-a>\frac{d}{2}$.

$${}_2F_1^{(d)}(a,b;c;y_1,y_2)=c_0\frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)}\int_0^1\int_0^1\prod_{i=1}^{2}(1-y_ix_i)^{-b}$$

$$\cdot \prod_{i=1}^{2} x_i^{a-(1+d/2)} \prod_{i=1}^{2} (1-x_i)^{c-a-(1+d/2)} |x_1 - x_2|^d$$

$$\cdot {}_2f_1(d/2, b; d; -\frac{(x_1-x_2)(y_1-y_2)}{\prod_{i=1}^{2}(1-x_iy_i)}) dx_1 dx_2 \qquad (17)$$

if $a > \frac{d}{2}, c-a > \frac{d}{2}$.

Finally, we have

Proposition 4.10.

$${}_1\mathcal{F}_0^{(d)}(\frac{rd}{2}; x_1, \ldots, x_r | y_1, \ldots, y_r) = \prod_{i,j=1}^{r} (1 - x_iy_j)^{-d/2}.$$

Proof. On the one hand, by Proposition 2.1 in [S], we have

$$\prod_{i,j=1}^{r} (1 - x_iy_j)^{-d/2} = \sum_{\kappa} J_\kappa(X; 2/d) J_\kappa(Y; 2/d) j_\kappa^{-1}.$$

On the other hand, by §1 (iv), we have

$$J_\kappa(I_r; 2/d) = (2/d)^k (\frac{rd}{2})_\kappa.$$

Hence, by the definitions, we have

$$\begin{aligned} &{}_1\mathcal{F}_0^{(d)}(\frac{rd}{2}; x_1, \ldots, x_r | y_1, \ldots, y_r) \\ &= \sum_{\kappa} (\frac{rd}{2}) \frac{(2/d)^k}{J_\kappa(I_r; 2/d)} J_\kappa(X; 2/d) J_\kappa(Y; 2/d) j_\kappa^{-1} \\ &= \prod_{i,j=1}^{r} (1 - x_iy_j)^{-d/2}. \end{aligned}$$

Corollary 4.11.

$${}_2F_1^{(d)}(a, \frac{rd}{2}; c; y_1, \ldots, y_r) = c_0 \frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)}$$

$$\cdot \int_0^1 \cdots \int_0^1 \prod_{i,j=1}^{r} (1 - x_iy_j)^{-d/2} \prod_{i=1}^{r} x_i^{a-q_0} \prod_{i=1}^{r} (1-x_i)^{c-a-q_0} dV(X, d, r)$$

if $a > \frac{d}{2}(r-1), c-a > \frac{d}{2}(r-1)$.

§5 Generalized Laplace Transform

Suppose that $f(x_1, x_2)$ is defined on $\mathbf{R}_+^2$, the generalized Laplace transform of f is defined by

$$\mathcal{L}(f)(y_1, y_2) =$$
$$c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0^{(d)}(-x_1, -x_2|y_1, y_2) f(x_1, x_2)|x_1 - x_2|^d dx_1 dx_2. \quad (1)$$

By Proposition 4.5 , we also have

$$\mathcal{L}(f)(y_1, y_2) = c_0 \int_0^\infty \int_0^\infty e^{-(x_1y_1 + x_2y_2)}$$
$$\cdot f(x_1, x_2) {}_1f_1(d/2; d; (x_1 - x_2)(y_1 - y_2))|x_1 - x_2|^d dx_1 dx_2. \quad (2)$$

Let S_L denote the space of all functions defined on $\mathbf{R}_+^2$ such that $f(x_1, x_2) = f(x_2, x_1)$ and $\mathcal{L}(|f|)(y_1, y_2) < \infty$.

As in the classical case, we have

Proposition 5.1. *If* $f(x_1, x_2) \in S_L$, *and* $\mathcal{L}(f) \equiv 0$, *then* $f \equiv 0$ *a.e.*

To prove the proposition, we need some lemmas. For simplicity, we write $\int_{-1}^1 f(s)dm(s)$ for $\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(d/2)} \int_{-1}^1 f(s)(1 - s^2)^{d/2-1}$.

Lemma 5.2.

$${}_0\mathcal{F}_0^{(d)}(1 + x_1, 1 + x_2|y_1, y_2) = e^{y_1 + y_2} {}_0\mathcal{F}_0^{(d)}(x_1, x_2|y_1, y_2). \quad (3)$$

Proof. By (4.14), we have

$${}_0\mathcal{F}_0^{(d)}(X|Y) = \int_{-1}^1 e^{1/2\gamma_s(Y,X)} dm(s).$$

So

$$ {}_0\mathcal{F}_0^{(d)}(1+x_1, 1+x_2|y_1, y_2) = \int_{-1}^{1} e^{1/2\gamma_s(Y,I+X)} dm(s) = $$

$$ \int_{-1}^{1} e^{1/2\gamma_s(Y,I)+1/2\gamma_s(Y,X)} dm(s) = e^{y_1+y_2} {}_0\mathcal{F}_0^{(d)}(x_1, x_2|y_1, y_2). $$

Lemma 5.3.

$$ C_\kappa^{(d)}(y_1, y_2) e^{y_1+y_2} = k! \sum_{s=k}^{\infty} \sum_{\sigma, |\sigma|=s} \binom{\sigma}{\kappa} \frac{C_\sigma^{(d)}(Y)}{s!}. \tag{4} $$

Proof. By Lemma 5.2, we have

$$ \begin{aligned} \sum_\kappa (e^{y_1+y_2} C_\kappa^{(d)}(Y)) \frac{C_\kappa^{(d)}(X)}{k! C_\kappa^{(d)}(I)} &= e^{y_1+y_2} {}_0\mathcal{F}_0^{(d)}(X|Y) \\ &= {}_0 \mathcal{F}_0^{(d)}(I+X|Y) \\ &= \sum_\sigma \frac{C_\sigma^{(d)}(X+I) C_\sigma^{(d)}(Y)}{C_\sigma^{(d)}(I)|\sigma|!} \\ &= \sum_\sigma \sum_{k=0}^{|\sigma|} \sum_{\kappa, |\kappa|=k} \binom{\sigma}{\kappa} \frac{C_\kappa^{(d)}(X) C_\sigma^{(d)}(Y)}{C_\kappa^{(d)}(I)|\sigma|!} \\ &= \sum_\kappa [k! \sum_{s=k}^{\infty} \sum_{\sigma, |\sigma|=s} \binom{\sigma}{\kappa} \frac{C_\sigma^{(d)}(Y)}{s!}] \frac{C_\kappa^{(d)}(X)}{k! C_\kappa^{(d)}(I)}. \end{aligned} $$

Equating the coefficients of $C_\kappa^{(d)}(X)$, we obtain the result.

Let $\mathcal{P}$ be the vector space consisting of all symmetric polynomials in $y_1, \ldots, y_r$, and $\mathcal{P}_s = \{ P \in \mathcal{P} | P$ is of degree $s \}$

Set

$$ \Phi_\sigma^d(y_1, \ldots, y_r) = \frac{C_\sigma^{(d)}(Y)}{C_\sigma^{(d)}(I)}. \tag{5} $$

Lemma 5.4. *For each partition σ, there exists a polynomial P_σ of degree $|\sigma|$ such that*

$$ P_\sigma\left(\frac{\partial}{\partial y}\right) \Phi_\sigma^d(Y) \,|_{Y=0} = 1 $$

$$P_\sigma(\frac{\partial}{\partial y})\Phi^d_\kappa(Y)\mid_{Y=0}=0$$

if $\kappa \neq \sigma$.

Proof. For $P, Q \in \mathcal{P}, < P, Q >= P(\frac{\partial}{\partial y})Q(Y)\mid_{Y=0}$ is an inner product on $\mathcal{P}$.

Let $s = |\sigma|$ and $\mathcal{P}^-_s$ be the subspace of $\mathcal{P}_s$ spanned by $\Phi^d_\kappa, \kappa \neq \sigma, |\kappa| = s$, then $dim\mathcal{P}^-_s = dim\mathcal{P}_s - 1$ and $\Phi_\sigma \notin \mathcal{P}^-_s$.

So there exists a $P_\sigma \perp \mathcal{P}^-_s$ under $<,>$ and $< P_\sigma, \Phi^d_\sigma >= 1$. P_σ is the polynomial we are seeking.

Lemma 5.5. *Let* P_σ *be the polynomial given by lemma 5.4. , then*

$$P_\sigma(\frac{\partial}{\partial y})\Phi^d_\kappa(Y)\mid_{Y=I}=\binom{\kappa}{\sigma}$$

for κ *with* $|\kappa| \geq |\sigma|$.

Proof.

$$P_\sigma(\frac{\partial}{\partial y})\Phi^d_\kappa(Y)\mid_{Y=I}=P_\sigma(\frac{\partial}{\partial x})\Phi^d_\kappa(I+X)\mid_{X=0}$$

$$=P_\sigma(\frac{\partial}{\partial x})[\sum_{s=0}^{k}\sum_{\alpha,|\alpha|=s}\binom{\kappa}{\alpha}\Phi^d_\alpha(X)]$$

$$=\sum_{s=0}^{k}\sum_{\alpha,|\alpha|=s}\binom{\kappa}{\alpha}P_\sigma(\frac{\partial}{\partial x})\Phi^d_\alpha(X)\mid_{X=0}=\binom{\kappa}{\sigma}.$$

Lemma 5.6.

$$P_\sigma(\frac{\partial}{\partial y})_0\mathcal{F}^d_0(-X|Y)\mid_{Y=I}=(-1)^{|\sigma|}\frac{C^{(d)}_\sigma(X)}{|\sigma|!}e^{-(x_1+x_2)}. \qquad (6)$$

Proof. Lemma 5.3 and Lemma 5.6 give that

$$P_\sigma(\frac{\partial}{\partial y})_0\mathcal{F}^d_0(-X|Y)\mid_{Y=I}$$

$$= \sum_{\kappa, |\kappa| \geq |\sigma|} \binom{\kappa}{\sigma} \frac{C_\kappa^{(d)}(-X)}{k!} = (-1)^{|\sigma|} \frac{C_\sigma^{(d)}(X)}{|\sigma|!} e^{-(x_1+x_2)}.$$

Lemma 5.7. *If $f(x_1, x_2) = f(x_2, x_1)$ and for all partition κ,*

$$\int_0^\infty \int_0^\infty e^{-(x_1+x_2)} C_\kappa^{(d)}(x_1, x_2) f(x_1, x_2) |x_1 - x_2|^d dx_1 dx_2 = 0$$

then $f(x_1, x_2) \equiv 0$ a.e.

Proof. Let

$$F(z_1, z_2) = \int_0^\infty \int_0^\infty e^{-(x_1 z_1 + x_2 z_2)} f(x_1, x_2) |x_1 - x_2|^d dx_1 dx_2,$$

then, for $Re(z_1) > 0, Re(z_2) > 0$, $F(z_1, z_2)$ is a symmetric analytic function of z_1, z_2 . Moreover for all κ,

$$\begin{aligned} & C_\kappa^{(d)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) F(z_1, z_2) \,|_{z_1=1, z_2=1} \\ &= \int_0^\infty \int_0^\infty e^{-(x_1+x_2)} C_\kappa^{(d)}(x_1, x_2) f(x_1, x_2) |x_1 - x_2|^d dx_1 dx_2 = 0. \end{aligned}$$

So $F(z_1, z_2) \equiv 0$. Since the classical Laplace transform is one-to-one, we see that

$$f(x_1, x_2) |x_1 - x_2|^d \equiv 0.$$

Therefore, $f(x_1, x_2) \equiv 0$ a.e.

