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# FORGETFUL MAPPINGS BETWEEN TEICHMÜLLER SPACES

by

ALEKSANDAR BULATOVIC

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Frederick P. Gardiner
Chairman of Examining Committee

8/16/99
date

Executive Officer

Jozef Dodziuk

Linda Keen

Nik Lakic

Supervisory Committee

The City University of New York

# ABSTRACT

## FORGETFUL MAPPINGS BETWEEN TEICHMÜLLER SPACES

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Aleksandar Bulatovic

Adviser: Professor Frederick Gardiner

We study holomorphic motions and conformal metrics on hyperbolic Riemann surfaces. We state and prove the extension of holomorphic motions theorem for arbitrary hyperbolic Riemann surface. We then use the results obtained to compare the hyperbolic metric with the metric induced by extremal point shift mappings. We show that the lengths, induced by these metrics, of any rectifiable curve are equal. At the end of this part we prove that two metrics coincide if and only if fibers of "puncture forgetful" maps are in fact Teichmüller disks. We study certain extremal problems in the Riemann sphere with the unit lattice removed. We give a useful criteria for quasiconformal map, defined on the Riemann sphere with the unit lattice removed, to be extremal.

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# Part I

# The Shift metric

In this part we study holomorphic motions and conformal metrics on hyperbolic Riemann surfaces. We state and prove the extension of holomorphic motions theorem for arbitrary hyperbolic Riemann surface. We then use the results obtained to compare the hyperbolic metric with the metric induced by extremal point shift mappings. We show that the lengths, induced by these metrics, of any rectifiable curve are equal. At the end of this part we prove that two metrics coincide if and only if fibers of "puncture forgetful" maps are in fact Teichmüller disks.

## 1 Teichmüller Space Preliminaries

Let $R$ be a hyperbolic Riemann surface and $D$ be the unit disk. There is a Fuchsian group $\Gamma$ acting freely and properly discontinuously on $D$ such that $D/\Gamma$ is biholomorphically isomorphic to $R$. We denote the natural projection of $D \longrightarrow D/\Gamma$ by $p$. $p$ is a regular (unbranched) covering.

Let $\Omega$ be a plane region. A sense-preserving homeomorphism $f$ with locally integrable distributional derivatives is $K$-quasiconformal if its derivatives satisfy the inequality

$$|f_{\bar{z}}| \leq \frac{K-1}{K+1}\,|f_z| \qquad \text{almost everywhere in } \Omega$$

We denote by $K(f)$ or usually just by $K$ the least $L$ for which $f$ is $L$-quasiconformal. Let $R_1$ and $R_2$ be two Riemann surfaces and let $K \geq 1$. By definition, the homeomorphism $f : R_1 \longrightarrow R_2$ is $K$-quasiconformal if for every choice of holomorphic charts $g$ and $h$ on $R_1$ and $R_2$ the composition map $h \circ f \circ g^{-1}$ is $K$-quasiconformal.

Let $f$ be a quasiconformal map defined on a plane domain $\Omega$. By definition, the complex dilatation of $f$ is the function $\mu_f = f_{\bar{z}}/f_z$. This function is measurable and defined almost everywhere in $\Omega$. Thus

$$||\mu_f||_\infty = \frac{K(f)-1}{K(f)+1} < 1$$

Every measurable complex function on $\Omega$ with $||\mu_f||_\infty < 1$ is the complex dilatation for some quasiconformal mapping $f$ defined in $\Omega$. The complex dilatation $\mu$ of a quasiconformal-map $f : R_1 \longrightarrow R_2$ between Riemann surfaces is a tensor such that $\mu(z)\frac{d\bar{z}}{dz}$ remains invariant under the change of local coordinates. For any local parameter $\varphi$ we have $\mu^{\varphi}(\varphi(z))\frac{\overline{\varphi'(z)}}{\varphi'(z)} = \mu(z)$. This form is also called Beltrami differential. One may easily verify that $|\mu|$ is a well defined function on a Riemann surface. We now define $Belt(R)$ to be the space of essentially bounded Beltrami differentials i.e. $\left\{\mu(z)\frac{d\bar{z}}{dz} \mid ess\sup|\mu| = ||\mu||_\infty < \infty\right\}$. $Belt(R)$ is a complex Banach space. The open unit ball in $Belt(R)$ is denoted by $M(R)$. The elements of $M(R)$ are called Beltrami coefficients because every Beltrami coefficient is the coefficient of some quasiconformal homeomorphism $f$ satisfying the Beltrami equation $f_{\bar{z}} = \mu f_z$ equation. This statement is the second half of the following theorem.

**Theorem 1** *(Measurable Riemann mapping theorem) The complex dilatation of any quasiconformal map $f : R_1 \longrightarrow R_2$ belongs to $M(R_1)$. Conversely, given any Beltrami differential in $M(R_1)$ there is a Riemann surface $R_1^{\mu}$ with the same underlying topological space as $R_1$ such that the identity map from $R_1$ to $R_1^{\mu}$ has Beltrami coefficient $\mu$. Moreover if $\mu$ is the complex dilatation of $f : R_1 \longrightarrow R_2$ , then $f$ is a conformal mapping of $R_1^{\mu}$ onto $R_2$.*

The Teichmüller space $Teich(R)$ of a Riemann surface $R$ is the set of equivalence classes of quasiconformal mappings defined on $R$. Two maps $f_1 : R \longrightarrow R_1$ , $f_1 : R \longrightarrow R_2$ are equivalent if and only if there is a conformal map $c : R_1 \longrightarrow R_2$ such that the self homeomorphism $f_2^{-1} \circ c \circ f_1$ is homotopic to the identity map and the homotopy fixes every point on the boundary of $R$. Every such homotopy can be in fact replaced by an isotopy through quasiconformal mappings see [8]. Given any isotopy $f_t$, $t \in [0,1]$ of quasiconformal self-mappings of $R$ one can lift to an isotopy $\tilde{f}_t$ of quasiconformal self-mappings of the unit disk $D$. Since quasiconformal mappings are Hölder continuous every quasiconformal self-map of $D$ extends uniquely continuously to the closure $\bar{D}$. Thus for every $t \in [0,1]$ quasiconformal mapping $\tilde{f}_t$ extends continuously and uniquely to the closure $\bar{D}$. Moreover if this isotopy $\tilde{f}_t$ is an isotopy given in the definition above then $\tilde{f}_t(z) = z$ for every $z$ such that $|z| = 1$ after normalization.

Quasiconformal mappings that are equivalent to the identity map are called trivial. The following lemma describes trivial quasiconformal maps via their lifts to the universal covering space

**Lemma 2** *A quasiconformal mapping $f$ is trivial in the $Teich(R)$ if and only if it has a lift to a map such that the extension of this lift fixes each point on the boundary of the unit disk.*

If $\mu_f = \mu_g$ then the composition map $f \circ g^{-1}$ has a Beltrami coefficient that is identically equal to 0 i.e. on $R$. We conclude that $f \circ g^{-1}$ is a conformal map and that $[f] = [g]$. Theorem 1 shows that there is a surjective map from $M(R)$ to $Teich(R)$ that sends $\mu$ to the equivalence class of the identity map from $R$ to $R^{\mu}$. Thus we have a well defined map from $M(R) \to Teich(R)$ given by

$$\Phi(\mu_f) = [f]$$

Therefore we can define equivalence relation on $M(R)$ by stating that two Beltrami differentials $\mu_1$ and $\mu_2$ are equivalent if and only if $\Phi(\mu_1) = \Phi(\mu_2)$. $M(R)$ has its natural complex structure as the open unit ball in $L_\infty(R)$.This complex structure yields quotient complex structure on the Teichmüller space $Teich(R)$ since $M(R)$ has a complex structure as the open unit ball in a complex Banach space. In fact one of the fundamental results in the Teichmüller theory is the following

**Theorem 3** *The space $Teich(R)$ has a unique complex manifold structure such that* *the map $\Phi$ from $M(R)$ to $Teich(R)$ is holomorphic and every point of $M(R)$ is the image of some locally defined holomorphic cross section of $\Phi$.*

On every complex manifold there is a way to define an intrinsic pseudometric with the property that every holomorphic map is a weak contraction. The largest pseudometric with such property is called Kobayashi's pseudometric. Kobayashi's pseudometric on $T(R)$ turns out to be a complete metric, and we denote it by $d_k$.

On the other hand Teichmüller metric on $T(R)$ is defined by

$$d_T([f_1],[f_2]) = \inf \frac{1}{2} \log K(g) \tag{1}$$

where the infimum is taken over all quasiconformal mappings $g$ such that $[g \circ f_1] = [f_2]$. That the mappings $g$ that achieve the minimum are called the extremal quasiconformal mappings. The infimum in (1) is always achieved. This is a consequence of the following theorem.

**Theorem 4** *Let $\{f_n\}$ be a sequence of quasiconformal homeomorphisms of the extended complex plane $\hat{\mathbb{C}}$, normalized to fix 0,1 and $\infty$. Let $K(f_n)$ be the dilatation of $f_n$ and assume $K(f_n) \leq K_0$ for every $n$. Then there is a subsequence of $f_n$ converging uniformly on compact subsets of $\mathbb{C}$ to a normalized quasiconformal map $f$. Moreover $K(f) \leq K_0$.*

The proof of the previous theorem can be found in [15].

