72-1007

SAUKIN, Walter P., 1943-
THE NATURE OF THERMAL AND FLOW FIELDS IN
THE VICINITY OF THERMAL DISCONTINUITIES.

The City University of New York, Ph.D., 1971
Engineering, mechanical

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THE NATURE OF THERMAL AND FLOW FIELDS IN THE VICINITY OF THERMAL DISCONTINUITIES

by

WALTER P. SAUKIN

A dissertation submitted to the Graduate
Faculty in Engineering in partial fulfillment
of the requirement for the degree of Doctor
of Philosophy, The City University of New York.

1971

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

5/4/71
Date

Sheldon Weinbaum
Chairman of Examining Committee

4 May 1971
Date

Executive Officer

Latif M. Jiji
Robert Pfeffer
James R. Steven
C. M. Tchen
Sheldon Weinbaum, Chairman

Supervisory Committee

The City University of New York

I dedicate this to my loving wife, Olga and my mama and papa.

Acknowledgements

I would like to take this opportunity to thank the many people who have helped me with my dissertation.

I would like to give my deepest thanks to my advisor, Professor Sheldon Weinbaum, for his continuous guidance and encouragement. I shall long remember his friendly and tireless efforts on my behalf. I am also most grateful to my co-advisor, Professor Latif M. Jiji, whose personal understanding and advice were most reinforcing and helpful. I wish to thank my Doctoral Committee members, Professor C. M. Tchen, Professor Robert Pfeffer and Professor James R. Steven, for their interest and effort.

I would like to express my appreciation to The City College Fund and The City University of New York for the financial assistance which they gave me during my graduate career.

I am also grateful to my mother, may she rest in peace, my father, and my wife's family, all of whom provided me with encouragement when I really needed it.

For my wife, Olga, who gave of herself without question during my endeavor, words can not express what my heart feels.

Table of Contents

List of Figures

Nomenclature

$A, A', A_{1,2}$	Constants
C_n, C'_n, C''_n	nth arbitrary constant
G	Grashof number $\frac{g_x \beta_1 (T_1 - T_\infty) r_c^3}{\nu^2}$
$H_\nu^{(2)}$	Hankel function of second kind and order ν
K	coefficient of conductivity
$\overline{K}$	coefficient of isothermal compressibility
K_ν	modified Bessel function of the second kind and order ν
L	distance from the leading edge of the plate, Fig. 3
P'	pressure associated with fluid motion
P_∞	far field pressure
Pr	Prandtl number $\frac{\mu C_p}{K}$
R_∞	Reynolds number $\frac{U_\infty \rho_\infty L}{\mu}$
T	local temperature
T_1, T_2	constant temperature on a plate, Fig. 1, 2, 3, 4
T_∞	constant free stream temperature, Fig. 1, 2, 3, 4
U_∞	constant free stream velocity
a_o	constant
b	constant
C_p	constant pressure specific heat
g_x, g_y	x,y-component body forces respectively
k	complex variable
m, n	integers
r	local dimensionless radial distance
r_c	Stokes thermal radius
u	local dimensionless x-component of velocity

u_∞	far field x-component of velocity
u_o	x-component of velocity at the surface
$\bar{u}$	vector expression for general field velocity
u_c	characteristic x-component of velocity
v	local y-component of velocity
v_∞	far field y-component of velocity
v_c	characteristic y-component of velocity
v_o	y-component of velocity at the surface
v_ϕ, v_r	respectively velocity in tangential and radial directions
x,y	space coordinates
α	slope of velocity profile near plate
α_1	$\beta_1(T_1-T_\infty)$
β	slope of Pohlhausen solution near plate
β_1	coefficient of thermal expansion
Γ	gamma function
γ	$=(\frac{1}{2\lambda}-1)$
Δ	small positive constant
δ_T, δ_V	are respectively thermal and velocity boundary layer thicknesses
ΔT	largest temperature difference for the flow
∇	vector gradient operator
∇^2	Laplacian operator
$\nabla^2\nabla^2$	biharmonic operator
ε	small positive constant
ω	nondimensional form of local temperature $(\frac{T-T_1}{T_2-T_1})$
ω_n	n^{th} term of the iterative solution for temperature

θ	nondimensional form of local temperature $\left(\frac{T-T_\infty}{T_1-T_\infty}\right)$
θ_n	n^{th} term of the iterative solution for temperature
θ_{n_p}	n^{th} particular solution
$\hat{\theta}$	nondimensional form of local temperature $\left(\frac{T-T_\infty}{\Delta T}\right)$
● =	$\theta - (1+\beta y)$
$\bar{●}, \bar{\theta}$	Fourier transformation of ●, θ.
λ	values from 1 to 1/2, see Fig. 4
μ	viscosity
ν	kinematic viscosity
ξ	variable of integration
ρ_∞	freestream density
ϕ	angle measured counter clockwise, Fig. 1, 2, 3, 4
ϕ_1	constant angle, Fig. 1, 2
ψ	streamfunction
ψ_c	characteristic streamfunction value
ψ_n	n^{th} term in iterative expression for streamfunction
ψ_{n_c}	n^{th} complimentary solution
ψ_{n_p}	n^{th} particular solution
superscript*	its presence or lack of presence represents dimensional or nondimensional variables respectively.

Abstract

This investigation concerns the steady two-dimensional flow of a fluid in the vicinity of solid boundaries with discontinuous thermal boundary conditions. In particular we shall examine the local temperature field for forced convection near (i) the apex of an isothermal wedge of arbitrary angle, (ii) a wall ramp juncture with a step jump in temperature at their intersection and (iii) a step jump in temperature on a semi-infinite flat plate with a well established velocity and temperature boundary layer. The last problem (iv) looked into here is that of the thermal and velocity fields for natural convection near the apex of an isothermal vertical wedge of arbitrary angle. For problems (i), (ii), and (iii), locally valid series solutions for the thermal field are presented. While in the natural convection problem (iv), the solutions for the thermal field and streamfunction are truncated after two terms. The solution procedure is similar in concept to that first introduced by Carrier and Lin[10] for determining the velocity field near the leading edge of a flat plate. The generating functions and undetermined arbitrary constants that appear in the series solution are determined by matching with Oseen type approximations of the global problem for some of the simpler geometries. Of particular interest is the technique used for determining the unknown constants when the Fourier integral for the corresponding global problem cannot be inverted to obtain a local field solution. In these cases, the unknown constants are determined by matching with a global solution for the heat transfer or temperature along a boundary surface in the region of overlap. Such boundary solutions for the global problem are frequently easier to obtain than a solution in a region of space.

Chapter 1

Introduction

This investigation considers four related problems involving separately forced convection of an incompressible fluid and natural convection in the immediate vicinity of discontinuous changes in thermal boundary conditions

(1.1) Background for the Forced Convection Problems

Problem (i), see Fig. 1, is the symmetrical flow of a fluid at temperature T_∞ over an isothermal semi-infinite wedge of half angle ϕ_1 whose wall temperature is maintained at temperature T_1. The velocity field is assumed known and given by the idealized description of the symmetric potential flow solution for the flow past an infinite wedge of the same half angle. The present interest is to determine the temperature distribution in the neighborhood of the apex. This idealized treatment of the velocity field, which neglects vorticity diffusion near the boundary, obviously overestimates convective effects in the region of interest. The approximation for the velocity field should therefore be viewed in the same spirit as the equivalent Oseen type approximation of the energy equation presented in Carrier, Krook, and Pearson[8] and Arpaci[9] for the forced convection past an isothermal plate and channel of constant height, respectively, where slug velocity profiles are used. The isothermal plate case is in fact identical to the present problem with $\phi_1 = o$. This idealized formulation has the additional important purpose, in the present paper, of motivating the matching procedure for determining the unknown constants and the generating functions in the local series solution. The matching procedure so developed is also then applied to problems (ii), (iii), and (iv).

Problem (ii) treats forced convection over a semi-infinite forward facing ramp with wall temperature T_1 and angle ϕ_1 in which the incoming flow is parallel to and has the same temperature T_∞ as the planar wall upstream of the ramp, (see Fig. 2). The assumed velocity field, is idealized in the same manner given in problem (i). Its description also being given by the symmetric potential flow solution for flow past an acute angle wedge. The basic difference between problems (i) and (ii) is in the thermal boundary condition applied at $\phi = \pi$ which for problem (i) is the symmetry condition $\partial T^*/\partial\phi = o$ and for the problem (ii) a constant wall temperature T_∞. Using the above idealization of the velocity field, one observes that the upper half plane in problem (i) can also be viewed as the flow past a ramp with an adiabatic upstream wall.

Problem (iii), see Fig. 3, considers a uniform freestream flow at temperature T_∞ over a semi-infinite flat plate in which there is a step jump in temperature distribution from T_1 to T_2 at a distance sufficiently far downstream from the leading edge for well established velocity and temperature boundary layers to develop. In this problem, in contrast to problems (i) and (ii), we shall attempt to accurately model the detailed structure of the velocity boundary layer by using the series solution of the well established Blasius profile.

Previous analysis of forced convection with discontinuous boundary conditions have been tested primarily from the boundary layer point of view. Problem (i), symmetrical flow about an isothermal wedge, has been treated within the boundary layer framework by Tifford[1]. Similarly Pohlhausen[2] has treated the laminar forced convection over a semi-infinite isothermal flat plate. The semi-infinite plate with step jump in temperature has been considered by several authors[3-7] using approximate boundary layer techniques. The ramp problem shown in Fig. 2 for ϕ_1 o has not been considered previously.

In the large, the boundary layer solutions in the last paragraph have been based on integral techniques and similarity transformations applied to the governing boundary layer equations. This type of analysis is not applicable in the neighborhood of a discontinuous thermal boundary condition for in this localized region streamwise conduction cannot be neglected since the fluid experiences thermal gradients which are of the same order of magnitude in the transverse and longitudinal directions. Rubesin[7], in a preliminary investigation, points out the importance of streamwise conduction in a region which contains steep thermal gradients in the streamwise direction. The neglect of this conduction term results in the inability of the boundary layer solution to describe temperature disturbances upstream of the discontinuity and yields erroneous predictions for temperature and heat transfer up to some unknown point downstream of the discontinuity. Two important exceptions to the above boundary layer treatments are the investigations by Carrier, Krook, and Pearson[8] mentioned previously and Imai[15]. Imai[15] examines the uniform freestream flow past a semi-infinite flat plate in which the thermal boundary condition changes discontinuously from an adiabatic wall to an isothermal wall after a well established velocity boundary layer has been achieved. The problem differs from our problem (iii) in that our upstream wall boundary condition is isothermal and that the temperature of the upstream flow at infinity need not be the same as the upstream wall temperatures. Thus, problem (iii) is more general in that it includes an incoming thermal boundary layer.

(1.2) Background for the Natural Convection Problem

The last problem (iv) involves natural convection of a fluid whose far field temperature is T_∞ and velocity is zero about the leading edge of a semi-infinite isothermal vertical wedge at temperature T_1, Fig. 4. Here, both the thermal and velocity fields are of interest.

To date, natural convection about a semi-infinite isothermal vertical wedge has been treated from the boundary layer point of view. This type of analysis is not applicable in the leading edge region where the approximations of Prandtl boundary layer theory are not valid. The formulation of the wedge boundary layer problem is basically the same as that for its limiting case, the vertical plate, Ostrach[11], except that for the wedge the driving force in the longitudinal momentum equation is based on the component of the gravitational force in the direction parallel to the surface of the wedge, see Sparrow, Eichhorn, and Gregg[12]. Thus, all the boundary layer type investigations for the vertical plate are applicable in general to the vertical wedge with the proper modification of the buoyancy force.

The boundary layer type investigations do not apply to the non-similar flow in the neighborhood of the leading edge. For example, at the thermal leading edge, conduction in the streamwise direction results in thermal deviations upstream which in turn give rise to fluid motion upstream of the plate. Furthermore, the flow in the leading edge region is essentially a thermally induced slow flow in which both components of velocity are of the same order of magnitude and the normal pressure gradient may not be neglected. For these reasons, the boundary layer analysis is inapplicable to the leading edge flow.

No previous analytical study has satisfactorily treated the above effects on the leading edge temperature and velocity fields. Suriano and Yang[13] present finite difference solutions of the Navier-Stokes momentum and energy equations for natural convection over finite isothermal vertical and horizontal plates. Results are given for Rayleigh numbers (PrG) from less than one to three hundred based on two Prandtl numbers (Pr equals .72 and 10.0). The Rayleigh number is used to correlate field behavior. It was found that in the low Grashof number (G) range the energy transfer is dominated by heat conduction in the entire region covered by the numerical mesh, while thermal convection prevailed over most of the computed domain at the highest Grashof numbers. While these solutions provide a description of the overall flow pattern, the grid size and domain of numerical integration preclude an accurate description of the detail behavior in the leading edge region.

Yang and Jerger[14] also consider a finite isothermal vertical plate using a perturbation expansion of the Navier-Stokes equations based on a lowest order similar solution to the boundary layer equations. This procedure is satisfactory for taking into account the departures from similar behavior in the boundary layer region. However, such a solution does not apply in a region where the boundary layer equations are not valid to lowest order.

Scherberg[15] attempted to take into account the leading edge effects in the framework of boundary layer theory by assuming a finite velocity and temperature profile at the leading edge of the plate. The purpose of these initial conditions was to simulate the effect of conduction upstream of the plate. This approach is unsatifactory in that the boundary layer equations are not valid in this region and the assumed profiles are not known. Scherberg suggests that the proper technique for handling the problem is to develop a multiple matching akin to that used by Carrier and Lin[10] for the flat plate leading edge. This approach is developed herein.

(1.3) Method of Solution

In the present analysis, the thermal fields within a characteristic diffusion length of the discontinuity in thermal boundary conditions are described by an energy equation which includes the streamwise conduction term. For the natural convection problem, the flow field within a thermal characteristic diffusion length of the leading edge of the wedge is also determined based on governing equations which include the transverse momentum equation. The thermal diffusion length, termed the thermal Stokes radius, is deduced separately for each problem based on an order of magnitude analysis of the energy equation. This radius is defined, by analogy with the Stokes radius for momentum diffusion, as that radial distance within which conduction dominates convection as the principal mode of heat transfer. Thus, in each case, an approximate analytic solution to the local thermal field and in the natural convection problem an additional solution for the local streamfunction shall be shought in the form of a series based on Stokes like generating functions. The solution for higher order terms follows the iterative procedure developed by Carrier and Lin[10] for the velocity field near the leading edge of a flat plate and extended by several authors, Dean and Montagnon[16], Moffatt[17], Schwiderski and Lugt[18], and Weinbaum[19] for the Stokes like flow very near the apex of finite angle wedges.

In the Carrier and Lin analysis the unknown constants in the iterative series solution are determined by matching with the Blasius boundary layer solution at a radial distance comparable to a Stokes radius from the leading edge. This procedure cannot be justified unless one can first demonstrate that there is indeed a region of overlapping validity, near the leading edge, of the Stokes slow flow equations and the Prandtl boundary layer equations Furthermore, the boundary layer equations without interaction do not have elliptic features and thus are not able to describe upstream influences. Thus, if the leading edge flow were to approach the Blasius similar solution at some distance downstream of the leading edge after passing through a non-similar region where the parabolic boundary layer equations are valid the local solution at the leading edge would not be influenced by this downstream influence. These same arguments, of course, apply to the present problems. There is no reason to expect the locally valid series solution in the leading edge region to have a region of overlapping validity with existing boundary layer similarity or integral solutions.

Global solutions of elliptic problems even when linearized are often difficult to obtain and it is suggested in this paper that when it is not possible to match with a neighboring large scale solution in a region of overlapping validity the remaining constants in the local iterative solution can still be determined by matching with a solution which provides only local surface or centerline information from the large scale problem. This local boundary information from the global problem is also to be used to determine which generating functions are to be used in constructing the local iterative series solutions. This technique for determining the generating functions and the unknown constants in the local series solution is simpler than matching with local global field solutions since transform techniques more readily yield localized boundary solutions, as in the application of Weiner-Hopf technique, than field inversions. Thus, the iterative solution used in conjunction with a local transform solution for the heat transfer or temperature along a boundary can yield a complete or an approximate truncated solution, depending on the accuracy of the boundary information provided, for the local thermal field and, if the equations are coupled, a solution for the local velocity field can be obtained.

Chapter 2

Formulation

(2.1) Formulation of Forced Convection Problems

Consider first the problem of the two-dimensional flow of an incompressible fluid with a uniform velocity U_∞ and temperature T_∞ upstream over a semi-infinite flat plate with a step jump in its temperature distribution, see Fig. 3. The leading portion of the plate is maintained at temperature T_1 while the remainder of the plate is at temperature T_2. The region of the discontinuity is at a distance L sufficiently downstream of the leading edge of the plate for well established velocity and temperature boundary layers to obtain. The dimensional continuity, momentum, and energy equations assuming constant fluid properties μ, C_p, K, and negligible viscous dissipation for an incompressible flow are

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

$$\rho_\infty\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p'^*}{\partial x^*} + \mu\nabla^2 u^* \tag{2}$$

$$\rho_\infty\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p'^*}{\partial y^*} + \mu\nabla^2 v^* \tag{3}$$

$$\rho_\infty c_p\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right) = K\nabla^2 T^* + \left(u^*\frac{\partial p'^*}{\partial x^*} + v^*\frac{\partial p'^*}{\partial y^*}\right) \tag{4}$$

To nondimensionalize equations (1)-(4), two characteristic lengths must be introduced. One is characteristic of the velocity boundary layer thickness δ_v at the temperature discontinuity and a second r_c is characteristic of a local thermal diffusion length which is as yet undefined. Thus, for the momentum equations, we introduce the following nondimensional variables:

$$\frac{x^*}{L} = x \ , \quad \frac{y^*}{\delta_v} = y \ , \quad \frac{u^*}{U_\infty} = u \ , \quad \frac{v^* L}{U_\infty \delta_v} = v \ , \quad \frac{P'^*}{\rho_\infty U_\infty^2} = P'$$

If one assumes that in the region of interest inertial, pressure, and viscous forces are of the same order, then $R_\infty\left(\frac{\delta_v}{L}\right)^2$ must be of order unity. Thus for $R_\infty >> 1$, equations (2) and (3) reduce to the Blasius boundary layer equations for flow over a flat plate.

