PRICING COLLATERALIZED DEBT OBLIGATIONS WITH PURE JUMP LÉVY PROCESSES: A DYNAMIC BOTTOM-UP APPROACH

by

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Abstract

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The Gaussian copula model is the industry standard in pricing CDO tranches because of its easy implementation and speedy calibration. However, it has several well-known shortcomings: It leads to the so-called "correlation smile", generates symmetric and light-tailed asset return distributions and it is static. This dissertation proposes a dynamic bottom-up model based on a pure jump Lévy process, a path rarely taken in the credit pricing literature, and makes a comprehensive empirical analysis of bottom-up CDO pricing models. Owing to its ability to capture asymmetric heavy-tailed return distributions and to accommodate different degrees of dampening for positive and negative jumps, empirical evidence shows that the proposed model significantly outperforms the models commonly employed in the industry and frequently referenced in the literature in fitting CDX and iTraxx tranche spreads. As such, it constitutes an important addition to the credit pricing literature.

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Chapter 1

Introduction and Market Review of Collateralized Debt Obligations

The 2008 credit crunch demonstrated, rather painfully, the desperate need for a better understanding of Collateralized Debt Obligations and a more accurate valuation model for pricing Collateralized Debt Obligation tranche spreads. This dissertation is a step in that direction.

This section serves as an introduction to the mechanics of credit derivatives and Collateralized Debt Obligations, and presents a summary of their evolution through time.

1.1 Collateralized Debt Obligation Basics

Credit derivatives are over-the-counter (OTC) contracts that transfer credit risk associated with a specific reference entity or with a group of entities from one counterparty to another. These contracts may refer either to a single-name credit or to a portfolio of credits. Credit Default Swaps (CDSs) and Total Return Swaps (TRSs) are the most commonly traded instruments in the single-name credit derivatives markets. Collateralized Debt Obligations (CDOs) and Portfolio Default Swaps (PDSs) are the most popular instruments traded in the portfolio credit derivatives markets.

The chief building block of the credit derivatives market is the CDS, which is an OTC bilateral insurance contract that enables one to transfer credit risk of a reference entity such as a corporate or sovereign bond/loan. The buyer of a CDS, the protection buyer, pays a periodic premium to the protection seller until the maturity date of the contract or a specific credit event, whichever is sooner. In exchange, the protection seller agrees to pay the par value[1] of the underlying asset in the case of a default or another pre-specified credit event such as restructuring, bankruptcy or downgrade. CDSs are leveraged instruments which do not require initial funding by the investor.

A CDO is a structured transactions in which a Special Purpose Vehicle (SPV) is created from a pool of underlying assets in order to transfer credit risk associated with the underlying assets. Subsequently, the SPV issues notes with different levels of seniority, and the cash-flow generated/backed by the underlying pool of assets (i.e. coupon interest payments, notionals of maturing assets and the sale of unmatured assets) is transferred to the investors as coupon payments in a pre-set sequential basis in accordance with the purchased tranche. Tranches range from senior tranches to mezzanine tranches and finally to the equity/first-loss tranche. The seniority of the purchased tranche determines both the investor's risk-return appetite and also his/her claim on the portfolio in the case of a bankruptcy. Losses in the underlying pool first affect the equity tranche, followed by the mezzanine tranches, and finally the senior tranches. This schedule is known as the waterfall structure and it is very similar to the debt structure of a commercial bank whose outstanding debt exist with different levels of seniority such as depositors, senior debt, subordinated debt, preferred stock and equity.

The underlying/collateral pool of assets in a CDO may include, but is not limited to, high yield loans (e.g. U.S. domestic bank loans, foreign bank loans), high yield bonds (e.g. U.S. domestic bonds, Euro high yield bonds), emerging market debt, other credit derivatives (e.g.

[1] CDS contracts may either be physically or cash settled. In the former case, protection seller pays the par value of the reference asset in exchange for the physical delivery of the "deliverable obligation". If the contract is cash settled, protection seller pays only the difference between par value and the market price of the reference entity in cash.

CDSs, CDOs), special situation loans/distressed debt or any other Asset-Backed-Security (ABS)[2].

1.2 CDO Classes

Depending on the source of funding, CDOs may be structured as cash, synthetic, or a combination of both (i.e. hybrid CDOs). The reference portfolio of a cash CDO is made up of cash assets such as bonds or loans; the reference portfolio of a synthetic CDO consists of CDSs and the reference portfolio of a hybrid CDO combines cash assets as well as CDSs. Synthetic CDOs aim to transfer the credit risk without physically transferring the assets; thus have the advantage over cash CDOs in that they isolate credit risk from other financial risks such as interest rate and currency risk.

CDOs, whether cash or synthetic, may be grouped into two main categories, according to the originator's motivation: Balance-sheet CDOs and arbitrage CDOs. A balance-sheet CDO enables its originator to transfer assets and/or credit risk already extant on its balance-sheet to another entity, enabling the originator to more effectively manage both its portfolio credit risk and regulatory capital requirements. Arbitrage CDOs, on the other hand, are generally structured by asset management companies in order to exploit yield differentials between the return on assets in the CDO portfolio and the interest payments to the tranche investors, plus the expenses incurred funding the CDO, which include costs such as legal services, bridging loan facilities (i.e., warehouse facility costs) and rating agencies.

Cash arbitrage CDOs may be either cash-flow CDOs or market value CDOs. The manager of a cash-flow CDO is expected to transfer principal and interest payments generated by the underlying assets to the tranche investors in a timely fashion. These cash-flows are

[2] If the underlying pool consists only of loans then the instrument is called Collateralized Loan Obligation (CLO), if it consists only of bonds then the instrument is called Collateralized Bond Obligation (CBO), if it consists only of CDSs then the instrument is called synthetic Collateralized Debt Obligation and if it consists only of Mortgage-Backed securities (MBSs) then the instrument is called Collateralized Mortgage Obligation (CMO).

hedged and reinvested only in accordance with the guidelines of the CDO. By contrast, the CDO manager of a market-value CDO is expected to improve the market value, price volatility and liquidity of the underlying pool of assets through frequent trades and sales of the underlying assets.

Collateral managers of "managed CDOs" are vital in the success of a CDO deal. They are in charge of selecting the proper collateral assets, hedging their transactions, re-investing and selling the underlying assets. The collateral manager generally holds a portion of the equity tranche to increase the marketability of the CDO by signaling to the investors that the management has an incentive for the success of the CDO deal other than solely collecting management fees. Rating agencies are important participants of the CDO market as they provide information on the qualitative and quantitative performance of the CDO managers. Fitch ratings has its CDO Asset Manager Scores[3], Standard and Poor's (S&P) has its CDO Manager Focus [4] and Moody's has its Deal Score Reports[5].

1.3 Evolution of the CDO market

The Collateralized Obligations (COs) market dates back to 1983, when the first CMOs were issued by Salomon Brothers and First Boston. By mid-1983, the Federal Home Loan Mortgage Corporation (FHLMC) had issued its first CMO by creating paythrough structures in which the cash flow of the CMO was sliced into a number of tranches in order to meet the risk-return demands of various investors. Until the 1990s, CMOs remained the only form of CO. For more details see Matthews (2001) and Shapiro (1999).

By the beginning of 1990s, it became a common practice for commercial banks to securitize their loan portfolios by repacking them into marketable securities and removing them from their balance-sheets through CLOs. In the absence of synthetic CDOs, balance-sheet CLO

[3] Fitch bases its scores on company and management experience, staffing, procedures and controls, portfolio management, CDO administration and CDO performance.

[4] S&P bases its assessment on organizational background, CDO team depth, investment style and analysis approach, monitoring and operations, technology infrastructure, and performance track record.

[5] Moody's Deal Score measures the ratings performance of CDOs over time considering Moodys-adjusted O/C deterioration and violation/compliance with the Moodys WARF for each deal.

deals were the driving force of the CDO market. This practice not only enabled banks to manage their regulatory capital requirements in accordance with the Basel Capital Accord of 1988, but it also allowed them to transfer credit risks/costs associated with their loan portfolios. These deals enabled banks to acquire capital relief while increasing their lending capabilities. Throughout the 1990s, commercial banks in the US, Europe and Japan were able to close high volumes of CLO deals; this led to the formation of the CDO market as it exists today.

By 2000, the CDO market started to develop at an increasing pace with the growth of the CDS market, the introduction of CDS indices such as Credit Default Index (CDX) and iTraxx[6], and the emergence of active trading in standardized trances on those indices. With these developments, the volume of the CDO market shifted from cash-flow transactions to synthetic, and CDOs became a major instrument for market participants to exploit arbitrage opportunities in financial markets. In 1997, JP Morgan and the Swiss Bank Corporation closed the first synthetic CDO deal in Europe. By 2003, 92% of all European CDOs rated by Moody's were synthetic structures, Watts (2005).

1.4 Size of the Credit Derivatives and CDO markets

Since the late 1990's, the size of the global credit derivatives markets has increased at an astonishing rate in spite of numerous damaging credit events such as the 1998 Russian default, the defaults of Conseco, Enron and WorldCom and the downgrades of GM and Ford in 2005. Data on the size of credit derivatives, CDSs and CDOs does differ somewhat depending on the timing and methodology of data collection.

According to the British Bankers' Association (BBA) survey[7], the size of the global credit

[6] CDX indices, administered by CDS Index Company (CDSIndexCo) and marketed by Markit Group, contain North American and Emerging Market companies. CDX.NA.IG is based on 125 investment-grade North American companies whereas CDX.NA.HY is based on 100 North American companies that do not rank as investment grade. iTraxx, managed by the International Index Company (IIC), contains companies from Europe, Japan and Asia. iTraxx Europe is based on 125 European investment-grade companies whereas iTraxx Europe Crossover is based on 50 sub-investment grade European companies

[7] http://www.bba.org.uk/content/1/c4/76/71/Credit_derivative_report_2006_exec_summary.pdf

derivatives market increased from $180 billion in 1996 to $893 billion in 2000 and reached $20.207 trillion in 2006. According to the US Office for the Comptroller of the Currency[8], credit derivative contracts in US alone grew at a 100% compounded annual growth rate from 2003 to 2007 and reached $16.1 trillion in the third quarter of 2008.

Single-name CDSs have the biggest market share in credit derivatives markets. The International Swaps and Derivatives Association (ISDA) reports that the total value of CDS outstandings increased from $918.87 billion in 2001 to $17.10 trillion in 2005, reached to $62.17 trillion in 2007 and dropped to $26.26 trillion in 2010[9].

Global CDO issuance increased from $157.82 billion in 2004 to $251.26 billion in 2005, reached to $430.45 billion in 2007 and declined to $7.68 billion in 2010 according to Securities Industry and Financial Markets Association (SIFMA)[10]. Contrary to SIFMA's numbers, the IMF, in its April 2008 Global Financial Stability Report, estimates that the global issuance of CDOs grew from $150 billion in 2000 to about $1.2 trillion in 2007[11].

To put these numbers in perspective; according to estimates given in the IMF's Global Financial Stability Report of October 2008[12], by the end of 2007 the world GDP was $54.55 trillion, world stock market capitalization was $65.11 trillion, outstanding world public and private debt together was $79.82 trillion.

1.5 Reasons to Invest in CDOs

The driving force behind the dramatic increase in the volume of CDOs is the ever-increasing demand for CDO tranches from institutional investors such as commercial banks, pension funds, insurance companies, hedge funds, proprietary trading desks and mutual funds. There are several reasons why these institutions prefer to invest in CDOs rather than di-

[8] http://www.occ.gov/ftp/release/2008-152a.pdf
[9] http://www.isda.org/statistics/pdf/ISDA-Market-Survey-historical-data.pdf
[10] http://www.sifma.org/research/pdf/CDO_Data2008-Q4.pdf
[11] http://www.imf.org/External/Pubs/FT/GFSR/2008/01/pdf/text.pdf
[12] http://www.imf.org/external/pubs/ft/gfsr/2008/02/pdf/text.pdf

rectly building a portfolio with similar underlying assets.

Firstly, CDOs are instrumental in supplying high-credit-quality securities to fixed-income investors even though the underlying pool of assets may consist of lower-rated securities or other assets that would be unappealing individually due to their lack of liquidity. Secondly, well-managed CDOs offer custom exposure to a diversified and low correlated set of securities from various industries and countries that would otherwise be very costly and time-consuming to bring together. Finally, commercial banks, in accordance with the Basel regulations, have strong incentives to employ these instruments to more effectively manage their portfolio credit risk and reduce the costs associated with their capital requirements.

However, CDOs may also be structured as very risky instruments. Coupled with poor regulatory and rating practices, CDOs played an explosive role in spreading the already existing risks (especially those in the housing market) in the financial markets, as has been clearly proven during the 2008 credit crunch. Hence, a better valuation model in pricing CDO tranches is needed today more than ever; this dissertation aims to fill that gap by proposing a stochastic model.

The dissertation is organized as follows: Chapter 2 presents a brief review of the literature on credit pricing models. Since copula functions are the main blocks upon which CDO pricing models very often rely, a mathematical review of copula functions is also presented in that chapter. Chapter 3 makes an extensive empirical analysis of the bottom-up CDO pricing models commonly employed in the industry, as well as several other models most frequently referenced in the credit pricing literature. Chapter 4 introduces the proposed stochastic model with its extensions and calibrates them to CDX.NA.IG.13 and iTraxx EUR.9 series tranche upfront fees and running spreads.

Chapter 2

Literature Review of Credit Pricing Models and Mathematical Review of Copula Functions

This chapter provides a short introduction to the literature on credit pricing models. Since copula functions are the main building blocks upon which the CDO pricing models most often rely, a mathematical review of copula functions, their simulation procedures and their probability distributions is also presented here. The chapter ends with a review of different dependence structures that will be employed throughout this dissertation.

2.1 Literature Review

Single name credit risk modeling relies on three inputs. The first of these is the probability of default for a credit. The second input is the recovery rate for an individual name in the case of a default. These first two inputs make it possible to calculate the Loss Given Default (LGD) of the credit. The third and final input is the Exposure at Default (EAD) of the credit. When working with a portfolio of credit risk, such as a CDO, a fourth input must also be incorporated into the model: Some type of dependence structure between single name defaults within the portfolio, which enables one to calculate the joint distribution of correlated default probabilities. There are two main schools of thought on default processes in the credit risk literature: Structural models and Reduced Form models.

2.1.1 Structural Models

Structural models attempt to describe the default process by modelling the change in a company's structural characteristics (e.g. assets and liabilities), assuming that default is not a sudden or unexpected event, given that information on a company's structural characteristics is publicly available. These models assume that the evolution of a companys structural characteristics follow a diffusion process. Applying the Black & Scholes (1973) model to corporate bond pricing, Merton (1974) models default as a one-time event, occurring when a company's equity value is less that its outstanding debt at exactly the time of servicing its debt. Defaults can only occur at maturity of the debt, not sooner, as the companys debt is assumed to be composed entirely of a zero-coupon bond. Hence, at maturity, bondholder's payoff is simply the difference between the face value of the bond and a put option on the value of the firm, with a strike price equal to the face value of the bond.

Like Merton (1974) and Geske (1977), Black & Cox (1976) create a model in which the structural characteristics of a company determine both the default probability and also the recovery at default, while assuming non-stochastic interest rates. Black & Cox (1976), similar to Longstaff & Schwartz (1995) and Hull & White (1995), allow for the default of a company whenever its asset value hits an exogenously fixed barrier. Models with this structure are sometimes referred to as first-passage or default-barrier models, and they allow default to be an unexpected/surprise event. According to these models, default may occur at any time before the maturity of the debt. The stochastic model that will be proposed in Chapter 4 has similar properties. Longstaff & Schwartz (1995) allow for stochastic interest rates that are correlated with defaults, while incorporating an exogenous recovery rate estimated from historical data.

2.1.2 Reduced Form Models

Reduced form models, also known as intensity-based hazard rate models, are the ones that describe the default time as the stopping time generated by an exogenously given jump process. The hazard rate may be modeled as either a deterministic or a stochastic process. Reduced form models do not condition the default of a firm on its assets/liabilities; instead, they consider default times as unpredictable sudden surprises with non-zero probability over a pre-specified period. Although the firm's value is not incorporated explicitly into the reduced form models, the parameters of the hazard rate are calibrated from market data.

A well-known example of a reduced form model is the Poisson and Cox processes given in Lando (1998); where an inhomogeneous Poisson process $N(t)$ with a non-negative intensity $\lambda(\mathbf{.})$ satisfies;

$$\mathbb{P}[N_t - N_s = k] = \frac{1}{k!}\left(\int_s^t \lambda(u)du\right)^k \exp\left(-\int_s^t \lambda(u)du\right) \quad \text{k=0,1,...}$$

assuming $N_0 = 0$, $\mathbb{P}[N_t = 0] = \exp\left(-\int_0^t \lambda(u)du\right)$ and the first stopping time τ of process $N(t)$ is;

$$\tau = \inf\{t : \int_0^t \lambda(u)du \geq E_1\}$$

where E_1 is a unit exponential random variable. The first default time is defined as: $\tau = \inf\{t \in R^+ | N(t) > 0\}$ where the probability of survival, $S(t)$, and probability of default, $F(t)$, is given by:

$$\begin{aligned} S(t) &= \mathbb{P}[\tau > t] = \mathbb{E}\left[\exp\left(\int_0^t \lambda(u)du\right)\right] \\ F(t) &= \mathbb{P}[\tau \leq t] = \mathbb{E}\left[1 - \exp\left(\int_0^t \lambda(u)du\right)\right] \end{aligned}$$

A similar approach will be employed in Chapters 3 and 4 to estimate instantaneous risk-neutral default probabilities of the underlying credits in a CDO from their CDS spreads.

While Jarrow & Turnbull (1995) model default employing the hazard rate function, with the assumption that recovery rate is an exogenous fraction of the value of an equivalent default-free bond, Jarrow, Lando & Turnbull (1997) model default using a transition matrix[1] and represent default dynamics by a Markov chain. Duffie & Singleton (1999) incorporate stochastic spreads and accounts for stochastic changes in the market price of default risk with the assumption that recovery rate is an exogenous fraction of the market value of the pre-default bond. Reduced form models generally incorporate exogenous recovery rate dynamics that are independent from default time processes.

2.1.3 Hybrid models

Hybrid models incorporate the main principles of both reduced form models and structural models. In Madan & Unal (1998) default probability is a function of firm value process and an independent interest rate movement. Similarly, Madan & Unal (2000) model default probability with the help of a two-factor hazard rate function in which the structural characteristics of a company capture the likelihood of a default, and stochastic interest rates affect the structural characteristics of a company. Duffie & Lando (2001) form a structural model with incomplete accounting information; hence, their model shapes up to be a reduced form model inside a structural model where the firm value process is a geometric Brownian Motion and default is an unpredictable event with an endogenously defined intensity. Based on incomplete information, Giesecke & Goldberg (2004) assume that investors cannot observe the random default barrier that is independent of the firm value which follows an observable geometric Brownian motion. In Giesecke & Goldberg (2004), in the spirit of structural models, default is triggered when the value of the firm falls below the random barrier; as with the reduced form models, however, the timing of default is inaccessible. Zhou (2001) models firm value as a jump-diffusion process, which allows for a firm to default instantaneously due to a surprise jump in its value. This model also links recovery rates to the firm value at default so that the variation in recovery rate is endogenously generated.

[1] Transition matrixes are prepared by the three rating agencies and provide information on the probability of a rating downgrade/upgrade as well as the likelihood of a default

2.1.4 Recovery Rates

Structural models assume stochastic recovery rates. In structural models, the debt holder receives the remaining firm value at the time of default and the recovery rates are endogenously determined. Reduced form models, on the other hand, exogenously define recovery rates. There are three major recovery assumptions employed by the reduced form models: First, the recovery of face value (RFV), where the recovery rate is assumed to be a pre-determined fraction of the face value of the defaultable bond; Second, the recovery of treasury (RT), where the recovery rate is assumed to be a pre-determined fraction of the value of a default-free equivalent of the defaultable bond; and third, the recovery of market value (RMV), where the recovery rate is assumed to be a pre-determined fraction of the pre-default market value of the defaultable bond.

2.2 Mathematical Review of Copula Functions

Copula functions are widely used for linking individual default probabilities to the portfolio loss distribution function. They have the advantage of separating the dependence structure from the marginal behavior and they allow one to work with cumulative probabilities instead of quantiles. This section presents a mathematical review of copula functions and follows presentations by Schönbucher (2003), Schönbucher & Rogge (2003), McNeil, Frey & Embrechts (2005), Embrechts, Lindskog & McNeil (2001), Meneguzzo & Vecchiato (2004), Nelsen (1999), Schmidt (2006) and Cherubini, Luciano & Vecchiato (2004).

Definition 2.2.1. *A n-dimensional copula C on $\mathbb{R}^n$ is a distribution function on $[0,1]^n$ with standard uniform marginal distributions. C is a mapping of the unit hypercube into the unit interval $C : [0,1]^n \to [0,1]$ and it has the following properties:*

1. *$C(u_1, ..., u_n)$ is increasing in each component u_i for all $i \in \{1, ..., n\}$*

2. *$C(1, ..., 1, u_i, 1, ..., 1) = u_i$ for all $i \in \{1, ..., n\}, u_i \in [0,1]$*

3. For all $(a_1, ..., a_n), (b_1, ..., b_n) \in [0,1]^n$ *with* $a_i \leq b_i$ *we have*

$$\sum_{i_1}^{2}\sum_{i_2}^{2}...\sum_{i_n}^{2}(-1)^{i_1+i_2+...+i_n}C(u_{1i_1}, u_{2i_2}, ..., u_{ni_n}) \geq 0$$

where $u_{j1} = a_j$ *and* $u_{j2} = b_j$ *for all* $j \in \{1, ..., n\}$

2.2.1 Sklar's Theorem

Sklar's theorem states that one can compute a joint distribution function from a copula and compute a copula from a joint distribution function.

Theorem 2.2.1. ***(Sklar 1959)*** *Let* $X_1, ..., X_n$ *be random variables with marginal distribution functions* $F_1, ..., F_n$ *and joint distribution function* F*. Then there exists an n-dimensional copula* C *such that for all* $x \in \mathbb{R}^n$*:*

$$F(x_1, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{2.1}$$

C *is the distribution function of* $(F_1(x_1), F_2(x_2), ..., F_n(x_n))$*. If* $F_1(x_1), F_2(x_2), ..., F_n(x_n)$ *are continuous, then* C *is unique. Otherwise* C *is uniquely determined on* $RanF_1 \times ... \times RanF_n$*, where* $RanF_i$ *denotes the range of* F_i *for* $i = 1, ...n$*.*

An immediate corollary of Sklars theorem is;

Corollary 2.2.2. *Let* F *be an n-dimensional distribution function with continuous marginal distribution functions* $F_1, ..., F_n$ *and* C *be the copula function satisfying equation 2.1. Denote* $F^{-1}(t)$ *as generalized inverse of* F *defined as* $F^{-1}(t) = \inf\{x \in \mathbb{R} | F(x) \geq t\}$ *for all* $t \in [0,1]$*, using the convention* $inf\phi = -\infty$*. Then for any* $u_i \in [0,1]^n$*;*

$$C(u_1, ..., u_n) = F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n))$$

Hence, in a setting described as in 2.2.1, it is possible to write;

$$F(x_1,...,x_n) = \mathbb{P}[X_1 \leq x_1,...,X_n \leq x_n] \quad (2.2)$$
$$= \mathbb{P}[F_1(X_1) \leq F_1(x_1),...,F_1(X_n) \leq F_1(x_n)] \quad (2.3)$$

denote $X_i = F_i^{-1}(U_i)$ for $0 \leq u_i \leq 1$ and $i = 1,...,n$. Then

$$\begin{aligned} F(x_1,...,x_n) &= \mathbb{P}[F_1^{-1}(U_1) \leq x_1,...,F_n^{-1}(U_n) \leq x_n] \\ &= \mathbb{P}[U_1 \leq F_1(x_1),...,U_n \leq F_1(x_n)] \\ &= C(F_1(x_1),...F_n(x_n)) \end{aligned}$$

If the marginal distributions are continuous then C is unique. Denote $x_i = F_i^{-1}(u_i)$ for $0 \leq u_i \leq 1$ and $i = 1,...,n$. Then equation 2.2 becomes;

$$F(F_1^{-1}(u_1),...,F_n^{-1}(u_n)) = \mathbb{P}[X_1 \leq F_1^{-1}(u_1),...,X_n \leq F_n^{-1}(u_n)] \quad (2.4)$$
$$= \mathbb{P}[F_1(X_1) \leq u_1,...,F_n(X_n) \leq u_n] \quad (2.5)$$
$$= C(u_1,...u_n) \quad (2.6)$$

Equations 2.4 and 2.6 will be instrumental in generating the portfolio default probabilities in the next chapter.

2.2.2 Copula Densities

Probability density of a function is given by the derivative of the cumulative distribution function as long as the cumulative distribution function is absolutely continuous (i.e. differentiable everywhere).

Definition 2.2.2. *Let $c(F_1(x_1),...F_n(x_n))$ denote the multivariate density of the copula function $C(F_1(x_1),...F_n(x_n))$. Then;*

$$f(x_1, ..., x_n) = \frac{\partial^n (C(F_1(x_1), ...F_n(x_n)))}{\partial F_1(x_1)...\partial F_n(x_n)} \prod_{i=1}^{n} f_i(x_i) \tag{2.7}$$

$$= c(F_1(x_1), ...F_n(x_n)) \prod_{i=1}^{n} f_i(x_i) \tag{2.8}$$

$$c(F_1(x_1), ...F_n(x_n)) = \frac{\partial^n (C(F_1(x_1), ...F_n(x_n)))}{\partial F_1(x_1)...\partial F_n(x_n)} = \frac{f(x_1, ..., x_n)}{\prod_{i=1}^{n} f_i(x_i)} \tag{2.9}$$

where $f_i(x_i) = \frac{\partial F_i(x_i)}{\partial x_i}$ *is the standard univariate probability density function.*

2.2.3 Elliptical Copulae

Gaussian and student's t copula fall into the category of Elliptical Copulae.

Gaussian Copula

Definition 2.2.3. *Let* Σ *be a symmetric, positive definite matrix with* $diag(\Sigma) = (1, ..., 1)^T$ *and* Φ_Σ *the standardized multivariate normal distribution with correlation matrix* Σ*. Then the* n*-dimensional Gaussian copula is defined as:*

$$C_\Sigma^G(u_1, ..., u_n) = \Phi_\Sigma[\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)] \tag{2.10}$$

where Φ^{-1} *is the inverse of the standard univariate normal distribution function* Φ*.*

Hence, the Gaussian copula can be expressed as:

$$C_\Sigma^G(u_1, ..., u_n) = \int_{-\infty}^{\Phi^{-1}(u_1)} ... \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}\right) dx_1 dx_2 ... dx_n$$

The Gaussian copula generates the standard Gaussian joint distribution function, whenever the margins are standard normal.

Gaussian Copula Density

The Gaussian copula density may be found using equation 2.9 and the definition of Gaussian copula given in equation 2.10:

$$
\begin{aligned}
f^G(x_1,...,x_n) &= c_\Sigma^G(F_1(x_1),...F_n(x_n))\prod_{i=1}^{n} f_i^G(x_i) \\
\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) &= c_\Sigma^G(\Phi(x_1),...,\Phi(x_n))\prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_i^2\right)\right) \\
c_\Sigma^G(\Phi(x_1),...,\Phi(x_n)) &= \frac{\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right)}{\prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_i^2\right)\right)}
\end{aligned}
$$

where $|\Sigma|$ is the determinant of Σ. Denoting $u_i = \Phi(x_i)$ and $\varsigma = (\Phi^{-1}(u_1),...,\Phi^{-1}(u_n))^T$, the density becomes:

$$
c_\Sigma^G(u_1,...,u_n) = \frac{1}{|\Sigma|^{\frac{1}{2}}}\exp\left(\frac{1}{2}\varsigma^T(\Sigma^{-1}-I)\varsigma\right)
$$

Gaussian Copula Simulation

The following steps are followed to simulate a n-dimensional Gaussian copula C_Σ^n with a correlation matrix Σ.

- Find the Cholesky decomposition A of Σ
- Simulate n independent random variates $\mathbf{z} = (z_1,...,z_n)^T$ from $N(0,1)$
- Set $\mathbf{x} = A\mathbf{z}$
- Set $u_i = \Phi(x_i)$ with $i = 1,...,n$ and where Φ denotes the univariate standard normal distribution function
- $(u_1,...,u_n)^T = (F_1(t_1),...,F_n(t_n))^T$ where F_i denotes the ith margin

Figure 4.6 shows Gaussian bivariate copula simulation and probability density with 3000 samples.

Student's t Copula

Definition 2.2.4. *Let Σ be a symmetric, positive definite matrix with $diag(\Sigma) = (1, ..., 1)^T$ and t_v the standard univariate cumulative Student's t distribution function with v degrees of freedom and $T_{\Sigma,v}$ the n-dimensional cumulative Student's t distribution with v degrees of freedom and a covariance matrix Σ. Then Student's t copula is defined as:*

$$C^t_{\Sigma,v}(u_1, ..., u_n) = T_{\Sigma,v}(t_v^{-1}(u_1), ..., t_v^{-1}(u_n)) \tag{2.11}$$

where $t_v^{-1}u$ denotes the inverse of standard univariate t cumulative distribution function.

Hence, Student's t copula may be expressed as;

$$C^t_{\Sigma,v}(u_1, ..., u_n) = \int_{-\infty}^{t_v^{-1}(u_1)} \int_{-\infty}^{t_v^{-1}(u_2)} ... \int_{-\infty}^{t_v^{-1}(u_n)} \frac{\Gamma(\frac{v+n}{2})}{\Gamma(\frac{v}{2})\sqrt{(v\pi)^n|\Sigma|}} \times (1 + \frac{\mathbf{x}^T\Sigma^{-1}\mathbf{x}}{v})^{-\frac{v+n}{2}} dx_1 dx_2 ... dx_n$$

Student t distribution has the advantage of converging to Gaussian distribution as $v \to \infty$. Moreover, it has fatter tails compared to the Gaussian distribution; smaller the degrees of freedom, the fatter the tails are.

Student's t Copula Density

The student's t copula density may be found using equation 2.9 and the definition of student's t copula given in equation 2.11. Denote $\varsigma_i = t_v^{-1}(u_i)$, then:

$$\begin{aligned} f^t(x_1, ..., x_n) &= c^t_{\Sigma,v}(F_1(x_1), ...F_n(x_n)) \prod_{i=1}^n f_i^t(x_i) \\ c^t_{\Sigma,v}(u_1, ..., u_n) &= |\Sigma|^{-\frac{1}{2}} \frac{\Gamma\left(\frac{v+n}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \left(\frac{\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right)}\right)^n \frac{\left(1 + \frac{1}{v}\varsigma^T\Sigma^{-1}\varsigma\right)^{-\frac{v+n}{2}}}{\prod_{i=1}^n \left(1 + \frac{\varsigma_i^2}{v}\right)^{-\frac{v+1}{2}}} \end{aligned}$$

Student's t Copula Simulation

The following steps are followed to simulate a n-dimensional Student's t copula $C^t_{\Sigma,v}$ with v degrees of freedom and a correlation matrix Σ.