Proof of Proposition 5.1

For each partition σ, let P_σ be the polynomial given by Lemma 5.4, then

$$\begin{aligned} 0 &= P_\sigma\left(\frac{\partial}{\partial y}\right) \mathcal{L}(f)(y_1, y_2) \,|_{y_1=1, y_2=1} \\ &= c_0 \int_0^\infty \int_0^\infty P_\sigma\left(\frac{\partial}{\partial y}\right) {}_0\mathcal{F}_0^{(d)}(-x_1, -x_2 | y_1, y_2) f(x_1, x_2) |x_1 - x_2|^d dx_1 dx_2 \\ &= c_0 \int_0^\infty \int_0^\infty \frac{(-1)^{|\sigma|}}{|\sigma|} e^{-(x_1+x_2)} C_\sigma^{(d)}(x_1, x_2) f(x_1, x_2) |x_1 - x_2|^d dx_1 dx_2 = 0. \end{aligned}$$

Now the proposition follows from Lemma 5.7.

As a corollary of (1.8) and Proposition 4.5, we have

Lemma 5.8.

$$\int_{SO(d+1)} e^{-(k.(x_1c_1+x_2c_2)|y_1c_1+y_2c_2)}dk$$
$$= {}_1f_1(d/2;d;(x_1-x_2)(y_1-y_2))e^{-(x_1y_1+x_2y_2)} \qquad (7)$$

for $d = 1, 2, \ldots$.

It is known that the Laplace transform of a ${}_pf_q$ function is a ${}_{p+1}f_q$ function. A similar result can be established.

First, we shall prove some lemmas. Through our following work we use the symbol c to denote constants whose values change from line to line, but are independent of the relevant parameter.

Lemma 5.9. *There exists a constant C such that*

$$\left|\frac{\Gamma(Re(w))}{\Gamma(w)}\right| \leq Ce^{\frac{\pi}{2}|Imw|} \qquad (8)$$

for all $Re(w) \geq 1$.

Proof. It is a standard fact that (e.g. see [T])

$$\Gamma(w) = e^{(w-1/2)\log w - w + 1/2\log 2\pi + o(\frac{1}{|w|})}$$

as $|w| \to \infty$, uniformly for $-\pi \mid +\delta \leq arg\, w \leq \pi - \delta, \delta > 0$.

Let $w = x + iy$, we have

$$|\Gamma(w)| \approx e^{(x-1/2)\log|x+iy| - y\, arg\, w - x},$$

$$| \Gamma(Rew) | \approx e^{(x-1/2)\log x - x}.$$

Since $Re(w) \geq 1, x - 1/2 > 0, \log \frac{x}{\sqrt{x^2+y^2}} \leq 0$.

So

$$e^{(x-1/2)\log \frac{x}{\sqrt{x^2+y^2}}} \leq 1. \tag{9}$$

Hence

$$\left| \frac{\Gamma(Re(w))}{\Gamma(w)} \right| \leq Ce^{(x-1/2)\log \frac{x}{\sqrt{x^2+y^2}} + y \arg w} \leq Ce^{y \arg w}$$

for some constant C.

The following lemma gives the Laplace transform of Φ_κ^d for positive integer d.

Lemma 5.10. *For $d = 1, 2, ... y_1, y_2 > 0, Re(a) > d/2$*

$$c_0 \int_0^\infty \int_0^\infty e^{-(x_1 y_1 + x_2 y_2)} \Phi_\kappa^d(x_1, x_2)_1 f_1(d/2; (x_1 - x_2)(y_1 - y_2)) \\ \cdot (x_1 x_2)^{a-q_0} |x_1 - x_2|^d dx_1 dx_2 = \Gamma_d(a + \kappa) \Phi_\kappa^d(y_1^{-1}, y_2^{-1})(y_1 y_2)^{-a} \tag{10}$$

where $q_0 = 1 + d/2$.

Proof. For $d = 1, 2, ..., Re(a) > d/2$, it is known that (See [FK]) for $y = y_1 c_1 + y_2 c_2$,

$$\int_{\Omega_{d+1}} e^{-(x|y)} \Phi_\kappa(x) det(x)^{a-q_0} dx = \Gamma_{\Omega_{d+1}}(a + \kappa) \Phi_\kappa(y^{-1}) det(y)^{-a}. \tag{11}$$

By (1.9), Lemma 5.8 and (11) give

$$\begin{aligned} &\Gamma_d(a + \kappa) \Phi_\kappa^d(y_1^{-1}, y_2^{-1})(y_1 y_2)^{-a} \\ = \ & c_0 \int_0^\infty \int_0^\infty \int_{SO(d+1)} e^{-k.(x_1 c_1 + x_2 c_2)|y_1 c_1 + y_2 c_2)} dk \end{aligned}$$

$$\cdot\Phi_\kappa^d(x_1,x_2)(x_1x_2)^{a-q_0}dV(X,d,2)$$

$$= c_0\int_0^\infty\int_0^\infty e^{-(x_1y_1+x_2y_2)}$$

$$\cdot\Phi_\kappa^d(x_1,x_2)(x_1x_2)^{a-q}{}_1f_1(\frac{d}{2};(x_1-x_2)(y_1-y_2))dV(X,d,2).$$

More generally, we have

Proposition 5.11. *For all* $d>0, y_1, y_2>0, Re(a)>d/2$

$$c_0\int_0^\infty\int_0^\infty e^{-(x_1y_1+x_2y_2)}\Phi_\kappa^d(x_1,x_2){}_1f_1(d/2;(x_1-x_2)(y_1-y_2))$$
$$\cdot(x_1x_2)^{a-q_0}|x_1-x_2|^d dx_1dx_2=\Gamma_d(a+\kappa)\Phi_\kappa^d(y_1^{-1},y_2^{-1})(y_1y_2)^{-a} \quad (12)$$

where $q_0=1+d/2$.

Proof. Let $a=d/2+c$, then $Re(c)>0$, we need to show that

$$c_0\int_0^\infty\int_0^\infty e^{-(x_1y_1+x_2y_2)}\Phi_\kappa^d(x_1,x_2){}_1f_1(d/2;(x_1-x_2)(y_1-y_2))(x_1x_2)^{c-1}$$
$$\cdot|x_1-x_2|^d dx_1dx_2=\Gamma_d(d/2+c+\kappa)\Phi_\kappa^d(y_1^{-1},y_2^{-1})(y_1y_2)^{-(c+d/2)}. \quad (13)$$

We denote the integral in (13) by $G(d)$. Now it is enough to show that

$$\frac{\Gamma(d/2)c_0G(d)}{\Gamma(\frac{d+1}{2})\Gamma_d(d/2+c+\kappa)}=\Phi_\kappa^d(y_1^{-1},y_2^{-1})(y_1y_2)^{-(c+d/2)}\frac{\Gamma(d/2)}{\Gamma(\frac{d+1}{2})} \quad (14)$$

if $y_1y_2\geq 1$ and

$$(y_1y_2)^{c+d/2}\frac{\Gamma(d/2)c_0G(d)}{\Gamma(\frac{d+1}{2})\Gamma_d(d/2+c+\kappa)}=\Phi_\kappa^d(y_1^{-1},y_2^{-1})\frac{\Gamma(d/2)}{\Gamma(\frac{d+1}{2})} \quad (15)$$

if $y_1y_2<1$.

Since the proofs of (14) and (15) are the same , we only prove (14). We note that both sides of (14) are analytic functions of d on the half plane

$Re(d) > 0$. For $Re(d) \geq 1$, by (3.1), we can see that

$$\Phi^d_\kappa(y_1^{-1}, y_2^{-1})\frac{\Gamma(d/2)}{\Gamma(\frac{d+1}{2})} =$$
$$\frac{2^{k_2-k_1}}{\sqrt{\pi}}(y_1y_2)^{-k_2}\int_{-1}^{1}[\frac{1}{y_1}+\frac{1}{y_2}+(\frac{1}{y_1}-\frac{1}{y_2})s]^{k_1-k_2}(1-s^2)^{d/2-1}ds$$

is bounded as d varies, so is $(y_1y_2)^{-(d/2+c)}$. Hence, the right side of (14) is bounded as d varies on $Re(d) \geq 1$. In the following, we let $y = min\{y_1, y_2\}$ and assume that $Re(d) \geq 1$.

By (4.14), it follows easily that

$$_0\mathcal{F}_0^{(d)}(X|Y) \leq ce^{-y(x_1+x_2)}|d|. \tag{16}$$

For $x_1, x_2 > 0, -1 \leq s \leq 1$,

$$|x_1 + x_2 + (x_1 - x_2)s|^{k_1-k_2} \leq 2^{k_1-k_2}(x_1 + x_2)^{k_1-k_2}. \tag{17}$$

By (17) and (3.1), we have

$$e^{-\frac{y}{2}(x_1+x_2)}(x_1x_2)^{c-1}|\Phi^d_\kappa(x_1, x_2)||\frac{\Gamma(d/2)}{\Gamma(\frac{d+1}{2})}| \leq c. \tag{18}$$

By the Selberg integral formula , we have

$$\int_0^\infty\int_0^\infty e^{-\frac{y}{2}(x_1+x_2)}|x_1 - x_2|^{Red}dx_1dx_2 = \left(\frac{2}{y}\right)^{Red+2}\Gamma(Red + 1). \tag{19}$$

By Stirling's formula,

$$|\frac{\Gamma(d/2 + 1)}{\Gamma(d/2 + k_1 + c)}| \leq c. \tag{20}$$

Now we have , by (1.10), (4.7) and (20)

$$
\begin{aligned}
&\left| \frac{\Gamma(d/2)c_0G(d)}{\Gamma(\frac{d+1}{2})\Gamma_d(d/2+c+\kappa)} \right| \\
&= \left| \frac{\Gamma(d/2)G(d)\Gamma(d/2+1)(2\pi)^{d/2}}{\Gamma(\frac{d+1}{2})(2\pi)^{d/2}\Gamma(d/2+c+k_1)\Gamma(c+k_2)\Gamma(d+1)} \right| \\
&= \left| \frac{\Gamma(d/2+1)}{\Gamma(d/2+c+k_1)} \right| \frac{1}{|\Gamma(c+k_2)|} \frac{1}{|\Gamma(d+1)|} \left| \frac{\Gamma(d/2)G(d)}{\Gamma(\frac{d+1}{2})} \right| \\
&\leq c \frac{1}{|\Gamma(d+1)|} \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0^{(d)}(-X|Y) \\
&\cdot \left| \frac{\Gamma(d/2)}{\Gamma(\frac{d+2}{2})} \right| \cdot \left| \Phi_\kappa^d(x_1,x_2)(x_1x_2)^{c-1} \right| \left| |x_1-x_2|^d \right| dx_1dx_2.
\end{aligned}
$$

By (16), (18) and (19)

$$
\begin{aligned}
&\leq \quad c\frac{|d|}{\Gamma(d+1)} \int_0^\infty \int_0^\infty e^{-\frac{y}{2}(x_1+x_2)}|x_1-x_2|^{Red}dx_1dx_2 \\
&\leq \quad c\frac{|d|}{|\Gamma(d+1)|}|\Gamma(Red+1)| \left(\frac{2}{y}\right)^{Red+2}.
\end{aligned}
$$

By (8), for some δ, $0 < \delta < \frac{\pi}{2}$,

$$
\leq ce^{(\pi/2+\delta)|d|} \left(\frac{2}{y}\right)^{Red}.
$$

Now, by Lemma 5.10 and Corollary 1.2, (14) is true for all $d > 0, c > 1$. Again both sides of (14) are analytic functions of c, therefore, (14) is true for all $d > 0, Re(c) > 0$. We have finished the proof.