For the energy equation (4), the following set of nondimensional variables is used where the same characteristic length r_c is used to nondimensionalize the x^* and y^* coordinates in the vicinity of the temperature discontinuity

$$\frac{u^*}{u_c} = u \ , \quad \frac{v^*}{v_c} = v \ , \quad \frac{x^*}{r_c} = x \ , \quad \frac{y^*}{r_c} = y \ , \quad \frac{T^* - T_\infty}{\Delta T} = \hat{\Theta}$$

The characteristic reference velocities u_c and v_c are obtained from the Blasius solution and are of order $\frac{u_\infty r_c}{\delta_v}$ and $\frac{u_c r_c}{L}$ respectively where δ_v is of order $\frac{L}{R_\infty^{1/2}}$. Using these definitions and the fact that $\frac{\partial P'}{\partial x} = \frac{\partial P'}{\partial y} = 0$ the energy equation in nondimensional form may be written as

$$R^{3/2} Pr \left(\frac{r_c}{L}\right)^2 \left(u \frac{\partial \hat{\Theta}}{\partial x} + \frac{r_c}{L} v \frac{\partial \hat{\Theta}}{\partial y}\right) = \nabla^2 \hat{\Theta} \tag{5}$$

In the same sense that the Stokes radius for momentum transport is defined as the distance from a vorticity source in which vorticity diffusion dominates vorticity convection, the thermal Stokes radius is defined as the distance from a thermal source in which conduction heat transfer dominates convection heat transfer. Thus, one defines r_c by equating the coefficients of the conduction and convection terms in equation (5)

$$r_c = \frac{\delta_v}{R^{1/4} Pr^{1/2}} \tag{6}$$

Equation (5) with the definition of r_c given by (6) becomes to order $\frac{r_c}{L}$

$$u \frac{\partial \Theta}{\partial x} = \nabla^2 \Theta \tag{7}$$

where for convenience θ is defined as $\left(\frac{T - T_\infty}{T_1 - T_\infty}\right)$

It is observed that the qualitative behavior of the flow in the region of interest is determined by the relative magnitude of the thermal Stokes radius and the velocity boundary layer thickness. This, to the author's knowledge, is the first time that a forced convection problem with a thermal discontinuity has been formulated using these two length scales.

In addition to the governing equation (7), θ must satisfy the wall boundary conditions

$$x < 0 \quad y = 0 \quad \theta = 1 \;, \quad x > 0 \quad y = 0 \quad \theta = \left(\frac{T_2 - T_\infty}{T_1 - T_\infty}\right) \tag{8}$$

and match with the Pohlhausen solution for the incoming thermal boundary layer at distances large compared to r_c upstream of x=0.

The order of magnitude analysis presented above can be readily generalized for problems (i) and (ii) when the wall ramp juncture is embedded in a well established velocity boundary layer on either an adiabatic or isothermal wall. The only modification is that velocity components would be of the same order of magnitude near the juncture and both terms in $\bar{u} \cdot \nabla\theta$ would be retained provided compression work can be neglected. The later is tantamount to requiring the Eckert number $\frac{u_\infty^2}{C_p \Delta T} << 1$. Thus, for a finite ramp angle, the dimensional energy equation in polar coordinates becomes

$$\frac{\rho_\infty C_p}{K}\left(v_r^* \frac{\partial T^*}{\partial r^*} + \frac{v_\phi^*}{r^*}\frac{\partial T^*}{\partial \phi}\right) = \nabla^2 T^* \tag{9}$$

where v_r^* and v_ϕ^* are radial and tangential velocity components, respectively.

Suppose next that v_r^* and v_ϕ^* in equation (9) are given by the idealized description for the symmetric potential flow past a wedge whose half angle is ϕ_1. The solution for v_r^* and

$v_\phi{}^*$ satisfying inviscid boundary conditions is given in Landau and Lifshits[20] as

$$v_r^* = A\left(\frac{\pi}{\pi-\phi_1}\right) r^{*\left(\frac{\pi}{\pi-\phi_1}-1\right)} \cos\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right)$$

$$v_\phi^* = -A\left(\frac{\pi}{\pi-\phi_1}\right) r^{*\left(\frac{\pi}{\pi-\phi_1}-1\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right) \qquad (10)$$

Equation (9) is still valid for this idealized velocity field; however, the thermal Stokes radius must be redefined since the potential flow has no characteristic length equivalent to the outer length scale δ_v used when a velocity boundary layer was present. One can still introduce a dimensionless $r = \frac{r^*}{r_c}$ substitute (10) into (9) and nondimensionalizing θ as before

$$\frac{\rho_\infty c_p A}{K}\left(\frac{\pi}{\pi-\phi_1}\right) r_c^{\left(\frac{\pi}{\pi-\phi_1}\right)} \left[r^{\left(\frac{\pi}{\pi-\phi_1}-1\right)} \cos\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right)\frac{\partial\Theta}{\partial r} - r^{\left(\frac{\pi}{\pi-\phi_1}-2\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right)\frac{\partial\Theta}{\partial\phi}\right] = \nabla^2\Theta \qquad (11)$$

One can again define a thermal Stokes radius by equating the coefficients of the convection and conduction terms in (11). r_c defined in this manner is

$$r_c = \left(\frac{K(\pi-\phi_1)}{\pi \Delta c_p \rho_\infty}\right)^{\left(\frac{\pi-\phi_1}{\pi}\right)} \tag{12}$$

If the flow in Fig. 1 is that past a wedge rather than a ramp with an adiabatic upstream wall, then the order of magnitude analysis of the real viscous problem can no longer be patterned directly after the analysis presented for problem (iii). The characteristic length for the vortical region in this case is the momentum Stokes radius which should be of the same order as r_c if P_r is of $0(1)$. Thus, in the region of interest r^* $0(r_c)$, the dynamic balance is between pressure and viscous forces. The appropriate velocity field is that of a slow flow past a semi-infinite wedge. These local flow solutions, which are the counterpart of the local thermal field solutions presented herein, have been developed in references[16-19] but no satisfactory technique has yet been discovered for determining the unknown constants in the solution for these finite wedge angle cases. In contrast, the simplified potential flow description (10) provides for the possibility of matching with an approximate global solution to determine all constants in the local solution at least for the limiting case $\phi_1 = o$.

With the definition of r_c given by (12), the governing energy equation (11) becomes

$$r^{\left(\frac{\pi}{\pi-\phi_1}-1\right)}\cos\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right)\frac{\partial\Theta}{\partial r}-r^{\left(\frac{\pi}{\pi-\phi_1}-2\right)}\sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right)\frac{\partial\Theta}{\partial\phi}=\nabla^2\Theta \tag{13}$$

The wall boundary conditions for equation (13) for problem (i) are

$$\frac{\partial\Theta}{\partial\phi}=0 \quad \phi=\pi \quad , \quad \Theta=1 \quad \phi=\phi_1 \tag{14}$$

and for problem (ii) are

$$\Theta=0 \quad \phi=\pi \quad , \quad \Theta=1 \quad \phi=\phi_1 \tag{15}$$

In addition, one must satisfy appropriate conditions at infinity.

The solution to equation (7) with boundary conditions (8) and equation (13) with boundary conditions (14) and (15) and appropriate conditions at infinity shall be sought in series form based on an iterative procedure which is the thermal counterpart of that used by Carrier and Lin[10] for the velocity field near the leading edge of a semi-infinite flat plate. The energy equations (7) and (13) may be expressed in operator form as

$$\mathcal{L}_\Theta(\Theta)=\mathcal{L}_\Theta'(\Theta) \tag{16}$$

where

$$\mathcal{L}_\Theta = \nabla^2 \quad , \quad \mathcal{L}_\Theta' = \bar{u}\cdot\nabla \tag{17}$$

Since the region of interest $r \ll 1$ is conduction dominated, the series solutions to problems (i), (ii), and (iii)

$$\Theta = \Theta_0 + \Theta_1 + \Theta_2 + \ . \ . \ . \tag{18}$$

have their lowest order θ_0 term based on

$$\mathcal{L}_\Theta(\Theta_0) = 0 \tag{19}$$

and boundary conditions (14), (15), and (8) respectively. Having θ_0, θ_1 is generated based on

$$\mathcal{L}_\Theta(\Theta_1) = \mathcal{L}_\Theta'(\Theta_0) \tag{20}$$

and for problems (i), (ii), and (iii), must satisfy respectively

$$\frac{\partial \Theta_1}{\partial \phi} = 0 \quad \phi = \pi \quad , \quad \Theta_1 = 0 \quad \phi = \phi_1 \tag{21}$$

$$\Theta_1 = 0 \quad \phi = \pi \quad , \quad \Theta_1 = 0 \quad \phi = \phi_1 \tag{22}$$

and

$$\Theta_1 = 0 \quad \phi = \pi \quad , \quad \Theta_1 = 0 \quad \phi = 0 \tag{23}$$

In general, Θ_n for (i), (ii), and (iii) has governing equations

$$\mathcal{L}_\Theta(\Theta_n) = \mathcal{L}'_\Theta(\Theta_0 + \Theta_1 \ldots \Theta_{n-1}) - \mathcal{L}'_\Theta(\Theta_0 + \Theta_1 \ldots \Theta_{n-2}) \tag{24}$$

and the same homogeneous boundary conditions (21), (22), and (23) as for θ_1.

The resulting locally valid series solution for each problem contains an infinite number of homogeneous solutions which satisfy equation (19) and the homogeneous boundary conditions (21), (22), and (23). These homogeneous solutions introduce an infinite array of unknown constants. How these constants are determined shall be outlined in section (2.3) of this chapter.

(2.2) Formulation of the Natural Convection Problem

This last problem (iv) under consideration is that of two-dimensional, steady state, symmetrical, natural convection about the leading edge of an isothermal, semi-infinite vertical wedge (Fig. 4). Boundary conditions at the surface of the wedge are the no slip velocity conditions and the isothermal wall temperature T_1. Very far from the wedge, $u_\infty = v_\infty = 0$ and the temperature is constant at T_∞. Furthermore, the fluid's coefficient of thermal expansion β_1, coefficient of viscosity μ, specific heat C_p, coefficient of conductivity K, and coefficient of isothermal compressibility $\bar{K}$ are considered constant. Gravitational force acts vertically downward. In addition, it is assumed that:

(1) there are only small deviations in relative density differences $\frac{\rho^* - \rho_\infty}{\rho_\infty}$,

(2) the liquid satisfies $\left|\frac{P_\infty \bar{K}}{\beta_1 T_\infty}\right| \ll 1$ or in the case of a gas which obeys the ideal gas law $\left(\frac{P^* - P_\infty}{P_\infty}\right) \ll \left(\frac{T^* - T_\infty}{T_\infty}\right)$,

(3) $\alpha_1 = \beta_1(T_1 - T_\infty) \ll 1$, and

(4) hydrostatic pressure is constant.

In view of the above assumptions, the fluid conservation equations and the equation of state reduce to:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{25}$$

$$\rho_\infty\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial P'^*}{\partial x^*} - \alpha_1 \rho_\infty g_x \Theta + \mu \nabla^2 u^* \tag{26}$$

$$\rho_\infty\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial P^*}{\partial y^*} + \mu \nabla^2 v^* \tag{27}$$

$$\rho_\infty c_p\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right) = K\nabla^2 T^* + \left(u^*\frac{\partial P'^*}{\partial x^*} + v^*\frac{\partial P'^*}{\partial y^*}\right) + 2\mu\left(\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial y^*}\right)^2\right) + \mu\left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right) \tag{28}$$

$$\rho^* = \rho_\infty(1 - \alpha_1 \Theta) \tag{29}$$

p'* is the pressure associated with the fluid motion.

To determine the relative importance of each of the terms, a nondimensional analysis of the above equations is performed by introducing the following set of nondimensional variables

$$\frac{u^*}{u_c} = u \qquad \frac{x^*}{r_c} = x \qquad \frac{r^*}{r_c} = r \qquad \left(\frac{T^* - T_\infty}{T_1 - T_\infty}\right) = \Theta$$

$$\frac{v^*}{u_c} = v \qquad \frac{y^*}{r_c} = y \qquad \frac{p'^* r_c}{\mu u_c} = P$$

The characteristic velocity u_c and length r_c have purposely been left unspecified and will be determined next by requiring an appropriate balance of terms in the dimensionless equations. It is also noted that the characteristic length r_c will be the same for both the energy and momentum equations. This is in accord with well-known behavior provided the Prandtl number ($Pr = \frac{\mu C_p}{K}$) is of order unity.

If one requires that the pressure, viscous, and buoyancy terms in the nondimensional x-component of the momentum equation be of the same order, u_c must be defined by

$$u_c = \frac{g_x \beta (T_1 - T_\infty) r_c^2}{\nu} \tag{30}$$

where r_c is still to be determined. Using this definition for u_c and assuming that the kinetic energy is very much less than the enthalpy, the governing equations reduce to the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{31}$$

$$G\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P'}{\partial x} - \Theta + \nabla^2 u \tag{32}$$

$$G\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P'}{\partial y} + \nabla^2 v \tag{33}$$

$$G\Pr\left(u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y}\right) = \nabla^2 \Theta \tag{34}$$

Here,

$$G = \frac{g_x \beta (T_1 - T_\infty) r_c^3}{\nu^2}$$

is the local Grashof number which is based on some characteristic radius for thermal conduction defined below.

If the governing equations (32) and (33) are combined to eliminate the pressure terms and if the velocities are defined in terms of a nondimensional stream function

$$\psi = \frac{\psi^*}{\psi_c} = \frac{\psi^*}{u_c r_c} \qquad u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

then equations (31), (32), and (33) reduce to one equation

$$G\left(\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}-\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3}-\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial y^2\partial x}\right)-\frac{\partial\Theta}{\partial x}=\nabla^2\nabla^2\psi \tag{35}$$

At this point, it is inquired under what conditions conduction will be the dominant mode of heat transfer. It is noted from equations (34) and (35) that the convection terms on the left hand side of these equations are of lower order than the remaining terms when $G < o(1)$. Thus, r_c is defined by setting the local Grashof number equal to unity and solving for the characteristic thermal diffusion length r_c.

$$r_c = \left(\frac{\nu^2}{g_x\beta(T_1-T_\infty)}\right)^{1/3} \tag{36}$$

Definition (36) shall henceforth be referred to as the thermal Stokes radius for natural convection. It is noted that when the dimensionless radius $r < 1$ conduction dominates and there exists a thermally induced slow flow with higher order corrections due to the effects of convection.

The governing equations are equations (34) and (35) with u_c and r_c as defined in equations (30) and (36) respectively. The solution must also satisfy local boundary conditions of

$$v=\frac{\partial u}{\partial y}=\frac{\partial\Theta}{\partial y}=0 \qquad \phi=0 \quad , \qquad u=v=0 \quad \Theta=1 \qquad \phi=\lambda\pi \tag{37}$$

and be consistent with a far field solution.

Solutions for the streamfunction and the temperature in the near field, shall be sought in series form based in principle on the method first used by Carrier and Lin[10]. The series solution for θ has its lowest order solution based on pure conduction while that for the streamfunction is based on a thermally induced Stokes slow flow. The form of θ, ψ, and this iterative procedure are outlined as follows:

$$\Theta = \Theta_0 + \Theta_1 + \ldots$$
$$\psi = \psi_0 + \psi_1 + \ldots$$

where θ_0 and ψ_0 satisfy

$$0 = \nabla^2 \Theta_0 \tag{38}$$

$$-\frac{\partial \Theta_0}{\partial x} = \nabla^2 \nabla^2 \psi_0 \tag{39}$$

and the local boundary conditions:

$$\Theta_0 = 1 \quad \phi = \lambda\pi \quad , \quad \frac{\partial \Theta_0}{\partial y} = 0 \quad \phi = 0 \tag{40}$$

$$u = v = 0 \quad \phi = \lambda\pi \quad , \quad v = \frac{\partial u}{\partial y} = 0 \quad \phi = 0 \tag{41}$$

Equations (34) and (35) respectively can be written in operator form as

$$\mathcal{L}'_\Theta(\psi, \Theta) = \mathcal{L}_\Theta(\Theta) \tag{42}$$

$$\mathcal{L}'_\psi(\psi, \Theta) = \mathcal{L}_\psi(\psi) \tag{43}$$

In general

$$\mathcal{L}'_{\theta}(\theta_o+\ldots\theta_{n-1},\psi_o+\ldots\psi_{n-1})-\mathcal{L}'_{\theta}(\theta_o+\ldots\theta_{n-2},\psi_o+\ldots\psi_{n-2})=\mathcal{L}_{\theta}(\theta_n) \qquad (44)$$

$$\mathcal{L}'_{\psi}(\theta_o+\ldots\theta_n,\psi_o+\ldots\psi_{n-1})-\mathcal{L}'_{\psi}(\theta_o+\ldots\theta_{n-1},\psi_o+\ldots\psi_{n-2})=\mathcal{L}_{\psi}(\psi_n) \qquad (45)$$

where θ_n and ψ_n must satisfy homogeneous local boundary conditions when $n > o$

$$\theta_n = 0 \quad \phi = \lambda\pi \quad , \quad \frac{\partial\theta_n}{\partial y} = 0 \quad \phi = 0 \qquad (46)$$

$$u_n = v_n = 0 \quad \phi = \lambda\pi \quad , \quad v_n = \frac{\partial u_n}{\partial y} = 0 \quad \phi = 0 \qquad (47)$$

The resulting series solutions for θ and ψ generated in this fashion contain an infinite number of homogeneous solutions satisfying equation (38) and homogeneous boundary conditions (46). Along with these homogeneous solutions are associated an infinite number of unknown constants. The manner in which these constants are to be determined is outlined next in section (2.3).

(2.3) Matching Procedure for Determination of Constants

The resulting locally valid series solutions for problems (i), (ii), (iii), and (iv) are determined to within the constants which correspond to the infinite number of homogeneous solutions which satisfy equation (38) and the boundary conditions satisfied by their respective θ_n, (21), (22), (23) and (46). While no rigorous procedure has yet been developed for matching local Stokes like solutions to outer inertial flows, it is known from simple non-linear exact solutions of the Navier-Stokes equation, for example, the iterative solution presented in Weinbaum[19] for the two-dimensional stagnation point flow, that the values of the unknown constants in the local solution that are not determined by local boundary conditions depend on global characteristics of the flow such as its total energy. Except perhaps for very special situations where exact solutions in the large can be obtained, one is led to approximate techniques such as Oseen type linearizations for determining the essential qualitative features of the complete flow field. These approximate global solutions must still have a region of overlapping validity with the local Stokes like solution if the solutions are to be matched and the unknown constants in the local solution determined. For such a region of overlap to exist, the approximate global equation must reduce to the governing equation for the local flow as one approaches within a characteristic diffusion length of the leading edge. Thus, one is able to establish a region of overlapping validity of a Stokes like region with an outer boundary layer flow for the two-dimensional stagnation point flow, but not for the Carrier and Lin[10] problem of the uniform flow past the leading edge of a semi-infinite flat plate. In the first case, the viscous terms in the Stokes like region are the same as those that appear in the boundary layer equations of motion, whereas in the second case the same viscous terms are not common to both the near field and boundary layer equations.

