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SUBGROUPS OF FINITELY PRESENTED SOLVABLE LINEAR GROUPS

by

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INTRODUCTION

The complexity of the subgroup structure of finitely presented solvable groups has only recently begun to emerge (e.g. see the papers of Gilbert Baumslag [2] - [6]). Among the results is the following theorem of Baumslag [2]: Every finitely generated metabelian group can be embedded in a finitely presented metabelian group. This indicates that the subgroup structure of finitely presented solvable groups is surprisingly complex. In fact in [3] Baumslag has demonstrated the existence of a finitely presented solvable group which is not residually finite.

Despite this knowledge many open problems remain, e.g.: Is a finitely presented solvable group hopfian? In this regard it is worth noting that Philip Hall's old question (see [11]) as to whether a finitely presented solvable group necessarily satisfies the maximal condition for normal subgroups is still open - V.N. Remeslennikov's assertion in [16] is unfortunately incorrect because his group does satisfy the maximal condition for normal subgroups.

The object of this paper is to throw some additional light on the nature of finitely presented solvable groups by proving the following:

Theorem 1:

Let R be a commutative associative ring with 1 and n a positive integer. Further, let G be a finitely generated multiplicative group of $n \times n$ triangular matrices over R. Then there exists a commutative associative ring S with 1

and a finitely presented multiplicative group H of n×n triangular matrices over S with G embeddable in H. Moreover if R is also a domain, then S can be taken to be the field of fractions of R.
(The analogue of this Theorem for associative algebras has recently been obtained by Gilbert Baumslag [7].)

Remeslennikov [17] has pointed out that if G is any finitely generated metabelian group there exists a commutative associative ring R with 1 such that G is embedded in the multiplicative group of 2×2 triangular matrices over R. Hence for n equal to 2, Theorem 1 reduces to Baumslag's Theorem [2].

Theorem 1 admits a slight, but pleasing, generalization for solvable linear groups:

<u>Theorem 2</u>:

Every finitely generated solvable linear group can be embedded in a finitely presented solvable linear group. Here a linear group is simply a multiplicative group of non-singular matrices with coefficients in some commutative field.

There are some other consequences of Theorem 1 that are worth observing. First of all we note that every countable free nilpotent group of any given class c has a faithful representation as a linear group (e.g. see B.A.F. Wehrfritz [19], p.34). It is not difficult to embed this linear representation into a finitely generated solvable linear group and thereby prove

Corollary 1:

A free nilpotent group of any given class c and countable rank can be embedded in a finitely presented solvable linear group.

Similarly using the appropriate matrix representations due to D.I. Eidel'kind [8] in the first case and C.K. Gupta [9] in the second, one can also prove the further corollaries

Corollary 2:

A finitely generated free group in the product variety $\underline{\underline{N}}_c\underline{\underline{A}}$ can be embedded in a finitely presented linear group in the same variety (here $\underline{\underline{N}}_c$ is the variety of all nilpotent groups with class less than or equal to c and $\underline{\underline{A}}$ is the variety of all abelian groups).

Corollary 3:

The free center-by-metabelian group of rank 3 can be embedded in a finitely presented solvable linear group.

It should be noted that the result of D.I. Eidel'kind [8] was also proved by N.S. Romanovskii [18] and C.K. Gupta and N.D. Gupta [10].

CHAPTER 1

PRELIMINARIES

In this chapter we establish the general algebraic definitions and notation used in the sequel, and state some of the basic results that will be required.

Section 1. Presentations of Groups

Given a subset X of a group G, gp(X) denotes the subgroup of G generated by the elements of X (if X is empty, gp(X) = 1). We denote by $gp_G(X)$ the normal subgroup of G generated by the elements of X, that is, the least normal subgroup of G containing X. Thus $gp_G(X) = gp(g^{-1}xg \mid x \in X, g \in G)$. A subgroup H of G is finitely generated if there is a finite subset X of H with H = gp(X). A normal subgroup H of G is finitely generated as a normal subgroup of G if $H = gp_G(X)$ for some finite subset X of H. On occasion we shall briefly write $H \leq G$ to indicate that H is a subgroup of G and $H \trianglelefteq G$ to indicate that H is a normal subgroup of G.

The following notational conventions will be observed: if h, k are elements of a group G and H, K are subgroups of G, then

$h^k = k^{-1}hk$, (the conjugate of h by k)

$H^k = \{h^k \mid h \in H\}$,

$[h, k] = h^{-1}k^{-1}hk$, (the commutator of h and k)

$[H, K] = gp([h, k] \mid h \in H, k \in K)$.

The remainder of this section is devoted to giving groups algebraic descriptions via presentations. A standard source for the notions to be discussed is Magnus, Karrass and Solitar [13].

Definition 1.1: A group F is a free group freely generated by the subset X if

(i) $F = gp(X)$

(ii) for any group H and any set mapping $\theta: X \longmapsto H$ there exists a homomorphism $\psi: F \longmapsto H$ which agrees with θ on X.

The set X is termed a free basis for F. Property (ii) is described as the universal mapping property for the set X, with respect to all groups. The cardinality of the set X is an invariant of the free group F, termed the rank of F. We note that if X is any abstract set there exists a free group F freely generated by X.

Let F be a free group freely generated by the set X, and $\psi: F \longmapsto G$ a homomorphism of F onto a group G defined by $x \longmapsto g_x$, $(x \in X)$. Let N denote the kernel of ψ. Then ψ induces an isomorphism of the quotient F/N onto G given by $xN \longmapsto g_x$, $(x \in X)$. Suppose R is a subset of N with $N = gp_F(R)$. The expression $\langle X; R\rangle$ is called a presentation for G under the mapping $x \longmapsto g_x$, $(x \in X)$ and we say $\langle X; R\rangle$ presents G. The set X is usually termed a set of generators for G subject to the defining relators R. The presentation $\langle X; R\rangle$ is a finite presentation if both X and R are finite sets. We call a group G finitely presentable (or finitely presented) if G has a finite presentation.

Notice that $\langle X; R\rangle$ presents the factor group F/N under the mapping $x \longmapsto xN$, $(x \in X)$. Hence we can think of the presentation $\langle X; R\rangle$ as defining a group by simply letting F be a free group, freely generated by the set X and then writing $\langle X; R\rangle = F/gp_F(R)$.

There are some consequences of these definitions which will be helpful in the sequel. First of all let F be a free group freely generated by the set X, and let $\theta: X \longmapsto H$ be a mapping of X into a group H. Then θ extends to a homomorphism ψ of F into H given by $w(x)\psi = w(x\psi) = w(x\theta)$ for any element w(x) in F. Let R be a subset of F such that $r(x\theta) = 1$ for every element r(x) of R. Then the mapping θ extends to a homomorphism of $\langle X; R\rangle$ into H.

Next, suppose G has the presentation $\langle X; R\rangle$ under the mapping $x \longmapsto g_x$, $(x \in X)$. Let N be a normal subgroup of G and suppose $N = gp_G(Y)$ for some subset Y of N. Let F be the free group on the set X. For each $y \in Y$, select an element $s_y \in F$ in the preimage of y under the homomorphism given by $x \longmapsto g_x$, $(x \in X)$ and put $S = \{ s_y \mid y \in Y\}$. Then the factor G/N has the presentation $\langle X; R, S\rangle$ under the mapping $x \longmapsto g_xN$, $(x \in X)$. Therefore if G is finitely presented and N is finitely generated as a normal subgroup of G we see that G/N is a finitely presented group.