- Find the Cholesky decomposition A of Σ
- Simulate n i.i.d. $\mathbf{z} = (z_1, ..., z_n)^T$ from $N(0, 1)$
- Simulate a random variate s from χ^2_v independent of $\mathbf{z}$
- Set $\mathbf{y} = A\mathbf{z}$
- Set $\mathbf{x} = \sqrt{(v/s)}\mathbf{y}$
- Set $u_i = T_v(x_i)$ with $i = 1, ..., n$ and where T_v denotes the univariate Student t distribution function
- $(u_1, ..., u_n)^T = (F_1(t_1), ..., F_n(t_n))^T$ where F_i denotes the ith margin

Figure 4.7 shows student's t bivariate copula simulation and probability density with 3000 samples.

2.2.4 Archimedean Copulae

Clayton, Gumbel and Frank copulas fall into the category of Archimedean Copulae.

Definition 2.2.5. *Let φ be a continuous and strictly decreasing function from $[0, 1]$ to $[0, \infty]$ such that $\varphi(1) = 0$. Denote the pseudo-inverse and ordinary inverse of φ as $\varphi^{[-1]}$ and φ^{-1}, respectively. $\varphi^{[-1]}$ is continuous and decreasing on $[0, \infty]$, and strictly decreasing on $[0, \varphi(0)]$ and follows:*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(u), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t \leq \infty \end{cases}$$

If φ is convex then the function φ is called a generator of the copula.

Note that $\varphi^{[-1]}(\varphi(u)) = u$ on $[0,1]$. When $\varphi(0) = \infty \Rightarrow \varphi^{[-1]} = \varphi^{-1}$ and φ is said to be a strict generator.

Theorem 2.2.3. ***(Kimberling 1974)*** *Let φ be a strict generator. For a completely monotone*[2] *φ^{-1} on $[0,\infty)$, the n-dimensional Archimedean copula from $[0,1]^n$ to $[0,1]$ is:*

$$C(u_1, ..., u_n) = \varphi^{-1}(\varphi(u_1) + ... + \varphi(u_n))$$

So, to generate n-dimensional Archimedean copulae one needs generators as described in 2.2.5 and 2.2.3 as well as their inverses. Inverse generators can be derived from inverse Laplace transforms of the distribution function of a random variable.

Theorem 2.2.4. ***(Feller 1971)*** *A function ψ on $[0,\infty)$ is the Laplace*[3] *transform of a distribution function Λ if and only if ψ is completely monotonic and $\psi(0) = 1$.*

Clayton Copula

Definition 2.2.6. *Let the copula generator be $\varphi(u) = u^{-\alpha} - 1$ and inverse generator be $\varphi^{-1}(t) = (1+t)^{-\frac{1}{\alpha}}$; it is completely monotonic if $\alpha > 0$. The Clayton n-copula is therefore;*

$$C(u_1, ..., u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-\frac{1}{\alpha}} \quad \textit{with } \alpha > 0 \tag{2.12}$$

[2]A function g(t) is completely monotonic on an interval J if it is continuous and satisfies

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0, \text{ for } k \in \mathbb{N}, t \in J$$

[3]Let Y be anon-negative random variable with distribution function $G(y)$ and density function $g(y)$ (if it exists). Then;

- The Laplace transform of Y is defined as

$$\mathfrak{L}_Y(t) := \mathbb{E}[e^{-tY}] = \int_0^\infty e^{-ty} dG(y) = \int_0^\infty e^{-ty} g(y) dy =: \mathfrak{L}_g(t), \forall \geq 0$$

- Let $\psi : \mathbb{R}_+ \to [0,1]$. If a solution exists, the inverse Laplace transform $\mathfrak{L}_\psi^{[-1]}$ of ψ is defined as the function $\chi : \mathbb{R}_+ \to [0,1]$ which solves

$$\mathfrak{L}_\chi(t) = \int_0^\infty e^{-ty} \chi(y) dy = \psi(t), \forall \geq 0$$

- The distribution of Y is uniquely characterized by its Laplace transform

Clayton Copula Density

Clayton copula density can be found using equation 2.9 and the definition of Clayton copula given in equation 2.12:

$$\frac{\partial^n C(u_1, ..., u_n)}{\partial u_1 \partial u_2 ... \partial u_n} = \alpha^n \frac{\Gamma\left(\frac{1}{\alpha} + n\right)}{\Gamma\left(\frac{1}{\alpha}\right)} \left(\prod_{i=1}^{n} u_i^{-\alpha-1}\right) \left(\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right)^{-\frac{1}{\alpha}-n}$$

where Γ indicates the Euler function.

Clayton Copula Simulation

Set $u_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right)$, where G is the distribution function of Y and $\widehat{G}$ is its Laplace transform.

- Simulate n iid uniform variable x_i
- Simulate a variable Y such that $\widehat{G}$ is the inverse generator: Simulate $Y = \Gamma(1/\alpha, 1)$
- Calculate $u_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right) = \left(1 + \left(-\frac{\ln(x_i)}{Y}\right)\right)^{-\frac{1}{\alpha}}$

Figure 4.8 shows Clayton bivariate copula simulation and probability density with 3000 samples.

Gumbel Copula

Definition 2.2.7. *Let the copula generator be* $\varphi(u) = (-\ln(u))^\alpha$ *and inverse generator be* $\varphi^{-1}(t) = \exp\left(-t^{\frac{1}{\alpha}}\right)$*; it is completely monotonic if* $\alpha > 1$*. The Gumbel n-copula is therefore;*

$$C(u_1, ..., u_n) = \exp\left[-\left[\sum_{i=1}^{n}(-\ln u_i)^\alpha\right]^{\frac{1}{\alpha}}\right] \text{ with } \alpha > 1 \qquad (2.13)$$

Gumbel Copula Simulation

Set $u_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right)$, where G is the distribution function of Y and $\widehat{G}$ is its Laplace transform.

- Simulate n iid uniform variable x_i
- Simulate a variable Y such that $\widehat{G}$ is the inverse generator: Simulate Y=Stable$(\kappa, \beta, \gamma, \delta)$ with $\kappa = 1/\alpha$, $\beta = 1$, $\gamma = \left(\cos\left(\frac{\pi}{2\alpha}\right)\right)^{\alpha}$ and $\delta = 0$
- Calculate $u_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right) = \exp\left(-\left(-\frac{\ln(x_i)}{Y}\right)^{\frac{1}{\alpha}}\right)$

Figure 4.10 shows Gumbel bivariate copula simulation and probability density with 3000 samples.

Frank Copula

Definition 2.2.8. *Let the copula generator be $\varphi(u) = -\ln\left(\frac{\exp(-\alpha u)-1}{\exp(-\alpha)-1}\right)$ and inverse generator be $\varphi^{-1}(t) = -\frac{1}{\alpha}\ln(1 - e^{t}(1 - e^{-\alpha}))$; it is completely monotonic if $\alpha > 0$. The Frank n-copula is therefore;*

$$C(u_1, ..., u_n) = -\frac{1}{\alpha}\ln\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right] \text{ with } \alpha > 0 \text{ when } n \geq 3 \tag{2.14}$$

Frank Copula Copula Density

Frank copula density can be found using equation 2.9 and the definition of Frank copula given in equation 2.14. However, the n-dimensional case is algebraically cumbersome, so a 4-variate case will be presented here:

Example 2.2.1. *Let $w_i = e^{-\alpha u_i} - 1$ for $i = 1, ..., 4$. Then;*

$$
\begin{aligned}
\frac{\partial w_i}{\partial u_i} &= -\alpha e^{-\alpha u_i} = -\alpha(w_i+1) \quad \textit{for } i = 1, ...4 \\
\frac{\partial C}{\partial w_i} &= -\frac{1}{\alpha}\frac{(e^{-\alpha})^3)}{(e^{-\alpha}-1)^3 + w_1w_2w_3w_4}\frac{w_2w_3w_4}{(e^{-\alpha}-1)^3} \\
\frac{\partial C}{\partial u_1} &= \frac{\partial C}{\partial w_i}\left(\frac{\partial w_i}{\partial u_i}\right) = \frac{(w_1+1)w_2w_3w_4}{(e^{-\alpha}-1)^3 + w_1w_2w_3w_4} \\
\frac{\partial^2 C}{\partial u_1 \partial u_2} &= \frac{\partial}{\partial u_2}\left(\frac{\partial C}{\partial u_1}\right) = -\alpha(w_1+1)(w_2+1)w_3w_4 \times \\
&\times \frac{(e^{-\alpha}-1)^3}{[(e^{-\alpha}-1)^3 + w_1w_2w_3w_4]^2} \\
\frac{\partial^3 C}{\partial u_1 \partial u_2 \partial u_3} &= \frac{\partial}{\partial u_3}\left(\frac{\partial^2 C}{\partial u_1 \partial u_2}\right) = \alpha^2(w_1+1)(w_2+1)(w_3+1)w_4(e^{-\alpha}-1)^3 \times \\
&\times \frac{(e^{-\alpha}-1)^3 - w_1w_2w_3w_4}{[(e^{-\alpha}-1)^3 + w_1w_2w_3w_4]^3} \\
\frac{\partial^4 C}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} &= \frac{\partial}{\partial u_4}\left(\frac{\partial^3 C}{\partial u_1 \partial u_2 \partial u_3}\right) = -\alpha^3(w_1+1)(w_2+1)(w_3+1)(w_4+1)(e^{-\alpha}-1)^3 \\
&\times \frac{[(e^{-\alpha}-1)^6 - 4(e^{-\alpha}-1)^3 w_1w_2w_3w_4 + w_1^2w_2^2w_3^2w_4^2]}{[(e^{-\alpha}-1)^3 + w_1w_2w_3w_4]^4}
\end{aligned}
$$

Frank Copula Copula Simulation

Set $u_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right)$, where G is the distribution function of Y and $\widehat{G}$ is its Laplace transform.

- Simulate n iid uniform variable x_i
- Simulate a variable Y such that $\widehat{G}$ is the inverse generator: Simulate Y from Logarithmic series with $\theta = 1 - \exp(-\alpha)$
- Calculate $u_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right) = -\frac{1}{\alpha}\ln(1 - e^{-\left(-\frac{\ln(x_i)}{Y}\right)}(1 - e^{-\alpha}))$

Figure 4.9 shows Frank bivariate copula simulation and probability density with 3000 samples.

2.2.5 Marshall-Olkin Copula

Bivariate Marshall-Olkin Copula

Denote the lifetimes of two components that are subject to fatal shocks as X_1 and X_2. Assume that the shocks follow three independent Poisson processes with parameters $\lambda_1, \lambda_2, \lambda_{12} \geq 0$, where the index indicates whether the shocks effect only component 1, only component 2 or both. Then the corresponding times Z_1, Z_2 and Z_{12} of occurrence of these shocks are independent exponential random variables. The probability that the components live longer than x_1 and x_2 is given by:

$$H(x_1, x_2) = \mathbb{P}[X_1 > x_1, X_2 > x_2] = \mathbb{P}[Z_1 > x_1]\mathbb{P}[Z_2 > x_2]\mathbb{P}[Z_{12} > \max(x_1, x_2)]$$

The univariate survival functions for X_1 and X_2 are $F_1(x_1) = \exp(-(\lambda_1 + \lambda_{12})x_1)$ and $F_2(x_2) = \exp(-(\lambda_2 + \lambda_{12})x_2)$. Denote $\alpha_i = \lambda_{12}/(\lambda_i + \lambda_{12})$ for $i = 1, 2$ then,

$$\widehat{C}(u_1, u_2) = u_1 u_2 \min(u_1^{-\alpha_1} u_2^{-\alpha_2}) = \min(u_1^{1-\alpha_1} u_2, u_1 u_2^{1-\alpha_2})$$

where $u_i = F_i(x_i)$ for $i = 1, 2$. Then the Marshall-Olkin copula or Generalized Cuadras-Augé copula is given by:

$$C_{\alpha_1,\alpha_2}(u_1, u_2) = \min(u_2 u_1^{1-\alpha_1}, u_1 u_2^{1-\alpha_2}) = \begin{cases} u_2 u_1^{1-\alpha_1}, & u_1^{\alpha_1} \geq u_2^{\alpha_2} \\ u_1 u_2^{1-\alpha_2}, & u_1^{\alpha_1} \leq u_2^{\alpha_2} \end{cases}$$

Marshall-Olkin copulas are absolutely continuous and singular as its density is:

$$\frac{\partial^2}{\partial u_1 \partial u_2} C_{\alpha_1,\alpha_2}(u_1, u_2) = \begin{cases} u_1^{-\alpha_1}, & u_1^{\alpha_1} > u_2^{\alpha_2} \\ u_2^{-\alpha_2}, & u_1^{\alpha_1} < u_2^{\alpha_2} \end{cases}$$

Marshall-Olkin Copula Simulation

Following the notation given in section 2.2.5; denote $l := |S| = 2^n - 1$ nonempty subsets of $\{1, ..., n\}$ in some arbitrary way, $s_1, ..., s_l$ and set $\lambda_k := \lambda_{sk}$ (the parameter of Z_{sk}) for

$k = 1, ..., l$. Then, the following algorithm generates random variates from Marshall-Olkin copula.

- Simulate $l = 2^n - 1$ independent random variates $v_1, ..., v_l$ from $U(0, 1)$
- Set $x_i = \min_{1 \le k \le l, i \in s_k, \lambda_k \neq 0}(-\ln v_k / \lambda_k)$ for $i = 1, ..., n$
- Set $\Lambda_i = \sum_{k=1}^{l} 1\{i \in s_k\}\lambda_k$ for $i = 1, ..., n$
- Set $u_i = \exp(-\Lambda_i x_i)$ for $i = 1, ..., n$

Then $(x_1, ..., x_n)^T$ is an n-variate from n-dimensional Marshall-Olkin distribution and $(u_1, ..., u_n)^T$ is an n-variate from the corresponding Marshall-Olkin n-copula with the shock intensity Λ_i felt by component i.

2.2.6 Measure of Dependence

Dependence between different observations, such as default of individuals names in a CDO, is the central consideration in pricing CDOs. Hence, it is crucial to have a stable correlation measure to be able to capture skewed, fat/light tailed dependencies.

Pearson Linear Correlation

Pearson's linear correlation is a simple measure that captures the strength of dependency between two random variables. For two random variables X and Y, the Pearson linear correlation is defined as follows:

$$\rho_{XY} = \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{(\mathbb{E}[X^2] - \mathbb{E}[X]^2)(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2)}}$$

Kendall's tau

Kendall's tau may be expressed in terms of a bivariate copula and it is independent of the marginal distributions. For two random variables X and Y with their corresponding bivariate copula $C(u_1, u_2)$, Kendall's tau is defined as follows:

$$\tau_{XY} = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1$$

where $\tau_{XY} \in [-1, 1]$. $\tau_{XY} = 0$ corresponds to independence, $\tau_{XY} = -1$ and $\tau_{XY} = 1$ corresponds to minimum and maximum dependence, respectively.

Kendall's tau for Clayton copula is $\frac{\alpha}{\alpha+2}$ and for Gumbel copula it is $1 - \frac{1}{\alpha}$. For Frank copula Kendall's tau is $D_k(x) = \frac{k}{x^k} \int_0^x \frac{t^k}{e^t - 1} dt$, where $D_k(x)$ is the Debye function, Embrechts et al. (2001). For Marshall-Olkin copula Kendall's tau is $\alpha/(2 - \alpha)$.

Spreman's rho

Spreman's rho is the correlation between ranks of two random variables and it may be expressed in terms of a bivariate copula. For two random variables X and Y with their corresponding bivariate copula $C(u_1, u_2)$, Spreman's rho is defined as follows:

$$\rho_{XY}^S = 12 \int_0^1 \int_0^1 u_1 u_2 dC(u_1, u_2) - 3 = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3$$

2.2.7 Tail Dependence

Although Spreman's rho and Kendall's tau are useful concepts, they fail to measure the absolute magnitude of two random variable takes. Tail dependence offers an important supplementary metric. Tail dependence is a measure of extremely large co-movements in the upper right quadrant (i.e. upper tail dependence) and lower left quadrant (i.e. lower tail dependence) of a bivariate distribution.

For two random variables X and Y with their marginal distribution functions F_X and F_Y, the coefficient of upper tail dependence is defined as follows:

$$\lambda_U = \lim_{u \to 1} \mathbb{P}[Y > F_Y^{-1}(u) | X > F_X^{-1}(u)]$$

similarly, the coefficient of lower tail dependence is defined as follows:

$$\lambda_L = \lim_{u \to 1} \mathbb{P}[Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)]$$

where $\lambda_U \in [0, 1]$ and $\lambda_L \in [0, 1]$ provided that the limit exists. When $\lambda_U = 0$, X and Y are said to be asymptotically independent in the upper tail; similarly, when $\lambda_L = 0$, X and Y are said to be asymptotically independent in the lower tail. When $\lambda_U \in (0, 1]$ X and Y are said to be asymptotically dependent in the upper tail. Using copula functions definition and properties;

$$\begin{aligned}
\lambda_U &= \lim_{u \to 1} \mathbb{P}[Y > F_Y^{-1}(u) | X > F_X^{-1}(u)] \\
&= \lim_{u \to 1} \frac{1 - \mathbb{P}[X \leq F_X^{-1}(u)] - \mathbb{P}[Y \leq F_Y^{-1}(u)] + \mathbb{P}[X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u)]}{1 - \mathbb{P}[X \leq F_X^{-1}(u)]} \\
&= \lim_{u \to 1} \frac{(1 - 2u + C(u, u))}{1 - u}
\end{aligned}$$

A similar procedure for lower tail dependence gives:

$$\lambda_L = \lim_{u \to 1} \frac{C(u, u)}{u}$$

Using copula definitions, these equalities lead to explicit terms for bivariate Gaussian, student's t, Clayton, Gumbel and Frank copula upper/lower tail dependencies. Gaussian bivariate copula has asymptotic lower and upper tail independence. Student's t copula has both upper and lower tail dependence with equal coefficients:

$$\lambda_L = \lambda_U = 2t_{v+1}\left(- \sqrt{\frac{(v + 1)(1 - \rho)}{1 + \rho}} \right)$$

The lower tail dependence of a bivariate Clayton copula is given as $2^{-1/\alpha}$ while it has an independent upper tail. The Frank family has no upper or lower tail dependence. The Gumbel copula has upper tail dependence given as $2 - 2^{1/\alpha}$.

Chapter 3

Empirical Analysis of CDO Pricing Models

This chapter presents the valuation basics of CDSs and CDOs and makes an extensive empirical analysis of the CDO pricing models commonly employed in the industry, as well as several other models frequently referenced in the literature. These bottom-up models may be classified into two main groups. The first group comprises industry benchmark models that represent the default dependence structures with copula functions. Included in this group are all of the Elliptical and Archimedean copula models with Monte Carlo simulation, as well as their factor representations. The second group includes stochastic correlation models with independent correlation and symmetric but dependent correlation assumptions, the Random Factor Loadings model, and symmetric/asymmetric normal inverse Gaussian models. These latter models are shown to fit market quotes better than industry benchmark models as they bring more tail dependence and introduce a stochastic correlation parameter.

The notation that will be used throughout this dissertation is as follows: Assume that there are $i = 1, ..., n$ credits (i.e. CDSs) in a synthetic CDO with the associated vector of default times $(\tau_1, ..., \tau_n)$, where τ_i's are continuous random variables. At time t, i^{th} credits probability density function is $f_i(t)$ for all $(t_1, ..., t_n) \in \mathbb{R}^n$ and its probability distribution function is defined as:

$$p_i(t) = \mathbb{E}_t[\mathbf{1}_{(\tau_i \leq t)}] = F_i(t) = \mathbb{P}(\tau_i \leq t) \tag{3.1}$$

This function is also known to be the unconditional default probability of obligor i before time t. Similarly, i^{th} credits survival function is defined as:

$$1 - p_i(t) = \mathbb{E}_t[\mathbf{1}_{(\tau_i > t)}] = S_i(t) = 1 - F_i(t) = \mathbb{P}(\tau_i > t)$$

Each credit contract in the portfolio starts at time t_0 with payment dates t_k for $k = 1, .., K$, maturity t_K, notional A_i and a constant recovery rate of δ_i.

3.1 Literature Review

Section 2.1 presented a review of models used in the single-name credit risk literature. Equipped with the default probability/recovery rate of each credit in the portfolio and a dependence structure between single-name defaults, one can model the portfolio loss distribution. There are two schools of thought regarding the modeling of multi-name loss distribution in the literature: Top-down approach and bottom-up approach.

The top-down approach starts by modeling the loss process of the entire portfolio, which is generally assumed to be composed of homogenous assets. These models may be calibrated and implemented very quickly, but generally they do not provide detailed characteristics of the individual credit dynamics. For more on top-down models, see Bennani (2006), Errais, Gieseke & Goldberg (2006), Schönbucher (2005), Gieseke & Tomecek (2005), Gieseke & Goldberg (2005) and Longstaff & Rajan (2006).

On the other hand, the bottom-up approach starts by modeling individual credit dynamics and proposing a default correlation structure, and then calculates the portfolio loss process. Because they can govern heterogeneous assets within the portfolio, these models are capable of pricing exotic correlation products. But this flexibility comes at the cost of calibration and implementation speed increases, especially for large portfolios. Since bottom-up models have a wider use, this dissertation provides a large empirical analysis of the bottom-up models in this chapter and proposes a dynamic bottom-up model in the next one.

3.2 Primer on CDSs

There are three methods used to estimate instantaneous default probability of a credit. First, it can be estimated from historical data by using S&P's[1], Moody's[2] or Fitch's[3] database for historical default rates. Second, one can employ a Merton (1974) style structural model. Finally, one can bootstrap the implied default rates from market-observable data such as defaultable bonds or CDS spreads. The bootstrapping methodology will be employed throughout this dissertation.

3.2.1 Hazard Rate Function

Following Li (2000), assume a filtered probability space $(\Omega, \Im, (\Im_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a risk-neutral probability measure. τ_i is the $(\Im_t)$ random stopping time for the i^{th} credit. The hazard rate function which is the instantaneous default probability in interval $(t, t+\Delta t)$ for a security that already has attained age t, is defined as:

$$\begin{aligned} h_i(t) &= \lim_{\Delta t \to 0} \mathbb{P}(t < \tau_i \leq t + \Delta t | \tau_i > t) \\ &= \lim_{\Delta t \to 0} \frac{\mathbb{P}(t < \tau_i \leq t + \Delta t, \tau_i > t)}{\mathbb{P}(\tau_i > t)} = \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} f_i(u)du}{\int_t^{\infty} f_i(u)du} \\ &= \frac{f_i(t)}{1 - F_i(t)} = \frac{\frac{\partial}{\partial t} F_i(t)}{1 - F_i(t)} = -\frac{\partial}{\partial t} \log(1 - F_i(t)) \end{aligned}$$

solving the differential equation;

---

[1] S&P's RatingsDirect Service and Annual Global Corporate Default Study and Rating Transitions.
[2] Moody's Structured Finance Default Risk Service.
[3] Fitch's Rating Transition and Default Studies.

$$F_i(t) = 1 - \exp\left(-\int_0^t h_i(u)du\right) \tag{3.2}$$

$$S_i(t) = 1 - F_i(t) = \exp\left(-\int_0^t h_i(u)du\right) \tag{3.3}$$

$$f_i(t) = h_i(t)\exp\left(-\int_0^t h_i(u)du\right) \tag{3.4}$$

3.2.2 CDS Valuation

The buyer of a CDS, the protection buyer, pays a periodic premium to the protection seller up to the maturity date of the contract or a credit event, whichever is sooner. In exchange, the protection seller agrees to pay the face value of the underlying asset in case of a default or a pre-specified credit event such as restructuring, bankruptcy or downgrade. Payments by the protection buyer to the protection seller are known as the "premium leg" of the CDS contract. Payments from the protection seller to the protection buyer are known as the "default leg" of the contract.

Following the notation in Galiani (2003), the premium of the i^{th} CDS, v_i, makes the value of the contract equal to zero at the beginning of the contract. As long as the reference entity does not default, at each payment date t_k, the protection buyer pays to the protection seller $v_i A_i \Delta_k$, where Δ_k is the year fraction between t_{k-1} and t_k. The value of the premium leg at time t is:

$$\begin{aligned} PL_i &= \sum_{k=1}^{K} \mathbb{E}_t\left[B(t_0,t_k)\Delta t_k v_i A_i[\mathbf{1}_{(\tau_i>t_k)}]\right] = v_i A_i \sum_{k=1}^{K} B(t_0,t_k)\Delta t_k \mathbb{E}_t[\mathbf{1}_{(\tau_i>t_k)}] \\ &= v_i A_i \sum_{k=1}^{K} B(t_0,t_k)\Delta t_k[1-F_i(t_k)] = v_i A_i \sum_{k=1}^{K} B(t_0,t_k)\Delta t_k \exp\left(-\int_{t_0}^{t_k} h_i(u)du\right) \end{aligned}$$

where $B(t_0, t_k)$ is the default-free zero coupon bond priced at time t_0 with maturity t_k. In the case of a default at time $\tau_i \leq t_K$, the protection seller pays the face value of the underlying security reduced by the recovery rate and receives a premium payment accrued since the last premium payment: $DL_i = DP_i - AP_i$. The value of the default premium and accrued premium at time t is:

$$
\begin{aligned}
DP_i &= \mathbb{E}_t[B(t_0,\tau_i)(1-\delta_i)A_i[\mathbf{1}_{(\tau_i \le t_K)}]] = A_i B(t_0,\tau_i)(1-\delta_i)\mathbb{E}_t[\mathbf{1}_{(\tau_i \le t_K)}] \\
&= (1-\delta_i)A_i \int_{t_0}^{t_K} B(t_0,u)F_i(du) \\
&= (1-\delta_i)A_i \int_{t_0}^{t_K} B(t_0,u)h_i(u)\exp\left(-\int_{t_0}^{u} h_i(s)ds\right)du
\end{aligned}
$$

and

$$
\begin{aligned}
AP_i &= \sum_{k=1}^{K} \mathbb{E}_t\left[\frac{\tau_i - t_{k-1}}{t_k - t_{k-1}}\Delta t_k B(t_0,\tau_i)v_i A_i[\mathbf{1}_{(t_{k-1}<\tau_i \le t_k)}]\right] \\
&= v_i A_i \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} \frac{\tau_i - t_{k-1}}{t_k - t_{k-1}}\Delta t_k B(t_0,u)F_i(du) \\
&= v_i A_i \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} \frac{u - t_{k-1}}{t_k - t_{k-1}}\Delta t_k B(t_0,u)h_i(u)\exp\left(-\int_{t_0}^{u} h_i(s)ds\right)du
\end{aligned}
$$

The CDS premium, v_i^*, makes the value of the contract equal to zero at the beginning of the contract: $PL_i(v^*) - DL_i(v^*) = 0$. Therefore, the break even CDS spread is:

$$
v_i^* = \frac{(1-\delta)\int_{t_0}^{t_K} B(t_0,u)F_i(du)}{\sum_{k=1}^{K} B(t_0,t_k)\Delta t_k \exp\left(-\int_{t_0}^{t_k} h_i(u)du\right) + \int_{t_{k-1}}^{t_k} \frac{u - t_{k-1}}{t_k - t_{k-1}}\Delta t_k B(t_0,u)F_i(du)}
$$

3.3 Pricing Basket Default Swaps

Pricing j-to-default baskets follows a similar routine as in plain vanilla CDS valuation given in section 3.2.2. Following Galiani (2003) and denoting the distribution of the τ_j as $F_j(t) = \mathbb{P}[\tau_j \le t]$, the protection leg, default payment, accrued premium and default leg $(DL_i = DP_i - AP_i)$, for j-to- default basket is given as follows:

$$\begin{aligned}
PL_i &= \mathbb{E}_t\Big[\sum_{k=1}^{K} B(t_0,t_k)\Delta t_k v_i A_i[\mathbf{1}_{(\tau_j>t_k)}]\Big] = v_i A_i \sum_{k=1}^{K} B(t_0,t_k)\Delta t_k[1-F_j(t_k)] \\
DP_i &= \mathbb{E}_t\Big[\sum_{i=1}^{n} B(t_0,\tau_j)(1-\delta_i)A_i[\mathbf{1}_{(\tau_j\le t_K)}]\Big] = \sum_{i=1}^{n}(1-\delta_i)A_i \int_{t_0}^{t_K} B(t_0,t)F_j dt \\
AP_i &= \mathbb{E}_t\left[\sum_{k=1}^{K} \frac{\tau_j - t_{k-1}}{t_k - t_{k-1}}\Delta t_k B(t_0,\tau_j)A_i v_i[\mathbf{1}_{(t_{k-1}<\tau_j\le t_k)}]\right] \\
&= v_i A_i \sum_{k=1}^{K}\int_{t_{k-1}}^{t_k} \frac{u-t_{k-1}}{t_k-t_{k-1}}\Delta t_k B(t_0,u)F_j du
\end{aligned}$$

The premium v_i^* makes the value of the contract equal to zero at the beginning of the contract: $PL_i(v^*) - DL_i(v^*) = 0$. So, the break even default swap spread is:

$$v_i^* = \frac{\sum_{i=1}^{n}(1-\delta_i)\int_{t_0}^{t_K} B(t_0,t)F_j dt}{\sum_{k=1}^{K} B(t_0,t_k)\Delta t_k[1-F_j(t_k)] + \int_{t_{k-1}}^{t_k} \frac{u-t_{k-1}}{t_k-t_{k-1}}\Delta t_k B(t_0,u)F_j du}$$

3.4 Primer on CDOs

3.4.1 A CDO example

The most actively traded indices are CDX North America Investment Grade Series (CDX.NA.IG) and iTraxx Europe. Each index is composed of 125 names; hence, each company accounts for 0.8 percent of the exposure. Figure 4.1 shows 5 year tranche bid-offer quotes for the CDX.NA.IG Series 13 and iTraxx Europe Series 9 on November 25, 2009. Table 4.1 shows 5 year tranche mid-market quotes for the CDX.NA.IG Series 13 and iTraxx Europe Series 9 on the same day.

Quotes are comprised of an upfront fee (in percentage points) and a running spread (in basis points). The protection buyer pays the protection seller a one-time upfront fee (percentage of the principal) at the inception of the trade as well as a fixed running spread on the outstanding principal. Running spreads are reported in basis points annually and paid quarterly on the 20th of March, June, September, and December of each calendar year.

The day-count convention is ACT/360. A standard contract on CDX and on iTraxx is 10 million dollars and 10 million euros, respectively.