Once we have established Proposition 5.11, the following proposition follows immmediately by expanding the ${}_pF_q^{(d)}$ functions and integrating term by term.

Proposition 5.12. *If $y_1, y_2 > 0$, then*

$$
c_0 \int_0^\infty \int_0^\infty e^{-(x_1y_1+x_2y_2)}{}_pF_q^{(d)}(a_1,...,a_p;b_1,,b_q;X)_1f_1(d/2;(x_1-x_2)(y_1-y_2))
$$

$$\cdot(x_1x_2)^{a-q}|x_1-x_2|^d dx_1 dx_2 = \Gamma_d(a)(y_1y_2)^{-a}{}_{p+1}F_q^{(d)}(a_1,\ldots,a_p,a;b_1,\ldots,b_q;\frac{1}{y_1},\frac{1}{y_2})$$

for $p < q$, $Re(a) > d/2$; *or* $p = q$, $Re(a) > d/2, \frac{1}{y_1} < 1, \frac{1}{y_2} < 1$.

§6 Relation with the Heckman-Opdam Hypergeometric Functions

In this section, we shall establish the relation between ${}_2F_1^{(d)}$ and the hypergeometric functions introduced by Heckman and Opdam. Throughout this section r is 2.

Let

$$L = \Delta_2 - \mu_\kappa. \tag{1}$$

Then $C_\kappa^{(d)}(y_1, y_2)$ satisfies $LC_\kappa^{(d)}(y_1, y_2) = 0$.

Put

$$a_1 = y_1 + y_2,$$

$$a_2 = y_1 y_2. \tag{2}$$

Then (1) becomes

$$(a_1^2 - 2a_2)\frac{\partial^2}{\partial a_1^2} + 2a_1a_2\frac{\partial^2}{\partial a_1 \partial a_2}$$
$$+2a_2^2\frac{\partial^2}{\partial a_2^2} + d(a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2}) - \mu_\kappa. \tag{3}$$

Now substituting

$$u = 1/2a_1a_2^{-1/2}, \tag{4}$$

$$v = a_2^{1/2} \tag{5}$$

into (3) gives

$$-\frac{1}{2}[(1-u^2)\frac{\partial^2}{\partial u^2} - v^2\frac{\partial^2}{\partial v^2} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - d(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}) + 2\mu_\kappa. \tag{6}$$

Let $P^{\gamma,\gamma}_{k_1-k_2}(x)$ be the classical Jacobi polynomials, where $\gamma = \frac{d-1}{2}$, then $v^k P^{\gamma,\gamma}_{k_1-k_2}(u)$ satisfies (6). Therefore,

$$(y_1y_2)^{k/2}P^{\gamma,\gamma}_{k_1-k_2}(\frac{1}{2}(y_1+y_2)(y_1y_2)^{-1/2})$$

satisfies (1). Since $(y_1y_2)^{k/2}P^{(\gamma,\gamma)}_{k_1-k_2}(\frac{1}{2}(y_1+y_2)(y_1y_2)^{-1/2})$ is a symmetric polynomial with $y_1^{k_1}y_2^{k_2}$ as its term of highest weight, by Corollary 3.2 , we get

$$C^{(d)}_\kappa(y_1,y_2) = C(y_1y_2)^{k/2}P^{\gamma,\gamma}_{k_1-k_2}(\frac{1}{2}(y_1+y_2)(y_1y_2)^{-1/2})$$

for some constant C.

Set

$$R^{(\gamma,\gamma)}_n(x) = \frac{P^{(\gamma,\gamma)}_n(x)}{P^{(\gamma,\gamma)}_n(1)}$$

and

$$Z^\gamma_{k_1,k_2}(a_1,a_2) = \frac{(2\gamma+1)_{k_1-k_2}}{(\gamma+1/2)_{k_1-k_2}}a_2^{\frac{k_1+k_2}{2}}R^{(\gamma,\gamma)}_{k_1-k_2}(\frac{1}{2}a_1a_2^{-1/2}),$$

then,

$$\begin{aligned}C^{(d)}_\kappa(y_1,y_2) &= C^{(d)}_\kappa(1,1)(y_1y_2)^{k/2}R^{(\gamma,\gamma)}_{k_1-k_2}(\frac{1}{2}(y_1+y_2)(y_1y_2)^{-1/2})\\ &= C^{(d)}_\kappa(1,1)\frac{(\gamma+1/2)_{k_1-k_2}}{(2\gamma+1)_{k_1-k_2}}Z^\gamma_{k_1,k_2}(y_1+y_2,y_1y_2).\end{aligned}$$

Since

$$C^{(d)}_\kappa(1,1) = \frac{k!d_\kappa}{(1+d/2)_\kappa} = \frac{k!(d)_{k_1-k_2}(k_1-k_2+d/2)}{k_2!(k_1-k_2)!(1+d/2)_{k_1}(d/2)},$$

we have

$$C^{(d)}_\kappa(y_1,y_2) = \frac{k!(\gamma+3/2)_{k_1-k_2}}{k_2!(k_1-k_2)!(\gamma+3/2)_{k_1-k_2}}Z^\gamma_{k_1,k_2}(y_1+y_2,y_1y_2).$$

Thus we have proved the following result

Lemma 6.1.

$$C_\kappa^{(d)}(y_1, y_2) = \frac{k!(\gamma + 3/2)_{k_1-k_2}}{k_2!(k_1 - k_2)!(\gamma + 3/2)_{k_1-k_2}} Z^{\gamma}_{k_1,k_2}(y_1 + y_2, y_1y_2)$$

where $d = 2\gamma + 1$.

Let $R_{n,k}^{\alpha,\beta,\gamma}$ be the orthogonal polynomials in two variables defined in [K3]. As an immediate consequence of Lemma 6.1, Lemma 2, (4.27) and (4.36) in [K1], we have

Proposition 6.2. *If* $2\gamma + 1 = d$, *then*

$$R_{n,n}^{\alpha,\beta,\gamma}(y_1 + y_2, y_1y_2) =$$
$$\quad {}_2F_1^{(d)}(-n, n + \alpha + \beta + (\gamma + 1/2) + 1; \alpha + (\gamma + 1/2) + 1; y_1, y_2).$$

In [K3], a similar result was proved for $\gamma = 0$.

From (3.3) in [K3], we have

$$R_{n,k}^{\alpha,\beta,\gamma}(y_1 + y_2, y_1y_2) = \frac{P_{n,k}^{\alpha,\beta,\gamma}(2 - 2(y_1 + y_2), 1 - 2(y_1 + y_2) + 4y_1y_2)}{P_{n,k}^{\alpha,\beta,\gamma}(2, 1)}.$$

For α, β,$\gamma > -1$, $\alpha+\beta+\gamma+3/2 > 0$ and $\alpha+\gamma+3/2 > 0$, let $D_1^{\alpha,\beta,\gamma}$, $D_-^{\alpha,\beta,\gamma}$ and $D_+^{\alpha,\beta,\gamma}$ be defined as in [K2],

$$D_1^{\alpha,\beta,\gamma} = (1 - x^2)\frac{\partial^2}{\partial x^2} + (1 - y^2)\frac{\partial^2}{\partial y^2} + [\beta - \alpha - (\alpha + \beta + 2)x$$
$$+(2\gamma + 1)\frac{1 - x^2}{x - y}]\frac{\partial}{\partial x} + [\beta - \alpha - (\alpha + \beta + 2)x + (2\gamma + 1)\frac{1 - y^2}{y - x}]\frac{\partial}{\partial y},$$

$$D_-^\gamma = D_-^{\alpha,\beta,\gamma} =$$

$$\frac{1}{2(x-y)^{2\gamma+1}}\left[\frac{\partial}{\partial x}\left((x-y)^{2\gamma+1}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left((x-y)^{2\gamma+1}\frac{\partial}{\partial x}\right)\right],$$

$$D_+^{\alpha,\beta,\gamma} = ((1-x)(1-y))^{-\alpha}$$

$$\cdot((1+x)(1+y))^{-\beta} \circ D_-^\gamma((1-x)(1-y))^{\alpha+1}((1+x)(1+y))^{\beta+1}.$$

Put

$$D_2^{\alpha,\beta,\gamma} = D_+^{\alpha,\beta,\gamma} \circ D_+^{\alpha,\beta,\gamma}.$$

Let $x = 1-2y_1, y = 1-2y_2$. In (y_1, y_2) coordinates, we have

$$D_1^{\alpha,\beta,\gamma} = \sum_{i=1}^{2} y_i(1-y_i)\frac{\partial^2}{\partial y_i^2} + (2\gamma+1)\frac{1}{y_1-y_2}[y_1(1-y_1)\frac{\partial}{\partial y_1}$$

$$-y_2(1-y_2)\frac{\partial}{\partial y_2}] - (\alpha+\beta+2)\sum_{i=1}^{2} y_i\frac{\partial}{\partial y_i} + (\alpha+1)\sum_{i=1}^{2}\frac{\partial}{\partial y_i},$$

$$D_1^\gamma = \frac{1}{8(y_1-y_2)^{2\gamma+1}}\{\frac{\partial}{\partial y_1}[(y_1-y_2)^{2\gamma+1}\frac{\partial}{\partial y_2}] + \frac{\partial}{\partial y_2}[(y_1-y_2)^{2\gamma+1}\frac{\partial}{\partial y_1}]\},$$

$$D_+^{\alpha,\beta,\gamma} =$$

$$16(y_1y_2)^{-\alpha}[(1-y_1)(1-y_2)]^{-\beta} D_-^\gamma \circ (y_1y_2)^{\alpha+1}[(1-y_1)(1-y_2)]^{\beta+1}.$$

In [K2], it was shown that

$$D_i^{\alpha,\beta,\gamma} P_{n,k}^{\alpha,\beta,\gamma}(x,y) = \lambda_i(n,k) P_{n,k}^{\alpha,\beta,\gamma}(x,y)$$

$i = 1, 2,$

where

$$\lambda_1(n,k) = -n(n+\alpha+\beta+2\gamma+2) - k(k+\alpha+\beta+1),$$

$$\lambda_2(n,k) = k(k+\alpha+\beta+1)(n+\gamma+3/2)(n+\alpha+\beta+3/2).$$

Therefore, by Proposition 6.2,

$$D_i^{\alpha,\beta,\gamma}(y_1,y_2)_2F_1^{(d)}(-n,n+\alpha+\beta+(\gamma+1/2)+1;\alpha+(\gamma+1/2)+1;y_1,y_2)$$

$$=\lambda_i(n,n)_2F_1^{(d)}(-n,n+\alpha+\beta+(\gamma+1/2)+1;\alpha+(\gamma+1/2)+1;y_1,y_2)$$

$i=1,2.$

More generally, we have

Proposition 6.3.

$$D_i^{\alpha,\beta,\gamma}(y_1,y_2)_2F_1^{(d)}(-n,n+\alpha+\beta+(\gamma+1/2)+1;\alpha+(\gamma+1/2)+1;y_1,y_2)$$

$$=\lambda_i(n,n)_2F_1^{(d)}(-n,n+\alpha+\beta+(\gamma+1/2)+1;\alpha+(\gamma+1/2)+1;y_1,y_2)\ (6)$$

for all $n\in\mathbf{C}, i=1,2.$

Proof. We consider both sides of (6) as an analytic function of n. When $n=0,1,2,...$, (6) is true, we will prove our proposition by using Carlson's Theorem. Some estimates are needed.