Finally, let G be a group and N a normal subgroup of G.

It will be necessary to compute a presentation for G from presentations for G/N and N. To this end suppose that G/N has the presentation $\langle X;\, R\rangle$ under the mapping $x \longleftrightarrow g_xN$, $(x \in X)$ and that N has the presentation $\langle Y;\, S\rangle$ under the mapping $y \longmapsto n_y$, $(y \in Y)$. Let $\{\bar{g}_x \mid x \in X\}$ be a set of representatives of the cosets g_xN, $(x \in X)$. Since $N \trianglelefteq G$ there is an element $A_{x,y}$ in N so that

$$\bar{g}_x^{-1} n_y \bar{g}_x = A_{x,y}, \; (x \in X,\; y \in Y).$$

Moreover for each $r = r(x)$ in R we have

$$N = r(g_xN) = r(\bar{g}_xN) = r(\bar{g}_x)N$$

under the mapping $x \longleftrightarrow g_xN$, $(x \in X)$. So we can choose elements B_r of N such that

$$r(\bar{g}_x) = B_r, \; (r \in R).$$

Let F be the free group on the set Y. For each $x \in X$, $y \in Y$ choose $T_{x,y} \in F$ in the preimage of $A_{x,y}$ and for each $r \in R$ choose $U_r \in F$ in the preimage of B_r under the homomorphism of choose $T_{x,y} \in F$ in the preimage of $A_{x,y}$ and for each $r \in R$ presentation

$$\langle X,\; Y;\; S,\; x^{-1}yxT_{x,y}^{-1}\; (x \in X,\; y \in Y),\; rU_r^{-1}\; (r \in R)\rangle$$

under the mapping $x \longmapsto \bar{g}_x$, $(x \in X)$, $y \longmapsto n_y$, $(y \in Y)$. Thus if both G/N and N are finitely presented groups then G is also finitely presented.

Section 2. Solvable Groups of Matrices and Metabelian Groups

Let G be a group. Subgroups $\delta_n G$ of G are defined inductively by the rule:

$\delta_0 G = G$, and

$\delta_n G = [\delta_{n-1} G, \delta_{n-1} G]$ for positive integers n.

$\delta_n G$ is called the n-th derived group of G. $\delta_1 G$ is also called the commutator subgroup of G and is usually written as G'. We also write G'' in place of $\delta_2 G$.

Definition 1.2: A group G is called solvable if $\delta_d G = 1$ for some integer d.

If G is a solvable group, the least integer d for which $\delta_d G = 1$ is called the derived depth of G. If G'' = 1, G is said to be metabelian.

A large class of solvable groups is provided by the multiplicative groups of matrices over commutative rings. Specifically, let R be a commutative associative ring with 1 and n a positive integer. Denote by GL(n,R) the multiplicative group of units in the ring of all n×n matrices with coefficients in R. A matrix is triangular if all its entries above the main diagonal are zero. The set of triangular matrices in GL(n,R) forms a subgroup which is denoted by Tr(n,R). It is easy to check that if G is a subgroup of Tr(n,R), then G is solvable and $\delta_n G = 1$. On the other hand it will be of interest to note the

Theorem 1.3: (V.N. Remeslennikov [17])

If G is a metabelian group then there exists a commutative associative ring R with 1 such that G is embedded in Tr(2,R).

A particularly important class of groups which have representations as matrices over commutative rings are the linear groups.

<u>Definition 1.4</u>: A group G is said to be a linear group if there exists a commutative field k and a positive integer n with G embedded in GL(n,k).

The following theorem of A.I. Mal'cev shows that solvable linear groups are almost triangular.

<u>Theorem 1.5</u>: (A.I. Mal'cev [14])

Let k be an algebraically closed commutative field and n a positive integer. Further, let G be a solvable subgroup of GL(n,k). Then there exists a subgroup H of G and an element x of GL(n,k) such that H is of finite index in G and H^x is a subgroup of Tr(n,k).

Metabelian groups will play a central role in the results to follow. Anticipating this importance we shall record some of the properties of these groups.

<u>Definition 1.6</u>: A group G is said to satisfy the maximal condition for normal subgroups if every ascending chain of normal subgroups of G stabilizes in a finite number of steps.

Equivalently, G satisfies the maximal condition for normal subgroups if every normal subgroup of G is finitely generated as a normal subgroup.

<u>Theorem 1.7</u>: (P. Hall [11])

Every finitely generated metabelian group satisfies the maximal condition for normal subgroups.

Notice that if G is a finitely presented metabelian group and N is a normal subgroup of G, then, as G satisfies the maximal condition for normal subgroups, the factor group G/N must be finitely presented. Thus the class of finitely presented metabelian groups is an image closed class.

Philip Hall also points out in [11] that if N is a normal subgroup of a group G and if H and K are subgroups of G such that $H \leq K$, $HN = KN$, and $H \cap N = K \cap N$ then $H = K$. So an infinite tower of groups in G implies an infinite tower in either N or G/N if not both. It follows from this fact that if both N and G/N satisfy the maximal condition for normal subgroups so does G.

The last result to be recorded in this section is a technical lemma due to G. Baumslag concerning defining relators in certain metabelian groups. Before stating this result we establish some useful notation:
let G be a group and a, t elements of G. Let x be an indeterminate over the integers Z and

$$f(x) = c_0 + c_1 x + \cdots + c_d x^d$$

be an element of the polynomial ring Z[x]. Then we define

$$a^{f(t)} = a^{c_0}(a^{c_1})^{t} \cdots (a^{c_d})^{t^d} \quad \text{in } G.$$

We can now record

<u>Lemma 1.8</u>: (G. Baumslag [2])

Let a, b, t, u be elements of a group and let d be a positive integer. Suppose that

$$[t, u] = 1$$

and that

$$[a, b^{t^n}] = 1 \quad \text{whenever} \quad -d \leq n \leq d.$$

In addition suppose that

$$a^u = a^{f(t)}, \quad b^u = b^{f(t)}$$

where $f(x) = 1 + c_1x + \cdots + c_{d-1}x^{d-1} + x^d$ is an element of the polynomial ring $Z[x]$. Then

$$[a, b^{t^i u^j}] = 1 \quad \text{for all integers } i, j.$$

Section 3. Some Commutative Algebra

In this section we record some notions from commutative algebra. Throughout the remainder of this paper a ring will be taken to mean a commutative associative ring with 1 unless specified otherwise. The facts about these rings which are recorded here may be readily found in Atiyah and Macdonald [1].

Let R be a ring and X a subset of R. By rg(X) we mean the least subring of R containing the set $X \cup \{1\}$. A ring R is said to be finitely generated if R = rg(X) for some finite subset X of R.

