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Observer-controller design for linear and nonlinear output-feedback systems

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OBSERVER-CONTROLLER DESIGN FOR LINEAR AND NONLINEAR OUTPUT-FEEDBACK SYSTEMS

by

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Abstract

OBSERVER-CONTROLLER DESIGN FOR LINEAR AND NONLINEAR OUTPUT-FEEDBACK SYSTEMS

by

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This Thesis solves several of the problems in the designing of observer-controllers for linear and nonlinear output-feedback systems. The solution to the linear ouput-feedback regulator problem is first formulated for continuous time systems. The problem of forcing the estimation error and system state to zero in finite time using a linear controller is then presented. The above problem is solved by appropriately selecting a proper Lyapunov-like function together with a required boundary condition. Parameters N and M are introduced for the controller and observer design respectively in order to shape the state and error trajectories.

The solution to the linear continuous output-feedback regulation problem is then extended to include aclass of nonlinear systems for which deadbeat response is also acchieved. A detailed analysis is then given on the structure of this class of nonlinear systems. There are design

situations where the gains of the observer and controller are constrained and thus a deadbeat response becomes impractical. In such cases, the design technique is modified. It is will shown that the parameters N and M together with the terminal time can be appropriately selected to lead to a design procedure for generating the required gains. Such gains are shown to produce a good compromise among state component excursions and input magnitude without the great expense of speed of response. Several examples are given to demonstrate the design procedures.

Analogously design procedures are presented for linear discrete systems. A new technique is presented for producing deadbeat response to the output-feedback regulation problem. Modification schemes are formulated for cases where the gains of the observer and controller are constrained and thus a deadbeat response is not feasible. A new scheme is presented for achieving near deadbeat response in output-feedback system without time consuming matrix inversions. Finally a design procedure is presented to place all the eigenvalues of the estimator error dynamic system and the overall closed-loop system at the origin for single input, single output discrete systems.

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Anthony C. D. Louison

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1. INTRODUCTION AND BACKGROUND MATERIAL

1.1 INTRODUCTION:

The effectiveness of any system lies in its ability to carry out a specific function. For example, forcing a spaceship to follow a prescribed trajectory. Often in order to satisfy this requirement, the system's trajectory must be monitored and controlled to reach a desired state in the fastest possible time without exceeding a maximum overshoot requirement. To force a system's state to a predetermined value along a prescribed path, using a state-feedback control law requires knowledge of the entire state vector. Due to the physical nature of many systems, only incomplete state measurements are available. Thus control of such systems based on the separation principle in [7] first requires estimation of their state and then use of a control law that operates on the system's state estimates.

Over the past decade, great strides in this field have led to various techniques for designing the state estimator (observer) and forcing function (input) required for both linear and nonlinear systems. A brief review of this work will now be given. Most modern methods utilize a performance index or cost function from which gains of the control law are calculated by maximizing or minimizing the performance index. However using this method [21] the state estimator derived for linear systems had an exponentially decaying observation error since its gain was constant. As a result the state estimates were not exact after a finite time interval. Hence the forcing function did not produce a deadbeat response for the feedback system since the control was a function of the state of the state-estimator and not of the system's state. Thus the work of [21] cannot be used to treat the problems related to producing a deadbeat response when a controller has only incomplete state measurement. The two problems are: first, the design of an observer which will give estimates of the state for a prescribed accuracy in the fastest possible time and second the design of a controller which will drive the system to steady state and does not lead to excessively large overshoot.

Due to the difficulty in modelling nonlinear systems, it is almost impossible to develop any scheme which will control in general every nonlinear system. Thus it is much more practical and effective to consider different classes of nonlinear systems and the various techniques used to estimate their states or control them. Achieving deadbeat response for nonlinear output-feedback systems has proven to be an intractible problem. The following paragraphs present a survey of the research on both state reconstruction and control of linear and nonlinear deterministic systems.

Although nonlinear controllers frequently demonstrate superior performance relative to linear controller as shown by R. W. Bass and R. F. Webber [2], because of the flexibility of the specifications of the controller parameters, the design procedure of [2] was limited to state feedback systems (see section 3.3 below). A straight-forward design procedure for a nonlinear controller for nonlinear systems was achieved by K. Watanabe and D. H. Himmelblau [37]. They based their procedure upon an approximation using a discretized analogue model. The parameters of the model were estimated using a noniterative method in which the system model is discretized by a proper integration formula and then a set of resulting algebraic equations solved simultaneously. Although the parameter estimates are noise sensitive, the technique requires only a modest amount of computation. Very good state-feedback results were achived for nonlinear systems which can be decomposed into strictly linear and nonlinear components such that the control law can be designed to control each component separately.

One of the most effective methods used to study the stability of nonlinear systems is the Lyapunov stability principle [10]. Since its advent, several researchers have used it in the design of both controllers and observers for nonlinear systems without using approximate models. Stanley B. Gershwin and David H. Jacobson [9] have used this approach in the development and extension of linear deadbeat control response to nonlinear deadbeat state-feedback control response. The class of nonlinear systems considered was of the type where some linear elements were present. The characteristics of the linear portion were responsible for determining the terminal time, while the strictly nonlinear part had certain negative definite properties. A Lyapunov-like approach was used to achieve stability of the closed-loop system. The design did not take into account constraints on the maximum value of state variables or on the allowed input.

Another significant example on the application of Lyapunov Stability analysis is seen in [17] in the development of an exponential observer for nonlinear systems. Here R. Kou, David L. Elliot and Tzyh Jong utilized it to establish a significant condition under which certain classes of nonlinear systems can be observed. They showed that under certain conditions there exists an observer gain which produces a Lyapunov-like function that may be used to establish required asymptotic convergence of the observation error. Unlike many other design, ie, [32], the output of the system is not required to be a linear combination of the state components. However, convergence of the error is dependent on the gradient of the nonlinearity and can in some cases be very slow.

There are certain classes of nonlinear systems where the nonlinearity is a function of the observed variable only. Such a class was considered by S. R. Kou, T. Tarn and D. L. Elliot [18]. They showed that if the system is completely observable and the initial error lies within some known region, an approximately finite-time observer can be designed for that system. The observer is designed such that the nonlinearity is eliminated from the dynamic error equation. The convergence of the error state is controlled to any degree of accuracy desired within any terminal time by pole placement due to the fact that the error equation becomes linear. One drawback of the technique is that it leaves no alternative for cases where the region of the initial error is not known as an a priori condition.

Many classes of nonlinear systems are such that their strictly nonlinear elements have certain interesting properties. Examples are the trigonometric functions which are bounded and other functions, whose Taylor series expansion about an equilibrium point contains dominating elements. In such cases the control of the nonlinear system is closely associated with the control of an associated linear system. Such a case can be seen in [10] for controlling nonlinear systems and in [18] for observing a special class of nonlinear system.

In contrast to the methods described above for nonlinear continuous systems, the control theory for nonlinear discrete systems is still poorly developed. A major problem of discrete system control is forcing the system state to zero in a minimum number of steps. Such controls are called deadbeat. For linear systems, this corresponds to placement of all closed-loop poles at the origin. Of course the placement of multiple poles in one spot is a sensitive design problem but, nevertheless, rapid settling time will result even if exact deadbeat response is not realized in practice. In the last two decades the problem has been attacked from a state-space point of view and early contributions appeared in [11] and [12]. Solution to the problem using nilpotency of the system matrix appeared in [19], [30] and [20]. In 1976 B. Leden [23] established a connection between deadbeat control and optimal control for linear, time-invariant discrete systems. Here the integrand of the performance index contains only state penalty terms and the state is required to be zero at a prescribed terminal time. An explicit solution to the singular Riccati equation, associated with the optimization problem is given. Properties of the time varying gain matrices, generating the optimum policies are presented in [23]. In particular, necessary and sufficient conditions for each of these gain matrices to yield a time-invariant deadbeat controller are given.

During the period 1979 - 1980, B. Lewis [24] working on the application of the Moore- Penrose pseudo inverse to the solution of the singular discrete-time Riccati equation, presented a closed-form solution, which unfortunately does not lend itself to computation since the matrices used increase in size with increasing time. However it leads to a new geometric treatment of the study of linear quadratic optimal control. The work of Lewis was pursued by A. Emami-Naeini and G. F. Franklin [6]. They presented a new approach for forcing the state of a linear system to zero in a minimum number of steps. The problem was formulated as a solution to a steady state optimal control problem with no cost on the control. No special assumption on the open-loop system matrix and the ratio of the number of states to control is required. Stable numerical techniques were presented for solving for the feedback gain.

1.2. MODELLING THE SYSTEM:

This section presents the state space models to be used throughout this thesis. Models for both linear and nonlinear continuous time systems are given. Models for discrete time systems are analogous to those for continuous time systems and are thus omitted.

1.2.1 STATE CONTROLLERS:

Many systems can be described by a set of simultaneous equations of the form

$$\dot{x} = f[x(t),u(t),t] \qquad (1.1)$$

where t is the time variable, x(t) is a real n-dimensional time varying column vector which denotes the state of the system, and u(t) is a real m-dimensional column vector which indicates the input variable or control variable. The function f is a real vector-valued function. For many systems the choice of the state follows naturally from the physical structure and (1.1) which will be called the state differential equation, usually follows directly from the physical laws that govern the system. Of general interest is a feedback law of the form

$$u(t) = g(x(t)) \qquad (1.2)$$

which when applied to the process (1.1) results in an asymptotically stable closed-loop system

$$\dot{x} = f[x(t),g(x(t)),t] \tag{1.3}$$

Thus the origin is considered the target set and the control law (1.2) is assumed to be such that

$$\lim_{t \to \infty} x_g(t) = 0 \tag{1.4}$$

where $x_g(t)$ denotes the trajectory of the asymptotically stable closed-loop system (1.3). Of particular interest are control laws that result in

$$\lim_{t \to t_1} x_g(t) = 0 \tag{1.5}$$

where t_1 is some finite time. Such response is said to be a deadbeat response. Formulation of control laws (1.2) that result in property (1.5) is examined in chapter 2 for linear systems and extended to nonlinear systems in chapter 3.

A broad class of nonlinear systems take the form

$$\dot{x} = A(t)x(t) + f(x(t)) + B(t)u(t) \tag{1.6}$$

where A(t) is an nxn matrix, B(t) is an nxm matrix and f(x(t)) is a nonlinear function of the state x(t). Such a process may be dominated by the properties of its A(t) and B(t) matrices. The control law to be designed below takes the form

$$u(t) = -K(t)x(t) + g(x(t)) \tag{1.7}$$

where K(t) is an mxn matrix called the gain of the system, and g(x(t)) is a real vector-valued function. The system described by (1.6) reduces to a linear system if f(x) equals zero and in (1.7) g(x(t)) becomes zero. The problem in controller design lies in finding an appropriate function u(t) which will accomplish a prescribed goal. The techniques for finding such functions are described in a latter section.

1.2.2 FULL-ORDER OBSERVERS:

Consider the nonlinear time-varying system represented by the differential equation

$$\mathring{x}(t) = f(x(t)) \tag{1.8}$$

where x(t) is the nx1 state vector, and $f: R^n \rightarrow R^n$ is continuously differentiable in the domain R^n and satisfies a condition ensuring existence of a solution on $[0, \infty)$. The state vector is not available for direct measurement. Only a measurement of the form

$$y(t) = h(x(t)) \qquad (1.9)$$

is available, where y(t) is in R^r and is the output of (1.8) and h: $R^n \to R^r$ is continuous. The problem is to design a dynamic system called an observer or state-reconstructor to reconstruct or estimate the state of (1.8) by using (1.9) as input to the observer such that the state of the designed observer can be used as an estimate of the state x(t).

An observer for (1.8) and (1.9) can be represented in general as

$$\dot{z}(t) = f(z(t)) + p(y(t),h(z(t))) \qquad (1.10)$$

where z(t) is the n-dimensional state variable (or output) of the observer and p: $R^r x R^n \to R^n$ is once continuously differentiable in both variables and

$$p(y(t),h(z(t))) = 0 \qquad (1.11)$$

if h(x(t)) = h(z(t)). Note that observer (1.10) contains a model of (1.8) and a correction signal p that is to be found so that the error x(t) - z(t) goes to zero as quickly as possible. When (1.8) and (1.9) are linear, they can be represented in the following form

$$\overset{\circ}{x}(t) = A(t)x(t) \qquad (1.12)$$

$$y(t) = C(t)x(t) \qquad (1.13)$$

The state estimator can now be represented in the following form

$$\overset{\circ}{z}(t) = A(t)z(t) + G(t)[y(t) - C(t)z(t)] \qquad (1.14)$$

where A(t) is an nxn matrix, C(t) is an rxn matrix and G(t) is an nxr matrix called the gain of the state estimator or observer. The system described by (1.14) is called a full-order observer because the states of x(t) and z(t) are of the same dimension. If we define the reconstruction error e(t) by

$$e(t) = x(t) - z(t) \qquad (1.15)$$

then combination of (1.12), (1.13) and (1.14) yields

$$\overset{\circ}{e}(t) = [A(t) - G(t)C(t)]e(t) \qquad (1.16)$$

Equation (1.16) is referred to as the error equation of the linear observer. The above development can be easily ex-

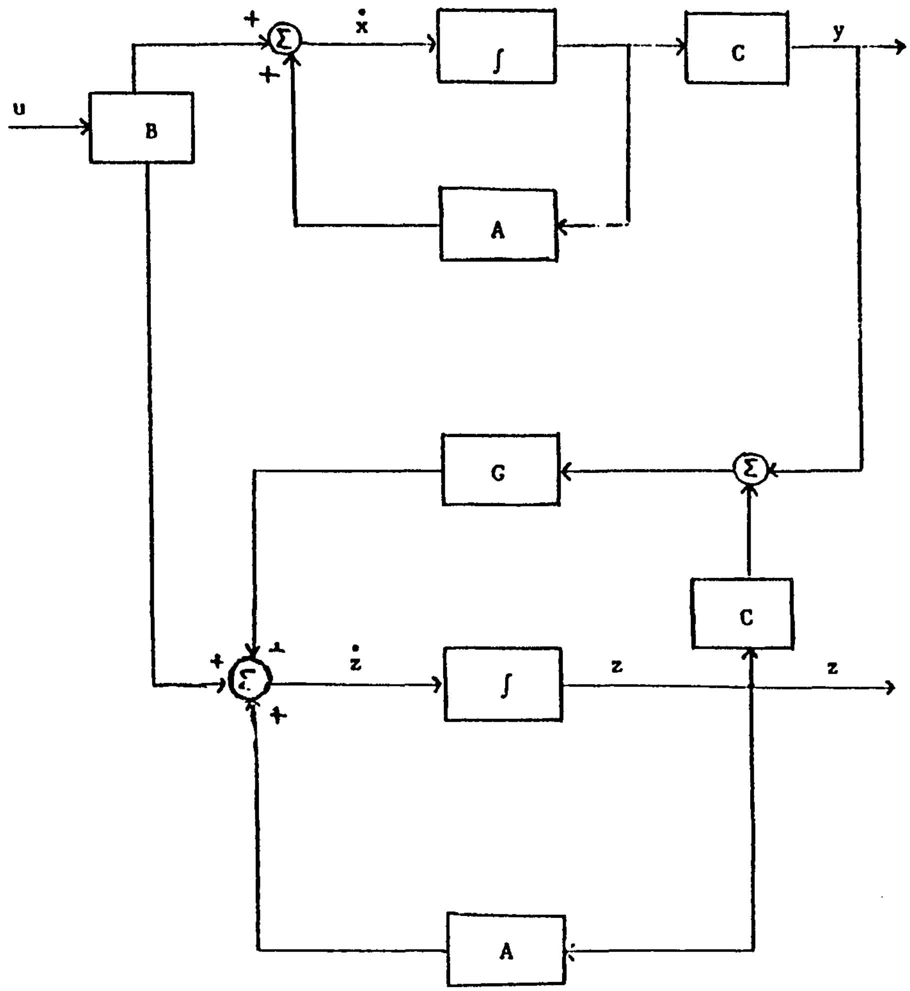


Fig. 1.1 Full-order Observer

tended to forced systems by including the input term u(t) into (1.12) and (1.14). The configuration for the full-order observer for forced linear systems is shown in Fig. 1.1.

1.2.3 REDUCED-ORDER OBSERVERS:

The full-order observer possesses a certain degree of redundancy. The reason for this is that the full-order observer constructs an estimate of the entire state, but the output of the plant which is made up of linear combinations of the state components, is available for direct measurement. Hence it is unnecessary to construct a full-order dynamic observer to estimate the plant state.

The following form was first developed by Luenberger [25] and a detailed treatment can be found in [5]. We give here a brief summary of the form of the reduced-order observer. Consider the system described by

$$\mathring{x}(t) = Ax(t) + Bu(t) \tag{1.17}$$

$$y(t) = Cx(t) \tag{1.18}$$

Assume that C has full rank an introduce an (n-r) dimensional vector w(t)

$$w(t) \quad = \quad C'x(t) \tag{1.19}$$

such that $[C, C']^T$ is nonsingular. Then

$$x(t) \quad = \quad \begin{bmatrix} C \\ C \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ w(t) \end{bmatrix} = L_1 y(t) + L_2 w(t) \tag{1.20}$$

Let $\hat{w}(t)$ and $\hat{x}(t)$ represent estimates of $w(t)$ and $x(t)$ respectively. Then

$$\hat{x}(t) = L_1 y(t) + L_2 \hat{w}(t) \tag{1.21}$$

Diffferentiating (1.19) and using (1.20) yields

$$\dot{w}(t) = C'AL_2 w(t) + C'AL_1 y(t) + C'Bu(t) \tag{1.22}$$

Since $y(t)$ carries no information about $w(t)$, new information must come from the derivative of $y(t)$. An observer for (1.22) takes the form

$$\mathring{\hat{w}}(t) = C'AL_2 \hat{w}(t) + C'AL_1 y(t) + C'Bu(t) +$$

$$K[\mathring{y}(t) - CAL_1 y(t) - CBu(t) - CAL_2 \hat{w}(t)] \tag{1.23}$$

where K is the observer's gain matrix. In realization there is no need to take the derivative of $y(t)$. Various techniques for designing K are shown in the next section.

1.3 TECHNIQUES FOR DESIGNING CONTROLLERS AND OBSERVERS

This section gives several standard and recent techniques for designing control laws and observer correction signals described in the last section. Since linear systems theory plays an integral part in the development and construction of many types of nonlinear processes, standard techniques for the design of linear systems are described first.

1.3.1 POLE PLACEMENT

Consider the time-invariant form of (1.6) when the nonlinear part is absent, ie,

$$\overset{\circ}{x}(t) = Ax(t) + Bu(t) \tag{1.24}$$

A linear state-feedback control law

$$u(t) = -Kx(t) \tag{1.25}$$

gives the closed-loop system

$$\overset{\circ}{x}(t) = [A - BK]x(t) \tag{1.26}$$

Solution to (1.26) is

$$x(t) = \exp[(A - BK)t]x(0) \tag{1.27}$$

Similarly the solution to (1.16) when A, C and G are constant matrices is

$$e(t) = \exp[(A - GC)t]e(0) \tag{1.28}$$

Thus if the matrices $[A - BK]$ and $[A - GC]$ are stable matrices then the states of $x(t)$ and $e(t)$ will approach zero asymptotically. It is well known that if the pair $[A,B]$ is completely controllable and the pair $[A,C]$ is completely observable, then the eigenvalues of $[A - BK]$ and $[A - GC]$ can be arbitrarily placed in the complex plane, by proper selection of the K and G matrices. This process of selecting the gain matrices based upon eigenvalues locations is called pole placement.

1.3.2 INTEGRAL QUADRATIC PERFORMANCE INDEX

An alternate design procedure for observers and controllers is based on a performance index. This technique can be used for both linear and nonlinear systems where an "optimal" design is desired. Several forms of the integral quadratic performance index are used. Here only the simplest form used for linear optimal systems is presented. Consider finding an optimal gain K for (1.26). A suitable performance index is of the form

$$J = \int_t^{t_1} [x^T(\tau)R_1(\tau)x(\tau) + u^T(\tau)R_2(\tau)u(\tau)]d\tau + x^T(t_1)P_1x(t_1)$$

$$= \int_t^{t_1} x^T(\tau)[R_1(\tau)+K^T(\tau)R_2(\tau)K(\tau)x(\tau)d\tau + x^T(t_1)P_1x(t_1) \quad (1.29)$$

where R_1 and R_2 and P_1 are positive definite matrices. Equation (1.29) can be written as

$$J = x^T(t)P(t)x(t) \quad (1.30)$$

where P(t) satisfies the matrix differential equation

$$-\mathring{P}(t)=R_1(t)+K^T(t)R_2(t)K(t)+P(t)[A-BK(t)]+[A-BK(t)]^TP(t) \quad (1.31)$$

with $P(t_1) = P_1$. It is shown in [19] that the optimum gain is

$$K^o(t) = R_2^{-1}(t)B^TP(t) \quad (1.32)$$

where P(t) satisfies

$$-\mathring{P}(t) = R_1(t) - P(t)BR_2^{-1}(t)B^TP(t) + P(t)A + A^TP(t) \quad (1.33)$$

1.3.3 OPTIMAL NONLINEAR FEEDBACK CONTROL:

The methods described in the preceding section for minimization of a quadratic performance criterion can be extended to minimization of integrals containing quartic, hexadic or even higher- terms in the state variables [2]. This leads respectively to cubic, quintic or higher-order state feedback. Such higher-order feedback is often necessary in order to impose inequality constraints upon the state variables. The technique described below is that of R. W. Bass and R. F. Webber [2]. The procedure imposes a mean amplitude constraint on the control. By defining a certain Lyapunov function whose derivative is forced to take a required form, an exact solution of the problem is obtained. The case considered in [2] is a completely controllable system with a single control variable.

Consider the system described by the following differential equation

$$\overset{\circ}{x} = Ax + a\psi \qquad (1.34)$$

where x is the system state vector, A is an nxn plant matrix, a is the actuator vector, and ψ is the scalar control law to be chosen in the feedback form $\psi = \psi(x)$. Control laws are admissible only if they produce asymptotic stability of the equilibrium state $x = 0$; in particular, it is required that

$$x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

The problem can be stated as follows: Find the control law in (1.34) to minimize the unconstrained performance criterion

$$\Phi = \int_0^\infty \{\Xi(x) + 0.5\psi^2(x) + 0.5\, [\psi_{n1}(x)]^2\}dt \qquad (1.35)$$

with

$$\Xi = \sum_{\nu = 1}^{\infty} (1/2\nu)\xi_{2\nu}(x) \qquad (1.36)$$

where

$$\xi_2 = x \,.\, Cx \qquad (C^* = C > 0) \qquad (1.37)$$

is a given positive-definite homogeneous quadratic form and where each given ξ_2 is a positive semidefinite homogeneous multinomial form of degree 2ν, ($\nu = 2,3,4,....$) and

with

$$\psi_{n1}(x) = \sum_{\nu=1}^{\infty} (1/2\nu)a.\text{grad}\ \emptyset_{2\nu}(x) \qquad (1.38)$$

where

$$\emptyset_2 = x \,.\, Bx \qquad (B^* = B > 0) \qquad (1.39)$$

is a positive-definite quadratic form, where B is to be found and $\emptyset_2(x)$ is a positive semidefinite homogeneous multinomial of degree 2ν ($\nu = 2,3,4....$) to be constructed.

The matrix B is found by minimizing (1.35) with $\psi_{n1} = 0$. The optimal control law is given by

$$\psi = g\,.x\,, \qquad g = -Ba \tag{1.40}$$

where B satisfies the (equilibrium) matrix Riccati equation

$$BA + A^*B - Baa^*B = -C \tag{1.41}$$

Rewrite (1.34)

$$\mathring{x} = Ax + a = \hat{A}\,x \tag{1.42a}$$

where $$\hat{A} = A - aa^*B \tag{1.42b}$$

and where $\hat{A}$ is a known stability matrix. This $\hat{A}$ and $\{\xi_{2\nu}: \nu = 2,3,..\}$. are then used to construct $\{\emptyset_{2\nu}: \nu = 2,3,..\}$..by solving the partial differential equation

$$\hat{A}x\,.\,\mathrm{grad}\,\emptyset_{2\nu}(x) = -\xi_{2\nu}(x) \tag{1.43}$$

The optimal control law for (1.34) relative to (1.35) is given by

$$\psi(x) = g.x + \psi_{n1}(x) \tag{1.44}$$

1.3.4 LYAPUNOV-LIKE METHODS:

The methods previously described in sections 1.2.1 and 1.3.2 are suited for the control of linear systems. A method which is inherently suited for nonlinear systems is one which uses the properties of Lyapunov stability analysis and optimization. Consider (1.1). Define the Lyapunov-like function V(x,t),

$$V(x,t) = x^T S(t) x \qquad (1.45)$$

where S(t) is a positive definite time-varying symmetrical matrix. Differentiating (1.45) along the trajectory of (1.1) using (1.2) yields

$$\dot{V}(x,t) = 2x^T S(t)\dot{x} + \dot{x}^T S(t) x$$

$$= 2x^T S(t) f[x(t), g(x(t)), t] + x^T \dot{S}(t) x \qquad (1.46)$$

It is well known [10] that if V(x,t) satisfies the following conditions:

i) V(x,t) has continuous first partial derivatives with respect to x and t and V(0,t) = 0;

ii) V(x,t) is positive definite

iii) $\dot{V}(x,t) = \partial V(x,t)/\partial t + (\text{grad } V(x,t))^T f(x,g,t) < 0$

iv) V(x,t) is bounded by a nondecreasing continuous scalar function for all t and

v) $V(x,t) \rightarrow \infty$ as $||x|| \rightarrow \infty$

then the equilibrium state x_e is globally uniformly asymptotically stable. Two important questions are, one, what is the relationship between g(x(t)) and S(t) and two, for what family of g(x(t)) does (1.5) hold? If there exists a relationship between u(t) and S(t), then one may investigate the effect of S(t) on the trajectory of (1.1) and on the maximum magnitude of u in (1.2). A key point in this thesis is the finding of such a relationship for a broad class of nonlinear swystems and to examine the effect of S(t) on the state trajectory with constraint on the input u(t).

1.3.5 LINEARIZATION ABOUT EQUILIBRIUM :

It is common practice in engineering to often consider only small deviations from the operation point (ie., equilibrium state). This is done by expressing a nonlinear function in a Taylor series about the operating point. Consider the nonlinear differential equation

$$\overset{\circ}{x}(t) = f(x(t)) \tag{1.47}$$

Let $y = x - x_e$ where x_e is taken as an equilibrium state and f is analytic in the neighborhood of x_e. Then

$$\overset{\circ}{y}(t) = f(x_e,t) + [F(t) + G(y,t)][x - x_e]$$

$$= [F(t) + G(y,t)]y(t) \tag{1.48}$$

where F(t) is the Jacobian matrix of f evaluated t x_e, t; and G(y,t) is a matrix such that $||G(y,t)||/||y||$ tends to zero with $||y|| \to 0$. If F(t) is bounded and the linear part of (1.47) is uniformly asymptotically stable then x_e is (locally) uniformly asymptotically stable. Many non-linear systems possess this characteristic and stable regulators and stable observers can be design for them by only considering their linear elements. Other methods retain first-order and second-order terms of the Taylor series to get an approximate design procedure.

This thesis solves the problem of state estimation and state control for linear and nonlinear systems. It outlines an algorithm which generates the observer and controller gains which produce exact state estimates and forces the system's state to the zero-state in a prescribed terminal time for a large class on linear and nonlinear continuous time systems. Chapter 2 solves the problem of the output-feedback finite design for linear continuous time systems. The problem is divided into three subproblems. First, the state of the system is assumed known and a controller which will result in a deadbeat state response is presented. Second, a dynamic observer is designed as a state estimator which will give exact state estimates in finite time. And third, the above schemes are combined to produce a deterministic output-feedback controlled system. In chapter 3, the theory is extended to nonlinear output-feedback systems. In chapter 4 the effect of the designed parameters N, M and t_f are examined. In chapter 5 an analogous algorithm is developed to produce exact state estimates and deadbeat responses in discrete time linear systems. A new algorithm is developed which produces near deadbeat response without matrix inversion. A conclusion and extensions are given in chapter 6.

2. FINITE-TIME OUTPUT-FEEDBACK CONTROLLER DESIGN FOR LINEAR CONTINUOUS TIME SYSTEMS

This chapter solves the continous time linear output-feedback linear regulator finite time response problem. The solution is divided into three sections. In section 2.1 the system state is assumed known and a controller is designed which forces the system state to the origin in finite time. Section 2.2 demonstrates the design of an observer which forces the estimation error to zero in finite time. Finally sections 2.1 and 2.2 are combined to produce a finite time output-feedback regulator design. The output feedback control depends on parameters N and M which can be used to shape the error and system state trajectories. A detailed treatment of the effect of these parameters is given in chapter 4.

2.1 CONTROLLER DESIGN:

Consider the time-invariant linear system given by the following differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

where $x(t) \in R^n$ is the state of the system, $u(t) \in R^m$ is the input of the system, and A and B are nxn and nxm matrices, respectively.

Definition 1. System (2.1) is said to be deadbeat if there exists an input $u^*(t)$ and a finite time t_f such that

$$\lim_{t \to t_f} ||x(t)|| = 0 \tag{2.2}$$

where $||\circ||$ denotes the Euclidean norm. Let the input be given by

$$u(t) = -F(t)x(t) \tag{2.3}$$

where F(t) is an mxn matrix. Now to find the appropriate F(t) which will satisfy the requirement (2.2), define the scalar V(x,t) such that

$$V(x,t) = x^T(t)S(t)x(t) \tag{2.4}$$

where S(t) is a time-varying symmetrical nxn matrix and T denotes the transpose. Differentiating (2.4) along the trajectory of (2.1) and using (2.3) yields

$$\mathring{V}(x,t) = 2x^T(t)S(t)[A - BF(t)]x(t) + x^T(t)\mathring{S}(t)x(t) \tag{2.5}$$

Let
$$F(t) = 0.5B^TS(t) \tag{2.6}$$

where

$$-N\dot{S}(t) = S(t)A + A^T S(t) - S(t)BB^T S(t) \qquad (2.7)$$

and N is a scalar, $N \geq 1$. Then substituting (2.6) into (2.5) yields

$$\dot{V}(x,t) = -[N - 1]x^T(t)\dot{S}(t)x(t) \qquad (2.8)$$

It is obvious that if S(t) is nonsingular then (2.7) can be written as

$$\dot{S}^{-1}(t) = (A/N)S^{-1}(t) + S^{-1}(t)(A/N)^T - BB^T/N \qquad (2.9)$$

Using the boundary condition

$$S^{-1}(t_f) = 0 \qquad (2.10)$$

and if $S^{-1}(t)$ exists for $t < t_f$ then the solution to (2.9) becomes

$$S^{-1}(t) = (1/N)\int_t^{t_f} \Phi(t/N,\tau/N)BB^T\Phi^T(t/N,\tau/N)d\tau \qquad (2.11)$$

where $\Phi(t,\tau)$ is the state transition matrix of (2.1). The condition that $S^{-1}(t)$ be invertible (positive-definite) for $t \in [t_0,t_f)$ is

$$(1/N)\int_t^{t_f} \Phi(t/N,\tau/N)BB^T\Phi^T(t/N,\tau/N)d\tau > 0 \qquad (2.12)$$

which is a form of the Kalman's condition [13] for complete controllability of (2.1) from (x_0,t_0) to $(0,t_f)$ for all x_0 and $[t_0,t_f)$. The value of N has a great impact on the state trajectory for $t < t_f$. It can be shown that $\dot{S}(t)$ is positive semidefinite (see Appendix B) if (2.12) holds. Thus if the [A,B] pair is completely controllable, S(t) is positive-definite for $t < t_f$ and (2.8) reduces to

$$\dot{V}(x,t) \leq 0 \tag{2.13}$$

Thus from Gershwin and Jacobson First Controllability Theorem (see Appendix A) we conclude that the input given by (2.3) transfers the state of (2.1) from (x_0,t_0) to $(0,t_f)$. Computer simulation and graphical results are shown in a later section.

2.2 OBSERVER DESIGN:

Consider the linear system given by (2.1). The state x(t) is not available for direct measurement. The only measurement is

$$y(t) = Cx(t) \tag{2.14}$$

where $y(t) \in R^r$ is the output vector and C is an rxn matrix. Given the system (2.1) and (2.14) is completely observable, the problem is to design a dynamic system (observer) for (2.1) by using y(t) and u(t) as inputs to the observer such that, independent of the initial state of (2.1), the state of this designed dynamic system can be used as an estimate of the state of (2.1). Furthermore the state of this observer or state estimator must be identical with that of (2.1) after some finite time.

In general, the state estimator for (2.1) and (2.14) can be expressed in the form [25]

$$\dot{z}(t) = Az(t) + K(t)[y(t) - Cz(t)] + Bu(t) \tag{2.15}$$

where z(t) is the n-dimensional state variable (or output) of the observer and K(t) is an nxr matrix called the gain of the observer. Substituting (2.14) into (2.15) yields

$$\dot{z}(t) = Az(t) + K(t)C[x(t) - z(t)] + Bu(t) \tag{2.16}$$

Define the estimation error e(t) as follows

$$e(t) = x(t) - z(t) \tag{2.17}$$

<u>Definition 2</u>: A state estimator of a given dynamic system is said to be deadbeat if there exists a specific finite time t_f such that

$$\lim_{t \to t_f} ||e(t)|| = 0 \tag{2.18}$$

The value of t_f is arbitrary and does not depend on the initial condition of the system and thus affects only the trajectory of the observer.

Differentiating (2.17) and using (2.1) and (2.16) gives

$$\mathring{e}(t) = [A - K(t)C]e(t) \tag{2.19}$$

The following shows a new technique in designing K(t) such that the system (2.19) satisfies (2.18). Since (2.1), (2.14) is assumed to be completely observable, by the duality principle [20], the $[A^T,C^T]$ pair is completely controllable. This duality principle is the key idea behind this new technique.

Let W(t) be a time-varying symmetric nxn matrix. Define the scalar V(e,t) such that

$$V(e,t) = e^T(t)W(t)e(t) \tag{2.20}$$

Taking the derivative of (2.20) along the trajectory of (2.19) gives

$$\mathring{V}(e,t) = e^T(t)[\mathring{W}(t)+W(t)A+A^TW(t)-2W(t)K(t)C]e(t) \tag{2.21}$$

Let

$$2W(t)K(t) = C^T \tag{2.22}$$

where

$$M\mathring{W}(t) = W(t)A + A^TW(t) - C^TC \tag{2.23}$$

where M is a positive scalar parameter. Substituting (2.22) and (2.23) into (2.21) results in

$$\mathring{V}(e,t) = [M + 1]e^T(t)\mathring{W}(t)e(t) \tag{2.24}$$

It is obvious that if $\mathring{W}(t)$ is negative semidefinite, then $\mathring{V}(e,t)$ is negative semidefinite from (2.24). Using the bounndary condition

$$W(t_f) = 0 \tag{2.25}$$

the solution to (2.23) becomes

$$W(t) = (1/M)\int_{t}^{t_f} \Phi^T(t/M,\tau/M)C^TC\Phi(t/M,\tau/M)d\tau \qquad (2.26)$$

where $\Phi(t,\tau)$ is the state transition matrix of (2.1). The condition that W(t) be invertible (positive definite) for $t \in [t_0,t_f)$ is

$$W(t) = (1/M)\int_{t}^{t_f} \Phi^T(t/M,\tau/M)C^TC\Phi(t/M,\tau/M)d\tau \;>\; 0 \qquad (2.27)$$

which is a form of the Kalman condition [13] for completely controllability of the dual of (2.1), (2.14). Since (2.1), (2.14) is assumed completely observable, its dual is completely controllable and thus (2.27) holds and as a consequence W(t) is positive definite for $t < t_f$). Also C is nonzero for all t since (2.1),(2.14) is completely observable. Thus K(t) exists and is given by

$$K(t) = 0.5W^{-1}(t)C^T \qquad (2.28)$$

This gain can be shown to satisfy all the condition of the Gershwin and Jacobson First Controllability Theorem (see Appendix B). M has a similar effect on the error trajectory as N has on the system state trajectory and a full discussion on its effect is given later. $\dot{W}(t)$ can be shown (see Appendix B) to be negative semidefinite. Thus (2.24) reduces to

$$\dot{V}(e,t) \leq 0 \qquad (2.29)$$

Therefore the gain given by (2.28) transfers the state of (2.19) from (e_0,t_0) to $(0,t_f)$ (see Appendix B, Lemma 5).

2.3 THE STRUCTURE OF THE OUTPUT-FEEDBACK CONTROL SYSTEM:

In this section we consider the problem of regulating a linear continuous system with incomplete state measurement. Consider the system described by (2.1), (2.14) which is completely controllable and observable. Also consider the observer as given by (2.16). Using (2.3), where z replaces x, in (2.16) yields

$$\dot{z}(t) = [A - BF(t) - K(t)C]z(t) + K(t)y(t) \qquad (2.30)$$

Substitution of

$$z(t) = x(t) - e(t) \qquad (2.31)$$

into (2.30) yields

$$\dot{x}(t) = [A - BF(t)]x(t) + BF(t)e(t) \qquad (2.32)$$

From the observer design we noticed that there exists a K(t) such that (2.18) is satisfied. This gain is independent of the input, since u(t) was not part of the error equation (2.19). Therefore after time t_{fo} where t_{fo} is the observation interval, (2.32) reduces to

$$\dot{x}(t) = [A - BF(t)]x(t) \tag{2.33}$$

From the controller design we noticed that there also exists an F(t) which satisfies (2.2). Combining the results of the controller and observer designs we can conclude that if (2.1),(2.14) is completely controllable and observable, then the state of (2.1) can be driven to zero from any unknown state. The gains which accomplish this transfer are given by the following equations

$$K(t,t_{fo}) = 0.5W^{-1}(t,t_{fo})C^T \quad t \le t_{fo} \tag{2.34}$$

$$u(t,t_f) = -0.5B^T S(t,t_f)z(t) \quad t_{fo} \le t \le t_f \tag{2.35}$$

where

$$W(t,t_{fo}) = (1/M)\int_t^{t_{fo}} \Phi^T(t/M,\tau/M)C^T C\Phi(t/M,\tau/M)d\tau \tag{2.36}$$

$$S^{-1}(t,t_f) = (1/N)\int_t^{t_f} \Phi(t/N,\tau/N)BB^T\Phi^T(t/N,\tau/N)d\tau \tag{2.37}$$

where $t_f - t_{fo}$ is the control time. N and M are positive constants and $t_{fo} < t_f$. Graphical results are shown in a later section.

2.4 REDUCED-ORDER OBSERVERS:

The new technique developed for deadbeat output-feedback design can be applied to the construction of reduced-order output-feedback control systems. In this section we will show an application of the new method to the construction of a reduced-order observer for producing exact state estimates. Consider (2.1), (2.14). Since (2.14) provides us with r linear equations in the unknown x(t), it is only necessary to construct n-r linear combinations of the components of the state. This method was first considered by Luenberger (1964, 1966) and here follows the derivation of Cummings (1969) [5].

Consider the time-invariant case for simplicity. Following the procedure as described by (1.17) through (1.23) from the background material above, define the error $e'(t)$ as follows

$$e'(t) = w(t) - \hat{w}(t) \tag{2.38}$$

Differentiating (2.38) and using (1.22) and (1.23) yields

$$\dot{e}'(t) = [C'AL_2 - K(t)CAL_2]e'(t) \tag{2.39}$$

Assuming that the [A,C] pair is completely reconstructible, then $[C'AL_2, CAL_2]$ pair is completely reconstructible. And thus by theduality principle the $[(C'AL_2)^T, (CAL_2)^T]$ pair is completely controllable. Let the scalar $V(e',t)$ be defined by

$$V(e',t) = e'^T(t)\hat{W}(t)e'(t) \qquad (2.40)$$

where $\hat{W}(t)$ is a time varying symmetric (n-r)x(n-r) matrix. Taking the derivative of $V(e',t)$ along the trajectory of (2.39), one gets

$$\overset{\circ}{V}(e',t) = e'(t)[\overset{\circ}{\hat{W}}(t) + \hat{W}(t)(C'AL_2) + (C'AL_2)^T\hat{W}(t) -$$

$$= 2\hat{W}(t)K(t)CAL_2]e'(t) \qquad (2.41)$$

Let

$$2\hat{W}(t)K(t) = (CAL_2)^T \qquad (2.42)$$

where

$$\hat{M}\overset{\circ}{\hat{W}}(t) = \hat{W}(t)(C'AL_2) + (C'AL_2)^T\hat{W}(t) - (CAL_2)^T(CAL_2) \qquad (2.43)$$

where $\hat{M}$ is a positive scalar. Substituting (2.42) and (2.43) into (2.41) yields

$$\overset{\circ}{V}(e',t) = [\hat{M} + 1]e'^T(t)\overset{\circ}{\hat{W}}(t)e'(t) \qquad (2.44)$$

Using the boundary condition

$$\hat{W}(t_f) = 0 \tag{2.45}$$

the solution to (2.43) becomes

$$\hat{W}(t,t_f) = (1/\hat{M})\int_t^{t_f} \Phi^T(p/\hat{M})(CAL_2)^T(CAL_2)\Phi(p/\hat{M})d\tau \tag{2.46}$$

where $p = t - \tau$ and $\Phi(t - \tau)$ is the state transition matrix of (1.22).

Using an argument analogous to that developed in the last section for the full-order observer, one can conclude that $\hat{W}(t)$ is indeed positive definite (see Appendix B). Since $\hat{W}(t)$ exists for $t < t_f$, then $K(t)$ is given by

$$K(t) = 0.5\hat{W}^{-1}(t)(CAL_2)^T \tag{2.47}$$

In realization there is no need to take the derivatives of $y(t)$ and $K(t)$. To show this define

$$q(t) = \hat{w}(t) - K(t)y(t) \tag{2.48}$$

From (2.47) $\quad \mathring{K}(t) = -0.5\hat{W}^{-1}(t)\mathring{\hat{W}}(t)\hat{W}^{-1}(t)(CAL_2)^T$

$$= -\hat{W}^{-1}(t)[\hat{W}(t)(C'AL_2) + (C'AL_2)^T\hat{W}(t) - (CAL_2)^T(CAL_2)]K(t)/\hat{M}$$

$$= -[(C'AL_2)K(t)+\hat{W}^{-1}(t)(C'AL_2)^T(CAL_2)^T/2-2K(t)(CAL_2)K(t)]/\hat{M} \tag{2.49}$$

$$\text{Then} \quad \mathring{q}(t) = [C'AL_2 - K(t)CAL_2]q(t) + [C'AL_2K(t)(1+1/\hat{M}) + C'AL_1 - K(t)CAL_1 - K(t)CAL_2K(t)(1 + 2/\hat{M}) + \hat{W}^{-1}(t)(C'AL_2)^T(CAL_2)^T]y(t) + [C'B - K(t)CB]u(t) \tag{2.50}$$

This equation does not contain $\mathring{y}(t)$ or $\mathring{K}(t)$. The state estimate is

$$\hat{x}(t) = L_2q(t) + [L_1 + L_2K(t)]y(t) \tag{2.51}$$

The parameter M plays an important part in shaping the error trajectory in a manner similar to the parameter M of the full-order observer discussed earlier.

2.5 EXAMPLES AND GRAPHICAL RESULTS:

EXAMPLE 2.1: Consider the system given by

$$\mathring{x}_1 = u \tag{2.52}$$

$$\mathring{x}_2 = -x_1 - x_2 \tag{2.53}$$

This example illustrates the design of a finite-time controller for a prescribed t_f and various values of the parameter N. The state transition matrix of the system is

$$\Phi(t,\tau) = \begin{bmatrix} 1 & 0 \\ \exp(\tau - t) - 1 & \exp(\tau - t) \end{bmatrix} \tag{2.54}$$

And the solution to (2.9),(2.10) is

$$\dot{S}^{-1}(t,t_f) = \begin{bmatrix} R & 0 \\ \exp(R)-R-1 & 0.5\exp(2R)-2\exp(R)+R+1.5 \end{bmatrix} \tag{2.55}$$

where

$$R = (t_f - t)/N \tag{2.56}$$

Therefore, from (2.3) and (2.36)

$$u(t) = - \frac{s_{11}x_{11} + s_{12}x_2}{2} \tag{2.57}$$

where $$S(t,t_f) = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \tag{2.58}$$

This example was chosen to show the effect of N on the system's trajectory. The system was simulated using $t_0 = 0$, t_f = 1sec., $x_1(0) = 1$ and $x_2(0) = 2$. Fig. 2.1 shows the system's trajectory for different values of N. For this example it is obvious that as N increases the overshoot increases. Noticeable also is the time at which the maximum overshoot occurs. Table 1.1 shows the maximum value of $x_1(t)$ and the time it occurs for different values of N. For this example, the maximum overshoot is directly proportional to N while the time it occurs is inversely proportional to N. Fig. 2.1 also shows the relationship between the different trajectories for the same instant of time for different values of N.

N	Time(secs)	$x_1(t)$ max.	$x_2(t)$ corr
1	0.5675	1.5577	0.5424
2	0.3000	3.1204	0.8457
3	0.2050	4.8057	9890
4	0.1600	6.526	0.9891
5	0.1300	8.2632	1.0181

Table 1.1 The effect of N on the Maximum overshoot and the time it occurs. The corresponding value of $x_2(t)$ is also shown. 0.1sec corresponds to 1 time unit in Fig. 2.

