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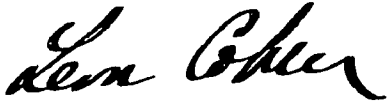
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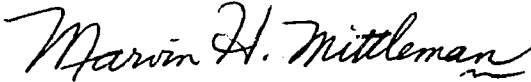
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1972

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Abstract

RANDOM FORCE IN GRAVITATIONAL SYSTEMS

by

Afaq Ahmad

Adviser: Professor Leon Cohen

A new method for numerically integrating the equations of motion of N particles acting under their mutual gravitational attraction is developed and used to study various aspects of the statistics of the gravitational field.

The integration scheme takes advantage of the fact that the force on a particle can be divided into two parts: the irregular force, which is due to the particles in the vicinity of the star in question, and the mean field force, due to the rest of the system. The irregular component changes much more quickly than the mean field component; the latter does not have to be revised so often. Since for systems with large numbers of particles most of the stars constitute distant stars, and there is a considerable saving in computer time. This has allowed us to integrate systems with up to 1000 stars. The scheme is presented in detail and results relating to the efficiency of the scheme with respect to the controls of the program are given.

The feasibility of being able to integrate a large number of particles has allowed us to perform numerical experiments on the theory of Chandrasekhar and von Neumann which is the most fully developed theory on the statistics of the gravitational field. In particular, we have investigated four fundamental aspects of the theory. The probability distribution of the force, the distribution of the time rate of change of the force, the existence of dynamical friction, and the autocorrelation function for force.

Some of the results of the theory have been rederived for the finite- N case, as contrasted to the infinite-particle constant density limit of Chandrasekhar and von Neumann.

The autocorrelation function for force has been derived in a new way, taking into account the constraints for small and large distances. When this is done, the autocorrelation function drops faster than the inverse power of the time. The experimental results seem to indicate that it is indeed the case that the autocorrelation function has a stronger time dependence than one over the time.

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PART ONE

I. INTRODUCTION

Mathematically, the N-body gravitational problem involves the solution of N second order coupled differential equations, one corresponding to every particle or star in the system,

$$\frac{d^2 \vec{r}_i}{dt^2} = - \sum_{j=1}^{N'} m_j \frac{\vec{r}_{ij}}{r_{ij}^3} \quad i=1, N \quad (1)$$
$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

where $\vec{r}_i$ and m_i are the position and mass of the i-th star respectively, and G is the constant of universal gravitation. This system is non-linear and strongly coupled. There are two facts that lead to difficulty in numerically integrating the system given by (1). The first difficulty is related to the fact that each of the equations in (1) have N singularities corresponding to an actual collision between two stars. Although actual collisions are not of practical importance, near collisions or encounters and the formation of binaries are quite common. In these situations the relative force between two stars varies very rapidly. This forces most conventional numerical integration schemes to take very small integration steps, leading to a slowdown of the computation. The second difficulty with the numerical evolution of the N-body problem has to do with the fact that a complete recalculation of the forces in the whole

system requires N^2 calculations. Since such calculations are periodically necessary for all N-body integration schemes, we are severely limited by this in the number of stars that can be evolved for dynamically significant times. These problems limit the usefulness of standard integration schemes, such as Runge-Kutta method for the numerical solution of the N-body gravitational problem and has forced investigators in the field to look for new approaches. To fully appreciate the nature of the problem involved and the techniques devised to overcome these we will briefly describe the recent history of the work done in this field, emphasizing only those points relevant to our scheme.

With the advent of fast electronic computers in the early sixties it became possible to numerically integrate systems containing relatively large numbers of particles. The first such effort was made by von Horner (1960) who evolved clusters of up to 16 stars. In his scheme positions and velocities of all stars are updated simultaneously by a timestep which is the same for all stars. This timestep is selected to be small enough to be appropriate for the pair of stars undergoing the strongest interaction at the time. After each timestep, all the forces are calculated analytically using (1), and the cycle is repeated by finding a new timestep. This method works reasonably well for systems with few stars in the absence of any strong interactions. When there is a pair of particles in tight binary formation, this method is very inefficient since it

requires the recalculation of force for all stars with a frequency that is only really necessary for the components of the binary.

In order to overcome the above difficulty, Aarseth (1963) introduced individual timesteps for each star. This allowed separate treatment of each star for purposes of updating and calculation of the force. The only coupling between the stars occurred at the time of the force calculation. Thus, if there are two stars undergoing strong interaction requiring frequent recalculation of the forces of these stars, one does not compute the forces on all the stars in the system as is the case for von Horner's scheme. The introduction of individual timesteps means that the stars are no longer synchronized during the integration. To find the stars that must be updated before all others, one must search for the star that has been left farthest behind. To do this, one looks for the future times when the force on each star is to be recalculated and the star with the smallest value is chosen; that is, the star for which the force recalculation is most urgently needed. This procedure insures that the force on any star is recalculated slightly before its validity expires. Also, when the force on a star has to be recalculated, all other stars in the system must have their positions updated (to a low order) to synchronize with that star. The additional computing time necessary for the synchronization and the search for the star left farthest behind is smaller than the time wasted in the recalculation of the forces of the whole system at each step. This allowed Aarseth (1968)

to increase the number of stars that could be evolved to 250.

As was pointed out earlier, the singularities in (1) lead to difficulties when two stars come very close together. Thus, for example, it is common to have a binary which is so tight that the amount of computer time necessary to update it is a very large fraction of the total computer time. In some cases, such binaries can slow the progress of the whole system to a point where it becomes prohibitive. Another difficulty with such close binaries or in a particularly close hyperbolic encounter is that they force a timestep which is so small that the roundoff error may become significant. The complete elimination of the difficulties caused by this singularity in the equation of motion has not been solved, although some progress has been made by transforming the equation of motion in such a manner as to eliminate a singularity. Such treatment involves a change of variables and is called regularization. The regularization can be done successfully for a binary. Recently, Szebehely and Peters (1967) have incorporated binary regularization into an integration scheme for the N-body problem. The basic idea is to regularize the closest interacting pair while treating the rest of the system in the conventional manner. Aarseth (1972) has since incorporated regularization into his scheme and has used it to integrate up to 500 stars. Although the method of regularization is mathematically very elegant and has recently been successfully applied, there are two things that must be pointed out. First, even in its most advanced form

(due to Aarseth) it is still restricted to two-particle regularization and is of limited value for frequent cases where more than two stars are undergoing close interaction. Secondly, since regularization involves a complex change of variable, rather elaborate programming is necessary to properly combine the rectangular and the regularized coordinate systems.

Another approach to overcoming the problem of tight binaries and close encounters is due to Pines (1961). This involves using analytical orbital equations for pairs modified to take into account the presence of an external field. Aarseth (1970) has used this scheme to integrate 500 stars although he finds regularization a more efficient method for dealing with this problem. This is expected because the scheme of Pines is based on the slow time variation of the parameters of the binary due to the perturbing influence of a time-dependent external field. Since the external field that the binary may experience in a real N-body problem can be quite large, especially if it is caused by a nearby third body, the Pines scheme has limited value for the integration.

The other basic difficulty with N-body integration schemes, namely the fact that the time needed to calculate the forces for the whole system is proportional to the square of the number of particles in the system, severely limits the number of particles that can be handled. At first sight, this difficulty appears to be basic to the problem and to the inter-particle nature of the force of gravity, but we shall describe a means of avoiding the recalculation of the force at fre-

quent intervals. Before we discuss the ways in which this can be done, we must digress briefly to examine the nature of forces acting on a star in a stellar system.

Chandrasekhar (1960) has pointed out that the force acting on a star in a stellar system can be divided into two parts. The first part is the force due to the random encounters with the other stars in the vicinity of the star in question. This force fluctuates rapidly and is usually called the "random" or "irregular" force. The other part of the force is due to the rest of the stars in the stellar system. This force varies much more slowly than the random force and is called the "regular" or "mean field" force.

This suggests one way to get around the problem of N^2 calculations for force if a method can be found to compute these two forces separately without directly using (1). This could be done by completely abandoning the exact N-body approach and using Monte-Carlo techniques to find the random force and Poisson's equation or the fluid dynamic equation to find the mean field. Such methods have recently become popular in the N-body problem (Henon 1972, Spitzer and Hart 1971, Larson 1970). Another approach is to still use the exact N-body dynamics but to modify the method of integration to take advantage of the difference in time scales of the regular and the irregular forces.

We have developed a new method of integrating the N-body problem which exploits the difference in the rates of change regular and irregular forces in a direct manner. We start

out with an "arbitrary" but practical definition for irregular force which is useful for numerical integration. Our functional definition of the irregular force shall be the force on a particle due to the particles which are within a sphere of radius R circumscribed about the particle. Although a precise value of R is wanting we do expect it to be of the order of the nearest neighbor distance or interparticle distance. We shall, in the next chapter, discuss the effects of choosing different values of R . The regular force is, of course, the force due to the rest of the system.

As in previous schemes, we will have timesteps for each particle. But we will have two timesteps, one each for the irregular and the regular force. The timesteps determine when the force on the particle has to be recalculated. Now let us contrast our two timesteps to the single timestep method. The average timestep for the irregular force will, of course, be smaller than the timestep of the single timestep scheme. But the timestep for the regular force will be much greater. Hence, the irregular force will have to be recalculated more often in comparison to the single timestep scheme, while the recalculation of the mean field force will be done with much less frequency than the total force would have to have been done. But for systems with a large number of particles, the stars which contribute to the regular force contain the major fraction of the stars in the system; we expect a considerable saving in the regular force calculation and hence, in computing time. This method has allowed

us to integrate systems containing up to 1000 stars. There are, of course, a number of difficulties to be overcome as to how to handle the fact that the particles which constitute the regular force for a given particle do not remain the same. We shall, in the rest of this part, give a detailed description of the scheme and also numerical results relating to the efficiency of the scheme with respect to the different control parameters. This type of scheme has been suggested in the literature (Aarseth and Hoyle 1964) and has been called the "double individual" timestep method by Wielen (1967).

A scheme with some similarities to ours is the category scheme of Henon and Hayli (1967). In this scheme, the stars in the system are grouped into categories. The stars in the same category have near equal timesteps. In calculating the new force on a particle, only the contributions from stars of equal or lower categories are considered. But we should point out that not all particles of lower categories are indeed responsible for the recalculation of the force. Thus, to take an extreme example, consider two binaries very far away from each other. Suppose both of the binaries (4 particles) fall into the first category. When recalculating the force on one of the particles, the other three particles would be considered; but, of course, it is clear that only the particle forming the binary is necessary. Also, when we recalculate the force in a category which is not the lowest, all particles with lower categories necessarily contribute. The advantage of our scheme is that only those particles

which are indeed responsible for the rapid change in the irregular force are used for recalculation.

II. METHOD OF INTEGRATION

1. Introduction

The basic prescription for updating the positions and velocities of the particles is the familiar Taylor series which relates the position and velocity of a star at time $T + \Delta T$ to its position and velocity at the earlier time T

$$\begin{aligned}\vec{r}_i(T + \Delta T) &= \vec{r}_i + \vec{v}_i \Delta T + \frac{1}{2} \frac{\vec{F}_i}{m_i} \Delta T^2 + \frac{1}{m_i} \sum_{k=1}^l \frac{\vec{F}^{(k)}}{(k+2)!} \Delta T^{k+2} \\ \vec{v}_i(T + \Delta T) &= \vec{v}_i + \frac{\vec{F}_i}{m_i} \Delta T + \frac{1}{m_i} \sum_{k=1}^l \frac{\vec{F}^{(k)}}{(k+1)!} \Delta T^{k+1}\end{aligned}\quad (1)$$

where $\vec{F}$ is the total force on the particle and $F^{(k)}$ is the k -th derivative of the force at time T . This updating is valid if

$$|\vec{F}^{(\ell)}| \gg |\vec{F}^{(\ell+1)}| \Delta T \quad (2)$$

It is obvious that the condition can be satisfied by choosing a sufficiently small ΔT , irrespective of the order of integration l . The saving in time spent on evolving N -body systems occurs if we can use large ΔT and small l . For the same errors of integration, it has been found by Wielen (1967) that the gain in efficiency is considerable as l is increased from 1 to 4. For $l > 4$, the increase in the speed is not sufficient to justify additional programming. The most

straightforward method of computing the derivatives $F^{(k)}$ is to use the analytic expressions obtained by direct differentiation of (1) of Chapter I with respect to time. This is quite straightforward and one gets the following results for different time derivatives of the force per unit mass

$$\begin{aligned}
 \dot{\vec{F}}_i &= - \sum_j m_j p_{ij}^3 (\vec{v}_{ij} - 3 \vec{r}_{ij} q_{ij} p_{ij}^2) \\
 \ddot{\vec{F}}_i &= - \sum_j m_j p_{ij}^3 \{ \ddot{\vec{F}}_j - 6 \vec{v}_{ij} p_{ij}^2 q_{ij} + 3 \vec{r}_{ij} p_{ij}^2 (5 q_{ij}^2 p_{ij}^2 \\
 &\quad - v_{ij}^2 - \vec{r}_{ij} \cdot \ddot{\vec{F}}_j) \} \\
 \ddot{\vec{F}}_i &= - \sum_j m_j p_{ij}^3 \{ \ddot{\vec{F}}_j - 9 \ddot{\vec{F}}_j q_{ij} p_{ij}^2 - \vec{v}_{ij} p_{ij}^2 [9 v_{ij}^2 + \\
 &\quad + 9 \vec{r}_{ij} \cdot \ddot{\vec{F}}_j - 45 q_{ij}^2 p_{ij}^2] - \vec{r}_{ij} p_{ij}^2 [3 \vec{r}_{ij} \cdot \ddot{\vec{F}}_j + 9 \vec{v}_{ij} \cdot \ddot{\vec{F}}_j - \\
 &\quad 45 q_{ij} p_{ij}^2 (\vec{r}_{ij} \cdot \ddot{\vec{F}}_j) - 45 q_{ij} v_{ij}^2 p_{ij}^2 + 105 p_{ij}^4 q_{ij}^3] \} \\
 \vec{v}_{ij} &= \dot{\vec{r}}_{ij} , \quad \ddot{\vec{F}}_j = \ddot{\vec{v}}_j , \quad q_{ij} = \vec{r}_{ij} \cdot \vec{v}_{ij} , \quad p_{ij} = r_{ij}^{-1}
 \end{aligned} \tag{3}$$

where dots represent differentiation with respect to time. This method of finding $\dot{\vec{F}}^{(k)}$ has been used by Gonzalez and Lecar (1968), but is found to be slow for large particle numbers because of the repeated sums involved in these expressions. A more efficient way of finding the force derivatives is to utilize the knowledge of the value of the force at a certain number of previous times to compute these derivatives numerically. In the individual timestep method, the updating (1) is done individually for each star using different ΔT . After each updating, the force $\vec{F}$ is calculated using Newton's equation. This force calculation

requires N different square roots. This makes it especially desirable to have as large a ΔT as possible. We are usually prevented from taking a very large ΔT because of the condition (2). The strong dependence of the higher force derivatives on the inter-particle distance suggests that a large value for $\vec{F}^{(1+1)}$ is primarily caused by the particles in the near vicinity of the particle in question.

2. Sphere of Influence

We surround each particle i with a sphere of radius R_i and divide the total force, $\vec{F}_i$, on the particle into two parts: the irregular force, $\vec{S}_i$, which is the sum of the forces due to the particles within R_i and the regular force $\vec{K}_i$, due to the rest of the system. If a star is approaching the i -th star with a high velocity it will cause a rapid change in the irregular force and it is therefore included as a neighbor even though it lies outside R_i . Also, a particle approaching with a high velocity is very likely to come within the sphere of influence of the star and hence, should be included as a neighbor. The search for such particles is made within a sphere of $2R_i$. A useful criterion can be found by using a straight line approximation for the relative orbit. Thus, if $\vec{v}_0$ and $\vec{r}_0$ are the velocity and the relative distance respectively, we can find the impact parameter d_0 and the time of the nearest approach t_0

$$d_0 = r_0 \sin \theta$$

$$t_0 = -\frac{r_0}{v_0} \cos \theta$$

(4)

where θ is the angle between $\vec{v}_0$ and $\vec{r}_0$

The approaching star is included as a neighbor if

$$t_0 < \tau$$

and

$$d_0 < \delta$$

where δ is an arbitrarily chosen small distance and τ is the time before a new total force calculation is made. If computer storage is at a premium, it is prudent to limit the maximum number of stars to be neighbors. For systems containing 1000 stars or less, a choice of R_i which gives approximately a maximum of 15 neighbors has been found to be sufficient. Initially all R_i s are taken equal but in later stages of evolution, high density in some regions usually forces a decrease in some of the R_i s. In a later section we shall describe the effects on the integration of choosing different maximum values of R_i .

3. Time Steps

The basic advantage of this separation of forces is that exact computation of the regular force needs to be done at rather large intervals of time ΔT_i . At any time in between these calculations, one need only use an extrapolating polynomial to find the instantaneous value of this force as explained later. The regular timestep is found from

$$\Delta T_i = \beta \frac{|\vec{K}_i|}{|\dot{\vec{K}}_i|} \quad (5)$$

This criterion restricts the relative change in the magnitude of the force to a value which is approximately equal to β . This gives quite satisfactory results except when $\vec{K}_i$ is very small. One then must use an upper bound for T_i as suggested by Aarseth (1963).

$$\Delta T_i < \frac{\beta r_i}{\langle v \rangle} \quad (6)$$

where $\langle v \rangle$ is the average speed of the particles in the cluster.

The irregular force $\vec{S}_i$ has to be calculated much more frequently than the regular force. This however needs little computing time because it involves the contribution due to only the few neighbors. How frequently one must calculate the irregular force depends on the strength of the interactions. There is usually a dominant neighbor and we must insure that t_i , the interval between two successive irregular force calculations, is short enough so that the force due to this neighbor does not change appreciably. Thus, the irregular timestep, δt_i , is found, following Gonzalez and Lecar (1968).

$$\delta t_i = \alpha \min \{ r_{ij}^{3/2} \} \quad (7)$$

where j includes all neighbors. This, however, does not take into account a fast-approaching neighbor. An upper bound for δt_i is taken to be

$$\delta t_i < \frac{\eta \alpha}{m_i^{1/2}} \min \left(\frac{r_{ij}}{v_{ij}} \right) \quad (8)$$

η is a constant best taken as 3.3 from experience.

The reason for using (7) and (8) for finding the irregular force timestep becomes clear if we analyze the expression for the derivative of the force due to just one neighbor. For definiteness we consider the second derivative. From (3) it is seen that it reduces to

$$\ddot{\vec{F}} = \frac{Gm}{r^5} \left\{ \vec{r} \left(2Gm + 15 \frac{(\vec{r} \cdot \vec{v})}{r^2} - 3v^2 \right) - 6(\vec{r} \cdot \vec{v}) \vec{v} \right\} \quad (9)$$

where $\vec{r}$ is the distance from the neighbor and $\vec{v}$ its velocity with respect to the star.

Defining the time scales t_1 and t_2

$$t_1 = \frac{r^{3/2}}{\sqrt{Gm}}, \quad t_2 = \frac{r}{v} \quad (10)$$

We can now rewrite the force derivative as

$$\ddot{\vec{F}} = \frac{r}{t_1^2} \left\{ \left[\frac{2}{t_1^2} + 3 \frac{(5 \cos^2 \theta - 1)}{t_2^2} \right] \hat{r} - 6 \frac{\cos \theta}{t_2^2} \hat{v} \right\} \quad (11)$$

where $\hat{r}$ and $\hat{v}$ are unit vectors along the relative position and velocity respectively and θ is the angle between them. It is obvious from the expression (11) that, for a particular value of r , t_1 and t_2 are the proper time scales. This analysis can easily be extended to the higher order derivatives. Another criterion for calculation of the timestep,

used by Aarseth (1968), is to find the solution of

$$\Delta T^3 = \eta \frac{|\vec{F}| + |\vec{F}| \Delta T + \frac{1}{2} |\ddot{\vec{F}}| \Delta T^2}{\frac{1}{6} |\ddot{\vec{F}}| + \frac{1}{24} |\dddot{\vec{F}}| \Delta T} \quad (12)$$

for Δt .

4. Method of divided differences

Numerical evaluation of irregular and regular force derivatives is needed for our scheme. These derivatives are required for updating the positions and velocities using (1), since the total force derivatives are simply given by

$$\vec{F}^{(k)} = \vec{S}^{(k)} + \vec{K}^{(k)} \quad k = 1, \dots, 4 \quad (13)$$

As pointed out earlier there are two possible methods by which one can find the derivatives of the forces at a certain time. We could use the analytic formulas (3) which require only the knowledge of the positions and velocities of the particle at that time. Alternatively, a difference scheme can be used to find these derivatives, which requires knowledge of the force at a few previous times. For this scheme we have used a difference method since it is found to be more efficient for large numbers of stars. As the timesteps are taken to be variable the most suitable scheme is found to be the divided difference method which has the additional advantage of semi-iteration, which is not possible with

Lagrangian interpolation methods.

Consider a fourth order divided difference polynomial relating force (regular or irregular) at time T to its value at earlier times

$$\begin{aligned}\vec{F}(T) = & \vec{F}(T_0) + \vec{D}[T_0, T_1](T-T_0) + \vec{D}_2[T_0, T_2](T-T_0)(T-T_1) \\ & + \vec{D}_3[T_0, T_3](T-T_0)(T-T_1)(T-T_2) \\ & + \vec{D}_4[T, T_4](T-T_0)(T-T_1)(T-T_2)(T-T_3)\end{aligned}\quad (14)$$

where T_0, T_1, T_2 , and T_3 are the times, in decreasing order, of the last four exact force calculations. It can easily be shown that (14) is satisfied if the divided differences are defined as follows

$$\begin{aligned}\vec{D}[T_0, T_1] &= \frac{\vec{F}(T_0) - \vec{F}(T_1)}{T_0 - T_1} \\ \vec{D}_2[T_0, T_2] &= \frac{\vec{D}[T_0, T_1] - \vec{D}[T_1, T_2]}{T_0 - T_2} \\ \vec{D}_3[T_0, T_3] &= \frac{\vec{D}_2[T_0, T_2] - \vec{D}_2[T_1, T_2]}{T_0 - T_3} \\ \vec{D}_4[T, T_4] &= \frac{\vec{D}_3[T, T_2] - \vec{D}_3[T_0, T_3]}{T - T_3}\end{aligned}\quad (15)$$

We can now compare (14) with a Taylor expansion of the force

$$\vec{F}(T) = \vec{F}(T_0) + \sum_{k=1}^4 \frac{\vec{F}^{(k)}}{k!} (T-T_0)^k \quad (16)$$

to obtain the derivatives of forces in terms of the divided

differences

$$\begin{aligned}
 \vec{F}^{(1)} &= \vec{D} + T_1' \vec{D}_2 + T_1' T_2' \vec{D}_3 + T_1' T_2' T_3' \vec{D}_4 \\
 \vec{F}^{(2)} &= 2! \{ \vec{D}_2 + (T_1' + T_2') \vec{D}_3 + (T_1' T_2' + T_2' T_3' + T_1' T_3') \vec{D}_4 \} \\
 \vec{F}^{(3)} &= 3! \{ \vec{D}_3 + (T_1' + T_2' + T_3') \vec{D}_4 \} \\
 \vec{F}^{(4)} &= 4! \vec{D}_4
 \end{aligned} \tag{17}$$

where $T_k' = T_k - T_0$ for $k=1,2,3$

The advantage to using (14) and (15) is the simple manner in which the divided differences at two successive times are related. This results from the definition of the divided difference. From (15) it is clear that the successive divided differences require two divided differences of one lower order. One of them is calculated immediately before, as indicated by (15), while the other is the divided difference at the end of the previous force calculation.

5. Semi-Iteration

It should be noted that to obtain the last term in (14) a semi-iteration procedure is used. This is necessary because the calculation of $\vec{D}_4$ requires a knowledge of the force at time T , which, of course, is unknown at time T_0 , the time of other $\vec{D}_k$ calculations. The force at T can only be found after the position of the star is known. To circumvent this difficulty, we use a semi-iteration procedure. To describe the semi-iteration procedure, it is easier to consider (14)

first, as it applies to the irregular force polynomial. Similar consideration applies to the regular force polynomial. The position and velocity of the star is updated using (1) up to the $\vec{F}^{(3)}$ term using the values of the derivatives which only have contributions up to $\vec{D}_3$. This new position is now used to find the force for that star at time T. Now one can find $\vec{D}_4$ and use that to get a better value for the position and the velocity by adding the $\vec{F}^{(4)}$ term and also adding the $\vec{D}_4$ correction for the lower force derivatives.

6. Computer Storage

At the start of the integration or at any other time during it, a certain number of quantities have to be stored for every star in the system. For completeness they are listed here for the i-th star

t_i, t_i', t_i'', t_i'''	times of the last four irregular force calculations, in decreasing order
T_i, T_i', T_i'', T_i'''	times of the last four regular force calculations, in decreasing order
$\vec{r}_i, \vec{v}_i$	position and velocity, respectively, at time t_i
$\vec{F}_i, \vec{F}_i^{(1)}$	total force and its first derivative at time t_i
$\vec{K}_i, \vec{D}_{1i}^k, \vec{D}_{2i}^k, \vec{D}_{3i}^k$	regular force and its divided differences computed at time t_i
$\vec{S}_i, \vec{D}_{1i}^s, \vec{D}_{2i}^s, \vec{D}_{3i}^s$	irregular force and its divided differences computed at time t_i
$\delta t_i, \Delta T_i$	irregular and regular timesteps
R_i	radius of sphere defining neighborhood

n_{ij} matrix of neighbor n_{ij} is the j -th
neighbor of the i -th star

n_{bi} total number of neighbors

The divided differences are stored rather than the force derivatives for purely computational considerations. But the total force and its derivatives are stored because they are frequently needed for synchronization. It should be noted that it requires approximately twice as many storage locations per particle as the scheme used by Aarseth (1968) and Wielen (1967) but the increase in the speed of computation, even for a moderate number of stars, outweighs this drawback. We will later discuss how these quantities are obtained at the very start of a computation. It is important to remember that at any time after the time of the start, $t=0$, the stars are not synchronized. This means that t_i , which is the time of the last irregular force calculation and also the time for which the position and the velocity are stored, are all different. Similarly, the T_i 's are also not synchronized. The actual scheme then proceeds according to the flow chart given in Figure (1). Following is a description of the main integration routine.

7. Updating

Since the particles are not synchronized, we choose the particle whose position and velocity are to be revised first by finding the minimum of $t_i + \delta t_i$. That is, the particle whose irregular force needs recalculation before all others.

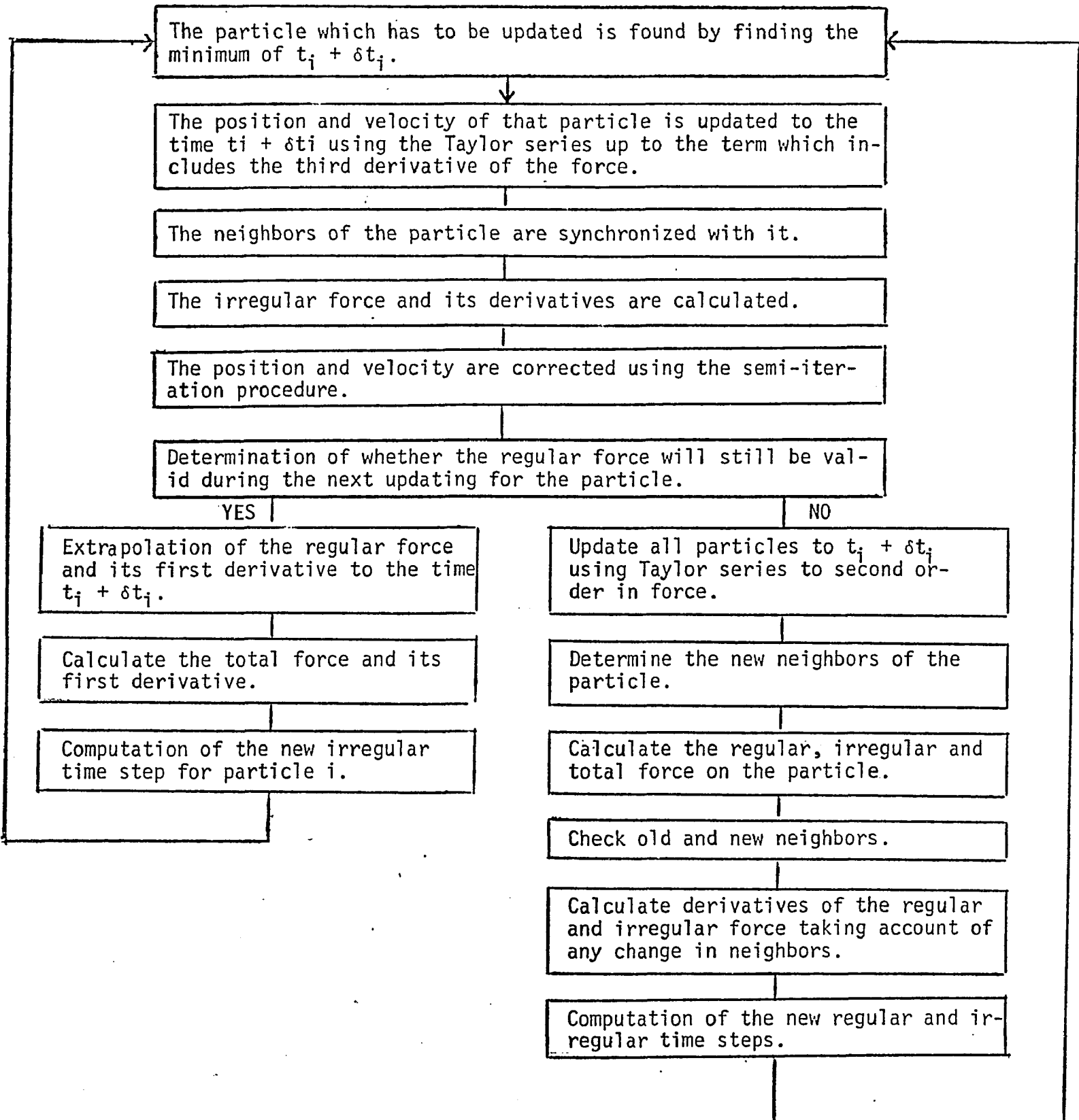


Figure 1

In a previous stage a check has already been made to assure that the regular force is still good at this stage. The position and velocity are updated to the time $t_i + \delta t_i$ using the Taylor expansion (1), but carrying it out to only the third order in force. The fourth order term is taken into account at a later stage by the semi-iteration procedure described above.

As the positions of the neighbors of the i -th star are not synchronized with it, we update these to the time t_i

$$\vec{r}_k = \vec{r}_k(t_k) + \vec{v}_k \Delta T + \frac{1}{2m_k} \vec{F} \Delta T^2 + \frac{1}{6m_k} \vec{F}^{(1)} \Delta T^3$$

$$\Delta T = t_i + \delta t_i - t_k \quad (18)$$

for each neighbor k , and t_k is the time for which the position $\vec{r}_k$ is known. We use a low order extrapolation for $\vec{r}_k$ here because it is found that it is more efficient to control the accuracy by α than to go to the higher order for synchronization. We can now use these $\vec{r}_k$ s to calculate the irregular force and its derivatives. The positions and velocities of i are now corrected by semi-iteration which involves adding the contribution of $\vec{F}_i$ and also the additional contribution from the lower force derivatives due to the $\vec{D}^4$ term as shown by (17). The semi-iteration correction to the position and velocity,

$$\Delta \vec{r}_s \text{ and } \Delta \vec{v}_s, \text{ respectively, are}$$

$$\Delta \vec{r}_s = \frac{\delta t_i^3}{m_i} \left[\frac{\vec{F}_i^{(4)}}{6!} \delta t_i^3 + \frac{\Delta \vec{F}_i^{(3)}}{5!} \delta t_i^2 + \frac{\Delta \vec{F}_i^{(2)}}{4!} \delta t_i + \frac{\Delta \vec{F}_i^{(1)}}{3!} \right]$$

$$\Delta \vec{v}_s = \frac{\delta t_i^2}{m_i} \left[\frac{\vec{F}_i^{(4)}}{5!} \delta t_i^3 + \frac{\Delta \vec{F}_i^{(3)}}{4!} \delta t_i^2 + \frac{\Delta \vec{F}_i^{(2)}}{3!} \delta t_i + \frac{\Delta \vec{F}_i^{(1)}}{2!} \right] \quad (19)$$

where the $\Delta \vec{F}_i$ s are semi-iteration corrections to the lower force derivatives given by

$$\begin{aligned}\Delta \vec{F}_i^{(1)} &= t_1' t_2' t_3' \vec{D}_4 \\ \Delta \vec{F}_i^{(2)} &= 2! (t_1' t_2' + t_2' t_3' + t_1' t_3') \vec{D}_4 \\ \Delta \vec{F}_i^{(3)} &= 3! (t_1' + t_2' + t_3') \vec{D}_4\end{aligned}\tag{20}$$

and

$$\vec{F}_i^{(4)} = 4! \vec{D}_4\tag{21}$$

where t_1' , t_2' , and t_3' are the times of the last three irregular force calculations measured from t_i , and $\vec{D}_4$ is the fourth divided difference.