Recall that if R_i, $(i \in I)$ is an indexed collection of rings we may form their direct sum $\bigoplus_{i \in I} R_i$ with addition and multiplication defined coordinatewise on their Cartesian product. Thus if $\mathfrak{a}_i$, $(i \in I)$ is a collection of ideals of a ring R, the mapping from R into the direct sum $\bigoplus_{i \in I} R/\mathfrak{a}_i$ of the quotient rings $R/\mathfrak{a}_i$, $(i \in I)$ given by $r \longmapsto \sum_{i \in I} (r + \mathfrak{a}_i)$, $(r \in R)$ is a ring homomorphism, whose kernel is equal to the intersection $\bigcap_{i \in I} \mathfrak{a}_i$ of the ideals $\mathfrak{a}_i$, $(i \in I)$.

Let R be a ring. An element r of R is a zero divisor if $ar = 0$ for some non-zero element a of R; otherwise r is called a non-zero divisor. If the only zero divisor of R is 0, R is said to be a domain. An element r is called nilpotent if $r^n = 0$ for some positive integer n. Finally, an element s of R is termed a unit if s is invertible in R, that is, if there exists an element s^{-1} of R with $s^{-1}s = 1$. A unit s of R has finite order if $s^n = 1$ for some positive integer n, the least such integer being the order of s. If a unit does not have finite order, it is said to have infinite order.

Notice that if r is nilpotent and w is any element of R then $1 - wr$ is a unit: if $r^n = 0$, then $(wr)^n = 0$; and so $1 = 1 - (wr)^n = (1 + wr + \cdots + (wr)^{n-1})(1 - wr)$.

<u>Definition 1.9</u>: An ideal $\mathfrak{q}$ of a ring R is called a primary ideal if every zero divisor in the quotient ring $R/\mathfrak{q}$ is nilpotent.

An ideal $\mathfrak{a}$ of a ring R is said to have a primary decomposition if there exists a finite number of primary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_p$ of R with $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_p$. We record

<u>Theorem 1.10</u>: (Lasker- Noether)

If R is a finitely generated ring every ideal has a primary decomposition.

In particular, the zero ideal (0) of a finitely generated ring has a primary decomposition, and this fact shall be required below.

Next, we recall the notion of localization. Let R be a ring, and let S be a multiplicatively closed set of non-zero divisors in R containing 1. A ring RS^{-1} is a localization of R with respect to S if there is an injective ring homomorphism $R \longmapsto RS^{-1}$ such that the image of S in RS^{-1} is contained in the group of units of RS^{-1}. That RS^{-1} exists is established by the construction which follows. Let $\sim$ be the equivalence relation on $R \times S$ defined by:

$$(r, s) \sim (r', s') \quad \text{if and only if } rs' = sr'.$$

We denote the set of equivalence classes in $R \times S$ by RS^{-1} and the equivalence class containing (r, s) by rs^{-1}. RS^{-1} is turned into a ring by defining addition and multiplication in the usual way:

$$rs^{-1} + r_1 s_1^{-1} = (rs_1 + sr_1)(ss_1)^{-1},$$
$$rs^{-1} \cdot r_1 s_1^{-1} = rr_1(ss_1)^{-1}.$$

The mapping $r \longmapsto r1^{-1}$, $(r \in R)$ is an injective ring homomorphism of R into RS^{-1}. Abusing the notation we have just introduced we denote $r1^{-1}$ by r and $1s^{-1}$ by s^{-1}: notice that for every element s in S, $s \cdot s^{-1} = 1$ in RS^{-1}. If R is a domain and $S = R \setminus \{0\}$, the ring RS^{-1} is the field of fractions of R.

A result we shall need below is due to G. Baumslag. We record this as

<u>Theorem 1.11</u>: (G. Baumslag [2])

If R is a finitely generated ring and s is a unit of infinite order, then there exists a non-zero divisor f in

R of the form

$$f = 1 + c_1 s + \cdots + c_{d-1} s^{d-1} + s^d$$

where d, $c_1, \ldots, c_{d-1}$ are integers and d is positive.

We shall also require for the sequel, the concept of an R-module. For this purpose we shall assume that R is an associative ring with 1 which may not be commutative. Recall that

<u>Definition 1.12</u>: An R-module M is an abelian group (written additively) together with an action $R \times M \longmapsto M$, the image of (r, m) denoted by rm, $(r \in R,\ m \in M)$, satisfying

(i) $r(m + n) = rm + rn$, $(r \in R,\ m, n \in M)$

(ii) $(r + s)m = rm + sm$, $(r, s \in R,\ m \in M)$

(iii) $(rs)m = r(sm)$, $(r, s \in R,\ m \in M)$

(iv) $1m = m$, $(m \in M)$.

Let M be an R-module and X a subset of M. By R-mod(X) we mean the least R-submodule of M containing the set X. An R- module M is said to be finitely generated if M = R-mod(X) for some finite subset X of M. An R-module M is said to satisfy the maximal condition for R-submodules if every ascending chain of R-submodules of M stabilizes in a finite number of steps. Equivalently, M satisfies the maximal condition for R-submodules if every R-submodule of M is finitely generated as an R-module. The theorem we wish to record is

<u>Theorem 1.13</u>:

Let R be a finitely generated commutative ring. If M is a finitely generated R-module, then M satisfies the maximal condition for R-submodules.

These notions can be utilized to examine the normal subgroup structure of certain groups. Recall that with any multiplicative group G there is associated its integral group ring ZG. An element of ZG is a formal sum $\sum n_g \cdot g$, g ranging over the elements of G, where the integer n_g is equal to zero for all but a finite number of g. Addition and multiplication are defined in ZG by

$$\sum m_g \cdot g + \sum n_g \cdot g = \sum (m_g + n_g) \cdot g ,$$

$$(\sum m_g \cdot g)(\sum n_g \cdot g) = \sum (\sum_h m_{gh^{-1}} n_h) \cdot g .$$

The element n of Z is identified with the element $n \cdot 1$ of ZG and the element g of G is identified with the element $1 \cdot g$ of ZG, so that Z and G are to be regarded as subsets of ZG.

Notice that ZG is an associative ring with 1 and if $G = gp(X)$, then $ZG = rg(X \cup X^{-1})$ where $X \cup X^{-1}$ denotes the elements of X and their inverses. Observe also that if G is an abelian group then ZG is a commutative ring.

Let G be a multiplicative group with a multiplicative subgroup H and an abelian normal subgroup M which is written additively. Then we can turn M into a ZH-module as follows: since $M \trianglelefteq G$, for each $h \in H$ and $m \in M$ we can define the element hm of M by $hm = m^{h^{-1}}$. We then extend this action to all of ZH additively, that is, if $r = \sum n_h h$ is an element of ZH and $m \in M$ we define the element rm of M by the rule $rm = \sum n_h(hm)$. The verification that this action of ZH on M gives M the structure of a ZH-module is straight-forward.

Suppose $M = gp_G(Y)$ for some subset Y of M. Then $M = gp(y^g \mid y \in Y, g \in G)$ and so, as we are writing M additively, the elements of M have the form

$$m = \sum_{y,\,g} n_{y,g} y^g ,$$

where the integer $n_{y,g}$ is equal to zero for all but a finite number of pairs y, g. Turn M into a ZG-module in the manner described above. Then with this structure we see that every element of M has the form

$$\begin{aligned} m &= \sum_{y,\,g} n_{y,g}(g^{-1}y) \\ &= \sum_{y} r_y y , \end{aligned}$$

where the element r_y of ZG is equal to zero for all but a finite number of y. It follows therefore that M is equal to ZG-mod(Y). Hence we see that M is finitely generated as a normal subgroup of G if and only if M, as a ZG-module, is finitely generated as a ZG-module.