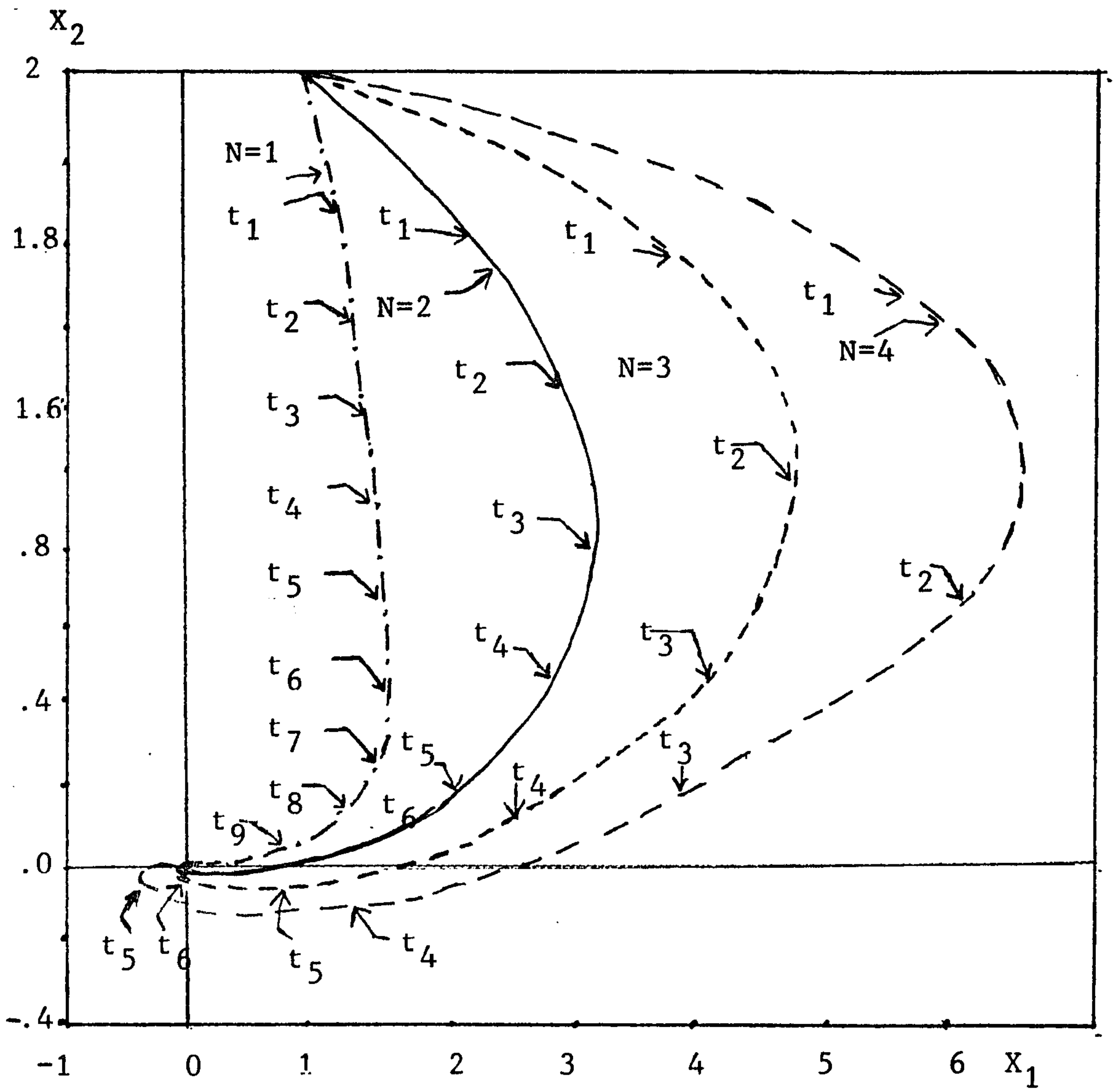


Fig. 2.1 Trajectory of X_1 and X_2 for N = 1, 2, 3 and 4 respectively. t_b means b time units. 0.1sec. = 1 time unit

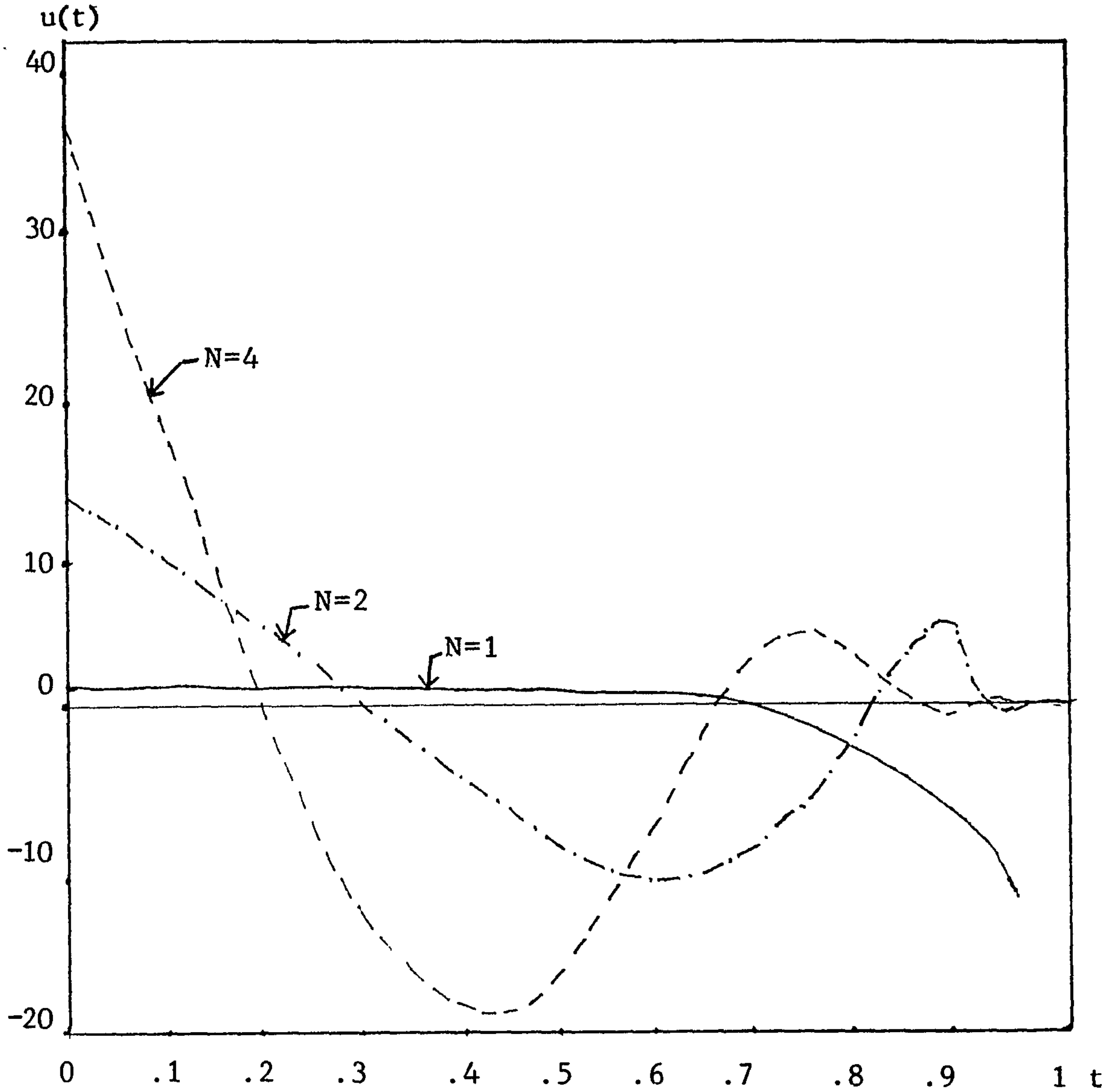


Fig. 2.2 Input versus time(secs.) for N=1,2 and 4

EXAMPLE 2.2: Given the following dynamic system

$$\dot{x} = Ax \tag{2.59}$$

$$y = Cx \tag{2.60}$$

where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1, & 0 \end{bmatrix}$ (2.61)

Example 2.2 illustrates the design of a finite-time observer for a prescribed t_f and various values of the parameter M. It is obvious that the [A, C] pair is of full rank and thus the system is completely observable. As a result the $[A^T, C^T]$ pair is completely controllable. Application of the algorithm ofsection 2.2gives the solution to (2.23), (2.25) as

$$W(t,t_f) = \begin{bmatrix} 0.25[R+\sin(R)] & 0.25[1-\cos(R)] \\ 0.25[1-\cos(R)] & 0.25[R-\sin(R)] \end{bmatrix} \tag{2.62}$$

where $R=2[t_f - t]/M$. The derivative of (2.62) is given by

$$\dot{W}(t,t_f) = -1/\sqrt{M} \begin{bmatrix} \cos(R/2) & 0 \\ -\sin(R/2) & 0 \end{bmatrix} \begin{bmatrix} \cos(R/2) & -\sin(R/2) \\ 0 & 0 \end{bmatrix} / \sqrt{M} \tag{2.63}$$

Also the determinant of $W(t,t_f)$ is positive for $t < t_f$. Thus $W(t,t_f)$ is positive definite for $t < t_f$. Thus $K(t,t_f)$

exists and from (2.28) is given by

$$K(t,t_f) = 0.5W^{-1}(t,t_f)C^T \tag{2.64}$$

Figs. 2.3 and 2.4 show the system and observer responses for t_f =1 sec. and M = 1, 2 and 4 respectively. Figs. 2.5 and 2.6 show the responses for t_f = 2 secs. and M = 1, 2 and 4 respectively.

<u>EXAMPLE 2.3</u>: Let

$$\dot{x} = Ax + Bu \tag{2.65}$$

$$y = Cx \tag{2.66}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $C \begin{bmatrix} 1, & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (2.67)

This example illustrates the design of a finite-time observer-controller for prescribed t_{fo} and t_f, N and M. The system is both controllable and observable. An observer for (2.65) is given by

$$\dot{z} = Az + K(t)[y - Cz] + Bu \tag{2.68}$$

For this example, from (2.26), $W(t,t_{fo})$ is given by

$$W(t,t_{fo}) = \begin{bmatrix} 0.25[I+\sinh(I)] & 0.25[1-\cosh(I)] \\ 0.25[1-\cosh(I)] & -0.25[I-\sinh(I)] \end{bmatrix} \tag{2.69}$$

where $I = 2(t_{fo} - t)/M$. Since $\Phi(t,\tau)$, the state transition matrix is symmetrical and $C^T = B$, it follows from (2.37) that

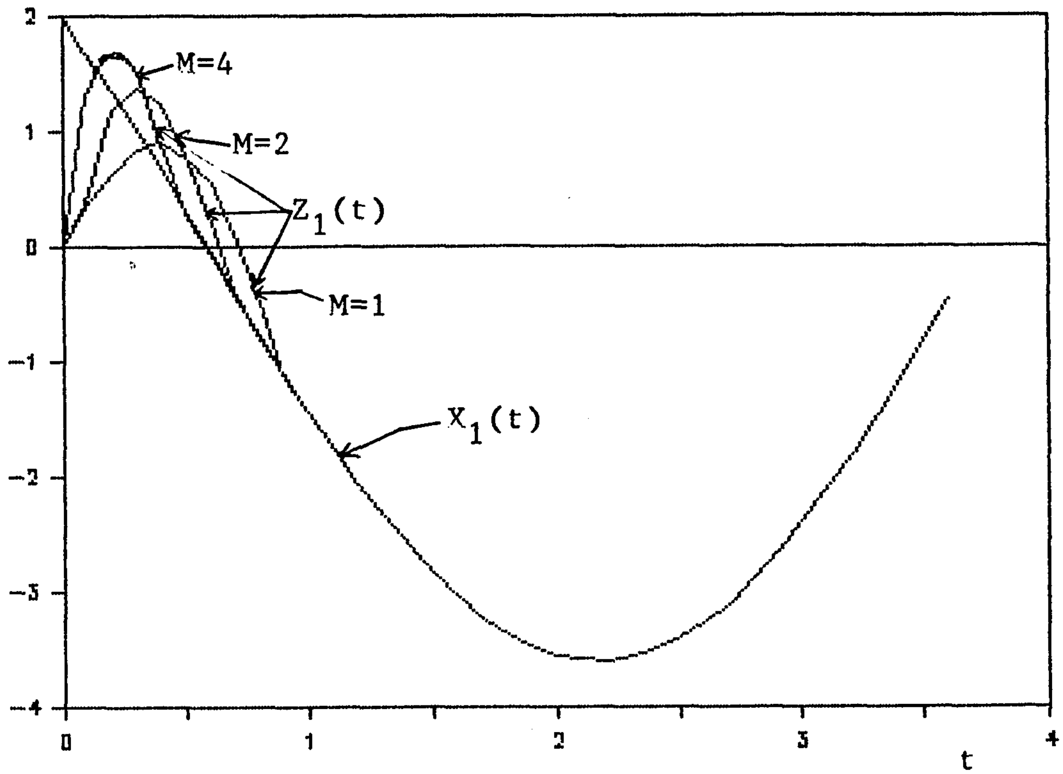
$$S^{-1}(t,t_f) = \begin{bmatrix} 0.25[J+\sinh(J)] & 0.25[1-\cosh(J)] \\ 0.25[1-\cosh(J)] & -0.25[J-\sinh(J)] \end{bmatrix} \qquad (2.70)$$

where $J = 2(t_f - t)/N$. The input and observer gain are thus given by

$$u(t,t_f) = -0.5B^T S(t,t_f) \qquad (2.71)$$

$$K(t,t_{fo}) = 0.5W^{-1}(t,t_f)C^T \qquad (2.72)$$

where $t_f > t_{fo}$. The system and observer were simulated using t_{fo} = 1sec., t_f = 2secs., $x(0) = [3, -4]^T$, $z(0) = [0, 0]^T$, N = 2 and M = 2. Graphical results for the controlled system are shown in Figs. 2.7 and 2.8.



Time in Seconds

Fig. 2.3 X_1 and Z_1 for M = 1, 2 and 4 respectively

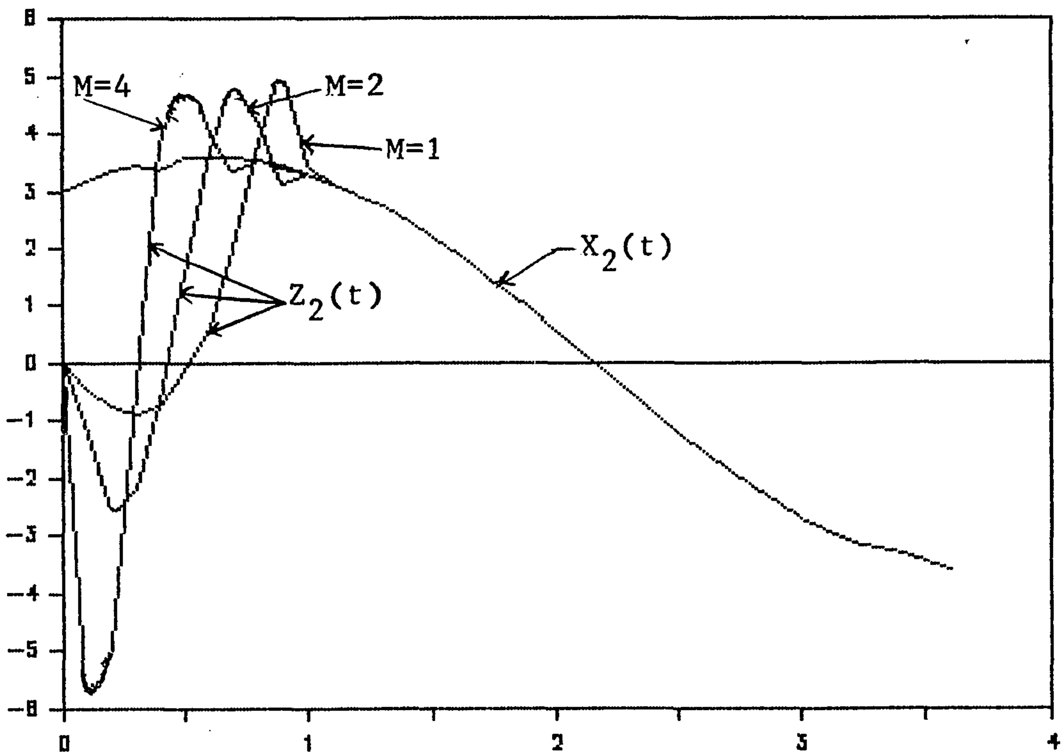


Fig. 2.4 X_2 and Z_2 for M = 1, 2 and 4 respect.

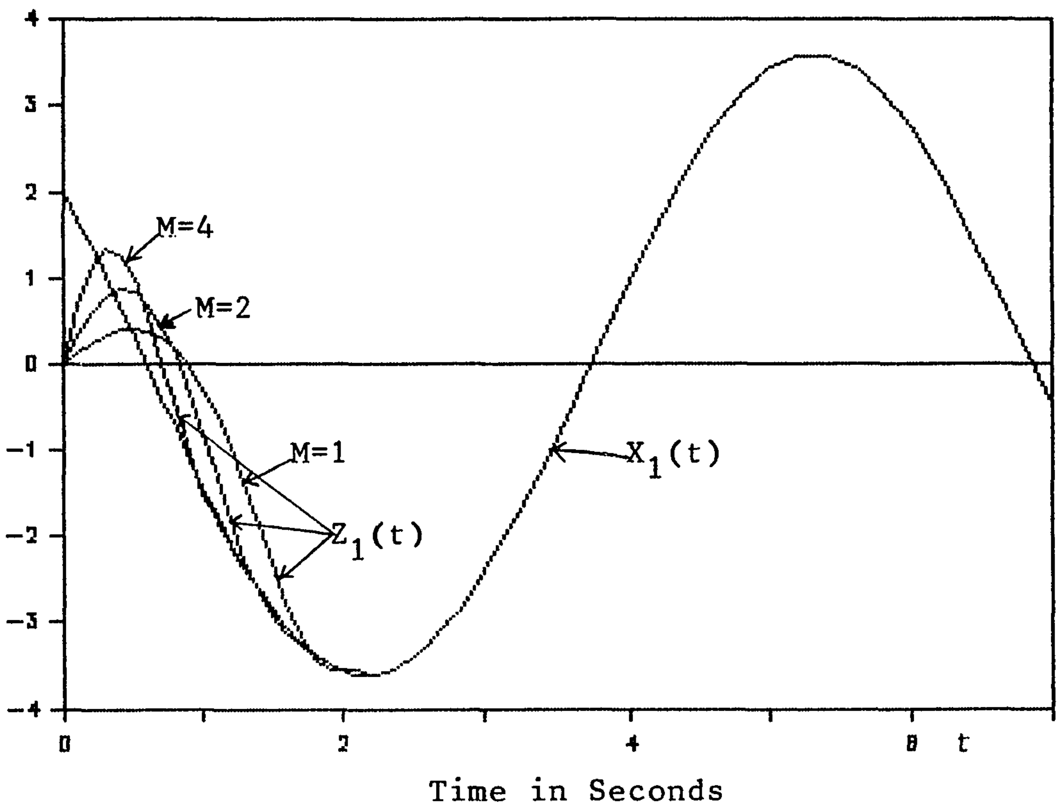


Fig. 2.5 X_1 and Z_1 for t_f=2 and M=1, 2 and 4 resp.

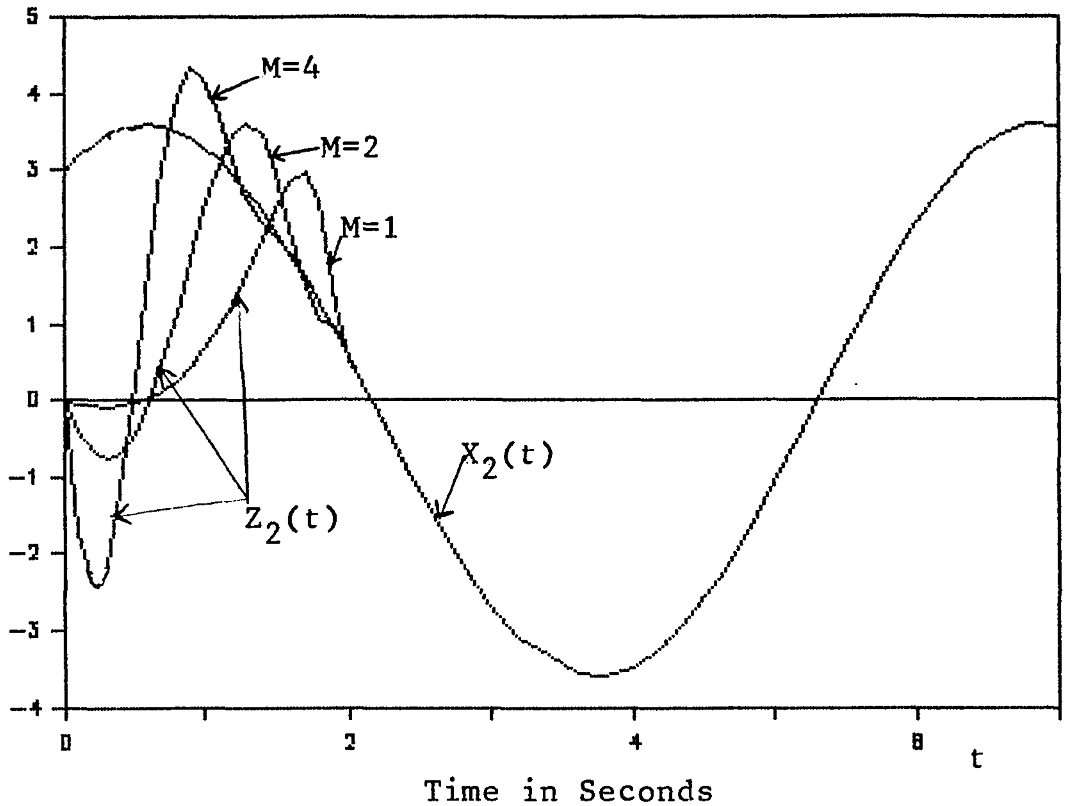


Fig. 2.6 X_2 and Z_2 for t_f=2 and M=1, 2 and 4 resp.

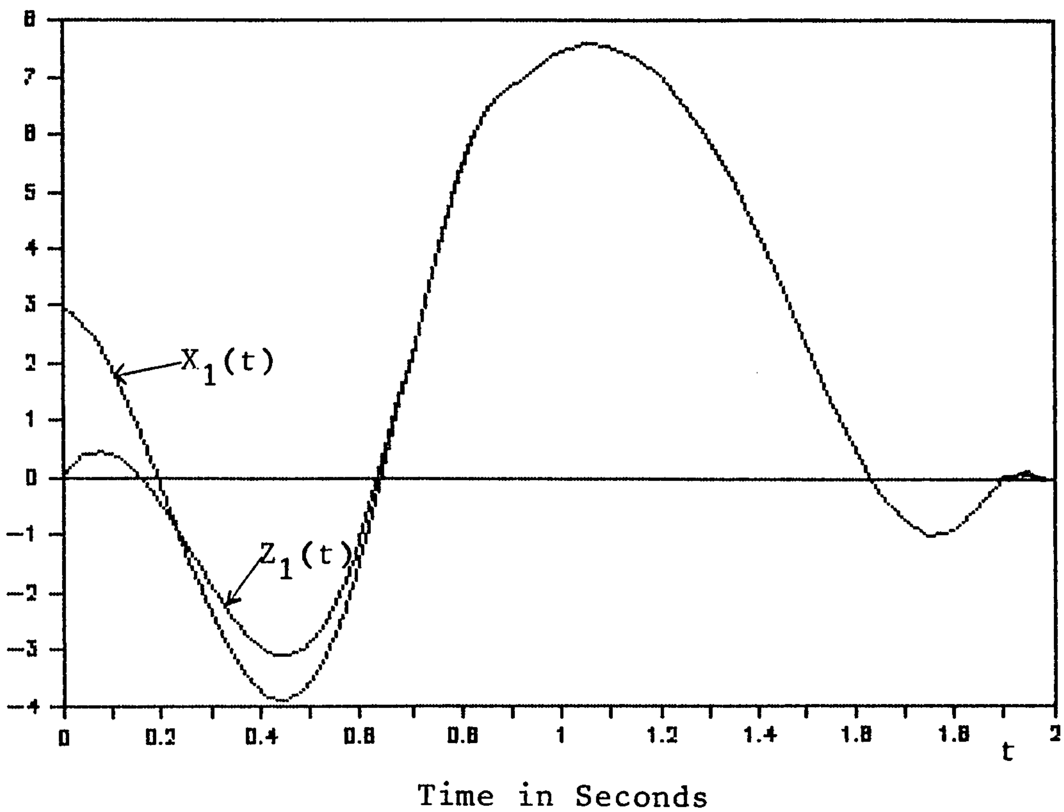


Fig. 2.7 Responses of X_1 and Z_1 for the controlled system with t_{fo} = 1sec., t_{fc} = 2secs. and N = M = 2.

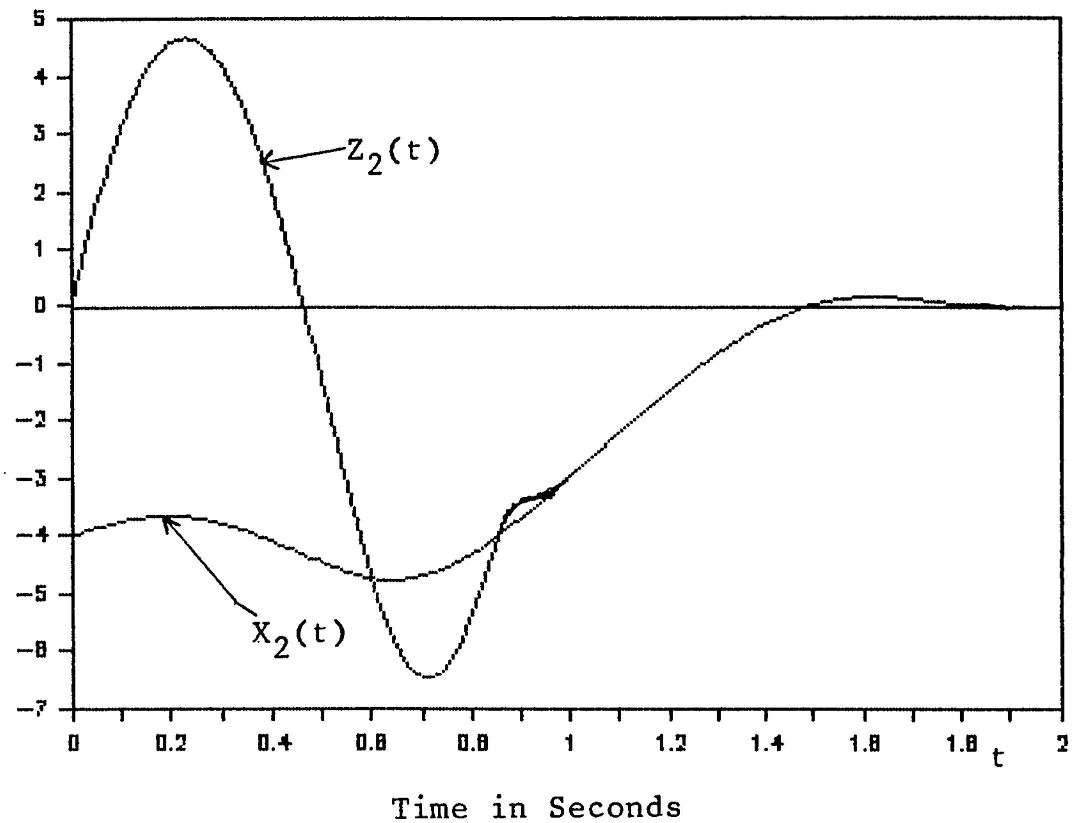


Fig. 2.8 Responses of $X_2(t)$ and $Z_2(t)$ for the controlled system with t_{fo}=1sec., t_{fc}=2secs. and N = M = 2.

3. DESIGN OF NONLINEAR OUTPUT-FEEDBACK FINITE-TIME REGULATORS

A family of nonlinear control laws is developed to bring a nonlinear dynamic system from an arbitrary initial state to the zero state in finite-time when only limited output measurements are available. Design parameters are found for an observer-controller which can be used to shape both the system's state trajectory and the dynamic observer's error trajectory. Numerical simulation studies demonstrate properties of the nonlinear finite-time output-feedback regulator.

In this chapter, we consider the problem of observing the output and regulating the state of a class of nonlinear systems in finite time. Controllability of nonlinear systems, assuming complete state measurement, is examined in section 3.1 where the work of Gershwin and Jacobson [9] is extended. In section 3.2 we design a nonlinear deadbeat observer, a dynamic state reconstructor that yields exact state estimates in finite time. Sufficient conditions for observability of certain nonlinear systems are established. The finite-time nonlinear observers in section 3.2 are distinct from the class of observers considered in [18]. An example is given to demonstrate the new nonlinear deadbeat observer design technique. In section 3.3, the finite-time observer and the finite-time controller are combined to produce a finite-time output-feedback regulator. Numerical examples are presented to illustrate properties of the state and error trajectories.

3.1 CONTROLLER DESIGN:

Consider the nonlinear time-invariant system described by

$$\dot{x} = Ax + f(x) + Bu \tag{3.1}$$

where x is an n-dimensional state vector, u is the r-dimensional input vector, f is a nonlinear function such that $f: R^n \rightarrow R^n$, and A and B are nxn and nxr matrices respectively. The controllability of this class of systems was examined by [9]. We will extend the results of [9] as follows: Let the input u be given by

$$u = -0.5B^T S(t)x + g(x) \tag{3.2}$$

where S(t) satisfies

$$-N\dot{S}(t) = S(t)A + A^T S(t) - S(t)BB^T S(t) \tag{3.3}$$

g(x) is a nonlinear function to be specified below, and N is a positive scalar. Note that this is in contrast to the method of [9] where N is assumed to be equal to unity. Let us define the scalar V(x,t) as follows

$$V(x,t) = x^T S(t) x \tag{3.4}$$

Differentiating (3.4) along the trajectory of (3.1) and using (3.2) and (3.3) results in

$$\mathring{V}(x,t) = x^T[(1-N)\mathring{S}(t)]x + 2x^T S(t)[f(x) + Bg(x)] \tag{3.5}$$

Assume the boundary condition

$$S^{-1}(t_f) = 0 \tag{3.6}$$

Then from (3.3) and (3.6) the closed form solution for $S^{-1}(t)$ becomes

$$S^{-1}(t) = (1/N)\int_t^{t_f} \exp[A(t-\tau)/N]BB^T\exp[A^T(t-\tau)/N]d\tau$$

$$= N\int_t^{t_f} \exp[A(t-\tau)/N]\frac{BB^T}{NN}\exp[A^T(t-\tau)/N]d\tau \tag{3.7}$$

We state the following Lemma: If the system given by

$$\mathring{x} = Ax + Bu \tag{3.8}$$

is completely controllable, then the system given by

$$\mathring{x} = (A/N)x + (B/N)u \tag{3.9}$$

is completely controllable.

Thus using (3.3),(3.6),(3.7) and the result in [9] we find that

$$\mathring{S}(t) = -S(t)\mathring{S}^{-1}(t)S(t) \geq 0 \qquad (3.10)$$

and from (3.5) we conclude that for $N \geq 1$, the first term of the right hand side of (3.5) is negative semidefinite. Thus (3.5) reduces to

$$\mathring{V}(x,t) \leq 2x^T S(t)[f(x) + Bg(x)] \qquad (3.11)$$

Next assume there exists an nxn matrix $H(x)$ and an r-dimensional nonlinear vector-valued function $g(x)$ such that

$$f(x) + Bg(x) = -H(x)x \qquad (3.12)$$

and
$$S(t)H(x) + H^T(x)S(t)$$

is positive definite for all x and t. Then (3.11) becomes

$$\mathring{V}(x,t) \leq 0 \qquad (3.13)$$

From (3.6) and (3.13) we conclude that all the conditions of the controllability theorem of [9] are satisfied for $N \geq 1$. Hence, if (3.12) holds, the input given by (3.2) accomplishes the transfer of the state of (3.1) from (x_0, t_0) to $(0, t_f)$.

3.2 FINITE-TIME OBSERVER DESIGN:

In this section we develop sufficient conditions under which a class of nonlinear systems is completely observable. The procedure is a variation of that described in the previous section for controller design. Here we express the observer gain as the sum of two components. The first component is characterized by the properties of the strictly linear part of the nonlinear observer. The second component is related to the strictly nonlinear part of the observer.

Consider the class of systems given by (3.1) where the output measurement is assumed to be

$$y = Cx \tag{3.14}$$

where $y \in R^r$ and C is an rxn matrix. An observer for (3.1) and (3.14) is given by

$$\mathring{z} = Az + f(z) + K[y - Cz] + Bu \tag{3.15}$$

where $z \in R^n$ is the output of the observer and K is an nxr observer gain matrix written as

$$K = K_1 + K_2 \tag{3.16}$$

From (3.1), (3.15) and (3.16) we find

$$\dot{x} - \dot{z} = [A - K_1C](x - z) + [f(x) - K_2Cx] - [f(z) - K_2Cz] \tag{3.17}$$

Let W(t) be a matrix which satisfies the following conditions

$$M\dot{W}(t) = W(t)A + A^TW(t) - C^TC \tag{3.18}$$

$$W(t_f) = 0 \tag{3.19}$$

where M is a positive scalar. Now we establish the following:

<u>THEOREM</u>: If 1) the [A,C] pair is completely observable

2) there exists a K_2 such that $W[\nabla f - K_2C]$ is uniformly negative definite

3) the solution to (3.1) exists and is unique.

Then there exists a K given by (3.16) such that the state of (3.17) can be transferred from $[(x_0-z_0),t_0]$ to $[0,t_f]$ in a prescribed finite time t_f.

<u>PROOF</u>: The proof is constructive, combines the approaches of [9] and [17] and leads to the design of the observer's gain K. Define the scalar function V((x-z),t) such that

$$V((x-z),t) = (x - z)^T W(t)(x - z) \tag{3.20}$$

Take the derivative of (3.20) along the trajectory of (3.17) to yield

$$\overset{\circ}{V}(x-z,t) = (x - z)^T[\dot{W}(t) + W(t)(A - K_1C) + (A - K_1C)^T W(t)]*$$

$$(x - z)+2(x - z)^T W(t)[f(x) - K_2Cx) - (f(z) - K_2Cz)] \tag{3.21}$$

Let $$2W(t)K_1 = C^T \tag{3.22}$$

Substitute (3.22) and (3.18) into (3.21) to give

$$\overset{\circ}{V}(x-z,t) = (x - z)^T(1+M)W(t)(x - z) +$$

$$2(x - z)^T W(t)[(f(x) - K_2Cx) - (f(z) - K_2Cz)] \tag{3.23}$$

The solution to (3.18) using (3.19) becomes

$$W(t) = (1/M)\int_t^{t_f} \exp[A^T(t-\tau)/M]C^T C\exp[A(t-\tau)/M]d\tau \tag{3.24}$$

It is well known by the duality principle [20] that if the [A,C] pair is completely observable then the $[A^T, C^T]$ is completely controllable. Note that the positive-definite of W(t) is equivalent to the condition for controllability of the $[A^T/M, C^T/M]$ pair. But since the $[A^T, C^T]$ pair is completely controllable, the $[A^T/M, C^T/M]$ is also controllable. Thus $W(t)$ exists and is positive definite for $t < t_f$. Hence K_1 exists and is given by

$$K_1 = 0.5W^{-1}(t)C^T \tag{3.25}$$

Then, as in the analysis of the previous section,

$$\mathring{W}(t) = -(1/M)\exp[A^T(t-t_f)/M]C^TC\exp[A(t-t_f)/M] \le 0 \tag{3.26}$$

Thus from (3.26) and $M > 0$, (2.23) reduces to

$$\mathring{V}(x-z,t) \le 2(x-z)^TW(t)[(f(x)-K_2Cx) - (f(z)-K_2Cz)] \tag{3.27}$$

By the fundamental theorem of integral calculus for vector-valued functions of several variables (Ortega and Rheinboldt, 1970), we get from property 2)

$$2(x - z)^TW(t)[(f(x) - K_2Cx) - (f(z) - K_2Cz)]$$

$$= 2(x - z)^TW(t)\int_0^1(\nabla f - K_2C)Q_p(x - z)dp$$

$$= -\varepsilon||x - z||^2 \le 0 \tag{3.28}$$

where $\varepsilon > 0$, $Q_p = px + (1 - p)z$ and ∇f is the Jacobian of $f(x)$. Substituting (3.29) into (3.27) yields

$$\overset{\circ}{V}(x,z,t) \;=\; \leq \;0 \qquad (3.29)$$

Let $r(t)$ be a vector-valued function such that

$$\lim_{t \to t_f} r(t) \;\neq\; 0 \qquad (3.30)$$

Then $\lim_{t \to t_f} V(K(t)r(t),t) = \lim_{t \to t_f} r^T(t)K^T(t)W(t)r(t)$

$$= \lim_{t \to t_f} r^T(t)[0.5CW^{-1}(t) + K_2^T][0.5C^T + W(t)K_2] = \infty \qquad (3.31)$$

Therefore all the conditions of the Controllability Theorem of [9] are satisfied. And the gain given by (3.16) accomplishes the transfer of $[(x_0 - z_0),t_0]$ to $[0,t_f]$. This concludes the proof. The procedure can be summarized as follows: For a prescribed t_f, the gain K_1 is calculated first by using (3.25). Given $W(t)$ from (3.24) and the Jacobian of f one seeks K_2 such that $W(t)[\nabla f - K_2C]$ is uniformly negatively definite.

<u>EXAMPLE 3.1</u>: Given the system (3.1) and (3.14) where

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } f(x) = \begin{bmatrix} 0 \\ -x_2^3 \end{bmatrix} \quad (2.32)$$

we illustrate the design of a finite-time observer for prescribed t_f and M. From (3.24) we get

$$W(t,t_f,M) = \begin{bmatrix} 0.25[R + \sin(R)] & 0.25[1 - \cos(R)] \\ 0.25[1 - \cos(R)] & 0.25[R - \sin(R)] \end{bmatrix} \quad (3.33)$$

where $R = 2[t_f - t]/M$. It is obvious that the [A,C] pair is completely observable and $W(t,t_f,M)$ is positive definite for $t < t_f$.

$$x^T\overset{\circ}{W}(t)x = -[x_1\cos(R/2) - x_2\sin(R/2)]^2 \leq 0 \quad (3.34)$$

for all x and t. Thus $K(t,t_f,M)$ exists and is given by

$$K_1(t,t_f,M) = 0.5W^{-1}(t,t_f,M)C^T \quad . \quad (2.35)$$

To establish property 2) of the theorem, note that the Jacobian of f(x) is

$$\nabla f = \begin{bmatrix} 0 & 0 \\ 0 & -3x_2^2 \end{bmatrix} \quad (3.36)$$

If we select

$$K_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{3.37}$$

then
$$(\nabla f - K_2C)_{sym} = \begin{bmatrix} -4 & 0 \\ 0 & -3x_2^2 \end{bmatrix} \tag{3.38}$$

which is uniformly negative semidefinite. The finite-time observer gain is thus given by

$$K(t,t_f,M) = 0.5W^{-1}(t,t_f,M)C^T + \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{3.39}$$

The dynamic response of system (3.1), (3.14) with parameters specified in (3.32) and the finite-time observer (3.15) with gain (3.39) was simulated using $t_0 = 0$, $t_f =$ 1s., $x(0) = [2, 3]^T$ and $z(0) = [0, 0]^T$. Figs. 3.1 and 3.2 show the results for M = 1.5. Clearly the state reconstruction error has been reduced to zero in the prescribed 1 sec..

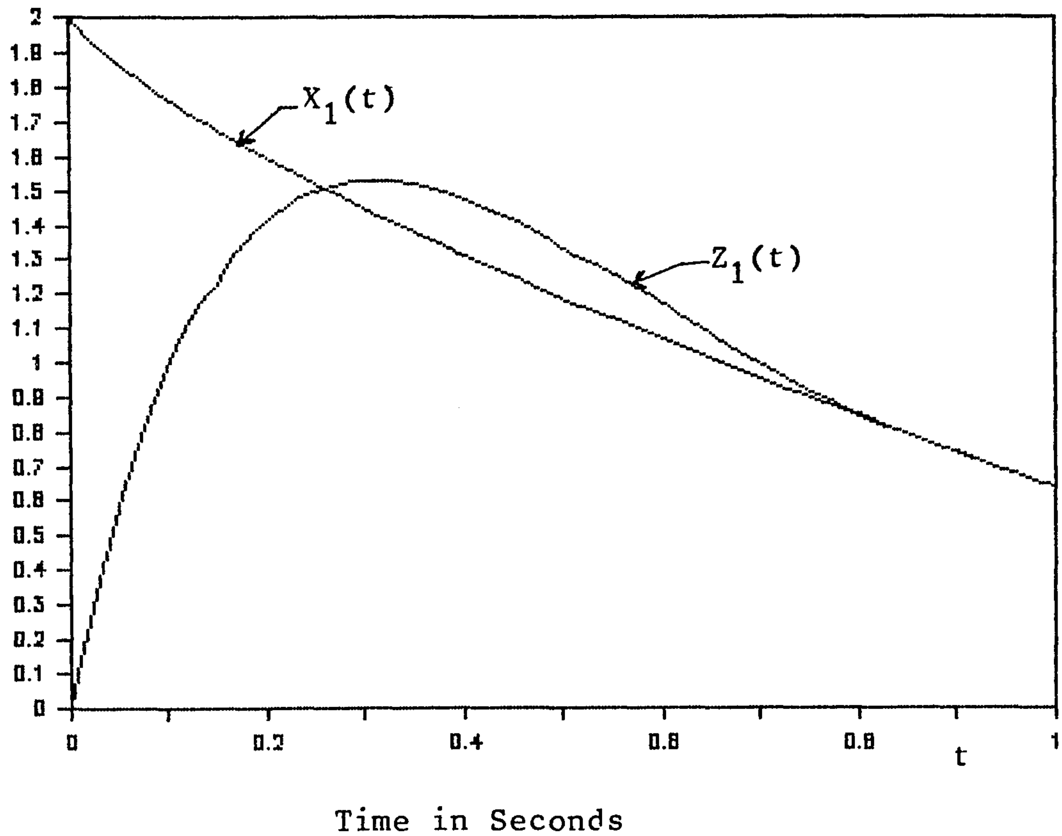


Fig. 3.1 Responses of X_1 and Z_1 for $M = 1.5$.

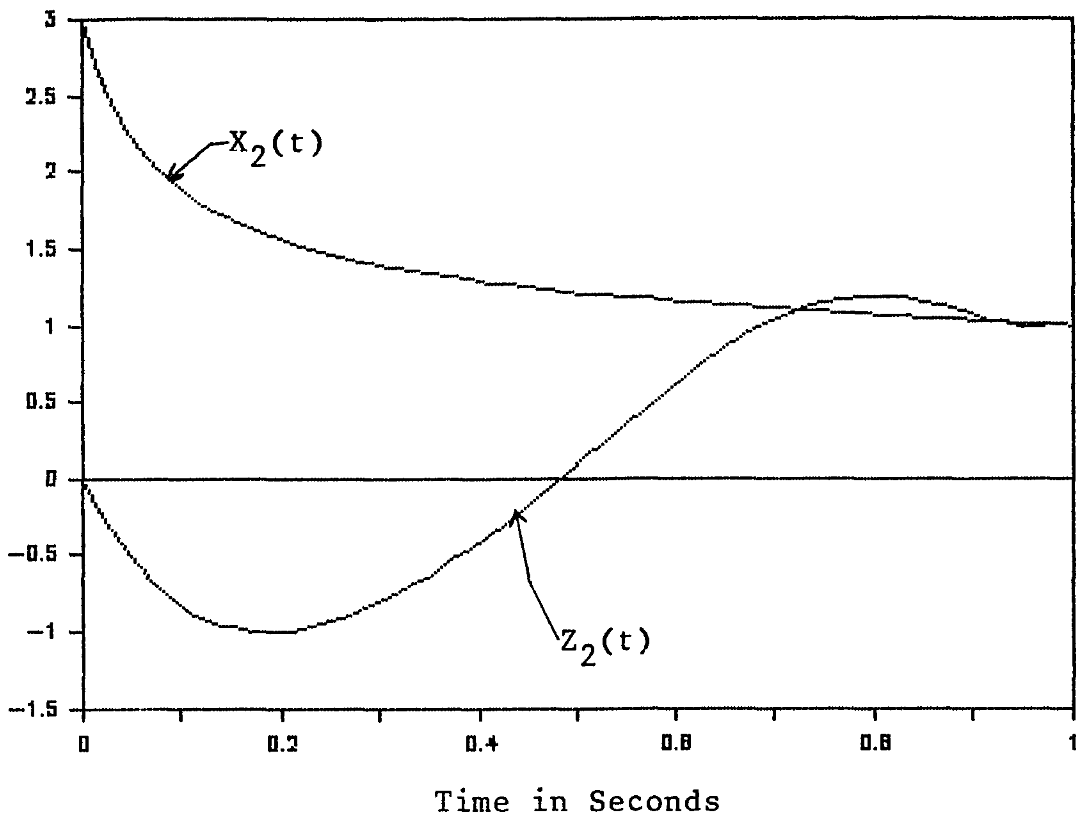


Fig. 3.2 Responses of X_2 and Z_2 for $M = 1.5$

3.3 NONLINEAR OUTPUT-FEEDBACK FINITE-TIME REGULATION:

The last two sections can be combined to produce an output-feedback finite-time regulator. To demonstrate this, consider the system given by (3.1), (3.14). A finite-time observer of the form (3.15) is designed so that the reconstruction error e,

$$e = x - z \tag{3.40}$$

has the property that $e(t_{fo}) = 0$, where t_{fo} is prescribed and determines the observation time required for complete state reconstruction. Let the input be given by (3.2). Substituting $z = x - e$ into (3.15) and using (3.2) yields

$$\dot{x} = A(x-e) + f(x-e) + K(W,x-e)Ce + B[0.5B^T S(x-e)+g(x-e)]+\dot{e} \tag{3.41}$$

Hence if f(x) and g(x) satisfy the requirements of the controller design discussed earlier, then the state of (3.42) will be transferred from $[x(t_{fo}),t_{fo}]$ to $(0,t_{fc})$. Thus $t_{fo} + t_{fc}$ is the finite time for complete state regulation based on the output measurements y.

EXAMPLE 3.2: Consider the nonlinear system (3.1),(3.14) where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1, & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad f(x) = \begin{bmatrix} 0 \\ -x_2{}^3 \end{bmatrix}$$

and

$$x = [x_1, x_2] \tag{3.43}$$

The uncontrolled system is unstable. It is obvious that the [A,B] and [A,C] pairs are completely controllable and observable, respectively. Using the theories developed earlier for exact state estimation and control we obtain

$$K_2 = [4, 0]^T \tag{3.44}$$

and for

$$g(x) = -x_1 x_2{}^2 \tag{3.45}$$

$$2x^T S(t)[f(x) - Bg(x)] = -2x_2{}^2 x^T S(t) x \le 0 \tag{3.46}$$

The dynamic response of the output-feedback system was simulated using t_{fo} = 1s., t_{fc} = 1s., $x_1(0) = 3$, $x_2(0) = -4$, $z_1(0) = 0$ and $z_2(0) = 0$. For this example we used N = M = 2. Graphical results are shown in Figs. 3.3 and 3.4. The state estimator takes the form (3.15) where

$$K = 0.5W^{-1}(t)C^T + [4, 0]^T \tag{3.47}$$

and where W(t) is evaluated using (3.18). The control law is

$$u = -0.5B^T S(t)z - z_1 z_2^2 \tag{3.48}$$

where $S^{-1}(t)$ is evaluated using (3.3). From Figs. 3.3 and 3.4 it is seen that state regulation is achieved at the end of the prescribed 2 second interval.