The simplification introduced here is that the solution to the global equation does not have to be determined even in a local region of space to establish the remaining unknown constants and homogeneous functions that are to appear in the local iterative solution. These constants and functions can be identified by matching with the global solution along any convenient boundary that emanates at the origin and extends through the near field. The advantage of this procedure is that transform techniques quite often yield local solutions for a function or its normal derivative along a boundary more readily than they do field solutions. This is exemplified by the Weiner-Hopf technique where at an intermediate step in seeking the field solution, local boundary information is determined. Thus, the local information along a line, obtained from a global problem, when combined with a local iterative procedure may be used to determine a complete or an approximate truncated near field solution, depending on the global boundary information provided.

Chapter 3

Temperature Field In A Flow Over An Isothermal Wedge

The solution for the temperature field shall be sought in the iterative form (18) where the lowest order term θ_o satisfies equation (19) and wall boundary conditions (14). Solving by the standard method of separation of variables, one obtains

$$\theta_o = 1 + \sum_{n=1}^{\infty} C_n r^{\left(\frac{2n-1}{2}\right)\left(\frac{\pi}{\pi-\phi_1}\right)} \cos\left(\frac{2n-1}{2}\left(\frac{\pi}{\pi-\phi_1}\right)(\phi-\pi)\right) \qquad (48)$$

In (48), θ_o is given by the first term unity while the terms in the infinite series satisfy the homogeneous equation for (24) and the homogeneous boundary conditions (21). To establish which of the homogeneous solutions are required for each θ_n in the iterative solution and to determine the constants in these solutions, local boundary information from the global Oseen problem must first be obtained.

(3.1) Global Oseen Problem for $\phi_1 = o$

The global Oseen problem for the temperature distribution over the flat plat is given by equation (13) with $\phi_1 = o$

$$\frac{\partial\theta}{\partial x} = \nabla^2\theta \qquad (49)$$

and boundary conditions

$$
\begin{array}{ll}
r \to \infty & \Theta = 0 \\
\phi = \pi & \dfrac{\partial \Theta}{\partial y} = 0 \\
\phi = 0 & \Theta = 1
\end{array}
\tag{50}
$$

The solution to (49) and (50) is given in Carrier, Krook, and Pearson[8] as

$$\Theta = 1 - \operatorname{erf}\left(r^{1/2} \sin(\phi/2)\right) \tag{51}$$

The centerline temperature upstream of the plate along $\phi = \pi$ can be expressed as the following series summation for $r < 1$

$$\Theta\Big|_{\phi=\pi} = 1 - \frac{2}{\sqrt{\pi}} r^{1/2} + \frac{2}{3\sqrt{\pi}} r^{3/2} - \frac{2}{10\sqrt{\pi}} r^{5/2} + \; . \; . \; . \; . \tag{52}$$

This additional boundary information will be used to determine constants in the series solution for $\phi_1 = o$ and the generating functions for all ϕ_1.

(3.2) Solution for θ_1

θ_o is equal to unity and thus generates no inhomogeneous term in equation (20). θ_1, therefore, has no particular solution. However, one observes from equation (52) that the global solution requires a term which behaves as $r^{1/2}$ as $r \to 0$ for the limiting case of $\phi_1 = 0$. This requirement can be satisfied by retaining a homogeneous solution for θ_1 given by the term n=1 in the infinite series in equation (48). The value of C_1 is obtained by evaluating θ_1 at $\phi=\pi$ and matching with the coefficient of the $r^{1/2}$ term in (52) for the limiting case of $\phi_1=0$. C_1 determined in this manner is given by $C_1 = -\frac{2}{\sqrt{2\pi}}$.

For non-zero values of ϕ_1, one anticipates that the behavior of the solution should vary smoothly as ϕ_1 approaches zero. It is, therefore, reasonable to conclude, although obviously not rigorously established, that the n=1 term in the infinite series in (48) is required for compatibility with the far field solution for all values of ϕ_1. Thus, the form of the solution for all ϕ_1 is given by

$$\Theta_1 = C_1 r^{\frac{1}{2}\left(\frac{\pi}{\pi-\phi_1}\right)} \cos\left(\tfrac{1}{2}\left(\tfrac{\pi}{\pi-\phi_1}\right)(\phi-\pi)\right) \tag{53}$$

The unknown constant C_1, in principle, can be determined by matching with the upstream centerline temperature distribution which obtains from the global formulation of the finite wedge angle cases. Such a solution has not been obtained by the authors but is being studied further. However, the value of C_1 for the limiting case $\phi_1=0$ provides some qualitative estimate of the order of magnitude of this constant coefficient for the finite wedge angle cases.

For problem (i), with $\phi_1=0$, the local iterative solution obviously provides no new information that is not already contained in the global solution (51). The importance of considering this limiting case is that it (a) motivates the selection of the generating functions for the locally valid solutions for the finite wedge angle cases for which there are no global solutions and (b) it demonstrates how a complete locally valid field solution can be obtained when only a local boundary solution for an unknown function or its normal derivative is available from the related global problem. As shall be seen later, this situation transpires for problems (ii) and (iii).

(3.3) Solution to θ_2 and Higher Order θ_n

Substituting for θ_o and θ_1 in equation (24), the governing equation for θ_2 becomes

$$\nabla^2\Theta_2 = c_1\left(\tfrac{1}{2}\right)\left(\frac{\pi}{\pi-\phi_1}\right)r^{\left(\frac{3}{2}\left(\frac{\pi}{\pi-\phi_1}\right)-2\right)}\cos\left(\tfrac{1}{2}\left(\frac{\pi}{\pi-\phi_1}\right)(\phi-2\phi_1+\pi)\right) \tag{54}$$

θ_2 like θ_1 must obey homogeneous boundary conditions (21). The particular solution to (54) is

$$\Theta_{2p} = \left(\frac{\pi-\phi_1}{\pi}\right)\left(\frac{c_1}{4}\right)r^{\frac{3}{2}\left(\frac{\pi}{\pi-\phi_1}\right)}\cos\left(\tfrac{1}{2}\left(\frac{\pi}{\pi-\phi_1}\right)(\phi-2\phi_1+\pi)\right) \tag{55}$$

and satisfies the homogeneous boundary conditions (21) automatically.

However, this does not imply that an additional homogeneous solution is not required for compatibility with the outer flow behavior. One observes that the coefficient of θ_{2p} for the limiting case $\phi_1 = 0$ does not match with the third term in the global solution for θ along $\phi = \pi$ (52). A homogeneous solution which behaves as $r^{3/2}$

in the near field is needed. The solution corresponds to the n=2 homogeneous solution in (48). The value of C_2, for the limiting case of $\phi_1=0$, is obtained by evaluating θ_2 at $\theta=\pi$ and matching with the coefficient of the $r^{3/2}$ term in (52). C_2 is thus found to be equal to $-\frac{2}{12\sqrt{\pi}}$.

Since it is reasonable to expect that the behavior of the solution varies smoothly as ϕ_1 approaches zero, one also requires that the n=2 term be retained for compatibility with the far field solution for all values of ϕ_1. Thus, the form of the solution θ_2 for ϕ_1 in general is

$$\Theta_2 = r^{\frac{3}{2}\left(\frac{\pi}{\pi-\phi_1}\right)}\left[C_2\cos\left(\tfrac{3}{2}\left(\tfrac{\pi}{\pi-\phi_1}\right)(\phi-\pi)\right)+\left(\tfrac{\pi-\phi_1}{\pi}\right)\left(\tfrac{C_1}{4}\right)\cos\left(\tfrac{1}{2}\left(\tfrac{\pi}{\pi-\phi_1}\right)(\phi-2\phi_1+\pi)\right)\right] \quad (56)$$

In theory, the unknown constant C_2 in expression for θ_2 (56), as C_1 in the expression for θ_1, can be found by matching with the corresponding term in the upstream centerline solution for the temperature distribution provided by the global formulation of the finite wedge angle cases.

In general, θ_n must satisfy equation (24) and homogeneous boundary conditions (21). The resulting particular solution could like θ_{2p} require an additional homogeneous term in order to be compatible with the far field behavior and would be determined in a manner entirely analogous to that just shown for θ_2. Thus, the local iterative solution becomes

$$\begin{aligned}\Theta = 1 &+ C_1 r^{\frac{1}{2}\left(\frac{\pi}{\pi-\phi_1}\right)} \cos\left(\tfrac{1}{2}\left(\tfrac{\pi}{\pi-\phi_1}\right)(\phi-\pi)\right) \\ &+ r^{\frac{3}{2}\left(\frac{\pi}{\pi-\phi_1}\right)}\Big[C_2 \cos\left(\tfrac{3}{2}\left(\tfrac{\pi}{\pi-\phi_1}\right)(\phi-\pi)\right) \\ &\qquad + \left(\tfrac{\pi-\phi_1}{\pi}\right)\left(\tfrac{C_1}{4}\right) \cos\left(\tfrac{1}{2}\left(\tfrac{\pi}{\pi-\phi_1}\right)(\phi-2\phi_1+\pi)\right)\Big] + \cdots \end{aligned} \tag{57}$$

The important point illustrated by the above analysis is that a series solution satisfying local boundary conditions is not in itself complete and that additional generating functions can be required for the local solution to be consistent with far field behavior. The nature of the solution for ϕ_1 in general shall be illustrated by examining the results for the case $\phi_1=0$. In Fig. 5, the first and second order iterative solutions for the case $\phi_1=0$ are plotted and compared with the exact solution (51) of the Oseen equation. Of particular interest is the extent of the region in which the leading terms in the local solution provide a good description. The isotherms for θ 0.5 appear to be well represented even for values of r of order unity. The convergence of the series however is slow at least for the first few terms. The convective correction θ_1 is observed to wash the isotherms downstream. However, θ_1 is an overestimate of the convective contribution and θ_2 must compensate. For problems (ii) and (iii), it will not be possible to obtain exact field solutions for the global problems. However, the agreement between the local and global solutions in Fig. 5 provide cogent evidence as to the validity of the matching procedure used in determining the local solution for all three problems.

Chapter 4

Temperature Field For The Flow Over A Ramp With A Step Jump Temperature Distribution

The leading term Θ_o in the iterative series expression for the local temperature distribution must satisfy equation (19) and boundary conditions (15). Solving by separation of variables, one obtains

$$\Theta_o = \frac{\phi-\pi}{\phi_1-\pi} + \sum_{n=1}^{\infty} C_n r^{\left(\frac{\pi}{\pi-\phi_1}\right)n} \sin\left(n\left(\frac{\pi}{\pi-\phi_1}\right)(\phi-\phi_1)\right) \tag{58}$$

The first term in expression (58) is taken as Θ_o while the remaining terms in the infinite sum, which satisfy the homogeneous equation for (24) and the homogeneous boundary conditions (22), may be required for higher order Θ_n terms. In order to establish which of the homogeneous functions are necessary for each Θ_n and how their respective coefficients are to be determined, we again turn to the related global Oseen problem.

(4.1) Global Oseen Problem for $\phi_1 = 0$

The limiting case of ϕ_1=0 corresponds to uniform flow over a semi-infinite flat plate which has a step jump in its temperature distribution. The governing equation for the temperature distribution is given by equation (13) with ϕ_1=0 or

$$\frac{\partial\Theta}{\partial x} = \nabla^2\Theta \tag{59}$$

and has boundary conditions

$$
\begin{aligned}
&y \to \infty && \Theta \to 0 \\
&x < 0 \quad y = 0 && \Theta = 0 \\
&x > 0 \quad y = 0 && \Theta = e^{-bx}
\end{aligned}
\tag{60}
$$

where the limit as $b \to o$ is to be taken at an appropriate point in the analysis. The solution to the boundary value problem (59) and (60) has not been given previously and is outlined below

Defining the Fourier transform by

$$\bar{\Theta}(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Theta(x,y)\, e^{ikx}\, dx \tag{61}$$

One obtains for equation (59)

$$\frac{d^2\bar{\Theta}}{dy^2} - (k^2 - ik)\bar{\Theta} = 0 \tag{62}$$

The transformed boundary conditions (60) are

$$
\begin{aligned}
&y = 0 && \bar{\Theta}(k,0) = \frac{1}{\sqrt{2\pi}}\left(\frac{i}{k + bi}\right) \\
&y \to \infty && \bar{\Theta}(k,\infty) = 0
\end{aligned}
\tag{63}
$$

The solution to (62) satisfying the boundary condition (63) is given by

$$\bar{\Theta}(k,y) = \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{i}{k+bi}\right) e^{-y(k^2-ik)^{1/2}} \tag{64}$$

The branch line for $k^{1/2}$ is taken as the negative imaginary axis and the branch line for $(k-i)^{1/2}$ goes from i along the positive imaginary axis to infinity.

For $x<0$, the inversion of (64) reduces to an integration around the branch line in the upper half plane and gives after some manipulation

$$\Theta(x,y) = \frac{e^{x/2}}{\pi}\int_0^\infty\left(\frac{-1}{\xi\sqrt{1+4\xi^2}} + \frac{1}{\xi}\right)e^{x/2\sqrt{1+4\xi^2}}\sin(y\xi)\,d\xi \tag{65}$$

Similarly for $x > o$ the contribution to Θ arises from the integration around the branch in the lower half plane and gives the result

$$\Theta(x,y) = -\frac{e^{x/2}}{\pi}\int_0^\infty\left(\frac{1}{\xi\sqrt{1+4\xi^2}} + \frac{1}{\xi}\right)e^{-x/2\sqrt{1+4\xi^2}}\sin(y\xi)\,d\xi \quad + \quad 1 \tag{66}$$

The local heat transfer rate for $x<o$ and $x>o$ may be obtained by differentiating expressions (65) and (66) with respect to y, integrating, and taking the limit as y goes to zero, respectively. The resulting expressions are

$$\left.\frac{\partial\Theta}{\partial y}\right|_{\substack{y=0\\x<0}} = \frac{-e^{x/2}}{2\pi}\left[K_0\left(\frac{x}{2}\right) + K_1\left(\frac{x}{2}\right)\right] \tag{67}$$

and

$$\left.\frac{\partial\Theta}{\partial y}\right|_{\substack{y=0\\x<0}} = \frac{e^{x/2}}{2\pi}\left[K_1\left(\frac{-x}{2}\right) - K_0\left(\frac{-x}{2}\right)\right] \tag{68}$$

where K_0 and K_1 are modified Bessel functions of the zeroth and first kind. The series expansions for (67) and (68) when $|x|<1$ are

$$\left.\frac{\partial\Theta}{\partial y}\right|_{\substack{y=0\\x>0}} = \frac{-1}{\pi x} + \frac{\ln(\frac{x}{4})}{2\pi} - \frac{.4228}{2\pi} + \frac{x}{8\pi}\ln\left(\frac{x}{4}\right) \quad . \; . \; . \tag{69}$$

$$\left.\frac{\partial\Theta}{\partial y}\right|_{\substack{y=0\\x<0}} = \frac{-1}{\pi x} + \frac{\ln(\frac{-x}{4})}{2\pi} - \frac{.4228}{2\pi} + \frac{x}{8\pi}\ln\left(\frac{-x}{4}\right) \quad . \; . \; . \tag{70}$$

(4.2) Solution for θ_1

Having θ_o, θ_{1p} must satisfy equation (20), where θ_o is given by the leading term in (58), and homogeneous boundary conditions (22). θ_{1p} is thus found to be

$$\Theta_{1p} = \frac{1}{2\pi}\ln(r)\, r^{\left(\frac{\pi}{\pi-\phi_1}\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right) \tag{71}$$

One notes that if the local heat transfer rate obtained for the iterative solution, for the limiting case $\phi_1=0$, is to match with the local heat transfer of the global problem at $\phi=\pi$ (70) then the n=1 homogeneous solution in (58) is necessary in addition to θ_{1p} for $\left.\frac{\partial\Theta}{\partial y}\right|_{\phi=\pi}$ to be compatible with the second term in (70). The coefficient C_1

of the homogeneous solution used in θ_1 is evaluated for $\phi_1=0$ by matching $\left.\frac{\partial\Theta_1}{\partial\phi}\right|_{\phi=\pi}$ with the second term in (70). C_1 determined in this way is given as $C_1 = \frac{\ln(1/4)}{2\pi}$.

As in the previous problem, one anticipates that the behavior of the solution for non-zero values of ϕ_1 should vary smoothly as ϕ_1 approaches zero. It is thus concluded that the n=1 term in the infinite series in (58) is required for compatibility with the far field solution for all values of ϕ_1. Therefore, the form of the solution for θ_1 for all ϕ_1 is given by

$$\Theta_1 = \frac{1}{2\pi}\ln(C_1 r)\, r^{\left(\frac{\pi}{\pi-\phi_1}\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right) \tag{72}$$

For finite ϕ_1, the solution for the unknown constant C_1 could in theory be obtained using the same procedure as suggested in problem (i). Thus, it is observed that consideration of the limiting case $\phi_1=0$ motivates the selection of the generating functions for the locally valid solutions for the finite ramp angle cases for which there are no global solutions and gives a qualitative estimate as to the order of magnitude of the coefficients of the homogeneous functions in (58) which are to be incorporated in the iterative solution.

(4.3) Solution for θ_2 and Higher Order θ_n

Substituting θ_0 and θ_1 into equation (24), one obtains a particular solution for θ_2 which automatically satisfies homogeneous boundary conditions (22). This particular solution is of order $r^2 \ln r$ and has its partial derivative with respect to y of order $r \ln r$. Thus, when this term is compared, for $\phi_1 = o$, with the local heat transfer information (70) provided by the global formulation, one observes that it corresponds to the fourth term in the series and that the third term has not yet been accounted for. This is rectified by including a second n=1 homogeneous term in the series summation for the heat transfer at y=0 to be compatible with (70). The coefficient of the n=1 homogeneous solution is equal to $-\frac{.4228}{2\pi}$.

Once again anticipating that the behavior varies smoothly with ϕ_1, one writes for θ_2

$$\Theta_2 = C_1'' r^{\left(\frac{\pi}{\pi-\phi_1}\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right) \tag{73}$$

C_1'' must be determined separately for each ϕ_1 by matching with local heat transfer information provided by the associated global formulation.