Section 4. Varieties of Groups

Let F be a free group freely generated by a set X. A word w(x) is an element of F. If G is a group and $\theta: X \longmapsto G$ a mapping of X into G, then the image $w(x\theta)$ of the word w(x) under the homomorphism defined by θ is called a value of the word $w(x)$ in G. A word $w(x)$ is a law (or identical relation) in the group G if the only possible value of $w(x)$ in G is 1. Equivalently, if $w(x)$ has the form $w(x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ distinct elements of X, $w(x_1, \ldots, x_n)$ is a law in G if $w(g_1, \ldots, g_n) = 1$ for every choice of elements $g_1, \ldots, g_n$ of G.

<u>Definition 1.14</u>: A variety $\underline{\underline{W}}$ of groups is the class of all groups that satisfy a given set W of laws.

The verbal subgroup W(G) of a group G corresponding to a set W of words is the subgroup generated by all values in G of the words of W:

$$W(G) = gp(w(g_1, \ldots, g_n) \mid w(x_1, \ldots, x_n) \in W,\ g_1, \ldots, g_n \in G).$$

Notice that the words $w \in W$ are laws in G if and only if $W(G) = 1$.

<u>Definition 1.15</u>: A group G is a relatively free group in the variety $\underline{\underline{W}}$ (or a free $\underline{\underline{W}}$-group) freely generated by a subset X if

(i) $G = gp(X)$

(ii) for any group H in the variety $\underline{\underline{W}}$ and any set mapping $\theta: X \longmapsto H$ there exists a homomorphism $\psi: G \longmapsto H$ which agrees with θ on X.

The set X is termed a free $\underline{\underline{W}}$-basis for G. Property (ii) is described as the universal mapping property for the set X, with respect to all groups in $\underline{\underline{W}}$. The cardinality of the set X is an invariant of the free $\underline{\underline{W}}$-group G, termed the rank of G.

As a consequence of this definition, if G is a free

$\underline{\underline{W}}$-group, G has a representation as a factor group of a free group F of appropriate rank by the verbal subgroup W(F): $G \cong F/W(F)$.

A variety of groups is also characterized by the closure properties with respect to isomorphisms, subgroups, factor groups and (unrestricted) Cartesian products. It follows that if $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are varieties, the class of all groups that have a normal subgroup in $\underline{\underline{U}}$ and the factor group in $\underline{\underline{V}}$ forms a variety. We call this the product variety $\underline{\underline{U}}\,\underline{\underline{V}}$ of $\underline{\underline{U}}$ and $\underline{\underline{V}}$; if G is any group the corresponding verbal subgroup of G is U(V(G)). This multiplication of varieties is associative; we may, therefore, dispense with brackets in products of three or more varieties; and we may write, unambiguously, $\underline{\underline{U}}^n$, for positive integers n, for the repeated product of $\underline{\underline{U}}$ by itself.

We denote by $\underline{\underline{A}}$ the variety of abelian groups. Then $\underline{\underline{A}}^2$ is the variety of metabelian groups (taken in the wide sense to include the abelian groups); and, more generally, $\underline{\underline{A}}^d$ is the variety of all solvable groups of derived depth at most d. If G is any group, the verbal subgroup $A^d(G)$ of G corresponding to the variety $\underline{\underline{A}}^d$ is equal to $\delta_d G$, the d-th derived group of G.

We shall also be interested in the variety $\underline{\underline{N}}_c$ of all groups which are nilpotent of class at most a given positive integer c and the variety $C(\underline{\underline{A}}^2)$ of center-by-metabelian groups. The corresponding verbal subgroups which characterize these varieties are defined as follows. If G is any group define subgroups $\gamma_n G$ of G inductively by the rule:

$\gamma_1 G = G$, and

$\gamma_n G = [\gamma_{n-1} G, G]$ for $n \geq 2$.

$\gamma_n G$ is called the n-th lower central subgroup of G. The verbal subgroup $N_c(G)$ corresponding to the variety $\underline{\underline{N}}_c$ is equal to $\gamma_{c+1} G$. The variety $C(\underline{\underline{A}}^2)$ is easily characterized by the property: a group G is in the variety $C(\underline{\underline{A}}^2)$ if and only if $[G'', G] = 1$.

CHAPTER 2

EMBEDDING THEOREMS FOR SOME SOLVABLE GROUPS

Section 1. Subgroups of Tr(n,R)

Let R denote a commutative associative ring with 1 and n a positive integer. We shall examine special subgroups of Tr(n,R).

Definition 2.1: Let X be a set of units in R:
$G(n,X) = \{(a_{ij}) \in Tr(n, rg(X \cup X^{-1})) \mid a_{ii} \in gp(X)\ (1 \leq i \leq n)\}$.
Here $rg(X \cup X^{-1})$ is a subring with 1 of R generated by the elements of X and their inverses, and gp(X) is the subgroup of the group of units of R generated by the elements of X.

We now examine the structure of G(n,X). First notice that G(1,X) is isomorphic to gp(X) in the obvious way. Let $n \geqslant 2$. If A is an element of G(n,X) then we can write

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & & & \\ \vdots & & A' & \\ a_n & & & \end{pmatrix},$$

where $A' \in G(n-1,X)$ is simply the lower right $(n-1)\times(n-1)$ submatrix of A. By straight-forward multiplication of the lower triangular matrices we see that the mapping given by $A \longmapsto A'$, $(A \in G(n,X))$ is a homomorphism of G(n,X) onto G(n-1,X) with kernel

$$K(n,X) = \left\{ A \in G(n,X) \mid A = \begin{pmatrix} a_1 & 0 \ \cdots \ 0 \\ a_2 & \\ \vdots & 1_{n-1} \\ a_n & \end{pmatrix} \right\}.$$

(1_k is the k×k identity matrix in Tr(k,R).)

We record these observations as

Lemma 2.2:

G(1,X) is naturally isomorphic to gp(X) and for $n \geq 2$, $K(n,X) \trianglelefteq G(n,X)$ with $G(n,X)/K(n,X) \cong G(n-1,X)$.

Next, for $n \geq 2$, let $\overline{G}(n-1,X)$ be the subgroup of G(n,X) given by

$$\overline{G}(n-1,X) = \left\{ \begin{pmatrix} 1 & 0 \ \cdots \ 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} \;\middle|\; A \in G(n-1,X) \right\}.$$

Notice that the mapping given by

$$A \longmapsto \begin{pmatrix} 1 & 0 \ \cdots \ 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}, \quad (A \in G(n-1,X))$$

is an isomorphism from G(n-1,X) onto $\overline{G}(n-1,X)$. Further, we easily see that $G(n,X) = gp(K(n,X), \overline{G}(n-1,X))$ and $K(n,X) \cap \overline{G}(n-1,X) = 1_n$. So we have proved

Lemma 2.3:

$G(1,X)$ is naturally isomorphic to $gp(X)$ and for $n \geq 2$, $K(n,X) \trianglelefteq G(n,X)$, $G(n,X) = gp(K(n,X), \bar{G}(n-1,X))$, $K(n,X) \cap \bar{G}(n-1,X) = 1_n$ and $\bar{G}(n-1,X)$ is naturally isomorphic to $G(n-1,X)$.