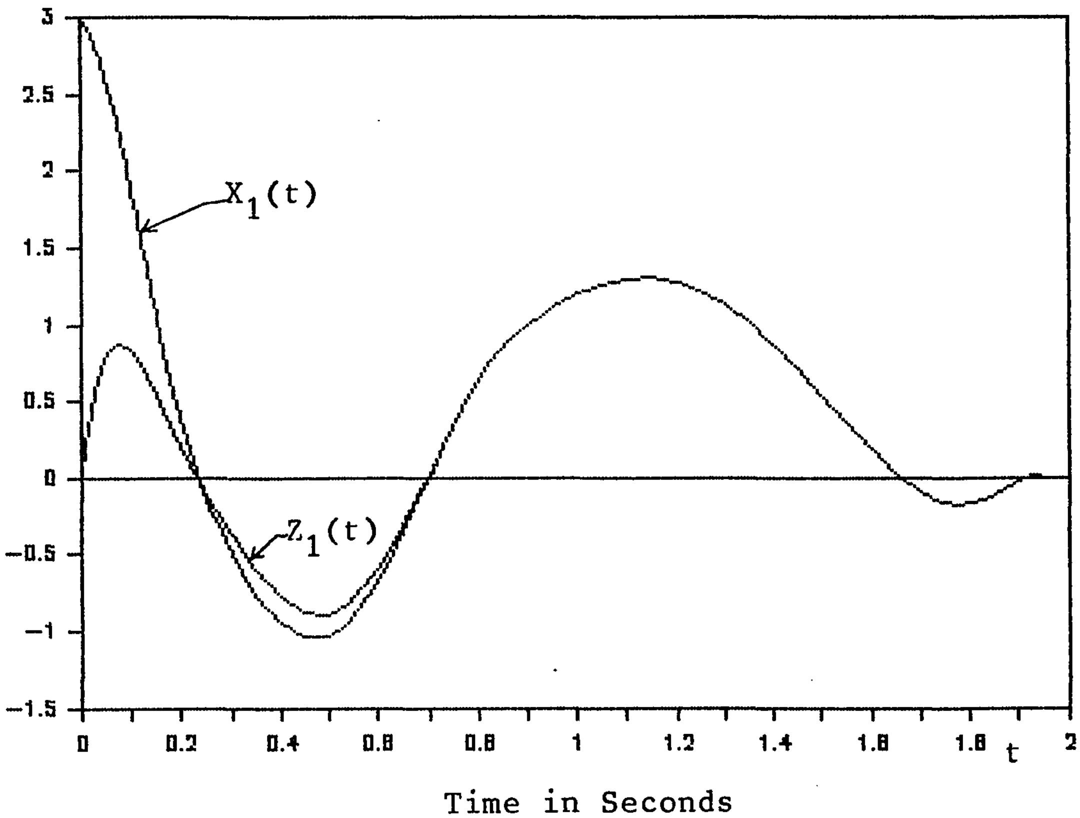
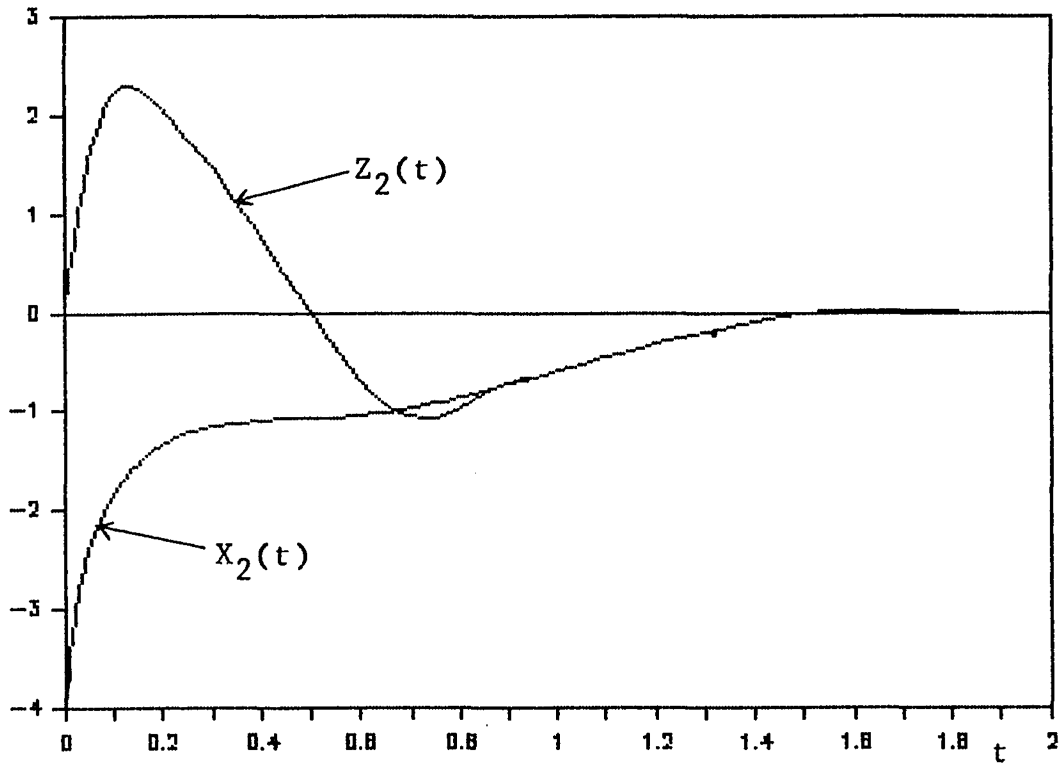


Fig. 3.3 Time Responses of X_1 and Z_1 of the Controlled System for N=M=2, t_{fo}=1 and t_{fc} = 2.



Time in Seconds

Fig. 3.4 time Responses of X_2 and Z_2 of the Controlled System for N = M = 2, t_{fo} = 1 and t_{fc} = 2.

3.4 ANALYSIS OF A CLASS OF SINGLE INPUT SINGLE OUTPUT NONLINEAR SYSTEMS:

PROBLEM STATEMENT 1: Given the nonlinear system (3.1), where u is a scalar input, find the class of nonlinear functions f(x) and g(x) such that

$$2x^T S(t)[f(x) + Bg(x)] \leq 0 \tag{3.49}$$

where u is given by (3.2) and S(t) satisfies (3.3), (3.6). It is assumed that the [A,B] pair is completely controllable.

SOLUTION: Since the [A,B] pair is completely controllable, then there exists a nonsingular matrix Q [41] which transforms (3.1) into

$$\dot{y} = A_1 y + f_1(y) + B_1 u \tag{3.50}$$

where

$$y = Qx \tag{3.51}$$

$$A_1 = QAQ^{-1} \tag{3.52}$$

$$B_1 = QB \tag{3.53}$$

$$f_1(y) = Qf(x) \tag{3.54}$$

and $$Q = [Q_1 \; Q_1A \; \ldots\ldots, \; Q_1A^{n-1}]^T \tag{3.55}$$

where $$Q_1 = [\;0\;0\;\ldots\ldots\ldots 1][B \; AB \; \ldots\ldots \; A^{n-1}B]^{-1} \tag{3.56}$$

The matrix A_1 takes the canonical form

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots\ldots & 0 & 0 \\ 0 & 0 & 1 & 0 \;\ldots & 0 & 0 \\ \cdot & & & & & \\ 0 & 0 & \ldots\ldots & & 0 & 1 \\ -a_1 & -a_2 & \ldots\ldots & & & -a_n \end{bmatrix} \tag{3.57}$$

and $$B_1 = [\;0\;0\;0\;\ldots\ldots\;0\;1]^T \tag{3.58}$$

From (3.2) it follows

$$u = -0.5B_1^T(Q^{-1})^T SQ^{-1}y + g(Q^{-1}y) \tag{3.59}$$

Let $$S_1 = (Q^{-1})^T SQ^{-1} \tag{3.60}$$

Then $$u = -0.5B_1^T S_1 y + g_1(y) \tag{3.61}$$

where $$g_1(y) = g(Q^{-1}y) \tag{3.61}$$

and $$N\mathring{S}_1^{-1} = A_1S_1^{-1} + S_1A_1^T - B_1B_1^T \tag{3.62}$$

with $$S_1^{-1}(t_f) = 0 \tag{3.63}$$

The condition required by (3.49) reduces to finding the class of f_1 and g_1 such that

$$2y^T S_1 [f_1(y) + B_1 g_1(y)] \le 0 \tag{3.64}$$

Let write $$f_1(y) = [f_{11}, f_{12}, \ldots\ldots f_{1n}]^T \tag{3.65}$$

Then $$f_1(y) + B_1 g_1(y) = [f_{11}, f_{12}, \ldots\ldots f_{1n-1}, f_1 + g_1(y)]^T \tag{3.66}$$

Therefore if the nonlinear function $f_1(y)$ is such that

$$f_{1i}(y) = -\alpha(y) y_i \quad 1 \le i \le n-1, \ \alpha(y) \ge 0 \text{ for all } y \tag{3.67}$$

Then for $$g_1(y) = -f_{1n} - \alpha(y) y_n \tag{3.68}$$

$$f_1(y) + B_1 g_1(y) = -\alpha(y) y \tag{3.69}$$

then $$-2y^T S_1 [f_1(y) + B_1 g_1(y)] = -2\alpha(y) y^T S_1 y < 0 \tag{3.70}$$

Since S_1 is positive definite for all $t < t_f$. Therefore

$$f(x) = Q^{-1} f_1(Qx) \tag{3.71}$$

$$g(x) = g_1(Qx) \tag{3.72}$$

PROBLEM STATEMENT 2: Given the nonlinear system (3.1), (3.14), find a class of nonlinear functions f(x) such that there exists a K_2 which satisfies

$$W(t)[\nabla f - K_2 C] \leq 0 \tag{3.73}$$

where W(t) satisfies (3.18),(3.19).

SOLUTION: If all the eigenvalues of $[\nabla f - K_2 C]$ are negative, then(3.73) is satisfied since W(t) is positive definite. Since the [A, C] pair is assumed completely observable, then there exists a nonsingular matrix P which transforms (3.1), (3.14) and (3.15) into

$$\dot{x}^* = A^* x^* + f^*(x^*) + B^* u \tag{3.74}$$

$$y = C^* x^* \tag{3.75}$$

$$\dot{z}^* = A^* z^* + f^*(x^*) + K^* C^* [y - C^* z^*] + B^* u \tag{3.76}$$

$$\text{where} \quad x^* = P^{-1} x \qquad z^* = P^{-1} z \quad f^*(\circ) = P^{-1} f(\circ) \tag{3.77}$$

$$P = [P_1, AP_1, \ldots\ldots A^{n-1} P_1] \tag{3.78}$$

$$P_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{3.79}$$

$$A^* = P^{-1}AP \qquad C^* = CP \qquad B^* = P^{-1}B \qquad K^* = P^{-1}K \tag{3.80}$$

The matrices A^* and C^* take the form

$$A^* = \begin{bmatrix} 0 & 0 & \cdots\cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots\cdots & 0 & -\alpha_1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -\alpha_{n-1} \end{bmatrix} \qquad C^* = [0,\ 0,\ \ldots 0, 1] \tag{3.81}$$

Let $K^* = K_1^* + K_2^*$ and $K_2^* = [K^*_{21}, K^*_{22} \ldots K^*_{2n}]$ (3.82)

Now let us examine the structure of the W(t) matrix under a sim ilarity transformation. Let

$$W^* = P^TWP \tag{3.83}$$

Then from (3.18)

$$\mathring{W}^*(t) = P^T\mathring{W}(t)P = P^T[W(t)A + A^TW(t) - C^TC]P$$

$$= W^*(t)P^{-1}AP + P^TA^TP^{T^{-1}}W^*(t) - P^TC^TCP$$

$$= W^*(t)A^* + A^{T^*}W^*(t) - C^{T^*}C^* \tag{3.84}$$

Then $W(t)[\nabla f(x)-K_2C] = P^{T^{-1}}W^*(t)P^{-1}[P\nabla f^*(x^*)P^{-1}-PK^*_2C^*P^{-1}]$

$$= P^{T^{-1}}W^*(t)[\nabla f^*(x^*) - K^*_2C^*]P^{-1} \tag{3.85}$$

Therefore the matrix $W^*(t)[\nabla f^*(x^*) - K^*_2 C^*]$ and $W(t)[\nabla f - K_2 C]$ are similar and thus have the same eigenvalues. Consequently we seek an $f^*(x^*)$ such that there exists a K^*_2 such that $[\nabla f^*(x^*) - K^*_2 C^*]$ is seminegative definite. Then by (3.82) we have

$$\nabla f^* - K^*_2 C^* = \begin{bmatrix} \partial f^*_1/\partial x^*_1 & \partial f^*_1/\partial x^*_2 \cdots & \partial f^*_1/\partial x^*_n - K^*_{21} \\ \partial f^*_2/\partial x^*_1 & \partial f^*_2/\partial x^*_2 \cdots\cdots & \partial f^*_2/\partial x^*_n - K_{22} \\ \vdots & & \\ \partial f^*_n/\partial x^*_1 & \partial f^*_n/\partial x^*_2 \cdots\cdots & \partial f^*_n/\partial x^*_n - K_{2n} \end{bmatrix} \tag{3.86}$$

Therefore from (3.86) the following is obvious:

Case 1: If ∇f^* is nonpositive definite, then $K^*_2 = 0$.

Case 2: If f^*_i is a function of x^*_i only and $\partial f^*_i/\partial x^*_i \le 0$ $1 \le i < n$, then $K^*_2 = [0, 0, \quad 0, \partial f^*_n/\partial x^*_n + \alpha]^T$ where α is any positive constant.

Case 3: If f^*_i can be expressed as the sum of functions of x^*_i and x^*_n such that $\partial f^*_i/\partial x^*_i \le 0$ for $1 \le i < n$. Then $K^*_2 = [\partial f^*_1/dx^*_n, \partial f^*_2/\partial x^*_n \ldots \partial f^*_{n-1}/\partial x^*_n, \partial f^*_n/\partial x^*_n + \alpha]^T$. The gain K_2 can now be gotten by the use of (3.80).

If f(x) simultaneously satisfies the conditions of (3.49) and (3.73), then a deadbeat output-feedback regulator can be designed for the system given (3.1),(3.14). An example of such a case can be seen in example 3.2.

EXAMPLE 3.3: Consider the system given by (3.1),(3.14) where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad C = [1,\ 0] \text{ and } \quad f(x) = \begin{bmatrix} -x_1^{\,2} \\ -(2x_1+x_2)^3+3x_1^{\,2} \end{bmatrix} \tag{3.87}$$

Using (3.77) to (3.80) leads to

$$P = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } f^*(x^*) = \begin{bmatrix} -x_1^{*3} + x_2^{*2} \\ -x_2^{*2} \end{bmatrix} \tag{3.88}$$

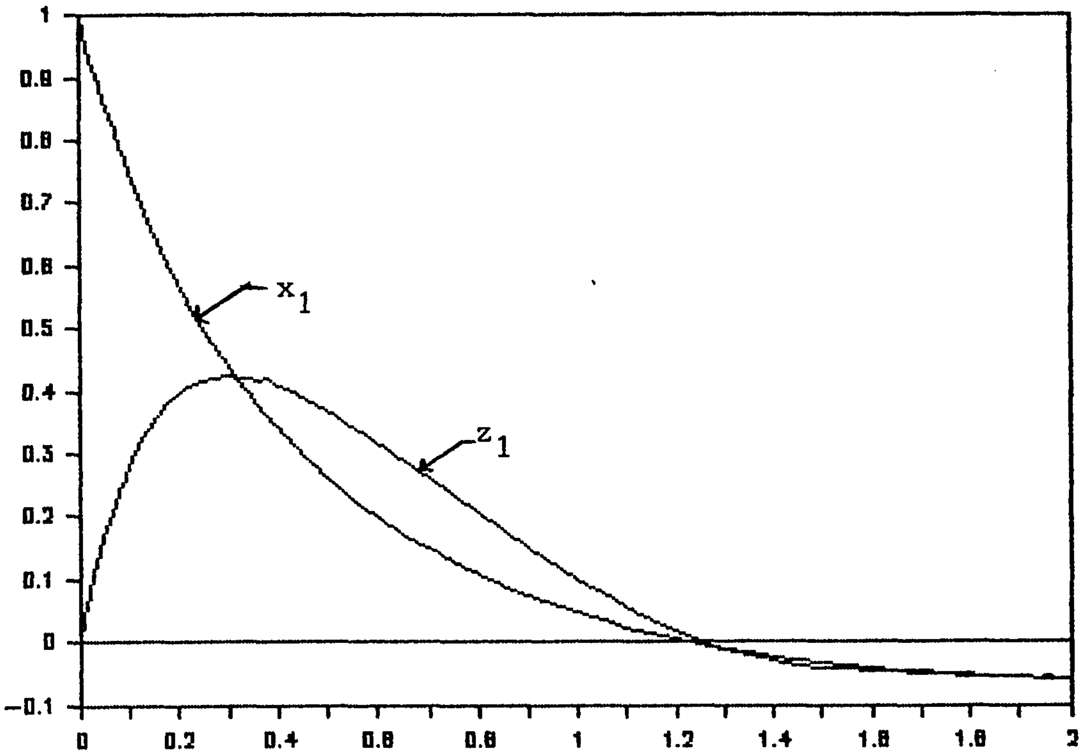
$$\text{Thus} \quad \nabla f^* - K^*_2 C^* = \begin{bmatrix} -3x_1^{*2} & 2x_2^*-K_{21}^* \\ 0 & -2x_2^* -K_{22}^* \end{bmatrix} \tag{3.89}$$

Therefore the required gain is

$$K_2^* = [\ 2x_2^*, \quad -2x_2^* + 4]^T \tag{3.90}$$

From (3.80) we get $\quad K_2 = [2x_1,\ 2x_1+4\ ]^T$ (3.91)

The system was simulated using $x(0)=[1,\ -2]^T$ and $z(0) = [0,\ 0]^T$. Figs. 3.5 and 3.6 show the responses of the state and state estimator for $t_f = 2$ seconds $M = 2$. Note that after the two seconds time interval, exact state estimator is indeed achieved.



TIME IN SECONDS

Fig. 3.5 Time Responses of x_1 and z_1 Ex. 3.3

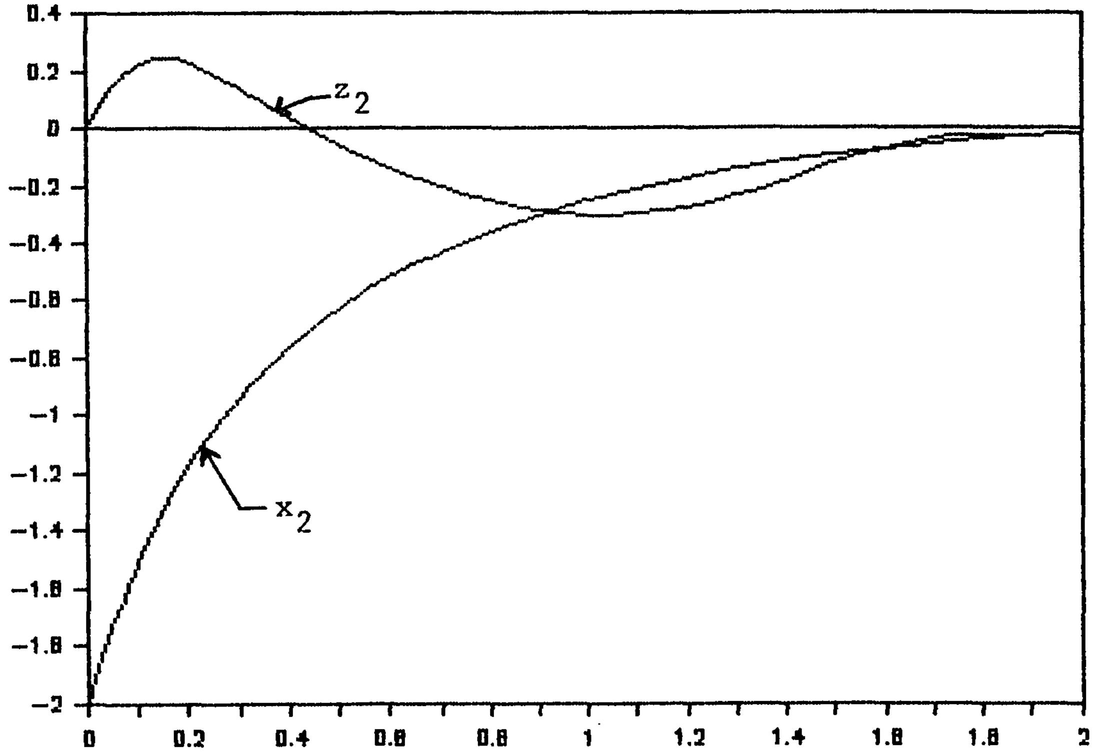


Fig. 3.6 Time Responses of x_2 and z_2 for Ex. 3.3.

We have established sufficient conditions for the observability and controllability of a class of nonlinear systems. The theorems are constructive in that they lead to the design of observers and controllers which result in exact state estimates and zero state values in a finite time t_f. The theory was then applied to the regulator problem with incomplete state measurements. A detailed analysis of a class of single input, single output systems was given. Numerical examples demonstrated the application of these techniques.

4. EFFECT OF SYSTEM PARAMETERS ON OBSERVER-CONTROLLER DESIGN

4.1 TRIMMING THE STATE AND INPUT MAGNITUDE:

4.1.1 THE EFFECT OF N

This section investigates the effect of N on the state and input magnitudes for a linear time-invariant system. It is shown that the use of a time-varying N can prevent large state component excursions and large input magnitudes. This is accomplished by curve fitting N such that it enhances the good qualities of a small value at one time interval and the good qualities of a large value at another time interval and thus eliminating large state and input excursions that usually occur at the beginning or end when a an N of constant value is used.

Reconsider the linear time-invariant system given by (2.1).

Let $$V(x,t) = x^T S(t) x \tag{4.1}$$

where S(t) satisfies

$$-N(t)\dot{S}(t) = S(t)A + A^T S(t) - S(t)BB^T S(t) \tag{4.2}$$

where N(t) is a positive scalar. Then for u(t) given by

$$u(t) = -0.5B^T S(t)x \tag{4.3}$$

$$\dot{V}(x,t) = x^T S(t)[Ax + Bu(t)] + [Ax + Bu(t)]^T S(t)x + x^T \dot{S}(t)x$$

$$= x^T[\dot{S}(t) + S(t)A + A^T S(t) - S(t)BB^T S(t)]x$$

$$= -[N(t) - 1]x^T \dot{S}(t)x \qquad (4.4)$$

If S(t) is invertible, then (4.2) can be written as

$$\dot{S}^{-1}(t) = N^{-1}(t)AS^{-1}(t) + S^{-1}(t)A^T N^{-1}(t) - N^{-1}(t)BB^T \qquad (4.5)$$

Using the boundary condition

$$S^{-1}(t_f) = 0 \qquad (4.6)$$

the solution to (4.5) (see Appendix B) becomes

$$S^{-1}(t) = \int_0^{p_1} \exp(-Ap)BB^T \exp(-A^T p)dp \qquad (4.7)$$

where p_1 is given by

$$p_1 = \int_t^{t_f} N^{-1}(\lambda)d\lambda \qquad (4.8)$$

Thus given that the [A,B] pair is completely controllable, $S^{-1}(t)$ is invertible for $t < t_f$, since $p_1 > 0$ for all $t < t_f$. Thus S(t) exists and is positive definite for all t <

t_f. Differentiate (4.7) with respect to t to yield

$$\dot{S}^{-1}(t) = -N^{-1}(t)\exp(-Ap_1)BB^T\exp(-A^Tp_1) \leq 0 \qquad (4.9)$$

And thus $$\dot{S}(t) = -S(t)\dot{S}^{-1}(t)S(t) \geq 0 \qquad (4.10)$$

Therefore for $N(t) \geq 1$, (4.4) reduces to

$$\dot{V}(x,t) \leq 0 \qquad (4.11)$$

Thus by Gershwin and Jacobson First Controllability condition, the input given by (4.3) transfers the state of (2.1) from (x_0,t_0) to $(0,t_f)$. We will now examine the effect and advantage of using a time-varying N over one of constant value. First, let us examine the changes of the norm of u as t varies. From (4.3)

$$d(u^Tu)/dt = 2u^T\dot{u} = 0.25x^TSBB^T(\dot{S}x + S\dot{x})$$

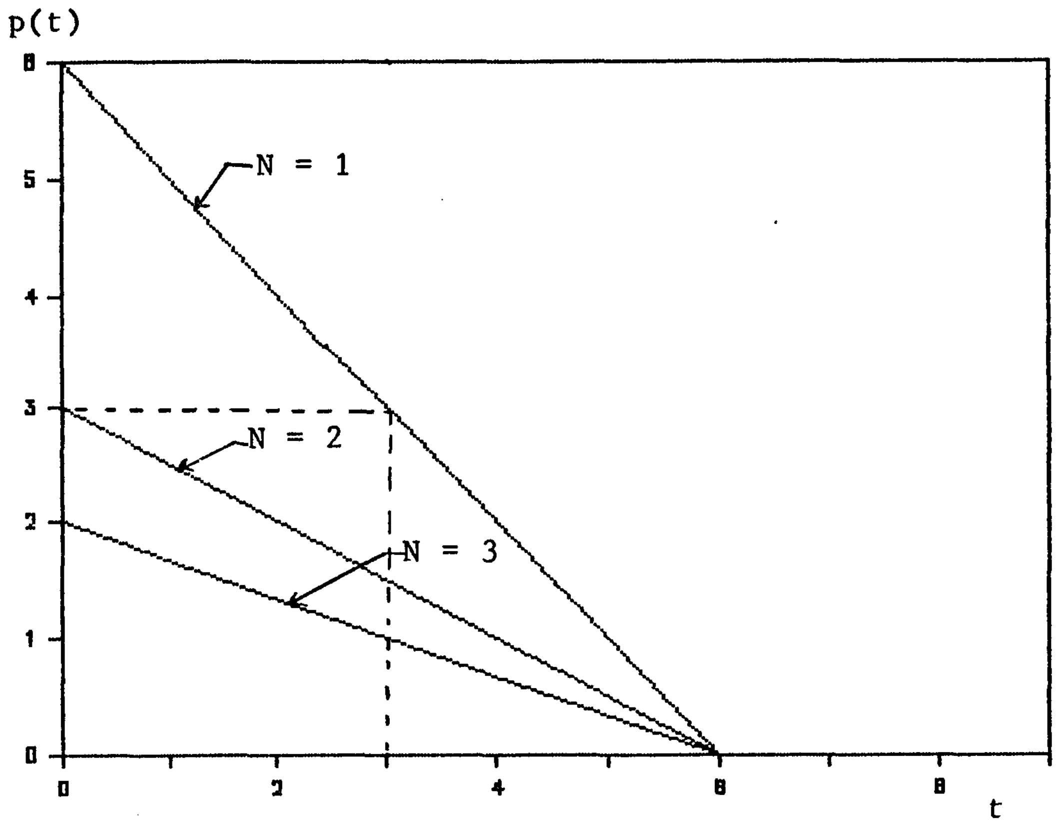
$$= 0.5x^TSBB^T[\dot{S} + 0.5(SA + A^TS - SBB^TS)]x$$

$$= -0.125[N(t) - 2]x^T(\dot{S}BB^TS + SBB^T\dot{S})x \qquad (4.12)$$

Since the determinant of $SBB^T\dot{S} = 0$, then $d(u^Tu)/dt$ can be positive, negative or zero for different instants of time. It is tempting to conclude that for $N(t) = 2$ that u^Tu has a constant value. Simulation studies has shown that this in true only when S is a scalar where there is no cancellation of the (N(t) - 2) term in (4.12). For any given A, B and t_f, the matrix S^{-1} and thus S is completely determined by p_1. Therefore by proper selection of p_1 we can specify both the inital norm of the controller gain and its rate of change for $0 < t < t_f$. This is in effect shaping the state and input norms. When N is a constant, p_1 is given by

$$p_1(t) = (t_f - t)/N \tag{4.13}$$

Fig. 4.1 shows $p_1(t)$ for $0 < t < t_f$ and different values of con stant N. It is obvious that as N increases, the range of $p_1(t)$ decreases. Let us consider an unstable A matrix. To examine the advantage of a time-varying N over one of constant value, we will first investigate the effect of different constant values of N on the state trajectory and input norms. First consider a constant N of small value. From (4.13) we conclude that $p_1(0)$ will be large in comparison to large values of constant N and very small values of p_1 occur at a much later time than when large values of constant N are used. Since S^{-1} is inversely proportional to S, from (4.7) we conclude that the gain for all $0 \leq t < t_f$ for small values of constant N will be smaller than those of larger values of constant N. Thus for any fixed $||x(0)||$, an upper bound on $||u(0)||$ will be smaller for small values of constant N than large values of constant N. It has been found that small initial gains result in slow



Time in Seconds

Fig. 4.1 p(t) for N = 1, 2 and 3 with t_f=6 state trajectory movement and thus small excursions in the state components. Thus for small excursions in the state components, a small values of constant N are advantageous. The slow movement in the state trajectory means that for an unstable A matrix the the maximum $||x(t)||$ will occur at a later time for small values of constant N than for large values of constant N. The gain is increasing with t for all values of N. Thus the slow movement of the state trajectory means that through the product of S and x large $||u||$ will occur. This large input will occur in the latter part of the process. Therefore we conclude that small values of constant N result in small inputs and small state excursion in the early part of the process and large inputs in the latter part of the process.

On the other hand large values of constant N result in larger initial gains than smaller values of constant N. It has been found that large initial gains result in fast state trajectory movement and for an unstable A matrix large excursions in the state components. Therefore from (4.3) as N increases an upper bound on $||u(0)||$ increases for fixed $||x(0)||$. Due to the rapid movement of the state trajectory the state will be very close to the origin by the time the gains get very large. Thus in the later part of the process the $||u||$ will be small in comparison to that of small values of constant N. Thus the disadvantages of large constant values of N occurs early in the process while the advantages occurs in the latter part of the process.

Hence to compromise between input and state components excursions, one must take advantage of the desired qualities of large and small values of N and stay away from their bad effects. To accomplish this result we use an N that varies with time. A very good choice of N(t) is one whose values are close to one up to the point where the maximum norm of the state occurs and then increases to keep the input small by speeding up the state trajectory. But since N(t) must be known before S(t) is known, there is no way of changing N(t) by measurement of the state. A simple algorithm for generating N(t) is a follows: If A is a stable matrix then fast state trajectory is expected for all N(t), thus set N(t) = 1 to limit maximum $||x||$. If A is an unstable matrix then set $1 \leq N(t) \leq 2$ since $N(t) \geq 1$ for stability and N = 2 from many simulations seems to be the smallest of large constant values of N which possess good properties in the latter part process. Fig. 4.2 shows a typical case. Let $p_1(t)$ be given by

$$p_1(t) = a_d t^d + a_{d-1} t^{d-1} + \dots\dots\dots + a_1 t + a_0 \qquad (4.14)$$

Then $$N(t) = -1/(dp_1/dt) \qquad (4.15)$$

where a_d and d are constants to be selected. For any arbi-

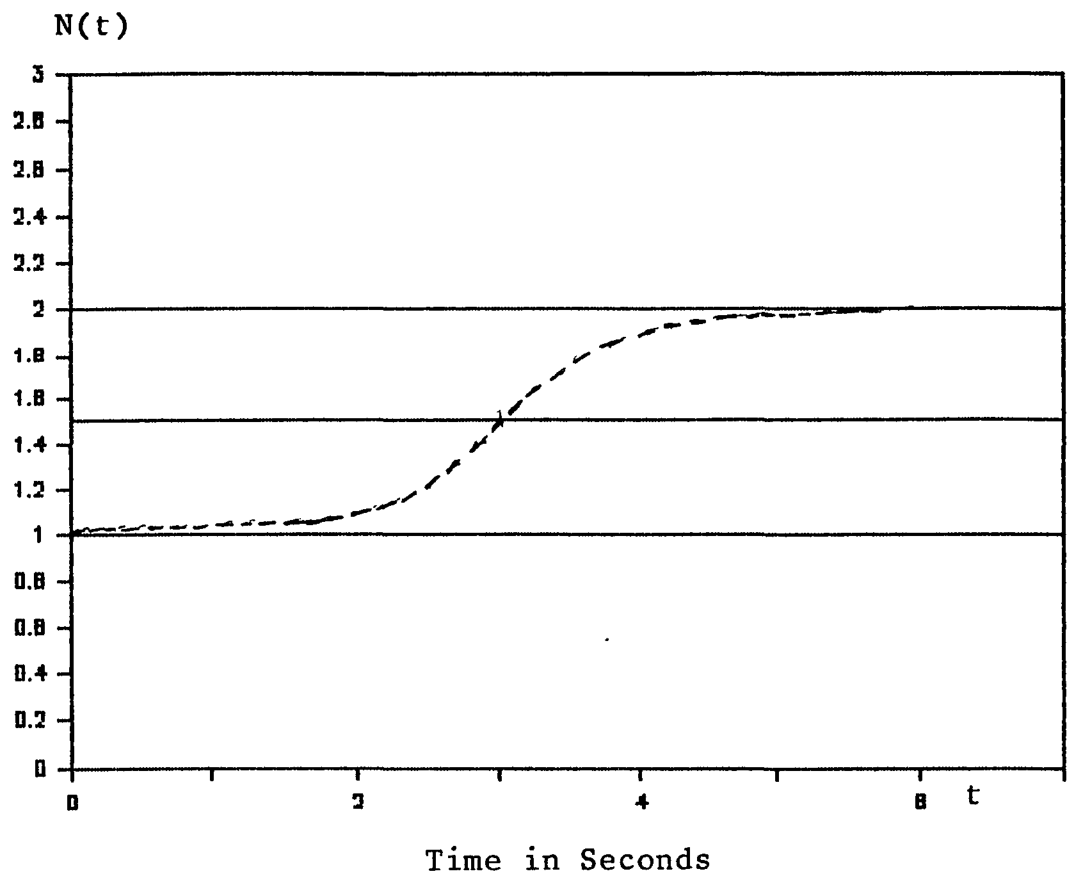


Fig. 4.2 A Typical Trajectory of N(t) t_f=6.

trary d select a_d such that for $t < t_f/2$ $N(t)$ is close to 1 while for $t > t_f/2$ $N(t)$ behaves as $N(t) = 2$ and for $t = t_f/2$ $N(t)$ is equal to or close to 1.5. The following example demonstrates the procedure for selecting an $N(t)$ for eliminating large state and input norms.

<u>EXAMPLE 4.1</u>: Consider the time-invariant linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad (4.16)$$

The eigenvalues of A are 0 and +1. Therefore we expect state and input of large norms in the early part of the process for large values of constant N and small input norms as t approaches t_f. The opposite is true for small constant values of N. The following trajectory was chosen for $p_1(t)$: $p_1(0) = 0.75$, $p_1(t_f/2) = (3/8)t_f$ and $p_1(t_f) = 0$. This results in $p_1(t)$ and $N(t)$ as given by (4.17) and (4.18) for d = 2.

$$p_1(t) = (t^2 - 4t_f t + 3t_f^{\,2})/(4t_f) \qquad (4.17)$$

$$N(t) = 2t_f/(2t_f - t) \qquad (4.18)$$

Figs. 4.3 and 4.4 show the trajectory of the state and input responses for t_f= 2 secs., for $x(0) = [1, 2]^T$ and N = 1, N = 2 and N(t) as given by (4.18). The time varying N does act to reduce the overshoot in the state components in respect to large constant values of N and simultaneously produce the fast response required for small input norms. The trimming of the input norm is clearly seen in Fig. 4.4.

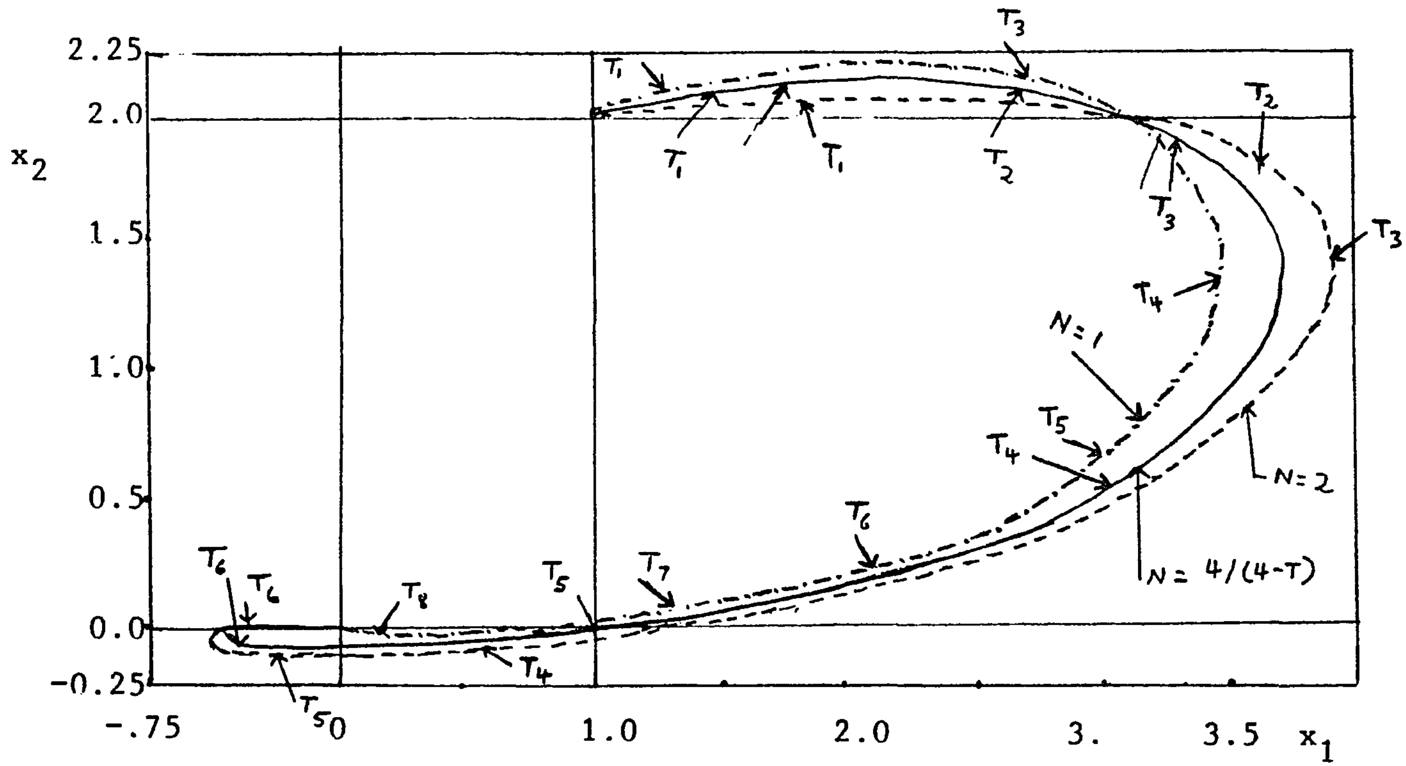


Fig. 4.3 Trajectories of x_1 verses x_2 for different values of N. Relative positions of the trajectories are also shown for the same time interval

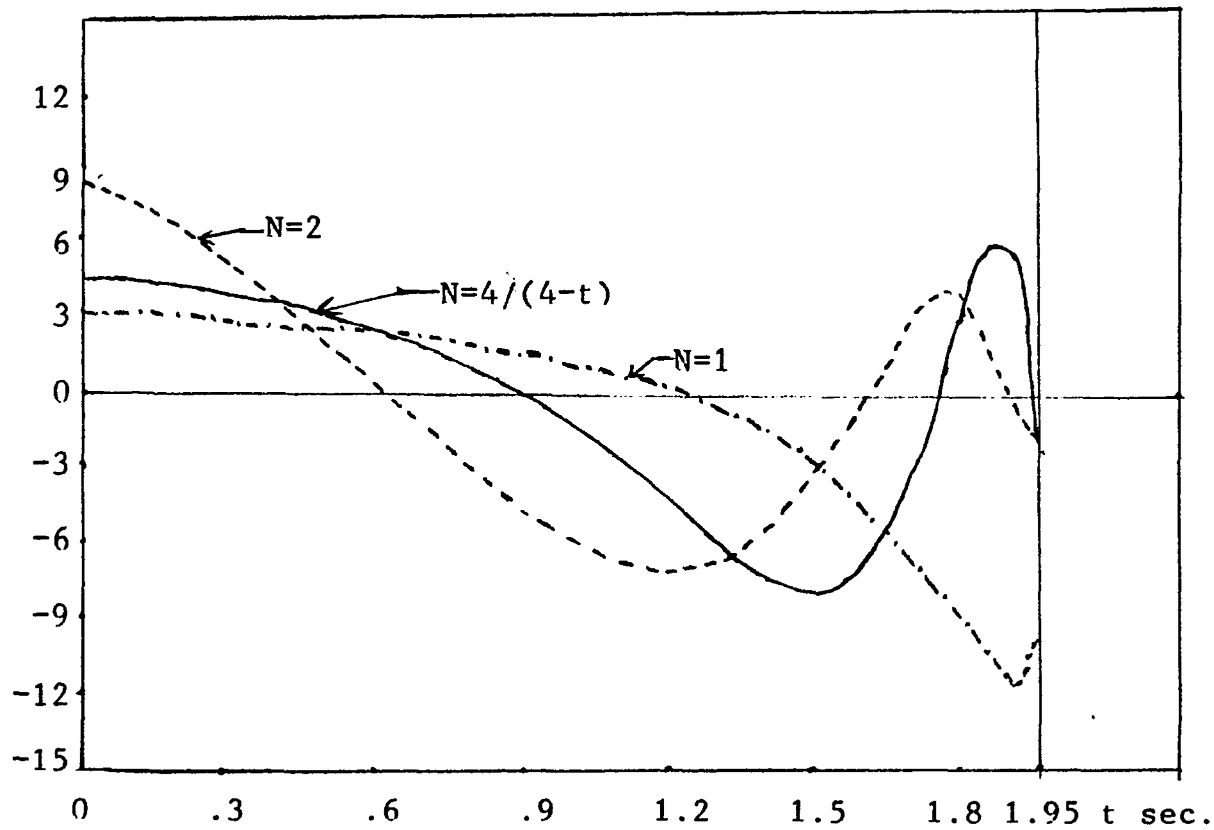


Fig. 4.4 Input as a function of time

4.1.2 THE EFFECT OF t_f ON THE STATE AND INPUT:

In the previous section we examined the effect of N(t) on the state and input magnitude and found a technique of designing N(t) to compromise between state and input magnitudes for any given terminal time t_f. In most design the terminal time is also an important factor. This section examines the effect of t_f on both the input and state magnitudes. The main objective here is to find the smallest t_f which will yield small inputs and state excursions. The procedure takes into consideration that the N(t) used has the qualities described in the last section.

Define F(t) as follows

$$F(t) = -0.5B^T S(t) \tag{4.19}$$

Then from (4.7), for any fixed t_f

$$d(S^{-1})/dt = -N^{-1}(t)\exp(-Ap_1)BB^T\exp(-A^Tp_1) \leq 0 \tag{4.20}$$

And thus $$dS/dt = -Sd(S^{-1})/dtS \geq 0 \tag{4.21}$$

Also for any fixed t_f, p_1 decreases with increasing t for $N(t) > 0$ for all t. Thus increasing t decreases $||S^{-1}||$ and increases $||S||$. And from (4.19) we conclude that if $t_1 < t_2 < t_f$, then $||F(t_1)|| < ||F(t_2)||$. Similarly for every fixed t and $N(t) > 0$ p_1 increases with increase in

t_f. Thus we conclude that the norm of F for a fixed t decreases with t_f. Since the smallest $||F||$ occurs at $t = 0$, we will examine S with respect to t_f at $t = 0$ and $N = 1$. For $t = 0$ and $N = 1$, (4.7) reduces to

$$S^{-1}(t_f) \;=\; \int_0^{t_f} \exp(-Ap)BB^T\exp(-A^Tp)dp \qquad (4.22)$$

which is the solution to the differential equation

$$dS^{-1}/dt_f \;=\; -AS^{-1} - S^{-1}A^T + BB^T \qquad (4.23)$$

with

$$S^{-1}(0) \;=\; 0 \qquad (4.24)$$

Define $||C|| = \sqrt{\text{Trace}(C^TC)}$. We now examine $||S^{-1}||$ and $||S||$ as t_f varies. To determine the effect of t_f on the state trajectory, define

$$M(t_f) \;=\; [A - 0.5BB^TS(t_f)]$$

$$\;=\; [A - BF(t_f)] \qquad (4.25)$$

For any fixed x(0) the state x(t) is completely determined by M. Thus $||M(t_f)||$ will give us a scalar measure of how increasing t_f will affect the state norm. If the real part of the eigenvalues of A are less than zero, then from

(4.22), $||S^{-1}(\infty)|| = \infty$, and thus $||F(\infty)|| = 0$ and $||M(\infty)|| = ||A||$. On the other hand if at least one of the eigenvalues of A has positive real part, then the $||S^{-1}(\infty)||$ approaches a finite limit. Consequently $||F(\infty)||$ and $||M(\infty)||$ approach finite nonzero limits. The most effective t_f is the point on the $||M||$ verses t_f curve where incrasing t_f has little or no effect on $||M||$. This point occurs at the knee of $||M||$ verses t_f curve. Since t_f determines the initial norm of F, this curve determines the smallest gain at $t = 0$ and thus the smallest state excursion achievable without the expense of long terminal time.

The procedure can be summarized as follows: Solve (4.23), (4.24). Select t_f at the point of the $||M||$ verses t_f curve where there is no appreciable change in $||M||$. This gives the best compromise among t_f $||u||$ and $||x||$ for any N.

EXAMPLE 4.2: Consider (2.1) where A and B are given by

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \qquad B = [1,\ 0\ ]^T \tag{4.26}$$

The eigenvalues of A are 0, -1. Thus $||M||$ will approach $||A||$ and $||F||$ will approach zero as t_f goes to infinity. Fig. 4.5 shows the response for $0.5 < t_f < 7$ secs.. The most effective t_f is about 2.5 secs..

EXAMPLE 4.3 In contrast to (4.26) we select

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \qquad B = [1,\ 2]^T \tag{4.27}$$

The eigenvalues of A are $(5 \pm j\sqrt{3})/2$. Thus one expects nonzero limit for $||F||$. This means that the gain can not be arbitrarily small. Fig. 4.6 shows the response. The best t_f is about 1.5 secs.

