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SPHERICAL SHELLS.

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NON-LINEAR DEFORMATION

OF THIN, SHALLOW, SPHERICAL SHELLS

by

Maria Cristina Sinay

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Abstract

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Maria C Sinay

Adviser: Professor Harry E. Rauch

In this thesis a theoretical and numerical analysis of large axisymmetric deflections and stresses of a thin, shallow spherical shell is presented.

We consider the shell as an initially clamped, flat, circular plate which has suffered a deformation ω_o, thus the geometrical characteristics of the shell are represented by the parameter λ defined in chapter 1.

In considering large deflection bendings, i.e., deflections up to several thickness of the plate, we are led to a system of two coupled differential equations, each one of them an inhomogeneous form of the Bessel equation of index 1. An approximate solution of the system is represented as a finite sum of n eigenfunctions of the Bessel operator of index 1, (n modes). This yields a non-linear algebraic system of coupled equations which is solved numerically. The numerical results are used to calculate the deflection of the shell as well as bending and membrane stresses.

We find that if $\lambda > \lambda_* = 3.4001$, buckling pressures p_b are determined even for a one-mode solution. When $\lambda < \lambda_*$, two modes are necessary if λ is very close to λ_*, otherwise, the buckling pressure becomes too large to be determined even with a ten-modes solution.

Our results agree in the range $3.4 < \lambda < 5$ with those found by most of previous authors. When $\lambda > 5$, three different kinds of dependence of p_b on λ have been obtained in other papers. The buckling pressure increases monotonically, [7] *, tends monotonically

* Numbers in brackets refer to bibliography listed in page 51

to one,[2],or behave in an oscillatory way,[6],[18].

Our results agree with those of Chen,[7] and Budiansky,[6] for $5 < \lambda < 6$, and with those of Budiansky for $\lambda > 6$.

To
Beatriz
and
Laura

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Introduction

A theoretical and numerical analysis will be presented for the elastic,axisymmetric deflections and stresses of a shallow portion of an axisymmetrical spherical shell,clamped along a circular boundary,and subjected to a uniform,normal pressure.

The term thin shell is applied to bodies bounded by two curved surfaces,where the distance between the surfaces is assumed to be small compared to other dimensions in the problem.

The shell is assumed to be constructed of homogeneous, isotropic and elastic material,i.e.,the shell returns to its original undeformed shape when the pressure is removed,and Hooke's law is valid,namely,strains are linear functions of the stresses.

Experimental results show that as the pressure increases from zero,the shell deforms continuously until a critical pressure is reached,buckling pressure,where the shell jumps to a nonspherical shape,reaching the so called buckling state.

The response curve pressure-maximum deflection for a spherial cap subjected to a uniform,normal pressure is frequently assumed to be similar to that of fig. 2.This curve implies that for all $p > p_b$ and $p < p_s$ there is only one equilibrium state while for each p in $p_s < p < p_b$ there are three equilibrium states and the cap must buckle at some p on this interval.

The edge support of the shell is assumed to clamp it in such a way that not only the deflection is zero at the edge,but radial displacements and rotation around the edge are not possible.

The shell under consideration has a circular boundary of radius R,and its thickness is called h.

We shall be working with thin shells,so,as we said above, h is small compared with other dimensions of the problem.In particular $h \ll R$.No prior assumption is made about the ratio h/ R but we shall see later that when all the quantities are written in dimensionless variables,the results are independent of it.

The literature on the determination of critical pressures for spherical shells is vast.Studies done by Von Karman and Tsien,[12], Friedrichs,[10],Yoshimura and Uemura,[25],Mushtari and Surkin,[15], and Feodosiev,[9],involve the determination of buckling pressures by minimization of a potential energy expression for the shell with respect to a special class of deflection functions.

The works of Biezeno,[4],Kaplan and Fung ,[11],are based on integrations of non linear differential equations corresponding to those which are used in this thesis.

Archer,[2],Wilson and Spier,[24],have employed a Gaussian procedure,whereas Thurston,[20],has utilized Newton's method. Mescall,[14],applied the Newton's method to obtain a system of linear correctional equations for an initial approximate solution and the Gaussian procedure for the solution of the finite difference; Way,[22],applied the method of power series and Keller and Reiss,[13], the method of finite differences.

We shall find an approximate solution of the problem in the form of finite Fourier-Bessel series.After substitution of these series in the differential equations and boundary conditions representing the states of the shell,the problem is reduced to a system of non linear,coupled algebraic equations in a finite number of unknowns.These equations can be viewed as truncations of an infinite system corresponding to an exact solution.

We shall find that the method,outlined in chapter 1,is simpler in principle than other author's methods,and can be used to solve many related problems,[16].

Since the solutions obtained by this method can be decomposed into what we shall call modes,we can see how the different modes contribute to the results. Furthermore,this method allows us to impose all the boundary conditions at once which is an improvement over some of the methods mentioned above.

1. Mathematical model

We consider the axisymmetric deformations of a clamped shell of thickness h, that result from a uniform, inwardly directed, normal pressure P

Let R be the radius of the shell's base and H its high, (see Fig. 1 for the geometry of the shell). We shall study the case in which $H/R \ll 1$, (shallow), and $h \ll R$, (thin).

Let

$$(1.1) \qquad \omega_o = - H\left[1 - \left(\frac{r}{R}\right)^2\right]$$

be the vertical displacement of a point at a distance r from the vertical axis when P = 0, and let

$$(1.2) \qquad \omega_T = \omega + \omega_o$$

be the vertical displacement of the point for a non-zero pressure P

The differential equations that describe the deformation of the shell have been derived by several authors, [21], [17]. They can be written as, (See Appendix A for their derivation):

$$(1.3) \qquad D\left[r\beta'' + \beta' - \frac{\beta}{r} \right] = - P\frac{r^2}{2} + \psi\beta - \frac{2H\psi r}{R^2}$$

$$(1.4) \qquad r\psi'' + \psi' - \frac{\psi}{r} = -\frac{Eh\beta^2}{2} + \frac{2EhH\beta r}{R^2}$$

In (1.3), (1.4) we have used the following notation:

$\beta = -\dfrac{d\omega}{dr}$ $\qquad \psi = h\dfrac{d\Phi}{dr}$ $\qquad$ (Φ stress function)

E: Young's modulus

μ: Poisson's ratio

$$(1.5) \qquad D = \frac{E h^3}{12(1 - \mu^2)}$$

To complete the formulation of the problem, conditions at the center $r = 0$ and at the edge $r = R$ are required. From the symmetry of the deformation and the regularity of the radial membrane stress at the center, we obtain:

(1.6) $\beta(0) = 0$

(1.7) $\psi(0) = 0$

The fact that the shell is clamped implies the following condition at the edge:

(1.8) $\beta(R) = 0$

In addition, we require that no radial displacement at the edge be possible, i.e.:

(1.9) $\psi'(R) - \frac{\mu}{R}\psi(R) = 0$

We now define the geometrical parameter λ and the dimensionless variables θ and ϕ as follows :

(1.10) $\lambda^4 = \frac{48\ H^2(1-\mu^2)}{h^2}$; $x = \frac{r}{R}$

(1.11) $\theta(x) = \frac{\lambda^2 R\ \beta(Rx)}{2\ H}$; $\phi(x) = \frac{12(1-\mu^2)\ R\ \psi(Rx)}{E\ h^3}$

Substituting β and ψ in (1.3),(1.4) by (1.11) we obtain the boundary value problem

(1.12) $\theta'' + \frac{\theta'}{x} - \frac{\theta}{x^2} = -\lambda^2\phi - 2\ p\ x + \frac{\theta\ \phi}{x}$

(1.13) $\phi'' + \frac{\phi'}{x} - \frac{\phi}{x^2} = \lambda^2\theta - \frac{\theta^2}{2x}$

$(\ )' = d/dx$, $0 \leq x \leq 1$

(1.14) $\theta(0) = 0$

(1.15) $\theta(1) = 0$

(1.16) $\phi(0) = 0$

(1.17) $\phi'(1) - \mu\phi(1) = 0$

where

$$(1.18) \qquad p = \frac{6\sqrt{3(1-\mu^2)^3}\ R^4}{E\ h^4}\ P$$

The left hand sides in equations (1.12),(1.13) are the Bessel's operator of order 1,

$$(1.19) \qquad B[f] = f'' + \frac{f'}{x} - \frac{f}{x^2}$$

applied to θ and ϕ respectively,thus we have

$$(1.20) \qquad B(\theta) + \lambda^2\phi + 2\ p\ x - \frac{\theta\ \phi}{x} = 0$$

$$(1.21) \qquad B(\phi) - \lambda^2\theta + \frac{\theta^2}{2\ x} = 0$$

In order to solve the equations (1.20),(1.21) subjected to (1.14)-(1.17) we apply the Bubnov - Galerkin variant of Rayleigh - Ritz method, [16]. To find approximate solutions of (1.20),(1.21) we assume formal finite eigenfunction expansions of θ and ϕ consisting in the sum of n terms,(modes), of the form $a_i J_1(k_i x)$,where J_1 is the Bessel function of order 1, k_i is its i-th positive zero and a_i is a coefficient to be determined.Introducing these expansions in the boundary value problem,carrying out the multiplications,and applying the orthogonality properties of the Bessel functions,one obtains a coupled, finite quadratic algebraic system of equations in the unknown coefficients,which can be solved numerically.

Thus we assume

$$(1.22) \qquad \theta(x) = \sum_{i=1}^{n} a_i\ J_1(k_i x)$$

$$(\ J_1(k_i) = 0)$$

$$(1.23) \qquad \phi(x) = b_o x + \sum_{i=1}^{n} b_i\ J_1(k_i x)$$

We observe that (1.14)-(1.17) are automatically satisfied, and the term $b_o x$ in (1.23) is consistent with an eigenfunction expansion since

$$B[\, x \,] = 0 \qquad (\text{see } (1.19))$$

Substituting ϕ and Φ in (1.20),(1.21) by (1.22),(1.23), multiplying both equations by $xJ_1(k_m x)$ and integrating from 0 to 1 we obtain, for every m:

$$(1.24) \quad \frac{-k_m^2\, a_m\, J_2^2(k_m)}{2} + \frac{\lambda^2\, b_o\, J_2(k_m)}{k_m} + \frac{\lambda^2\, b_m\, J_2^2(k_m)}{2} +$$

$$\frac{2\, p\, J_2(k_m)}{k_m} - \frac{b_o\, a_m\, J_2^2(k_m)}{2} - \sum_{\ell,i=1}^{n} a_\ell b_i\, j_{\ell im} = 0$$

$$(1.25) \quad -\, k_m^2\, b_m\, J_2^2(k_m) - \lambda^2\, a_m\, J_2^2(k_m) + \sum_{\ell,i=1}^{n} a_\ell a_i\, j_{\ell im} = 0,$$

where

$$(1.26) \quad j_{\ell im} = \int_0^1 J_1(k_\ell x)\, J_1(k_i x)\, J_1(k_m x)\, dx$$

In addition,(1.17) yields

$$(1.27) \quad b_o = \frac{1}{1-\mu} \sum_{i=1}^{n} b_i k_i\, J_2'(k_i)$$

We now summarize some relevant properties of the Bessel functions which have been used in obtaining (1.24)-(1.27),[8].

$$(1.28) \quad \int_0^1 x\, J_1(k_m x)\, J_1(k_i x)\, dx = \tfrac{1}{2}\delta_{mi}\, J_2^2\,(k_i)\ ,\ (\delta_{mi} \text{ Kroenecker's delta})$$

$$\frac{d}{dx}J_1(k_i x) = k_i\left[J_o(k_i x) - \frac{J_1(k_i x)}{k_i x} \right] \tag{1.29}$$

hence:

$$\left.\frac{d}{dx}J_1(k_i x)\right|_{x=1} = - k_i\ J_2(k_i)$$

From (1.25) we have

$$b_m = - \frac{\lambda^2}{k_m^2} a_m + \frac{1}{k_m^2\ J_2^2(k_m)} \sum_{i,\ell=1}^{n} a_i a_\ell\ j_{i\ell m} \quad \text{(for every m)} \tag{1.30}$$

Substituting b_i in (1.27) by (1.30) and introducing the resulting expression,as well as (1.29),in (1.24) we obtain

$$\frac{(k_m^4 + \lambda^4)\ J_2(k_m)\ a_m}{4\ k_m} + \sum_{i=1}^{n}\Bigg\{\frac{(.5\ \lambda^4 - .25\ \lambda^2\ J_2(k_m)\ k_m a_m)}{1 - \mu} \cdot \frac{J_2(k_i) a_i}{k_i} \tag{1.31}$$

$$- \sum_{\ell=1}^{n} \Bigg[\frac{\lambda^2 j_{i\ell m}}{J_2(k_m)}\left(\frac{1}{4k_m} + \frac{k_m}{2\ k_\ell^2}\right) a_i a_\ell +$$

$$\sum_{s=1} \left(\left(\lambda^2 - \frac{J_2(k_m)k_m a_m}{2}\right) \frac{j_{s\ell i}\ a_s a_\ell}{2(1-\mu)k_i J_2(k_i)}\right. -$$

$$\left.\frac{k_m\ j_{i\ell m}\ a_s a_i}{2J_2(k_m)J_2^2(k_\ell)k_\ell^2} \sum_{\nu=1}^{n} j_{s\nu\ell}\ a_\nu\ )\Bigg]\Bigg\} = p \qquad \text{(for every m)}$$

2. Approximate expansions of the deflection, radial and tangential membrane and bending stress functions

In order to obtain approximate expansions of the shell's deflection and stress functions, we recall that β and ψ were defined in (1.3),(1.4) by

$$(2.1) \qquad \beta = - d\omega/dr \quad ; \quad \psi = h\, d\Phi / dr,$$

thus, introducing (1.1),(1.10),(1.22) and (1.23) in (1.2) we have:

$$(2.2) \qquad \omega_T(Rx) = \sum_{i=1}^{n} \frac{a_i}{k_i} [J_o(k_i x) + J_2(k_i)] - \frac{\lambda^2}{2}(1 - x^2)$$

$$(2.3) \qquad \omega(Rx) = \sum_{i=1}^{n} \frac{a_i}{k_i}[J_o(k_i x) + J_2(k_i)]$$

In particular, the central deflection of the shell is given by

$$(2.4) \qquad \omega(0) = \sum_{i=1}^{n} \frac{a_i}{k_i}[1 + J_2(k_i)]$$

We define the radial membrane stress following Volmir,[21], by

$$(2.5) \qquad \sigma_r^*(r) = \psi(r)/hr,$$

which, in terms of x, and using (1.22),(1.23), becomes

$$(2.6) \qquad \sigma_r(Rx) = -\lambda^2[\frac{1}{1-\mu} \sum_{i=1}^{n} \frac{J_2(k_i)}{k_i} a_i + \sum_{i=1}^{n} \frac{J_1(k_i x)}{k_i^2 x} a_i] + \frac{1}{1-\mu} \sum_{i,\ell,s=1}^{n} \frac{j_{i\ell s}\, a_i a_\ell}{k_s J_2(k_s)} + \sum_{i,\ell,s=1}^{n} \frac{J_1(k_i x)\, j_{is\ell}\, a_s a_\ell}{x\, k_s^2 J_2^2(k_s)} \qquad x \neq 0$$

And, since

(2.7) $$\lim_{x \to 0} \frac{J_1(k_i x)}{x} = \tfrac{1}{2}k_i$$

$$\sigma_r(0) = -\lambda^2 \left[\frac{1}{1-\mu} \sum_{i=1}^{n} \frac{J_2(k_i)}{k_i} a_i + \sum_{i=1}^{n} \frac{a_i}{2k_i} \right] +$$

$$\frac{1}{1-\mu} \sum_{i,\ell,s=1}^{n} \frac{j_{i\ell s}\, a_i a_\ell}{k_s\, J_2(k_s)} + \sum_{i,\ell,s-1}^{n} \frac{j_{i\ell s}\, a_i a_\ell}{2\, k_s\, J_2^2(k_s)}$$

i.e.,

(2.8) $$\sigma_r(0) = -\lambda^2 \sum_{i=1}^{n} \left[\frac{J_2(k_i)}{1-\mu} + \tfrac{1}{2} \right] \frac{a_i}{k_i} +$$

$$\sum_{i,\ell,s=1}^{n} \left[\frac{1}{1-\mu} + \frac{1}{2\, J_2(k_s)} \right] \frac{j_{i\ell s}\, a_i a_\ell}{k_s\, J_2(k_s)}$$

In (2.6),(2.8) we have used

(2.9) $$\sigma_r = \frac{12\, R^2 (1-\mu^2)}{E\, h^2}\, \sigma_r^*$$

Also according to Volmir,[21],we have that the tangential membrane stress is

(2.10) $$\sigma_e^*(r) = \frac{1}{h} \frac{d\psi}{dr}$$

and therefore

(2.11) $$\sigma_e(Rx) = -\frac{\lambda^2}{1-\mu}\sum_{i=1}^{n}\frac{J_2(k_i)}{k_i}a_i + \frac{1}{1-\mu}\sum_{i,\ell,s=1}^{n}\frac{j_{i\ell s}}{k_i J_2(k_i)}a_s a_\ell -$$

$$\lambda^2\sum_{i=1}^{n}\frac{J_o(k_i x)\ a_i}{k_i} + \sum_{i,\ell,s=1}^{n}\frac{J_o(k_i x)\ j_{i\ell s}\ a_\ell a_s}{k_i\ J_2^2(k_i)} +$$

$$\lambda^2\sum_{i=1}^{n}\frac{J_1(k_i x)\ a_i}{k_i^2\ x} - \sum_{i,\ell,s=1}^{n}\frac{J_1(k_i x)\ j_{i\ell s}\ a_\ell a_s}{x\ k_i^2\ J_2^2(k_i)} \qquad x \neq 0$$

Where

(2.12) $$\sigma_e = \frac{12(\ 1 - \mu^2)\ R^2}{E\ h^2}\ \sigma_e^*$$

Applying (2.7) we obtain

(2.13) $$\sigma_e(0) = \frac{-\lambda^2}{1-\mu}\sum_{i=1}^{n}\frac{J_2(k_i)}{k_i}a_i + \frac{1}{2}\sum_{i,\ell,s=1}^{n}\frac{j_{i\ell s}\ a_\ell a_s}{k_i\ J_2^2(k_i)} -$$

$$-\frac{\lambda^2}{2}\sum_{i=1}^{n}\frac{a_i}{k_i} + \frac{1}{1-\mu}\sum_{i,\ell,s=1}^{n}\frac{j_{i\ell s}\ a_\ell a_s}{k_i\ J_2(k_i)}$$

It follows from (2.8) and (2.13) that

(2.14) $$\sigma_r(0) = \sigma_e(0)$$

The radial bending stress $\sigma^*_{r,b}$ and the tangential bending stress $\sigma^*_{e,b}$ calculated at the face of the plate where they reach their maximum, namely, $z = h/2$, are

(2.15) $$\sigma^*_{r,b} = \frac{6\,D}{h^2}\left[\frac{d\beta}{dr} + \mu \frac{\beta}{r} \right]$$

and

(2.16) $$\sigma^*_{e,b} = \frac{6\,D}{h^2}\left[\mu \frac{d\beta}{dr} + \frac{\beta}{r} \right]$$

Proceeding as above, we have

(2.17) $$\sigma_{r,b}(Rx) = \sum_{i=1}^{n} \left[k_i J_o(k_i x) - (1 - \mu)\frac{J_1(k_i x)}{x} \right] a_i \qquad x \neq 0$$

(2.18) $$\sigma_{r,b}(0) = \frac{1+\mu}{2} \sum_{i=1}^{n} k_i a_i$$

and

$$\sigma_{e,b}(Rx) = \frac{E\,h}{1-\mu^2}\left[\frac{H}{\lambda^2 R^2\, x} \sum_{i=1}^{n} a_i J_1(k_i x) + \frac{\mu H}{\lambda^2 R^2} \sum_{i=1}^{n} \frac{dJ_1(k_i x)}{dx}\, a_i \right]$$

(2.19) $$\sigma_{e,b}(Rx) = \sum_{i=1}^{n} \left[\frac{J_1(k_i x)}{x}(1-\mu) + k_i J_o(k_i x)\,\mu \right] a_i \qquad x \neq 0$$

(2.20) $$\sigma_{e,b}(0) = \frac{1+\mu}{2} \sum_{i=1}^{n} a_i k_i$$

Where

(2.21) $$\sigma_{r,b} = \frac{\sqrt{48(1-\mu^2)^3}}{E\,h^2} R^2 \sigma^*_{r,b}$$

and

(2.22) $$\sigma_{e,b} = \frac{\sqrt{48(1-\mu^2)^3}}{E\,h^2} R^2 \sigma^*_{e,b}$$

We now observe that at the boundary $x = 1, (r=R)$, we have

(2.23) $$\sigma_r(R) = -\frac{\lambda^2}{1-\mu} \sum_{i=1}^{n} \frac{J_2(k_i)a_i}{k_i} + \frac{1}{1-\mu} \sum_{i,\ell,s=1}^{n} \frac{j_{i\ell s} a_i a_\ell}{k_s\, J_2(k_s)}$$

(2.24) $$\sigma_e(R) = \frac{\mu}{1-\mu} \left[-\lambda^2 \sum_{i=1}^{n} \frac{J_2(k_i)}{k_i} a_i + \sum_{i,\ell,s=1}^{n} \frac{j_{i\ell s} a_i a_\ell}{k_i\, J_2(k_i)} \right]$$

Therefore:

(2.25) $$\mu\, \sigma_r(R) = \sigma_e(R)$$

And, since

(2.26) $$\sigma_{r,b}(R) = -\sum_{i=1}^{n} J_2(k_i)\, k_i a_i$$

and

(2.27) $$\sigma_{e,b}(R) = -\mu \sum_{i=1}^{n} J_2(k_i) k_i a_i$$

we obtain

(2.28) $$\mu\, \sigma_{r,b}(R) = \sigma_{e,b}(k)$$

Equation (2.25) and (2.28) can now be used to verify numerical results

3. Approximate solution of the boundary value problem.

§3.1 Analysis of a one-mode solution.

In order to gain insight on the method of solution outlined in chapter 2, we shall now analyze a one-mode approximate solution in the form (1.22),(1.23), of the boundary value problem (1.12)-(1.17).

The order of the approximation is n=1, that is

$$\theta(x) = a_1 J_1(x) \tag{3.1.1}$$

$$\Phi(x) = b_1\left[\frac{k_1 J_2(k_1)}{1-\mu}\, x + J_1(k_1 x)\right] = b_1[2.204656\, x + J_1(k_1 x)] \tag{3.1.2}$$

Although no claims on the accuracy of this approximate solution can be made, some of its salient features, such as the dependence on λ, can be used toward a better understanding of an approximate n-mode solution with $n > 1$.

The analysis is restricted to a steel shell for which $\mu = .3$; the constants $k_1 = 3.83171$ and $J_2(k_1) = .40276$ were taken from Archer, [2], and Abramowits, [1]; $j_{111} = .082111$ was calculated using different FORTRAN programs, it has been verified with the corresponding one displayed in [1]. A detailed explanations of the calculations is given, in a more general context, in § 3.2.

For n = 1, equation (1.31) is reduced to

$$p = [5.664534 + .101358\, \lambda^4]a_1 - .135843\, \lambda^2 a_1^2 + .042791\, a_1^3. \tag{3.1.3}$$

From (2.4) we have

$$\omega(0) = \frac{a_1}{k_1}[\,1 + J_2(k_1)\,] = .3660924\, a_1 \tag{3.1.4}$$

thus,

$$p = [15.472962 + .276865\, \lambda^4]\omega(0) - .371062\, \lambda^2\omega^2(0) + .116887\, \omega^3(0). \tag{3.1.5}$$

The advantage of this last equation resides in the fact that the conclusions drawn from it can be interpreted in terms of a

pressure-maximum deflection relation and its dependence on λ.

We now observe that p is a cubic polynomial in $\omega(0)$,((3.1.5)), with coefficients which are functions of λ, therefore, depending on the value of λ, p can be a monotone increasing function of $\omega(0)$, it can have an inflection point or extrema. This later case is characterized by the fact that the equation

$$\frac{\partial p}{\partial \omega(0)} = 0 \tag{3.1.6}$$

posseses real solutions. The discriminant of this quadratic equation in $\omega(0)$ is

$$\Delta = .040601\ \lambda^4 - 5.425760 \tag{3.1.7}$$

Let

$$\lambda_* = 3.40001 \tag{3.1.8}$$

be the positive root of (3.1.7), then, if $\lambda < \lambda_*$, the pressure is a monotone increasing function of $\omega(0)$ for every $\omega(0)$, and if $\lambda > \lambda_*$, there exist two values of $\omega(0)$, ω_s and ω_b, for which p attains a local minimum and maximum respectively. The physical phenomenon can be better understood if the roles of the parameters p and $\omega(0)$ are interchanged, that is, if one studies the dependence of $\omega(0)$ on p. We have sketched in fig.3 this dependence for $\lambda > \lambda_*$. One can observe that $\omega(0) = 0$, (the shell is not deformed), if $p = 0$, and it is an increasing function of the pressure until the critical value p_b is reached, for which $\omega(0) = \omega_b$. If the pressure is increased beyond this point, a sudden jump to a higher value of the deflection takes place, this is the so called <u>buckling of the shell</u>. On the other hand, if the point (p, $\omega(0)$) with $p > p_b$ is in the upper branch of fig.3 (<u>buckled branch</u>), and p is decreased, $\omega(0)$ decreases continuously until p_s is reached, with $\omega(0) = \omega_s$. If p is decreased still further, i.e., p becomes less than p_s, a jump to a smaller value of $\omega(0)$ occurs. this phenomenon is known as <u>snapping of the shell</u>

The qualitative aspects of these two phenomena can actually be observed in nature. For instance, if a thin, shallow spherical cap is buckled, warmed up to about 37°C and put on a cold plate, (approximately 0°C), the released potential energy at the snapping point makes the shell to jump up to several inches high, depending on the material the shell is made of. In this experience, the variation of pressure has been simulated by a change in temperature.

In addition to the phenomena explained above, we can also find an explicit dependence of the buckling and snapping pressures on λ, that is, solving (3.1.6) for $\lambda > \lambda_*$, and substituting $\omega(0)$ in (3.1.5) we have

$$p_b = .015977\,\lambda^6 + 16.373159\,\lambda^2 + .233774\,\tilde{\Delta}^{1/2} \tag{3.1.9}$$

$$p_s = .015977\,\lambda^6 + 16.373159\,\lambda^2 - .233774\,\tilde{\Delta}^{1/2} \tag{3.1.10}$$

where

$$\tilde{\Delta} = .3301190\,\lambda^4 - 44.125129 \tag{3.1.11}$$

We can see in fig. 4, where these two equations have been plotted, that p_b is an increasing function of λ. It follows from (1.10) that λ can be increased either by increasing the rise H of the shell or by decreasing its thickness h. Since in the derivation of (1.3), (1.4), terms of order $4\,r^2H^2/R^4$ have been neglected, changes in H should be accompanied by changes in R in order to keep $H/R \ll 1$, thus, is preferable to think variations in λ as variations in thickness rather than in the geometry. The monotonicity of p_b with respect to λ is then consistent with the fact that for a given material, (fixed μ), the thinner the shell, the more elastic it is and therefore, it can stand greater loads before buckling.

The state of the shell is unstable for $p_s \leq p \leq p_b$, that is, small perturbations can have large effects on the shell. Since p_s is a decreasing function of λ, the thinner the shell, the wider the instability region is. Moreover, it can be shown that if

$$\lambda > \lambda^* = 5.443475, \tag{3.1.12}$$

the snapping point corresponds to a negative pressure, which means that if $(p, \omega(0))$ is in the buckled branch, and p is diminished, the shell will not recover its original, undeformed shape, even for $p = 0$, it can only be done by reverting the pressure to an outward one.

Fig. 2 was first proposed by Von Karman and Tsien,[12], as the response curve of the shell.Latter studies,[10],[13],[19], have shown that this graph does no quite correspond to reality.

Experimental data indicates that for some types of shells, buckling occurs for pressures which are smaller than p_b.

Friedrich,[10],have determined that there exists an intermidiate value of the pressure, $p_s < p_E < p_b$ for which buckling takes place,and H.Keller and E.Reiss,[13], have justified the buckling at pressures smaller than p_b studying neighboring states of the shell with different potential energies.

This shows that a one-mode solution and its response curve can result in an oversimplified analysis of the problem,however,some of its features are still important for a numerical study of approximate solutions with more than one mode since it provides the order of magnitude of p_b and the general behavior which one should expect of the numerical results.Furthermore,this analysis shows that the theory might fail for large values of λ since,as we saw above, the shell does not return from a buckled state to its undeformed form for $p = 0$ if $\lambda > \lambda^*$ contradicting the assumption of elastic shell for which the equations where derived,i.e.,for $\lambda > \lambda^*$,(or latter refinements of it),the limit of pressure for which the properties of an elastic shell hold,is exceeded before buckling.

We wish to point out that is precisely for values of $\lambda > 5$ that different authors differ in their respective results.