We now check whether the next time around the regular force would need revision. This is done by checking $t_i + 2\delta t_i$ against $T_i + \Delta T_i$. If the latter is greater, then the regular force polynomial and its derivative are extrapolated to $t_i + \delta t_i$. The new t_i for that particle is determined and the loop is begun again by finding the new particle with the minimum $t_i + \delta t_i$.

If the regular force polynomial has to be revised, then all the particles are synchronized to $t_i + \delta t_i$ using (1) only to the term containing the first derivative of the force. The neighbors of i are determined and any change of the previous neighbors is noted. If there are no new neighbors and all of the old neighbors are still neighbors, one straight-

forwardly calculates the forces and the derivatives. As the local distribution of stars about a certain star changes in time it is probable that there will be changes in the neighbors. For the success of the scheme it is crucial that this change be properly taken into account. It is an easy matter to check the new neighbors against the old and identify those stars which have either moved into the neighborhood of the star or moved out of it. Of course, the regular and the irregular forces have to be corrected for this change, which requires addition or subtraction of the contributions of these stars to the forces. Difficulty arises, however, in the calculation of derivatives which in our scheme are calculated by a difference scheme; and if the irregular or regular forces are not caused by the same stars at two times, one cannot use the difference scheme for finding the derivatives directly. If there has been a change in neighbors, the forces and the derivatives are first calculated assuming there has been no change. Then these are corrected by using analytic formulas to calculate contributions from old neighbors which are no longer neighbors and/or the new neighbors. These are then subtracted and added, respectively. $\text{New } \Delta T_i$ and δt_i are calculated for the particle and the cycle is repeated.

As the above method is not self-starting, a separate initial routine is used to start.

8. Starting Mechanism

The initial conditions can be given to the integration routine to be evolved in time. This routine, however, needs the force and its three derivatives to start and continue the integration. The simplest way to find these is to use the analytic formulas given by (3). The force and its first derivative are found first and they are used to find the second and the third derivative of the force. During these calculations the neighbor matrix n is also found, where n_{ij} is the j -th neighbor of the i -th particle. Similarly, the regular and irregular timesteps ΔT_i and δt_i , respectively, are also found. All this information is stored on a disc or a tape to be used as input for the main integration scheme. This method eliminates the need for a specialized starting procedure.

One possible starting procedure, used by Aarseth (1968), requires the positions and velocities of all stars in the system to be updated by a few small, equal timesteps. The forces at each of these times are calculated analytically, and these are then used to find the derivatives by numerical differentiation. We have found the analytical calculation of all the derivatives in the beginning to be more convenient.

UNITS AND SCALING

We can now proceed with a computer simulation of a real cluster of mass M and radius R given in any well defined units. It is, however, of some advantage, for the purposes of comparison and computational ease, to have a system of units which gives simple values for some of the basic physical quantities. Many such systems are currently used in the literature. We use a system where

$$M = 1, \quad G = 1, \quad \bar{R} = 2 \quad (22)$$

$\bar{R}$ is a "radius" of the cluster given by

$$E = -\frac{1}{4} \frac{GM^2}{\bar{R}} \quad (23)$$

The relationship between the model units and the astronomical units is then as follows

$$\begin{aligned} y_2 &= 1.897 \times 10^{-8} \left(\frac{M}{R^3} \right)^{1/2} && \text{model unit of time} \\ p_c &= 2/\bar{R} && \text{model unit of distance} \\ \odot &= 1/M && \text{model unit of mass} \end{aligned}$$

where $\bar{R}$ is the radius in parsecs and M is in solar mass. In our units the total energy is

$$E = -.125 \quad (24)$$

and assuming the virial theorem for an equilibrium cluster

the potential energy V and the kinetic energy T are

$$V = 2 E = .25$$

$$T = -E = .125 \quad (25)$$

The crossing time for the cluster T_c is given by

$$T_c = \frac{\bar{R}}{\bar{V}} = G M^{5/2} (-2E)^{-3/2} = 8 \quad (26)$$

where $\bar{V}$ is the root mean square velocity for the cluster.

INITIAL CONDITIONS

Computer generation of initial velocities and positions to conform to a given distribution function $f(\vec{r}, \vec{v})$ requires considerable programming and therefore it is best done in a separate program. The following describes such a program written for generating any spherically symmetric distribution which is a function of the energy only.

1. Mass Distribution

The mass density $\rho(r)$ can be found by

$$\rho(\vec{r}) = \int f(\vec{r}, \vec{v}) d\vec{v} \quad (27)$$

For spherically symmetric systems in equilibrium for which the distribution function is a function of energy E only, (27) reduces to

$$\rho(\vec{r}) = 4\pi \int f(\epsilon) v^2 dv \quad (28)$$

where

$$\epsilon = \frac{1}{2} v^2 + \psi(r)$$

where ψ is the gravitational potential at r . Combining (28) with Poisson's equation we get

$$\frac{1}{r^2} \frac{d^2}{dr^2} (r^2 \frac{d\psi}{dr}) = (4\pi)^2 G \int_0^\infty f(\frac{1}{2} v^2 + \psi) v^2 dv \quad (29)$$

To generate a cluster with density $\rho(r)$ we need $m(r)$, the mass interior to the radius. This can be done by numerically solving (29) by a method such as Taylor Series expansion, which gives ψ and its derivatives. The mass interior to r , $m(r)$, can then be found by

$$m(r) = - \frac{r^2}{G} \frac{d\psi}{dr} \quad (30)$$

This $m(r)$ is inverted to yield $r(m)$, or radius containing a certain mass. This can now be used to generate the radius r_i for the i -th star

$$r_i = r_i \left(\frac{i}{N} M \right) \quad i = 1, \dots, N \quad (31)$$

where M is the total cluster mass and N is the total number of stars. Once a radius r_i has been assigned to each star, it is only necessary to give it a random direction in space. This is most easily done by picking three random numbers, a , b , and c between -1 and $+1$ such that

$$\begin{aligned} r_0 &= \sqrt{a^2 + b^2 + c^2} < 1 \\ x_i &= \frac{a r_i}{r_0} \\ y_i &= \frac{b r_i}{r_0} \\ z_i &= \frac{c r_i}{r_0} \end{aligned} \quad (32)$$

To avoid very close pairs which may lead to numerical instabilities in the beginning, it is advisable to check for

such, since they are not dynamically balanced with proper angular momentum and may lead to very close collisions, causing large errors in energy. An additional advantage in removing these close pairs is that the final total potential energy is very close to the theoretical value. Since the radii generated by (31) are in ascending order, it is necessary to check $\vec{r}_i$ against the positions of only a few preceding particles. If $\vec{r}_i$ is very close to one of these, then its direction is again chosen according to (32), and a new check for close pairs is made. This process is repeated until no such close pair is found. This minimum separation distance between two stars is taken to be a small fraction (about a tenth) of the average nearest neighbor distance.

Once the stars have been distributed according to the above scheme, the coordinates of the center of mass R_c are found by

$$\vec{R}_c = \frac{1}{N} \sum_{i=1}^N \vec{r}_i \quad (33)$$

It is advantageous to have a system which has its center of mass at the origin and hence, the system is translated by the distance

$$\vec{r}'_i = \vec{r}_i - \vec{R}_c \quad (34)$$

It is important to have a small $\vec{R}_c$ to prevent the transformation (34) from altering the original density profile. This

can be done by generating many systems by the above method and selecting the one with the smallest value of $|\vec{R}_c|$. As a further insurance against distortion of the density by the center of mass correction we could use the following transformation

$$\vec{r}'_i = \vec{r}_i - p \vec{R}_c \quad (35)$$

where p is a random number between 0 and 2. Unlike (34), this system does not yield a system with the center of mass at the origin. This correction, however, can be applied repeatedly and one can make the center of mass as close to the origin as desired by this fast converging process.

2. Velocity Distribution

Once the stars have been distributed inside the sphere of radius R as described above, we must assign velocities to them according to

$$\rho_{\vec{v}} = f(\vec{r}, \vec{v}) \quad (36)$$

The velocity of the center of mass $\vec{V}_c$ given by

$$\vec{V}_c = \frac{1}{N} \sum_{i=1}^N \vec{v}_i \quad (37)$$

is usually non-zero and must be corrected to zero by a method analogous to the center of mass correction.

Although spherically symmetric distribution functions

have no angular momentum, the cluster generated on a computer has an angular momentum

$$\vec{L} = \sum_{i=1}^N m_i \vec{r}_i \times \vec{v}_i \quad (38)$$

which is finite. If it is desirable to have a cluster without angular momentum, then the following transformation can be applied

$$\vec{v}_i' = \vec{v}_i - \vec{r}_i \times \vec{\omega} \quad (39)$$

where $\vec{\omega}$ is the rotational velocity found by solving

$$\vec{L} = \mathbb{I} \vec{\omega} \quad (40)$$

and $\mathbb{I}$ is the usual moment of inertia tensor. It is clear that (39) leaves the center of mass at rest if it was initially coincident with the origin.

This completes the simulation of the initial conditions for a given distribution function. It must be emphasized that although some restrictions are placed on completely random generation, the restrictions and the corrections are rather slight and do not cause the system to depart considerably from the theoretical model. The potential and kinetic energies and the virial ratio of the simulated cluster are found to be within a few percent of their theoretically predicted values, even for $N=100$.

A particularly convenient set of equilibrium distribution

functions are the Emden polytropes. We shall therefore discuss their generation in some detail.

POLYTROPIC SPHERES

The properties of polytropic gas spheres have been extensively studied. Following Camm (1950) we consider

$$f(\vec{r}, \vec{v}) = A(E_0 - E')^n \quad E' = \frac{1}{2}v^2 + \psi < E_0 \quad (41)$$

where E' and ψ are the binding and potential energies per unit mass, respectively. A and E_0 are arbitrary constants which we can relate to the mass M and the radius R of a given cluster of stars. The mass density $\rho(\vec{r})$ is given by

$$\begin{aligned} \rho(\vec{r}) &= \int f(\vec{r}, \vec{v}) d^3\vec{v} \\ &= 4\pi A \int_0^{v_{\max}} \left(\frac{v_{\max}^2}{2} - \frac{v^2}{2} \right)^n v^2 dv \end{aligned} \quad (42)$$

with

$$v_{\max} = \sqrt{2(E_0 - \psi)}$$

setting

$$y = v/v_{\max}$$

$$\rho = A(E_0 - \psi)^{n+3/2} I_n$$

where

$$I_n = 4\pi 2^{3/2} \int_0^1 (1-y^2)^n y^2 dy \quad (44)$$

The Poisson equation gives

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\psi}{dr} = 4\pi G A (E_0 - \psi)^{n+3/2} I_m \quad (45)$$

Let us define a dimensionless variable

$$\theta_m = \frac{E_0 - \psi}{E_0 - \psi_0} \quad (46)$$

where ψ_0 is the central potential.

Thus

$$\begin{aligned} \theta_m(0) &= 1 \\ \theta'_m(0) &= 0 \end{aligned}$$

Also setting

$$x = \alpha r \quad (47)$$

where

$$\alpha^2 = 4\pi G A (E_0 - \psi_0)^{n+1/2} \quad (48)$$

we get

$$\frac{1}{x^2} \frac{d}{dx} x^2 \frac{d\theta_m}{dx} = -\theta_m^{n+3/2} \quad (49)$$

This is Emden's Equation with index

$$m = n + \frac{3}{2} \quad (50)$$

It is well known that for $m < 5$, the sphere given by (49) has

both finite radius and finite mass. For $m=5$, this represents the Plummer model which has finite mass though infinite radius. For $m>5$, both the mass and the radius are infinite. The index n must also be greater than -1 for finite density. The density is given by

$$\rho(\vec{r}) = \rho_c \Theta_m^{n + \frac{3}{2}} \quad (51)$$

where ρ_c , the central density, is

$$\rho_c = A I_m (E_0 - \psi_0)^{n + \frac{3}{2}} \quad (52)$$

We can identify the first zero x_0 of the Emden function as the radius of the cluster. We will also need

$$y_0 = - \left(x^2 \frac{d\Theta_m}{dx} \right)_{x=0} \quad (53)$$

Obviously the scaling factor is

$$\alpha = \frac{x_0}{R} \quad (54)$$

(This is not strictly true for the Plummer model where x_0 , R are infinite. Here, however, one can define , an arbitrary scaling factor.)

Since

$$\rho(R) = 0 \quad (55)$$

combining (46) and (51) we get

$$E_0 = \psi(R) = - \frac{GM}{R} \quad (56)$$

Also

$$y_0 = -x^2 \left. \frac{d\theta_m}{dx} \right|_{x=x_0} = \frac{\alpha \pi^2}{(E_0 - \psi_0)} \left. \frac{d\psi}{dr} \right|_{r=R} \quad (57)$$

Using Gauss' Law

$$y_0 = \frac{\alpha M G}{(E_0 - \psi_0)} \quad (58)$$

But (54) and (57) give

$$\psi_0 = -\frac{M G}{R} \left(1 + \frac{x_0}{y_0} \right) \quad (59)$$

Now A can easily be related to M and R by combining (59), (56) and (48)

$$A = \frac{1}{4\pi I_m} \left(\frac{R}{x_0} \right)^{n-3/2} \left(\frac{y_0}{M G} \right)^{n+1/2} \quad (60)$$

Thus, for a given n we can find a constant A for a cluster of mass M and radius R, since x_0 and y_0 are tabulated in the literature for many polytropes (Chandrasekhar 1967).

It is well known that a polytropic sphere of index m has the gravitational potential energy V given by

$$V = -\frac{3}{5-m} \frac{G M^2}{R} \quad (61)$$

The virial theorem then gives the total energy

$$E = \frac{V}{2} = -\frac{3}{2(5-m)} \frac{G M^2}{R} \quad (62)$$

1. Velocity Distribution

The velocity distribution $\eta(\vec{v})$ for a given $f(\vec{r}, \vec{v})$ can be found by

$$\eta(\vec{v}) = \int f(\vec{r}, \vec{v}) d\vec{r} \quad (63)$$

For the polytropes, we have

$$\eta(\vec{v}) = 4\pi A \int_0^{R_m} r^2 dr (E_0 - \frac{1}{2}v^2 - \psi)^n \quad (64)$$

where R_m is given by

$$E_0 - \frac{1}{2}v^2 = \psi(R_m) \quad (65)$$

Although this system of equations cannot, in general, be solved analytically, one could get a $\eta(v)$ for any value of v by numerical methods.

2. Energy Distribution

The spectrum of binding energy for different polytropes can be found by a general formula which gives the probability density $P(F)$ for a function F of $\vec{r}$ and $\vec{v}$

$$P(F) = \int f(\vec{r}, \vec{v}) \delta(F - F(\vec{r}, \vec{v})) d\vec{r} d\vec{v} \quad (66)$$

Thus, taking (41) for $f(r,v)$ and binding energy E' for F

$$P(E') = 16A\pi^2 \int_0^{v_m} v^2 \delta(E' - \frac{1}{2}v^2 - \psi) (\frac{v_m^2 - v^2}{2})^n dv$$

$$v_m = \sqrt{2(E_0 - \psi)} \quad (67)$$

integrating over the velocities we get

$$P(E') = 16A\pi^2 \sqrt{2} (E_0 - E')^n \int_0^{R_{max}} \frac{1}{v_m} r^2 dr \quad (68)$$

with the restrictions

$$E' < E_0$$

and

$$\psi < E'$$

The first inequality is an obvious consequence of (41); the second inequality gives the upper limit of the r integration by

$$\psi(R_{max}) = E' \quad (69)$$

As in the case of the velocity distribution, the system of equations (68) and (69) has, in general, no analytic solution and must be numerically solved. This is quite straightforward. One must first invert equation (69) for a particular E' to find R_{max} and then integrate (68) to this upper limit.

3. The Canonical Model

The model we have chosen to test the different aspects of the program described above is a special case of (41) for $n=0$. This is the Emden polytrope of index $3/2$.

$$\begin{aligned} f(E) &= A & E < E_0 \\ &= 0 & E > E_0 \end{aligned}$$

where

$$E_0 = - \frac{GM}{R} \tag{70}$$

In our units it can be shown using (60) and (61)

$$R = 2^{4/7}$$

$$A = .01217 \tag{71}$$

NUMERICAL RESULTS

1. Computing time as a function of the number of particles

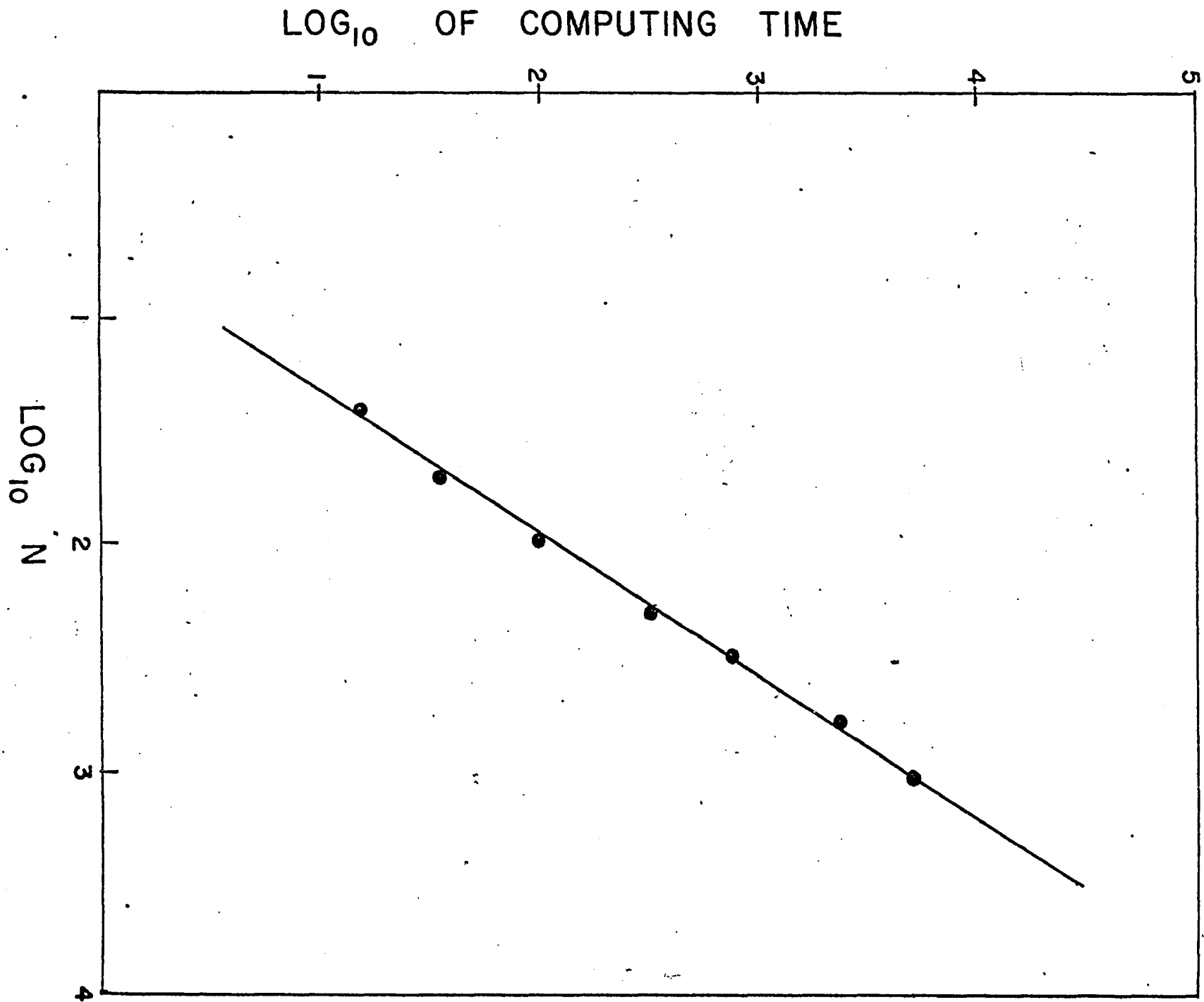
Due to the N^2 nature of gravitational interaction, computing time for schemes based on individual timesteps increases at least as fast as N^2 . Investigators have reported a dependence varying from N^2 to N^3 . But these estimates of computing time are not based on extensive studies, for only a few values of N are used and the number of cases integrated for each N is small. A careful study would require an enormous amount of computing time not commensurate with the interest in knowing the precise dependence. Figure (2) shows the N -dependence of computing time for the method described above. Many cases were integrated for the low values of N , but only a few for $N=600$ and $N=1000$. The logarithm of computing time per crossing time is plotted against the logarithm of the number of particles. The time for the 100-body case is taken to be 100 units of time. For the cases considered, the energy error is constant within a factor of 2.

If we assume a power law for computing time, C_T , versus the number of particles, then, approximately

$$C_T = \gamma \left(\frac{N}{100} \right)^{1.6} \quad (72)$$

where γ is about 40 seconds for the CDC 6600 per crossing time and relative energy error of about 10^{-4} .

Figure 2



2. Computing time as a function of the number of neighbors

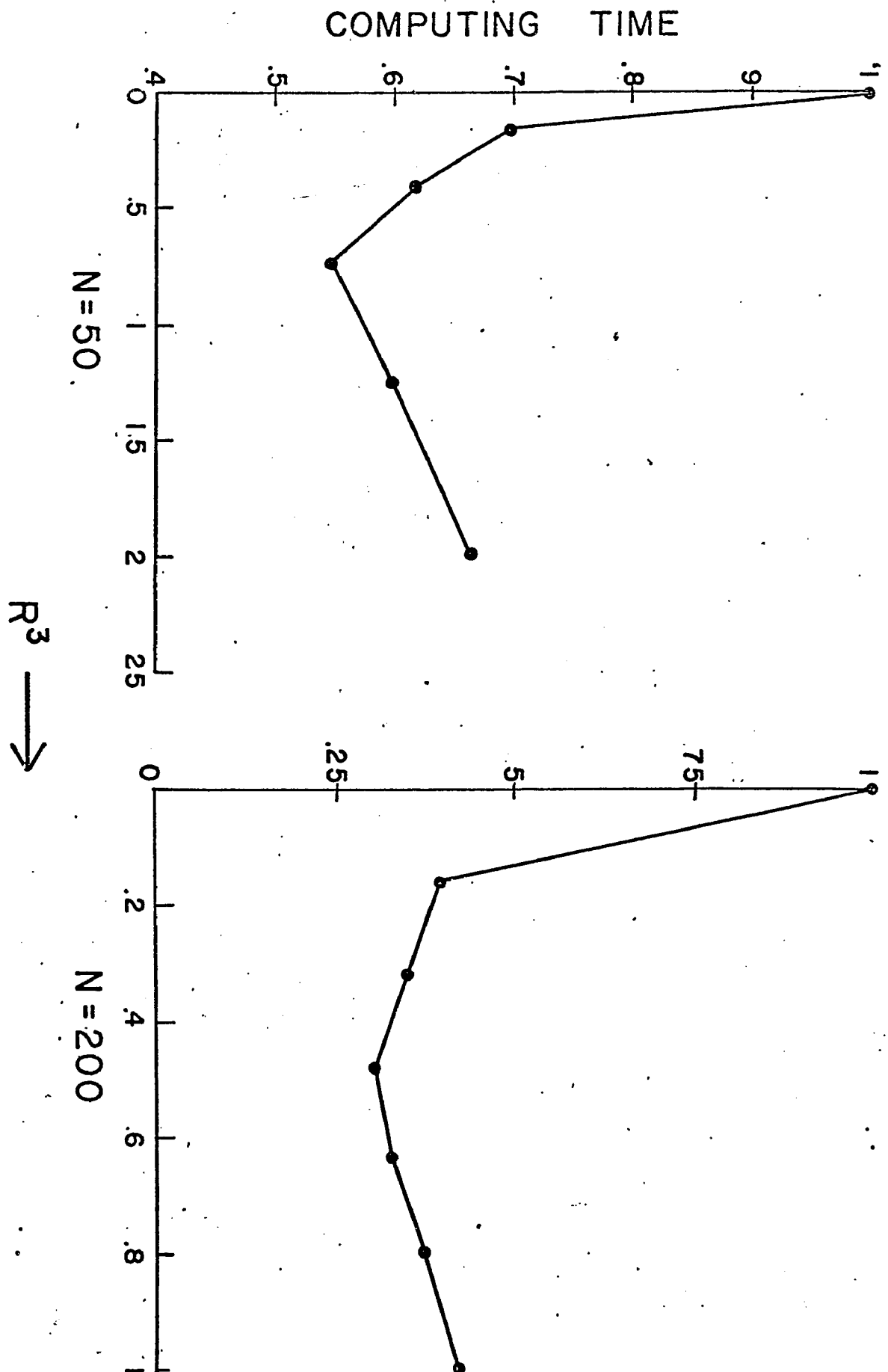
As the operational definition of irregular force is the force due to the particles within a sphere of radius R surrounding each particle, there obviously must be an optimum value of R depending on the average density. Choosing too small a value for R would result in the regular force's changing too rapidly and hence, too frequent a recalculation of the regular force. This is due to particles which are near the particles in question, but nonetheless being included as part of the regular force. Choosing an R too large increases the number of neighbors to the point that the time spent in updating the irregular force overwhelms the time saved by separation of the total force into two parts.

Figure (3) shows a set of integrations for $N=50$ and 200 where all the parameters of the program were kept the same except for the maximum allowed number of neighbors. The number of neighbors is controlled by choosing different values of R^3 . In each case, the figures show, as expected, that there is an optimum value of R which minimizes computing time.

3. Error Analysis

The analysis of the factors which contribute to errors in the numerical integration of the N -body problem is extremely difficult. As Miller has shown, two systems with slightly different initial conditions or two systems with identical initial conditions but integrated slightly differently, will

Figure 3



become increasingly divergent in phase space. The basic reason is the numerical instability of the problem, principally due to the errors caused by close encounters of two bodies (Miller 1964, 1971). The obvious measure of error, microscopic reversibility, is hence much too stringent a test and the impossibility of it in actual practice makes it a useless criterion. It is generally accepted that the microscopic instability is not reflected in the macroscopic properties of the solution. Although this may be more a belief than a proven fact, there are indications that this may be so (Standish 1968).

As an alternative for using microscopic reversibility as a criterion for error, the total energy of the system has been adopted by most investigators for measuring the errors of integration. But the use of energy as a measure is not totally satisfactory either. This is again due to the microscopic instabilities, although the effect is not so great on the total energy as it is on positions and velocities. As an example of the situations which may arise, we cite the following situation. Two cases with the same initial conditions may be integrated on the same computer, one in single precision and the other in double precision; nonetheless, the case done with double precision may show a larger error. This is usually due to the fact that single precision integration may have missed a close encounter and hence its energy remained relatively stable. Due to the above facts, one must therefore not expect exact relationships between the controls of the pro-