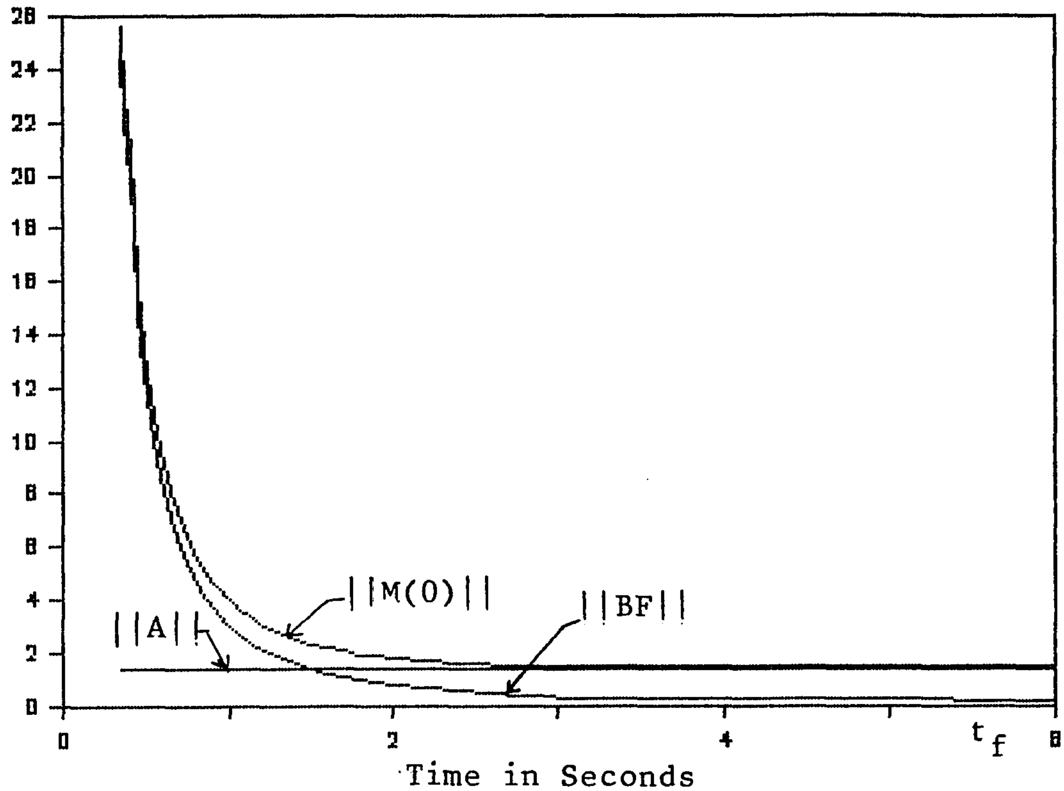
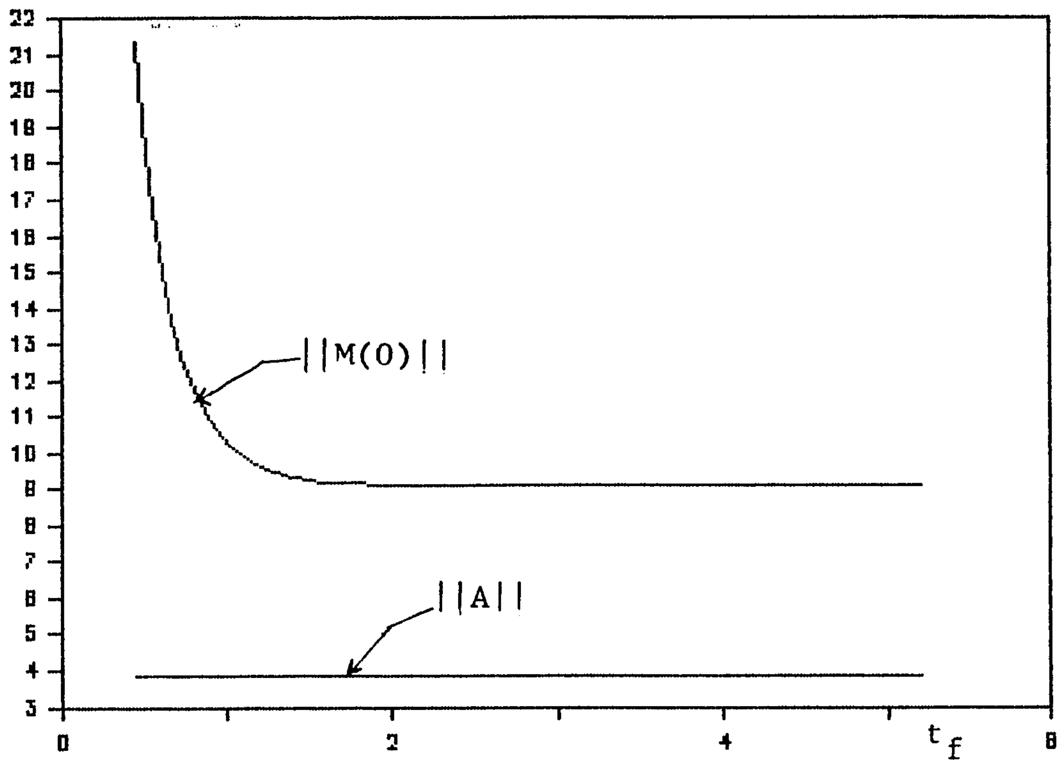


Fig. 4.5 $||A||$, $||M(0)||$ and $||BF||$ verses t_f



Time in Seconds

Fig. 4.6 $||M(0)||$ and $||A||$ verses t_f.

4.2 THE DESIGN OF GAIN-CONSTRAINED LINEAR FEEDBACK CONTROLLERS

A linear observer-controller is developed to implement regulation of a linear system where a priori constraints are specified on the controller and observer gain matrices. The compensator's gain matrices are generated through the solution of matrix differential equations. Conditions are developed to examine whether or not a regulation accuracy specification can be met with a class of constant gain-constrained observer-controllers. Using this class of compensators, the time required to drive the system's state from any given initial state to a prescribed region is estimated. Numerical simulation studies demonstrate the procedure.

It is well known [9], that relatively large controller gains result when a linear state-feedback control law is used to transfer the state of a linear system from its initial state to the zero-state in a finite time. Hence, if a linear feedback controller is restricted to use gains that are constrained by an a priori magnitude bound, the state of the controlled process will be transferred to only a finite neighborhood of the zero-state in finite time. For this gain-constrained linear controller, it is of interest to have a set of relations or a design procedure that explicitly relates a priori controller gain magnitude constraints to the regulation accuracy achievable in finite time by a linear state-feedback control system.

Similarly it is has been shown in chapter 2 that arbitrarily small reconstruction error may result in finite time with linear full-order observers whose gains may be prohibitively large. Again is of interest to characterize the reconstruction accuracy achievable in finite time with gain-constrained observers.

The design of gain-constrained observers and controllers is the subject of the research described below. A gain-constrained state-feedback regulator problem is formulated in 4.2.1. Properties of the time-varying matrix S which is used to generate the gain matrix of the state-feedback control law are described in section 4.2.2. A design procedure for the gain-constrained state-feedback regulator is presented in section 4.2.3 along with a numerical example to illustrate the procedure of finding the required constant gain matrix and estimating the time required to be within a pre-specified region of the process zero-state. In section 4.2.4 the procedure for gain-constrained state-feedback regulator design is extended to the output-feedback regulation of a linear system using a gain-constrained observer-controller. An example of the performance of the resulting output-feedback system is also given in section 4.2.4.

4.2.1 PROBLEM FORMULATION:

Given the linear time-invariant system

$$\dot{x} = Ax + Bu \qquad (4.28)$$

where x and u are of dimensions n and r, respectively, and $x(0) \in R_1$. If a region R_2, containing the zero-state, is specified, where $R_2 \subset R_1$, find (a) the control law

$$u = -F(t)x \qquad (4..29)$$

subject to the constraint

$$||F_{ij}(t)|| \leq F_{max} \quad i= 1..., r, \; j= 1...,n \text{ for all } t \qquad (4..30)$$

and (b) an estimate of the time t_b such that

$$x(t) \in R_2 \subset R_1 \qquad (4.31)$$

for all $t \geq t_b$.

In the work described below we first establish properties of a feedback system (4.28,(4.29) where the controller (time-varying) gain is

$$F(t,t_f,N) = 0.5B^T S(t,t_f,N) \qquad (4.32)$$

and where $S(t,t_f,N)$ satisfies (2.7),(2.10). This class of systems was first studied in [9]. We now summarize some properties of matrix S and relate those to the stability of a class of systems with constant gain feedback control. Then the problem posed above is solved in section 4.2.3.

4.2.2 PROPERTIES OF THE TIME-VARYING S MATRIX:

The following is a summary of some of the properties of the time-varying S matrix which satisfies (2.7),(2.10).

LEMMA 1: If the [A,B] pair of the system (4.28) is completely controllable, then for S satisfying (2.7),(2.10) the following holds

$$(1) \quad S(t,t_f,N) > 0 \qquad (4.33)$$

$$(2) \quad \partial S/\partial t \geq 0 \qquad (4.34)$$

$$(3) \quad \partial S/\partial t_f \leq 0 \qquad (4.35)$$

$$(4) \quad \partial S/\partial N \geq 0 \qquad (4.36)$$

PROOF: Rewrite (2.11) as

$$S^{-1}(t,t_f,N) = \int_0^{(t_f-t)/N} \exp(-A\tau)BB^T\exp(-A^T\tau)d\tau \qquad (4.37)$$

which is positive definite because of the complete controllability assumption. Differentiating (4.37) with respect to t gives

$$\partial S^{-1}/\partial t = -(1/N)\exp[A(t-t_f)]BB^T\exp[A^T(t-t_f)/N] \leq 0 \quad (3.38)$$

But $$\partial S/\partial t = -S[\partial S^{-1}/\partial t]S \quad (4.39)$$

and thus (4.38) we conclude $$\partial S/\partial t \geq 0 \quad (4.40)$$

Similarly

$$\partial S/\partial t_f = -(1/N)S\exp[A(t-t_f)/N]BB^T\exp[A^T(t-t_f)/N]S \leq 0 \quad (4.41)$$

$$\partial S/\partial N = (1/N^2)(t_f-t)S\exp[A(t-t_f)/N]BB^T\exp[A^T(t-t_f)/N] \geq 0 \quad (4.42)$$

From the previous lemma we conclude the following: That the minimum $||K(t,t_f,N)||$ for a fixed t_f and fixed N occurs at t = 0 since $||S(t_1,t_f,N)|| \leq ||S(t_2,t_f,N)||$ for $t_1 < t_2$. Hence from (4.35), the $\min||F(t,t_f,N)|| = ||F(0, \infty, N)||$. Thus (4.35), (4.35) and (4.37) show that the norm of the gain F can be decreased by by setting t = 0 and either increasing t_f or decreasing N. The gain given by (4.32) was shown in chapter 2 to drive the state of (2.1) from any initial state to the zero state in time t_f

for $N \geq 1$. We next show that the use of constant gain control (4.29), (4.32) for $t = t_1 < t_f$ and any positive N results in an stable closed-loop system.

LEMMA 2: The control law given by

$$u(t) = -F(t_1,t_f,N)\ x(t) = -0.5B^TS(t_1,t_f,N)x(t) \quad (4.43)$$

where $S(t_1,t_f,N)$ is the solution to (2.7),(2.10) at time $t = t_1 < t_f$ produces a stable feedback system when applied to (4.28) where $t_1 < t_f$ and $N > 0$.

PROOF: Consider the Lyapunov-like function

$$V(x,t) = x^TS(t_1,t_f,N)x \quad (4.44)$$

Since $S(t,t_f,N)$ is positive definite for all $t < t_f$ and $N > 0$,

$$V(x,t) > 0 \quad (4.45)$$

Differentiate (4.44) along the trajectory of (4.28) using (4.43) and (2.7) to give

$$\dot{V}(x,t) =$$

$$x^T[S(t_1,t_f,N)A+A^TS(t_1,t_f,N)-S(t_1,t_f,N)BB^TS(t_1,t_f,N)]x$$

$$= \quad -Nx^T \dot{S}(t,t_f,N)\Big|_{t=t_1} x \leq 0 \qquad (4.46)$$

This lemma shows that unlike the case discussed in [9] where an $N = 1$ is required for the system state to reach the origin in finite time t_f when a time-varying gain matrix is used, here it is seen that even a value of $0 < N < 1$ can produce stability when a constant gain is used.

4.2.3 THE GAIN-CONSTRAINED REGULATOR:

The following section investigates the solution to the problem (4.28) - (4.31) outlined in the previous section. We have noticed that the properties of the S matrix are completely specified by the values of t_f and N. We now seek to find values of N and t_f such that (4.30) can be satisfied for a given F_{max}. Since the problem has no restriction on the maximum magnitude of the state allowed, we use a constant gain matrix and since the gain of smallest norm for every fixed t_f and N occurs at $t = 0$, we seek t_f and N such that

$$\max|F_{ij}(0,t_f,N)| \leq F_{max} \qquad (4.47)$$

For $t = 0$, (4.37) becomes

$$S^{-1}(0,t_f,N) = \int_0^{t_f/N} \exp(-A\tau)BB^T\exp(-A^T\tau)d\tau = S^{-1}(t_f/N) \quad (4.48)$$

For $p = t_f/N$, (4.48) becomes

$$S^{-1}(p) = \int_0^{p} \exp(-A\tau)BB^T\exp(-A^T\tau)d\tau \quad (4.49)$$

This is the solution to

$$dS^{-1}/dp = -AS^{-1} - S^{-1}A^T + BB^T \quad (4.50)$$

$$S^{-1}(0) = 0 \quad (4.51)$$

The problem reduces to finding a finite p* such that

$$\max_{i,j}|F_{ij}(p^*)| \leq F_{max} \quad (4.52)$$

Since $||S^{-1}(p)||$ increases with p, $||S(p)||$ decreases with p, and thus if

$$\max_{i,j}|F_{ij}(\infty)| < F_{max} \quad (4.53)$$

then there will exists a finite p* to satisfy (4.52). On the other hand if the prescribed F_{max} is such that

$$\max_{i,j} |F_{ij}(\infty)| \quad \geq \quad F_{max} \tag{4.54}$$

then there will be no constant gain feedback control law of the form (4.43) obtained through the solution of (2.7), (2.10) that will also satisfy the gain constraint (4.30).

A numerical procedure for finding a gain constrained state-feedback regulator control law (4.43) is as follows:

(1) Numerically integrate (4.50), (4.51) for $0 < p < p_a$ to obtain a non-singular $S^{-1}(p_a)$.

(2) Invert $S^{-1}(p_a)$ to obtain $S(p_a)$.

(3) Solve for F(p) using

$$F(p) \quad = \quad 0.5B^TS(p) \tag{4.55}$$

(4) If $\max|F_{ij}(p_a)| < F_{max}$, then numerically solve

$$dS/dp \quad = \quad SA + A^TS - SBB^TS \tag{4.56}$$

backward in p starting from $S(p_a)$ until (4.52) is satisfied. This would result in the gain of largest normm that satisfies the constraint (4.52). Otherwise solve (4.56) forward in p. If p* is the first time such that (4.52) is satisfied then F(p*) is the required constant gain matrix.

(5) If the solution to (4.56) reaches a steady-state $S(\infty)$

such that (4.54) is satisfied, then there is no solution of the form (4.55) satisfying the gain constraint.

Note from (4.49) that when all the eigenvalues of A are inside the left half plane, then

$$\lim_{p \to \infty} S^{-1}(p) = \infty \tag{4.57}$$

and thus

$$S(\infty) = 0 \tag{4.58}$$

and from (4.55)

$$F(\infty) = 0 \tag{4.59}$$

and thus the problem of the gain-constrained regulator always has a solution of the form (4.43) for a stable A matrix. On the other hand when some of the eigenvalues of A are inside the right half plane then as p approaches infinity, $S^{-1}(p)$ will approach some finite value. Thus S will be nonzero, and from (4.55) F will be nonzero. Hence for unstable matrix A, numerical solution is required to test whether a solution of the form (4.43) satisfying a given constraint F_{max} exists.

Once the required F is found, the time t_b can be estimated by examination of the eigenstructure of the [A - BF] matrix. The closed-loop system becomes

$$\dot{x} = [A - BF]x \tag{4.60}$$

whose solution is given by

$$x(t) = \exp[(A - BF)t]x(0) \tag{4.61}$$

Suppose the matrix [A - BF] has m distinct characteristic values λ_i, i = 1,2,..,m. Let the multiplicity of each characteristic value in the characteristic polynomial [A - BF] be given by m_i. There exists a transformation T [20] such that

$$[A - BF] = TJT^{-1} \tag{4.62}$$

and
$$\exp[(A - BF)t] = T\exp(Jt)T^{-1} \tag{4.63}$$

where J is a matrix in the Jordan form. We now find a bound on $||\exp(Jt)||$ as follows: Let λ_{max} satisfy Re λ_i < λ_{max} < 0. Since [A - BF] is a stable matrix all it eigenvalues have negative real parts, then it can be shown (see Appendix B) that

$$||\exp(Jt)||^2 = \sum_{i=1}^{m} ||\exp(J_i t)||^2 \le c_1{}^2\exp(2\lambda_{max}t) \tag{4.64}$$

where C_1 is given by

$$C_1 = [\sqrt{n} + \sum_{i=1}^{m} \sum_{r=1}^{m_i-1} \sqrt{r}\ [(m_i-r)/[(\lambda_{max}-Re\lambda_i)e]]^{(m_i-r)}/(m_i-r)!] \tag{4.65}$$

Let R_2 be the region such that

$$||x(t)|| \leq x_{min} \tag{4.66}$$

Thus to satisfy (4.66) we require that

$$||x(t)|| \leq ||T||\,||T^{-1}||C_1 \exp(\lambda_{max}t)||x(0)|| \leq x_{min} \tag{4.67}$$

Solving (4.67) we find the required t_b is found to satisfy

$$t_b \geq [-\lambda_{max}]^{-1} \ln[||x(0)||C_2/x_{min}] \tag{4.68}$$

where C_2 is given by $\quad C_2 = ||T||\,||T^{-1}||\,|C_1|| \quad$ (4.69)

EXAMPLE 4.4: Consider the system given by (2.1) where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.70}$$

The eigenvalues of A are -1 and +1. Thus the open-loop system is unstable. It is obvious that the [A,B] pair is completely controllable. Let F_{max} be given as 3 while R_2 is specified by $||x(t)|| \leq 10^{-3}||x(0)||$. Steady-state solution to (4.56) leads to

$$S(\infty) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \tag{4.71}$$

From (4.55) $\qquad F(\infty) = [1, 1] \qquad$ (4.71)

Since $F_{max} \geq 1$, solution to (4.50),(4.51) leads to $p^* = 1.26$ and $F = [3.00, 1.928]$. Then

$$A - BF = \begin{bmatrix} 0 & 1 \\ -2 & -1.928 \end{bmatrix}$$

Since the eigenvalues of [A - BF] are distinct, the values of C_2 is found simply to be equal to 2.934. Thus by (4.68) $t_b \geq 8.282$. The system was simulated with $F = [3.00, 1.928]$ and $x(0) = [2, -1]^T$. It is found that the time required to reach R_2 is 6.7 seconds whereas the calculated $t_b \geq 8.282$ secs. indicates only that the state will never leave R_2 after 8.282 seconds. Fig. 4.7 shows the state trajectory.

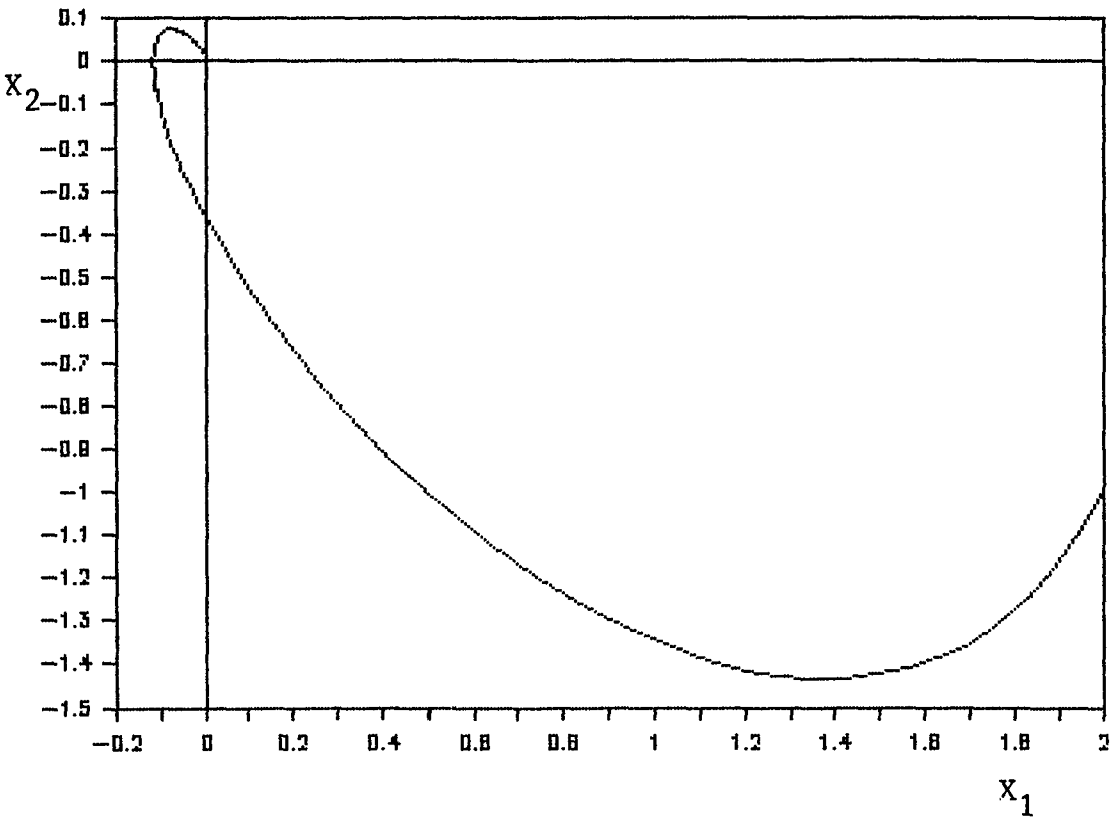


Fig. 4.7 Trajectory of $X_1(t)$ and $X_2(t)$

4.2.4 THE OUTPUT-FEEDBACK REGULATOR:

The controller design procedure of the last section can be easily extended for the case where the complete state is unavailable for direct measurement. This section deals with the design of a dynamic state estimator and the form of the output-feedback control system where there are constraints on the both the controller and observer gain matrices. In addition it gives an estimate of the time required to bring the state estimation error from any initial value to a given fraction of its initial norm and also the time required to reach a prescribed region enclosing the zero-state of the system. Since the differential equation used to generate the observer's gain is the dual of the differential equation used to generate the controller's gain matrix, a detailed treatment of its properties is not given here.

Consider the system given by (2.1),(2.14). A state estimator for (2.1),(2.14) is given by (2.15). The estimator error equation is given by (2.19).

LEMMA 3: If the [A,C] pair of (2.19) is completely observable, then for (2.23),(2.25) the following holds

(1) $$W^{-1}(t,t_f,M) > 0 \qquad (4.72)$$

(2) $$\partial W^{-1}/\partial t \geq 0 \qquad (4.73)$$

$$(3) \qquad \partial W^{-1}/\partial t_f \quad \leq \quad 0 \qquad (4.74)$$

$$(4) \qquad \partial W^{-1}/\partial M \quad \geq \quad 0 \qquad (4.75)$$

The proof is analogous to that of lemma 1 and is thus omitted. Let the observer gain be given by (2.28).

<u>LEMMA 4</u>: If the [A,C] pair is completely observable, then the gain given by

$$K = 0.5W^{-1}(t_1,t_f,M)C^T \qquad (4.76)$$

where $W(t_1,t_f,M)$ is the solution to (2.23),(2.25) at time $t = t_1 < t_f$ produces stability of the error in (2.19). Again the proof is analogous to lemma 2 and is thus omitted.

Although x(0) and e(0) are unknown, the region $\bar{R}_1$ of possible initial observation errors is assumed known as is the region R_1 of initial states x(0). The regulation problem can be reformulated as follows: Let $\bar{R}_2$ be specified as

$$||e(t)|| \leq D||e(0)|| \qquad (4.77)$$

where $D > 0$ is an a priori parameter. Then given the time-invariant system (2.1),(2.14) where $e(0) \in \bar{R}_1$ and $\bar{R}_2 \subset \bar{R}_1$, find (a) the gain K subject to

$$|K_{ij}| \leq K_{max} \qquad (4.78)$$

(b) the time t_{bo} such that (4.77) is satisfied for all $t \geq t_{bo}$, and (c) the feedback control law

$$u(t) = -Fz(t) \tag{4.79}$$

subject to the constraint

$$|F_{ij}| \leq F_{max} \tag{4.80}$$

and (d) the time t_{bc} such that

$$||x(t)|| \leq H||x(0)|| \text{ for all } t \geq t_{bc} \tag{4.81}$$

where H, D, F_{max} and K_{max} are specified parameters. Note that the algorithm for generating F has already been given. In a manner analogous to that described for generating F, we observe that at time t = 0, the solution to (2.23), (2.25) can be written as

$$W(0,t_f,M) = \int_0^{t_f/M} \exp(-A^T\tau)C^TC\exp(-A\tau)d\tau = W(t_f/M) \tag{4.82}$$

Now let $q = t_f/M$. Then (4.82) becomes

$$W(q) = \int_0^{q} \exp(-A^T\tau)C^TC\exp(-A\tau) \tag{4.83}$$

which is the solution to

$$dW/dq = -WA - A^TW + C^TC \quad (4.84)$$

$$W(0) = 0 \quad (4.85)$$

Since the S matrix has dual properties to W^{-1}, we now outline thenumerical procedure for finding a gain-constrained K matrix as follows:

(1) Numerically integrate (4.84),(4.85) for $0 < q < q_a$ to obtain a non-singular $W(q_a)$.

(2) Invert $W(q_a)$ to obtain $W^{-1}(q_a)$.

(3) Solve for K(q) using

$$K(q) = 0.5W^{-1}(q)C^T \quad (4.86)$$

(4) If $|K_{ij}(q_a)| < K_{max}$, then numerically solve

$$dW^{-1}/dq = AW^{-1} + W^{-1}A^T - W^{-1}C^TCW^{-1} \quad (4.87)$$

backward in q. This would result in the largest gain that satisfies the constraint (4.78).

Otherwise solve (4.87) forward in q. If q* is the first time that (4.78) is satisfied, then K(q*) is the required constant observer gain matrix.

(5) If the solution to (4.87) reaches a steady-state solution such that (4.78) is not satisfied, then there

is no solution of the form (4.76) satisfying the gain constraint.

Analogously to the controller design, we also conclude that if the real part of the eigenvalues of A are less than or equal to zero, there will always exist a K such that (4.78) can be satisfied. On the other hand if at least one of the eigenvalues of A has positive real part, a numerical solution is required to test whether or not a solution of the form (4.76) satisfying a given constraint K_{max} exists.

Once the required K is generated, the time t_{bo} is found in analogous manner to t_b to satisfy

$$t_{bo} \geq [-\lambda_{max}]^{-1} \ln[D_2/D] \tag{4.88}$$

where λ_{max} is greater than or equal to the real part of the largest eigenvalue of [A - KC] and D_2 is similarly defined for [A-KC] as C_2 is defined for [A-BF]. To examine the response of the output-feedback system, write z = x - e and substitute (4.79) into (2.1) to yield

$$\dot{x} = (A - BF)x + BFe \tag{4.89}$$

The solution to (4.89) using (2.19) where K(t) is constant and is given by (4.76) becomes

$$x(t) = \exp[(A-BF)t]x(0) +$$

$$\int_0^t \exp[(A-BF)(t-\tau)]BF\exp[(A-KC)\tau]e(0)d\tau \qquad (4.90)$$

Let $C^* < 0$ and $D^* < 0$ be greater than or equal to the real part of the largest eigenvalues of $[A-BF]$ and $[A-KC]$ respectively. Then

$$||x(t)|| \leq C_2\exp(C^*t)||x(0)|| +$$

$$C_2D_2||BF||||e(0)||\int_0^t \exp[C^*(t-\tau)+D^*\tau]d\tau$$

$$= C_2\exp(C^*t)||x(0)|| +$$

$$C_2D_2\exp(C^*t)||BF||||e(0)||[\exp[(D^*-C^*)t]-1]/(D^*-C^*) \qquad (4.91)$$

Assume further thaT $z(0) = 0$ and hence $e(0) = x(0)$. Thus to satisfy (4.81) we require that

$$C_2\exp(C^*t_{bc}) + C_2D_2||BF||\exp(C^*t_{bc})*$$

$$[\exp[(D^* - C^*)t_{bc}] - 1]/(D^* - C^*) \leq H \qquad (4.92)$$

which can be written as

$$H \geq [C_2 + C_2D_2||BF||/(C^* - D^*)]\exp(C^*t_{bc}) +$$

$$[C_2D_2||BF||/(D^* - C^*)]\exp(D^*t_{bc}) \qquad (4.93)$$

<u>EXAMPLE 4.5</u>: Consider the system given by (2.1), (2.14) where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [\ 1,\ 0\ ] \qquad (4.94)$$

In addition let $K_{max} = 20$, $F_{max} = 9$, and $D = H = 10^{-4}$. Note that the open-loop system is unstable. Numerical so-solution of (2.23), (2.25) and (4.87) leads to

$$K(q^*) = K(0.4) = [5.066,\ 19.44] \qquad (4.95)$$

D_2 and C_2 are found to be 4.169 and 1.00 respectively for

$$F(p^*) = F(0.6) = [8.868,\ 3.419] \qquad (4.96)$$

C^* is found to be -1.72 while $D^* = -2.533$ and $||BF|| =$ 9.5043. And from (4.88) $t_{bo} = 4.2$ s. and from (4.92) t_{bc} = 7.5 secs. The system was simulated with $x(0) = [2, -1]^T$ and $z(0) = [0, 0]$. It is found that the first time (4.77) is satisfied is t = 4.22 seconds. Also the first time the state reached the region specified by (4.81) is found to be 6.3 secs. Figs. 4.8 and 4.9 show the response of the observer and state as a function of time.

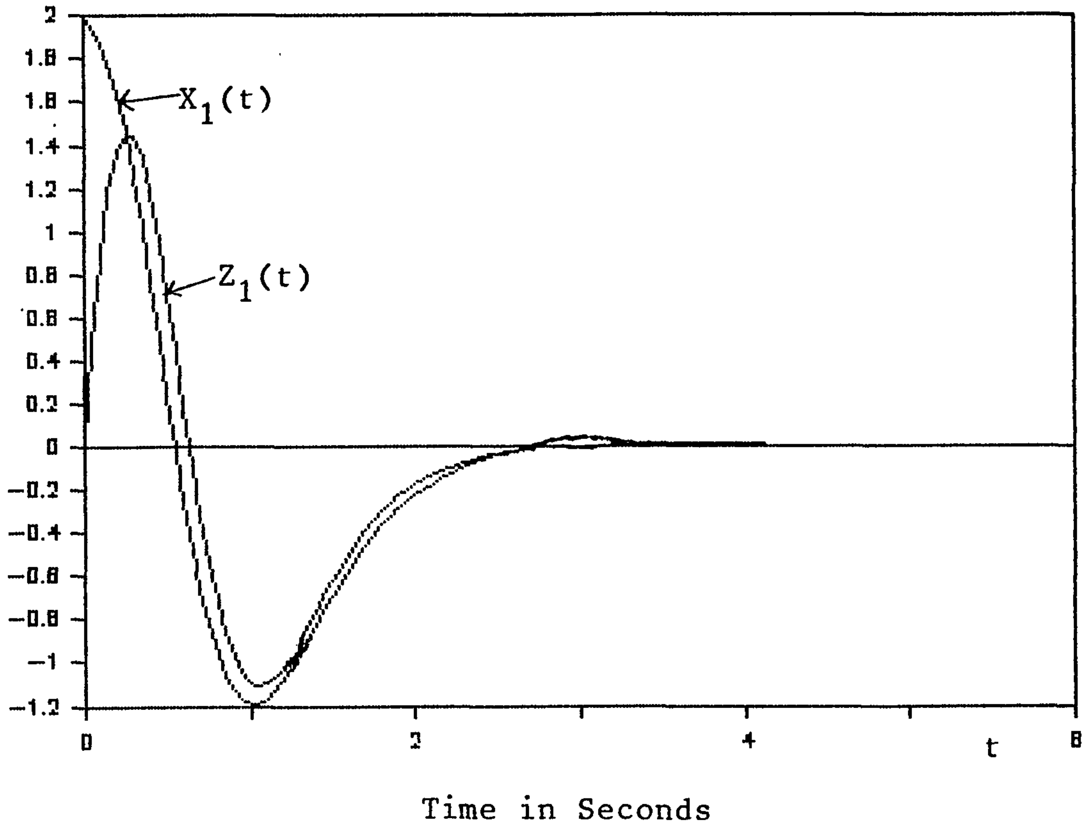


Fig. 4.8 Responses of X_1 and Z_1 as a function of t.

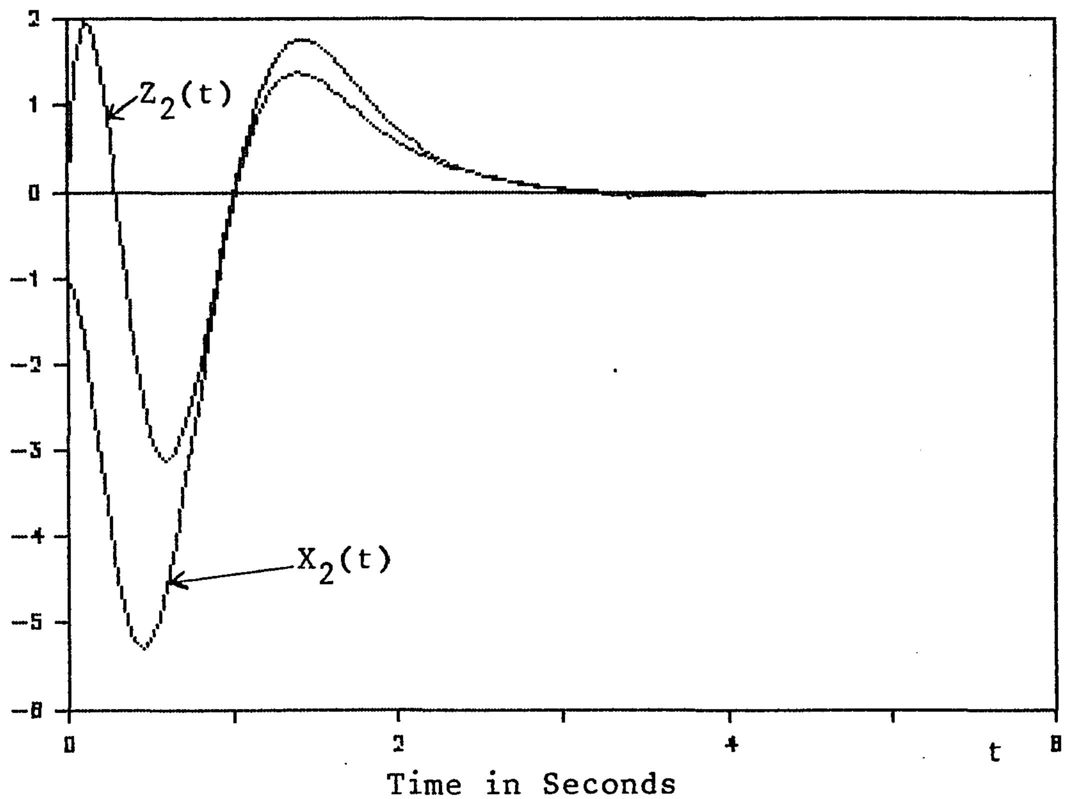


Fig. 4.9 Time Responses of X_2 and Z_2.

The class of gain-constrained observer-controllers designed above to meet regulation and state estimation accuracy requirements did not impose any constraint on the state and input excursions during the control interval. In situations where large transient state and reconstruction error excursions are unacceptable it appears that the use of time-varying observer and controller gains could be employed to satisfy a set of a proiri trajectory constraints. This is the subject of the next section.

4.3 TRADE-OFF TECHNIQUE AMONG STATE COMPONENTS EXCURSIONS INPUT NORM AND SPEED OF RESPONSE UNDER GAIN CONSTRAINT:

The following modified technique shows how to design a state feedback controller which will simultaneously take into consideration constraints on the norm of the input and state vector and the time taken to reach a desired region containing the zero-state given a constraint on the maximum norm of the allowed feedback gain. This method can be considered a form of a 'good' trade-off among input, state and terminal time under a restricted gain norm. In the last section the values of N and M did not play an integral part in the generation of the controller and observer constant gains, ie any N and M greater than zero worked. It will now be shown that when state and input norm restrictions are required, the values of N and M do play an important part.

4.3.1 CONTROLLER DESIGN:

The general objective in any controller design is to develop a feedback control law which will force the state of a system to a desired terminal state along the best possible trajectory, ie, one which will not result in prohibitively large input magnitude or state magnitude. Most designs which utilize a constant gain feedback control law suffer from two main disadvantages. If the magnitude of

the gain is small, the time required to reach a desired state norm is usually very long. On the other hand, if the gain magnitude is increased to reduce the time response, the input and state norms increase significantly. For example, consider the linear system given below in example 4.6.

EXAMPLE 4.6:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad (4.97)$$

Two sets of constant gains were selected such that the poles of the closed-loop system were located at -2.0, -2.5 and -4.0, -5.0 respectively. This resulted in constant gain norms of 6.02 and 21.0 respectively. Figs 4.10, 4.11 and 4.12 show the time responses for the state components

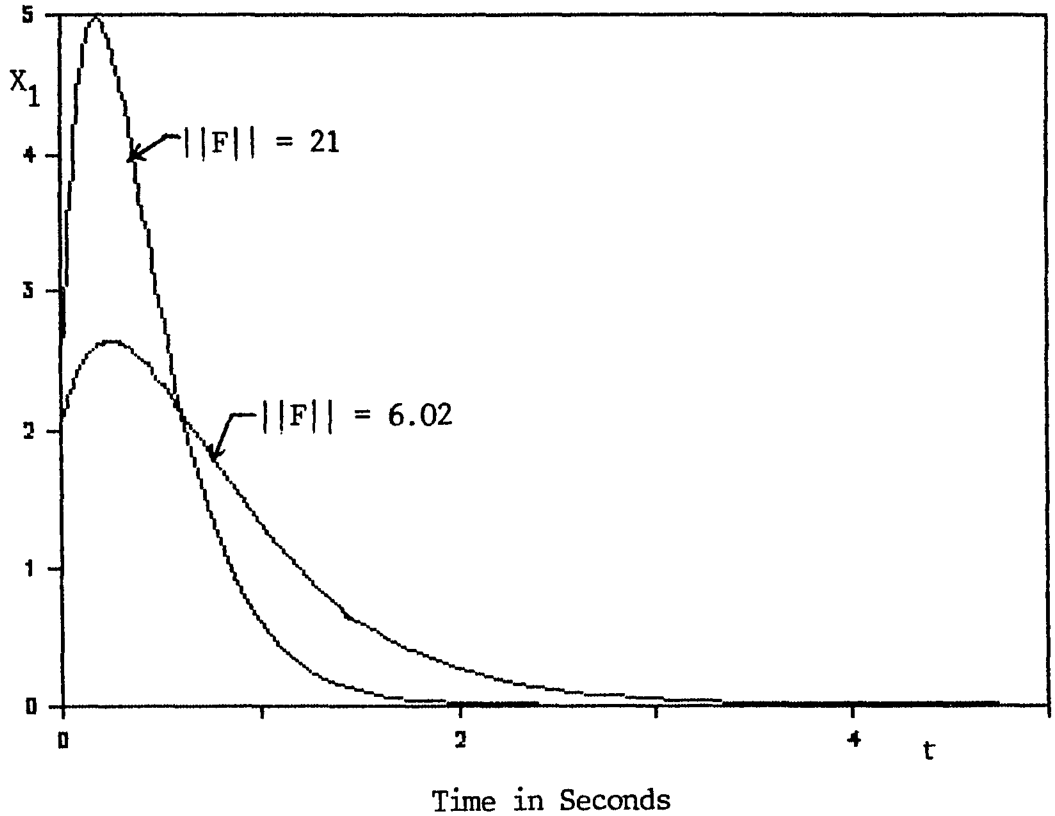


Fig. 4.10 X_1 as a function of time

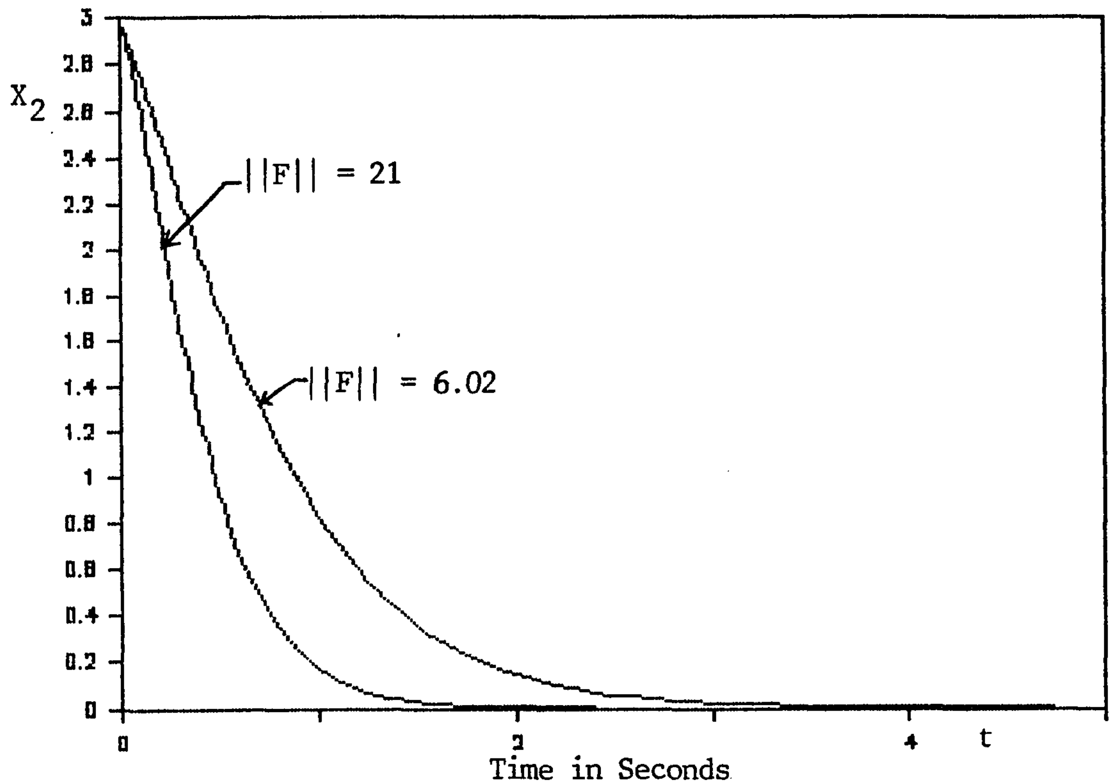


Fig. 4.11 X_2 as a function of time.

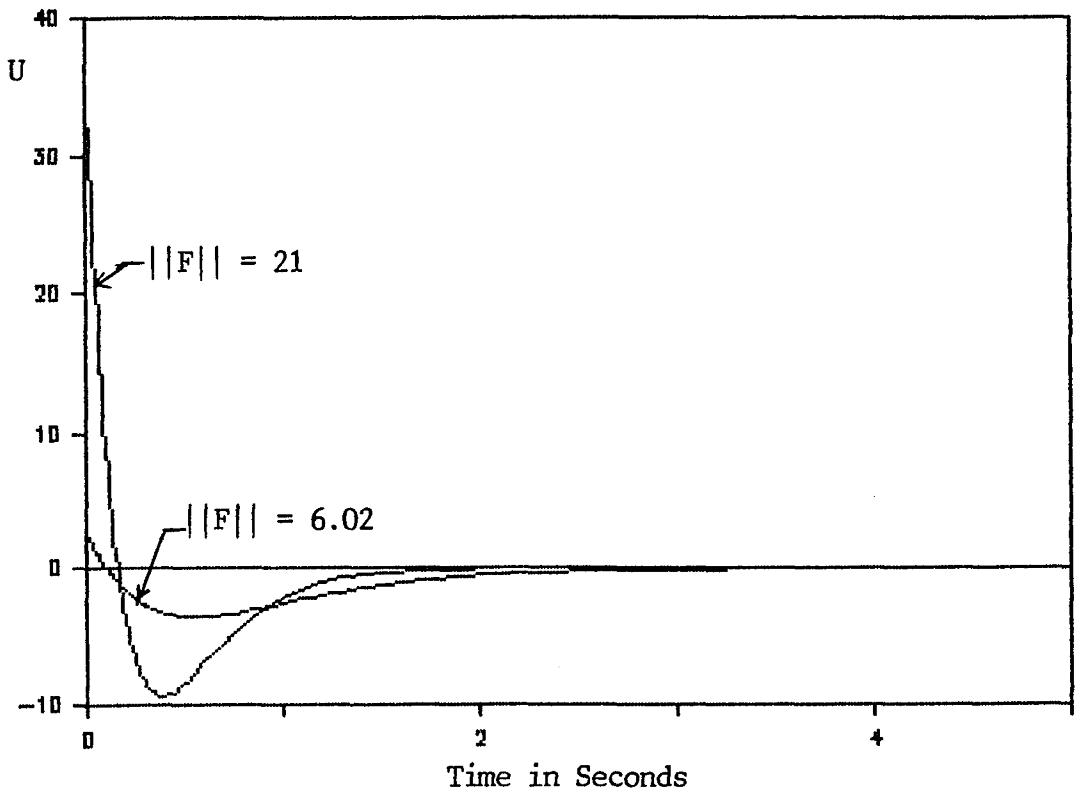


Fig. 4.12 U as a function of time.

and input. The time taken to reach a factor of 10^{-3} of the initial state norm was 4.36 secs. and 2.38 secs. respectively. Note the large overshoot with large gain.

What one needs is a feedback control law whose gain varies with time in such a manner that gains of large norms do not affect the input and state norm drastically, but serve mainly to decrease the speed of response. We now describe a technique which will take advantage of the good qualities of the transient response resulting from a gain of small norm, ie, small input norm and decreased state component excursion, and the good qualities of the transient response resulting from a gain of large norm, ie, fast system response time.

<u>**PROBLEM STATEMENT:**</u> Given the time-invariant continuous time system (2.1). Find (a) the control law

$$u = \begin{cases} -F(t)x & 0 \le t \le t_r \\ -F(t_r)x & t > t_r \end{cases} \tag{4.98}$$

where F and S satisfy (2.6) and (2.9), (2.10) respectively subject to

$$||F(t)|| \le F_m \quad 0 < t < t_r \tag{4.99}$$

where F_m is a given gain constraint and N, t_r and t_f are constants to be found in a manner such that one accomplishes a good compromise among state excursions, input magnitude and speed of response and (b) the time t_b such that

$$||x(t)|| < x_d \qquad (4.100)$$

for all $t > t_b$ where x_d is a given bound that specifies a desired region including the zero-state.

<u>SOLUTION</u>: Since A and B are given the problem outlined above reduces to finding N and t_f such that (4.99) can be satisfied. Different values of N and t_f generate different gain time functions of gains for $t < t_f$. Even for an arbitrary N, (2.9) and (2.10) must be solved for different values of t_f. This results in a time consuming computational process. We now describe an alternate algorithm for generating F(t) through F(p) as define by (4.55), ie keeping t and N constant and varying t_f. It will be shown that t_r is directly proportional to N, while N can be selected based upon the eigenvalues of A.

Comparing (2.9) and (2.10) with (4.50) and (4.51) shows something very interesting. As p varies from zero to infinity, the $||F(p)||$ verses p is identical to the $||F(t)||$ verses t as t varies from to infinity to zero with $S^{-1}(\infty) = 0$. Since (2,9), (2.10) is impractical to solve for t_f equal to infinity, we must use (4.50),(4.51). Note also that (4.50), (4.51) generates all possible gain time functions that produce finite time responses in (2.1) and the gains generated must be used in the reversed time order in the control law (4.98). That is the gain must be stored.

The following discussion shows how to generate the gain time function F(t) without directly solving for t_f. This will be achieved by solving (4.50),(4.51) and relating the function $\hat{F}(p)$ to be defined to F(t).