Claim 1. There exist positive constants C and δ which only depend on A such that

$$\left(\frac{u+k}{A+k}\right)^k\le Ce^{(1+\delta)}$$

for all u and $k=0,1,2,....$

Proof.

$$\begin{aligned}\left(\frac{u+k}{A+k}\right) &= \left(1+\frac{u-A}{A+k}\right)^k=\left(1+\frac{u-A}{A+k}\right)^{\frac{k+A}{u-A}(u-A)\frac{k}{k+A}}\\ &\le e^{(u-A)\frac{k}{k+A}}\le Ce^{(1+\delta)u}.\end{aligned}$$

Let

$$I_\kappa = \frac{\Gamma(u+k_1)\Gamma(u+d/2+k_2)\Gamma(u+D+k_1)\Gamma(u+D+d/2+k_2)}{\Gamma(B+k_1)\Gamma(B-d/2+k_2)\Gamma(8u+k_1)\Gamma(8u-d/2+k_2)}.$$

Claim 2. There exist constants C and $N \in \mathbf{Z}^+$ such that

$$|I_\kappa| \leq C$$

for all $u > N$ and κ.

Proof. This follows from Stirling's formula and Claim 1.

Put $B = \alpha+\gamma+3/2, D = |\alpha+\beta+\gamma+3/2|, d = 2\gamma+1$ and $u = |n|$.

It is easy to see that there exist positive numbers ϵ and K with $K < \pi$ such that

$$\prod_{i=1}^{2}(1-|y_i|)^{-8u} \leq e^{Ku} \tag{7}$$

for all $|y_1|, |y_2| < \epsilon$.

Now

$$\begin{aligned}
&|\ {}_2F_1^{(d)}(-n, n+\alpha+\beta+\gamma+3/2, \alpha+\gamma+3/2; y_1, y_2)\ | \\
&\quad \leq \sum_\kappa \frac{|(-n)_\kappa(n+\alpha+\beta+\gamma+3/2)_\kappa|}{|(\alpha+\gamma+3/2)_\kappa|} \cdot \frac{C_\kappa^{(d)}(|y_1|,|y_2|)}{k!} \\
&\quad \leq \sum_\kappa \frac{(u)_{k_1}(u+d/2)_{k_2}(u+D)_{k_1}(u+D+d/2)_{k_2}}{|(B)_{k_1}||(B-d/2)_{k_2}|} \cdot \frac{C_\kappa^{(d)}(|y_1|,|y_2|)}{k!} \\
&\quad \leq \sum_\kappa |I_\kappa|(8u)_\kappa \frac{C_\kappa^{(d)}(|y_1|,|y_2|)}{k!} \leq Ce^{Ku}.
\end{aligned}$$

The last inequality follows from Claim 2 and (7).

Similarly, we can show that the L.H.S. of (6.6) has the same estimate. Again by Corollary 1.2, we have proved the proposition.

Let $\mathbf{R}$ be a root system, W the Weyl group of R and $k \in \mathbf{C}^m$. G.J. Heckman and E.M. Opdam introduced the algebra of differential operators $\mathbf{D}(k)$ and the hypergeometric functions associated with $\mathbf{R}$

$$F(\lambda, k; h) = \sum_{w \in W} c(w\lambda, k)\phi(w\lambda + \rho, k; h), \lambda \in \hbar$$

which essentially generalize the algebra of the radial part of the invariant differential operators and the spherical functions on a symmetric space. When ${}_2F_1^{(d)}$ corresponds to a bounded symmetric domain, it is known that ${}_2F_1^{(d)}$ is a hypergeometric function in the sense of Heckman and Opdam. In the case of $r = 2$, this turns out to be still true for all positive d.

More precisely, we consider the root system BC_2. A triple of complex numbers $k = (k_1, k_2, k_3)$ is assigned to the three orbits of the roots under the Weyl group by order of increasing length.

For $\alpha, \beta, \gamma > -1, \alpha + \gamma + 3/2 > 0, \beta + \gamma + 3/2 > 0$, we have

Proposition 6.4. *If* $k_1 = 2(\alpha - \beta), k_2 = 2\gamma + 1$, *and* $k_3 = 2\beta + 1$, *then* $D_1^{\alpha,\beta,\gamma}$ *and* $D_1^{\alpha,\beta,\gamma}$ *generate* $\mathbf{D}(k)$.

Proof. See [K3]

Let $\alpha = c - d/2 - 1, \beta = a + b - c$, and $\gamma = d/2 - 1/2$. If α, β,$\gamma > -1$, $\alpha + \beta + \gamma + 3/2 > 0$ and $\alpha + \gamma + 3/2 > 0$, then, by Proposition 6.3, ${}_2F_1^{(d)}(a, b; c; y_1, y_2)$ is an eigenfunction of both $D_1^{\alpha,\beta,\gamma}$ and $D_1^{\alpha,\beta,\gamma}$. Now as a consequence of this fact, Proposition 6.4 and Theorem 6.9 in [HO], we obtain

Proposition 6.5. *If* $r = 2, d$ *is any positive number, then* ${}_2F_1^{(d)}$ *is a* *hypergeometric function in the sense of Heckman and Opdam, i.e.,*

$$ {}_2F_1^{(d)}(a,b;c;z_1,z_2) = F(\lambda,k;z_1,z_2) $$

for a suitable choice of λ.

§7 Asymptotic Behavior of ${}_{p+1}F_p^{(d)}$

From Proposition 1.5, we know that ${}_{p+1}F_p(Y)$ is convergent for Y with $|y_i| < 1, i = 1, ..., r$. In this section, we study the asymptotic behavior of ${}_{p+1}F_p$ as $Y \to I$. It turns out that some new phenomena appear when $r > 1$.

Let

$$\begin{aligned} q_0 &= 1 + \frac{d}{2}(r-1), \\ d_\kappa &= \prod_{1 \le i < j \le r} \frac{k_i - k_j + \frac{d}{2}(j-i)}{\frac{d}{2}(j-i)} \frac{B(k_i - k_j, \frac{d}{2}(j-i-1)+1)}{B(k_i - k_j, \frac{d}{2}(j-i+1))}. \end{aligned}$$

for all κ with $l(\kappa) \le r$.

Proposition 7.1.

$$\left(\frac{2}{d}\right)^k J_\kappa(1, ..., 1; 2/d) j_\kappa^{-1} = \frac{d_\kappa}{(q_0)_\kappa}. \tag{1}$$

Proof. For a partition κ, let $s(\kappa)$ be the positive integer such that

$$k_1 \ge \cdots \ge k_{s(\kappa)} > k_{s(\kappa)+1} = \cdots = k_r = 0.$$

We will prove (1) by induction on $s(\kappa)$.

Let $s = s(\kappa)$. When $s = 1$, a direct calculation gives (1).

Now we assume that (1) is true for all partition λ with $s(\lambda) \le s - 1$.

Suppose $s > 1$. Let

(a) $l_i = k_i - k_s$, if $i \le s$,

(b) $l_i = 0$, if $i > s$

and $\lambda = (l_1, ..., l_r)$.

Then λ is a partition of $k - sk_s$ with $s(\lambda) \leq s-1$, hence

$$\left(\frac{2}{d}\right)^l J_\lambda(1, ..., 1; 2/d) j_\lambda^{-1} = \frac{d_\lambda}{(q_0)_\lambda} \tag{2}$$

with $l = k - sk_s$.

Let

$$A = \prod_{i=1}^{s} \prod_{j=1}^{k_s} [1 + (r-i)d/2 + k_i - j],$$

$$B = \prod_{1 \leq i < s+1 \leq j \leq r} \left[\frac{k_i + \frac{j-i}{2}d}{k_i - k_s + \frac{j-i}{2}d} \cdot \prod_{n=1}^{k_s} \frac{k_i - n + \frac{j-i+1}{2}d}{k_i - n + \frac{j-i-1}{2}d + 1} \right].$$

Claim 1.

$$(q_0)_\kappa = (q_0)_\lambda A. \tag{3}$$

Proof.

$$\begin{aligned}
(q_0)_\lambda &= \prod_{i=1}^{r} (1 + \frac{r-i}{2}d)_{l_i} \\
&= \prod_{i=1}^{s-1} [(1 + \frac{r-i}{2}d) \cdots (1 + \frac{r-i}{2}d + k_i - k_s - 1)], \\
(q_0)_\kappa &= \prod_{i=1}^{r} (1 + \frac{r-i}{2}d)_{k_i} \\
&= \prod_{i=1}^{s} (1 + \frac{r-i}{2}d) \cdots (1 + \frac{r-i}{2}d + k_i - k_s - 1) \\
&\quad \cdot (1 + \frac{r-i}{2}d + k_i - k_s) \cdots (1 + \frac{r-i}{2}d + k_i - 1) \\
&= (q_0)_\lambda \prod_{i=1}^{s-1} (1 + \frac{r-i}{2}d + k_i - k_s) \cdots (1 + \frac{r-i}{2}d + k_i - 1) \\
&\quad \cdot (1 + \frac{r-s}{2}d) \cdots (1 + \frac{r-s}{2}d + k_s - 1) \\
&= (q_0)_\lambda \prod_{i=1}^{s} \prod_{j=1}^{k_s} [1 + (r-i)d/2 + k_i - j] = (q_0)_\lambda A
\end{aligned}$$

Claim 2.

$$d_\kappa = d_\lambda B. \tag{4}$$

Proof.

$$\begin{aligned}
d_\lambda &= \prod_{1\le i<j\le r} \frac{l_i - l_j + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(l_i - l_j, \frac{j-i-1}{2}d+1)}{B(l_i - l_j, \frac{j-i+1}{2}d)} \\
&= \prod_{1\le i<j\le s} \frac{k_i - k_j + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i - k_j, \frac{j-i-1}{2}d+1)}{B(k_i - k_j, \frac{j-i+1}{2}d)} \\
&\quad \cdot \prod_{1\le i<s+1\le j\le r} \frac{k_i - k_s + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i - k_s, \frac{j-i-1}{2}d+1)}{B(k_i - k_s, \frac{j-i+1}{2}d)}, \\
d_\kappa &= \prod_{1\le i<j\le s} \frac{k_i - k_j + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i - k_j, \frac{j-i-1}{2}d+1)}{B(k_i - k_j, \frac{j-i+1}{2}d)} \\
&\quad \cdot \prod_{1\le i<s+1\le j\le r} \frac{k_i + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i, \frac{j-i-1}{2}d+1)}{B(k_i, \frac{j-i+1}{2}d)} \\
&= d_\lambda \prod_{1\le i<s+1\le j\le r} \frac{k_i + \frac{j-i}{2}d}{k_i - k_s + \frac{j-i}{2}d} \\
&\quad \cdot \frac{B(k_i, \frac{j-i-1}{2}d+1)}{B(k_i, \frac{j-i+1}{2}d)} \frac{B(k_i - k_s, \frac{j-i+1}{2}d)}{B(k_i - k_s, \frac{j-i-1}{2}d+1)} \\
&= d_\lambda \prod_{1\le i<s+1\le j\le r} \left[\frac{k_i + \frac{j-i}{2}d}{k_i - k_s + \frac{j-i}{2}d} \cdot \prod_{n=1}^{k_s} \frac{k_i - n + \frac{j-i+1}{2}d}{k_i - n + \frac{j-i+1}{2}d + 1} \right] \\
&= d_\lambda B.
\end{aligned}$$