Figure 4a

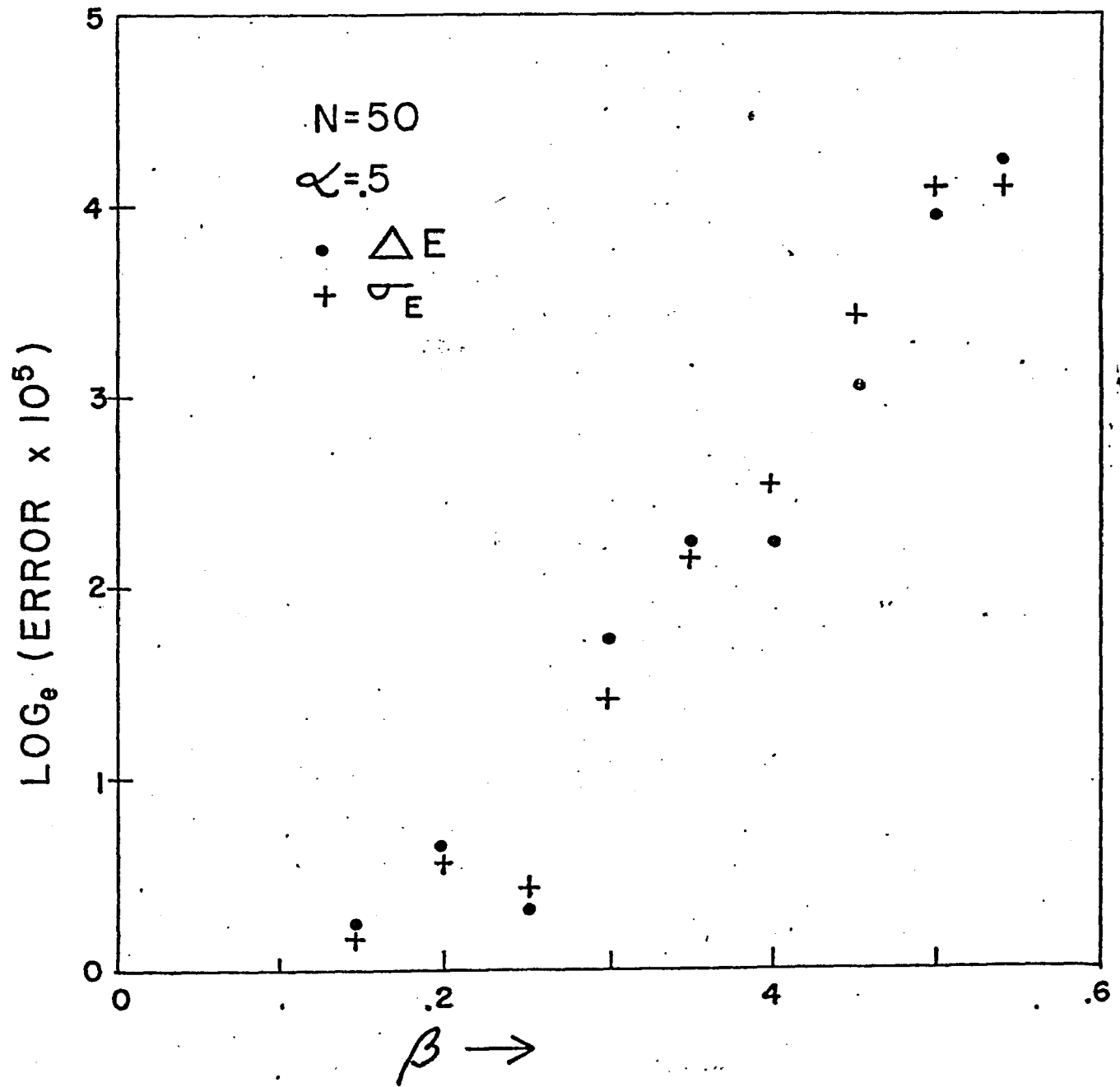


Figure 4b

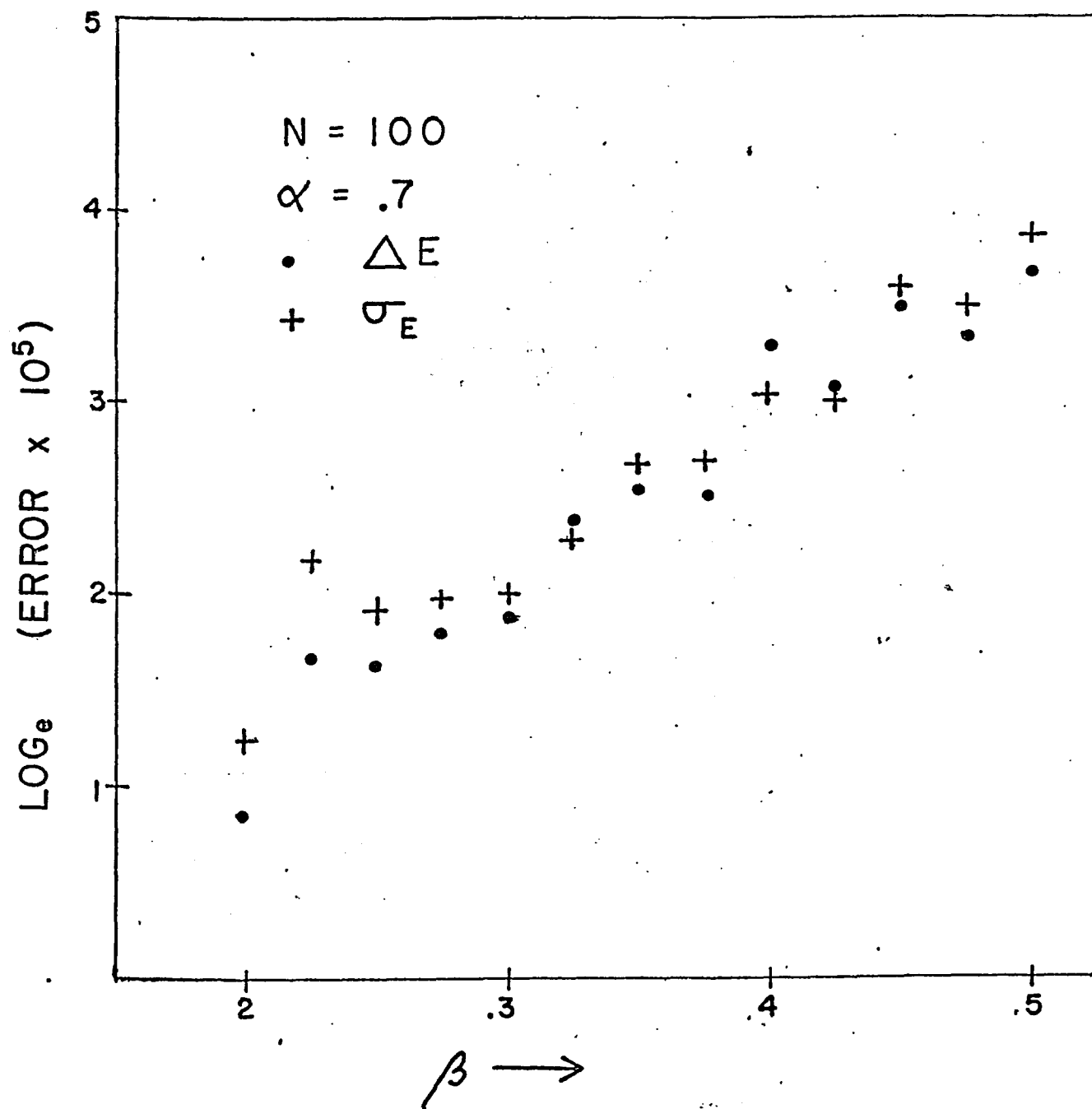


Figure 5a

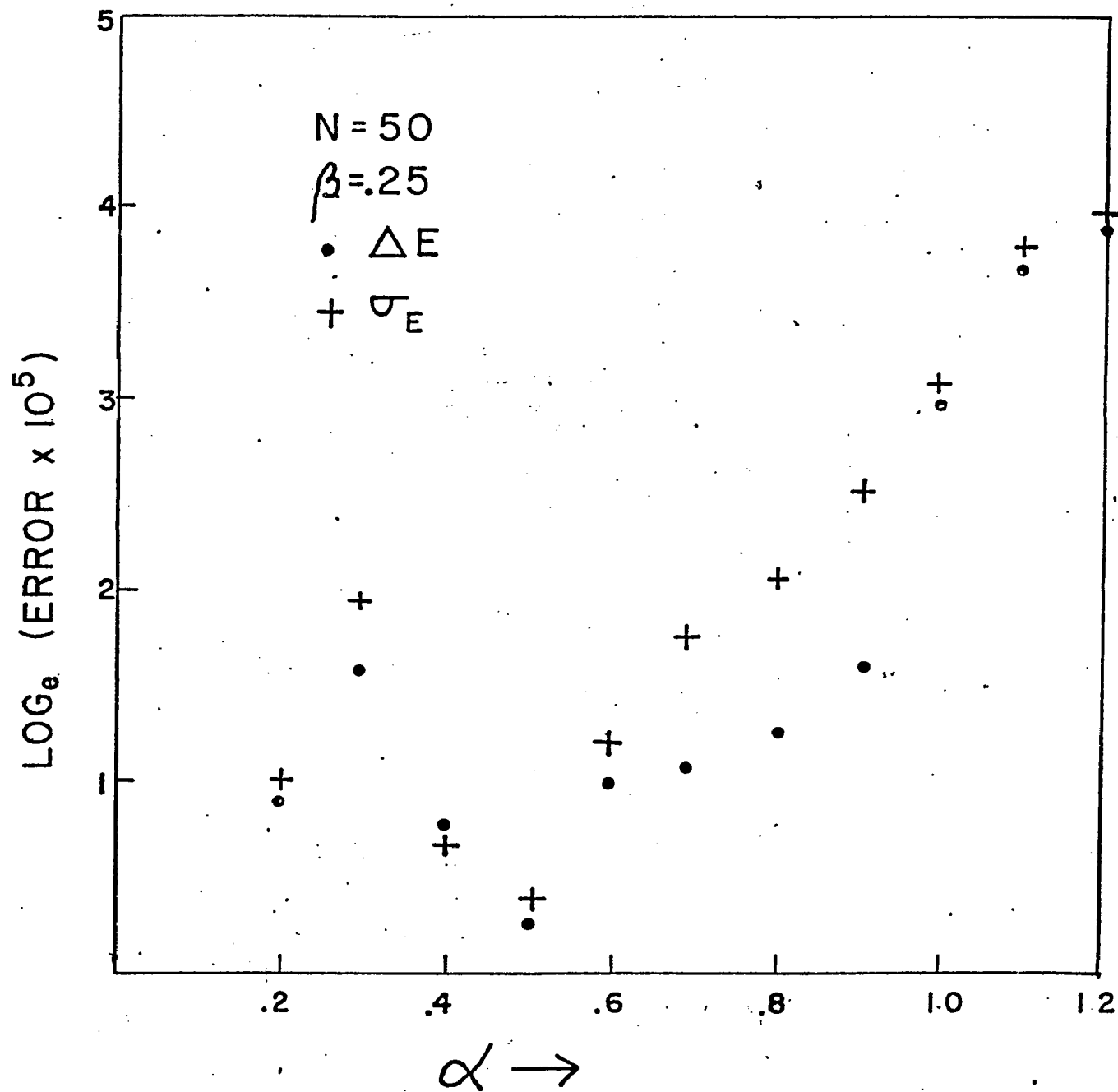
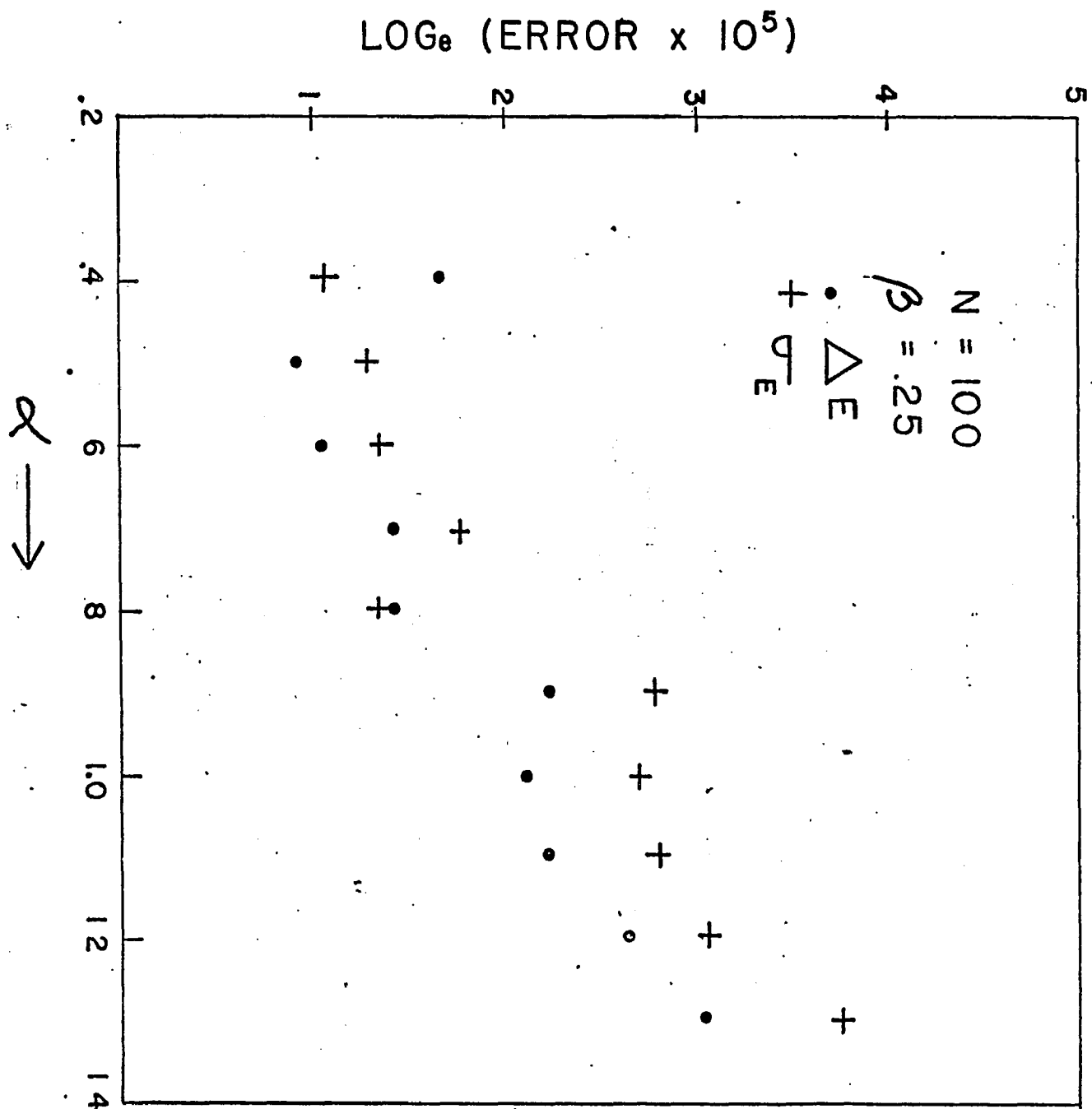


Figure 5b



gram and resulting errors. At best, one would hope to find a general trend which may be a useful guide in practice.

A detailed error analysis was made by Wielen (1967) for his method, which uses single timesteps. In our scheme, we have two primary error control parameters, α and β , appearing in the regular and irregular timesteps. To study the dependence of the error parameters, we kept one of the two parameters constant while integrating different cases of varying the other. In each instance the integrations were done for one crossing time and the energy was sampled eight times. We have used two different criteria for the energy error

$$\Delta E = \frac{1}{n} \sum_{i=1}^n |E_{i+1} - E_i| \quad (73)$$

and

$$\sigma_E = \frac{1}{n} \left[\sum_{i=1}^n (E_i - \bar{E})^2 \right]^{1/2} \quad (74)$$

where E_i is the energy at each of the samplings, $\bar{E}$ the average of the E_i s and n is the number of samples taken. In (73) $n=8$ but in (74) $n=9$ as the initial energy, $-.1250$ in our system of units, is used as one of the samples.

Figures (4a), (4b), (5a) and (5b) show the relative error as a function of the irregular and regular timestep control parameters appearing in (5) and (7). The figures show a

general trend even though there is quite a high scatter. The trend is reliable and has appeared in most of the cases we have seen. Of course, there may be times when the relative error becomes very large, but after some time it returns to its average relative error. This is a well known phenomenon in N-body integrations and is usually due to a very close encounter (Standish 1968).

PART TWO

III. INTRODUCTION

Our aim in the following is to study various aspects of the stochastic nature of the gravitational field. When a gravitational system contains a great many stars, solution of Newton's equation is both practically impossible and unnecessary. As in the study of a gas the natural description is a statistical mechanical one. At the present time there is no complete theory of the statistical mechanics for a collection of stars. Recently there have been some attempts to derive the kinetic equations for the gravitational case starting from first principles. The basic idea is to start from the Liouville equations for N particles and use the BBGKY hierarchy (Gilbert 1967, 1970, 1972; Lerche 1971) or the method of Prigogine (1968) to derive equations of evolution for the low order distribution functions. Undoubtedly, in the future, these methods will yield results, but at present these are at a formalistic stage and have not been developed to the point where they yield any specific results. Chandrasekhar and von Neumann (1941, 1943), on the other hand, have developed a statistical theory for the gravitational field which has yielded a wealth of specific results and physical effects. It is at present the most complete theory and has the great advantage that its predictions are sharp and thus can be compared with results obtained from numerical experiments. This must be balanced against the drawback that their theory is not derived from first principles. Chandrasekhar and von Neumann

make certain statistical assumptions about some of the basic variables. No doubt, refinements will have to be made in the future, but if the theory is essentially correct, it could be used to attack the problem of globular clusters both theoretically and by Monte Carlo methods. Our purpose is to experimentally verify some aspects of the theory. We have concentrated on the crucial aspects of it, particularly those which bear on the assumptions that they have made. In particular, we will investigate the probability distribution of force, the distribution of time rate of change of force, the existence of dynamical friction and the auto-correlation function. We will also, in some instances, derive some new theoretical results for certain aspects of it (particularly for the finite-N case) and give simpler derivations of the basic physical phenomena.

We will now give a brief qualitative description of the theory and discuss more specific details in subsequent chapters. The starting point is to divide the force on a star into two parts: one part due to the effect of the system as a whole, and another part due to the stars in the immediate vicinity of the star in question. The force due to the smoothed out distribution of stars, the mean field force, changes deterministically and can be derived from a time-dependent potential. On the other hand, the force due to the nearby stars oscillates rapidly due to chance encounters and the relatively rapid fluctuations in the distribution of the nearby stars. The mean field force can, of course, be studied

by Newton's equations or their fluid dynamic counterpart, but the random force must be studied stochastically.

The theory of Chandrasekhar and von Neumann deals both with the probability distribution of the random force and the joint distribution of the random force with other dynamical variables such as the rate of change of the force or the random force at a different time. Since the gravitational force depends only on the relative positions of the other stars, obtaining the probability distribution for it depends only on the probability distribution of the relative distances of the other stars. Chandrasekhar and von Neumann assume that the stars surrounding the star on which the force is being calculated are distributed from a uniform probability density. This is the most crucial assumption of the theory. It implies no correlation among the stars, either in positions or in positions and velocities. They also take the number of stars to be infinite but keep the density constant. With these assumptions it is straightforward to derive the probability distribution of force which is known as the Holtsmark distribution. We shall, in the next Chapter, derive the probability distribution without the latter assumption.

It is clear that the probability distribution of the force by itself cannot provide a total description. Equally important is the joint probability of force and the time rate of change of force

$$\vec{f} = \dot{\vec{F}} = -G \sum_{j=1}^N m_j \left(\frac{\vec{v}_j}{r_j^3} - 3 \frac{(\vec{v}_j \cdot \vec{r}_j)}{r_j^5} \vec{r}_j \right)$$

(1)

Since the velocity of the field stars appears in (1), a further assumption has to be made about their distribution to derive the joint distribution of $\vec{F}$ and $\vec{f}$. Chandrasekhar and von Neumann assume that the distribution of the velocities is independent of the position and is spherical in velocities. Mathematically, the problem now becomes the following. Given the probability distribution of position and velocities for an arbitrary star i

$$P(\vec{\pi}_i, \vec{v}_i) = P_1(\vec{\pi}) P_2(|\vec{v}|) \quad (2)$$

derive the probability distributions of the relevant physical quantities which are functions of $\vec{r}$ and $\vec{v}$.

Besides deriving the joint probability distribution of $\vec{F}$ and $\vec{f}$, Chandrasekhar (1943) also derived the joint distribution of $\vec{F}$ and $\vec{F}_t$ where $\vec{F}_t$ is the force on the star at time t later. To do this, he further assumed the field particles move in straight line orbits. We shall point out that due to the mathematical complexity of the problem, explicit expressions for the joint distribution have never been obtained; but expectation values of relevant physical quantities can be obtained since they very often involve integrations over the distributions which can in most cases be done explicitly.

Having the above mentioned probability distribution Chandrasekhar and von Neumann go on to derive a number of physical phenomena, among the most important of which are

1 - Dynamical Friction: This is one of the most remarkable predictions of the theory. The result is that the stars experience a retarding deceleration in the direction of motion. More specifically, it is shown that

$$\begin{aligned} \left\langle \frac{d|\vec{F}|}{dt} \right\rangle &> 0 & \vec{v} \cdot \vec{F} &> 0 \\ \left\langle \frac{d|\vec{F}|}{dt} \right\rangle &< 0 & \vec{v} \cdot \vec{F} &< 0 \end{aligned} \quad (3)$$

Although the derivation of this result by Chandrasekhar and von Neumann (1943) is extremely lengthy and mathematically intricate, its physical cause can be derived quite readily. We shall do so in a later section. Having shown the existence of dynamical friction, Chandrasekhar (1960) calculated the so-called coefficient of dynamical friction and used it to derive the escape rate of stars from globular clusters.

2 - Mean Lifetime: By considering the first moment of f^2 , an explicit expression is obtained for the mean lifetime of a force of a given strength. From the mean lifetime one can obtain the time of relaxation of a stellar system.

3 - Auto-correlation Function: The auto-correlation function plays a very significant role in any stochastic theory. The theory predicts the remarkable results that

$$\langle \vec{F}(t_1) \cdot \vec{F}(t_2) \rangle \sim \frac{1}{t_2 - t_1}$$

We shall give a simplified derivation of this result and do a numerical experiment to test it. We shall also give a derivation which takes into proper account the cutoffs and show that it then falls off as $1/t^3$ for large times.

4 - Dissociation of Binary: The correlation of forces at two different points in space in a stellar system was used by Chandrasekhar to derive the mean lifetime of binary stars.

Although many physical results have been obtained using the theory, it has never been carried to its full potential. Assuming that the theory is correct, it is in principle capable of solving the evolutionary problem of stellar dynamics. Thus, the governing equation would be of similar nature as the Langevin equation in the theory of Brownian motion. We hope the numerical experiments we have performed will revive interest in applying this theory. In the next Chapter we begin with the most fundamental result, the probability distribution of the random force.

IV. DISTRIBUTION OF RANDOM FORCE

1. Holtmark Distribution

As mentioned in the introduction, the basic assumption used by Chandrasekhar and von Neumann (1941) in deriving the probability distribution for the random force is that the particles causing the random force on a test particle are uniformly distributed about it without any correlation either among themselves or with the test particle. This also implies that the density in configuration space is the same for all ranges of the velocity of field particles. The assumption is not strictly correct and gives rise to the infinite second moment of the force for the Holtmark distribution. Chandrasekhar (1941) himself pointed out that a rigorous treatment of the statistical properties of the random force must be done in phase space. Recent efforts (Gilbert 1970, Lerche 1971) have been made in this direction, but the theories have not been advanced to a point where a comparison can be made, either with the theory of Chandrasekhar and von Neumann (1941) or with the results of numerical experiments. The second basic assumption is that the number of field stars can be allowed to go to infinity while keeping the density constant.

We shall now derive the probability distribution for force, $W_N(\vec{F})$, with the assumption that the field stars are distributed from a uniform probability density, but without allowing the number of field stars to go to infinity. The interest in doing so is two-fold. Besides the theoretical

interest it will reflect on how good the assumption of an infinite number of stars is. Also, since the numerical experiments deal with finite numbers of stars, it is imperative to understand the finite N effect on the Holtsmark distribution. Furthermore, as we shall discuss in a later section, if the theory of Chandrasekhar and von Neumann is correct, it is a very powerful method of devising a Monte Carlo scheme in which the finite N Holtsmark will be needed.

2. Holtsmark Distribution for a finite number of Field Particles

We will use the method of characteristic functions to derive $W_N(\vec{F})$. The characteristic function $C(\vec{\theta})$ of a probability density $P(\vec{F})$, is its Fourier transform

$$C(\vec{\theta}) = \int e^{i\vec{\theta} \cdot \vec{F}} P(\vec{F}) d\vec{F} \quad (1)$$

Therefore, knowing $C(\vec{\theta})$ is sufficient, in principle, to obtain $P(\vec{F})$ by Fourier inversion.

Let us define

$$\vec{F} = \sum_{i=1}^N \vec{F}_i \quad (2)$$

where $\vec{F}_i$ are independently chosen from the same distribution $P(\vec{F})$. Use of the characteristic function allows us to relate the characteristic function associated with the probability of F to the characteristic function for one of the $\vec{F}_i$ in a

simple manner. Let $C_N(\vec{\theta})$ be the characteristic function for the distribution of $\vec{F}$. We then have

$$C_N(\vec{\theta}) = \int d\vec{F}_1 \dots d\vec{F}_N e^{i \vec{\theta} \cdot \sum_{i=1}^N \vec{F}_i} P(\vec{F}_1, \dots, \vec{F}_N) \quad (3)$$

But as the $\vec{F}_i$ s are taken from the same distribution

$$P(\vec{F}_1, \dots, \vec{F}_N) = P(\vec{F}_1) \dots P(\vec{F}_N) \quad (4)$$

This allows us to express the integral in (3) as the product of N identical integrals.

$$C_N(\vec{\theta}) = \left[\int d\vec{F} P(\vec{F}) e^{i \vec{\theta} \cdot \vec{F}} \right]^N \quad (5)$$

or

$$C_N(\vec{\theta}) = C^N(\vec{\theta}) \quad (6)$$

This relation is particularly suited for the problem of finding the distribution of the random force. For, now one can identify the individual $\vec{F}_i$ s in (2) as the force arising from the interaction of the test particle with just one field particle, and $\vec{F}$ then is the total random force.

The whole problem of finding the random force can now be

divided into three steps. First we must find, using the gravitational force law and a given space density $\rho(r)$, the distribution function for force due to one particle, $P(\vec{F})$. Secondly, we must find the Fourier transform of $P(F)$ to get the one-particle characteristic function $C(\vec{\theta})$. This immediately gives the N-particle characteristic function $C_N(\vec{\theta})$. Finally, we can find the probability distribution for the total random force by simply taking the inverse Fourier transform of $C_N(\vec{\theta})$. In the following paragraph we give the details of such a calculation for the random force distribution at the center of the sphere containing N uniformly distributed stars within a radius R.

Let the test star be located at the origin and let the single field star be randomly placed inside a sphere of radius R circumscribed about the origin. Assuming the field particle to be uniformly distributed, the probability distribution for its position $\rho(r)$ is

$$\rho(\vec{r}) = \frac{1}{V} \quad (7)$$

where V is the volume of the sphere

$$V = \frac{4}{3} \pi R^3 \quad (8)$$

The probability distribution for the force $\vec{F}_i$ due to the field star is obtained from the relation

$$P(\vec{F}) d\vec{F} = \rho(\vec{\pi}) d\vec{\pi} \quad (9)$$

Using Newton's law of gravitation

$$\vec{F} = \frac{Gm\vec{\pi}}{\pi^3} \quad (10)$$

it is easily seen

$$P(\vec{F}) d\vec{F} = \frac{(Gm)^{3/2} d\vec{F}}{2V F^{9/2}} \quad (11)$$

$$\frac{GM}{R^2} < F$$

It should be noted that the distribution (11) depends only on the magnitude of the force. This is expected from the spherical symmetry of our density (7).

The characteristic function for this one-particle probability of force is thus given by

$$C(\vec{\theta}) = \int e^{i\vec{\theta} \cdot \vec{F}} P(\vec{F}) d\vec{F} \quad (12)$$

Without any loss of generality, we can take $\vec{\theta}$ to be along the

z-axis. We then have

$$C(\vec{\theta}) = 2\pi \int_{-1}^1 \int_{\frac{GM}{R^2}}^{\infty} e^{i\theta F\mu} P(\vec{F}) F^2 dF d\mu \quad (13)$$

where μ is the cosine of the angle between $\vec{\theta}$ and $\vec{F}$

Using (11) for $P(\vec{F})$ and performing the angular integration, we get

$$C(\vec{\theta}) = \frac{2\pi}{V} (Gm)^{3/2} \int_{\frac{GM}{R^2}}^{\infty} \frac{\sin \theta F}{\theta F^{7/2}} dF \quad (14)$$

Making the substitution

$$z = \theta F$$

we get

$$C(\vec{\theta}) = \frac{2\pi (Gm)^{3/2}}{V} \theta^{3/2} \int_{\frac{GM\theta}{R^2}}^{\infty} \frac{\sin z}{z^{7/2}} dz \quad (15)$$

$C(\vec{\theta})$ as given by (15) is then the characteristic function for the random force due to just one field particle. The characteristic function for force due to N such particles, C_N , is simply

$$C_N(\vec{\theta}) = C(\vec{\theta})^N$$

(16)

and the probability distribution for the total force due to N particles is just the inverse Fourier transform of $C_N(\vec{\theta})$

$$P(\vec{F}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{\theta} \cdot \vec{F}} C_N(\vec{\theta}) d\vec{\theta} \quad (17)$$

Again taking $\vec{\theta}$ along the z -axis and performing a straightforward angular integration in (17) we obtain

$$P(\vec{F}) = \frac{2}{(2\pi)^2} \int_0^\infty \frac{\sin(\theta F)}{\theta F} C_N(\vec{\theta}) \theta^2 d\theta \quad (18)$$

As the probability $P(\vec{F})$ is a function of the magnitude of the force only, it is sufficient to consider the distribution for the magnitude $W_N(F)$

$$\begin{aligned} W_N(F) &= 4\pi F^2 P(\vec{F}) \\ &= \frac{2F}{\pi} \int_0^\infty \theta \sin(\theta F) C_N(\vec{\theta}) d\theta \end{aligned} \quad (19)$$

Following Chandrasekhar, we define a dimensionless variable, β ,

$$\beta = \frac{F}{Q} \quad (20)$$

where

$$Q = \left(\frac{4}{15}\right)^{2/3} 2\pi G m n^{2/3} \quad (21)$$

and n is the number density of the particles given by N/V . Then β measures the force in units of the normal force Q . We can express (19) in terms of the probability distribution for β .

$$P_N(\beta) = \frac{2\beta}{\pi} \int_0^{\infty} y \sin \beta y C^N(y/Q) dy \quad (22)$$

where

$$C(y/Q) = \frac{3}{2} \epsilon^{3/2} \int_{\epsilon}^{\infty} \frac{\sin z}{z^{7/2}} dz \quad (23)$$

is given by

$$\epsilon = \frac{1}{2} \left(\frac{25}{\pi} \right)^{1/3} \frac{y}{N^{2/3}} \quad (24)$$

The integral in (23) can be reduced by several integrations by parts to yield

$$\begin{aligned} C(y/Q) = & \frac{3}{5} \frac{\sin \epsilon}{\epsilon} + \frac{2}{5} \cos \epsilon - \frac{2}{5} \sqrt{2\pi} \epsilon^{3/2} \\ & - \frac{4}{5} \epsilon \sin \epsilon + \frac{4}{5} \epsilon^{3/2} \int_0^{\infty} \frac{\cos z}{z^{1/2}} dz \end{aligned} \quad (25)$$

The last term in (25) contains the Fresnel type integral for which an approximation is known

$$\int_0^x \frac{\omega y}{y^{1/2}} dy \cong \sqrt{\frac{2\pi}{x}} \sum_{i=0}^6 a_i t^i$$

$$t = 16 - x^2, \quad a_0 = .18410, \quad a_1 = .011025$$

$$a_2 = 1.0204 \times 10^{-3}, \quad a_3 = 3.2733 \times 10^{-5}$$

$$a_4 = 5.4512 \times 10^{-7}, \quad a_5 = 5.2443 \times 10^{-9} \quad (26)$$

$$a_6 = 5.1008 \times 10^{-11}$$

This approximation is well suited for a numerical evaluation of $W_N(\beta)$ because it reduces (22) which involves a double integration to a single integration.

To study the N-dependence of the distribution $W_N(\beta)$, it is useful to expand $C(y/Q)$ in inverse powers of N. This can be done by expanding (25) in powers of ϵ which yields

$$C(y/Q) = 1 - \frac{2}{5} \sqrt{2\pi} \epsilon^{3/2} + \frac{\epsilon^2}{2} \quad (27)$$

Expressing ϵ in terms of N from (24) we get

$$C(y/Q) = 1 - \frac{y^{3/2}}{N} + \frac{1}{8} \left(\frac{25}{\pi} \right)^{2/3} \frac{y^2}{N^{4/3}} \quad (28)$$

For the case where N is infinite we have

$$C^N(y/Q) \simeq \left(1 - \frac{y^{3/2}}{N}\right)^N \quad (29)$$

$$C^N(y/Q) = e^{-y^{3/2}} \quad N \rightarrow \infty \quad (30)$$

and substituting (30) into (22) we get

$$P(\beta) = \frac{2\beta}{\pi} \int_0^\infty y e^{-y^{3/2}} \sin \beta y dy \quad (31)$$

which is the Holtzmark distribution.

