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A MODELLING APPROACH OF
TWO-ECHELON INVENTORY SYSTEMS

by

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ABSTRACT

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To provide good service to customers, many firms use multi-echelon inventory systems. There are two types of inventory replenishment policies in multi-echelon inventory models. The one-for-one replenishment policy has commonly been applied. The batch order policy is much more complicated and more challenging. In our study, several typical multi-echelon inventory models are presented. The differences among these models have been elaborated. We proposed a new approach to model two-echelon inventory systems.

Most two-echelon (Q, R) studies have similar assumptions. The major difference lies in the modelling of the depot demand process. In our study, the depot demand is approximated by a Poisson process. We investigate the steady state of the two-echelon inventory system. The depot demand process in computer simulations is tested with several goodness of fit tests against a Poisson process and a normal process. We also run the run-ups test and correlation test for the independence of base orders at the depot. The

number of bases and the base order size affect the approximation of the demand process at the depot. As the number of bases increases and/or the base order size decreases, the approximation of the demand process at the depot by a Poisson process is improved. We compare the test power of these goodness of fit tests. Our conclusion is that the most commonly used χ^2 test has a very low test power and should not be used alone.

The inventory system performance is not very sensitive to the demand process. We compare our simulation results with the results of our analytical model when the approximation of depot demand process by a Poisson process is not good. We conclude that the results with our approach of modelling the two-echelon inventory systems are consistent with simulation results.

In order to obtain the global minimum total inventory cost, the total inventory cost should be a convex function of the decision variables. Conditions of the convexity of the total inventory cost are studied in detail. We point out that certain parts of the inventory system cost, such as the holding cost at bases, are not convex with respect to some decision variables. The sufficient condition of the convexity of the total holding cost with respect to the depot reorder point is obtained. Since the proof of the convexity of the total cost is very long, we put the complete proof in the appendix. An algorithm to obtain the optimal values of decision variables and the minimum total inventory cost is developed. Numerical examples show that computer simulations and our approach to model the inventory system produce very close results.

We study the effect of system parameters on the minimum total cost and the optimal values of decision variables. A table to show the general trend of changes in decision variables with respect to changes of the system parameter is obtained. Most results in that table are consistent with what we expected. Because of the interactions among the system decision variables and the assumption that decision variables only have integer values, some results of our algorithm do not follow the general trend. These intuitive results are explained in detail at the related chapter. We also determine the priority of the decision variables on the system total cost. Possible future research topics are discussed at the end of our study.

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INTRODUCTION AND LITERATURE REVIEW

1.1 GENERAL ISSUES IN TWO-ECHELON INVENTORY SYSTEMS

Two-echelon inventory systems are generally used to provide products and services for customers who are distributed over an extensive geographical region. These systems are usually characterized by a lower echelon, consisting of bases that serve as the first level of product support to customers. The higher echelons are distribution centers (or depots) for the replenishment of stock to bases.

The purpose of this study is to develop a new approach to model two-echelon inventory systems. We use computer simulations to establish the accuracy of our approach. There are several issues that make two-echelon inventory systems more challenging than single echelon inventory systems.

The first issue is the depot demand process. It is a unique issue of the multi-echelon inventory systems. At each base, customers arrive independently of other bases. Customers' demand at each base is assumed to follow a Poisson distribution. When bases use (Q, R) inventory policy, the inter-arrival between base orders from the same base at the depot is an Erlang process. The demand process at the depot is the summation of base ordering processes. That is, the demand processes at bases (lower echelon), together with the ordering policy followed at each base, decide the demand process at the depot (higher echelon).

Second, the type of items affects the complexity of the analysis. Products with a limited lifetime are perishable items. If perishable items have been stored in the inventory system after certain periods, those perishable items have to be discarded, which will increase the cost of the inventory system. Computer software and fashion items can be considered as perishable items. Old version software and out-of-fashion items have very little value. Repairable items also complicate the analysis of the inventory system. To deal with repairable items, the inventory system has to consider the repair facility, the waiting time and the service time of failed items. That is the reason why perishable or repairable items make inventory systems more difficult to analyze than do consumable items.

The third major issue in two-echelon inventory systems is the lateral transshipment among bases. As we know, the demand at each base is independent of the demand at other bases. Sometimes, a base runs out of stock and is expecting to receive its orders from the depot while customers who arrive at the base are backlogged. At the same time, other bases of the inventory system may have inventory. It is reasonable to supply those waiting customers with other base inventory. We call this practice the lateral transshipment among bases.

Lateral transshipment results in higher customer satisfaction. However, lateral transshipment has its transportation cost and greater coordination of base inventory management is required. With lateral transshipment among bases, the demand at any base should include the possible demand from other bases' customers if those bases run out of stock.

The fourth issue is the assumption of backorders. When a base runs out of stock, some customers may wait and their demands are backlogged while other customers may go to its competitors. That is the assumption of partial backorders. With partial backorders, the demand rate at a base will change according to the inventory level at the base.

1.2 LITERATURE OF TWO-ECHELON (S-1, S) INVENTORY MODELS

The reorder points of each base and the depot are not negative in (S-1, S) models. With an (S-1, S) policy, an order of one unit is placed at each occurrence of a demand or at each occurrence of outdating of a unit in perishable item models. Compared with the later discussed two-echelon (Q, R) inventory models, the order quantities of bases and the depot in (S-1, S) models are one unit. The (S-1, S) policy is a special case of the general two-echelon inventory systems. Among two-echelon inventory systems, (S-1, S) models are widely studied.

(S-1, S) models have been applied to lower demand and high value items, where a unit is very expensive to justify such a frequent replenishment policy. Zipkin (1986) points out that two-echelon inventory models often assume an (S-1, S) policy. **Table 1.1** summarizes the assumptions and approaches of nine papers, selected from many papers on the topic. METRIC, the Multi-Echelon Technique for Recoverable Item Control System, was designed by Sherbrooke (1967) for aircraft engine maintenance. Aircraft engines are expensive and the maintenance cost for aircraft engines is also very high.

In METRIC, demand is modeled by a compound Poisson distribution with a mean value estimated by a Bayesian procedure. The Bayesian procedure is being implemented continuously to update the parameters of the prior distribution (Muckstadt, 1973). METRIC assesses the inventory system performance of any allocation of stock among bases and the depot in terms of availability of aircraft engines and the investment of the inventory system.

Muckstadt (1973) developed the MOD-METRIC model for aircraft maintenance to permit the consideration of a hierarchical parts structure. The differences between METRIC and MOD-METRIC are related to the necessity of items and the nature of the demand process in aircraft maintenance. In METRIC, backorders for a module needed to repair an engine and backorders for an engine needed for an aircraft are assumed to be equally undesirable. Muckstadt modified this assumption. He suggests that backorders of engines make an aircraft inoperational, but backorders for a module only delay the repair work of an engine. The backorders for an aircraft engine are more serious and should be assigned higher cost than the backorders for a module in an engine repair process.

Table 1.1 Major (S-1, S) Models of Two-Echelon Systems

Model	Transit time	stockout	Lateral transshipment	Repairable/ Perishable	Solution
METRIC Sherbrooke (1967)	expon.	full backorders	no	repairable	approx.
Mod-Metric Muckstadt (1973)	expon.	full backorders	no	repairable	approx.
Schmidt & Nahmias (1985)	expon.	no backorders	no	perishable	exact
Lee (1987)	expon.	full backorders	yes	repairable	approx.
Moinzadeh (1989)	expon.	partial backorders	no	repairable	approx.
Axsater (1990a)	expon.	full backorders	yes	repairable	approx.
Axsater (1999b)	expon.	full backorders	no	repairable	exact
Svoronos & Zipkin (1991)	expon.	full backorders	no	repairable	approx.
Dada (1992)	expon.	no backorders	priority rule	no	approx.

With perishable items, the stochastic process corresponding to the number of units in inventory is no longer a Markov process. The number of items changes because of the aging nature of the perishable items even though there is no demand or supply for the items. The system state vector should be expanded to keep track of the age of all items in stock. In the Schmidt and Nahmias Model (1985), each unit is assumed to have a fixed shelf life.

A potential application of the Schmidt and Nahmias model is a situation where equipment is subject to shutdown due to failure or routine maintenance. In such an application, failure of the equipment can be considered the result of the aging process. Routine maintenance is considered as normal repair work.

Galliher, Morse and Simond (1959) point out that in terms of mathematical computation difficulty, systems with full backorders are simpler than systems with partial backorders. Moinzadeh (1989) considers an (S-1, S) inventory system with partial backorders and constant supply time.

In lateral transshipment models, when a failed item is brought to a base and that base is out of stock, the base will issue an emergency lateral transshipment request to other bases that have stock. If there is more than one base that has stock, several prioritized source rules may be applied to choose the source of the emergency transshipment. The base that supplies the transshipment will request a replenishment order from the depot to restore its inventory level. If all bases are backlogged, there will be no lateral transshipment.

Sherbrooke (1967) shows that when lateral transshipment is ignored, we do not need to consider the transportation costs among bases and between bases and depot. The total transportation cost is not a function of the inventory policy. That is, the transportation cost in those cases does not change with different reorder points at the depot and bases.

The lateral transshipment improves customer satisfaction and reduces the time that a demand is backlogged. The level and cost of backorders in the system are reduced. On the other hand, the lateral transshipment has its own cost, such as the cost of issuing the lateral resupply order, the search cost for locating among bases that have stock and the transportation cost of the lateral transshipment. Also, by releasing an item for lateral transshipment, the source base ability to meet its customer demand is reduced.