The next theorem relates the Kobayashi's and Teichmüller's metric.

**Theorem 5** *Teichmüller's metric on $Teich(R)$ is complete and equal to Kobayashi's metric.*

Another important tool in the study of Teichmüller theory are integrable holomorphic quadratic differentials. Quadratic differentials are tensors $\varphi$ such that the form $\varphi(z)\,dz^2$ is invariant under the change of complex coordinates. This means that for two local parameters $z$ and $\zeta$ we have $\varphi^{\zeta}(\zeta(z))(\zeta'(z))^2 = \varphi^{z}(z)$ whenever such composition makes sense. We define $Q(R)$ to be the Banach space of integrable holomorphic quadratic differentials i.e. those $\varphi(z)\,dz^2$ such that $\varphi^{z}(z)$ is holomorphic in local parameter $z$ and the norm

$$||\varphi|| = \int_R |\varphi|$$

is finite. An important result in theory of quadratic differentials is the main inequality of Reich and Strebel

**Theorem 6** *Let $R$ be a hyperbolic Riemann surface and let $\varphi$ be an integrable holomorphic differential. Let $f$ be quasiconformal self-mapping of $R$ which is homotopic to the identity and $\mu$ be the Beltrami coefficient of $f$. Then*

$$||\varphi|| \leq \int_R |\varphi| \frac{\left|1-\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2}$$

There is a natural pairing between $Q(R)$ and $Belt(R)$. This pairing is defined by $<\varphi,\mu>=\int_R \mu\varphi$. It is well defined since $\mu\varphi$ is an integrable 2-form on $R$. From the Hahn-Banach and Riesz representation theorem one shows that the dual space $Q(R)^* = Belt(R)/N(R)$ where $N(R)$ is the space of infinitesimally trivial Beltrami differentials. Beltrami differential $\mu \in Belt(R)$ is infinitesimally trivial if $\int_R \mu\varphi = 0$ for all $\varphi \in Q(R)$. These differentials are called infinitesimally trivial because of the following theorem.

**Theorem 7** *$\mu$ is an infinitesimally trivial Beltrami differential if, and only if, there exists a holomorphic curve $\sigma_t$ of trivial Beltrami differentials for which*

$$||\sigma_t - t\mu||_\infty = O\left(t^2\right)$$

see for example [2]

## 2 Extending Holomorphic Motions

A holomorphic motion $h_t(z)$ of points $z$ of a set $\Lambda$ contained in the extended complex plane $\bar{C}$ is a mapping $h_t$ defined for each $|t| < \epsilon$, $\epsilon > 0$ with the following properties

- $h_0(z) = z$ whenever $z \in \Lambda$.
- $h_t$ is an injection for each $|t| < \epsilon$.
- $h_t(z)$ is a holomorphic function of $t$ for each fixed $z \in \Lambda$.

Although no continuity is assumed the remarkable lambda lemma of Mañe, Sad and Sullivan [16] asserts that $(t, z) \longrightarrow h_t(z)$ is continuous and has unique extension to a holomorphic motion of the closure $\bar{\Lambda}$ and each mapping $z \rightarrow h_t(z)$ has a quasiconformal extension to $\bar{C}$. Holomorphic motions have found many applications to complex dynamics and Teichmüller theory. However, it was not known if every such motion can be extended to a holomorphic motion of the extended plane $\bar{\mathbf{C}}$. The existence of a local extension was first obtained by Sullivan and Thurston in [27]. They proved that every holomorphic motion defined for $|t| < 1$ can be extended to a holomorphic motion of the whole Riemann sphere, defined for $|t| < \epsilon_0$ where $\epsilon_0$ is the universal constant independent of the motion itself. This result was improved by Bers and Royden in [4], who obtained a canonical extension over $|t| < \frac{1}{3}$. The question as to whether the extension could be defined over the full disk $|t| < 1$ is solved affirmatively by Slodkowski in [20].

**Theorem 8** *(Slodkowski) Let $D$ be the open unit disk and $(t, z) \longrightarrow h_t(z) : D \times \Lambda \longrightarrow \bar{C}$ be a holomorphic motion. Then there is a holomorphic motion $(t, z) \longrightarrow H_t(z) : D \times \bar{C} \longrightarrow \bar{C}$ such that $H_t(z) = h_t(z)$ for all $z \in \Lambda$ and $t \in D$. Moreover $z \longrightarrow H_t(z)$ is quasiconformal map and $K(H_t) \leq \frac{1+|t|}{1-|t|}$.*

Many authors have considered the previous results simply as results about holomorphic motions of the Riemann sphere. My purpose here is to state and prove a similar result for arbitrary hyperbolic Riemann surface. One problem that arises from such considerations is the problem of how to deal with the boundary of the underlying Riemann surface. We will prove that every holomorphic motion $h_t$ of a non compact set must fix every limit point of this set located on the ideal boundary. We first proceed with the following definition.

A holomorphic motion $(t, z) \longrightarrow h_t(z) : D \times \Lambda \longrightarrow R$ of a set $\Lambda$ contained in the same Riemann surface $R$ is a mapping $h_t$ defined for each $|t| < 1$ with the following properties

- $h_0(z) = z$ whenever $z \in \Lambda$
- $h_t$ is one to one selfmapping of $R$ for each $|t| < 1$
- $h_t(z)$ is a holomorphic function of $t$ for each fixed $z \in \Lambda$

Here is the result about motion of a set on Riemann surface.

**Theorem 9** *Let $\Delta = \{|z| < 1\}$ and $D = \{|t| < 1\}$. Assume $(t, z) \longrightarrow h_t(z) : D \times \Lambda \longrightarrow R$ is a holomorphic motion. Then there exists a holomorphic motion $(t, z) \longrightarrow H_t(z)$ : such that $H_t(z) = h_t(z)$ for all $z \in \Lambda$. Moreover, it is possible to choose $H_t(z)$ so that $H_t$ has a lift $\tilde{H}_t : D \times \Delta \longrightarrow \Delta$, and the unique extension of $H_t$ to $\bar{\Delta}$ fixes points $z$ on $\partial\Delta$.*

**Proof.** Let $p : \Delta \longrightarrow R$ be a covering map, $\Gamma$ the group of covering transformations, $\omega$ a fundamental domain of $\Gamma$ and $z_0 \in \Lambda$ such that $p(0) = z_0$. We can assume that $0 \in \tilde{\Lambda}$, where $\tilde{\Lambda} = p^{-1}(\Lambda)$. Given a holomorphic $h_t : D \times \Lambda \to R$, we can lift it to a map $\tilde{h}_t(\zeta)$ from $\tilde{\Lambda} \cap \omega$ to $\Delta$. Let $L(\Gamma)$ be the limit set of $\Gamma$. Define a holomorphic motion of $\Delta^c \cup \tilde{\Lambda}$ as follows

$$f_t(\zeta) = \begin{cases} \zeta & \text{if} \quad \zeta \in \Delta^c \\ \gamma\left(\tilde{h}_t(w)\right) & \text{if} \quad \zeta = \gamma(w) \text{ for some } w \in \tilde{\Lambda} \cap \omega \end{cases}$$

We now verify that $f_t(\zeta)$ is a $\Gamma$-compatible holomorphic motion. First of all, the definition shows that for a fixed $\zeta \in \Delta^c \cup \tilde{\Lambda}$, $f_t(\zeta)$ is a holomorphic function in $t$. The definition of $f_t$ shows that $f_0(\zeta) = \zeta$ for all $\zeta \in \left(\tilde{\Lambda} \cap \omega\right) \cup \Delta^c$. Let $f_t(\zeta_1) = f_t(\zeta_2)$. If $\zeta_1$ and $\zeta_2$ are in $\Delta^c$ then at any time $t$, $f_t$ is identity and therefore $f_t(\zeta_1) = \zeta_1$ and $f_t(\zeta_2) = \zeta_2$ which implies $\zeta_1 = \zeta_2$. On the other hand, If $\zeta_1$ and $\zeta_2$ are in $\tilde{\Lambda}$ and if $f_t(\zeta_1) = f_t(\zeta_2)$, then $\gamma_1\left(\tilde{h}_t\left(\gamma_1^{-1}(w_1)\right)\right) = \gamma_2\left(\tilde{h}_t\left(\gamma_2^{-1}(w_2)\right)\right)$ for $w_1, w_2 \in \tilde{\Lambda} \cap \omega$. But this means that $\tilde{h}_t(w_1) = \tilde{h}_t(w_2)$ since $\tilde{h}_t$ is a lifting map. Finally, this implies $w_1 = w_2$, since $w_1, w_2 \in \tilde{\Lambda} \cap \omega$ and since $p$ restricted to $\tilde{\Lambda} \cap \omega$ is bijection and $h_t$ is injection. Let $\zeta_1 \in \Delta^c$ and $\zeta_2 \in \tilde{\Lambda}$. If $f_t(\zeta_1) = f_t(\zeta_2)$ then $\zeta_1 = f_t(\zeta_2)$. On the other hand $|f_t(\zeta_2)| < 1$ for all time $t$ whereas $|\zeta_1| \geq 1$ and we get the contradiction. We conclude from the reasoning above that $f_t$ is injection for any $t \in D$.