3. Asymptotic Behavior

It is instructive to study the behavior of $W_N(\beta)$ for very high and very low values of β . For small values of β we can expand the sine in (22) in powers of β . We get

$$W_N(\beta) \simeq A_N \beta^2 + B_N \beta^3 \quad (32)$$

$$\beta \rightarrow \infty$$

A_N and B_N are functions of N only, given by

$$\begin{aligned}
 A_N &= \frac{2}{\pi} \int_0^{\infty} y^2 C^N(y/Q) dy \\
 B_N &= -\frac{1}{3\pi} \int_0^{\infty} y^4 C^N(y/Q) dy
 \end{aligned} \tag{33}$$

It must be noted, however, that this result is not valid for $N=1$ where β has a lower limit. To study the behavior of $W_N(\beta)$ for high values of β we change the variable of integration in (22)

$$y' = \beta y \tag{34}$$

We get

$$W_N(\beta) = \frac{2}{\pi\beta} \int_0^{\infty} y' \sin y' C^N\left(\frac{y'}{\beta Q}\right) dy' \tag{35}$$

Expanding C in inverse powers of β by (28)

$$C(y'/\beta Q) \rightarrow 1 - \frac{y'^{3/2}}{\beta^{3/2} N} \quad \beta \rightarrow \infty \tag{36}$$

and

$$W_N(\beta) = \frac{2}{\pi\beta} \int_0^{\infty} y' \sin y' \left(1 - \frac{y'^{3/2}}{\beta^{3/2} N}\right)^N dy' \tag{37}$$

but

$$\left(1 - \frac{y^{3/2}}{N\beta^{3/2}}\right)^N \rightarrow 1 - \frac{y^3}{\beta^{3/2}} \quad \beta \rightarrow \infty \quad (38)$$

or

$$W_N(\beta) = \frac{15}{8} \sqrt{\frac{2}{\pi}} \frac{1}{\beta^{5/2}} \quad \beta \rightarrow \infty \quad (39)$$

where we have used

$$\int_0^\infty y \sin y \, dy = 0$$

and

$$\int_0^\infty y^{5/2} \sin y \, dy = -\frac{15}{8} \sqrt{\frac{\pi}{2}}$$

We therefore see that all $W_N(\beta)$ have the same asymptotic behavior for large forces. Furthermore, this is precisely the one-particle distribution (11) derived earlier. The physical reason for this fact is that the higher forces are due mostly to just one particle. An important consequence of (39) is that the second moment of β is infinite for all functions W_N . This is due to improper inclusion of binaries or close approaches in the theory. To exclude binaries, we must define a minimum separation distance that two stars are allowed to have

$$r_{\min}(\nu) = \frac{2Gm}{\nu^2} \quad (40)$$

where v is the relative speed of the pair. To see how this restriction modifies the distribution function, we consider the distribution for force due to the nearest neighbor (Chandrasekhar 1941).

$$W(\beta) = \frac{15}{8} \sqrt{\frac{2}{\pi}} \beta^{-5/2} e^{-\frac{5}{2\sqrt{2}\pi} \beta^{-3/2}} \quad (41)$$

This function also has asymptotic behavior (39) and therefore gives rise to a divergent second moment. If the restriction (40) is applied to the space density of the nearest neighbors we get the following modified distribution for the force due to the nearest neighbors

$$W'(\beta) = \frac{15}{8} \sqrt{\frac{2}{\pi}} \beta^{-5/2} e^{-\frac{5}{2\sqrt{2}\pi} \beta^{-3/2}} \left(\beta^{-3/2} - \beta_{\max}^{-3/2} \right) \quad \beta < \beta_{\max}$$

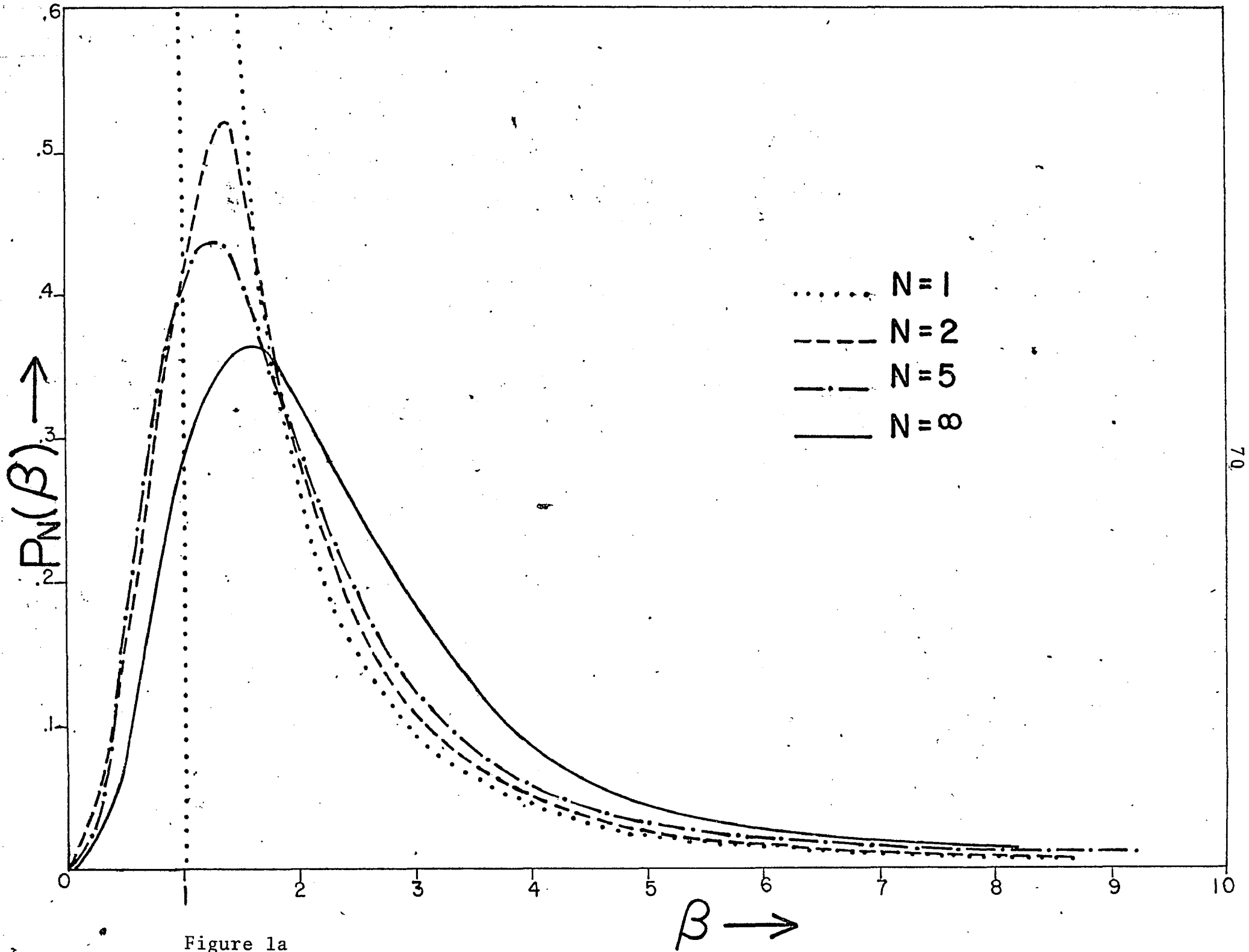
$$= 0 \quad \beta > \beta_{\max}$$

where

$$\beta_{\max} = \frac{v_T^4}{9GQm}$$

where v_T is the average relative root mean square velocity. It is clear that the second moment of force for this modified distribution is finite.

The probability distributions $W_N(\beta)$ were numerically evaluated on a computer using a simple integration method and the approximation (26), for various values of N . They are plotted in Figure (1). The case $N=1$ is different from all



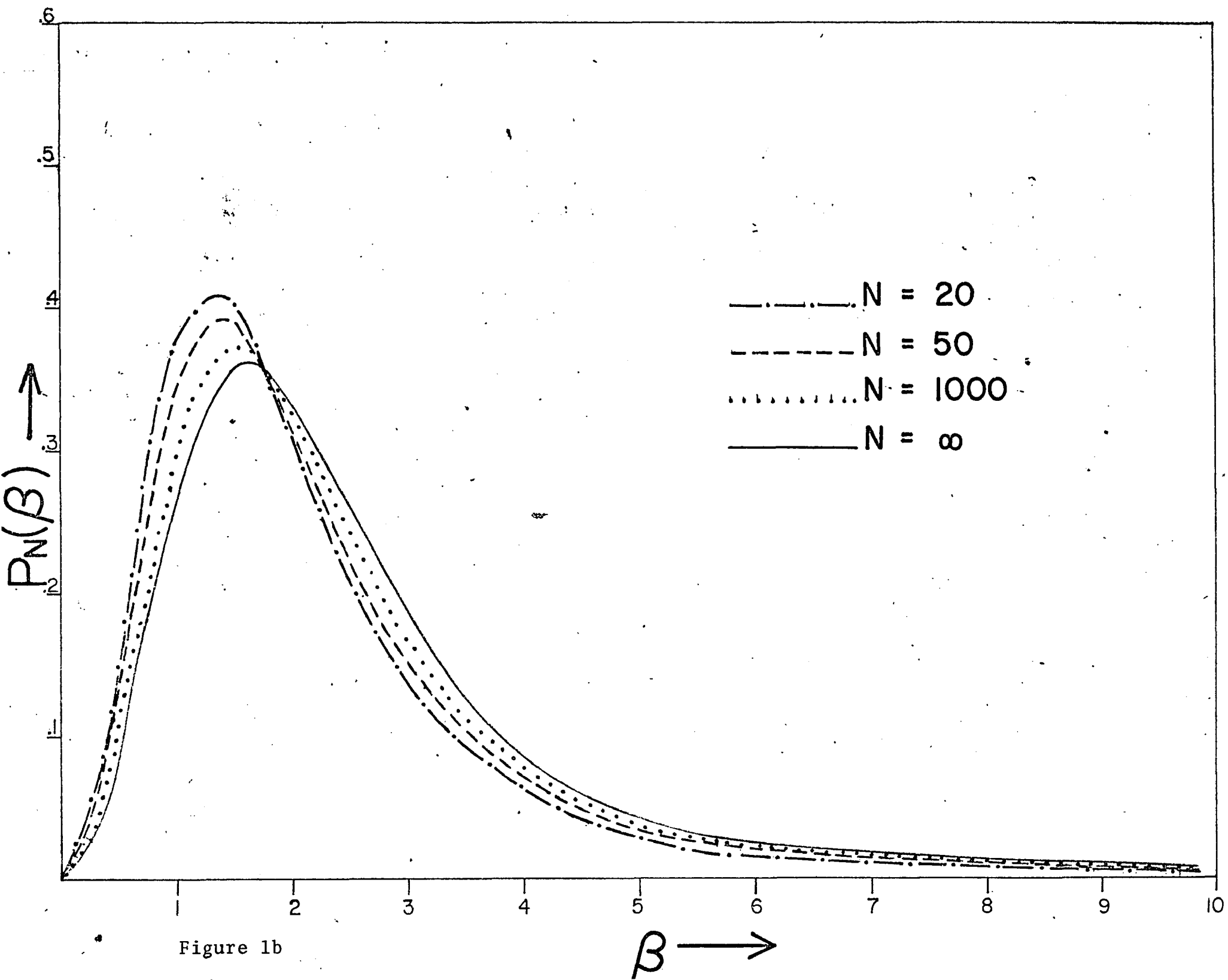


Figure 1b

others in that it has a sharp cutoff at $\beta = .998$. This is due to the fact that when there is only one particle in a sphere of finite radius, the magnitude of its force at the origin cannot be less than Gm/R^2 as is evident from (11). For $N > 1$, no such restriction exists and configurations are possible for all magnitudes of forces. Although it is difficult to obtain a precise N -dependence of $W_N(\beta)$, it is possible to make a few qualitative remarks about the curves of Figure (1). The maximum of W_N decreases with increasing N , while the β at which the maximum occurs increases with increasing N . Hence, for the $N = \infty$ case, the maximum is lowest of all and occurs at the largest β .

EXPERIMENTAL RESULTS

1. Method of Experiment

To study the distribution of the random force in a system of gravitational bodies, the integration scheme described in Chapter II was used. We evolved systems containing 100, 300, and 1000 particles for a few crossing times. Since the crossing time is the time in which a star with an average velocity traverses the system, it insures that particles are thoroughly mixed during this interval. Since the calculation of β for a star involves a knowledge of the space density in its neighborhood, it is necessary to have initial conditions which are in or close to equilibrium. This insures that there are no violent changes in the density of the cluster. For most of our cases we have used the Emden polytrope of index 1.5, which has been described in detail in the second chapter.

During the evolution of the system on the computer, such quantities as the positions, the velocities, and the total forces were sampled every 1/64 of a crossing time and stored on a tape. This tape was then used to study the distribution of the random force.

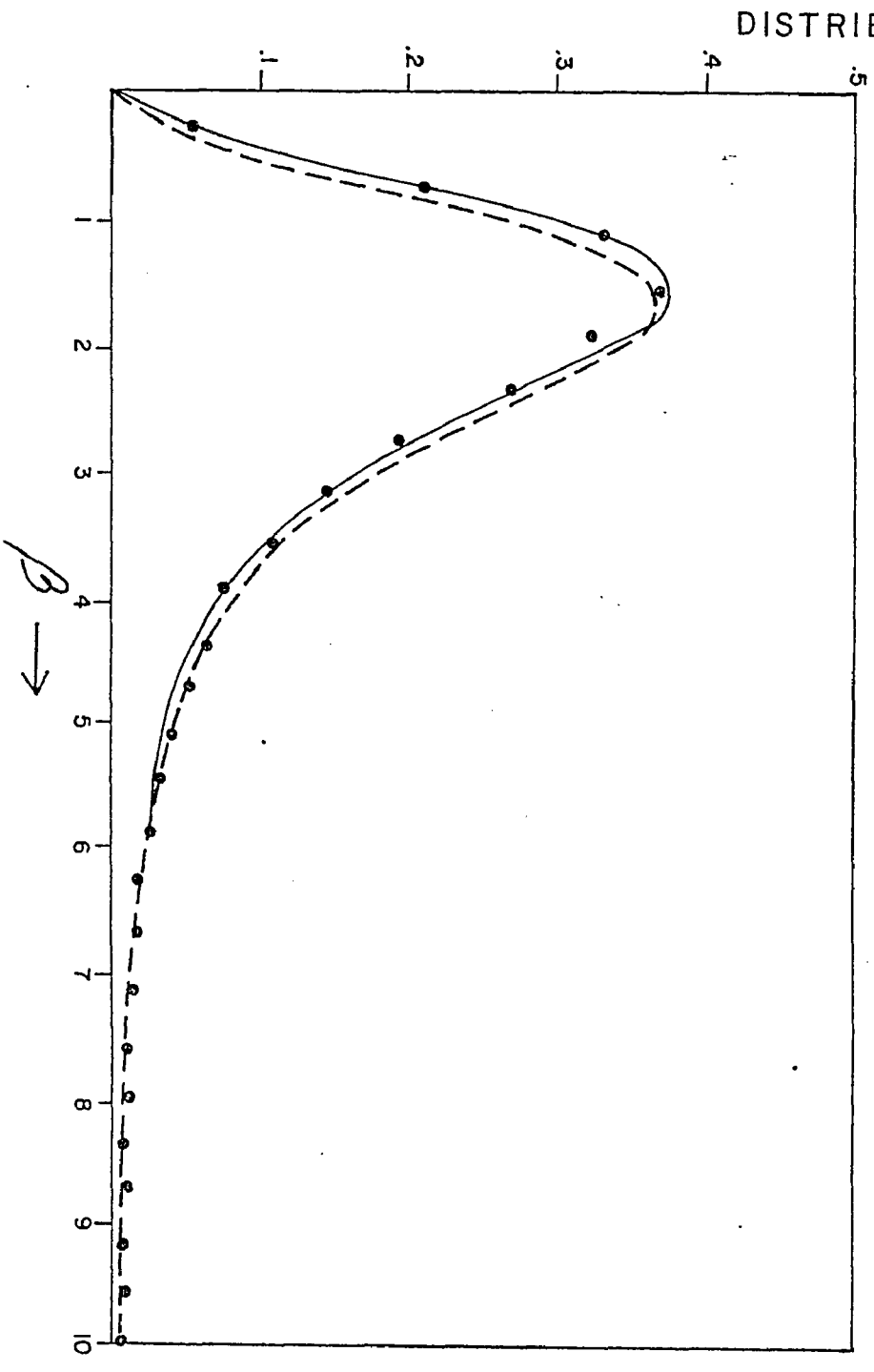
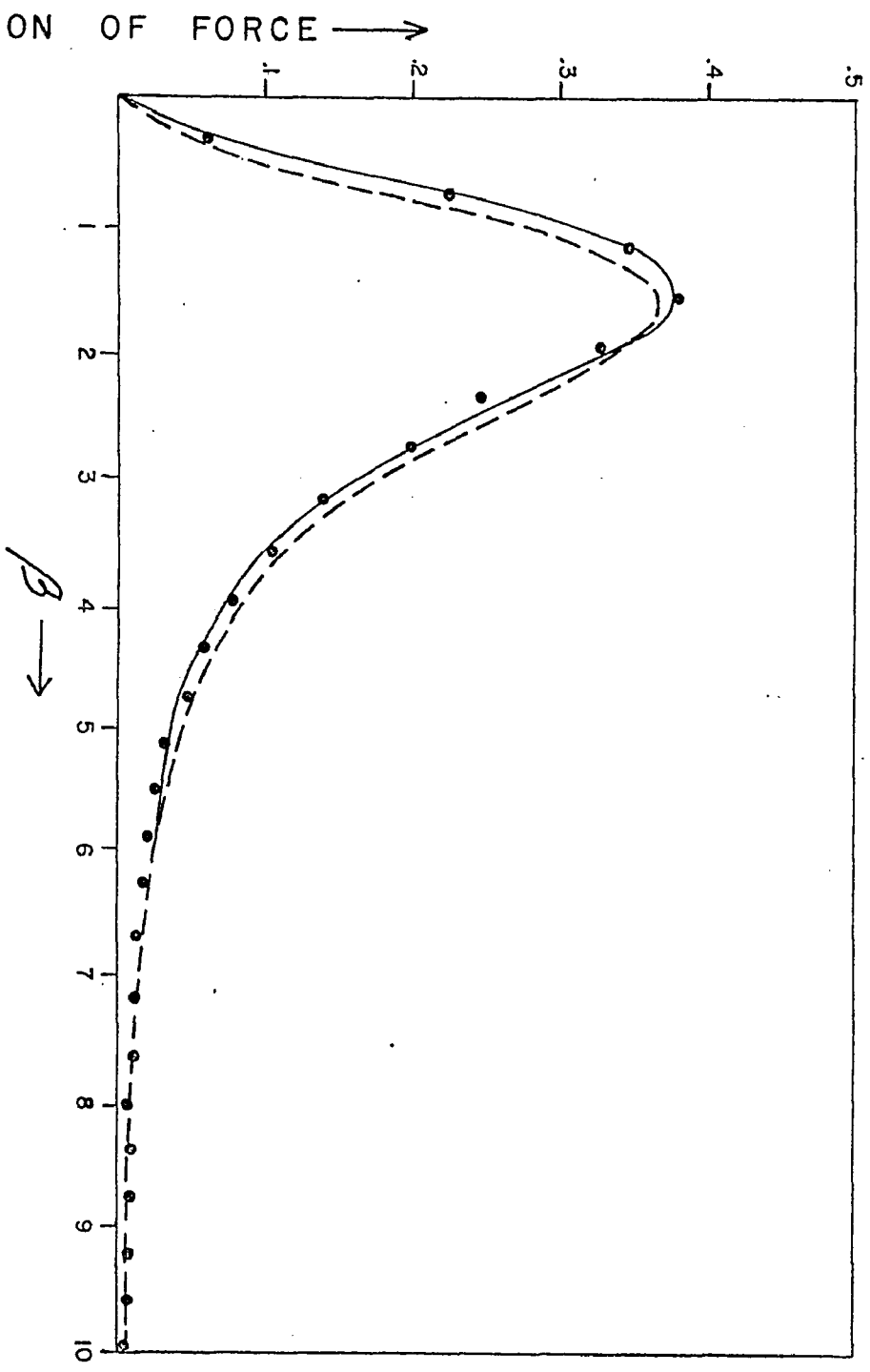
The total force which is stored on the tape can be used to give the random force if we simply subtract the mean field force from it. In spherically symmetric systems with large numbers of particles, the mean field force can be simply taken to be

$$\vec{F}_{\text{mean}} = -G \frac{M(r)}{r^3} \vec{r}$$

where $\vec{r}$ is the position of the star and $M(r)$ is the mass interior to it. Since our systems are spherically symmetric, (42) is a good approximation to the mean field. We tried two different methods of obtaining $M(r)$. One, by merely counting the particles interior to the one in question, and the other, by integrating the density distribution which was averaged for a particular quarter crossing time. Both methods gave substantially the same results. To find β , however, we should also have the density of stars in the system at different distances from the origin. A time average density is calculated for fractions of the crossing time (usually a quarter). This density is then used to find $\vec{\beta}$ for each star for the samples that are contained in that interval. Since the density of the system is not uniform, this implies different values of Q for different parts of the cluster. We are now in a position to study the probability distribution of the forces. To improve the statistics, results for different samples in a certain fixed interval are combined.

2. Comparison with the Theory

Special care must be taken when a distribution of random forces obtained from a numerical experiment, such as the one described above, are to be compared with the theoretical distribution. Since the number of particles is finite here, we cannot compare this with the Holtsmark distribution (31). The difficulty of comparing the experimental results with finite-



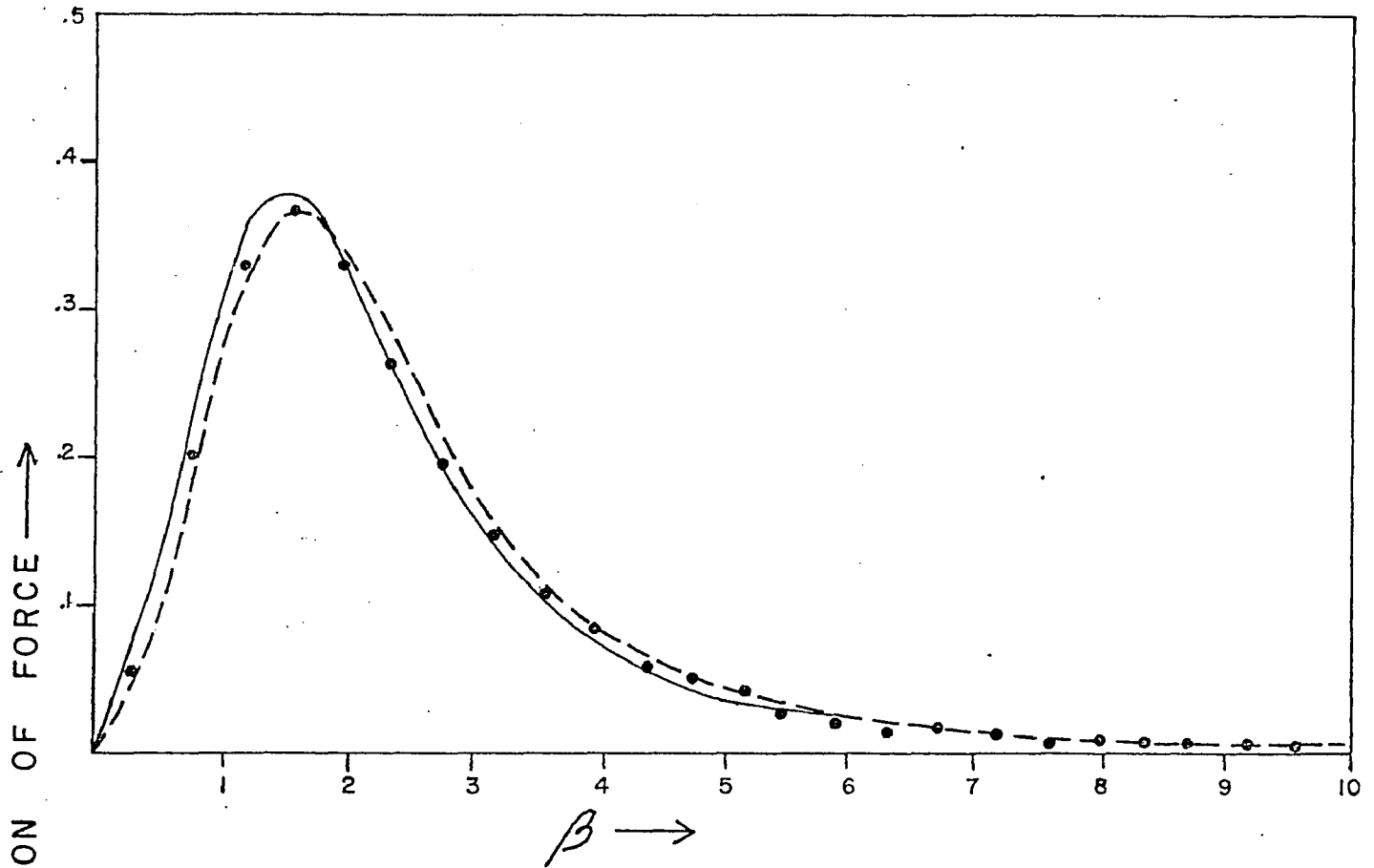


Figure 2c

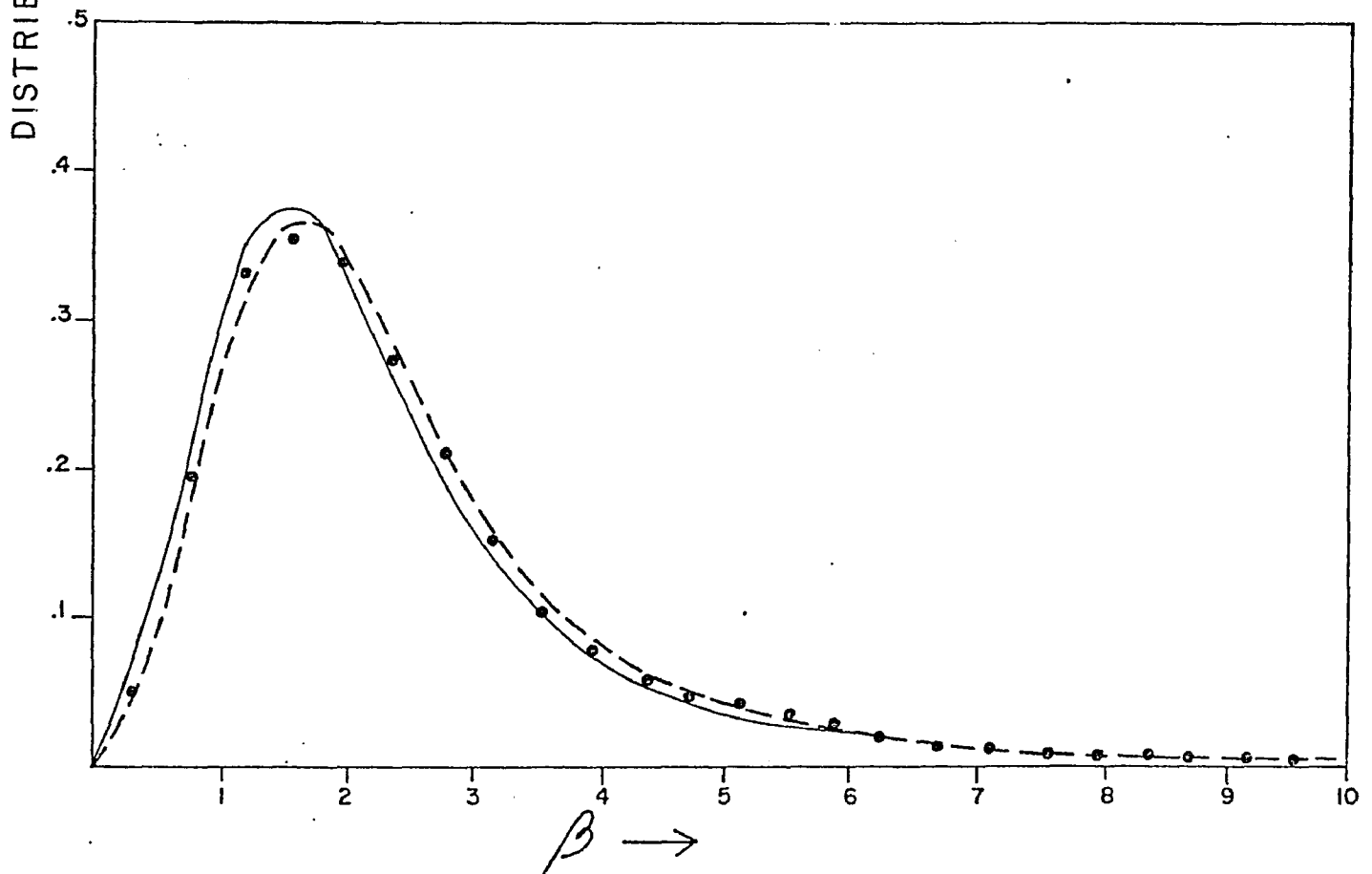
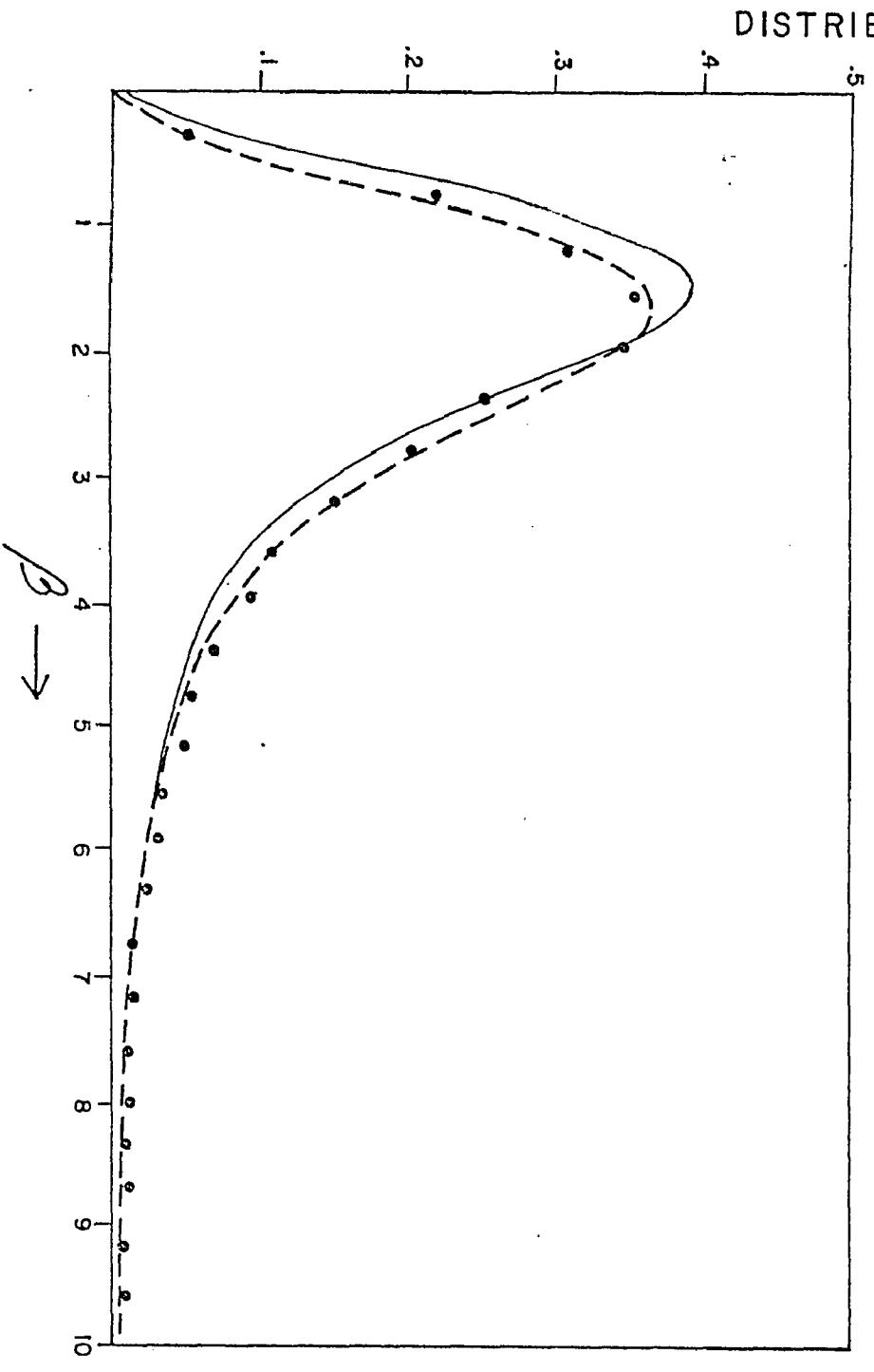
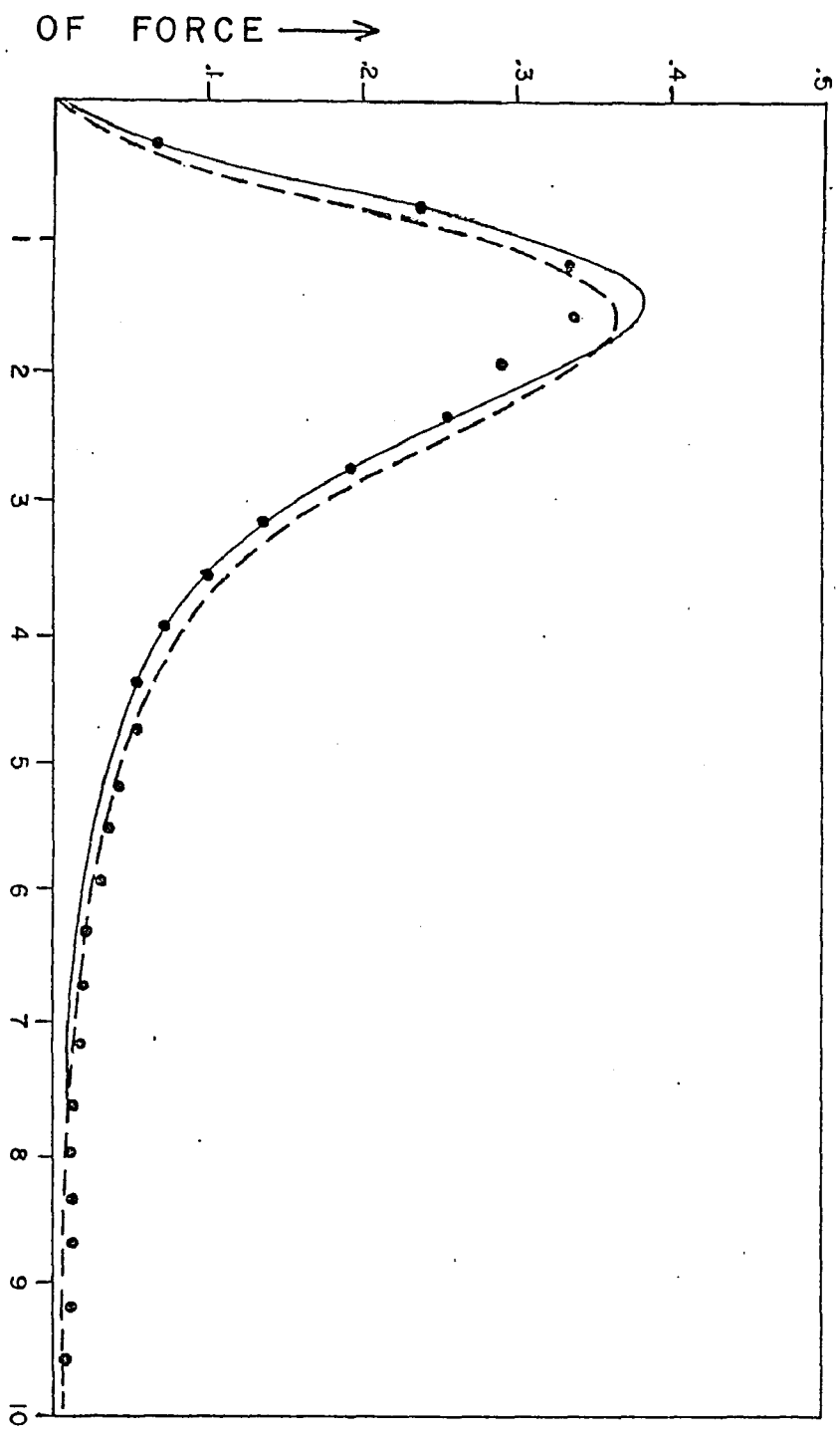