In Lee's model (1987), the demand at all bases is placed in a pooling group and is distributed according to a Poisson process with the same repair arrival rates. The transportation time from the depot to each base is fixed and the same. Lee uses three lateral transshipment source rules to determine the base as the source for emergency lateral transshipment: **1.** Random source rule: the source base is chosen randomly from the bases with stock; **2.** Priority source rule 1: the base with the maximum stock is chosen as the source base. If there are ties, the source base is chosen randomly among the ties; **3.**

Priority source rule 2: the base with the maximum stock is chosen as the source base. If there are ties, the base with the smallest number of outstanding orders is chosen. If there are still ties, the random rule is used to break the tie.

The number of backorders and the quantity of lateral transshipment are the two key performance measurements of interest in the Lee model (1987). He derives the approximation to their expected values in his model. Although Lee lists three different transshipment source rules, he only applies the random transshipment source rule in his analytical model. With random transshipment source rule and identical bases, Lee's approach is to approximate the outstanding orders at each base as if there were no lateral transshipment among bases.

Axsater's (1990a) approach to the lateral transshipment issue assumes that a base demand rate depends on the inventory status of the base. With positive on-hand inventory, a base faces possible lateral transshipment requests from other bases in addition to the demand from its own customers. When a base does not have on-hand inventory, the demand in that base is the backlogged demand. Although Axsater (1990) and Lee (1987) consider similar models, Lee focuses on the analysis of the number of outstanding orders and whereas Axsater emphasizes in modeling the base demand. Axsater's approach can also be used in the non-identical bases models.

In Dada's model (1992), each base stocks exactly one unit. Dada combines all n bases into an aggregate center that stocks n units, resulting in a one-warehouse, one-base model. He incorporates the priority shipments by specifying the system response when a base is out of stock. In Dada's model: **1.** the exact model provides the basis for the approximation; **2.** An aggregate model is used to derive error bounds on approximating system performance; **3.** The approximation is chosen so that the properties of an aggregate model create a bound on the resulting errors.

There are several special issues in (S-1, S) systems. One important issue is to model the demand process. A compound Poisson process incorporates more parameters. However, a compound Poisson process makes the system analysis difficult. For models where the variance to mean ratio is less than three, Sherbrooke (1967) finds that the steady state probabilities for a Poisson process and a compound Poisson process are almost identical. His finding makes a simple Poisson demand process a reasonable substitute for a compound Poisson demand.

The second issue is the treatment of service time. The service time is the time to repair an item. The rate at which a channel's service is accomplished may vary with the number of busy channels because of the speed-up of the repair work.

The third issue is the transit time. In most models, the transit time between the base and depot is assumed to be constant or identically independent random variables.

Svoronos and Zipkin (1991) assume that the transportation between each base and depot represents a transit system. The transit times are determined by a queuing system that the system can not control. With the increasing of the outstanding orders, the transit time becomes longer.

According to Schmidt and Nahmias (1985), traditional approach used in (S-1, S) model analysis is to consider the stochastic process corresponding to the number of units in stock. Such an approach focuses on the steady-state behavior of the inventory levels, and then uses the steady-state distribution to find out the average costs. Axsater (1990b) uses an inventory cost function without considering steady states of the inventory system. Axsater's approach is more efficient at finding the optimal inventory policy. However, Axsater's (1990b) approach is inappropriate for cost functions that are expressed as nonlinear functions of the inventory levels and/or backorders.

1.3 LITERATURE OF TWO-ECHELON (Q, R) INVENTORY MODELS

When the ordering cost is relatively high to the holding cost and/or when demand rate is high, an (S-1, S) model is not appropriate. With the (S-1, S) policy, each base and the depot will replenish its inventory as soon as a customer picks up a product. The high frequency of ordering activity at bases and the depot will result in huge ordering cost. According to Moinzadeh and Lee (1986), relatively little work has been done on multi-echelon (Q, R) inventory models, where Q is the ordering quantity and R is the reorder point at the depot and bases. Grave (1996) indicates that two-echelon (Q, R) models are much harder and the progress has been slower.

For the two-echelon (Q, R) inventory systems, customers arrive at bases and the depot is used to supply bases' orders. In a two-echelon (Q, R) inventory system, each base has its ordering quantity and the reorder point. When the inventory position at a base reaches its reorder point, the base sends a base order request to the depot. If the depot has on-hand inventory, the depot supplies the base order with its inventory. Otherwise, the base order is backlogged at the depot. When the depot inventory position reaches its reorder point, the depot will send its order request to the outside supplier. The demand process observed at the depot is a superposition of the ordering processes of bases. Because of the complex interrelationships among ordering quantities and reorder points at the depot and bases, it is difficult to get the optimal solutions to all decision variables simultaneously. Generally, all studies of two-echelon (Q, R) inventory systems deal with the same model. The differences among these studies are how to approach the system.

In the analysis of multi-echelon (Q, R) inventory models, the queuing theory is very useful. Galliher, Morse, Simond (1959) compare the queuing system with the inventory system. The analogy is between ordering quantities in the inventory system and servers in the queuing system, the number of outstanding orders and the number of busy servers in the queuing system with infinite number of servers.

The assumptions of most two-echelon (Q, R) inventory models are the same. The transit time at bases and the depot is constant. When a base is stockout, demands are backlogged and there is no lateral transshipment among bases. **Table 1.2** summarizes major papers of two-echelon (Q, R) inventory models.

Table 1.2 Major (Q, R) Models of Two-Echelon Systems

Model	product	Solution
Moinzadeh & Lee (1986)	repairable /consumable	approximation.
Svoronos &Zipkin (1988)	repairable	approximation.
Axsater (1993)	consumable	exact /approximation
Aggarwal & Moinzadeh (1994)	repairable	approximation

In the Moinzadeh and Lee model (1986), failed items are sent to the depot for repair. An order is placed by the base to the depot simultaneously to replenish the stock. The transit time from the depot to each site is fixed and the same for all bases. All failed items can be repaired and used indefinitely.

Their approach is to determine the batch size as the first step, via some power approximation scheme. Given the batch size, an upper boundary on the stocking level at the depot can be found. Using the upper boundary, the depot stocking level is then obtained by a "one-pass" search. One of Moinzadeh and Lee findings in the simulation analysis is that it is reasonable to treat the number of outstanding orders of any base at the depot and the number of failed units waiting for service at the depot from the same base as independent variables.

Svoronos and Zipkin (1988) use a decomposition technique adapted from the METRIC model. They approximate each facility as a single location inventory system. The key innovation in their models is the use of second-moment of demand distribution in the approximation of the depot demand process. But they do not consider the ordering cost in their cost function. As we know, the ordering cost makes the difference between the (Q, R) policy and the much simpler (S-1, S) policy.

In the Svoronos and Zipkin (1988) model, they assume that the holding costs per unit at base and depot are the same. They prove the convexity of total cost function with such an assumption. We feel that the assumption of equal holding cost at the depot and bases is not reasonable. With economies of scale, the holding cost per unit at the depot should be less than that at bases. With different holding costs per unit at the depot and bases, the convexity of the total cost to the reorder point may only be true under certain conditions. We prove the sufficient conditions for the convexity of the total cost to the reorder points of bases and the depot.

A slightly different approach is followed by Deuermeyer and Schwarz (1981). The differences between the Deuermeyer and Schwarz model (1981) and the Svoronos and Zipkin model are: 1. Svoronos and Zipkin calculate the exact variance of the depot leadtime demand whereas Deuermeyer and Schwarz use the asymptotic limit that results in underestimates of the depot backorders, the customer waiting time at bases and the base backorders; 2. Svoronos and Zipkin use the mixture of two translated Poisson distributions to approximate the leadtime demand whereas Deuermeyer and Schwarz use a normal distribution to approximate leadtime demand. Svoronos and Zipkin find out that the normal approximation overstates the expected depot backorders when the depot reorder point is small.

Axsater (1993) studies the same inventory model that has been studied by Svoronos and Zipkin (1988), Lee and Moinzadeh (1987), Moinzadeh and Lee (1986). Axsater approximates the system in three different ways: 1. Every Qth demand generates a retailer order. The same approximation is used by Deuermeyer and Schwarz; 2. A retailer order occurs with probability $1/Q_B$ for each demand where Q_B is the batch size of the base; 3. The cost is a weighted sum of the costs obtained from approximation 1 and 2 with the weights of 1/n and 1-1/n respectively, where n is the number of bases in the system. Svoronos and Zipkin (1988) and Axsater (1993) develop their models on identifying which depot order meets which customer's demand.

1.4 DESCRIPTION OF THE SYSTEMS TO BE STUDIED AND SIMULATIONS

We study the two-echelon inventory system with (Q, R) policy at the depot and bases. **Figure 1.1** shows the two-echelon inventory system we study.

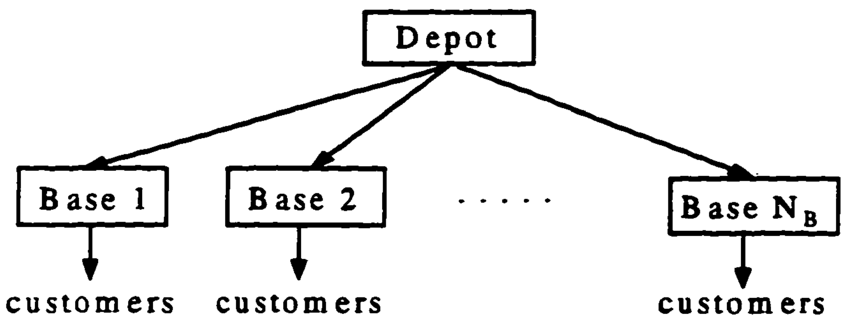

Figure 1.1 The General Two-Echelon Inventory System

In our two-echelon inventory systems, there are N_B bases and one depot. Each base receives its supply from the depot. The depot orders its supply from the outside supplier. When a base runs out of stock, its customers will be backlogged and later served on the "first come, first served" (FCFS) basis.