Furthermore, for any $g \in \Gamma$ we have $f_t(g(\zeta)) = \gamma\tilde{h}_t(w)$ where $g(\zeta) = \gamma(w)$. We write

$$\gamma\tilde{h}_t(w) = gg^{-1}\gamma\tilde{h}_t(w) = gf_t\left(g^{-1}\gamma(w)\right) = gf_t(\zeta)$$

and this proves that $f_t$ is a $\Gamma$-compatible holomorphic motion.

We now can use the Slodkowski's theorem 8 as modified by [6, Earle, Kra and Kruskal] to extend this motion to a $\Gamma$-compatible holomorphic motion of the whole sphere. To get an extension on the Riemann surface we put $H_t = p \circ f_t$. Since $f_t$ is $\Gamma$-compatible $H_t$ is well defined and is the desired extension. ■

We state the following immediate corollary of the theorem above.

**Corollary 10** *Suppose that $\Omega$ is a plane domain and $\{z_n\}$ is a sequence that converges to a point $z \in \partial\Omega$ . If $f_t$ is a holomorphic motion of $\Omega$ to itself then $f_t(z_n) \to z$ for every $|t| < 1$.*

**Proof.** We set $\Lambda = \{z_n | n \in N\}$. The restriction of $f_t$, $f_t : \Lambda \times D \to \Omega$ is a holomorphic motion. According to the theorem above there is an extension that fixes every point on the ideal boundary of $\Omega$. Hence, the projection of this holomorphic motion $h_t$ via covering transformation fixes every point on $\partial\Omega$ see [8]. Since $h_t(z_n) = f_t(z_n)$ we conclude $f_t(z_n) \to z$ for every $|t| < 1$. ■

# 3 Teichmüller Shifts

In this section we discuss some special quasiconformal mappings called point shift mappings see [26]. Related results can be found in [24], [23], [22], [17]. We begin by stating the Strebel's [25] frame mapping theorem. We first define boundary dilatation $H([f])$ of any class $[f]$ in $Teich(R)$.

Assume $S$ is a compact subset of $R$. Let $[f] \in Teich(R)$. Define $K([f], S)$ to be the infimum of all $K(g|_{R-S})$ such that $g \in [f]$. If $S$ increases then $K([f], S)$ decreases. Define the boundary dilatation of the class $[f]$ to be $\lim_{S \subset R} K([f], S)$. Note that if $R$ is compact or compact with finitely many punctures then $H([f]) = 1$ for every $[f]$ in $Teich(R)$. We recall that $K([f])$ is $\inf_{g \in [f]} K(g)$. The following theorem is due to Strebel.

**Theorem 11** *Suppose that $H([f]) < K([f])$. Then there is $f_0 \in [f]$ whose Beltrami coefficient has the form $k\frac{|\varphi|}{\varphi}$ for some $0 < k < 1$ and $\varphi$ is an integrable holomorphic quadratic differential on $R$. Moreover $k$ and $\varphi$ are uniquely determined (up to multiplication by a positive constant) by the equivalence class $[f]$, and $k\frac{|\varphi|}{\varphi}$ is the Beltrami coefficient of the unique extremal representative of the Teichmüller equivalence class $[f]$.*

Let $R$ be a Riemann surface and let $z_0$ and $z_1$ two points on $R$. A shift from $z_0$ to $z_1$ is a quasiconformal selfmapping $f$ of $R$ such that $f(z_0) = z_1$ and $f$ is trivial in $Teich(R)$. A Teichmüller shift from $z_0$ to $z_1$ is an extremal quasiconformal map among all shifts from $z_0$ to $z_1$ i.e. extremal among all quasiconformal mappings $g$ for which $g(z_0) = z_1$ and $g$ is homotopic to the identity on $R$.

**Theorem 12** *Let $R$ be a hyperbolic Riemann surface, and $z_0$ and $z_1$ be two arbitrary distinct points in $R$. There is a Teichmüller shift from $z_0$ to $z_1$ such that its Beltrami coefficient has the form $k\frac{|\varphi|}{\varphi}$ for some $0 < k < 1$ where $\varphi$ is an integrable holomorphic quadratic differential on $R\setminus\{z_0\}$ with $||\varphi|| = 1$ and $\varphi$ has a simple pole at $\{z_0\}$.*

**Proof.** Denote $R\setminus\{z_0\}$ by $\dot{R}$. Let $f$ be a Teichmüller shift from $z_0$ to $z_1$. The quasiconformal mapping $f$ has the smallest possible dilatation among all shifts from $z_0$ to $z_1$. Since $f$ is trivial in $Teich\,(R)$ then there is an isotopy $F_t$, $t \in [0,1]$ through quasiconformal mappings such that $F_0 = id$ and $F_1 = f$. Let $\alpha(t)$ be the path from $z_0$ to $z_1$ defined by $\alpha(t) = F_t(z_0)$ for $t \in [0,1]$.

For every $t \in [0,1]$ there is $\delta(t)$ such that one can place a closed hyperbolic disk of radius $\delta(t)$ into $R$ centered at $\alpha(t)$. Since the image of $\alpha$ is a compact set there are finitely many such disks whose union covers the path $\alpha$. We may assume that all disks $\Delta_i$ are having the same radius $\delta = \min_{t\in[0,1]} \delta(t)$ and that they are centered at $\{\alpha(t_i)\}$, $0 = t_0 \leq t_1 \leq .... \leq t_n = 1$. We can take further refinement of this partition so that each closed disk $\Delta_i$ around $\alpha(t_i)$ contains $\alpha(t_{i+1})$. For each disk $\Delta_i$ we have a conformal map $c_i$ that maps it onto the closed unit disk $D = \{|\zeta| \leq 1\}$. We may assume $c_i(\alpha(t_i)) = 0$.

Consider the following holomorphic motion

$$G_t(z) = \begin{cases} t & \text{if } \; z = 0 \\ z & \text{if } \; |z| \geq 1 \end{cases}$$

This motion can be extended to a holomorphic motion of the whole Riemann sphere. In particular one obtains a holomorphic motion $G_t$ that fixes every point on the boundary of the unit disk. Now for each $i$, we define isotopies $g_t^i$ on $R$ for $t \in [t_i, t_{i+1}]$ as follows.

$$g_t^i(z) = \begin{cases} c_i^{-1}\left(G_{c_i(\alpha(t))}(c_i(z))\right) & \text{if } \; z \in \Delta_i \\ z & \text{if } \; z \notin \Delta_i \end{cases}$$

Thus

$$g_{t_i}^i(z) = c_i^{-1}\left(G_{c_i(\alpha(t_i))}(c_i(z))\right) = z$$

and

$$g_{t_{i+1}}^i(z) = c_i^{-1}\left(G_{c_i(\alpha(t_{i+1}))}(c_i(z))\right)$$

Set $g_i(z) = g_{t_i+1}^i(z)$. Note now that

$$g_i(\alpha(t_i)) = g_{t_i+1}^i(\alpha(t_i)) = \alpha(t_{i+1}).$$

and that $g_i(z)$ is identity off of $\Delta_i$. Let $g = g_n \circ g_{n-1} \circ ... \circ g_1$. We conclude the following

$$\begin{aligned} g(z_0) &= g_n(g_{n-1}(...g_0(\alpha(0)))) \\ &= g_n(g_{n-1}(...g_1(\alpha(t_1)))) \\ &= ... \\ &= g_n(\alpha(t_{n-1})) \\ &= \alpha(t_n) \\ &= z_1 \end{aligned}$$

and $g(z) = z$ whenever $z \notin \cup_{i=0}^n \Delta_i$. We further conclude that $g$ is a shift from $z_0$ to $z_1$ and that $\mu_g = 0$ for every $z \notin \cup_{i=0}^n \Delta_i$. Since the union $\cup_{i=0}^n \Delta_i$ is compact we conclude that $H(g) = 1$. In order to apply the frame mapping theorem it remains to show that $g$ is in the same homotopy class as $f$ i.e. that they are in the same Teichmüller class in $Teich\left(\dot{R}\right)$.