In a similar fashion, θ_{np} must satisfy equation (24) and homogeneous boundary conditions (22). In addition, $\left.\frac{\partial \Theta_p}{\partial y}\right|_{\phi=\pi}$ is required to be consistent with the corresponding term in the local heat transfer rate provided by the global solution (70) when $\phi_1 = o$. This requirement, as before, will establish whether θ_{np} requires an additional homogeneous solution in order to fulfill far field compatibility criteria. Thus, θ for arbitrary ϕ_1 is written as

$$\Theta = \frac{\phi-\pi}{\phi_1-\pi} + \frac{1}{2\pi}\ln(C_1' r)\, r^{\left(\frac{\pi}{\pi-\phi_1}\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right) + C_1'' r^{\left(\frac{\pi}{\pi-\phi_1}\right)} \sin\left(\frac{\pi}{\pi-\phi_1}(\phi-\phi_1)\right) \quad . \quad . \quad . \tag{74}$$

The nature of the temperature distribution for ϕ_1 in general will be examined by considering the results for the limiting case of $\phi_1=0$, Fig. 6. θ_o, the local pure conduction solution, has radial isotherms with $\theta_o=\frac{1}{2}$ along the vertical axis $\alpha=\pi/2$. θ_1, the first correction for convection, shows that the isotherms are washed downstream.

Chapter 5

Temperature Field In Flow Over A Flat Plate With A Step Jump In Temperature Distribution

The two previous problems were formulated using a potential flow approximation for the velocity field in the energy equation. The important objectives in addition to obtaining a qualitative picture of the local thermal field was to establish the validity of the matching procedure for determining both the generating functions and constants for the local field solutions. The present problem shall be considered in a more exact fashion. In principal, the full series solution for the Blasius velocity profile could be used to represent u in equation (7); however, the leading term will be sufficient to obtain an accurate description provided $\frac{r_c}{\delta_v} \ll 1$. Thus, equation (7) becomes

$$\alpha y \frac{\partial \Theta}{\partial x} = \nabla^2 \Theta \tag{75}$$

where $\alpha = 1.66$ is the slope of the velocity profile at the plate. For the sake of convenience, x, y, and r shall be scaled and henceforth be considered as

$$x \rightarrow \frac{x}{\alpha^{1/2}} \quad , \quad y \rightarrow \frac{y}{\alpha^{1/2}} \quad , \quad r \rightarrow \frac{r}{\alpha^{1/2}}$$

Thus, equation (75) is written as

$$y\frac{\partial \Theta}{\partial X} = \nabla^2 \Theta \tag{76}$$

The leading term θ_o in the iterative series solution sought satisfies the governing equation (19) and wall boundary conditions (8). The procedure of separation of variables yields

$$\Theta_o = \left(\frac{T_2 - T_\infty}{T_1 - T_\infty}\right) + \left(\frac{T_1 - T_2}{T_1 - T_\infty}\right)\frac{1}{\pi}\phi + \sum_{n=1}^{\infty} C_n r^n \sin(n\phi) \tag{77}$$

θ_o is taken as the first two terms in (77) and, as in the previous problems (i) and (ii), the infinite sum in (70) satisfies the homogeneous equation (24) for higher order θ_n and homogeneous boundary conditions (23). The unknown coefficients similarly have to be determined from the local boundary information obtained from the global Oseen formulation of the problem.

(5.1) Global Oseen Problem

Equation (76) can also be treated as a global Oseen problem provided $\frac{r_c}{\delta_v} \ll 1$ so that the thermal Stokes radius is imbedded in the linear portion of the Blasius velocity profile. Similarly, the far field temperature distribution removed from the discontinuity is given by the linear portion of the well-established Pohlhausen temperature profile. The related problem of a plate which is adiabatic rather than isothermal upstream of x=0 has been formulated in an analogous manner by Imai[21].

In view of the above discussion, it is convenient to define a new dimensionless temperature $\circledast$ related to θ by

$$\circledast = \Theta - (1 + \beta y) \tag{78}$$

where $\beta = \frac{-(.332)^{1/2}}{R_{\infty}^{1/4}\, Pr^{1/6}}$ and the bracketed quantity is the linear approximation to the Pohlhausen solution near the wall but at distance large compared to r_c. The governing equation (76) becomes on substituting (78)

$$y \frac{\partial \circledast}{\partial x} = \nabla^2 \circledast \tag{79}$$

The boundary conditions are

$$\begin{array}{lll} y \to \infty & & \circledast = 0 \\ x < 0 & y = 0 & \circledast = 0 \\ x > 0 & y = 0 & \circledast = \left(\frac{T_2 - T_1}{T_1 - T_{\infty}}\right) e^{-bx} \end{array} \tag{80}$$

where the limit $b \to 0$ is taken at an appropriate point in the analysis. The parameter b is introduced as a mathematical device for purposes of having a well-defined Fourier transform along the x axis.

The solution shall be sought by Fourier transforms. The transform of equation (79) is

$$\frac{d^2\bar{\Theta}}{dy^2} + ik(ik+y)\bar{\Theta} = 0 \tag{81}$$

where $\bar{\Theta}$ is the usual transform variable. The boundary conditions (80), when written in terms of the transform variable $\bar{\Theta}$, become

$$\begin{aligned} y &\rightarrow \infty & \bar{\Theta}(k,\infty) &= 0 \\ y &= 0 & \bar{\Theta}(k,0) &= \left(\frac{T_2-T_1}{T_1-T_\infty}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{i}{k+bi}\right) \end{aligned} \tag{82}$$

The solution to equation (81) which satisfies the far field ($y \rightarrow \infty$) boundary condition (82) is

$$\bar{\Theta} = A\left[k^{1/3}(k-iy)\right]^{1/2} H^{(2)}_{1/3}\left(\tfrac{2}{3}\left(k^{1/3}(k-iy)\right)^{3/2}\right) \tag{83}$$

where A is a complex constant of integration. The branch cut for k goes from zero to $-i\infty$ and that for $(k-iy)$ from iy to $+i\infty$ in the complex k plane. The boundary conditions at y=0 (82) yields A and hence the complete solution in the transform domain is

$$\bar{\Theta} = \left(\frac{T_2-T_1}{T_1-T_\infty}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{i}{k+bi}\right)\frac{(k-iy)^{1/2}}{(k-i\epsilon)^{1/2}}\,\frac{H^{(2)}_{1/3}\left(\frac{2}{3}k^{1/2}(k-iy)^{3/2}\right)}{H^{(2)}_{1/3}\left(\frac{2}{3}k^{1/2}(k-i\epsilon)^{3/2}\right)} \tag{84}$$

In (84), we have introduced ε, an arbitrary small positive quantity, as a mathematical device to establish the inversion contour and we will take the limit $\varepsilon \to 0$ later in the analysis. The inversion of (84) to obtain the thermal field solution is not easily accomplished. However, the heat transfer at the wall can be obtained readily for both small and large values of x. To do so, we evaluate $\left.\frac{\partial \bar{\Phi}}{\partial y}\right|_{y=0}$

$$\left.\frac{\partial \bar{\Phi}}{\partial y}\right|_{y=0} = \left(\frac{T_2 - T_1}{T_1 - T_\infty}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{-i^{2/3}}{k+bi}\right) k^{1/2}(k-i\varepsilon)^{1/2} \frac{H^{(2)}_{2/3}\left(\frac{2}{3}k^{1/2}(k-i\varepsilon)^{3/2}\right)}{H^{(2)}_{1/3}\left(\frac{2}{3}k^{1/2}(k-i\varepsilon)^{3/2}\right)} \tag{85}$$

For the heat transfer rate at the plate far downstream ($x \gg 1$), the principal contribution to the inversion integral for (85) comes for $|k| \ll 1$. Thus, for $x \gg 1$, $\left.\frac{\partial \bar{\Phi}}{\partial y}\right|_{y=0}$ is

$$\begin{aligned} \left.\frac{\partial \bar{\Phi}}{\partial y}\right|_{y=0} \approx \left(\frac{T_2 - T_1}{T_1 - T_\infty}\right)\left(\frac{1}{2\sqrt{2\pi}}\right)\left(\frac{-i^{2/3}}{\cos 60}\right)\Bigg[& \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\left(\tfrac{1}{3}\right)^{-1/3} k^{-2/3} \\ & + \frac{\left(\frac{1}{3}\right)^{1/3} i^{2/3}\,\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})}\frac{(k-i\varepsilon)}{k^{1/3}} + \frac{\Gamma(\frac{2}{3})}{3}\left(\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})^2} \right. \\ & \left. - \frac{1}{\Gamma(\frac{5}{3})}\right) i^{4/3}(k-i\varepsilon)^2 + \left(\tfrac{1}{3}\right)^{2/3}\Gamma(\tfrac{2}{3})\,A'(k-i\varepsilon)^2 k^{1/3} \Bigg] \end{aligned} \tag{86}$$

where

$$A' = \left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{5}{3})}\right) - \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})}\left[\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})^2} - \frac{1}{\Gamma(\frac{5}{3})}\right] \tag{87}$$

and Γ denotes the gamma function. The Fourier inversion integral for (86) can be evaluated in terms of known functions

$$\left.\frac{\partial T}{\partial y}\right|_{\substack{y=0\\x>>0}} = (T_1 - T_\infty)\beta - (T_2 - T_1)\frac{(3)^{5/6}}{2\pi}\Gamma\left(\tfrac{2}{3}\right)x^{-1/3}$$

$$- (T_2 - T_1)\left(\tfrac{1}{3}\right)^{4/3}\tan(60)\frac{\Gamma\left(\tfrac{2}{3}\right)^2}{\pi\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{4}{3}\right)}x^{-5/3} \qquad (88)$$

$$- (T_2 - T_1)\left(\frac{1+0}{81\pi}\right)\tan(60)\frac{\Gamma\left(\tfrac{2}{3}\right)\Gamma\left(\tfrac{1}{3}\right)A'}{(3)^{8/3}}\, x^{-13/3}$$

where it is noted that when $T_1 = T_\infty$ the leading term in our solution (88) vanishes and the second term matches with the leading term in the Leveque solution (22). The two additional terms, which indicate increased heat transfer to $(T_1 > T_2)$ or from $(T_2 > T_1)$ the plate, are due to the effects of streamwise conduction and are not present in the Leveque boundary layer solution.

The expression for local heat transfer $(|x| < 1)$ is obtained by approximating for large values of k

$$\left.\frac{\partial \bar{\Theta}}{\partial y}\right|_{y=0} = \left(\frac{T_2 - T_1}{T_1 - T_\infty}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\left[\frac{-i(k-i\epsilon)^{1/2}}{k^{1/2}} - \frac{1}{4k(k-i\epsilon)} + \cdot \cdot \quad \right] \qquad (89)$$

Direct inversion of the second term diverges due to excessive contribution to the inversion integral when $|k| << 1$. Since (89) is an approximation for $|k| >> 1$, the second term in (89) can by approximated by introducing a cut off a_o provided a_o is small compared to unity. Thus, one can

$$
\begin{aligned}
|k| > a_o \quad & \left(\frac{T_2-T_1}{T_1-T_\infty}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{-1}{4k(k-i\epsilon)}\right) \\
|k| < a_o \quad & 0
\end{aligned}
\tag{90}
$$

The inversion of the first two terms in (89) with modification (90) yields

$$
\left.\frac{\partial\Theta}{\partial y}\right|_{\substack{y=0\\|x|<1}} = \left(\frac{T_2-T_1}{T_1-T_\infty}\right)\left(-\frac{1}{\pi x} - \frac{1}{4\pi a_o} + \frac{|x|}{8} - \frac{a_o|x|^2}{8\pi}\right) \tag{91}
$$

The cut off point a_o shall be determined by requiring the near field solution for heat transfer (91) and the first three terms in the far field heat transfer rate (88) which must be redefined in terms of Θ as in (91), to be equal at the point r=1 where both can be considered to approximate the solution equally well. The resulting solution for a_o is .17. Fig. (7) illustrates that when a_o=.17 the resulting near field solution for heat transfer rate closely approximates the far field solution, based on the first three terms in (88), in the vicinity of r=1. If more terms are required in (89), they are obtained in a similar manner. Thus,

$$
\begin{aligned}
\left.\frac{\partial\Theta}{\partial y}\right|_{\substack{y=0\\|x|<1}} = {} & \left(\frac{T_1-T_2}{T_1-T_\infty}\right)\left(\frac{1}{\pi}\right)\left(\frac{1}{x}\right) + \left(\beta + \left(\frac{T_1-T_2}{T_1-T_\infty}\right)\left(\frac{1}{.68\pi}\right)\right) \\
& - \left(\frac{T_1-T_2}{T_1-T_\infty}\right)\frac{|x|}{8} + \left(\frac{T_1-T_2}{T_1-T_\infty}\right)\left(\frac{.17}{8\pi}\right)|x|^2
\end{aligned}
\tag{92}
$$

(5.2) Solution for θ_1

Substituting θ_0 into equation (20), results in an equation which has a particular solution of order r^2 which automatically satisfies homogeneous boundary conditions (23). When the y derivative of this particular solution is taken and compared with the local heat transfer information (92), provided by the global formulation of the problem, the particular solution is seen to be compatible with the third term while the second term is still not accounted for. However, if one lets θ_1 be the n=1 homogeneous solution in (77), one finds that $\left.\frac{\partial\Theta_1}{\partial y}\right|_{y=0}$ can be matched with the second term in (92) in which case the coefficient C_1 must equal $\beta + \left(\frac{T_1-T_2}{T_1-T_\infty}\right)\left(\frac{1}{68\pi}\right)$ Thus, θ_1 is given by

$$\Theta_1 = \left(\beta + \left(\frac{T_1-T_2}{T_1-T_\infty}\right)\left(\frac{1.244}{\pi}\right)\right) r \sin\phi \tag{93}$$

(5.3) Solution for θ_2 and Higher Order θ_n

After substituting θ_0 and θ_1 into equation (24), a particular solution is found for (24) which again satisfies homogeneous boundary conditions (23). This θ_{2p} is found to be

$$\Theta_{2p} = \frac{r^2}{8\pi}\left(\frac{T_1-T_2}{T_1-T_\infty}\right)\left(\cos(2\phi) - 1 + \phi\sin(2\phi)\right) \tag{94}$$

On comparing $\left.\frac{\partial\Theta_{2p}}{\partial y}\right|_{y=0}$ with the third term in the global approximation for the local heat transfer rate (92), it is seen that the n=2 homogeneous function in (77) is necessary. On matching coefficients, one finds $C_2 = -\frac{1}{16}\left(\frac{T_1-T_2}{T_1-T_\infty}\right)$.

Higher order θ_n can be determined in an analogous manner. One must be careful, however, in calculating these higher order θ_n to account for the higher order terms in the Blasius solution which were neglected in the linear approximation (76). This higher order correction for the Blasius profile does not effect the first three terms in (95) obtained above to order θ_2 and will not effect θ_3. Thus, to order θ_3 one finds

$$
\begin{aligned}
T = T_2 + (T_1-T_2)\left(\frac{1}{\pi}\right)\phi &+ \left((T_1-T_\infty)\beta + (T_1-T_2)\left(\frac{1}{.69\pi}\right)\right)r\sin\phi \\
&+ \left(\frac{T_1-T_2}{8\pi}\right)\left[\cos(2\phi) - 1 + \phi\sin(2\phi) - \frac{\pi}{2}\sin(2\phi)\right]r^2 \qquad (95) \\
&+ \frac{(.17)(T_1-T_2)}{24\pi}r^3\sin 3\phi + \ldots
\end{aligned}
$$

Since we shall want to examine the local solution when T_∞ is greater than, less than, or equal to T_1, it is convenient to display the results in terms of the dimensionless temperature ω.

$$
\begin{aligned}
\omega = \left(1-\frac{\phi}{\pi}\right) &+ \left(\frac{(T_1-T_\infty)}{(T_2-T_1)}\beta - \frac{1}{.69\pi}\right)r\sin\phi \\
&- \frac{r^2}{8\pi}\left(\cos(2\phi) - 1 + \phi\sin(2\phi) - \frac{\pi}{2}\sin(2\phi)\right) \qquad (96)
\end{aligned}
$$

The first term in ω, ω_o, represents the local pure conduction solution in which the isotherms are radial lines which vary monotonically near $r=0$ from unity at $\phi=0$ to o at $\phi=\pi$ with $\omega_0=\frac{1}{2}$ at $\phi=\frac{\pi}{2}$. In the discussion of ω_1 and ω_2 which follows, $\frac{\beta}{(T_2-T_1)}$ is treated as a constant and T_∞ varied relative to T_1 in the grouping $\left(\frac{T_1-T_\infty}{T_2-T_1}\right)\beta$. This reflects the influence of the far field temperature on the behavior of the local temperature distribution. The second and third terms, ω_1 and ω_2 respectively, reflect the influence of convection on the local pure conduction solution in which case the radial isotherms are seen in Figs. 8 and 9 for far field temperatures of $T_\infty < T_1$ and $T_\infty = T_1$ respectively to be washed downstream. Fig. 10 illustrates the case of T_∞ being significantly larger than T_1 with the result of the isotherms close to the plate being convected downstream due to the cooler liquid close to the wall being washed downstream while farther away from the wall significantly warmer fluid is washed downstream causing isotherms to move upstream. When T_∞ is very much greater than T_1, all the isotherms move upstream and a better overall view of the variation in field temperature is observed, Fig. 11. Deviations between ω_o and $\omega_o+\omega_1$ are larger in regions further away from the plate, where convection dominates, than those deviations corresponding to regions close to the plate or near the discontinuous boundary condition where conduction dominates. This implies that in order to get a better approximation to the thermal field further away more terms need to be taken. Next we would like to examine the heat transfer rate at the plate as T_2 varies relative to T_1. In view of this, it is convenient to define the temperature in terms of θ. The heat transfer rate goes to infinity as r goes to zero because of the mathematically idealized step jump in temperature considered to exist at the wall. Here, the relative temperature difference $(\frac{T_2-T_1}{T_1-T_\infty})$ shall be used as a parameter. Considering (T_1-T_∞) to be positive and constant, the local pure conduction solution predicts that heat transfer will be out of the wall for $x>0$ and into the wall for $x<0$ when $(T_2-T_1)>0$ [see equation (92)]. The convective contribution of θ_1 tends to have a cooling effect, in this case increasing the heat transfer rate out of the wall for $x>0$ and decreasing it into the wall for $x<0$. The opposite behavior would be observed when $(T_2-T_1)<0$. While for $T_2=T_1$, the heat transfer is just that predicted by the Pohlhausen solution and given as θ. The behavior of $\left.\frac{\partial(\Theta_0+\Theta_1)}{\partial y}\right|_{y=0}$ for $T_2 \gtrless T_1$ is illustrated in Fig. 12. Once again, in order to improve the approximation for the heat transfer rate, more θ_n terms should be taken.