When the depot runs out of stock, base orders are backlogged and later fulfilled on the FCFS basis. The outside supplier has enough items to satisfy the demand from the depot. It never runs out of stock.

Figure 1.2 Inventory System Process

Figure 1.2 shows the input, the process and the output of our models. We will define the elements of the inventory system process in Chapter 5 and Chapter 6.

1.4.1 Assumptions of Proposed Inventory System

a. The demands in bases are independent, identical Poisson processes.

The Poisson process is known as a good approximation of the demand process when customers arrive infrequently and each customer demands one item. In some inventory studies, demand is approximated as a normal process. There are some limitations with normal approximated demand. In theory, a normally distributed random variable may have the value from $(-\infty,\infty)$. However, it is impossible that the demand is negative.

A normal approximation of demand also has some problems in the computer simulation of the inventory system. In our simulation, the simulation clock is advanced to the next event time. When the normal random variable is used to decide the interval, there may be a negative time advance. If a normal distribution is used in the approximation of the demand process, truncating negative values of the normal random variable is required.

b. The leadtimes at the depot and bases are constant. All bases have the same leadtime.

The leadtime at the depot is the transportation time of shipping items from the outside supplier to the depot. The base leadtime is the transportation time of shipping items from the depot to each base. The assumption of the constant leadtime is very important to the FCFS policy.

Our inventory system notifies customers of the time when they will receive their items, based on the outstanding orders at the depot and bases and number of waiting customers. With constant base leadtime, we are sure that bases receive their orders in the same sequence when they send out their base orders. Any customer whose demand had been assigned to an early base order will receive his or her item before those who later arrived at the same base.

If the leadtime is a random variable, the FCFS policy may be violated. In a random variable base leadtime two-echelon inventory system, the replenishment time of each base order is determined with certain types of probability distributions such as normal distribution, exponential distribution, etc. When the base leadtime is a random variable, customers whose demand had been assigned to early base orders may receive their items after some customers who ordered items later and were put into a later base order.

For the same reason, the depot leadtime should also be constant to enforce the FCFS policy. Otherwise, the base orders backlogged and assigned to a depot order may arrive at the depot later than base orders assigned to a later depot order. The late arrival of base orders at the depot causes a violation of the FCFS customer service policy at bases.

c. When a base runs out of stock, customer demand is backlogged.

In our two-echelon inventory system, there are leadtimes at the depot and bases when the depot or a base requests a replenishment. The replenishment will not arrive at the requested facility immediately.

We use the reorder points at the depot and bases to determine when an order will be initiated at a base or the depot. A negative reorder point at the depot and/or bases is considered in our study. A negative reorder point means that a facility only sends out an order to its supplier when there are some outstanding orders at the facility. Here, the facility can be a base or the depot.

In either case, customers may be backlogged because of the stockout at a base. When the inventory system backlogs its customers, there are certain costs involved. The cost of goodwill and offering incentives to retain unhappy customers are two types of the backorders cost. As the customer waiting time increases and/or more customers are waiting for the items, the backorders cost increases. We assume that the backorders cost depends on the amount of backorders and the customer waiting time at bases.

We only consider the backorders cost at bases. When the depot runs out of stock, base orders are backlogged at the depot. It seems that the backorders cost at the depot should be considered. However, because of the relationship between the customer waiting time at the base and the depot backorders, we do not need to consider the depot backorders cost.

When the depot runs out of stock, those bases that put orders to the depot will wait longer than the base leadtime to receive their orders from the depot. The actual replenishment time depends on the number of outstanding base orders at the depot and the number of orders in the transportation process from the outside supplier to the depot. When the depot reorder point is positive, the replenishment time is between base leadtime and the sum of the depot leadtime and the base leadtime. The replenishment time is longer when the depot reorder point is negative. The replenishment time affects the customer backorders cost at bases. Therefore, the effect of the depot stockout on the two-echelon inventory system is reflected in the customer backorders cost at bases.

1.4.2 Topics to be Studied

In our study, we cover several important topics in the two-echelon inventory system. We develop both the analytical model and computer simulations.

a. Demand Process at the Depot

The way to approach the demand process at the depot constitutes the difference among two-echelon (Q, R) inventory system studies. We assume that the demand process at each base is a Poisson process. When bases use the (Q_B, R_B) policy, each base sends one order to the depot after it receives Q_B customers orders, where Q_B is the base ordering quantity. The ordering process from each base to the depot is a point process with the interval between the base orders distributed as a Q_B-stages Erlang process.

The demand at the depot is a point process which is the superposition of the ordering processes of bases. It is very difficult to study the depot demand process. All two-echelon inventory models use some types of approximations. We use the Poisson process to approximate the depot demand process. Franken's (1963) paper lays the theoretical basis for such an approximation. In our study, we investigate in detail how "good" our approximation is. The results of our simulation and the tests of goodness of fit based on simulation results are extensively discussed.

b. Steady State Probabilities of the Two-echelon Models

The steady state probabilities of on-hand inventory, backorders at the depot and bases help us to understand the performance of the two-echelon inventory system. We analyze the inventory system by exploring the relationship between the status of the depot and that of bases. That relationship has not been fully explored previously. We get the steady state probabilities that depot and bases have on-hand inventory and backorders with different decision variables such as the reorder points at bases and the depot and the ordering quantities of bases and the depot.

c. The Expected Holding Costs at Bases and the Depot and the Backorders Cost at Bases and Convexity Analysis

We determine these costs by using the steady state probabilities of the inventory system. We also study the conditions of the convexity of the total cost function. Convexity is an important issue when we want to get the global optimum solution of the total cost function instead of a local optimum solution. The total cost of the two-echelon inventory system is not a linear function of decision variables, such as the reorder points at bases and the depot and the ordering quantities of the depot and bases.

d. An Algorithm to Get the Optimal Values of the Decision Variables and the Minimum Total Cost

As we know, it is very difficult to find out the optimal values of the decision variables and the total cost of the two-echelon inventory system. Most two-echelon inventory models get the optimum reorder points at bases and the depot (Axsater, 1994). With the Poisson approximation of the depot demand process, the total cost is a function of the system parameters and the four decision variables of the two-echelon inventory system. The system parameters are the unit holding cost, the ordering cost, and the leadtime at the depot and the base. The backorders cost at the base and the customer arrival rate at each base are also system parameters. Since our two-echelon inventory system has more than one base, the number of bases in the system is a system parameter. The four decision variables in our study are the reorder points and the ordering quantities at the depot and bases.

We develop an algorithm that searches every direction of changes for our four decision variables to minimize the total cost of our system. Four decision variables of the two-echelon inventory model are ordering quantities and reorder points at the depot and bases. In the searching process, we consider the interrelationship among the decision variables and limitations of the system parameters on the decision variables. The four decision variables and the minimum total cost are functions of the system parameters. We also study the effect of these parameters on the minimum total cost and decision variables.

e. Computer Simulation of the Inventory Systems

Computer simulation plays an important role in the study of multi-echelon inventory systems. The complicated interactions between inventory policies and customer behavior at bases, as well as the inventory policy followed by the depot can be explored by performing extensive simulations.

In the study, we use Borland C 4.0 to program the analytical model and the simulation. We also use the Simscript II.5, a simulation language in the simulation programming. With the simulation results, we conduct several goodness-of-fit tests for the depot demand approximation. We also compare the total cost of our analytical inventory model with that of the simulations.

1.5 NOTATION

bin(y,m,p): Binomial probability of having m events in the total of y events with the probability of p.

$$bin(y,m,p)=\binom{y}{m}(1-p)^{y-m}p^{m}$$

B_B: Annual backorders cost per unit at a base.

$E^B{}_1$: Expected on-hand inventory at a base

$E^B{}_2$: Expected backorders at a base

E^D_I: Expected on-hand inventory at the depot

H_B: Holding cost per unit at a base

H_D: Holding cost per unit at the depot

L_B: Constant transportation time from the depot to bases, which is identical for all bases

L_D: Constant transportation time from outsider suppliers to the depot,

N_B: Number of bases in the inventory system

$P(x,\mu)$: Cumulative Poisson probability of x or more events with the expected number of arrivals μ

P^D_{out}: Probability that the depot is backlogged

$P^B_{\infty}(i)$: Probability that a base has i orders backlogged in the depot

Q_B: Ordering quantity of a base in term of units

Q_D: Ordering quantity of the depot in term of base orders.