Let

$$\begin{aligned}
C_1 &= \prod_{i=1}^{s} \prod_{j=1}^{k_s} [r - (i-1) + \frac{2}{d}(k_i - k_s + j - 1)], \\
C_2 &= \prod_{i=1}^{s} \prod_{j=1}^{k_s} [s - i + \frac{2}{d}(1 + k_i - j)], \\
C_3 &= \prod_{i=1}^{s} \prod_{j=1}^{k_s} [s - i + 1 + \frac{2}{d}(k_i - j)].
\end{aligned}$$

From (iv) in §1 and [Mc], we have

$$J_\kappa(I_r;2/d) = J_\lambda(I_r;2/d)C_1. \tag{5}$$

For a partition κ, let

$$h^*(\kappa) = \prod_{s\in\kappa} h^*_\kappa(s),$$
$$h_*(\kappa) = \prod_{s\in\kappa} h^\kappa_*(s).$$

Then, a computation yields

$$h^*(\kappa) = h^*(\lambda)C_2,$$

$$h_*(\kappa) = h_*(\lambda)C_3.$$

By §1 (vii), we have

$$j_\kappa = h^*(\kappa)h_*(\kappa).$$

Hence

$$\begin{aligned} j_\kappa &= j_\lambda C_2 C_3, \\ \frac{J_\lambda(I_r;2/d)}{j_\lambda} &= \frac{J_\kappa(I_r;2/d)}{j_\kappa}\frac{C_2C_3}{C_1}. \end{aligned} \tag{6}$$

By Claim 1 , Claim 2 , (2) and (6), we have

$$\begin{aligned} \frac{d_\kappa}{(q)_\kappa} &= \frac{d_\lambda}{(q)_\lambda}\cdot\frac{B}{A} \\ &= (2/d)^l\frac{J_\lambda(I_r;2/d)}{j_\lambda}\frac{B}{A} \\ &= (2/d)^{k-sk_s}\frac{J_\kappa(I_r;2/d)}{j_\kappa}\frac{C_2C_3}{C_1}\frac{B}{A}. \end{aligned}$$

Therefore, it is enough to show that

$$\frac{B}{A} = (2/d)^{sk_s} \frac{C_1}{C_2 C_3}.$$

Let

$$\begin{aligned} B_1 &= \prod_{i=1}^{s} \prod_{j=s+1}^{r} \frac{k_i + \frac{j-i}{2} d}{k_i - k_s + \frac{j-i}{2} d}, \\ B_2 &= \frac{\prod_{i=1}^{s} \prod_{j=s+2}^{r+1} [k_i - k_s + \frac{j-i}{2} d]}{\prod_{i=1}^{s} \prod_{j=s}^{r-1} [k_i + \frac{j-i}{2} d]}. \end{aligned}$$

Then,

$$\begin{aligned} B &= B_1 \prod_{i=1}^{s} \prod_{j=s+1}^{r} \prod_{n=1}^{k_s} \frac{[k_i - n + \frac{j-i+1}{2} d]}{[k_i - n + \frac{j-i-1}{2} d]} \\ &= B_1 \prod_{n=1}^{k_s} \frac{\prod_{i=1}^{s} \prod_{j=s+2}^{r+1} [k_i - n + \frac{j-i}{2} d]}{\prod_{i=1}^{s} \prod_{j=s}^{r-1} [k_i - n + 1 + \frac{j-i}{2} d]} \\ &= B_1 \prod_{i=1}^{s} \frac{\prod_{j=s+2}^{r+1} \prod_{n=1}^{k_s} [k_i - n + \frac{j-i}{2} d]}{\prod_{j=s}^{r-1} \prod_{n=0}^{k_s-1} [k_i - n + \frac{j-i}{2} d]} \\ &= B_1 \prod_{i=1}^{s} \frac{\prod_{j=s+2}^{r+1} \prod_{n=1}^{k_s-1} [k_i - n + \frac{j-i}{2} d]}{\prod_{j=s}^{r-1} \prod_{n=1}^{k_s-1} [k_i - n + \frac{j-i}{2} d]} \\ &\quad \cdot \prod_{i=1}^{s} \frac{\prod_{j=s+2}^{r+1} [k_i - k_s + \frac{j-i}{2} d]}{\prod_{j=s}^{r-1} [k_i + \frac{j-i}{2} d]} \\ &= B_1 B_2 \frac{\prod_{i=1}^{s} \prod_{n=1}^{k_s-1} [k_i - n + \frac{r-i}{2} d][k_i - n + \frac{r+1-i}{2} d]}{\prod_{i=1}^{s} \prod_{n=1}^{k_s-1} [k_i - n + \frac{s-i}{2} d][k_i - n + \frac{s+1-i}{2} d]}. \end{aligned}$$

Since

$$B_1 B_2 = \frac{\prod_{i=1}^{s} [k_i + \frac{r-i}{2} d] \prod_{i=1}^{s} [k_i - k_s + \frac{r+1-i}{2} d]}{\prod_{i=1}^{s} [k_i + \frac{s-i}{2} d] \prod_{i=1}^{s} [k_i - k_s + \frac{s+1-i}{2} d]},$$

we have

$$B = \frac{\prod_{i=1}^{s} \prod_{j=1}^{k_s} [k_i - j + 1 + \frac{r-i}{2} d] \prod_{i=1}^{s} \prod_{j=1}^{k_s} [k_i - j + \frac{r+1-i}{2} d]}{\prod_{i=1}^{s} \prod_{j=1}^{k_s} [k_i - j + 1 + \frac{s-i}{2} d] \prod_{i=1}^{s} \prod_{j=1}^{k_s} [k_i - j + \frac{s+1-i}{2} d]}.$$

Then

$$\frac{B}{A} = \frac{\prod_{i=1}^{s}\prod_{j=1}^{k_s}[k_i - j + \frac{r+1-i}{2}d]}{\prod_{i=1}^{s}\prod_{j=1}^{k_s}[k_i - j + 1 + \frac{s-i}{2}d]\prod_{i=1}^{s}\prod_{j=1}^{k_s}[k_i - j + \frac{s+1-i}{2}d]}$$
$$= (2/d)^{sk_s}C_1C_2^{-1}C_3^{-1}.$$

This finishes the proof.

Corollary 7.2.

$$_pF_q^{(d)}(a_1, ..., a_p; b_1, ..., b_p; y_1, \ldots, y_r)$$
$$= \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(a_1)_\kappa \cdots (a_p)_\kappa} \frac{d_\kappa}{(q)_\kappa} \cdot \frac{C_\kappa^{(d)}(y_1, \ldots, y_r)}{C_\kappa^{(d)}(1, ..., 1)}.$$

Once having Corollary 7.2, we can use the Γ- function to give the asymptotic behavior of $_{p+1}F_p^{(d)}$.

Set

$$I_\alpha(t) = \sum_{\kappa}[\prod_{j=1}^{l}(k_j + 1)^\alpha][\prod_{1\leq p<q\leq l}(k_p - k_q + 1)^d]t^k$$

with $l(\kappa) \leq l$.

First we have the following lemma

Lemma 7.3

(i) *If* $\alpha + (l-1)d + 1 < 0$, *then,* $I_\alpha(t)$ *is bounded;*

(ii) If $\alpha + 1 > 0$,

$$I_\alpha(t) \approx (1-t)^{-[l\alpha + l + (l-1)\frac{l}{2}d]};$$

(iii) If $\alpha + (l-j)d + 1 = 0$,

$$I_\alpha(t) \approx (1-t)^{-(j-1)\frac{j}{2}d}\log\frac{1}{1-t};$$

(iv) If $\alpha+(l-j)d+1>0>\alpha+(l-j-1)d+1$,

$$I_\alpha(t)\approx(1-t)^{-j[\alpha+1+ld-\frac{j+1}{2}d]}.$$

(By $A(x)\approx B(x)$, we mean that there exist two positive numbers C_1 and C_2 such that $C_1\le\frac{A(x)}{B(x)}\le C_2$ as x varies.)

Proof. On the one hand,

$$\begin{aligned}
I_\alpha(t) &= \sum_{k_l=0}^{\infty}\Big[\sum_{\substack{k_1\ge\cdots\ge k_{l-1}\\ k_{l-1}\ge k_l}}\Big(\prod_{j=1}^{l-1}(k_j+1)^\alpha\Big)\Big(\prod_{i=1}^{l-1}(k_i-k_l+1)^d\Big)\\
&\quad\cdot\Big(\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d\Big)t^{k_1+\dots+k_{l-1}}\Big](k_l+1)^\alpha t^{k_l}\\
&\le \sum_{k_l=1}^{\infty}\Big[\sum_{k_1\ge\dots\ge k_{l-1}\ge 0}\Big(\prod_{j=1}^{l-1}(k_j+1)^{\alpha+d}\Big)\\
&\quad\cdot\Big(\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d\Big)t^{k_1+\dots+k_{l-1}}\Big]k_l^\alpha t^{k_l}\\
&= \Big[\sum_{k_1\ge\dots\ge k_{l-1}}\Big(\prod_{j=1}^{l-1}(k_j+1)^{\alpha+d}\Big)\\
&\quad\cdot\Big[\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d\Big)t^{k_1+\dots+k_{l-1}}\Big]\Big[\sum_{k_l=0}^{\infty}k_l^\alpha t^{k_l}\Big]\\
&\ \ \vdots\\
&\le \Big(\sum_{k_1=0}^{\infty}k_1^{\alpha+(l-1)d}t^{k_1}\Big)\Big(\sum_{k_2=0}^{\infty}k_2^{\alpha+(l-2)d}t^{k_2}\Big)\\
&\quad\cdots\Big(\sum_{k_{l-1}=0}^{\infty}k_{l-1}^{\alpha+d}t^{k_{l-1}}\Big)\Big(\sum_{k_l=0}^{\infty}k_l^\alpha t^{k_l}\Big).
\end{aligned}$$

On the other hand,

$$I_\alpha(t)\ \ge\ \sum_{k_l=0}^{\infty}\Big\{\sum_{\substack{k_1\ge\cdots\ge k_{l-1}\\ k_{l-1}\ge k_l}}\Big(\prod_{j=1}^{l-1}(k_j-k_l+1)^\alpha\Big)$$

$$\cdot\Big(\prod_{1\le p<q\le l-1}[(k_p-k_l)-(k_q-k_l)+1]^d\Big)$$

$$\Big(\prod_{j=1}^{l-1}(k_j-k_l+1)^d\Big)t^{(k_1-k_l)+\cdots+(k_{l-1}-k_l)+(l-1)k_l}\Big\}k_l^{\alpha}t^{k_l}$$

$$= \sum_{k_l=0}^{\infty}\Big[\sum_{k_1\ge\cdots\ge k_{l-1}\ge 0}\prod_{j=1}^{l-1}(k_j+1)^{\alpha+d}\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d t^{k_1+\cdots+k_{l-1}}\Big]k_l^{\alpha}t^{l\,k_l}$$

$$= \Big[\sum_{k_1\ge\cdots\ge k_{l-1}\ge 0}\prod_{j=1}^{l-1}(k_j+1)^{\alpha+d}$$

$$\cdot\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d t^{k_1+\cdots+k_{l-1}}\Big]\Big[\sum_{k_l=0}^{\infty}k_l^{\alpha}t^{l\,k_l}\Big]$$

$$\vdots$$

$$\ge C\Big(\sum_{k_1=1}^{\infty}k_1^{\alpha+(l-1)d}t^{k_1}\Big)$$

$$\cdot\Big(\sum_{k_2=1}^{\infty}k_2^{\alpha+(l-2)d}t^{k_2}\Big)\cdots\Big(\sum_{k_{l-1}=1}^{\infty}k_{l-1}^{\alpha+d}t^{k_{l-1}}\Big)\Big(\sum_{k_l=1}^{\infty}k_l^{\alpha}t^{k_l}\Big).$$

Hence

$$I_{\alpha(t)}\approx\Big(\sum_{m=1}^{\infty}m^{\alpha+(l-1)d}t^m\Big)\Big(\sum_{m=1}^{\infty}m^{\alpha+(l-2)d}t^m\Big)\cdots\Big(\sum_{m=1}^{\infty}m^{\alpha}t^m\Big). \qquad (7)$$

Let

$$I_{\alpha,j}(t)=\sum_{m=1}^{\infty}m^{\alpha+(l-j)d}t^m,$$

for $-1<t<1$, we have

(a) if $\alpha+(l-j)d+1<0$, then, $I_{\alpha,j}(t)$ is bounded;

(b) if $\alpha+(l-j)d+1>0$, then,

$$I_{\alpha,j}(t)\approx(1-t)^{-[\alpha+(l-j)d+1]};$$

(c) if $\alpha+(l-j)d+1=0$, then,

$$I_{\alpha,j}(t)\approx\log\frac{1}{1-t}.$$

Now the lemma follows immediately from (7), (a), (b) and (c).