§3.2 Analysis of n-mode solutions,($n \leq 10$)

We have seen in chapter 2 that the boundary value problem (1.12)-(1.17) is reduced to the non-linear algebraic system of coupled equations represented by (1.31) for $1 \leq m \leq n$ for each fixed n. It remains now to solve this system of n equations in the n unknowns a_i $1 \leq i \leq n$. In order to solve it numerically for a given λ,it is necessary to have as data the zeros k_i of the Bessel function J_1,the values of J_2 at k_i and the triple integrals $j_{i\ell m}$ $1 \leq i,\ell,m \leq n$ ((1.26)).

These number can be found in the literature,for instance, Archer,[2],and Abramowits,[1],however,the limited scope of these tables does not allow to attempt finding a solution of (1.31) for n greater than 8.Therefore,independent calculations were done so it would be possible to solve (1.31) for $n \leq 15$.Furthermore,computing programs were designed to deal with the problem for $n \leq 20$ in case it was needed.

In a preliminary step,we calculated the values of the Bessel functions $J_\nu(k_i x)$, $\nu = 0,1,2$,$1 \leq i \leq 15$,with $0 \leq x \leq 1$ at intervals of length .01 using the subroutine BESJN from Los Alamos Laboratories as the package programs to compute them in the IBM 370 at CUNY was found to be inadequate with regard to precision. The results were compared with the corresponding ones in [1].

BESJN was given as argument to the subroutine SIMPSN,also from Los Alamos,to compute $j_{i\ell m}$. We found total agreement of our calculations with those,that for $1 \leq i,\ell,m \leq 8$ have been published by Archer.

Finally,the roots k_i and the values $J_2(k_i)$ were taken from [1].

In a second step,we proceeded to find numerical solutions of (1.31) for fixed λ and $n \leq 10$ using the FORTRAN program BUCKLD, (Appendix C).

As we pointed out in § 3.1,several solutions of (1.31) must be expected for p within some ranges,therefore,in order to determine all of them,the coefficient a_1 was chosen as the independent parameter.The reasoning is that different values of it can lead to the same value of p while, for a given p, it is impossible for the computer to determine more than one value of a_1 at the time.

The program BUCKLD consists of the main program, the double precision function F, the subroutine ORIGIN, and it calls the subroutine ZSYSTM of the IMSL library.

The system (1.31) was given in the function

(3.2.1) F(X,K,PAR,IER),

where X is a n-dimensional array with X(1) = p and X(I) = a_i, $2 \leq i \leq n$ if $n \geq 2$, K is a pointer which indicates the k-th equation in (1.31). The variable PAR is a scalar used to pass the value of a_1.

The function F was verified to actually represent the system (1.31) with an equivalent program written in the symbolic manipulation computer language ALTRAN, (ALgebra TRANslator), [5], which was run at the Courant Institute of Mathematical Sciences, NYU, (Appendix C).

The subroutine

(3.2.2) ZSYSTM(F,EPS,NSIG,N,X,IMAX,WA,PAR,IER)

solves a system of N simultaneous non-linear equations in N unknowns using an iterative method. The subroutine returns an approximate solution X if the norm of the system F is less than EPS, or an error message if the iteration does not converge after IMAX steps. Finally, the subroutine ORIGIN was used to calculate the deflection and stress functions at the origin.

4. Discussion of results

We have seen in the analysis of a one-mode solution in § 3.1 that buckling of the shell can be expected from the first mode for $\lambda > \lambda_* \sim 3$, and that a negative value of snapping pressure is obtained if $\lambda > \lambda^* \sim 5$, furthermore, when λ is in this region, the results in the literature are qualitatively different. Chosing the intermidiate value $\lambda = 4$, (Tables I-X), we have found that the cubic-like relationship between pressure and deflection at the centre, ($\omega(0)$), holds for solutions with more than one mode, (fig. 5).

A good convergence is observed for solutions containing two or more modes in the unbuckled branch of fig. 2 ; a three-modes solution shows little difference with a 10-modes one in the unbuckled and middle branch, and solutions with five or more modes closely agree with the 10-modes solution in the whole range p-$\omega(0)$

Let us define the relative rate of convergence as

$$(4.1) \qquad C_r(m) = 100 \, \frac{\max \left| p_m - p_{10} \right|}{p_{10}} ,$$

where p_{10} in the denominator is taken equal to the value where $|p_m - p_{10}|$ is maximum for each m, i.e., for each p in the region under consideration, the m-modes solution is at most $C_r(m)$ % off the corresponding values of the 10- modes solution. We then find that $C_r(4) = 8.25\%$ for $p = 1.971930$, $\omega(0)=13.690367$, while $C_r(5)= 2.87\%$ for $p=1.8496272$, $\omega(0)=13.735734$.

Approximate values for the buckling and snapping pressures, p_b and p_s, are displayed in Table XI. It can be observed in it that the difference between the values of p_b corresponding to four and five modes solutions is of order 10^{-4} and it decreases to 10^{-6} for the 9 and 10 modes.

The convergence of the snapping is not so good, seven and eight modes were necessary in order to obtain a difference of order 10^{-4}. The corresponding values of the deflection, ω_b and ω_s as well as their differences are displayed in table XII. Once again, the convergence of ω_b is better than the convergence of ω_s

The analysis of the relation pressure-deflection was completed with the calculation of the shell's deflection $\omega(x)$

for $0 \leq x \leq 1$, (fig. 6), and the shape of the shell at five different pressures, p_i $1 \leq i \leq 5$. $p_1 \sim \frac{2}{3}p_b$ and it corresponds to an unbuckled state, $p_2 \sim p_b$, $p_3 > p_b$ and therefore the shell is buckled, $p_4 \sim p_s$ and finally, $p_s < p_5 < p_b$ corresponds to a point in the middle branch of fig. 2 We now observe that for $p = p_1$ the shell still has a parabolic shape; when $p = p_2 \sim p_b$, the shell becomes almost flat in its central part, more specificaly, the average slope for the central half of the shell is approximately .021, or an angle of 1.29^{o} . When p is greater than p_b, for instance p_3, the shell presents a wave-like shape in a clearly buckled state, moreover, for this pressure, which is approximately 2 times p_b, the concavity of the shell is reverted in 76 % of the radius, corroborating the fact that rapid changes in the shape of the shell take place in a boundary layer near the clamped edge.

Stress functions were analyzed along the same lines, i.e., we found their dependence at the origin on the pressure p and their behavior along the radius for the five representative values of p, p_i $1 \leq i \leq 5$, chosen for the analysis above.

It can be observed in fig. 8 that the tangential bending stress at the origin increases with the pressure in the unbuckled branch of fig. 2 , it also increases with deacreasing pressures in the middle branch, while it remains in a narrow range for buckled states. It must be pointed out that the curves representing the pressure-tangential bending stress at the centre of the shell fan around some middle value which is approximately 55 Stresses corresponding to solutions with an even number of modes tend to decrease remaining below 55 with increasing pressure, while those corresponding to solutions with an odd number of modes increase beyond 55. Fig. 9 shows that in a pre-buckling state, the value taken on at the centre of the shell remains almost constant along the radius; for unstable states corresponding to pressures smaller than p_b, the values of the stress function tend to values of the same order of the one of the unbuckled state as $x \rightarrow 1$, but for $p = p_3 > p_b$, the values of the stress sharply decay to zero in the outer 33 % of the radius.

In addition to these properties, we can observe that the curves corresponding to unstable states present an oscillatory character near the center. Similar comment can be made about the radial bending stress.

The different behavior of the curves for solutions with an even or odd number of modes is also found in fig. 11 where we have plotted the radial membrane stress at the center of the shell vs. the pressure.

In order to determine the behavior of p_b on λ, we calculated approximate solutions of (1.31) for different values of λ. Thus we found that for $\lambda = 3.4 < \lambda_*$, buckling is found considering an at least two-modes solution. It was not possible to determine a buckling pressure for $\lambda = 3.2$ even with a ten-modes solution. For λ between 3 and 5, our results agree with those of previous authors. For $\lambda > 5$ but less than 6, our values are close to those obtained by Chen, [7], and Budiansky, [6]. Outside this range, i.e., $\lambda > 6$, the magnitudes of the numbers involved in the problem made, for most values of λ, fail the iterative process due to overflow, however, for those for which our program run succesfully, the results closely agree with those of Budiansky. See fig. 14 for a comparison of results from different authors and this thesis.

We think that a different approach should be tried for large values of λ, perhaps considering higher order terms in the derivation of the equations might allow to find solutions of the problem for middle size λ, while this, together with an asymptotic analysis could yield a solution for $\lambda >> 7$. Then, if these solutions would be valid in overlapping intervals of λ, they could be matched.

Appendix A

§A.1 Derivation of equations (1.3) and (1.4).

We consider the axisymmetric, finite, deformations of a clamped shell of thickness h, that result from a uniform, inwardly directed, normal pressure P.

Let R be the radius of the shell's base and H its high. We study the case in which $H/R \ll 1$, (shallow), and $h \ll R$, (thin).

Let

$$\omega_o = -H\left[1-\left(\frac{r}{R}\right)^2\right] \qquad \text{(A.1.1)}$$

be the vertical displacement of a point at a distance r from the vertical axis when P = 0, and let

$$\omega_T = \omega + \omega_o \qquad \text{(A.1.2)}$$

be the vertical displacement of the point for a non-zero pressure P.

According to Volmir, [21], the coupled system of two partial differential equations that describe the deformation of the shell are

$$\frac{D}{h}\nabla^2\nabla^2\omega_T = L(\omega_T,\Phi) + \frac{P}{h} \qquad \text{(A.1.3)}$$

$$\frac{1}{E}\nabla^2\nabla^2\Phi = -\frac{1}{2}L(\omega_T,\omega_T), \qquad \text{(A.1.4)}$$

where the operator L is given by

$$L(\omega_T,\Phi) = \frac{\partial^2\omega_T}{\partial r^2}\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2}\right) + \left(\frac{1}{r}\frac{\partial\omega_T}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega_T}{\partial\varphi^2}\right)\frac{\partial^2\Phi}{\partial r^2} - 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\omega_T}{\partial\varphi}\right) \qquad \text{(A.1.5)}$$

Since Φ and ω_T are independent of φ, introducing (A.1.5) in (A.1.3), (A.1.4) we have:

(A.1.6) $$D\, r \frac{d}{dr}\left(\frac{1}{r} \frac{d}{dr}\left(r \frac{d\omega_T}{dr}\right)\right) = h \frac{d\omega_T}{dr}\frac{d\Phi}{dr} + \frac{P\, r^2}{2} + C_1$$

(A.1.7) $$r \frac{d}{dr}\left(\frac{1}{r} \frac{d}{dr}\left(r \frac{d\Phi}{dr}\right)\right) = - \frac{E}{2}\left(\frac{d\omega_T}{dr}\right)^2 + C_2$$

C_1, C_2 constants.

Since the shell is complete we can assume the existence of both $d\omega_T / dr$ and $d\Phi / dr$ at $r = 0$, moreover, the axisymmetry implies that both derivatives are zero, therefore, $C_1 = C_2 = 0$.

Introducing (A.1.1),(A.1.2) in (A.1.6),(A.1.7), we have:

(A.1.8) $$D\, r \left[\frac{d^3\omega}{dr^3} + \frac{1}{r}\frac{d^2\omega}{dr^2} - \frac{1}{r^2}\frac{d\omega}{dr} \right] = \frac{P\, r^2}{2} + h \frac{d\omega}{dr}\frac{d\Phi}{dr} + h \frac{d\omega_o}{dr}\frac{d\Phi}{dr}$$

(A.1.9) $$r \frac{d^3\Phi}{dr^3} + \frac{d^2\Phi}{dr^2} - \frac{1}{r}\frac{d\Phi}{dr} = - \frac{E}{2}\left[\left(\frac{d\omega}{dr}\right)^2 + 2 \frac{d\omega}{dr}\frac{d\omega_o}{dr} \right]$$

In (A.1.8),(A.1.9) we have neglected the term

$$\left(\frac{d\omega_o}{dr}\right)^2 = \frac{4\, r^2\, H^2}{R^4}$$

in virtue of the shallowness of the shell, ($H / R \ll 1$).

Let

$$\beta(r) = - \frac{d\omega}{dr}$$

$$\psi(r) = h \frac{d\Phi}{dr}$$

thus, equations (A.1.8),(A.1.9) become:

(A.1.10) $$D\left[r\, \beta'' + \beta' - \frac{\beta}{r} \right] = - \frac{P\, r^2}{2} + \psi\beta - \frac{2H\, \psi\, r}{R^2}$$

(A.1.11) $$r\, \psi'' + \psi' - \frac{\psi}{r} = - E \frac{\beta^2 h}{2} + \frac{2\, E\, H\, h\, \beta\, r}{R^2}$$

$(\ )' = d / dr$.

§ A.2 The geometrical parameter λ and the dimensionless variables θ,ϕ

Let $x = r/R$, $0 \leq x \leq 1$, then

$$\frac{d\beta}{dr} = \frac{1}{R}\frac{d\beta}{dx} = \beta' \quad , \quad \frac{d\psi}{dr} = \frac{1}{R}\frac{d\psi}{dx} = \psi'$$

With this change of variables, equations (A.1.10),(A.1.11) are transformed into:

$$\text{(A.2.1)} \qquad x\,\beta'' + \beta' - \frac{\beta}{x} = \frac{12(1-\mu^2)R}{Eh^3}\left[-\tfrac{1}{2}PR^2x^2 + \psi\beta - \frac{2H\psi x}{R} \right]$$

$$\text{(A.2.2)} \qquad x\psi'' + \psi' - \frac{\psi}{x} = EhR\left[-\tfrac{1}{2}\beta^2 + \frac{2H\beta x}{R} \right]$$

Let

$$\text{(A.2.3)} \qquad \beta(Rx) = A(R)\theta(x)$$

$$\psi(Rx) = B(R)\phi(x).$$

Introducing (A.2.3) in (A.2.1),(A.2.2) we get

$$\text{(A.2.4)} \qquad A(R)\left[x\,\theta'' + \theta' - \frac{\theta}{x} \right] = \frac{12(1-\mu^2)R}{Eh^3}\left[-\tfrac{1}{2}PR^2x^2 + A(R)B(R)\theta(x)\phi(x) - \frac{2Hx}{R}B(R)\phi(x) \right]$$

$$\text{(A.2.5)} \qquad B(R)\left[x\,\phi'' + \phi' - \frac{\phi}{x} \right] = EhR\left[-\tfrac{1}{2}A^2(R)\theta^2(x) + \frac{2HA(R)\theta(x)}{R} \right]$$

Let us define

$$\text{(A.2.6)} \qquad A^2(r) = \frac{h^2}{12(1-\mu^2)R^2} \quad ; \quad B(R) = \frac{Eh^3}{12(1-\mu^2)R}$$

Substituting A(R),B(R) in (A.2.4),(A.2.5) by (A.2.6), we obtain

$$\text{(A.2.7)} \qquad x\theta'' + \theta' - \frac{\theta}{x} = -\frac{12R^4x^2\sqrt{3(1-\mu^2)^3}}{Eh^4}\,p + \theta\phi - \frac{H}{h}\sqrt{48(1-\mu^2)}\,x\phi$$

$$\text{(A.2.8)} \qquad x\phi'' + \phi' - \frac{\phi}{x} = -\tfrac{1}{2}\theta^2 + \frac{H}{h}\sqrt{48(1-\mu^2)}\,\theta$$

Finally, let us introduce the geometrical parameter λ as follows:

(A.2.9) $$\lambda^4 = \frac{48(1 - \mu^2)H^2}{h^2}$$

Then, equations (A.2.7), (A.2.8) become

(A.2.10) $$x\theta'' + \theta' - \frac{\theta}{x} = - 2\, px^2 + \theta\phi - \lambda^2\phi$$

(A.2.11) $$x\phi'' + \phi' - \frac{\phi}{x} = - \frac{\theta^2}{2} + \lambda^2\theta\, x,$$

where

(A.2.12) $$p = \frac{6\sqrt{3(1 - \mu^2)^3}\, R^4}{E\, h^4}\, P = \frac{\lambda^4}{q_o}\, P$$

The parameter

(A.2.13) $$q_o = \frac{8Eh^2H^2}{3(1-\mu^2)\, R^4}$$

is the classical buckling pressure of a complete spherical shell with radius equal to the radius of curvature of the shell under consideration in this thesis at the origin, namely $\rho = R^2 / 2H$.

§A.3 Derivation of equations (2.3) and (2.4)

Assuming that we have computed the coefficients a_i and b_i, for $i = 1,2...n$, where n has been fixed previously, we can write down our approximate solutions (1.22) and (1.23). We can proceed further to calculate the approximate deflection of the plate at any point at radial distance r from the center of the plate, as well as the approximate stresses, both, membrane and bending, for fixed pressure p.

Since the deflection at the edge of the plate is zero, i.e.,

$$\omega^*(R) = \omega^*_T(R) = 0,$$

we have

(A.3.1) $$\omega^*_T(r) = \omega^*_T(r) - \omega^*_T(R) = \int_R^r \frac{d\omega^*}{ds} ds + \omega_o(r) =$$

$$\int_R^r (-\beta(s)) ds + \omega_o(r) = \int_r^R \beta(s) ds + \omega_o(r)$$

In addition, since

(A.3.2) $$\frac{d}{dx}[\, x^{-\nu} J_\nu(x) \,] = - x^{-\nu} J_{\nu+1}(x),$$

introducing (1.11) and (1.22) in (A.3.1) we obtain

(A.3.3) $$\omega^*_T(Rx) = \frac{2H}{\lambda^2} \int_x^1 \sum_{i=1}^{n} a_i J_1(k_i s)\, ds - H(\, 1 - x^2) =$$

$$\frac{2H}{\lambda^2} \sum_{i=1}^{n} \frac{a_i}{k_i} [\, J_o(k_i x) - J_o(k_i) \,] - H(\, 1 - x^2)$$

Let us define the non dimensional deflection as follows:

(A.3.4) $$\omega_T(Rx) = \frac{\lambda^2}{2H} \omega^*_T(Rx).$$

Since

(A.3.5) $$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x),$$ (see [1], pp 361) we have

$$(A.3.6)\qquad \omega_T(Rx) = \sum_{i=1}^{n} \frac{J_o(k_i x) + J_2(k_i)}{k_i} a_i - \frac{\lambda^2}{2}(1 - x^2)$$

hence, the shell's dimensionless deflection is

$$(A.3.7)\qquad \omega(Rx) = \sum_{i=1}^{n} \frac{J_o(k_i x) + J_2(k_i)}{k_i} a_i \quad , \left(\omega = \frac{\lambda^2}{2H} \omega^* \right)$$

In particular:

$$(A.3.8)\qquad \omega(0) = \sum_{i=1}^{n} \frac{1 + J_2(k_i)}{k_i} a_i$$

§ A.4 Derivation of equations (2.6) and (2.8)

The radial membrane stress, σ_r^*, is defined as, [21],

$$(A.4.1)\qquad \sigma_r^*(r) = \psi(r) / hr$$

Introducing (1.11) and (1.23) in (A 4.1) we get

$$(A.4.2)\qquad \sigma_r^*(Rx) = \frac{Eh^2 \phi(x)}{12x(1-\mu^2)R^2} = \frac{Eh^2}{12(1-\mu^2)R^2} \left[b_o + \sum_{i=1}^{n} \frac{b_i J_1(k_i x)}{x} . \right]$$

Substituting (1.27) and (1.30):

$$(A.4.3)\qquad \sigma_r^*(Rx) = \frac{Eh^2}{12R^2(1-\mu^2)} \left[- \frac{\lambda^2}{1-\mu} \sum_{i=1}^{n} \frac{J_2(k_i)}{k_i} a_i + \right.$$

$$\frac{1}{1-\mu} \sum_{i=1}^{n} \frac{j_{i\ell s}\, a_i a_\ell}{k_s J_2(k_s)} - \lambda^2 \sum_{i=1}^{n} \frac{J_1(k_i x)}{x\, k_i^2} a_i +$$

$$\left. \sum_{i,\ell,s=1}^{n} \frac{J_1(k_s x)\, j_{i\ell s}}{x k_s^2 J_2^2(k_s)} a_i a_\ell \right] \qquad (\text{for all } x \neq 0)$$

If we now define

$$\text{(A.4.4)} \qquad \sigma_r(Rx) = \frac{12R^2(1-\mu^2)}{Eh^2}\,\sigma_r^*(Rx),$$

we obtain:

$$\text{(A.4.5)} \qquad \sigma_r(Rx) = -\lambda^2\Big[\frac{1}{1-\mu}\sum_{i=1}^{n}\frac{J_2(k_i)}{k_i}a_i + \sum_{i=1}^{n}\frac{J_1(k_i x)}{k_i^2\,x}a_i\Big] +$$

$$\frac{1}{1-\mu}\sum_{i,\ell,s=1}^{n}\frac{j_{i\ell s}\,a_i a_\ell}{k_s\,J_2(k_s)} + \sum_{i,\ell,s=1}^{n}\frac{J_1(k_s x)\;j_{i\ell s}a_i a_\ell}{x\,k_s^2\,J_2^2(k_s)} \qquad (x \neq 0)$$

The stress function is undefined at the center of the shell because the expression $J_1(x)/x$ approaches the form $0/0$ as x tends to 0 However, σ_r can be extended by continuity to $x = 0$ since

$$\frac{J_1(x)}{x} = \Big[\frac{x}{2} - \frac{x^3}{2^3 2!} + \frac{x^5}{2^5 2!3!} - \ldots\Big]/x = \tfrac{1}{2} - x^2/\,2^3 2! + \ldots$$

thus,

$$\text{(A.4.6)} \qquad \lim_{x\to 0}\frac{J_1(k_i x)}{x} = \frac{k_i}{2}$$

So, for $x = 0$, one can use (A.4.6) to compute the radial membrane stress as follows:

$$\text{(A.4.7)} \qquad \sigma_r(0) = -\lambda^2\sum_{i=1}^{n}\Big[\frac{J_2(k_i)}{1-\mu} + \tfrac{1}{2}\Big]\frac{a_i}{k_i} +$$

$$\sum_{i,\ell,s=1}^{n}\Big[\frac{1}{1-\mu} + \frac{1}{2J_2(k_i)}\Big]\frac{j_{i\ell s}}{k_s J_2(k_s)}\,a_i a_\ell\,,$$

where

$$\sigma_r = \frac{12R^2(1-\mu^2)}{E\,h^2}\,\sigma_r^*$$

is the dimensionless stress function.

§ A.5 Derivation of equations (2.11) and (2.13)

The tangential membrane stress, σ_e^*, is defined as

$$\text{(A.5.1)} \qquad \sigma_e^*(r) = \frac{1}{h}\frac{d\psi}{dr}$$

Introducing (1.11) and (1.23) in (A.5.1) we get:

$$\text{(A.5.2)} \qquad \sigma_e^*(Rx) = \frac{E\,h^2}{12(1-\mu^2)R^2}\left[\, b_o + \sum_{i=1}^{n} b_i \frac{d}{dx} J_1(k_i x) \,\right]$$

Substituting (1.27) and (1.30) and using (1.29) in (A.5.2) we obtain:

$$\text{(A.5.3)} \qquad \sigma_e(Rx) = \sum_{i=1}^{n} \left\{ \frac{k_i J_2(k_i)}{1-\mu} + k_i J_o(k_i x) - \frac{J_1(k_i x)}{x} \right\} \cdot$$

$$\left\{ -\frac{\lambda^2}{k_i^2} a_i + \frac{1}{k_i^2 J_2^2(k_i)} \sum_{s,\ell=1}^{n} j_{si\ell}\, a_s a_\ell \right\}, \qquad (x \neq 0)$$

where

$$\sigma_e(Rx) = \frac{12(1-\mu^2)R^2}{Eh^2}\,\sigma_e^*(Rx)$$

Using (A.4.6), the tangential membrane stress at the center of the shell, i.e., $x = 0$, results:

$$\text{(A.5.4)} \qquad \sigma_e(0) = -\frac{\lambda^2}{1-\mu} \sum_{i=1}^{n} \frac{J_2(k_i)}{k_i} a_i + \frac{1}{1-\mu} \sum_{i,\ell,s=1}^{n} \frac{j_{si\ell}\, a_i a_\ell}{k_s J_2(k_s)} -$$

$$-\frac{\lambda^2}{2} \sum_{i=1}^{n} \frac{a_i}{k_i} + \frac{1}{2} \sum_{i,\ell,s=1}^{n} \frac{j_{si\ell}\, a_i a_\ell}{k_s J_2^2(k_s)}$$

§ A.6 Derivation of equations (2.18), (2.19) and (2.20).

The bending stresses, radial, $\sigma^*_{r,b}$ and circunferential, $\sigma^*_{\theta,b}$, are calculated at the face of the plate where they attain their maximum, i.e., at $z = h/2$. We have

$$\text{(A.6.1)} \qquad \sigma^*_{r,b} = \frac{6D}{h^2}\left[\frac{d\phi}{dr} + \mu\frac{\phi}{r}\right]$$

Introducing (1.11), (1.22) and (1.29) we get

$$\text{(A.6.2)} \qquad \sigma^*_{r,b}(Rx) = \frac{Eh}{1-\mu^2}\left\{\frac{H}{\lambda^2R^2}\sum_{i=1}^{n}\frac{d}{dx}J_1(k_ix)a_i + \frac{\mu H}{\lambda^2R^2}\sum_{i=1}^{n}\frac{J_1(k_ix)a_i}{x}\right\} =$$

$$= \frac{EhH}{(1-\mu^2)R^2\lambda^2}\left\{\sum_{i=1}^{n}k_i\,J_o(k_ix)a_i - (1-\mu)\sum_{i=1}^{n}\frac{J_1(k_ix)a_i}{x}\right\} \qquad x \neq 0$$

If we define the dimensionless radial bending stress as

$$\text{(A.6.3)} \qquad \sigma_{r,b} = \frac{(1-\mu^2)R^2\lambda^2}{EhH}\,\sigma^*_{r,b}\,,$$

we get

$$\text{(A.6.4)} \qquad \sigma_{r,b}(Rx) = \sum_{i=1}^{n}k_iJ_o(k_ix)a_i - (1-\mu)\sum_{i=1}^{n}\frac{J_1(k_ix)a_i}{x} \qquad x \neq 0$$

Applying (A.4.6), we compute the radial bending stress at the center of the shell,

$$\text{(A.6.5)} \qquad \sigma_{r,b}(0) = \sum_{i=1}^{n}\left[k_i - \tfrac{1}{2}(1-\mu)k_i\right]a_i = \frac{1+\mu}{2}\sum_{i=1}^{n}k_ia_i$$

The tangential bending stress is given by

$$\text{(A.6 6)} \qquad \sigma^*_{\theta,b}(r) = \frac{6D}{h^2}\left[\frac{\phi}{r} + \mu\frac{d\phi}{dr}\right]$$

Introducing (1.11),(1.22) and (1.29) we obtain

$$(A.6.7) \qquad \sigma^*_{e,b}(Rx) = \frac{Eh}{1-\mu^2}\left\{\frac{H}{\lambda^2R^2}\sum_{i=1}^{n}\frac{J_1(k_ix)}{x}a_i + \frac{\mu H}{\lambda^2R^2}\sum_{i=1}^{n}\frac{d}{dx}J_1(k_ix)\ a_i\right\} =$$

$$\frac{EhH}{(1-\mu^2)\lambda^2R^2}\sum_{i=1}^{n}\left[\frac{J_1(k_ix)}{x}(1-\mu) + k_iJ_o(k_ix)\mu\right]a_i$$

If we define the dimensionless tangential bending stress as

$$(A.6.8) \qquad \sigma_{e,b}(Rx) = \frac{(1-\mu^2)R^2\lambda^2}{EhH}\sigma^*_{e,b}(Rx)$$

we then get

$$(A.6.9) \qquad \sigma_{e,b}(Rx) = \sum_{i=1}^{n}\left[(1-\mu)\frac{J_1(k_ix)}{x} + \mu k_iJ_o(k_ix)\right]a_i \qquad x\neq 0$$

Finally,we use (A.4.6) to compute the nondimensional tangential bending stress at the center of the shell,

$$(A.6.10) \qquad \sigma_{e,b}(0) = \sum_{i=1}^{n}\left[\tfrac{1}{2}(1-\mu)k_i + \mu k_i\right]\cdot a_i = \frac{1+\mu}{2}\sum_{i=1}^{n}k_ia_i$$

Appendix B
Derivation of equation (1.31)

As we deduced in chapter 1, the state of the shell under study is represented mathematically by the boundary value problem

(B.1) $$\theta'' + \frac{\theta'}{x} - \frac{\theta}{x^2} = -\lambda^2\phi - 2px + \frac{\theta\phi}{x}$$

(B.2) $$\phi'' + \frac{\phi'}{x} - \frac{\phi}{x^2} = \lambda^2\theta - \frac{\theta^2}{2x}$$

(B.3) $\theta(0)=0$

(B 4) $\theta(1)=0$

(B.5) $\phi(0)=0$

(B.6) $\phi'(1) - \mu\phi(1)=0$

We pointed out in chapter 1 that the left hand side of equations (B.1),(B.2) are the Bessel operator of index 1, namely:

(B.7) $$B[f] = f'' + \frac{f'}{x} - \frac{f}{x^2} ,$$

thus, we can rewrite (B.1),(B.2) as

(B.8) $$B[\theta] = \frac{1}{x}\left[(x\theta')' - \frac{1}{x}\theta\right] = -\lambda^2\phi - 2px + \frac{\theta\phi}{x}$$

(B.9) $$B[\phi] = \frac{1}{x}\left[(x\phi')' - \frac{1}{x}\phi\right] = \lambda^2\theta - \frac{\theta^2}{2x}$$

From the theory of Bessel functions, [8], we know that if $f = J_1(kr)$, then $B[f] = -k^2 f$ where k^2 can be called the formal eigenvalue associated with the formal eigenfunction f

We now reason as follows: if equations (1.20) and (1.21) have solutions they could certainly be expanded in an infinite series of Bessel functions since the set $\left\{J_1(k_i x)\right\}_{i=1}^{\infty}$ is complete in [0,1] It is also an orthonormal set in the same interval with respect to the weight function x. The boundary condition (3 4) suggests as a tentative choice

(B.10) $$\theta(x) = \sum_{i=1}^{\infty} a_i J_1(k_i x)$$

if we ask the k_i to be such that $J_1(k_i) = 0$ for every n,(with this, each term of the series will satisfy the two boundary conditions for θ, namely (B.3) and (B.4)).