R_B: Reorder points at bases in term of units

R_D: Reorder point at the depot in term of base orders

S_B: Ordering cost at a base

S_D: Ordering cost at the depot

TC: Total cost of the two-echelon inventory system

λ_B: Customer arrival rate at each base

$\lambda_D = \lambda_B N_B / Q_B$: Base order arrival rate at the depot

$\mu = \lambda L$: Expected number of demands at a system during the leadtime L

$\mu_B = \lambda_B L_B$: Expected number of demands at a base during the leadtime L_B

$\mu_D=\lambda_D L_D$: Expected number of base orders at the depot during the leadtime L_D

$\rho(i)$: Probability that the inventory position is i

$\psi^B{}_1(x)$: Probability that a base has x units of on-hand inventory

$\psi^B{}_2(y)$: Probability that a base has y units of backorders

$\psi^D{}_1(x)$: Probability that the depot has x base orders of on-hand inventory

$\psi^D{}_2(y)$: Probability that the depot has y base orders of backorders

CHAPTER 2

DEPOT ANALYSIS OF TWO-ECHELON (Q, R) MODEL

2.1 ANALYSIS OF BASIC (Q, R) MODEL

Our study of the two-echelon inventory model is based on the study of a single-echelon (Q, R) inventory model. In the single-echelon inventory model, there is no depot and there is only one base to serve customers. Hadley (1963) studied a (Q, R) model with a Poisson demand process and constant leadtime. Formulas of the expected on-hand inventory, the expected backorders with the reorder point $R \geq 0$ can be found in his book. He also got the steady state probabilities that the inventory system has on-hand inventory or backorders.

We develop formulas of the steady state probability of the (Q, R) model, the expected on-hand inventory and expected backorders when the reorder point R is negative. We feel that this extension is necessary. In a (Q, R) inventory model, the system parameters, such as the leadtime, customer arrival rate and the cost factors, determine the values of the decision variables, such as the ordering quantity and the reorder point. When a system has a negative reorder point, we assume that $Q+R > 0$. After the inventory system receives its order, the system will have some on-hand inventory if there is no demand during the replenishment time. We feel that this assumption is reasonable. Most companies have facilities to store their inventories. If Q+R is less than one for a company, the company does not need any facility to store inventory. The backorders of the inventory system increase as the system operates. This is not the model we study.

If the backorders cost is very small compared to the holding cost, the inventory system should have a negative reorder point to reduce the holding cost of the system. In this case, the manager of the inventory system sends out a replenishment order to the outside supplier when there are some outstanding orders from his customers.

We study the (Q, R) model with a Poisson demand and constant leadtime. When the inventory system runs out of stock, customers' orders are fully backlogged. Of course, the system has the backorders cost. The backorders cost depends on the duration of backorders and number of backorders. We also assume that customers receive their goods or services with the rule of "first come, first served" (FCFS). As we indicated before, the constant leadtime guarantees such a rule. If the leadtime is randomly distributed, the FCFS rule may be violated.

The idea of inventory position of the inventory system is widely used in our study. In an inventory system, the inventory position at any time is defined as the on-hand inventory plus any outstanding orders minus possible backorders. The initial inventory position of our model is Q+R, where Q is the ordering quantity and R is the reorder point. We assume that customers demands are fully backlogged if the inventory system runs out of stock. With such an assumption, the inventory position of our model will be reduce by one when a customer arrives. When the inventory position reaches the reorder point, our system orders Q units from the outside supplier. By doing this, the inventory position is Q+R units though the on-hand inventory is not Q+R units now. The inventory position of our model changes within the range of [R+1, R+Q].

Hadley (1963) proves that with full backorders and Poisson demand, the probability that the inventory position at any of the following states: Q+R, Q+R-1,..., R+1 is 1/Q. ρ(j) is the probability that the inventory position is j units at time t.

$$\rho(j)=\frac{1}{Q} \quad where: j=R+1...,R+Q$$

This conclusion is very important to our study. With constant leadtime, we can get the steady state probabilities of on-hand inventory and backorders of the system.

For a (Q, R) inventory system with constant leadtime L, orders sent out by the inventory system to the outside supplier have an important feature. After the system sends out an order to the outsider supplier, it takes leadtime L for the system to receive the order. Orders sent out by the system before the time t-L will be received by the system at time t. Any orders sent out by the system after the time t-L will not be received by the system at time t. We use this finding to get the steady state probabilities of on-hand inventory and backorders in the system.

2.1.1 When Reorder Point R≥0

With our previous discussion of the constant leadtime and the inventory position, the on-hand inventory of the inventory system is between [R+1, Q+R] at time t, if there is no demand during [t-L, t] and the reorder point R≥0. With Hadley's finding, the probability that the on-hand inventory is any value between [R+1, Q+R] is 1/Q.

a. On-hand Inventory Analysis

On-hand inventory is the number of units available to customers in the inventory system. As we know, if the inventory position is j units at time t-L and there is no demand during [t-L, t], the system on-hand inventory is j units at time t. However, if the demand during [t-L, t] is j-x units, the system on-hand inventory is x units at time t. Of course, we assume that j is greater than x. Based on Hadley's finding, the probability that the system inventory position is j units is 1/Q. $\psi_1(x)$ is defined as the probability of x units of on-hand inventory at time t.

$$\begin{aligned} &\psi_1(x)=\sum_{j=x}^{Q+R}\rho(j)p(j-x,\mu)=\frac{1}{Q}[1-P(R+Q+1-x,\mu)] \\ &\qquad where\ \ R+1\le x\le R+Q \\ &\psi_1(x)=\sum_{j=1}^{Q}\rho(R+j)p(R+j-x,\mu)=\frac{1}{Q}[P(R+1-x,\mu)-P(R+Q+1-x,\mu)] \\ &\qquad where\ \ x\le R+1,\ \ \mu=\lambda L \end{aligned} \tag{2.1}$$

$p(n, \mu)$ is the Poisson probability that the demand is n units where μ is the demand rate. $P(n, \mu)$ is the cumulative Poisson probability that there are n or more units demand where μ is the demand rate.

$$p(n,\mu)=\frac{\mu^n}{n!}e^{-\mu}$$
$$P(n,\mu)=\sum_{i=n}^{\infty} p(i,\mu) \qquad (2.2)$$

b. Backorders Analysis

We know that if the inventory position is j units at time t-L and there is no demand during [t-L, t], the system on-hand inventory is j units at time t. However, if the demand during [t-L, t] is j+x units, the system will not have on-hand inventory time t. Instead, there are j backorders at time t. $\psi_2(y)$ is defined as the probability of y backorders in the system at time t.

$$\psi_2(y)=\sum_1^Q \rho(R+j)p(y+R+j,\mu)=\frac{1}{Q}[P(y+R+1,\mu)-P(y+R+Q+1,\mu)] \quad y\geq 0 \qquad (2.3)$$

2.1.2 When Reorder Point R < 0

The inventory system reorder point R may be negative. An example of negative reorder point is that a firm will only order or produce Q items when it is backlogged. When the reorder point is negative, we have the assumption of Q+R > 0.

With negative reorder point R, the on-hand inventory of the system is between [0, Q+R] at time t, if there is no demand during [t-L, t]. Based on Hadley's finding, the probability that the system has j units of on-hand inventory is 1/Q in this case.

a. On-hand Inventory Analysis

If the inventory position is $j \geq 0$ units at time t-L and the demand during [t-L, t] is j-x units, the inventory system will have x units of on-hand inventory at time t. $\psi_1(x)$ is the probability of x units of on-hand inventory at time t.

$$\psi_1(x) = \sum_{j=x}^{Q+R} p(j)p(j-x,\mu) = \frac{1}{Q}[1 - P(Q+R+1-x,\mu)] \qquad (2.4)$$
$$\textit{where: } 0 \leq x \leq Q+R$$

b. Backorders Analysis

With negative reorder point R, the system backorders are between [1,-R] at time t, if there is no demand during [t-L, t]. Based on Hadley's finding, the probability that there are j backorders at time t is 1/Q in this case. $\psi_2(y)$ is the probability of y units of backorders at time t.

If the inventory position is j units at time t-L and the demand during [t-L, t] is j+y units, the inventory system will have y backorders at time t.

$$\Psi_2(y)=\sum_{j=-y}^{Q+R} \rho(j)p(j+y,\mu)=\frac{1}{Q}[1-P(y+Q+R+1,\mu)] \quad \textit{where: } 0\le y\le -R-1$$
$$\Psi_2(y)=\sum_{j=1}^{Q} \rho(R+j)p(y+R+j,\mu) \qquad (2.5)$$
$$=\frac{1}{Q}[P(y+R+1,\mu)-P(y+R+Q+1,\mu)] \quad \textit{where: } y\ge -R$$

In this section, we show the steady state probabilities that the inventory system has on-hand inventory or backorders at any time. Hadley (1963) develops the first part of this section, where the reorder point is positive. We develop the second part of this section, where the reorder point is negative. These formulas developed in the one inventory facility are used in our analysis of the two-echelon inventory system which is the focus of our study. We realize that the negative reorder point assumption is especially necessary in the two-echelon inventory model study.

2.2 THE APPROXIMATION OF THE DEPOT DEMAND PROCESS

For two-echelon (Q, R) inventory systems, one of the most important issues is the approximation of the depot demand process. When bases send their orders to the depot, these orders constitute the depot demand process. The depot demand process is a point process which takes only the positive integer values. The approximation of the depot demand process is also the major difference among two-echelon (Q, R) inventory studies.

We use a Poisson distribution to approximate the depot demand process. By using the Poisson process approximation of the depot demand, the base analysis is relatively straightforward. In the rest of this chapter, we first study the feasibility of the approximation of the depot demand process by a Poisson distribution. In Chapter 5, we will use the simulation results and goodness of fit tests based on the simulation results to show the feasibility of such an approximation.

In our two-echelon (Q, R) inventory model, customers arrive at each base according to a Poisson distribution. The customer arrival process at each base is independent of customer arrival processes at other bases in the same system. Each base uses (Q_B, R_B) replenishment policy. When the inventory position at a base reaches the reorder point R_B, the base orders Q_B units as a base order from the depot. After another Q_B customers arrive at the same base, the base will send another base order of Q_B units to the depot. The base ordering process is a renewal process. The interval between orders from the same base is distributed as an Erlang process with Q_B stages. There are N_B bases in our inventory system. The depot demand process is the superposition of N_B base ordering processes.