Proposition 7.4. *Let* $\gamma = \sum_{i=1}^{p+1} a_i - \sum_{i=1}^{p} b_i$. *If for all* κ

$$\frac{(a_1)_\kappa \cdots (a_{p+1})_\kappa}{(b_1)_\kappa \cdots (b_p)_\kappa} > 0.$$

For $-1 < y_i < 1, i = 1, ..., r$, *we have*

(i) if $\gamma > d/2(r-1)$, *then*

$${}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; y_1, \ldots, y_r) \approx \prod_{i=1}^{r} (1 - y_i)^{-r};$$

(ii) if $\gamma < -d/2(r-1)$, *then there exists a constant* C *such that*

$${}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; y_1, \ldots, y_r) \leq C;$$

(iii) if $\gamma = d(-\frac{r-1}{2} + j - 1), j = 1, \ldots, r$, *then, for* $y_1 = \cdots = y_r = t, -1 < t < 1$,

$${}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; t_1, \ldots, t) \approx (1 - t)^{-(j-1)\frac{1}{2}d} \log \frac{1}{1-t};$$

(iv) if $d(-\frac{r-1}{2} + j - 1) < \gamma < (-\frac{r-1}{2} + j)d, j = 1, \ldots, r-1$, *then, for* $y_1 = \cdots = y = t, -1 < t < 1$,

$${}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; t, \ldots, t) \approx (1 - t)^{-j[\gamma + d/2(r-j)]}.$$

Proof. By Corollary 7.2,

$$\begin{aligned} &{}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; Y) \\ &\quad = \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_{p+1})_\kappa}{(b_1)_\kappa \cdots (b_p)_\kappa} \frac{d_\kappa}{(q)_\kappa} \frac{C_\kappa(Y)}{C_\kappa(I_r)}. \end{aligned}$$

First,

$$\begin{aligned} d_\kappa &= \prod_{1\le i<j\le r} \frac{\Gamma(d/2(j-i-1)+1)}{\Gamma(d/2(j-i+1))} \\ &\cdot \prod_{1\le i<j\le r} \frac{k_i-k_j+d/2(j-i)}{d/2(j-i)} \frac{\Gamma(k_i-k_j+d/2(j-i+1))}{\Gamma(k_i-k_j+d/2(j-i-1)+1)}. \end{aligned}$$

By Stirling's formula, as κ varies

$$d_\kappa \approx \prod_{1\le i<j\le r} (k_i-k_j+1)^d. \tag{8}$$

Secondly, if $|\frac{(A)_\kappa}{(B)_\kappa}| > 0$, again by Stirling's formula, as κ varies,

$$|\frac{(A)_\kappa}{(B)_\kappa}| \approx \prod_{j=1}^r (k_j+1)^{A-B}.$$

Hence, as κ varies,

$$\frac{(a_1)_\kappa \cdots (a_{p+1})_\kappa}{(b_1)_\kappa \cdots (b_p)_\kappa (q)_\kappa} \approx \prod_{j=1}^r (k_j+1)^{\gamma-d/2(r-1)-1}. \tag{9}$$

(a) If $\gamma > d/2(r-1)$, then

$$\frac{(\gamma)_\kappa}{(q)_\kappa} \approx \prod_{j=1}^r (k_j+1)^{\gamma-d/2(r-1)-1}.$$

Thus

$${}_{p+1}F_p(a_1,\ldots,a_{p+1};b_1,\ldots,b_p;Y) \approx \sum_\kappa \frac{(\gamma)_\kappa}{(q)_\kappa} d_\kappa \frac{C_\kappa(Y)}{C_\kappa(I_r)} = \prod_{i=1}^r (1-y_i)^{-\gamma}.$$

(b) If $\gamma < -d/2(r-1)$, let $t = max\{|y_1|,\cdots,|y_r|\}$, then

$$|C_\kappa^{(d)}(y_1,\ldots,y_2)| \le C_\kappa^{(d)}(t,\ldots,t).$$

So, by (i) in Lemma 7.3,

$$|{}_{p+1}F_p(a_1,\ldots,a_{p+1};b_1,\ldots,b_p;Y)| = I_{\gamma-\frac{d}{2}(r-1)-1}(t) \le C.$$

That is (ii).

(c) If $-\frac{d}{2}(r-1) \leq \gamma \leq \frac{d}{2}(r-1)$, then, Lemma 7.3 gives (iii) and (iv).

§8 Generalized Laguerre Polynomials and Hankel Transformation

The classical Laguerre polynomials are given by

$$L_k^\gamma(x) = (\gamma+1)_k \sum_{s=0}^{k} \binom{k}{s} \frac{(-x)^s}{(\gamma+1)_s}$$

for $\gamma > -1$.

It is known that the classical Laguerre polynomials have the following properties

(i) (Generating function)

$$(1-z)^{-\gamma-1} \sum_{l=0}^{\infty} 1/l!(x\frac{z}{z-1})^l = \sum_{k=0}^{\infty} \frac{L_k^\gamma(x) z^k}{k!} \tag{1}$$

for $|z| < 1$;

(ii) L_k^γ are orthogonal on $(0,\infty)$ with respect to the weight function $e^{-x}x^\gamma$, in fact

$$\int_0^\infty e^{-x} x^\gamma L_k^\gamma(x) L_j^\gamma(x) dx = \delta_{jk} k! \Gamma(\gamma+1+k); \tag{2}$$

(iii)

$$e^{-x} L_k^\gamma(x) = \frac{1}{\Gamma(\gamma+1)} \int_0^\infty e^{-y} y^{\gamma+k} {}_0f_1(\gamma+1; -xy) dy. \tag{3}$$

In the first part of this section, we shall generalize these results to the case of two variables.

Definition. The generalized Laguerre polynomial $L_\kappa^\gamma(x_1, x_2; d)$ corresponding to the parameter d and the partition κ of k is defined , for $\gamma > -1$, by

$$L_\kappa^\gamma(x_1,x_2;d) = (\gamma+q_0)_\kappa C_\kappa^{(d)}(1,1)\sum_{s=0}^{k}\sum_{\sigma}\binom{\kappa}{\sigma}_2 \frac{C_\sigma^{(d)}(-x_1,-x_2)}{(\gamma+q)_\sigma C_\sigma^{(d)}(1,1)} \qquad (4)$$

where $q_0 = 1 + d/2$.

In the following, we simply write $L_\kappa^\gamma(x_1,x_2)$ for $L_\kappa^\gamma(x_1,x_2;d)$, $C_\kappa(x_1,x_2)$ for $C_\kappa^{(d)}(x_1,x_2)$ and $\binom{\kappa}{\sigma}$ for $\binom{\kappa}{\sigma}_2$.

First we have

Proposition 8.1. *If $y_1, y_2 > 0$, then*

$$\begin{aligned} & c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0^{(d)}(-x_1,-x_2|y_1,y_2)(x_1,x_2)^\gamma L_\kappa^\gamma(x_1,x_2)|x_1-x_2|^d dx_1dx_2 \\ = \; & \Gamma_d(\gamma+q_0+\kappa)(y_1y_2)^{-(\gamma+q_0)}C_\kappa^{(d)}(1-1/y_1,1-1/y_2). \qquad (5)\end{aligned}$$

Proof. By (4), the L.H.S. of (5) is equal to

$$\begin{aligned} & (\gamma+q_0)_\kappa C_\kappa(1,1)\sum_{s=0}^{k}\sum_{\sigma}\binom{\kappa}{\sigma}\frac{(-1)^s}{(\sigma+q_0)_\sigma C_\sigma(1,1)} \\ & \cdot c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0^{(d)}(-X|Y)(x_1x_2)^\gamma C_\sigma(X)|x_1-x_2|^d dx_1dx_2 \\ & = (\gamma+q_0)_\kappa C_\kappa(1,1)\sum_{s=0}^{k}\sum_{\sigma}\binom{\kappa}{\sigma}\frac{C_\sigma(-1/y_1,-1/y_2)}{C_\sigma(1,1)}\Gamma_d(\gamma+q_0)(y_1y_2)^{-(\gamma+q_0)} \\ & = \Gamma_d(\gamma+q_0+\kappa)(y_1y_2)^{-(\gamma+q_0)}C_\kappa(1-1/y_1,1-1/y_2).\end{aligned}$$

(1) has the following generalization

Proposition 8.2. *If $x_1, x_2 > 0$, then*

$$\begin{aligned} & \prod_{i=1}^{2}(1-z_i)^{-(\gamma+q_0)}{}_0\mathcal{F}_0^{(d)}(-x_1,-x_2;\frac{z_1}{1-z_1},\frac{z_2}{1-z_2}) \\ & = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{L_\kappa^\gamma(X)C_\kappa(Z)}{k!C_\kappa(1,1)} \qquad (6)\end{aligned}$$

for $|z_i| < 1, i = 1, 2$.

Proof. The L.H.S. of (6) is an analytic function of x_1, x_2, z_1 and z_2 in the domain $D = \{(x_1, x_2, z_1, z_2)|$ all $x_1, x_2, |z_1| < 1, |z_2| < 1\}$. When it is expanded in a series of x_1, x_2, z_1 and z_2, the series is absolutely convergent. Therefore, the L.H.S of (6) can be written as

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{L}_{\kappa}^{\gamma}(X)}{k!}\frac{C_{\kappa}(Z)}{C_{\kappa}(1,1)} \tag{7}$$

with $\tilde{L}_{\kappa}^{\gamma}(x_1, x_2) = \tilde{L}_{\kappa}(x_2, x_1)$.