Let us define M(p) as follows

$$M(p) = [A - B\hat{F}(p)] \tag{4.101}$$

where

$$\hat{F}(p) = 0.5B^T S(p) \tag{4.102}$$

Then

$$dM(p)/dp = -B^T dF(p)/dp \tag{4.103}$$

From (4.98) the closed-loop system (2.1) can be written as

$$\overset{\circ}{x} = [A - BF(t)]x \tag{4.104}$$

For any given initial state, the trajectory of (4.104) is completely specified by [A - BF(t)]. Since M(p) and [A - BF(t)] will be directly related, we conclude that M(p) completely specifies the trajectory of (4.104). From (101), since A and B are constant matrices, $||M||$ will behave similar to $||\hat{F}||$. Thus $||M||$ can be used as a scalar measure of the effect of p on both the system state excursion and input norm. $||S^{-1}||$ is monotonically increasing while $||M||$ is monotonically decreasing with increase in p. A typical response for $||M||$ is shown in Fig. 4.13. The curve can be divided into basically three regions A, B and C as shown in the figure. In region A, one observes rapid changes in $||M||$. Since the gains generated are actually used in reversed time order, if the $||x||$ is not very small in this region large input magnitudes will occur. Thus in order to limit large input norms, it best to avoid this region or ensure that state of small norm accompany this region. Region C shows that further increase in p has little effect on $||M||$. Since this region affects $||F(t)||$ for t = 0, it determines the state components maximum excursions. Region B is the region where the best compromise among input norm, state excursion and speed of response occurs since no rapid changes in $||M||$ or extremely slow changes in $||M||$ occur there.

Let p_1 and p_2 be the times when gains of norms $\hat{F}_1$ and $\hat{F}_2$ occur where $\hat{F}_1$ and $\hat{F}_2$ represent the norms of largest and smallest gains respectively in region B. Then from (4.49) it is clear that for fixed $N = N_1$, the time interval $p_2 - p_1$ in the p domain would be $N_1(p_2 - p_1)$ in the t_f domain.

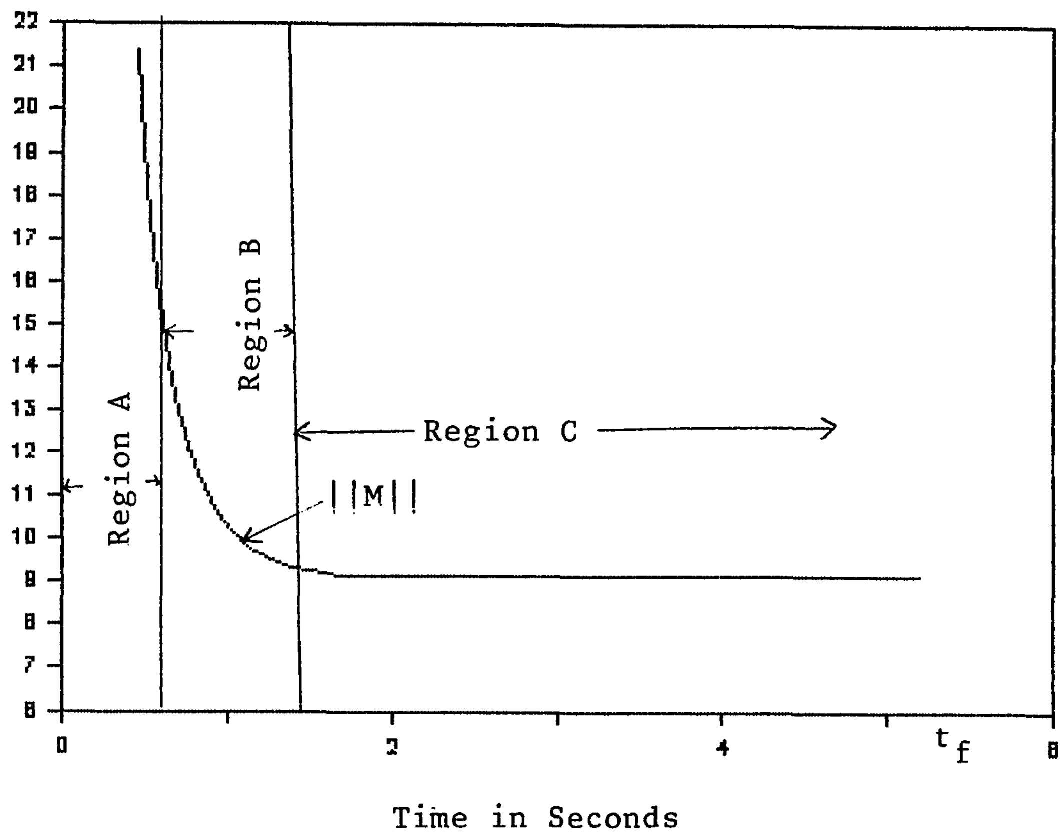


Fig. 4.13 Typical Response of ||M(0)||.

Thus we conclude that if (4.50), (4.51) was solved in terms of t_f and constant N, the interval between the occurences of gains of fixed norms can be controlled by the value of N used. Note that the shortest interval between gains of fixed norms occurs when N = 1. But for N = 1, the state trajectory is moving the slowest and thus for unstable A matrices very large input norm may result. Thus the value of N must be selected in accordance with the structure of the A matrix. The real part of the smallest eigenvalue of A also affect the numerical solution of (4.50),(4.51). If the difference between the real part of the largest eigenvalue of A and the smallest eigenvalue is great, then some components of S^{-1} may be very small, and thus affect the numerical stability of the solution of (4.50), (4.51). Therefore for computational purposes, the value of N selected must depend not only on the real part of the largest eigenvalue of A but also on the real part of the smallest eigenvalue of A.

The design procedure for generating the required F can be stated as follows: Let α and β be the real part of the largest and smallest eigenvalues of A respectively.

(a) Solve for α and β. If $\alpha \le 0$ set $N = 1 + (\alpha - \beta)/10$ else set $N = 1 + \alpha$.

(b) Using the value of N from (a) numerically solve

$$dS^{-1}/dt_f = -(A/N)S^{-1} - S^{-1}(A^T/N) + BB^T/N \qquad (4.105)$$

$$S^{-1}(0) = 0 \tag{4.106}$$

up to a nonsingular solution.

(c) Invert S^{-1} and switch to (4.107) given below

$$dS/dt_f = S(A/N) + (A^T/N)S - SBB^TS/N \tag{4.107}$$

$$\hat{F}(t_f) = 0.5B^TS \tag{4.108}$$

(d) Solve (4.107) up to the first time $||\hat{F}(t_f)|| \leq F_m$. Denote this time as t_1. Continue the solution of (4.107) storing all subsequent gains using (4.108) until there is little or no change in $||\hat{F}(t_f)||$ or $||M||$. Denote the value of t_f when the last gain is stored as t_2. Then t_r is equal to $t_2 - t_1$. and set $F(0) = \hat{F}(t_2)$ and $F(t_r) = \hat{F}(t_1)$. If the first time the gain constraint is satisfied occurs in a region where no appreciale change occurs, set $t_r = 0$ and $F(0) = \hat{F}(t_1)$. This means a constant gain control law will be used. Thus F(t) is given by

$$F(t) = \begin{cases} \hat{F}(t_2-t) & 0 \leq t \leq t_r \\ \hat{F}(t_1) & t \geq t_r \end{cases} \tag{4.109}$$

(e) If the steady state solution of (4.107) does not satisfy the gain constraint, then this technique can not be used to meet the design procedure or the gain magnitude constraint can not produce asymptotic stability in (2.1).

Once the gains are found and stored, an estimate of the time t_b can be found. The closed-loop form of (2.1) using (4.98) is now given by

$$\dot{x} = \begin{cases} [A - BF(t)]x & 0 \le t \le t_r \\ [A - BF(t_r)]x & t > t_r \end{cases} \tag{4.110}$$

where t_r is the time interval for the time-varying gain before a constant gain control law is used. The relationship between t_b and t_r is highly dependent on $x(0)$, F_m and x_d. And thus t_r may be greater than or less than t_b. Since there is no simple closed-form solution to (4.110) for $0 < t < t_r$, some form of estimate must be used for $||x||$. Since N is selected proportional to α, the interval $[0, t_r]$ will be sufficient long such that even for large F_m, the rate of change of $||F(t)||$ with respect to t will be very small. Divide the interval $[0, t_r]$ into q parts and assume the gain is held constant in each interval. Thus the gain F(t) is estimated as $F(0), F(t_r/q), \ldots, F[(p-1)t_r/q]$. An estimate of $x(t_r)$ is now given by

$$\hat{x}(t_r) = \prod_{i=0}^{q-1} \exp[(A - BF(it_r/q))t_r/q]x(0) \tag{4.111}$$

Using the estimation technique developed in the section on gain-constrained controller design for finding a bound on $||\exp(A-BF)||$ yields

$$||x(vt_r/q)|| \leq \prod_{i=0}^{v-1} C_i \exp[\lambda_{max}(A-BF(it_r/v))t_r/q]\,||x(0)|| \tag{4.112}$$

where $v = 1,\ldots,q$ and λ_{max} is the real part of the largest eigenvalue of and $C_i > 0$. The smallest value of v for which the right side of (4.111) is less than x_d determines t_b, ie, $t_b \geq v^* t_r/q$ where v^* is the smallest value of v for which (4.100) is satisfied. If there exists no v for which (4.100) is satisfied then $t_r < t_b$. For $t > t_r$, the solution to (4.110) is given by

$$\hat{x} = \exp[(A - BF(t_r))(t-t_r)]x(t_r) \tag{4.113}$$

And thus to satisfy (4.100) we require that

$$||x(t)|| \leq C_1 \exp[\lambda_{max}(A-BF(t_r))(t-t_r)]\,||x(t_r)|| \leq x_d \tag{4.114}$$

where C_1 is defined analogously to C_i of (4.112). Solving (4.114) yields

$$t_b \geq t_r + (1/\lambda_{max}(A-BF(t_r))\ln[x_d/(C_1||x(t_r)||] \quad (4.115)$$

where $||x(t_r)||$ is given by (4.111) when $v = q$.

Consider example 4.6 where this time the gain used is time-varying. This example is given to compare the response of the system (2.1) to a time varying as opposed to a constant gain control law. The eigenvalues of A are $\pm j$. Since $\alpha = 0$, we set $N = 1$. In order to compare the new algorithm with example 4.3, we set $F_m = 21$. Application of the algorithm on page 123 results in $F_1 = 5.045, F_2 = -18.88$ at $t_f = 0.4$ secs. and minimum trade-off gain $F_1 = 0.75$ $F_2 = -0.26$ at $t_f = 2$ secs.. This results in $t_r = 2 - 0.4 = 1.6$ secs. The system was simulated with $x(0) = [2, 3]^T$. Figs. 4.14 - 4.16 show the state and input responses. t_b was found to be 3.29 secs. to reach a factor of 10^{-3} of the initial state norm.The state excursion are very much that of a small constant gain while the time response is closely related to that of a gain of large norm. Therefore we conclude that using appropriate time-varying gains, can reduce the time response markedly without the expense of large state excursions and input norms.

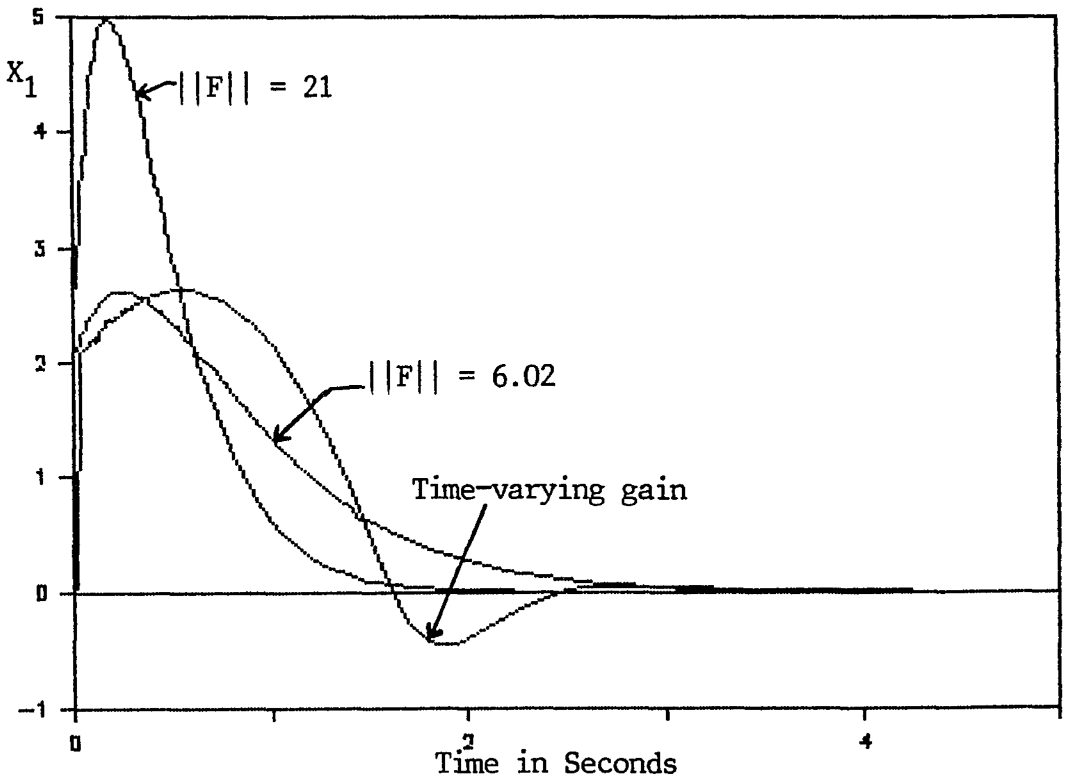


Fig. 4.14 X_1 as function of time Different gains.

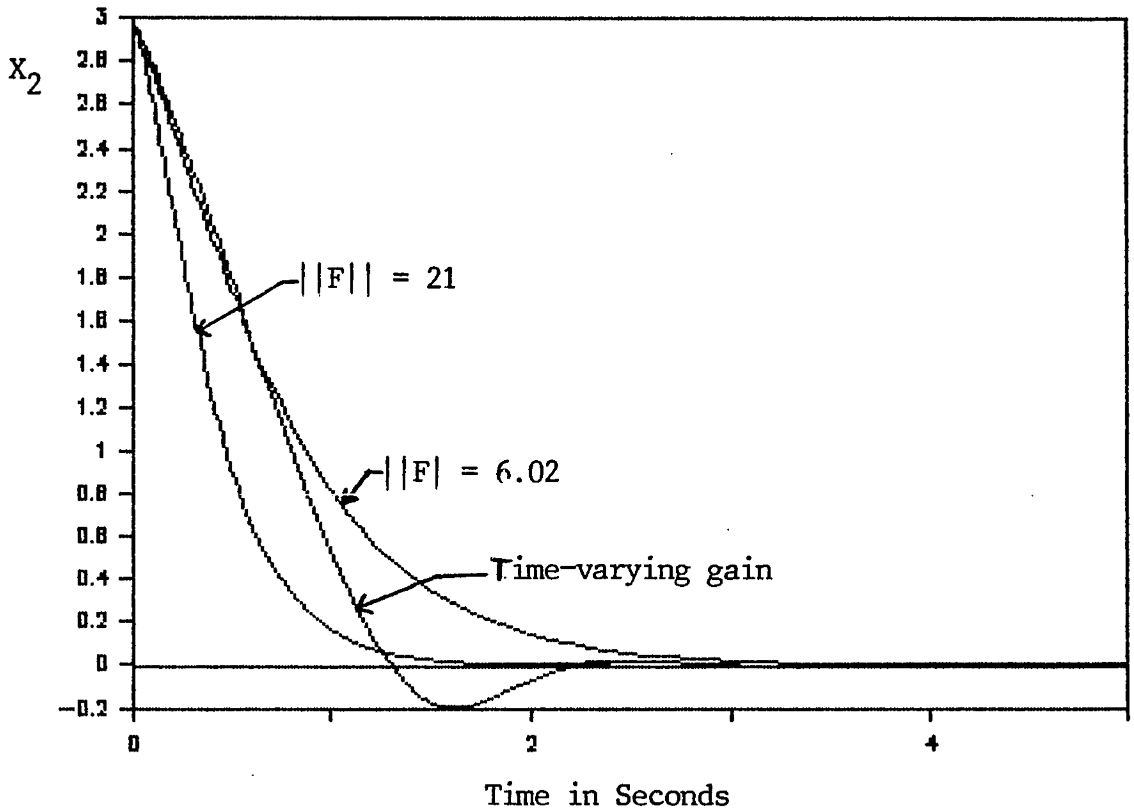


Fig. 4.15 X_2 as a function of time for diff. gains.

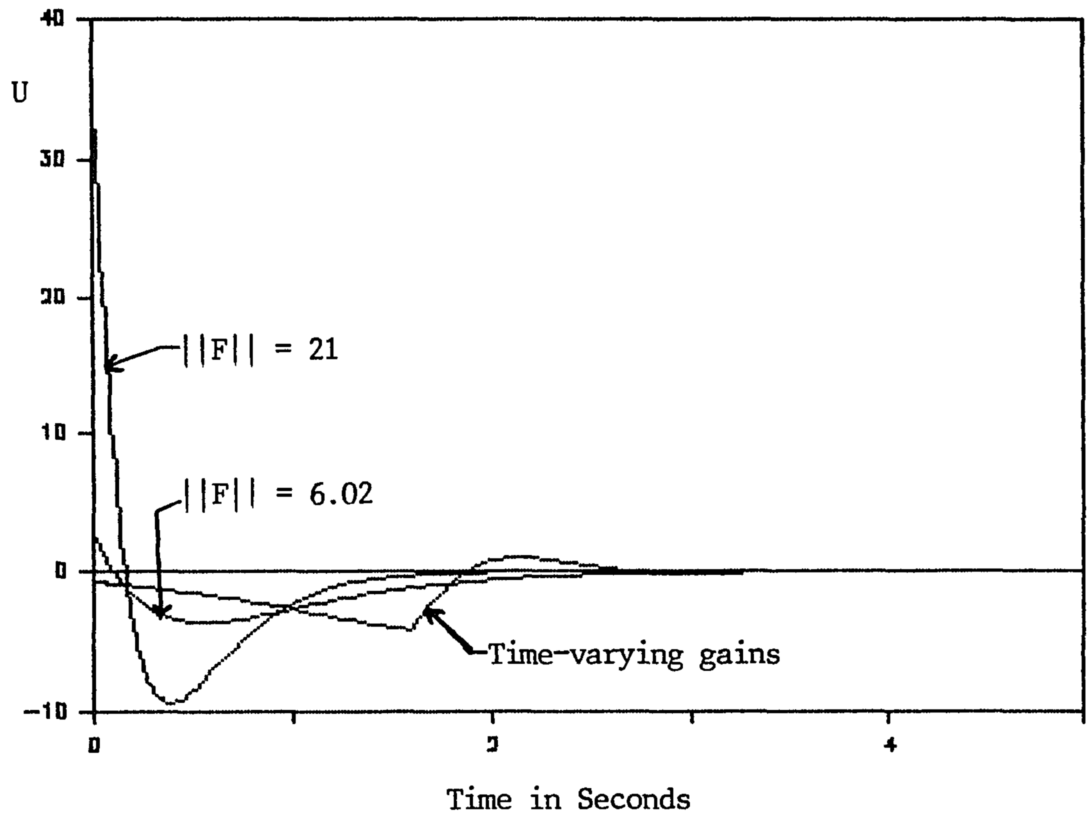


Fig. 4.16 U as a function of time for diff. gains.

4.3.2 OBSERVER DESIGN:

Since the state estimates will be used during the application of the output feedback design, it is essential that the estimation error be kept small during the whole observation period. This will avoid the state components having large excursions. Analogous to the discussion on the controller design, large gain results in short observation period but large error component excursions and thus may cause large input and state excursions. On the other hand small gain will definitely produce a long observation period. The solution to both short observation and small error lies in using a gain whose norm increases with time such that the smaller norms occur during the period when the norm of the observation is increasing, thus keeping the observation error small while the larger norms occur during the phase when the norm of the error is decreasing, thus reducing the observation period.

The design of the state estimator which will result in a good compromise among error component excursions and response time is analogous to that of the controller design discussed earlier. Thus we will state the algorithm for generating the observer's gain. Let it be required that $\max|K_{ij}(t)| \leq K_{max}$. The design procedure for generating the required K can be stated as follows: Let α and β be the real part of largest and smallest eigenvalues of A respectively.

(a) Solve for α and β. If $\alpha \le 0$ then set $M = 1 + (\alpha-\beta)/10$, else set $M = 1 + \alpha$.

(b) Using the value of M found in (a) numerically solve

$$dW/dt_f = -W(A/M) - (A/M)^T W + C^T C/M \tag{4.116}$$

$$W(0) = 0 \tag{4.117}$$

up to a nonsingular solution.

(c) Invert W and switch to (4.118) given below

$$dW^{-1}/dt_f = (A/M)W^{-1} + W^{-1}(A/M) - W^{-1}C^T C W^{-1} \tag{4.118}$$

$$\hat{K}(t_f) = 0.5W^{-1}C^T \tag{4.119}$$

(d) Solve (4.118) up to the first time $|\hat{K}_{ij}| \le K_m$. Denote this time as t_1. Continue the numerical solution of (4.118) storing all subsequent gain using (4.118) un til there is little or no change in $||\hat{F}(t_f)||$ or $||A - \hat{K}(t_f)||$. Denote the value of t_f when the last gain is stored as t_2. Then the time interval for which a time- varying gain will be used is $t_{ro} = t_2 - t_1$. From then on a constant gain $\hat{K}(t_2)$ will be used. Then K(t) is given as

$$K(t) = \begin{cases} \hat{K}(t_2 - t) & 0 \le t \le t_{ro} \\ \hat{K}(t_1) & t \ge t_{ro} \end{cases} \tag{4.120}$$

(d) If steady state solution of (4.118) does not satisfy the gain constraint, then this technique can not be used to meet the design procedure or the gain magnitude constraint can not produce stability in the error equation

$$\mathring{e} = [A - KC]e \tag{4.121}$$

4.4 APPLICATIONS

4.4.1 CONTROL OF A NUCLEAR PLANT

The rapidly growing need for power together with the ever decreasing supply of conventional fuel makes nuclear power an inevitable alternative for the future. The significant role of nuclear fission and eventually, fusion is an extremely touchy issue when the ecology of the society is taken into account. Consideration of the cost-effectiveness of generation and the effects of pollution points to the need for high performance operation. The performance of these nuclear plants depends highly upon the quality of their control. In this section we examine the nature of a nuclear plant and regulation of the neutron population required to maintain a desired nuclear energy supply.

4.4.1 PLANT DYNAMIC: A detailed derivation of the neutronic state equations can be found in [27] and [28]. Here the pertinent equations are given. The net change in neutron population over one generation is governed by

$$dn/dt = \frac{(\delta k - \beta)}{L} n + \sum_{i=1}^{6} \lambda_i c_i \qquad (4.122)$$

$$dc_i/dt = (\beta_i/L)n - \lambda_i c_i \qquad i = 1.....6 \qquad (4.123)$$

where δk is the reactivity, L is the mean prompt neutron generation time, n is the neutron population, λ_i is the decay constant for the i^{th} group of precursors and c_i is the population of the i^{th} precursor group. The precurors are a small group of unstable fission products which produce a small portion of neutrons. For many nuclear fission processes, it is possible to change the reactivity quite rapidly. Let the rate of change of the reactivity be the control u(t). Then

$$d\delta k/dt = u(t) \tag{4.124}$$

For a single precursor group (4.123) becomes

$$dc/dt = (\beta/L)n - \lambda c \tag{4.125}$$

The objective here is to bring the precursor state from an initial equilibrium state $c_o = \beta n_o/L\lambda$ and $\delta k = 0$ to some desired equilibrium state $c_f = n_f/L\lambda$ with $\delta k = 0$.

Let $x_1 = (n - n_e)/n_e$, $x_2 = (c - c_e)/c_e$ and $x_3 = \delta k$ where c_e is an equilibrium precursor level and $n_e = L\lambda c_e/\beta$. Then (4.122),(4.124) and (4.125) can be written as

$$\mathring{x} = Ax + f(x) + Bu \tag{4.126}$$

$$\text{where} \quad A = \begin{bmatrix} -\beta/L & \beta/L & 1/L \\ \lambda & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.127}$$

$$f(x) = \begin{bmatrix} x_1 x_3/L \\ 0 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.128}$$

The eigenvalues of A are $0, 0, -(\lambda + \beta/L)$. The values of L for slow breeders is of the order of milliseconds and microseconds for fast breeders. Thus the smallest eigenvalue of A can be located very far in the left half plane. For this reason we rewrite (4.126) as

$$\mathring{x} = \hat{A}x + \hat{f}(x) + Bu \tag{4.129}$$

were

$$\hat{A} = \begin{bmatrix} (\gamma-\beta)/L & (\beta-\gamma/L & (1-\gamma)/L \\ \lambda & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\hat{f}(x) = \begin{bmatrix} (x_3-\gamma)x_1/L + \gamma x_2/L + \gamma x_3/L \\ 0 \\ 0 \end{bmatrix} \tag{4.130}$$

and γ is to be selected such that the largest eigenvalue is not far into the right half plane while the smallest eigenvalues are not far into the left half plane. This will ensure that the smallest gain component is not negligible in respect to the other while the steady state gain components are not too large.

The [A,B] pair is completely controllable. Let the input be given by

$$u(t) = -F(t)x(t) \tag{4.131}$$

where
$$F(t) = 0.5B^T S(t) \tag{4.132}$$

$$N\dot{S}^{-1} = \hat{A}S^{-1} + S^{-1}\hat{A}^T - BB^T \tag{4.133}$$

$$S^{-1}(t_f) = 0 \tag{4.134}$$

Then if

(i) $\hat{f}(0) = 0$ and $\hat{f}(x)$ is continuous about $x = 0$ (4.135)

(ii) the eigenvalues of $(\hat{A} - BF)$ have negative real part

(iii) $|\hat{f}(x)|/|x| \to 0$ as $|x| \to 0$ where $|x| = |x_1|+|x_2|+|x_3|$.

Then (4.126) is locally stable and (4.131) drives the state of (4.126) to the origin in time $t = t_f$.

In controlling this plant a gain constraint is imposed. Thus we seek a control law which will drive the state of the system to the origin in such a manner as described in section 4.3. The parameters of the plant are $\beta = 1$, $L = 3*10^{-4}$/sec and $\lambda = 0.1$/sec. Thus we set $\gamma = 0.9999$. The resulting eigenvalues of $\hat{A}$ are now -0.433, 0.0 and 0.00 while that of A are 0,0 -3333.33. Thus we select N = 1. Let it be required that the maximum gain component F_m =275. Solution of (4.105) - (4.108) results in t_1 = 1.49 sec. $\hat{F}(1..49)$ = [21.46, 264.29, 2.863], t_2 = 6.01 and $\hat{F}(6.01)$ =

[0.833, 3.537,0.580]. This results in t_r = 4.52secs. The system was simulated with x(0) = [-0.5, -0.5, 0]. Fig. 4.17 show $||\hat{F}||$ as a function of t_f. Note that after 6.0 secs. the curve levels and thus decreasing the gain serves only to increase the response time. The control law (4.109) was then applied to (4..126) and Figs. 4.18 - 4.20 show the state response while Fig. 4.21 shows the input response. Note that the maximum state components excursions and input magnitude as seen from Figs. 4.18 - 4.21, occurred before the maximum applied gain. Thus even gains of greater norms can be used without affecting the state and input norms while decreasing the time response.

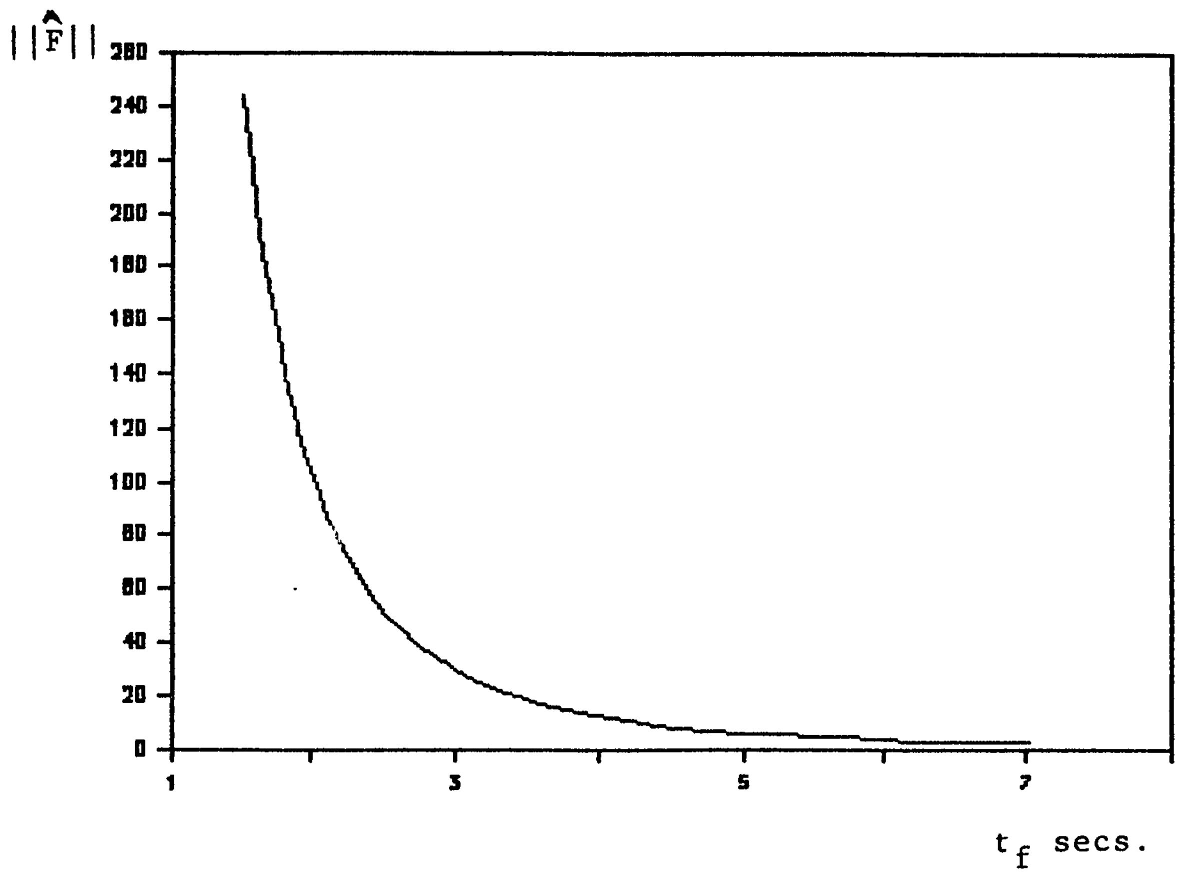


Fig. 4.17 $||\hat{F}||$ as a function of t_f(secs.)

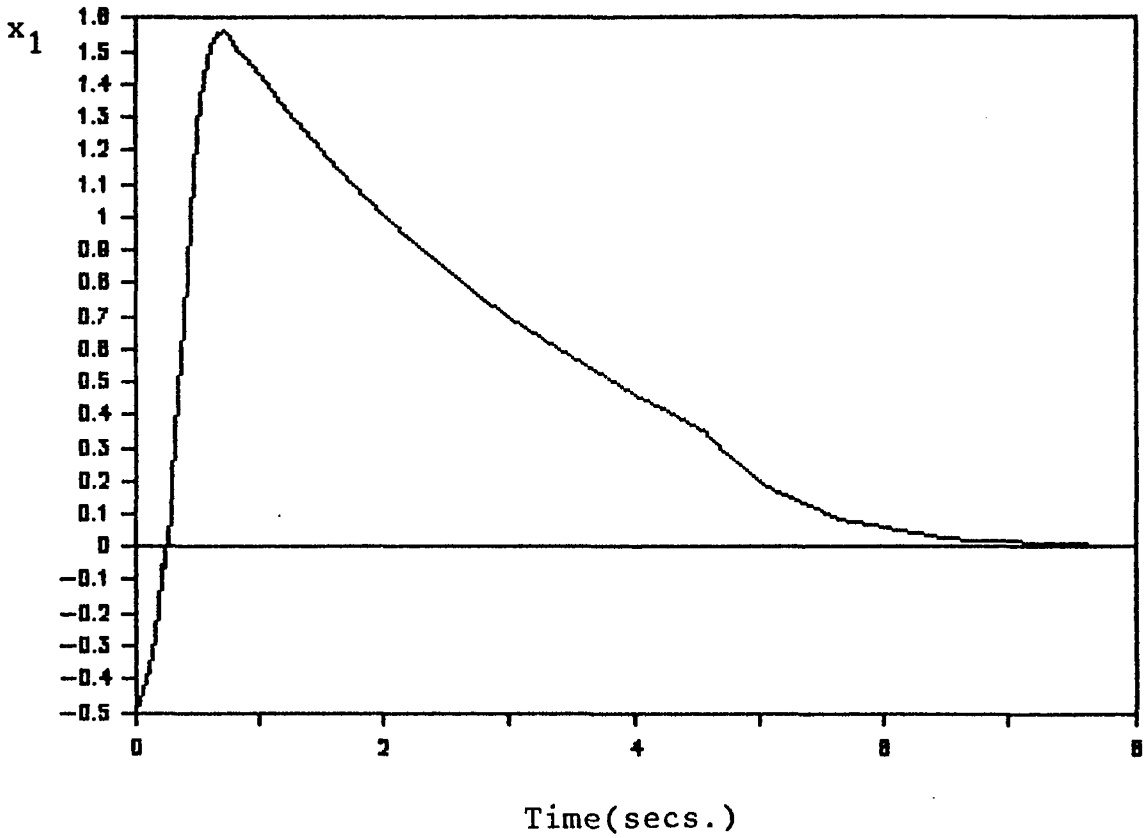


Fig. 4.18 x_1 as a function of Time (secs.)

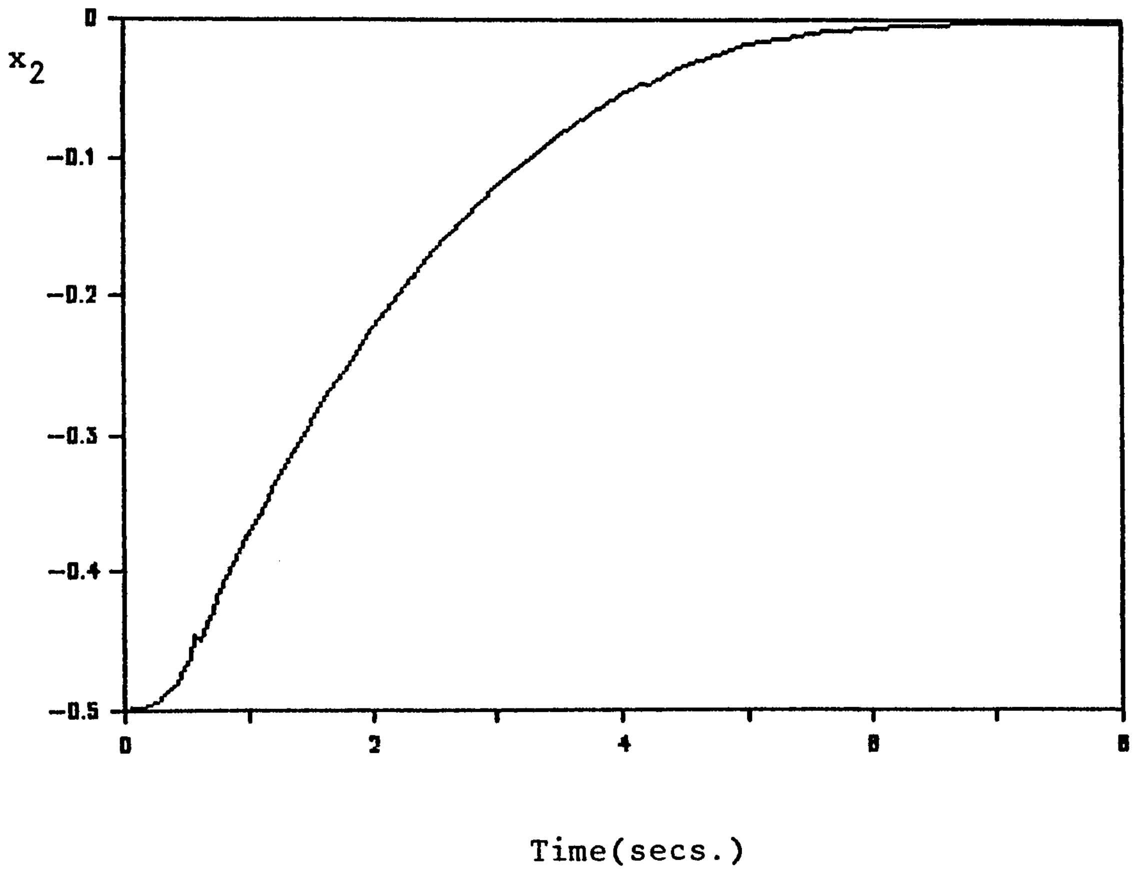


Fig. 4.19 x_2 as a function of Time (secs.)

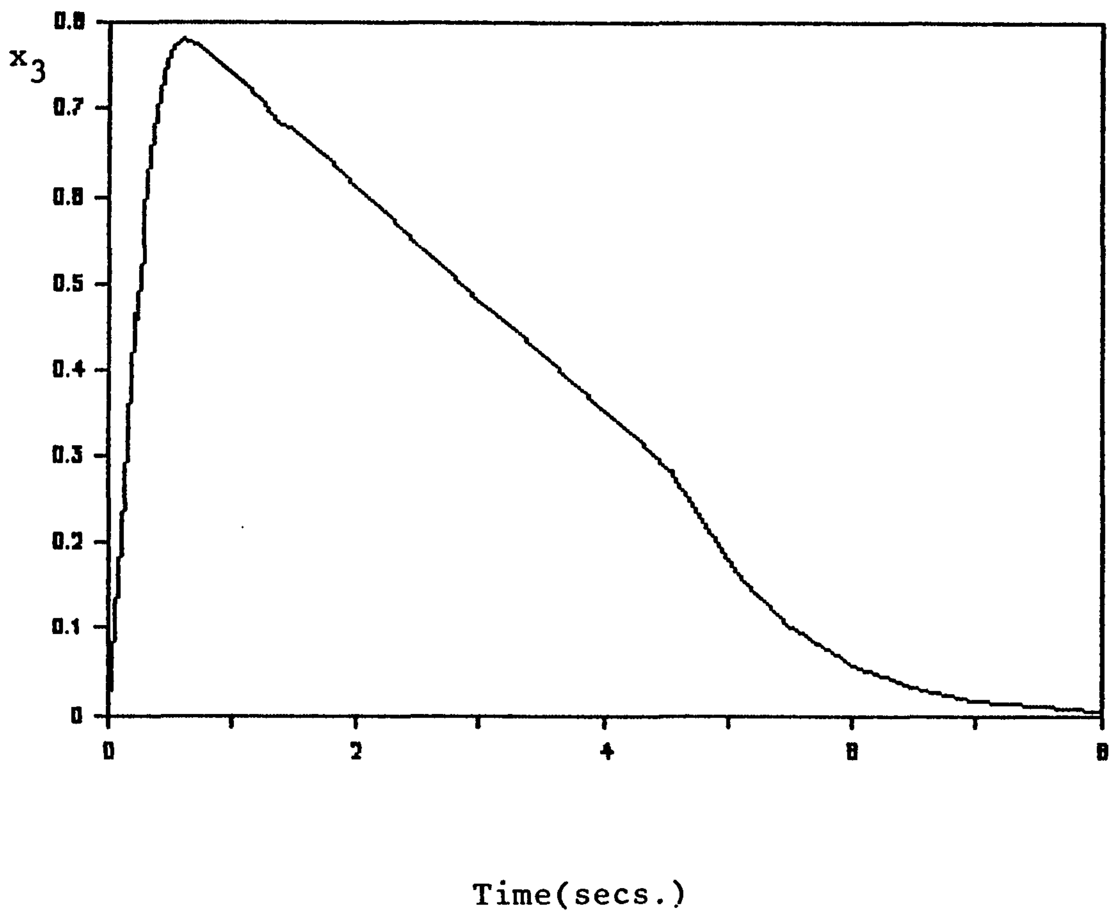
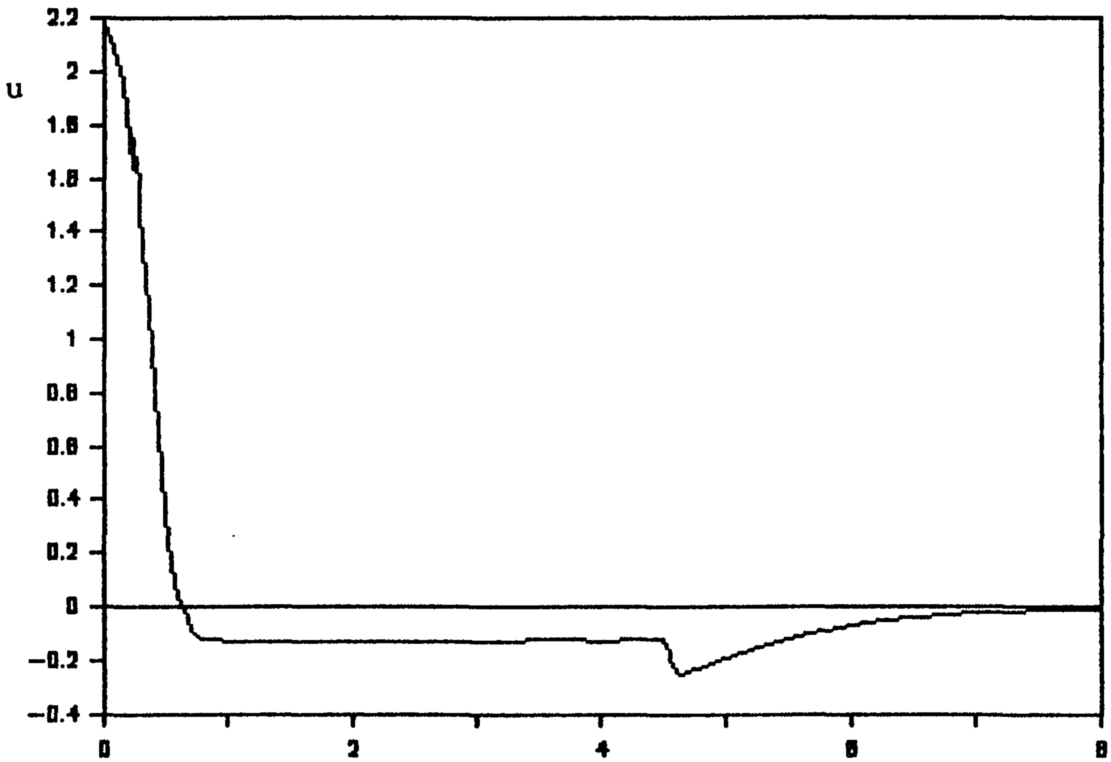


Fig. 4.20 x_3 as a function of time Secs.)



Time (secs.)

Fig. 4.2d Input u as a function of Time(secs.)

4.4.2 OBSERVER DESIGN:

Consider the case where only the reactivity and precursor levels can be measured. The objective here is to design a state estimator to estimate the neutron population x_1. The output y is given by

$$y = Cx = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \qquad (4.136)$$

Note that the [A,C] pair is completely observable. If u = 0 in (4.126) then the reactivity x_3 will remain constant. Thus if $|x_3(t)| \le \gamma_1$, then one can rewrite (4.126) as follows:

$$\mathring{x} = \bar{A}x + \bar{f}(x) + Bu \qquad (4.137)$$

$$\text{where} \quad \bar{A} = \begin{bmatrix} (2\gamma_1-\beta)/L & \beta/L & 1/L \\ \lambda & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{f}(x) = \begin{bmatrix} (x_3-2\gamma_1)x_1/L \\ 0 \\ 0 \end{bmatrix} \qquad (4.138)$$

Let the form of the state estimator be given by

$$\mathring{z} = Az + f(z)\ K(t)[y - Cz] + Bu \qquad (4.139)$$

$$\text{where} \qquad K = 0.5W^{-1}(t)C^T \qquad (4.140)$$

$$M\overset{\circ}{W}(t) = W\bar{A} + \bar{A}^T W(t) - C^T C \qquad (4.141)$$

$$W(t_f) = 0 \qquad (4.142)$$

Then $\quad \overset{\circ}{x} - \overset{\circ}{z} = (\bar{A} - KC)(x - z) + \bar{f}(x) - \bar{f}(z)$

$$= (\hat{A} - KC)(x - z) + \int_0^1 \nabla\hat{f}W_p(x - z)dp \qquad (4.143)$$

It can be easily shown that

$$x^T\nabla f(x)x = -2[\gamma_1 - x_3]x_1{}^2/1 \leq 0 \qquad (4.144)$$

Thus the gain given by (4.140) drives the state of (4.143) from $[(x_0 - z_0),t_0]$ to $[0,t_f]$.

Similar to the control design let us impose a gain constraint $K_m = 150$ on $|K_{ij}|$. Let $\gamma_1 = 0.895$, then the eigenvalues of $\bar{A}$ are 0, 0.8 and -351. Thus we set M = 10. Solution of (4.116),(4.117) with $A = \bar{A}$ results in t_1 = 0.4sec.,

$$\hat{K}(0.4) = \begin{bmatrix} 120.70 & 0.20 & 12.51 \\ 145.49 & 15.10 & 0.20 \end{bmatrix}^T \qquad (4.145)$$

$t_2 = 0.73$,

$$\hat{K}(0.63) = \begin{bmatrix} 77.71 & 0.21 & 7.98 \\ 89.96 & 9.6 & 0.21 \end{bmatrix}^T \qquad (4.146)$$

Thus t_r = 0.23sec. The system was simulated with x(0) = $[0.5, 0.5, 0.5]^T$, $z(0) = [0.0, 0.0, 0.0]^T$ and K(t) as given by (120). Fig. 4.21 shows $||K||$ as a function of t_f. Note that after t_f = 0.73sec. the curve starts to level off. Figs.4.23 - 4.25 show the state and estimator responses.

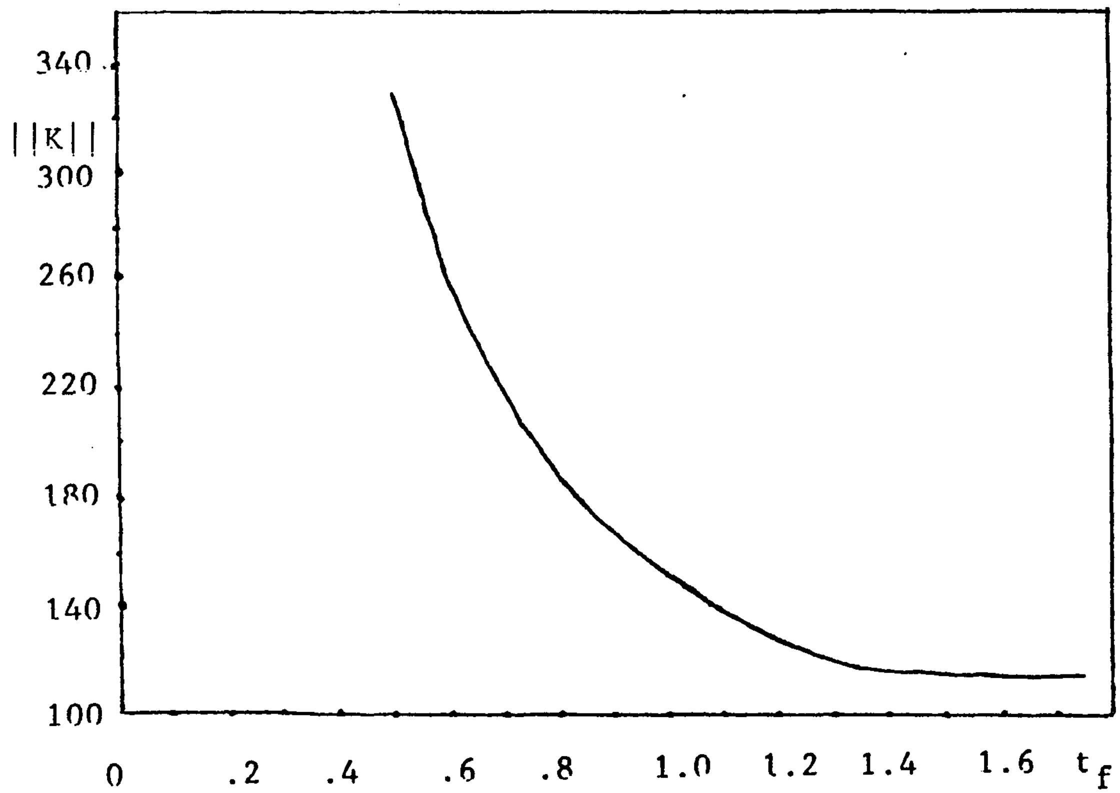


Fig. 4.22 ||K|| verse Time(secs.)