We now turn to an analysis of the groups $K(n,X)$. Let $n \geq 2$. For each i, $(2 \leq i \leq n)$ and element $a \in rg(X \cup X^{-1})$ let $A_i(a)$ be the $n \times n$ lower triangular matrix with entry a in the i-th row and first column, 1 everyplace on the main diagonal, zero elsewhere. For each element $b \in gp(X)$ let $D(b)$ denote the $n \times n$ diagonal matrix with entry b in the first position on the main diagonal and the entry 1 thereafter. Obviously these matrices are contained in $K(n,X)$. We list some of the easily verified properties of these matrices:

$$
(1)\left\{
\begin{array}{ll}
D(b_1)^{n_1} \cdots D(b_r)^{n_r} = D(b_1^{n_1} \cdots b_r^{n_r}) & \\
\qquad\qquad (n_1, \ldots, n_r \in Z) & \\
[D(b), D(b')] = 1_n & \\
A_i(a_1)^{n_1} \cdots A_i(a_r)^{n_r} = A_i(n_1 a_1 + \cdots + n_r a_r) & \\
\qquad\qquad (n_1, \ldots, n_r \in Z,\ 2 \leq i \leq n) & \\
[A_i(a), A_{i'}(a')] = 1_n \ (2 \leq i, i' \leq n) & \\
A_i(a)^{D(b)} = A_i(ab) \qquad (2 \leq i \leq n), &
\end{array}
\right.
$$

and

$$D(b)A_2(a_2)\cdots A_n(a_n) = \begin{pmatrix} b & 0 \cdots 0 \\ a_2 & \\ \vdots & 1_{n-1} \\ a_n & \end{pmatrix} .$$

If we put $A_i = A_i(1)$ for each i, $(2 \leq i \leq n)$, it follows that we have proved

Lemma 2.4:

Let $n \geq 2$. Then $K(n,X) = gp(A_2, \ldots, A_n, D(x) \mid x \in X)$ is a metabelian group.

Corollary 2.5:

Let $n \geq 2$. Let X, X' be sets of units of R with $gp(X) \leq gp(X')$. Then $K(n,X) \leq K(n,X')$.

Proof

From above, $K(n,X) = gp(A_2, \ldots, A_n, D(x) \mid x \in X)$ and $K(n,X') = gp(A_2, \ldots, A_n, D(x') \mid x' \in X')$. Because $gp(X) \leq gp(X')$ we see that the subgroup $gp(D(x) \mid x \in X)$ of $K(n,X)$ is a subgroup of the subgroup $gp(D(x') \mid x' \in X')$ of $K(n,X')$. Hence

$$\begin{aligned} K(n,X) &= gp(A_2, \ldots, A_n, D(x) \mid x \in X) \\ &\leq gp(A_2, \ldots, A_n, D(x') \mid x' \in X') \\ &= K(n,X'). \end{aligned}$$

Corollary 2.6:

Let X, X' be sets of units of R with $gp(X) \leq gp(X')$. Then $G(n,X) \leq G(n,X')$ for every positive integer n.

Proof

The proof is by induction on n. If $n = 1$, we clearly have $G(1,X) \leq G(1,X')$. Let $n \geqslant 2$ and assume inductively that $G(k,X) \leq G(k,X')$ whenever k is less than n. Recall from Lemma 2.3 that for $n \geqslant 2$,

$$G(n,X) = gp(K(n,X), \overline{G}(n-1,X))$$

with $\overline{G}(n-1,X)$ isomorphic to $G(n-1,X)$ under the mapping

$$A \longmapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}, \quad (A \in G(n-1,X)).$$

Inductively, $G(n-1,X) \leq G(n-1,X')$ and so we see that $\overline{G}(n-1,X) \leq \overline{G}(n-1,X')$. By Corollary 2.5, $K(n,X) \leq K(n,X')$. Therefore

$$\begin{aligned} G(n,X) &= gp(K(n,X), \overline{G}(n-1,X)) \\ &\leq gp(K(n,X'), \overline{G}(n-1,X')) \\ &= G(n,X'). \end{aligned}$$

It is interesting to note that the normal subgroup structure of $G(n,X)$ is determined by $gp(X)$.

Theorem 2.7:

Let n be a positive integer. Then $G(n,X)$ satisfies the maximal condition for normal subgroups if and only if $gp(X)$ is finitely generated.

Proof

First suppose $gp(X)$ is finitely generated. We show

$G(n,X)$ satisfies the maximal condition for normal subgroups by induction on n. If $n = 1$, $G(1,X) \cong gp(X)$ is a finitely generated abelian group and the result is obvious. Let $n \geqslant 2$ and assume that $G(k,X)$ satisfies the maximal condition for normal subgroups whenever k is less than n. By Lemma 2.2, we have $K(n,X) \trianglelefteq G(n,X)$ with $G(n,X)/K(n,X)$ isomorphic to $G(n-1,X)$. Inductively $G(n-1,X)$ satisfies the maximal condition for normal subgroups. Using Corollary 2.5 and Lemma 2.4 jointly, since $gp(X)$ is finitely generated, it follows that $K(n,X)$ is a finitely generated metabelian group. Invoking Theorem 1.7, $K(n,X)$ satisfies the maximal condition for normal subgroups. Therefore, since both $K(n,X)$ and $G(n,X)/K(n,X)$ satisfy the maximal condition for normal subgroups we see that $G(n,X)$ also satisfies the maximal condition for normal subgroups.

Conversely, assume $G(n,X)$ satisfies the maximal condition for normal subgroups. Since $G(1,X) \cong gp(X)$ and for any $n \geqslant 2$, $G(n,X)/K(n,X) \cong G(n-1,X)$ we see that $gp(X)$ is a homomorphic image of $G(n,X)$. Therefore, $gp(X)$ satisfies the maximal condition for normal subgroups: for an infinite tower of normal subgroups in $gp(X)$ would imply an infinite tower of normal subgroups in $G(n,X)$. But $gp(X)$ is abelian and so it must be finitely generated.

We next investigate subsets of units X of R for which the group $G(n,X)$ is finitely presented. Clearly we must require that $gp(X)$ be a finitely generated group and we notice that $G(1,X)$ is finitely presented whenever X is finite. To say more we need the following lemma.

Lemma 2.8:

Let $n \geq 2$. Let X be a finite set of units of R of the form

$$X = \{ s_1, \ldots, s_q, s_{q+1}, \ldots, s_r, f_1, \ldots, f_q \}$$

where $s_1, \ldots, s_q$ are of infinite order, $s_{q+1}, \ldots, s_r$ are of finite order and for each i, $(1 \leq i \leq q)$, f_i has the form

$$(2) \qquad f_i = 1 + c_{i,1}s_i + \cdots + c_{i,d_i-1}s_i^{d_i-1} + s_i^{d_i}$$

where d_i, $c_{i,1}, \ldots, c_{i,d_i-1}$ are integers and d_i is positive.