Figure 2d

300-Body Case



300-Body Case

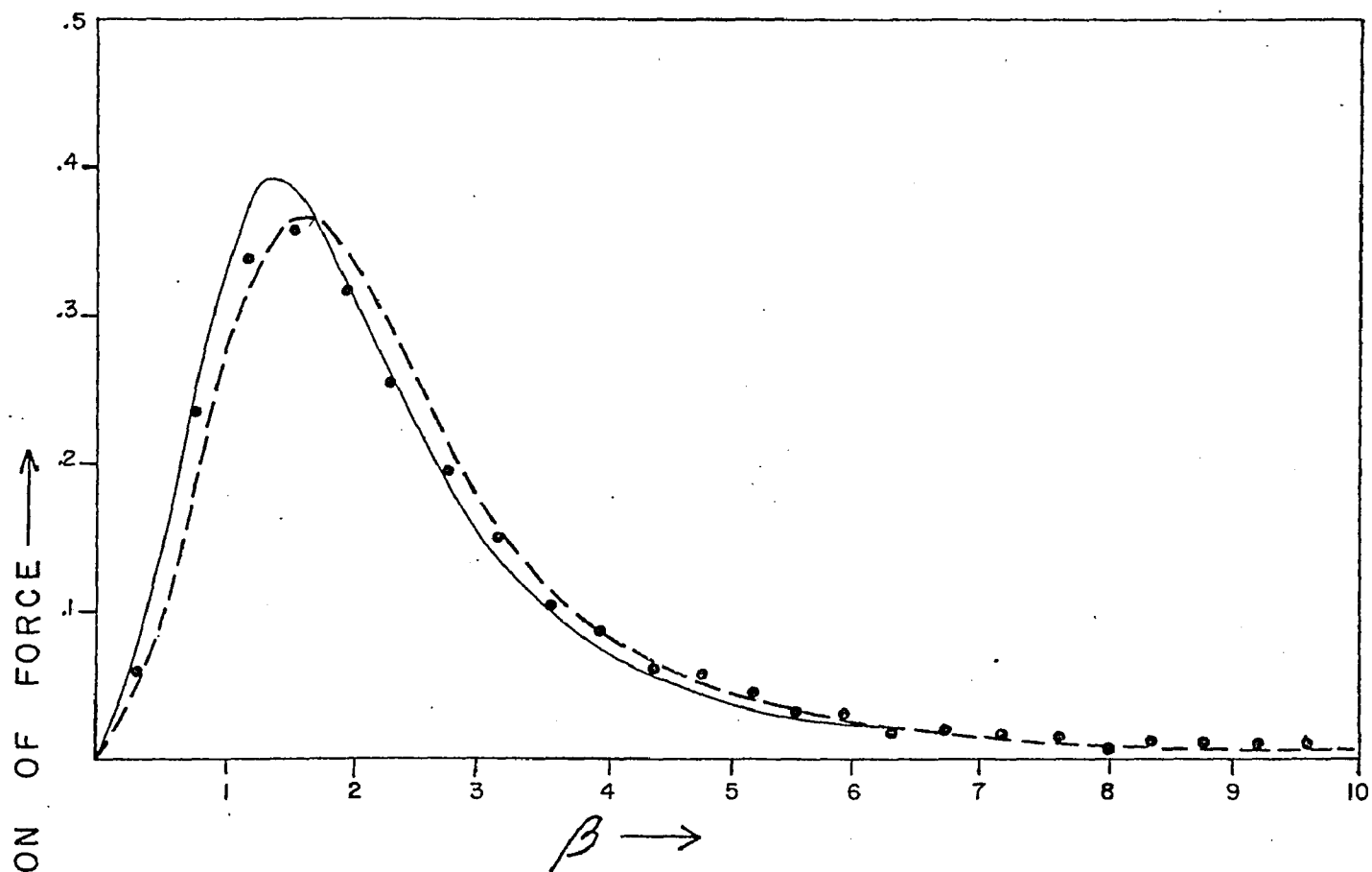


Figure 3c

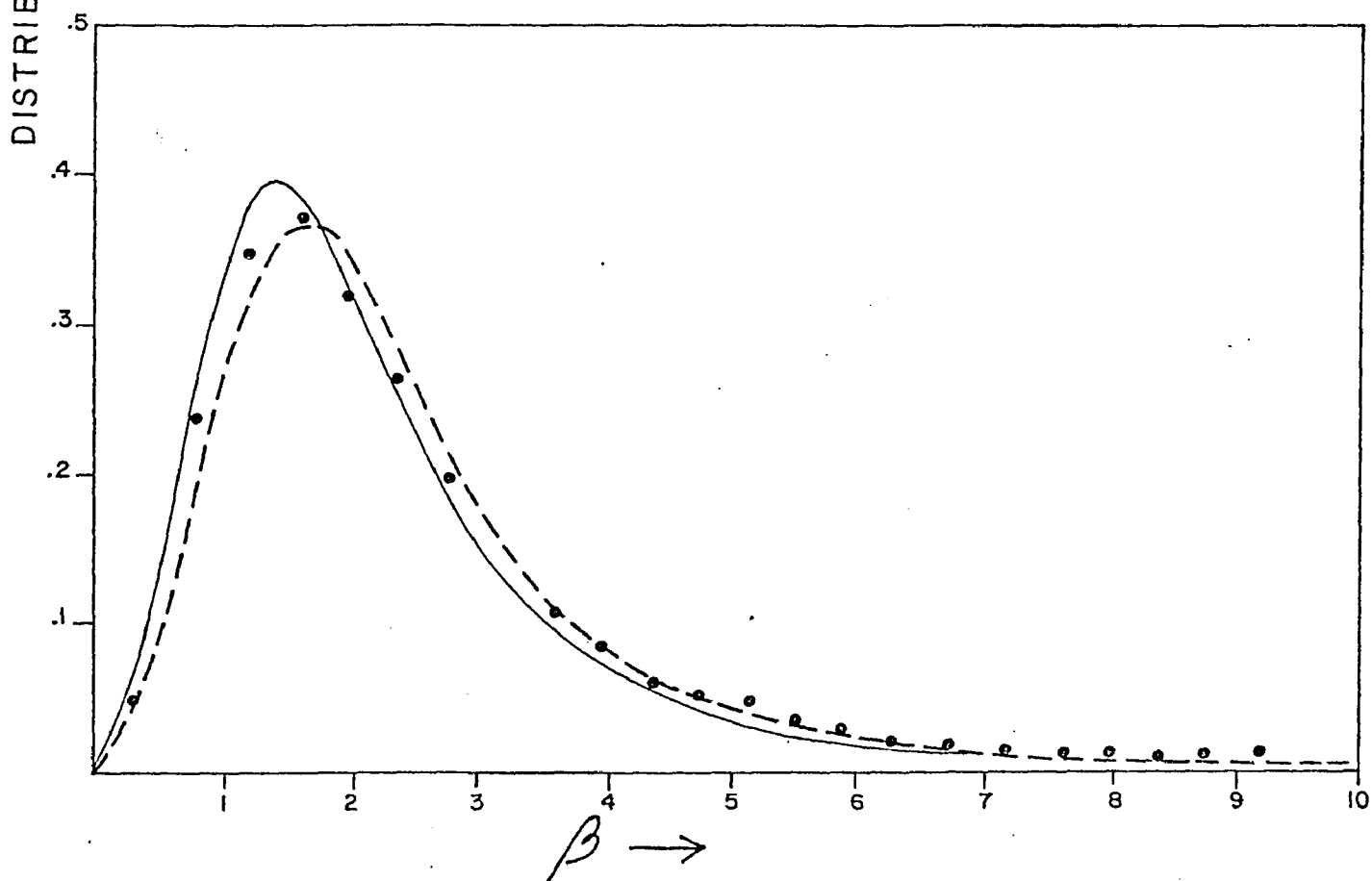


Figure 3d

100-Body Case

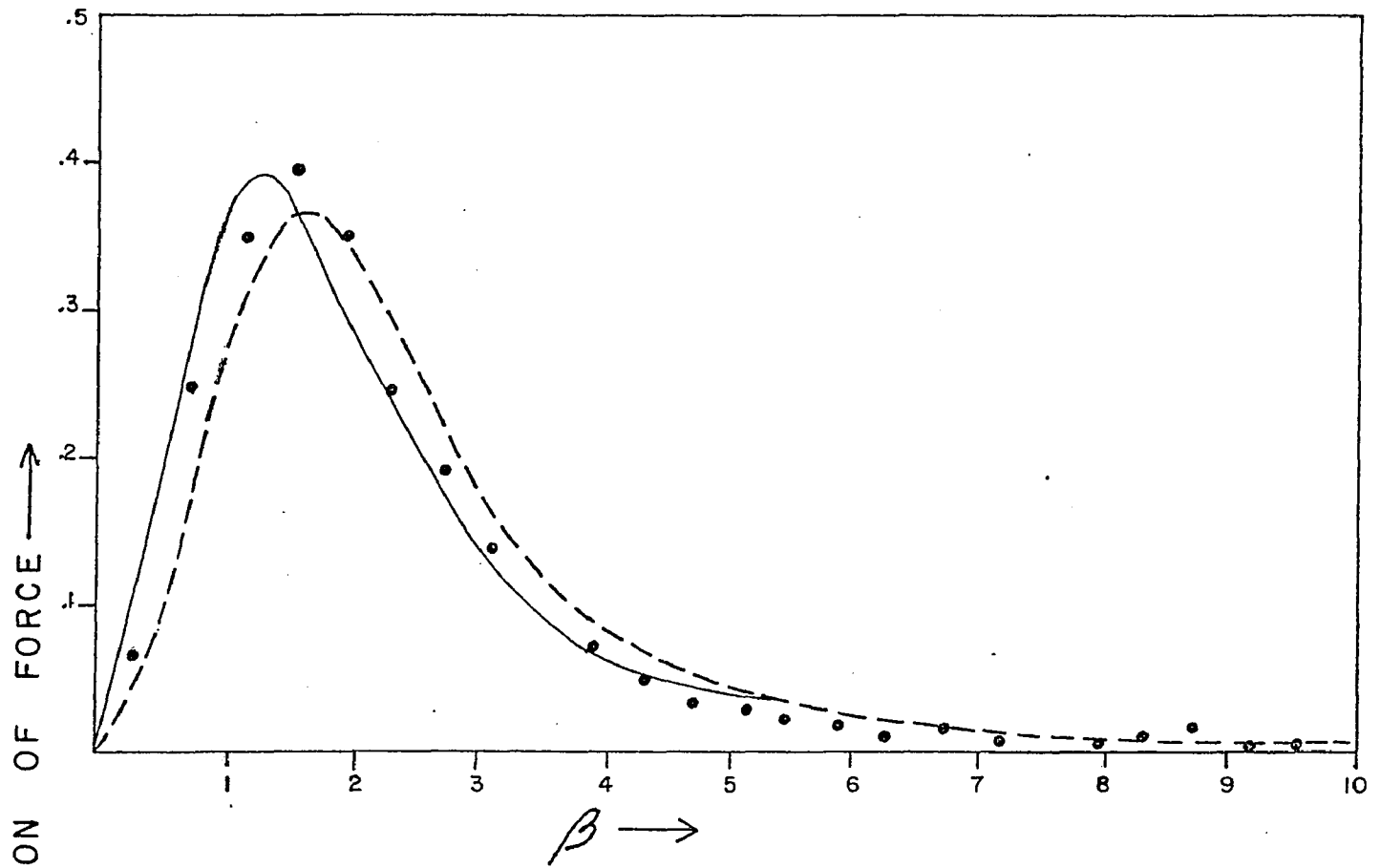


Figure 4a

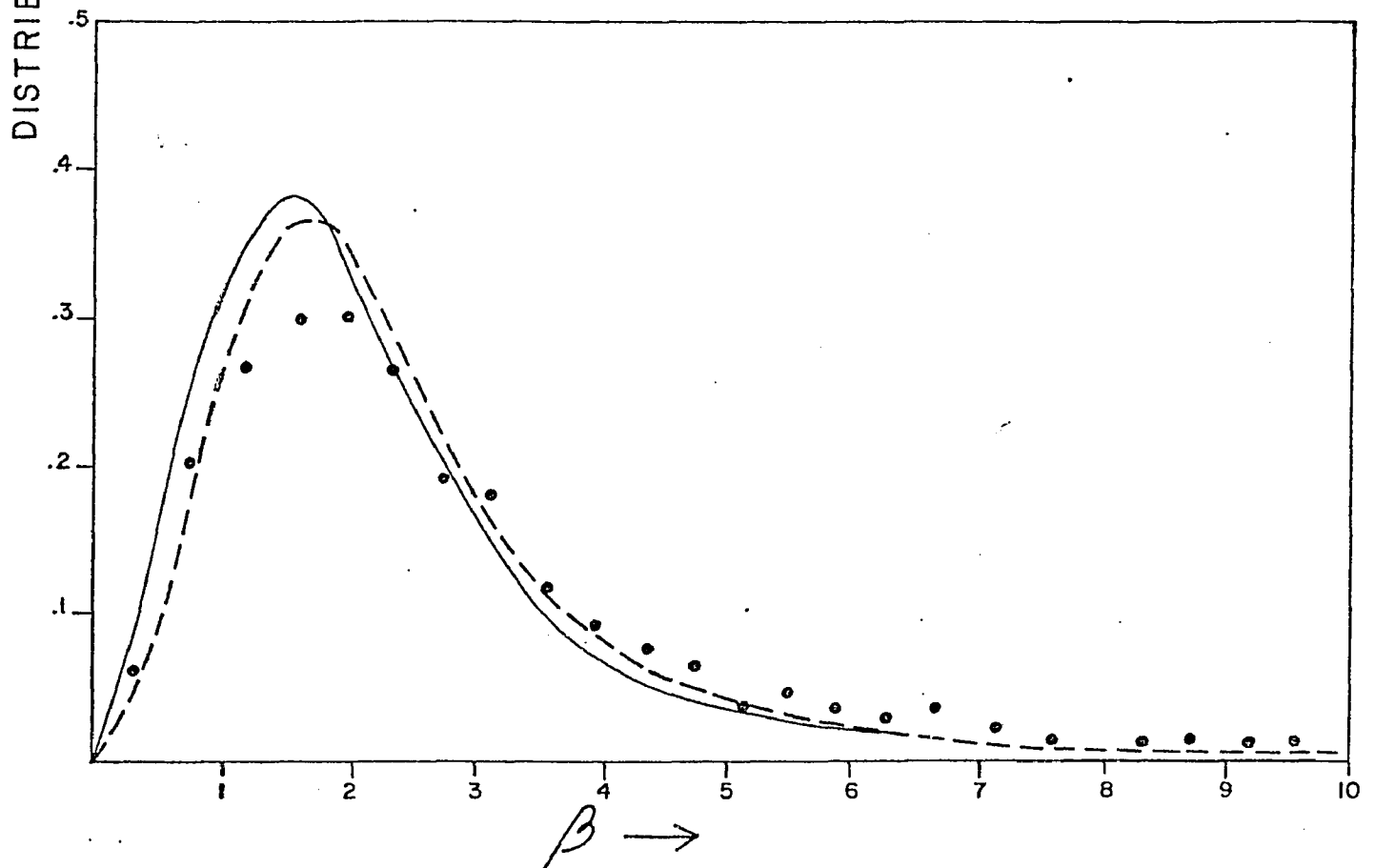


Figure 4b

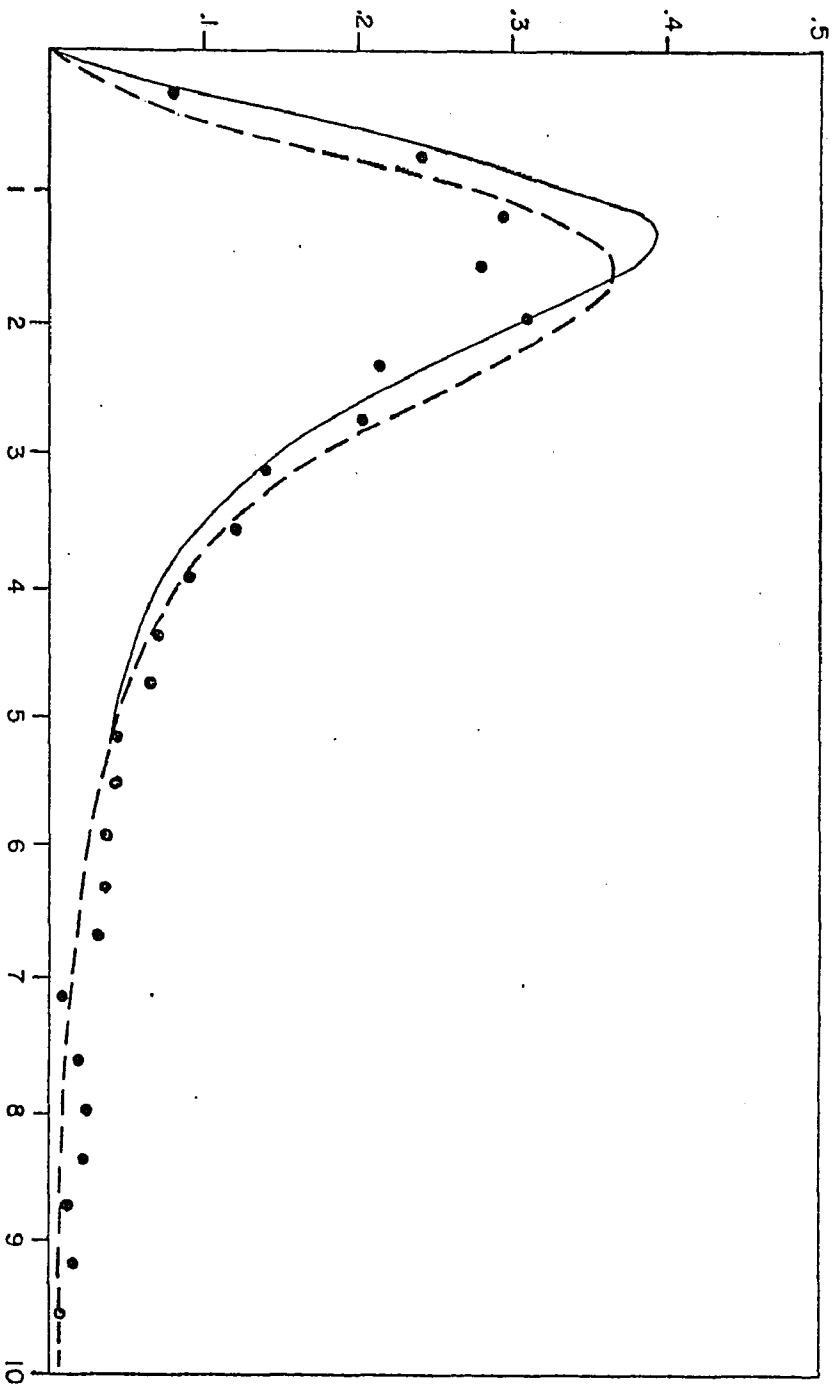


Figure 4c

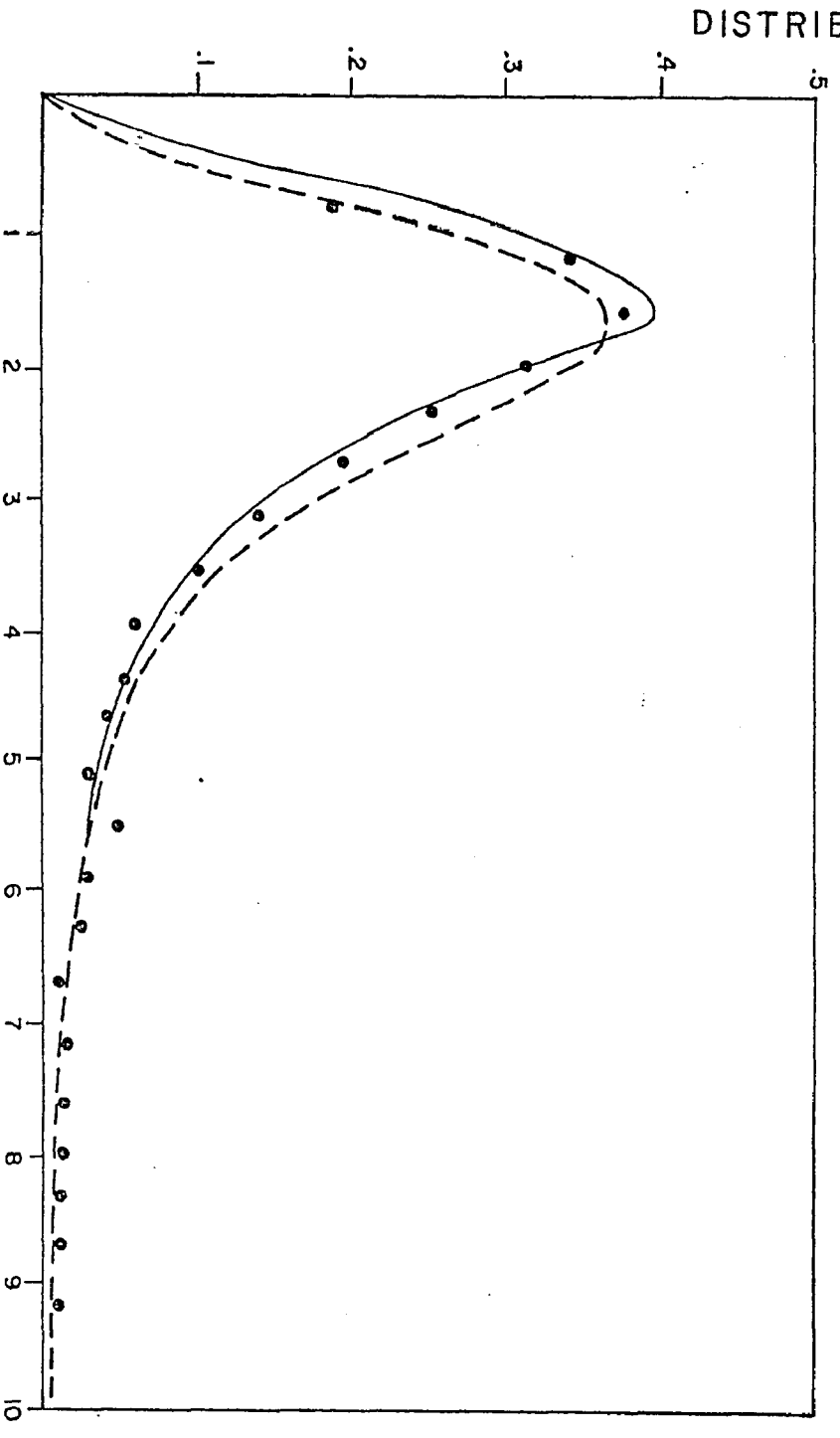


Figure 4d

particle Holtsmark distributions (22) is that the number of particles that give rise to the random force, not to be confused with the total number of particles in the system, is not known. This has forced us to generate clusters randomly from the same space density distribution as the evolved case. This is a static cluster and has no effect of dynamic evolution and no correlations between the stars. The model cluster so simulated satisfies the assumptions of Chandrasekhar, and we therefore expect the random force distribution that we obtain for them to be close to some $W_N(\rho)$. The simplest way to generate a cluster which has the same density distribution as the evolved cluster is to keep the distance of the stars from the origin the same but randomly choose a new direction. We have tried other methods resulting in the same results.

A number of experiments were performed on clusters with different initial conditions and total number of stars. We shall present here only representative results of three clusters containing 100, 300, and 1000 stars. The results of the random force distribution in the cluster containing 1000 stars are presented in Figure (2). These show the distribution of the random forces in four intervals of one-half the crossing time. The solid curves are the distributions of the force in the simulated theoretical cluster of the same density profile, and has no dynamic effects as explained earlier. The experimental results must be compared with this simulated curve and not with the Holtsmark distribution, which is merely plotted for visual orientation. The distribution of force for the

simulated cluster is quite similar to the finite-particle distribution W_N where N is of the order of 10. The close agreement with the simulated cluster is excellent in general. There is, however, a slight tendency for the experimental curve to have more of the higher forces, and the probability for low forces is somewhat lower. We will discuss the possible explanation of this effect later.

The results for the cluster containing 300 stars are given in Figure (3). The average of the distributions over a whole crossing time are plotted. Again, the experimental points must be compared with the distribution for the random force in the simulated cluster in those intervals. Here too the general trend of the experimental results is in close agreement with the simulated curve. It might be noted that both the experimental points and the simulated curve here are slightly higher than was the case in the cluster containing 1000 stars. This indicates, from the comparison with the finite-particle distribution in Figure (1), that fewer particles are giving rise to the random force. In the experimental results, there is a slight tendency towards low probability for the low forces and high probability for the high forces as compared with the theoretical curve. This is particularly so for the first two crossing times.

Only in the case of the cluster containing 100 particles are there any significant discrepancies between the distributions of the simulated and the experimental curves. The results for this case are plotted in Figure (4) for four intervals of

half a crossing time each. Even in this case, the agreement in the first and the fourth half-crossing time is quite good.

From the results of Figures (2) to (4) we see that although the agreement between the theoretical simulated distribution and the corresponding experimental is quite close, on occasion there is a tendency for the experimental clusters to have more stars with higher forces. This is indicated by the fact that the experimental force distribution is shifted slightly to the right and has a lower maximum than the distribution for the simulated clusters.

A possible cause for high forces could be the existence of binaries. A search for binaries was made in some cases where there was a significant deviation from the theoretical curve. A consistent definition of a binary in a cluster is not available. We have used a fairly liberal criterion to classify a pair of stars as a binary. A pair was counted as a binary if the mutual relative energy was negative and the separation was a fourth of the cluster radius. To eliminate weakly bound pairs, we required a minimum lifetime of an eighth of a crossing time. Even with this liberal criterion, the number of binaries in both the 300- and the 1000-body clusters were, on the average, about 10. In spite of the small number, the distribution of forces was tested for these cases after exclusion of the binaries. As would be expected, the resulting distribution was quite similar to the case when binaries were included. It is not intended to imply that the binaries are unimportant in the N-body theory. The results of