The superposition of renewal processes is not a renewal process unless these renewal processes are Poisson processes (Cinlar, 1972). Our theoretical basis of the approximation of the depot demand process by a Poisson process is based on Franken's article (1963). Franken (1963) proves that under certain conditions, the superposition of independent renewal processes can be approximated by a Poisson distribution. The approximation error decreases proportionally as the number of renewal processes increases. S. Albin (1982) shows that the superposition can be satisfactorily approximated by a Poisson process when less than ten independent point processes with Erlang distributed renewal interval are involved. Our simulation results and several goodness-of-fit tests in Chapter 5 show that such an approximation is reasonable.

We can find out the probability distribution of the base ordering process to the depot with Erlang distributed intervals. A common way to do that includes two steps. (Cox, 1970)

Step 1. find out the cumulative Erlang distribution function F(t) by integration.

Step 2. find out the point process with the convolution of F(t).

We use a different approach to find out the base ordering process distribution, which is simple and can be easily explained

We define $P[N(t)=n,Q_B,\lambda]$ as the probability that a base sends n orders to the depot with the ordering quantity of Q_B units during interval t. The customer arrival rate at the base is λ.

Proposition 2.1: Define $P[N(t)=n,Q_B,\lambda]$ as the probability that a base sends n orders to the depot with the ordering quantity of Q_B units during interval t, where customers arrival at the base is a Poisson process with the arrival rate of λ. The base ordering process to the depot is a point process with the probability distribution as:

$$P[N(t)=n,\ t,\ Q_B,\ \lambda]=\sum_{i=nQ_B}^{(n+1)Q_B-1}\frac{(\lambda t)^i}{i!}e^{\lambda t} \qquad (2.6)$$

Proof: We prove proposition 2.1 by proving the distribution of interval corresponding to the above point process is an Erlang process.

$$\begin{aligned} P(T>t)=P[N(t)=0,t,Q_B,\lambda]&=\sum_{i=0}^{Q_B-1}\frac{(\lambda t)^i}{i!}e^{\lambda t}\\ F(t)=1-P(T>t)&=1-\sum_{i=0}^{Q_B-1}\frac{(\lambda t)^i}{i!}e^{\lambda t}\\ f(t)=\frac{dF(t)}{dt}&=\frac{\lambda(\lambda t)^{Q_B-1}}{(Q_B-1)!}e^{\lambda t}\end{aligned} \qquad (2.7)$$

Therefore, f(t) is the probability density function of an Erlang process with Q_B stages.

Proposition 2.2: As time $t \to \infty$, the rate of the base orders to the depot will be λ/Q_B.

Proof: the moment generating function of the Erlang process $\phi(s)$ is:

$$\phi(s)=\frac{\lambda^{Q_B}}{(s+\lambda)^{Q_B}} \qquad (2.8)$$

The renewal function of the Erlang process m(t) is

$$m(s)=\frac{\phi(s)}{s[1-\phi(s)]}=\frac{\lambda^{Q_B}}{s[(s+\lambda)^{Q_B}-\lambda^{Q_B}]}=\frac{\lambda}{s^2Q_B}-\frac{Q_B-1}{2sQ_B}+o(\frac{1}{s})$$

$$m(t)=\frac{\lambda t}{Q_B}-\frac{Q_B-1}{2Q_B}+o(1) \qquad (2.9)$$

where $o(t)$ *is a finite such that* $limit_{t\to\infty}\frac{o(t)}{t}=0$

The rate of renewal process is $limit_{t\to\infty}\frac{m(t)}{t}=\frac{\lambda}{Q_B}$

Since we proved the proposition 2.2, we are sure that the base order rate to the depot is λ/Q_B. With N_B bases in the inventory system, the base orders arrival rate at the depot is $\lambda N_B/Q_B$.

Franken Finding (1963): Let $M_1, M_2, \ldots, M_n$ be independent renewal processes, all having the same distribution function F(t) for their intervals. Suppose $H(t)=E\{M_i(t)\}$ is such that $nH(t)=E\{N(t)\}$ for some renewal process N. If $F(0) < 1/2$, then for any interval A we can expand the cumulative probability distribution of $M_1(A)+..+M_n(A)$ as

$$\sum_{k\leq x}\phi(k)=\sum_{k\leq x}[\psi(k)+\sum_{i=1}^{m}\frac{\gamma_i(k)}{n^i}]+o(\frac{1}{n^{m+1}}) \qquad (2.10)$$

where ψ is the Poisson distribution with parameter H(A) and the $\gamma_i(k)$ are polynomials in k.

Proposition 2.3: The base ordering processes at the depot can be approximated by a Poisson process when each base has a (Q_B, R_B) policy and the customers arrival process at each base is a Poisson distribution independent of other bases' customer arrival processes.

Proof: For the point process with Erlang distribution of interval time, we have

$$F(0)=1-P(T>0)=\sum_{i=1}^{Q_B-1}\frac{(\lambda t)^i}{i!}e^{-\lambda t}\Big|_{t=0}=0<\frac{1}{2} \qquad (2.11)$$

That is, the demand process at the depot can be approximated by a Poisson process. The approximation error is a function of $1/N_B$, where N_B is the number of bases in the system.

When the depot demand process is approximated by a Poisson distribution, we can find out the steady state probabilities that the depot has on-hand inventory or backorders at time t by using the results of Section 2.1.

Proposition 2.4 When the demand at the depot is distributed as a Poisson process, the inventory position at the depot in uniformly distributed. That is,

$$p(R_D+j)=\frac{1}{Q_D} \quad where: \; j=1,\ldots,Q_D \qquad (2.12)$$

Proof: The demand at the depot is approximated by a Poisson process with demand rate of $\lambda_B N_B/Q_B$. Base orders at the depot are fully backlogged and the depot uses a (Q_D, R_D)

policy. Based on Hadley's (1963) study, the inventory position at the depot is uniformly distributed.

After we approximate the base ordering process at the depot by a Poisson process, the only effect of bases on the depot is the base ordering quantity Q_B. When we study depot performance, we do not need to know the inventory level at bases.

2.3 DEPOT ON-HAND INVENTORY ANALYSIS

In the depot inventory analysis, we use the Poisson process to approximate the demand process at the depot. $\psi_1^D(x)$ is defined as the probability that there are x base orders of on-hand inventory at the depot.

2.3.1 When $R_D \geq 0$

$$\begin{aligned} \psi_1^D(x) &= \frac{1}{Q_D}[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \quad \textit{where} \quad 0 \leq x < R_D+1 \\ &= \frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad \textit{Where} \ R_D+1 \leq x \leq R_D+Q_D \end{aligned} \tag{2.13}$$

2.3.2 When $R_D < 0$

$$\psi_1^D(x) = \frac{1}{Q_D}[1-P(Q_D+R_D+1-x,\mu_D)] \quad \textit{where:} \ 0 \leq x \leq Q_D+R_D \tag{2.14}$$

The depot expected on-hand inventory is: $E_1^D=\sum_1^{R_D+Q_D} x\psi_1^D(x)$

2.4 DEPOT BACKORDERS ANALYSIS

$\psi_2^D(y)$ is defined as the probability that the depot has y backorders.

2.4.1 When $R_D \geq 0$

$$\psi_2^D(y)=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \quad y\geq 0 \tag{2.15}$$

2.4.2 When $R_D<0$,

$$\begin{aligned}\psi_2^D(y)&=\frac{1}{Q_D}[1-P(y+Q_D+R_D+1,\mu_D)] && where\colon\ 0\leq y\leq -R_D-1\\ &=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] && where\colon\ y\geq -R_D-1\end{aligned} \tag{2.16}$$

P^D_{out} is defined as the probability that base orders to the depot are backlogged.

$$P_{out}^D=\sum_{y=1}^{\infty} \psi_2^D(y) \tag{2.17}$$

In the analysis of the depot inventory, we isolate the depot from bases. The only effect of bases on the depot is Q_B.

CHAPTER 3

BASE ANALYSIS OF THE TWO-ECHELON (Q, R) MODEL

Contrary to the effect of bases on the depot performance, the base performance depends on the inventory level at the depot. The availability of inventory at the depot determines whether a base will receive its order in the leadtime L_B. If the depot has on-hand inventory when a base sends an order to the depot, the base will receive its order in the leadtime L_B. On the other hand, if the depot runs out of stock when a base sends an order, the base order will be backlogged at the depot.

In this chapter, we study the inventory level at bases in two cases: 1. The base reorder point $R_B \geq 0$; 2. The base reorder point $R_B < 0$, where the analysis of base is slightly different.

3.1 BASE ON-HAND INVENTORY ANALYSIS

The on-hand inventory at each base is affected by the demand, the reorder point and the ordering quantity of the base. Because the depot affects the base inventory replenishment process, a base may not receive its orders if the depot runs out of stock. The effect of depot inventory level on bases inventory availability complicates the analysis of the two-echelon inventory systems.