For ϕ the choice will be

$$\phi(x) = b_o x + \sum_{i=1}^{\infty} b_i J_1(k_i x) \tag{B.11}$$

Each term satisfies (B.5) and they will also satisfy (B.6) if we require

$$b_o + \sum_{i=1}^{\infty} \frac{d}{dx} J_1(k_i x)\, b_i \quad - \quad \mu b_o = 0 \qquad \text{at } x = 1 \tag{B.12}$$

The undetermined coefficients a_i's and b_i's are formally given in the Fourier-Bessel manner, i.e.,

$$a_i = \frac{2}{R^2 J_2^2(k_i R)} \int_0^R xJ_1(k_i x)\theta(x)dx \tag{B.13}$$

$$b_i = \frac{2}{R^2 J_2^2(k_i R)} \int_0^R xJ_1(k_i x)\phi(x)dx \tag{B.14}$$

Purely formally, one could find the a_i's and b_i's as follows: we insert the expansions for θ and ϕ into equations (1.12) and (1.13), multiply each term of the resulting equations by $xJ_1(k_i x)$ and integrate from 0 to R for each i=1,2...

We arrive at a doubly infinite set of coupled quadratic equations for the a_i's and the b_i's. We note that equation (1.13) allows us to eliminate the b_i's and we can thus obtain a singly infinite set of cubic equations in the a_i's.

Since there is no way that we can deal with the full infinite set of equations at once, we will attempt to find approximate solutions for θ and ϕ in the form of finite series of Bessel functions and obtain finite set of equations in a finite number of variables, equations which may be viewed as truncations of the infinite set. A paper by Rauch, [16], dealing with a similar problem, suggests just such a solution

Since

(B.15) $$\frac{d}{dx}J_1(k_i x) = k_i[\ J_o(k_i x) - \frac{J_1(k_i x)}{k_i x}\ ]$$ ([8],pp 170)

then

(B.16) $$\frac{d}{dx}J_1(k_i x)\Big|_{x=1} = k_i J_o(k_i)$$

Also, since

(B.17) $$\frac{d}{dx}J_{\nu+1}(x) = 2\nu\ J_\nu(x) - x\ J_{\nu-1}(x)$$ ([8],pp 170,(4))

then

(b.18) $$J_o(k_i) = -\ J_2(k_i)$$

which introduced in (B.16) yields

(B.19) $$\frac{d}{dx}J_1(k_i x)\ \Big|_{x=1} = -\ k_i J_2(k_i)$$

which replaced in (B.12) gives

(B.20) $$b_o = \frac{1}{1-\mu}\sum_{i=1}^{\infty} b_i k_i\ J_2(k_i)$$

Now, we proceed to use the method of Boubnov-Galerkin, [16], i.e., we introduce θ and ϕ as defined in (1.22) and (1.23) in (1.12) and (1.13), we multiply each term by $xJ_1(k_m x)$, integrate from 0 to R, and set the resulting expressions equal to zero.

We will use the following properties which can be deduced from Churchill, [8],

(B.21) $$\int_0^1 x^2 J_1(k_i x)\ dx = \frac{J_2(k_i)}{k_i}$$

(B.22) $$\int_0^1 x\ J_1(k_i x)\ J_1(k_m x)\ dx = \frac{\delta_{im}\ J_2^2(k_m)}{2}$$

where δ_{im} is Kroenecker's delta.

Equation (1.12) yields

(B.23)
$$0 = -k_i^2 a_i \int_0^1 J_1(k_i x)J_1(k_m x)\; x\, dx + \lambda^2 \int_0^1 [b_o x + \sum_{i=1} b_i J_1(k_i x)]$$

$$J_1(k_m x)\; x\, dx + 2p \int_0^1 x^2 J_1(k_m x) dx -$$

$$\int_0^1 [\sum_{i=1}^{n} a_i J_1(k_i x)][\; b_o x + \sum_{\ell=1}^{n} b_\ell J_1(k_\ell x)] J_1(k_m x)\; dx =$$

$$-\frac{k_m^2\; J_2^2\,(k_m)}{2}\, a_m + \lambda^2\; \frac{J_2(k_m)}{k_m}\, b_o + \lambda^2\, \frac{J_2^2(k_m)}{2}\, b_m \;+$$

$$2p\, \frac{J_2(k_m)}{k_m} - \tfrac{1}{2}\, J_2^2(k_m) b_o a_m - \sum_{i,\ell=1}^{n} j_{i\ell m} a_i b_\ell \qquad \text{(for every m)}$$

Equation (1.13) yields:

(B.24)
$$0 = \int_0^1 -k_i^2 \sum_{\ell=1} b_\ell J_1(k_\ell x) J_1(k_m x)\; x\, dx - \lambda^2 a_i \int_0^1 J_1(k_m x) J_1(k_i x) x dx \;+$$

$$\tfrac{1}{2} \int_0^1 [\; \sum_{\ell=1}^{n} a_\ell J_1(k_\ell x)]^2\; J_1(k_m x)\; dx =$$

$$= -\tfrac{1}{2}\, k_m^2\; J_2^2(k_m) b_m - \tfrac{1}{2}\lambda^2\; J_2^2(k_m) a_m + \tfrac{1}{2} \sum_{i,\ell=1}^{n} j_{i\ell m} a_i b_\ell$$

where

$$j_{i\ell m} = \int_0^1 J_1(k_i x) J_1(k_\ell x) J_1(k_m x)\; dx$$

From (B.24) we get

(B.25) $$b_m = -\frac{\lambda^2}{k_m^2} a_m + \frac{1}{k_m^2 J_2^2(k_m)} \sum_{i,\nu=1}^{n} j_{i\nu m} a_i a_\nu \quad \text{(for every } m)$$

which substituted in (B.23) together with (1.27) gives

(B.26) $$p = \frac{(k_m^4 + \lambda^4) J_2(k_m) a_m}{4k_m} + \frac{\lambda^2}{2(1-\mu)} [\lambda^2 - \tfrac{1}{2} J_2(k_m) k_m a_m]$$

$$\sum_{i=1}^{n} \frac{J_2(k_i) a_i}{k_i} - \frac{\lambda^2}{4k_m J_2(k_m)} \sum_{i,\nu=1}^{n} j_{i\nu m} a_i a_\nu \; -$$

$$\frac{\lambda^2 k_m}{2J_2(k_m)} \sum_{i,\nu=1}^{n} \frac{j_{i\nu m} a_i a_\nu}{k_\nu^2} - \frac{1}{2(1-\mu)} [\lambda^2 - \frac{J_2(k_m) k_m a_m}{2}]$$

$$\sum_{s,i,\nu=1}^{n} \frac{j_{s\nu i}\, a_s a_\nu}{k_i\, J_2(k_i)} + \frac{k_m}{2\, J_2(k_m)} \sum_{i,t,s,\nu=1}^{n} \frac{j_{i\nu m} j_{st\nu} a_t a_s a_\nu}{k_\nu^2\, J_2^2(k_\nu)}$$

Appendix C

§ C.1 Fortran program BUCKLD

It computes deflections and stresses at the origin

```
IV G LEVEL  21                    MAIN                  DATE = 78158          00/13/29

          EXTERNAL F
          REAL*8 K(15),T(15,15,15),A10,DA10,Y(15),F,EPS,WA(170),W,STEP,J(15)
         1,LAMBDA,MU,RTBENO,RTMEMO,X(3),Z(15,100)
          COMMON/BESSEL/J,K,T
          COMMON/PARAM/LAMBDA,MU
          COMMON MODE
C         X(1)=W,X(2)=RTBENO,X(3)=RTMEMO
C         Z(I,J)=A(I) AFTER J INCREMENTS OF AI
999       READ 1,LAMBDA,MU,LAST
1         FORMAT(2F10.6,50X,I2)
          IF(LAST.EQ.99)GO TO 1000
          READ 2,A10,DA10,EPS,MAXMOD,MAXA10,NSIG,IMAXO
2         FORMAT(3F10.6,4I5)
          PRINT 100
100       FORMAT(1H1)
          READ 3,(K(I),J(I), I=1,MAXMOD)
3         FORMAT(2F10.6)
          PRINT 3,K(I),J(I)
          DO 4 I=1,MAXMOD
          DO 5 M=I,MAXMOD
          DO 6 L=M,MAXMOD
          READ 7,TRIPLE
7         FORMAT(1F10.6)
          PRINT 7,TRIPLE
          T(I,M,L)=TRIPLE
          T(M,I,L)=TRIPLE
          T(I,L,M)=TRIPLE
          T(M,L,I)=TRIPLE
          T(L,I,M)=TRIPLE
          T(L,M,I)=TRIPLE
6         CONTINUE
5         CONTINUE
4         CONTINUE
          PRINT 100
          PRINT 101,LAMBDA,MU,A10,DA10,EPS,NSIG,MAXMOD,IMAXO
101       FORMAT(//10X,'SHELL PARAMETERS'//10X,'LAMBDA=',F10.6//10X,'MU=',
         1F10.6//10X,'INITIAL GUESS:A10=',F10.6//10X,'INCREMENT FOR A10:DA10=
         2',F10.6//10X,'FIRST ERROR ALLOWES:EPS=',F10.6//10X,'SECOND ERROR
         3ALLOWED:NSIG=',I5//10X,'NUMBER OF MODES:MAXMOD=',I5//10X,'NUMBER
         4OF ITERATIONS ALLOWED:IMAXO=',I5)
          PRINT 100
          PRINT 102
102       FORMAT(//43X,'F I R S T   M O D E'//16X,'PRESSURE',13X,'AI',16X,
         1'DEFLECTION',12X,'RTBENO',13X,'RTMEMO'//)
          MODE=1
          DO 8 L=1,MAXA10
          IF(L.NE.50)GO TO 81
          PRINT 100
81        STEP=FLOAT(L-1)
          AI=A10+DA10*STEP
          Y(1)=F(1.,1,AI,IER)/100.
          Z(1,L)=Y(1)
          CALL ORIGIN(Y,AI,X)
          PRINT 9,L,Y(1),AI,(X(I),I=1,3)
          WRITE(7,888) Y(1),(X(I),I=1,3)
888       FORMAT(4F15.6)
```

```
9       FORMAT(1X,I3,6X,5(F15.6, 5X))
8       CONTINUE
        IF(MAXMOD.EQ.1)GO TO 999
        Y(1)=Z(1,1)
        DO 10 M=2,MAXMOD
        MODE=M
        PRINT 100
        PRINT 103,M
103     FORMAT(//23X,'MODE NUMBER',2X,I2//16X,'PRESSURE',16X,'DEFLECTION'
       1,12X,'RTBENO',13X,'RTMEMO'//)
        Y(M)=0.0
        DO 12 L=1,MAXA10
        IF(L.NE.40)GO TO 121
        PRINT 100
121     IMAX=IMAXO
        STEP=FLOAT(L-1)
        A1=A10+DA10*STEP
        CALL ZSYSTM(F,EPS,NSIG,M,Y,IMAX,WA,A1,IER)
        IF(IER.EQ.129)GO TO 500
        IF(IER.EQ.130)GO TO 502
        Z(1,L)=A1
        DO 122 I=2,MODE
122     Z(I,L)=Y(I)
        IER=MODE+1
        IF(IER.GT.15)GO TO 128
        DO 123 I=IER,15
123     Z(I,L)=0.
128     CALL ORIGIN(Y,A1,X)
        PRINT 14,L,Y(1),(X(I),I=1,3)
14      FORMAT(1X,I3,6X,4(F15.6,10X),2X,I5)
        WRITE(7,888) Y(1),(X(I),I=1,3)
12      CONTINUE
        PRINT 100
        PRINT 124,(L,(Z(I,L),I=1,8),L=1,80)
124     FORMAT(1X,I3,2X,8F15.6)
        PRINT 125,(L,(Z(I+8,L),I=1,7),L=1,80)
125     FORMAT(1X,I3,2X,7F15.6)
        DO 129 I=2,M
129     Y(I)=Z(I,1)
10      CONTINUE
        GO TO 999
500     PRINT 501,IMAX,M
501     FORMAT(//10X,'FAILURE TO CONVERGE WITHIN',1X,I5,1X,'ITERATIONS'/
       110X,'MODE NUMBER',1X,I3)
        GO TO 1000
502     PRINT 503,IMAX,M
503     FORMAT(//10X,'SINGULAR JACOBIAN AFTER',1X,I5/10X,'MODE NUMBER',
       11X,I3)
1000    PRINT 100
        STOP
        END
```

:V G LEVEL 21 ORIGIN DATE = 78158

```
      SUBROUTINE ORIGIN(Y,PAR,X)
      REAL*8 Y(15),X(3),PAR,J(15),K(15),MU,LAMBDA,T(15,15,15),U
      COMMON/BESSEL/J,K,T
      COMMON/PARAM/LAMBDA,MU
      COMMON MODE
      U=Y(1)
      Y(1)=PAR
      X(1)=0.0
      X(2)=0.0
      X(3)=0.0
      DO 1 N=1,MODE
      S=0.0
      DO 2 LL=1,MODE
      DO 3 I=1,MODE
    3 S=S+T(1,LL,N)*Y(LL)*Y(I)
    2 CONTINUE
      X(3)=X(3)+S*(1./(1.-MU)+.5/J(N))/(K(N)*J(N))
      X(3)=X(3)-LAMBDA**2*(J(N)/(1.-MU)+.5)*Y(N)/K(N)
      X(2)=X(2)+.5*(1.+MU)*K(N)*Y(N)
      X(1)=X(1)+Y(N)*(1.+J(N))/K(N)
    1 CONTINUE
      Y(1)=U
      RETURN
      END
```

```
G LEVEL  21                  F                    DATE = 78158          00/13/29
          DOUBLE PRECISION FUNCTION F(X,L,PAR,IER)
          REAL*8 F,X(15),PAR,COE(20),J(15),K(15),LAMBDA,MU,T(15,15,15)
         1,TE(15),S,V
          COMMON/BESSEL/J,K,T
          COMMON/PARAM/LAMBDA,MU
          COMMON MODE
          IF(MODE.GT.1) GO TO 100
          F=((K(1)**4+LAMBDA**4)/2.+LAMBDA**4/(1.-MU))*J(1)*PAR/(2.*K(1))
          COE(1)=1.5+1./(1.-MU)
          COE(1)=COE(1)*T(1,1,1)/(2.*J(1)*K(1))
          COE(1)=COE(1)+(J(1)**2)/(4.*(1.-MU))
          TE(1)=COE(1)*(LAMBDA*PAR)**2
          COE(2)=T(1,1,1)/(4.*(1.-MU))
          COE(2)=COE(2)+T(1,1,1)**2/(2.*K(1)*J(1)**3)
          TE(2)=COE(2)*PAR**3
          F=F-TE(1)+TE(2)
          RETURN
  100     V=0.0
          DO 1 N=1,MODE
          DO 2 LL=1,MODE
          DO 3 I=1,MODE
          S=0.0
          DO 4 JJ=1,MODE
          IF(JJ.GT.1) GO TO 10
          COE(1)=T(I,JJ,LL)*PAR
          GO TO 11
   10     COE(1)=T(I,JJ,LL)*X(JJ)
   11     S=S+COE(1)
    4     CONTINUE
          IF(N.GT.1.AND.I.GT.1) GO TO 20
          IF(N.EQ.1.AND.I.EQ.1) GO TO 30
          IF(N.EQ.1.AND.I.GT.1) GO TO 40
          COE(2)=.5*K(L)*T(N,LL,L)*PAR*X(N)
          GO TO 21
   30     COE(2)=.5*K(L)*T(N,LL,L)*PAR**2
          GO TO 21
   40     COE(2)=.5*K(L)*T(N,LL,L)*PAR*X(I)
          GO TO 21
   20     COE(2)=.5*K(L)*T(N,LL,L)*X(N)*X(I)
   21     COE(3)=COE(2)*S/((K(LL)*J(LL))**2*J(L))
          IF(L.GT.1) GO TO 50
          COE(4)=.5*(LAMBDA**2-.5*J(L)*K(L)*PAR)/(1.-MU)
          GO TO 51
   50     COE(4)=.5*(LAMBDA**2-.5*J(L)*K(L)*X(L))/(1.-MU)
   51     IF(LL.GT.1.AND.I.GT.1) GO TO 60
          IF(LL.EQ.1.AND.I.EQ.1) GO TO 70
          IF(LL.EQ.1.AND.I.GT.1) GO TO 80
          COE(5)=T(I,LL,N)*X(LL)*PAR/(K(N)*J(N))
          GO TO 81
   80     COE(5)=T(I,LL,N)*X(I)*PAR/(K(N)*J(N))
          GO TO 81
   70     COE(5)=T(I,LL,N)*PAR**2/(K(N)*J(N))
          GO TO 81
   60     COE(5)=T(I,LL,N)*X(LL)*X(I)/(K(N)*J(N))
   81     COE(6)=COE(3)-COE(4)*COE(5)
          V=V+COE(6)
```

```
FORTRAN IV G LEVEL  21                    E                    DATE = 78138

      3     CONTINUE
            COE(7)=.25/K(L)+.5/K(LL)**2
            IF(LL.GT.1.AND.N.GT.1) GO TO 90
            IF(LL.EQ.1.AND.N.EQ.1) GO TO 95
            IF(LL.EQ.1.AND.N.GT.1) GO TO 931
            COE(8)=LAMBDA**2*T(N,LL,L)*PAR*X(LL)*COE(7)/J(L)
            GO TO 91
    931     COE(8)=LAMBDA**2*T(N,LL,L)*PAR*X(N)*COE(7)/J(L)
            GO TO 91
     95     COE(8)=LAMBDA**2*T(N,LL,L)*PAR**2*COE(7)/J(L)
            GO TO 91
     90     COE(8)=LAMBDA**2*T(N,LL,L)*X(N)*X(LL)*COE(7)/J(L)
     91     V=V-COE(8)
      2     CONTINUE
            IF(N.GT.1) GO TO 105
            COE(9)=J(N)*LAMBDA**2*COE(4)*PAR/K(N)
            GO TO 101
    105     COE(9)=J(N)*LAMBDA**2*COE(4)*X(N)/K(N)
    101     V=V+COE(9)
      1     CONTINUE
            IF(L.GT.1) GO TO 110
            COE(10)=.25*(K(L)**4+LAMBDA**4)*J(L)*PAR/K(L)
            GO TO 111
    110     COE(10)=.25*(K(L)**4+LAMBDA**4)*J(L)*X(L)/K(L)
    111      V=V+COE(10)
            F=V-100.*X(1)
            RETURN
            END
```

§C.2 Fortran program RADIUS

It computes the deflection and stresses along the radius

```
IV C LEVEL  21                    MAIN                DATE = 78178          23/00/37

          REAL*8 LAMBDA,MU,P,X,W,RBENX,TBENX,RMEMX,TMEMX,TRIPLE,B(3,15,101)
         1,ZK(15),T(15,15,15),A(15),TER(2),PART(11)
          PRINT 100
   100    FORMAT(1H1)
          DO 2 J=1,15
          DO 3 L=1,101
          READ 4,(B(I,J,L),I=1,3)
   4       FORMAT(5X,3F15.10)
   3      CONTINUE
   2      CONTINUE
          READ 5,(ZK(I),I=1,15   )
   5      FORMAT(1F10.6)
          DO 6 IX1=1,15
          DO 7 IX2=IX1,15
          DO 8 IX3=IX2,15
          READ 5,TRIPLE
          T(IX1,IX2,IX3)=TRIPLE
          T(IX1,IX3,IX2)=TRIPLE
          T(IX2,IX1,IX3)=TRIPLE
          T(IX2,IX3,IX1)=TRIPLE
          T(IX3,IX1,IX2)=TRIPLE
          T(IX3,IX2,IX1)=TRIPLE
   8      CONTINUE
   7      CONTINUE
   6      CONTINUE
          READ 1,LAMPDA,MU
   1      FORMAT(2F10.6)
   985    READ 10,MODE,P,LAST
   10     FORMAT(I5,F10.6,55X,I2)
          IF(LAST.EQ.99)GO TO 1000
          PRINT 110,LAMBDA,MU,MODE,P
   110    FORMAT(//10X,'SHELL PARAMETERS'//10X,'LAMBDA=',F10.6//10X,'MU=',
         1F10.6////10X,'MODE=',I3//10X,'PRESSURE=',F15.6)
          PRINT 100
          PRINT 120
   120    FORMAT(//17X,'PRESSURE',5X,'DEFLECTION',7X,'RADIAL',5X,'TANGENTIAL'
         1',7X,'RADIAL',7X,'TANGENTIAL'/47X,'BENDING',8X,'BENDING',8X,'MEMBR
         2ANE',5X,'MEMERANE'//77X,'STRESS',7X,'STRESS'//)
          READ 11,(A(I),I=1,MODE)
   11     FORMAT(5F16.6)
          DO 12 I=2,101
          X=.01*(I-1)
          W=0.0
          RBENX=0.
          TBENX=0.
          RMEMX=0.
          TMEMX=0.
          DO 13 N=1,MODE
          TER(1)=0.
          TER(2)=0.
          DO 14 L=1,MODE
          DO 15 M=1,MODE
          PART(1)=T(M,L,N)/(ZK(M)*B(3,M,101)*(1.-MU))
          PART(2)=T(M,L,N)*B(2,M,I)/((ZK(M)*B(3,M,101))**2*X)
          TER(1)=TER(1)+A(L)*(PART(1)+PART(2))
          PART(3)=T(M,L,N)*B(1,M,I)/(ZK(M)*B(3,M,101)**2)
```

```
IV G LEVEL  21                  MAIN                  DATE = 78178
          TER(2)=TER(2)+A(L)*(PART(1)-PART(2)+PART(3))
   15     CONTINUE
   14     CONTINUE
          PART(4)=B(1,N,1)/ZK(N)
          PART(5)=B(3,N,101)/ZK(N)
          W=W+A(N)*(PART(4)+PART(5))
          PART(6)=ZK(N)*B(1,N,1)
          PART(7)=(1.-MU)*B(2,N,1)/X
          RBENX=RBENX+A(N)*(PART(6)-PART(7))
          PART(8)=MU*PART(6)
          TBENX=TBENX+A(N)*(PART(8)+PART(7))
          PART(9)=(-LAMBDA**2)*PART(5)/(1.-MU)
          PART(10)=(-LAMBDA**2)*B(2,N,1)/(ZK(N)**2*X)
          RMEMX=RMEMX+A(N)*(PART(9)+PART(10)+TER(1))
          PART(11)=-LAMBDA**2*PART(4)
          TMEMX=TMEMX+A(N)*(PART(9)+PART(10)+PART(11)+TER(2))
   13     CONTINUE
          PRINT 20,X,W,RBENX,TBENX,RMEMX,TMEMX
   20     FORMAT(10X,6F15.6)
          WRITE(7,50) X,W,RBENX,TBENX,RMEMX,TMEMX
   50     FORMAT(6F12.6)
          IF(I.NE.51)GO TO 12
          PRINT 100
   12     CONTINUE
          GO TO 985
 1000     STOP
          END
```

§ C.3 Fortran program JILK

It computes $J_i(k_\ell x)$, $0 \le x \le 1, i=0,1,2$, $1 \le \ell \le 15$
and the triple integrals $j_{i\ell k}$ $1 \le i,\ell,k \le 15$

```
FORTRAN IV G LEVEL  21                    MAIN                    DAT

              EXTERNAL PRODC
              REAL*8 K(15),T
              DIMENSION WA(200)
              COMMON/BESSEL/K,T
              COMMON/INDEX/IX1,IX2,IX3
              READ 1,(K(I),I=1,15)
        1     FORMAT(1F10.6)
              PRINT 100
      100     FORMAT(1H1)
              GO TO 371
      372     DO 2 I=1,15
              DO 3 L=1,101
              X=.01*(L-1)*K(I)
              CALL BESJN(X,0,WA,BF0)
              CALL BESJN(X,1,WA,BF1)
              CALL BESJN(X,2,WA,BF2)
              PRINT 4,BF0,BF1,BF2
        4     FORMAT(5X,3F15.10)
        3     CONTINUE
              PRINT 100
        2     CONTINUE
      371     DO 30 IX1=1,15
              DO 40 IX2 =IX1,15
              DO 50 IX3 =IX2,15
              TRIPLE=SIMPSN(PRODC,0.,1.,.0000001)
              PRINT 150,TRIPLE,IX1,IX2,IX3
      150     FORMAT(10X,F15.6,5X,3I5)
       50     CONTINUE
       40     CONTINUE
       30     CONTINUE
              STOP
              END
```

```
      SUBROUTINEBESJN(X,N,T,B)
      EQUIVALENCE(ATEX,TWOX),(C(1),B7,T1),(FT,AP),(N1,IC)
     1,(S,APS),(T2,C4),(BET,ENU)
      DIMENSIONAO(6),BO(6)
      DIMENSIONC(4)
      DIMENSIONT(N)
      LOGICALSET
      DATAPH/1.570796/
      DATAAO(1),AO(2),AO(3),AO(4),AO(5),AO(6)/
     1-1.1476882112E+11,98.124896E+6,-142304.,424.,-4.,1./
      DATABO(1),BO(2),BO(3),BO(4),BO(5),BO(6)/
     1-3.150778311E+9,3435232.,-6867.2,3.333333333E+1,-1.,.125/
      SET=.TRUE.
      Y=ABS(X)
      XN=IABS(N)
      N2=XN
      IF(XN.LE.20.)GOTO1
      PRINT 750
750   FORMAT(10X,'BESJN THE N SUPPLIED IS GREATER THAN 20')
      STOP
751   RETURN
    1 IF(Y.NE.0.)GOTO10
    3 IF(N2.EQ.0)GOTO5
    4 B=0.
      RETURN
    5 B=1.
      RETURN
   10 IF(Y.LT.50.)GOTO100
    6 IF(N2.EQ.0)GOTO150
    7 IF(Y/XN.LT.3.)GOTO100
 8    ATEX=8.*Y
      ASYM1=-(XN+.5)*PH+Y
      C(3)=SIN(ASYM1)
      C(2)=COS(ASYM1)
      C(1)=-C(3)
      C(4)=-C(2)
      IC=2
      AP=17.
      APS=33.
      ASYM1=C(1)
      FNU=1.
      FN2=2*N
      DO15I=1,17
      FMP=((FN2-APS)/AP)*((FN2+APS)/ATEX)
      ASYM1=C(IC)+ASYM1*FMP
      AP=AP-1.
      APS=APS-2.
   15 IC=1+MOD(IC,4)
      ASYM1=(ASYM1       )/SQRT(PH*Y)
   14 IF(MOD(N2,2).EQ.0)GOTO18
   17 IF(N.LT.0)ASYM1=-ASYM1
   20 IF(X.LT.0.)ASYM1=-ASYM1
   18 B=ASYM1
      RETURN
  100 IF(Y.LE..001)GOTO200
      TWOX=2.*Y
```

```
V G LEVEL   21                    BESJN
         N1=(6+MAX1(TWOX,XN)/2)*2
         B7=2./Y
         T(1)=0
         T(2)=1.E-19
         DO101I=2,N1
         FT=N1-I
         FT=FT*B7
 101     T(I+1)=T(I)*FT-T(I-1)
         SET=.FALSE.
         IF(Y.GE.50.)GOTO150
 102     S=0.
         N3=N1-2
         DO103I=2,N3,2
 103     S=S+T(I)
         J=N1-N2
         ASYM1=T(J)/(2.*S+T(N1))
         GOTO14
 150     BET=A0(1)
         T1=(8.*Y)**2
         T2=B0(1)
         DO151I=2,6
         BET=A0(I)+(BET/T1)
 151     T2=B0(I)+(T2/T1)
         T2=T2*(8.*Y)-PH/2.
         FT=BET*COS(T2)
         ASYM1=FT/SQRT(PH*Y)
         IF(SET)GOTO14
         J=N1-N2
         ASYM1=T(J)*ASYM1/T(N1)
         GOTO14
 200     T1=Y/2.
         T2=T1**2
         S=0.
         DO201I=1,3
 201     S=(S+(-1.)**I)*T2/FLOAT(N2+4-I)
         ASYM1=(1.+S)
         IF(N2.EQ.0)GOTO18
         N1=1
         DO202I=1,N2
 202     N1=I*N1
         S=N1
         ASYM1=ASYM1*(T1**N2)/S
         GOTO14
         END
```

```
EVEL   21                    PRODC
         FUNCTION PRODC(X)
         REAL*8 K(15),T
         DIMENSION WA(200)
         COMMON/INDEX/IX1,IX2,IX3
         COMMON/BESSEL/K,T
         CALL BESJN(K(IX1)*X,1,WA,BF1IX1)
         CALL BESJN(K(IX2)*X,1,WA,BF1IX2)
         CALL BESJN(K(IX3)*X,1,WA,BF1IX3)
         PRODC=BF1IX1*BF1IX2*BF1IX3
         RETURN
         END
```

```
      FUNCTION SIMPSN(ARG,Y1,Y2,FERR)
C     SIMPSN INTEGRATION ROUTINE WRITTEN AS FORTRAN IV FUNCTION J.SMITHTH
      DIMENSION F2T(20),FMT(20),F3T(20),F4T(20),FBT(20),
     1DXT(20),X1T(20),X2T(20),ART(20),EPST(20),ES2T(20),
     2ES3T(20),LEG(20),SUM1(20),SUM2(20)
C       INITIAL SET-UP
        A=Y1
        EPS=FERR
        B=Y2
        DA=B-A
        FA=ARG(A)
        FM=4.*ARG((A+B)*.5)
        FB=ARG(B)
        AREA=1.0
        EST=1.0
        L=1
C       BEGIN SIMPSON
      1 DX=DA/3.
        X1=A+DX
        X2=X1+DX
        F1=4.*ARG(A+.5*DX)
        F2=ARG(X1)
        F3=ARG(X2)
        F4=4.*ARG(A+2.5*DX)
        DX6=DX/6.
        EST1=(FA+F1+F2)*DX6
        EST2=(F2+FM+F3)*DX6
        EST3=(F3+F4+FB)*DX6
        AREA=AREA-ABS(EST)+ABS(EST1)+ABS(EST2)+ABS(EST3)
        SUM=EST1+EST2+EST3
C       TEST FOR CONVERGENCE
        IF(ABS(EST-SUM)-EPS*AREA)2,2,3
      2 IF(EST-1.0)6,3,6
      3 IF(L-20)5,6,6
      5 L=L+1
        LEG(L)=3
C       STORE PARAMETERS FOR SIMPSON II AND III
        F2T(L)=F2
        FMT(L)=FM
        F3T(L)=F3
        F4T(L)=F4
        FBT(L)=FB
        DXT(L)=DX
        X1T(L)=X1
        X2T(L)=X2
        ART(L)=AREA
        EPST(L)=EPS/1.7
        ES2T(L)=EST2
        ES3T(L)=EST3
C       RETURN TO SIMPSON I
        DA=DX
        FM=F1
        FB=F2
        EST=EST1
        EPS=EPST(L)
        GO TO 1
```

```
IV G LEVEL  21                        SIMPSN
         6  IF(LEG(L)-2)9,8,7
         7  SUM1(L)=SUM
            LEG(L)=2
   C        RETURN TO SIMPSON II
            A=X1T(L)
            DA=DXT(L)
            FA=F2T(L)
            FM=FMT(L)
            FB=F3T(L)
            AREA=ART(L)
            EST=ES2T(L)
            EPS=EPST(L)
            GO TO 1
         8  SUM2(L)=SUM
            LEG(L)=1
   C        RETURN TO SIMPSON III
            A=X2T(L)
            DA=DXT(L)
            FA=F3T(L)
            FM=F4T(L)
            FB=FBT(L)
            AREA=ART(L)
            EST=ES3T(L)
            EPS=EPST(L)
            GO TO 1
         9  SUM=SUM1(L)+SUM2(L)+SUM
            L=L-1
            IF(L-1)11,11,6
        11  SIMPSN = SUM
            RETURN
            END
```