Chapter 6

Natural Convection In The Vicinity Of An Isothermal Vertical Wedge

The solution for θ_o which satisfies equation (38) and the boundary conditions (40) is obtained by the standard procedure of separation of variables:

$$\Theta_0 = 1 + \sum_{n=1}^{\infty} C_n r^{\left(\frac{2n-1}{2\lambda}\right)} \cos\left[\left(\frac{2n-1}{2\lambda}\right)\phi\right] \tag{97}$$

The first term in (97) is the steady state pure conduction solution θ_o while the infinite sum contains the homogeneous solutions that satisfy the homogeneous equation for (44) and homogeneous boundary conditions (46).

Having θ_o and substituting for it in equation (39), the governing equation for ψ_o has no particular part and is simply the homogeneous biharmonic equation. Thus, the uniform temperature field of unity for θ_o is observed to generate no fluid motion.

(6.1) Solution for θ_1

With θ_o being equal to unity and ψ_o being equal to zero, the governing equation for θ_1 (44) reduces to Laplaces equation. Thus, θ_1 requires no particular solution. However, a homogeneous solution could be necessary based on compatibility requirements with the far field behavior. The choice of which of the homogeneous functions in the infinite sum (97) is to be used as θ_1, is deduced by examining the limiting case of $\lambda=\frac{1}{2}$ corresponding to a horizontal wall. Here, if the horizontal wall where $\theta=1$, is considered to be at a large but finite distance above another horizontal wall, where $\theta=0$, the basic temperature distributions would be linear and the n=1 term would be used. In this case, due to the fact that the upper wall is at a temperature greater than that of the lower wall, the temperature stratification is stable and no natural convection occurs. Hence, the coefficient of C_1 would be equal to the temperature gradient based on pure conduction between the plates. Furthermore, if the lower plate was moved to infinity, C_1 would go to zero. Based on this reasoning and the fact that the solution is expected to vary continuously as λ varies from $\frac{1}{2}$ to 1, the eigenfunction corresponding to the lowest eigenvalue n=1 in (97) is selected as θ_1 for any wedge

$$\theta_1 = C_1 r^{(1/2\lambda)} \cos\left(\frac{1}{2\lambda}\phi\right) \tag{98}$$

The choice of the lowest eigenvalue is also consistent with the modified Oseen linearization solution for the vertical plate which is presented later in this analysis.

(6.2) Solution for ψ_1

θ_o, θ_1, and ψ_o which is zero, are now used in generating the term ψ_1 in the series solution for ψ. ψ_1 must satisfy equation (45) and local homogeneous boundary conditions (41). Substituting for $\theta_0 \to \theta_1$ equation (45) becomes

$$C_1\left(\frac{1}{2\lambda}\right) r^{\left(\frac{1}{2\lambda}-1\right)} \sin\left[\left(\frac{1}{2\lambda}-1\right)\phi\right] = \nabla^2\nabla^2\psi_1 \tag{99}$$

It is to be noted that when $\lambda = \frac{1}{2}$ the driving term becomes zero and there is no flow. The particular solution to equation (99) is

$$\psi_{1p} = \frac{C_1 r^{\left(\frac{1}{2\lambda}+3\right)}}{32\left(\frac{1}{2\lambda}+1\right)} \sin\left(\left(\frac{1}{2\lambda}-1\right)\phi\right) \tag{100}$$

The homogeneous solutions to the biharmonic equation, Weinbaum[19], are

$m \neq 2$

$$\psi_{1c} = r^m \left[A_1\sin(m\phi) + A_2\sin((m-2)\phi) + A_3\cos(m\phi) + A_4\cos((m-2)\phi) \right] \tag{101}$$

$m = 2$

$$\psi_{1c} = r^2 \left[A_1\sin(2\phi) + A_2\cos(2\phi) + A_3\phi + A_4 \right] \tag{102}$$

Since the flow is symmetric, the streamfunction must be antisymmetric or $A_3 = A_4 = 0$ in equation (101) and $A_2 = A_4 = 0$ in equation (102). It is observed that ψ_{1p} is also antisymmetric. If solutions (101) and (102) are to negate the contributions of the particular solution (100) to the no slip boundary condition at the wall, m must equal $\frac{1}{2}\lambda + 3$.

$$\psi_1 = \frac{C_1 r^{(\frac{1}{2\lambda}+3)}}{32(\frac{1}{2\lambda}+1)} \sin((\tfrac{1}{2\lambda}-1)\phi) + r^{(\frac{1}{2\lambda}+3)}\Big(A_1 \sin((\tfrac{1}{2\lambda}+3)\phi) + A_2 \sin((\tfrac{1}{2\lambda}+1)\phi)\Big) \tag{103}$$

The constants A_1 and A_2 are determined by satisfying the no slip conditions at the wall. Thus, the governing equations for A_1 and A_2 are

$$0 = \frac{C_1 r^{(\frac{1}{2\lambda}+3)}}{32(\frac{1}{2\lambda}+1)} \sin((\tfrac{1}{2\lambda}-1)\lambda\pi) + r^{(\frac{1}{2\lambda}+3)}\Big(A_1 \sin((\tfrac{1}{2\lambda}+3)\lambda\pi) + A_2 \sin((\tfrac{1}{2\lambda}+1)\lambda\pi)\Big) \tag{104}$$

$$0 = \frac{C_1}{32}\left(\frac{\frac{1}{2\lambda}-1}{\frac{1}{2\lambda}+1}\right) r^{(\frac{1}{2\lambda}+3)} \cos(\tfrac{1}{2\lambda}-1)\lambda\pi + r^{(\frac{1}{2\lambda}+3)}\Big((\tfrac{1}{2\lambda}+3)A_1 \cos((\tfrac{1}{2\lambda}+3)\lambda\pi) + (\tfrac{1}{2\lambda}+1)A_2 \cos((\tfrac{1}{2\lambda}+1)\lambda\pi)\Big) \tag{105}$$

The solution for A_1 and A_2 when $\frac{1}{2} < \lambda \leq 1$ is

$$A_1 = \frac{(\frac{1}{2\lambda})\left(\frac{1}{32(\frac{1}{2\lambda}+1)}\right)C_1}{\frac{1}{2\lambda}+1+(4\cos^2(\lambda\pi)-1)} \tag{106}$$

$$A_2 = \frac{\frac{-C_1}{32(\frac{1}{2\lambda}+1)}\cos\lambda\pi - A_1\cos(3\lambda\pi)}{\cos\lambda\pi} \tag{107}$$

and the general expression for ψ_1 becomes

$$\psi_1 = \frac{C_1 r^{(\frac{1}{2\lambda}+3)}}{32(\frac{1}{2\lambda}+1)}\sin((\tfrac{1}{2\lambda}-1)\phi) + r^{(\frac{1}{2\lambda}+3)}\Bigg[\frac{(\frac{1}{2\lambda})\left(\frac{C_1}{32(\frac{1}{2\lambda}+1)}\right)}{(\frac{1}{2\lambda}+1+(4\cos^2(\lambda\pi)-1))}\sin((\tfrac{1}{2\lambda}+3)\phi)$$
$$-\frac{C_1}{32(\frac{1}{2\lambda}+1)}\left(1+\frac{\cos(3\lambda\pi)}{\cos(\lambda\pi)}\left(\frac{1}{2\lambda}\right)\left(\frac{}{\frac{1}{2\lambda}+1+(4\cos^2(\lambda\pi)-1)}\right)\sin((\tfrac{1}{2\lambda}+1)\phi)\right)\Bigg] \quad (108)$$

It is observed at this point that the solutions for θ and ψ given by $(\theta_0+\theta_1)$ and $(\psi_0+\psi_1)$ respectively are determined to within the constant C_1. Before C_1 can be determined, local boundary information must first be provided by a global formulation of the problem.

(6.3) Global Oseen Problem for $\lambda = 1$

The global Oseen problem for the temperature distribution corresponding to natural convection over the leading edge of an isothermal, semi-infinite, vertical, flat plate has, as a first approximation, its governing equation given as

$$-C\frac{\partial\Theta}{\partial x} = \nabla^2\Theta \quad (109)$$

where C is some mean convective velocity which will be defined shortly. This equation satisfies the requirement that it reduce to the near field equation (38) as one approaches within a thermal Stokes radius of the leading edge. The boundary conditions are

$$\Theta = 1 \qquad \phi = \pi$$

$$\frac{\partial \Theta}{\partial \phi} = 0 \qquad \phi = 0 \tag{110}$$

$$\Theta_\infty = 0 \qquad r \to \infty$$

The solution, as given in Carrier, Krook, and Pearson[8], is

$$\Theta = \mathrm{erfc}\left(c^{1/2} r^{-1/2} \cos\left(\frac{\phi}{2}\right)\right) \tag{111}$$

and the centerline temperature distribution upstream of the plate along $\phi=0$ is

$$\Theta\Big|_{\phi=0} = 1 - \frac{2c^{1/2}}{\sqrt{\pi}} r^{1/2} + \frac{2c^{3/2}}{3\sqrt{\pi}} r^{3/2} - \frac{2c^{5/2}}{10\sqrt{\pi}} r^{5/2} + \ldots \tag{112}$$

When the global solution for the centerline temperature distribution upstream of the plate (112) is compared with that predicted by $(\theta_0+\theta_1)$ with $\lambda=1$ in the local iterative solution as $r\to 0$, they are found to be compatible and upon equating the coefficients of the $r^{1/2}$ term $C_1=-2\sqrt{\frac{c}{\pi}}$ This serves to illustrate that the remaining unknown constant C_1 in θ_1 (98) and ψ_1 (108) must be determined for each wedge case separately by matching the local iterative solution for temperature or heat transfer along a boundary with that predicted by a global formulation of the problem.

The definition of C in equation (109) is somewhat arbitrary. One qualitatively reasonable criterion for its determination is to require that the difference between the nonlinearized term and the linearized term in equation (109) to be zero when integrated along the centerline from the leading edge upstream to infinity. This criterion is similar in spirit to that used in previous evaluations of the modified Oseen constant; e.g., Carrier and Lewis[23] and Weinbaum[24]. Accordingly, we shall define C by

$$\int_0^\infty \left(u\frac{\partial\Theta}{\partial x} + c\frac{\partial\Theta}{\partial x} \right)\bigg|_{\phi=0} dr = 0 \tag{113}$$

Since no total field expression exists for u, an approximate expression shall be constructed. As $r\to 0$, u must behave as the first nonzero term $-\frac{\partial\psi}{\partial y}$ in the iterative near field solution (108) where $\lambda=1$. While as r goes to infinity, u is required to behave as e^{-cr} since Θ and u are coupled and Θ decays approximately as e^{-cr}. Thus,

$$u \approx \left(-\frac{2}{27}\sqrt{\frac{5}{\pi}}\right) r^{3/2} e^{-cr} \tag{114}$$

Substituting this value of u in equation (113) and performing the integration, one finds C=.181 and C_1 in θ_1 (98) and ψ_1 (108), when $\lambda=1$, is equal to -.48.

(6.4) Higher Order θ_n and ψ_n

The calculation of higher order θ_n and ψ_n involves fundamental difficulties that were not encountered in the three forced convection problems considered previously. In the latter three problems, the convective term in the global energy equation for the limiting case $\phi_1=0$ was the same as the convective term used to generate the iterative series solution. Thus, when the global solution for the centerline temperature or heat transfer was expanded about r=0, a term by term matching could be completed with the inner field solution. In the present case, the centerline temperature distribution (112) near r=0 upstream of the plate is based on equation (109). There is no reason to expect that if the full convective term $\bar{u}\cdot\nabla\theta$, on which the hierarchy of equations (44) is based, were used instead of the Oseen linearization $c\frac{\partial\theta}{\partial x}$, the solution for θ at $\phi=0$ would take the form (112) as r approaches zero. In view of the above, it is interesting to observe that the homogeneous solutions to equation (44), which comprise the infinite sum in (97), do correspond on a one-to-one basis n=1, 2, 3, etc. with the global series solution (112). Apparently, the solutions of Laplaces equation (97) with the homogeneous boundary conditions (46) are the correct generating functions for an iterative series solution to the linearized equation (109). Thus, the solutions for θ_1 equation (98) and ψ_1 equations (108) correspond to an Oseen linearization of the temperature field. By the same token, it would be fortuitous if the generating functions in (97) should also match on a term-by-term basis with the series solution for $r\ll 1$ of the full non-linear global problem. This implies that the Stokes like generating functions do not in all cases form a complete set and that additional generating functions may be required to represent the non-linear behavior of the convective terms.

In light of the above, it would not be meaningful as a rational procedure to calculate higher order θ_n and ψ_n from (44) and (45) using the lower order solutions θ_1 and ψ_1 obtained in (98) and (108). We shall simply give the results obtained by solving (44) subject to (46). The solution for θ_2 for $\frac{1}{2} < \lambda \leq 1$ and $\lambda \neq \frac{5}{6}$ is given by

$$\begin{aligned}\Theta_2 = \Theta_{2p} - \frac{(\gamma+1)C_1}{\cos\left(\frac{3\pi}{2(\gamma+1)}\right)}\Bigg[&(\gamma+1)A_1\cos\left(\frac{3\pi}{2(\gamma+1)}\right) + \cos\left(\frac{\pi}{2(\gamma+1)}\right)\left(\frac{(\gamma+1)A_2}{\left(1+\frac{4}{2(\gamma+1)}\right)}\right.\\ &\left.+ \frac{[(\gamma+1)^3-(\gamma+1)^2]C_1}{32(\gamma+2)[(\gamma+1)^2+3(\gamma+1)+2]}\right) + \frac{[(\gamma^2+1)+7(\gamma+1)+4]C_1}{128(\gamma+2)^2(\gamma+3)(2\gamma+3)}\cos\left(\frac{\pi}{2(\gamma+1)}\right)\\ &+ \frac{(\gamma+5)A_2}{8(\gamma+2)(\gamma+3)}\cos\left(\frac{\pi}{2(\gamma+1)}\right)\Bigg] r^{(2\gamma+5)}\cos((2\gamma+5)\phi)\end{aligned} \tag{115}$$

and when $\lambda = \frac{5}{6}$

$$\begin{aligned}\Theta_2 = \Theta_{2p}\Big|_{\lambda=5/6} + \frac{18C_1}{25\pi}\cos\left(\frac{5\pi}{6}\right)&\left[\frac{195}{832}A_2 + \frac{1475}{292{,}864}C_1\right]\cdot\\ &\left[\ln(r)\,r^{21/5}\cos\left(\tfrac{21}{5}\phi\right) - r^{21/5}\phi\sin\left(\tfrac{21}{5}\phi\right)\right]\end{aligned} \tag{116}$$

where θ_{2p} is given by

$$\begin{aligned}\theta_{2p} = -(\gamma+1)C_1 r^{(2\gamma+5)} \Big[& \frac{A_1}{4(\gamma+1)}\cos(3\phi) + \left(\frac{A_2}{4(\gamma+3)} + \frac{\gamma C_1}{32(\gamma+1)(4\gamma^2+20\gamma+24)}\right)\cos\phi \\ & + \frac{(\gamma^2+9\gamma+12)C_1}{128(\gamma+1)(2\gamma^3+13\gamma^2+27\gamma+18)}\sin(\gamma\phi)\sin((\gamma+1)\phi) \\ & - \frac{\gamma C_1}{128(2\gamma^3+13\gamma^2+27\gamma+18)}\cos(\gamma\phi)\cos((\gamma+1)\phi) \\ & + \frac{(\gamma+5)A_2}{8(\gamma+2)(\gamma+3)}\sin((\gamma+2)\phi)\sin((\gamma+1)\phi) \\ & - \frac{(\gamma+1)A_2}{8(\gamma+2)(\gamma+3)}\cos((\gamma+2)\phi)\cos((\gamma+1)\phi) \Big] \end{aligned} \tag{117}$$

and $\gamma=(\frac{1}{2\lambda}-1)$. It is to be noted that when $\lambda=1$ θ_2 behaves as r^4 and hence does not match with the expansion in fractional powers of r that obtains from the Oseen type solution (112).

The solution for the temperature distribution and streamfunction, given by $(\theta_0+\theta_1)$ and $(\psi_0+\psi_1)$ respectively, offer a good qualitative picture of the basic behavior of the fluid for all the wedge cases. In order to illustrate the nature of the thermal and flow fields described by $(\theta_0+\theta_1)$ and $(\psi_0+\psi_1)$ in general, let us consider the results for the limiting case of an isothermal vertical plate. The lowest order solutions for the temperature and velocity fields are shown in Figs. 13 through 15. Fig. 13 shows that the steady state pure conduction solution of unity is washed downstream due to the presence of convection. It is also to be noted that at a distance of a thermal Stokes radius along the center line upstream of the leading edge the fluid has already realized approximately 52 percent of the total temperature drop between the plate and the far field. Fig. 14 illustrates that entrainment of fluid must occur in an intermediate region upstream since for r 1 the fluid moves upward and around the plate. In the vicinity of the plate, the fluid tends to stagnate. However, moving away from the surface, the velocity of the fluid increases [see Fig. 15]. This type of behavior comes as no surprise. Intuitively, the fluid would be expected to reach a maximum velocity in some intermediate region upstream of the leading edge and to decay to zero in the near and far field as r goes to zero and infinity respectively.