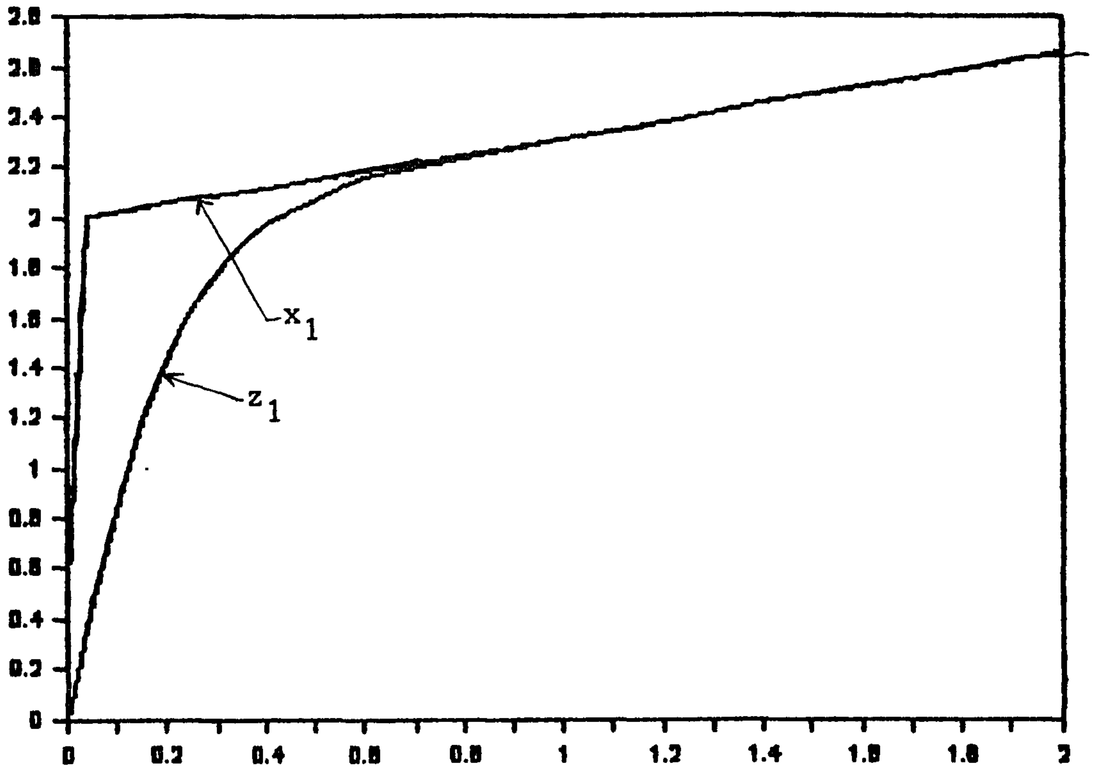


Fig. 4.23 x_1 and z_1 as functions of Time(secs.)

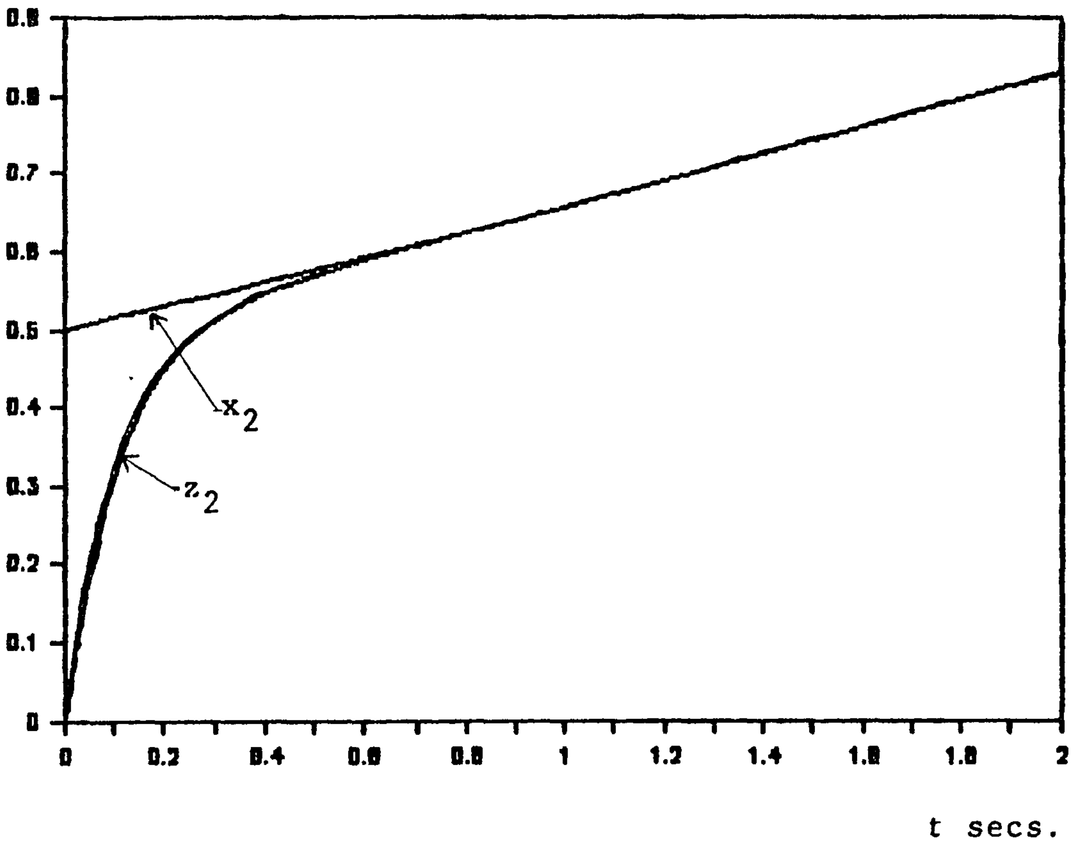


Fig. 4.24 x_2 and z_2 as functions of Time.

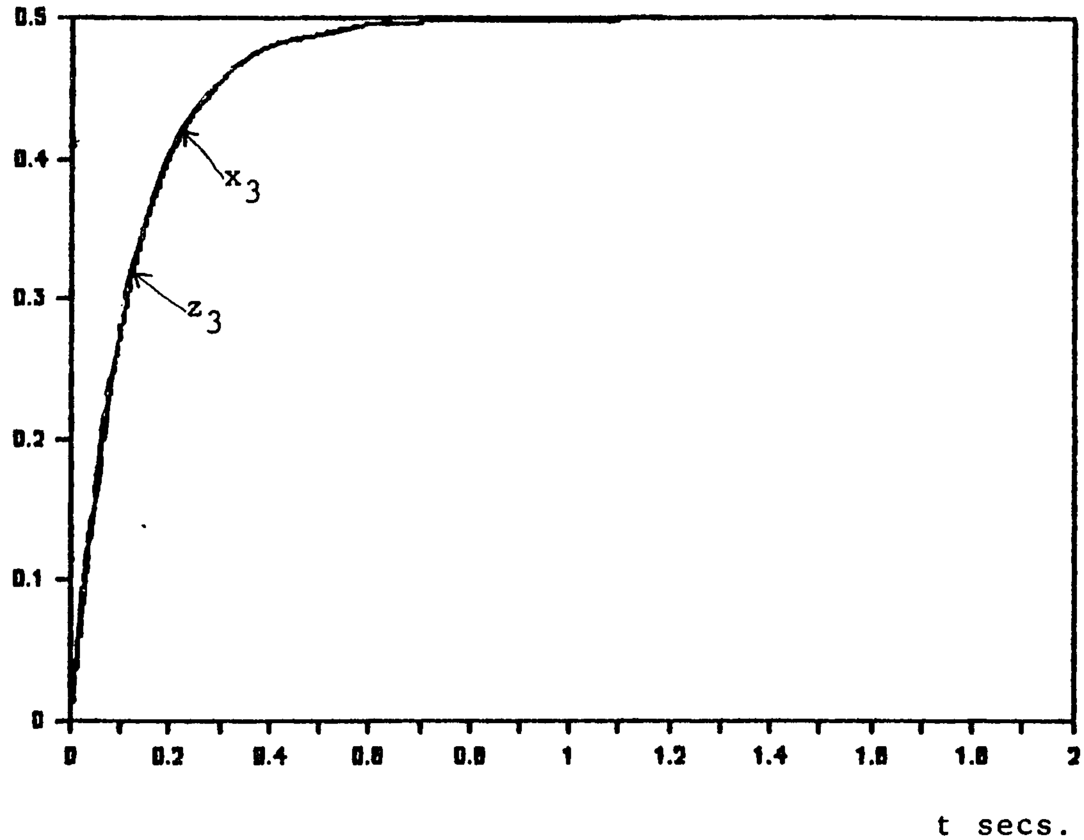


Fig. 4.25 x_3 and z_3 as functions of Time.

4.4 .3 OBSERVER-CONTROLLER DESIGN:

We now combine the design of the controller and observer to produce an output-feedback control system. Due to the fact that the system is only locally controllable, we require that identification be achieved quickly and without large excursion in the estimation error. Let input u be now given by

$$u = -F(t)z \tag{4.147}$$

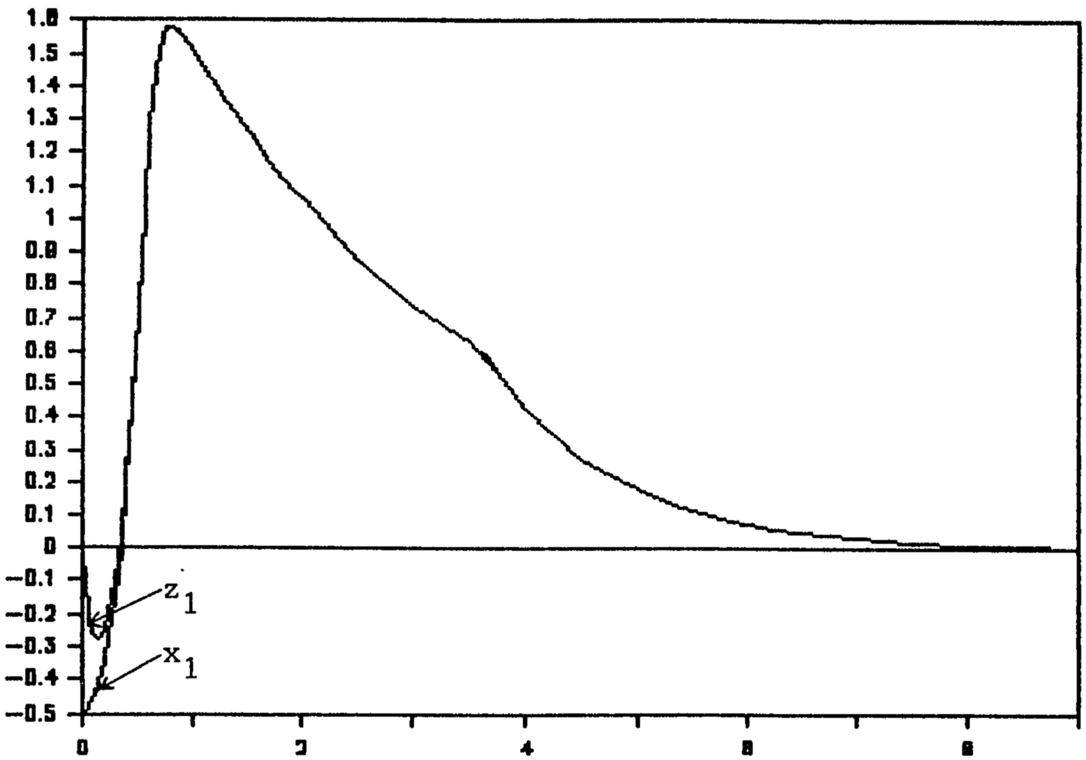
where F is defined by (4.109). Then the output-feedback can be written as

$$\mathring{x} = Ax + f(x) - BF(t)z \tag{4.148}$$

$$y(t) = Cx \tag{4.149}$$

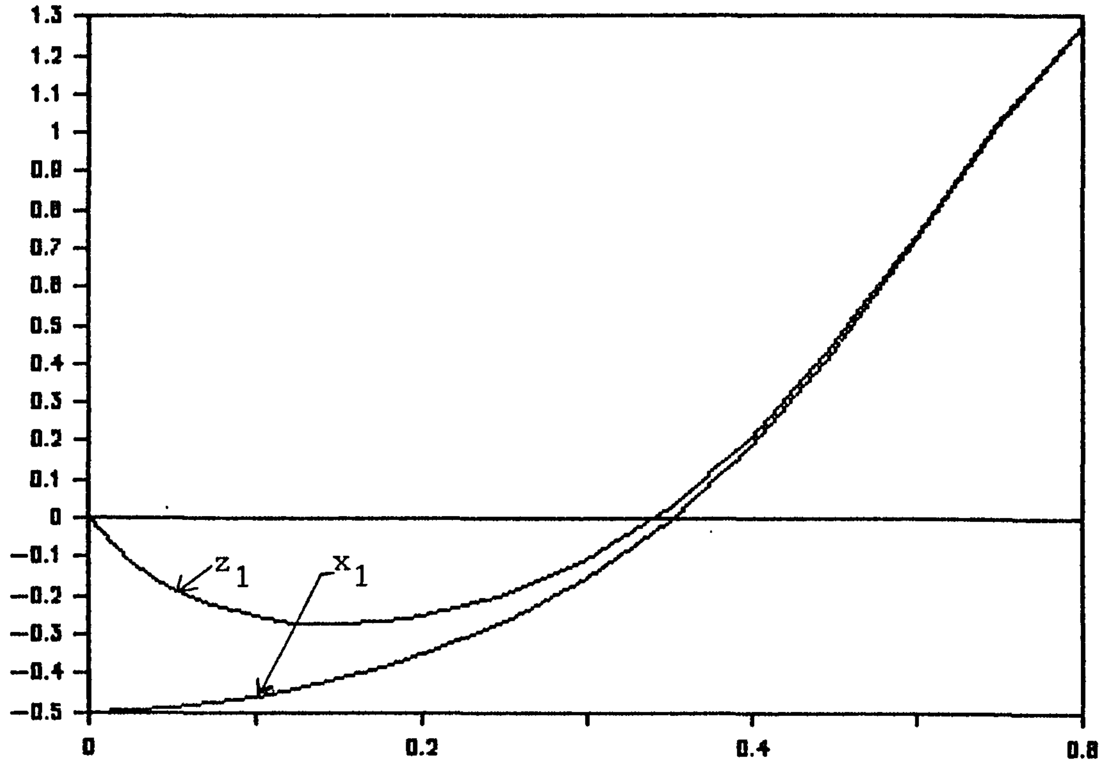
$$\mathring{z} = Az + f(z) - BFz + K(t)[y - Cz] \tag{4.150}$$

where K(t) is defined by (4.140). For this design we choose a constant gain K as given by (4.146) for the observer and a time-varying gain F(t) as described in 4.4.1 for the controller. The system (4.148) - (4.150) was simulated with $x(0) = [-0.5,-0.5,0]^T$, $z(0) = [0, 0, 0]^T$. Figs. 4.26 - 4.28 show the state and estimator responses while Fig. 4.28 show the input of the output-feedback control system.



Time (secs.)

Fig. 4.26 Responses of x_1 and z_1



Time(secs.)

Fig. 4.26(a) Responses of x_1 and z_1

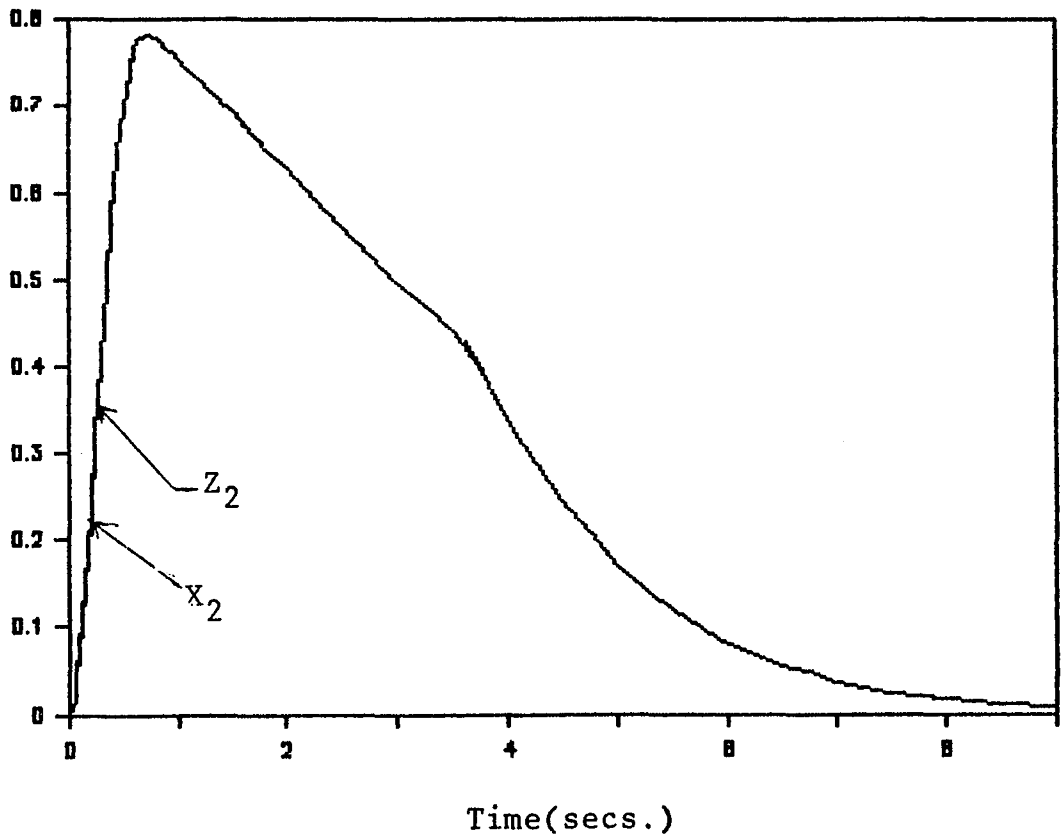
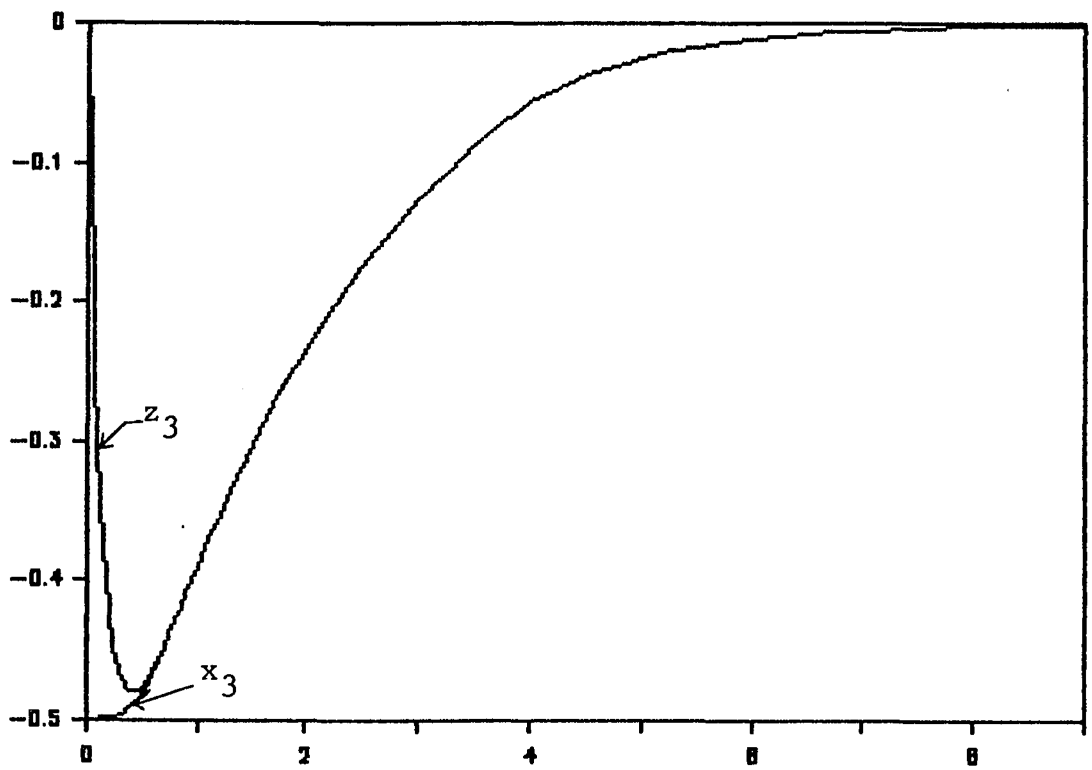
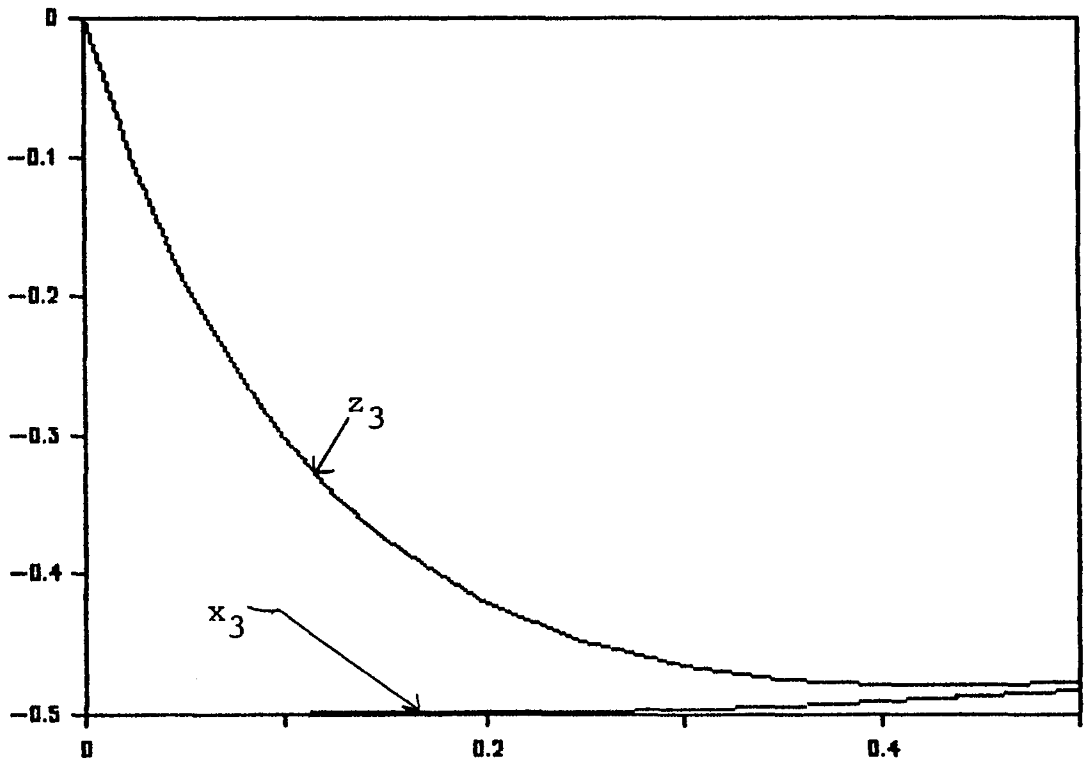


Fig. 4.27 Responses of x_2 and z_2



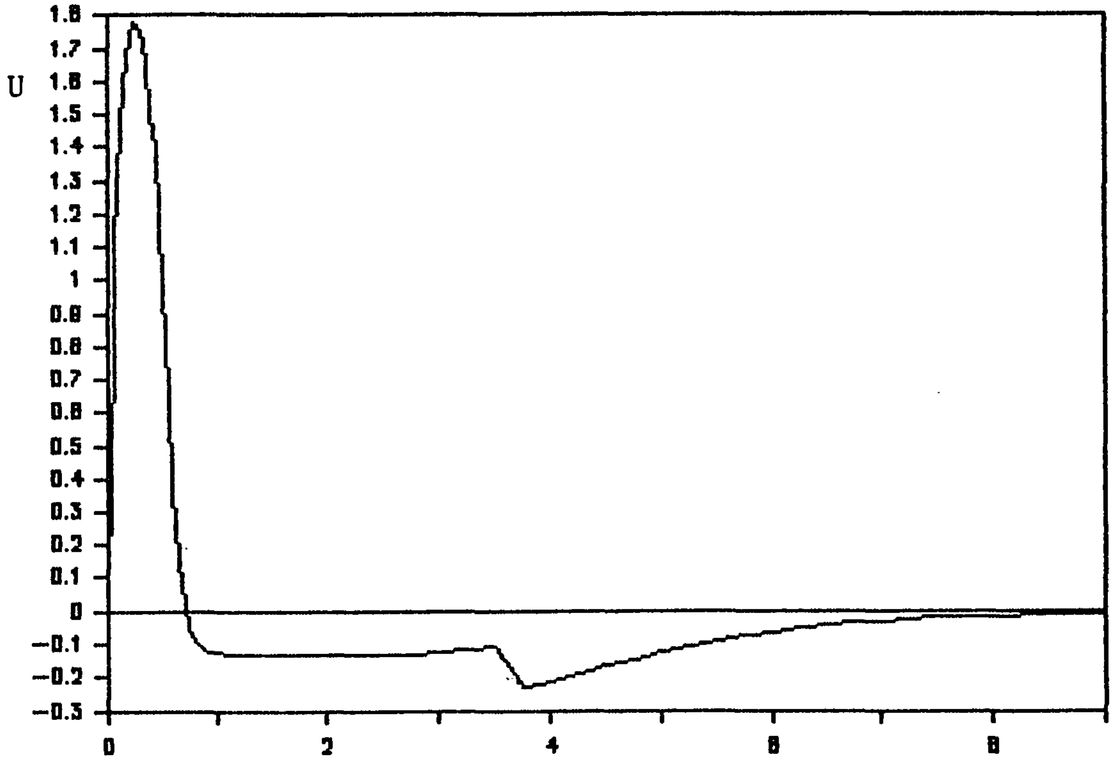
Time (secs.)

Fig. 4.28 Responses of x_3 and z_3



Time (secs.)

Fig. 28(a) Responses of x_3 and z_3



Time (secs.)

Fig. 4.29 Input u as a function of Time

This chapter has accomplished several modified designs. First we showed that even if the terminal time t_f is fixed, the state and input norms can be trimmed by appropriately assigning values for N(t) according to the eigenvalues of the system A matrix. Second, we showed how to designed a constant gain control law when there is a given constraint on the maximum norm of the gain of the system. And third, we described a new algorithm to generate a time varying gain control law to inhibit extreme state excursions and large input norm without sacrificing response time. The algorithm was then applied to several problems where it was shown to have several advantages over a constant gain control laws.

5. OBSERVER-CONTROLLER DESIGN FOR LINEAR DISCRETE SYSTEMS

It is well known [6], [20], [23] and [24] that several solutions exist which result in deadbeat responses in linear discrete systems. The following technique utilizes the methods of Lyapunov stability principles to generate a family of solutions to the obsever-controller design for linear systems. The technique is then modified to yield a deadbeat response as a special case. A theorem is formulated and proven which gives sufficient conditions for a nonlinear discrete system to be completely controllable. In section 5.2 the theorem is applied to the design of controllers, state estimators and output-feedback regulators for linear discrete systems. The solutions utilize two parameters M and N which can be used to shape the state estimator error and state trajectories respectively. It will be shown that as the values of these parameters increase, the controller and observer matrices increase in magnitude. Thus minimum gain magnitudes can be set by correct selection of N and M. Although N and M can be taken in general as time-varying, for simplicity only the time-invariant case is considered here. Several examples are given to demontrate the application of the techniques.

Although deadbeat responses are very desirable, in general they result in excessively large gain magnitudes and thus wide excursions in the state components of the system and also large estimator error magnitude. Therefore such responses are unrealizable in many practical situations where there are physical constraints on both the state excursions and gain magnitudes. Section 5.3 demonstrates how the new technique can be applied to gain-constrained observer-controller design. Some useful properties of matrices are discussed in this section. A modified design is achieved in section 5.4. Here matrix inversion is replaced by scalar division. Several examples are given in this section to illustrate the design of the gain-constrained output-feedback regulator. Section 5.5 shows a simple technique for designing the control law for a single input system which will result in a deadbeat response.

5.1 CONTROLLABILITY THEORY:

Consider the nonlinear discrete system given by

$$x(k+1) = f(x,u(x,k),k) \tag{5.1}$$

where x is the n-dimensional state vector, u is the r-dimensional input vector, k is the time variable and f is some nonlinear function such that $f: R^n \rightarrow R^n$.

Theorem 5.1: If a scalar function V(x,k) exists such that 1) for all continuous c(k) (n-valued vector function)

$$\lim_{k \to k_f} c(k) \neq 0 \Rightarrow \lim_{k \to k_f} V(c(k),K) = \infty \tag{5.2}$$

where k_f is some finite terminal time.

and if a control law $u^*(.) \subset U$ exists such that

2) along the trajectory of (5.1)

$$\Delta V = V(x,k+1) - V(x,k) \leq M < \infty \tag{5.3}$$

3) the solution to

$$x(k_0) = x_0 \tag{5.4}$$

$$x(k+1) = f(x,u^*,k) \tag{5.5}$$

exists and is unique,

then the system (5.1) is completely controllable from (x_0,k_0) to $(0,k_f)$ and $u^*(x,k)$ accomplishes the transfer.

<u>**PROOF**</u> : For $k < k_f$

$$V(x,K) = V(x_0,k_0) + V(x,k) - V(x_0,k_0)$$

$$= V(x_0,k_0) + \sum_{i=k_0+1}^{k} [V(x(i),i)-V(x(i-1),i-1)]$$

$$\leq V(x_0,k_0) + M(k - k_0) \tag{5.6}$$

Therefore $$\lim_{k \to k_f} V(x,K) \le V(x_0,k_0) < \infty \qquad (5.7)$$

And thus from 1) we conclude that

$$x(k_f) = \lim_{k \to k_f} x(k) = 0 \qquad (5.8)$$

This completes the proof.

5.2 OBSERVER-CONTROLLER DESIGN FOR LINEAR DISCRETE SYSTEMS:

5.2.1 CONTROLLER DESIGN:

Consider the linear time-invariant discrete system

$$x(k+1) = Ax(k) + Bu(k) \qquad (5.9)$$

where x and u are defined as before and A and B are nxn and nxr matrices, respectively. Let the input be given by

$$u(k) = -F(k)x(k) \qquad (5.10)$$

where F is an rxn matrix called the controller gain matrix. Then the closed-loop system becomes

$$x(k+1) = [A - BF(k)]x(k) \qquad (5.11)$$

Let us assume further that A is invertible. Define the scalar function V(x,k) as follows

$$V(x,k) = x^T(k)\ S(k)x(k) \tag{5.12}$$

where $$S^{-1}(k) = A^{-1}[S^{-1}(k+1)/N + BB^T]A^{T^{-1}} \tag{5.13}$$

$$S^{-1}(k_f) = 0 \tag{5.14}$$

where N is a positive scalar and k_f is some finite terminal time. Let F(k) satisfy

$$B = AS^{-1}(k)F^T(k) \qquad k < k_f \tag{5.15}$$

If S(k) exists then F is uniquely determined by

$$F(k) = B^T A^{T^{-1}} S(k) \tag{5.16}$$

If the [A,B] pair is completely controllable, then the matrix $[B, AB, \ldots\ldots\ldots\ldots, A^{n-1}B]$ is of full rank. Also (5.15) is linear in F. If $S^{-1}(k)$ is singular, then solving (5.15) for F will notresult in a unique solution. In this case many components of F can be arbitrarily selected. Solution of (5.13),(5.14) is given by

$$S^{-1}(k_f-k,N) = N\sum_{i=0}^{k-1} N^{i-k}A^{i-k}BB^TA^{T^{i-k}} \tag{5.17}$$

$$= N^{1-k} A^{-k} \sum_{i=0}^{k-1} N^i A^i B B^T A^{T^i} A^{T^{-k}} \qquad (5.18)$$

The matrix $S^{-1}(k)$ is invertible if

$$\sum_{i=0}^{k-1} N^i A^i B B^T A^{T^i} \quad > \quad 0 \qquad (5.19)$$

which is the controllability condition [20]. This condition is always satisfied for $k \geq n$. Thus for $k < n$, $S^{-1}(k_f-k,N)$ may be singular. Therefore for $k \geq n$, $S(k_f-k, N)$ exists and (5.13) can be written as

$$S(k) = NA^T S(k+1)A - NA^T S(k+1)B[I_r + NB^T S(k+1)B]^{-1} B^T S(k+1)AN \qquad (5.20)$$

Equation (5.16) can now be written as

$$F(k) \quad = \quad [I_r + NB^T S(k+1)B]^{-1} B^T S(k+1)AN \qquad (5.21)$$

From (5.20) and (5.21)

$$S(k) \quad = \quad NA^T S(k+1)[A - BF(k)] \qquad (5.22)$$

Taking the difference of (5.12) along the trajectory of (5.11) gives

$$V = x^T(k+1)S(k+1)x(k+1) - x^T(k)S(k)x(K)$$

$$= x^T(k)[A^TS(k+1)A-A^TS(k+1)B[I_r+NB^TS(k+1)B]^{-1}B^TS(k+1)AN -$$

$$S(k)-NA^TS(k+1)B[I_r+NB^TS(k+1)B]^{-1}[I_r+NB^TS(k+1)B]^{-1}*..$$

$$B^TS(k+1)AN]x(k) \qquad = x^T(k)[(-1+1/N)S(k) - F^TF]x(k)$$

$$= -x^T(k)[(1-1/N)S(k) + S(k)A^{-1}BB^TA^{T^{-1}}S(k)]x(k)$$

$$\leq 0 \qquad \text{for } N \geq 1 \tag{5.23}$$

Thus the input given by (5.10) will produce stability for $N = 1$ and asymptotic stability in (5.9) for $k \geq n$. Let us examine the case for $0 < k \leq n$. In this region $S^{-1}(k)$ may be singular. Note that (5.15),(5.13) can be written as

$$S^{-1}(k+1) = N[AS^{-1}(k)A^T - BB^T]$$

$$= N[A - BF(k)]S^{-1}(k)A^T \tag{5.24}$$

The solution to (5.24) is given by

$$S^{-1}(k)=N^k[A-BF(k-1)][A-BF(k-2)]...[A-BF(0)]S^{-1}(0)A^{T^k} \tag{5.25}$$

Since (5.14) is satisfied, we conclude that for $k_f \geq n$,

$$[A-BF(k_f-1)][A-BF(k_f-2)] \ldots\ldots [A-BF(0)]x(0) = 0 \quad (5.26)$$

Thus from (5.11) we conclude that

$$x(k_f)=[A-BF(k_f-1)][A-BF(k_f-2)] \ldots\ldots [A-BF(0)]x(0)=0 \quad (5.27)$$

Therefore the gain given by (5.15) produces a deadbeat response in (5.9).

5.2.2 OBSERVER DESIGN:

In many instances, the state of (5.9) is unavailable for complete measurement. When such a case occurs, the control law (5.10) can not be used. Thus one must first find estimates of the state and use these estimates to control the system. The following technique describes how to design the observer.

Let's assume that the output of the system is given by

$$y(k) = Cx(k) \quad (5.28)$$

where $y \in R^m$ is the output and C is an mxn matrix. An observer for (5.9),(5.28) is given by

$$z(k+1) = Az(k) + G(k)[y(k) - Cz(k)] + Bu(k) \quad (5.29)$$

where $z \in R^n$ is the output of the observer and G is an nxm matrix called the observer's gain matrix. Let usassume further that the [A,C] pair is completely observable. Define the observation error e as

$$e(k) = x(k) - z(k) \tag{5.30}$$

Then from (5.9),(5.28) and (5.29)

$$e(k+1) = [A - G(k)C]e(k) \tag{5.31}$$

For any initial state, the solution to (5.31) is given by

$$\begin{aligned} e(k) &= [A-G(k-1)C][A-G(k-2)C]....[A-G(1)C][A-G(0)C] \\ &= R(k)e(0) \end{aligned} \tag{5.32}$$

where $R(k) = [A-G(k-1)C][A-G(k-2)C]..[A - G(1)C][A - G(0)C]$ (5.33)

Thus if there is a $k = k_f$ such that $R(k_f) = 0$, then it is obvious that $e(k_f) = 0$. Define the scalar function V(e,k) as

$$V(e,k) = e^T(k)W^{-1}(k)e(k) \tag{5.34}$$

where $W^{-1}(k)$ satisfies

$$W^{-1}(k+1) \;=\; A^{T^{-1}}[W^{-1}(k)/M + C^{T}C]A^{-1} \qquad (5.35)$$

$$W^{-1}(0) \;=\; 0 \qquad (5.36)$$

where M is a positive scalar. Although M can be taken as time- varying, we will only consider the time-invariant case here. The solution to (5.35),(5.36) is given by

$$W^{-1}(k,M) \;=\; M\sum_{i=1}^{k} M^{-i}A^{T^{-i}}C^{T}CA^{-i}$$

$$= \; M^{1-K}A^{T^{-k}}\sum_{i=1}^{k} M^{i-1}C^{T}CA^{i-1}A^{-k} \qquad (5.37)$$

The matrix $W^{-1}(k,M)$ is invertible if

$$\sum_{i=1}^{k} M^{i-1}A^{T^{-1}}C^{T}CA^{i-1} \;>\; 0 \qquad (5.38)$$

which is exactly the observability condition [20] for linear discrete systems. Since it is assumed that the [A,C] pair is completely observable, then there exists $1 < k \leq n$ such that $W^{-1}(k,M)$ is invertible. Let us write (5.35) as

$$W^{-1}(k) \;=\; M[A^{T}W^{-1}(k+1)A - C^{T}C] \qquad (5.39)$$

Now define G(k) such that

$$C^T = A^T W^{-1}(k+1)G(k) \tag{5.40}$$

If W(k+1) exists, then

$$G(k) = W(k+1)A^{T^{-1}}C^T \tag{5.41}$$

Substituting (5.40) into (5.39) leads to

$$W^{-1}(k) = MA^T W^{-1}(k+1)[A - G(k)C] \tag{5.42}$$

Using matrix inversion, (5.35) can be written as

$$W(k+1) = MAW(k)A^T - MAW(k)C^T[I_m + MCW(K)C^T]^{-1}CW(k)A^T M \tag{5.43}$$

Substitution of (5.43) into (5.41) yields

$$G(k) = MAW(k)C^T[I_m + MCW(k)C^T]^{-1} \tag{5.44}$$

Taking the difference of (5.34) along the trajectory of (5.31) yields

$$\Delta V = e^T(k)[(A-G(k)C)^T W^{-1}(k+1)(A-G(k)C) - W^{-1}(k)]e(k)$$

$$= e^T(k)[(A - G(k)C)^T A^{T^{-1}} W^{-1}(k) - W^{-1}(k)]e(k)$$

$$= -e^T(k)[(1 - 1/M)W^{-1}(k) + C^T G^T(k)A^{T^{-1}} W^{-1}(k)]e(k)$$

$$= -e^T(k)[(1 - 1/M)W^{-1}(k) + MC^T[I_m + MCW(K)C^T]^{-1}C]e(k)$$

$$< 0 \quad \text{for } M > 1 \tag{5.45}$$

Thus the gain given by (5.5), (5.44) produce asymptotic stability in (5.31) for all k such that W(k) is positive definite.We now examine the condition will result in a deadbeat observer design. From (5.38) we know there exist a $1 < k \leq n$ such that $W^{-1}(k)$ is invertibe. Let the first time that $W^{-1}(k)$ is invertible be k_f. Then solving (5.42) k steps backward from k_f results in

$$W^{-1}(k_f-k) = M^kA^{T^{-k}}W^{-1}(k_f)[A-G(k_f-1)C][A-G(k_f-2)C]*...$$

$$......[A-G(k_f-k)C] \tag{5.46}$$

and thus for $k = k_f$ (5.46) becomes

$$W^{-1}(0) = M^kA^{T-1}W^{-1}(k_f)[A-G(k_f-1)C][A-G(k_f-2)C]..[A-G(0)C] \tag{5.47}$$

Thus from (5.33) and (5.36) we conclude that there is a finite $k_f \leq n$ such that the gain given by (5.40) forces the estimator's error to zero.

5.2.3 OUTPUT FEEDBACK REGULATION:

The theories developed in 5.2.1 and 5.2.2 can be combined to produce an output-feedback system. To demonstrate this, let the input be given by

$$u(k) = -F(k)z(k) \tag{5.48}$$

Using (5.48) and substituting $z(k) = x(k)-e(k)$ into (5.29) leads to

$$x(k+1) = [A - BF(k)]x(k) + BF(k)e(k) \tag{5.49}$$

Shown in 5.2.2, there exists a sequence of observer gain matrices which reduces the observation error e to zero in at most n steps. Therefore after n steps, (5.49) reduces to

$$x(k+1) = [A - BF(k)]x(k) \tag{5.50}$$

But as seen in 5 2.1, we observed that there exists a sequence of gain matrices F(k) which forces the state x to zero in at most n steps. Thus the system (5.9) can be driven to zero in at most 2n steps using the input (5.48). The gain F is generated from (5.13), (5.14) and (55.15) with $k_f = 2n$. It is worthwhile to note that the gain $F(k_f-k_1)$ where $k_1$is the smallest value of k such that

$S(k_f-k)$ exists will drive the state of (5.9) to the origin in at most k_1 steps. The gain assign all the poles of $[A - BF(k_1)]$ to the origin. Similarly, the constant gain $G(k_1)$ where k_1 is the smallest k such that W(k) exists drive the estimation error to zero by assigning all the eigenvalues of $[A-G(k_1)]$ to the origin.

<u>EXAMPLE 5.1</u>: Consider the system given by (5.9) where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{5.51}$$

Solution of (5.13),(5.14) and (5.15) for $k_f=3$ and N = 1 results in $F_2(2) = 1$, $F_2(1) = 1$, $F_1(1) + F_3(1) = 0$, and $F(0) = [-1,1,1]$. The systems was simulated with $F(2) = [0, 1, 0]$, $F(1) = [0,1,0]$ and $F(0)=[-1,1,1]$. Figs 5.1, 5.2, and 5.3 show the time responses of the state components. Note that $[A - BF(0)]^3 = 0$. Thus the constant gain $F(0) = [-1,1,1]$ can be used to drive the state of (5.9) to the origin in 3 steps.

<u>EXAMPLE 5.2</u>: Consider the system (5.9),(5.28) where A is given as in example 5.1 and

$$C = [1,0,0] \qquad u = 0 \tag{5.52}$$

Gain G was calculated from (5.35), (5.36) and (5.40). Of the many possible sequences for G(0) and G(1), the following sequence was chosen $G(0) = [-1, 0, 0]^T$ and $G(1) = [0, -1, 0]^T$ while G(2) is uniquely found to be $G(2) = [2, -5, -2]^T$. The system was simulated with $x(0) = [1, 1, -1]^T$ and $z(0) = [0, 0, 0]^T$. Figs 5.4, 5.5, and 5.6 show the result graphically. M = 1 was used. It is of interest to note that $[A - G(2)C]^3 = 0$. Thus the constant gain G(2) can be used to drive the estimator error to zero in 3 steps.

<u>EXAMPLE 5.3</u>: Consider system (5.9),(5.28) where A and B are given as in example 5.1 and C is given in example 5.2. The observer gain used were the same as in example 5.2. With $k_f = 6$, the controller gain for each k was calculated backward and stored using (5.13), (5.14) and (5.15). The system was simulated using $x(0)= [1,2,3]^T$ N = M = 1 and $z(0) = [0, 0, 0]^T$. Figs. 5.7, 5.8 and 5.9 show the responses graphically.