3.4.2 Tranche Loss Distribution

The loss-given-default (LGD) of obligor i is $(1-\delta_i)A_i$. Let $TL(t)$ denote the total portfolio loss at time t and $TL_{(K_A,K_D)}(t)$ denote the total loss at time t on a given tranche of a CDO with attachment and detachment points K_A and K_D, respectively. Total portfolio loss at time t and its expectation is given with the help of equation 3.1:

$$\begin{aligned} TL(t) &= \sum_{i=1}^{n}(1-\delta_i)A_i[\mathbf{1}_{(\tau_i \le t)}] \\ \mathbb{E}_t[TL(t)] &= \sum_{i=1}^{n}(1-\delta_i)A_i\mathbb{E}_t[\mathbf{1}_{(\tau_i \le t)}] = \sum_{i=1}^{n}(1-\delta_i)A_iF_i(t) \end{aligned}$$

so the total portfolio loss is a pure jump process. Total percentage loss at time t on a given tranche of a CDO and its expectation can be expressed as:

$$TL_{(K_A,K_D)}(t) = \frac{1}{K_D-K_A} \times \begin{cases} 0 & \text{if } TL(t) \le K_A \\ TL(t)-K_A & \text{if } K_A \le TL(t) \le K_D \\ K_D-K_A & \text{if } TL(t) \ge K_D \end{cases} \tag{3.5}$$

$$= \frac{1}{K_D-K_A} \times \max\Big(\min(TL_i, K_D)-K_A, 0\Big) \tag{3.6}$$

$$\mathbb{E}_t[TL_{(K_A,K_D)}] = \frac{1}{K_D-K_A} \times \sum_{i=1}^{n}\max\Big(\min(TL_i, K_D)-K_A, 0\Big) \times F_i(t) \tag{3.7}$$

Hence, as long as the loss distribution function $F_i(t)$ of the reference portfolio is modeled, CDO tranches may be easily priced. A model for linking default probabilities of each entity in the portfolio with each other will be presented in section 3.6.

3.4.3 Tranche Break Even Spread

The value of the premium leg of a tranche is the present value of all expected spread payments on whatever notional is left:

$$PL_{(K_A,K_D)} =$$

if we discretize,

$$\backsimeq \sum_{i=1}^{n} \triangle t_i B(t_0, t_{i-1}) w_{(K_A,K_D)} \Big(1 - \mathbb{E}_t[TL_{(K_A,K_D)}(t_{i-1})]\Big)$$

where $\triangle t_i = t_i - t_{i-1}$ and $w_{(K_A,K_D)}$ is the tranche spread. The value of the default leg may be calculated as the expected value of the discounted default payments:

$$DL_{(K_A,K_D)} = \mathbb{E}_t\left[\int_{t_0}^{t_n} B(t_0,s)dTL_{(K_A,K_D)}(s)\right] = \int_{t_0}^{t_n} B(t_0,s)\mathbb{E}_t[dTL_{(K_A,K_D)}(s)]$$

if we discretize,

$$\backsimeq \sum_{i=1}^{n} B(t_0,t_i)\Big(\mathbb{E}_t[TL_{(K_A,K_D)}(t_i)] - \mathbb{E}_t[TL_{(K_A,K_D)}(t_{i-1})]\Big)$$

Then the break even spread $w^*_{(K_A,K_D)}$ is calculated by equating the protection leg to premium leg: $DL_{(K_A,K_D)} - PL_{(K_A,K_D)}(w^*_{(K_A,K_D)}) = 0$:

$$w^*_{(K_A,K_D)} = \frac{\sum_{i=1}^{n} B(t_0,t_i)\Big(\mathbb{E}_t[TL_{(K_A,K_D)}(t_i)] - \mathbb{E}_t[TL_{(K_A,K_D)}(t_{i-1})]\Big)}{\sum_{i=1}^{n} B(t_0,t_{i-1})\Delta t_i\Big(1 - \mathbb{E}_t[TL_{(K_A,K_D)}(t_{i-1})]\Big)}$$

where $\mathbb{E}_t[TL_{(K_A,K_D)}]$ is given in equation 3.7.

3.5 Pricing CDOs with Monte Carlo Simulation

The copula definitions and simulations procedures presented in section 2.2 form the basis for pricing CDOs using the Monte Carlo simulation routine. The pricing algorithm can be summarized as given in Galiani (2003).

Repeat the following routine S times, where $s = 1, 2, ..., S$;

- Generate n dimensional vector of correlated variables using simulation procedures descried in sections 2.2.3 and 2.2.4 depending the choice of copula, where n is the total size of the portfolio.
- For each obligor transform the univariate random variable, u_i into default time τ_i using the following equality: $\tau_i = \frac{-\ln(u_i)}{\lambda}$ for $i = 1, .., n$.
- For the sth simulation, sort τ^s in ascending order such that $\tau_i^s \leq T$ and form a vector default times $\Upsilon^s = (\tau_1^s, \tau_2^s, ..., \tau_L^s)$.
- Calculate the Default Payments, DP^s:
  - Calculate total portfolio loss: $TL^s(T) = \sum_{i=1}^n A_i(1 - \delta_i)\mathbf{1}_{\{\tau_i^s \leq T\}}$
  - If $TL^s < K_A$ then $DP^s = 0$, else select the default trigger vector $\Upsilon^s = (\tau_\gamma^s, ..., \tau_L^s)$ where $\tau_\gamma^s = inf\{t > 0 | TL(t) \geq K_A\}$ and $\tau_L^s = inf\{t > 0 | TL(t) \leq K_D\}$.
  - For a realization of $\tau_r^s \in \Upsilon^s$, calculate default payments as $DP_r^s = B(0, \tau_r^s)(1 - \delta_r)A_r$
  - Sum default payments for all realizations: $\sum_{r=\gamma}^L DP_r^s$
- Calculate the Premium Payments, PL^s, for each premium date:
  - Calculate total portfolio loss at each payment date: $TL^s(t_i) = \sum_{i=1}^n A_i(1 - \delta_i)\mathbf{1}_{\{\tau_i^s \leq t_i\}}$
  - Calculate Premium Leg: $PL^s = \sum_{i=1}^n \triangle t_i B(t_0, t_i)$ $\times \min(\max(0, K_D - TL(t_i)^s), K_D - K_A)$
- Fair spread $w^* = \frac{1}{S}\sum_{s=1}^S \left(\frac{DP}{PL}\right)^s$

Table 4.3 shows Gauss, student's t, Clayton, Frank, Gumbel and Marshall-Olkin copulae fitted to CDX with 10,000 Monte Carlo simulations. Similarly, Table 4.4 shows the same copulae fitted to iTraxx with 10,000 Monte Carlo simulations.

3.6 Pricing CDOs with Gaussian Copulas

Section 3.2 showed how to bootstrap survival/default probabilities of each credit in a CDO from their CDS spreads. Copula functions will be used to link several one-dimensional survival curves to a single portfolio level survival curve. Copula functions have long been used in survival analysis and actuarial sciences, but came to the forefront of the finance literature with Li (1998) and Li (2000).

Let $F_{portfolio}(t)$ represent the unconditional joint default function of default times of the portfolio such that $F_{portfolio}(t) = P(\tau_1 \leq t, ..., \tau_n \leq t)$. Also let $S_{portfolio}(t)$ represent the unconditional joint survival function of default times of the portfolio such that $S_{portfolio}(t) = P(\tau_1 > \mathrm{t}, ..., \tau_\mathrm{n} > \mathrm{t})$. Unconditional marginal default function and unconditional marginal survival function is as defined in equations 3.2 and in 3.3, respectively. Then, following the copula approach one may find the unconditional joint default function of default times of the portfolio:

$$F_{portfolio}(t) = P(\tau_1 \leq t, ..., \tau_n \leq t) = C[F_1(t), ..., F_n(t)] \tag{3.8}$$

and using Skalar's theorem 2.2.1;

$$C[F_1(t), ..., F_n(t)] = F_{portfolio}[F_1^{-1}(t), ..., F_n^{-1}(t)]$$

The market standard is to use Gaussian copulas as defined in equation 2.10. Then equation 3.8 becomes;

$$\begin{aligned} C_\Sigma^G[F_1(t), ..., F_n(t)] &= \Phi_\Sigma[\Phi^{-1}(F_1^{-1}(t)), ..., \Phi^{-1}(F_n^{-1}(t))] \\ F_{portfolio}(t) &= \Phi_\Sigma^n[\Phi^{-1}(F_1^{-1}(t)), ..., \Phi^{-1}(F_n^{-1}(t))] \end{aligned}$$

where Φ_Σ^n is the n-dimensional Gaussian distribution function with correlation matrix Σ.

3.6.1 One Factor Gaussian Copula

The next two sections follow the presentation in Gregory & Laurent (2003), Burtschell, Gregory & Laurent (2005*b*) and Schönbucher (2003). Assume a Gaussian vector $(V_i, ..., V_n)$ of individual risk process where $V_i = \rho_i Y + w_i \varepsilon_i$. Y is the systemic risk, ε_i is the idiosyncratic risk of the i^{th} firm, ρ_i represents the sensitivity of V_i to Y and w_i represents the sensitivity of V_i to ε_i. It is assumed that Y and ε_i are $i.i.d$ with $\Phi(0,1)$ for $i = 1, ..., n$. Due to the stability of normal distributions under convolution, V_i's follow normal distribution with zero-mean and unit-variance as well:

$$
\begin{aligned}
\mathbb{E}[V_i] &= \rho_i \mathbb{E}[Y] + w_i \mathbb{E}[\varepsilon_i] = 0 \\
var[V_i] &= \rho_i^2 var[Y] + w_i^2 var[\varepsilon_i] + 2\rho_i w_i cov[Y, \varepsilon_i] \\
&= \rho_i^2 + w_i^2 \\
\text{hence,} \qquad w_i &= \sqrt{1-\rho_i^2} \\
\text{and} \qquad V_i &= \rho_i Y + \sqrt{1-\rho_i^2}\varepsilon_i
\end{aligned}
$$

And the correlation between V_i and V_j is given by:

$$
\begin{aligned}
corr[V_i, V_j] &= \frac{cov[V_i, V_j]}{\sqrt{var[V_i]var[V_j]}} = cov[V_i, V_j] \\
cov[V_i, V_j] &= E[(\rho_i Y + \sqrt{1-\rho_i^2}\varepsilon_i)(\rho_j Y + \sqrt{1-\rho_j^2}\varepsilon_j)] = \rho_i \rho_j
\end{aligned}
$$

Hence, conditional on the systemic factor, Y, the default times are independent and only idiosyncratic factors matter.

Similar to Merton (1974)'s structural model, the probability of each obligor's default before maturity is exactly the probability of the decline of the individual risk process (i.e. the change in the value of the assets of the firm) below a pre-specified barrier k_i. Hence, the equation 3.1 becomes $p_i(t) = \mathbb{P}(\tau_i \leq t) = \mathbb{P}(V_i \leq k_i)$.

The default thresholds are chosen so that they produce risk neutral default probabilities implied by the quoted CDS spreads: $p_i(t) = \mathbb{P}(\tau_i \leq t) = \mathbb{P}(V_i \leq k_i) = \Phi(k_i)$, hence $k_i = \Phi^{-1}(F_i(t))$.

The default probability of each obligor conditioning on the common factor Y taking a value y is calculated as follows:

$$\begin{aligned} p_i(t|y) &= \mathbb{P}(\tau_i \leq t|Y=y) = \mathbb{P}(V_i \leq k_i|Y=y) && (3.9)\\ &= \mathbb{P}(\rho_i Y + \sqrt{1-\rho_i^2}\varepsilon_i \leq k_i|Y=y) && (3.10)\\ &= \mathbb{P}\left(\varepsilon_i \leq \frac{k_i - \rho_i Y}{\sqrt{1-\rho_i^2}}|Y=y\right) = \Phi\left[\frac{k_i-\rho_i y}{\sqrt{1-\rho_i^2}}\right] && (3.11)\\ \text{where,} \qquad k_i &= \Phi^{-1}(F_i(t)) && (3.12) \end{aligned}$$

There is no upper or lower tail dependence when $\rho_i < 1$. $\rho_i = 0$ is associated with independent default times, whereas $\rho_i = 1$ is associated with comonotonic default times. The unconditional default probabilities may be found by integrating out the common risk factor Y, so, equation 3.11 becomes:

$$p_i(t|Y) = \mathbb{E}\left[\Phi\left[\frac{k_i-\rho_i Y}{\sqrt{1-\rho_i^2}}\right]\right] = \int_{-\infty}^{\infty} \Phi\left[\frac{k_i-\rho_i Y}{\sqrt{1-\rho_i^2}}\right]\varphi(y)dy \qquad (3.13)$$

where $\varphi(y) = \frac{1}{\sqrt{2\Pi}}e^{-y^2/2}$. Using equations 3.11 and 3.13, portfolio unconditional default probability may be expressed as:

$$
\begin{aligned}
F_{portfolio}(t) &= P(\tau_1 \le t, ..., \tau_n \le t) = \mathbb{E}\left[\prod_{i=1}^{n} \Phi\left[\frac{k_i - \rho_i Y}{\sqrt{1-\rho_i^2}}\right]\right] \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\left[\frac{k_i - \rho_i Y}{\sqrt{1-\rho_i^2}}\right] \varphi(y) dy
\end{aligned}
$$

3.6.2 Finite Size Homogeneous Portfolios

In the case that the underlying credits have exactly the same recovery rate, default barrier, notional and correlation to common risk factor, the portfolio is said to be "homogeneous": $\delta_i = \delta, k_i = k, A_i = A$ and $\rho_i = \rho$ for all $i = 1, ..., n$. The conditional default probability of all issuers in the portfolio is given by employing Vasicek (1987)'s model:

$$
\begin{aligned}
p(t|y) &= \mathbb{P}(\tau \le t|Y = y) = \mathbb{P}(V \le k|Y = y) \\
&= \mathbb{P}(\rho Y + \sqrt{1-\rho^2}\varepsilon \le k|Y = y) = \mathbb{P}\left(\varepsilon \le \frac{k - \rho Y}{\sqrt{1-\rho^2}}|Y = y\right) \\
&= \Phi\left[\frac{k - \rho y}{\sqrt{1-\rho^2}}\right]
\end{aligned}
$$

The probability of having exactly c out of n issuers at time t conditional on the realization of $Y = y$ can be calculated using binomial distribution, $\Pi_c^n(t|y)$:

$$
\begin{aligned}
\Pi_c^n(t|y) &= \binom{n}{c} p(t|Y)^c (1 - p(t|Y))^{n-c} \\
&= \binom{n}{c} \left(\Phi\left[\frac{k - \rho y}{\sqrt{1-\rho^2}}\right]\right)^c \left(1 - \Phi\left[\frac{k - \rho y}{\sqrt{1-\rho^2}}\right]\right)^{n-c}
\end{aligned}
$$

The unconditional probability of having exactly c out of n issuers at time t is the average of the conditional probability of c out of n defaults, averaged over the possible realizations of Y weighted with the probability density function $\varphi(y)$:

$$\Pi_c^n(t|y) = \left(\begin{array}{c} n \\ c \end{array} \right) \int_{-\infty}^{\infty} \left(\Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}} \right] \right)^c \left(1 - \Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}} \right] \right)^{n-c} \varphi(y)dy$$

As the portfolio at hand is homogenous, the probability of having c out of n issuers is equal to the probability of percentage portfolio loss $TL(t)$ being $TL_c(t) = \frac{c}{n}(1-\delta)$. Then the total percentage loss of the portfolio at time t is given by:

$$\mathbb{P}[TL(t) = TL_c(t)] = \left(\begin{array}{c} n \\ c \end{array} \right) \int_{-\infty}^{\infty} \left(\Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}} \right] \right)^c \left(1 - \Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}} \right] \right)^{n-c} \varphi(y)dy$$

and for $\theta \in [0,1]$, the cumulative probability of the percentage portfolio loss not exceeding θ is:

$$F_n(\theta) = \mathbb{P}[TL(t) \leq \theta] \qquad (3.14)$$

$$= \sum_{c=0}^{[\theta n]} \left(\begin{array}{c} n \\ c \end{array} \right) \int_{-\infty}^{\infty} \left(\Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}} \right] \right)^c \left(1 - \Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}} \right] \right)^{n-c} \varphi(y)dy \qquad (3.15)$$

3.6.3 Large and Homogeneous Portfolios

Large portfolio limit approximation was first proposed by Vasicek (1991). Let the number of the underlying credits be large, $n \to \infty$, and denote $s = \Phi\left[\frac{k-\rho y}{\sqrt{1-\rho^2}}\right]$ then equation 3.15 becomes:

$$F_n(\theta) = \sum_{c=0}^{[\theta n]} \left(\begin{array}{c} n \\ c \end{array} \right) \int_0^1 s^c(1-s)^{n-c} d\Phi\left(\frac{\sqrt{1-\rho^2}\Phi^{-1}(s) - k}{\rho} \right)$$

By the law of large numbers;

$$\lim_{n\to\infty} \sum_{c=0}^{[\theta n]} \left(\begin{array}{c} n \\ c \end{array} \right) s^c(1-s)^{n-c} = \begin{cases} 0, \text{ if } \theta < s \\ 1, \text{ if } \theta > s \end{cases}$$

The cumulative loss distribution of a large portfolio may be expressed as:

$$F_\infty(\theta) = \Phi\left(\frac{\sqrt{1-\rho^2}\Phi^{-1}(\theta) - k}{\rho}\right)$$

3.6.4 Finite Size Inhomogeneous Portfolios

Loss distribution of a finite and inhomogeneous portfolio can be calculated via the semi-analytic recursion method proposed by Andersen, Sidenius & Basu (2003):

$$\begin{aligned}
\Pi_c^1(t|y) &= \begin{cases} 1 - p_1(t|y), \text{if } c = 0 \\ p_1(t|y), \text{if } c = 1 \end{cases} \\
\Pi_c^2(t|y) &= \begin{cases} (1 - p_1(t|y))(1 - p_2(t|y)), \text{if } c = 0 \\ \Pi_1^1(t|y)(1 - p_2(t|y)) + \Pi_0^1(t|y)p_2(t|y), \text{if } c = 1 \\ p_1(t|y)p_2(t|y), \text{if } c = 2 \end{cases} \\
\Pi_c^3(t|y) &= \begin{cases} (1 - p_1(t|y))(1 - p_2(t|y))(1 - p_3(t|y)), \text{if } c = 0 \\ \Pi_1^2(t|y)(1 - p_3(t|y)) + \Pi_0^2(t|y)p_3(t|y), \text{if } c = 1 \\ \Pi_2^2(t|y)(1 - p_3(t|y)) + \Pi_1^2(t|y)p_3(t|y), \text{if } c = 2 \\ p_1(t|y)p_2(t|y)p_3(t|y), \text{if } c = 3 \end{cases}
\end{aligned}$$

$$\vdots$$

Hence, conditional portfolio default distribution may be written as:

$$\begin{aligned}
\Pi_0^{n+1}(t|y) &= \Pi_0^n(t|y)(1 - p_{n+1}(t|y)) \\
\Pi_c^{n+1}(t|y) &= \Pi_c^n(t|y)(1 - p_{n+1}(t|y)) + \Pi_{c-1}^n(t|y)p_{n+1}(t|y) \quad \text{for } c = 1, ..., n \\
\Pi_{n+1}^{n+1}(t|y) &= \Pi_n^n(t|y)p_{n+1}(t|y)
\end{aligned}$$

and unconditional portfolio default distribution may be written as:

$$\Pi_c^{n+1}(t) = \mathbb{E}[\Pi_c^{n+1}(t|y)] = \int_{-\infty}^{+\infty} \Pi_c^{n+1}(t|y)\varphi(y)dy$$

3.6.5 Fast Fourier Transform (FFT)

A more efficient way to calculate loss distribution of a finite and inhomogeneous portfolio is to use the Fast Fourier Transformation(FFT) as outlined in Carr & Madan (1998). Given that loss given default (LGD) of obligor i is $(1-\delta_i)A_i$ and total portfolio loss at time t is $TL(t) = \sum_{i=1}^{n}(1-\delta_i)A_i[\mathbf{1}_{(\tau_i \le t)}]$, conditional on the common factor Y, the characteristic function of the portfolio loss is given by:

$$\Phi_{TL(t)}(u) = \mathbb{E}[e^{iuTL(t)}]$$

assuming conditional independence of default times upon factor Y,

$$\begin{aligned}\Phi_{TL(t)}(u|y) &= \mathbb{E}[\mathbb{E}[e^{iuTL(t)}|Y=y]] \\ \mathbb{E}[e^{iuTL(t)}|Y=y] &= \prod_{i=1}^{n}\mathbb{E}[e^{iu(1-\delta_i)A_i[\mathbf{1}_{(\tau_i \le t)}]}|Y=y]\end{aligned}$$

If there is a default, $(1-\delta_i)A_i[\mathbf{1}_{(\tau_i \le t)}]$ equals to $(1-\delta_i)A_i$ with probability $p_i(t|y)$ and if there is no default it is equal to 0 with probability $1-p_i(t|y)$. Given the conditional default probabilities as in equation 3.11:

$$\mathbb{E}[e^{iu[(1-\delta_i)A_i[\mathbf{1}_{(\tau_i \le t)}]]}|Y=y] = e^{iu[(1-\delta_i)A_i]}p_i(t|y) + e^{iu[(1-\delta_i)A_i]0}(1-p_i(t|y))$$

Hence, the conditional characteristic function of the portfolio loss is:

$$\Phi_{TL(t)}(u|y) = \prod_{i=1}^{n}(1+p_i(t|y)[e^{iu(1-\delta_i)A_i}-1]$$

By integrating out the common factor Y, unconditional characteristic function may be calculated as:

$$\Phi_{TL(t)}(u) = \mathbb{E}[\Phi_{TL(t)}(u|y)] = \int_{-\infty}^{+\infty}\prod_{i=1}^{n}(1+p_i(t|y)[e^{iu(1-\delta_i)A_i}-1]\varphi(y)dy \tag{3.16}$$

In order to find the portfolio loss distribution, the unconditional characteristic function of 3.16 must be inverted via FFT. Figure 4.2 shows the probability density of portfolio loss using binomial distribution, large portfolio approximation, semi analytic recursion and Fast Fourier Transform methods for CDX and iTraxx indices employing the Gaussian factor copula model.

3.6.6 Drawbacks of the Gaussian Factor Copula Model

The Gaussian copula model is the industry standard in pricing CDO tranches because of its easy implementation and speedy calibration; however, this model has several well-known shortcomings. Its symmetric and light-tailed return distribution prevents the model from capturing the dependence structure when extreme events occur. Moreover, its static default dependence structure fails to incorporate changing market conditions and company credit profiles. Nor does the Gaussian one-factor copula framework allow one to price instruments such as options on CDOs, forward-starting CDO tranches that depend on the dynamic evolution of loss distributions from their initial values. Further, it assumes a single flat correlation for all tranches, when in practice market implied correlations for equity and most-senior tranches are typically higher than mezzanine tranches, leading to a so-called "correlation smile".

Table 4.2 shows the inability of Gaussian copula model to fit market upfront fee/running spread with a flat correlation of 0.3655 for CDX and 0.4550 for iTraxx. The flat correlation is the result of a least squares estimate that fits all five tranches at the same time. On the other hand, market implied correlation insures that the upfront fee and running spread of each tranche is matched exactly. Figure 4.3 shows the correlation smile for CDX and iTraxx indices.

Attempts to improve the Gaussian copula model aim to generate asymmetric and heavy-tailed return distributions, and to bring stochasticity to the correlation parameter.

3.7 CDO Pricing using Factor Copulas with Alternative Distributions

This section covers factor CDO pricing models using student's t copula, double t copula, Marshall-Olkin copula and Archimedean copulae.

3.7.1 Student's t Copula

In contrast to the Gaussian copula with independent upper and lower tail dependence, the student t copula has upper and lower tail dependence with equal coefficients. Assume that individual risk process $(V_1, ..., V_n)$ follows student's t distribution with v degrees of freedom given by:

$$V_i = \sqrt{\frac{v}{W}}\left(\rho Y + \sqrt{1-\rho^2}\varepsilon_i\right)$$

where Y and ε_i are still standard normal variables. W follows inverse Gamma distribution with parameters $\frac{v}{2}$ while $\frac{v}{W} \sim \chi^2_v$. When $i \neq j$, $\text{cov}(V_i, V_j) = \frac{v}{v-2}\rho^2$ for $v > 2$. The conditional default probabilities are given by:

$$p_i(t|Y, W) = \Phi\left(\frac{-\rho Y + \sqrt{\frac{W}{v}}t_v^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right)$$

Default times are given by $\tau_i = F_i^{-1}(t_v(V_i))$, where t_v is the distribution function of the standard univariate student's t. More details can be found in Andersen et al. (2003).

3.7.2 Double t Copula

Although the double t copula has upper and lower tail dependence with equal coefficients, it is more time consuming to calibrate the asset return distribution, as t distribution is not stable under convolution. Assume that individual risk process $(V_1, ..., V_n)$ follows sum of two variables both following student's t distribution.

$$V_i = \rho\sqrt{\frac{v-2}{v}}Y + \sqrt{1-\rho^2}\sqrt{\frac{\widetilde{v}-2}{\widetilde{v}}}\varepsilon_i$$

where both Y and ε_i follow student's t distribution with v and $\widetilde{v}$ degrees of freedom, respectively. As student's t distribution is not stable under convolution, the V_i term fails to follow student's t distribution. The conditional default probabilities are given by:

$$p_i(t|Y) = t_{\widetilde{v}}\left(\sqrt{\frac{\widetilde{v}}{\widetilde{v}-2}} \times \frac{H_i^{-1}(F_i(t)) - \rho\sqrt{\frac{v-2}{v}}Y}{\sqrt{1-\rho^2}}\right)$$

The default times are given by $\tau_i = F_i^{-1}(H_i(V_i))$, where H_i is the distribution function of V_i. Tail dependence coefficient is: $\lambda_U = \lambda_L = \left(1 + \left(\frac{\sqrt{1-\rho^2}}{\rho}\right)^v\right)^{-1}$ when $y = z$. Tail factor of $Y >$ tail factor of ε_i when $v < \widetilde{v}$. Tail factor of $Y <$ tail factor of ε_i (i.e. there is no tail dependence between default times) when $v > \widetilde{v}$. More details can be found in Hull & White (2004) and Burtschell et al. (2005*b*). Figure 4.4 shows the Portfolio Loss Distribution for CDX and iTraxx with Gaussian, student's t and double t using the optimized parameters reported in Tables 4.5 and in 4.6, respectively.

3.7.3 Archimedean Copulae

The Frank copula has independent upper and lower tail dependence. The Clayton copula has lower tail dependence only when $\delta > 0$ and it has no upper tail dependence. The Gumbel copula has upper tail dependence but no lower tail dependence. Assume that individual risk process $(V_1, ..., V_n)$ is given as follows:

$$V_i = \widehat{G}\left(-\frac{\ln(x_i)}{Y}\right)$$

where $\widehat{G}$ is the Laplace transform of the distribution function of Y as described in section 2.2.4. The conditional default probabilities are given by:

$$p_i(t|Y) = \exp(Y(1 - F_i(t)^{-}\alpha))$$

The default times are given by $\tau_i = F_i^{-1}(V_i)$. The copula of default times is the joint distribution of V_i's and given by:

$$Q(V_1 < u_1, ..., V_1 < u_1) = C(u_1, ..., u_n) = \varphi^{-1}(\varphi(u_1) + ... + \varphi(u_n))$$

The explicit forms of this function for Clayton, Gumbel and Frank copulas are given in section 2.2.4.

3.7.4 Marshall-Olkin Copula

The Marshall-Olkin copula has upper and lower tail dependence with equal coefficients. Individual risk process $(V_1, ..., V_n)$ is given as follows:

$$V_i = \min(Y, \varepsilon_i)$$

as described in section 2.2.5, Y and ε_i are exponentially distributed random variables with parameters $\alpha, 1-\alpha \in]0,1[$. The corresponding survival copula is:

$$C(u_1, ..., u_n) = \min(u_1^{\alpha}, ..., u_n^{\alpha}) \prod_{i=1}^{n} u_i^{1-\alpha}$$

The default times are defined as $\tau_i = S_i^{-1}(\exp(-\min(Y, \varepsilon_i)))$. The conditional default probabilities are independent of Y and they are given as:

$$p_i(t|Y) = 1 - (1_{Y > -\ln S_i(t)} S_i(t)^{1-\alpha})$$

In this case, there is upper and lower tail dependence with the coefficient α. Table 4.5 shows factor Gauss, student's t, Double t and Marshall-Olkin copulae fitted to CDX and Table 4.6 shows the fit for iTraxx.

3.8 Factor Copula CDO Pricing with Alternative Correlation Structures

3.8.1 Gaussian Copula with Stochastic Correlation

Burtschell, Gregory & Laurent (2005*a*) bring stochasticity to the correlation parameter and propose stochastic correlation models with independent correlation and symmetric but dependent correlation assumptions. The individual risk process $(V_1, ..., V_n)$ is given as:

$$V_i = \tilde{\rho}_i Y + \sqrt{1-\tilde{\rho}_i^2}\varepsilon_i$$

where $\tilde{\rho}_i$'s are some random variables taking values in $[0,1]$ independent from Gaussian random variables Y, ε_i for $i = 1, ..., n$. Hence, conditioning on $\tilde{\rho}_i$, V_i's are still standard Gaussian variables. The default times are still given by $\tau_i = F_i^{-1}(\Phi(V_i))$ and the conditional default probabilities may be written as:

$$p_i(t|Y) = \int_0^1 \Phi\left[\frac{k_i - \rho Y}{\sqrt{1-\rho^2}}\right] dF_{\tilde{\rho}_i}(\rho)$$

where $k_i = \Phi^{-1}(F_i(t))$ and $dF_{\tilde{\rho}_i}(\rho)$ is the distribution function of $\tilde{\rho}$. There are two forms that the stochastic correlation model may take. The first assumes two stochastic but independent correlation parameters. The correlation structure and individual risk process are:

$$\tilde{\rho}_i = (1-B_i)\rho + B_i\beta \tag{3.17}$$

$$V_i = \big((1-B_i)\rho + B_i\beta\big)Y + \sqrt{1-\big((1-B_i)\rho + B_i\beta\big)^2}\varepsilon_i \tag{3.18}$$

$$= (1-B_i)(\rho Y + \sqrt{1-\rho^2}\varepsilon_i) + B_i(\beta Y + \sqrt{1-\beta^2}\varepsilon_i) \tag{3.19}$$

where B_i's are Bernoulli random variables independent from Gaussian random variables Y, ε_i for $i = 1, ..., n$. $\rho, \beta \in [0,1]$ are constants. When $B_i = 0$ then $\tilde{\rho}_i = \rho$ and the given name is correlated to Y with ρ. When $B_i = 1$ then $\tilde{\rho}_i = \beta$ with $q = \mathbb{P}(B_i = 1)$ and the given name is correlated to Y with β.

The conditional individual default probability is given by;

$$p_i(t|Y) = (1-q)\Phi\left[\frac{k_i - \rho Y}{\sqrt{1-\rho^2}}\right] + q\Phi\left[\frac{k_i - \beta Y}{\sqrt{1-\beta^2}}\right]$$

where $k_i = \Phi^{-1}(F_i(t))$. This model is the sum of two one-factor Gaussian copulae, which allows one to incorporate a state of independence by setting $\beta = 0$. According to this model, names defaulting in this state do so with no other impact on the other names.

The second form that stochastic correlation model takes is a symmetric, stochastic but dependent structure; the correlation structure and individual risk process are:

$$\tilde{\rho}_i = (1-B_s)(1-B_i)\rho + B_s \tag{3.20}$$

$$V_i = \big((1-B_s)(1-B_i)\rho + B_s\big)Y + \big((1-B_s)((1-B_i)\sqrt{1-\rho^2} + B_i)\big)\varepsilon_i \tag{3.21}$$

where $\varepsilon_1, ..., \varepsilon_n$, Y, B_s and $B_1, ..., B_n$ are independent. $\varepsilon_1, ..., \varepsilon_n$ and Y are standard Gaussian variables. B_s and $B_1, ..., B_n$ are Bernoulli random variables with $q = \mathbb{P}(B_i = 1), q_s = \mathbb{P}(B_s = 1)$.