We now prove that $g$ and $f$ are in the same Teichmüller class in $Teich\left(\dot{R}\right)$. Define $g_t(z)$ as follows

$$g_t(z) = \begin{cases} g_t^n(g_{n-1}(...(g_1(z)))) & t \in [t_{n-1}, t_n = 1] \\ g_t^{n-1}(g_{n-2}(...(g_1(z)))) & t \in [t_{n-2}, t_{n-1}] \\ \vdots & \\ g_t^1(z) & t \in [0 = t_0, t_1] \end{cases}$$

We observe that $g_t(z)$ fixes points outside $\cup_{i=0}^n \Delta_i$ and that $g_0 = id$, $g_1 = g$. Another important observation is that for $t \in [t_{i-1}, t_i]$ we have

$$\begin{aligned} g_t(z_0) &= g_t^i(g_{i-1}(...(g_1(z_0)))) \\ &= g_t^i(g_{i-1}(...(g_1(\alpha(0))))) \\ &= g_t^i(\alpha(t_{i-1})) \\ &= \alpha(t) \\ &= F_t(z_0) \end{aligned}$$

Let $h_t = F_t^{-1} \circ g_t$. We get that $h_0$ is the identity map and $h_1 = f^{-1} \circ g_1$. Moreover $h_t(z_0) = F_t^{-1} \circ g_t(z_0) = F_t^{-1} \circ F_t(z_0) = z_0$. Since $g_t$ and $F_t$ are fixing the boundary points of $R$, so is $h_t$. Therefore we proved that $f$ and $g_1$ are in the same Teichmüller class of $Teich\left(\dot{R}\right)$. It now follows that $H([f]) = 1$ since we had earlier that $H(g_1) = 1$.

Since $z_0$ is different from $z_1$, $K$ has to be bigger then 1. Thus $H < K$ and the frame mapping theorem can now be applied. Since $K > 1$ we get a unique $h \in [f]$ as an element in $Teich\left(\dot{R}\right)$ whose Beltrami differential has the form $k\frac{|\varphi|}{\varphi}$ for some $0 < k < 1$ and $\varphi$ is an integrable holomorphic quadratic differential on $R\backslash\{z_0\}$. Since $f$ is an extremal map among all shifts it is also extremal in its class. It follows now that $f = h$.

We recall here that an arbitrary quasiconformal map defined on a Riemann surface with a puncture extends uniquely over that puncture.

If $\varphi$ does not have a pole at $z_0$ then $\varphi$ is holomorphic on all of $R$. Thus $\left[k\frac{|\varphi|}{\varphi}\right] \in Teich(R)$ is trivial which is impossible unless $k = 0$. This contradiction proves that $\varphi$ has a simple pole at $z_0$. ■

The proof of the previous theorem do not rule out the possibility that there may be several extremal Teichmüller shifts from $z_0$ to $z_1$. If there is more than one Teichmüller shift from $z_0$ to $z_1$ then any two such shifts necessarily represent different classes in $Teich\left(\dot{R}\right)$. Every such shift will have the Beltrami coefficient of the form $k\frac{|\varphi|}{\varphi}$ with the same $k$ and some $\varphi \in Q\left(\dot{R}\right)$. The quadratic differential $\varphi$ for which $k\frac{|\varphi|}{\varphi}$ is the extremal representative is called a point shift differential.

# 4 Invariant Metrics on Riemann Surfaces

In this section we study invariant metrics on a Riemann surface $R$. A complete Riemannian metric $d$ on a Riemann surface $R$ is an invariant metric if every conformal mapping $c$ is an isometry with respect to the metric $d$. Every hyperbolic Riemann surface $R$ carries the Poincaré metric $d_R$, and $d_R$ is the unique complex invariant metric with curvature constantly equal to $-1$. We will denote this metric simply by $d$ whenever it is clear which Riemann surface is the domain for the metric. This metric is equal to the integral of its infinitesimal form. From the Riemann surface point of view there are many other invariant metrics. Here we compare $d$ to another metric which is invariant under conformal mappings but is defined using Teichmüller shifts.

Let $R$ be a hyperbolic Riemann surface. The shift metric $d_s$ on $R$ is defined as follows. $d_s(z,w) = \frac{1}{2}\log\frac{1+k}{1-k}$ where $k\frac{|\varphi|}{\varphi}$ is the Beltrami coefficient of a Teichmüller shift from $z$ to $w$ . The well known inequality $K(f \circ g) \leq K(f)\,K(g)$ for any two quasiconformal mappings $f$ and $g$ implies $d_s$ is transitive. $K(f) = K(f^{-1})$ implies $d_s$ is symmetric. This two properties imply that $d_s$ is a pseudo metric. To show $d_s$ is a metric it remains to show $z \neq w$ implies $d_s(z,w) \neq 0$. Assume $d_s(z,w) = 0$. $K(f) = 1$ if and only if $f$ is conformal. But if $f$, the Teichmüller shift from $z$ to $w$ is conformal then it can be lifted to a conformal self mapping of the unit disk i.e. a Mobius transformation. Since $f$ is trivial in $Teich(R)$ we have that $f$ is homotopic to the identity mapping, relative to the ideal boundary, and therefore it has a lift that fixes every point on the boundary of the unit disk. We conclude that $f$ has to be the identity map, in which case $z = w$. Moreover for any conformal selfmap $c$, we have $d_s(c(z), c(w)) = d_s(z,w)$. To prove this, let $f$ be the Teichmüller $z$ to $w$ shift whose Beltrami coefficient is $k\frac{|\varphi|}{\varphi}$. The Beltrami coefficient of $f \circ c^{-1}$ is $k\frac{|\varphi \circ c^{-1}|}{\varphi \circ c^{-1}} \times \left(\frac{|(c^{-1})'|}{(c^{-1})'}\right)^2$ and so is a Beltrami coefficient of $c \circ f \circ c^{-1}$ since postcomposition by conformal maps does not change the complex dilatation. This implies that $c \circ f \circ c^{-1}$ is a Teichmüller $c(z)$ to $c(w)$ shift.

Our next proposition compares the $d_s$−metric to the Poincaré metric.

**Theorem 13** *For any hyperbolic Riemann surface $R$, the $d_s$−metric is always less then or equal to the Poincaré metric. Moreover, if there are two points $p_1$ and $p_2$ in $R$ for which $d_s(p_1,p_2) = d(p_1,p_2)$, then the two metrics are identically equal on $R$.*

**Proof.** Let $p : D \longrightarrow R$ be a covering map. We think of $p$ as a holomorphic motion of $p(0) = z$ on $R$. Theorem 9 shows that this motion can be extended to a holomorphic motion $f_t$ of the whole Riemann surface such that each point on the ideal boundary remain fixed at any time $t$. Moreover the Beltrami differentials $\mu_t$ of $f_t$ depend holomorphically on $t$ and $[\mu_t] = [0]$ in $Teich(R)$. By Schwarz's lemma $||\mu_t||_\infty \leq |t|$ and the definition of $d_s$ shows that $\tanh(d_s(z,w)) \leq ||\mu_t||_\infty$ for any $t$ such that $p(t) = w$. Note that if $p(t) = w$ then $f_t(z) = f_t(p(0)) = f^{\mu(t)}(p(0)) = w$ and so $f^{\mu(t)}$ competes for Teichmüller $z$ to $w$ shift. Hence, $d_s(p(0), z) \leq |t|$ for any $t$ with $p(t) = w$. This implies $d_s(z,w) \leq d(z,w)$. Using Schwarz's lemma we conclude that equality for a pair of points on $R$ implies that two metrics are identically equal on $R$ ■

The following lemma tells us that the two metrics are equal on the thrice punctured sphere. For the proof see [1].

**Lemma 14** *If* $R = \hat{C}\setminus\{-1,1,\infty\}$ *then* $d_s = d$

**Remark 1** *Since the fiber over* $[0]$ *in* $Teich\left(\hat{C}\setminus\{-1,1,\infty\}\right)$ *is a Teichmüller disk, this lemma is a consequence of theorem 23 proved in section 7.*

Let $R = D$. From the work of Teichmüller [28] it follows that

$$d_s(z_1,z_2) = \phi(d(z_1,z_2)) \text{ where } \phi(t) = 4\tanh^{-1}\left(e^{-\nu(\tanh(t))}\right)$$

where $\nu(s)$ denotes the modulus for $0 \leq s < 1$ of the unit disk $D$ slit from 0 to $s$. We know from Theorem 13 that $d_s \leq d$. In fact we have a strict inequality. One can show by direct calculation that $\phi(t)/t$ is strictly decreasing in $0 < t < \infty$ see for example [12, Gehring] . Here I present a different proof using quadratic differentials.