Then $K(n,X)$ is a finitely presented group.

Proof

To show $K(n,X)$ is finitely presented we shall construct a finitely presented metabelian group M which has $K(n,X)$ as a homomorphic image. Since the class of finitely presented metabelian groups is an image closed class (see Theorem 1.7), this will insure that $K(n,X)$ is finitely presented

Because $X = \{s_1, \ldots, s_q, s_{q+1}, \ldots, s_r, f_1, \ldots, f_q\}$, Lemma 2.4 shows that

$$(3) \qquad K(n,X) = gp(A_2, \ldots, A_n, D(s_i), D(f_j) \mid 1 \leq i \leq r, \; 1 \leq j \leq q).$$

Suppose for definiteness that the units $s_{q+1}, \ldots, s_r$ are respectively of finite order $e_{q+1}, \ldots, e_r$. We shall present M on the generators

$$(4) \qquad Y = \{a_2, \ldots, a_n, t_1, \ldots, t_q, t_{q+1}, \ldots, t_r, u_1, \ldots, u_q\}$$

where the integers n, q, r are the integers occurring in the description of K(n,X) (see (3)).

The defining relations of M are of four kinds. First we have the power relations

$$(5) \qquad t_i^{e_i} = 1 \qquad (q+1 \leq i \leq r).$$

Where here the positive integer e_i is the order of the unit s_i, $(q+1 \leq i \leq r)$. Next we have the commutativity relations

$$(6) \quad \begin{cases} [t_i, t_j] = 1 & (1 \leq i, j \leq r) \\ [t_i, u_j] = 1 & (1 \leq i \leq r, 1 \leq j \leq q) \\ [u_i, u_j] = 1 & (1 \leq i, j \leq q). \end{cases}$$

Thirdly we have the commutativity relations for the conjugates of the generators a_i:

$$(7) \quad \begin{cases} [a_i, a_j^w] = 1 \qquad \text{where } 2 \leq i, j \leq n \text{ and} \\ \quad w \in \{t_1^{n_1} \cdots t_r^{n_r} \mid -d_i \leq n_i < d_i \ (1 \leq i \leq q), \\ \qquad\qquad 0 \leq n_i < e_i \ (q+1 \leq i \leq r)\}. \end{cases}$$

Here the positive integer d_i occurring in (7) is simply the "degree" of the element f_i (see (2)), $(1 \leq i \leq q)$.

Finally we have the defining relations giving the action of the elements u_i on the elements a_j:

(8) $$a_j^{u_i} = a_j^{f_i(t_i)} \qquad (2 \leq j \leq n,\ 1 \leq i \leq q),$$

where $f_i(x) = 1 + c_{i,1}x + \cdots + c_{i,d_i-1}x^{d_i-1} + x^{d_i}$ is an element of the polynomial ring $Z[x]$ (again see (2)).

We emphasize that M is the group generated by the set Y given by (4) subject to the defining relations (5), (6), (7) and (8). M is patently finitely presented. We now make use of Lemma 1.8 to prove M is metabelian. To this end denote again by $a_2, \ldots, a_n, t_1, \ldots, u_q$ the images of the elements of the generating set Y in the group M and put $A = gp_M(a_2, \ldots, a_n)$. It follows from the definition (4) of Y and the defining relations (6) that the factor group M/A is abelian. So it suffices to prove that A is abelian.

It follows from (6) that A is generated by the elements

(9) $$a_i^w,\ (2 \leq i \leq n,\ w \in \{t_1^{n_1} \cdots u_q^{m_q} \mid n_1, \ldots, m_q \in Z\}).$$

So we have to prove that the elements in (9) commute. Notice that $[x, y]^z = [x^z, y^z]$ is a commutator relation which holds for any elements x, y, z in any group. It follows from this remark and the defining relations (5) and (6) that in order to prove that the elements in (9) commute it is enough to prove that

$$(10)\quad \begin{cases} [a_i, a_j^{w}] = 1 & \text{where } 2 \leq i,\ j \leq n \text{ and} \\ \quad w \in \{t_1^{n_1}\cdots u_q^{m_q} \mid n_1, \ldots, m_q \in Z, \\ \qquad\qquad 0 \leq n_i < e_i\ (q+1 \leq i \leq r)\}. \end{cases}$$

In fact (10) follows easily by a repeated application of Lemma 1.8. Indeed, in view of the defining relations (6), (7) and (8) of M, Lemma 1.8 can first be applied to show that

$$[a_i, a_j^{w}] = 1 \qquad \text{for } 2 \leq i,\ j \leq n \text{ and}$$
$$w \in \{t_1^{n_1}\cdots t_q^{n_q} t_{q+1}^{n_{q+1}}\cdots t_r^{n_r} u_1^{m_1} \mid n_1, m_1 \in Z,$$
$$-d_i \leq n_i \leq d_i\ (2 \leq i \leq q),$$
$$0 \leq n_i < e_i\ (q+1 \leq i \leq r)\}.$$

We emphasize that here the first and last exponents n_1 and m_1 are allowed to range over all the integers, but that the other exponents are still restricted. However on applying Lemma 1.8 q-1 more times it follows that (10) holds, as required. Therefore M is a finitely presented metabelian group.

Now define a map θ from the generating set Y into K(n,X) by the rule

$$\theta: \begin{cases} a_i \longmapsto A_i & (2 \leq i \leq n) \\ t_i \longmapsto D(s_i) & (1 \leq i \leq r) \\ u_i \longmapsto D(f_i) & (1 \leq i \leq q). \end{cases}$$

It follows, upon examining (2), the relations satisfied in K(n,X) (see (1)) and the defining relations (5), (6), (7) and (8) of M, that θ extends to a homomorphism from M onto K(n,X). Hence K(n,X) is finitely presented as desired.

We can now prove

Theorem 2.9:

Let X be a finite set of units of R of the form

$X = \{s_1, \ldots, s_q, s_{q+1}, \ldots, s_r, f_1, \ldots, f_q\}$

where $s_1, \ldots, s_q$ are of infinite order, $s_{q+1}, \ldots, s_r$ are of finite order and for each i, $(1 \le i \le q)$, f_i has the form

$$f_i = 1 + c_{i,1}s_i + \cdots + c_{i,d_i-1}s_i^{d_i-1} + s_i^{d_i}$$

where $d_i, c_{i,1}, \ldots, c_{i,d_i-1}$ are integers and d_i is positive. Then G(n,X) is a finitely presented group for every positive integer n.

Proof

The proof is by induction on n. If n = 1, G(1,X) is isomorphic to gp(X) and so is finitely presented because X is finite. Let $n \geq 2$ and assume inductively that G(k,X) is finitely presented whenever k is less than n. From Lemma 2.2 we have $K(n,X) \trianglelefteq G(n,X)$ and G(n,X)/K(n,X) is isomorphic to G(n-1,X). Inductively, G(n-1,X) is finitely presented. In view of Lemma 2.8 above, K(n,X) is also finitely presented. It follows then that G(n,X) is a finitely presented group.

We now prove the main result.