In this structure marginal distribution of $\tilde{\rho}_i$ is discrete:

$$\tilde{\rho}_i = \begin{cases} 0, & \text{with probability } q(1-q_s) \\ \rho_i, & \text{with probability } (1-q)(1-q_s) \\ 1, & \text{with probability } q_s \end{cases}$$

The correlation parameters are positively correlated and conditional on Y and B_s; the distribution of $\tilde{\rho}_i$ is discrete, taking value B_s with probability q and taking value $(1-B_s)\rho + B_s$ with probability $1-q$. The conditional individual default probability is given by:

$$p_i(t|Y) = (1-B_s)\left((1-q)\Phi\left(\frac{k_i(t) - \rho Y}{\sqrt{1-\rho^2}}\right) + qF_i(t)\right) + B_s 1_{k_{i(t)\geq Y}}$$

Instead of calculating unconditional portfolio loss distributions as described in Burtschell et al. (2005*a*) with probability generating functions, our calculations will employ characteristic functions and FFT inversions as described in section 3.6.5.

3.8.2 Gaussian Copula with Local Correlation

Turc, Very & Benhamou (2005) define a local correlation parameter as a function of the common factor Y: $\rho(Y)$ taking values in $[0, 1]$. The individual risk process $(V_1, ..., V_n)$ is not Gaussian unless $\rho(Y)$ is constant and may be expressed as:

$$V_i = -\rho(Y)Y + \sqrt{1 - \rho^2(Y)}\varepsilon_i$$

The marginal distribution of V_i may be written as:

$$H(x) = \mathbb{P}[V_i \leq x] = \int \Phi\left(\frac{x + \rho(y)y}{\sqrt{1 - \rho^2(y)}}\right)\varphi(y)dy$$

where Φ is the Gaussian distribution function and $\varphi()$ is the Gaussian density function. The default times are $\tau_i = F^{-1}(H(V_i))$. The conditional default probability may be written as:

$$p_i(t|Y) = \Phi\left(\frac{H^{-1}(F(t)) + \rho(Y)Y}{\sqrt{1 - \rho^2(Y)}}\right)$$

3.8.3 Gaussian Copula with Random Factor Loadings

Another model similar to local correlation is the Random Factor Loadings (RFL) model proposed by Andersen & Sidenius (2005). This model addresses the fact that the market loss distribution is more kinked than what Gauss model predicts and overcomes that by incorporating random factor loadings into the model. The individual risk process $(V_1, ..., V_n)$ is given as follows:

$$V_i = a_i(Y)Y + v\varepsilon_i + m$$

where v, m are some positive constants and Y, ε_i are i.i.d. standard normal random variables independent of Y for $i = 1, ..., n$. Factor loadings, $a_i(Y)$, follow a two-point distribution:

$$a_i(Y) = \begin{cases} w, \quad Y \leq \theta & \text{with probability } \phi(\theta) \\ z, \quad Y > \theta & \text{with probability } 1 - \phi(\theta) \end{cases}$$

where $\theta \in \mathbb{R}$. If $w > z$ than the factor loadings decrease as Y increases and asset values couple more strongly to the economy in bad times than in good. When $w = z$, the model becomes the constant factor loading Gaussian copula. w and z are such that V_i has a zero mean and unit variance; hence they are given by: $v = \sqrt{1 - Var[a_i(Y)Y]}$ and $m = -\mathbb{E}[a_i(Y)Y]$. Using Lemma 5 on page 38 of Andersen & Sidenius (2005):

$$
\begin{aligned}
\mathbb{E}[a_i(Y)Y] &= \mathbb{E}[w1_{Y\leq\theta}Y + z1_{Y>\theta}Y] = [-w\phi(\theta) + z\phi(\theta)] \\
m &= -[-w\phi(\theta) + z\phi(\theta)] = \phi(\theta)(w - z) \\
\mathbb{E}[a_i(Y)^2Y^2] &= \mathbb{E}[w^2 1_{Y\leq\theta}Y^2 + z^2 1_{Y>\theta}Y^2] = w^2\big(\Phi(\theta) - \theta\phi(\theta)\big) + z^2(\theta\phi(\theta) + (1 - \Phi(\theta))) \\
v &= \sqrt{1 - w^2(\Phi(\theta) - \theta\phi(\theta)) + z^2(\theta\phi(\theta) + (1 - \Phi(\theta))) - (-w\phi(\theta) + z\phi(\theta))^2}
\end{aligned}
$$

Using Lemma 1 on page 35 of Andersen & Sidenius (2005), the marginal distribution of V_i may be written as:

$$
\begin{aligned}
H(x) &= \mathbb{P}[V_i \leq x] =]\Phi_2\left[\frac{x - m}{\sqrt{v^2 + w^2}}, \theta; \frac{w}{\sqrt{v^2 + w^2}}\right] + \Phi\left[\frac{x - m}{\sqrt{v^2 + z^2}}\right] \\
&\quad -\Phi_2\left[\frac{x - m}{\sqrt{v^2 + z^2}}, \theta; \frac{z}{\sqrt{v^2 + z^2}}\right]
\end{aligned}
$$

where Φ_2 is the bivariate Gaussian cdf and the default times are given by $\tau_i = F^{-1}(H(V_i))$. The conditional default probability may be written as:

$$
\begin{aligned}
p_i(t|Y) &= \mathbb{E}\left[\mathbb{P}\left(\varepsilon_i \leq \frac{k_i - (w1_{Y\leq\theta}Y + z1_{Y>\theta}Y) - m}{v}\right)\right] \\
&= \mathbb{E}\left[\Phi\left[\frac{k_i - (w1_{Y\leq\theta}Y + z1_{Y>\theta}Y) - m}{v}\right]\right]
\end{aligned}
$$

Depending on the size and homogeneity of the portfolio, portfolio loss distribution may be calculated using methodologies outlined in sections 3.6.2, 3.6.3, 3.6.4 and 3.6.5.

3.9 CDO Pricing with Normal Inverse Gaussian Copula

Kalemanova, Schmidt & Werner (2005) applied NIG distribution in pricing CDO's. Normal Inverse Gaussian (NIG) is a normal variance-mean mixture where the mixing density is the inverse Gaussian distribution allowing the tails of the distribution to decrease slower than normal distribution. This model is capable of producing heavy-tailed asymmetric return distributions and it is relatively faster to calibrate as NIG distribution is stable under convolution.

A random variable X follows a NIG distribution with parameters α, β, μ and δ (shape, asymmetry, location and scale parameters, respectively). Given that $Y \sim IG(\delta\eta, \eta^2)$ with $\eta := \sqrt{(\alpha^2 - \beta^2)}$, then $X|Y = y \sim \Phi(\mu + \beta y, y)$ with $0 \leq |\beta| < \alpha$ and $\delta > 0$ [4]. The most important properties of the NIG distribution are its scaling[5] and closure under convolution for independent variables[6]. Individual risk processes $(V_1, ..., V_n)$ are given as follows:

$$V_i = \rho Y + \sqrt{1-\rho^2}\varepsilon_i$$

where Y and ε_i have independent NIG distributions for $i = 1, ..., n$.

$$\begin{aligned} Y &\sim NIG\left(\alpha, \beta, \frac{-\beta\eta^2}{\alpha^2}, \frac{\eta^3}{\alpha^2}\right) \\ \varepsilon_i &\sim NIG\left(\frac{\sqrt{1-\rho^2}}{\rho}\alpha, \frac{\sqrt{1-\rho^2}}{\rho}\beta, -\frac{\sqrt{1-\rho^2}}{\rho}\frac{\beta\eta^2}{\alpha^2}, \frac{\sqrt{1-\rho^2}}{\rho}\frac{\eta^3}{\alpha^2}\right) \end{aligned}$$

where $\eta = \sqrt{(\alpha^2 - \beta^2)}$. Using the scaling property and stability under convolution of NIG distribution, and conditioning on the systemic factor Y:

[4] It's moment generating function, mean, variance, skewness and kurtosis are given in the following fashion: $M(t) = \mathbb{E}[e^{xt}] = e^{\mu t}\frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{e^{\delta\sqrt{\alpha^2-(\beta+t)^2}}}$, $\mathbb{E}[X] = \mu + \frac{\delta\beta}{\eta}$, $var(X) = \delta\frac{\alpha^2}{\eta^3}$, $S(X) = 3\frac{\beta}{\alpha\sqrt{\delta\eta}}$ and $K(X) = 3 + 3\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\delta\eta}$

[5] $X \sim NIG(\alpha, \beta, \mu, \delta) \Rightarrow cX \sim NIG\left(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta\right)$

[6] For two random variables X and Y: $X \sim NIG(\alpha, \beta, \mu_1, \delta_1), Y \sim NIG(\alpha, \beta, \mu_2, \delta_2) \Rightarrow X + Y \sim NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$

$$
\begin{aligned}
\rho Y &\sim NIG\Big(\frac{\alpha}{\rho}, \frac{\beta}{\rho}, \frac{-\beta\eta^2\rho}{\alpha^2}, \frac{\eta^3\rho}{\alpha^2}\Big) \\
\sqrt{1-\rho^2}\varepsilon_i &\sim NIG\Big(\frac{\sqrt{1-\rho^2}}{\rho\sqrt{1-\rho^2}}\alpha, \\
&\quad \frac{\sqrt{1-\rho^2}}{\rho\sqrt{1-\rho^2}}\beta, -\frac{(\sqrt{1-\rho^2})^2}{\rho}\frac{\beta\eta^2}{\alpha^2}, \frac{(\sqrt{1-\rho^2})^2}{\rho}\frac{\eta^3}{\alpha^2}\Big) \\
\sqrt{1-\rho^2}\varepsilon_i &\sim NIG\Big(\frac{\alpha}{\rho}, \frac{\beta}{\rho}, -\frac{(1-\rho^2)}{\rho}\frac{\beta\eta^2}{\alpha^2}, \frac{(1-\rho^2)}{\rho}\frac{\eta^3}{\alpha^2}\Big) \\
V_i = \rho Y + \sqrt{1-\rho^2}\varepsilon_i &\sim NIG\Big(\frac{\alpha}{\rho}, \frac{\beta}{\rho}, -\frac{\beta\eta^2\rho}{\alpha^2} - \frac{(1-\rho^2)}{\rho}\frac{\beta\eta^2}{\alpha^2}, \frac{\eta^3\rho}{\alpha^2} + \frac{(1-\rho^2)}{\rho}\frac{\eta^3}{\alpha^2}\Big)
\end{aligned}
$$

Denoting $F_{NIG(s)}(x) = F_{NIG}\Bigg(x; s\alpha, s\beta, -s\frac{\beta\eta^2}{\alpha^2}, s\frac{\eta^3}{\alpha^2}\Bigg)$. V_i's follow NIG distribution;

$$
V_i \sim NIG\Big(\frac{\alpha}{\rho}, \frac{\beta}{\rho}, -\frac{1}{\rho}\frac{\beta\eta^2}{\alpha^2}, \frac{1}{\rho}\frac{\eta^3}{\alpha^2}\Big) \Rightarrow V_i \sim NIG_{\left(\frac{1}{\rho}\right)}
$$

The third and fourth parameters make sure that the expected value of V_i is zero and its variance is one[7]. Hence, the default barrier k_i follows;

$$
k_i = F^{-1}_{NIG\left(\frac{1}{\rho}\right)} F_i(t)
$$

Using the scaling property of NIG, the conditional default probability may be written as:

$$
p_i(t|Y) = F_{NIG\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}\left(\frac{k-\rho Y}{\sqrt{1-\rho^2}}\right)
$$

Depending on the size and homogeneity of the portfolio, portfolio loss distribution may be calculated using methodologies outlined in sections 3.6.2, 3.6.3, 3.6.4 and 3.6.5. Table 4.7 shows Stochastic Correlation, RFL and NIG models fitted to CDX and Table 4.8 shows the fit for iTraxx. Figure 4.5 shows the Portfolio Loss Distribution for CDX and iTraxx with Stochastic Correlation, RFL and NIG using the optimized parameters reported Tables 4.7 and in 4.8, respectively.

[7] $\mathbb{E}[V_i] = \frac{-\beta\eta^2}{\alpha^2} + \frac{\eta^3}{\alpha^2}\frac{\beta}{\eta} = 0, var(V_i) = \frac{\eta^3}{\alpha^2}\frac{\alpha^2}{\eta^3} = 1$

Chapter 4

CDO Pricing with Lévy Processes

This chapter introduces Lévy processes and presents a review of Lévy processes used in the credit pricing literature. It also introduces the Correlated Dampened Power Law model with its special cases and calibrates them to CDX and iTraxx tranche upfront fees and running spreads.

4.1 Lévy Processes

This section introduces Lévy processes and presents important definitions/theorems that will be employed throughout this chapter.

Definition 4.1.1. *(Lévy Process) Let $X = (X_t, t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, F, \mathbb{P})$ filtered by the filtration $\{F_t, t \geq 0\}$. Then X_t is said to be a Lévy process if:*

1. $X_0 = 0$ *(a.s)*

2. *X has independent increments: For each $n \in N$ and each $0 \leq t_1 < t_2 \leq ... < t_{n+1} < \infty$, the random variables $(X_{t_{j+1}} - X_{t_j})$ are independent for $1 \leq j \leq n$.*

3. *X has stationary increments: The distribution of the increment $X_{t+s} - X_t$ over interval $[t, t+s]$ does not depend on t, but only on the length s of the interval.*

4. *X is stochastically continuous: For all $\epsilon > 0$ and for all $s \geq 0$,*

$$\lim_{t \to s} P(|X_t - X_s| > \epsilon) = 0$$

The best-known Lévy processes are Wiener and Poisson processes.

Definition 4.1.2. *(Characteristic Function)The characteristic function ϕ of a distribution or a random variable X is the Fourier-Stieltjes transform of the distribution function $F(x) = P(X \leq x)$;*

$$\phi_X(u) \equiv \mathbb{E}[e^{iuX}] = \int_{-\infty}^{+\infty} e^{iux} dF(x) = e^{-t\psi(u)}$$

where $\psi(u)$ is the characteristic exponent.

Infinite divisibility is an important property of Lévy process.

Definition 4.1.3. *(Infinitely Divisible Distribution) A probability distribution F of a random variable X on $\mathbb{R}^d$ is said to be infinitely divisible if for any integer $n \geq 2$, there exist n i.i.d. random variables $Y_1, ..., Y_n$ such that $Y_1 + ... + Y_n$ has distribution F. For each $n \in N$, we can write,*

$$X = Y_1^{(n)} + ... + Y_n^{(n)}$$

where $Y_i^{(n)}$,i=1,...,n, are i.i.d. random variables, all following a law with characteristic function $\phi(z)^{1/n}$.

Theorem 4.1.1. *(Infinitely Divisible Lévy process) If $(X_t)_{t \geq 0}$ is a Lévy process, then for every t, X_t has an infinitely divisible distribution F. Conversely, if F is an infinitely divisible distribution then there exists a Lévy process (X_t) such that the distribution of X_1 is given by F.*

The Lévy-Khintchine theorem allows getting the explicit form of the characteristic function of a Lévy process.

Theorem 4.1.2. *(Lévy-Khintchine Theorem) If X_t is a Lévy process with characteristic function $\phi_X(u) \equiv \mathbb{E}[e^{iuX}] = e^{-t\psi(u)}$, where $\psi(u)$ is the characteristic exponent. The char-*

acteristic exponent satisfies the Lévy-Khintchine formula:

$$\psi(u) = -iu\mu + \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}^0} \left(1 - e^{iux} + iux\boldsymbol{1}_{|x|<1}\right) \upsilon(x)dx$$

where $\mu \in \mathbb{R}, \sigma^2 \geq 0,\ \boldsymbol{1}_{|x|<1}$ *is the indicator function and* v *is the Lévy measure.*

The Lévy-Khintchine theorem demonstrates that Lévy processes are compromised of three distinct components: A linear deterministic part characterized by μ, a Brownian part characterized by σ^2, and a pure-jump part characterized by $v(dx)$. A Lévy process is said to be fully represented by its corresponding Lévy/characteristic triplet $[\mu, \sigma^2, v(dx)]$.

Definition 4.1.4. *(Lévy Measure)* v *is the Lévy measure on* $\mathbb{R}\backslash\{0\}$ *is the expected number of jumps per unit of time and satisfies:*

$$\int_{-\infty}^{+\infty} \min\{1, x^2\}v(dx) = \int_{-\infty}^{+\infty} (1 \wedge x^2)v(dx) < \infty$$

when the Lévy measure is expressed in form $v(dx)$*, then it is called the Lévy density.*

Definition 4.1.5. *(Cumulant Exponent) Given that the constraint* $\int_{\mathbb{R}^0} x^2 \boldsymbol{1}_{|x|<1}\upsilon(x)dx < \infty$ *and the integral can be carried out explicitly, the cumulant exponent is defined as:*

$$\kappa(s) \equiv \frac{1}{t}\mathbb{E}[e^{sX_t}] = s\mu + \frac{s^2\sigma^2}{2} + \int_{\mathbb{R}^0} \left(e^{sx} - 1 + sx\boldsymbol{1}_{|x|<1}\right) \upsilon(x)dx$$

Definition 4.1.6. *(Infinite/Finite Variation) Given a Lévy process* X_t *with its characteristic triplet* $[\mu, \sigma^2, v(dx)]$*, the process is said to be of finite variation if*

$$\sigma^2 = 0 \text{ and } \int_{-1}^{1} |x|v(dx) < \infty$$

if not, it is said to be of infinite variation.

Definition 4.1.7. *(Infinite/Finite Activity) Given a Lévy process* X_t *with its characteristic triplet* $[\mu, \sigma^2, v(dx)]$*, the process is said to be of finite activity if*

$$\int_{-\infty}^{+\infty} v(x)dx < \infty$$

if not, it is said to be of infinite activity.

More details on Lévy processes can be found in Cont & Tankov (2004), Applebaum (2004) and Schoutens & Cariboni (2009).

4.2 Lévy Processes Used in the Credit Pricing Literature

There is a striking similarity between the "volatility smile" from the Black-Scholes model in option pricing and the "correlation smile" from Gaussian copula in credit pricing. Both smiles are related to the underlying normal distribution assumption. Lévy jump processes, which can generate return distributions with heavy tails, have been extensively employed in the option pricing literature to explain the volatility smile; yet only a handful studies in the credit pricing literature have attempted to explain the correlation smile or used Lévy processes to generate heavier tailed return distributions.

Albrecher, Ladoucette & Schoutens (2007) assume that $X = \{X_t, t \in [0,1]\}$ and $X_{(i)} = \{X_t^i, t \in [0,1]\}$ for $i = 1, 2, ..., n$ to be an iid Lévy process based on the same mother infinitely divisible distribution L. For $\rho \in [0,1]$, the generic one-factor individual risk process is given by:

$$A_i(T) = X_\rho + X_{1-\rho}^{(i)}$$

Several Lévy processes are tested within this functional structure. Firstly, they use the standard normal distribution as the mother infinitely divisible distribution L, where the associated Lévy process is the standard Brownian Motion $W = \{W_t, t \in [0,1]\}$. The individual risk process is given by $A_i(t) = W_\rho + W_{1-\rho}^{(i)}$, where W_ρ follows a Normal$(0, \rho)$ distribution and $W_{1-\rho}^{(i)}$ follows a Normal$(0, 1-\rho)$ distribution. Adding these independent random variables leads to a standard normally distributed asset return distribution: $A_i \sim N(0,1)$.

Secondly, they incorporate a Gamma[1] process, $G = \{G_t, t \geq 0\}$, with shape parameter a and

[1] The characteristic function of the Gamma distribution, $\Gamma(k, \delta)$, with parameters k and δ is: $\phi_{X_t}^{\text{Gamma}}(u) = (1 - \delta i u)^{-k}$

scale parameter $\sqrt{a}$ where the driving Lévy process is denoted as: $X_t = \sqrt{a}t - G_t, \quad t \in [0,1]$. Thirdly, they incorporate an Inverse Gaussian[2] process, $I = \{I_t, t \geq 0\}$, with parameters $a > 0$ and $b = a^{1/3}$ where the driving Lévy process is denoted as $X_t = \mu t - I_t, \quad t \in [0,1]$ with $\mu = a^{2/3}$. Hence, in their setting there is a deterministic up-trend with random downward shocks. Finally, they implement the Variance Gamma process of Madan, Carr & Chang (1998). They find that amongst the tested models, the shifted Gamma performs the best in fitting iTraxx tranches. Garcia & Goossens (2007) tests a very similar shifted Gamma model and also studies the effect of interpolating the base expected loss curve.

Baxter (2006) tests several variations of the Gamma process, including the Brownian Gamma model, where two Brownian noise terms are added to the Gamma process; and the Catastrophe Gamma model, where a low-intensity high-impact Poisson process is added to the Gamma process to capture disaster scenarios. Baxter (2006) also tests the Variance Gamma model and extends this model with the Brownian Variance Gamma and Catastrophe Variance Gamma models. He concludes, however, that the symmetric up and down jumps of the Brownian Variance Gamma and Catastrophe Variance Gamma models do not perform well in fitting CDX and iTraxx tranches. Similarly, Moosbrucker (2006) tests several versions of the Gamma process and Variance Gamma process, and also creates a Variance Gamma copula model. Brunlid (2006) proposes three different hyperbolic copulae: A normal inverse Gaussian copula, a variance gamma copula and a skewed student's t copula, induced from a one factor Lévy model. Table 4.9 shows Inverse Gaussian and Gamma processes fitted to CDX and Table 4.10 shows the fit for iTraxx.

4.3 Dampened Power Law (DPL)

Wu (2006)'s stylized model has been implemented in option and currency pricing, but it has never been calibrated for credit pricing. The DPL is not only capable of accommodating different degrees of dampening for positive and negative jumps, but it is also a "compre-

[2] The characteristic function of the Inverse Gaussian distribution, IG(a,b), with parameters a and b is: $\phi^{\mathrm{IG}}_{X_t}(u) = \exp\left(\frac{a}{b}\left[1 - \sqrt{1 - \frac{2b^2 iu}{a}}\right]\right)$

hensive" model in the sense that it is readily converted, for different parameter values, to well-known models used widely in option pricing but as yet not implemented to CDO pricing.

Wu (2006) assumes that the arrival rate of jumps of size x in asset returns follows a power law, dampened by an exponential function. The Lévy density of the DPL, defined on $\mathbb{R}_0$, for a pure jump process X_t is given by:

$$\upsilon(x) = \begin{cases} \gamma^{+} \frac{\exp(-\beta^{+}|x|)}{|x|^{\alpha+1}}, & \text{for } x > 0 \\ \gamma^{-} \frac{\exp(-\beta^{-}|x|)}{|x|^{\alpha+1}}, & \text{for } x < 0 \end{cases} \tag{4.1}$$

with parameters $\alpha \in (0, 2]/1$, $\beta^{\pm}, \gamma^{\pm} \in \mathbb{R}^{+}$. $\beta^{\pm}$ are measures of exponential decay that must be greater than 0 to ensure that return innovation X_t has finite moments. The exponential dampening in 4.1 makes sure that as the absolute jump size $|x|$ decreases its frequency increases. Since α is assumed to be the same for positive and negative jumps, the difference between γ^{+} and γ^{-} determines the asymmetry in the model.

Wu (2006) and Carr, Geman, Madan & Yor (2002) show that $-1 < \alpha < 0$ corresponds to a finite activity process capable of capturing large but rare jumps, $0 < \alpha < 1$ corresponds to an infinite activity process with finite variation capable of capturing both large and small jumps and $1 < \alpha < 2$ corresponds to an infinite variation process capable of capturing many small jumps. Hence, the DPL allows both infinite/finite activity as well as infinite/finite variation. Moreover, thanks to different scaling coefficients governing positive and negative jumps, it is capable of separating upside and downside jumps in a given asset's return distribution.

The cumulant exponent and the characteristic exponent of the DPL are found by plugging $\upsilon(x)$ in 4.1 into the Lévy-Khintchine equality given in 4.1.2 and solving the integral:

$$\kappa^{\text{DPL}}(s) = \gamma^{+}\Gamma(-\alpha)\left[(\beta^{+}-s)^{\alpha}-(\beta^{+})^{\alpha}\right] + \gamma^{-}\Gamma(-\alpha)\left[(\beta^{-}+s)^{\alpha}-(\beta^{-})^{\alpha}\right]+sC(h)$$

$$\psi^{\text{DPL}}(u) = -\gamma^{+}\Gamma(-\alpha)\left[(\beta^{+}-iu)^{\alpha}-(\beta^{+})^{\alpha}\right] - \gamma^{-}\Gamma(-\alpha)\left[(\beta^{-}+iu)^{\alpha}-(\beta^{-})^{\alpha}\right]-iuC(h)$$

the j^{th} cumulant is given by:

$$\kappa_1^{\text{DPL}} \equiv \frac{\partial\kappa(s)}{\partial s}\Big|_{s=0} = \Gamma(1-\alpha)\left[\gamma^{+}(\beta^{+})^{\alpha-1}-\gamma^{-}(\beta^{-})^{\alpha-1}\right]+C(h)$$

$$\kappa_j^{\text{DPL}} \equiv \frac{\partial^j\kappa(s)}{\partial s^j}\Big|_{s=0} = \Gamma(j-\alpha)\left[\gamma^{+}(\beta^{+})^{\alpha-j}-\gamma^{+}(-1)^{j}(\beta^{-})^{\alpha-j}\right]$$

$$j = 2,3,...$$

the condition $\beta^{\pm} > 0$ is necessary to make sure moments of the DPL are finite. If either one of these parameters is equal to 0, than only moments less that $\alpha < 2$ are finite. Having finite moments is a necessary condition for incorporating the correlation parameter into the DPL model. The characteristic function of the DPL is calculated as:

$$\phi_{X_t}^{\text{DPL}}(u) = e^{-t\left(-\gamma^{+}\Gamma(-\alpha)\left[(\beta^{+}-iu)^{\alpha}-(\beta^{+})^{\alpha}\right]-\gamma^{-}\Gamma(-\alpha)\left[(\beta^{-}+iu)^{\alpha}-(\beta^{-})^{\alpha}\right]-iuC(h)\right)}$$

4.4 General Framework

4.4.1 Asset Return Dynamics

Let X_t be a one-dimensional infinitely divisible pure jump Lévy process, such as the DPL, defined on a probability space $(\Omega, F, \mathbb{P})$. X_t captures the uncertainty of the economy with the corresponding Lévy triplet $(\mu, \sigma, v(x))$ where $v(x)$ is the density of the process. The firm value process of the i^{th} credit in the CDO can be expressed as:

$$S_t^i = S_0^i \exp[\mu^i t + X_t^i - \kappa^i(1)t] \tag{4.2}$$

$$\ln S_t^i = \ln S_0^i + (\mu^i t + X_t^i - \kappa^i(1)t) \tag{4.3}$$

where μ^i is the instantaneous drift of the firm value process and $\kappa^i(1)$ is a convexity adjustment of X_t^i ensuring the term $\exp[X_t^i - \kappa^i(1)t]$ forms a martingale; hence, $X_t^i - \kappa^i(1)t = \psi_X^i(u) + iu\kappa_X^i(1)$. The default thresholds are chosen so that they produce risk-neutral default probabilities implied by the quoted CDS spreads:

$$p_i(t) = \mathbb{P}(\ln S_t^i \leq k_i) = F_{\ln S_t^i}(k_i)$$

hence, $k_i = F_{\ln S_t}^{-1}(p_i(t))$. $F_{\ln S_t}$ and $F_{\ln S_t}^{-1}$ are the distribution functions of $\ln S_t$ and its inverse, respectively. There are several potential Lévy processes that X_t can be. Here, a modified version of the DPL as well as its special cases will be used in pricing CDO tanche's[3]. Asset return dynamics given in 4.3 will be modified depending on whether jumps are assumed to be unified or separated.

4.4.2 Correlation

Depending on the parametrization of the Lévy process at hand, several methods are available for incorporating the correlation term into Lévy processes. Following Baxter (2006):

Theorem 4.4.1. *For any Lévy process $X(t)$, any integer n, and any non-negative correlation ρ, thanks to the stationarity and independent increment properties of Lévy processes we can construct a set of n Lévy processes $X_1(t), X_2(t), ..., X_n(t)$, such that each $X_i(t)$ has the same distribution as $X(t)$, and the correlation between $X_i(t)$ and $X_j(t)$ is ρ for all distinct*

[3]For ease of readability, the i superscript is dropped for the rest of this chapter

i and j. If $X(t)$ has finite second moments, then

$$\begin{aligned} var(X_i(t)) &= \sigma^2 t, \quad \textit{for some } \sigma \\ cov(X_i(t), X_j(t)) &= \sigma^2 \rho t \\ corr(X_i(t), X_j(t)) &= \rho \end{aligned}$$

Both Moosbrucker (2006) and Baxter (2006) model correlation in this intuitive way. For example, Baxter (2006) proposes a multivariate Gamma process of form $X_i(t) = X_g(\rho t) + \widetilde{X_i}((1-\rho)t)$ with a global factor X_g and idiosyncratic factors $\widetilde{X_i}$ for $i = 1, ..., n$. Both of these components are assumed to be iid copies of the Lévy process $X(t)$. Thus the correlation between processes X_i and X_j is simply $\rho_{ij} = corr(X_i, X_j) = \rho$.

In order to incorporate the correlation parameter, the DPL model as well as its special cases will be modified by employing a similar structure. For the rest of this chapter, components of the Lévy process are assumed to be iid copies of the same Lévy process $X(t)$ with the same pairwise correlation parameter $\rho \in [0, 1]$. Hence, we re-define the γ^+ and γ^- parameters in the following way:

$$\gamma^+ = \gamma(1-\rho) \quad (4.4)$$

$$\gamma^- = \gamma\rho \quad (4.5)$$

for any γ, which is the measure of overall level of activity. ρ exists thanks to the finiteness of second moments of the DPL.

4.4.3 Separating Arrival Rate of Negative and Positive Jumps

The arrival rate of negative and positive jumps of the DPL model can be separated. The Lévy process with separated jumps is given as:

$$\exp[X_t^+ - t\kappa^+(1)]\exp[X_t^- - t\kappa^-(1)]$$

where X_t^+ and X_t^- are independent processes of only positive and negative jumps of X_t, respectively, and κ^+ and κ^- are the corresponding convexity adjustments. In this structure, the pricing dynamics given in 4.3 is modified as follows:

$$\ln S_t = \ln S_0 + \mu t + X_t^+ + X_t^- - \kappa^+(1)t - \kappa^-(1)t$$

The characteristic function of independent processes for positive and negative jumps of X_t^+ and X_t^- is calculated as follows:

$$\begin{aligned}\phi_{X_t^+}(u) &= e^{-t\left(-\gamma^+\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha\right]-iuC_Q(h)\right)}\\ \phi_{X_t^-}(u) &= e^{-t\left(-\gamma^-\Gamma(-\alpha)\left[(\beta_Q^-+iu)^\alpha-(\beta_Q^-)^\alpha\right]-iuC_Q(h)\right)}\end{aligned}$$

4.5 Special Cases of the DPL

The DPL is readily converted, for different parameter values, to well-known models used widely in option pricing but as yet not implemented to CDO pricing.