3.1.1 When $R_B \geq 0$

If at time t-L_B, the depot is not backlogged or the depot is backlogged with y orders, but none of these y orders comes from the base i we are concerned, base i performance at time t is independent of the depot. The probability that the depot is not stockout is 1-P^D_{out}. Let $P_{so}^B(0)$ be the probability that all base i outstanding orders will be received by base i.

$$P_{so}^B(0)=\sum_{y=1}^{\infty}(1-\frac{1}{N_B})^y\psi_2^D(y)=\sum_{y=1}^{\infty} bin(y,0,\frac{1}{N_B})\psi_2^D(y) \tag{3.1}$$

$\psi_1^B(x)$ is defined as the probability that x units are on hand at base i.

a. When all base i outstanding orders will arrive during [t-L_B, t]

$$\begin{aligned}\psi_1^B(x)&=[1-P_{out}^D+P_{so}^B(0)]\sum_{j=x}^{R_B+Q_B}\rho(j)p(j-x,\mu_B)\\ &=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(R_B+Q_B+1-x,\mu_B)] \quad where\ R_B+1\leq x\leq R_B+Q_B\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\psi_1^B(x)&=[1-P_{out}^D+P_{so}^B(0)]\sum_{1}^{Q_B}\rho(R_B+j)p(R_B+j-x,\mu_B)\\ &=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \quad where\ 0\leq x<R_B+1,\end{aligned} \tag{3.2b}$$

b. When some of base i orders are backlogged at the depot at time t-L_B

We define $P_{SO}^B(m)$ as the probability that m base orders from base i at time t-L_B are backlogged at the depot.

$$
\begin{aligned}
P_{so}^{B}(m) &= \sum_{y=m}^{\infty} bin(y,m,\frac{1}{N_B})\psi_2^{D}(y) \quad \textit{when } m\geq 1 \\
&= \sum_{1}^{\infty} bin(y,0,\frac{1}{N_B})\psi_2^{D}(y) \quad \textit{when } m=0
\end{aligned}
\tag{3.3}
$$

We consider the general case that R_B may be greater than Q_B. Those m outstanding orders of base i at the depot will not be received by the base i at t. The probability that base i has x units of on-hand inventory is:

$$
\begin{aligned}
\psi_1^{B}(x) = {} & \frac{1}{Q_B}\sum_{m=1}^{[\frac{R_B-x}{Q_B}]^*} p_{so}^{B}(m)[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
& +\frac{1}{Q_B}p_{so}^{B}([\frac{R_B-x}{Q_B}+1]^*)[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^*Q_B-x,\mu_B)] \\
& \textit{where } [\frac{R_B-x}{Q_B}]^* \textit{ is the largest integer } \leq \frac{R_B-x}{Q_B},\ 1\leq x\leq R_B
\end{aligned}
\tag{3.4}
$$

After we consider **Part a and Part b**, the probability that a base has x units of on-hand inventory if the reorder point of the base $R_B \geq 0$ is:

$$
\psi_1^{B}(x) = \frac{1-P_{out}^{D}+p_{so}^{B}(0)}{Q_B}[1-P(R_B+Q_B+1-x,\mu_B)] \quad \textit{where } R_B+1\leq x\leq R_B+Q_B \tag{3.5a}
$$

$$\psi_1^B(x)=\frac{1-P_{out}^D}{Q_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^*} p_{so}^B(m)[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \qquad (3.5b)$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B-x}{Q_B}+1]^*)[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^*Q_B-x,\mu_B)] \quad where: 1\le x\le R_B$$

3.1.2 When $R_B<0$

If the base reorder point is negative, the base only sends an order to the depot when there are some outstanding customers waiting for the product. The on-hand inventory at the base is zero when the base sends an order to the depot. If any order from the base i is backlogged at the depot at time t-L_B, base i does not have any on-hand inventory at time t.

From the above discussion, we conclude that base i only has on-hand inventory at time t if the depot was not running out of stock or none of backorders at the depot was from base i at time t-L_B. The probability that a base has x units of on-hand inventory at time t is:

$$\psi_1^B(x)=[1-P_{out}^D+p_{so}^B(0)]\sum_{j=x}^{R_B+R_B}\rho(j)p(j-x,\mu_B)$$
$$=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(Q_B+R_B+1-x,\mu_B)] \quad where:\ 0\le x\le Q_B+R_B \qquad (3.6)$$

After we consider the formulas of probability of the base on-hand inventory with $R_B \geq 0$ and $R_B < 0$ in (3.5a), (3.5b) and (3.6), we get the general formula of the probability of on-hand inventory at base i.

$$\psi_1^B(x) = \frac{1-P_{out}^D + p_{so}^B(0)}{Q_B}[1-P(R_B+Q_B+1-x,\mu_B)] \tag{3.7a}$$
$$\text{where } R_B \geq 0,\ R_B+1 \leq x \leq R_B+Q_B \vee R_B<0, x \leq R_B+Q_B$$

$$\psi_1^B(x) = \frac{1-P_{out}^D}{Q_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^{*}} p_{so}^B(m)[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \tag{3.7b}$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B-x}{Q_B}+1]^{*})[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^{*}Q_B-x,\mu_B)]$$
$$\text{where } R_B \geq 0,\ 0 \leq x < R_B+1,$$

The expected on-hand inventory at a base, $E^B{}_1$:

$$E_1^B = \frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B}) \tag{3.8a}$$
$$[\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\sum_{x=R_B-mQ_B+1}^{R+Q_B-mQ_B} x[1-P(R_B+Q_B+1-x-mQ_B,\mu_B)]] \text{ where: } R_B \geq 0$$

$$E_1^B = \frac{1-\sum_{y=1}[1-(1-\frac{1}{N_B})^y]\psi_2^D(y)}{Q_B}\sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \qquad (3.8b)$$

$$\textit{Where } R_B<0$$

3.2 BASE BACKORDERS ANALYSIS

Within a two-echelon inventory system, bases receive their orders from the depot instead of the outsider supplier directly. Since the availability of depot inventory affects the base replenishment process, the expected backorders at a base will be higher than that at a similar base that receives its supplies directly from outside supplier.

As we did with the analysis of on-hand inventory at bases, we consider two cases: **1.** the reorder point of each base $R_B \geq 0$; **2.** the reorder point of each base $R_B < 0$. Since the logic we use in developing the base backorders is similar to the one we used in developing the base on-hand inventory, the process will not be examined in detail.

3.2.1 When $R_B \geq 0$

a. If the depot has on-hand inventory or the base orders are backlogged at the depot but none of backlogged base orders is from base i at time t-L_B, base i performance at time t is independent of the depot. The probability that y units are backlogged at base i, $\psi_2^B(y)$:

$$\psi_2^B(y)=[1-P_{out}^D+P_{so}^B(0)]\sum_{j=1}^{Q_B}\rho(R_B+j)p(y+R_B+j,\mu_B)$$
$$=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[P(y+R_B+1,\mu_B)-P(y+R_B+Q_B+1,\mu_B)] \quad \textit{where } y\geq 0, R_B\geq 0 \tag{3.9}$$

b. When some of base i orders are backlogged in the depot at time t-L_B, we need to consider the number of base i orders backlogged at the depot. The probability that y units are backlogged at base i, $\psi_2^B(y)$, is:

$$\psi_2^B(y)=\frac{1}{Q_B}\sum_{m=1}^{[\frac{y+R_B}{Q_B}]^+} p_{so}^B(m)[P(y+R_B+1-mQ_B,\mu_B)-P(y+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B+y}{Q_B}+1]^+)[1-P(y+R_B+Q_B+1-[\frac{R_B+y}{Q_B}+1]^+Q_B,\mu_B)] \tag{3.10}$$
$$\textit{where } y\geq 0,\ m\geq 1,\ R_B\geq 0$$

Considering **Part a** and **Part b**, when $R_B\geq 0$, the probability that a base has y backorders is:

$$\psi_2^B(y)=\frac{1-P_{out}^D}{Q_B}[P(y+R_B+1,\mu_B)-P(y+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{y+R_B}{Q_B}]^+} p_{so}^B(m)[P(y+R_B+1-mQ_B,\mu_B)-P(y+R_B+Q_B+1-mQ_B,\mu_B)] \tag{3.11}$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B+y}{Q_B}+1]^+)[1-P(y+R_B+Q_B+1-[\frac{R_B+y}{Q_B}+1]^+Q_B,\mu_B)] \quad \textit{where } y\geq 0$$

the expected backorders in a base, when the reorder point $R_B \geq 0$, is:

$$E_2^B=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=1}^{y} bin(y,m,\frac{1}{N_B})\sum_{z=mQ_B}^{\infty} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \tag{3.12}$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})$$
$$\sum_{z=mQ_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

3.2.2 When $R_B<0$

a. If the depot has on-hand inventory or the depot runs out of stock but none of base i orders are backlogged in the depot at time t-L_B, base i runs as if it is independent of the depot. The probability that z units are backlogged at base i at time t is:

$$\psi_2^B(z)=[1-P_{out}^D+P_{so}^B(0)]\sum_{j=z}^{Q_B+R_B}\rho(j)p(z+j,\mu_B)$$
$$=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(z+R_B+Q_B+1,\mu_B)] \quad where\ 0\leq z\leq -R_B-1 \tag{3.13a}$$

$$\psi_2^B(z)=[1-P_{out}^D+P_{so}^B(0)]\sum_{i=1}^{Q_B}\rho(R_B+i)p(z+R_B+i,\mu_B)$$
$$=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[P(z+R_B+1_B,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \quad where\ z\geq -R_B \tag{3.13b}$$

b. When some of base i orders are backlogged in depot at time t-L_B, the probability that base i has z backorders at time t is:

$$\psi_2^B(z)=\frac{1}{Q_B}\sum_{m=1}^{[\frac{z+R_B}{Q_B}]^*} p_{so}^B(m)[P(z+R_B+1-mQ_B,\mu_B)-P(z+Q_B+R_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{z+R_B}{Q_B}+1]^*)[1-P(z+R_B+1-[\frac{R_B+z}{Q_B}+1]^*Q_B,\mu_B)] \quad z\geq -R_B \tag{3.14}$$