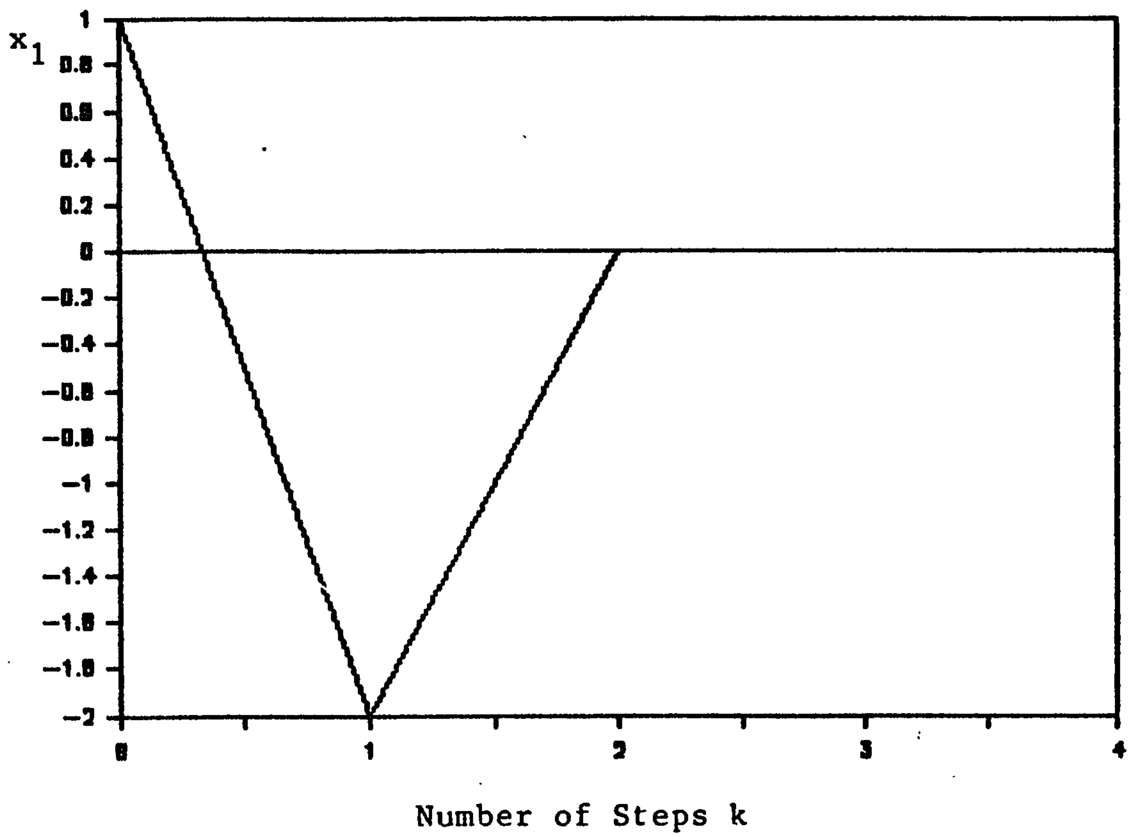


Fig. 5.1 Time Response of $x_1(k)$ For Ex. 5.1

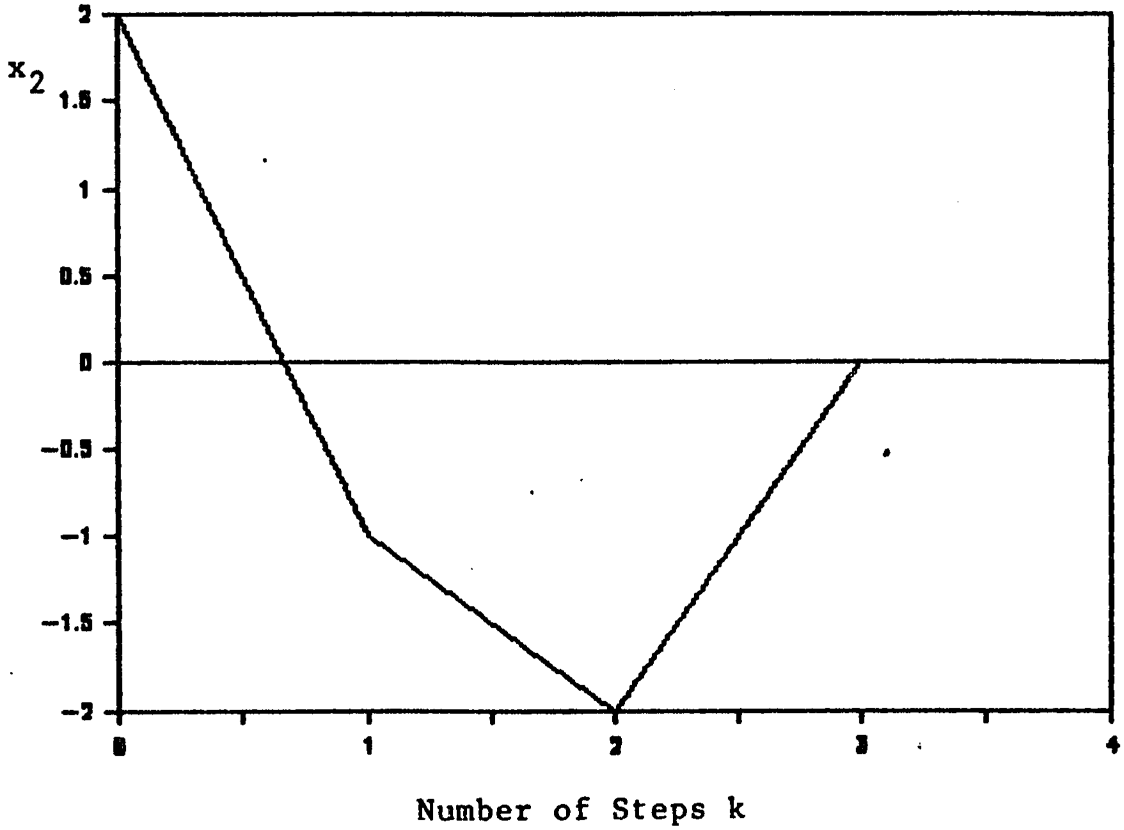


Fig. 5.2 Time Response of $x_2(k)$ For Ex. 5.1

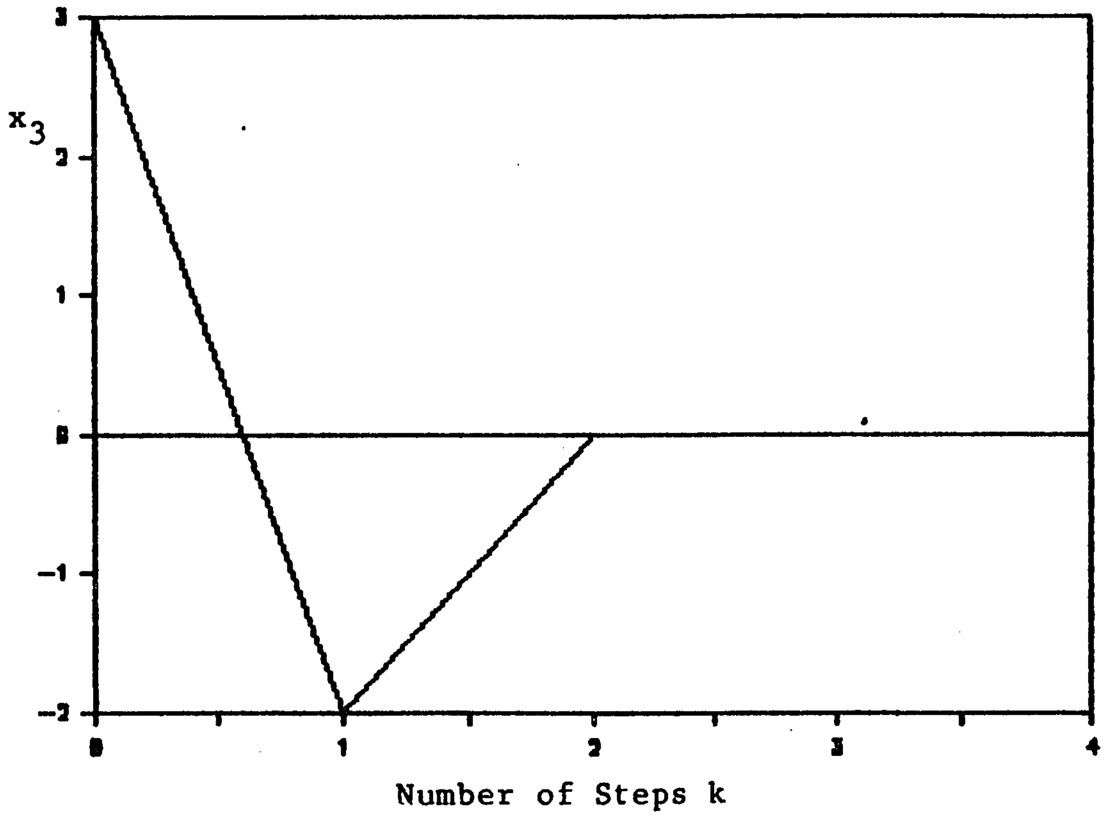


Fig. 5.3 Responses of $x_3(k)$ For Ex. 5.1

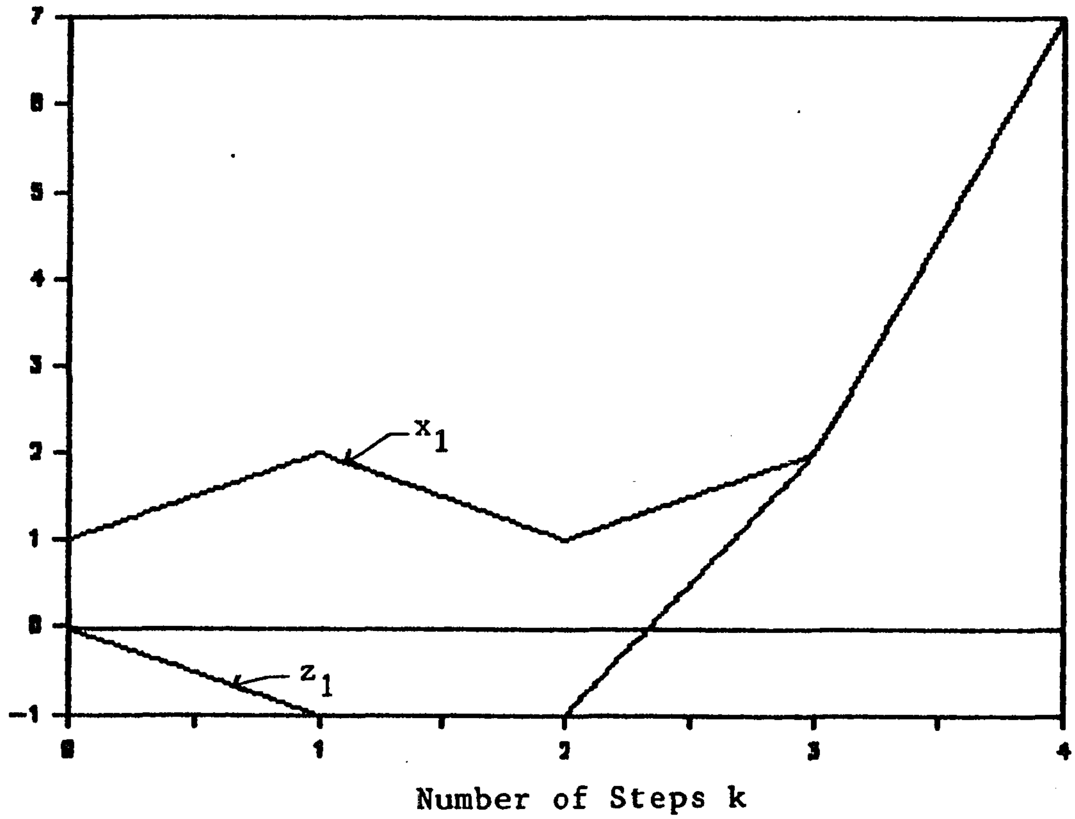


Fig. 5.4 Responses of x_1 and z_1 For Ex. 5.2

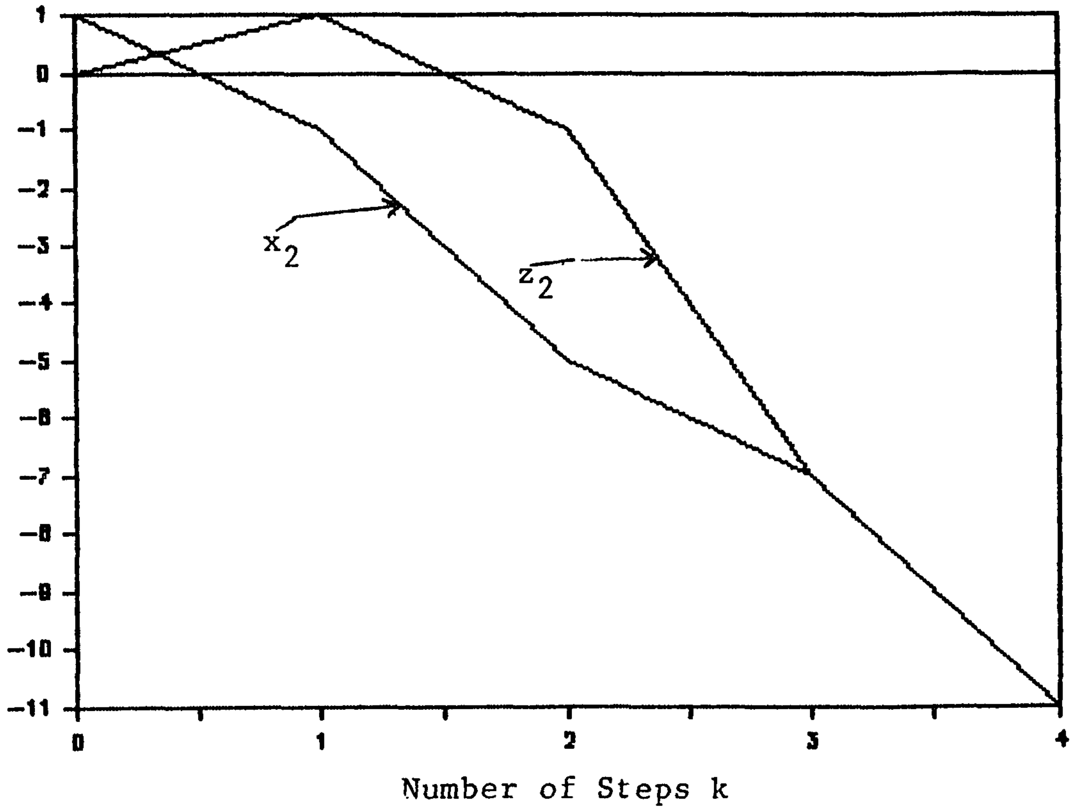


Fig. 5.5 Responses of x_2 and z_2 For Ex. 5.2

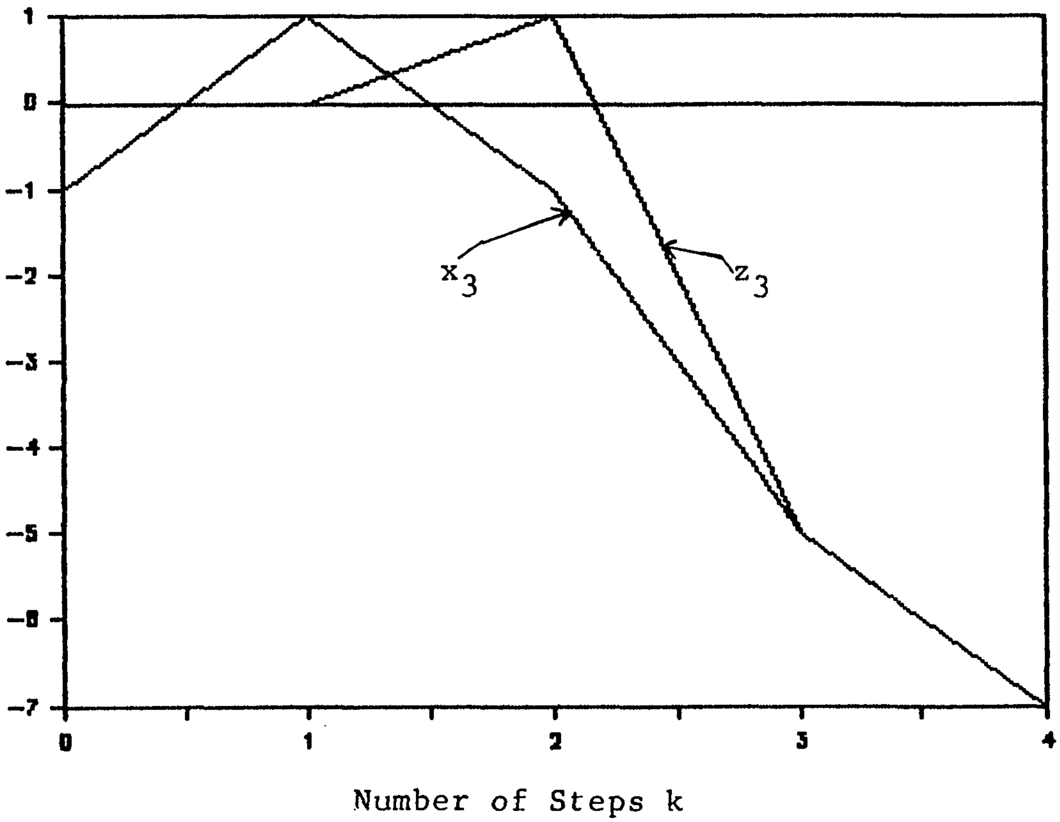


Fig. 5.6 Responses of x_3 and z_3 For Ex. 5.2

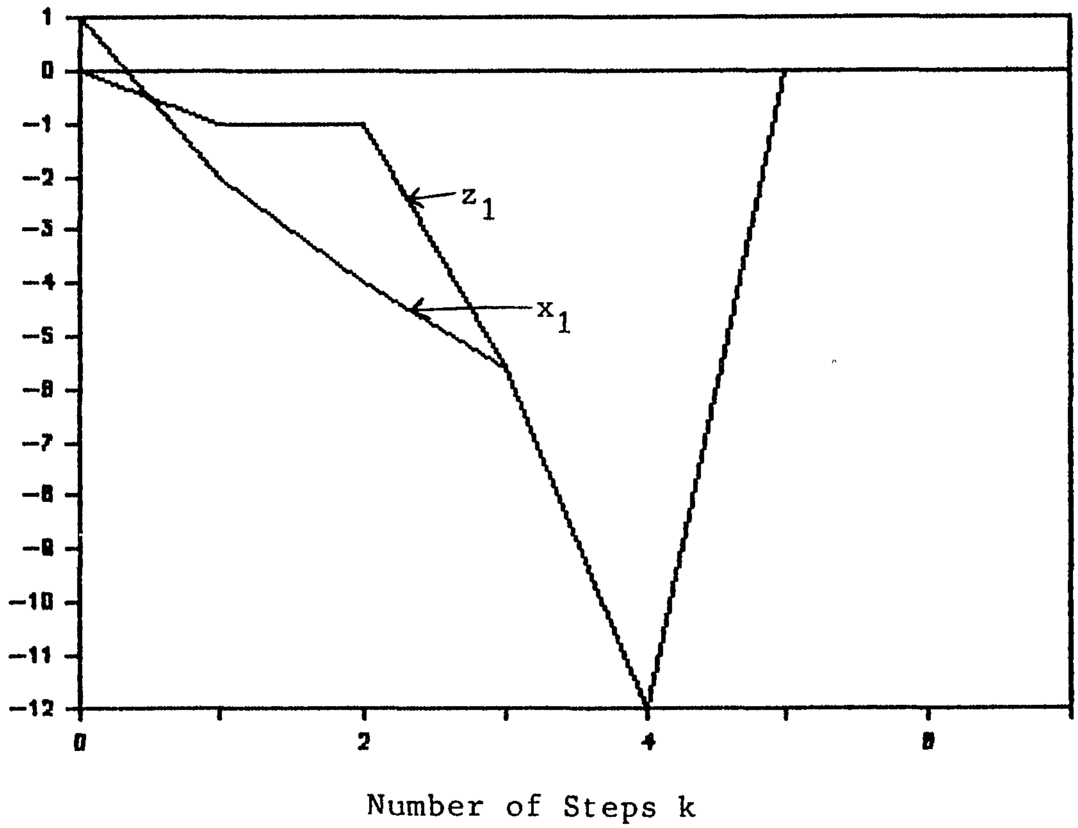


Fig. 5.7 Responses of x_1 and z_1 For Ex. 5.3

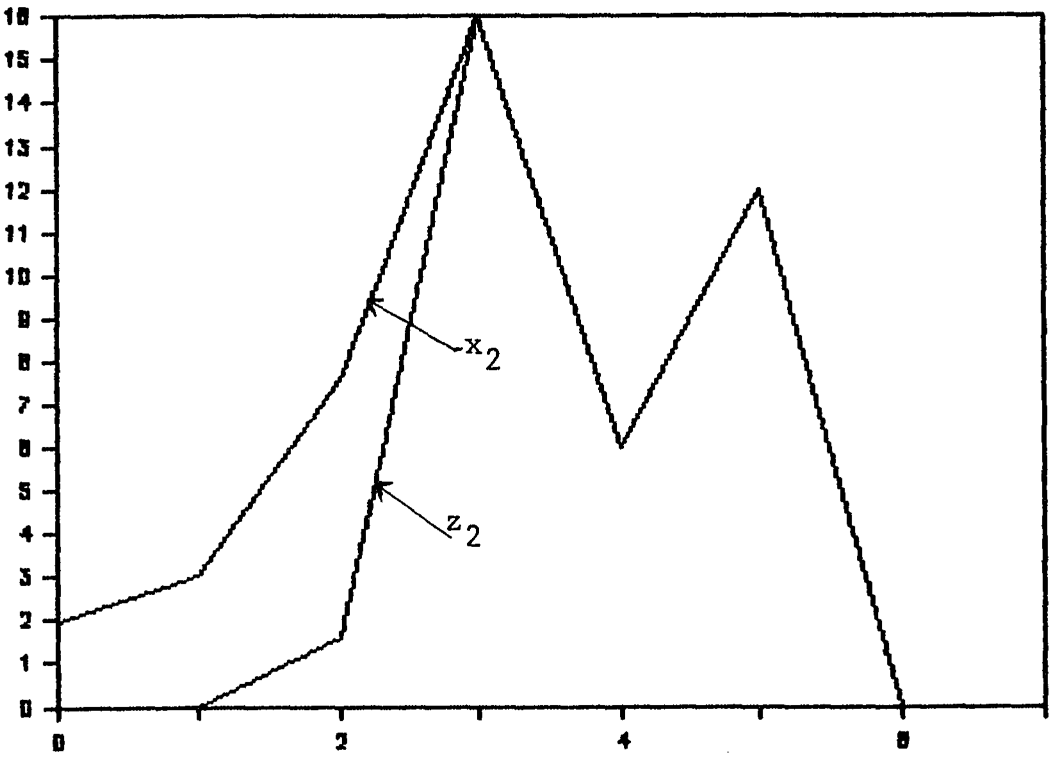


Fig. 5.8 Responses of x_2 and z_2 For Ex. 5.3

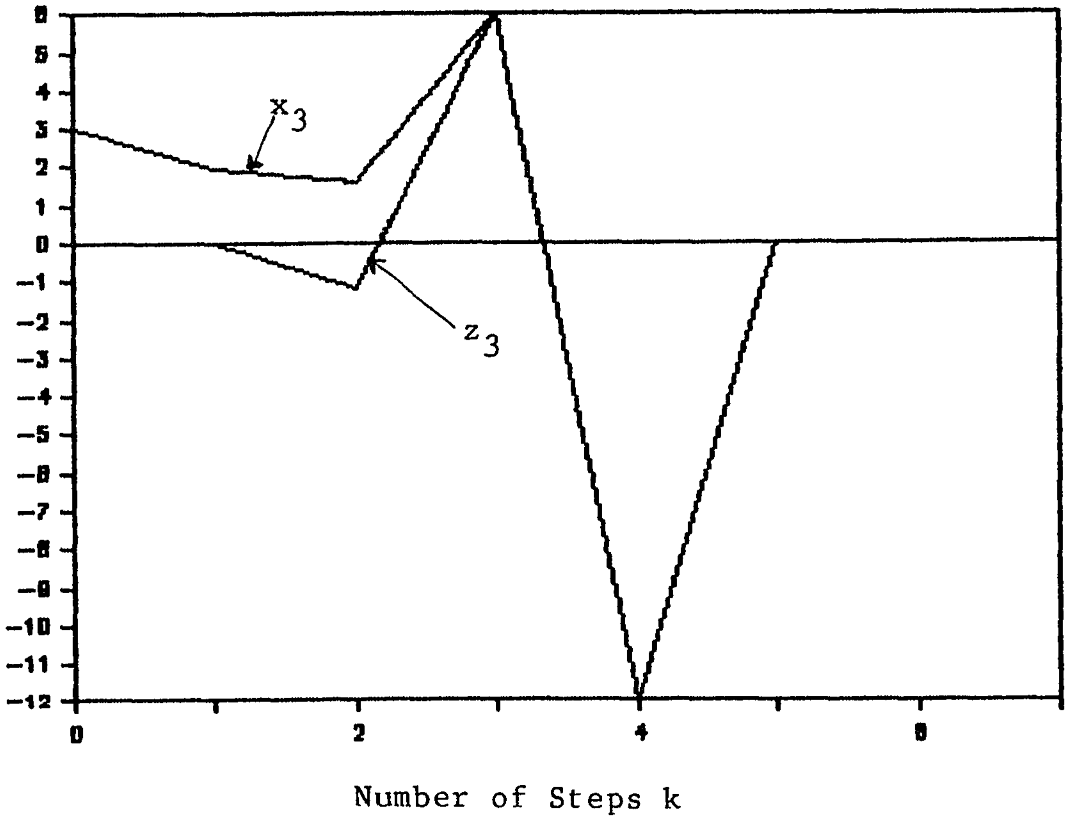


Fig. 5.9 Responses of x_3 and z_3 For Ex. 5.3

5.3 GAIN-CONSTRAINED OBSERVER-CONTROLLER DESIGN:

Relatively large controller gains result when a linear state- feedback control law is used to transfer the state of a linear discrete system from its initial state to the zero-state in the minimum number of steps. Hence, if a linear feedback controller is restricted to use gains that are constrained by an a priori magnitude bound, the state of the controlled process will be transferred to only a finite neighborhood of the zero-state in a finite number of steps. For this gain-constrained linear controller, it is of interest to have a set of relations or a design procedure that explicitly relates a priori controller gain magnitude constraints to the regulation accuracy achievable in a finite number of steps by a linear state-feedback control system.

Similarly arbitrary small reconstruction error may result in a finite number of steps with linear full-order observers whose gains may be prohibitively large. Again it is of interest to characterize the reconstruction accuracy achievable in a finite number of steps with gain-constrained observers.

The design of gain-constrained observers and controllers is the subject of the research described below. A gain-constrained state-feedback regulator problem is first formulated. Properties of the time-varying matrices used to generate the observer and controller gain matrices are then described. These properties are then used to generate the gains for the gain-constrained observer and controller design.

5.3.1 PROBLEM FORMULATION:

Given the time-invariant controllable discrete time linear system (5.9) such that $x(0) \in R_1$. If a region R_2, containing the zero-state, is specified, where $R_2 \subset R_1$, find (a) the control law

$$u(k) = -F(k)x(k) \tag{5.53}$$

subject to the constraint

$$|F_{ij}(k)| \leq F_{max} \quad i=1\ldots.r,\ j=1\ldots..,n \quad \text{for all } k \tag{5.54}$$

and (b) an estimate of the time k_b such that

$$x(k) \in R_1 \subset R_1 \tag{5.55}$$

for all $k \geq k_b$.

5.3.2 PROPERTIES OF $S(k_f,k,N)$ AND $W(k,k_f,N)$

The following are some of the properties of the S and W matrices which satisfy (5.13,(5.14) and (5.35),(5.36). For every nonsingular matrix S(k)

$$\nabla S(k) = S(k+1) - S(k) = -S(k)[S^{-1}(k+1) - S^{-1}(k)]S^{-1}(k+1)$$

$$= -S(k)\nabla S^{-1}(k)S(k+1) \qquad (5.56)$$

LEMMA 5.1: If the [A,B] pair of (5.9) is completely controllable and $S(k_f,k,N)$ exists, where $S(k_f,k,N)$ satisfies (5.13),(5.14) then

$$\text{(a)} \qquad \nabla S(k) \geq 0 \qquad (5.57)$$

$$\text{(b)} \qquad \nabla S(k_f) \leq 0 \qquad (5.58)$$

$$\text{(c)} \qquad \nabla S(N) \geq 0 \qquad (5.59)$$

PROOF:

$$S^{-1}(k_f,k,N) = N \sum_{i=1}^{k_f-k} N^{-i}A^{-i}BB^TA^{T^{-i}} \qquad (5.60)$$

Therefore $\quad \nabla S^{-1}(k) = S^{-1}(k_f,k+1,N) - S^{-1}(k_f,k,N)$

$$-N^{k-k_f+1}A^{k-k_f}BB^TA^{T^{k-k_f}} \leq 0 \qquad (5.61)$$

And thus from (5.56) $\nabla S(k) \geq 0$ (5.62)

Similarly $\nabla S^{-1}(k_f) = N^{k-k_f}A^{k-k_f-1}BB^TA^{T^{k-k_f-1}} \geq 0$ (5.64)

And thus $\nabla S(k_f) \leq 0$ (5.65)

Also $$\nabla S^{-1}(N) = \sum_{i=1}^{k-k_f}[(N+1)^{1-i}-N^{1-i}]\ A^{-i}BB^TA^{T^{-i}} \leq 0 \qquad (5.66)$$

And therefore $S^{-1}(N) \geq 0$ (5.67)

This completes the proof.

THEOREM 5.2: If the [A,B] pair is completely controllable, and $S^{-1}(k+1)$ is positive definite, then the input given by

$$u(k) = -F(k_1)x(k) \qquad (5.68)$$

where the constant gain $F(k_1)$ is the solution to (5.16) at $k = k_1$, produces stability for N = 1 and asymptotic stability for $N \geq 1$ in (5.9).

PROOF: Let V(x,k) be given by

$$V(x,k) = x^T(k)S(k_1+1)x(k) \qquad (5.69)$$

Thus V(x,k) is positive definite. Taking the difference of

(5.69) along the trajectory of (5.9) using (5.16),(5.20) and (5.21) at $k = k_1$ yields

$$V = x^T(k)[(A - BF(k_1))^T S(k_1+1)(A - BF(k_1)) - S(k_1+1)]x(k)$$

$$= x^T(k)[A^T S(k_1+1)A - A^T S(k+1)BF(k_1) - F^T(k_1)B^T S(k_1+1)A +$$

$$F^T(k_1)B^T S(k_1+1)BF(k_1) - S(k_1+1)]x(k)$$

$$= x^T(k)[A^T S(k_1+1)A - A^T S(k_1+1)B(I_r+NB^T S(k_1+1)B)^{-1}B^T S(k_1+1)AN$$

$$- A^T S(k_1+1)B(I_r+NB^T S(k_1+1)B)^{-2}B^T S(k_1+1)AN^2 - S(k_1+1)]x(k)$$

$$= x^T(k)[S(k_1)/N - S(k_1+1) - S(k_1)A^{-1}BB^T A^{T^{-1}} S(k_1)]x(k) \leq 0 \quad (70)$$

And thus by [5] the input given by (5.68) produces stability in (5.9). This completes the proof.

This theorem shows that each gain matrix of the sequence of gain matrices used to produce a deadbeat response can in itself generate a stabe system if S^{-1} is nonsingular.

LEMMA 2: If the [A,C] pair is completely observable and $W^{-1}(k,M)$ is nonsingular, then

$$\nabla W(k) \leq 0 \quad (5.71)$$

and $$\nabla W(M) \geq 0 \tag{5.72}$$

The proof is analogous to that of lemma 1 and is thus left out.

<u>**THEOREM 5.3**</u>: If the [A, C] pair is completely observable and $W^{-1}(k,M)$ is nonsingular, and $M > 1$ then the constant gain

$$G(k_1) = W(k_1+1)A^{T^{-1}}C^T \tag{5.73}$$

produces stability for $M = 1$ and asymptotic stability for $M \geq 1$ of the estimator error (5.31).

<u>PROOF</u>: Define the scalar function $V(e,k)$ as

$$V(e,k) = e^T(k)W^{-1}(k_1+1)e(k) \tag{5.74}$$

Then $V(e,k) > 0$. Taking the difference of (5.74) along the trajectory of (5.31) yields

$$\nabla V = e^T(k)[(A-G(k_1)C)^T W^{-1}(k_1+1)(A-G(k_1)C) - W^{-1}(k_1+1)]e(k)$$

$$= e^T(k)[(A - G(k_1)C)^T A^{T^{-1}} W^{-1}(k_1)/M - W^{-1}(k_1+1)e(k)$$

$$= -e^T(k)[W^{-1}(k_1+1) - W^{-1}(k_1)/M + C^T G^T(k_1)A^{T^{-1}} W^{-1}(k_1)/M]$$

$$=-e^T(k)[W^{-1}(k_1+1)-W^{-1}(k_1)/M + C^T[I_m+ MCW^{-1}(k_1)C^T]^{-1}C]e(k)$$

$$\leq 0 \tag{5.75}$$

And thus from [5.5] the gain given by (5.73) produces stability in (5.31). This completes the proof.

The previous lemmas and theorems clearly shows that asymptotic stability can be achieved for (5.9) and (5.31) where the control and observers gains are constants. From (5.21) and (5.44) one can see that the parameters N and M act as convergence factors. They determine the minimum magnitudes of the controller gain and observer gain. That is to say the eigenvalues of [A-BF(k)] and [A-G(k)C] will remain within certain boundary of the origin for fixed values of N and M. The larger the values of N and M, the closer to the origin will the eigenvalues of [A-BF(k)] and [A-G(k)C]. Thus as N and M take on very large values, near deadbeat can be produced using any constant controller gain and observer gain matrices. This is because F(k) and G(k) converge to their final values faster as N and M increase.

5.3.2 GAIN-CONSTRAINED CONTROLLER DESIGN:

The following describes the solution to the problem outlined in (5.53) - (5.55). Since the problem has no restriction on the maximum magnitude of the state allowed during the control interval prior to k_b we will use a constant gain control law. Since N affects the minimum values of the controller gain matrix, to ensure every possible gain which produces stability is examined, we will set k = 0 and N = 1. The problem reduces to finding a k_f such that

$$\max |F_{ij}(k_f,0,1)| \leq F_{max} \tag{5.76}$$

For N = 1 and k = 0, (5.60) reduces to

$$S^{-1}(k_f) = \sum_{i=1}^{k_f} A^{-1}BB^TA^{T^i} \tag{5.77}$$

Note that (5.75) is the solution to the following difference eqution

$$S^{-1}(k_f+1) = A^{-1}S^{-1}(k_f)A^{T^{-1}} + A^{-1}BB^TA^{T^{-1}} \tag{5.78}$$

$$S^{-1}(0) = 0 \tag{5.79}$$

Since the [A,B] pair is assumed completely controllable, then there exists a $k_f > n$ such that $S^{-1}(k_f)$ is invertible. Also for all $k_f < \infty$, the constant gain given by

$$F(k_f) = B^T A^{T^{-1}} S(k_f) \tag{5.80}$$

produce asymptotic stability in (5.9). Since the norm of $F(k_f)$ decreases with an increase in k_f, then the norm of F approaches a minimum as k_f approaches infinity. For invertible $S^{-1}(k_f)$, (5.78) can be written as

$$S(k_f+1) = A^T S(k_f)A - A^T S(k_f)B[I_r + B^T S(k_f)B]^{-1} B^T S(k_f)A \tag{5.81}$$

The design procedure for finding F which satisfies (5.76) can be stated as follows:

1) Numerically iterate (5.78),(5.79) to a nonsingular $S^{-1}(k_f)$.

2) Solve for F using (5.80). Check whether or not (5.76) is satisfied. If (5,76) is satisfied, then this gain is the required constant gain.

3) If not switch to (5.81) and continue the iteration using (5.80) to check if (5.76) is satisfied.

4) If (5.81) converges to a gain which does not satisfy (5.76) then this design procedure can not be used to meet the speification.

Once the required F is found, the time k_b can be estiamated by examination of the eigenstructure of the [A - BF] matrix. The closed-loop system becomes

$$x(k+1) = [A - BF]x(k) \tag{5.82}$$

whose solution is given by

$$x(k) = [A - BF]^k x(0) \tag{5.83}$$

Suppose the matrix [A - BF] has m distinct characteristic values λ_i, i = 1,2,...,m. Let the multiplicity of each characteristic value in the characteristic polynomial of [A - BF] be given by m_i. Then there exists a transformation T [20] such that

$$[A - BF] = TJT^{-1} \tag{5.84}$$

and

$$[A - BF]^k = TJ^kT^{-1} \tag{5.85}$$

where J is a matrix in Jordan canonical form. We now find a bound on $||J^k||$ as follows: Let λ_{max} satisfy $|\lambda_i| \leq \lambda_{max} < 1$. Since all the eigenvalues of [A - BF] have eigenvalues of moduli less than one, define $||C|| = \sqrt{\text{Trace } C^TC}$, then

$$||J^k||^2 = \sum_{i=1}^{m} ||J_i{}^k||^2$$

$$= \sum_{i=1}^{m} m_i |\lambda_i|^{2k} + \sum_{i=1}^{m} \sum_{r=1}^{m_i-1} (m_i - r) \binom{k}{r}^{2k} |\lambda_i|^{2(k - r)}$$

$$\leq [\sqrt{n} + \sum_{i=1}^{m} \sum_{r=1}^{m_i-1} \sqrt{(m_i - r)} |\lambda_i|^{-r} \left[{k_c \atop r} \right] [|\lambda_i|/\lambda_{max}]^{k_c}]^{2} \lambda_{max}{}^{2k}$$

$$= C_1 (\lambda_{max})^{2k} \qquad (5.85)$$

$$C_1 = \left[\sqrt{n} + \sum_{i=1}^{m} \sum_{r=1}^{m_i-1} \sqrt{(m_i - r)} |\lambda_i|^{-r} \left\{ {k_c \atop r} \right\} \left[|\lambda_i|/\lambda_{max}\right]^{k_c} \right] \qquad (5.87)$$

and

$$k_c = \text{integervalue}\left\{(r-1+|\lambda_i|/\lambda_{max})/(1-|\lambda_i|/\lambda_{max})\right\} \qquad (5.87a)$$

Let R_2 be the region such that

$$||x(k)|| \leq x_{min} \qquad (5.88)$$

To satisfy (5.85) we require

$$||x(k)|| \leq ||T||||T^{-1}||C_1\lambda_{max}{}^k||x(0)|| \leq x_{min} \quad (5.89)$$

Solution to (5.85) results in

$$K_b \geq -\ln[C_1||T||||T^{-1}||||x(0)||/x_{min}]/\ln(\lambda_{max} \quad (5.90)$$

If the $\max|\lambda_i|$ is of multiplicity one, then λ_{max} can be taken as $|\lambda_i|$.

5.3.3 GAIN-CONSTRAINED OUTPUT-FEEDBACK REGULATOR:

The design procedure of 5.3 can be easily extended to the case where the complete state is unavailable for direct measurement. This section deals with the design of a dynamic state estimator and the form of the output-feedback control system where there are constraints on both the controller and observer gain matrices. In addition it gives an estimate of the time required to bring the state estimator error from any initial value to a given fraction of its initial norm and also the number of steps required to reach a prescribed region enclosing the zero state of the system. Since the difference equation used to generate the observer's gain is the dual of the difference equation used to generate the controller's gain, a detailed treatment of its properties is not given here.

The problem of 5.3.1 can now be reformulated as follows: Let $\bar{R}_2$ be specified as

$$||e(k)|| \leq D||e(0)|| \tag{5.91}$$

where D > 0 is an a priori condition. Then given the time-invariant system (5.9),(5.28) where $e(0) \in \bar{R}_1$ and $\bar{R}_2 \subset \bar{R}_1$, where $\bar{R}_1$ the region of possible initial observation error is assumed known, find (a) the gain G subject to

$$|G_{ij}| \leq G_{max} \tag{5.92}$$

where G_{max} is specified.

(b) the time k_{bo} such that (5.91) is satisfied for all $k \geq k_{bo}$.

(c) the feedback control law

$$u(k) = -Fz(k) \tag{5.93}$$

subject to the constraint

$$|F_{ij}| \leq F_{max} \tag{5.94}$$

where H, D, F_{max} and G_{max} are specified quantities. Note that section 5.3.2 has already described the technique for generating F. From lemmma 5.2, theorem 5.3 and the similarity between (5.78) and (5.35), the algorithm for generating a G to satisfy (5.92) is given as follows:

(a) Iterate (5.35),(5.36) up to a nonsingular $W^{-1}(k)$,

(b) Using (5.41) solve for G,

(c) Check to see if (5.92) is satisfied,

(d) If (5.92) is satisfied, then this gain is the required one,

(e) If not switch to (5.43) and continue the iteration using (5.44) until (5.92) is satiisfied.

(f) If G converges to a value which is greater than G_{max}, then the design procedure can not be used to meet the required specification. In order that the widest range of gains are examined the value of M must be set equal to 1.

Once the required G is found, the number of steps k_{bo} is found in an analogous manner to k_b of 5.3 to satisfy

$$k_{bo} \geq -\ln[D_1||T_1||||T_1^{-1}||D]/\ln(\lambda_{max}) \qquad (5.96)$$

where λ_{max} is greater than or equal to eigenvalue of [A-GC] with the largest moduli and D_1 and T_1 are analogous to C_1 and T respectively for the matrix [A - GC]. To examine the response of the output-feedback system, write z = x - e and substitute into (5.9) to yield

$$x(k+1) = (A - BF)x(k) + BFe(k) \qquad (5.97)$$

The solution to (5.31) with G constant is given by

$$e(k) = (A - GC)^k e(0) \tag{5.98}$$

And thus the solution to (5.97) is given by

$$x(k) = (A-BF)^k x(0) + \sum_{i=0}^{k-1} (A-BF)^{k-i-1} BF(A-GC)^i e(0) \tag{5.99}$$

Let $|\lambda\ (A-BF)| \leq \lambda_c < 1$, and $|\lambda(A-GC)| \leq \lambda_o < 1$, then for

$$[A - BF]^k = T_c J_c^k T_c^{-1} \tag{5.100}$$

then $$||(A - BF)^k|| \leq ||T_c||\,||T_c^{-1}||C_1\lambda_c^k = C_2\lambda_c^k \tag{5.101}$$

where $$C_2 = C_1||T_c||\,||T_c^{-1}|| \tag{5.102}$$

and C_1 is given as in (5.86).

$$[A - GC]^k = T_o J_o^k T_o^{-1} \tag{5.103}$$

then

$$||(A - GC)^k|| \leq ||T_o||\,||T_o^{-1}||D_1\lambda_o^k = D_2\lambda_o^k \tag{5.104}$$

where $$D_2 = D_1||T_o||\,||T_o^{-1}|| \tag{5.105}$$

and where D_1 is analogously defined as C_1. From (5.99)

$$||x(k)|| \le C_2||x(0)||\ \lambda_c^{k} + \sum_{i=1}^{k-1} C_2\lambda_c^{k-i-1}||BF||D_2\lambda_o^{k}||e(0)||$$

$$= C_2||x(0)||\lambda_c^{k} + C_2D_2||BF||\,||e(0)||[\lambda_o^{k} - \lambda_c^{k}]/(\lambda_o-\lambda_c) \qquad (5.106)$$

Assume further that $z(0) = 0$ and hence $e(0) = x(0)$. Thus to satisfy (5.95) we require that

$$H \ge [C_2 + C_2D_2||BF||/(\lambda_c-\lambda_o)]\lambda_c^{k} - [C_2D_2||BF||/(\lambda_c-\lambda_o)]\lambda_o^{k} \qquad (5.107)$$

EXAMPLE 5.4: Consider the system given by (5.9) where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (5.108)$$

Let it be required that $x_{min} = 10^{-3}\ ||x(0)||$ and $F_{max} = 1$. Then from the solution of (5.78) and (5.79) and (5.80) result in the required $F = [-0.7,\ 1]$. This results in

$$A - BF = \begin{bmatrix} 0 & 1 \\ -0.3 & 1 \end{bmatrix} \qquad (5.109)$$

This results in $C_1||T||\,||T^{-1}|| = 8.222$ and $\lambda_{max} = (0.3)^{1/2}$. Let $x_{min} = 10^{-3}||x(0)||$. From (5.90) k_b is found to satisfy $k_b \le 15$. The system was simulated with $x(0) = [1,\ 2]^T$. The actual k_b is found to be greater than or equal to 13. Fig. 5.10 shows the trajectory of the simulated system.

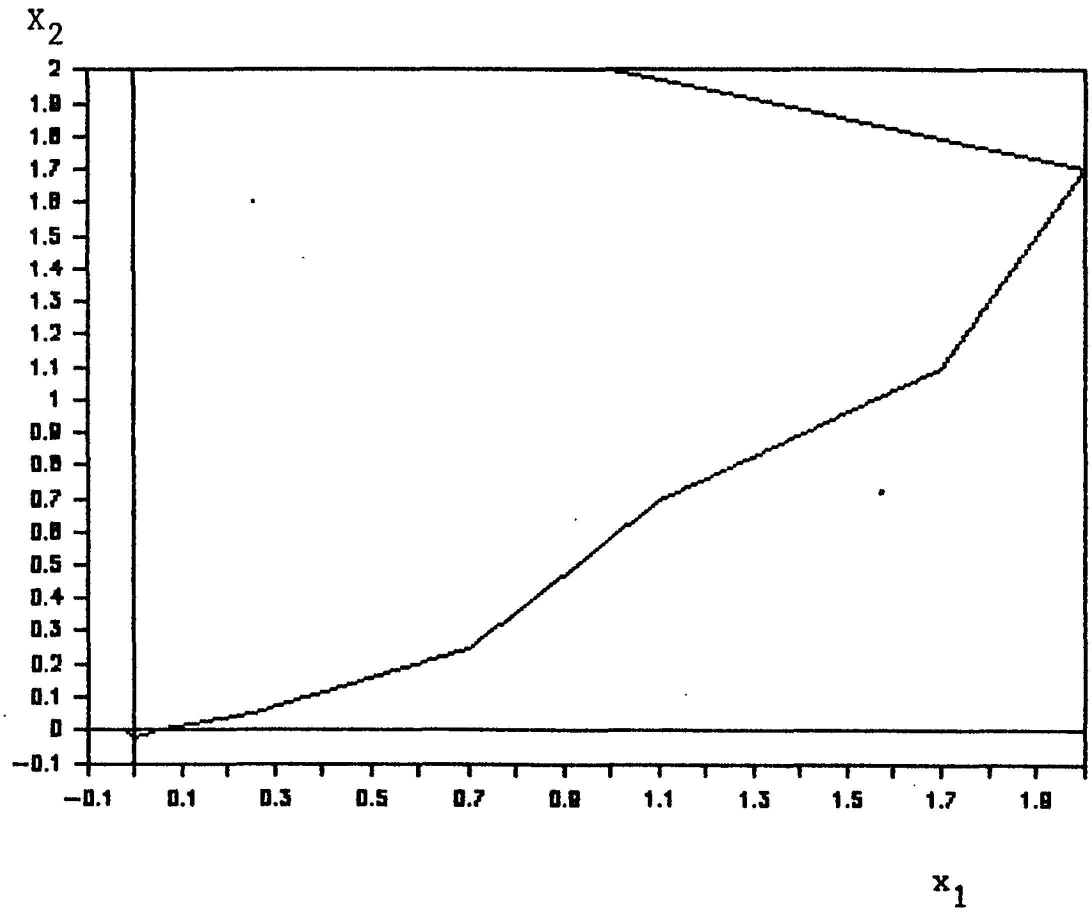


Fig. 5.10 Trajectory of x_1 and x_2 For Ex. 5.4

EXAMPLE 5.5 Consider the system given by (5.9),(5.28) where A and B are given as in example 4 and where

$$C = [1, 0] \tag{5.110}$$

where it is required that $H = 10^{-4}$, $D = 10^{-3}$, $F_{max} = 1$ and $G_{max} = 2$. Application of the theories developed in 5.3 and 5.4 result in $F = [-0.7, 1]$ and $G = [1.3333, 1.83333]^T$, $\lambda_c = (0.3)$ and $\lambda_o = (1/6)$, $C_2 = 8.2219$, $D_2 = 23$ and $k_{bc} \geq 26$. The system was simulated with $x(0) = [1, 2]^T$ and $z(0) = [0,0]^T$. Figs. 5.11 and 5.12 show the result. For this simulation k_{bc} was found to satisfy $k_{bc} \geq 20$.