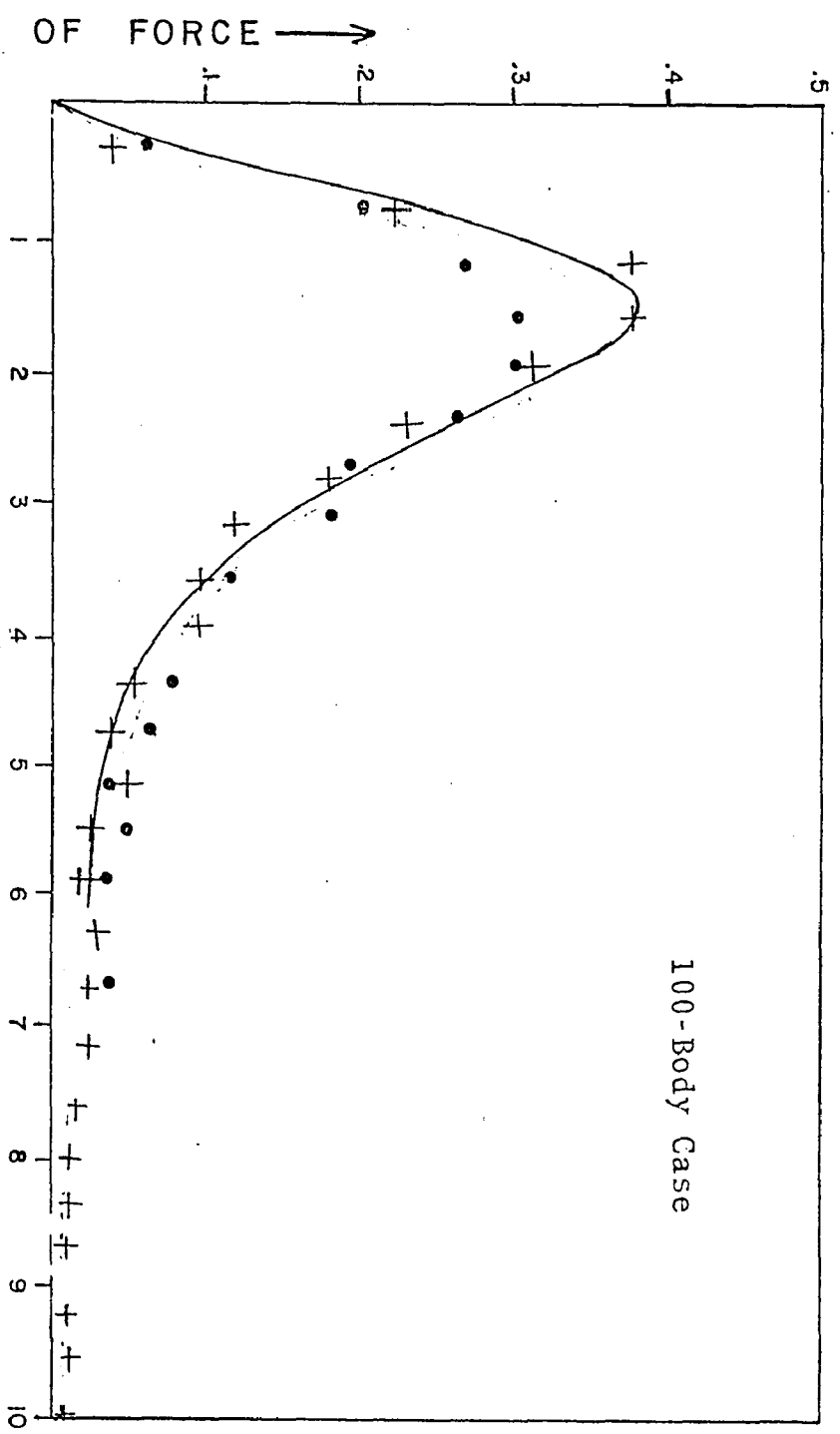


Figure 5

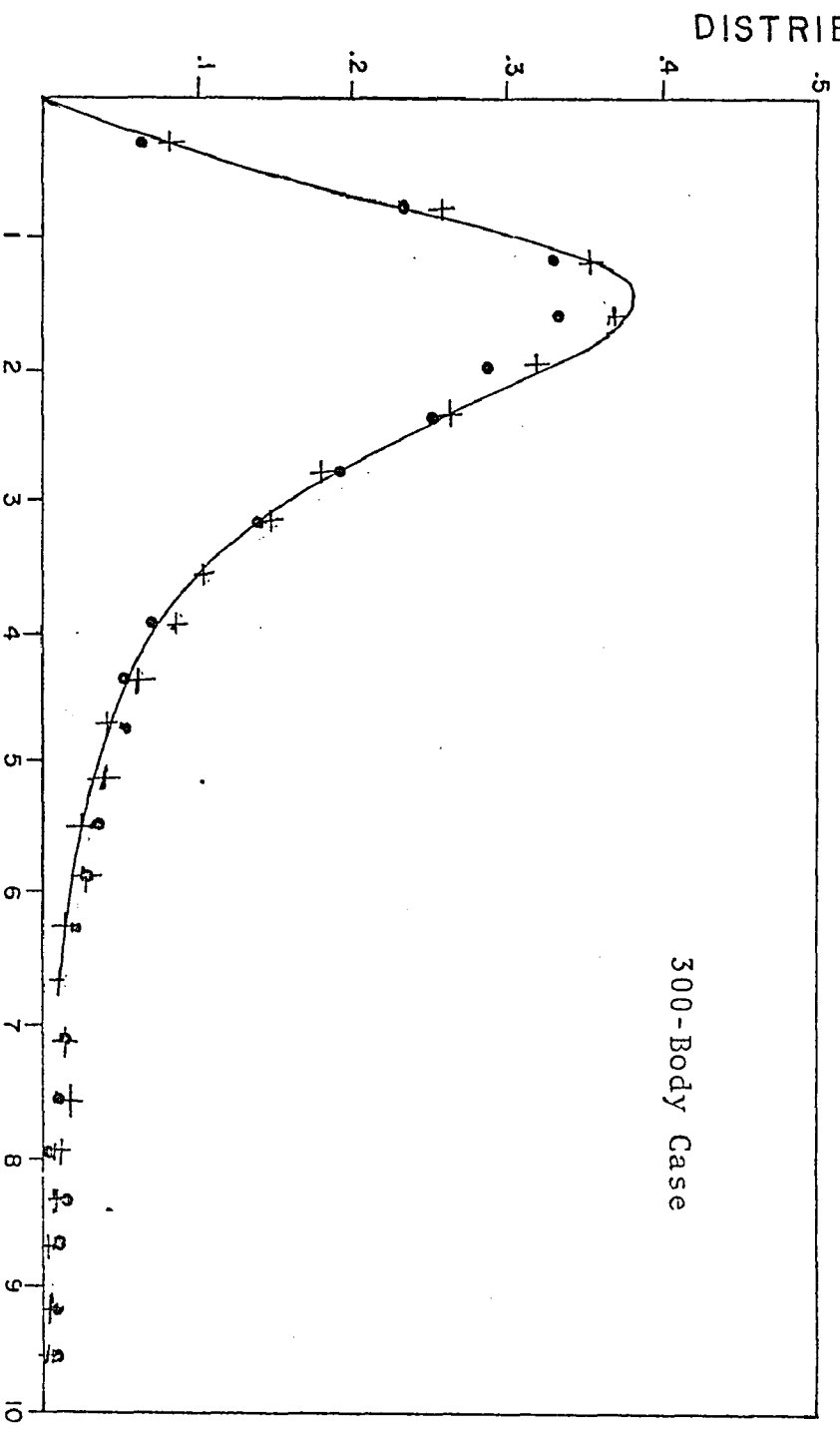


Figure 6

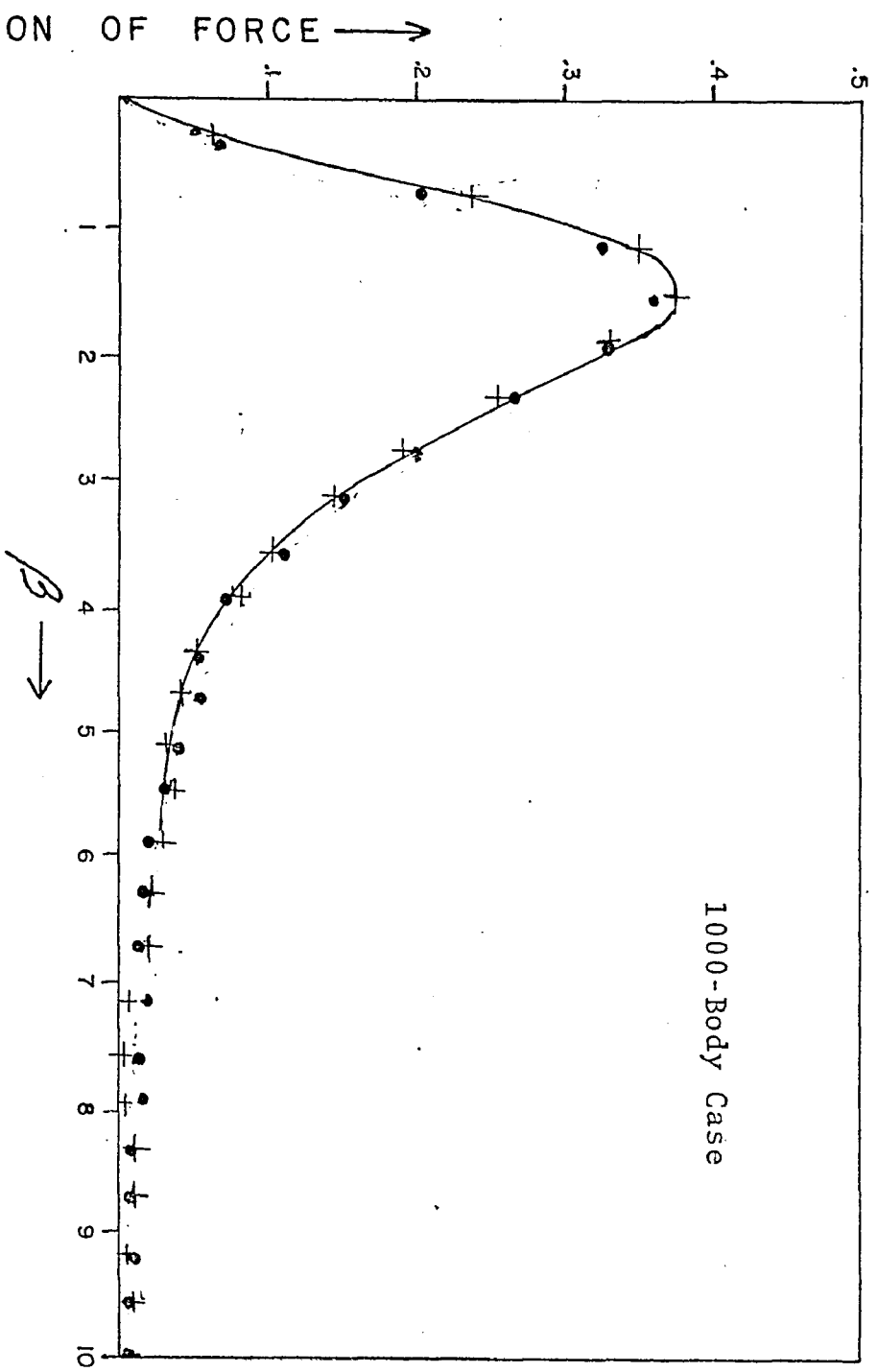
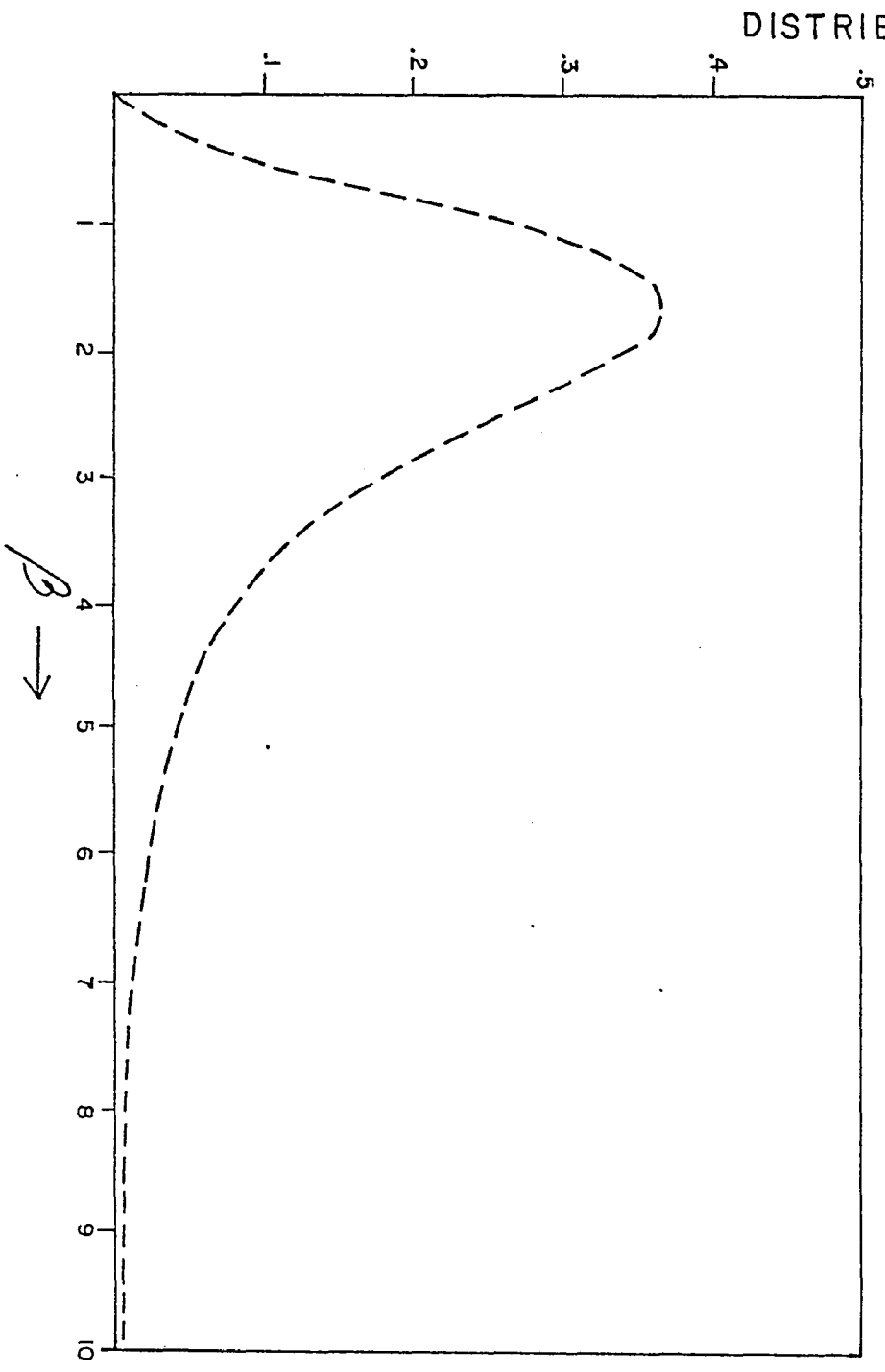


Figure 7



Aarseth (1968) indicate their importance, especially in cases where a mass spectrum is present.

Another possible cause of high forces may be what has been called "polarization". The cause of this is the fact that a star has the tendency to modify the distribution of surrounding stars by its presence. Gilbert (1970) predicted that this would enhance the force experienced by a star. The increase in force can be thought of as an increased effective mass. For the special case of Henon's isochrone model, Gilbert finds, for a star at the center of the cluster, this effective mass to be 1.75 times the real mass. Although this result refers to a very special case, an increase in mass for stars in our clusters would certainly improve the agreement between the theoretical curve and the experimental results, although the smallness of their deviations would indicate that the factor would be much smaller than 1.75.

A simple way to check whether there is any modification of the surrounding particles is to calculate the force at randomly selected points where there are no particles present. The distribution of the random force at these points can then be compared with the experimental results. To do this, we randomly selected points in the cluster, and the force due to all the stars was calculated, from which the random force was found by the method described earlier. The results of these experiments are given in Figures (5), (6), and (7) for 100, 300, and 1000 stars respectively.

The solid curve here is the force distribution for the

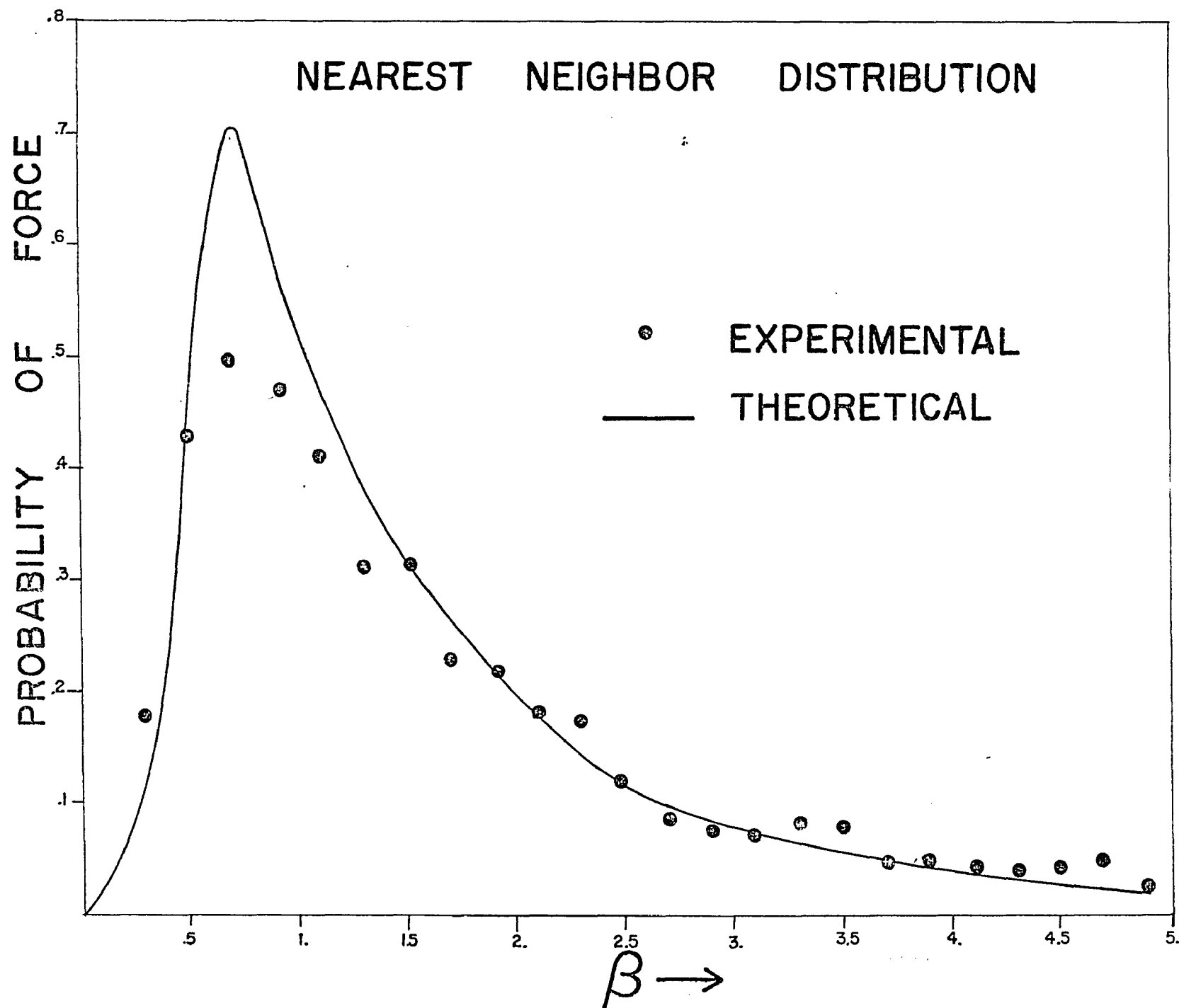


Figure 8

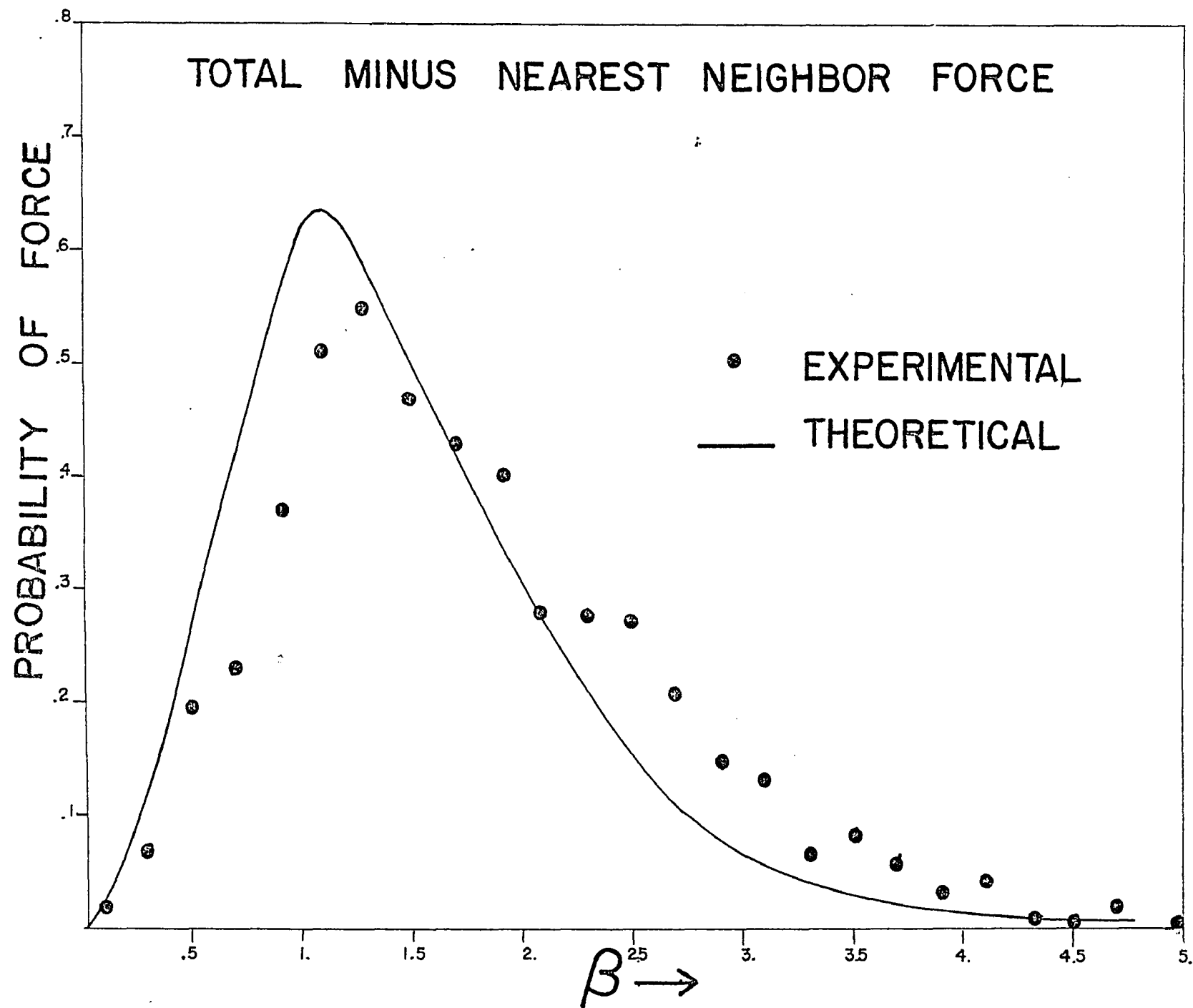


Figure 9

simulated cluster. The crosses represent the distribution of random points and the dots, the distribution due to stars. The distribution of the force at random points is very close to the simulated curve in each case. The discrepancy between the distribution of force at points where there are particles and the points where there are none indicates that there might be a modification of the field due to the presence of a star at a point.

We tested for the possibility that when the discrepancy between the theoretical curve and the experimental points appears, the effect is due to the possible enhancement of the force due to the nearest neighbors. This would be true, for example, if the particles in the system were undergoing numerous close collisions. The experiment we performed was to compare the distribution of the force for nearest neighbors for the real system with the "theoretical" distribution obtained by randomizing as described before. Figure (8) illustrates the results for one of the 100-body cases. Not surprisingly, the discrepancy still exists. We should point out that the simulated theoretical curve agrees almost identically with the theoretical nearest neighbor distribution given by (41).

But although the nearest neighbor accounts for some of the discrepancy, it does not account for all of it. What we also did was to look at the distribution of the total random force minus the nearest neighbor force. This is illustrated by Figure (9). As can be seen, there is a discrepancy there also. As the effect is not due, therefore, to the nearest neighbor

DISTRIBUTION OF ANGLES

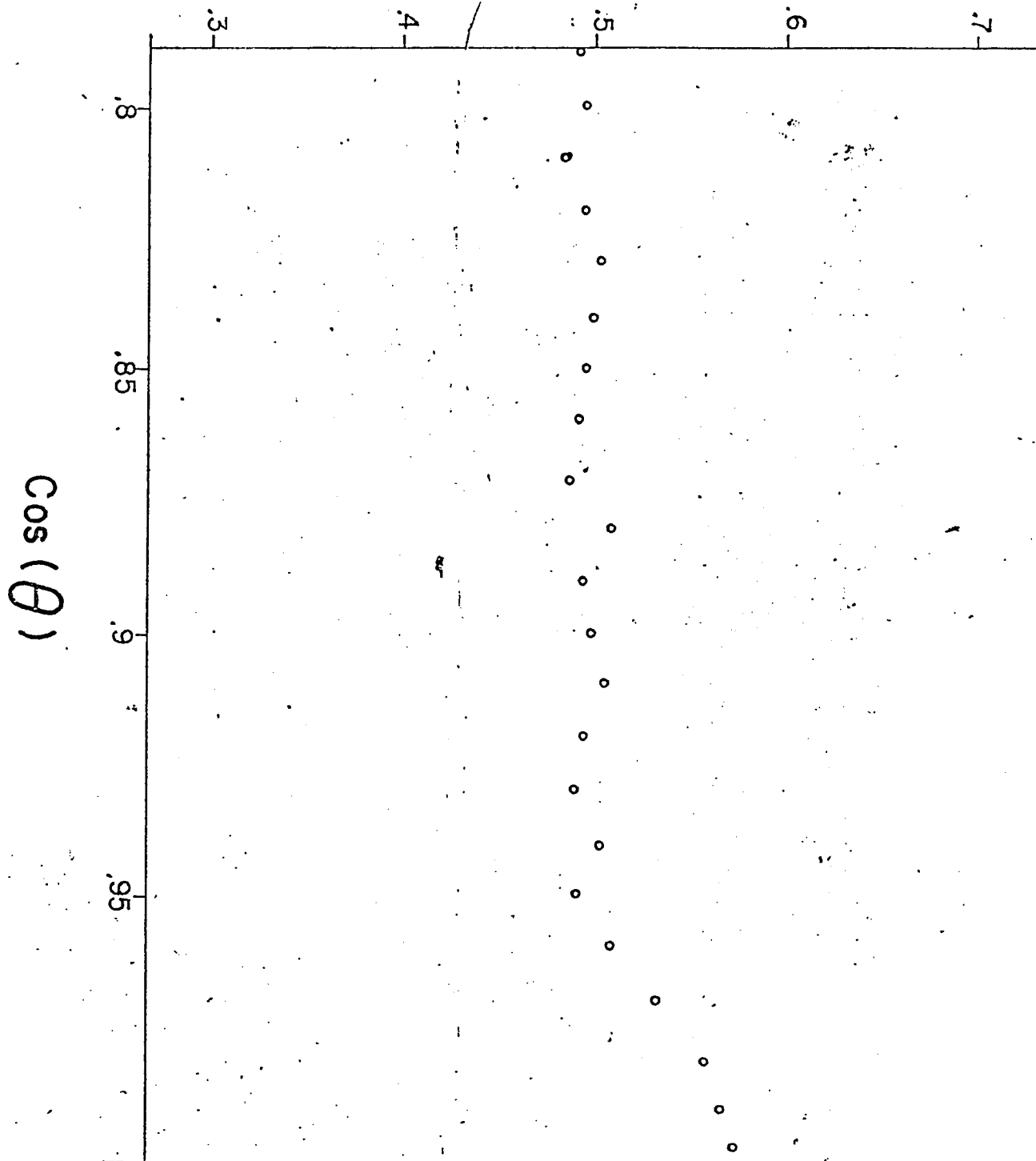


Figure 10

only, it may be due to clumping, or the formation of sub-systems.

Some recent studies have been concerned with the existence of correlation effects in gravitational systems (Gilbert 1968, 1970, 1972; Lerche 1971). Miller (1971a) has investigated possible correlation effects in a cluster evolved on a computer. In one of his experiments he studied the distribution of the cosines of angles between position vectors of pairs of particles. If there are no correlations or clusterings of stars, this angular distribution is uniform between -1 and $+1$, with a probability density of $.5$. We performed this experiment with our 300-body case for a period when there was a significant difference between the experimental and the theoretical curves (first crossing time). We calculated the cosines for pairs within a shell, because doing this for all the pairs in the cluster might hide possible clustering by including many particles with small angular separation but very distant from each other. Figure (10) shows the distribution of cosines for the 300-body case. The maximum of this distribution near 1 clearly indicates a correlative effect, possibly caused by the clustering of stars. This clustering may give rise to an increased probability for higher forces.

In conclusion, it must be emphasized that most of our results clearly conform to Chandrasekhar and von Neumann's theory. It is all the more remarkable in view of the simple assumption on which it is based.

Numerical experiments on the Holtsmark distribution have been reported by Hunger, Larenz and Wilke (1965) for the plasma case. Although they did not evolve the N-body system using the actual force law, they used straight line trajectories. Furthermore, their results are rather peculiar and at variance with ours. They obtained progressively worse agreement with the Holtsmark distribution as the number of particles increased. We performed similar experiments, that is, without evolution, and did not obtain their results. Since they did not describe their experimental procedure, we have no way of determining the cause of the discrepancy.

V. DISTRIBUTION OF THE TIME RATE OF CHANGE OF THE FORCE

The Holtsmark distribution is not sufficient for a description of many important physical quantities relating to the gravitational field. For example, the specification of the mean lifetime for a given force $\vec{F}$ not only requires the distribution of $\vec{F}$ but also the distribution of the time rate of change of $\vec{F}$, say, $\vec{f}$. Also, the main result of Chandrasekhar and von Neumann's theory (1943), the prediction of dynamical friction (to be discussed in the next chapter) depends on considerations of the distribution functions for both $\vec{F}$ and $\vec{f}$. The joint distribution of $\vec{F}, \vec{f}$ cannot be explicitly derived although the single distribution of $\vec{f}$ can be given. We will at this point give a brief outline of the basic method used by Chandrasekhar and von Neumann to derive the statistical properties of various quantities.

What we seek is the joint distribution of $\vec{F}$ and $\vec{f}$, given the probability distribution for position and velocity of the field particles. As in the derivation of the Holtsmark distribution, we circumscribe a sphere of radius R about the particles under consideration. Suppose that there are N particles distributed within the sphere. Then the force and its time derivative are given by

$$\vec{F} = -G \sum_{i=1}^N \frac{m_i}{r_i^3} \vec{r}_i \quad (1)$$

and

$$\vec{f} = -G \sum_{i=1}^N \frac{m_i}{\pi^3} \left(\vec{v}_i - 3 \frac{\vec{v}_i \cdot \vec{n}_i}{\pi^2} \vec{n}_i \right) \quad (2)$$

where $\vec{v}$ is the relative velocity of the i-th field star with respect to the test star. The joint probability of $\vec{F}$ and $\vec{f}$, $W(\vec{F}, \vec{f})$, can be given by

$$W(\vec{F}, \vec{f}) = \frac{1}{(2\pi)^6} \iint A(\vec{p}, \vec{\sigma}) d\vec{p} d\vec{\sigma} e^{-i\vec{p} \cdot \vec{F} - i\vec{\sigma} \cdot \vec{f}} \quad (3)$$

where A is the joint characteristic function

$$A(\vec{p}, \vec{\sigma}) = \langle e^{i\vec{F} \cdot \vec{p} + i\vec{\sigma} \cdot \vec{f}} \rangle \quad (4)$$

or

$$A(\vec{p}, \vec{\sigma}) = \iint e^{i\vec{F}(\vec{n}) \cdot \vec{p} + i\vec{\sigma} \cdot \vec{f}(\vec{n}, \vec{v})} P(\vec{n}, \vec{v}) d\vec{n} d\vec{v} \quad (5)$$

$P(r, v)$ is the joint probability distribution of position and velocity for the N field particles.

Before (5) can be evaluated, we must specify $P(\vec{r}, \vec{v})$. The assumption is that the particles are uncorrelated and also that each particle's velocity and position are uncorrelated. We therefore have

$$P(\vec{\pi}, \vec{v}) = \prod_{i=1}^N P'(\vec{\pi}_i, \vec{v}_i) \quad (6)$$

where $P'(r, v)$ is the single particle probability distribution.