4.5.1 $\alpha = 1$ (ONE)

When $\alpha = 1$ cumulant exponent converges to a different value and Wu (2006) shows that the Lévy density becomes:

$$\upsilon(x) = \begin{cases} \gamma^+ \frac{\exp(-\beta^+|x|)}{|x|^2}, & \text{for } x > 0 \\ \gamma^- \frac{\exp(-\beta^-|x|)}{|x|^2}, & \text{for } x < 0 \end{cases}$$

with parameters $\beta^{\pm}, \gamma^{\pm} \in \mathbb{R}^+$. The cumulant exponent and the characteristic exponent are:

$$\begin{aligned}
\kappa^{\text{ONE}}(s) &= \gamma^{+}(\beta^{+}-s)\ln(1-s/\beta^{+})+\gamma^{-}(\beta^{-}+s)\ln(1+s/\beta^{-})+sC \\
\psi^{\text{ONE}}(u) &= -\gamma^{+}(\beta^{+}-iu)\ln(1-iu/\beta^{+})-\gamma^{-}(\beta^{-}+iu)\ln(1+iu/\beta^{-})-iuC
\end{aligned}$$

the characteristic function of this Lévy process is:

$$\phi_{X_t}^{\text{ONE}}(u) = e^{-t(-\gamma^{+}(\beta^{+}-iu)\ln(1-iu/\beta^{+})-\gamma^{-}(\beta^{-}+iu)\ln(1+iu/\beta^{-})-iuC)}$$

4.5.2 α-stable (STBL)

When $\beta^{+} = \beta^{-} = 0$, the DPL becomes the Mandelbrot (1963) and Fama (1965) model. α-stable distributions are shown to produce return distribution with tails decayed following the power law . The Lévy density of the α-stable model is:

$$\upsilon(x) = \begin{cases} \gamma^{+}\frac{1}{|x|^{\alpha+1}}, & \text{for } x > 0 \\ \gamma^{-}\frac{1}{|x|^{\alpha+1}}, & \text{for } x < 0 \end{cases}$$

with parameters $\alpha \in (0,2]/1$, $\gamma^{\pm} \in \mathbb{R}^{+}$. The cumulant exponent and the characteristic exponent are:

$$\begin{aligned}
\kappa^{\text{STBL}}(s) &= -\sec\left(\frac{\pi\alpha}{2}\right)\left((s\nu^{+})^{\alpha}+(s\nu^{-})^{\alpha}\right) \\
\text{where} \quad \nu^{+} &= \left(\frac{\gamma^{+}\Gamma(\alpha/2)\Gamma(1-\alpha/2)}{2\Gamma(1+\alpha)}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad \nu^{-} = \left(\frac{\gamma^{-}\Gamma(\alpha/2)\Gamma(1-\alpha/2)}{2\Gamma(1+\alpha)}\right)^{\frac{1}{\alpha}} \\
\psi^{\text{STBL}}(u) &= \sec\left(\frac{\pi\alpha}{2}\right)\left((iu\nu^{+})^{\alpha}+(iu\nu^{-})^{\alpha}\right)
\end{aligned}$$

the characteristic function of the α-stable is:

$$\phi_{X_t}^{\text{STBL}}(u) = e^{-t\sec\left(\frac{\pi\alpha}{2}\right)\left((iu\nu^{+})^{\alpha}+(iu\nu^{-})^{\alpha}\right)}$$

4.5.3 Finite Moment Log Stable Model (FMLS)

Carr & Wu (2003) assume that $\beta^+ = \beta^- = 0$ and $\gamma^+ = 0$, hence they allow only negative jumps. The return distribution of this model is asymmetric with a fat left tail and a thin right tail. The Lévy density of this model is given by:

$$\upsilon(x) = \gamma^- \frac{1}{|x|^{\alpha+1}}$$

with parameters $\alpha \in (0, 2]/1$, $\gamma^- \in \mathbb{R}^+$. The cumulant exponent and the characteristic exponent are:

$$\begin{aligned} \kappa^{\text{FMLS}}(s) &= -(s\nu)^\alpha \sec\left(\frac{\pi\alpha}{2}\right), \quad \text{where} \quad \nu = \left(\frac{\gamma^-\Gamma(\alpha/2)\Gamma(1-\alpha/2)}{2\Gamma(1+\alpha)}\right)^{\frac{1}{\alpha}} \\ \psi^{\text{FMLS}}(u) &= (iu\nu)^\alpha \sec\left(\frac{\pi\alpha}{2}\right) \end{aligned}$$

the characteristic function of FMLS is:

$$\phi_{X_t}^{\text{FMLS}}(u) = e^{-t(iu\nu)^\alpha \sec\left(\frac{\pi\alpha}{2}\right)}$$

4.5.4 CGMY

When $\gamma^+ = \gamma^-$, the DPL becomes the Carr et al. (2002) model. This model is a pure jump Lévy model that can accommodate both infinite activity and finite activity as well as infinite and finite variation. The Lévy density of the CGMY model is given by:

$$\upsilon(x) = \begin{cases} \gamma \frac{\exp(-\beta^+|x|)}{|x|^{\alpha+1}}, & \text{for } x > 0 \\ \gamma \frac{\exp(-\beta^-|x|)}{|x|^{\alpha+1}}, & \text{for } x < 0 \end{cases}$$

with parameters $\alpha \in (0, 2]/1$, $\beta^\pm, \gamma \in \mathbb{R}^+$. The cumulant exponent and the characteristic exponent of the CGMY are:

$$\kappa^{\text{CGMY}}(s) = \gamma\Gamma(-\alpha)\left[(\beta^{+}-s)^{\alpha}-(\beta^{+})^{\alpha}+(\beta^{-}+s)^{\alpha}-(\beta^{-})^{\alpha}\right]+sC(h)$$
$$\psi^{\text{CGMY}}(u) = -\gamma\Gamma(-\alpha)\left[(\beta^{+}-iu)^{\alpha}-(\beta^{+})^{\alpha}+(\beta^{-}+iu)^{\alpha}-(\beta^{-})^{\alpha}\right]-iuC(h)$$

the characteristic function of CGMY is:

$$\phi_{X_t}^{\text{CGMY}}(u) = e^{-t\left(-\gamma\Gamma(-\alpha)\left[(\beta^{+}-iu)^{\alpha}-(\beta^{+})^{\alpha}+(\beta^{-}+iu)^{\alpha}-(\beta^{-})^{\alpha}\right]-iuC(h)\right)}$$

4.5.5 Double Exponential (DE)

When $\gamma^{+}=\gamma^{-}$ and $\alpha=-1$, the DPL becomes the Kou & Wang (2004) model. This model is shown to produce high peak and heavy tails in asset return distribution. The Lévy density of the double exponential model is given by:

$$v(x)=\begin{cases}\gamma\exp\left(-\beta^{+}x\right), & \text{for } x>0\\ \gamma\exp\left(-\beta^{-}|x|\right), & \text{for } x<0\end{cases}$$

with parameters $\beta^{\pm},\gamma\in\mathbb{R}^{+}$. The cumulant exponent and the characteristic exponent of the double exponential are:

$$\kappa^{\text{DE}}(s) = \gamma\left[(\beta^{+}-s)^{-1}-\frac{1}{\beta^{+}}+(\beta^{-}+s)^{-1}-\frac{1}{\beta^{-}}\right]$$
$$\psi^{\text{DE}}(u) = -\gamma\left[(\beta^{+}-iu)^{-1}-\frac{1}{\beta^{+}}+(\beta^{-}+iu)^{-1}-\frac{1}{\beta^{-}}\right]$$

the characteristic function of double exponential is:

$$\phi_{X_t}^{\text{DE}}(u) = e^{-t\left(-\gamma\left[(\beta^{+}-iu)^{-1}-\frac{1}{\beta^{+}}+(\beta^{-}+iu)^{-1}-\frac{1}{\beta^{-}}\right]\right)}$$

4.5.6 Variance Gamma (VG)

When $\gamma^+ = \gamma^-$ and $\alpha = 0$, the DPL becomes the Madan et al. (1998) and Madan & Seneta (1990) model. This model is a pure jump Lévy model that allows only infinite activity with finite variation. The Lévy density of the Variance Gamma model is given by:

$$\upsilon(x) = \begin{cases} \gamma \frac{\exp(-\beta^+|x|)}{|x|}, & \text{for } x > 0 \\ \gamma \frac{\exp(-\beta^-|x|)}{|x|}, & \text{for } x < 0 \end{cases}$$

with parameters $\beta^{\pm}, \gamma \in \mathbb{R}^+$. The cumulant exponent and the characteristic exponent of the variance gamma process are:

$$\begin{aligned} \kappa^{\text{VG}}(s) &= -\gamma \ln(1 - s/\beta^+)(1 + s/\beta^-) \\ &= -\gamma \left(\ln(\beta^+ - s) - \ln \beta^+ + \ln(\beta^- + s) - \ln \beta^-\right) \\ \psi^{\text{VG}}(u) &= \gamma \ln(1 - iu/\beta^+)(1 + iu/\beta^-) \\ &= \gamma \left(\ln(\beta^+ - iu) - \ln \beta^+ + \ln(\beta^- + iu) - \ln \beta^-\right) \end{aligned}$$

the characteristic function of the Variance Gamma is:

$$\begin{aligned} \phi^{\text{VG}}_{X_t}(u) &= e^{-t\left(\gamma \ln(1-iu/\beta^+)(1+iu/\beta^-)\right)} \\ &\quad \times e^{-t\left(-\gamma iu(ln(1-1/\beta^+)(1+1/\beta^-))\right)} \end{aligned}$$

In Madan et al. (1998)'s notation, $\eta_p = \frac{1}{\beta^+}$ and $\eta_n = \frac{1}{\beta^-}$, where $\eta_p - \eta_n = \theta\nu$ and $\eta_p\eta_n = \frac{\sigma^2\nu}{2}$.

4.5.7 Symmetric (SYM)

When $\gamma^+ = \gamma^-$ and $\beta^+ = \beta^- = \beta \neq 0$, the DPL becomes the Madan et al. (1998) and Koponen (1995) model. This model has been calibrated by Schoutens & Cariboni (2009) in pricing CDOs but with a different assumption of asset return dynamics than what will be presented here. This model produces symmetric return distribution where $\gamma^+ = \gamma^-$ control the kurtosis of the distribution. The Lévy density of this model is given by:

$$\upsilon(x) = \gamma \frac{\exp(-\beta|x|)}{|x|^{\alpha+1}}$$

with parameters $\alpha \in (0,2]/1,\ \beta, \gamma \in \mathbb{R}^+$. The cumulant exponent and the characteristic exponent of this model are:

$$\begin{aligned}\kappa^{\mathrm{SYM}}(s) &= \gamma\Gamma(-\alpha)\left[(\beta - s)^\alpha - \beta^\alpha\right] + sC(h) \\ \psi^{\mathrm{SYM}}(u) &= -\gamma\Gamma(-\alpha)\left[(\beta - iu)^\alpha - \beta^\alpha\right] - iuC(h)\end{aligned}$$

the characteristic function of this model is:

$$\phi^{\mathrm{SYM}}_{X_t}(u) = e^{-t(-\gamma\Gamma(-\alpha)[(\beta - iu)^\alpha - \beta^\alpha] - iuC(h))}$$

4.6 Model Calibration

The empirical performance of the DPL and its special cases are tested in pricing CDO tranches. Different pricing dynamics and Lévy processes are assumed depending on the assumption of whether $\gamma^+ = \gamma^-$ or $\gamma^+ \neq \gamma^-$. This section explains the details of the calibration procedure for each model. Table 4.11 shows the matrix of calibrated models based on Lévy processes.

4.6.1 $\gamma^+ \neq \gamma^-$ with no Brownian Motion

Simple Model

For Lévy processes when $\gamma^+ \neq \gamma^-$ such as the Dampened Power Law, $\alpha = 1$, α-stable and Finite Moment Log Stable, the Lévy process is simply X_t and the pricing dynamics is:

$$\ln S_t = \ln S_0 + (\mu t + X_t - \kappa(1)t)$$

where $\kappa(1)t$ is the convexity adjustment. The default probability of each obligor is calculated as follows:

$$\begin{aligned}\mathbb{P}[\ln S_t \leq k_t] &= \mathbb{P}[\ln S_0 + (\mu t + X_t - \kappa(1)t) < k_t] \\ &= P[X_t < \theta_t]\end{aligned}$$

where $\theta_t = k_t - (\ln S_0 + \mu t - \kappa(1)t)$ and $k_t = F^{-1}_{\ln S_t}$. Denoting F_{X_t} as the distribution function of process X_t, the default probability is:

$$p(t) = F_{X_t}(\theta_t)$$

Hence, the distribution function of both processes, F_{X_t}, and also the distribution function of the pricing dynamics, $F_{\ln S_t}$, must to be calculated.

The characteristic function of return is given as follows:

$$\phi^{\Omega}_{\ln S_t}(u) = e^{iu(\ln S_0 + \mu t)} e^{-t(\psi^{\Omega}(u))} e^{-iu(\kappa^{\Omega}(1))t}$$

where Ω =DPL, ONE, STBL, FMLS. This structure is tested for DPL, $\alpha = 1$, α-stable and FMLS models. For example when Ω =DPL, the characteristic function of return becomes:

$$\begin{aligned}\phi^{\mathrm{DPL}}_{\ln S_t}(u) &= e^{iu(\ln S_0 + \mu t)} e^{-t\left(-\gamma^{+}\Gamma(-\alpha)\left[(\beta^{+} - iu)^{\alpha} - (\beta^{+})^{\alpha}\right] - \gamma^{-}\Gamma(-\alpha)\left[(\beta^{-} + iu)^{\alpha} - (\beta^{-})^{\alpha}\right] - iuC(h)\right)} \\ &\quad \times e^{-t\left(iu\gamma^{+}\Gamma(-\alpha)\left[(\beta^{+} - 1)^{\alpha} - (\beta^{+})^{\alpha}\right] + iu\gamma^{-}\Gamma(-\alpha)\left[(\beta^{-} + 1)^{\alpha} - (\beta^{-})^{\alpha}\right] + iuC(h)\right)}\end{aligned}$$

Table 4.12 shows DPL, ONE, STBL and FMLS models fitted to CDX and Table 4.13 shows the fit for iTraxx.

Separate Negative and Positive Jumps

For Lévy processes when $\gamma^+ \neq \gamma^-$ such as the DPL, $\alpha = 1$ and α-stable, the Lévy process can be separated to incorporate different arrival rates of negative and positive jumps. The Lévy process X_t that will be tested becomes:

$$X_t^+ + X_t^-$$

The pricing dynamics is given by:

$$\ln S_t = \ln S_0 + \mu t + X_t^+ + X_t^- - \kappa^+(1)t - \kappa^-(1)t$$

where X_t^+ and X_t^- are independent processes of only positive and negative jumps of X_t, respectively, and κ^+ and κ^- are the corresponding convexity adjustments. This set-up is tested for the DPL, $\alpha = 1$ and α-stable. The default probability of each obligor is calculated as follows:

$$\begin{aligned}
\mathbb{P}[\ln S_t < k_t] &= \mathbb{P}[\ln S_0 + \mu t + X_t^+ + X_t^- - \kappa^+(1)t - \kappa^-(1)t < k_t] \\
&= \mathbb{P}[X_t^+ + X_t^- < \theta_t] \\
&= \mathbb{P}[X_t^- < \theta_t - X_t^+]
\end{aligned}$$

where $\theta_t = k_t - (\ln S_0 + (\mu - \kappa^+(1) - \kappa^-(1))t)$ and $k_t = F^{-1}_{\ln S_t}(p(t))$. Denoting $F_{X_t^+}$ and $F_{X_t^-}$ as the distribution function of processes X_t^+ and X_t^-, respectively, the conditional default probability is:

$$
\begin{aligned}
p(t|y) &= \mathbb{P}[X_t^- < \theta_t - y | y = X_t^+] \\
&= F_{X_t^-}(\theta_t - y)
\end{aligned}
$$

and the unconditional default probability is found by integrating out the y variable.

$$
p(t) = \int_{-\infty}^{+\infty} F_{X_t^-}(\theta_t - y) dF_{X_t^+}
$$

Hence, the distribution function of processes, $F_{X_t^-}$, $F_{X_t^+}$ and also pricing dynamics, $F_{\ln S_t}$, must be calculated.

Now, $\Omega =$DPL, ONE, STBL. Denote the cumulant exponent, characteristic exponent and characteristic function of process Ω corresponding to process X_t^+ as $\kappa^{\Omega+}$, $\psi^{\Omega+}$ and $\phi^{\Omega+}$, respectively. Similarly, denote the cumulant exponent, characteristic exponent and characteristic function of process Ω corresponding to process X_t^- as $\kappa^{\Omega-}$, $\psi^{\Omega-}$ and $\phi^{\Omega-}$, respectively. The characteristic functions of X_t^+, X_t^- and the characteristic function of return is given by:

$$
\begin{aligned}
\phi_{X_t^+}^{\Omega} &= \phi_{X_t}^{\Omega+}(u) \\
\phi_{X_t^-}^{\Omega} &= \phi_{X_t}^{\Omega-}(u) \\
\phi_{\ln S_t}^{\Omega}(u) &= e^{iu(\ln S_0 + \mu t)} e^{-t(\psi^{\Omega+}(u))} e^{-iu(\kappa^{\Omega+}(1))t} e^{-t(\psi^{\Omega-}(u))} e^{-iu(\kappa^{\Omega-}(1))t}
\end{aligned}
$$

where $\Omega =$DPL, ONE, STBL. For example when $\Omega =$ONE, characteristic functions of X_t^+, X_t^- and the characteristic function of return are:

$$
\begin{aligned}
\phi^{\text{ONE}}_{X_t^+} &= e^{-t(-\gamma^+(\beta^+-iu)\ln(1-iu/\beta^+))} \\
\phi^{\text{ONE}}_{X_t^-} &= e^{-t(-\gamma^-(\beta^-+iu)\ln(1+iu/\beta^-))} \\
\phi^{\text{ONE}}_{\ln S_t}(u) &= e^{iu(\ln S_0+\mu t)}e^{-t(-\gamma^+(\beta^+-iu)\ln(1-iu/\beta^+))}e^{-t(iu\gamma^+(\beta^+-1)\ln(1-1/\beta^+))} \\
&\quad \times e^{-t(-\gamma^-(\beta^-+iu)\ln(1+iu/\beta^-))}e^{-t(iu\gamma^-(\beta^-+1)\ln(1+1/\beta^-))}
\end{aligned}
$$

Table 4.16 shows DPL, ONE and STBL models fitted to CDX and Table 4.17 shows the fit for iTraxx.

4.6.2 $\gamma^+ \neq \gamma^-$ with Brownian Motion(s)

Simple Model

The second set of models add an independent standard Brownian Motion as a noise term to the above Lévy processes. The extended Lévy process becomes:

$$X_t + \sigma W_t$$

where W_t is an independent standard Brownian Motion which adds an orthogonal diffusion component to the model. Pricing dynamics is given by:

$$\ln S_t = \ln S_0 + (\mu - \frac{1}{2}\sigma^2)t + X_t + \sigma W_t - \kappa(1)t$$

where $\kappa(1)t$ and $\frac{1}{2}\sigma^2 t$ are convexity adjustments for the Lévy process X_t and standard Brownian Motion W_t, respectively. The default probability of each obligor is calculated as follows:

$$\begin{aligned}\mathbb{P}[\ln S_t < k_t] &= \mathbb{P}[(\ln S_0 + (\mu - \frac{1}{2}\sigma^2 - \kappa(1))t + X_t + \sigma W_t < k_t] \\ &= \mathbb{P}[L_t < \theta_t]\end{aligned}$$

where $L_t = X_t + \sigma W_t$, $k_t = F^{-1}_{\ln S_t}(p(t))$ and $\theta_t = k_t - (\ln S_0 + (\mu - \frac{1}{2}\sigma^2 - \kappa(1))t)$. Denoting F_{L_t} as the distribution function of process L_t, the default probability is:

$$p(t) = F_{L_t}(\theta_t)$$

Hence, the distribution function of both the process, F_{L_t}, and also pricing dynamics, $F_{\ln S_t}$, must be calculated.

The characteristic functions of the extended processes and return are:

$$\begin{aligned}\phi^{\Omega}_{L_t}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t} \times \phi^{\Omega}_{X_t}(u) \\ \phi^{\Omega}_{\ln S_t}(u) &= e^{iu(\ln S_0 + \mu t - \frac{1}{2}\sigma^2 t)} e^{-t(\psi^{\Omega}(u))} e^{-iu(\kappa^{\Omega}(1))t} e^{-\frac{1}{2}\sigma^2 u^2 t}\end{aligned}$$

where Ω =DPL, ONE, STBL, FMLS. This structure is tested for DPL, $\alpha = 1$, α-stable and FMLS models. For example when Ω =STBL, characteristic functions of the extended processes and return becomes:

$$\begin{aligned}\phi^{\mathrm{STBL}}_{L_t}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t} e^{-t \sec\left(\frac{\pi\alpha}{2}\right)\left((iu\nu^+)^\alpha + (iu\nu^-)^\alpha\right)} \\ \phi^{\mathrm{STBL}}_{\ln S_t}(u) &= e^{iu(\ln S_0 + \mu t - \frac{1}{2}\sigma^2 t)} e^{-t \sec\left(\frac{\pi\alpha}{2}\right)\left((iu\nu^+)^\alpha + (iu\nu^-)^\alpha\right)} e^{iut \sec\left(\frac{\pi\alpha}{2}\right)\left((\nu^+)^\alpha + (\nu^-)^\alpha\right)} e^{-\frac{1}{2}\sigma^2 u^2 t}\end{aligned}$$

Table 4.12 shows DPL, ONE, STBL and FMLS models fitted to CDX and Table 4.13 shows the fit for iTraxx.

Separate Negative and Positive Jumps

The second set of models add two independent standard Brownian Motions as noise terms to the above Lévy processes. The extended Lévy process becomes:

$$X_t^+ + \sqrt{\sigma}W_t^+ + X_t^- + \sqrt{1-\sigma}W_t^-$$

where X_t^+ and X_t^- are independent processes of only positive and negative jumps of X_t, respectively. W_i^+ and W_t^- are independent standard Brownian Motions. This specification is convenient as it allows one to easily compare the Lévy processes against Gaussian copula when $\gamma^+ = \gamma^- = 0$. The pricing dynamics is given by:

$$\ln S_t = \ln S_0 + (\mu - \frac{\sigma + (1-\sigma)}{2})t + X_t^+ + \sqrt{\sigma}W_t^+ + X_t^- + \sqrt{1-\sigma}W_t^- - \kappa^+(1)t - \kappa^-(1)t$$

where $\kappa^+(1)t$ and $\kappa^-(1)t$ are convexity adjustments for the processes X_t^+ and X_t^-, respectively. The convexity adjustment for the standard Brownian Motions add up to $-\frac{1}{2}t$. This set-up is tested for DPL, $\alpha = 1$ and α-stable. The default probability of each obligor is calculated as follows:

$$\begin{aligned}
\mathbb{P}[\ln S_t < k_t] &= \mathbb{P}[\ln S_0 + (\mu - \frac{\sigma + (1-\sigma)}{2})t + X_t^+ + \sqrt{\sigma}W_t^+ \\
&\quad + X_t^- + \sqrt{1-\sigma}W_t^- - \kappa^+(1)t - \kappa^-(1)t < k_t] \\
&= \mathbb{P}[X_t^- + \sqrt{1-\sigma}W_t^- < \theta_t - (X_t^+ + \sqrt{\sigma}W_t^+)] \\
&= \mathbb{P}[L_t^- < \theta_t - L_t^+]
\end{aligned}$$

where $\theta_t = k_t - (\ln S_0 + \left(\mu - \frac{1}{2} - \kappa^+(1) - \kappa^-(1)\right)t)$, $k_t = F^{-1}_{\ln S_t}(p(t))$, $X_t^+ + \sqrt{\sigma}W_t^+ = L_t^+$ is the global term and $X_t^- + \sqrt{1-\sigma}W_t^- = L_t^-$ is the idiosyncratic term. The conditional default probability is:

$$
\begin{aligned}
p(t|y) &= \mathbb{P}[L_t^- < \theta_t - y | y = L_t^+] \\
&= F_{L_t^-}(\theta_t - y)
\end{aligned}
$$

and the unconditional default probability is found by integrating out the y variable.

$$
p(t) = \int_{-\infty}^{+\infty} F_{L_t^-}(\theta_t - y) dF_{L_t^+}
$$

Hence, the distribution function of processes, $F_{L_t^-}$, $F_{L_t^+}$ and the pricing dynamics, $F_{\ln S_t}$, must be calculated.

Now, Ω =DPL, ONE, STBL. The characteristic functions of L_t^+, L_t^- and the characteristic function of return are:

$$
\begin{aligned}
\phi^{\Omega}_{L_t^+}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t} \phi^{\Omega+}_{X_t}(u) \\
\phi^{\Omega}_{L_t^-}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t} \phi^{\Omega-}_{X_t}(u) \\
\phi^{\Omega}_{\ln S_t}(u) &= e^{iu(\ln S_0 + \mu t - \frac{1}{2}t)} e^{-t(\psi^{\Omega+}(u))} e^{-iu(\kappa^{\Omega+}(1))t} e^{-t(\psi^{\Omega-}(u))} e^{-iu(\kappa^{\Omega-}(1))t} \\
&\quad \times e^{\left(-\frac{1}{2}\sigma^2 u^2\right)t} e^{\left(-\frac{1}{2}(1-\sigma^2)u^2\right)t}
\end{aligned}
$$

where Ω =DPL, ONE, STBL. For example when Ω =DPL, characteristic functions of L_t^+, L_t^- and the characteristic function of return are:

$$
\begin{aligned}
\phi_{L_t^+}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t} e^{-t\left(-\gamma^+\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha\right]\right)} \\
\phi_{L_t^-}(u) &= e^{-\frac{1}{2}(1-\sigma^2)u^2 t} e^{-t\left(-\gamma^-\Gamma(-\alpha)\left[(\beta^-+iu)^\alpha-(\beta^-)^\alpha\right]\right)} \\
\phi_{\ln S_t}(u) &= e^{iu(\ln S_0+\mu t-\frac{1}{2}t)} e^{-t\left(-\gamma^+\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha\right]\right)} \\
&\quad \times e^{-t\left(iu\gamma^+\Gamma(-\alpha)\left[(\beta^+-1)^\alpha-(\beta^+)^\alpha\right]\right)} \\
&\quad \times e^{-t\left(-\gamma^-\Gamma(-\alpha)\left[(\beta^-+iu)^\alpha-(\beta^-)^\alpha\right]\right)} \\
&\quad \times e^{-t\left(iu\gamma^-\Gamma(-\alpha)\left[(\beta^--1)^\alpha-(\beta^-)^\alpha\right]\right)} e^{\left(-\frac{1}{2}\sigma^2u^2\right)t} e^{\left(-\frac{1}{2}(1-\sigma^2)u^2\right)t}
\end{aligned}
$$

Table 4.16 shows DPL, ONE and STBL models fitted to CDX and Table 4.17 shows the fit for iTraxx.

4.6.3 $\gamma^+ = \gamma^-$ with no Brownian Motion

For Lévy processes when $\gamma^+ = \gamma^-$ such as CGMY, Double Exponential, Variance Gamma and Symmetric Case, two sets of models are tested. The first set is an addition of two Lévy processes where the first process is with parameter γ^+ and the second one with parameter γ^-, defined as in 4.4 and 4.5. The Lévy process that will be tested becomes:

$$
X_t^g + X_t^i
$$

where X_t^g is the Lévy processes with parameter γ^+ and X_t^i is the Lévy processes with parameter γ^-. The pricing dynamics is given by:

$$
\ln S_t = \ln S_0 + \mu t + X_t^g + X_t^i - \kappa^g(1)t - \kappa^i(1)t
$$

where $\kappa^g(1)t$ and $\kappa^i(1)t$ are convexity adjustments for the processes X_t^g and X_t^i, respectively. This set-up is tested for CGMY, Double Exponential, Variance Gamma and Symmetric Cases. The default probability of each obligor is calculated as follows:

$$
\begin{aligned}
\mathbb{P}[\ln S_t < k_t] &= \mathbb{P}[\ln S_0 + \mu t + X_t^g + X_t^i - \kappa^g(1)t - \kappa^i(1)t < k_t] \\
&= \mathbb{P}[X_t^g + X_t^i < \theta_t] \\
&= \mathbb{P}[X_t^i < \theta_t - X_t^g]
\end{aligned}
$$

where $\theta_t = k_t - (\ln S_0 + (\mu - \kappa^g(1) - \kappa^i(1))t)$ and $k_t = F_{\ln S_t}^{-1}(p(t))$. Denoting $F_{X_t^g}$ and $F_{X_t^i}$ as the distribution function of processes X_t^g and X_t^i, respectively, the conditional default probability is:

$$
\begin{aligned}
p(t|y) &= \mathbb{P}[X_t^i < \theta_t - y | y = X_t^g] \\
&= F_{X_t^i}(\theta_t - y)
\end{aligned}
$$

and the unconditional default probability is found by integrating out the y variable.

$$
p(t) = \int_{-\infty}^{+\infty} F_{X_t^i}(\theta_t - y) dF_{X_t^g}
$$

Hence, the distribution function of processes, $F_{X_t^i}$, $F_{X_t^g}$ and also the pricing dynamics, $F_{\ln S_t}$, must be calculated.

Now, Ω =CMGY, DE, VG, SYM. Denote the cumulant exponent, characteristic exponent and characteristic function of process Ω with parameter γ^+ as $\kappa^{\Omega+}$, $\psi^{\Omega+}$ and $\phi^{\Omega+}$, respectively. Similarly, denote the cumulant exponent, characteristic exponent and characteristic function of process Ω with parameter γ^- as $\kappa^{\Omega-}$, $\psi^{\Omega-}$ and $\phi^{\Omega-}$, respectively. The characteristic functions of X_t^g, X_t^i and the characteristic function of return are given by:

$$
\begin{aligned}
\phi^{\Omega}_{X^g_t} &= \phi^{\Omega+}_{X_t}(u) \\
\phi^{\Omega}_{X^i_t} &= \phi^{\Omega-}_{X_t}(u) \\
\phi^{\Omega}_{\ln S_t}(u) &= e^{iu(\ln S_0+\mu t)} e^{-t(\psi^{\Omega+}(u))} e^{-iu(\kappa^{\Omega+}(1))t} e^{-t(\psi^{\Omega-}(u))} e^{-iu(\kappa^{\Omega-}(1))t}
\end{aligned}
$$

where Ω =CMGY, DE, VG, SYM. This structure is tested for CGMY, double exponential, variance gamma and symmetric models. For example when Ω =SYM, characteristic functions of X^g_t, X^i_t and the characteristic function of return are:

$$
\begin{aligned}
\phi^{\mathrm{SYM}}_{X^g_t} &= e^{-t\left(-\gamma^{+}\Gamma(-\alpha)[(\beta-iu)^{\alpha}-\beta^{\alpha}]-iuC(h)\right)} \\
\phi^{\mathrm{SYM}}_{X^i_t} &= e^{-t\left(-\gamma^{-}\Gamma(-\alpha)[(\beta-iu)^{\alpha}-\beta^{\alpha}]-iuC(h)\right)} \\
\phi^{\mathrm{SYM}}_{\ln S_t}(u) &= e^{iu(\ln S_0+\mu t)} e^{-t\left(-\gamma^{+}\Gamma(-\alpha)[(\beta-iu)^{\alpha}-\beta^{\alpha}]\right)} e^{-t\left(iu\gamma^{+}\Gamma(-\alpha)[(\beta-1)^{\alpha}-\beta^{\alpha}]\right)} \\
&\quad \times e^{-t\left(-\gamma^{-}\Gamma(-\alpha)[(\beta-iu)^{\alpha}-\beta^{\alpha}]\right)} e^{-t\left(iu\gamma^{-}\Gamma(-\alpha)[(\beta-1)^{\alpha}-\beta^{\alpha}]\right)}
\end{aligned}
$$

Table 4.14 shows CGMY, DE, VG and SYM models fitted to CDX and Table 4.15 shows the fit for iTraxx.