Considering formula (3.13a), (3.13b) and (3.14), the probability that z customers are backlogged at a base, when $R_B<0$, is

$$\psi_2^B(z)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(z+R_B+Q_B+1,\mu_B)] \quad where \quad 0\leq z\leq -R_B-1 \tag{3.15a}$$

$$\psi_2^B(z)=\frac{1-P_{out}^D}{Q_B}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{z+R_B}{Q_B}]^*} p_{so}^B(m)[P(z+R_B+1-mQ_B,\mu_B)-P(z+Q_B+R_B+1-mQ_B,\mu_B)] \tag{3.15b}$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B+z}{Q_B}+1]^*)[1-P(z+R_B+Q_B+1-[\frac{R_B+z}{Q_B}+1]^*Q_B,\mu_B)] \quad where \quad z\geq -R_B$$

The expected backorders at a base, when $R_B<0$, are:

$$\begin{aligned} E_2^B = & \frac{1-\sum_{y=1}^{\infty} \psi_2^D(y)}{Q_B} \sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\ & + \frac{1-\sum_{y=1}^{\infty} \psi_2^D(y)}{Q_B} \sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\ & + \frac{1}{Q_B} \sum_{y=1}^{\infty} \psi_2^D(y) \sum_{m=0}^{y} bin(y,m,\frac{1}{N_B}) \\ & [\sum_{z=\max(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\ & + \sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \end{aligned} \quad (3.16)$$

With the expected on-hand inventory and backorders in the depot and bases and the steady state probability of the inventory system, we determine the effect of the four decision variables: Q_B, Q_D, R_B, R_D, on the performance of the two-echelon systems.

The proof of the uniform distribution of inventory position at the depot and the bases are in **Appendix 1 and Appendix 2**.

CHAPTER 4

CONVEXITY ANALYSIS OF THE TWO-ECHELON INVENTORY SYSTEM

The convexity of the objective function is a major issue in the two-echelon (Q, R) inventory systems. One of the objectives in two-echelon inventory system management is to minimize the total cost of inventory systems. The total cost of a two-echelon inventory system includes the setup cost at the depot and the bases, the backorders cost at bases, and the holding costs at the depot and the bases.

In previous two-echelon inventory system studies (Svoronos 1986, Svoronos and Zipkin 1988), the convexity is proven only to the decision variable of reorder points when the holding costs per unit at a base and the depot are the same. With the economy of scale at the depot facility, the unit holding cost at the depot should be lower than that at a base. Convexity analysis will help us to prove that we get the global optimum, instead of a local optimum, solution.

We obtain the conditions under which the convexity of the total cost with respect to the reorder points at the depot and the bases is held. The total cost consists of the setup cost at the depot and the bases, the backorders cost at the bases and the holding cost at the depot and the bases. We prove the convexity of each part of the total cost with respect to R_B and R_D whenever it is possible. We also prove that the expected on-hand inventory at the depot is a convex function of the ordering quantity of the depot Q_D. The conclusion of the total cost convexity is given at the end of the chapter. Since the setup costs at the depot and the bases are not a function of R_B and R_D, we do not consider the setup cost in the convexity analysis. In the numerical example at the end of this chapter, each part of the total cost is shown to be convex with respect to Q_B and Q_D in the same conditions as to R_B and R_D.

In this chapter, we give the overview of the convexity proof. Since the proof is very complicated, we put the complete proof in **Appendix 3**.

The total cost of our two-echelon inventory system is:

$$TC = N_B\left(\frac{\lambda_B}{Q_B}S_B + H_B E_1^B + B_B E_2^B\right) + \frac{\lambda_D}{Q_D}S_D + H_D Q_B E_1^D$$

In the total cost function, $\lambda_B S_B/Q_B$ is the expected ordering cost at each base. $H_B E^B_1$ is the expected holding cost at each base. $B_B E^B_2$ is the expected backorders cost at each base. $H_D Q_B E^D_1$ is the expected depot holding cost. $\lambda_D S_D/Q_D$ is the expected ordering cost at each base. In this study, the two-echelon inventory system has identical N_B bases.

The objective function is a minimized non-linear integer programming. Because it is an integer programming, we use the direct search method to find out the optimal solution.

Let us introduce the difference operator Δ (Saatty, 1970), which is defined by

$$\Delta f(x)=f(x+1)-f(x)$$
$$\Delta^i f(x)=\Delta^{i-1}[f(x+1)-f(x)]$$

a relative minimization is achieved at x_0 *when:*

$$\Delta f(x_0-1)\le 0,\ \Delta f(x_0)\ge 0,\ \Delta^2 f(x_0)\ge 0$$

We use the first derivative operator and the second derivative operator of the total cost function with respect to R_B and R_D to check the issue of the convexity. When the second derivative operator of the total cost function is positive, the total cost is a convex function to the decision variable.

Analogous to the derivatives, we will prove:

$$\Delta^2_{R_B} TC(Q_B,Q_D,R_B,R_D)\ge 0$$
$$\Delta^2_{R_D} TC(Q_B,Q_D,R_B,R_D)\ge 0$$

4.1 CONVEXITY OF THE EXPECTED DEPOT ON-HAND INVENTORY WITH RESPECT TO R_D

4.1.1 When $R_D \ge 0$

With the Poisson approximation of the base orders at the depot, Chapter 2 gives $\psi_I^D(x)$, the probability that there are x base orders of on-hand inventory at the depot, in (2.13).

$$\psi_1^D(x)=\frac{1}{Q_D}[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \quad where \quad 0\le x<R_D+1$$
$$=\frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad Where \; R_D+1\le x\le R_D+Q_D$$

The expected depot on-hand inventory $E_1^D(R_D)$ is

$$E_1^D(R_D)=\sum_{x=1}^{x=R_D+Q_D} x\psi_1^D(x)$$
$$=\frac{1}{Q_D}\sum_{x=1}^{x=R_D} x[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)]$$
$$+\frac{1}{Q_D}\sum_{x=R_D+1}^{x=R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)] \tag{4.1}$$

$$\Delta E_1^D(R_D)=E_1^D(R_D+1)-E_1^D(R_D)$$
$$=\frac{1}{Q_D}\sum_0^{R_D}[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)]$$
$$+\frac{1}{Q_D}\sum_{R_D+1}^{R_D+Q_D}[1-P(R_D+Q_D+1-x,\mu_D)]>0 \tag{4.2}$$

$$\Delta^2 E_1^D(R_D)=\Delta E_1^D(R_D+1)-\Delta E_1^D(R_D)$$
$$=\frac{1}{Q_D}[P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)]>0 \tag{4.3}$$

4.1.2 When $R_D<0$

With the Poisson approximation of the depot demand process, $\psi_1^D(x)$, the probability that there are x base orders of on-hand inventory at the depot, is given in (2.14).

$$\psi_1^D(x)=\frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad \textit{Where } 1\le x\le R_D+Q_D$$

The expected on-hand inventory at the depot $E_1^D(R_D)$ is

$$E_1^D(R_D)=\sum_{x=1}^{x=R_D+Q_D} x\psi_1^D(x)=\frac{1}{Q_D}\sum_{x=1}^{x=R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)] \qquad (4.4)$$

$$\Delta E_1^D(R_D)=E_1^D(R_D+1)-E_1^D(R_D)=\frac{1}{Q_D}\sum_{0}^{R_D+Q_D}[1-P(R_D+Q_D+1-x,\mu_D)]>0 \qquad (4.5)$$

$$\Delta^2 E_1^D(R_D)=\Delta E_1^D(R_D+1)-\Delta E_1^D(R_D)=\frac{1}{Q_D}[1-P(R_D+Q_D+2,\mu_D)]>0 \qquad (4.6)$$

From 4.1.1 and 4.1.2, we get following conclusions. The expected depot on-hand inventory increases as R_D increases. It is a convex function of R_D.

4.2 CONVEXITY OF THE EXPECTED DEPOT ON-HAND INVENTORY WITH RESPECT TO Q_D

4.2.1 When $R_D \geq 0$

With the Poisson approximation of the depot demand process, $\psi_1^D(x)$, the probability that there are x base orders of on-hand inventory at the depot, is given in (2.13).

$$\psi_1^D(x)=\frac{1}{Q_D}[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \quad where \quad 0\leq x<R_D+1$$
$$=\frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad Where \ R_D+1\leq x\leq R_D+Q_D$$

The expected depot on-hand inventory is given in (4.1).

$$E_1^D(Q_D)=\sum_{x=1}^{x=R_D+Q_D} x\psi_1^D(x)$$
$$=\frac{1}{Q_D}\sum_{x=1}^{x=R_D} x[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)]$$
$$+\frac{1}{Q_D}\sum_{x=R_D+1}^{x=R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)]$$

$$\Delta E_1^D(Q_D)=\frac{1}{2}+\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D(Q_D+1)}P(R_D+Q_D+1-x,\mu_D)$$
$$>\frac{1}{2}+\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D(Q_D+1)}=\frac{1}{2}+\frac{Q_D-1-2Q_D}{2(Q_D+1)}=0 \qquad (4.7)$$

$$\Delta^2 E_1^D(Q_D) > (Q_D^2+Q_D)[P(R_D+Q_D+1,\mu_D)-P(R_D+Q_D+2,\mu_D)] > 0 \qquad (4.8)$$