**Theorem 15** *If* $R = D$ *then* $d_s < d$.

**Proof.** Since both metrics are invariant under Möbius transformations it is enough to show that $d_s(0,r) < d(0,r) = \frac{1}{2}\log\frac{1+r}{1-r}$ for every $r \in (0,1)$ . We take a convex fundamental region $\omega$ for $\hat{C}\setminus\{-1,1,\infty\}$ such that $(0,1) \subset \omega$. For the sake of simplicity assume $p(0) = 0$ where $p$ is a covering map $D \longrightarrow \hat{C} \setminus \{-1, 1, \infty\}$ with $\Omega$ as a fundamental region. Let $f$ be the Teichmüller 0 to $p(r)$ shift on $\hat{C} \setminus \{-1, 1, \infty\}$. We know that Beltrami coefficient of $f$ is of the form $\mu_f = k\frac{|\varphi|}{\varphi}$ for $\varphi \in Q\left(\hat{C} \setminus \{-1, 0, 1, \infty\}\right)$. Lift this map to a map $\tilde{f}$ that sends 0 to $r$. Mapping $\tilde{f}$ is a quasiconformal map whose Beltrami coefficient satisfies $||\tilde{\mu}|| = \left|\left|\mu_f\right|\right| = d_{\hat{C} \setminus \{-1,1,\infty\}}(0, p(r))$. The last equality follows from the previous observation that $d_s$ is equal to $d$ on any thrice punctured sphere. On the other hand $d_{\hat{C} - \{-1,1,\infty\}}(0, p(r)) = d(0, r)$ and consequently $||\tilde{\mu}|| = \frac{1}{2}\log\frac{1+r}{1-r}$. Mapping $\tilde{f}$ fixes points on $\partial D$ since the unique extension of $f$ to the whole Riemann sphere fixes $\{-1, 1, \infty\}$. Therefore quasiconformal map $\tilde{f}$ is a candidate for Teichmüller 0 to $r$ shift. We also have $\tilde{\mu} = \left(\frac{1}{2}\log\frac{1+r}{1-r}\right) \cdot \frac{|\tilde{\varphi}|}{\tilde{\varphi}}$ where $\tilde{\varphi}$ is a holomorphic lift of $\varphi$. Since $\tilde{\varphi}$ is not integrable it follows that $\tilde{f}$ cannot be a Teichmüller shift. Therefore $d_s(0, r) < ||\tilde{\mu}|| = \frac{1}{2}\log\frac{1+r}{1-r}$. ■

It is going to be important for us to be able to compare the two metrics in both ways. So far we know that the shift metric is always smaller then the Poincaré metric. In this paragraph we consider the question of the local equivalence of the two metrics. The following lemma asserts that the two metrics are uniformly locally equivalent.

We recall here that the injectivity radius $\tau$ of the Riemann surface $R$ is the greatest radius such that the euclidean disk $D(0, \tau)$ can be conformally embedded in $R$ for every point $z \in R$ being the image of 0 under a conformal mapping $c$.

**Lemma 16** *Let $R$ be arbitrary hyperbolic Riemann surface, $z$ be a point on it and $m$ be a real number such that $m < 1$. There is a real number $\epsilon_0 > 0$ smaller then the injectivity radius and otherwise independent of $R$ and $z$ such that for any $w$ for which $d_R(z, w) \leq \epsilon_0$ we have*

$$m \cdot d_R(z, w) \leq d_{s,R}(z, w) \leq d_R(z, w)$$

*where $d_R$ is the Poincaré metric and $d_{s,R}$ is the shift metric on $R$.*

**Proof.** We first denote the Poincaré metric on the unit disk by $d$ and the shift metric in the unit disk by $d_s$. Let $p : D \to R$ be a covering map. We know that

$d_s(z_1, z_2) = \phi(d(z_1, z_2))$ where $\phi(t) = 4\tanh^{-1}\left(e^{-\nu(\tanh(t))}\right)$ and $\nu(s)$ denotes the modulus for $0 \leq s < 1$ of the unit disk $D$ slit from 0 to $s$. Gehring proved in [12] that $\phi(t)/t$ is strictly decreasing in $0 < t < \infty$ . Also note that $\phi(t)/t \longrightarrow 1$ as $t \longrightarrow 0^+$.

Since $\phi(t)$ is a continuous function there is $\epsilon_0 > 0$ such that $\phi(t)/t \geq m$ for every $t \leq \epsilon_0$. If $\epsilon_0$ is not smaller then the injectivity radius then take $\epsilon_0$ to be equal to the injectivity radius.

If $w$ and $z$ be the points on a Riemann surface $R$ such that $d_R(z, w) \leq \epsilon_0$ then there is a Teichmüller shift $f$ that realizes the distance $d_{s,R}(z, w)$ . Note that $f(z) = w$. We first lift this mapping to a quasiconformal selfmapping $\tilde{f}$ of the unit disk $D$. Then we choose $\tilde{z}$ such that $p(\tilde{z}) = z$. Put $\tilde{w} = \tilde{f}(\tilde{z})$. We have $p(\tilde{w}) = w$ by the definition of $\tilde{f}$. The hyperbolic distance from $\tilde{z}$ to $\tilde{w}$ is the same as hyperbolic distance on $R$ from $z$ to $w$. We have $d_R(z, w) = d(\tilde{z}, \tilde{w})$. The quasiconformal mapping $\tilde{f}$ competes for a Teichmüller shift from $\tilde{z}$ to $\tilde{w}$. Hence, $d_s(\tilde{z}, \tilde{w}) \leq \frac{1}{2}\log\frac{1+||\mu_{\tilde{f}}||_\infty}{1-||\mu_{\tilde{f}}||_\infty}$. Since $||\mu_{\tilde{f}}||_\infty = ||\mu_f||_\infty$ , we get

$$d_s(\tilde{z}, \tilde{w}) \leq \frac{1}{2}\log\frac{1+||\mu_{\tilde{f}}||_\infty}{1-||\mu_{\tilde{f}}||_\infty} = \frac{1}{2}\log\frac{1+||\mu_f||_\infty}{1-||\mu_f||_\infty} = d_{s,R}(z, w)$$

On the other hand $d(\tilde{z}, \tilde{w}) \leq \epsilon_0$ and therefore $\phi(d(\tilde{z}, \tilde{w}))/d(\tilde{z}, \tilde{w}) \geq m$. This implies $d_s(\tilde{z}, \tilde{w}) \geq m \cdot d(\tilde{z}, \tilde{w})$. Combining all the inequalities we arrived at

$$d_{s,R}(z, w) \geq d_s(\tilde{z}, \tilde{w}) \geq m \cdot d(\tilde{z}, \tilde{w}) = m \cdot d_R(z, w)$$

Combining this with theorem 13 we obtain the desired conclusion. ■

The following is an immediate corollary of the lemma above.

**Corollary 17** *Given a Riemann surface $R$ and two points $z$ and $w$ sufficiently close to each other. There is a unique Teichmüller shift $f$ from $z$ to $w$ such that its Beltrami coefficient has the form $k\frac{|\varphi|}{\varphi}$ for some $0 < k < 1$ where $\varphi$ is an integrable holomorphic quadratic differential on $R\backslash\{z_0\}$ with $||\varphi|| = 1$ and has a simple pole at $\{z_0\}$.*

**Proof.** Let $\alpha$ be an arc from $z$ to $w$. Two arcs $\alpha$ and $\beta$ are equivalent if and only if $\alpha(0) = \beta(0) = z$, $\alpha(1) = \beta(1) = w$ and $\alpha$ followed by $\beta^{-1}$ is homotopic to a point.

By $[\alpha]$ we mean an equivalence class of $\alpha$ joining $z$ to $w$. The proof of the theorem 1 shows that there is a one to one correspondence between the equivalence classes $[\alpha]$ and the isotopy classes of shift maps in $R\setminus\{z\}$. Since in each class $[\alpha]$ there exists exactly one geodesic we may identified the geodesics joining $z$ to $w$ and the isotopy classes of shift maps.

Theorem 12 asserts that there is at least one Teichmüller shift $f$ from $z$ to $w$. Let $k = \tanh d_s(z,w)$. Choose $\epsilon \le \epsilon_0/2$ as in the lemma above such that for every $w$ with $d_R(z,w) \le \epsilon$ there is just one hyperbolic geodesic $\alpha$ that realizes the distance from $z$ to $w$ and such that $d_{s,R}(z,w) \ge \frac{1}{2} d_R(z,w)$. If there is more then one quadratic differential that correspond to the same Teichmüller shift problem, it must correspond to a different homotopy class say $[g]$. The length of the hyperbolic geodesic in that homotopy class is bigger then $\delta$ where $\delta$ is an injectivity radius. Let $p$ be a covering map and let $p(0) = z$. Lift the maps $f$ to $\tilde{f}$ and $g$ to $\tilde{g}$. We must have $\tilde{g}(0) \ne \tilde{f}(0) = \tilde{w}$. Note that $M \ge d(0, \tilde{g}(0)) \ge \delta$. The theorem above asserts that the two metrics are equivalent on any compact set $S$ with the constant depending on set $S$. Thus

$$d_s(0, \tilde{g}(0)) \ge m d(0, \tilde{g}(0)) \ge m\delta$$

where $m$ is fixed for as long as $M$ stays fixed. $\tilde{g}$ can not be an extremal map in the unit disk shifting 0 to $\tilde{g}(0)$ therefore

$$\frac{1}{2}\log\frac{1+||\mu_{\tilde{g}}||}{1-||\mu_{\tilde{g}}||} \ge d_s(0, \tilde{g}(0)) \ge m\delta$$

. Furthermore since $||\mu_g|| = ||\mu_{\tilde{g}}||$ we conclude

$$\frac{1}{2}\log\frac{1+||\mu_{\tilde{g}}||}{1-||\mu_{\tilde{g}}||} \ge m\delta$$

This is a contradiction to the fact that $w$ can be arbitrary close to $z$ which makes $k = ||\mu_f||$ arbitrary close to 0. ■

## 5 The length of arcs in metric $d_s$

In this section we prove that the lengths measured in the Poincaré metric and the shift metric are equal.