4.6.4 $\gamma^{+} = \gamma^{-}$ with two Brownian Motions

The second set of models add two independent standard Brownian Motions as noise terms to the above Lévy processes. The extended Lévy process becomes:

$$
X^g_t + \sqrt{\sigma} W^g_t + X^i_t + \sqrt{1-\sigma} W^i_t \tag{4.6}
$$

where X^g and X^i are independent global and idiosyncratic factors, respectively. W^g_i and W^i_t are independent global and idiosyncratic standard Brownian Motions, respectively. When $\gamma^{+} = \gamma^{-}$, the specification in 4.6 is convenient as it allows one to easily compare the Lévy processes against Gaussian copula when $\gamma^{+} = \gamma^{-} = 0$. The pricing dynamics is given by:

$$\ln S_t = \ln S_0 + (\mu - \frac{\sigma + (1-\sigma)}{2})t + X_t^g + \sqrt{\sigma} W_t^g + X_t^i + \sqrt{1-\sigma} W_t^i - \kappa^g(1)t - \kappa^i(1)t$$

where $\kappa^g(1)t$ and $\kappa^i(1)t$ are convexity adjustments for the processes X_t^g and X_t^i, respectively. The convexity adjustment for the standard Brownian Motions add up to $-\frac{1}{2}t$. This set-up is tested for CGMY, Double Exponential, Variance Gamma and Symmetric Cases. The default probability of each obligor is calculated as follows:

$$\begin{aligned}
\mathbb{P}[\ln S_t < k_t] &= \mathbb{P}[\ln S_0 + (\mu - \frac{\sigma + (1-\sigma)}{2})t + X_t^g + \sqrt{\sigma} W_t^g \\
&\quad + X_t^i + \sqrt{1-\sigma} W_t^i - \kappa^g(1)t - \kappa^i(1)t < k_t] \\
&= \mathbb{P}[X_t^i + \sqrt{1-\sigma} W_t^i < \theta_t - (X_t^g + \sqrt{\sigma} W_t^g)] \\
&= \mathbb{P}[L_t^i < \theta_t - L_t^g]
\end{aligned}$$

where $\theta_t = k_t - (\ln S_0 + \left(\mu - \frac{1}{2} - \kappa^g(1) - \kappa^i(1)\right) t)$, $k_t = F^{-1}_{\ln S_t}(p(t))$, $X_t^g + \sqrt{\sigma} W_t^g = L_t^g$ is the global term and $X_t^i + \sqrt{1-\sigma} W_t^i = L_t^i$ is the idiosyncratic term. The conditional default probability is:

$$\begin{aligned}
p(t|y) &= \mathbb{P}[L_t^i < \theta_t - y | y = L_t^g] \\
&= F_{L_t^i}(\theta_t - y)
\end{aligned}$$

and the unconditional default probability is found by integrating out the y variable.

$$p(t) = \int_{-\infty}^{+\infty} F_{L_t^i}(\theta_t - y) dF_{L_t^g}$$

Hence, the distribution function of processes, $F_{L_t^i}$, $F_{L_t^g}$ and also pricing dynamics, $F_{\ln S_t}$, must be calculated.

Now, Ω =CMGY, DE, VG, SYM. The characteristic functions of L_t^g, L_t^i and the characteristic function of return are:

$$\begin{aligned}
\phi^{\Omega}_{L_t^g}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t}\phi^{\Omega+}_{X_t}(u) \\
\phi^{\Omega}_{L_t^i}(u) &= e^{-\frac{1}{2}\sigma^2 u^2 t}\phi^{\Omega-}_{X_t}(u) \\
\phi^{\Omega}_{\ln S_t}(u) &= e^{iu(\ln S_0+\mu t-\frac{1}{2}t)}e^{-t(\psi^{\Omega+}(u))}e^{-iu(\kappa^{\Omega+}(1))t}e^{-t(\psi^{\Omega-}(u))}e^{-iu(\kappa^{\Omega-}(1))t} \\
&\quad \times e^{\left(-\frac{1}{2}\sigma^2u^2\right)t}e^{\left(-\frac{1}{2}(1-\sigma^2)u^2\right)t}
\end{aligned}$$

where Ω =CMGY, DE, VG, SYM. This structure is tested for CGMY, double exponential, variance gamma and symmetric models. For example when Ω =CGMY, characteristic functions of L_t^g, L_t^i and the characteristic function of return are:

$$\begin{aligned}
\phi_{L_t^g}(u) &= e^{-\frac{1}{2}\sigma^2u^2t}e^{-t\left(-\gamma^+\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha+(\beta^-+iu)^\alpha-(\beta^-)^\alpha\right]-iuC(h)\right)} \\
\phi_{L_t^i}(u) &= e^{-\frac{1}{2}(1-\sigma^2)u^2t}e^{-t\left(-\gamma^-\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha+(\beta^-+iu)^\alpha-(\beta^-)^\alpha\right]-iuC(h)\right)} \\
\phi_{\ln S_t}(u) &= e^{iu(\ln S_0+\mu t-\frac{1}{2}t)}e^{-t\left(-\gamma^+\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha+(\beta^-+iu)^\alpha-(\beta^-)^\alpha\right]\right)} \\
&\quad \times e^{-t\left(iu\gamma^+\Gamma(-\alpha)\left[(\beta^+-1)^\alpha-(\beta^+)^\alpha+(\beta^-+1)^\alpha-(\beta^-)^\alpha\right]\right)} \\
&\quad \times e^{-t\left(-\gamma^-\Gamma(-\alpha)\left[(\beta^+-iu)^\alpha-(\beta^+)^\alpha+(\beta^-+iu)^\alpha-(\beta^-)^\alpha\right]\right)} \\
&\quad \times e^{-t\left(iu\gamma^-\Gamma(-\alpha)\left[(\beta^+-1)^\alpha-(\beta^+)^\alpha+(\beta^-+1)^\alpha-(\beta^-)^\alpha\right]\right)}e^{\left(-\frac{1}{2}\sigma^2u^2\right)t}e^{\left(-\frac{1}{2}(1-\sigma^2)u^2\right)t}
\end{aligned}$$

Table 4.14 shows CGMY, DE, VG and SYM models fitted to CDX and Table 4.15 shows the fit for iTraxx.

4.6.5 Data and Estimation

The Correlated Dampened Power Law (CDPL), as well as its special cases are calibrated to CDX.NA.IG.13 and iTraxx EUR.9 spreads read from Bloomberg on November 25, 2009. The vector of estimated parameters is $\Upsilon = [\mu, \alpha, \gamma^{\pm}, \beta^{\pm}]$ and it differs depending on the model under investigation.

The Fast Fourier Transform (FFT) method of Carr & Madan (1998) is employed to invert the characteristic function of each Lévy process and pricing dynamics. Fourier resolution of 2^{10} is used with spacing 0.25. The probability density function of each Lévy process and pricing dynamics is integrated using the trapezoidal rule to find the cumulative distribution function of each Lévy process and pricing dynamics. The default barrier of each credit is found by linearly interpolating the asset returns inverse cumulative distribution function. Similarly, the conditional portfolio loss distribution is found by linear interpolation. The unconditional portfolio loss distribution is found by integrating out the common factor/process taking a specific value by employing the trapezoidal rule.

The objective function, Mean Absolute Error, is minimized by employing the Nelder-Mead routine and it is defined as follows:

$$\Delta(\mu, \alpha, \gamma^{\pm}, \beta^{\pm}) = \frac{1}{K} \sum_{k=1}^{K} |w^k_{Market} - w^k_{Model}| \tag{4.7}$$

where K is the number of tranches, w^k_{Market} is the upfront fee for k^{th} tranche read from the market and w^k_{Model} is the upfront fee for k^{th} tranche estimated by the model.

Generally, the CDX/iTraxx spreads are reported as an upfront fee with a fixed running spread for the equity tranche and running spreads for all other tranches. In consequence, most of the time CDO pricing literature minimizes a version of equation 4.7 by inputting running spreads. However, the CDX and iTraxx data pulled from Bloomberg on November 25, 2009 shows upfront fees except for the two most senior tranches of iTraxx. Hence, upfront fees are chosen to be minimized instead of running spreads here. The two most senior tranches of iTraxx running spreads are converted to upfront fees during optimization, and optimal upfront fees are converted to running spreads in reporting.

4.7 Results

The CDPL model, as well as its special cases (i.e. ONE, STBL, FMLS, CGMY, DE, VG, SYM) have been calibrated to fit the CDX.NA.IG.13 and iTraxx EUR.9 series tranche upfront fees and running spreads. Their performance is compared to three sets of models commonly employed in the industry and frequently referenced in the literature.

The first group comprises industry benchmark models where the default dependence structures are represented by a copula function. Elliptical and Archimedean copula models (i.e. Gaussian, student's t, double t, Clayton, Frank, Gumbel and MO) with Monte Carlo simulation as well as their one-factor representations are in this group. Thanks to its upper and lower tail dependence, student's t copula outperforms the tail-independent Gaussian copula model in both Monte Carlo simulation and in factor model. As the degrees of freedom of student's t copula increases, its results approach to Gaussian copula findings. Owing to its upper tail dependence, the Gumbel copula not only fits better than the lower tail-dependent Clayton copula and tail-independent Frank copula models, but also better than all other copula models tested with Monte Carlo simulation. Amongst the Elliptical and Archimedean copula models tested, the Marshall-Olkin and Clayton copulae have the highest MAE's. Double t copula, with its symmetric heavy-tailed distribution, performs significantly better than the Gaussian copula.

The second group includes the Stochastic Correlation model with independent correlation and symmetric but dependent correlation assumptions, the Random Factor Loadings model, and symmetric/asymmetric Normal Inverse Gaussian models. The stochastic Correlation model with independent correlation parameter fits significantly better than its counterpart with symmetric and dependent structure. Moreover, the asymmetric Normal Inverse Gaussian model performs much better than the symmetric Normal Inverse Gaussian model. An important empirical result is that having a negative asymmetry parameter significantly improves the fit of the NIG model. Amongst the models tested in this section, Random Factor Loadings perform the best, whilst the Stochastic Correlation model with independent correlation parameter is a close second. Symmetric NIG is the worst performer amongst the tested models in this section.

The third group includes the most commonly implemented models based on Lévy processes in CDO pricing literature: Inverse Gaussian and Gamma processes. The results show that the Inverse Gaussian model fits much better to CDX and iTraxx tranches when compared to Lévy-based Gamma process model. Garcia, Goossens, Masol & Schoutens (2007) tests the SYM model, though with pricing dynamics different from what is tested here, and find that it performs better than the IG and Gamma models. Further, Baxter (2006) finds that the VG model performs better than the Gamma model; our results confirm both findings.

The CDPL, ONE, STBL and FMLS models tested with the assumption of unified jumps do not perform well. Although unified jump models perform significantly worse than their separated jump counterparts, the ranking of the empirical results is reasonable. As ONE and STBL are special cases of the CDPL, the CDPL performs better than both. Similarly, as the FMLS is a special case of the STBL, the STBL performs better than the FMLS.

Empirical findings show that adding a diffusion component to the unified jump models does not improve the fit of these models to CDX and iTraxx indices, except for the CDPL model. By contrast, for almost all the Lévy-based models tested with the assumption of separated jumps or sum of two Lévy processes, adding two diffusion components significantly improves the fit.

Amongst the CGMY, DE, VG and SYM models tested with the assumption of sum of two Levy processes, the CGMY model performs the best: It fits the CDX with a MAE of 0.2568% and iTraxx with a MAE of 0.5375%. These models significantly outperform models tested with the assumption of unified jumps. An argument similar to the one just made for the unified jump models can be made for these models as well: As DE and VG are special cases of the CGMY, the CGMY performs better than both. The worst performer here is the SYM model, tested previously by Garcia et al. (2007).

Most significantly, it is clear from the empirical findings that the CDPL with separated jumps significantly outperforms all of its special cases, all of the industry benchmark models and other tested models in the credit pricing literature. For the CDX index, the MAE attained with the industry benchmark Gaussian factor copula is 3.7890%. The best fit with the models already tested in the literature is attained with the Random Factor Loadings model with a MAE of 0.2579%. The CDPL fits the CDX tranches with a MAE of 0.1817%; hence, it decreases the MAE attained with the Gaussian factor copula model by 95.20%, with the RFL model by 29.54% and with the CGMY model by 29.25%.

Similarly, for the iTraxx index, the MAE attained with the industry benchmark Gaussian factor copula is 5.9474%. The best fit with the models already tested in the literature is attained with the IG model which has a MAE of 1.0193%. The CDPL with separated jumps fits the iTraxx tranches with a MAE of 0.4795%; hence, it decreases the MAE attained with the Gaussian factor copula model by 91.93%, with the IG model by 52.95% and with the CGMY model by 10.79%.

4.8 Conclusion

Thanks to its ability to capture asymmetric heavy-tailed return distributions and to accommodate different degrees of dampening for positive and negative jumps, the CDPL model significantly outperforms all of the industry benchmark models and other existing models in the credit pricing literature. The CDPL model decreases the mean absolute error of the Gaussian copula model by 95% and 91% for CDX and iTraxx indices, respectively. Similarly, the CDPL model decreases the mean absolute error of the best performing model in the literature by 30% for CDX and 52% for iTraxx indices. The empirical evidence shows that the CDPL model overcomes the major deficiencies of the existing CDO pricing models and it successfully fits the market quotes. As such, it constitutes an important addition to the credit pricing literature.

	CDX.NA.IG.13			iTraxx EUR.9		
	Tranche	Upfront Fee (%)	Running Spread (bps)	Tranche	Upfront Fee (%)	Running Spread (bps)
Equity	0-3%	59.06%	100	0-3%	35.75%	500
Mezzanine	3-7%	24.79%	100	3-6%	-1.37%	500
Junior	7-10%	9.33%	100	6-9%	-8.55%	500
Senior	10-15%	1.48%	100	9-12%		124.57
Super Senior	15-30%	-2.89%	100	12-22%		47.53

Table 4.1: 5 year mid-market quotes for CDX.NA.IG.13 and iTraxx.EUR.9 on 11/25/2009

Gaussian one factor copula model fitted to CDX with 100bps running spread for all five tranches and fitted to iTraxx with 500bps running spread for the first three tranches. Market implied correlation is the implied correlation that equates upfront fees of the Gaussian model to market upfront fees read from CDX. Similarly, market implied correlation is the implied correlation that equates upfront fees and running spreads of the Gaussian model to market upfront fees read from iTraxx. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w_{Market}^{k} - w_{Model}^{k}|$.

CDX.NA.IG.13				iTraxx EUR.9			
Tranche	Market UF	Model UF	Market Implied Correlation	Tranche	Market UF/RS	Model UF/RS	Market Implied Correlation
0-3%	59.06 %	59.06 %	0.3655	0-3%	35.75%	35.75%	0.4550
3-7%	24.79 %	29.15 %	0.5071	3-6%	-1.37%	8.89 %	0.7697
7-10%	9.33 %	15.64 %	0.0680	6-9%	-8.55%	-1.94 %	0.0584
10-15%	1.48 %	7.54 %	0.1324	9-12%	124.57bps	306.55bps	0.0983
15-30%	-2.89 %	-0.69 %	0.2231	12-22%	47.53bps	156.26bps	0.1716
MAE		3.7890%				5.9474%	
ρ		0.3655				0.4550	

Table 4.2: Gauss one factor copula market implied compound correlation for CDX and iTraxx.

Gauss, student's t, Clayton, Frank and Gumbel copulae upfront fee fitted to CDX with 10,000 Monte Carlo simulations with 100bps running spread for all five tranches. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w^k_{Market} - w^k_{Model}|$.

CDX.NA.IG.13								
Tranche	Market UF	Gauss Copula UF	Student's t Copula UF (Best Fit)	Student's t Copula UF	Student's t Copula UF	Clayton Copula UF	Frank Copula UF	Gumbel Copula UF
0-3%	59.06 %	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%
3-7%	24.79 %	29.52%	28.80%	29.72%	29.72%	36.16%	31.76%	17.83%
7-10%	9.33 %	16.39%	15.30%	16.42%	16.49%	22.18%	19.49%	8.47%
10-15%	1.48 %	8.29%	7.14%	8.23%	8.36%	10.14%	10.59%	4.49%
15-30%	-2.89 %	-0.30%	-0.92%	-0.27%	-0.15%	-3.24%	-1.67%	0.78%
MAE		4.2405%	3.5239%	4.2817%	4.3443%	6.6505%	5.4944%	2.8974%
ρ		0.3758	0.4348	0.4319	0.4093			
DoF			5	6	12			
α						2.8939	5.2707	1.3834

Table 4.3: Gauss, student's t, Clayton, Frank and Gumbel copulae fitted to CDX with 10,000 Monte Carlo simulations.

Gauss, student's t, Clayton, Frank and Gumbel copulae upfront fee and running spread fitted to iTraxx with 10,000 Monte Carlo simulations with 500bps running spread for the first three tranches. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w_{Market}^{k} - w_{Model}^{k}|$.

				iTraxx EUR.9				
		Gauss	Student's t	Student's t	Student's t	Clayton	Frank	Gumbel
	Market	Copula	Copula	Copula	Copula	Copula	Copula	Copula
	UF/RS	UF/RS	UF/RS	UF/RS	UF/RS	UF/RS	UF/RS	UF/RS
Tranche			(Best Fit)					
0-3%	35.75%	35.75%	35.75%	35.74%	35.75%	35.75%	35.75%	35.75%
3-6%	-1.37 %	9.73%	9.13%	10.00%	9.67%	15.32%	12.19%	-1.33%
6-9%	-8.55 %	-1.03%	-1.88%	-1.14%	-1.05%	4.33%	2.077%	-9.20%
9-12%	124.57bps	325.92bps	311.31bps	322.41bps	331.36bps	422.60bps	395.22bps	210.73bps
12-22%	47.53bps	171.38bps	160.60bps	169.79bps	175.19bps	165.24bps	189.59bps	142.57bps
MAE		6.5789%	6.0851%	6.5781%	6.6453%	9.5671%	8.4605%	1.7559%
ρ		0.4694	0.5820	0.5449	0.5165			
DoF			4	6	12			
α						4.4901	6.9802	1.4520

Table 4.4: Gauss, student's t, Clayton, Frank and Gumbel copulae fitted to iTraxx with 10,000 Monte Carlo simulations.

Factor Gauss, student's t, Double t and Marshall-Olkin copulae upfront fee fitted to CDX with 100bps running spread for all five tranches. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w^k_{Market} - w^k_{Model}|$.

CDX.NA.IG.13									
Tranche	Market UF	Gauss Copula UF	Student's t Copula UF (Best Fit)	Student's t Copula UF	Student's t Copula UF	Double t Copula UF (Best Fit)	Double t Copula UF	Double t Copula UF	Marshall-Olkin Copula UF
0-3%	59.06%	59.06%	57.53%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%
3-7%	24.79%	29.15%	26.84%	28.55%	29.12%	24.67%	24.55%	26.91%	6.80%
7-10%	9.33 %	15.64%	10.25%	12.73%	13.29%	11.92%	12.22%	13.92%	5.07%
10-15%	1.48 %	7.54%	1.26%	3.72%	4.41%	5.87%	6.12%	6.83%	5.07%
15-30%	-2.89%	-0.69%	-3.93%	-3.21%	-2.75%	0.21%	0.19%	-0.21%	5.07%
MAE		3.7890%	1.1531%	1.9477%	2.2737%	2.0401%	2.1732%	2.9510%	6.7599%
ρ		0.3655	0.2919	0.3649	0.4331	0.7360	0.6669	0.6337	
DoF			5	6	12	3-9	6-6	12-12	
α									0.4259

Table 4.5: Factor Gauss, student's t, Double t and Marshall-Olkin copulae fitted to CDX.

Factor Gauss, student's t, Double t and Marshall-Olkin copulae upfront fee and running spread fitted to iTraxx with 500bps running spread for the first three tranches. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w^k_{Market} - w^k_{Model}|$.

iTraxx EUR.9									
Tranche	Market UF/RS	Gauss Copula UF/RS	Student's t Copula UF/RS (Best Fit)	Student's t Copula UF/RS	Student's t Copula UF/RS	Double t Copula UF/RS (Best Fit)	Double t Copula UF/RS	Double t Copula UF/RS	Marshall-Olkin Copula UF/RS
0-3%	35.75%	35.75%	30.91%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%
3-6%	-1.37%	8.89%	1.56%	7.95%	8.36%	-0.75%	3.74%	6.35%	-12.89%
6-9%	-8.55%	-1.94%	-11.62%	-4.44%	-3.74%	-9.06%	-5.99%	-4.02%	-13.74%
9-12%	124.57bps	306.55bps	119.77bps	241.60bps	258.79bps	217.45bps	255.28bps	280.97bps	198.24bps
12-22%	47.53bps	156.26bps	49.18bps	95.66bps	110.83bps	146.50bps	151.18bps	154.07bps	198.24bps
MAE		5.9474%	2.2299%	4.1683%	4.6720%	1.9433%	3.6268%	4.7897%	5.3877%
ρ		0.4550	0.2978	0.4810	0.5426	0.7962	0.7306	0.7010	
DoF			4	6	12	3-3	6-6	12-12	
α									0.4489

Table 4.6: Factor Gauss, student's t, Double t and Marshall-Olkin copulae fitted to iTraxx.

Stochastic Correlation, Random Factor Loadings and Normal Inverse Gaussian models fitted to CDX with 100bps running spread for all five tranches. Stochastic(I) correspond to the model with independent correlation assumption as given in equation 3.19. Stochastic(II) correspond to the model with symmetric but dependent correlation assumption as given in equation 3.21. NIG(I) corresponds to the symmetric case with $\beta = 0$ and NIG(II) corresponds to the asymmetric case with $\beta \neq 0$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w^k_{Market} - w^k_{Model}|$.

			CDX.NA.IG.13			
Tranche	Market UF	Stochastic Correlation(I) UF	Stochastic Correlation(II) UF	RFL UF	NIG(I) UF	NIG(II) UF
0-3%	59.06 %	59.06%	59.05%	59.06%	59.06%	59.06%
3-7%	24.79 %	23.91%	24.79%	25.10%	24.79%	24.78%
7-10%	9.33 %	9.33%	9.32%	9.08%	-0.04%	3.08%
10-15%	1.48 %	2.09%	3.33%	1.48%	-2.94%	1.47%
15-30%	-2.89 %	-3.29%	-0.60 %	-3.61%	-3.76%	-1.19%
MAE		0.3782%	0.8287%	0.2579%	2.9342%	1.5871%
ρ, β, q		0.2980, 0.4373, 0.6707				
ρ, q, qs			0.3756, 0.4308, 0.3174			
w, z, θ				0.5899, 0.8498, 0.6039		
ρ, α, β					0.1056, 0.1277, 0	0.3761, 0.2643, -0.1885

Table 4.7: Stochastic Correlation, Random Factor Loadings and Normal Inverse Gaussian models fitted to CDX.

Stochastic Correlation, Random Factor Loadings and Normal Inverse Gaussian models fitted to iTraxx with 500bps running spread for the first three tranches. Stochastic(I) correspond to the model with independent correlation assumption as given in equation 3.19. Stochastic(II) correspond to the model with symmetric but dependent correlation assumption as given in equation 3.21. NIG(I) corresponds to the symmetric case with $\beta = 0$ and NIG(II) corresponds to the asymmetric case with $\beta \neq 0$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w^k_{Market} - w^k_{Model}|$.

iTraxx EUR.9						
Tranche	Market UF	Stochastic Correlation(I) RS/UF	Stochastic Correlation(II) RS/UF	RFL RS/UF	NIG(I) RS/UF	NIG(II) RS/UF
0-3%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%
3-6%	-1.37%	2.47%	-1.37%	1.12%	-1.37%	-1.39%
6-9%	-8.55%	-9.54%	-8.55%	-11.45%	-18.00%	-15.82%
9-12%	124.57bps	159.90bps	235.46bps	124.57bps	59.60bps	124.21bps
12-22%	47.53bps	57.57bps	160.82bps	37.64bps	40.48bps	74.76bps
MAE		1.3833%	2.0141%	1.1728%	2.5548%	1.7132%
ρ, β, q		0.2754, 0.4978, 0.8075				
ρ, q, qs			0.6983, 0.3743, 0.2913			
w, z, θ				0.6367, 0.9484, 0.9849		
ρ, α, β					0.2159, 0.1101, 0	0.4132, 0.1708, -0.0955

Table 4.8: Stochastic Correlation, Random Factor Loadings and Normal Inverse Gaussian models fitted to iTraxx.

Inverse Gaussian and Gamma models fitted to CDX with 100bps running spread for all five tranches. The left panel assumes there are no standard Brownian Motions. The Lévy process X_t is given as $X_t^g + X_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0 + \mu t + X_t^g + X_t^i - \kappa^g(1)t - \kappa^i(1)t$, where X_t^g is the Lévy processes with parameter γ^+ and X_t^i is the Lévy processes with parameter γ^- and $\kappa^g(1)t$ and $\kappa^i(1)t$ are the corresponding convexity adjustments. The right panel assumes there are two standard Brownian Motions. The Lévy process X_t is given as $X_t^g + \sqrt{\sigma}W_t^g + X_t^i + \sqrt{1-\sigma}W_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{\sigma + (1-\sigma)}{2})t + X_t^g + \sqrt{\sigma}W_t^g + X_t^i + \sqrt{1-\sigma}W_t^i - \kappa^g(1)t - \kappa^i(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w_{Market}^k - w_{Model}^k|$.

		CDX.NA.IG.13			
Tranche	Market UF	Gamma UF	Inverse Gaussian UF	Gamma UF	Inverse Gaussian UF
0-3%	59.06 %	59.06%	59.05%	59.06%	59.06%
3-7%	24.79 %	24.79%	25.09%	24.66%	23.76%
7-10%	9.33 %	9.17%	9.33%	10.07%	9.33%
10-15%	1.48 %	-0.27%	1.25%	2.60%	2.16%
15-30%	-2.89 %	-4.79%	-1.47%	-3.13%	-3.23%
MAE		0.7639%	0.3886%	0.4497%	0.4102%
ρ		0.7947	0.7421	0.6381	0.4916
γ		0.4471	0.1682	0.4844	0.1762
$\gamma^+ = \gamma(1-\rho)$		0.0917	0.0433	0.1752	0.0895
$\gamma^- = \gamma\rho$		0.3553	0.1248	0.3091	0.0866
k, δ		0.4851, 0.6495		0.5046,0.7552	
a, b			0.0846, 0.5710		0.1076, 0.5925
μ		0.6895	0.5343	0.5928	0.5367
σ		—	—	0.2745	0.2755

Table 4.9: Inverse Gaussian and Gamma models fitted to CDX.

Inverse Gaussian and Gamma models fitted to fitted to iTraxx with 500bps running spread for the first three tranches. The left panel assumes there are no standard Brownian Motions. The Lévy process X_t is given as $X_t^g + X_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0 + \mu t + X_t^g + X_t^i - \kappa^g(1)t - \kappa^i(1)t$, where X_t^g is the Lévy processes with parameter γ^+ and X_t^i is the Lévy processes with parameter γ^- and $\kappa^g(1)t$ and $\kappa^i(1)t$ are the corresponding convexity adjustments. The right panel assumes there are two standard Brownian Motions. The Lévy process X_t is given as $X_t^g + \sqrt{\sigma}W_t^g + X_t^i + \sqrt{1-\sigma}W_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{\sigma+(1-\sigma)}{2})t + X_t^g + \sqrt{\sigma}W_t^g + X_t^i + \sqrt{1-\sigma}W_t^i - \kappa^g(1)t - \kappa^i(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w_{Market}^k - w_{Model}^k|$.

	iTraxx EUR.9				
Tranche	Market UF/RS	Gamma UF/RS	Inverse Gaussian UF/RS	Gamma UF/RS	Inverse Gaussian UF/RS
0-3%	35.75%	35.75%	35.75%	35.75%	35.74%
3-6%	-1.37%	7.70%	6.08%	0.82%	0.52%
6-9%	-8.55%	-8.55%	-8.55%	-11.31%	-11.39%
9-12%	124.57bps	125.65bps	97.57bps	124.57bps	124.57bps
12-22%	47.53bps	4.26bps	64.45bps	38.55bps	39.81bps
MAE		2.2400%	1.8585%	1.0747%	1.0193%
ρ		0.7109	0.7854	0.4260	0.6917
γ		0.4219	0.2566	0.3104	0.1405
$\gamma^+ = \gamma(1-\rho)$		0.1219	0.0550	0.1781	0.0433
$\gamma^- = \gamma\rho$		0.2999	0.2015	0.1322	0.0972
k, δ		1.0257, 0.6733		1.0550, 0.6255	
a, b			0.0865, 0.0227		0.0657, 0.6817
μ		0.6535	0.5576	0.6606	0.6098
σ		—	—	0.2717	0.2945

Table 4.10: Inverse Gaussian and Gamma models fitted to iTraxx.

Matrix of Calibrated Models.

	Sum of Two Lévy Processes		Unified Jumps		Separated Jumps	
	With No Brownian Motion	With Two Brownian Motions	With No Brownian Motion	With One Brownian Motion	With No Brownian Motion	With Two Brownian Motions
$\gamma^+ = \gamma^-$	CGMY, DE, VG, SYM					
$\gamma^+ \neq \gamma^-$			DPL, ONE, STBL, FMLS		DPL, ONE, STBL	

Table 4.11: Matrix of Calibrated Models based on Lévy Processes.

Dampened Power Law (DPL), $\alpha = 1$ (ONE), Finite Moment Log Stable (FMLS) and α-stable (STBL) models upfront fee fitted to CDX with 100bps running spread for all five tranches. The left panel assumes unified jumps where there are no standard Brownian Motions as described in section 4.6.1: With Lévy process X_t and pricing dynamics $\ln S_t = \ln S_0 + (\mu t + X_t - \kappa(1)t)$, where κ is the convexity adjustment. The right panel assumes unified jumps where there is a single standard Brownian Motion as described in section 4.6.2: The Lévy process is given as $X_t + \sigma W_t$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{1}{2}\sigma^2)t + X_t + \sigma W_t - \kappa(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w^k_{Market} - w^k_{Model}|$. Parameter values in boldface are the assumed parameter values specific to the model tested.