Theorem 2.10:

Let R be a commutative associative ring with 1 and n a positive integer. Let G be a finitely generated subgroup of Tr(n,R). Then there exists a commutative associative ring S with 1 and a finitely presented subgroup H of Tr(n,S) with G embedded in H. Moreover if R is also a domain, then S can be taken to be the field of fractions of R.

Proof

Suppose $G = gp(g_1, \ldots, g_m)$ with m finite.

First assume that R is a domain. Let S be the field of fractions of R and consider the generators $g_1, \ldots, g_m$ of G as elements of Tr(n,S). Let X denote the set of all non-zero entries appearing in the generating matrices of G. Notice that X is a finite set of units of R and, as G(n,X) contains the generators of G, that G is embedded in G(n,X). Suppose that $X = \{s_1, \ldots, s_q, s_{q+1}, \ldots, s_r\}$ where $s_1, \ldots, s_q$ are of infinite order and $s_{q+1}, \ldots, s_r$ are of finite order. For each s_i, $(1 \leq i \leq q)$ let f_i be the unit in S given by $f_i = 1 + s_i$. Put

$$X' = \{s_1, \ldots, s_q, s_{q+1}, \ldots, s_r, f_1, \ldots, f_q\} ,$$

and observe that since X is a subset of X', G(n,X) is a subgroup of G(n,X') (see Corollary 2.6). Thus G is embedded in G(n,X'). Finally, by Theorem 2.9, we see, upon examining X', that G(n,X') is a finitely presented group.

In general, let X denote the set of all entries appearing in the matrices $g_1, \ldots, g_m, g_1^{-1}, \ldots, g_m^{-1}$ of G as a subgroup of Tr(n,R). Then G is a subgroup of Tr(n,rg(X)) and so we can assume that R is a finitely generated ring. By Theorem 1.10 (Lasker-Noether), there exists a primary decomposition of the zero ideal, say $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_p$. Let R_i, $(1 \leq i \leq p)$ be the quotient ring $R/\mathfrak{q}_i$. Then R is embedded in the direct sum R' of the rings R_i, $(1 \leq i \leq p)$. For each ring R_i, $(1 \leq i \leq p)$ let S_i be the multiplicatively closed set of all non-zero divisors of R_i and embed R_i into the localization $R_iS_i^{-1}$ of R_i with respect to S_i. Then R_i is embedded in $R_iS_i^{-1}$ in such a way that the images of elements of S_i in $R_iS_i^{-1}$ are units of $R_iS_i^{-1}$. Further, every zero divisor r_i of R_i is nilpotent and therefore has the form $r_i = 1 - (1 - r_i)$, where $1 - r_i$ is a unit of R_i, $(1 \leq i \leq p)$. Now form the direct sum R'' of the localizations $R_iS_i^{-1}$, $(1 \leq i \leq p)$ and notice that the direct sum R' of the rings R_i, $(1 \leq i \leq p)$ is embedded in R''. Thus R has been embedded in R''. The crucial observation is that the image in R'' of any element of R can be written as a finite sum of units of R''.

We can now consider the generators $g_1, \ldots, g_m$ of G in Tr(n,R) as elements of Tr(n,R''). By the above construction of R'', every entry appearing below the main diagonal of the matrices $g_1, \ldots, g_m$ is a finite sum of units of R''. Therefore we can choose a finite set X_1' of units of R" such that the elements of R" appearing below the main diagonal of the generating matrices $g_1, \ldots, g_m$ are all contained in the subring $rg(X_1')$ of R". In addition let X_2' denote the set of all entries in R" appearing on the main diagonal of the generating matrices $g_1, \ldots, g_m$. Since the matrices $g_1, \ldots, g_m$ are elements of $Tr(n,R")$, we see that X_2' is a finite set of units of R". Now put $X' = X_1' \cup X_2'$ and notice that X' is a finite set of units of R". Because every entry appearing in the generating matrices $g_1, \ldots, g_m$ of G is an element of $rg(X')$ and X' is a set of units of R", we have G embedded in $G(n,X')$ as a subgroup of $Tr(n,R")$.

Therefore we can assume R is a finitely generated ring with a finite set X of units and G is embedded in $G(n,X)$ as a subgroup of $Tr(n,R)$. Suppose that X is given by

$$X = \{s_1, \ldots, s_q, s_{q+1}, \ldots, s_r\}$$

where $s_1, \ldots, s_q$ are of infinite order and $s_{q+1}, \ldots, s_r$ are of finite order. According to Theorem 1.11, for each s_i of infinite order, $(1 \leq i \leq q)$ there exists a non-zero divisor f_i in R of the form

$$f_i = 1 + c_{i,1}s_i + \cdots + c_{i,d_i-1}s_i^{d_i-1} + s_i^{d_i}$$

where $d_i, c_{i,1}, \ldots, c_{i,d_i-1}$ are integers and d_i is positive. Let F denote the semi-group with 1 of R generated by the elements $f_1, \dots, f_q$ and notice that F is a set of non-zero divisors of R. Let $S = RF^{-1}$ be the localization of R with respect to F. Then R is embedded in S. If we now consider the elements of

$$X' = \{s_1, \dots, s_q, s_{q+1}, \dots, s_r, f_1, \dots, f_q\}$$

as elements of S, then we see that X' is a set of units of S which contains X. Therefore we have G embedded in G(n,X') as a subgroup of Tr(n,S). Finally, by Theorem 2.9, upon examining X', we see that G(n,X') is a finitely presented group. This completes the proof.

It is worth noting that the group H of Theorem 2.10 actually has a normal series of subgroups

$$1 = H_0 \leq H_1 \leq \dots \leq H_n = H$$

in which the sucessive factor groups H_{i+1}/H_i , $(0 \leq i \leq n-1)$ are all finitely presented and metabelian.

Section 2. Metabelian Groups and Solvable Linear Groups

In this section we point out two interesting consequences of the results of Section 1. First we can prove

Theorem 2.11: (G. Baumslag [2])

Every finitely generated metabelian group can be embedded in a finitely presented metabelian group.

Proof

Let G be a finitely generated metabelian group. By

Theorem 1.3, there exists a commutative associative ring R with 1 such that G is embedded in Tr(2,R). Invoking Theorem 2.10, there exists a commutative associative ring S with 1 and a finitely presented subgroup H of Tr(2,S) with G embedded in H. Finally, we see that $H'' = 1$.

The proof of the next result involves properties of wreath products. Recall that a group W is the (standard) wreath product of its subgroups G and T if

(i) $W = gp(G, T)$

(ii) $B = gp_W(G)$, the normal closure of G in W, is the direct sum of its subgroups G^t, $(t \in T)$.

B is called the base group of W, and W is usually denoted by $G \wr T$. It can be shown that if H is a subgroup of K, then $H \wr T$ is a subgroup of $K \wr T$. An important property of wreath products which will be required is

<u>Theorem 2.12</u>: (L. Kaloujnine, M. Krasner [12])

Let G be a group with a normal subgroup N of finite index in G. Then G is embedded in the wreath product $N \wr G/N$ of N by G/N.

It is now easy to prove

<u>Theorem 2.13</u>:

Every finitely generated solvable linear group can be embedded in a finitely presented solvable linear group.