Chapter 7

Summary

Forced convection over an isothermal wedge, over a wall ramp juncture with a step jump in temperature, and over a flat plate with a step jump in temperature and various approximations for velocity field, have been considered for the purpose of resolving the detailed thermal structure of representative flows in the neighborhood of discontinuous thermal boundary conditions where ordinary boundary layer theory does not apply. Similarly, the thermal and flow fields for natural convection about the leading edge of an isothermal vertical wedge have been examined. Nondimensional analysis of the Navier-Stokes equations in each case shows that there is a localized region in the neighborhood of the discontinuous boundary conditions in each problem which is conduction dominated and the natural convection problem, in addition to having the vicinity of the leading edge of the vertical wedge conduction dominated, is also observed to experience a thermally induced slow flow. A characteristic diffusion length termed the thermal Stokes radius is introduced as a measure of the extent of the region in which conduction dominates. For the forced convection problems, local analytic series solutions to the thermal fields are developed by applying an iterative procedure based on Stokes like generating functions while for the natural convection problem, approximate truncated expressions for thermal and velocity fields are found using a similar approach. In the first two problems of convection over an isothermal wedge and over a wall ramp juncture with a step jump in temperature, the velocity field in the convective terms in the energy equation is idealized using the inviscid flow description for flow past finite wedge angles. The generating functions for the local series solutions and the unknown constants in the limiting case $\phi_1 = 0$ are determined by matching with solutions for either the centerline temperature or wall heat transfer that obtain from Oseen type linearizations of the related global problem for this simple limiting geometry. In the problem of forced convection over a flat plate with a step jump in temperature, the effect of the detailed structure of the incoming velocity and temperature boundary layer on the local thermal field is considered. The generating terms and available constants are similarly determined based on local heat transfer information derived from a detailed global analysis. This detailed global investigation also gave additional heat transfer results far downstream of the step jump in temperature at the plate. In the natural convection problem, an iterative procedure is used to develop approximate truncated expressions for the temperature and velocity fields. The generating functions to be used in developing the expressions for the thermal and velocity fields were determined based on local boundary information from a modified Oseen linearization of the governing energy equation in which the convective effect is expressed as $c \frac{\partial \theta}{\partial x}$. It is thus seen from these several examples how an iterative procedure alone or used in conjunction with local line information obtained from a global analysis may be used to provide useful information in the vicinity of discontinuous boundary conditions when the governing equations are coupled or uncoupled.

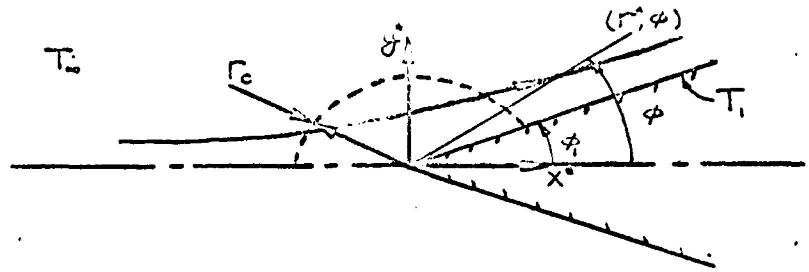

Figure 1 Schematic of Symmetrical Forced Convection Past a Wedge of Half Angle ϕ_1, $0 \leq \phi_1 < 90°$.

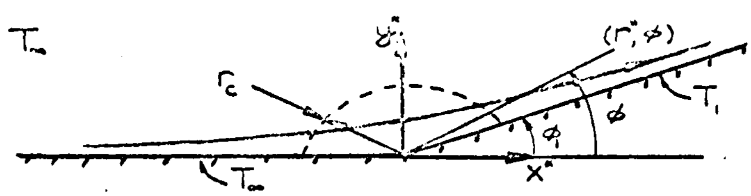

Figure 2 Schematic of Forced Convection Over a Wall Ramp Intersection, $0 \leq \phi_1 < 90°$.

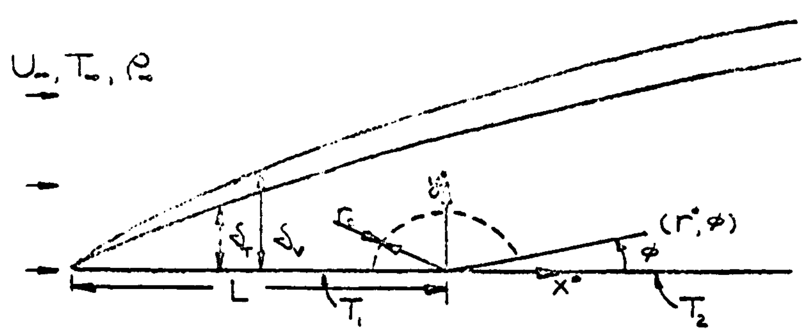

Figure 3 Schematic of Forced Convection Past a Semi-Infinite Flat Plate With a Well Established Temperature and Velocity Boundary Layer.

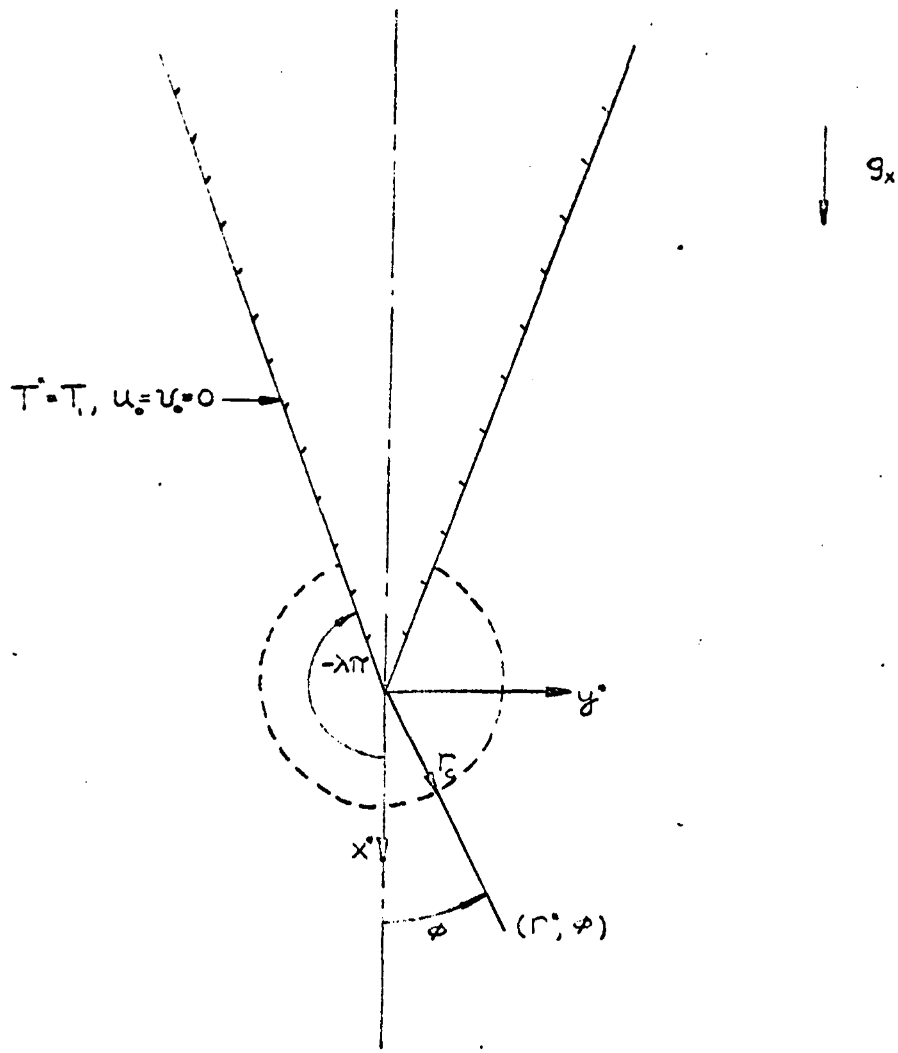

Figure 4 Schematic of the Leading Edge of a Vertical Wedge with an Arbitrary Exterior Angle. The Fluid Properties at $r^* = \infty$ are $T^* = T_\infty$, $\rho^* = \rho_\infty$ and $u_\infty = v_\infty = o$. While in general, $Pr = 1$, $T_1 > T_\infty$, and $1/2 \leq \lambda \leq 1$.

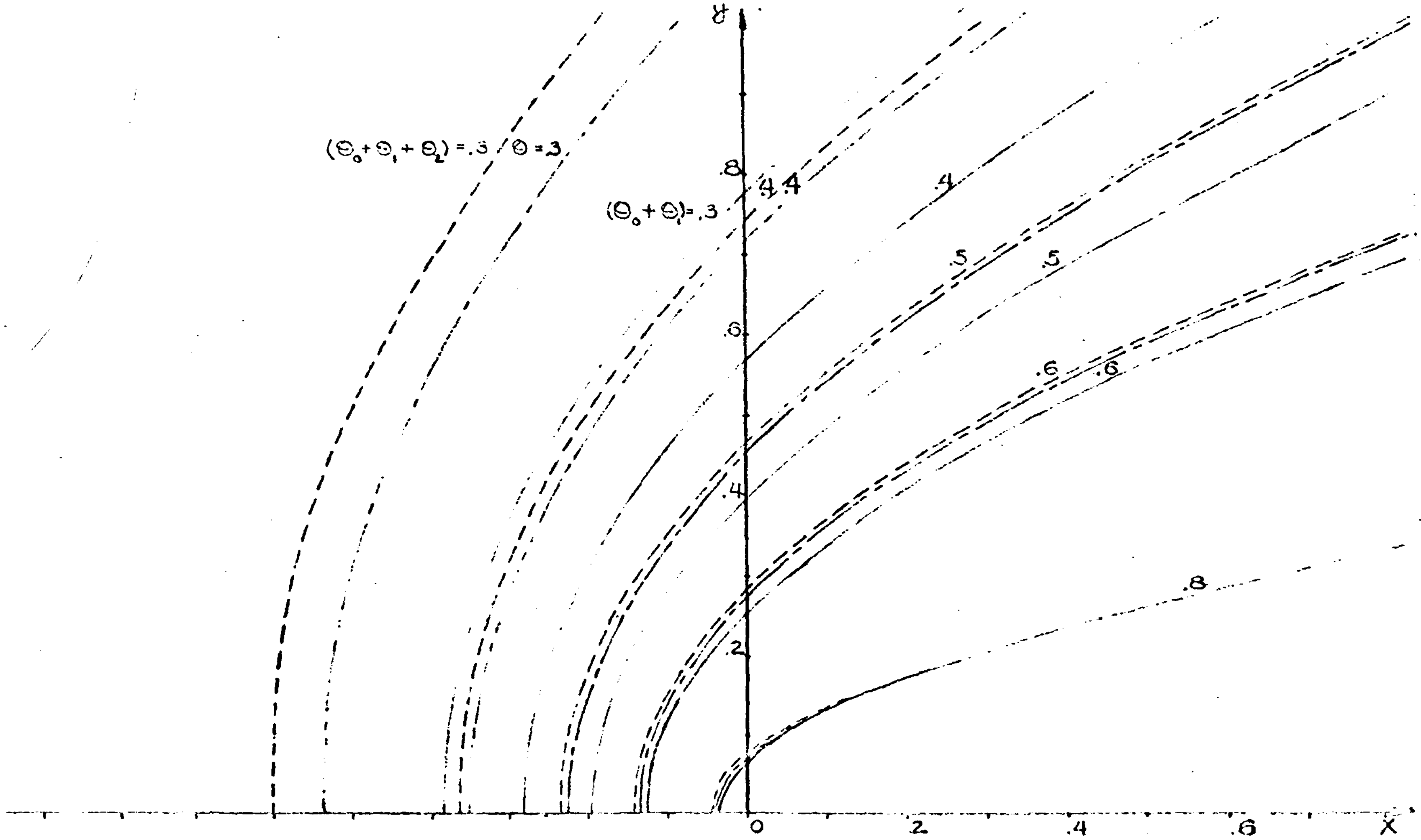

Figure 5 Temperature Distribution in Vicinity of Leading Edge of an Isothermal Semi-Infinite Flat Plate. Problem (i), $\phi_1 = o$, $\theta = o$ at Infinity, $(\theta_o+\theta_1)$——, $(\theta_o+\theta_1+\theta_2)$- - - -, (θ)-·-·-.

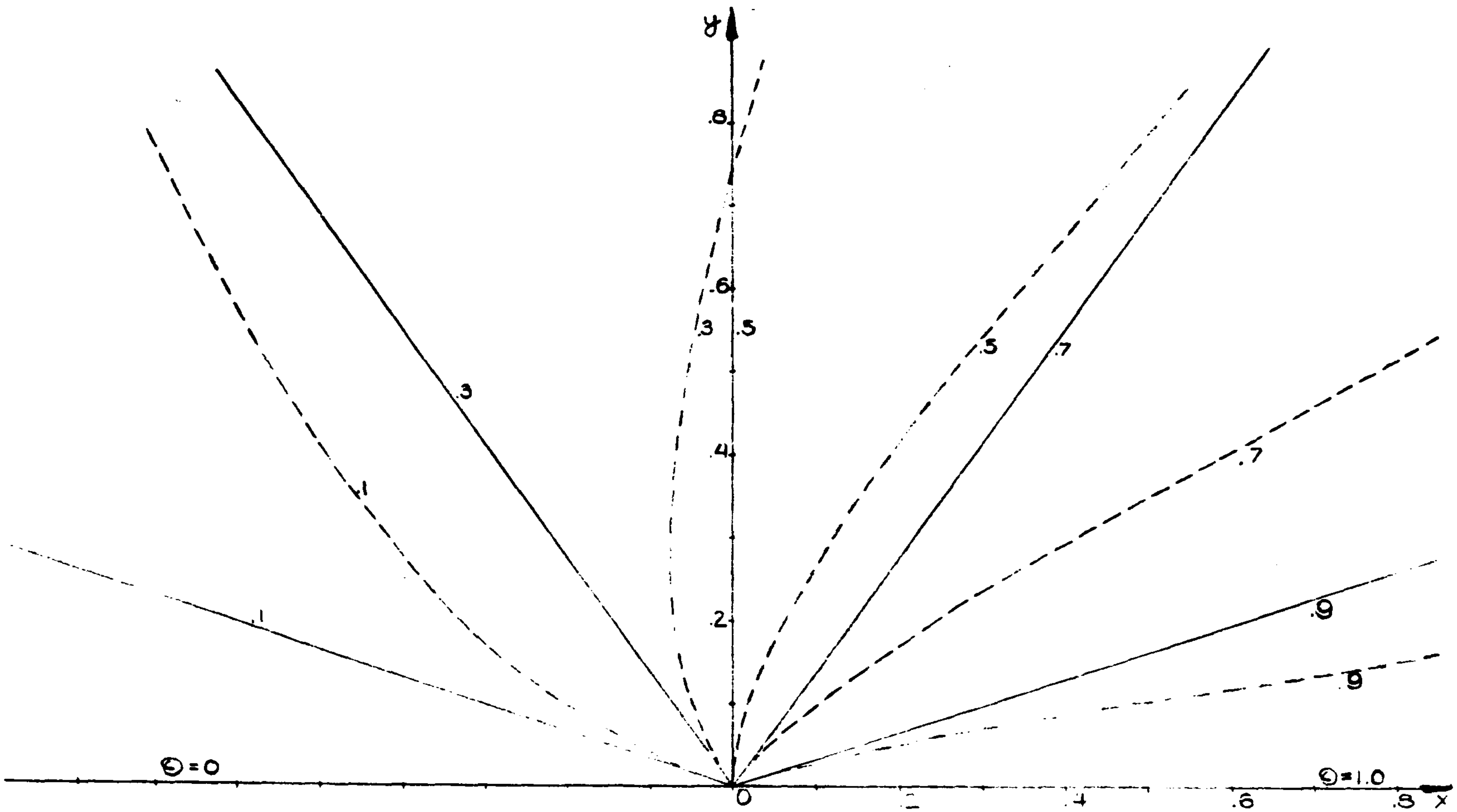

Figure 6 Temperature Distribution in Vicinity of Step Jump in Temperature for a Flat Plate Problem (ii), $\phi_1 = o$, $\theta = o$ at Infinity, θ_o ——, $\theta_o+\theta_1$ - - - -.

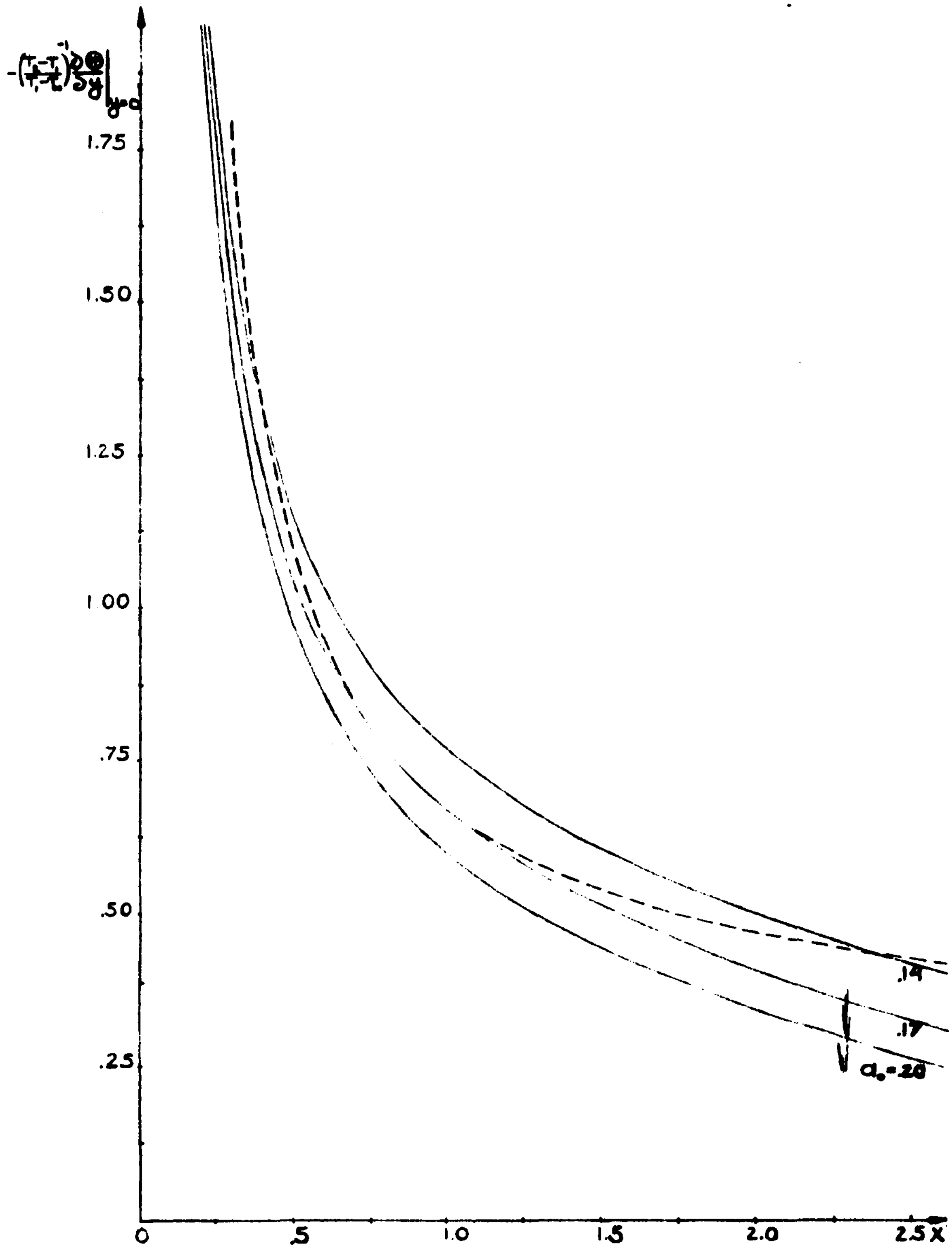

Figure 7 Heat Transfer Rate for Near and Far Field. Problem (iii), Near Field Heat Transfer —— with a_0 as Parameter, Far Field Heat Transfer - - - -.