§C.4 Altran program

It verifies the system of equations in BUCKLD for the 2-modes solution

```
ALTRAN VERSION 1 LEVEL 9

PROCEDURE MAIN
ALG(LA:10,P:3,J(2):5,K(2):5,A1:5,A2:5,T(2,2,2):5,MU:5)
ALGEBRAIC ARRAY(10) COE
ALGEBRAIC ARRAY(2) X
ALGEBRAIC V,S,PAR,F
INTEGER N,LL,MODE,I,JJ,L
MODE=2
X(1)=P
X(2)=A2
PAR=A1
DO L=1,2
IX100:    V=0
DO N=1,MODE
DO LL=1,MODE
DO I=1,MODE
S=0
DO JJ=1,MODE
IF(JJ.GT.1)GO TO IX10
ELSE      COE(1)=T(I,JJ,LL)*PAR
GO TO IX11
IX10:     COE(1)=T(I,JJ,LL)*X(JJ)
IX11:     S=S+COE(1)
DOEND
IF(N.GT.1.AND.I.GT.1)GO TO IX20
ELSE      IF(N.EQ.1.AND.I.EQ.1)GO TO IX30
ELSE      IF(N.EQ.1.AND.I.GT.1)GO TO IX40
ELSE      COE(2)=K(L)*T(N,LL,L)*PAR*X(N)/2
GO TO IX21
IX30:     COE(2)=K(L)*T(N,LL,L)*PAR**2/2
GO TO IX21
IX40:     COE(2)=K(L)*T(N,LL,L)*PAR*X(I)/2
GO TO IX21
IX20:     COE(2)=K(L)*T(N,LL,L)*X(N)*X(I)/2
IX21:     COE(3)=COE(2)*S/((K(LL)*J(LL))**2*J(L))
IF(L.GT.1)GO TO IX50
ELSE      COE(4)=(LA**2-J(L)*K(L)*PAR/2)/((1-MU)*2)
GO TO IX51
IX50:     COE(4)=(LA**2-J(L)*K(L)*X(L)/2)/(2-2*MU)
IX51:     IF(LL.GT.1.AND.I.GT.1)GO TO IX60
ELSE      IF(LL.EQ.1.AND.I.EQ.1)GO TO IX70
ELSE      IF(LL.EQ.1.AND.I.GT.1)GO TO IX80
ELSE      COE(5)=T(I,LL,N)*X(LL)*PAR/(K(N)*J(N))
GO TO IX81
IX80:     COE(5)=T(I,LL,N)*X(I)*PAR/(K(N)*J(N))
GO TO IX81
IX70:     COE(5)=T(I,LL,N)*PAR**2/(K(N)*J(N))
GO TO IX81
IX60:     COE(5)=T(I,LL,N)*X(LL)*X(I)/(K(N)*J(N))
IX81:     COE(6)=COE(3)-COE(4)*COE(5)
V=V+COE(6)
DOEND
COE(7)=1/(4*K(L))+K(L)/(2*K(LL)**2)
IF(LL.GT.1.AND.N.GT.1)GO TO IX90
ELSE      IF(LL.EQ.1.AND.N.EQ.1)GO TO IX95
ELSE      IF(LL.EQ.1.AND.N.GT.1)GO TO IX93
ELSE      COE(8)=LA**2*T(N,LL,L)*PAR*X(LL)*COE(7)/J(L)
GO TO IX91
IX93:     COE(8)=LA**2*T(N,LL,L)*PAR*X(N)*COE(7)/J(L)
```

```
GO TO IX91
IX95:     COE(8)=LA**2*T(N,LL,L)*PAR**2*COE(7)/J(L)
GO TO IX91
IX90:     COE(8)=LA**2*T(N,LL,L)*X(N)*X(LL)*COE(7)/J(L)
IX91:     V=V-COE(8)
DOEND
IF(N.GT.1)GO TO IX15
ELSE      COE(9)=J(N)*LA**2*COE(4)*PAR/K(N)
GO TO IX01
IX15:     COE(9)=J(N)*LA**2*COE(4)*X(N)/K(N)
IX01:     V=V+COE(9)
DOEND
IF(L.GT.1)GO TO IX00
ELSE      COE(10)=(K(L)**4+LA**4)*J(L)*PAR/(4*K(L))
GO TO IX1
IX00:     COE(10)=(K(L)**4+LA**4)*J(L)*X(L)/(4*K(L))
IX1:      V=V+COE(10)
F=V-100*X(1)
WRITE F
SKIP
DOEND
END
```

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TABLE I

FIRST MODE

	PRESSURE	A1	DEFLECTION	RTRENO	RTMENO
1	0.294816	1.000000	0.366093	2.490612	-4.348349
2	0.548729	2.000000	0.732185	4.981224	-8.412575
3	0.764308	3.000000	1.098277	7.471835	-12.192679
4	0.944119	4.000000	1.464370	9.962447	-15.688662
5	1.090730	5.000000	1.830462	12.453058	-18.900524
6	1.206709	6.000000	2.196555	14.943670	-21.828264
7	1.294622	7.000000	2.562647	17.434281	-24.471883
8	1.357039	8.000000	2.928740	19.924893	-26.831378
9	1.396525	9.000000	3.294832	22.415504	-28.906752
10	1.415649	10.000000	3.660924	24.906116	-30.698007
11	1.416978	11.000000	4.027017	27.396727	-32.205138
12	1.403080	12.000000	4.393109	29.887339	-33.428149
13	1.376521	13.000000	4.759202	32.377950	-34.367037
14	1.339870	14.000000	5.125294	34.868562	-35.021821
15	1.295694	15.000000	5.491386	37.359173	-35.392452
16	1.246560	16.000000	5.857481	39.849797	-35.478995
17	1.195037	17.000000	6.223573	42.340408	-35.281380
18	1.143690	18.000000	6.589666	44.831020	-34.799656
19	1.095089	19.000000	6.955758	47.321631	-34.033821
20	1.051800	20.000000	7.321850	49.812243	-32.983850
21	1.016391	21.000000	7.687943	52.302854	-31.649770
22	0.991429	22.000000	8.054035	54.793466	-30.031579
23	0.979482	23.000000	8.420128	57.284077	-28.129226
24	0.983117	24.000000	8.786220	59.774689	-25.942790
25	1.004902	25.000000	9.152312	62.265300	-23.472217
26	1.047405	26.000000	9.518405	64.755912	-20.717535
27	1.113192	27.000000	9.884497	67.246523	-17.678716
28	1.204831	28.000000	10.250590	69.737135	-14.355761
29	1.324890	29.000000	10.616682	72.227746	-10.748723
30	1.475937	30.000000	10.982775	74.718358	-6.857521
31	1.660538	31.000000	11.348867	77.208969	-2.682237
32	1.881261	32.000000	11.714959	79.699581	1.777184
33	2.140674	33.000000	12.081052	82.190192	6.520741
34	2.441344	34.000000	12.447144	84.680804	11.548407
35	2.785838	35.000000	12.813237	87.171415	16.860184
36	3.176725	36.000000	13.179329	89.662027	22.456097
37	3.616571	37.000000	13.545422	92.152638	28.336119
38	4.107945	38.000000	13.911514	94.643250	34.500278
39	4.653413	39.000000	14.277606	97.133861	40.948547
40	5.255543	40.000000	14.643699	99.624473	47.680952
41	5.916903	41.000000	15.009791	102.115084	54.697466
42	6.640059	42.000000	15.375884	104.605696	61.998117
43	7.427580	43.000000	15.741976	107.096307	69.582878
44	8.282033	44.000000	16.108069	109.586919	77.451749
45	9.205986	45.000000	16.474161	112.077530	85.604755
46	10.202006	46.000000	16.840253	114.568142	94.041872
47	11.272659	47.000000	17.206346	117.058753	102.763125
48	12.420515	48.000000	17.572438	119.549365	111.768488
49	13.648141	49.000000	17.938531	122.039976	121.057987

TABLE II

MODE NUMBER 2

PRESSURE	DEFLECTION	RTBENO	RTMEMO
0.325819	0.326307	0.671970	-4.305017
0.605241	0.659857	1.675072	-8.376456
0.841043	1.001405	3.043752	-12.206259
1.035933	1.351657	4.810259	-15.781618
1.192566	1.711189	7.000998	-19.084140
1.313604	2.080334	9.631126	-22.089633
1.401803	2.455070	12.697425	-24.769111
1.460149	2.846604	16.170425	-27.091734
1.491982	3.241733	19.988306	-29.030019
1.501058	3.642332	24.056247	-30.566566
1.491459	4.045773	28.254085	-31.700053
1.467347	4.449203	32.461434	-32.447778
1.432641	4.849928	36.525111	-32.843390
1.390735	5.245707	40.372514	-32.930980
1.344357	5.634863	43.917601	-32.757854
1.295581	6.016312	47.110174	-32.369403
1.245932	6.389418	49.921360	-31.805156
1.196536	6.753893	52.338036	-31.097481
1.148265	7.109685	54.357828	-30.271608
1.101864	7.456894	55.985278	-29.345741
1.058048	7.795713	57.229204	-28.332115
1.017578	8.126392	58.101015	-27.237631
0.981318	8.449215	58.613736	-26.064526
0.950280	8.764491	58.781515	-24.810919
0.925652	9.072552	58.619447	-23.471456
0.908823	9.373748	58.143593	-22.037407
0.901402	9.668454	57.371098	-20.497357
0.905228	9.957073	56.320342	-18.837511
0.922374	10.240036	55.011070	-17.041973
0.955141	10.517807	53.464442	-15.093436
1.006054	10.790878	51.702979	-12.973682
1.077836	11.059768	49.750380	-10.663939
1.173390	11.325013	47.631189	-8.145942
1.295765	11.587160	45.370356	-5.402006
1.448120	11.846750	42.992692	-2.416016
1.633690	12.104313	40.522306	0.826448
1.855754	12.360346	37.982042	4.337700
2.117556	12.615312	35.393007	8.127719
2.422486	12.869628	32.774201	12.204344
2.773659	13.123656	30.142288	16.573091
3.174299	13.377709	27.511505	21.237734
3.627532	13.632046	24.893687	26.200194
4.136424	13.886876	22.298405	31.461112
4.703983	14.142363	19.733157	37.019923
5.333157	14.398631	17.203611	42.875370
6.026845	14.655769	14.713856	49.025575
6.787899	14.913839	12.266658	55.468093
7.619131	15.172876	9.863691	62.200504
8.523319	15.432898	7.505751	69.220090
9.503211	15.693908	5.192944	76.524121

TABLE III

MODE NUMBER 3

PRESSURE	DEFLECTION	RTRENO	RTMEMO
0.333195	0.346544	1.762100	-4.530854
0.618243	0.695714	3.603466	-8.784503
0.857947	1.048336	5.565068	-12.753602
1.055038	1.405259	7.687661	-16.425040
1.212207	1.767349	10.006804	-19.779022
1.332203	2.134955	12.545793	-22.789317
1.417965	2.508120	15.307169	-25.425640
1.472801	2.886219	18.265072	-27.658993
1.500503	3.267932	21.362385	-29.469482
1.505336	3.651369	24.516351	-30.854241
1.491820	4.034376	27.632623	-31.831626
1.464377	4.414912	30.622597	-32.439426
1.426986	4.791358	33.416892	-32.728048
1.382971	5.162630	35.971056	-32.752323
1.334951	5.528171	38.264289	-32.564276
1.284915	5.887831	40.294435	-32.209817
1.234346	6.241783	42.072204	-31.726911
1.184357	6.590256	43.616234	-31.145647
1.135805	6.933760	44.949477	-30.489008
1.089387	7.272734	46.096812	-29.774141
1.045711	7.607661	47.083574	-29.012879
1.005349	7.939020	47.934738	-28.212841
0.968887	8.267276	48.674503	-27.378208
0.936953	8.592883	49.326145	-26.509684
0.910256	8.916266	49.912013	-25.605571
0.889608	9.237844	50.453624	-24.661065
0.875958	9.558020	50.971788	-23.669114
0.870416	9.877187	51.486749	-22.619950
0.874284	10.195733	52.018323	-21.501385
0.889081	10.514043	52.586002	-20.298696
0.916577	10.832500	53.209007	-18.994781
0.958813	11.151491	53.906288	-17.569600
1.018125	11.471368	54.696431	-16.001341
1.097161	11.792603	55.597475	-14.265596
1.196883	12.115483	56.626612	-12.336356
1.326564	12.440396	57.799797	-10.186284
1.483767	12.767691	59.131257	-7.787609
1.674307	13.097638	60.632970	-5.112598
1.902205	13.430525	62.314140	-2.134734
2.171623	13.766542	64.180769	1.170610
2.486804	14.105823	66.235368	4.825529
2.852002	14.448434	68.476877	8.848409
3.271425	14.794372	70.900802	13.254241
3.749188	15.143567	73.499573	18.054422
4.289282	15.495897	76.263061	23.256391
4.895549	15.851176	79.179194	28.864563
5.571682	16.209211	82.234613	34.880267
6.321223	16.569767	85.415286	41.302730
7.147580	16.932603	88.707054	48.129172
8.054035	17.297477	92.096074	55.355518

TABLE IV

MODL NUMBER 4

PRESSURE	DEFLECTION	RTEENO	RTMEMO
0.335717	0.341635	1.034048	-4.512920
0.622631	0.687141	2.335253	-8.755138
0.863559	1.037358	3.936104	-12.718341
1.061254	1.393096	5.867322	-16.388332
1.218448	1.755021	8.152605	-19.744018
1.337957	2.123495	10.801093	-22.757688
1.422825	2.498381	13.798344	-25.397572
1.476496	2.878875	17.098535	-27.633511
1.502928	3.263454	20.622326	-29.445247
1.506543	3.650023	24.264307	-30.830536
1.491589	4.036260	27.909464	-31.809119
1.463762	4.420011	31.452375	-32.420328
1.425866	4.799592	34.811254	-32.715592
1.381609	5.173905	37.932970	-32.749874
1.333578	5.542391	40.790477	-32.574190
1.283721	5.904914	43.376518	-32.232558
1.233483	6.261619	45.696854	-31.760905
1.183939	6.612825	47.764692	-31.186312
1.135914	6.958939	49.596710	-30.529201
1.090074	7.300405	51.210485	-29.804099
1.046993	7.637671	52.622958	-29.020440
1.007206	7.971170	53.849593	-28.183989
0.971245	8.301310	54.903974	-27.297403
0.939677	8.628475	55.797662	-26.360495
0.913122	8.953019	56.540215	-25.371458
0.892280	9.275270	57.139278	-24.326234
0.877953	9.595532	57.600726	-23.218829
0.871063	9.914050	57.928836	-22.042214
0.872678	10.231208	58.126453	-20.787421
0.884033	10.547140	58.195185	-19.443946
0.906555	10.862123	58.135589	-17.999986
0.941886	11.176387	57.947378	-16.441959
0.991912	11.490157	57.629629	-14.754987
1.058763	11.803650	57.181018	-12.922212
1.144937	12.117078	56.600056	-10.925654
1.253120	12.430652	55.885343	-8.746048
1.386386	12.744575	55.035826	-6.362596
1.548114	13.059046	54.051046	-3.754117
1.741977	13.374252	52.931365	-0.898710
1.971930	13.690367	51.678152	2.275093
2.242169	14.007548	50.293908	5.638490
2.557073	14.325926	48.762321	9.361078
2.921150	14.645607	47.148240	13.410718
3.336968	14.966664	45.397567	17.803124
3.815089	15.289137	43.537089	22.551168
4.354017	15.613032	41.574265	27.664907
4.960141	15.938326	39.516996	33.151701
5.637712	16.264968	37.373390	39.016340
6.390815	16.592882	35.151564	45.260759
7.223366	16.921977	32.859470	51.885289

TABLE V

MODE NUMBER 5

PRESSURE	DEFLECTION	RTBENO	RTMEMO
0.336795	0.345226	1.563221	-4.551534
0.624493	0.693376	3.253065	-8.824129
0.865920	1.045356	5.111390	-12.809277
1.063841	1.402046	7.178857	-16.492663
1.221017	1.764184	9.489575	-19.853230
1.340297	2.132199	12.063525	-22.863700
1.424760	2.506022	14.897623	-25.493182
1.477971	2.884913	17.958109	-27.713035
1.503895	3.267473	21.178613	-29.505023
1.507037	3.651543	24.467532	-30.869151
1.492089	4.035045	27.723940	-31.827164
1.463569	4.415876	30.855802	-32.419725
1.425483	4.792444	33.793055	-32.698803
1.381136	5.163727	36.492089	-32.719439
1.333100	5.529229	38.933200	-32.531879
1.283313	5.888856	41.114686	-32.180075
1.233209	6.242788	43.046630	-31.698993
1.183855	6.591372	44.745846	-31.116172
1.136071	6.935040	46.232346	-30.451707
1.090514	7.274261	47.527097	-29.720498
1.047750	7.609509	48.650741	-28.932685
1.008303	7.941248	49.622931	-28.094762
0.972693	8.269922	50.462068	-27.210340
0.941465	8.595951	51.185264	-26.280788
0.915215	8.919733	51.808439	-25.305238
0.894605	9.241645	52.346484	-24.281670
0.880391	9.562047	52.813453	-23.205846
0.873432	9.881282	53.272774	-22.072544
0.874717	10.199682	53.587440	-20.874951
0.885379	10.517567	53.920210	-19.605118
0.906716	10.835254	54.233779	-18.253543
0.940217	11.153052	54.540937	-16.809394
0.987576	11.471269	54.854694	-15.260419
1.050719	11.790210	55.188387	-13.592743
1.131826	12.110179	55.555734	-11.791117
1.233344	12.431477	55.970848	-9.838710
1.358009	12.754400	56.448196	-7.717711
1.508849	13.079235	57.002486	-5.408456
1.689189	13.406261	57.648492	-2.890514
1.902647	13.735734	58.400801	-0.142900
2.153109	14.067851	59.273506	2.855929
2.444707	14.402937	60.279840	6.127425
2.761775	14.741038	61.431798	9.692728
3.168804	15.082321	62.739760	13.571551
3.610387	15.426863	64.212169	17.781997
4.111161	15.774652	65.855277	22.340896
4.675758	16.125791	67.673006	27.262371
5.306755	16.480091	69.666915	32.558187
6.014633	16.837486	71.836293	38.238032
6.797753	17.197835	74.178346	44.309117

TABLE VI

MODE NUMBER 6

PRESSURE	DEFLECTION	RTBEND	RTMEMO
0.337331	0.343895	1.157018	-4.545342
0.625415	0.691081	2.551524	-8.813673
0.867081	1.042440	4.217334	-12.796309
1.065107	1.398826	6.186662	-16.478711
1.222265	1.760948	8.484720	-19.839513
1.341427	2.129200	11.122040	-22.851058
1.425716	2.503473	14.085434	-25.482047
1.478672	2.882984	17.330393	-27.703480
1.504353	3.266235	20.779376	-29.496837
1.507271	3.651176	24.329704	-30.861983
1.492138	4.035545	27.870117	-31.820657
1.463483	4.417264	31.299540	-32.413565
1.425311	4.794724	34.540413	-32.692735
1.380922	5.166892	37.543134	-32.713389
1.332864	5.533262	40.283270	-32.525513
1.283126	5.893735	42.755352	-32.173141
1.233080	6.248483	44.966362	-31.691597
1.183808	6.597846	46.930438	-31.108042
1.136126	6.942250	48.665134	-30.442823
1.090694	7.282158	50.189027	-29.710596
1.048070	7.618035	51.520299	-28.921222
1.008776	7.950339	52.675972	-28.081408
0.973329	8.279504	53.671548	-27.194280
0.942267	8.605940	54.520891	-26.261022
0.916176	8.930035	55.236236	-25.280368
0.895708	9.252153	55.828281	-24.249632
0.881600	9.572637	56.306303	-23.164852
0.874668	9.891813	56.678285	-22.019890
0.875929	10.209990	56.951047	-20.807601
0.886415	10.527467	57.130366	-19.519782
0.907393	10.844531	57.221088	-18.146828
0.940282	11.161460	57.227234	-16.677365
0.986695	11.478525	57.152088	-15.099435
1.048455	11.795991	56.998287	-13.399189
1.127616	12.114116	56.767895	-11.561868
1.226478	12.433153	56.462475	-9.571501
1.347606	12.753346	56.063153	-7.410797
1.493834	13.074927	55.630673	-5.061716
1.668272	13.398119	55.105454	-2.505268
1.874305	13.723126	54.507637	0.278171
2.115563	14.050131	53.837122	3.308157
2.395998	14.379292	53.093616	6.603917
2.719659	14.710739	52.276663	10.184560
3.090855	15.044567	51.385687	14.068225
3.514012	15.380825	50.420026	18.271315
3.993652	15.719564	49.378990	22.809216
4.534340	16.060739	48.261885	27.694651
5.140642	16.404307	47.068086	32.938889
5.817088	16.750183	45.797080	38.551003
6.568137	17.098255	44.448523	44.537781

TABLE VII

MODE NUMBER 7

PRESSURE	DEFLECTION	RTBENO	RTMEMO
0.337626	0.346071	1.481692	-4.557109
0.625920	0.692025	3.111546	-8.834577
0.867717	1.044921	4.930516	-12.823648
1.065796	1.401587	6.978101	-16.509794
1.222941	1.763759	9.286916	-19.871701
1.342036	2.121856	11.875021	-22.881942
1.426219	2.505790	14.736939	-25.509577
1.479048	2.884802	17.836113	-27.726131
1.504597	3.267419	21.103222	-29.513745
1.507396	3.651616	24.443923	-30.872976
1.492165	4.035160	27.755069	-31.826113
1.463438	4.416000	30.943029	-32.414268
1.425222	4.792556	33.936657	-32.689654
1.380811	5.163817	36.691589	-32.707816
1.332771	5.529295	39.187541	-32.517688
1.283029	5.888903	41.422318	-32.164128
1.235015	6.242826	43.405553	-31.681748
1.183783	6.591410	45.153646	-31.097794
1.136156	6.936090	46.686205	-30.432290
1.090764	7.274335	48.023796	-29.700268
1.046230	7.609619	49.186639	-28.911448
1.009014	7.941405	50.193917	-28.072263
0.973649	8.270136	51.063491	-27.185965
0.942671	8.596232	51.811833	-26.254093
0.916662	8.920080	52.454099	-25.275772
0.896270	9.242084	53.004258	-24.248487
0.882220	9.562571	53.475266	-23.168243
0.875342	9.881890	53.879243	-22.029617
0.876575	10.200367	54.227653	-20.826166
0.886995	10.518316	54.531471	-19.550161
0.907822	10.836043	54.801345	-18.192671
0.940444	11.153847	55.047738	-16.743300
0.986433	11.472071	55.281059	-15.191204
1.047560	11.790853	55.511764	-13.523569
1.125616	12.110629	55.750443	-11.726785
1.223428	12.431626	56.007876	-9.786217
1.342665	12.754119	56.295049	-7.686263
1.486857	13.078371	56.623146	-5.409865
1.658392	13.404636	57.003490	-2.940138
1.860720	13.733152	57.447447	-0.258965
2.097339	14.064136	57.966289	2.652311
2.371967	14.397783	58.571024	5.811397
2.688610	14.734258	59.272196	9.236600
3.051341	15.073663	60.079681	12.945439
3.464457	15.416182	61.002476	16.953997
3.932344	15.761783	62.048520	21.277630
4.459456	16.110513	63.224542	25.929659
5.050275	16.462349	64.535957	30.922985
5.709276	16.817236	65.986823	36.267485
6.440895	17.175082	67.579841	41.972203

TABLE VIII

MODE NUMBER 8

PRESSURE	DEFLECTION	RTBENO	RTMENO
0.337802	0.344505	1.214640	-4.554420
0.626221	0.692138	2.651726	-8.829995
0.868053	1.043796	4.346104	-12.817913
1.066203	1.400348	6.331061	-16.503580
1.223340	1.762515	8.632681	-19.865542
1.342394	2.130703	11.263130	-22.876244
1.426514	2.504809	14.209619	-25.504590
1.479267	2.884057	17.428852	-27.721927
1.504739	3.266958	20.844432	-29.510279
1.507468	3.651472	24.355047	-30.870120
1.492181	4.035353	27.851023	-31.823694
1.463413	4.416541	31.232960	-32.412080
1.425171	4.793450	34.424877	-32.687516
1.380748	5.165062	37.378505	-32.705794
1.332707	5.530887	40.070455	-32.515315
1.282974	5.890834	42.496046	-32.161355
1.232975	6.245085	44.662855	-31.678459
1.183769	6.593984	46.585487	-31.093715
1.136171	6.937962	48.281895	-30.427375
1.090835	7.277485	49.771028	-29.694402
1.048321	7.613025	51.071443	-28.904352
1.009148	7.945040	52.200561	-28.063987
0.973829	8.273971	53.174331	-27.176352
0.942898	8.600233	54.007115	-26.242816
0.916935	8.924218	54.711727	-25.262402
0.896584	9.246296	55.299522	-24.232785
0.882568	9.566818	55.780529	-23.149663
0.875709	9.886117	56.163596	-22.007707
0.876941	10.204510	56.456519	-20.800031
0.887327	10.522303	56.666180	-19.519096
0.908075	10.839754	56.798663	-18.155806
0.940556	11.157269	56.859358	-16.699653
0.986318	11.475010	56.853054	-15.139324
1.047107	11.793292	56.784016	-13.462443
1.124878	12.112384	56.656048	-11.655058
1.221813	12.432548	56.472544	-9.702338
1.340333	12.754040	56.236517	-7.588997
1.483104	13.077103	55.950621	-5.297771
1.653044	13.401969	55.617155	-2.811992
1.853318	13.728856	55.238052	-0.113798
2.087234	14.057958	54.814865	2.815124
2.358729	14.389449	54.348738	5.992552
2.671341	14.723471	53.840382	9.436075
3.029191	15.060137	53.290052	13.162717
3.436443	15.399525	52.697537	17.188338
3.897375	15.741676	52.062163	21.527387
4.416338	16.086595	51.382813	26.193307
4.997720	16.434253	50.657976	31.197982
5.645916	16.784586	49.885802	36.550773
6.365300	17.137502	49.064183	42.260464