Let $R$ be any hyperbolic Riemann surface. An arc $\alpha$ on a hyperbolic Riemann surface $R$ is a continuous map from $I = [0, 1]$ into $R$. We define the length of an arc $\alpha$ with respect to any metric $\delta$ on $R$ to be the following

$$length\,(\alpha, \delta) = \sup \sum_i \delta\left(\alpha\left(t_i\right), \alpha\left(t_{i-1}\right)\right) \tag{2}$$

where the supremmum is taken over all partitions $P : 0 = t_0 \leq t_1 \leq ... \leq t_n = 1$ of the unit interval. By placing $d$ and $d_s$ instead of $\delta$ in the formula above, we obtain the lengths of an arc $\alpha$ with respect to Poincaré metric and with respect to the shift metric. We denote $length\,(\alpha) = length\,(\alpha, d)$ and $length_s\,(\alpha) = length\,(\alpha, d_s)$

An arc alpha is rectifiable with respect to metric $m$, if $length\,(\alpha, m)$ is finite.

**Lemma 18** *Let $R$ be any hyperbolic Riemann surface and $\alpha$ be a rectifiable arc with respect to the Poincaré metric $d$ on $R$. Then the length of the arc $\alpha$ with respect to metric $d_s$ is finite.*

**Proof.** Let $P$ be the partition $0 = t_0 \leq ... \leq t_n = 1$ of the unit interval. We set $length_s\,(\alpha, P) = \sum_{i=1}^{n} d_s\left(z_i, z_{i+1}\right)$. If $S$ is another partition of the unit interval such that $P \subset S$ then the triangle inequality for the metric $d_s$ implies $length_s\,(\alpha, P) \leq length_s\,(\alpha, S)$. On the other hand the lemma 13 implies that

$$length_s\,(\alpha, P) \leq length\,(\alpha, P) \leq length\,(\alpha) = l$$

and since arc $\alpha$ is rectifiable we have $length_s\,(\alpha, P) \leq l < \infty$. Thus the supremmum of $length_s\,(\alpha, P)$ over all partitions $P$ is finite. ■

The following theorem asserts that lengths of curves measured in the two metrics are equal.

**Theorem 19** *Let $R$ be any hyperbolic Riemann surface and $\alpha$ be a rectifiable arc on $R$. Let $length\,(\alpha)$ be the length of $\alpha$ measured in hyperbolic metric and $length_s\,(\alpha)$ be the length measured in the shift metric. We then have*

$$length_s\,(\alpha) = length\,(\alpha)$$

**Proof.** Let $m$ be any positive real number such that $m < 1$. The lemma 16 implies the existence of a number $\epsilon_0$ such that $m \cdot d_R\,(z,w) \leq d_{s,R}\,(z,w) \leq d_R\,(z,w)$ whenever $d_R\,(z,w) < \epsilon_0$. If $\alpha$ is a rectifiable curve on $R$ we conclude first that

$$m \cdot length\,(\alpha, P) \leq length_s\,(\alpha, P) \leq length\,(\alpha, P)$$

for any partition $P : 0 = t_0 \leq ... \leq t_n = 1$ such that $d_R\,(\alpha\,(t_i)\,, \alpha\,(t_{i+1})) < \epsilon_0$. We now take the supremmum over all such partitions and get

$$m \cdot length\,(\alpha) \leq length_s\,(\alpha) \leq length\,(\alpha)$$

for any positive real number $m < 1$. We conclude the proof of the theorem by letting $m \longrightarrow 1$ in the inequality above. ■.

We now want to change the point of view and treat an arc as the set of points in $R$ under some continuous mapping $\alpha$ defined on the unit interval. We denote such image by $[\alpha]$. We would now like to define the length of $[\alpha]$ and denote it by $length\,([\alpha])$. The definition of a length above can not be used since it is possible to find different parametrizations of the same arc that will produce different lengths. To get around this problem and make it independent of the parametrization we set

$$length\,([\alpha]\,, \delta) = \inf_{\alpha} length\,(\alpha, \delta)$$

where the infimum is taken over all parametrization of an arc $[\alpha]$ . The following is a simple consequence of the theorem above

**Theorem 20** *Let $R$ be any hyperbolic Riemann surface and $\alpha$ be a rectifiable arc on $R$. Let $length\,([\alpha])$ be the length of $[\alpha]$ measured in hyperbolic metric and $length_s\,([\alpha])$ be the length measured in the shift metric. We then have*

$$length_s\,([\alpha]) = length\,([\alpha])$$

**Proof.** Take the infimum of both sides in the statement of the theorem 19 ■

# 6 Continuity property of $d_{s,R}$

In this section we prove an important continuity property of metric $d_{sR}$. Suppose $R$ is a Riemann surface and that we have a sequence $\{R_n\} \subset R$, $n = 0, 1, 2, \ldots$ of Riemann surfaces with the following properties. The closure of each $R_n$ is properly contained in $R_{n+1}$ and the union $\cup_{n=0}^{\infty} R_n = R$. It seems natural to ask whether $d_{s,R_n} \to d_{s,R}$ when $n$ tends to infinity. The following theorem answers the question affirmatively.

**Theorem 21** *Suppose that $R$ is a Riemann surface and $\{R_n\} \subset R$ is a sequence of Riemann surfaces with the properties described above. Then for any two points $z$ and $w$ in $R_0$ we have $d_{s,R_n}(z,w) \to d_{s,R}(z,w)$.*

**Proof.** Let $f$ be a Teichmüller shift map that realizes $d_{s,R}(z,w)$ for $z$ and $w$ in $R$. We may assume as well that $z$ and $w$ are sufficiently close to one another so that both points belong to $R_n$. Further assume that $z$ and $w$ are so close that 12 for each $n \in N$ we have a unique quadratic differential $\varphi_n \in Q(R_n)$ with $||\varphi|| = 1$ such that $f_n = f^{k_n \frac{|\varphi_n|}{\varphi_n}}$ is the Teichmüller shift that realizes $d_{s,R_n}(z,w)$. Each $f_n$ can be extended to a quasiconformal selfmapping on $R_{n+1}$ by defining $f_n$ to be the identity mapping on $R_{n+1} \backslash R_n$. Such an extension of $f_n$ falls in to the same Teichmüller class as $f_{n+1}$ in $Teich(R_{n+1})$. Since $f_{n+1}$ is extremal in its class it follows that $k_n \geq k_{n+1}$ and consequently $k_n \geq k_{n+1} \geq \ldots \geq k$.

Since $||\varphi_n|| = 1$ and $\varphi_n$ is defined on $R_n$ by passing to a subsequence if necessary, we conclude that $\varphi_n$ converge to an integrable holomorphic quadratic differential $\psi$ uniformly on compact subsets of $R$.

We first prove that $\varphi_n$ cannot be a degenerating sequence i.e. $\psi$ is not identically equal to 0. By the proof of the theorem 12 there is a mapping $g$ with the following properties. For each $n \in R_n$ $g$ restricted to $R_n$ is a quasiconformal selfmapping. For each $n \in R_n$ the restriction of mapping $g$ to $R_n$ is in the same Teichmüller class as $f_n$.

Moreover the restriction of $g$ to $R\backslash R_0$ is the identity mapping. If $\nu$ is the Beltrami coefficient of $g$ then by theorem 6 applied to $g^{-1}\circ f_n$ we have, see for example [9]

$$1 \leq \int_{R_n} |\varphi_n| \frac{\left|1-\mu_n \frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\mu_n|^2} \frac{\left|1+\nu\theta_n \frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\nu|^2}$$

where $\mu_n$ is a Beltrami coefficient of $f_n$ and $\theta_n = \left(1-\overline{\mu_n\varphi_n}/|\varphi_n|\right)\left(1-\mu_n\varphi_n/|\varphi_n|\right)^{-1}$ for every $n \in N$. Since $\mu_n = k_n \frac{|\varphi_n|}{\varphi_n}$ we get

$$K_n \leq \int_{R_n} |\varphi_n| \frac{\left|1+\nu \frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\nu|^2}$$

where $K_n = \frac{1+k_n}{1-k_n}$ for all $n \in N$. Hence,

$$\begin{aligned} K_n &\leq \int_S |\varphi_n| \frac{\left|1+\nu \frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\nu|^2} + \int_{R\backslash S} |\varphi_n| \\ &\leq \int_S |\varphi_n| \frac{\left|1+\nu \frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\nu|^2} + 1 \end{aligned}$$

where $S$ is a large compact subset of $R_0$ and $\nu = 0$ off $S$. The sequence $K_n$ is bounded below by $K$. We get that

$$K \leq K(g) \int_S |\varphi_n| + 1$$

and since $\varphi_n$ converges to $\psi$ uniformly on $S$ we conclude

$$K \leq K(g) \int_S |\psi| + 1$$

If $\psi = 0$ then $K = 1$ which is impossible since $z$ is different from $w$. This rules out the possibility that $\psi = 0$.