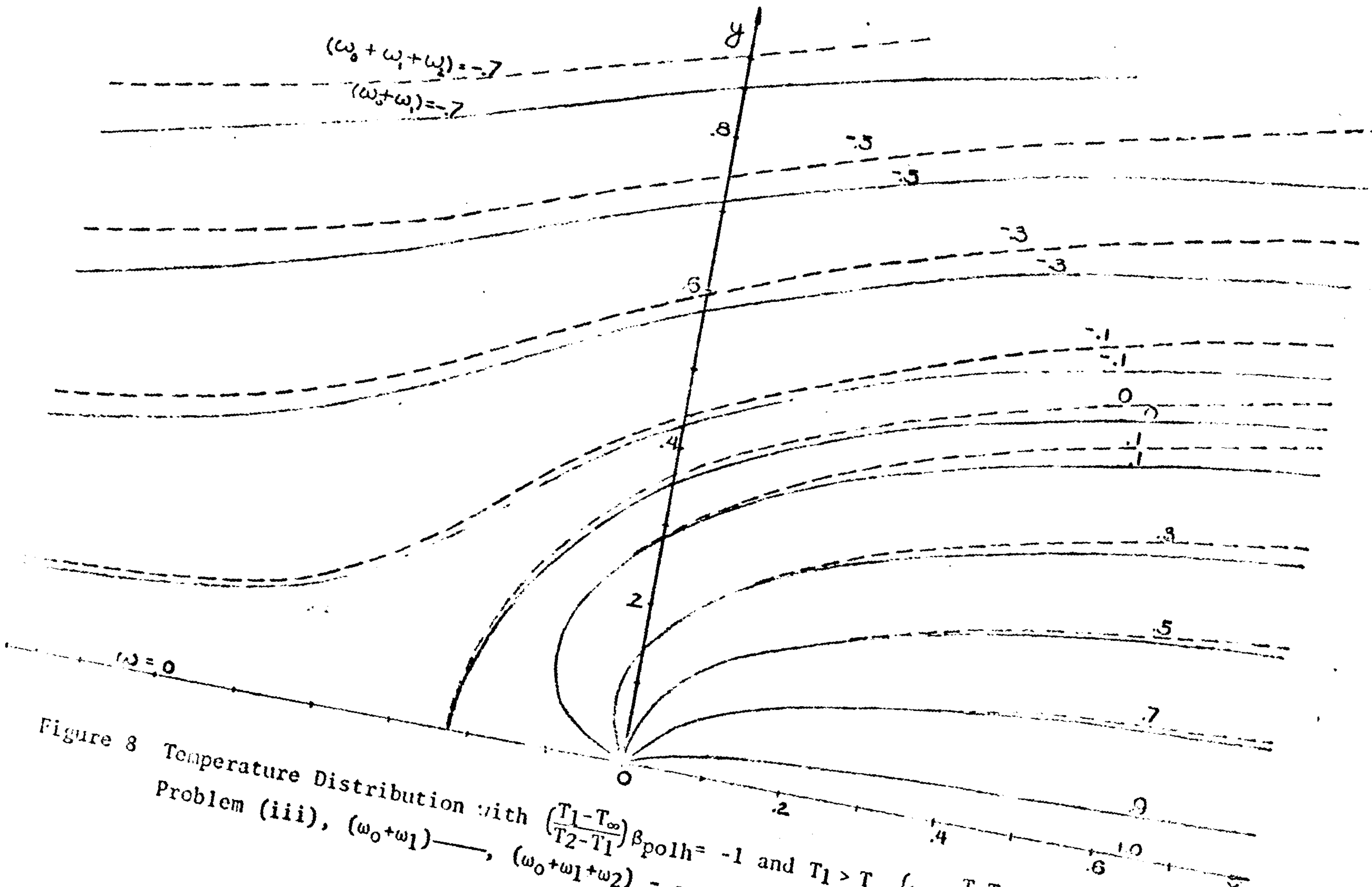

Figure 8 Temperature Distribution with $\left(\frac{T_1-T_\infty}{T_2-T_1}\right)\beta_{po}lh = -1$ and $T_1 > T_\infty$ $\left(\omega = \frac{T-T_1}{T_2-T_1}\right)$.
Problem (iii), $(\omega_0+\omega_1)$——, $(\omega_0+\omega_1+\omega_2)$ - - - - -.

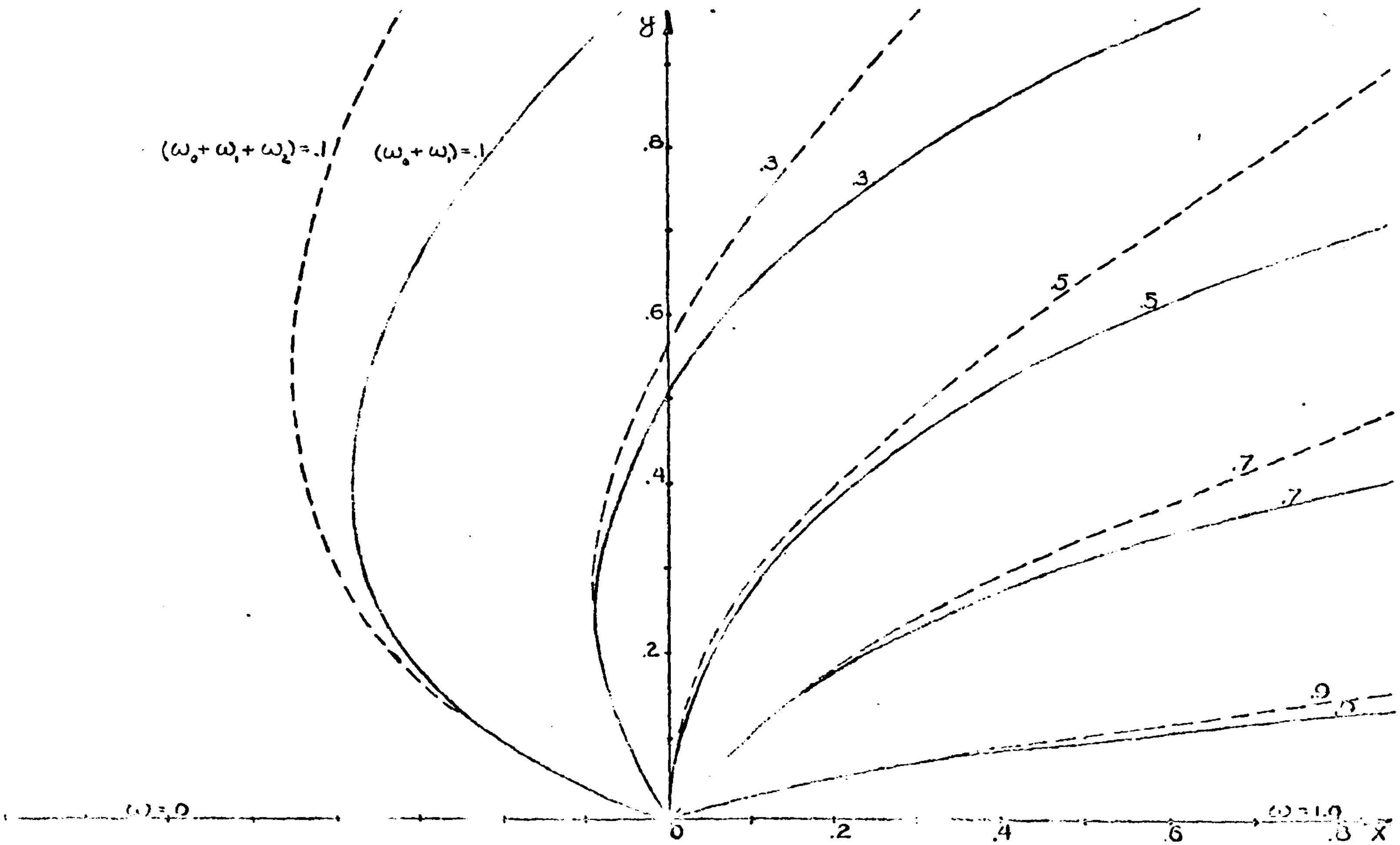

Figure 9 Temperature Distribution with $T_1 = T_\infty$ and $\omega = \left(\frac{T-T_1}{T_2-T_1}\right)$. Problem (iii), $(\omega_0+\omega_1)$———, $(\omega_0+\omega_1+\omega_2)$- - - -.

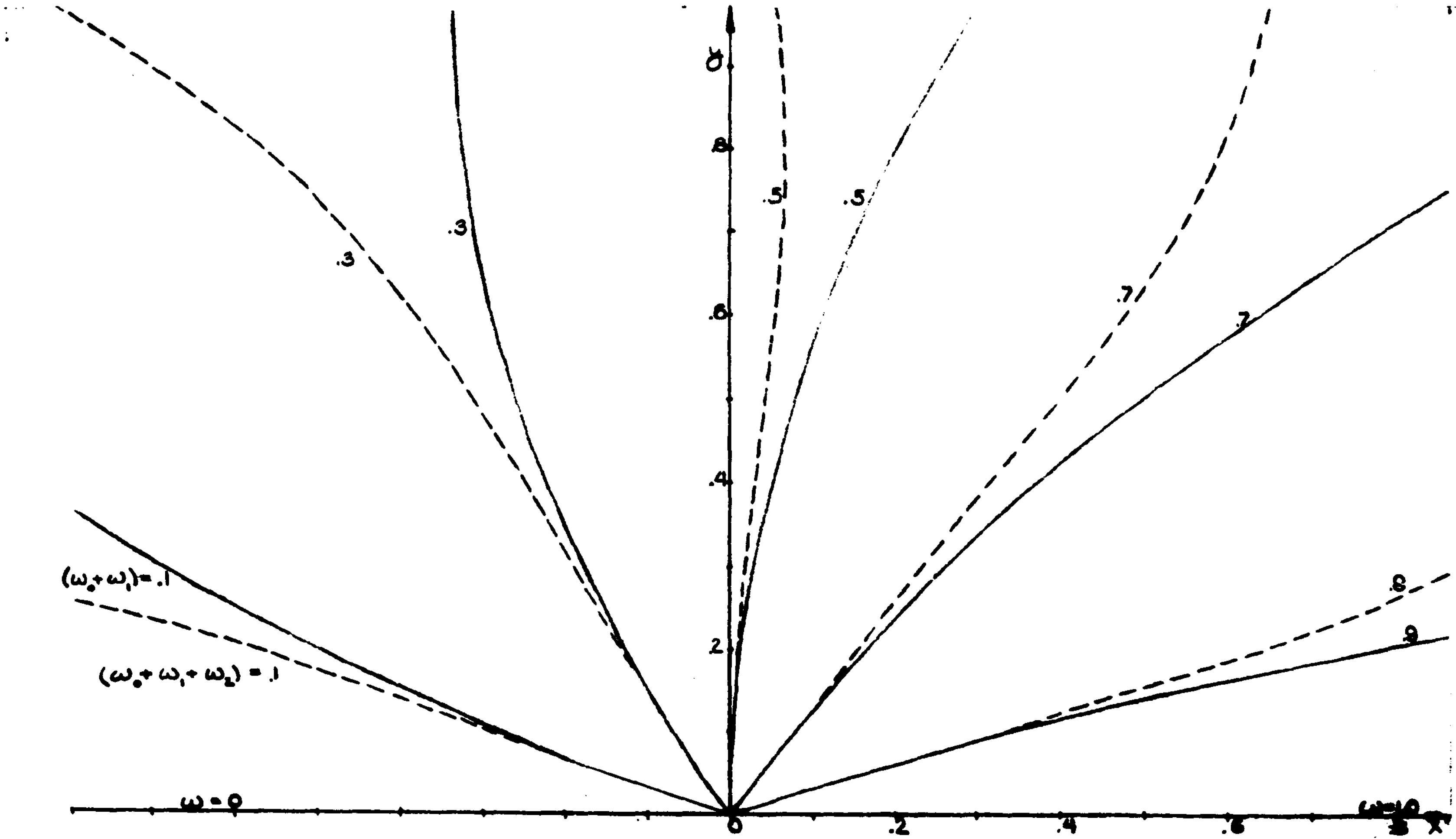

Figure 10 Temperature Distribution with $\left(\frac{T_1-T_\infty}{T_2-T_1}\right)\beta = .3$ or $T_\infty > T_1$ and $\omega = \left(\frac{T-T_1}{T_2-T_1}\right)$.
Problem (iii), $(\omega_0+\omega_1)$——,
$(\omega_0+\omega_1+\omega_2)$- - - -.

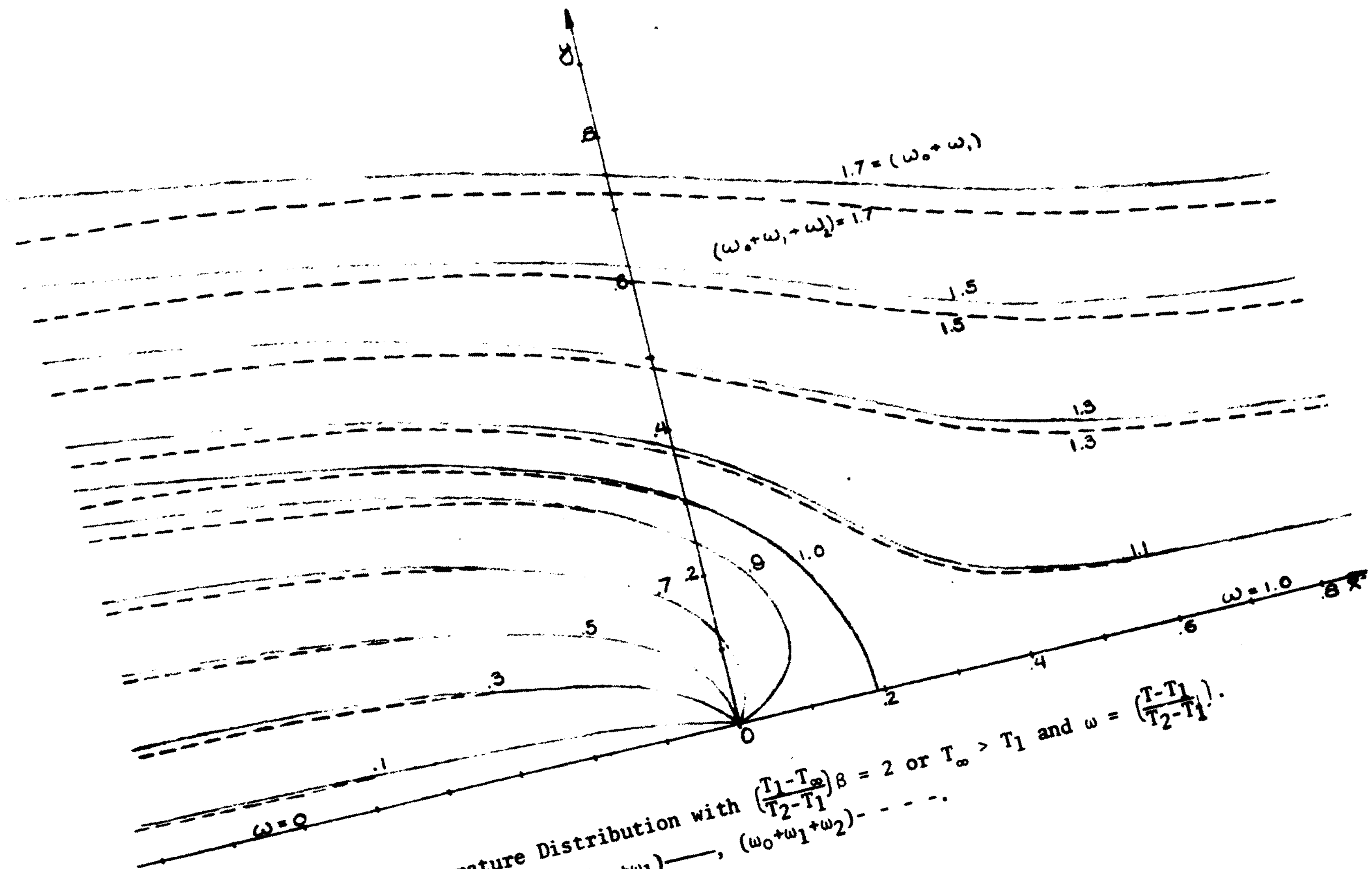

Figure 11 Temperature Distribution with $\left(\frac{T_1-T_\infty}{T_2-T_1}\right)\beta = 2$ or $T_\infty > T_1$ and $\omega = \left(\frac{T-T_1}{T_2-T_1}\right)$.
Problem (iii), $(\omega_o+\omega_1)$——, $(\omega_o+\omega_1+\omega_2)$- - - -.