TABLE IX

MODE NUMBER 9

PRESSURE	DEFLECTION	RTBENO	RTMEMO
0.337913	0.344976	1.439406	-4.559237
0.626410	0.692949	3.038538	-8.838521
0.868330	1.044828	4.837611	-12.829019
1.066459	1.401494	6.875307	-16.516151
1.223589	1.763679	9.183316	-19.878497
1.342616	2.131800	11.778639	-22.888610
1.426697	2.505764	14.654595	-25.515562
1.479403	2.884804	17.773304	-27.730922
1.504827	3.267447	21.064086	-29.516995
1.507513	3.651649	24.431336	-30.874517
1.492191	4.035150	27.770645	-31.825972
1.463397	4.416018	30.968162	-32.412590
1.425139	4.792556	34.011534	-32.686660
1.380709	5.163797	36.796141	-32.704291
1.332668	5.529255	39.321332	-32.513265
1.282941	5.889847	41.584602	-32.159015
1.232952	6.242758	43.595306	-31.676029
1.183761	6.591335	45.369580	-31.091389
1.136181	6.935014	46.926777	-30.425271
1.090866	7.274261	48.287207	-29.692581
1.048376	7.609553	49.470819	-28.903073
1.009229	7.941351	50.496516	-28.063149
0.973937	8.270098	51.381852	-27.176306
0.943035	8.596214	52.142965	-26.243621
0.917100	8.920095	52.794640	-25.264518
0.896774	9.242115	53.350434	-24.236106
0.882778	9.562629	53.822841	-23.154940
0.875931	9.881577	54.223463	-22.015492
0.877161	10.200482	54.563175	-20.811137
0.887527	10.518457	54.852289	-19.534359
0.908228	10.836205	55.100705	-18.176336
0.940625	11.154024	55.318041	-16.727269
0.986253	11.472203	55.513754	-15.175701
1.046841	11.791026	55.697237	-13.509980
1.124323	12.110772	55.877897	-11.716373
1.220857	12.431717	56.065208	-9.780782
1.336830	12.754125	56.268736	-7.688663
1.480873	13.078254	56.498136	-5.423049
1.649857	13.404349	56.765112	-2.967724
1.846156	13.732640	57.073355	-0.306317
2.081346	14.063339	57.438438	2.579236
2.350771	14.396630	57.867699	5.705560
2.660943	14.732675	58.370098	9.089594
3.015805	15.071600	58.954073	12.748048
3.419446	15.413499	59.627391	16.695910
3.876062	15.758432	60.397029	20.947319
4.385928	16.106418	61.269070	25.515150
4.965358	16.457443	62.248638	30.410807
5.606679	16.811461	63.339871	35.644127
6.318202	17.168393	64.545931	41.223624

TABLE X

MODE NUMBER 10

PRESSURE	DEFLECTION	RTBENO	RTMEMO
0.337986	0.344730	1.246891	-4.557870
0.626536	0.692577	2.707528	-8.836180
0.868487	1.044295	4.417441	-12.826077
1.066628	1.400907	6.410612	-16.512961
1.223754	1.763089	8.714012	-19.875326
1.342765	2.131253	11.339875	-22.885668
1.426818	2.505298	14.276662	-25.512991
1.479493	2.884450	17.481534	-27.728778
1.504885	3.267223	20.878795	-29.515256
1.507543	3.651580	24.367911	-30.873125
1.492198	4.035282	27.840052	-31.824823
1.463388	4.416277	31.196666	-32.411576
1.425120	4.792984	34.362545	-32.685668
1.380685	5.164394	37.290065	-32.703589
1.332644	5.530020	39.956344	-32.512516
1.282920	5.889776	42.357086	-32.157904
1.232937	6.243846	44.500172	-31.674508
1.183755	6.592576	46.400444	-31.089357
1.136187	6.936399	48.076069	-30.422806
1.090886	7.275792	49.546197	-29.689649
1.048411	7.611198	50.829591	-28.899439
1.009281	7.943107	51.943895	-28.058838
0.974007	8.271952	52.905297	-27.171224
0.943122	8.598149	53.728433	-26.237635
0.917205	8.922092	54.426417	-25.257515
0.896895	9.244154	55.010945	-24.228082
0.882912	9.564686	55.492431	-23.145630
0.876072	9.884025	55.880148	-22.004650
0.877302	10.202491	56.182373	-20.798480
0.887655	10.520394	56.406517	-19.519653
0.908327	10.838032	56.559254	-18.159174
0.940670	11.155696	56.646619	-16.707278
0.986714	11.473671	56.674113	-15.152223
1.046674	11.792235	56.646771	-13.482508
1.123974	12.111661	56.569231	-11.684354
1.220253	12.432215	56.445777	-9.743614
1.337881	12.754153	56.280366	-7.645172
1.479461	13.077726	56.076641	-5.373367
1.647838	13.403168	55.837922	-2.910582
1.845094	13.730699	55.567183	-0.240624
2.077540	14.060519	55.267019	2.653600
2.345703	14.392804	54.930595	5.789373
2.654305	14.727704	54.586606	9.184145
3.007239	15.065334	54.209225	12.853417
3.408536	15.405780	53.808064	16.812674
3.862338	15.749091	53.383182	21.075860
4.372860	16.095279	52.934036	25.655665
4.944361	16.444325	52.459554	30.563484
5.561111	16.796173	51.958155	35.808725
6.287369	17.150743	51.427824	41.400362

TABLE XI

PRESSURE

Buckling	Δ_b	Snapping	Δ_s
1.416978		.97482	
	.08408		-.073418
1.501058		.901402	
	004278		- 030986
1.505336		.870416	
	.001207		.000647
1 506543		.871063	
	.000494		.002369
1.507037		.873432	
	.000234		.001256
1.507271		.874688	
	.000125		.000654
1.507396		.875342	
	.000072		.000367
1.507468		.875709	
	.000045		.000222
1.507513		.875931	
	.000030		.000141
1.507543		.876072	

TABLE XII

DEFLECTION

Buckling	Δ_b	Snapping	Δ_s
4 027017		8 420128	
	-.384685		1 248326
3 642332		9 668454	
	009037		208733
3 651369		9 877187	
	- 001346		036903
3.650023		9 914090	
	.001520		- 032808
3.651543		9 881282	
	-.000367		.010531
3.651176		9.891813	
	.000440		-.009923
3.651616		9.881890	
	-.000144		.004227
3.651472		9.886117	
	.000177		-.004140
3.651649		9.881977	
	-.000069		.002026
3.651580		9.884025	

TABLE XIII

	DEFLECTION	RADIAL BENDING	TANGENTIAL BENDING	RADIAL MEMBRANE STRESS	TANGENTIAL MEMBRANE STRESS
0.010000	1.400658	6.417327	6.414487	-16.511999	-31.167121
0.020000	1.399910	6.436769	6.425775	-16.509535	-31.153800
0.030000	1.398681	6.466895	6.443504	-16.505417	-31.131547
0.040000	1.396947	6.504525	6.466152	-16.499633	-31.100306
0.050000	1.394712	6.545629	6.491791	-16.492191	-31.060037
0.060000	1.391970	6.585707	6.518205	-16.483063	-31.010623
0.070000	1.388720	6.620275	6.543126	-16.472234	-30.952005
0.080000	1.384958	6.645192	6.564405	-16.459707	-30.884132
0.090000	1.380683	6.657112	6.580169	-16.445454	-30.806925
0.100000	1.375893	6.653774	6.588979	-16.429466	-30.720366
0.110000	1.370589	6.634161	6.589954	-16.411749	-30.624499
0.120000	1.364774	6.598635	6.582783	-16.392288	-30.519225
0.130000	1.358450	6.548799	6.567746	-16.371095	-30.404727
0.140000	1.351624	6.487374	6.545651	-16.348169	-30.281038
0.150000	1.344301	6.417877	6.517739	-16.323518	-30.148258
0.160000	1.336486	6.344226	6.485537	-16.297150	-30.006484
0.170000	1.328189	6.270428	6.450717	-16.269085	-29.855860
0.180000	1.319415	6.200067	6.414908	-16.239339	-29.696494
0.190000	1.310172	6.136012	6.379555	-16.207926	-29.528495
0.200000	1.300466	6.080152	6.345708	-16.174858	-29.351961
0.210000	1.290298	6.033167	6.314323	-16.140159	-29.166974
0.220000	1.279676	5.994522	6.285408	-16.103838	-28.973597
0.230000	1.268603	5.962547	6.258835	-16.065912	-28.771884
0.240000	1.257081	5.934583	6.233963	-16.026391	-28.561868
0.250000	1.245112	5.907325	6.209824	-15.985283	-28.343559
0.260000	1.232699	5.877131	6.185218	-15.942597	-28.116997
0.270000	1.219844	5.840424	6.158860	-15.898339	-27.882205
0.280000	1.206551	5.794042	6.129522	-15.852512	-27.639219
0.290000	1.192823	5.735576	6.096135	-15.805120	-27.388070
0.300000	1.178665	5.663577	6.057938	-15.756183	-27.128859
0.310000	1.164085	5.577729	6.014503	-15.705698	-26.861667
0.320000	1.149090	5.478819	5.965799	-15.653679	-26.586606
0.330000	1.133691	5.368685	5.912169	-15.600128	-26.303832
0.340000	1.117896	5.249907	5.854277	-15.545066	-26.013489
0.350000	1.101718	5.125596	5.793036	-15.488512	-25.715762
0.360000	1.085167	4.999036	5.729472	-15.430471	-25.410823
0.370000	1.068255	4.873247	5.664628	-15.370967	-25.098832
0.380000	1.050995	4.750784	5.599452	-15.310021	-24.780018
0.390000	1.033397	4.633280	5.534644	-15.247657	-24.454524
0.400000	1.015472	4.521418	5.470619	-15.183892	-24.122513
0.410000	0.997232	4.414750	5.407447	-15.118755	-23.784172
0.420000	0.978686	4.311763	5.344815	-15.052268	-23.439592
0.430000	0.959838	4.210060	5.282111	-14.984437	-23.088942
0.440000	0.940704	4.106596	5.218444	-14.915301	-22.732353
0.450000	0.921291	3.997963	5.152758	-14.844879	-22.369949
0.460000	0.901609	3.880797	5.083943	-14.773191	-22.001883
0.470000	0.881667	3.752118	5.010942	-14.700251	-21.628285
0.480000	0.861478	3.609660	4.932892	-14.626090	-21.249347
0.490000	0.841056	3.452109	4.849193	-14.550730	-20.865278
0.500000	0.820415	3.279249	4.759572	-14.474191	-20.476293

0.510000	0.799569	3.091982	4.664124	-14.396499	-20.087631
0.520000	0.778539	2.892258	4.563289	-14.317682	-19.684597
0.530000	0.757340	2.682798	4.457796	-14.237767	-19.282451
0.540000	0.735995	2.466901	4.348609	-14.156791	-18.876550
0.550000	0.714521	2.247955	4.236764	-14.074778	-18.467174
0.560000	0.692940	2.029147	4.123297	-13.991763	-18.054649
0.570000	0.671272	1.813086	4.009119	-13.907793	-17.639342
0.580000	0.649536	1.601443	3.894666	-13.822895	-17.221536
0.590000	0.627753	1.394786	3.780869	-13.737116	-16.801569
0.600000	0.605942	1.192464	3.667068	-13.650493	-16.379740
0.610000	0.584119	0.992620	3.553016	-13.563063	-15.956320
0.620000	0.562306	0.792429	3.437940	-13.474874	-15.531637
0.630000	0.540519	0.588310	3.320781	-13.385964	-15.105973
0.640000	0.518776	0.376305	3.200313	-13.296375	-14.679615
0.650000	0.497098	0.152548	3.075290	-13.206151	-14.252905
0.660000	0.475504	-0.086371	2.944550	-13.115332	-13.826152
0.670000	0.454018	-0.342948	2.807183	-13.023966	-13.399743
0.680000	0.432661	-0.618489	2.662609	-12.932094	-12.974060
0.690000	0.411460	-0.912932	2.510674	-12.839766	-12.549556
0.700000	0.390443	-1.224746	2.351680	-12.747034	-12.126697
0.710000	0.369638	-1.551110	2.186393	-12.653949	-11.705985
0.720000	0.349075	-1.888125	2.015786	-12.560559	-11.287926
0.730000	0.328787	-2.231161	1.841366	-12.466930	-10.873109
0.740000	0.308804	-2.575414	1.664574	-12.373114	-10.462066
0.750000	0.289159	-2.916386	1.486868	-12.279182	-10.055394
0.760000	0.269884	-3.250375	1.309496	-12.185190	-9.653633
0.770000	0.251009	-3.575026	1.133339	-12.091206	-9.257320
0.780000	0.232564	-3.889568	0.958816	-11.997305	-8.867046
0.790000	0.214578	-4.195081	0.785765	-11.903547	-8.483204
0.800000	0.197079	-4.494460	0.613460	-11.810006	-8.106570
0.810000	0.180094	-4.792295	0.440614	-11.716756	-7.737402
0.820000	0.163650	-5.094438	0.265493	-11.623863	-7.376264
0.830000	0.147775	-5.407442	0.086092	-11.531405	-7.023702
0.840000	0.132497	-5.737937	-0.099692	-11.439454	-6.680252
0.850000	0.117847	-6.091897	-0.293831	-11.348084	-6.344406
0.860000	0.103856	-6.473845	-0.497900	-11.257371	-6.023041
0.870000	0.090561	-6.886205	-0.712876	-11.167398	-5.710600
0.880000	0.077999	-7.328724	-0.938931	-11.078244	-5.409935
0.890000	0.066211	-7.790137	-1.175311	-10.989988	-5.121815
0.900000	0.055243	-8.287930	-1.420210	-10.902727	-4.847146
0.910000	0.045138	-8.788532	-1.670833	-10.816550	-4.586829
0.920000	0.035944	-9.287645	-1.923453	-10.731548	-4.341779
0.930000	0.027707	-9.770784	-2.173548	-10.647827	-4.112958
0.940000	0.020472	-10.222107	-2.416059	-10.565489	-3.901202
0.950000	0.014280	-10.675358	-2.645658	-10.484638	-3.707594
0.960000	0.009168	-10.964832	-2.857056	-10.405381	-3.532687
0.970000	0.005166	-11.226383	-3.045305	-10.327831	-3.377225
0.980000	0.002246	-11.398385	-3.206133	-10.257098	-3.241733
0.990000	0.000573	-11.472468	-3.336186	-10.178284	-3.126571
1.000000	-0.000001	-11.444078	-3.433231	-10.106493	-3.032930

TABLE XIV

	DEFLECTION	RADI. BENDIN.	TANGENTIAL BENDING	RADIAL MEMBRANE STRESS	TANGENTIAL MEMBRANE STRESS
0.010000	3.650639	24.358127	24.362277	-30.871260	-72.341836
0.020000	3.647829	24.328757	24.345387	-30.866649	-72.305512
0.030000	3.643147	24.279522	24.317086	-30.858948	-72.244927
0.040000	3.636600	24.210061	24.277190	-30.848144	-72.160082
0.050000	3.628193	24.119866	24.225505	-30.834268	-72.051058
0.060000	3.617933	24.008389	24.161728	-30.817281	-71.917754
0.070000	3.605832	23.875173	24.085609	-30.797168	-71.760209
0.080000	3.591902	23.719751	23.996943	-30.773948	-71.578491
0.090000	3.576156	23.541822	23.895500	-30.747580	-71.372549
0.100000	3.558612	23.341267	23.781196	-30.718049	-71.142443
0.110000	3.539286	23.118113	23.653998	-30.685373	-70.888273
0.120000	3.518200	22.872659	23.513950	-30.649513	-70.610064
0.130000	3.495376	22.605332	23.361223	-30.610484	-70.307950
0.140000	3.470839	22.316787	23.196042	-30.568264	-69.982021
0.150000	3.444614	22.007783	23.018709	-30.522847	-69.632399
0.160000	3.416728	21.679154	22.829559	-30.474218	-69.259177
0.170000	3.387214	21.331792	22.629003	-30.422384	-68.862577
0.180000	3.356101	20.966515	22.417418	-30.367340	-68.442726
0.190000	3.323424	20.584097	22.195177	-30.309079	-67.999796
0.200000	3.289217	20.185198	21.962608	-30.247586	-67.533941
0.210000	3.253519	19.770295	21.720012	-30.182871	-67.045360
0.220000	3.216358	19.339765	21.467624	-30.114923	-66.534236
0.230000	3.177783	18.893822	21.209633	-30.043742	-66.000781
0.240000	3.137830	18.432570	20.934168	-29.969322	-65.445187
0.250000	3.096542	17.956066	20.653335	-29.891658	-64.867687
0.260000	3.053962	17.464312	20.363206	-29.810747	-64.268514
0.270000	3.010136	16.957387	20.063863	-29.726579	-63.647930
0.280000	2.965106	16.435365	19.755398	-29.639166	-63.006219
0.290000	2.918923	15.898533	19.437926	-29.548486	-62.343658
0.300000	2.871632	15.347255	19.111647	-29.454571	-61.660629
0.310000	2.823286	14.782127	18.776783	-29.357407	-60.957472
0.320000	2.773932	14.203857	18.433624	-29.257008	-60.234568
0.330000	2.723629	13.613432	18.082535	-29.153367	-59.492358
0.340000	2.672424	13.011798	17.723921	-29.046516	-58.731275
0.350000	2.620374	12.400091	17.358249	-28.936470	-57.951823
0.360000	2.567535	11.779490	16.985966	-28.823233	-57.154472
0.370000	2.513957	11.151045	16.607555	-28.706832	-56.339701
0.380000	2.459706	10.515948	16.223521	-28.587295	-55.508145
0.390000	2.404832	9.875075	15.834298	-28.464657	-54.660319
0.400000	2.349393	9.229281	15.440294	-28.338938	-53.796792
0.410000	2.293454	8.579331	15.041917	-28.210187	-52.918242
0.420000	2.237064	7.925684	14.639446	-28.078426	-52.025178
0.430000	2.180285	7.268831	14.233207	-27.943715	-51.118326
0.440000	2.123180	6.609178	13.823447	-27.806081	-50.198336
0.450000	2.065803	5.946977	13.410398	-27.665583	-49.265880
0.460000	2.008216	5.282592	12.994302	-27.522272	-48.321702
0.470000	1.950475	4.616378	12.575376	-27.376189	-47.366483
0.480000	1.892640	3.948802	12.153909	-27.227410	-46.401049
0.490000	1.834773	3.280516	11.730218	-27.075996	-45.426196
0.500000	1.776934	2.612266	11.304642	-26.922006	-44.442759

0.510000	1.719177	1.944914	10.877577	-26.765515	-43.451563
0.520000	1.661569	1.279580	10.449496	-26.606599	-42.453565
0.530000	1.604162	0.617342	10.020857	-26.445334	-41.449600
0.540000	1.547021	-0.040526	9.592216	-26.281815	-40.440700
0.550000	1.490198	-0.692868	9.164059	-26.116116	-39.427726
0.560000	1.433752	-1.338512	8.736904	-25.948329	-38.411643
0.570000	1.377743	-1.976329	8.311286	-25.778568	-37.393519
0.580000	1.322224	-2.605430	7.887629	-25.606918	-36.374265
0.590000	1.267251	-3.225012	7.466361	-25.433493	-35.354923
0.600000	1.212880	-3.834492	7.047827	-25.258399	-34.336486
0.610000	1.159160	-4.433517	6.632294	-25.081738	-33.319904
0.620000	1.106148	-5.021783	6.220030	-24.903640	-32.306260
0.630000	1.053893	-5.599187	5.811213	-24.724214	-31.296537
0.640000	1.002443	-6.165706	5.405995	-24.543581	-30.291729
0.650000	0.951851	-6.721251	5.004543	-24.361870	-29.292909
0.660000	0.902161	-7.265788	4.606983	-24.179197	-28.301040
0.670000	0.853422	-7.799115	4.213511	-23.995701	-27.317205
0.680000	0.805678	-8.320903	3.824326	-23.811502	-26.342421
0.690000	0.758974	-8.830691	3.439693	-23.626742	-25.377732
0.700000	0.713355	-9.327790	3.059935	-23.441560	-24.424229
0.710000	0.668862	-9.811390	2.685413	-23.256092	-23.482947
0.720000	0.625533	-10.280592	2.316504	-23.070470	-22.554884
0.730000	0.583413	-10.734323	1.953676	-22.884853	-21.641167
0.740000	0.542534	-11.171627	1.597333	-22.699374	-20.742757
0.750000	0.502938	-11.591517	1.247926	-22.514195	-19.860734
0.760000	0.464656	-11.993224	0.905805	-22.329449	-18.996029
0.770000	0.427720	-12.376163	0.571289	-22.145287	-18.149597
0.780000	0.392164	-12.739950	0.244640	-21.961876	-17.322431
0.790000	0.358016	-13.084532	-0.073993	-21.779344	-16.515368
0.800000	0.325303	-13.410094	-0.384555	-21.597853	-15.729315
0.810000	0.294052	-13.717111	-0.687058	-21.417557	-14.965102
0.820000	0.264283	-14.006244	-0.981616	-21.238592	-14.223485
0.830000	0.236023	-14.278227	-1.268346	-21.061118	-13.505304
0.840000	0.209289	-14.533862	-1.547416	-20.885280	-12.811297
0.850000	0.184101	-14.773881	-1.818986	-20.711221	-12.142178
0.860000	0.160477	-14.998809	-2.083157	-20.539088	-11.498680
0.870000	0.138432	-15.208912	-2.339954	-20.369033	-10.881517
0.880000	0.117981	-15.404094	-2.589310	-20.201191	-10.291367
0.890000	0.099136	-15.583884	-2.831051	-20.035696	-9.728858
0.900000	0.081912	-15.747321	-3.064819	-19.872705	-9.194712
0.910000	0.066318	-15.893091	-3.290167	-19.712350	-8.689528
0.920000	0.052360	-16.019534	-3.506538	-19.554760	-8.213850
0.930000	0.040047	-16.124658	-3.713236	-19.400083	-7.768269
0.940000	0.029384	-16.206314	-3.909519	-19.248446	-7.353278
0.950000	0.020372	-16.262378	-4.094633	-19.099979	-6.969259
0.960000	0.013013	-16.290731	-4.267814	-18.954807	-6.616586
0.970000	0.007303	-16.289575	-4.428368	-18.813061	-6.295506
0.980000	0.003236	-16.257442	-4.575702	-18.674861	-6.006179
0.990000	0.000805	-16.193345	-4.709365	-18.540315	-5.748637
1.000000	-0.000002	-16.096851	-4.829070	-18.409532	-5.522819

TABLE XV

	DEFLECTION	RADIAL BENDING	TANGENTIAL BENDING	RADIAL MEMBRANE STRESS	TANGENTIAL MEMBRANE STRESS
0.010000	7.941101	51.905116	51.921557	-28.058368	-119.509071
0.020000	7.935113	51.791029	51.855607	-28.063888	-119.477738
0.030000	7.925141	51.607884	51.748978	-28.073073	-119.425398
0.040000	7.911206	51.365534	51.606342	-28.085879	-119.351998
0.050000	7.893331	51.076261	51.433752	-28.102303	-119.257701
0.060000	7.871533	50.753750	51.237842	-28.122275	-119.142200
0.070000	7.845850	50.411813	51.025568	-28.145733	-119.005391
0.080000	7.816311	50.062830	50.803519	-28.172633	-118.847273
0.090000	7.782943	49.716745	50.577182	-28.202873	-118.667478
0.100000	7.745780	49.380091	50.350786	-28.236350	-118.465631
0.110000	7.704849	49.055269	50.126846	-28.272995	-118.241686
0.120000	7.660182	48.740521	49.905913	-28.312676	-117.995068
0.130000	7.611803	48.429929	49.686792	-28.355279	-117.725497
0.140000	7.559740	48.114212	49.466524	-28.400665	-117.432422
0.150000	7.504022	47.781478	49.240803	-28.448689	-117.115329
0.160000	7.444674	47.418510	49.004361	-28.499206	-116.773548
0.170000	7.381733	47.011927	48.751600	-28.552045	-116.406626
0.180000	7.315232	46.549491	48.476930	-28.607043	-116.013892
0.190000	7.245216	46.021326	48.175369	-28.664015	-115.594675
0.200000	7.171732	45.420632	47.842879	-28.722772	-115.148228
0.210000	7.094833	44.744236	47.476779	-28.783122	-114.673915
0.220000	7.014583	43.992952	47.075820	-28.844858	-114.170983
0.230000	6.931052	43.171101	46.640202	-28.907766	-113.638751
0.240000	6.844314	42.286086	46.171464	-28.971624	-113.076426
0.250000	6.754452	41.347423	45.672182	-29.036198	-112.483233
0.260000	6.661552	40.365663	45.145684	-29.101249	-111.858416
0.270000	6.565708	39.351390	44.595555	-29.166518	-111.201135
0.280000	6.467007	38.313324	44.025246	-29.231760	-110.510643
0.290000	6.365548	37.258686	43.437658	-29.296684	-109.785996
0.300000	6.261426	36.191169	42.834894	-29.361047	-109.026518
0.310000	6.154740	35.111296	42.217922	-29.424544	-108.231275
0.320000	6.045583	34.016024	41.586525	-29.486892	-107.399473
0.330000	5.934066	32.899546	40.939409	-29.547778	-106.530248
0.340000	5.820280	31.753287	40.274270	-29.606914	-105.622844
0.350000	5.704334	30.567514	39.588157	-29.664007	-104.676538
0.360000	5.586336	29.332149	38.877716	-29.718704	-103.690523
0.370000	5.466389	28.037728	38.139622	-29.770723	-102.664107
0.380000	5.344627	26.677136	37.371185	-29.819749	-101.596789
0.390000	5.221163	25.245596	36.570046	-29.865474	-100.488020
0.400000	5.096128	23.741932	35.735357	-29.907582	-99.337316
0.410000	4.969671	22.168755	34.867246	-29.945778	-98.144422
0.420000	4.841919	20.531831	33.966937	-29.979756	-96.908916
0.430000	4.713032	18.840268	33.036963	-30.009230	-95.630777
0.440000	4.583170	17.105455	32.080670	-30.033904	-94.309858
0.450000	4.452482	15.339734	31.102015	-30.053511	-92.946206
0.460000	4.321137	13.555843	30.105275	-30.067776	-91.540005
0.470000	4.189292	11.765292	29.094509	-30.076430	-90.091384
0.480000	4.057118	9.977589	28.073418	-30.079231	-88.600831
0.490000	3.924762	8.199653	27.044933	-30.075945	-87.068839
0.500000	3.792404	6.434951	26.010974	-30.066337	-85.495995

0.510000	3.660192	4.683523	24.972408	-30.050205	-83.883067
0.520000	3.578296	2.942779	23.929221	-30.027342	-82.231058
0.530000	3.396858	1.207286	22.880397	-29.997572	-80.540994
0.540000	3.266051	-0.529355	21.824518	-29.960741	-78.814311
0.550000	3.136016	-2.274220	20.759750	-29.916692	-77.052346
0.560000	3.006912	-4.032292	19.684484	-29.865306	-75.256795
0.570000	2.878905	-5.810387	18.597725	-29.8[illegible]504	-73.429710
0.580000	2.752146	-7.606593	17.499160	-29.740192	-71.573077
0.590000	2.626802	-9.419066	16.389770	-29.666342	-69.689276
0.600000	2.503035	-11.241160	15.271521	-29.584931	-67.780805
0.610000	2.381001	-13.062494	14.147906	-29.495958	-65.850296
0.620000	2.260874	-14.869249	13.023461	-29.399476	-63.900717
0.630000	2.142809	-16.645440	11.903561	-29.295545	-61.935048
0.640000	2.026963	-18.373956	10.794133	-29.184259	-59.956405
0.650000	1.913497	-20.037699	9.701799	-29.065746	-57.968112
0.660000	1.802547	-21.621448	8.630736	-28.940151	-55.973401
0.670000	1.694259	-23.112845	7.587507	-28.807657	-53.975779
0.680000	1.588761	-24.502627	6.575481	-28.668461	-51.978647
0.690000	1.486166	-25.787313	5.597240	-28.522796	-49.985515
0.700000	1.386595	-26.967067	4.653947	-28.370919	-48.000005
0.710000	1.290140	-28.046663	3.745315	-28.213100	-46.025622
0.720000	1.196883	-29.034461	2.869815	-28.049624	-44.065882
0.730000	1.106916	-29.940941	2.025165	-27.880828	-42.124485
0.740000	1.020301	-30.777660	1.208511	-27.707033	-40.204944
0.750000	0.937114	-31.555051	0.417227	-27.528613	-38.310951
0.760000	0.857404	-32.281439	-0.350893	-27.345921	-36.446012
0.770000	0.781227	-32.961060	-1.096883	-27.159353	-34.613663
0.780000	0.708640	-33.593188	-1.820281	-26.969329	-32.817509
0.790000	0.639682	-34.171623	-2.519051	-26.776251	-31.060869
0.800000	0.574398	-34.684586	-3.189315	-26.580558	-29.347131
0.810000	0.512819	-35.115345	-3.825574	-26.382699	-27.679466
0.820000	0.454969	-35.443432	-4.421080	-26.183112	-26.060742
0.830000	0.400869	-35.646261	-4.968176	-25.982264	-24.493813
0.840000	0.350522	-35.701471	-5.459108	-25.780612	-22.981117
0.850000	0.303916	-35.589111	-5.886672	-25.578609	-21.524755
0.860000	0.261025	-35.293988	-6.244955	-25.376713	-20.126556
0.870000	0.221806	-34.807925	-6.530071	-25.175371	-18.787967
0.880000	0.186195	-34.131433	-6.740810	-24.975015	-17.509997
0.890000	0.154107	-33.274926	-6.879055	-24.776057	-16.293204
0.900000	0.125452	-32.259095	-6.949930	-24.578914	-15.137978
0.910000	0.100113	-31.114704	-6.961940	-24.383962	-14.044177
0.920000	0.077960	-29.881370	-6.926640	-24.191552	-13.011310
0.930000	0.058862	-28.605607	-6.858039	-24.002037	-12.038779
0.940000	0.042685	-27.338371	-6.771994	-23.815721	-11.125677
0.950000	0.029286	-26.132248	-6.685307	-23.632894	-10.270910
0.960000	0.018541	-25.037822	-6.614630	-23.453816	-9.473400
0.970000	0.010330	-24.101062	-6.575601	-23.278742	-8.731996
0.980000	0.004553	-23.359876	-6.581762	-23.107891	-8.045598
0.990000	0.001128	-22.861885	-6.643789	-22.941459	-7.413160
1.000000	-0.000004	-22.562637	-6.768820	-22.779637	-6.833810