If $||\psi||$ is strictly less then 1 then we could construct another degenerating sequence $\tilde{\varphi}_n = \frac{\varphi_n - \psi}{||\varphi_n - \psi||}$ for $\mu$. Consequently, $||\psi|| = 1$.

Now extend $\varphi_n$ to $R$ by setting $\varphi_n(z)\,dz^2 = 0$ for $z \in R\backslash R_n$. The sequence $\{\varphi_n\}$ consists of integrable quadratic differentials in $R$. We have the obvious inequality

$$\int_R |\varphi_n - \psi| \leq ||\varphi_n|| - ||\psi|| + 2\int_{R-S} |\psi| + 2\int_S |\varphi_n - \psi|$$

Since both $||\varphi_n||$ and $||\psi||$ are equal to 1, this inequality implies that $\varphi_n$ converges to $\psi$ in norm.

Let $k_n \longrightarrow k_\infty$. We conclude that a subsequence of $f_n$ converges uniformly to $f_\infty$ whose Beltrami coefficient is equal to $k_\infty \frac{|\psi|}{\psi}$. But $f_\infty$ is a Teichmüller shift from $z$ to $w$ and in the same Teichmüller class as $f$ with respect to $Teich\left(\dot{R}\right)$. Since $f_\infty$ is of the Teichmüller form it follows that $f_\infty$ is uniquely extremal and therefore equal to $f$.

In particular, we obtain $\psi = \varphi$ and $k_\infty = k = d_{s,R}(z,w)$. ■

Similar techniques can be found in [11], [10] and [13].

# 7 Fibers of puncture forgetful mappings

Let $f$ be a quasiconformal mapping defined on $\dot{R}$, where $\dot{R} = R \backslash \{z\}$. We know there is a unique extension of $f$ across the puncture to the whole Riemann surface $R$. We denote such extension by $\tilde{f}$.

We now define $\pi$ to be the mapping from $Teich\left(\dot{R}\right)$ into $Teich(R)$ defined as follows

$$\pi\left([f]_{Teich(\dot{R})}\right) = \left[\tilde{f}\right]_{Teich(R)}$$

The mapping $\pi$ takes a class represented by a quasiconformal mapping $f$ defined on $\dot{R}$ and maps to a class represented by $\tilde{f}$. If two quasiconformal mappings $h_0$ and $h_1$ are in the same Teichmüller class in $\dot{R}$ then there is an isotopy $H_t$ through quasiconformal mappings that connects $h_0$ and $h_1$. The extensions of $h_0$ and $h_1$ across the puncture $z$, send $z$ to point $w$. We can extend the isotopy $H_t$ across the puncture $z$ by declaring $H_t(z) = w$ for all $t \in [0,1]$. This shows that the extensions of $h_0$ and $h_1$ are in the same Teichmüller class with respect to $Teich(R)$. We conclude that mapping $\pi$ is well defined. The mapping $\pi$ above is called puncture forgetful mapping

The fiber $\pi^{-1}([f])$ is defined to be an inverse image of $\pi$ at point $[f]$.

The Teichmüller disk is the set $\left\{\left[t\frac{|\varphi|}{\varphi}\right] \mid t \in [0,1]\right\}$ for some quadratic differential

$\varphi \in Q(R)$.

The following theorem is due to Bers [3].

**Theorem 22** *Let $\pi$ be the puncture forgetful mapping from*

$$\pi : Teich\left(\dot{R}\right) \to Teich\,(R)\,.$$

There exist a biholomorphic isomorphism from the fiber $\pi^{-1}([f])$ onto the unit disk.

Our next result gives us necessary and sufficient condition for metric $d_s$ to be equal to Poincaré metric on a given hyperbolic Riemann surface $R$.

**Theorem 23** *Let $R$ be a hyperbolic Riemann surface. And let $\pi : Teich\left(\dot{R}\right) \longrightarrow Teich\,(R)$ be the puncture forgetful map. The fiber $\pi^{-1}[0]$ is a Teichmüller disk if and only if metric $d_s$ coincide with Poincaré metric.*

**Proof.** Assume that $\pi^{-1}[0]$ is a Teichmüller disk. There exists a integrable holomorphic quadratic differential $\varphi$ such that $\pi^{-1}[0] = \left\{\left[t\frac{|\varphi|}{\varphi}\right] : t \in D\right\}$ We solve Beltrami equation for $\mu = t\frac{|\varphi|}{\varphi}$ and obtain a holomorphic motion $f^{t\frac{|\varphi|}{\varphi}}(z) = f^t$ of a Riemann surface through itself. Fix now $z_0 \in R$ and $z_1 \in R$. Because of theorem 12 Teichmüller $z_0$ to $z_1$ shift has to be $f^{t\frac{|\varphi|}{\varphi}}$ for some $t \in D$. On the other hand $f^t(z_0) : D \longrightarrow R$ is a holomorphic map from the unit disk to Riemann surface. We apply Schwarz's lemma and obtain the inequality

$$d\,(z_0, z_1) = d\left(f^0(z_0)\,, f^t(z_0)\right) \leq d_D\,(0, t)$$

whereas $d_D\,(0,t) = \frac{1}{2}\log\frac{1+t}{1-t} = d_{sR}\,(z_0, z_1)$. The inequality

$$d_s\,(z_0, z_1) \geq d\,(z_0, z_1)$$

follows and, since the reverse inequality is always true by theorem 13 we have

$$d_s\,(z_0, z_1) = d\,(z_0, z_1)\,.$$

Now, we assume that $d_{sR} = d_R$ . Take any covering map $p(t) : D \longrightarrow R$. We now consider $p(t)$ to be a holomorphic motion of a single point $p(0)$ and by theorem 9 we can extend this to a holomorphic motion $f^t$ of the whole Riemann surface. Observe that $\{f^t : t \in D\} = \pi^{-1}[0]$ Note that

$$f^t(p(0)) = p(t)$$

One actually obtains a holomorphic family of trivial Beltrami differential $\mu_t$ such that $f^{\mu_t} = f^t$. Thus we have a holomorphic map $t \to \mu_t$ from the unit disk in to the unit ball of $Belt(R)$. Applying Schwarz's lemma again we get

$$\frac{1}{2}\log\frac{1+||\mu_t||_\infty}{1-||\mu_t||_\infty} \leq \frac{1}{2}\log\frac{1+t}{1-t}$$

where equality for at least one $t$ implies equality for all $t \in D$ and $\mu_t = t\nu$ for some $\nu \in Belt(R)$ with $||\nu||_\infty = 1$. Observe further that $f^t$ is not necessarily a Teichmüller $p(0)$ to $p(t)$ shift and so $d_s(p(0), p(t)) \leq \frac{1}{2}\log\frac{1+||\mu_t||_\infty}{1-||\mu_t||_\infty}$. If $t \in D$ is sufficiently small then since $p$ is a covering map $d(p(0), p(t)) = d_D(0,t) = \frac{1}{2}\log\frac{1+t}{1-t}$. Since we assumed that $d_s = d$, it follows from

$$d_s(p(0), p(t)) \leq \frac{1}{2}\log\frac{1+||\mu_t||_\infty}{1-||\mu_t||_\infty} \leq \frac{1}{2}\log\frac{1+t}{1-t} = d(p(0), p(t)) \tag{3}$$

that $\frac{1}{2}\log\frac{1+||\mu_t||_\infty}{1-||\mu_t||_\infty} = \frac{1}{2}\log\frac{1+t}{1-t}$ . Moreover $\mu_t = t\nu$ for some $\nu \in Belt(R)$ with $||\nu||_\infty = 1$. But the inequality (3) also shows that $t\nu$ is a Teichmüller shift which means that $\nu = \frac{|\varphi|}{\varphi}$ for some holomorphic integrable quadratic differential $\varphi$. This proves that the fiber over [0] is a Teichmüller disk ■

# Part II

# Extremality in $Teich\left(\mathbb{C}\backslash Z \times Z\right)$

**[ This part is developed in collaboration with Prof. Miroslav Pavlovic]**

In this part We study certain extremal problems in the Riemann sphere with the unit lattice removed. We give a useful criteria for quasiconformal map, defined on the Riemann sphere with the unit lattice removed, to be extremal

## 8 Preliminaries

Let $\Gamma = \{n + im \mid n, m \in Z\}$ be the lattice generated by two unit vectors **1** and $i$. Every quasiconformal mapping $f$ defined on $\mathbb{C}\backslash\Gamma$ has a unique extension (up to a postcomposition by a Möbius transformation) to the whole complex plane $\mathbb{C}$. The Teichmüller space $Teich\,(C\backslash\Gamma)$ is defined to be the set of equivalence classes of quasiconformal mappings defined on $\mathbb{C}\backslash\Gamma$. The equivalence relation is defined as follows. Two quasiconformal mappings $f$ and $g$ are equivalent if the extension of $f \circ g^{-1}$ to $\mathbb{C}$ fixes $\Gamma$ pointwise. We say that $f$ is extremal in its class if and only if it has the least possible dilatation $K$ among all quasiconformal mappings belonging to its class. In other words for any $g \in [f]$ we have

$$K\,(g) \geq K\,(f) \tag{4}$$

We recall that $K\,(f) = \frac{1+||\mu_f||_\infty}{1-||\mu_f||_\infty}$. Hence, 4 is the same as $||\mu_g||_\infty \geq ||\mu_f||_\infty$.