<u>Proof</u>

Let G be a finitely generated solvable subgroup of GL(n,k). We can assume that k is algebraically closed. By

Theorem 1.5, there exists a subgroup H of G with finite index and an element x of GL(n,k) such that H^x is a subgroup of Tr(n,k). Replacing G by G^x if necessary, we can assume that H is a subgroup of Tr(n,k). Since G is finitely generated and H has finite index in G, H is also finitely generated and we can even assume that H is a normal subgroup of G. From Theorem 2.10, H is a subgroup of a finitely presented subgroup K of Tr(n,k). Since G/H is finite, G is embedded in the wreath product $H \wr G/H$ of H by G/H (see Theorem 2.12). Let $W = K \wr G/H$ be the wreath product of K by G/H and notice that $H \wr G/H$ is a subgroup of W . Therefore G has been embedded in W.

We observe that since G/K is finite, the base group B of W is the direct sum of finitely many isomorphic copies of K. Therefore B is a finitely presented group and so it follows that W is a finitely presented group. Next, since W is the wreath product of the solvable groups K and G/H, it too is solvable. Finally, K is a subgroup of GL(n,k) and G/H is finite so that W is isomorphic to a linear group over k (see B.A.F. Wehrfritz [19], p. 151).

Section 3. Solvable Relatively Free Groups

Theorem 2.14:

A free nilpotent group of any given class c and countable rank can be embedded in a finitely presented solvable linear group.

Proof

To begin, let x be a single indeterminate over Z and for each $r = 1, 2, \dots$ let $x_r = (x_r(i, j))$ be an element of $Tr(c+1,k)$ where k is the quotient field of the polynomial ring $Z[x]$ and

$$x_r(i,j) = \begin{cases} 0 & (i-j \geq 2 \text{ or } i-j \leq -1) \\ 1 & (i-j = 0) \\ x^{2^{cr+i-2}} & (i-j = 1), \end{cases}$$

(here i denotes the row and j the column of the entry $x_r(i,j)$). Then $F = gp(\, x_r \mid r = 1, 2, \dots\,)$ is a free nilpotent group of class c on the generating set $X = \{ x_r \mid r = 1, 2, \dots \}$ (see again B.A.F. Wehrfritz [19], p. 34).

We next embed F in a finitely generated subgroup G of $Tr(c+1,k)$. To this end, let $A_{i+1,i}$ be the matrix in $Tr(c+1,k)$ with entry 1 in the $(i+1)$-th row and i-th column, 1 everyplace on the main diagonal, zero elsewhere $(1 \leq i \leq c)$. For each j, $(1 \leq j \leq c)$ let D_j be the diagonal matrix in $Tr(c+1,k)$ with entry x in the j-th position on the main diagonal and 1 elsewhere. Let G be the subgroup of $Tr(c+1,k)$ generated by the elements $A_{i+1,i}$, $(1 \leq i \leq c)$ and D_j, $(1 \leq j \leq c)$. An easy calculation shows that for each $r = 1, 2, \dots$

$$x_r = A_{2,1}^{D_1^{2^{cr}}} A_{3,2}^{D_2^{2^{cr+1}}} \cdots A_{c+1,c}^{D_c^{2^{cr+c-1}}} .$$

Thus F is a subgroup of G. It follows that we have proved the Theorem.

C.K. Gupta and N.D. Gupta [10] pointed out that a relatively free group in the product variety $\underline{\underline{N}}_c\underline{\underline{A}}$ has a faithful representation in Tr(c+1,k) for some suitably chosen commutative field k. So it follows at once from Theorem 2.10 that we can prove

Theorem 2.15:

A finitely generated free group in the product variety $\underline{\underline{N}}_c\underline{\underline{A}}$ can be embedded in a finitely presented linear group in the same variety.

Similarly using the linear representation given by C.K. Gupta [9], one can also prove the further corollary to Theorem 2.13.

Theorem 2.16:

The free center-by-metabelian group of rank 3 can be embedded in a finitely presented solvable linear group.

CHAPTER 3

A COMMENT ON A PROBLEM OF PHILIP HALL

Philip Hall [11] asked whether a finitely presented solvable group necessarily satisfies the maximal condition for normal subgroups. In this chapter we remark that V.N. Remeslennikov's example [16] of a finitely presented solvable group G without the maximal condition for normal subgroups is incorrect. In view of the importance of the problem involved we shall indicate how Remeslennikov's example fails. To do this first recall that

$$G = \langle a, x, y;\ [x, y],\ a^y = a\,a^x,\ [a, a^x]^x = [a, a^x],\ [[a, a^x], a] \rangle.$$

It turns out that $G' = gp_G(a)$ is nilpotent of class 2.

Put $M = gp_G([a, a^x])$. The factor group G/M can be presented by

$$G/M = \langle \bar{a}, \bar{x}, \bar{y};\ [\bar{x}, \bar{y}],\ \bar{a}^{\bar{y}} = \bar{a}\,\bar{a}^{\bar{x}},\ [\bar{a}, \bar{a}^{\bar{x}}] \rangle$$

under the mapping $\bar{a} \longmapsto aM$, $\bar{x} \longmapsto xM$, $\bar{y} \longmapsto yM$. It follows from Lemma 1.8 that G/M is metabelian. Since G/M is finitely generated it satisfies the maximal condition for normal subgroups (Theorem 1.7). Therefore, G satisfies the maximal condition for normal subgroups if and only if M contains no infinite tower of normal subgroups of G (see the comment following Theorem 1.7).

Because $M = gp_G([a, a^x])$ and $G' = gp_G(a)$ is nilpotent of class 2, we see that M is an abelian normal subgroup of G whose elements commute with the elements of G'. Given any normal subgroup N of G contained in M we turn N into a ZG-module by extending the action of G on N, given by conjugating elements of N by elements of G, additively to all of ZG. We then notice that a normal subgroup N of G which is contained in M is finitely generated as a normal subgroup of G if and only if N, as a ZG-module, is finitely generated as a ZG-module. Therefore G satisfies the maximal condition for normal subgroups if and only if M, as a ZG-module, satisfies the maximal condition for ZG-submodules.

Put $H = gp(x, y)$. Then $G = gp(G', H)$. Notice that H is a 2-generator abelian group and so ZH is a finitely generated commutative ring. Let N be any ZG-submodule of the ZG-module M. We can also consider N as a ZH-module by extending the conjugation action of H on N to all of ZH. Observe, because the elements of M commute with the elements of G' and since $G = gp(G', H)$, that $N = ZG\text{-mod}(Y)$ if and only if $N = ZH\text{-mod}(Y)$ for some subset Y of N. It follows from this remark that M satisfies the maximal condition for ZG-submodules if and only if M, as a ZH-module, satisfies the maximal condition for ZH-submodules.

Finally notice that, because $G = gp(G', H)$,

$$M = gp_G([a, a^x])$$

$$= gp([a, a^x]^{g'h} \mid g' \in G', h \in H)$$
$$= gp([a, a^x]^h \mid h \in H),$$

and therefore $M = ZH\text{-mod}([a, a^x])$ is a 1-generator ZH-module. Hence, by Theorem 1.13, M satisfies the maximal condition for ZH-submodules. Thus we have proved that G satisfies the maximal condition for normal subgroups.

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