Now it suffices to show that

$$\mathcal{L}(\tilde{L}_{\kappa}^{\gamma}(x_1, x_2)(x_1x_2)^{\gamma})(y_1, y_2) = \mathcal{L}(L_{\kappa}^{\gamma}(x_1, x_2)(x_1x_2)^{\gamma})(y_1, y_2). \tag{8}$$

We observe that for $|z_i| < 1, i = 1, 2$,

$$\begin{aligned}&|\sum_{k=0}^{N}\sum_{\kappa}\frac{\tilde{L}_{\kappa}^{\gamma}(X)}{k!}\frac{C_{\kappa}(Z)}{C_{\kappa}(1,1)}|\\ &\quad \le \prod_{i=1}^{2}(1-|z_i|)^{-(\gamma+q_0)}{}_0\mathcal{F}_0(X|\frac{|z_1|}{1-|z_1|}, \frac{|z_2|}{1-|z_2|}).\end{aligned} \tag{9}$$

For any $y_1, y_2 > 0$, there exists a $\delta > 0$ such that if $|z_i| < \delta, i = 1, 2$,

$$\mathcal{L}_x({}_0\mathcal{F}_0(X|\frac{|z_1|}{1-|z_1|}, \frac{|z_2|}{1-|z_2|}))(y_1, y_2) < \infty. \tag{10}$$

(9) and (10) imply that we can integrate term by term in (7), hence, if $|z_i| < \delta, i = 1, 2$,

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\mathcal{L}(\tilde{L}_{\kappa}^{\gamma}(X)(x_1x_2)^{\gamma})(y_1, y_2)}{k!}\frac{C_{\kappa}(Z)}{C_{\kappa}(1,1)}\\ &\quad = \mathcal{L}_x(\prod_{i=1}^{2}(1-z_i)^{-(\gamma+q_0)}{}_0\mathcal{F}_0(-X|\frac{z_1}{1-z_1}, \frac{z_2}{1-z_2}))(y_1, y_2)\\ &\quad = \prod_{i=1}^{2}(1-z_i)^{-(\gamma+q_0)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\mathcal{L}((x_1x_2)^{\gamma}C_{\kappa}(-x_1, -x_2))(y_1, y_2)}{k!}\frac{C_{\kappa}(\frac{z_1}{1-z_1}, \frac{z_2}{1-z_2})}{C_{\kappa}(1,1)}.\end{aligned}$$

By (5.12) this is equal to

$$\prod_{i=1}^{2}(1-z_i)^{-(\gamma+q_0)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\Gamma_d(\gamma+q_0+\kappa)}{k!}C_\kappa(-\frac{1}{y_1},-\frac{1}{y_2})(y_1y_2)^{-(\gamma+q_0)}\frac{C_\kappa(\frac{z_1}{1-z_1},\frac{z_2}{1-z_2})}{C_\kappa(1,1)}$$
$$= \Gamma_d(\gamma+q_0)\prod_{i=1}^{2}(1-z_i)^{-(\gamma+q_0)}(y_1y_2)^{-(\gamma+q_0)}$$
$$\cdot C_\kappa(\frac{1}{y_1},\frac{1}{y_2})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\gamma+q_0)_\kappa}{k!}\frac{C_\kappa(\frac{z_1}{z_1-1},\frac{z_2}{z_2-1})}{C_\kappa(1,1)}$$
$$= \Gamma_d(\gamma+q_0)\prod_{i=1}^{2}(1-z_i)^{-(\gamma+q_0)}(y_1y_2)^{-(\gamma+q_0)}$$
$$\cdot\int_{-1}^{1}\{[1-\frac{1}{4}(\gamma_s(U,V)+l_s(U,V))][1-\frac{1}{4}(\gamma_s(U,V)-l_s(U,V))]\}^{-(\gamma+q_0)}dm(s).$$
$$= \Gamma_d(\gamma+q_0)(y_1y_2)^{-(\gamma+q_0)}$$
$$\int_{-1}^{1}\{[1-\frac{1}{4}(\gamma_s(Z,W)+l_s(Z,W))][1-\frac{1}{4}(\gamma_s(Z,W)-l_s(U,V))]\}^{-(\gamma+q_0)}dm(s).$$
$$= \Gamma_d(\gamma+q_0)(y_1y_2)^{-(\gamma+q_0)}{}_1\mathcal{F}_0(\gamma+q_0;1-1/y_1,1-1/y_2|z_1,z_2)$$
$$= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\Gamma_d(\gamma+q_0+\kappa)(y_1y_2)^{-(\gamma+q_0)}}{k!}C_\kappa(-\frac{1}{y_1},-\frac{1}{y_2})\frac{C_\kappa(Z)}{C_\kappa(1,1)}$$

where

$$u_i=\frac{1}{y_i},v_i=\frac{z_i}{z_i-1},w_i=1-\frac{1}{y_i},i=1,2,$$
$$U=(u_1,u_2),V=(v_1,v_2),W=(w_1,w_2).$$

In the second and the fourth equalities, we have used (4.15).

Therefore, by Proposition 8.1, we have

$$\begin{aligned}&\mathcal{L}(\tilde{L}_\kappa^\gamma(X)(x_1x_2)^\gamma)(y_1,y_2)\\ &= \Gamma_d(\gamma+q_0+\kappa)(y_1y_2)^{-(\gamma+q)}C_\kappa(-1/y_1,-1/y_2)\\ &= \mathcal{L}(L_\kappa^\gamma(X)(x_1x_2)^\gamma)(y_1,y_2).\end{aligned}$$

Next, we have

Proposition 8.3.

$$c_0 \int_0^\infty \int_0^\infty e^{-(x_1+x_2)}(x_1x_2)^\gamma L_\kappa^\gamma(X) L_\sigma^\gamma(X)|x_1 - x_2|^d dx_1 dx_2$$
$$= \delta_{\kappa\sigma} k! C_\kappa(1,1)\Gamma_d(\gamma + q_0 + \kappa). \quad (11)$$

Proof. By(5.3),

$$_0\mathcal{F}_0(-x_1, -x_2|\frac{z_1}{1-z_1}, \frac{z_2}{1-z_2})e^{-(x_1+x_2)} = {}_0\mathcal{F}_0(-x_1, -x_2|\frac{1}{1-z_1}, \frac{1}{1-z_2}). \quad (12)$$

Hence, by (6) and (12), we have

$$\sum_{k=0}^{\infty}\sum_{\kappa}\int_0^\infty \int_0^\infty \frac{L_\sigma^\gamma(X)L_\kappa^\gamma(X)}{k!C_\kappa(1,1)}$$
$$(x_1x_2)^\gamma \cdot e^{-(x_1+x_2)}|x_1 - x_2|^d dx_1 dx_2 C_\kappa(Z)$$
$$= \prod_{i=1}^{2}(1 - z_i)^{-(\gamma+q_0)} c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-x_1, -x_2|\frac{z_1}{1-z_1}, \frac{z_2}{1-z_2})$$
$$\cdot L_\sigma^\gamma(X)(x_1x_2)^\gamma e^{-(x_1+x_2)}|x_1 - x_2|^d dx_1 dx_2$$
$$= \prod_{i=1}^{2}(1 - z_i)^{-(\gamma+q_0)} c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-x_1, -x_2|\frac{1}{1-z_1}, \frac{1}{1-z_2})$$
$$\cdot L_\sigma^\gamma(X)(x_1x_2)^\gamma|x_1 - x_2|^d dx_1 dx_2.$$

By Proposition 8.1, this is equal to

$$\prod_{i=1}^{2}(1 - z_i)^{-(\gamma+q_0)}\Gamma_d(\gamma + q_0 + \kappa)\prod_{i=1}^{2}(1 - z_i)^{(\gamma+q_0)}C_\kappa(Z)$$
$$= \Gamma_d(\gamma + q_0 + \kappa)C_\kappa(Z).$$

We have proved the proposition.

The following proposition gives an integral representation for L_κ^γ.

Proposition 8.4.

$$e^{-(x_1+x_2)}L_\kappa^\gamma(x_1,x_2)=\frac{c_0}{\Gamma_d(\gamma+q_0)}\int_0^\infty\int_0^\infty e^{-(y_1+y_2)}(y_1y_2)^\gamma C_\kappa(Y)$$
$$\cdot{}_0\mathcal{F}_1(\gamma+q_0;-X|Y)|y_1-y_2|^d dy_1dy_2. \quad (13)$$

Proof. We shall prove (13) by showing that after being multiplied by $e^{-(x_1+x_2)}\cdot(x_1,x_2)^\gamma$, both sides of (13) have the same Laplace transform.

On the one hand, (5.3) gives

$${}_0\mathcal{F}_0(-X|Z)e^{-2(x_1+x_2)}={}_0\mathcal{F}_0(-X|2+z_1,2+z_2).$$

Thus

$$\mathcal{L}(e^{-(x_1+x_2)}(x_1,x_2)^\gamma L.H.S.)(z_1,z_2)$$
$$=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0(-X|2+z_1,2+z_2)(x_1,x_2)^\gamma L_\kappa^\gamma(x_1,x_2)|x_1-x_2|^d dx_1dx_2$$
$$=\Gamma_d(\gamma+q_0+\kappa)\prod_{i=1}^{2}(2+z_i)^{-(\gamma+q_0)}C_\kappa(\frac{z_1+1}{z_1+2},\frac{z_2+1}{z_2+2}).$$

On the other hand, for $x_1,x_2\geq 0$,

$$|{}_0\mathcal{F}_1(\gamma+q_0;-X|Y)|\leq {}_0\mathcal{F}_1(\gamma+q_0;X|Y).$$

It is easy to see that for $z_1,z_2>0$,

$$c_0c_0\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0(-X|Z)e^{-(x_1+x_2)}(x_1,x_2)^\gamma|x_1-x_2|^d$$
$$e^{-(y_1+y_2)}(y_1y_2)^\gamma C_\kappa(Y){}_0\mathcal{F}_1(\gamma+q_0;X|Y)dV(Y,d,2)dx_1dx_2<\infty. \quad (14)$$

As a consequence of (5.3) and (5.12),

$$\mathcal{L}(e^{-(x_1+x_2)}(x_1,x_2)^\gamma C_\sigma(x_1,x_2))(z_1,z_2)$$
$$=\Gamma_d(\gamma+q_0+\sigma)C_\sigma(\frac{1}{1+z_1},\frac{1}{1+z_2})[(z_1+1)(z_2+1)]^{-(\gamma+q_0)}. \quad (15)$$

Now by (15), the series expansion of ${}_0\mathcal{F}_1(\gamma+q_0;-X|Y)$ and Fubini's Theorem, we have

$$\mathcal{L}(e^{-(x_1+x_2)}(x_1,x_2)^\gamma R.H.S)$$
$$= c_0\int_0^\infty\int_0^\infty e^{-(y_1+y_2)}(y_1y_2)^\gamma C_\kappa(Y)$$
$$\cdot[(z_1+1)(z_2+1)]^{-(\gamma+q_0)}\sum_\sigma\frac{C_\sigma(-Y)}{s!}\frac{C_\sigma(\frac{1}{1+z_1},\frac{1}{1+z_2})}{C_\sigma(1,1)}|y_1-y_2|^d dy_1dy_2$$
$$=\prod_{i=1}^2(z_i+1)^{-(\gamma+q_0)}$$
$$c_0\int_0^\infty\int_0^\infty e^{-(y_1+y_2)}{}_0\mathcal{F}_0(\frac{1}{z_1+1},\frac{1}{z_2+1}|-Y)(y_1y_2)^\gamma C_\kappa(Y)|y_1-y_2|^d dy_1dy_2.$$

By Lemma 5.2 and Proposition 5.11, this is equal to

$$\prod_{i=1}^2(z_i+1)^{-(\gamma+q_0)}c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0(-Y|\frac{z_1+2}{z_1+1},\frac{z_2+2}{z_2+1})$$
$$\cdot(y_1y_2)^\gamma C_\kappa(Y)|y_1-y_2|^d dy_1dy_2$$
$$=\prod_{i=1}^2(z_i+1)^{-(\gamma+q_0)}\Gamma_d(\gamma+q_0+\kappa)C_\kappa(\frac{z_1+1}{z_1+2},\frac{z_2+1}{z_2+2})\prod_{i=1}^2(\frac{z_i+2}{z_i+1})^{-(\gamma+q_0)}$$
$$=\mathcal{L}(e^{-(x_1+x_2)}(x_1,x_2)^\gamma L.H.S).$$

The proof is complete.