The space $A(\Gamma)$ is defined to be a linear subspace of $L^1(\mathbb{C})$ consisting of all integrable analytic functions in $\mathbb{C} \setminus \Gamma$. A Beltrami coefficient $\mu \in L^\infty(\mathbb{C})$ is said to be extremal for $Q(\Gamma)$, or simply extremal, if $||\mu||_\infty$ equals the norm of the linear functional $\Lambda$ defined on $A(\Gamma)$ by

$$\Lambda(\phi) = \int \mu(z)\phi(z)|dz|^2 \quad \text{where } \phi \in Q(\Gamma)$$

Note that the last definition is equivalent to the following. A Beltrami coefficient $\mu$ is extremal if for any other $\nu$ for which $\int(\mu-\nu)\varphi = 0$ for all $\varphi \in Q(\Gamma)$ we have $||\nu||_\infty \geq ||\mu||_\infty$. In the Teichmüller theory the Banach space $Q(\Gamma)$ is referred to as the tangent space of the Teichmüller space $Teich(C\backslash\Gamma)$ at the base point $[id]$.

We described two different extremal problems. From the Teichmüller theory point of view the first type of extremal problem is a global one, while the second type is an infinitesimal one. The following result of Hamilton and Kruskal is the cornerstone in consideration of extremal problems mentioned above and it relates a global extremal problem to an infinitesimal extremal problem.

**Theorem 24** *Let $f$ be a quasiconformal mapping defined on a Riemann surface $R$ and $\mu_f$ be its Beltrami coefficient. Quasiconformal mapping $f$ is extremal in its Teichmüller class if and only if its Beltrami coefficient $\mu_f$ is extremal in the infinitesimal sense in $Q(R)$.*

Much of the related results can be found in [7], [5], [25], [19], [18], [17], [21].

# 9 A criterion for extremality in $Teich(\mathbb{C}\backslash\Gamma)$

We would like to present here a criterion for extremality specific to the Teichmüller space of the complex plane $\mathbb{C}$ minus the lattice $\Gamma$. We recall here that any quasiconformal mapping defined on $\mathbb{C}\backslash\Gamma$ extends the whole complex plane. The extension is unique up to a postcomposition by a Möbius transformation.

**Theorem 25** *Let $f$ be quasiconformal mapping defined on $\mathbb{C}$. Define $f_n$ to be*

$$f_n(z) = 2^{-n} f(2^n z).$$

*Suppose that $f_n$ converges uniformly to a quasiconformal mapping $g$. If $K(g) = K(f)$ then $f$ is extremal in its Teichmüller class in $Teich(C\backslash\Gamma)$*

**Proof.** Suppose that $f$ is not extremal. Then there exists a quasiconformal mapping $h$ defined on $\mathbb{C}$ such that $h$ is in the same Teichmüller class as $f$ and $K(h) < K(f)$. Consider the sequence $h_n = 2^{-n}h(2^n z)$. Postcomposition and precomposition by a Möbius transformation do not change the dilatation $K$. In particular, we get $K(h_n) = K(h)$. From the theory of quasiconformal mappings we know that if a family of quasiconformal mappings has a uniformly bounded dilatation then it is a normal family. We conclude that there is a uniform limit $h_\infty$ of some subsequence $h_{n_k}$. Moreover $K(h_\infty) \leq K(h)$.

On the other hand $h$ is in the same Teichmüller class as $f$. Therefore for every point $z$ in the lattice $\Gamma$, we have $f(z) = h(z)$. Moreover for every point $z$ in the lattice $2^{-n}\Gamma$ we have $f_n(z) = 2^{-n}f(2^n z) = 2^{-n}h(2^n z) = h_n(z)$. We conclude that $f_n$ and $h_n$ coincide on $2^{-n}\Gamma$. Hence, the limits $g$ and $h_\infty$ of two sequences have to coincide on $\cup_1^\infty 2^{-n}\Gamma$. But this union is a dense set in the complex plane and therefore $g = h_\infty$ everywhere. This implies

$$K(h) \geq K(h_\infty) = K(g) = K(f)$$

which is a contradiction to the fact that $f$ is not extremal. ■

The next thing we would like to prove is a similar statement for an infinitesimal problem.

**Theorem 26** *Let $\mu$ be a Beltrami coefficient defined on $\mathbb{C}$ with the property that there exists a $\sigma \in L^\infty(\mathbb{C})$ such that $||\sigma||_\infty = ||\mu||_\infty$ and*

$$\lim_{n\to\infty} \mu(2^n z) = \sigma(z) \quad a.e. \ z \in \mathbb{C}$$

Then $\mu$ is extremal in the infinitesimal sense i.e. for any other $\nu$ such that $\int \mu\varphi = \int \nu\varphi$ for all $\varphi \in Q(\Gamma)$ we have $||\nu||_\infty \geq ||\mu||_\infty$.

**Proof.** The following known fact plays the main role in the proof. Let $S$ be a dense subset of the complex plane. Then the space of all integrable rational functions with simple poles in $S$ is dense in $L^1(C)$.

We choose $S$ to be the union of lattices $2^{-n}\Gamma$, $n = 0, 1, 2, \ldots$. Now assume that $\nu$ is equivalent to $\mu$, i. e.,

$$\int \mu\varphi = \int \nu\varphi \quad \text{for all } \varphi \in Q(\Gamma).$$

Define two sequences of linear functionals on $L^1(\mathbb{C})$ in the following way:

$$\Lambda_n^\mu(\varphi) = \int \mu(2^n z)\varphi(z)|dz|^2$$

and similarly for $\nu$.

From the formula

$$\Lambda_n^\mu(\varphi) = \int \mu(w)\varphi(2^{-n}w)4^{-n}|dw|^2$$

it follows that $\Lambda_n^\nu = \Lambda_n^\mu$ on $Q(\Gamma)$. Moreover, let $\varphi$ be an integrable rational function with poles in some $2^{-s}\Gamma$. Then the poles of the function $\varphi(z/2^n)$, $n > s$, are in $\Gamma$ and therefore the above formula shows that $\Lambda_n^\nu\varphi = \Lambda_n^\mu\varphi$ for all $n > s$. It follows that

$$\lim_n \Lambda_n^\nu\varphi = \Lambda_\sigma(\varphi)$$

for every $\varphi \in A(2^{-s}\Gamma)$ and hence for every $\varphi \in L^1$. Hence

$$||\sigma||_\infty \leq \liminf ||\Lambda_n^\nu|| = ||\nu||_\infty.$$

This concludes the proof because $||\mu|| = ||\sigma||$. ■

We remark here that the infinitesimal criterion implies the global one but not conversely. This means that we could have used theorem 24 to deduce the last theorem but we presented the proof for the sake of developing different techniques in the area.

# 10 Some consequences of the criterion

In this section we derive some consequences of the criteria mentioned in the previous sections. Some related results can be found in [14]. A sequence $\{\varphi_n\} \in Q(\Gamma)$ is said to be a Hamilton sequence for a Beltrami coefficient $\mu \in L^\infty$ if $||\varphi_n||_1 = 1$ and

$$\int \mathrm{Re}(\mu\varphi_n) \to ||\mu||_\infty.$$

It is the fact that $\mu$ is extremal if and only if there is a Hamilton sequence for $\mu$. Here are the results.

**Theorem 27** *If $r$ is an arbitrary rational function, then $\mu(z) = |r(z)|/r(z)$ is extremal.*

**Theorem 28** *If $\mu = |r|/r$, where $r$ is an arbitrary rational function, then there exists a degenerate Hamilton sequence for $\mu$.*

**Proof.** . A Hamilton sequence $\varphi_n$ is degenerate if $\varphi_n(z) \to 0$ on compact subsets of $C \setminus \Gamma$. An element $\varphi$ on the unit sphere of $Q(\Gamma)$ is a Strebel differential if every Hamilton sequence for $|\varphi|/\varphi$ is non- degenerate. An easy consequence of Theorem 28 would be that if $\varphi$ is a rational function on the unit sphere of $Q(\Gamma)$, then $\varphi$ is not a Strebel differential.

Choose a positive integer $s$ so that the function $r(2^s z)$ does not belong to $Q(\Gamma)$. (This is possible because if $r$ is in $Q(\Gamma)$, then $r$ has at least three poles of order one.) Let $\mu_s(z) = |r(2^s z)|/r(2^s z)$. By Theorem 27 there exists a Hamilton sequence $\{\varphi_n\}$ for $\mu_s$. This sequence must be degenerate since otherwise $\mu_s = |\varphi|/\varphi$ for some $\varphi \in Q(\Gamma)$, which is impossible. Now it can be easily shown that the sequence $\varphi_n(z/2^s)4^{-s}$ is a degenerate Hamilton sequence for $|r|/r$.

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