CDX.NA.IG.13									
Tranche	Market UF	DPL UF	ONE UF	STBL UF	FMLS UF	DPL UF	ONE UF	STBL UF	FMLS UF
0-3%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%	59.04%
3-7%	24.79%	1.77%	4.64%	-1.64%	-2.33%	10.06%	-2.01%	-2.46%	-2.44%
7-10%	9.33%	-0.63%	-2.25%	-4.67%	-4.69%	5.06%	-4.68%	-4.69%	-4.69%
10-15%	1.48%	-0.71%	-4.32%	-4.69%	-4.69%	0.59%	-4.69%	-4.69%	-4.69%
15-30%	-2.89%	-2.89%	-4.68%	-4.69%	-4.69%	-4.18%	-4.69%	-4.69%	-4.69%
MAE		7.0343%	7.866%	9.6825%	9.8238%	4.2314%	9.7579%	9.8491%	9.850%
ρ		0.15504	0.2967	0.5582	0.0474	0.4243	0.8792	0.5341	0.0361
γ		0.5157	0.5977	0.2415	0.6197	0.5752	0.5368	0.4796	0.4917
$\gamma^+ = \gamma(1-\rho)$		0.4357	0.4203	0.1067	—	0.3311	0.0648	0.2234	—
$\gamma^- = \gamma\rho$		0.0799	0.1773	0.1348	0.0294	0.2441	0.4720	0.2562	0.0178
β^+		0.4093	0.3185	**0**	**0**	0.4745	0.7039	**0**	**0**
β^-		1.4137	1.4683	**0**	**0**	1.4240	0.8764	**0**	**0**
α		0.7018	**1**	1.3176	0.7522	0.1610	**1**	1.2707	0.7358
μ		0.4182	0.5005	0.7214	0.4733	0.6076	0.7385	0.6567	0.5155
σ		—	—	—	—	0.2952	0.00016	0.0012	0.6776

Table 4.12: DPL, ONE, STBL and FMLS models fitted to CDX.

Dampened Power Law (DPL), $\alpha = 1$ (ONE), Finite Moment Log Stable (FMLS) and α-stable (STBL) models upfront fee fitted to iTraxx with 500bps running spread for the first three tranches. The left panel assumes unified jumps where there are no standard Brownian Motions as described in section 4.6.1: With Lévy process X_t and pricing dynamics $\ln S_t = \ln S_0 + (\mu t + X_t - \kappa(1)t)$, where κ is the convexity adjustment. The right panel assumes unified jumps where there is a single standard Brownian Motion as described in section 4.6.2: The Lévy process is given as $X_t + \sigma W_t$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{1}{2}\sigma^2)t + X_t + \sigma W_t - \kappa(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w^k_{Market} - w^k_{Model}|$. Parameter values in boldface are the assumed parameter values specific to the model tested.

		iTraxx EUR.9							
Tranche	Market UF	DPL UF	ONE UF	STBL UF	FMLS UF	DPL UF	ONE UF	STBL UF	FMLS UF
0-3%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%
3-6%	-1.37%	-10.51%	-5.34%	-18.97%	-19.40%	10.60%	-3.39%	-19.22%	-19.48%
6-9%	-8.55%	-12.61%	-15.58%	-23.16%	-23.19%	12.54%	-15.01%	-23.17%	-23.19%
9-12%	124.57bps	99.55bps	18.12bps	0.003bps	0.0016bps	110.47bps	20.33bps	0.0028bps	0.0019bps
12-22%	47.53bps	70.70bps	0.89bps	0.00039bps	0.00000015bps	77.73bps	1.43bps	0.000036bps	0.000020bps
MAE		2.9835%	3.5633%	8.0464%	8.1374%	2.9602%	3.0278%	8.096%	8.1526%
ρ		0.5355	0.2829	0.5250	0.3008	0.5458	0.3123	0.4997	0.2002
γ		0.5310	0.6334	0.1487	0.5577	0.5226	0.6674	0.1036	0.6072
$\gamma^+ = \gamma(1-\rho)$		0.2466	0.4541	0.070	—	0.2373	0.4590	0.0518	—
$\gamma^- = \gamma\rho$		0.2844	0.1792	0.0781	0.1677	0.2852	0.2084	0.0517	0.1216
β^+		0.3956	0.3140	**0**	**0**	0.3972	0.2846	**0**	**0**
β^-		1.4547	1.3038	**0**	**0**	1.4492	1.4004	**0**	**0**
α		0.6823	**1**	1.4208	1.4579	0.6607	**1**	1.0404	1.0487
μ		0.5385	0.5492	0.8023	0.7686	0.5044	0.5096	0.7488	0.7300
σ		—	—	—	—	0.5288	0.9082	0.2179	0.5681

Table 4.13: DPL, ONE, STBL and FMLS models fitted to iTraxx.

CGMY, Double Exponential (DE), Variance Gamma (VG) and Symmetric (SYM) models upfront fee fitted to CDX with 100bps running spread for all five tranches. The left panel assumes there are no standard Brownian Motions as described in section 4.6.3: The Lévy process X_t is given as $X_t^g+X_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0+\mu t+X_t^g+X_t^i-\kappa^g(1)t-\kappa^i(1)t$, where X_t^g is the Lévy processes with parameter γ^+ and X_t^i is the Lévy processes with parameter γ^- and $\kappa^g(1)t$ and $\kappa^i(1)t$ are the corresponding convexity adjustments. The right panel assumes there are two standard Brownian Motions as described in section 4.6.4: The Lévy process X_t is given as $X_t^g+\sqrt{\sigma}W_t^g+X_t^i+\sqrt{1-\sigma}W_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0+(\mu-\frac{\sigma+(1-\sigma)}{2})t+X_t^g+\sqrt{\sigma}W_t^g+X_t^i+\sqrt{1-\sigma}W_t^i-\kappa^g(1)t-\kappa^i(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w_{Market}^k-w_{Model}^k|$. Parameter values in boldface are the assumed parameter values specific to the model tested.

	CDX.NA.IG.13								
Tranche	Market UF	CGMY UF	VG UF	DE UF	SYM UF	CGMY UF	VG UF	DE UF	SYM UF
0-3%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%	59.09%	59.06%	59.06%
3-7%	24.79%	24.79%	24.79%	24.80%	24.33%	24.44%	24.38%	24.32%	24.78%
7-10%	9.33%	9.23%	8.51%	8.59%	9.33%	9.32%	9.32%	9.33%	9.50%
10-15%	1.48%	2.43%	1.95%	2.09%	1.77%	1.79%	1.83%	1.91%	1.48%
15-30%	-2.89%	-2.61%	-2.77%	-2.81%	-3.66%	-3.51%	-3.47%	-3.39%	-4.11%
MAE		0.2659%	0.2814%	0.2881%	0.3044%	0.2568%	0.2758%	0.2798%	0.2806%
ρ		0.6917	0.6999	0.7043	0.7346	0.5114	0.4986	0.5713	0.6680
γ		0.5006	0.5422	2.5629	0.2819	0.7500	0.7346	2.5068	0.2914
$\gamma^+ = \gamma(1-\rho)$		0.1543	0.1626	0.7578	0.0748	0.3664	0.3682	1.0745	0.0967
$\gamma^- = \gamma\rho$		0.3463	0.3795	1.8051	0.2071	0.3836	0.3663	1.4323	0.1947
β^+		1.1913	1.2034	1.9643	0.5303	1.3220	1.4386	2.1920	0.5531
β^-		3.0104	3.2444	3.2606	0.5303	3.7957	3.6831	3.5426	0.5531
α		0.2084	**0**	**-1**	1.8034	0.1932	**0**	**-1**	1.7472
μ		0.6615	0.7341	0.7222	0.7806	0.7641	0.7155	0.7364	0.8227
σ		—	—	—	—	0.1901	0.2006	0.2000	0.2104

Table 4.14: CGMY, VG, DE and SYM models fitted to CDX.

CGMY, Double Exponential (DE), Variance Gamma (VG) and Symmetric (SYM) models upfront fee fitted to iTraxx with 500bps running spread for the first three tranches. The left panel assumes there are no standard Brownian Motions as described in section 4.6.3: The Lévy process X_t is given as $X_t^g + X_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0 + \mu t + X_t^g + X_t^i - \kappa^g(1)t - \kappa^i(1)t$, where X_t^g is the Lévy processes with parameter γ^+ and X_t^i is the Lévy processes with parameter γ^- and $\kappa^g(1)t$ and $\kappa^i(1)t$ are the corresponding convexity adjustments. The right panel assumes there are two standard Brownian Motions as described in section 4.6.4: The Lévy process X_t is given as $X_t^g + \sqrt{\sigma} W_t^g + X_t^i + \sqrt{1-\sigma} W_t^i$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{\sigma + (1-\sigma)}{2})t + X_t^g + \sqrt{\sigma} W_t^g + X_t^i + \sqrt{1-\sigma} W_t^i - \kappa^g(1)t - \kappa^i(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K} |w_{Market}^k - w_{Model}^k|$. Parameter values in boldface are the assumed parameter values specific to the model tested.

	iTraxx EUR.9								
Tranche	Market UF/RS	CGMY UF/RS	VG UF/RS	DE UF/RS	SYM UF/RS	CGMY UF/RS	VG UF/RS	DE UF/RS	SYM UF/RS
0-3%	35.75%	35.75%	35.89%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%
3-6%	-1.37%	-1.37%	-0.92%	-1.37%	0.83%	0.58%	-1.13%	-1.36%	0.66%
6-9%	-8.55%	-10.46%	-12.12%	-12.58%	-10.68%	-9.27%	-12.19%	-12.34%	-11.42%
9-12%	124.57bps	127.65bps	124.66bps	124.57bps	124.57bps	124.65bps	124.26bps	125.91bps	124.56bps
12-22%	47.53bps	70.83bps	48.19bps	53.03bps	17.60bps	47.71bps	46.70bps	51.34bps	37.54bps
MAE		0.5871%	0.8398%	0.8573%	1.1376%	0.5375%	0.7888%	0.8054%	1.0734%
ρ		0.6604	0.6852	0.6701	0.7166	0.5326	0.6059	0.1799	0.9983
γ		0.8767	2.6424	2.8868	0.9602	0.9822	0.4542	0.4865	0.19343
$\gamma^+ = \gamma(1-\rho)$		0.2976	0.8316	0.9522	0.2720	0.4590	0.1790	0.3989	0.000031494
$\gamma^- = \gamma\rho$		0.5790	1.8107	1.9345	0.6881	0.5231	0.2752	0.0875	0.1931
β^+		0.4372	2.1867	2.1898	1.4303	0.4795	1.5059	1.7278	0.4370
β^-		1.8880	2.5246	3.3393	1.4303	1.9877	1.1439	1.7424	0.4370
α		0.4184	**0**	**-1**	0.4707	0.3739	**0**	**-1**	1.7882
μ		0.4596	0.6684	0.7249	0.6855	0.4636	0.6971	0.7188	0.8737
σ		—	—	—	—	0.2346	0.2797	0.1116	0.5384

Table 4.15: CGMY, VG, DE and SYM models fitted to iTraxx.

Dampened Power Law (DPL), $\alpha = 1$ (ONE) and α-stable (STBL) models upfront fee fitted to CDX with 100bps running spread for all five tranches. The left panel assumes separate negative and positive jumps where there are no standard Brownian Motions as described in section 4.6.1: The Lévy process X_t is given as $X_t^+ + X_t^-$ and pricing dynamics is given by $\ln S_t = \ln S_0 + \mu t + X_t^+ + X_t^- - \kappa^+(1)t - \kappa^-(1)t$, where X_t^+ and X_t^- are independent processes of only positive and negative jumps of X_t, respectively, and κ^+ and κ^- are the corresponding convexity adjustments. The right panel assumes separate negative and positive jumps where there are two standard Brownian Motions as described in section 4.6.2: The Lévy process X_t is given as $X_t^+ + \sqrt{\sigma}W_t^+ + X_t^- + \sqrt{1-\sigma}W_t^-$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{\sigma+(1-\sigma)}{2})t + X_t^+ + \sqrt{\sigma}W_t^+ + X_t^- + \sqrt{1-\sigma}W_t^- - \kappa^+(1)t - \kappa^-(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w_{Market}^k - w_{Model}^k|$. Parameter values in boldface are the assumed parameter values specific to the model tested.

CDX.NA.IG.13							
Tranche	Market UF	DPL UF	ONE UF	STBL UF	DPL UF	ONE UF	STBL UF
0-3%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%	59.06%
3-7%	24.79%	24.55%	24.25%	20.04%	24.79%	24.35%	23.76%
7-10%	9.33%	9.33%	9.33%	9.33%	9.33%	9.32%	9.33%
10-15%	1.48%	1.74%	1.86%	3.37%	1.73%	1.82%	2.16%
15-30%	-2.89%	-3.33%	-3.48%	1.36%	-3.54%	-3.50%	-3.23%
MAE		0.1889%	0.3051%	2.1778%	0.1817%	0.2806%	0.4102%
ρ		0.3992	0.4641	0.4298	0.4033	0.30430	0.3632
γ		0.1917	0.1965	0.0327	0.2119	0.19318	0.00296
$\gamma^+ = \gamma(1-\rho)$		0.1152	0.10532	0.01868	0.1264	0.13440	0.00188
$\gamma^- = \gamma\rho$		0.0765	0.09121	0.01408	0.0854	0.05878	0.00107
β^+		0.7630	0.7596	**0**	0.7873	0.6825	**0**
β^-		1.3343	1.2911	**0**	1.3571	1.6234	**0**
α		0.7765	**1**	1.54119	0.8383	**1**	1.6802
μ		0.8998	0.7624	0.43311	0.6886	0.70636	0.5366
σ		—	—	—	0.2072	0.1916	0.2753

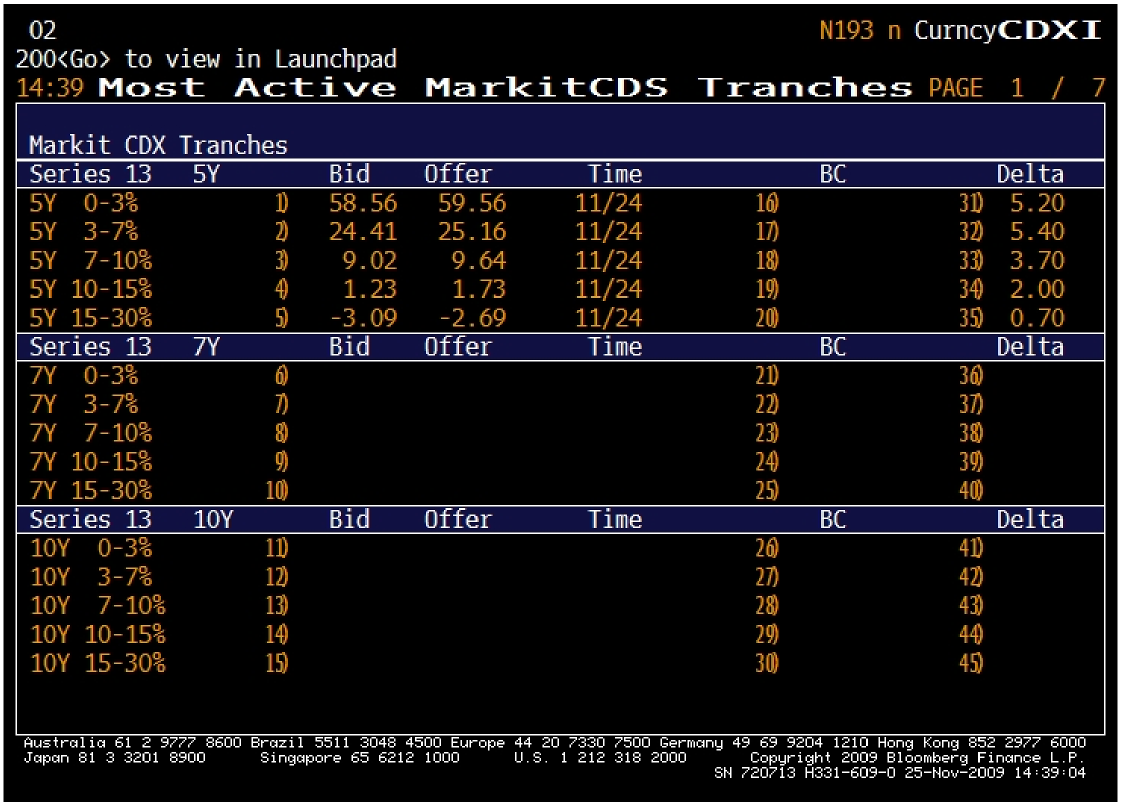
Table 4.16: DPL, ONE and STBL models fitted to CDX.

Dampened Power Law (DPL), $\alpha = 1$ (ONE) and α-stable (STBL) models upfront fee fitted to iTraxx with 500bps running spread for the first three tranches. The left panel assumes separate negative and positive jumps where there are no standard Brownian Motions as described in section 4.6.1: The Lévy process X_t is given as $X_t^+ + X_t^-$ and pricing dynamics is given by $\ln S_t = \ln S_0 + \mu t + X_t^+ + X_t^- - \kappa^+(1)t - \kappa^-(1)t$, where X_t^+ and X_t^- are independent processes of only positive and negative jumps of X_t, respectively, and κ^+ and κ^- are the corresponding convexity adjustments. The right panel assumes separate negative and positive jumps where there are two standard Brownian Motions as described in section 4.6.2: The Lévy process X_t is given as $X_t^+ + \sqrt{\sigma}W_t^+ + X_t^- + \sqrt{1-\sigma}W_t^-$ and pricing dynamics is given by $\ln S_t = \ln S_0 + (\mu - \frac{\sigma+(1-\sigma)}{2})t + X_t^+ + \sqrt{\sigma}W_t^+ + X_t^- + \sqrt{1-\sigma}W_t^- - \kappa^+(1)t - \kappa^-(1)t$. MAE is the Mean Absolute Error defined as $\frac{1}{K}\sum_{k=1}^{K}|w^k_{Market} - w^k_{Model}|$. Parameter values in boldface are the assumed parameter values specific to the model tested.

	iTraxx EUR.9						
Tranche	Market UF/RS	DPL UF/RS	ONE UF/RS	STBL UF/RS	DPL UF/RS	ONE UF/RS	STBL UF/RS
0-3%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%	35.75%
3-6%	-1.37%	-1.36%	-0.08%	-1.36%	-1.37%	-0.70%	-0.69%
6-9%	-8.55%	-9.87%	-11.71%	-9.33%	-12.00%	-11.74%	-12.05%
9-12%	124.57bps	111.80bps	128.96bps	210.73bps	124.57bps	129.52bps	124.56bps
12-22%	47.53bps	64.93bps	46.67bps	137.02bps	47.60bps	47.53bps	47.84bps
MAE		0.4795%	0.9383%	1.7385%	0.6915%	0.8171%	0.8374%
ρ		0.5314	0.3556	0.4504	0.3076	0.9836	0.2877
γ		0.1907	0.1991	0.1674	0.2142	0.3023	0.2121
$\gamma^+ = \gamma(1-\rho)$		0.0893	0.1283	0.0920	0.1483	0.0049	0.1510
$\gamma^- = \gamma\rho$		0.1013	0.0708	0.0754	0.0659	0.2974	0.0610
β^+		0.6472	1.2968	**0**	0.4620	0.4436	**0**
β^-		1.6735	1.6234	**0**	0.7241	0.6354	**0**
α		0.6795	**1**	1.7907	0.2860	**1**	1.8433
μ		0.8832	0.6977	0.3461	0.9811	0.6173	0.5852
σ		—	—	—	0.4309	0.5808	0.1198

Table 4.17: DPL, ONE and STBL models fitted to iTraxx.

Data



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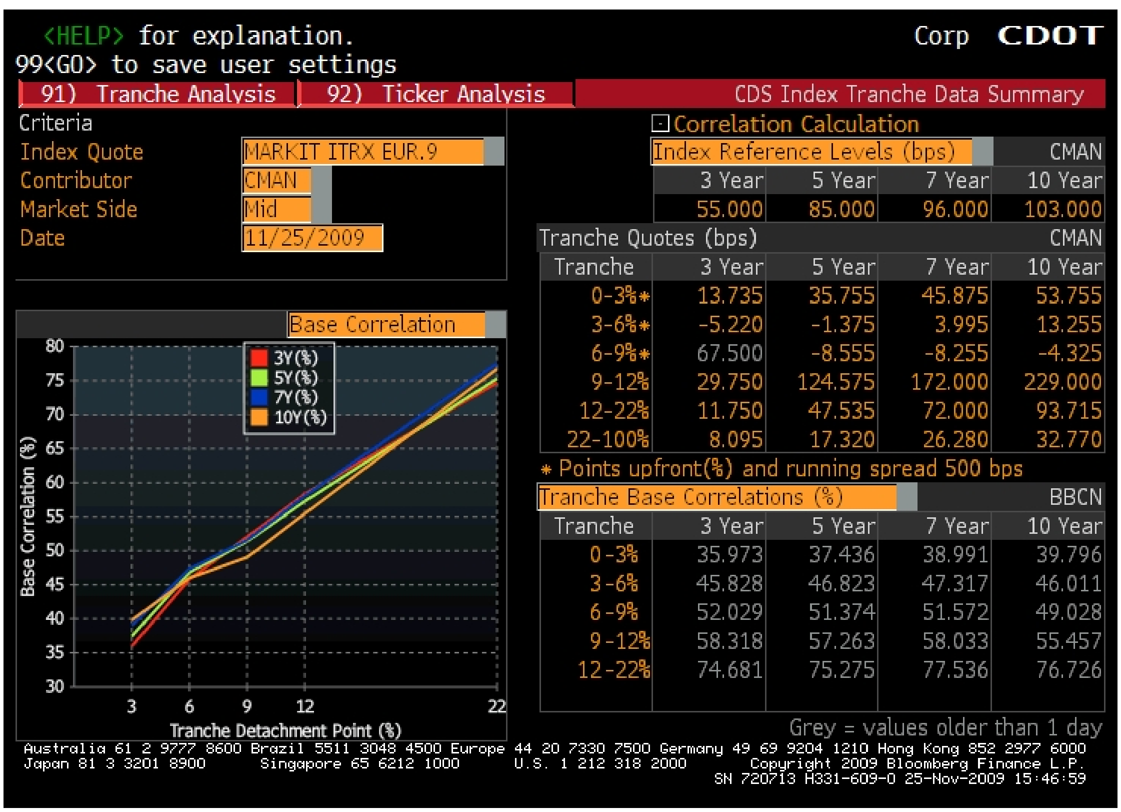
14:39 Most Active MarkitCDS Tranches PAGE 1 / 7

Markit CDX Tranches

Series 13 5Y		Bid	Offer	Time	BC	Delta
5Y 0-3%	1)	58.56	59.56	11/24	16)	31) 5.20
5Y 3-7%	2)	24.41	25.16	11/24	17)	32) 5.40
5Y 7-10%	3)	9.02	9.64	11/24	18)	33) 3.70
5Y 10-15%	4)	1.23	1.73	11/24	19)	34) 2.00
5Y 15-30%	5)	-3.09	-2.69	11/24	20)	35) 0.70
Series 13 7Y		**Bid**	**Offer**	**Time**	**BC**	**Delta**
7Y 0-3%	6)				21)	36)
7Y 3-7%	7)				22)	37)
7Y 7-10%	8)				23)	38)
7Y 10-15%	9)				24)	39)
7Y 15-30%	10)				25)	40)
Series 13 10Y		**Bid**	**Offer**	**Time**	**BC**	**Delta**
10Y 0-3%	11)				26)	41)
10Y 3-7%	12)				27)	42)
10Y 7-10%	13)				28)	43)
10Y 10-15%	14)				29)	44)
10Y 15-30%	15)				30)	45)

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2009 Bloomberg Finance L.P.
SN 720713 H331-609-0 25-Nov-2009 14:39:04



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99<GO> to save user settings
91) Tranche Analysis 92) Ticker Analysis CDS Index Tranche Data Summary

Criteria
Index Quote: MARKIT ITRX EUR.9
Contributor: CMAN
Market Side: Mid
Date: 11/25/2009

Correlation Calculation

Index Reference Levels (bps) CMAN	3 Year	5 Year	7 Year	10 Year
	55.000	85.000	96.000	103.000

Tranche Quotes (bps) CMAN

Tranche	3 Year	5 Year	7 Year	10 Year
0-3%*	13.735	35.755	45.875	53.755
3-6%*	-5.220	-1.375	3.995	13.255
6-9%*	67.500	-8.555	-8.255	-4.325
9-12%	29.750	124.575	172.000	229.000
12-22%	11.750	47.535	72.000	93.715
22-100%	8.095	17.320	26.280	32.770

* Points upfront(%) and running spread 500 bps

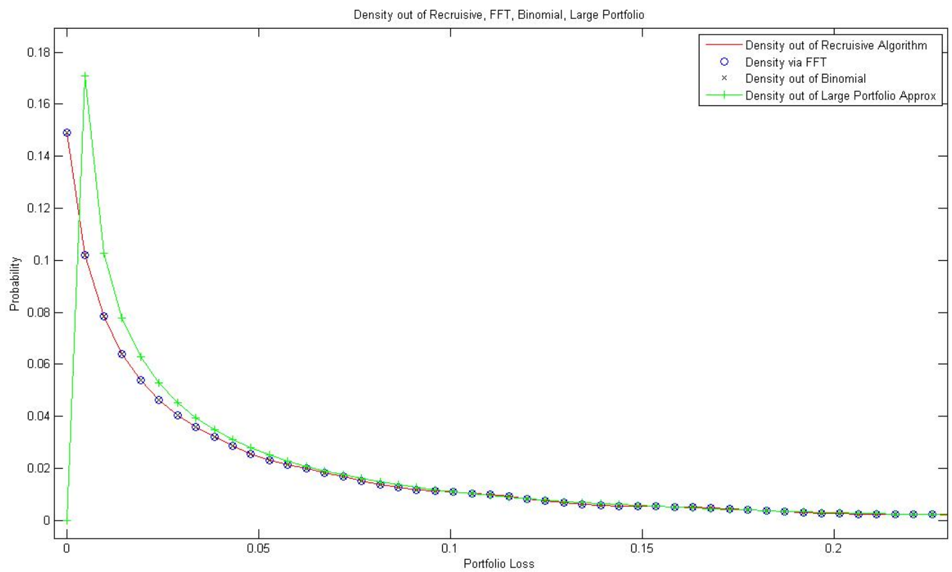
Tranche Base Correlations (%) BBCN

Tranche	3 Year	5 Year	7 Year	10 Year
0-3%	35.973	37.436	38.991	39.796
3-6%	45.828	46.823	47.317	46.011
6-9%	52.029	51.374	51.572	49.028
9-12%	58.318	57.263	58.033	55.457
12-22%	74.681	75.275	77.536	76.726

Grey = values older than 1 day
Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2009 Bloomberg Finance L.P.
SN 720713 H331-609-0 25-Nov-2009 15:46:59

Figure 4.1: CDX and iTraxx Data Bloomberg Screen Shots .

Probability density of Portfolio Loss using binomial distribution, large portfolio approximation, semi analytic recursion and Fast Fourier Transform methods for CDX (top) and iTraxx (bottom) with Gaussian one factor copula model and with correlation parameters 0.3655 and 0.4550, respectively.



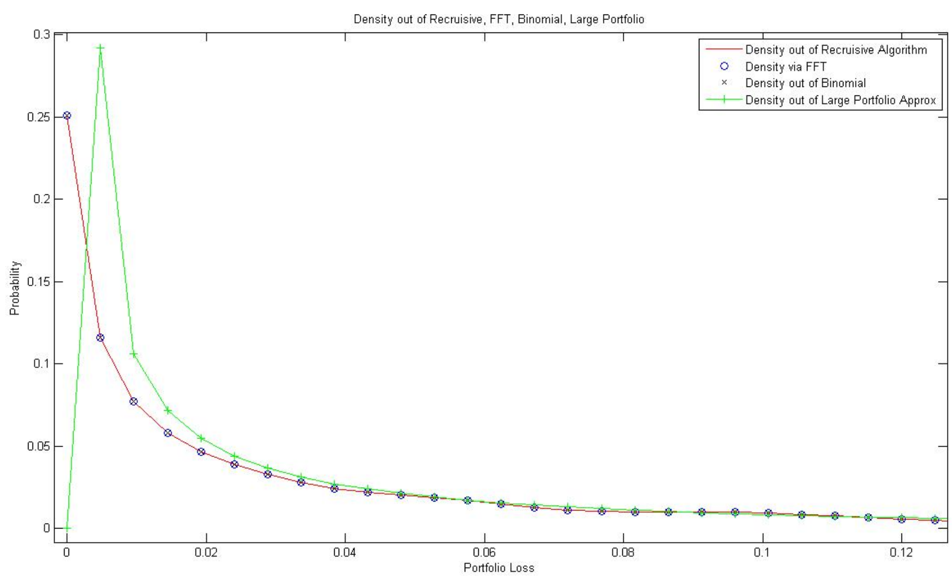
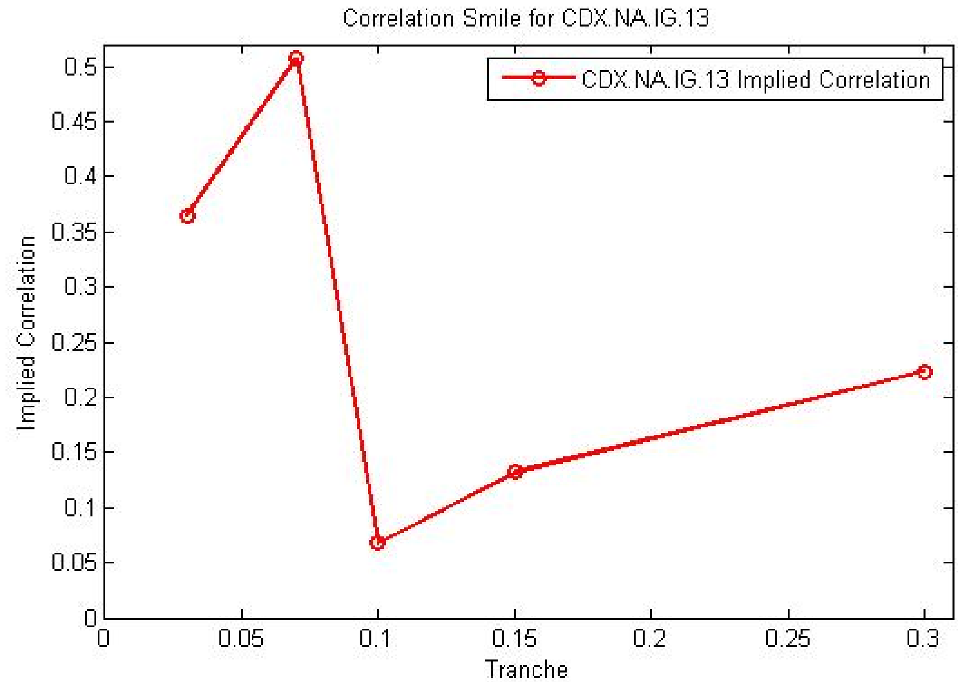


Figure 4.2: Probability density of Portfolio Loss .

The skew for the standardized tranches is shifted towards a higher subordination as the portfolio average spread is wider than the index spread.