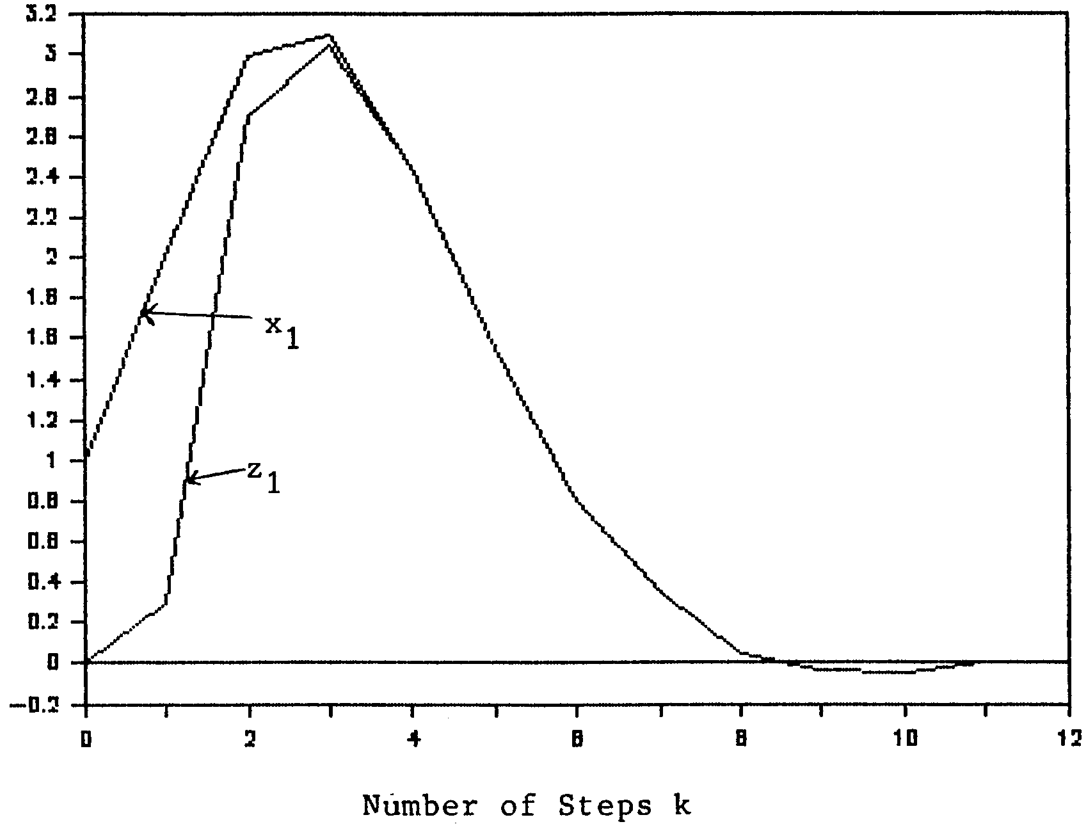
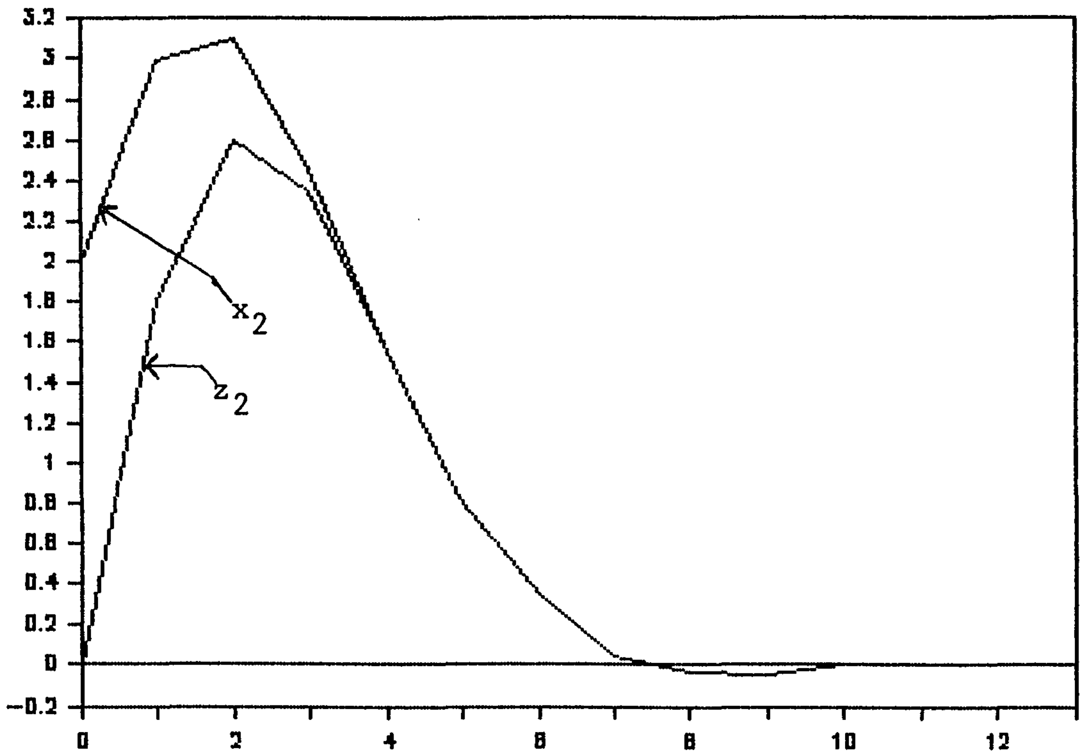


Fig. 5.11 Time Responses of x_1 and z_1 For Ex. 5.5



Number of Steps k

Fig. 5.12 Time Responses of x_2 and z_2 For Ex. 5.5

5.4 OBSERVER-CONTROLLER DESIGN WITHOUT MATRIX INVERSION:

In estimating and controlling the state of a linear discrete system, if the main objective is to transfer the process state to the origin in a minimum number of steps, this can be accomplished by placing the eigenvalues closed-loop system matrix at the origin. This will force both the estimator's error and system state at the origin in a minimum number of steps. Thus the process can be accomplished in at most 2n steps where n is the order of the system. Placement of all eigenvalues at the same location has proven to be too sensitive a design in respect to system noise. Several other techniques have accomplished deadbeat response using a discrete Riccati equation to generate the observer and controller gain matrices. These techniques have proven to be both optimal and deadbeat. It is noted that each of these techniques involves matrix inversion which is a time consuming computational process. The following method describes a way of accomplishing near deadbeat response without requiring matrix inversion. It is an application of the second method of Lyapunov for discrete systems where matrix inversion is replaced by scalar division. This scalar division acts as a normalization factor for the estimator gain matrix and analogously for the controller gain matrix. This divisor takes the form of the N^{th} root of the N power of the trace of a matrix where N is a positive integer. It is shown that as N approaches infinity, the estimator error converges more rapidly. Similar results hold for the closed-loop system state.

The design is divided into three parts. First, the state of the system is assumed known and the controller is designed which forces the state of the system to some specified region which includes the system's zero-state. Second, an observer is designed to accomplish the transfer of the observer's error to a required norm is designed. And third, the observer and controller methods are combined to lead to an output-feedback controlled system. Numerical examples are given to demonstrate both the observer and output-feedback controlled system.

5.4.1 CONTROLLER DESIGN:

Consider the linear time-invariant discrete system given by (5.9) where u is given by (5.10). Define the Lyapunov-like function V(x,k) such that

$$V(x,k) = x^T(k)S(k)x(k) \tag{5.111}$$

where S(k) satisfies

$$S(k) = A^T S(k+1)[I_n - BB^T S(k+1)/M(k+1)]^2 A \tag{5.112}$$

$$S(L) = I_n \tag{5.113}$$

where $$M(k+1) = \left[\text{Trace}[(B^T S(k+1)B)^N] \right]^{1/N} \quad (5.114)$$

and $N \geq 1$ and L is a finite positive scalar. Then

$$\nabla V = V(xk1,k+1) - V(x(k),k)$$

$$= x^T(k+1)S(k+1)x(k+1) - x^T(k)S(k)x(k)$$

$$= x^T(k)[(A-BF(k))^T S(k+1)(A-BF(k)) - S(k)]x(k) \quad (5.115)$$

Let $$F(k) = B^T S(k+1)A/M(k+1) \quad (5.116)$$

Substituting (5.116) into (5.115) yields

$$\nabla V = 0 \quad (5.117)$$

Thus if S(k) is nonnegative definite for all k, then the input given by (5.10),(5.116) will produce a stability in (5.9). Note that (5.112) can be written as

$$S(k) = (A - BF(k))^T S(k+1)(A-BF(k))$$

$$= A^T[I_n - BB^T S(k+1)/M(k+1)]^T S(k+1)[I_n - BB^T S(k+1)/M(k+1)] \quad (5.117)$$

$$S(L - 1) = A^T[I_n - BB^T/M(L)]^T[I_n - BB^T/M(L)]A$$

$$= A^T R_1^T R_1 A \tag{5.118}$$

where $$R_1 = [I_n - BB^T/M(L)] \tag{5.119}$$

And thus

$$S(L-2) = A^T[I_n - BB^T S(L-1)/M(L-1)]^T S(L-1)[I - BB^T S(L-1)/M(L-1)]$$

$$= A^{T^2} R_1^T [I_n - R_1 ABB^T A^T R_1/M(L-1)]^T [I_n - R_1 ABB^T A^T R_1] R_1 A^2$$

$$= A^{T^2} R_1^T R_2^T R_2 R_1 A^2 \tag{5.120}$$

Similarly it can be shown that

$$S(L-k) = A^{T^k} \left\{ \prod_{i=1}^{k} R_i^T \right\} \left\{ \prod_{i=1}^{k} R_i^T \right\}^T A^k \tag{5.121}$$

where

$$R_i = \left\{ I_n - \frac{\left[\prod_{j=1}^{i-1} R_j^T \right] \left(A^{i-1}B\right) \left(A^{i-1}B\right)^T \left[\prod_{j=1}^{i-1} R_j^T \right]}{M(L-i+1)} \right\} \tag{5.122}$$

and

$$M(L-i+1) = \left[Tr \left[\left(A^{i-1}B\right)^T \left\{ \prod R_j^T \right\} \left\{ \prod R_j^T \right\}^T \left(A^{i-1}B\right) \right]^N \right]^{1/N} \tag{5.123}$$

Given that (5.9) is completely controllable, then the rank of $[B, AB, \ldots, A^{n-1}B]$ is n and thus the R_i's exists and thus $S(L-k)$ is nonnegative definite. Thus the control law given by (5.10) where F is given by (5.116) drives the state of (5.9) from any initial state to the origin.

Now let us examine the effect of N on the system trajectory. Note that $B^T S(k+1)B$ is a symmetrical matrix. We have

$$M(k+1) = \left[Tr[(B^T S(k+1)B)^N]\right]^{1/N}$$

$$= \left[\sum_{i=1}^{r} \lambda_i^N \right]^{1/N} \qquad (5.124)$$

where λ_i is the i^{th} eigenvalue of $[B^T S(k+1)B]$. From (5.116) that the norm of F increases with increasing M. Thus set of gains that yields the fastest response occurs when M(k+1) is at its minimum value. As N gets larger for any k, M(k+1) gets smaller. And thus

$$\min M(k+1) = \lim_{N \to \infty} [Tr[B^T S(k+1)B]]^{1/N}$$

$$= \lim_{N \to \infty} \left[\sum_{i=1}^{r} \lambda_i^N \right]^{1/N}$$

$$= \lambda_{max}(B^T S(k+1)B) \tag{5.125}$$

Therefore the optimal gain is given by

$$F^o(k) = B^T S(k+1)A/\lambda_{max}(B^T S(k+1)B) \tag{5.126}$$

After L steps the solution to (5.9) becomes

$$x(L) = [A-BF(L-1)][A-BF(L-2)].......[A-BF(0)]x(0)$$

$$= Rx(0) \tag{5.127}$$

where $R = [A-BF(L-1)][A-BF(L-2)].....[A-BF(0)]$ (5.128)

Also after L steps backward, the solution to (5.118) becomes

$$S(0) = R^T R \tag{5.129}$$

Thus $\quad Tr(S(0)) = Tr(R^T R) = ||R||^2$ (5.130)

Thus from (5.127) and (5.130)

$$\sqrt{[Tr(S(0))]} \geq ||x(L)||/||x(0)|| \tag{5.131}$$

Let it be required that

$$||x(L)|| \leq x_{min} \tag{5.132}$$

where x_{min} is some region containing the zero state. We can satisfy (5.132) as follows

$$||x(L)|| \leq ||R||\,||x(0)|| \leq x_{min} \tag{5.133}$$

Thus $$[Tr(S(0))]^{1/2} \leq x_{min}/||x(0)|| \tag{5.134}$$

Therefore to satisfy the requirement (5.132), we should solve (5.116) and (5.118) using (5.113) storing the gain generated for each step backward until (5.134) is satisfied. Thus not only the required number of steps to satisfy (5.132) is known but also the gain.

5.4.2 OBSERVER DESIGN:

The design of the observer is analogous to that of the design of the controller discussed in the last section. The gain of the observer is generated forward in time. It requires no matrix inversion not memory. Consider (5.28) - (5.31). Let us assume that the [A,C] pair is completely observable. Define the Lyapunov-like function V(e,k) as follows:

$$V(e,k) = e^T(k)W(k)e(k) \tag{5.135}$$

where $W(k)$ satisfies

$$W^{-1}(k+1) = AW^{-1}(k)[I_n - C^T C W^{-1}(k)/Q(k)]^2 A^T \qquad (5.136)$$

$$W^{-1}(0) = I_n \qquad (5.137)$$

$$\nabla V = e^T(k)[(A-G(k)C)^T W(k+1)(A-G(k)C) - W(k)] \qquad (5.138)$$

where $$Q(k) = [Tr(CW^{-1}(k)C^T)^M]^{1/M} \qquad (5.139)$$

and $M \geq 1$. Let the gain G be given by

$$G(k) = AW^{-1}(k)C^T/Q(k) \qquad (5.140)$$

From (5.136) and (5.140)

$$W^{-1}(k+1) = (A - G(k)C)W^{-1}(k)(A - G(k)C)^T \qquad (5.141)$$

And thus $W^{-1}(k)$ inverse exists

$$W(k) = (A - G(k)C)^T W(k+1)(A - G(k)C) \qquad (5.142)$$

Thus from (5.138) $$\nabla V = 0 \qquad (5.143)$$

Let us investigate the solution to (5.136),(5.137). It can be easily shown that $W^{-1}(k)$ satisfies

$$W^{-1}(k) = A^k\left\{\prod R_i\right\}\left\{\prod R_i\right\}^T A^{k^T} \qquad (5.144)$$

$$R_i = \left[I_n - \frac{\left\{CA^{i-1}\prod_{j=1}^{i-1} R_j\right\}^T\left\{CA^{i-1}\prod_{j=1}^{i-1} R_j\right\}}{Q(i-1)}\right] \qquad (5.145)$$

$$R_1 = I_n - C^T C/Q(0) \qquad (5.146)$$

$$Q(i-1) = \left[TR\left\{\left(CA^{i-1}\prod_{j=1}^{i-1} R_j\right)\left(CA^{i-1}\prod_{j=1}^{i-1} R_j\right)^T\right\}^M\right]^{1/M} \qquad (5.147)$$

Thus analogous to that discussed for the controller design, we conclude that since the [A, C] pair is completely controllable, $W^{-1}(k)$ exists and is nonnegative definite. Thus W(k) exists and is positive-definite. Thus the gain given by (5.140) produce stability in the error equation (5.31).

The gain that produces the fastest response in (5.31) is

$$G^o(k) = AW^{-1}(k)C^T/\min Q(k)$$

$$= AW^{-1}(k)C^T/[\lambda_{max}(CW^{-1}(k)C^T)] \qquad (5.148)$$

The solutionm to (5.31) is given by

$$e(k) = [A-G(k-1)C][A-G(k-2)C]........[A-G(0)C]$$

$$= \underline{R}\, e(0) \qquad (5.149)$$

where $\underline{R} = [A-G(k-1)C][A-G(k-2)C]......[A-G(0)C]$ (5.150)

while the solution to (5.141),(137) is given by

$$W^{-1}(k) = \underline{R}\,\underline{R}^T \qquad (5.151)$$

$$Tr(W^{-1}(k)) = ||\underline{R}||^2 \qquad (5.152)$$

And as before to satisfy the error requirement such as

$$||e(k)|| \le e_{min} \qquad (1.153)$$

we set $||e(k)|| \le ||\underline{R}||\,||e(0)|| < e_{min}\, D||e(0)||$ (1.154)

where D is a predetermined fraction. Therefore from (5.152)

$$[Tr(W^{-1}(k))]^{1/2} < D \qquad (1.155)$$

The value of K when (5.155) is satisfied is the number of steps required to reach a predetermined region which includes the zero-error state. Note that when r= 1 and m =1, the method reduces to that of a deadbeat system, since the trace becomes the actual inverse. And thus both the estimator error and system state will be driven to zero in n or less steps.

5.4.3 THE OUTPUT-FEEDBACK REGULATOR:

The techniques developed for designing the controller and observer gains can be combined to form an output-feedback regulator. After the state estimation error has reached a prescribed region, the control law is put into effect by replacing x by z in (5.10). Since the observer's gain is generated forward in time, the gain after the error has reached a required norm can be easily generated, and thus the estimation error will remain within the prescribed region until the required state norm is achieved.

A much more effective algorithm is to apply the control law (5.10) with x replaced by z while observing the system in order to prevent large state norms during observation. After the estimation error has reached it required norm, the control law is then switched, using the sequence of gains starting at F(0) and proceeding to F(L-1). The required equations are

$$x(k+1) = Ax(k) - BF(k)z(k) \tag{5.156}$$

$$z(k+1) = Az(k) + G(k)[y - Cz(k)] - BF(k)z(k) \tag{1.157}$$

with $k = 0,1,\ldots.\underline{k}$, where $\underline{k}$ is the number of steps required to reach a prescribed error norm. After $\underline{k}$, the equations are switched to

$$x(k+1) = Ax(k) - BF(k^*)z(k) \tag{5.158}$$

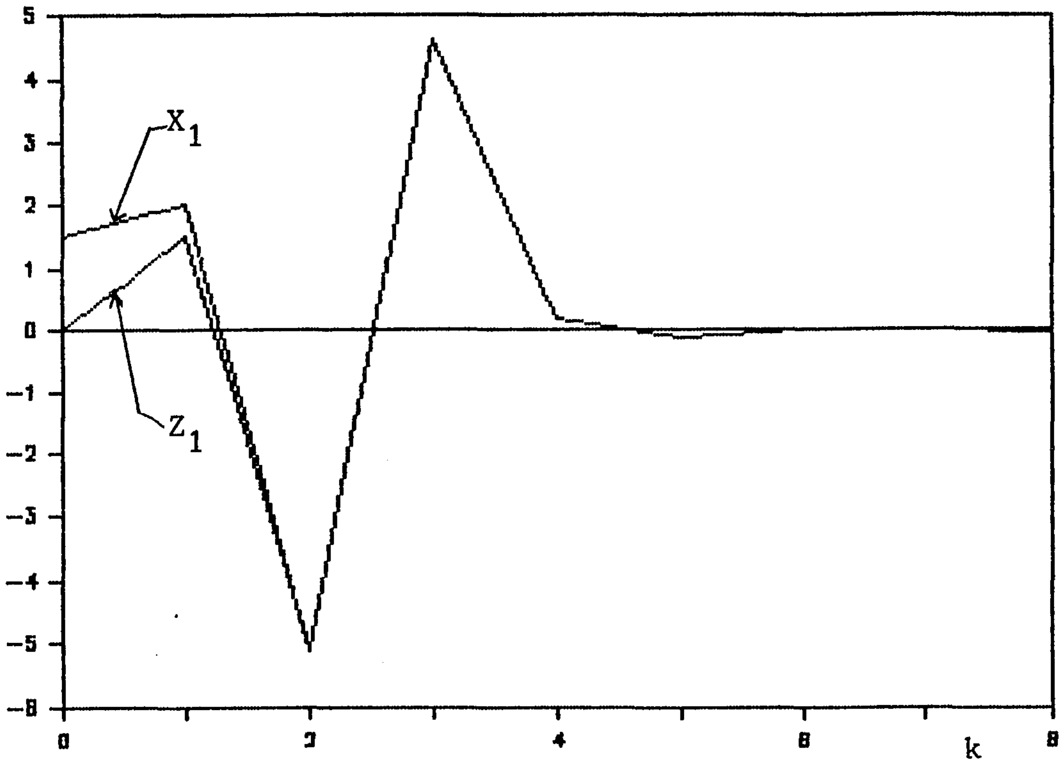
$$z(k+1) = Az(k) + G(k)[y - Cz(k)] - BF(k^*)z(k) \tag{5.159}$$

for $k^* = 0,1,\ldots\ldots.L.$

EXAMPLE 5.6: Consider the system given by (5.9) and (5.28) where

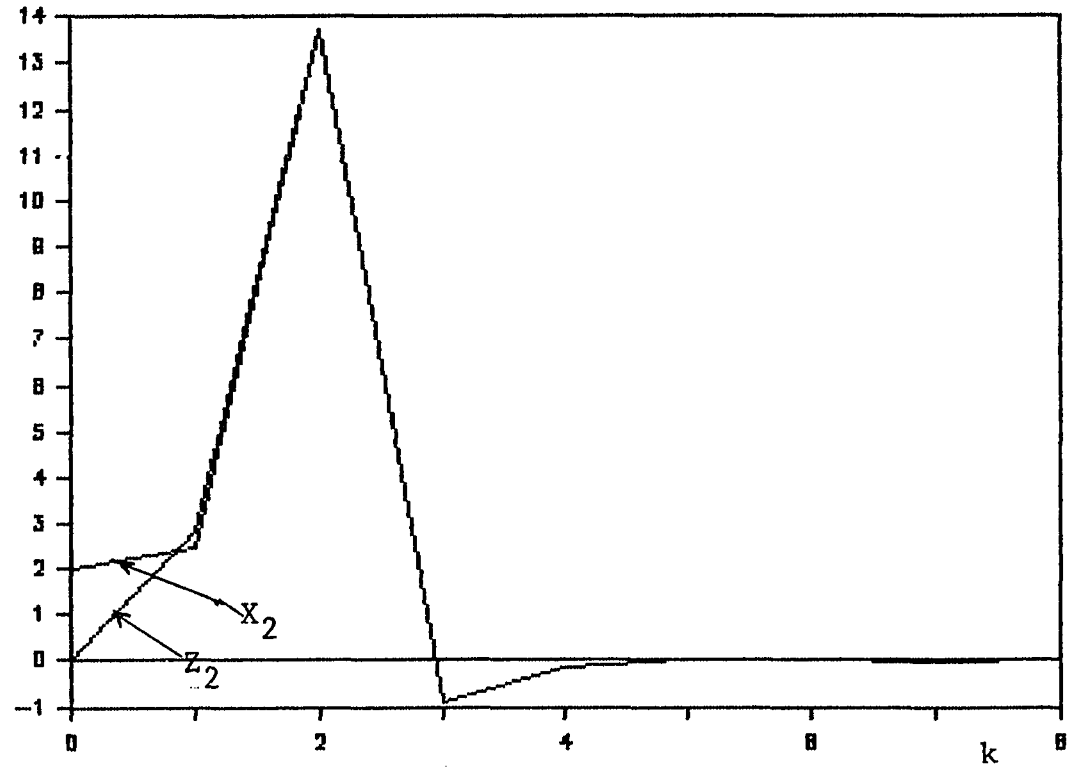
$$A = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.5 & 1.0 & 1.5 \end{bmatrix} \quad B = \begin{bmatrix} 0.0 & 1.0 \\ 1.0 & 0.0 \\ -1. & 1.0 \end{bmatrix} \quad C = \begin{bmatrix} 1.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 1.0 \end{bmatrix}$$

The system was simulated with $N = M = \infty$, $L = 6$, $x(0) = [1.5, 2.0, 2.5]^T$ and $z(0) = [0.0, 0.0, 0.0]^T$. Figs. 5.13 - 5.15 show the responses. Note that the response is near deadbeat.



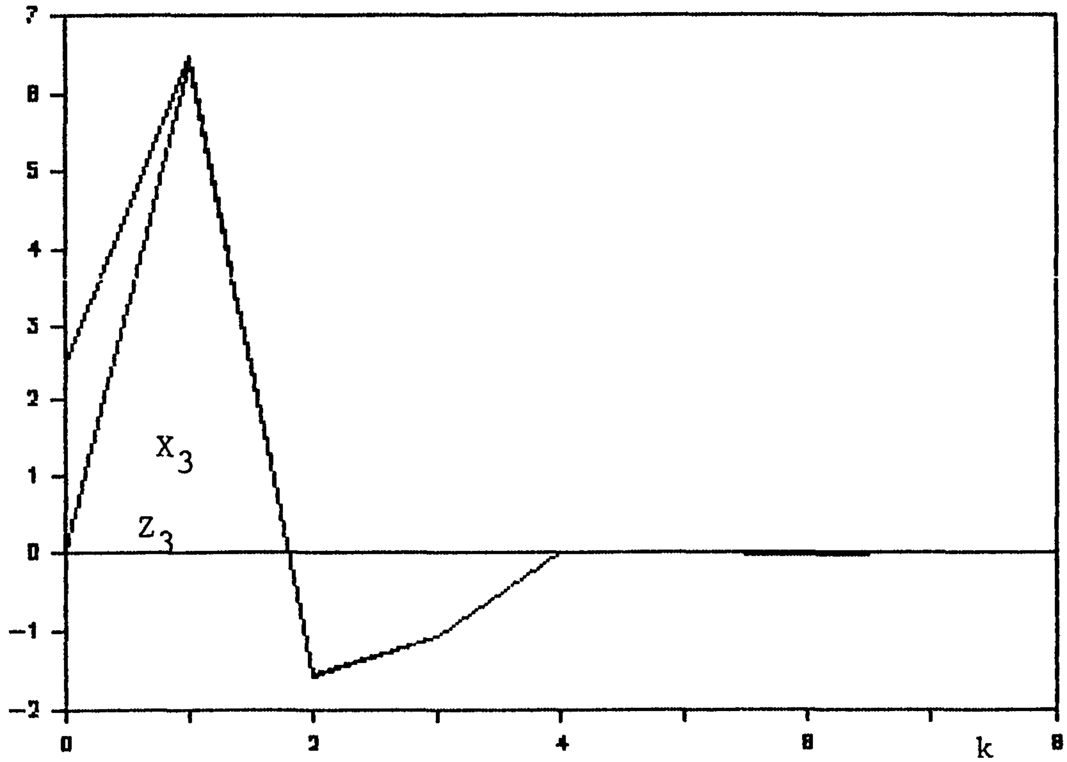
Number of Steps k

Fig. 5.11 Time Responses of X_1 and Z_1 for Ex. 5.6



Number of Steps k

Fig. 5.12 Time Responses of X_2 and Z_2 for Ex. 5.6



Number of Steps k

Fig. 5.13 Time Responses of X_3 and Z_3 for Ex. 5.6

5.5 OBSERVER-CONTROLLER DESIGN FOR SINGLE INPUT SINGLE OUTPUT SYSTEMS:

The following technique describes a design strategy for placing all the eigenvalues of a closed-loop single input system at the origin. The technique uses a factorization procedure which is directly related to the controllability condition of the system. The procedure is very simple in application and requires only one matrix inversion and no recursive relation, although it can be shown that there exists a Riccati equation which runs forward in time that will generate the same gain matrix.

5.5.1 CONTROLLER DESIGN:

Consider the linear discrete system given by

$$x(k+1) = Ax(k) + Bu(k) \qquad (5.160)$$

where x is the n-dimensional state vector, u is the single input scalar and A and B are nxn and nx1 matrices respectively. Let us assume that the [A,B] pair is completely controllable. Then there exists a nonsingular matrix Q such that

$$y_1(k+1) = A_1y(k) + B_1u(k) \qquad (5.161)$$

where $$B_1 = QB \tag{5.162}$$

$$A_1 = QAQ^{-1} \tag{5.163}$$

$$y = Qx \tag{5.164}$$

where $$Q = [Q_1,\ Q_1A, \dots\dots\dots Q_1A^{n-1}]^T \tag{5.165}$$

and $$Q_1 = [0\ 0\ 0\ \dots .0\ 1][B, AB, \dots\dots A^{n-1}B]^{-1} \tag{5.166}$$

The matrix A_1 takes the form

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots\dots & 0 & 0 \\ 0 & 0 & 1 & 0 \dots\dots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ a_1 & a_2 & \dots & & a_n & \end{bmatrix} \quad \text{and } B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.167}$$

Let the input be given by

$$u(k) = -F_1y(k) \tag{5.168}$$

Then (5.161) can be written as

$$y_1(k+1) = [A_1 - B_1F_1]y_1(k) \tag{5.169}$$

Let the gain F_1 be given by

$$F_1 = [B_1^TB_1]^{-1}B_1^TA_1 = [a_1\ a_2\ \dots\dots\ a_{n-1}\ a_n] \tag{5.170}$$

Then the matrix $[A_1 - B_1F_1]$ has the first (n-1) rows identical to that of A_1 and its n^{th} row equal to zero. Since all the eigenvalues of $[A_1 - B_1F_1]$ are equal to zero, then the system given by (5.169) state goes to zero in at most n steps. We now seek an F which will place all the eigenvalues of $[A - BF]$ in the same location as the eigenvalues of $[A_1 - B_1F_1]$. Let the input to (5.160) be given by

$$u(k) = -Fx(k) \tag{5.170}$$

Then from (5.160)

$$x(k+1) = [A - BF]x(k) \tag{5.171}$$

Let the gain F be given by

$$F = [B^TQ^TQB]^{-1}B^TQ^TQA \tag{5.172}$$

Then

$$x(k+1) = [A - B[B^TQ^TQB]^{-1}B^TQ^TQA]x(k)$$

$$= [Q^{-1}A_1Q - Q^{-1}B_1(B_1{}^TB_1)^{-1}B_1{}^TA_1Q]x(k)$$

$$= Q^{-1}[A_1 - B_1F_1]Qx(k) \tag{5.173}$$

Since $Q^{-1}[A_1 - B_1F_1]Q$ is similar to $[A_1 - B_1F_1]$, the control law given by (5.172) drives the state of (5.160) to the origin in at most n steps from any initial state.

5.5.2 OBSERVER DESIGN: Consider the system (5.160,(5.28)-(5.31) where m = 1. Assume that the [A, C] pair is completely observable. Then using an argument analogous to that described above for the controller design, there exists a nonsingular matrix P which transforms (5.160), (5.28) - (5.31) into

$$x(k+1) = A^* x^*(k) + B^* u \qquad (5.177)$$

$$y = C^* x^* \qquad (5.178)$$

$$z^*(k+1) = A^* z^*(k) + K^* C^*(y(k) - C^* z^*(k)) + B^* u \qquad (5.179)$$

$$e^*(k+1) = (A^* - K^* C^*) e^*(k) \qquad (5.180)$$

$$\text{where} \quad x^* = P^{-1}x \quad z^* = P^{-1}z \quad e^* = P^{-1}e \qquad (5.181)$$

$$P = [P_1, AP_1, A^2P_1 \ldots\ldots A^{n-1}P_1] \qquad (5.182)$$

$$P_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (5.182)$$

$$A^* = P^{-1}AP \quad C^* = CP \quad B^* \quad P^{-1}B \quad K^* = P^{-1}K \qquad (5.184)$$

The matrix A^* takes the form of A_1^T while C^* takes the form of B_1^T. Let the gain K^* be given by

$$K^* = A^* C^{*^T} [C^* C]^{-1} \qquad (5.185)$$

Then the eigenvalues of the matrix $[A^* - K^* C^*]$ are equal to zero. Thus the gain given by (5.185) forces the error $e^*(k)$ to zero in at most n steps. But note that

$$[A - KC] = [PA^* P^{-1} - PK^* C^* P^{-1}] = P[A - K^* C^*]P^{-1} \qquad (5.186)$$

Thus the matrix $[A - KC]$ and $[A^* - K^* C^*]$ are similar and there have the same eigenvalues. Thus the gain given by $K = PK^*$ forces the error given by

$$e(k+1) = (A - KC)e(k) \qquad (5.187)$$

to zero in at most n steps.

5.5.3 OUTPUT-FEEDBACK REGULATOR DESIGN:

The output-feedback regulator can now be designed as follows: Let the input to (5.160) be given by

$$u(k) = -Fz(k) \qquad (5.188)$$

where F satisfies (5.172).Let the observer gain be given by

$$K = APP^T C^T [CPP^T C^T]^{-1} \qquad (5.189)$$

Since the gain given by (5.189) drives e(k) to zero in at most n steps, then the input given (5.188) will drive the state of (5.160) to zero in at most 2n steps.

In this chapter we accomplished three major objectives. First, an observer-controller was designed for an output-feedback controlled system. The gains of the observer and control law was generated through matrix Riccati Equations. Gain-constrained design techniques were developed. Second, a technique was developed to circumvent matrix inversion. Parameter M and N were found which can be used to rate of convergence of the estimation error and system state. An Third, a simple but powerful technique was developed to place all the eigenvalues of the closed-loop system at the origin for single input single output controlled systems.

6. CONCLUSION AND EXTENSIONS

This thesis has solved several problems in the design of observers and controllers for linear and nonlinear systems. In chapter 2 we formulated the problems of designing controllers and observers for linear continuous time systems. There we extended the work of Gershwin and Jacobson [9] in designing controllers which produce finite time responses. In a manner analogous to the controller design a new technique was formulated and applied to the design of state estimators. By assigning an appropriate boundary condition to the singular Riccati equation a dynamic observer was designed which produced exact state estimates in finite time. The finite-time controller and the finite-time observer were then combined to produce a finite-time output feedback regulator. Many examples were given to demonstrate the properties and performances of the resulting feedback systems.

In chapter 3 the solution to an output-feedback finite-time regulator problem for a class of nonlinear continuous time systems was obtained. Unlike the class considered by [17] the nonlinearity was not restricted to contain only measurable state components. A detailed analysis of such class of nonlinear systems was given. Examples were given to demonstrate the performance of the dynamic state estimator and output-feedback regulator.

The effect on dynamic feedback system response of the parameters N and M, described in chapters 2 and 3, and the terminal time t_f were examined in chapter 4. These parameters were shown to play a major role in shaping the state and estimation error trajectories. It was shown that properly selected time-varying N and M can reduced excursions in the state and input norms while producing finite-time responses. In cases where there were restrictions on the norms of the largest controller and observer gains a new algorithm was developed for selecting either a constant gain control law or a time-varying gain controller which act as a compromise among state excursion, input norms and time response. An analogous algorithm was described for the gain in the observer design. An estimate of the time required to bring the estimator error from any initial norm to a required fraction of the initial norm was found. Similar results were also found for the state of the system. A detailed physical application for the state estimation and control of a nuclear plant was given.

The dual of the techniques described in chapters 2 and 4 for linear continuous time system was formulated and described for linear discrete time systems in chapter 5. In addition a technique was described to produce near deadbeat output-feedback response without the need for the time consuming matrix inversion many algorithms incorporate. Finally a very simple but powerful scheme was given to place all the eigenvalues of a closed-loop single-input single ouput linear system at the origin, thus producing a deadbeat output-feedback system.

In the discussion of the nonlinear systems of chapter 3, the termial time t_f was selected based upon the characteristics of the strictly linear components of the systems. An extension of this thesis would be to find the relationship between the terminal time and nature of the nonlinearity. Similarly the relationship between the parameters N an M and the characteristics of the nonlinearly can be sought.

We have noticed that time-varying N and M when chosen correctly, do have advantages over N and M of constant values. In both cases, the effect of these scalar parameters can be looked upon as being equivalent to the effect of diagonal matrices whose diagonal elements are equal. In such a case the parameters N and M have the same effect on all the eigenvalues of the A matrix. An interesting extension of this thesis is the following: suppose N and M were diagonal matrices whose diagonal elements were time-varying and different from each other, under what conditions can deadbeat response still be established? And if deadbeat responses can be established, what is the relationship between the elements of N and M and the eigenvalues of the A matrix.

The theory developed in this thesis is applicable to all forms of linear systems and to nonlinear systems whose linear components dominate the general behavior of the system. Also the nonlinear systems considered encompassed a broad class of nonlinear systems. A logical extension would be to find an even broader class of nonlinear systems which can possess the characteristics of chapter 3 and to find parameters analogous to N and M that affect the trajectories of these systems. Similarly, can an analogous theory be developed for its discrete-time counterpart? And finally, what applications can be found for this broader class of systems?

In conclusion, this thesis has contributed to the solution of the many practical problems of automatic control where constraints on dynamic response and on controller components must be satisfied. It is hoped that the theories developed here will found be beneficial to the development and enhancement of the human society.

APPENDIX A

Here is a restatement of Gershwin and Jacobson ' First Controllability Theorem'[9]. Given the system

$$\dot{x} = f(x,u,t) \tag{A.1}$$

Theorem (Controllability)

If a scalar function $V(x,t)$ exists such that

1) $V_x(x,t)$ and $V_t(x,t)$ exist, for all x, t, $t = t_f$

2) for all continuous $c(t)$ (n-vector function of t),

$$\lim_{t \to t_f} c(t) \neq 0 \quad \Rightarrow \quad \lim_{t \to t_f} V(c(t),t) = \infty \tag{A.2}$$

and if a control function $u^*(\circ) \in U$ exists such that

3) along the trajectories of (A.1), the full derivative of $V(x,t)$ satisfies

$$\dot{V} = V_t + V_x f(x,u^*,t) \le M < \infty \tag{A.3}$$

for every t, $t_0 < t < t_f$

4) the solution to

$$x(t_0) = x_0 \tag{A.4}$$

$$\dot{x} = f(x,u^*(x,t),t) \tag{A.5}$$

exists and is unique

then system (A.1) is controllable from (x_0,t_0) to $(0,t_f)$, and $u^*(x,t)$ accomplishes this transfer.

APPENDIX B

LEMMA 1: If S(t) satisfies

$$N\mathring{S}(t) + S(t)A + A^TS(t) - S(t)BB^TS(t) = 0 \qquad (B.1)$$

$$S^{-1}(t_f) = 0 \qquad (B.2)$$

and the system

$$\mathring{x}(t) = Ax(t) + Bu(t) \qquad (B.3)$$

where $$u(t) = -0.5B^TS(t) \qquad (B.4)$$

is completely controllable, then $S(t) > 0$ and $\mathring{S}(t) \geq 0$ for all $t < t_f$.

PROOF: Since (B.3) is completely controllable, then

$$\int_t^{t_f} \exp[A(t-\tau)]BB^T\exp[A^T(t-\tau)]d\tau > 0 \qquad (B.5)$$

The solution to (B.1), (B.2) is

$$S^{-1}(t,t_f,N) = N\int_t^{t_f} \exp[A(t-\tau)/N]\frac{B}{N}\frac{B^T}{N}\exp[A^T(t-\tau)/N]d\tau$$

$$= \int_0^{(t_f-t)/N} \exp(-A\tau)BB^T\exp(-A^T\tau)d\tau \quad > \quad 0 \qquad (B.6)$$

since the matrix $[B,\ AB,.A^2B.....A^{n-1}B]$ is column equivalent to $[B/N,\ AB/N^2,.....A^{n-1}B/N^n]$ and N is positive. Therefore $S(t) > 0$. From (B.6)

$$\partial S^{-1}(t)/\partial t \;=\; -(1/N)\exp[A(t-t_f)/N]BB^T\exp[A^T(t-t_f)/N] \le 0 \qquad (B.7)$$

But

$$\partial S^{-1}(t,t_f,N)/\partial t = -S^{-1}(t,t_f,N)[\partial S(t,t_f)/\partial t]S^{-1}(t,t_f,N) \qquad (B.8)$$

And from (B.6) and (B.7) it follows

$$\partial S(t,t_f,N)/\partial t \;\ge\; 0 \qquad (B.9)$$

Thus for any fixed t_f and fixed N, $dS(t,t_f,N)/dt \ge 0$.

<u>LEMMA 2</u>: If W(t) satisfies

$$M\dot{W}(t) \;=\; W(t)A \;+\; A^TW(t) \;-\; C^TC \qquad (B.10)$$

$$W(t_f) \;=\; 0 \qquad (B.11)$$

and A and C are constant matrices such that the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{B.12}$$

$$y = Cx \tag{B.13}$$

is completely observable, then $\dot{W}(t) \le 0$ and there exists a nonzero zero c(t) given by

$$c(t) = K(t)r(t) \tag{B.14}$$

where r(t) is any nonzero mx1 vector and

$$K(t) = 0.5W^{-1}(t)C^T \tag{B.15}$$

such that
$$\lim_{t \to t_f} V(c(t),t) = \infty \tag{B.16}$$

for
$$V(e,t) = e^T(t)W(t)e(t) \tag{B.17}$$

<u>**PROOF:**</u> The solution to (B.10), (B.11) is

$$W(t,t_f,M) = M\int_t^{t_f} \exp[A^T(t-\tau)/M]\frac{C^T}{M}\frac{C}{M}\exp[A(t-\tau)/M]d\tau > 0 \tag{B.18}$$

since (B.12), (B.13) is assumed completely observable. Then

$$\partial W/\partial t = -(1/M)\exp[A^T(t-t_f)/M]C^T C\exp[A(t-t_f)/M] \leq 0 \quad (B.19)$$

And for any fixed t_f and M we get

$$dW(t,t_f,M)/dt \leq 0 \quad (B.20)$$

$$\lim_{t \to t_f} V(c(t),t) = \lim_{t \to t_f} V(K(t)r(t),t)$$

$$= 0.25* \lim_{t \to t_f} r^T(t)K^T(t)W(t)K(t)r(t)$$

$$= 0.25* \lim_{t \to t_f} r^T(t)CW^{-1}(t)W(t)W^{-1}(t)C^T r(t)$$

$$= 0.25* \lim_{t \to t_f} r^T(t)CW^{-1}(t)C^T r(t) = \infty \quad (B.21)$$

LEMMA 3: If the system given by (B.3) is completely controllable, then for any fixed t_f and $t < t_f$,

$$dS(t,t_f,N)/dN \geq 0 \quad (B.22)$$

PROOF: From (B.6) yields

$$S^{-1}(t,t_f,N) = (1/N)\int_t^{t_f} \exp[A(t-\tau)/N]BB^T\exp[A^T(t-\tau)/N]d\tau$$

$$= \int_0^{(t_f-t)/N} \exp(-Ap)BB^T\exp(-A^Tp)dp \qquad (B.23)$$

where $$p = (\tau - t)/N \qquad (B.24)$$

Then

$$dS^{-1}/dN = (1/N^2)(t-t_f)\exp[A(t-t_f)/N]BB^T\exp[A^T(t-t_f)/N]$$

$$\leq 0 \qquad (B.25)$$

$$dS^{-1}(t,t_f,N)/dN = -S^{-1}(t,t_f,N)[dS(t,t_f,N)/dN]S^{-1}(t.t_f,N) \qquad (B.26)$$

Therefore $$dS(t,t_f,N)/dN \geq 0 \qquad (B.27)$$

LEMMA 4: The solution to

$$\mathring{S}^{-1} = N^{-1}(t)AS^{-1} + S^{-1}A^TN^{-1}(t) - BB^TN^{-1}(t) \qquad (B.28)$$

$$S^{-1}(t_f) = 0 \qquad (B.29)$$

$$S^{-1}(t,t_f) = \int_0^{p_1} \exp(-Ap)BB^T\exp(-A^Tp)dp \qquad (B.30)$$

$$p_1 = \int_t^{t_f} N^{-1}(\lambda)d\lambda \qquad (B.31)$$

PROOF: The solution to (B.28),(B.29) is

$$S^{-1}(t,t_f) = \int_t^{t_f} \Phi(t,\tau)BB^T N^{-1}(\tau)\Phi^T(t,\tau)d\tau \qquad (B.32)$$

where $$\dot{\Phi}(t,\tau) = N^{-1}(t)A\Phi(t,\tau) \qquad (B.33)$$

Since N(t) is a scalar, then solution to (B.33) is

$$\Phi(t,\tau) = \exp\int_\tau^t N^{-1}(\lambda)Ad\lambda \qquad (B.34)$$

let $$p = \int_t^\tau N^{-1}(\lambda)d\lambda \qquad (B.35)$$

Then from (B.32), (B.34) and (B.35)

$$S^{-1}(t,t_f) = \int_0^{p_1} \exp(-Ap)BB^T\exp(-A^Tp)dp \qquad (B.36)$$

FINDING AN UPPER BOUND FOR ||EXP(Jt)||

Suppose the matrix [A - BF] has m distinct characteristic values λ_i, i = 1,2,...,m. Let the multiplicity of each characteristic value in the characteristic polynomial

[A - BF] be given by m_i. There exists a transformation T [20] such that

$$\exp[(A - BF)t] = T\exp(Jt)T^{-1} \tag{B.37}$$

where J is a matrix in the Jordan form. An upper bound on $||\exp(Jt)||$ can be found as follows: Let λ_{max} satisfy Re $\lambda_i < \lambda_{max} < 0$. Since [A - BF] is a stable matrix, then all the eigenvalues of [A - BF] real part is negative. Then

$$||\exp(Jt)||^2 = \mathrm{Trace}[\exp(Jt)^T\exp(Jt)]$$

$$= \sum_{i=1}^{m} ||\exp(J_i t)||^2 \tag{B.38}$$

But each $\exp(J_i t)$ is given by [20]

$$\exp(J_i t) = \exp(\lambda_i t)\begin{bmatrix} 1 & t & 1/2t^2 \ldots\ldots & t^{m_i-1}/(m_i-1)! \\ 0 & 1 & t \ldots\ldots\ldots\ldots & t^{m_i-2}/(m_i-2)! \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots & 1 \end{bmatrix}$$

(B.39)

Therefore

$$\mathrm{Trace}[\exp(J_i t)^T \exp(J_i t)]$$

$$= [m_i + (m_i-1)t^2 + (m_i-2)t^4/4 + \quad + (t^{m_i-1}/(m_i-1)!)^2]e^{2\lambda_i t}$$

$$= m_i e^{2\lambda_i t} + \sum_{r=1}^{m_i-1} r[t^{m_i-r}/(m_i-r)!]^2 e^{2\lambda_i t}$$

$$= m_i e^{2\lambda_i t} + \sum_{r=1}^{m_i-1} r[t^{m_i-r} e^{(\lambda_i-\lambda_{max})t}/(m_i-r)!]^2 e^{2\lambda_{max} t}$$

$$= \left\{m_i + \sum_{r=1}^{m_i-1}\left[\sqrt{r}\; t^{m_i-r} e^{(\lambda_i-\lambda_{max})t}/(m_i-r)!\right]^2\right\} e^{2\lambda_{max} t} \qquad (B.40)$$

Let $$y(t) = t^{m_i-r} e^{(Re\lambda_i - \lambda_{max})t} \qquad (B.41)$$

then $$dy(t)/dt = t^{m_i-r} e^{(Re\lambda_i-\lambda_{max})t}[(m_i-r)t^{-1}-(\lambda_{max}-Re\lambda_i)] \qquad (B.42)$$

therefore $$\max y(t) = y(t)\Big|_{t = (m_i-r)/(\lambda_{max}-Re\lambda_i)}$$

$$= \left[(m_i-r)/[(\lambda_{max}- Re\lambda_i)e]\right]^{(m_i-r)} \qquad (B.43)$$

Thus from (B.40) we get

$$||\exp(J_i t)||^2$$

$$\leq \left[m_i + \sum_{r=1}^{m_i-1}\left\{\sqrt{r}\left[(m_i-r)/[(\lambda_{max}-\mathrm{Re}\lambda_i)e]\right]^{(m_i-r)}/(m_i-r)!\right\}^2\right]e^{2\lambda_{max}t}$$

(B.44)

From (B.38) and (B.44)

$$||\exp(Jt)||^2$$

$$\leq\left[\sqrt{n}+\sum_{i=1}^{m}\ \sum_{r=1}^{m_i-1}\left[\sqrt{r}\left\{(m_i-r)/[(\lambda_{max}-\lambda_i)e]\right\}^{m_i-r}/(m_i-r)!\right]^2 e^{2\lambda_{max}t}$$

(B.45)

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