TABLE XVI

	DEFLECTION	RADIAL BENDING	TANGENTIAL BENDING	RADIAL MEMBRANE STRESS	TANGENTIAL MEMBRANE STRESS
0.010000	9.839260	49.554136	49.554461	-22.956220	-132.346635
0.020000	9.833543	49.553899	49.554084	-22.960238	-132.313032
0.030000	9.824012	49.559829	49.556433	-22.966971	-132.257030
0.040000	9.810670	49.580682	49.565651	-22.976428	-132.178756
0.050000	9.793516	49.627055	49.587039	-22.988678	-132.078747
0.060000	9.772537	49.710215	49.626105	-23.003772	-131.957015
0.070000	9.747734	49.840788	49.688334	-23.021779	-131.813908
0.080000	9.719094	50.026950	49.778423	-23.042801	-131.650030
0.090000	9.686593	50.273407	49.899462	-23.066928	-131.465540
0.100000	9.650214	50.580427	50.052793	-23.094255	-131.260909
0.110000	9.609926	50.942996	50.237581	-23.124931	-131.036775
0.120000	9.565696	51.351055	50.450521	-23.159083	-130.793382
0.130000	9.517486	51.789462	50.686153	-23.196845	-130.531377
0.140000	9.465257	52.239134	50.936907	-23.238359	-130.251078
0.150000	9.408967	52.678013	51.193632	-23.283773	-129.952872
0.160000	9.348572	53.082552	51.446080	-23.333252	-129.636951
0.170000	9.284043	53.429373	51.683677	-23.386909	-129.303759
0.180000	9.215347	53.696642	51.895975	-23.444894	-128.953387
0.190000	9.142462	53.865701	52.073373	-23.507325	-128.585858
0.200000	9.065376	53.921997	52.207584	-23.574318	-128.201026
0.210000	8.984086	53.855707	52.292078	-23.645954	-127.798765
0.220000	8.898603	53.662282	52.322265	-23.722302	-127.378664
0.230000	8.808949	53.342002	52.295528	-23.803402	-126.940290
0.240000	8.715157	52.899436	52.211119	-23.889261	-126.482921
0.250000	8.617275	52.342446	52.069837	-23.979864	-126.005721
0.260000	8.515357	51.680967	51.873718	-24.075149	-125.507734
0.270000	8.409475	50.925701	51.625465	-24.175021	-124.987746
0.280000	8.299697	50.086550	51.328043	-24.279363	-124.444580
0.290000	8.186107	49.171882	50.984127	-24.387990	-123.876621
0.300000	8.068795	48.187281	50.595908	-24.500730	-123.282525
0.310000	7.947857	47.135366	50.164594	-24.617324	-122.660498
0.320000	7.823387	46.015380	49.690422	-24.737505	-122.008804
0.330000	7.695509	44.824060	49.172687	-24.860953	-121.325480
0.340000	7.564322	43.555487	48.609830	-24.987338	-120.608686
0.350000	7.429953	42.202901	47.999819	-25.116295	-119.856480
0.360000	7.292533	40.759569	47.340370	-25.247394	-119.066715
0.370000	7.152183	39.219663	46.679433	-25.380224	-118.237369
0.380000	7.009074	37.580319	45.865757	-25.514334	-117.366504
0.390000	6.863342	35.841313	45.048907	-25.649247	-116.452135
0.400000	6.715155	34.006575	44.179715	-25.784466	-115.492276
0.410000	6.564698	32.084214	43.260518	-25.919485	-114.485166
0.420000	6.412131	30.085681	42.294780	-26.053783	-113.420875
0.430000	6.257655	28.025934	41.287409	-26.186823	-112.321917
0.440000	6.101466	25.972114	40.244138	-26.318065	-111.162636
0.450000	5.943752	23.791951	39.171271	-26.446970	-109.949749
0.460000	5.784719	21.653013	38.075286	-26.572988	-108.682094
0.470000	5.624557	19.570858	36.962217	-26.695569	-107.358411
0.480000	5.463465	17.408047	35.837457	-26.814176	-105.978174
0.490000	5.301646	15.323460	34.705284	-26.928288	-104.540725
0.500000	5.139289	13.271306	33.568538	-27.037378	-103.045661

0.510000	4.976572	11.251138	32.428606	-27.140955	-101.492889
0.520000	4.813698	9.258765	31.285558	-27.238518	-99.882572
0.530000	4.650829	7.286018	30.138061	-27.329611	-98.215043
0.540000	4.488163	5.323026	28.984064	-27.413798	-96.491078
0.550000	4.325863	3.358460	27.820752	-27.490654	-94.711420
0.560000	4.164104	1.381745	26.645356	-27.559793	-92.877218
0.570000	4.003080	-0.615675	25.455592	-27.620873	-90.990042
0.580000	3.842957	-2.639648	24.249739	-27.673545	-89.051392
0.590000	3.683927	-4.691800	23.027368	-27.717539	-87.063198
0.600000	3.526174	-6.769616	21.789255	-27.752590	-85.027521
0.610000	3.369875	-8.866414	20.537472	-27.778466	-82.946580
0.620000	3.215234	-10.971578	19.275478	-27.794991	-80.822969
0.630000	3.062435	-13.071888	18.007565	-27.802002	-78.659322
0.640000	2.911661	-15.152712	16.738701	-27.799379	-76.458457
0.650000	2.763111	-17.199083	15.474102	-27.787040	-74.223463
0.660000	2.616956	-19.197844	14.218553	-27.764932	-71.957406
0.670000	2.473380	-21.138384	12.976253	-27.733042	-69.663752
0.680000	2.332555	-23.014013	11.750200	-27.691386	-67.345938
0.690000	2.194642	-24.822680	10.542066	-27.640020	-65.007670
0.700000	2.059808	-26.566617	9.352137	-27.579034	-62.652907
0.710000	1.928202	-28.252545	8.179249	-27.508549	-60.285612
0.720000	1.799962	-29.890350	7.021087	-27.428714	-57.909934
0.730000	1.675245	-31.491563	5.874755	-27.339735	-55.530375
0.740000	1.554177	-33.068079	4.736986	-27.241830	-53.151412
0.750000	1.436905	-34.629661	3.605086	-27.135277	-50.777867
0.760000	1.323559	-36.182734	2.477148	-27.020356	-48.414547
0.770000	1.214259	-37.728293	1.352910	-26.897406	-46.066429
0.780000	1.109157	-39.260799	0.234138	-26.766806	-43.738768
0.790000	1.008376	-40.767638	-0.875300	-26.628943	-41.436609
0.800000	0.912049	-42.229125	-1.969140	-26.484258	-39.165322
0.810000	0.820296	-43.619314	-3.039010	-26.333218	-36.930167
0.820000	0.733230	-44.907406	-4.074865	-26.176312	-34.736226
0.830000	0.650968	-46.059705	-5.065391	-26.014066	-32.588761
0.840000	0.573599	-47.042410	-5.998984	-25.847026	-30.492743
0.850000	0.501199	-47.824293	-6.864539	-25.675753	-28.452932
0.860000	0.433826	-48.379376	-7.652292	-25.500837	-26.473942
0.870000	0.371516	-48.689786	-8.354758	-25.322875	-24.560113
0.880000	0.314279	-48.747623	-8.967456	-25.142473	-22.715418
0.890000	0.262097	-48.556649	-9.489490	-24.960243	-20.943368
0.900000	0.214941	-48.132669	-9.923694	-24.776813	-19.247415
0.910000	0.172747	-47.503581	-10.276883	-24.592792	-17.630372
0.920000	0.135428	-46.707664	-10.559499	-24.408788	-16.094680
0.930000	0.102894	-45.792222	-10.784942	-24.225414	-14.642789
0.940000	0.075041	-44.810028	-10.968931	-24.043259	-13.275043
0.950000	0.051745	-43.816644	-11.120505	-23.862900	-11.994085
0.960000	0.032901	-42.866196	-11.280788	-23.684889	-10.800288
0.970000	0.018395	-42.008415	-11.441999	-23.509784	-9.694207
0.980000	0.008129	-41.284792	-11.626280	-23.338098	-8.675977
0.990000	0.002018	-40.725973	-11.844796	-23.170321	-7.745386
1.000000	-0.000005	-40.349759	-12.104971	-23.006928	-6.901974

TABLE XVII

	DEFLECTION	RADIAL BENDING	TANGENTIAL BENDING	RADIAL MEMBRANE STRESS	TANGENTIAL MEMBRANE STRESS
0.010000	15.063224	54.289986	54.255810	12.852262	-137.462569
0.020000	15.056961	54.525143	54.392155	12.839705	-137.450104
0.030000	15.046496	54.893789	54.608340	12.818954	-137.420064
0.040000	15.031799	55.363362	54.888750	12.790314	-137.394687
0.050000	15.012833	55.892686	55.213685	12.754211	-137.348476
0.060000	14.989535	56.435700	55.560488	12.711045	-137.286786
0.070000	14.961872	56.946661	55.905975	12.661304	-137.207284
0.080000	14.929798	57.383270	56.228212	12.605494	-137.108291
0.090000	14.893268	57.711419	56.508112	12.544003	-136.987762
0.100000	14.852261	57.908206	56.730988	12.477243	-136.844527
0.110000	14.806761	57.963366	56.887952	12.405509	-136.678490
0.120000	14.756771	57.880834	56.975926	12.328960	-136.489506
0.130000	14.702298	57.677221	56.997847	12.247722	-136.278728
0.140000	14.643368	57.380526	56.962072	12.161757	-136.047389
0.150000	14.580011	57.026893	56.881395	12.070937	-135.797333
0.160000	14.512256	56.656378	56.771504	11.975017	-135.530447
0.170000	14.440153	56.310093	56.649486	11.873781	-135.248961
0.180000	14.363727	56.024508	56.532006	11.766900	-134.954705
0.190000	14.283007	55.828709	56.433746	11.654046	-134.649063
0.200000	14.198015	55.741788	56.366241	11.534880	-134.332922
0.210000	14.108752	55.770229	56.336603	11.409154	-134.006819
0.220000	14.015215	55.908043	56.347189	11.276638	-133.670393
0.230000	13.917395	56.137701	56.395700	11.137175	-133.322962
0.240000	13.815261	56.431352	56.475294	10.990683	-132.963109
0.250000	13.708788	56.754640	56.575886	10.837127	-132.589303
0.260000	13.597944	57.069755	56.684887	10.676593	-132.199753
0.270000	13.482703	57.339497	56.788924	10.509165	-131.792583
0.280000	13.363030	57.531028	56.875190	10.335005	-131.366544
0.290000	13.238911	57.619112	56.932445	10.154279	-130.920111
0.300000	13.110336	57.588373	56.952700	9.967172	-130.453045
0.310000	12.977312	57.434942	56.931233	9.773853	-129.964902
0.320000	12.839847	57.166203	56.867453	9.574450	-129.456153
0.330000	12.697989	56.800054	56.764454	9.369064	-128.927301
0.340000	12.551755	56.361808	56.628597	9.157774	-128.379721
0.350000	12.401196	55.881964	56.468695	8.940596	-127.814719
0.360000	12.246362	55.392570	56.294569	8.717522	-127.233223
0.370000	12.087279	54.922520	56.116151	8.488497	-126.636442
0.380000	11.924029	54.495821	55.942270	8.253472	-126.024964
0.390000	11.756626	54.126532	55.779043	8.012406	-125.399070
0.400000	11.585108	53.818756	55.629601	7.765234	-124.758216
0.410000	11.409526	53.564987	55.493376	7.511962	-124.101418
0.420000	11.229871	53.346101	55.365825	7.252581	-123.426939
0.430000	11.046179	53.134131	55.239179	6.987191	-122.732830
0.440000	10.858480	52.894202	55.102814	6.715079	-122.016326
0.450000	10.666781	52.587712	54.944349	6.438826	-121.274962
0.460000	10.471120	52.174344	54.750731	6.156252	-120.505924
0.470000	10.271526	51.625783	54.509492	5.868397	-119.706472
0.480000	10.068052	50.908957	54.210057	5.575503	-118.874553
0.490000	9.860779	50.008444	53.844545	5.278108	-118.008124
0.500000	9.649789	48.918098	53.408369	4.976299	-117.105606

0.510000	9.435176	47.643128	52.900698	4.670484	-116.166040
0.520000	9.217087	46.199255	52.324088	4.361010	-115.188469
0.530000	8.995635	44.609917	51.684041	4.048189	-114.172376
0.540000	8.771003	42.904294	50.988363	3.732352	-113.117223
0.550000	8.543327	41.112148	50.245461	3.413802	-112.022111
0.560000	8.312782	39.260957	49.463463	3.092856	-110.885895
0.570000	8.079564	37.372013	48.649926	2.769859	-109.707125
0.580000	7.843835	35.456849	47.808245	2.445161	-108.483325
0.590000	7.605791	33.515477	46.939481	2.119138	-107.211546
0.600000	7.365614	31.535418	46.040426	1.792219	-105.887989
0.610000	7.123479	29.491909	45.103866	1.464877	-104.508166
0.620000	6.879599	27.350500	44.119363	1.137656	-103.067304
0.630000	6.634172	25.069421	43.073598	0.811146	-101.560129
0.640000	6.387404	22.603407	41.951687	0.486002	-99.981349
0.650000	6.139548	19.908859	40.738723	0.162936	-98.325861
0.660000	5.890841	16.947562	39.420919	-0.157316	-96.588894
0.670000	5.641576	13.691514	37.987312	-0.473949	-94.766513
0.680000	5.392063	10.125807	36.430654	-0.786167	-92.855311
0.690000	5.142633	6.250450	34.748338	-1.093144	-90.853063
0.700000	4.893673	2.081498	32.942794	-1.394053	-88.758348
0.710000	4.645572	-2.351042	31.021085	-1.688100	-86.570514
0.720000	4.398743	-7.005100	28.994370	-1.974505	-84.289587
0.730000	4.153650	-11.830456	26.877120	-2.252508	-81.916398
0.740000	3.910738	-16.774274	24.685069	-2.521394	-79.451874
0.750000	3.670491	-21.786251	22.434009	-2.780465	-76.897295
0.760000	3.433477	-26.823338	20.137864	-3.029068	-74.253724
0.770000	3.199864	-31.855871	17.807068	-3.266571	-71.522177
0.780000	2.970449	-36.869525	15.447670	-3.492343	-68.703765
0.790000	2.745591	-41.868035	13.060070	-3.705803	-65.799137
0.800000	2.525770	-46.872774	10.639335	-3.906365	-62.809518
0.810000	2.311451	-51.971514	8.175145	-4.093459	-59.736514
0.820000	2.103098	-57.063612	5.653099	-4.266562	-56.582384
0.830000	1.901209	-62.354158	3.056596	-4.425165	-53.351050
0.840000	1.706277	-67.847449	0.368600	-4.568821	-50.048136
0.850000	1.518810	-73.509543	-2.425967	-4.697150	-46.681515
0.860000	1.339354	-79.609804	-5.337466	-4.809900	-43.261859
0.870000	1.168472	-85.914592	-8.369400	-4.906884	-39.802890
0.880000	1.006760	-92.481195	-11.516750	-4.988115	-36.321358
0.890000	0.854823	-99.254961	-14.763985	-5.053778	-32.837026
0.900000	0.713317	-106.146824	-18.084659	-5.104235	-29.372870
0.910000	0.582882	-113.035975	-21.441805	-5.140082	-25.953982
0.920000	0.464151	-119.773720	-24.788779	-5.162148	-22.607135
0.930000	0.357765	-126.189458	-28.070783	-5.171473	-19.360311
0.940000	0.264323	-132.099844	-31.227734	-5.169335	-16.241232
0.950000	0.184354	-137.319196	-34.197297	-5.157205	-13.276614
0.960000	0.118345	-141.669871	-36.918144	-5.136728	-10.490925
0.970000	0.066678	-144.993132	-39.333261	-5.109697	-7.905775
0.980000	0.029638	-147.159669	-41.393640	-5.077988	-5.530490
0.990000	0.007392	-148.077292	-43.060859	-5.043532	-3.401562
1.000000	-0.000010	-147.696997	-44.309201	-5.008254	-1.502306

TABLE XVIII

.082111	1	1	1
.024521	1	1	2
-.001664	1	1	3
.000544	1	1	4
-.000240	1	1	5
.000125	1	1	6
-.000072	1	1	7
.000045	1	1	8
-.000030	1	1	9
.000021	1	1	10
-.000015	1	1	11
.000011	1	1	12
-.000008	1	1	13
.000006	1	1	14
-.000005	1	1	15
.046220	1	2	2
.016029	1	2	3
-.001235	1	2	4
.000442	1	2	5
-.000208	1	2	6
.000114	1	2	7
-.000069	1	2	8
.000045	1	2	9
-.000030	1	2	10
.000022	1	2	11
-.000016	1	2	12
.000012	1	2	13
-.000009	1	2	14
.000007	1	2	15
.033739	1	3	3
.011642	1	3	4
-.000966	1	3	5
.000362	1	3	6
-.000177	1	3	7
.000100	1	3	8
-.000062	1	3	9
.000041	1	3	10
-.000029	1	3	11
.000021	1	3	12
-.000015	1	3	13
.000012	1	3	14
-.000009	1	3	15
.025901	1	4	4
.009377	1	4	5
-.000790	1	4	6
.000304	1	4	7
-.000153	1	4	8
.000088	1	4	9
-.000056	1	4	10
.000037	1	4	11
-.000026	1	4	12
.000019	1	4	13
-.000014	1	4	14
.000011	1	4	15
.021006	1	5	5
.007759	1	5	6
-.000667	1	5	7
.000262	1	5	8
-.000133	1	5	9
.000078	1	5	10
-.000050	1	5	11
.000034	1	5	12
-.000024	1	5	13
.000018	1	5	14
-.000014	1	5	15
.017663	1	6	6
.006616	1	6	7
-.000577	1	6	8
.000230	1	6	9
-.000118	1	6	10
.000070	1	6	11
-.000045	1	6	12
.000031	1	6	13
-.000022	1	6	14
.000017	1	6	15
.015236	1	7	7
.005765	1	7	8
-.000508	1	7	9
.000204	1	7	10
-.000106	1	7	11
.000064	1	7	12
-.000041	1	7	13
.000029	1	7	14
-.000021	1	7	15
.013395	1	8	8
.005108	1	8	9
-.000454	1	8	10
.000184	1	8	11
-.000096	1	8	12
.000058	1	8	13
-.000038	1	8	14
.000026	1	8	15
.011950	1	9	9
.004586	1	9	10
-.000410	1	9	11
.000167	1	9	12
-.000088	1	9	13
.000053	1	9	14
-.000035	1	9	15
.010787	1	10	10
.004160	1	10	11
-.000374	1	10	12
.000153	1	10	13
-.000081	1	10	14
.000049	1	10	15
.009829	1	11	11
.003807	1	11	12
-.000344	1	11	13
.000141	1	11	14
-.000075	1	11	15
.009028	1	12	12
.003508	1	12	13
-.000316	1	12	14
.000131	1	12	15
.008347	1	13	13
.003254	1	13	14
-.000296	1	13	15
.007762	1	14	14
.003033	1	14	15
.007254	1	15	15
.036835	2	2	2
.033373	2	2	3
.011177	2	2	4
-.000958	2	2	5
.000372	2	2	6
-.000186	2	2	7
.000107	2	2	8
-.000067	2	2	9
.000045	2	2	10
-.000031	2	2	11
.000023	2	2	12
-.000017	2	2	13
.000013	2	2	14
-.000010	2	2	15
.027917	2	3	3
.024954	2	3	4
.006550	2	3	5
-.000770	2	3	6
.000312	2	3	7
-.000162	2	3	8
.000096	2	3	9
-.000061	2	3	10
.000042	2	3	11
-.000030	2	3	12
.000022	2	3	13
-.000017	2	3	14
.000013	2	3	15
.021999	2	4	4
.019854	2	4	5
.006919	2	4	6
-.000641	2	4	7
.000260	2	4	8
-.000141	2	4	9
.000085	2	4	10
-.000056	2	4	11
.000038	2	4	12
-.000028	2	4	13
.000021	2	4	14
-.000016	2	4	15
.018056	2	5	5
.016460	2	5	6
.005811	2	5	7
-.000548	2	5	8
.000232	2	5	9
-.000125	2	5	10
.000076	2	5	11
-.000050	2	5	12
.000035	2	5	13
-.000026	2	5	14
.000019	2	5	15
.015282	2	6	6
.014058	2	6	7
.005009	2	6	8
-.000479	2	6	9
.000205	2	6	10
-.000112	2	6	11
.000069	2	6	12
-.000046	2	6	13
.000032	2	6	14
-.000024	2	6	15
.013235	2	7	7
.012262	2	7	8
.004401	2	7	9
-.000425	2	7	10
.000183	2	7	11

-.000101	2	7	12
.000063	2	7	13
-.000042	2	7	14
.000030	2	7	15
.011665	2	8	8
.010871	2	8	9
.003925	2	8	10
-.000382	2	8	11
.000166	2	8	12
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.006010	13	15	15
.006134	14	14	14
.006032	14	14	15
.005853	14	15	15
.005776	15	15	15

TABLE XIX

P--

```
( LA**4*J(1)**4*J(2)**2*K(2)**2*A1*MU - 3*LA**4*J(1)**4*J(2)**2*K(2)**2*
A1 - 2*LA**4*J(1)**3*J(2)**3*K(1)*K(2)*A2 + LA**2*J(1)**5*J(2)**2*K(1)*
K(2)**2*A1**2 + LA**2*J(1)**4*J(2)**3*K(1)**2*K(2)*A1*A2 +
2*LA**2*J(1)**3*J(2)*K(1)*K(2)*A1**2*T(1,1,2) + 2*LA**2*J(1)**3*J(2)*
K(1)*K(2)*A1*A2*T(1,2,2) + 2*LA**2*J(1)**3*J(2)*K(1)*K(2)*A1*A2*
T(2,1,2) + 2*LA**2*J(1)**3*J(2)*K(1)*K(2)*A2**2*T(2,2,2) -
2*LA**2*J(1)**2*J(2)**2*K(1)**2*A1*A2*T(1,2,1)*MU + 2*LA**2*J(1)**2*
J(2)**2*K(1)**2*A1*A2*T(1,2,1) - 2*LA**2*J(1)**2*J(2)**2*K(1)**2*A2**2*
T(2,2,1)*MU + 2*LA**2*J(1)**2*J(2)**2*K(1)**2*A2**2*T(2,2,1) -
3*LA**2*J(1)**2*J(2)**2*K(2)**2*A1**2*T(1,1,1)*MU + 5*LA**2*J(1)**2*
J(2)**2*K(2)**2*A1**2*T(1,1,1) - LA**2*J(1)**2*J(2)**2*K(2)**2*A1*A2*
T(1,2,1)*MU + 3*LA**2*J(1)**2*J(2)**2*K(2)**2*A1*A2*T(1,2,1) -
3*LA**2*J(1)**2*J(2)**2*K(2)**2*A1*A2*T(2,1,1)*MU + 5*LA**2*J(1)**2*
J(2)**2*K(2)**2*A1*A2*T(2,1,1) - LA**2*J(1)**2*J(2)**2*K(2)**2*A2**2*
T(2,2,1)*MU + 3*LA**2*J(1)**2*J(2)**2*K(2)**2*A2**2*T(2,2,1) -
400*P*J(1)**3*J(2)**2*K(1)*K(2)**2*MU + 400*P*J(1)**3*J(2)**2*K(1)*
K(2)**2 + J(1)**4*J(2)**2*K(1)**4*K(2)**2*A1*MU - J(1)**4*J(2)**2*
K(1)**4*K(2)**2*A1 - J(1)**4*J(2)*K(1)**2*K(2)*A1**3*T(1,1,2) -
J(1)**4*J(2)*K(1)**2*K(2)*A1**2*A2*T(1,2,2) - J(1)**4*J(2)*K(1)**2*K(2)*
A1**2*A2*T(2,1,2) - J(1)**4*J(2)*K(1)**2*K(2)*A1*A2**2*T(2,2,2) -
J(1)**3*J(2)**2*K(1)*K(2)**2*A1**3*T(1,1,1) - J(1)**3*J(2)**2*K(1)*
K(2)**2*A1**2*A2*T(1,2,1) - J(1)**3*J(2)**2*K(1)*K(2)**2*A1**2*A2*
T(2,1,1) - J(1)**3*J(2)**2*K(1)*K(2)**2*A1*A2**2*T(2,2,1) +
2*J(1)**2*K(1)**2*A1**3*T(1,1,2)*T(1,2,1)*MU - 2*J(1)**2*K(1)**2*A1**3*
T(1,1,2)*T(1,2,1) + 2*J(1)**2*K(1)**2*A1**2*A2*T(1,1,2)*T(2,2,1)*MU -
2*J(1)**2*K(1)**2*A1**2*A2*T(1,1,2)*T(2,2,1) + 2*J(1)**2*K(1)**2*A1**2*
A2*T(1,2,1)*T(1,2,2)*MU - 2*J(1)**2*K(1)**2*A1**2*A2*T(1,2,1)*T(1,2,2) +
2*J(1)**2*K(1)**2*A1**2*A2*T(1,2,1)*T(2,1,2)*MU - 2*J(1)**2*K(1)**2*
A1**2*A2*T(1,2,1)*T(2,1,2) + 2*J(1)**2*K(1)**2*A1*A2**2*T(1,2,1)*
T(2,2,2)*MU - 2*J(1)**2*K(1)**2*A1*A2**2*T(1,2,1)*T(2,2,2) +
2*J(1)**2*K(1)**2*A1*A2**2*T(1,2,2)*T(2,2,1)*MU - 2*J(1)**2*K(1)**2*A1*
```

```
A2**2*T(1,2,2)*T(2,2,1) + 2*J(1)**2*K(1)**2*A1*A2**2*T(2,1,2)*T(2,2,1)*
MU - 2*J(1)**2*K(1)**2*A1*A2**2*T(2,1,2)*T(2,2,1) + 2*J(1)**2*K(1)**2*
A2**3*T(2,2,1)*T(2,2,2)*MU - 2*J(1)**2*K(1)**2*A2**3*T(2,2,1)*T(2,2,2) +
2*J(2)**2*K(2)**2*A1**3*T(1,1,1)**2*MU - 2*J(2)**2*K(2)**2*A1**3*
T(1,1,1)**2 + 2*J(2)**2*K(2)**2*A1**2*A2*T(1,1,1)*T(1,2,1)*MU -
2*J(2)**2*K(2)**2*A1**2*A2*T(1,1,1)*T(1,2,1) + 4*J(2)**2*K(2)**2*A1**2*
A2*T(1,1,1)*T(2,1,1)*MU - 4*J(2)**2*K(2)**2*A1**2*A2*T(1,1,1)*T(2,1,1) +
2*J(2)**2*K(2)**2*A1*A2**2*T(1,1,1)*T(2,2,1)*ML - 2*J(2)**2*K(2)**2*A1*
A2**2*T(1,1,1)*T(2,2,1) + 2*J(2)**2*K(2)**2*A1*A2**2*T(1,2,1)*T(2,1,1)*
MU - 2*J(2)**2*K(2)**2*A1*A2**2*T(1,2,1)*T(2,1,1) + 2*J(2)**2*K(2)**2*
A1*A2**2*T(2,1,1)**2*MU - 2*J(2)**2*K(2)**2*A1*A2**2*T(2,1,1)**2 +
2*J(2)**2*K(2)**2*A2**3*T(2,1,1)*T(2,2,1)*MU - 2*J(2)**2*K(2)**2*A2**3*
T(2,1,1)*T(2,2,1) ) / ( ( 4*J(1)**3*J(2)**2*K(1)*K(2)**2 ) * ( MU - 1 ) *
)
```