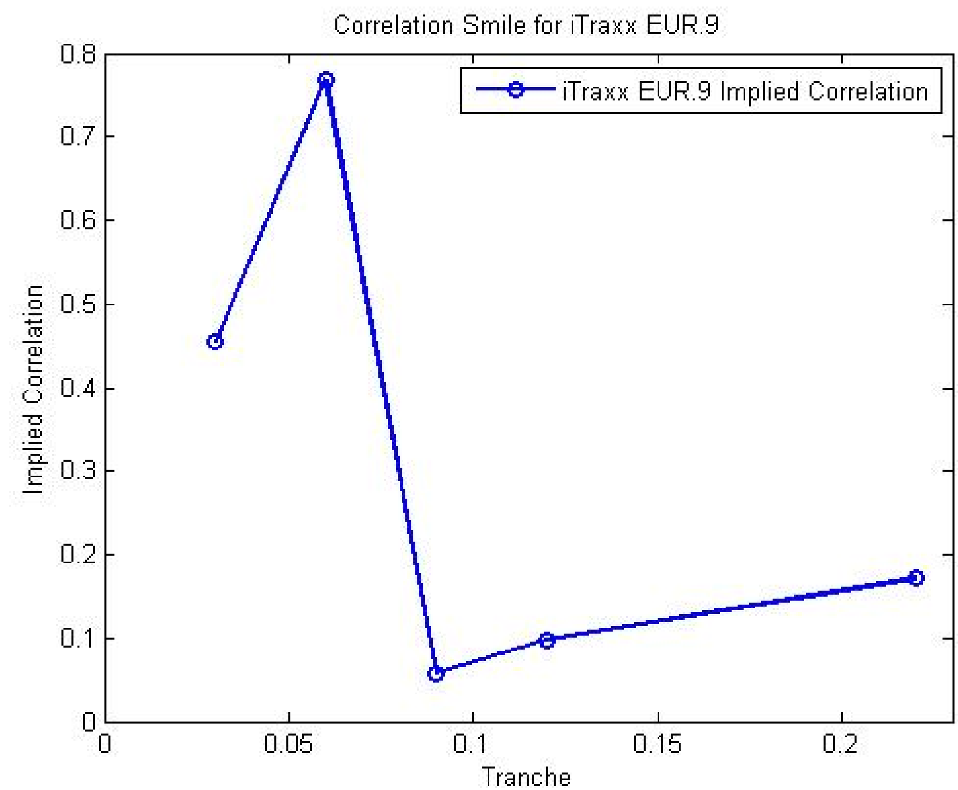


Figure 4.3: Correlation Smile for CDX and iTraxx.

Portfolio Loss Distribution for CDX(top), iTraxx(bottom) with Gaussian, student's t and double t with the optimized parameters reported in tables.

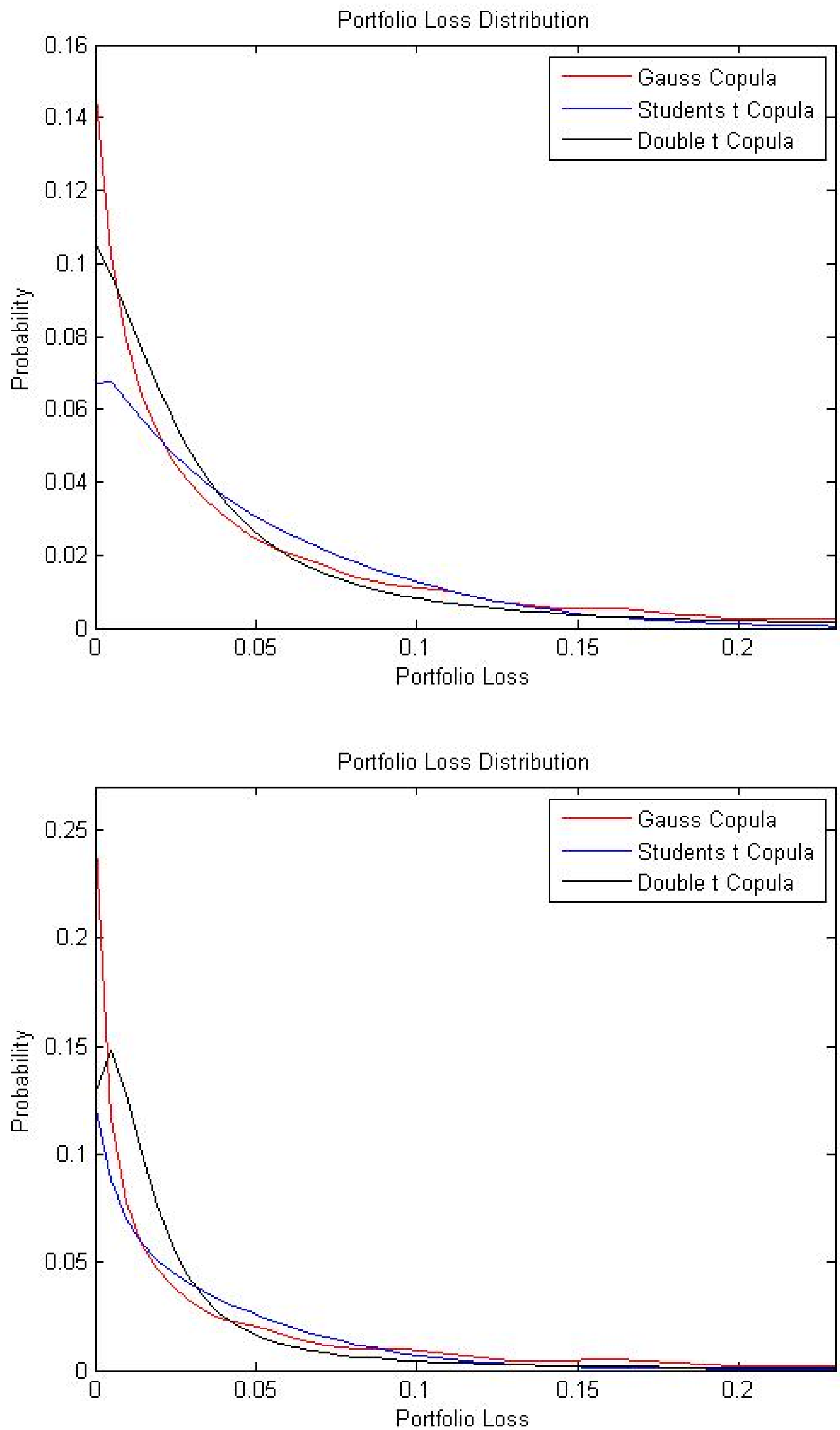
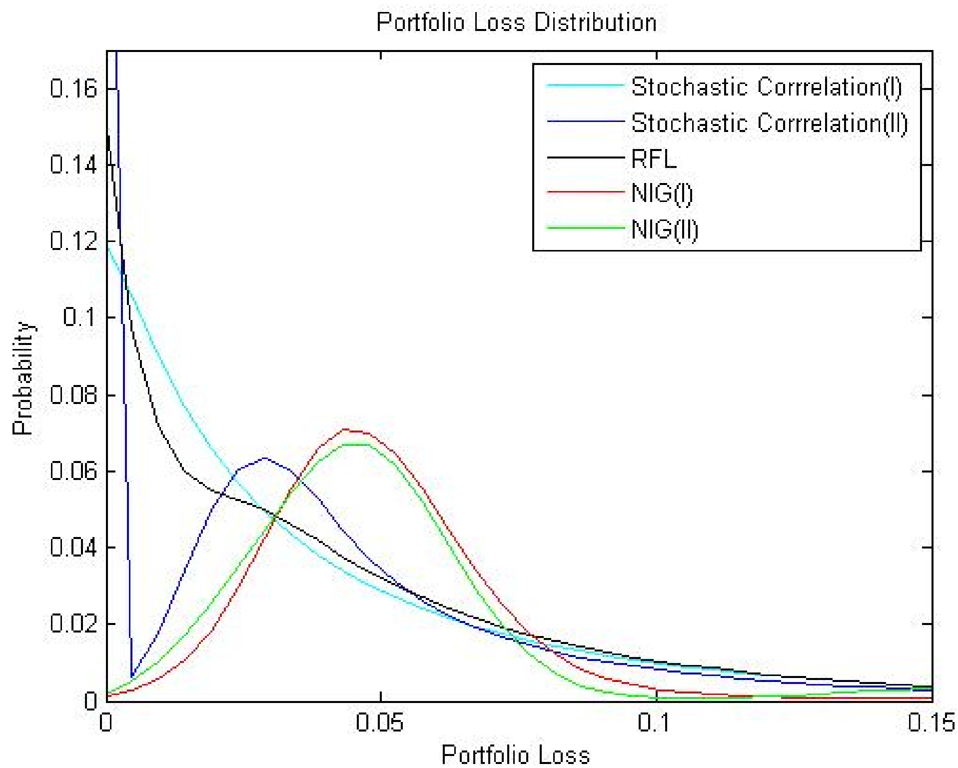


Figure 4.4: Probability density of Portfolio Loss.

Portfolio Loss Distribution for CDX(top), iTraxx(bottom) with Stochastic Correlation, RFL and NIG with the optimized parameters reported in tables.



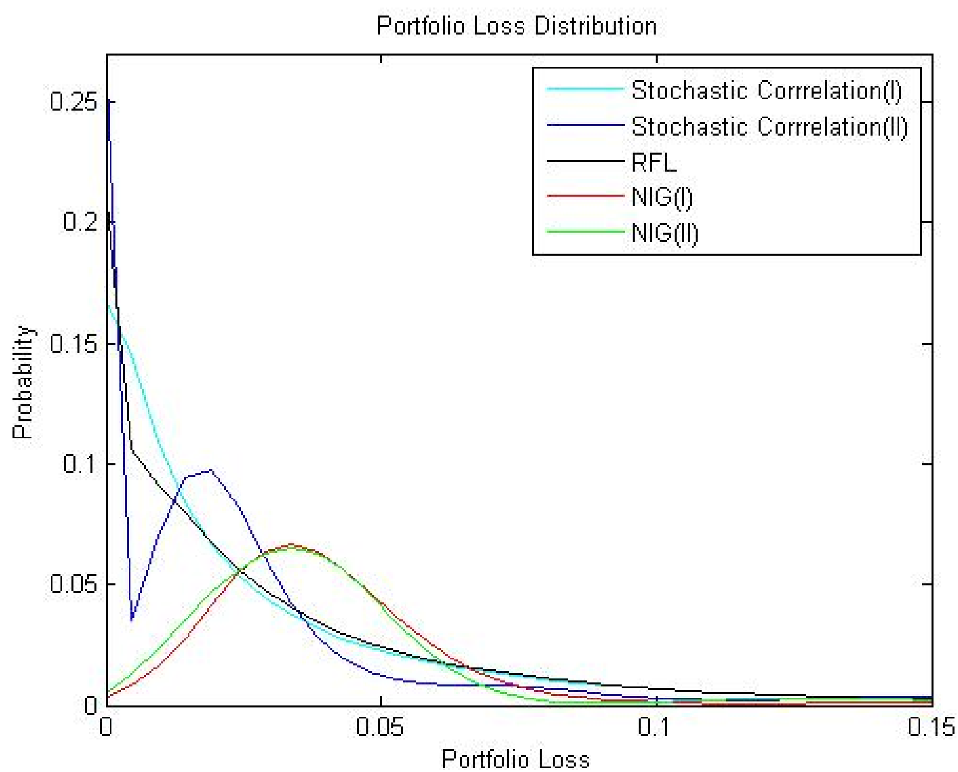


Figure 4.5: Probability density of Portfolio Loss

Blue histograms are the corresponding marginal histograms. ρ is the correlation parameter, K is Kendall's tau and S is Spearman's rho.

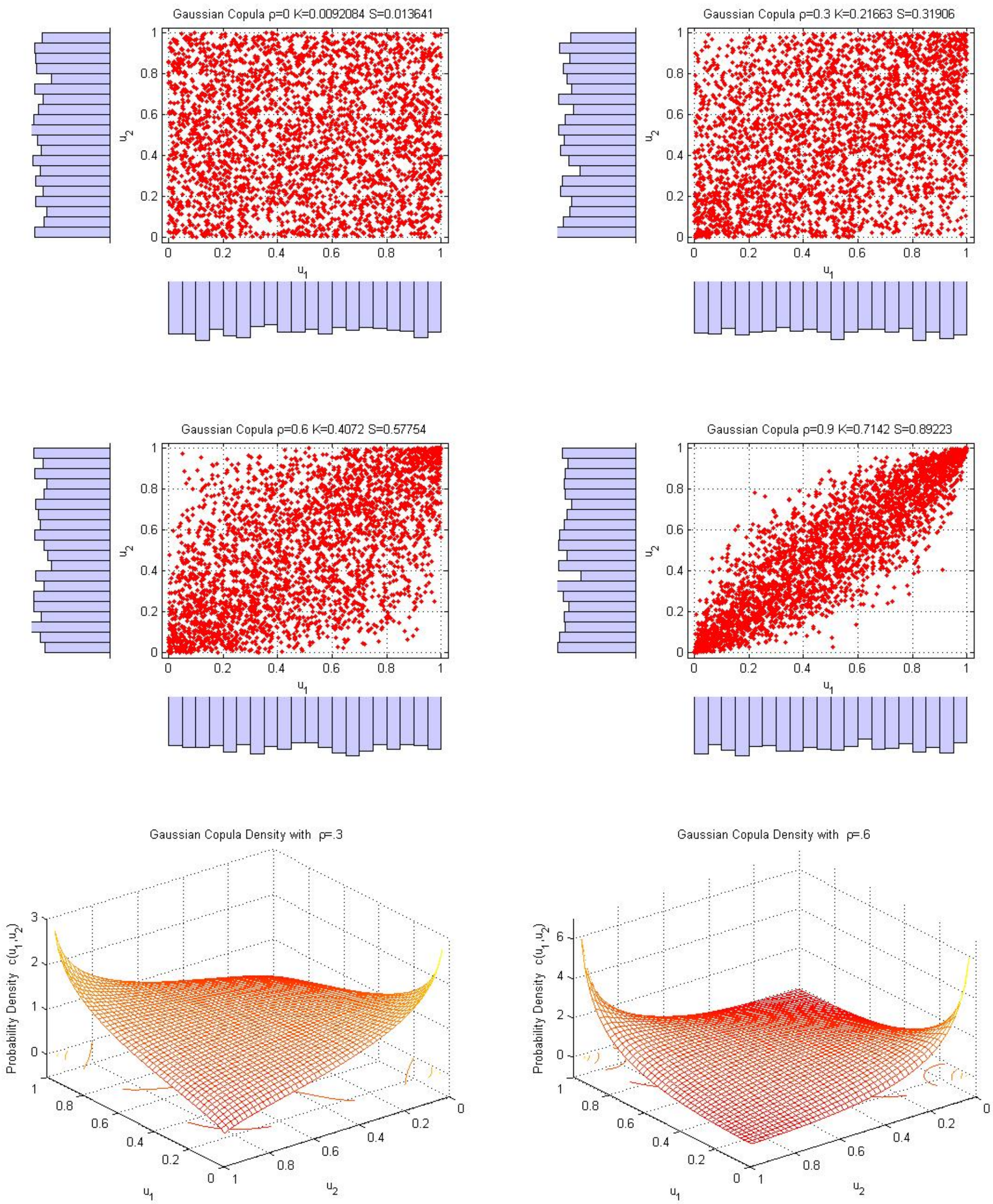


Figure 4.6: Gaussian bivariate copula simulation and probability density with 3000 samples
.

Blue histograms are the corresponding marginal histograms. ρ is the correlation parameter, K is Kendall's tau and S is Spearman's rho.

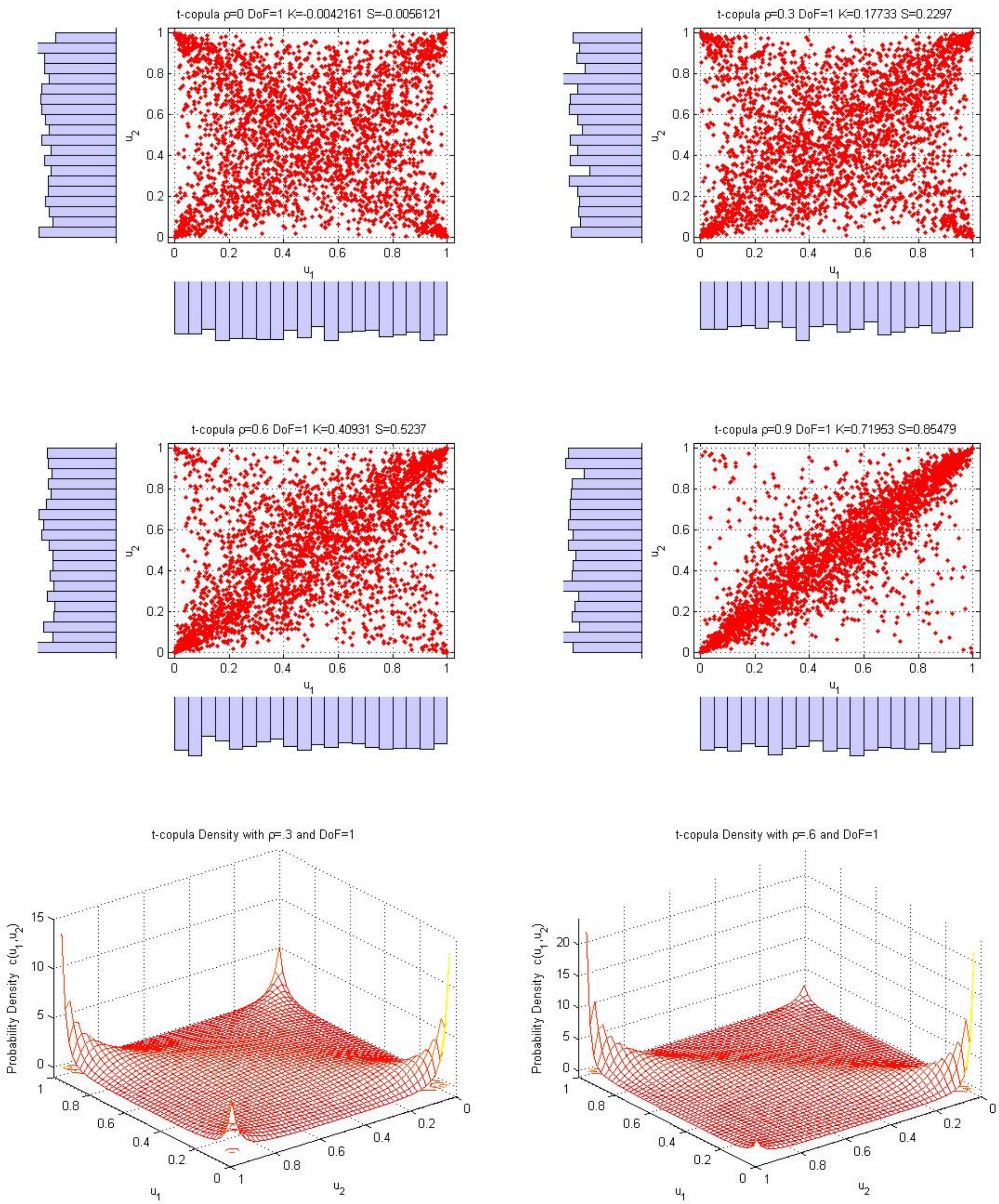


Figure 4.7: student's t bivariate copula simulation and probability density with 3000 samples
.

Blue histograms are the corresponding marginal histograms. ρ is the correlation parameter, K is Kendall's tau and S is Spearman's rho.

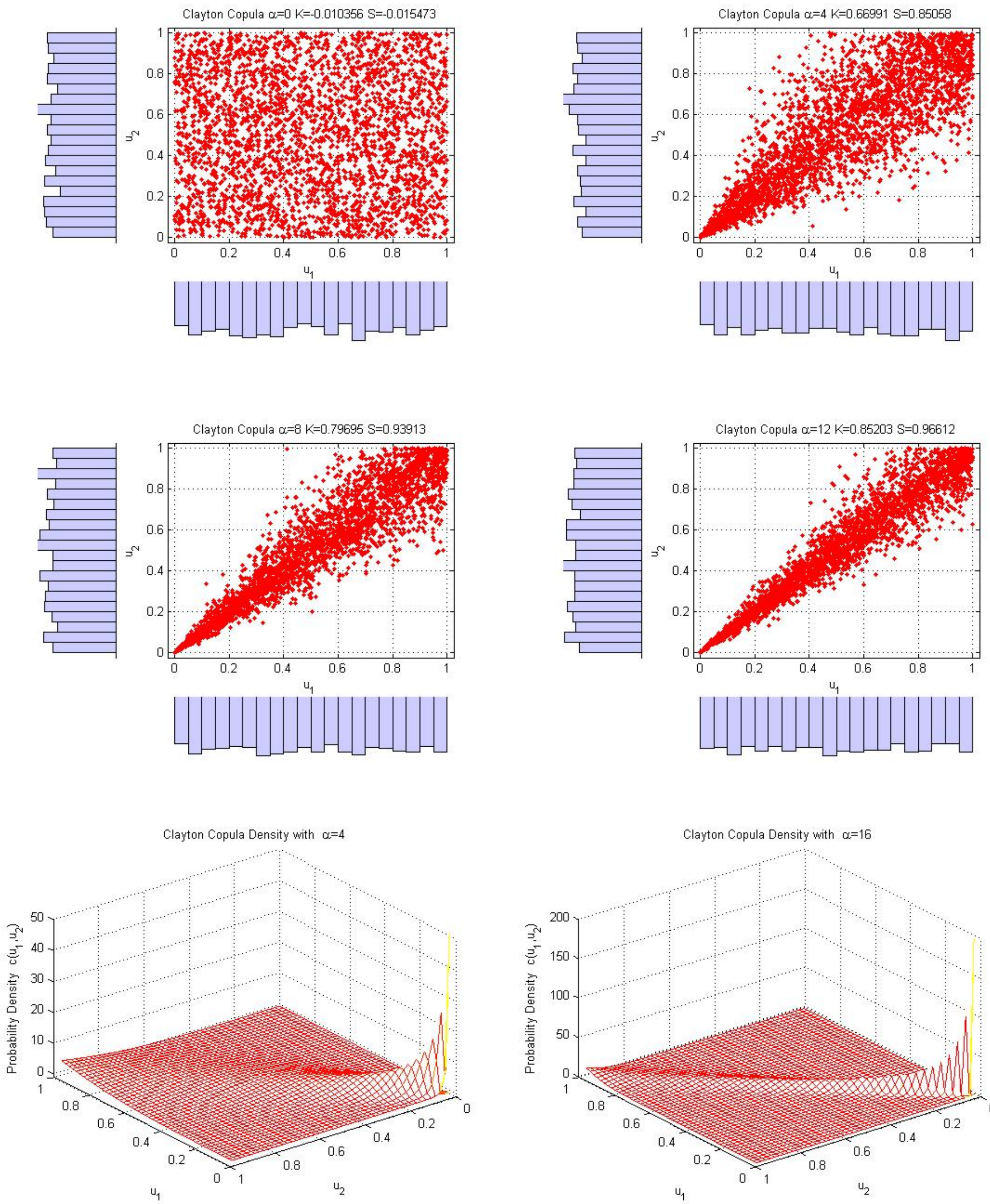


Figure 4.8: Clayton bivariate copula simulation and probability density with 3000 samples
.

Blue histograms are the corresponding marginal histograms. ρ is the correlation parameter, K is Kendall's tau and S is Spearman's rho.

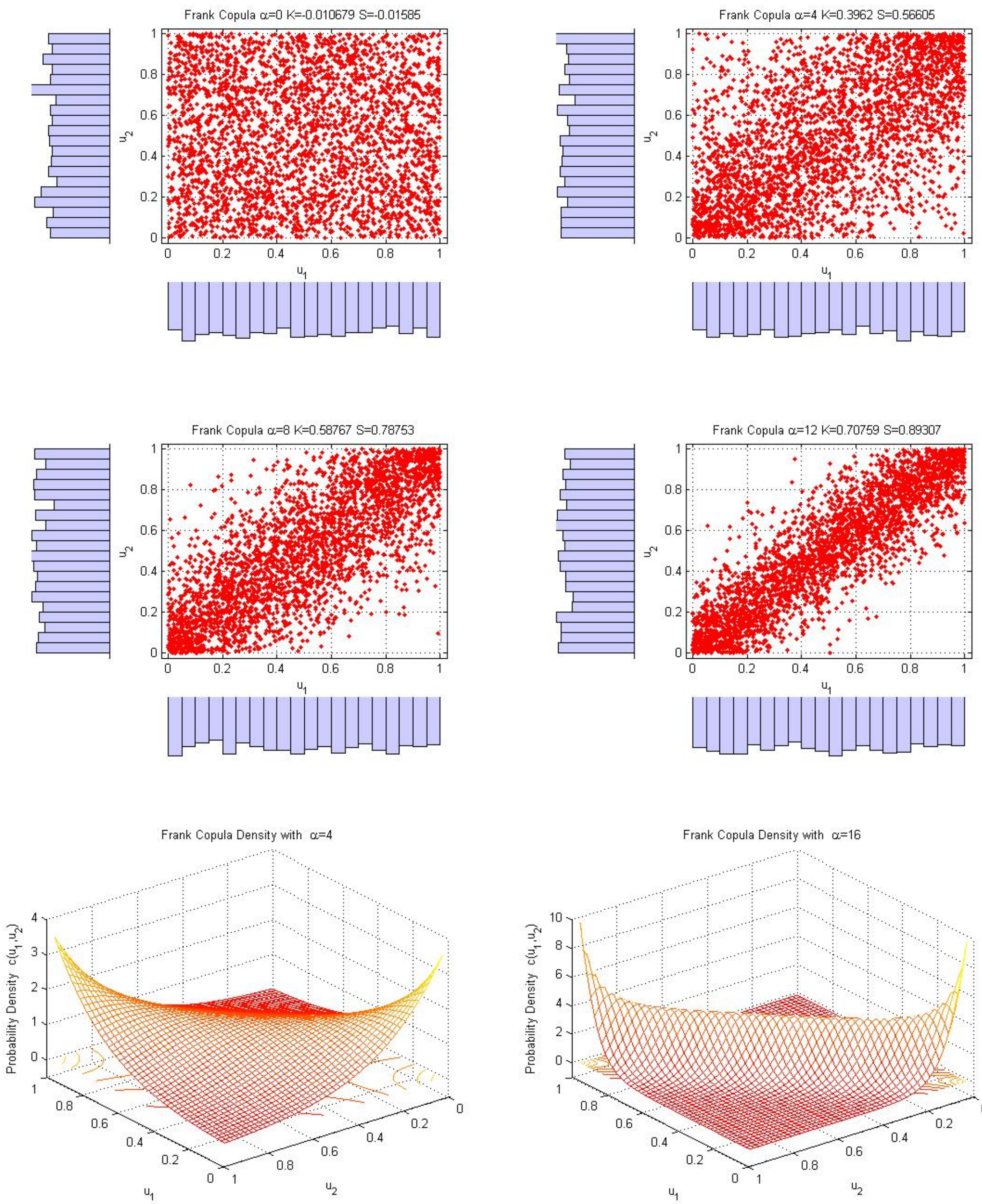


Figure 4.9: Frank bivariate copula simulation and probability density with 3000 samples .

Blue histograms are the corresponding marginal histograms. ρ is the correlation parameter, K is Kendall's tau and S is Spearman's rho.

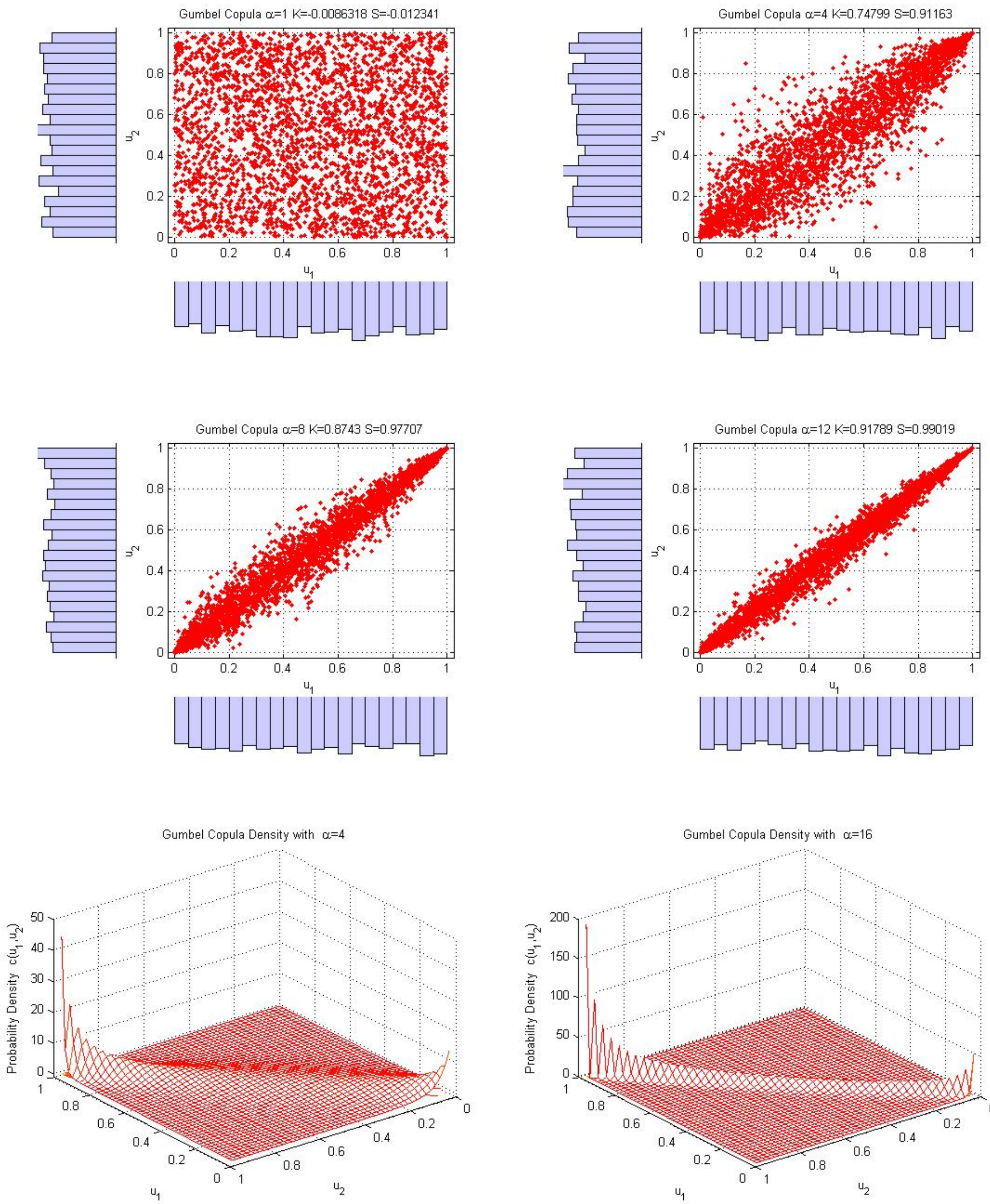


Figure 4.10: Gumbel bivariate copula simulation and probability density with 3000 samples
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