Let $L^2_\gamma(\mathbf{R}_+\times\mathbf{R}_+)=\{f$ is defined on $\mathbf{R}_+\times\mathbf{R}_+|f(x_1,x_2)=f(x_2,x_1)$ and $\int_0^\infty\int_0^\infty|f(x_1,x_2)|^2(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2<\infty\}$,

$$||f||_\gamma=[c_0\int_0^\infty\int_0^\infty|f(x_1,x_2)|^2(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2]^{1/2}$$
$$\mathcal{L}^\gamma_\kappa(x_1,x_2)=e^{-(x_1+x_2)}L^\gamma_\kappa(2x_1,2x_2).$$

As a consequence of (7.1) and (11), we have

$$||[\frac{2^{2\gamma+d+2}(q_0)_\kappa}{d_\kappa\Gamma_d(\gamma+q_0+\kappa)}]^{1/2}\frac{1}{k!}\mathcal{L}^\gamma_\kappa(X)||_\gamma=1. \qquad (16)$$

Moreover, we have

Proposition 8.5.

$$\{[\frac{2^{2\gamma+d+2}(q_0)_\kappa}{d_\kappa \Gamma_d(\gamma+q_0+\kappa)}]^{1/2}\frac{1}{k!}\mathcal{L}_\kappa^\gamma(X)\}$$

is an orthonormal basis in L^2_γ.

Proof. Suppose $f \in L^2_\gamma$ such that for all κ

$$\int_0^\infty \int_0^\infty f(x_1,x_2)\mathcal{L}_\kappa^\gamma(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2 = 0, \qquad (17)$$

then (17) implies

$$\int_0^\infty \int_0^\infty f(x_1,x_2)C_\kappa^{(d)}(x_1,x_2)e^{-(x_1+x_2)}(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2 = 0$$

for all κ.

By Lemma 5.7, $f(x_1,x_2) \equiv 0$ a.e.; together with (16), this proves the proposition.

Definition. If $f \in L^2_\gamma$ with compact support, we define the Hankel transform of f , for $\gamma > -1$, by

$$\mathcal{H}_\gamma f(y_1,y_2) = c_0 \int_0^\infty \int_0^\infty f(x_1,x_2)\,{}_0\mathcal{F}_1(\gamma+q_0;-X|Y) \cdot (x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2. \qquad (18)$$

We shall extend $\mathcal{H}_\gamma$ to the whole space L^2_γ. (cf. [D])

We define, for all $z_1, z_2 > 0$,

$$e_Z(x_1,x_2) = {}_0\mathcal{F}_0(-Z|X).$$

Proposition 8.6. *The closed linear space spanned by all $e_Z(\cdot)$ is L^2_γ.*

Proof. First, as a consequence of (4.14), we have

$$ {}_0\mathcal{F}_0^{(d)}(-Z|X) \leq 1 $$

for $x_1, x_2 > 0, z_1, x_2 > 0$. Hence

$$ ||e_Z||^2_\gamma \leq c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-Z|X)(x_1x_2)^\gamma |x_1 - x_2|^d dx_1 dx_2 < \infty. $$

Thus $e_Z(\cdot) \in L^2_\gamma$.

Secondly, if $f \in L^2_\gamma$ and orthogonal to all $e_Z\cdot$, then, by the injectivity of the generalized Laplace transformation, $f \equiv 0$ a.e.

Proposition 8.7. *We have*

$$ \mathcal{H}_\gamma(e_Z)(y_1, y_2) = \Gamma_d(\gamma + q_0)(z_1z_2)^{-(\gamma+q_0)} e_{(z_1^{-1}, z_2^{-1})}(y_1, y_2), \qquad (19) $$

$$ ||\mathcal{H}_\gamma(e_Z)||_\gamma = \Gamma_d(\gamma + q_0)||e_Z||_\gamma. \qquad (20) $$

Proof. First, since

$$ |{}_0\mathcal{F}_1^{(d)}(\gamma + q_0; -X|Y)| \leq {}_0\mathcal{F}_1^{(d)}(\gamma + q_0; X|Y) $$

and

$$ c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-Z|X) {}_0\mathcal{F}_1^{(d)}(\gamma + q_0; X|Y)(x_1x_2)^\gamma |x_1 - x_2|^d dx_1 dx_2 < \infty, $$

we see that $\mathcal{H}_\gamma(e_Z)$ is defined for all $z_1, z_2 > 0$. Moreover,

$$ \mathcal{H}_\gamma(e_Z)(y_1, y_2) = $$
$$ c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-Z|X) {}_0\mathcal{F}_1^{(d)}(\gamma + q_0; -X|Y)(x_1x_2)^\gamma |x_1 - x_2|^d dx_1 dx_2 $$
$$ = \sum_\kappa \frac{(-1)^k}{(\gamma + q_0)_\kappa} \frac{1}{k!} \frac{C_\kappa^{(d)}(Y)}{C_\kappa^{(d)}(I)} c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-Z|X) C_\kappa^{(d)}(X)(x_1x_2)^\gamma dV(X, d, 2). $$

By (5.12), this is equal to

$$\sum_{\kappa} \frac{(-1)^k}{(\gamma+q_0)_\kappa} \frac{1}{k!} \frac{C_\kappa^{(d)}(Y)}{C_\kappa^{(d)}(I)} \Gamma_d(\gamma+q_0+\kappa) C_\kappa^{(d)}(\frac{1}{z_1}, \frac{1}{z_2})(z_1 z_2)^{-(\gamma+q_0)}$$
$$= (z_1 z_2)^{-(\gamma+q_0)} \Gamma_d(\gamma+q_0) {}_0\mathcal{F}_0(-\frac{1}{z_1}, -\frac{1}{z_2}|y_1, y_2).$$

Secondly, by Proposition 8.5, to show (20), it is enough to show that

$$(\mathcal{H}_\gamma(e_Z), \mathcal{L}_\kappa^\gamma)_\gamma^2 = [\Gamma_d(\gamma+q_0)]^2 (e_Z, \mathcal{L}_\kappa^\gamma)_\gamma^2 \tag{21}$$

for all κ.

On the one hand, by Lemma 5.2 and Proposition 8.1,

$$(e_Z, \mathcal{L}_\kappa^\gamma)_\gamma$$
$$= c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-Z|X) e^{-(x_1+x_2)} L_\kappa^\gamma(2x_1, 2x_2)(x_1x_2)^\gamma |x_1-x_2|^d dx_1 dx_2$$
$$= c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-(Z+I)|X) L_\kappa^\gamma(2x_1, 2x_2)(x_1x_2)^\gamma |x_1-x_2|^d dx_1 dx_2$$
$$= (\frac{1}{2})^{d+2\gamma+2} \Gamma_d(\gamma+q_0+\kappa)[(\frac{1+z_1}{2})(\frac{1+z_2}{2})]^{-(\gamma+q_0)} C_\kappa^{(d)}(\frac{z_1-1}{z_1+1}, \frac{z_2-1}{z_2+1}).$$

That is

$$(e_Z, \mathcal{L}_\kappa^\gamma)_\gamma = \Gamma_d(\gamma+q_0+\kappa) \prod_{i=1}^{2} (1+z_i)^{-(\gamma+q_0)} C_\kappa^{(d)}(\frac{z_1-1}{z_1+1}, \frac{z_2-1}{z_2+1}). \tag{22}$$

On the other hand,by (19)

$$\mathcal{H}_\gamma(e_Z) = \Gamma_d(\gamma+q_0)(z_1z_2)^{-(\gamma+q_0)} e_{(\frac{1}{z_1}, \frac{1}{z_2})}.$$

Hence

$$(\mathcal{H}_\gamma(e_Z), \mathcal{L}_\kappa^\gamma)_\gamma$$
$$= \Gamma_d(\gamma+q_0)\Gamma_d(\gamma+q_0+\kappa)(z_1z_2)^{-(\gamma+q_0)}(\frac{1+z_1}{z_1} \frac{1+z_2}{z_2})^{-(\gamma+q_0)} C_\kappa^{(d)}(\frac{1-z_1}{1+z_1}, \frac{1-z_2}{1+z_2})$$
$$= \Gamma_d(\gamma+q_0)(-1)^k (e_Z, \mathcal{L}_\kappa^\gamma)_\gamma.$$

We have proved (21).

Theorem 8.8. (Generalized Tricomi Theorem) (cf. [D], [H])

$(1/\Gamma_d(\gamma+q_0))\mathcal{H}_\gamma$ is an involutive isometry of L^2_γ. Moreover, if

$$F(z_1,z_2)=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0(-Z|X)f(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2$$

and

$$G(z_1,z_2)=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0(-Z|X)g(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2$$

then $g=\mathcal{H}_\gamma f$ if and only if

$$G(z_1,z_2)=\Gamma_d(\gamma+q_0)(z_1z_2)^{-(\gamma+q_0)}F(\frac{1}{z_1},\frac{1}{z_2}).$$

Proof. The first part of Theorem 8.8 follows immediately from Proposition 8.7.

Next, put $g_1=\mathcal{H}_\gamma f$,

$$G_1(z_1,z_2)=c_0\int_0^\infty\int_0^\infty {}_0\mathcal{F}_0(-Z|X)g_1(x_1,x_2)(x_1x_2)^\gamma|x_1-x_2|^d dx_1dx_2$$

which exists as an absolutely convergent integral for $z_1,z_2>0$, since $g_1\in L^2_\gamma$.

$$G_1(z_1,z_2)=$$
$$(e_Z,\mathcal{H}_\gamma f)=\Gamma_d(\gamma+q_0)(z_1z_2)^{-(\gamma+q_0)}F(\frac{1}{z_1},\frac{1}{z_2}). \tag{23}$$

If $g=\mathcal{H}_\gamma f$, (23) gives

$$G(z_1,z_2)=\Gamma_d(\gamma+q_0)(z_1z_2)^{-(\gamma+q_0)}F(\frac{1}{z_1},\frac{1}{z_2}).$$

Conversely, the injectivity of Laplace transform and (22) imply

$$g = g_1 = \mathcal{H}_\gamma f$$

almost everywhere.

Finally we have

Proposition 8.9.

$$\mathcal{H}_\gamma(\mathcal{L}_\kappa^\gamma) = (-1)^k \Gamma_d(\gamma + q_0)\mathcal{L}_\kappa^\gamma.$$

Proof. Let $f = \mathcal{H}_\gamma(\mathcal{L}_\kappa^\gamma)$, $g = (-1)^k \Gamma_d(\gamma + q_0)\mathcal{L}_\kappa^\gamma$.

By (19), we have

$$\begin{aligned} c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-Z|X) g(x_1, x_2)(x_1 x_2)^\gamma |x_1 - x_2|^d dx_1 dx_2 \\ = \Gamma_d(\gamma + q_0)(z_1 z_2)^{-(\gamma+q_0)} \\ \cdot c_0 \int_0^\infty \int_0^\infty {}_0\mathcal{F}_0(-\frac{1}{z_1}, -\frac{1}{z_2}|X) f(x_1, x_2)(x_1 x_2)^\gamma |x_1 - x_2|^d dx_1 dx_2. \end{aligned}$$

Now Theorem 8.8 yields Proposition 8.9.

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