≡ F

```
= ( 2*LA**4*J(1)**3*J(2)**3*K(1)*K(2)*A1 - LA**4*J(1)**2*J(2)**4*
K(1)**2*A2*MJ + 3*LA**4*J(1)**2*J(2)**4*K(1)**2*A2 - LA**2*J(1)**3*
J(2)**4*K(1)*K(2)**2*A1*A2 - LA**2*J(1)**2*J(2)**5*K(1)**2*K(2)*A2**2 +
LA**2*J(1)**2*J(2)**2*K(1)**2*A1**2*T(1,1,2)*ML - 3*LA**2*J(1)**2*
J(2)**2*K(1)**2*A1**2*T(1,1,2) + 3*LA**2*J(1)**2*J(2)**2*K(1)**2*A1*A2*
T(1,2,2)*MJ - 5*_A**2*J(1)**2*J(2)**2*K(1)**2*A1*A2*T(1,2,2) +
LA**2*J(1)**2*J(2)**2*K(1)**2*A1*A2*T(2,1,2)*ML - 3*LA**2*J(1)**2*
J(2)**2*K(1)**2*A1*A2*T(2,1,2) + 3*LA**2*J(1)**2*J(2)**2*K(1)**2*A2**2*
T(2,2,2)*MJ - 5*_A**2*J(1)**2*J(2)**2*K(1)**2*A2**2*T(2,2,2) +
2*LA**2*J(1)**2*J(2)**2*K(2)**2*A1**2*T(1,1,2)*ML - 2*LA**2*J(1)**2*
J(2)**2*K(2)**2*A1**2*T(1,1,2) + 2*LA**2*J(1)**2*J(2)**2*K(2)**2*A1*A2*
T(2,1,2)*MJ - 2*_A**2*J(1)**2*J(2)**2*K(2)**2*A1*A2*T(2,1,2) -
2*LA**2*J(1)*J(2)**3*K(1)*K(2)*A1**2*T(1,1,1) - 2*LA**2*J(1)*J(2)**3*
K(1)*K(2)*A1*A2*T(1,2,1) - 2*LA**2*J(1)*J(2)**3*K(1)*K(2)*A1*A2*
T(2,1,1) - 2*LA**2*J(1)*J(2)**3*K(1)*K(2)*A2**2*T(2,2,1) +
```

```
400*P*J(1)**2*J(2)**3*K(1)**2*K(2)*MU - 400*P*J(1)**2*J(2)**3*K(1)**2*
K(2) - J(1)**2*J(2)**4*K(1)**2*K(2)**4*A2*MU + J(1)**2*J(2)**4*K(1)**2*
K(2)**4*A2 + J(1)**2*J(2)**3*K(1)**2*K(2)*A1**2*A2*T(1,1,2) +
J(1)**2*J(2)**3*K(1)**2*K(2)*A1*A2**2*T(1,2,2) + J(1)**2*J(2)**3*
K(1)**2*K(2)*A1*A2**2*T(2,1,2) + J(1)**2*J(2)**3*K(1)**2*K(2)*A2**3*
T(2,2,2) - 2*J(1)**2*K(1)**2*A1**3*T(1,1,2)*T(1,2,2)*MU +
2*J(1)**2*K(1)**2*A1**3*T(1,1,2)*T(1,2,2) - 2*J(1)**2*K(1)**2*A1**2*A2*
T(1,1,2)*T(2,2,2)*MU + 2*J(1)**2*K(1)**2*A1**2*A2*T(1,1,2)*T(2,2,2) -
2*J(1)**2*K(1)**2*A1**2*A2*T(1,2,2)**2*MU + 2*J(1)**2*K(1)**2*A1**2*A2*
T(1,2,2)**2 - 2*J(1)**2*K(1)**2*A1**2*A2*T(1,2,2)*T(2,1,2)*MU +
2*J(1)**2*K(1)**2*A1**2*A2*T(1,2,2)*T(2,1,2) - 4*J(1)**2*K(1)**2*A1*
A2**2*T(1,2,2)*T(2,2,2)*MU + 4*J(1)**2*K(1)**2*A1*A2**2*T(1,2,2)*
T(2,2,2) - 2*J(1)**2*K(1)**2*A1*A2**2*T(2,1,2)*T(2,2,2)*MU +
2*J(1)**2*K(1)**2*A1*A2**2*T(2,1,2)*T(2,2,2) - 2*J(1)**2*K(1)**2*A2**3*
T(2,2,2)**2*MU + 2*J(1)**2*K(1)**2*A2**3*T(2,2,2)**2 +
J(1)*J(2)**4*K(1)*K(2)**2*A1**2*A2*T(1,1,1) + J(1)*J(2)**4*K(1)*K(2)**2*
A1*A2**2*T(1,2,1) + J(1)*J(2)**4*K(1)*K(2)**2*A1*A2**2*T(2,1,1) +
J(1)*J(2)**4*K(1)*K(2)**2*A2**3*T(2,2,1) - 2*J(2)**2*K(2)**2*A1**3*
T(1,1,1)*T(1,1,2)*MU + 2*J(2)**2*K(2)**2*A1**3*T(1,1,1)*T(1,1,2) -
2*J(2)**2*K(2)**2*A1**2*A2*T(1,1,1)*T(2,1,2)*ML + 2*J(2)**2*K(2)**2*
A1**2*A2*T(1,1,1)*T(2,1,2) - 2*J(2)**2*K(2)**2*A1**2*A2*T(1,1,2)*
T(1,2,1)*MJ + 2*J(2)**2*K(2)**2*A1**2*A2*T(1,1,2)*T(1,2,1) - 2*J(2)**2*
K(2)**2*A1**2*A2*T(1,1,2)*T(2,1,1)*MU + 2*J(2)**2*K(2)**2*A1**2*A2*
T(1,1,2)*T(2,1,1) - 2*J(2)**2*K(2)**2*A1*A2**2*T(1,1,2)*T(2,2,1)*MU +
2*J(2)**2*K(2)**2*A1*A2**2*T(1,1,2)*T(2,2,1) - 2*J(2)**2*K(2)**2*A1*
A2**2*T(1,2,1)*T(2,1,2)*MU + 2*J(2)**2*K(2)**2*A1*A2**2*T(1,2,1)*
T(2,1,2) - 2*J(2)**2*K(2)**2*A1*A2**2*T(2,1,1)*T(2,1,2)*MU +
2*J(2)**2*K(2)**2*A1*A2**2*T(2,1,1)*T(2,1,2) - 2*J(2)**2*K(2)**2*A2**3*
```

```
T(2,1,2)*T(2,2,1)*MU + 2*J(2)**2*K(2)**2*A2**3*T(2,1,2)*T(2,2,1) ) /
( ( 4*J(1)**2*J(2)**3*K(1)**2*K(2) ) * ( MU + 1 ) )
```

Captions for Figures

1. Geometry of the thin, shallow spherical shell.
   Rise of the shell: H
   Radius at the base: R
   Thickness: h

2. Load deformation curve proposed by v. Karman and Tsien, (v.K-T curve)

3. Same as Fig. 2, with p as independent parameter. The broken lines indicate where the shell buckles and snaps according to a one-mode solution.

4. Buckling and snapping pressures as functions of λ given by a one-mode solution

5. v.K-T curves given by n-modes solutions with $1 \leq n \leq 10$ (*)

6. Deflection, (of the shell), $\omega(x)$ for the five displayed pressures.

7. Shell's shape for the five displayed pressures.

8. Tangential, (=radial) bending stress function at the centre of the shell vs. pressure for n-modes solutions with $1 \leq n \leq 10$ (*)

9. Tangential bending stress along the radius for the five pressures in Fig. 6

10. Same as 9 with radial bending stress.

11. Tangential, (= radial) membrane stress function at the centre of the shell vs. pressure for n-modes solutions with $1 \leq n \leq 10$ (*)

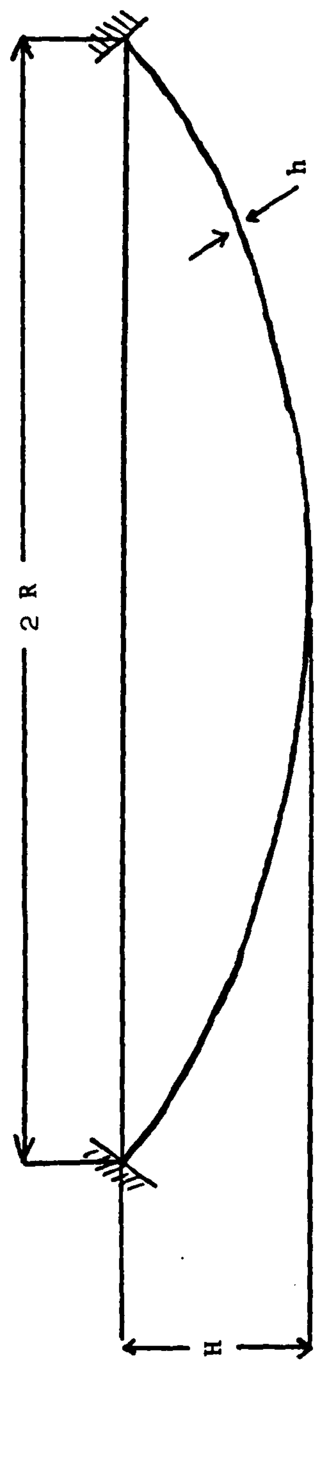
12. Tangential membrane stress along the radius for the five pressures in Fig. 6

13. Same as 12 with radial membrane stress

14. Comparison of results of this thesis with previous results.

(*) The number at the end of the curve indicates the number of modes in the approximate solution.

Fig. 1



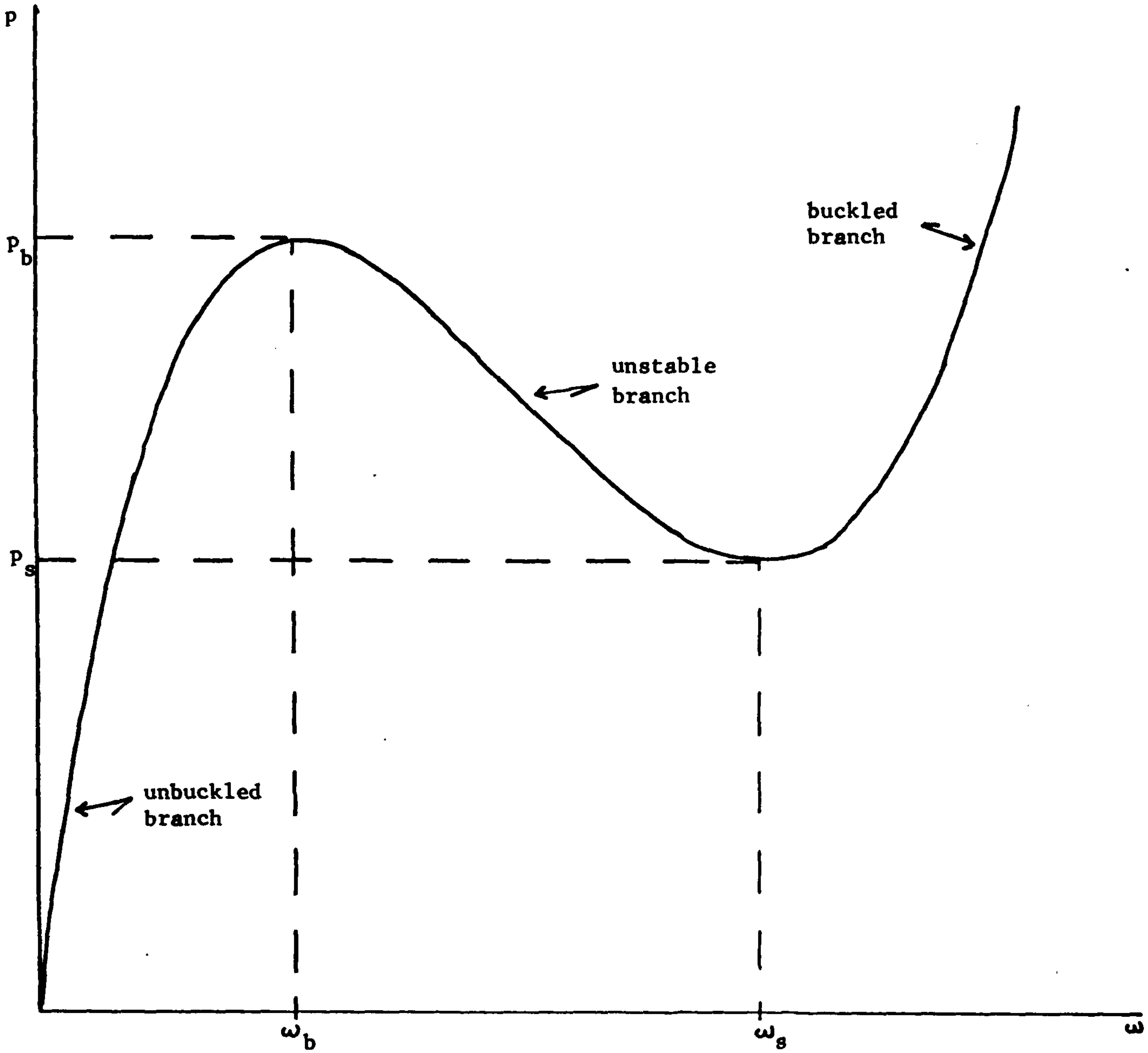

p
p_b
p_s
buckled
branch
unstable
branch
unbuckled
branch
ω_b
ω_s
ω

Fig. 2

Fig. 3

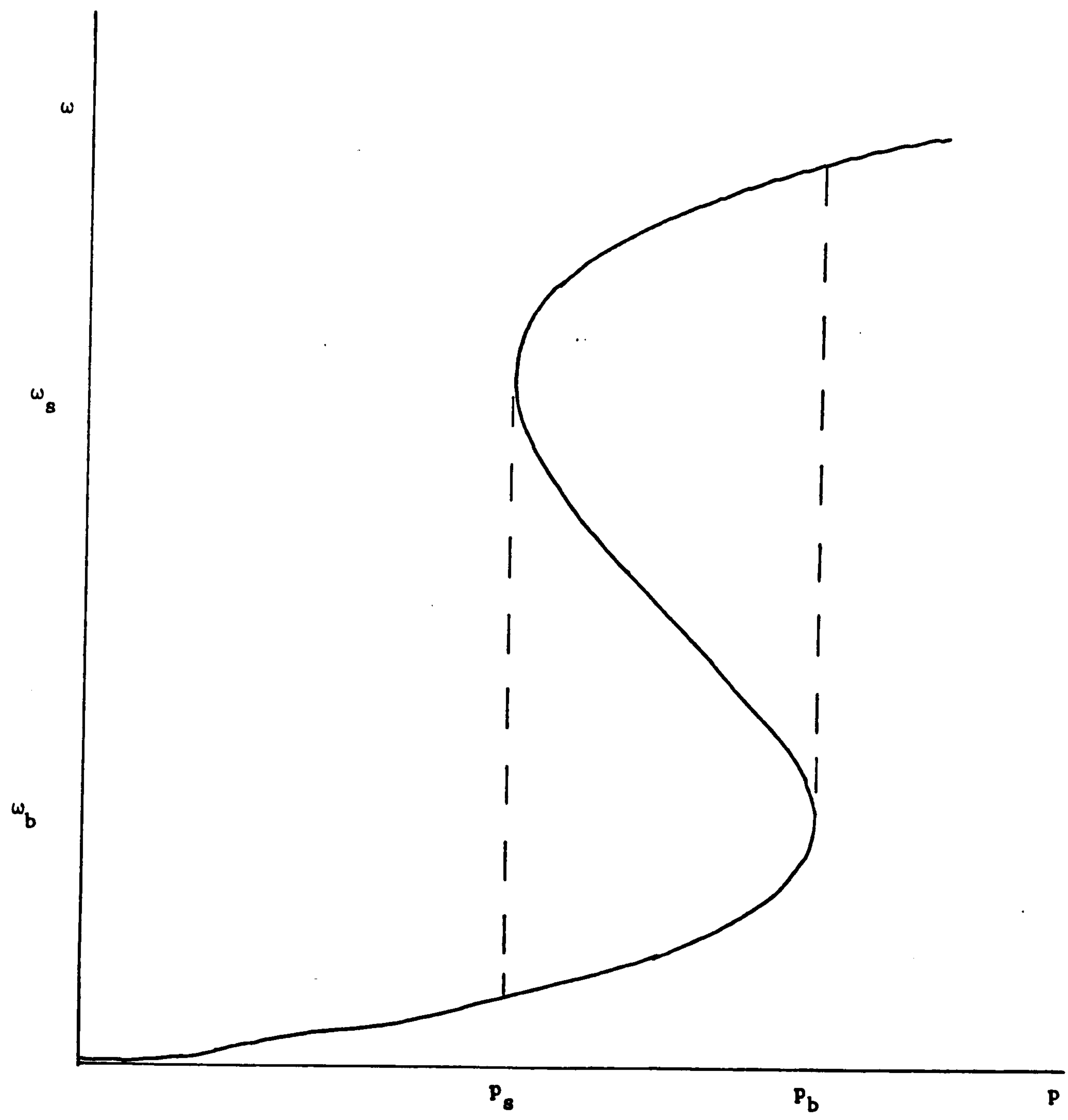


Fig. 4

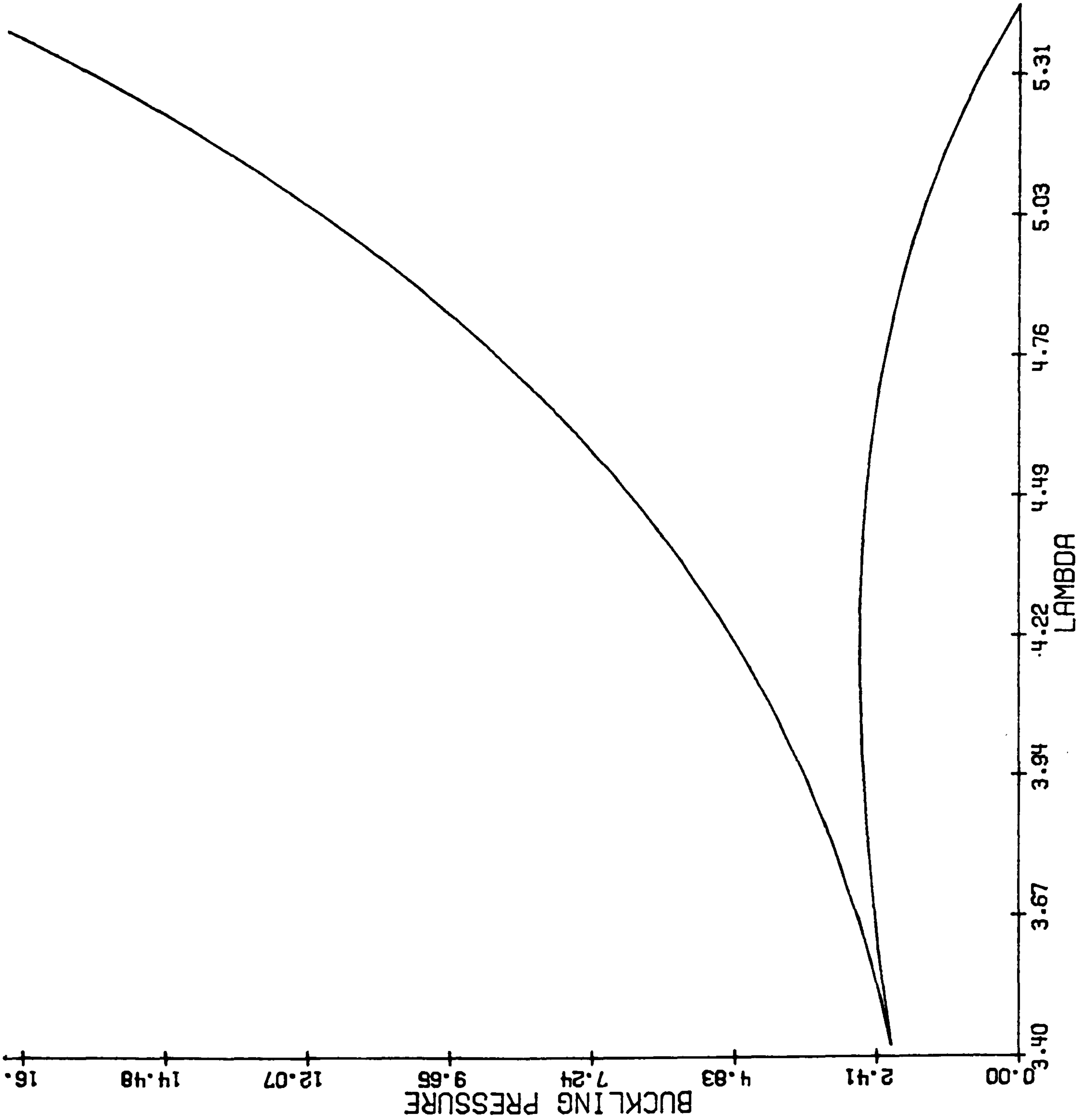
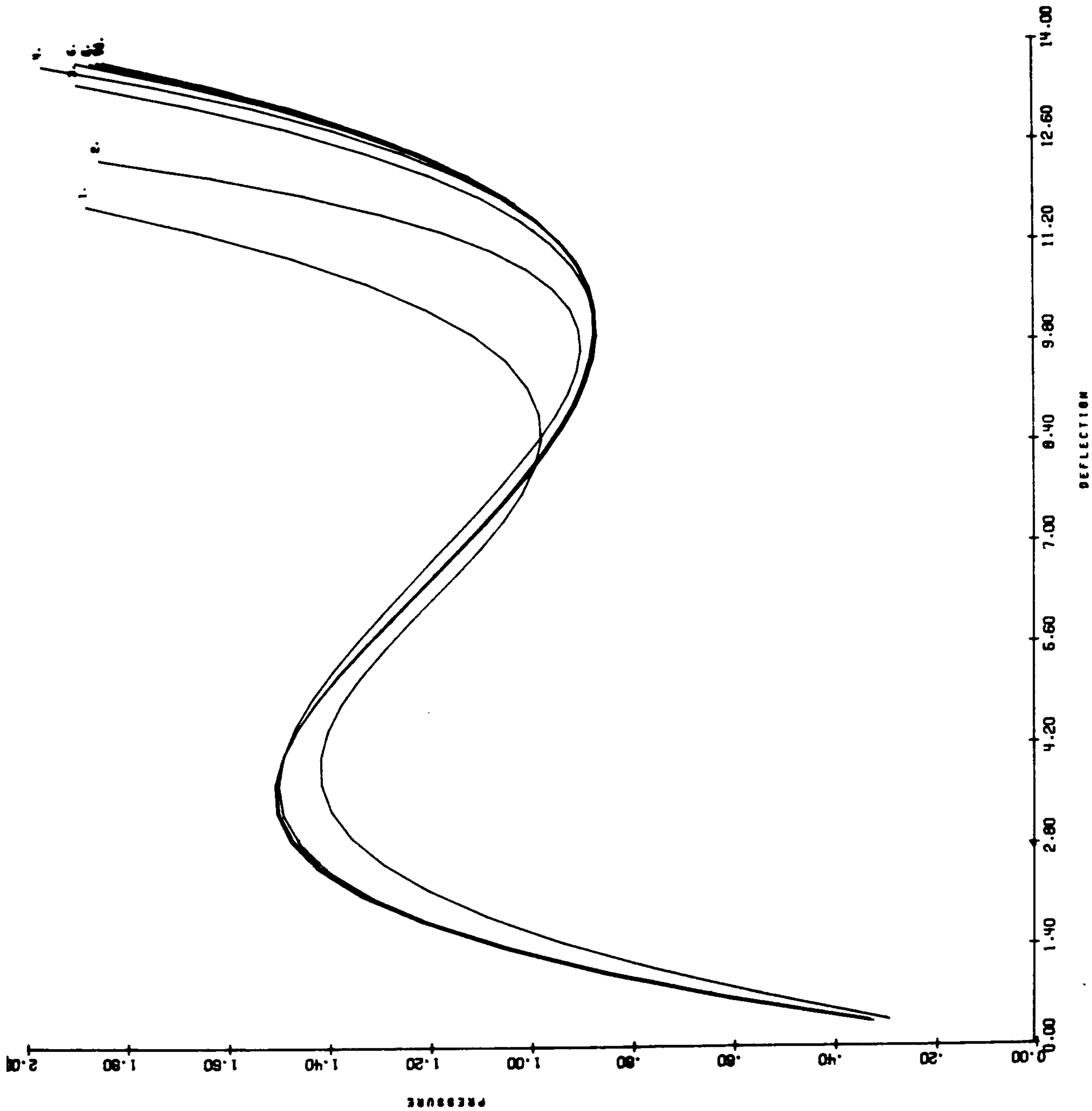

BUCKLING PRESSURE
0.00
2.41
4.83
7.24
9.66
12.07
14.48
16.
LAMBDA
3.40
3.67
3.94
4.22
4.49
4.76
5.03
5.31

Fig. 5


PRESSURE
DEFLECTION
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.20
.40
.60
.80
1.00
1.20
1.40
1.60
1.80
2.00
1.40
2.80
4.20
5.60
7.00
8.40
9.80
11.20
12.60
14.00

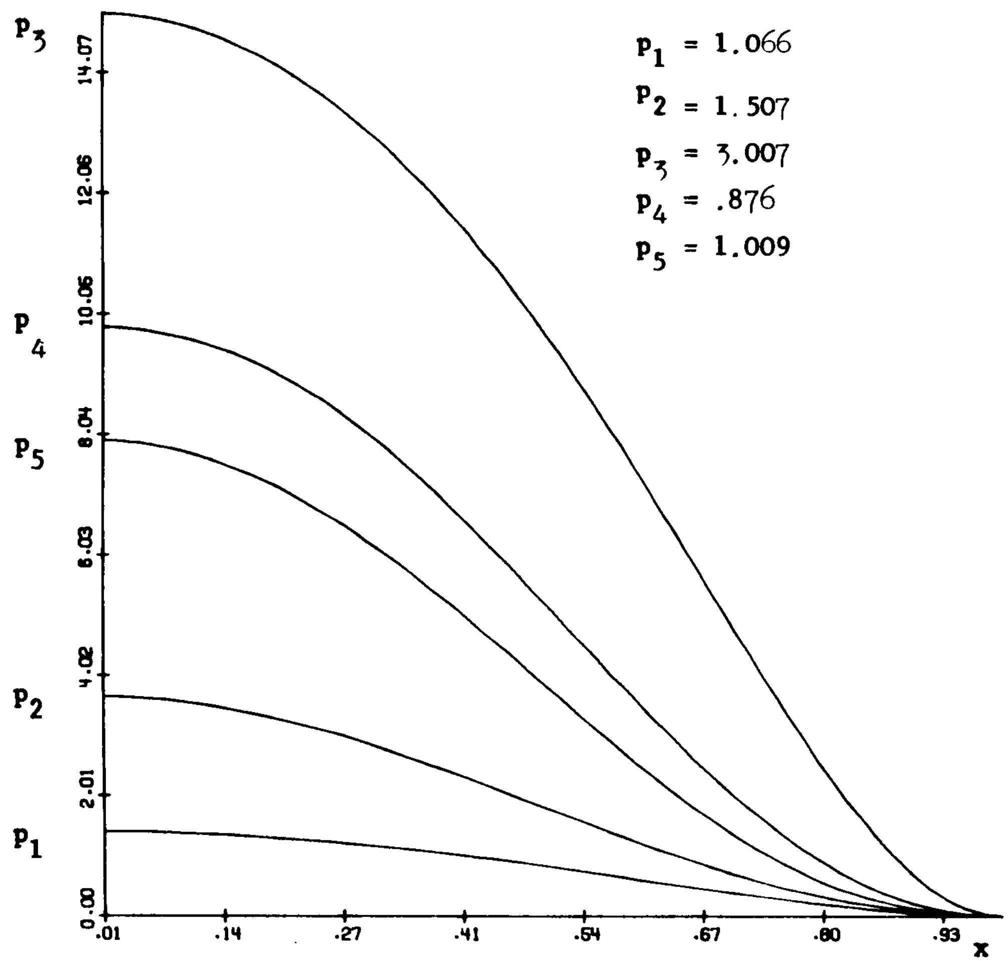

p3
p4
p5
p2
p1
14.07
12.06
10.06
8.04
6.03
4.02
2.01
0.00
.01
.14
.27
.41
.54
.67
.80
.93
x
p1 = 1.066
p2 = 1.507
p3 = 3.007
p4 = .876
p5 = 1.009