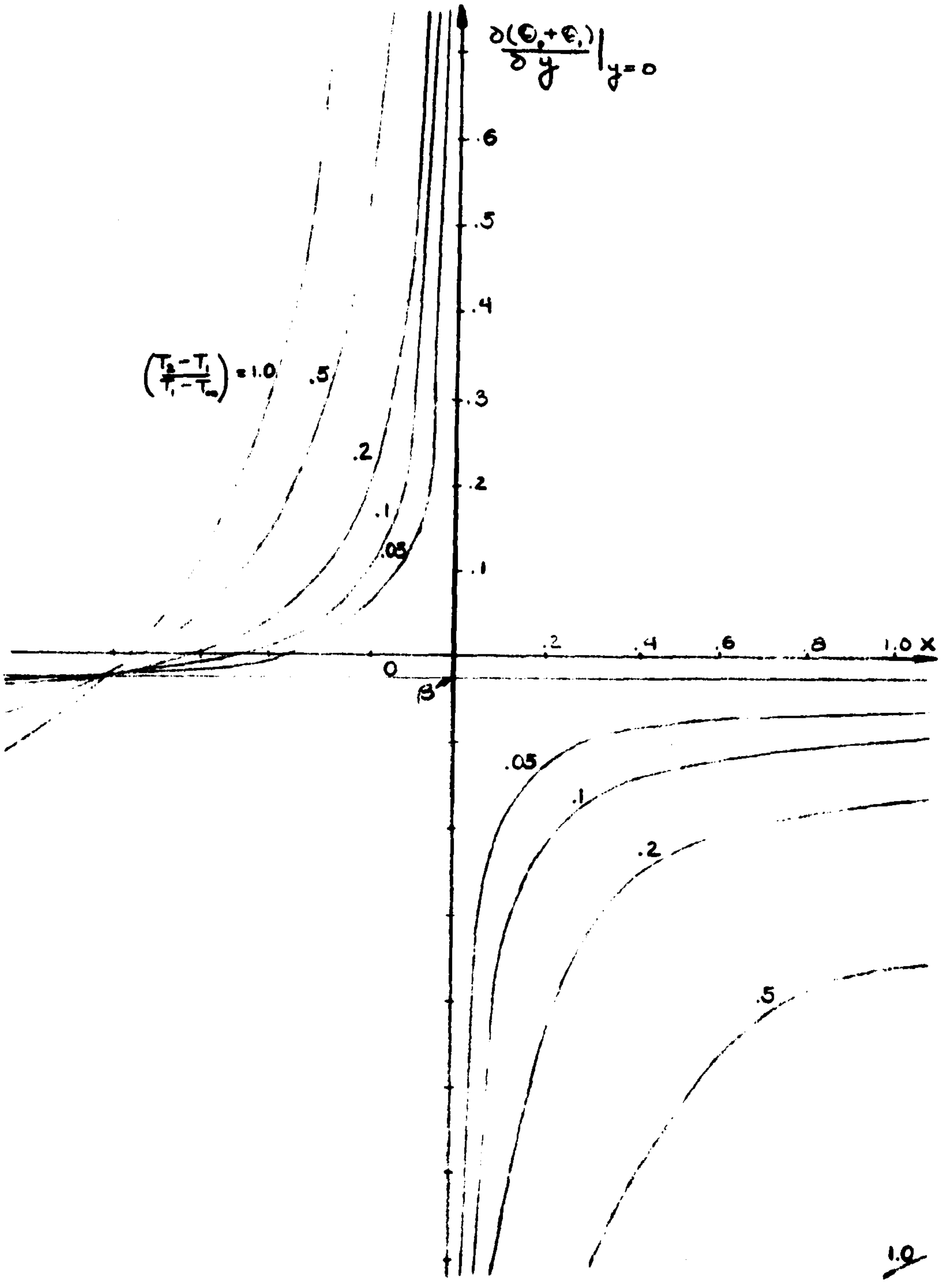

Figure 12 Heat Transfer Rate at the Plate with β = .027 and $\left(\frac{T_2-T_1}{T_1-T_\infty}\right)$ used as a Parameter with Values 0, .05, .1, .2, .5, 1. Problem (iii).

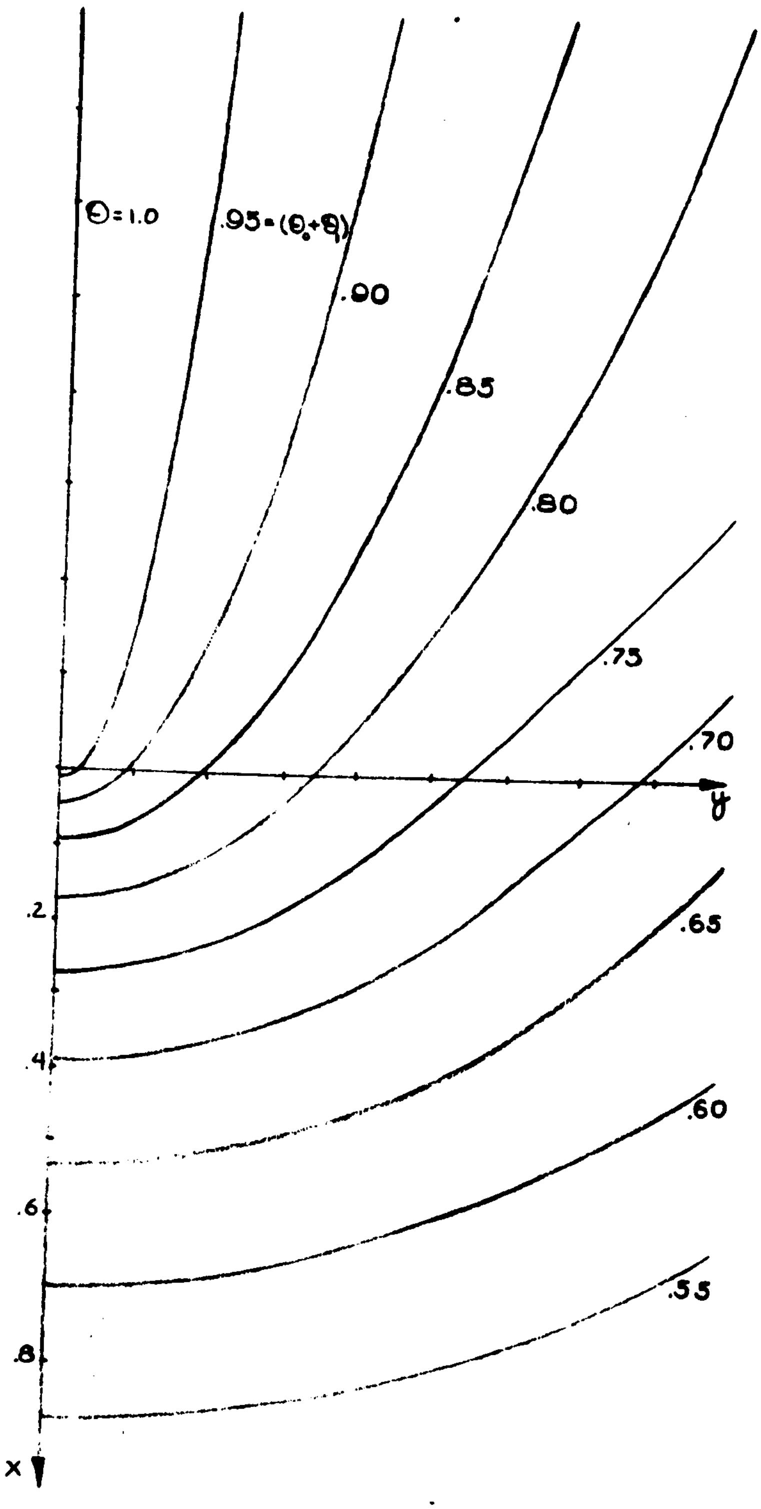

Figure 13 Isotherms based on $\theta_0+\theta_1$ for the Isothermal Vertical Plate. Problem (iv).

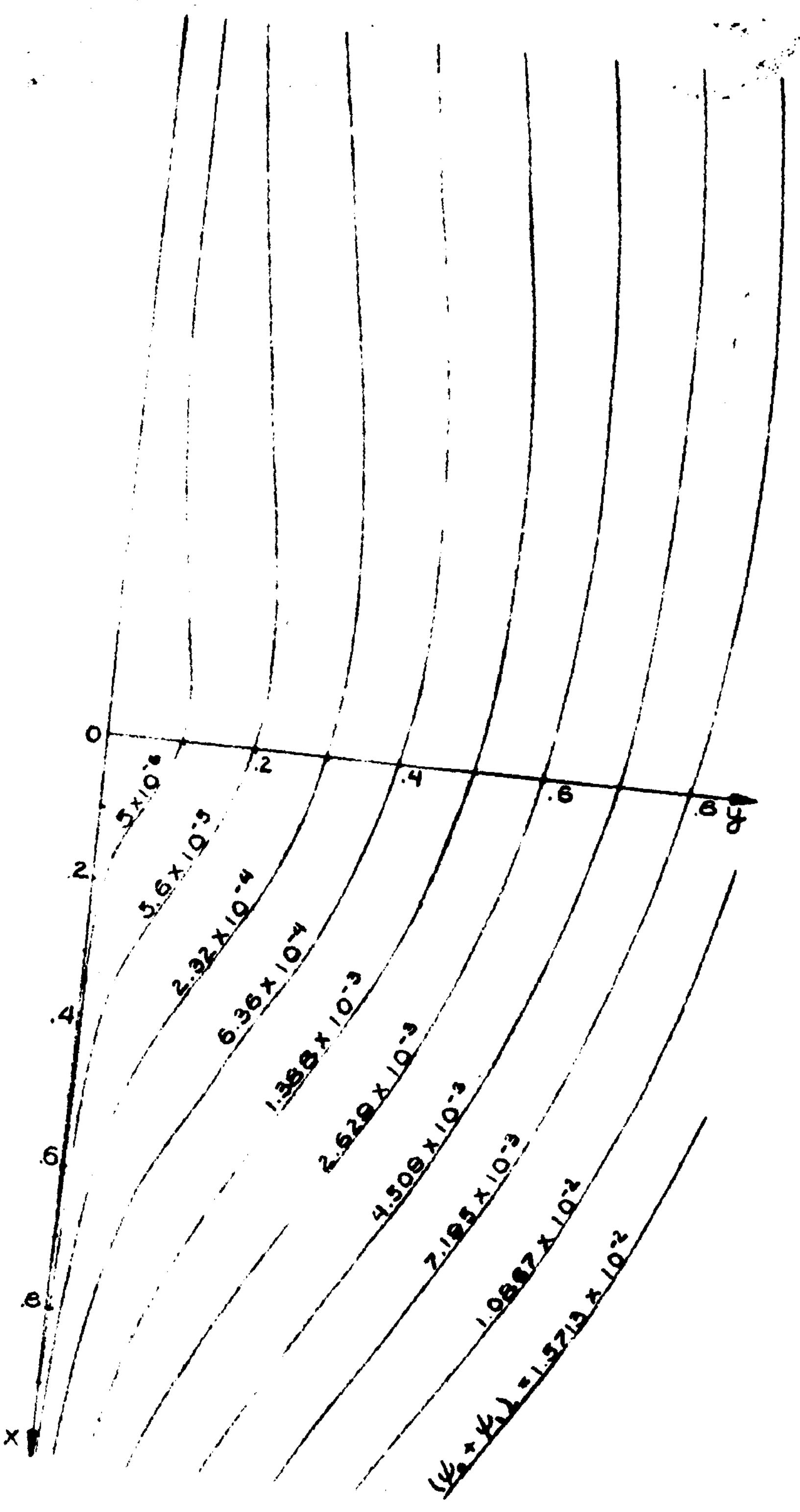

Figure 14 Streamlines Based on $\psi_0+\psi_1$ for the Isothermal Vertical Plate. Problem (iv).

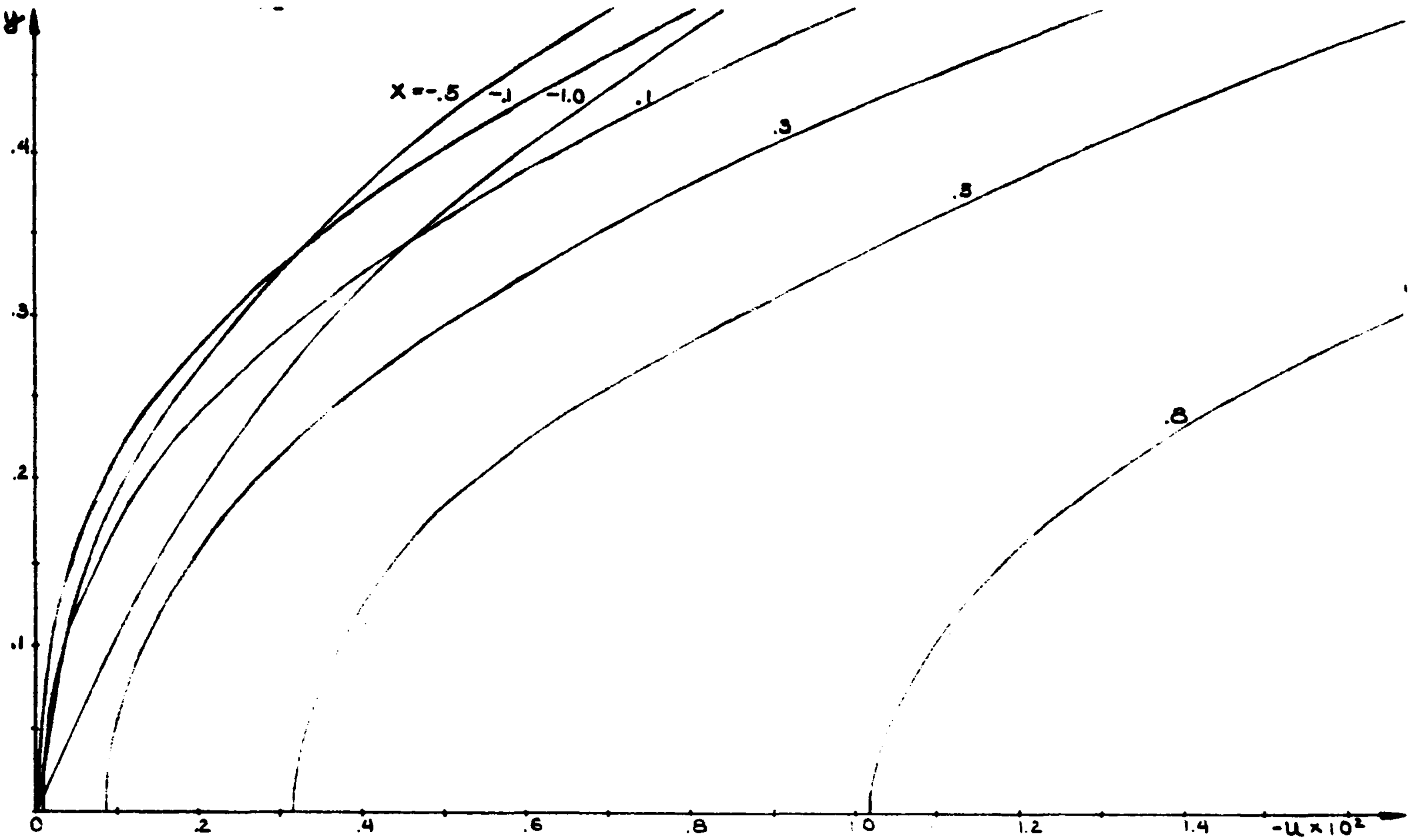

Figure 15 Profiles of the Longitudinal Velocity Component u at Several Sections for x Equal to a Constant. Problem (iv).

Bibliography

[1]Tifford, A. N., "Thermodynamics of the Laminar Boundary Layer of a Heated Body in a High Speed Gas Flow Field," Journal of Aerospace Science, 12, 1945, pp. 241.

[2]Pohlhausen, Tech. Mech. Thermodynam., Bul., 1, 1930, pp. 391.

[3]Leveque, "Heat Transfer Notes," by L. M. K. Boelter, Univer. of Calif. Press, 1946, pp. x-38.

[4]Rubesin, M. W., "An Analytic Investigation of Convective Heat Transfer From a Flat Plate Having a Stepwise Discontinuous Surface Temperature," ASME Paper 48-A-43, 1948.

[5]Eckert, E. R. G., "Introduction to the Transfer of Heat and Mass," McGraw Hill, New York, 1950, pp. 88.

[6]Lighthill, M. J., "Contributions to the Theory of Heat Transfer Through a Laminar Boundary Layer," Proceedings of the Royal Society, London Eng., Series A, Vol. 202, 1950, pp. 359-377.

[7]Gee, L. J. and Seban, R. A., "An Investigation of the Effect of a Step in the Surface Temperature on the Heat Transfer to a Laminar Boundary Layer," ASME Paper No. 54-SA-54, 1954.

[8]Carrier, G. F., Krook, M., and Pearson, C. E., Functions of a Complex Variable, 1966, pp. 376-379

[9]Arpaci, V. S., Conduction Heat Transfer, 1966, pp. 210.

[10]Carrier, G. F., and Lin, C. C., Quart. Appl. Math., 6, 1948, pp. 63-68.

[11]Ostrach, S., NACA Rept. 1111, 1953.

[12]Sparrow, E. M., Eichhorn, R., and Gregg, J. L., Physics of Fluids, 2, 1959, pp. 319-328.

[13]Suriano, F. J., and Yang, K-T., Int. J. Heat and Mass Transfer, 11, 1968, pp. 473.

[14]Yang, K-T., and Jerger, E. W., J. Heat Transfer, 86, 1964, pp. 107-115.

[15]Scherberg, M. G., Int. J. Heat and Mass Transfer, 5, 1962, pp. 1001.

[16]Dean, W. R. and Montagnon, P. E., Proc. Camb. Phil. Soc., 45, 1949, pp. 389.

[17]Moffatt, H. K., J. Fluid Mech., 18, 1964, pp. 1-17.

[18]Lugt, H. J., and Schwiderski, E. W., Proc. Royl. Soc., A285, 1965, pp. 382.

[19]Weinbaum, S., J. Fluid Mech. 33, 1968, pp. 38-63.

[20]Landau, L. D., and Lifshitz, E. M., Fluid Mechanics, 1959, pp. 27.

[21]Imai, I., "On the Viscous Flow Near the Trailing Edge of a Flat Plate," Proc. of the XI Int. Cong. of Appl. Mech. Julius Springer, Verlag, Berlin.

[22]Howarth, L., Modern Developments in Fluid Dynamics, High Speed Flow, Vol. 2, 1953, pp. 773.

[23]Carrier, G. F. and Lewis, J. A., Quart. Appl. Math., 7, 1949, pp. 228-234.

[24]Weinbaum, S., J. Fluid Mech., 18, 1964, pp. 409-437.

Vita

Walter P. Saukin was born August 16, 1943 in Newark, New Jersey and grew up in New York City.

He graduated Morris High School, Bronx, New York, and entered the civil engineering program at The City College of New York in 1961. He received his Bachelor's degree in Civil Engineering in January of 1966 and continued on at The City College for his master's degree which was received in January 1968. After his master's degree, he continued on as a full time student in the Doctoral Program at The City University of New York.

In February of 1968 he received a great emotional shock when his mother passed away. They were very close and it came as a great loss.

The most cherished moment of his life came on January 26, 1969 when he married Olga Czuwak. She meant nothing less than life itself to him.

In June of 1971, Mr. Saukin received his Ph.D. in Engineering from The City University of New York. In the future, he plans to do research in an academic setting. His greatest desire was to make his wife, and mother and father proud of him.