Furthermore, it is assumed that the positions of the particles are uniformly distributed and that the distribution of velocities is Gaussian. That is

$$\begin{aligned} P'(\vec{\pi}, \vec{v}) &= \frac{1}{V} \tau(\vec{v}) \\ &= \frac{V^3}{V} \frac{e^{-\frac{1}{2}v^2}}{(\pi)^{3/2}} \end{aligned} \quad (7)$$

where V is the volume of the sphere. Using similar logic that led to (16) of Chapter IV we have

$$A_N(\vec{\rho}, \vec{\sigma}) = A^N(\vec{\rho}, \vec{\sigma}) \quad (8)$$

Since

$$\iint P'(\vec{\pi}, \vec{v}) d\vec{\pi} d\vec{v} = 1$$

equation (8) can be written as

$$A_N(\vec{\rho}, \vec{\sigma}) = \left[1 + \frac{1}{V} \int e^{i\vec{\rho} \cdot \vec{F} + i\vec{\sigma} \cdot \vec{f}} \frac{\tau(\vec{v})}{V} d\vec{\pi} d\vec{v} \right]^N \quad (9)$$

If the limit is now taken as N and V tend to infinity, keep-

ing the number density, $n = N/V$ constant, we obtain

$$A(\vec{p}, \vec{\sigma}) = \lim_{N, V \rightarrow \infty} \left[1 - \frac{n}{N} \int (1 - e^{i\vec{p} \cdot \vec{F} + i\vec{\sigma} \cdot \vec{f}}) \tau d\vec{n} d\vec{v} \right]^N \quad (10)$$

$$= e^{-nC(\vec{p}, \vec{\sigma})} \quad (11)$$

where

$$C(\vec{p}, \vec{\sigma}) = \iint [1 - e^{i(\vec{p} \cdot \vec{F} + \vec{\sigma} \cdot \vec{f})}] \tau d\vec{n} d\vec{v} \quad (12)$$

To obtain $W(\vec{F}, \vec{f})$ one must hence evaluate (12). Unfortunately, this is not possible in terms of known functions. But Chandrasekhar and von Neumann were able to obtain a series expansion, up to the second order in $\vec{p}$ and $\vec{\sigma}$, for A . This allows the derivation of dynamical friction and the mean lifetime for a given force. Although $W(\vec{F}, \vec{f})$ cannot be obtained explicitly, the distribution of f alone can be. The distribution of f is, of course, equal to

$$\begin{aligned} P(\vec{f}) &= \int W(\vec{F}, \vec{f}) d\vec{F} \\ &= \frac{1}{(2\pi)^6} \int e^{-i\vec{F} \cdot \vec{p} - i\vec{f} \cdot \vec{\sigma}} A(\vec{p}, \vec{\sigma}) d\vec{p} d\vec{\sigma} d\vec{F} \\ &= \frac{1}{(2\pi)^3} \int \delta(\vec{p}) e^{-i\vec{f} \cdot \vec{\sigma}} A d\vec{p} d\vec{\sigma} \end{aligned}$$

$$P(\vec{f}) = \frac{1}{(2\pi)^3} \int e^{-i\vec{f} \cdot \vec{\sigma}} A(0, \vec{\sigma}) d\vec{\sigma} \quad (13)$$

A lengthy calculation yields the following result

$$P(\vec{f}) = \frac{1}{\pi^2} \frac{c^2}{(c^2 + f^2)^2} \quad (14)$$

where

$$c = \frac{2}{3} \pi^{3/2} G Q_0 m n \langle v \rangle \quad (15)$$

and where

$$Q_0 = 1 + \frac{1}{2\sqrt{3}} \log(2 + \sqrt{3}) = 1.38013$$

Now the distribution for the magnitude of f is given by

$$P(|\vec{f}|) = \frac{4}{\pi} \frac{f^2 c}{(c^2 + f^2)^2} \quad (16)$$

Since the probability distribution $P(f)$ depends only on f

$$P(|\vec{f}|) = 4\pi f^2 P(\vec{f}) \quad (17)$$

If $\vec{f}$ is expressed in units of c , and $\alpha = f/c$, then the

PROBABILITY OF $|f|$

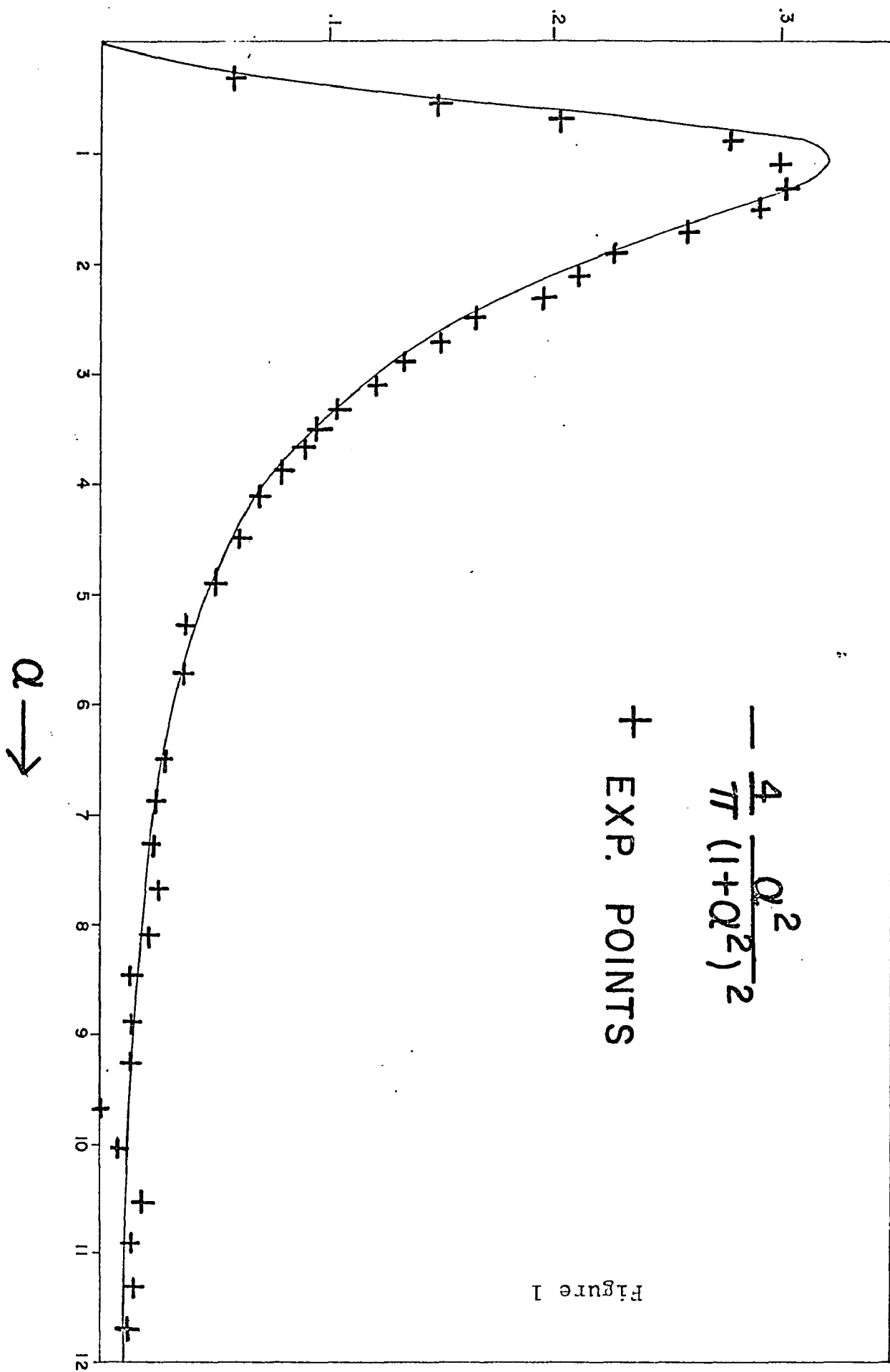


Figure 1

probability distribution of α becomes

$$P(\alpha) = \frac{4}{\pi} \frac{\alpha^2}{(1 + \alpha^2)^2} \quad (18)$$

To experimentally test the distribution given by (18), we divided our system into spherical shells and for each shell the density and the root mean square velocity of the particles was obtained. This allows the determination of c for each shell. The time derivative of the forces was obtained by numerical differentiation and then depending on which shell the particle was in, the time derivative was divided by the appropriate c . The statistics were then combined for various fractions of a crossing time. We also performed the experiment by surrounding each particle by a sphere which on the average contained about 20 particles and calculated the force derivatives directly from the analytical formula. We obtained substantially the same results. Figure (1) presents the results of this experiment for the 1000-body case. As can be seen, the experimental results are in excellent agreement with the theory.

VI. DYNAMICAL FRICTION

A remarkable result of the theory of Chandrasekhar and von Neumann (1943) is the prediction of the existence of dynamical friction in gravitational systems. By dynamical friction is meant the tendency of a particle to be preferentially decelerated in the direction of its motion. We shall, in this Chapter, describe experiments which demonstrate the existence of dynamical friction. As Chandrasekhar and von Neumann's derivation is long and complicated, we shall give a very simple derivation for the case of a test particle influenced only by one field particle. As we shall see, the essence of the result will be clear from this simple example. Furthermore, it will give us a feeling for the finite-N effect.

Before we proceed to our example, we shall briefly outline the derivation of Chandrasekhar and von Neumann. As mentioned in the previous Chapter, an explicit evaluation of $A(\vec{\rho}, \vec{v})$ is not possible but a series expansion up to the second order in $\vec{\rho}$ and $\vec{v}$ was obtained. It is

$$A(\vec{\rho}, \vec{v}) = e^{iC(\vec{\rho}, \vec{v})}$$

where

$$\begin{aligned} C(\vec{\rho}, \vec{v}) = & \frac{4}{15} (2\pi)^{3/2} G^{3/2} m \rho^{3/2} - \frac{2\pi i}{3} G m (\sigma_1 v_1 + \sigma_2 v_2 - 2\sigma_3 v_3) \\ & + \frac{1}{4} (2\pi)^{3/2} G^{1/2} m^{1/2} \langle u^2 \rangle \rho^{-3/2} (\sigma_1^2 + \sigma_2^2) \\ & + \frac{3}{28} (2\pi)^{3/2} G^{1/2} m^{1/2} \rho^{-3/2} [\sigma_1^2 (5v_1^2 + 4v_2^2 - 2v_3^2) \\ & + \sigma_2^2 (4v_1^2 + 5v_2^2 - 3v_3^2) + \sigma_3^2 (4v_3^2 - 2v_1^2 - 2v_2^2) \\ & - 8\sigma_2 \sigma_3 v_2 v_3 - 8\sigma_3 \sigma_1 v_3 v_1 + 2\sigma_1 \sigma_2 v_1 v_2] + O(\sigma^3) \end{aligned}$$

where the z-axes of $\vec{\sigma}$ and $\vec{v}$ are taken to be in the direction of $\vec{\rho}$ and where $\vec{u}$ and $\vec{v}$ are the velocities of the field star and the test star respectively. This expansion is sufficient for the calculation of the first two moments of $\vec{f}$. In particular, if $W(\vec{F}, \vec{f})$ is the joint distribution of $\vec{F}$ and $\vec{f}$, then the expected value of $\vec{f}$ for a given $\vec{F}$ is the following

$$\langle \vec{f} \rangle = \frac{\int \vec{f} W(\vec{F}, \vec{f}) d\vec{f}}{W(\vec{F})} \quad (1)$$

Substituting W from (3) of Chapter V, we obtain

$$\langle \vec{f} \rangle = -\frac{1}{8\pi} \int e^{-i\vec{\rho} \cdot \vec{F}} \frac{\partial}{\partial \vec{\sigma}} A(\vec{\rho}, \vec{\sigma}) \Big|_{\vec{\sigma}=\vec{v}} d\vec{\rho} d\vec{\sigma} \quad (2)$$

Evaluation of (2) using the series expansion for $A(\vec{\rho}, \vec{\sigma})$ yields the following result for the expected value of the time rate of change of the force for a given force strength, and a given test particle velocity $\vec{v}$

$$\begin{aligned} \langle \vec{f} \rangle_{\vec{F}, \vec{v}} &= \left\langle \frac{d\vec{F}}{dt} \right\rangle \\ &= -\frac{2}{3} \pi G m m B(\beta) \left(\vec{v} - \frac{\vec{v} \cdot \vec{F}}{F} \frac{\vec{F}}{F} \right) \end{aligned} \quad (3)$$

where

$$B(\beta) = \frac{3 \int_0^\beta H(\beta) d\beta}{\beta H(\beta)} - 1 \quad (4)$$

$$\beta = \frac{F}{Q} \quad (5)$$

and where $H(\beta)$ is the Holtsmark distribution.

Before showing how equation (3) implies the existence of dynamical friction, we shall derive an equation similar to (3) for a simple case. Consider a particle located momentarily at the origin and moving with velocity $\vec{v}$. Suppose there is another particle at location $\vec{r}$ and with a velocity $\vec{u}$. For the force and the time derivative of the force we have

$$\vec{F} = \frac{Gm\vec{r}}{r^3} \quad (6)$$

$$\dot{\vec{F}} = \dot{\vec{F}} = Gm \left(\frac{\vec{V}}{r^3} - 3 \frac{\vec{r}(\vec{r} \cdot \vec{V})}{r^5} \right) \quad (7)$$

where $\vec{V}$ is the relative velocity

$$\vec{V} = \vec{u} - \vec{v} \quad (8)$$

From (6) we have

$$\vec{r} = \frac{(Gm)^{1/2} \vec{F}}{F^{3/2}} \quad (9)$$

and

$$\mu = \frac{\sqrt{Gm}}{F^{1/2}} \quad (10)$$

Substituting (9) and (10) into equation (7)

$$\vec{f} = \frac{F^{3/2}}{(Gm)^{3/2}} \left(\vec{V} - \vec{F} \frac{\vec{V} \cdot \vec{F}}{F^2} \right) \quad (11)$$

Thus far, we have not brought in any probabilistic considerations, Equation (11) is purely dynamical, but the obvious similarity between (11) and (3) should be noted. Suppose there is a probability distribution of $\vec{F}$ and $\vec{u}$, $P(\vec{F}, \vec{u})$; then the expected value of f for a given $\vec{F}$ is

$$\langle \vec{f} \rangle_{\vec{F}, \vec{v}} = \frac{\int \vec{f} P(\vec{F}, \vec{u}) d\vec{u}}{\int P(\vec{F}, \vec{u}) d\vec{u}} \quad (12)$$

If we further assume that $\vec{F}$ and $\vec{u}$ are not correlated, then

$$P(\vec{F}, \vec{u}) = P(\vec{F}) P(\vec{u}) \quad (13)$$

and (12) reduces to

$$\langle \vec{f} \rangle_{\vec{F}, \vec{v}} = \int \vec{f} P(\vec{u}) d\vec{u} \quad (14)$$

$$\langle \vec{f} \rangle_{\vec{F}, \vec{v}} = \frac{F^{3/2}}{(Gm)^{3/2}} \left(\langle \vec{v} \rangle - \vec{F} \frac{\vec{F} \cdot \langle \vec{v} \rangle}{F^2} \right) \quad (15)$$

Now, if the probability distribution of $\vec{u}$ depends only on its magnitude, we have

$$\langle \vec{v} \rangle = \langle \vec{u} \rangle - \vec{v} \quad (16)$$

but

$$\langle \vec{u} \rangle = \int \vec{u} P(u) d\vec{u} \quad (17)$$

which equals zero by symmetry and therefore

$$\langle \vec{v} \rangle = -\vec{v}$$

Substituting in (15) we obtain

$$\langle \vec{f} \rangle_{\vec{F}, \vec{v}} = -\frac{F^{3/2}}{(Gm)^{1/2}} \left(\vec{v} - \frac{\vec{v} \cdot \vec{F}}{F^2} \vec{F} \right) \quad (18)$$

The only difference between (18) and (3) is the coefficient multiplying the expression in brackets. We should point out that, using the asymptotic values of $H(\beta)$, (18) and (3) are identical for high β . Also, it is worth noting that (18) is independent of the probability distribution of the force.

Let us write (3) and (18) in the form

$$\langle \vec{f} \rangle_{\vec{F}, \vec{v}} = -D(|\vec{F}|) \left(\vec{v} - \frac{\vec{v} \cdot \vec{F}}{F^2} \vec{F} \right) \quad (19)$$

where $D(|\vec{F}|)$ is the appropriate function multiplying (3) or (18). We first note that $D(|\vec{F}|)$ is a positive function of F . To illustrate more directly the effect of dynamical friction we proceed as Chandrasekhar and take the dot product of (19) with $\vec{F}$.

$$\vec{F} \cdot \langle \vec{f} \rangle_{\vec{F}, \vec{v}} = -2 D(F) \frac{\vec{v} \cdot \vec{F}}{F^2} \quad (20)$$

but

$$\vec{F} \cdot \langle \vec{f} \rangle_{\vec{F}, \vec{v}} = \vec{F} \cdot \langle \dot{\vec{F}} \rangle = |\vec{F}| \left\langle \frac{d|\vec{F}|}{dt} \right\rangle \quad (21)$$

and therefore we find that

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle = 2 D(F) \frac{\vec{v} \cdot \vec{F}}{F} \quad (22)$$

Now, since $D(F)$ is always positive, we have the remarkable result that

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle > 0 \quad \vec{v} \cdot \vec{F} > 0 \quad (23)$$

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle < 0 \quad \vec{v} \cdot \vec{F} < 0 \quad (24)$$

That is, if $\vec{F}$ has a positive component in the direction of $\vec{v}$, then the force will tend to increase; and conversely, if $\vec{F}$ has a negative component, it will tend to decrease.

We performed the experiment to test for dynamical friction in two different ways. As in Chapter V, the derivative of the force was calculated both by numerical differentiation and by surrounding the particle with a sphere containing from 10 to 20 particles and calculating the force derivative analytically. The results to be described were substantially the same in either case.

The experiments performed were to test directly for equation (22). As can be seen, $\langle \frac{d|\vec{F}|}{dt} \rangle$ is a function of three variables, the magnitudes of $\vec{v}$ and $\vec{F}$ and the angle included between $\vec{v}$ and $\vec{F}$. The equation was tested by integrating out two of the variables and examining $d|\vec{F}|/dt$ against the remaining one.

By $\langle d|\vec{F}|/dt \rangle_{\vec{F}}$ we shall mean equation (22) when the angle and velocity have been integrated out. Similarly for $\langle d|\vec{F}|/dt \rangle_{\vec{v}}$. Now, in integrating out the angle, one cannot average over the whole range of 0 to π since that would obviously give zero. What must be done is to average the cosine from 0 to $\pi/2$ and in the experiment use only those particles which have a cosine

within that range. In practice, of course, not to waste the statistics for half the particles when the cosine is between $\pi/2$ and π , the sign of $d|F|/dt$ is changed and is counted in the same statistics. This is under the assumption that the cosine between v and F is uniformly distributed, and this is indeed the case experimentally. Now

$$\langle \cos \theta \rangle_{0 \rightarrow \frac{\pi}{2}} = \frac{\int_0^{\pi/2} \cos \theta \sin \theta d\theta}{\int_0^{\pi/2} \cos \theta d\theta} = \frac{1}{2} \quad (25)$$

Therefore equation (22) can be written as a function of the magnitudes of velocity and force

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle_{F,v} = D(F) v \quad (26)$$

If one assumes a Maxwellian distribution for velocity, then the integration over velocity is straightforward

$$\begin{aligned} \langle v \rangle &= \frac{4j^3}{(\pi)^{1/2}} \int_0^{\infty} v^3 e^{-j^2 v^2} dv \\ &= \frac{2}{\sqrt{\pi} j} \end{aligned} \quad (27)$$

and where j is related to the average of the square of the velocity by

$$j^2 = \frac{3}{2} \frac{1}{\langle v^2 \rangle} \quad (28)$$

We thus have

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle_{|\vec{F}|} = \frac{4\sqrt{\pi}}{3J} G_{mn} B(\beta) \quad (29)$$

for the infinite-N constant density case, and

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle_{|\vec{F}|} = \frac{2}{15J} (\pi G_{mn})^{\frac{1}{2}} F^{3/2} \quad (30)$$

for the one-particle case. By dividing and multiplying equation (30) by $Q^{3/2}$, it can be expressed as

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle_{|\vec{F}|} = \frac{16\sqrt{2}\pi}{15J} G_{mn} \beta^{3/2} \quad (31)$$

The normalization we have chosen is to express $d|\vec{F}|/dt$ in units of $\frac{4}{3\sqrt{\pi}} \frac{G_{mn}}{J}$ in which case equations (29) and (31) become

$$\mathcal{F}_{\infty}(|\vec{F}|) = B(\beta) \quad (32)$$

$$\mathcal{F}_1 = \frac{4\sqrt{2}}{5\sqrt{\pi}} \beta^{3/2} = 2.0053 \beta^{3/2} \quad (33)$$

The experiment was performed by taking particles which fell between two spherical shells of the system. This is

necessary since the number density is not a constant throughout the cluster.

For each particle, $\mathcal{J}_\infty$ was calculated (taking into account the sign of $\vec{v} \cdot \vec{F}$ as discussed previously) and was put in the appropriate force box. The average for each force box was then taken. This then gives $\mathcal{J}_\infty(F)$ as a function of F . We should point out that the experiment was done both by calculating j via $\langle v^2 \rangle$ and by calculating $\langle v \rangle$ directly. This did not make any significant difference. Figure (1) illustrates the results. The solid curve is $B(\beta)$ corresponding to the theory of Chandrasekhar and the dashed curve is what is expected from the one-particle distribution.

To test for $\langle d|\vec{F}|/dt \rangle$ as a function of the velocity we run into a slight problem. In equation (26) one cannot integrate out the force since the integral of D with the appropriate distribution function yields infinity; this is true for both the infinite-particle case and for the one-particle case. What we have chosen to do therefore is to look at particles only up to a certain value of the force. For practical reasons we use a maximum of 10.7 for the force when it is expressed in units of Q . This involves 97% of the particles. The expected value of $B(\beta)$ in that range is given by

$$\langle B(\beta) \rangle = \frac{\int_0^{10.7} B(\beta) H(\beta) d\beta}{\int_0^{10.7} H(\beta) d\beta} = 3.89 \quad (34)$$

where $H(\beta)$ is the Holtsmark distribution. The integrals were,

of course, done numerically. For the one-particle case we have to evaluate the expected value of $\beta^{3/2}$ in that range with the probability distribution given by (39) of Chapter IV.

$$P_1(\beta) = \frac{15}{8} \frac{\sqrt{2}}{\sqrt{\lambda}} \beta^{-5/2} \quad (35)$$

hence

$$\langle \beta^{3/2} \rangle = \frac{\int_0^{10.7} \beta^{3/2} P_1(\beta) d\beta}{\int_0^{10.7} \beta P_1(\beta) d\beta} = 3.65170 \quad (36)$$

We therefore have for the N equal to infinity case

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle_v = bv \quad (37)$$

where

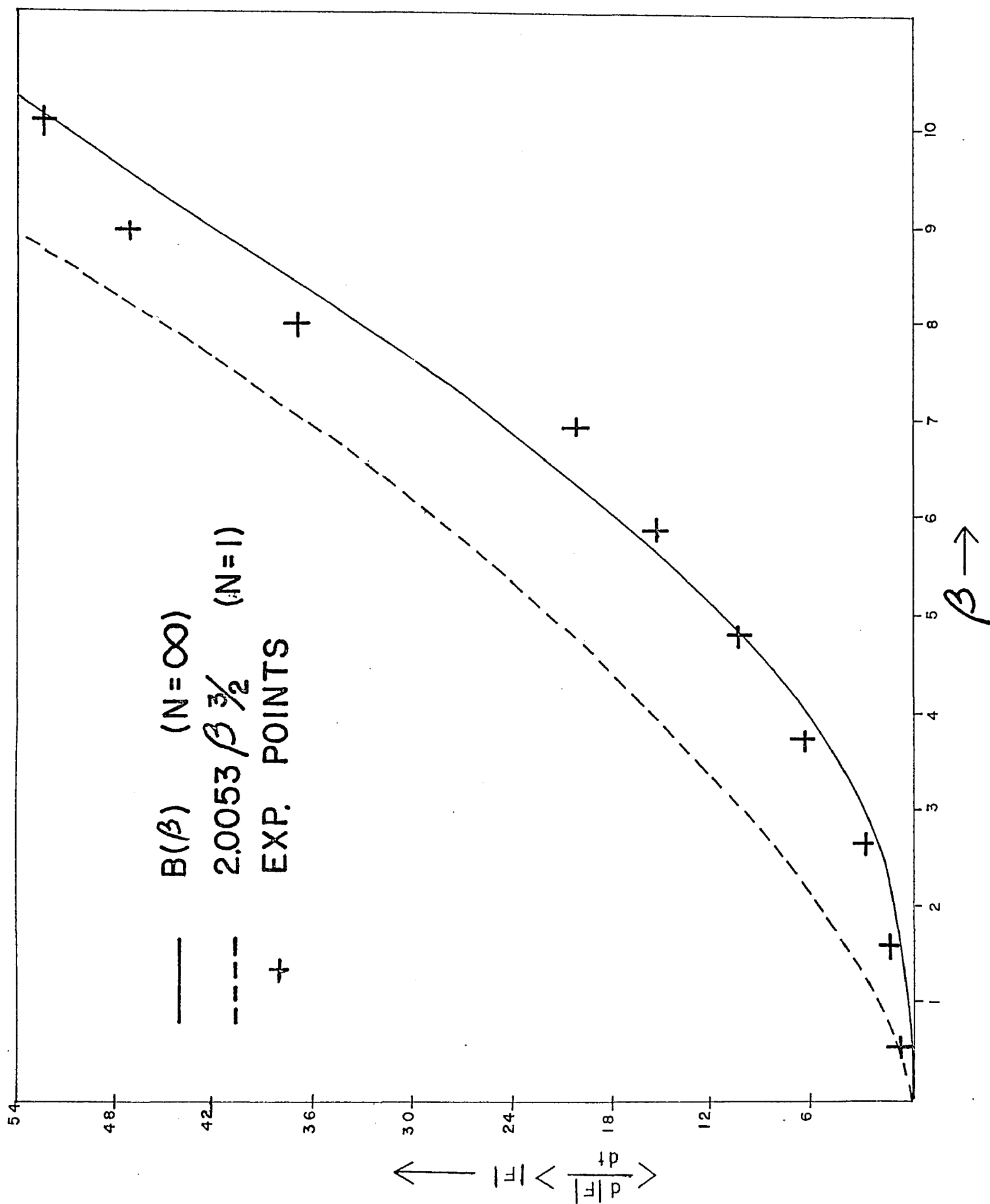
$$b = \frac{2}{3} \pi G m (3.89) n$$

and

$$\left\langle \frac{d|\vec{F}|}{dt} \right\rangle_v = 1.88bv \quad (38)$$

for the one-particle case. We have chosen the normalization such that the slope for the infinite-particle case is 1 and we therefore have for the infinite-particle case

Figure 1



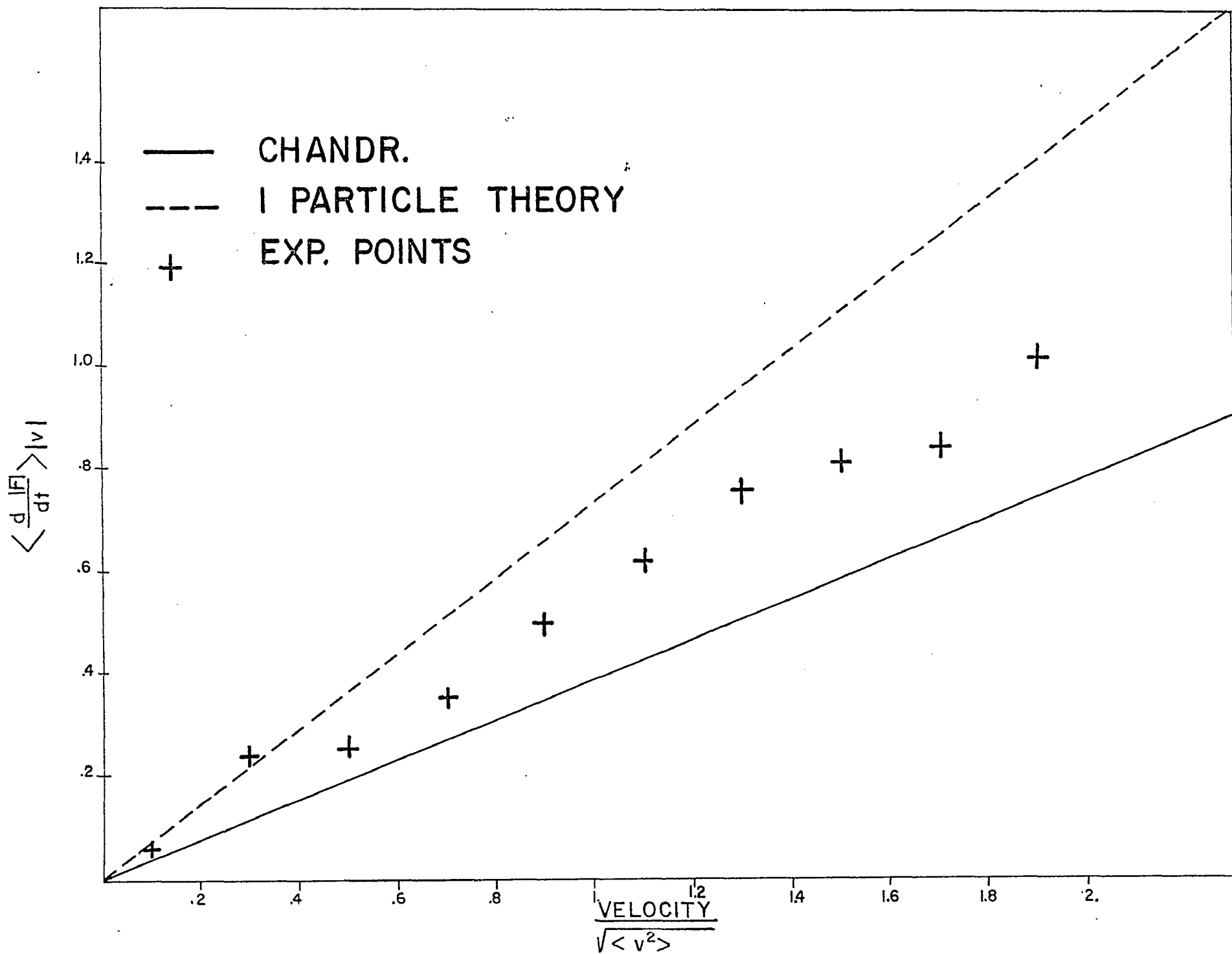


Figure 2

$$\mathcal{F}_{\infty}(v) = \frac{1}{b} \frac{d|\vec{F}|}{dt} = v$$

and

$$\begin{aligned} \mathcal{F}_1(v) &= \frac{1}{b} \frac{d|\vec{F}|}{dt} \\ &= 1.88 v \end{aligned}$$

(39)

The experiment to test for the v dependence is identical to that for the F dependence except the roles of F and v are interchanged. Furthermore, of course, a cutoff at $\beta=10.7$ was made. Figure (2) illustrates the result.

The slight discrepancy between the experimental points and Chandrasekhar's result must be attributed to the finite- N property of our system. But, as the figures illustrate, there is no question but that there exists dynamical friction in gravitational systems!

VII. AUTO-CORRELATION

The auto-correlation function plays a central role in the theory of stochastic processes. Chandrasekhar (1943) has derived the two-time correlation function for the random force, with the surprising result that for a large time difference it fell off only as $1/t$. This extremely long memory has been an odd feature of the theory (Prigogine 1966). In particular, the root mean square velocity deflection can be expressed as an integral of the correlation function with respect to time. But, of course, the $1/t$ dependence leads to a divergence. We shall, in this chapter, perform numerical experiments to explain the time behavior of the correlation function. We shall then derive the auto-correlation function by properly taking into account the cutoff at small and large distances. We will see that the cutoff for small distances prevents the divergence at zero time, and the cutoff for large distances makes the auto-correlation function fall off faster than $1/t$.

The customary definition of the auto-correlation function is

$$A(t_1, t_2) = \langle \vec{F}(t_1) \cdot \vec{F}(t_2) \rangle \quad (1)$$

where $\vec{F}(t_1)$ and $\vec{F}(t_2)$ are the forces at two different times. Chandrasekhar assumed that the particles moved in straight lines. The auto-correlation function is then only a function of $t=t_2-t_1$ and no generality is lost if we take $t_1=0$. We

hence have

$$A(t) = \langle \vec{F}(0) \cdot \vec{F}(t) \rangle \quad (2)$$

with

$$\begin{aligned} \vec{F}(0) &= \sum_{i=1}^N G \frac{m_i \vec{r}_i}{r_i^3} \\ \vec{F}(t) &= G \sum m_i \frac{\vec{r}_i + \vec{v}_i t}{|\vec{r}_i + \vec{v}_i t|^3} \end{aligned} \quad (3)$$

As we know from Chapter IV, the expected value of the square of the force diverges and therefore

$$A(0) = \infty$$

To avoid this and to make the theory appear as similar as possible to Brownian motion, Chandrasekhar defined the correlation coefficient as

$$R(t) = \left\langle \frac{\vec{F}(0) \cdot \vec{F}(t)}{F^2} \right\rangle$$

and more specifically, after a change in units, $R(t)$ is defined as

$$\bar{R}(\tau) = \left\langle \frac{\vec{\beta}(0) \cdot \vec{\beta}(\tau)}{\beta^2} \right\rangle \quad (4)$$

where β is the force measured in units of Q , and τ is the time in units of t_0 where

$$\begin{aligned}
 t_0 &= \ell_J \\
 \ell &= \left(\frac{15}{4}\right)^{1/3} \frac{n^{-1/3}}{(2\pi)^{1/3}} \\
 J &= \sqrt{\frac{3}{2}} \frac{1}{V_T}
 \end{aligned} \tag{5}$$

n is the number density of the stars and V_T is the average root mean square velocity of the stars. A lengthy derivation leads to the following result for the auto-correlation for a star with a force β and a velocity v

$$\begin{aligned}
 R(\tau, \beta, v) &= \frac{\sqrt{2\pi} 15}{\beta^2 H(\beta)} \int_0^\infty e^{-\left(\frac{x}{\beta}\right)^{3/2}} x J_{3/2}(x) \gamma\left(\frac{x}{\tau\beta^2}, v\right) dx \\
 v &= Jv
 \end{aligned} \tag{6}$$

where

$$\gamma(y, v) = \frac{1}{\sqrt{\pi}} \int_0^\infty d\lambda e^{-\lambda^2 v^2} \frac{\lambda}{v} \sin(2\lambda v) \alpha\left(\frac{y}{\lambda^2}\right) d\lambda$$

where

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

and

$$\alpha(x) = \int_0^x \frac{J_{3/2}(y)}{y^2} dy$$

For the purpose of comparison with numerical experiments, it is more convenient to use the function $\bar{R}$ which can be obtained

from R by averaging out the force and velocity over their respective distributions. In the following we briefly give different steps in the derivation of (6) by Chandrasekhar.

The problem here is, given the density of the stars per unit volume n and the distribution of relative velocities V , to find the joint distribution for $F(0)$ and $F(t)$ given by (3).

An expression for this joint distribution can be written down, but analytically, it is not in very useful form. Since we are only interested in the average value of the normalized scalar product of the force at two different times it is more useful for us to obtain an expression for the characteristic function. Using the method similar to the one used in deriving the characteristic function for the joint distribution for the random force and its first derivative in Chapter V, Chandrasekhar found the characteristic function to be

$$A(\vec{\rho}, \vec{\sigma}) = e^{iC(\vec{\rho}, \vec{\sigma})} \quad (7)$$

where

$$C(\vec{\rho}, \vec{\sigma}) = n \iint \left[1 - e^{iGm \left\{ \frac{\vec{n} \cdot \vec{\rho}}{n^3} + \frac{(\vec{n} + \vec{v}t) \cdot \vec{\sigma}}{|\vec{n} + \vec{v}t|^3} \right\}} \right] X(V) d\vec{n} d\vec{v} \quad (8)$$

where n is the number density of stars, assumed uniform, and V is the relative velocity with the distribution $X(V)$. Since we are only interested in the average value of $\vec{F}(t)$ in the

direction of $\vec{F}(0)$, which is

$$\left\langle \frac{\vec{F} \cdot \vec{F}(t)}{F^2} \right\rangle_F = \frac{1}{F} \langle |\vec{F}(t)| \cos \theta \rangle \quad (9)$$

where θ is the angle between $\vec{F}$ and $\vec{F}(t)$, we only need an expansion for $C(\vec{\rho}, \vec{\sigma})$ up to the terms linear in components of $\vec{\sigma}$.