Fig. 6

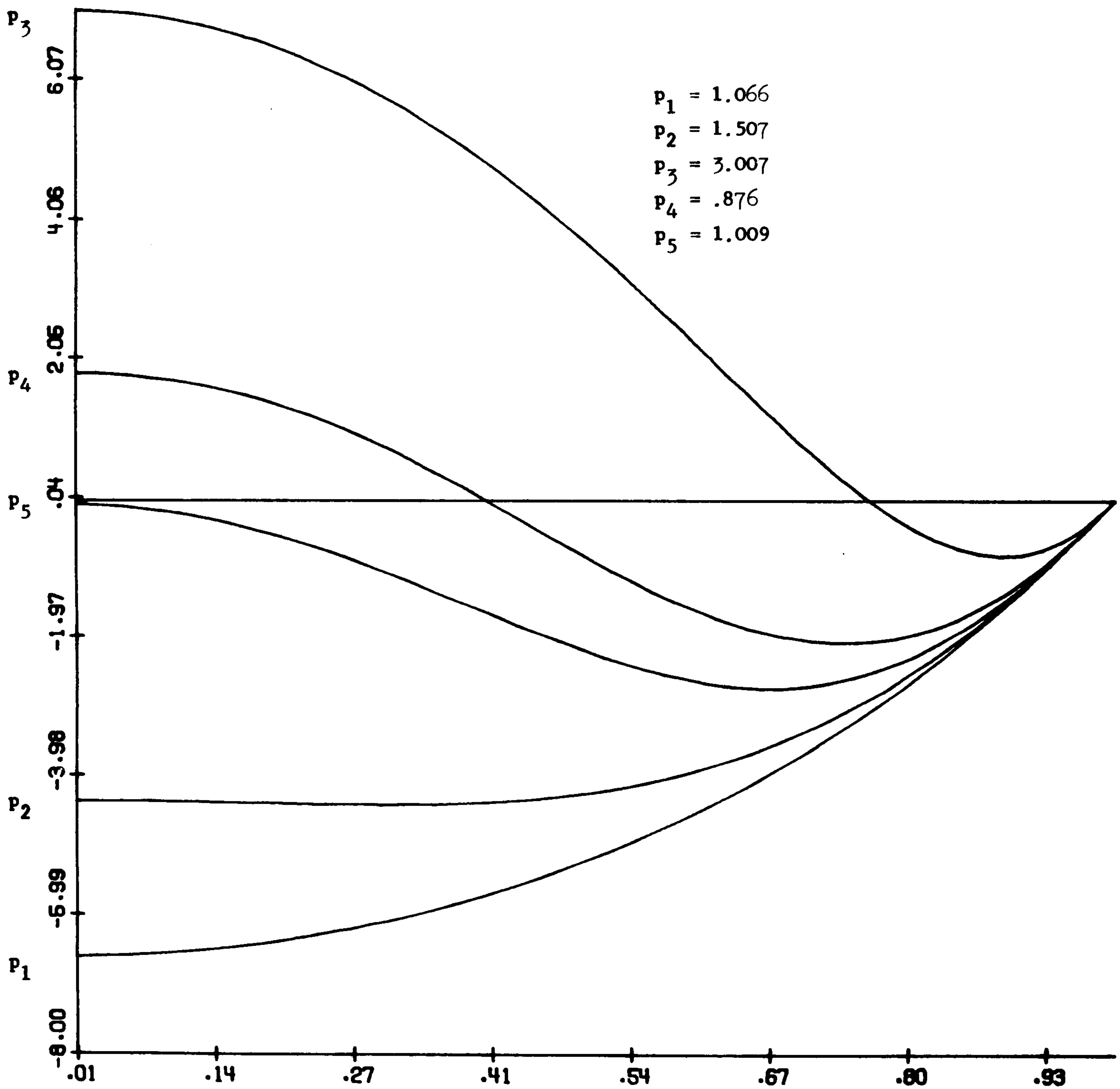

p1 = 1.066
p2 = 1.507
p3 = 3.007
p4 = .876
p5 = 1.009
p3
p4
p5
p2
p1
6.07
4.06
2.05
.04
-1.97
-3.98
-5.99
-8.00
.01
.14
.27
.41
.54
.67
.80
.93

Fig. 7

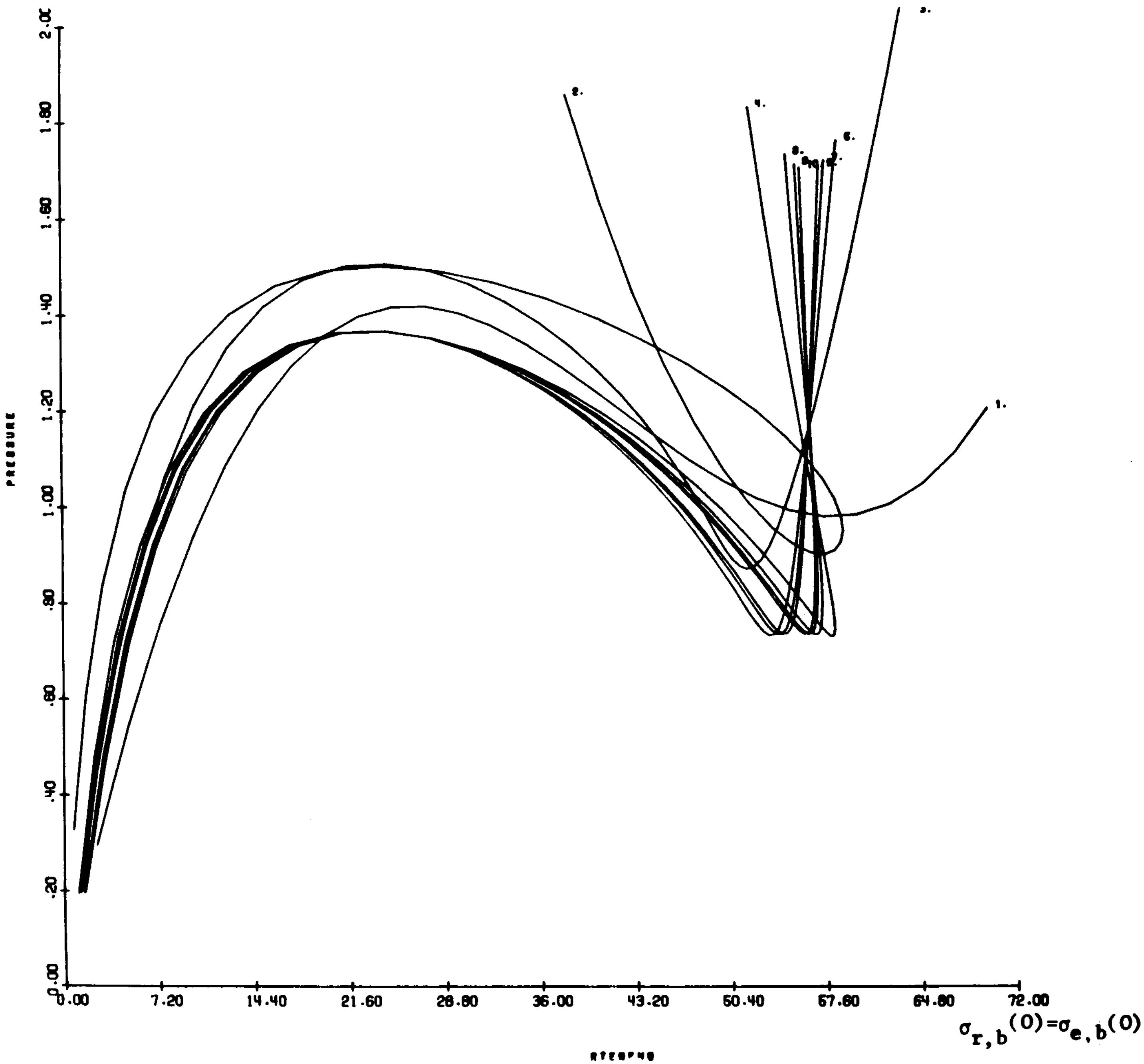
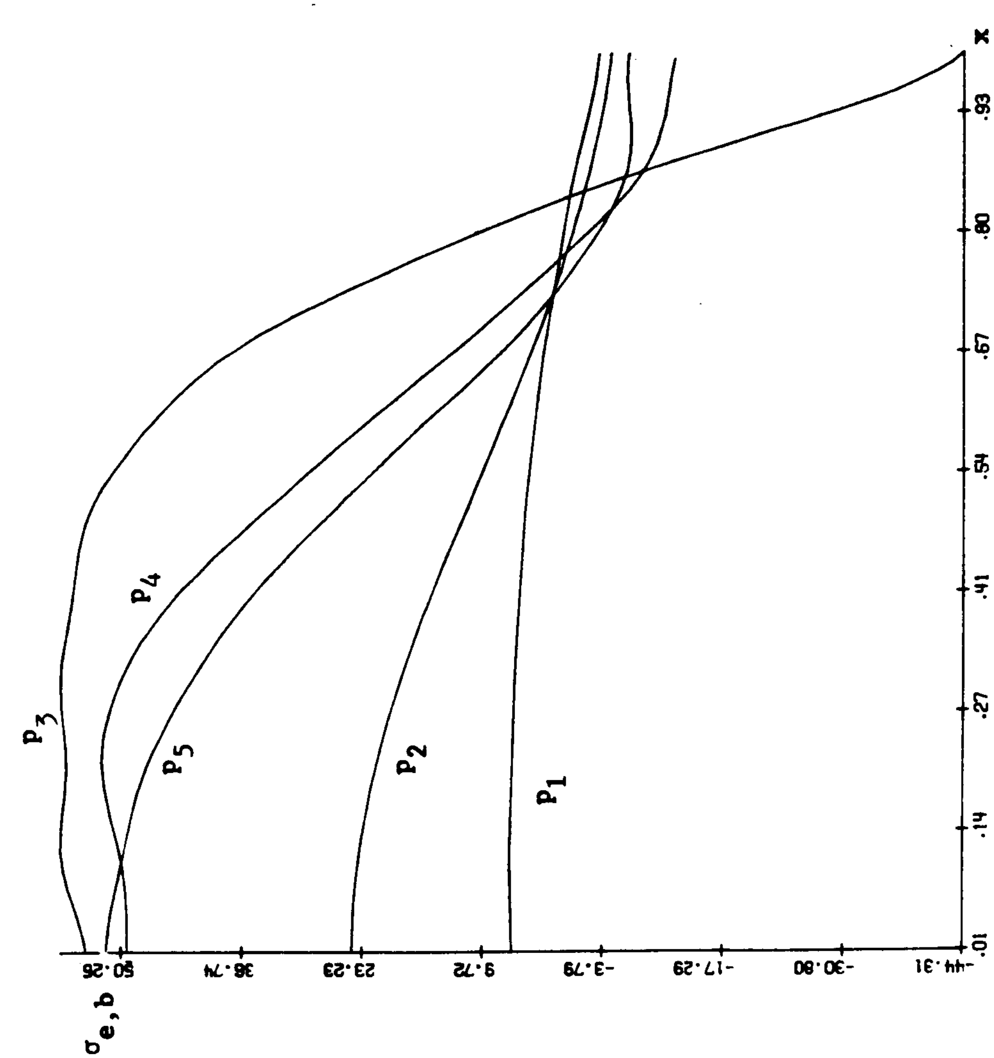

PRESSURE
0.00
.20
.40
.60
.80
1.00
1.20
1.40
1.60
1.80
2.00
0.00
7.20
14.40
21.60
28.80
36.00
43.20
50.40
57.60
64.80
72.00
$\sigma_{r,b}(0)=\sigma_{e,b}(0)$
1.
2.
3.
4.
6.
8.

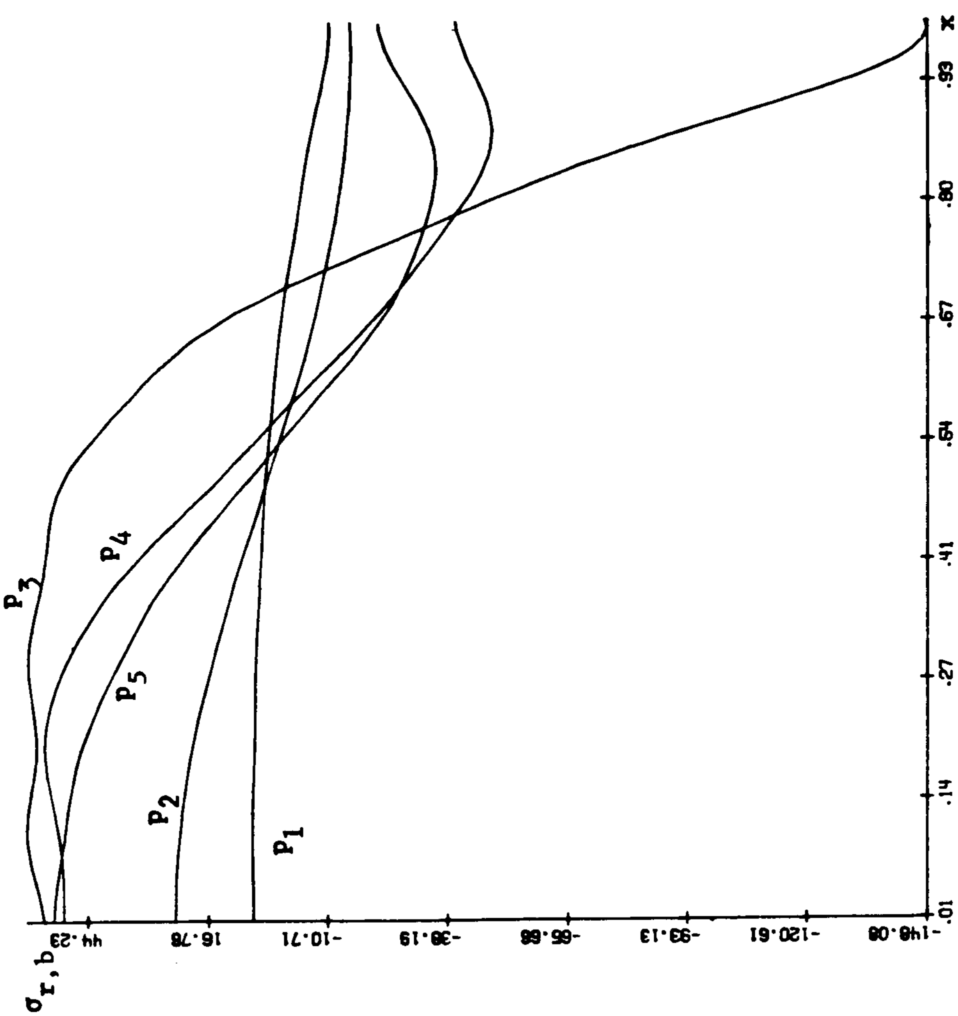
Fig. 8

Fig. 9



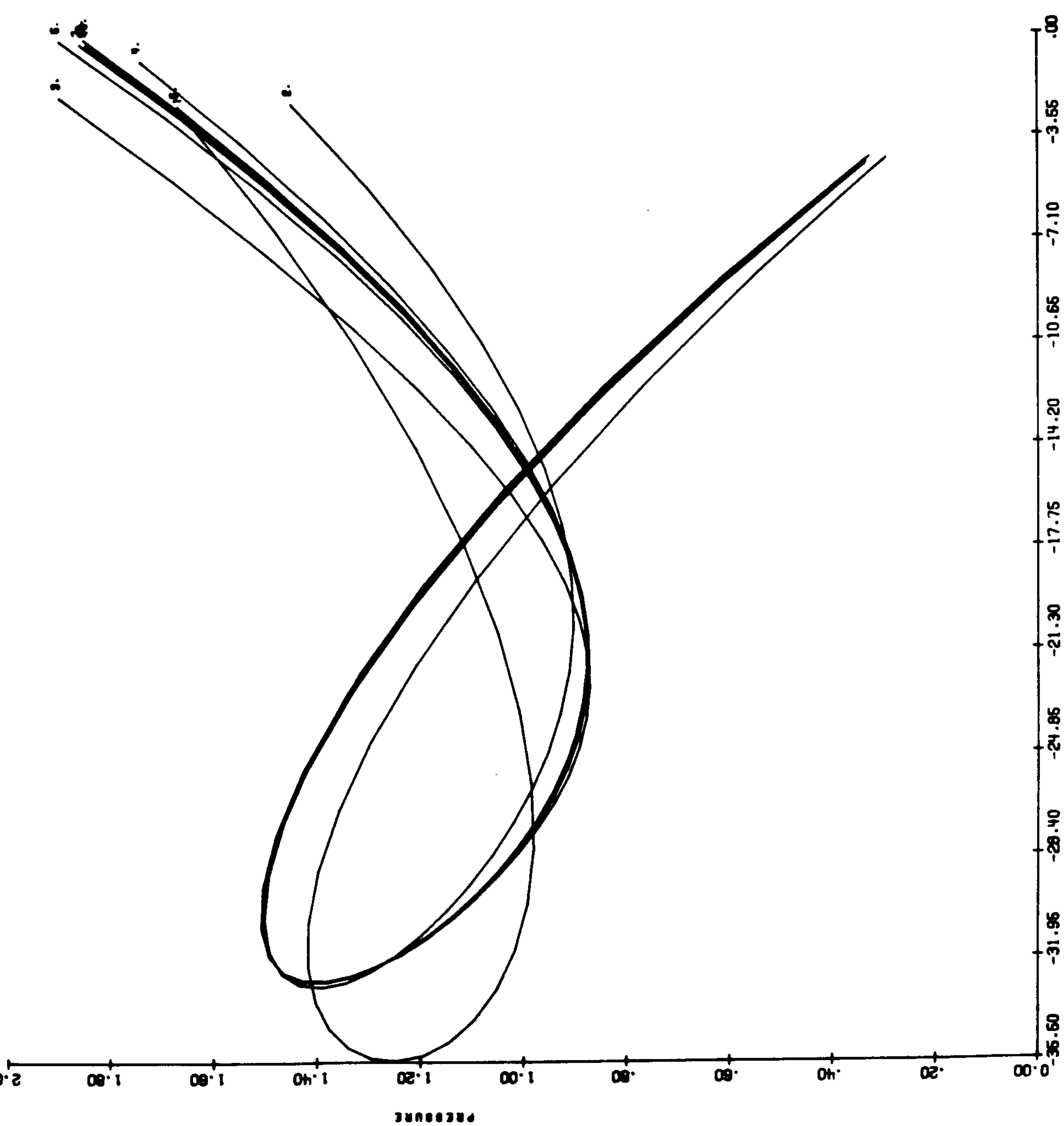
$\sigma_{e,b}$
50.26
36.74
23.23
9.72
-3.79
-17.29
-30.80
-44.31
.01
.14
.27
.41
.54
.67
.80
.93
x
P_1
P_2
P_3
P_4
P_5

Fig. 10



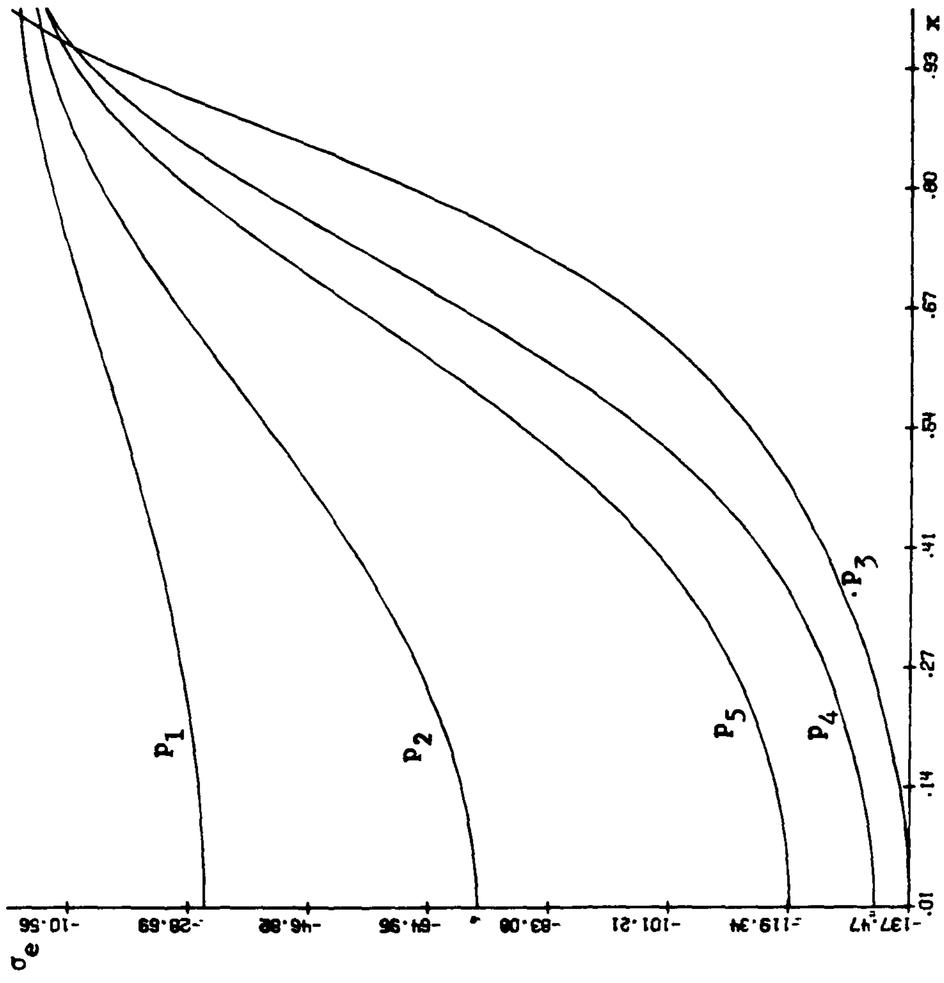
σr,b
44.23
16.78
-10.71
-38.19
-65.66
-93.13
-120.61
-148.08
.01
.14
.27
.41
.54
.67
.80
.93
x
P1
P2
P3
P4
P5

Fig. 11



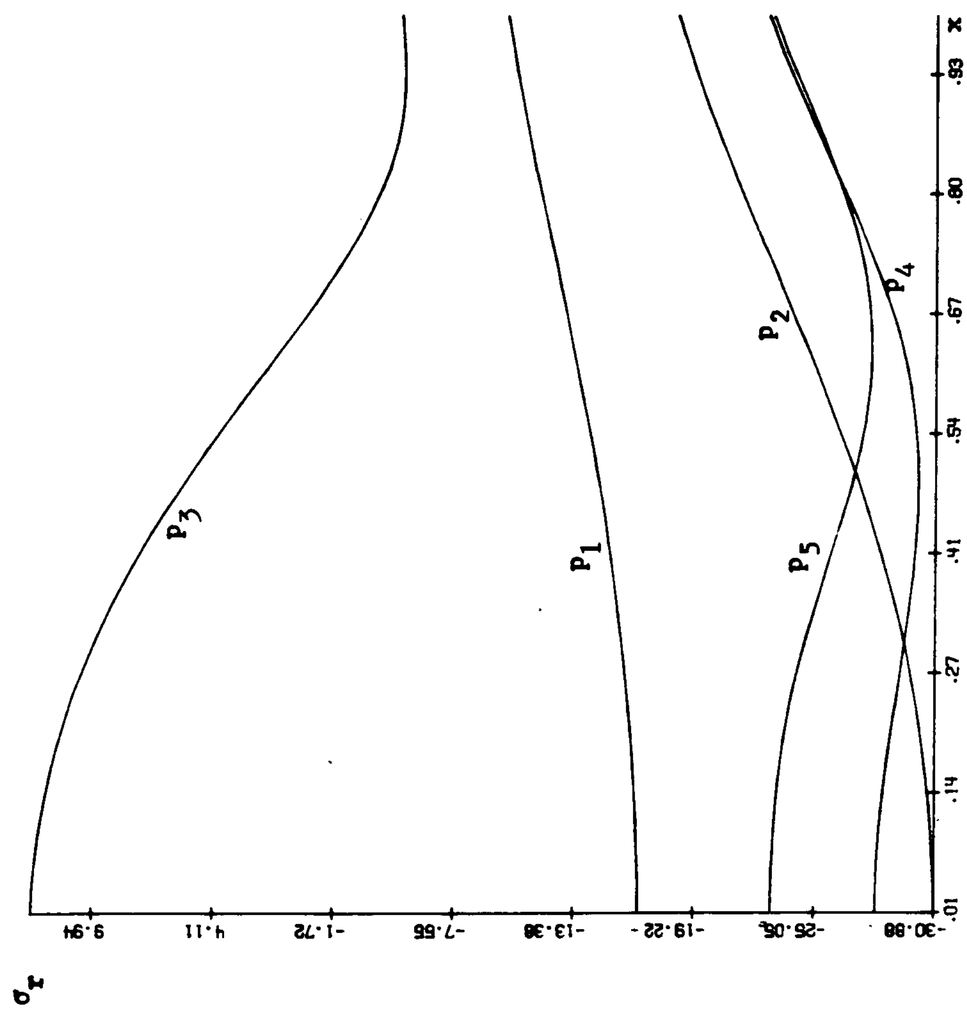
$\sigma_r(0) = \sigma_e(0)$

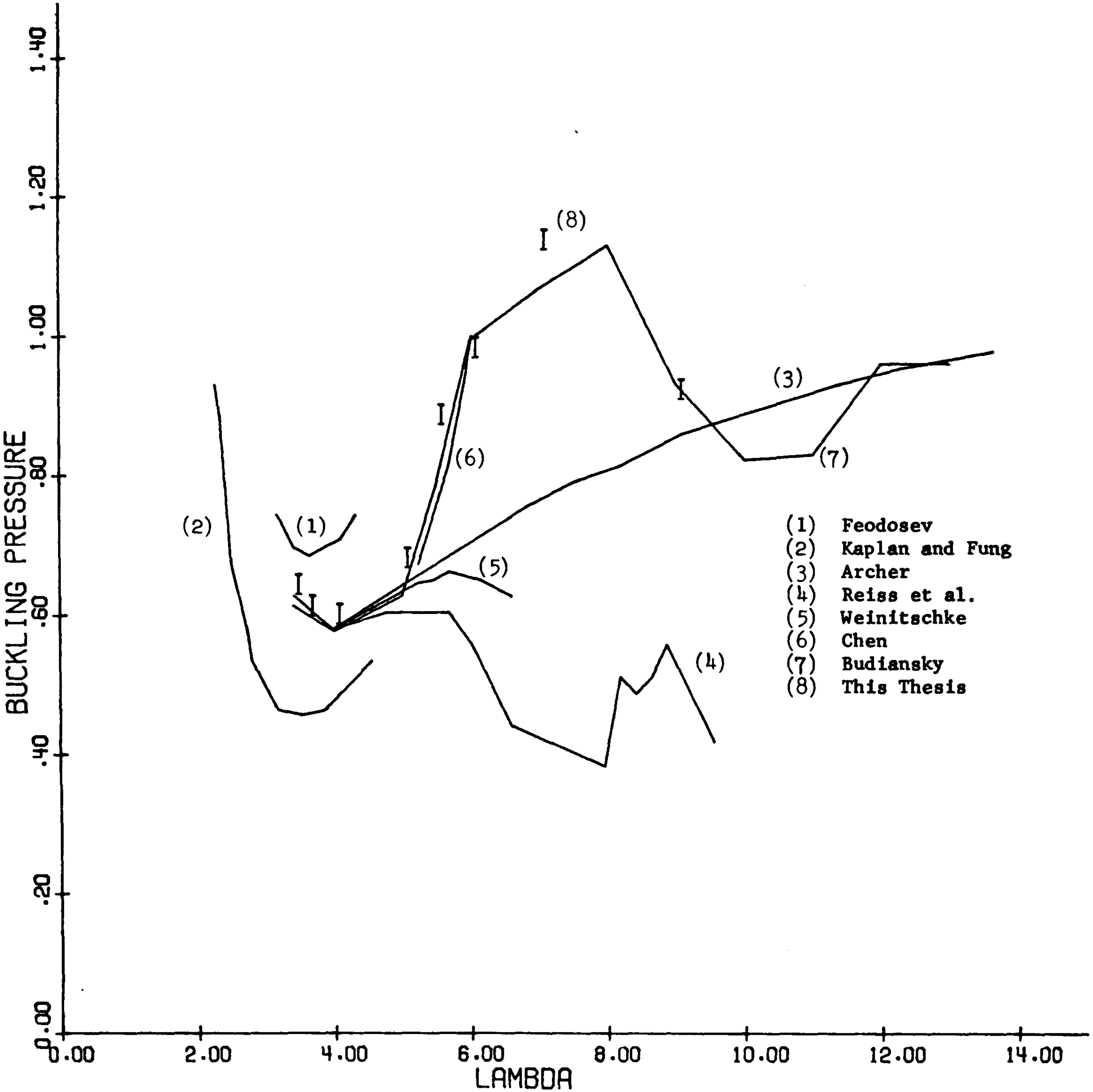
Fig. 12



σe
-10.56
-28.69
-46.82
-64.96
-83.08
-101.21
-119.34
-137.47
x
.01
.14
.27
.41
.54
.67
.80
.93
P1
P2
P3
P4
P5

Fig. 13


σ_r
9.94
4.11
-1.72
-7.66
-13.38
-19.22
-25.06
-30.88
.01
.14
.27
.41
.54
.67
.80
.93
x
P_1
P_2
P_3
P_4
P_5



BUCKLING PRESSURE
0.00
.20
.40
.60
.80
1.00
1.20
1.40
0.00
2.00
4.00
6.00
8.00
10.00
12.00
14.00
LAMBDA
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(1) Feodosev
(2) Kaplan and Fung
(3) Archer
(4) Reiss et al.
(5) Weinitschke
(6) Chen
(7) Budiansky
(8) This Thesis

Fig. 14