For this, Chandrasekhar found

$$C(\vec{\rho}, \vec{\sigma}) = \frac{4}{16} (2\pi G m)^{3/2} n \rho^{1/2} + D(\vec{\rho}, \vec{\sigma}, t) + O(\sigma^2) \quad (10)$$

where

$$D(\vec{\rho}, \vec{\sigma}, t) = -i G m n \iint e^{i G m \frac{\vec{n} \cdot \vec{\rho}}{|\vec{n}|^3}} \chi(\vec{V}) \left(\frac{\vec{\sigma} \cdot (\vec{n} + \vec{V}t)}{|\vec{n} + \vec{V}t|^3} \right) d\vec{n} d\vec{V} \quad (11)$$

Taking the Gaussian distribution for the field star velocities $\vec{u}$

$$P(\vec{u}) = \frac{J^3}{(\pi)^{3/2}} e^{-J^2 u^2}, \quad J^2 = \frac{3}{2 \langle u^2 \rangle} \quad (12)$$

and remembering

$$\vec{V} = \vec{u} - \vec{v} \quad (13)$$

where $\vec{v}$ is the velocity of the test star, we get for the dis-

tribution of the relative velocity

$$X(\vec{V}) = \frac{J^3}{(\bar{\lambda})^{3/2}} e^{-J^2 |\vec{V} + \vec{v}|} \quad (14)$$

Using this distribution $X(\vec{V})$, we can now evaluate $\langle \vec{F} \cdot \vec{F}(t) \rangle / F^2$ as a function of $F(0)$ and the test particle velocity v . We then obtain the final result of Chandrasekhar for R after some detailed calculations.

To find $\bar{R}(\tau)$ we must average out R over the proper distribution of test star velocities and forces. When this is done, assuming a Gaussian distribution for the velocities and a Holtsmark distribution for the forces, the expression can be reduced to a double integral

$$\bar{R}(\tau) = \int_0^\infty \varphi(\tau\lambda) \Theta(\lambda) d\lambda \quad (15)$$

where

$$\frac{2}{3} \tau^2 \int_0^\infty e^{-y^{3/2}} \frac{J_{3/2}(y/\tau^2\lambda^2)}{y^2} dy$$

and

$$\Theta(\lambda) = \frac{15}{4\sqrt{2\pi}} \lambda^2 e^{-\lambda^2/2}$$

The details of the calculations leading to (15) are given in Appendix A. The integral (15) was numerically evaluated on a computer for different values of τ and the results were com-

pared with those of the numerical experiment. It can easily be seen, by expanding $J_{3/2}$ in inverse powers of τ , that the asymptotic behavior of $\bar{R}(\tau)$ for large times is

$$\bar{R}(\tau) \sim \frac{1}{\tau} \quad \tau \rightarrow \infty$$

To calculate the auto-correlation function for clusters evolved on a computer, the forces of all the stars were sampled every 1/64 of a crossing time. The random force in units of Q, β , was found by the method described in Chapter IV. As the correlation function R is given as a function of τ , we must also convert our time to units of t_0 by

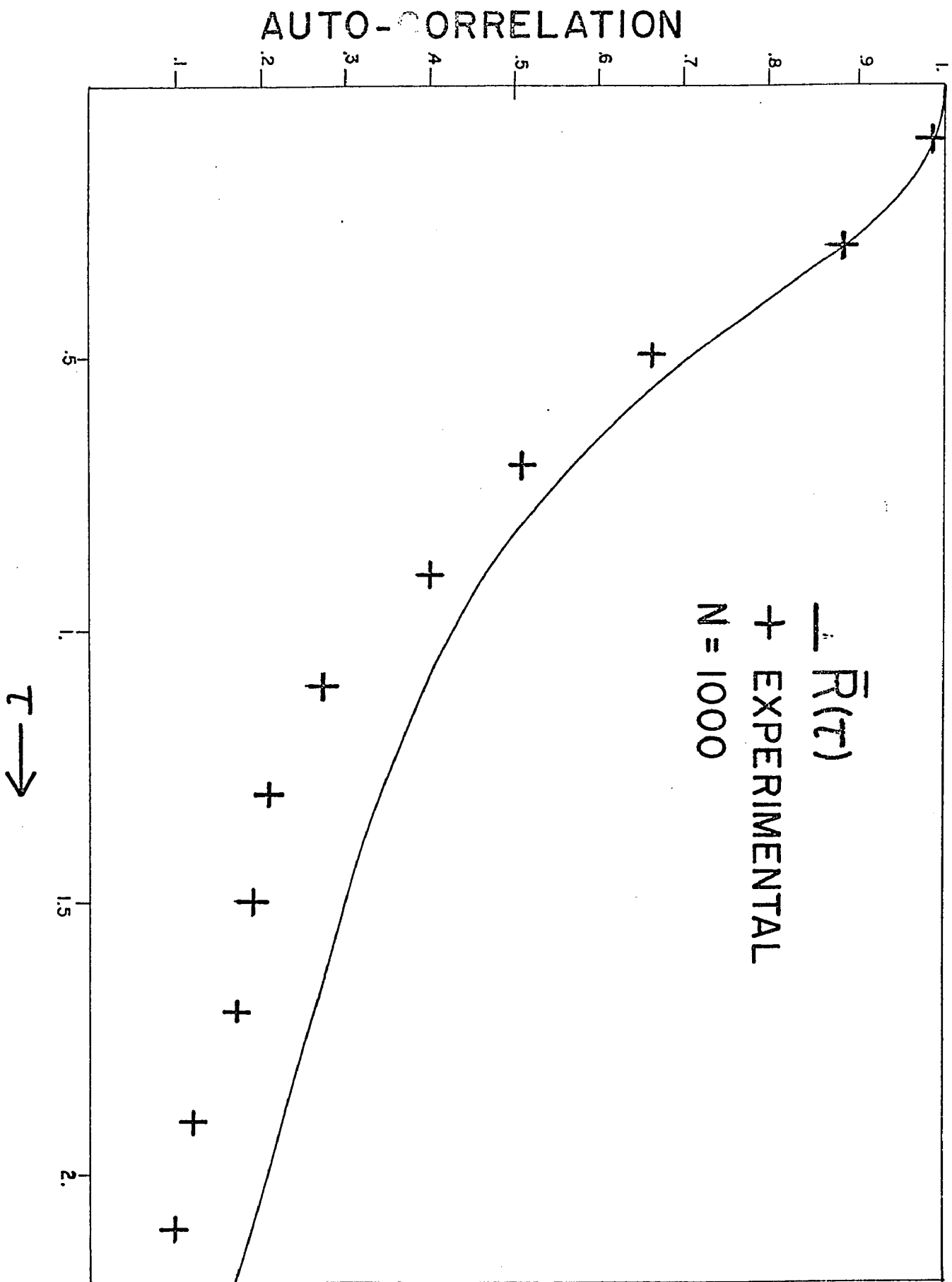
$$\tau = \frac{t}{t_0}$$

The conversion time constant, t_0 , depends on the local density and the local root mean square velocity. The average of these two quantities was found over a quarter of the crossing time as a function of the distance from the cluster center. The value of the correlation function $\bar{R}(\tau)$ was calculated for different values of τ by finding the average value of $\vec{\beta} \cdot \vec{\beta}(\tau) / \beta^2$. To do this, we look at the random force on a particle at two different times, t_1 and t_2 . In terms of τ , these two times are separated by

$$\tau = \frac{(t_2 - t_1)}{\bar{t}_0}$$

where $\bar{t}_0$ is the average value of t_0 for that particle in the interval from t_1 to t_2 . The τ obtained in this way is not discrete and we must find the $\langle \vec{\beta} \cdot \vec{\beta}(\tau) / \beta^2 \rangle$ not for one particular τ but for a small interval about τ . The time was divided into

Figure 1



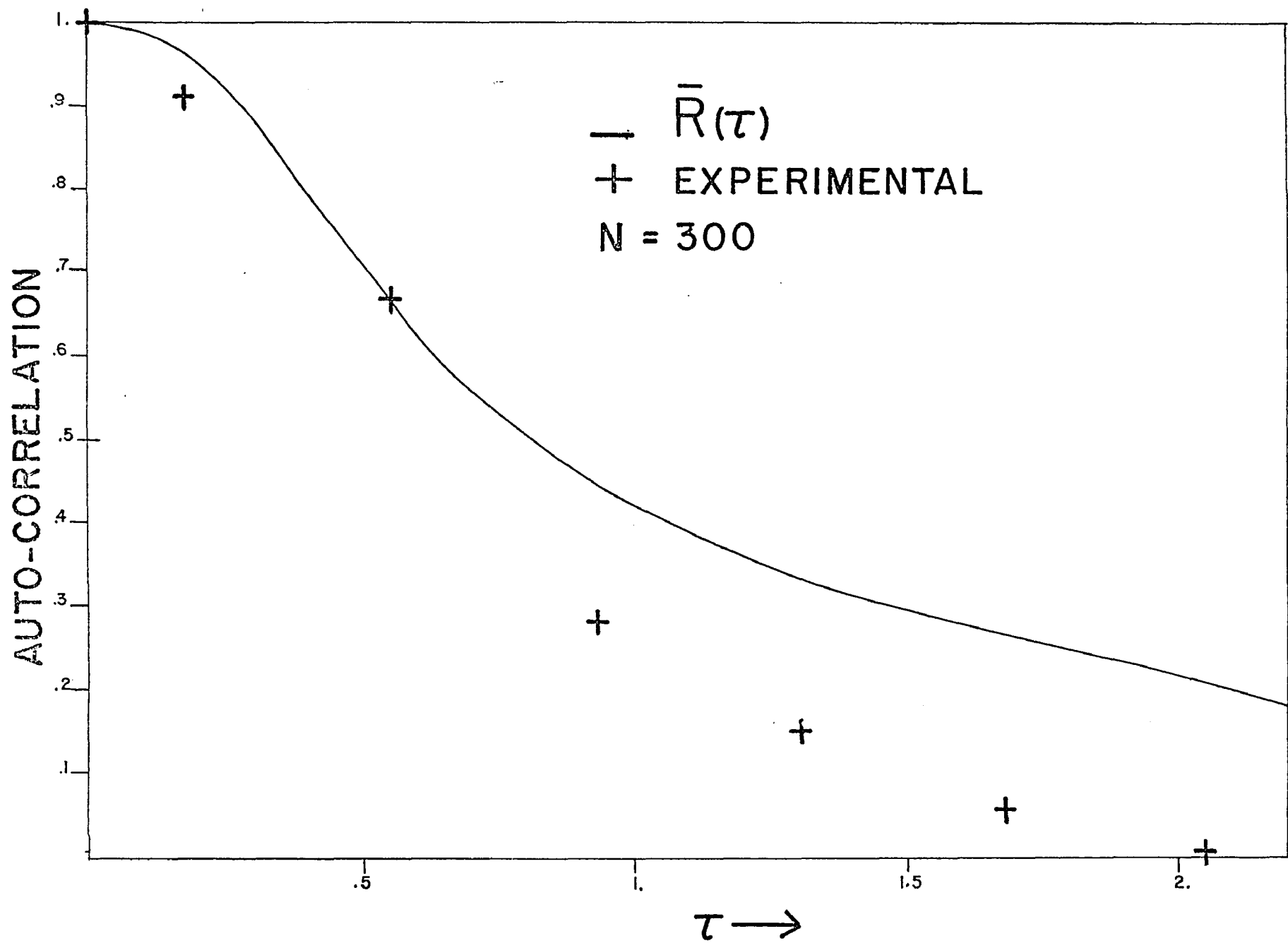


Figure 2

intervals of .2 and a separate average was kept for $\bar{R}(\tau)$ for each interval. This average was identified as the value of the correlation function at the middle of that interval.

The results of the numerical experiment are plotted for clusters containing 1000 and 300 particles, in Figures (1) and (2), respectively. The theoretical value $\bar{R}(\tau)$ is also plotted for comparison. It is seen that for the low value of τ the two curves agree well. For the high values of τ , however, the experimental values of $\bar{R}(\tau)$ are lower than the theoretical. The reason for this discrepancy is not clear at this stage.

A more direct comparison with experiment can be made if we compare the data with the auto-correlation function as defined by (2), and using the assumption of the theory of Chandrasekhar. Also the evaluation of A as defined by (1) is very much simpler in comparison with the derivation of $\bar{R}$. But in deriving A, as defined by (1), we shall assume a cutoff both for small and large distances.

Substituting F and F(t) in (1) we have for A(t)

$$\begin{aligned}
 A(t) &= G^2 m^2 \left\langle \left(\sum_{i=1}^N \frac{\vec{\pi}_i}{\pi_i^3} \right) \cdot \left(\sum_{j=1}^N \frac{(\vec{\pi}_j + \vec{v}_j t)}{|\vec{\pi}_j + \vec{v}_j t|^3} \right) \right\rangle \\
 &= G^2 m^2 \left\langle \sum_{i=1}^N \frac{\vec{\pi}_i \cdot (\vec{\pi}_i + \vec{v}_i t)}{\pi_i^3 |\vec{\pi}_i + \vec{v}_i t|} \right\rangle
 \end{aligned} \tag{16}$$

where we have dropped the cross terms for which $i \neq j$ because

the averaging process makes the dot product of two uncorrelated vector quantities vanish. The sum in (16) consists of N identically distributed quantities and hence we can write

$$A(t) = G^2 m^2 N \left\langle \frac{\vec{n} \cdot (\vec{n} + \vec{V}_t)}{n^3 |\vec{n} + \vec{V}_t|} \right\rangle \quad (17)$$

We can now evaluate the averaged term in (17) by using a one-particle distribution for the space density and a Gaussian distribution for the relative velocities

$$A(t) = G^2 m^2 n \int d\vec{V} \chi(\vec{V}) D(\vec{V}, t) \quad (18)$$

where n is the number density of the stars.

$$D(\vec{V}, t) = \int \frac{\vec{n} \cdot (\vec{n} + \vec{V}_t)}{n^3 |\vec{n} + \vec{V}_t|} d\vec{n} \quad n < R \quad (19)$$

where R is the radius of the system. To perform the integration in (19), we shall impose a restriction that the field particles cannot come arbitrarily close to the rest particle. For this purpose, we define a distance ϵ

$$\epsilon \ll R$$

which will then serve as the lower limit of integration in (19). The distance ϵ may be a function of the relative velocity.

We then have

$$D(\vec{V}, t) = 2\pi \int_0^\pi \int_\epsilon^R \frac{(r + Vt \cos \theta) \sin \theta d\theta dr}{[r^2 + V^2 t^2 + 2rVt \cos \theta]^{3/2}} \quad (20)$$

where θ is the angle between the $\vec{r}$ and $\vec{V}$ directions, and $\vec{V}$ is taken to be in the z-direction. After some elementary integrations we find the following:

$$D(\vec{V}, t) = 2\pi \left[\frac{\epsilon + Vt - |\epsilon - Vt|}{V\epsilon t} - \frac{R + Vt - |R - Vt|}{VRt} \right] \quad (21)$$

or

$$\begin{aligned} D(\vec{V}, t) &= 4\pi \left(\frac{1}{\epsilon} - \frac{1}{R} \right) & t < \frac{\epsilon}{V} \\ &= 4\pi \left(\frac{1}{Vt} - \frac{1}{R} \right) & \frac{\epsilon}{V} < t < \frac{R}{V} \\ &= 0 & t > \frac{R}{V} \end{aligned} \quad (22)$$

We can now write an expression for $A(t)$ using (21) and (18). Where we have $X(V)$ to be spherically symmetric, we note that since

$$D(\vec{V}, t) = \frac{4\pi}{Vt} \quad \epsilon \rightarrow 0 \quad R \rightarrow \infty$$

we get

$$A(t) \sim 4\pi G^2 m^2 n \int \frac{d\vec{V} X(\vec{V})}{Vt} \quad (23)$$

a result found by Lee (1968) without using our restrictions. With $\epsilon=0$ and $R=\infty$, we have the $1/t$ dependence again. But it is at once clear that both difficulties mentioned about the

behavior of $A(t)$ can be avoided by choosing finite values for ϵ and for R ! To perform the integration in (22), we take, as Chandrasekhar does, a Gaussian for $\vec{V}$

$$\chi(\vec{V}) = \frac{j'^3}{\pi^{3/2}} e^{-j'^2 V^2} \quad (24)$$

The j' for the relative velocities is related to the j for field particles by

$$j' = \frac{j}{\sqrt{2}} \quad (25)$$

We should point out that the assumption of Maxwellian distribution is not necessary for what follows. The same results would hold for any distribution which is a function of the absolute velocity only.

The correlation function then equals

$$A(t) = 16 (\pi)^{1/2} j'^3 G^2 m^2 n \left[\left(\frac{1}{\epsilon} - \frac{1}{R} \right) \int_0^{\epsilon/t} V^2 e^{-j'^2 V^2} dV + \int_{\epsilon/t}^{R/t} \left(\frac{1}{Vt} - \frac{1}{R} \right) V^2 e^{-j'^2 V^2} dV \right] \quad (26)$$

In writing (26), we have neglected any possible dependence of ϵ on the velocity. After some elementary manipulations and an integration by parts, we have for (26)

$$A(t) = 8 G \sqrt{\pi} j'^3 m^2 n \left[\frac{1}{\epsilon} \int_0^{\epsilon/t} e^{-j'^2 V^2} dV - \frac{1}{R} \int_0^{R/t} e^{-j'^2 V^2} dV \right] \quad (27)$$

If we let $\epsilon \rightarrow 0$ and $R \rightarrow \infty$, we recover the $1/t$ dependence

$$A(t) \sim \frac{8J' \sqrt{\pi} G^2 m^2 n}{t} \quad \epsilon \rightarrow 0, R \rightarrow \infty \quad (28)$$

However, for finite ϵ and $t \rightarrow 0$, we have

$$A(t) \sim \frac{8J' \pi G^2 m^2 n}{\sqrt{2} \epsilon} \quad \begin{matrix} \epsilon \neq 0 \\ t \rightarrow 0 \end{matrix} \quad (29)$$

and similarly for finite R

$$A(t) \rightarrow \frac{8J' \sqrt{\pi} G^2 m^2 n R^2}{3t^3} \quad (30)$$

Thus, as long as R is kept finite, we have a much weaker dependence on time.

For the purpose of comparison with experiment, it will be more convenient to use dimensionless units as before. Expressing the force in terms of Q and ϵ and R in terms of the unit length l as defined by (5)

$$l = \frac{t_0}{J}, \quad \epsilon_0 = \frac{\epsilon}{l}, \quad R_0 = \frac{R}{l}, \quad v = V_J'$$

we have from (27),

$$\begin{aligned} G(\tau) &= \langle \vec{\beta}(0) \cdot \vec{\beta}(\tau) \rangle \\ &= \frac{15}{\sqrt{2} \pi} \left[\frac{1}{\epsilon_0} \int_0^{\epsilon_0/\sqrt{2}\tau} e^{-v^2} dv - \frac{1}{R_0} \int_0^{R_0/\sqrt{2}\tau} e^{-v^2} dv \right] \quad (31) \end{aligned}$$

Again

$$G(\tau) = \frac{15}{2\pi} \frac{1}{\tau}$$

$$\begin{aligned} \epsilon_0 &\rightarrow 0 \\ R_0 &\rightarrow \infty \end{aligned} \quad (32)$$

But

$$G(\tau) = \frac{15}{2\sqrt{2}\pi} \frac{1}{\epsilon_0} \quad \tau \rightarrow 0 \quad \epsilon_0 \neq 0$$

(33)

Similarly, for finite R_0

$$G(\tau) \sim \frac{5}{8\pi} \frac{R_0^2}{\tau^3} \quad \begin{aligned} \tau &\rightarrow \infty \\ R_0 &\neq \infty \end{aligned} \quad (34)$$

We can estimate R_0 to be the average interparticle distance. An estimate for ϵ_0 can be obtained by using the condition that two stars should not have a negative binding energy. Calling that distance ϵ

$$\begin{aligned} \epsilon &= \frac{2 G m}{V^2} \\ \bar{\epsilon} &= \frac{2 G m}{\langle V^2 \rangle} \end{aligned} \quad (35)$$

Using the virial theorem and using the fact that $\langle V^2 \rangle = 2 \langle v^2 \rangle$ we find

$$\bar{\epsilon} \sim \frac{R}{N}$$

AUTO - CORRELATION

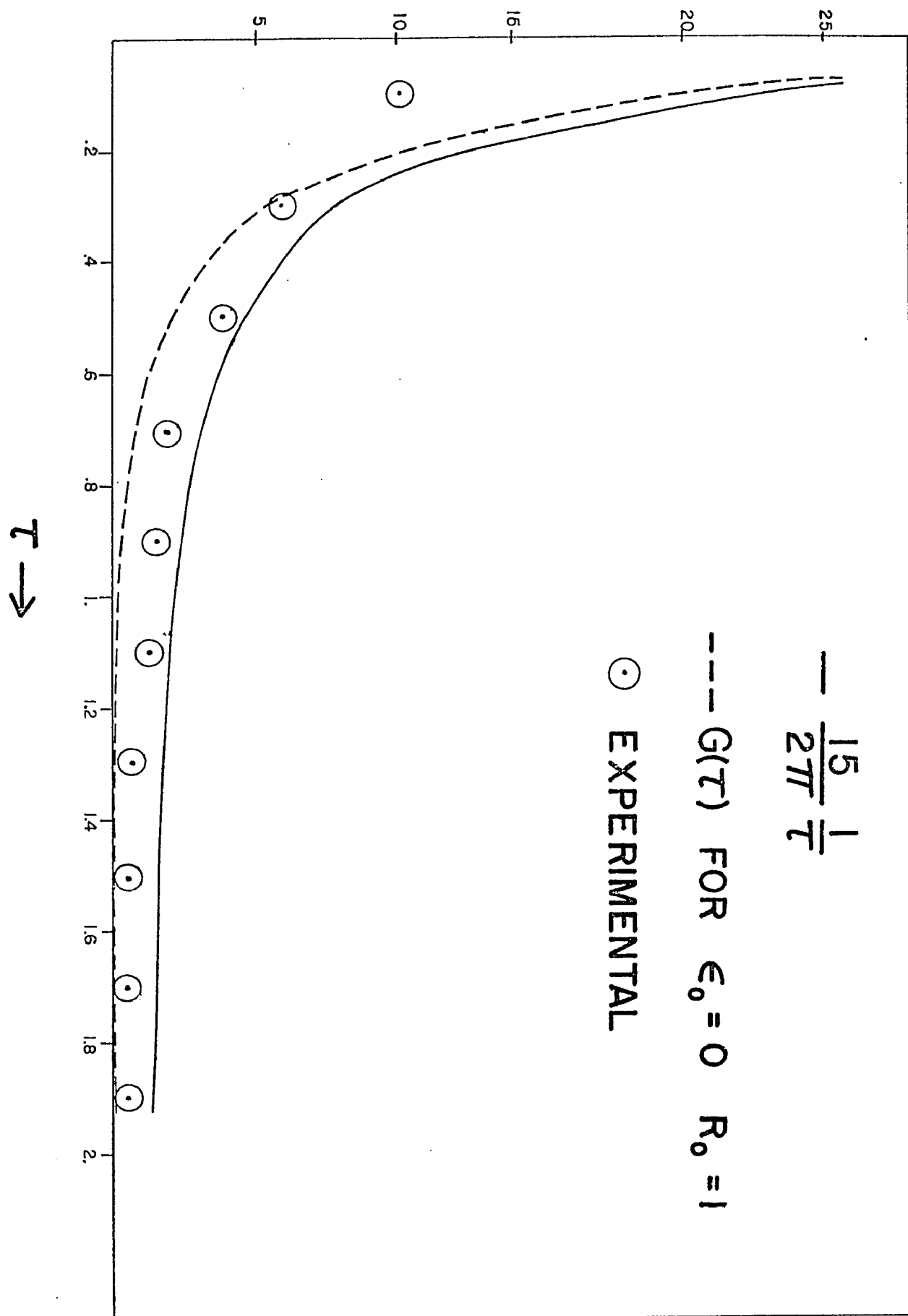


Figure 3

and

$$\bar{\epsilon}_0 \sim \frac{1}{N^{2/3}} \quad (36)$$

For the purpose of comparison with the numerical experiment, $G(\tau)$ was evaluated for different values of τ . To do this, we took R_0 to be unity and ϵ_0 to be zero, since for $N=1000$,

$$\bar{\epsilon}_0 \sim 10^{-2}$$

We calculated the function $G(\tau)$ for a cluster containing 1000 stars by a method identical to the one described earlier for $R(\tau)$. The results are presented in Figure (3). The solid curve is the theoretically obtained value $G(\tau)$ for $\epsilon_0=0$ and $R_0=\infty$. This has a simple behavior of $1/\tau$ as given by (32). The dashed curve is $G(\tau)$ for $\epsilon_0=0$ but $R_0=1$. Of course, this choice of R_0 is somewhat arbitrary. The problem of choosing an appropriate R_0 is similar to the one encountered in the calculation of the relaxation time. A deeper analysis will have to be done, perhaps along the lines of Gilbert (1968) and Lerche (1970) to obtain a proper expression for the auto-correlation of force. But it is clear from the experimental results that it falls off faster than predicted by the $1/t$ dependence.

VIII. CONCLUSION

The N-body integration scheme has allowed us to experimentally examine various results of the statistical theory of Chandrasekhar and von Neumann. This was possible due to the ability of the scheme to integrate a large number of particles.

We have tested the most fundamental results of the theory and found, on the whole, excellent agreement. In the experiment on the Holtsmark distribution, we did find, on occasion, some discrepancy between theory and experiment; but the agreement got progressively better as the number of particles in the system increased. Systems containing large numbers of stars may have peculiarities which become less important as the number of particles in the system increases. For example, a proportionately large number of stars may have a tendency to be part of a subsystem in clusters with low numbers of particles. But for the 1000-body case, which is closer to real clusters, there is no question that the Holtsmark distribution is of sufficient accuracy.

Similar remarks pertain to the distribution of the time-derivative of the force.

We have also shown that one of the most significant results of the theory, namely, the prediction of the existence of dynamical friction in gravitational systems, is indeed correct.

Our results on the auto-correlation function for force indicates that it falls off faster than predicted and we have

shown that this is most probably due to not taking the cutoff into proper account. We have shown, theoretically, that as long as a finite cutoff at infinity is imposed, the autocorrelation function drops as one over the time cubed as compared to one over the time, otherwise.

Chandrasekhar's theory is very powerful for it gives explicit expressions for statistical distributions for important physical quantities. Although the mathematics is, at certain stages quite involved, it is a very workable theory from a calculational standpoint. The theory allows the study of the evolutionary problem for globular clusters from a theoretical and a computer simulation approach. From a theoretical point of view, equations similar to the ones used for Brownian motion can be studied. We can write the equation of motion as

$$\frac{d\vec{v}}{dt} = \vec{F}_m + \vec{F}_r \quad (1)$$

where $\vec{F}_m$ and $\vec{F}_r$ are the mean and the random forces respectively. For spherical systems, the mean field force can simply be written as

$$\vec{F}_r = - G \frac{M(r)}{r^3} \vec{r} \quad (2)$$

where $M(r)$ is the mass interior to r . The random force is probabilistic and governed by the Holtsmark distribution.

Since $\vec{F}_r$ is a random variable, equation (1) is, therefore a stochastic equation which can in principle be solved for the probability distribution of significant random variables such as density, velocity, etc. In practice, however, equation (1) is quite difficult to solve mainly because the mean field force is complicated and also because the random force is not only a function of time as it is for Brownian motion but depends on the local density. But there are some approximation schemes (e.g., moment equations) which might render some progress in its solution.

Another possibility is to devise a Monte Carlo scheme using the Holtzmark distribution for the random force. There are many variants, but for spherical systems the most obvious is to proceed as follows. At each stage of the simulation the mean field force can be readily obtained by merely counting the particles interior to the particle which is being evolved at the particular moment. The random force on the particle is naturally obtained from the Holtzmark distribution. The sum of the two gives the total force on the particle, and the particle can be updated for a time given by the mean lifetime of the random force. One can introduce different levels of sophistication into the above simple scheme. Thus, for example, one can choose both the force and the time rate of change of the force from a joint distribution and hence, go one term higher in the Taylor series. Also, instead of updating for a mean lifetime, one can choose the lifetime probabilistically. Our experiments have shown

that one can use with considerable confidence the probability distributions given by the theory of Chandrasekhar and von Neumann. The inclusion of the proper distribution for the random force should make the simulation much more physically realistic than current Monte Carlo schemes do.

Appendix A

We have

$$\bar{R}(\tau) = \int H(\beta) R(\tau, \beta, v) P(v) d\beta dv$$

A 1

where $H(\beta)$ is the Holtmark distribution and $P(v)$ is the distribution of velocities taken to be Maxwellian

$$P(v) = \frac{4}{(\pi)^{1/2}} v^2 e^{-v^2}$$

A 2

where

$$v = \int v$$

Integrating over β first and defining

$$I(v, \tau) = \int_0^\infty H(\beta) R(\tau, \beta, v) d\beta$$

A 3

we have

$$\bar{R}(\tau) = \frac{4}{(\pi)^{1/2}} \int_0^\infty v^2 e^{-v^2} I(v, \tau) dv$$

A 4

Expressing the Bessel function appearing in the expression for $R(\tau)$ in terms of the trigonometric functions, the integration over β gives

$$I(v, \tau) = \frac{15}{4} \int_0^\infty e^{-y^{3/2}} y^{1/2} \mathcal{L}\left(\frac{y}{\tau^2}, v\right)$$

A 5

where

$$2\left(\frac{z}{\tau^2}, \nu\right) = \int_0^\infty \varphi(\lambda, \nu) \alpha\left(\frac{z}{\tau^2 \lambda^2}\right)$$

A 6

and

$$\varphi(\lambda, \nu) = \frac{1}{\pi^{1/2}} \frac{\lambda}{\nu} e^{-\lambda^2 - \nu^2} \sinh 2\lambda\nu$$

A 7

$$\alpha(\kappa) = \int_0^\kappa \frac{J_{3/2}(z)}{z} dz$$

A 8

We have for $I(\nu, \tau)$

$$I(\nu, \tau) = \frac{15}{4} \int_0^\infty \varphi(\lambda, \nu) \chi(\tau\lambda) d\lambda$$

A 10

where

$$\chi(\tau\lambda) = \int_0^\infty e^{-y^{3/2}} y^{1/2} \alpha\left(\frac{y}{\tau^2 \lambda^2}\right)$$

A 11

Integrating $\chi(\tau\lambda)$ by parts

$$\chi(\tau\lambda) = \frac{2}{3} \int_0^\infty e^{-y^{3/2}} \frac{\tau^2 \lambda^2 J_{3/2}\left(\frac{y}{\tau^2 \lambda^2}\right)}{y^2}$$

A 12

we finally get

$$\bar{R}(\tau) = \int_0^{\infty} X(\tau\lambda) \theta(\lambda) d\lambda \quad \text{A 13}$$

where

$$\theta(\lambda) = \frac{15}{(\pi)^{1/2}} \int_0^{\infty} v^2 e^{-v^2} \varphi(\lambda, v) dv \quad \text{A 14}$$

This can be further reduced by substituting for φ from (A 7)

$$\theta(\lambda) = \frac{15}{4} \lambda e^{-\lambda^2} \int_0^{\infty} v e^{-2v^2} \sinh(2\lambda v) dv \quad \text{A 15}$$

The integral in (A 15) is a known definite integral and we finally get

$$\theta(\lambda) = \frac{15}{4\sqrt{2\pi}} \lambda^2 e^{-\lambda^2/2} \quad \text{A 16}$$

As no further reduction seems possible, the value of $\bar{R}(\tau)$ must be obtained by evaluating the double integral in (A 13) numerically for different values of τ .

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