4.2.2 When $R_D<0$

With the Poisson approximation of the demand process at the depot. $\psi_1^D(x)$, the probability that there are x base orders of on-hand inventory at the depot is given in (2.14).

$$\psi_1^D(x)=\frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad Where\ 1\le x\le R_D+Q_D$$

The expected on-hand inventory at the depot $E_1^D(R_D)$ is given in (4.4).

$$E_1^D(Q_D)=\sum_{x=1}^{x=R_D+Q_D} x\psi_1^D(x)=\frac{1}{Q_D}\sum_{x=1}^{x=R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)]$$

$$\Delta E_1^D(Q_D)=\frac{1}{Q_D(Q_D+1)}\sum_0^{R_D+Q_D}(Q_D-x)[1-P(R_D+Q_D+1-x,\mu_D)]$$

$$Since\ R_D<0, that\ is\ R_D+Q_D<Q_D. \qquad (4.9)$$

$$\sum_0^{R_D+Q_D}(Q_D-x)>0\ \ We\ have\ \Delta E_1^D(Q_D)>0$$

$$\Delta^2 E_1^D(Q_D) = (Q_D^2+Q_D)[1-P(R_D+Q_D+2,\mu_D)]-2\sum_0^{R_D+Q_D}(Q_D-x)[1-P(1,\mu_D)]>0 \quad (4.10)$$

From 4.2.1 and 4.2.2, we have following conclusions. The expected depot on-hand inventory increases as Q_D increases. The expected on-hand inventory in the depot is a **convex** function of Q_D.

4.3 CONVEXITY OF THE EXPECTED BASE ON-HAND INVENTORY WITH RESPECT TO R_D

The reorder point at the depot R_D affects the probability that the depot runs out of stock. The stockout at the depot affects the performance of bases. We discover the conditions under which the convexity of the total holding cost is held with R_D. This has not been done by any other study.

4.3.1 When $R_B<0$:

The expected on-hand inventory at a base is given in (3.8b)

$$E_1^B(R_D) = \frac{1-\sum_{y=1}[1-(1-\frac{1}{N_B})^y]\psi_2^D(y)}{Q_B}\sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$

a. When $R_D \geq 0$:

$$\Delta E_1^B(R_D)=\frac{\sum_{y=1}^{\infty}(1-\frac{1}{N_B})^{y-1}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]}>0 \qquad (4.11)$$

$$\Delta^2 E_1^B(R_D)>-\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B}) \qquad (4.12)$$

b. When $R_D<0$:

$$\Delta E_1^B(R_D)=\frac{\sum_{y=1}^{-R_D-1}(1-\frac{1}{N_B})^{y-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_B Q_D * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} + \frac{\sum_{y=-R_D}^{\infty}(1-\frac{1}{N_B})^{y-1}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} \qquad (4.13)$$

$$\Delta^2 E_1^B(R_D)>-\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B}) \qquad (4.14)$$

4.3.2 When $R_B \geq 0$:

The expected on-hand inventory at a base is given in (3.8a).

$$E_1^B = \frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})[\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]$$

a. When $R_D \geq 0$:

$$\Delta^2 E_1^B(R_D) > -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B}) \qquad \textbf{(4.15)}$$

b. When $R_D < 0$

$$\Delta^2 E_1^B(R_D) > -\frac{1-P(R_D+Q_D+1,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B}) \qquad \textbf{(4.16)}$$

When $R_D \geq 0$, the second difference operator of the system total holding cost is:

$$\Delta^2[H_D Q_B E_1^D(R_D)+N_B H_B E_1^B(R_D)] >$$
$$\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{Q_D}[H_D Q_B - H_B(1+\frac{R_B}{Q_B})] \qquad \textbf{(4.17a)}$$

When $R_D<0$, the second difference operator of the system total holding cost is:

$$\Delta^2[H_D Q_B E_1^D(R_D)+N_B H_B E_1^B(R_D)] > \frac{1-P(R_D+Q_D+2,\mu_D)}{Q_D}[H_D Q_B - H_B(1+\frac{R_B}{Q_B})] \quad (4.17b)$$

The expected base holding cost is **not** a convex function of the depot reorder point R_D. However, the total expected holding cost (the holding costs at the depot and bases) increases as R_D increases and is a **convex** function of R_D, when the depot holding cost per base order is as large as the base holding cost of $(1+ R_B/Q_B)$ units. This is the sufficient condition of the convexity of the total expected holding cost with respect to R_D.

The conclusion that the expected holding cost at bases is not a convex function of the depot reorder point R_D is explainable. When R_D is very small, bases often face stockout at the depot. The probability of on-hand inventory at bases is small because of small R_D value. As R_D increases, the expected base on-hand inventory increases. When R_D is very large, the probability of stockout at the depot is very small. In that case, the base expected on-hand inventory is determined mainly by the base decision variables Q_B and R_B and increases slowly as R_D increases than when R_D is small. The rate of increasing expected base on-hand inventory with respect to R_D decreases as R_D increases.

4.4 CONVEXITY OF THE EXPECTED BASE BACKORDERS WITH R_D

The reorder point at the depot R_D affects the probability that the depot runs out of stock. The stockout at the depot affects the performance of bases. We find out the convexity of the expected base backorders cost with respect to R_D.

4.4.1 When $R_B \geq 0$.

We analyze the convexity issue in two cases: **1.** $R_B \geq Q_B$,. and **2.** $R_B < Q_B$. It is easier to prove the convexity with $R_B < Q_B$. We give the proof with $R_B \geq Q_B$ here. The expected backorders at a base is given in (3.12).

$$
\begin{aligned}
E_2^B = & \frac{1-\sum_{y=1}^{\infty} \psi_2^D(y)}{Q_B} \sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
& + \frac{1}{Q_B} \sum_{y=1}^{\infty} \psi_2^D(y) \sum_{m=0}^{y} bin(y,m,\frac{1}{N_B}) [\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& + \sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]
\end{aligned}
$$

a. When $R_D \geq 0$:

$$\Delta E_2^B(R_D) = -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B}$$
$$[\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]+\sum_{z=0}^{Q_B-R_B-1} z[1-P(z+R_B+1,\mu_B)]]$$
$$+\frac{\sum_{y=2}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_B Q_D}$$
$$[\sum_{m=0}^{y-1} bin(y-1,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \tag{4.18}$$
$$+\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$
$$-\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})[\sum_{(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]<0$$

$$\Delta^2 E_2^B(R_D) \geq \frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B}$$
$$[\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]+\sum_{z=0}^{Q_B-R_B-1} z[1-P(z+R_B+1,\mu_B)]]$$
$$+\frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{Q_B Q_D}$$
$$[\sum_{m=0}^{1} bin(1,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \tag{4.19}$$
$$-2\sum_{m=0}^{2} bin(2,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$
$$+\sum_{m=0}^{3} bin(3,m,\frac{1}{N_B})[\sum_{(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B)]$$
$$+\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]>0$$

b. When $R_D<0$

$$\Delta E_2^B(R_D)=\frac{1-P(R_D+Q_D+2,\mu_D)}{Q_DQ_B}$$

$$[\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]-\sum_{m=0}^{1} bin(1,m,\frac{1}{N_B})$$

$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$+\frac{\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B}[\sum_{m=0}^{y-1} bin(y-1,m,\frac{1}{N_B})$$

$$[\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$-\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \tag{4.20}$$

$$+\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$

$$+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_BQ_D}[\sum_{m=0}^{y-1} bin(y-1,m,\frac{1}{N_B})$$

$$[\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$-[\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})[\sum_{m=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]<0$$

$$\Delta^2E_2^B(R_D)>0 \tag{4.21}$$

When $R_B\geq Q_B$, the expected backorders at bases is a convex function of R_D. It is relative easier to prove the same conclusion when $R_B<Q_B$.

4.4.2. When $R_B<0$,

The expected backorders at a base is given in (3.16).

$$E_2^B=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=0}^{-R_B-1}z[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=-R_B}^{\infty}z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}bin(y,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

a. When $R_D\geq 0$:

$$\Delta E_2^B(R_D)=\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{Q_DQ_B}$$
$$[\sum_{z=0}^{-R_B-1}z[1-P(z+R_B+Q_B+1,\mu_B)]+\sum_{z=-R_B}^{\infty}z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\sum_{m=0}^{1}bin(1,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$
$$+\frac{1}{Q_BQ_D}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \quad (4.22)$$
$$[\sum_{m=0}^{y-1}bin(y-1,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$
$$-[\sum_{m=0}^{y}bin(y,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]]<0$$

$$\Delta^2 E_2^B(R_D)>0 \tag{4.23}$$

b. When $R_D<0$:

$$\begin{aligned}
\Delta E_2^B(R_D)=&\frac{1-P(R_D+Q_D+2,\mu_D)}{Q_BQ_D}\\
&[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)]\\
&+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]\\
&-\sum_{m=0}^{1} bin(1,m,\frac{1}{N_B})\\
&[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]\\
&+\frac{1}{Q_DQ_B}\sum_{y=2}^{-R_B-1}[1-P(z+R_D+Q_D+1,\mu_D)][\sum_{m=0}^{y-1} bin(y-1,m,\frac{1}{N_B})\\
&[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]\\
&-\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]\\
&+\frac{1}{Q_DQ_B}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]\\
&[\sum_{m=0}^{y-1} bin(y-1,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B-mQ_B,\mu_B)]\\
&+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]\\
&-\sum_{m=0}^{y} bin(y,m,\frac{1}{N_B})[\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]]<0
\end{aligned} \tag{4.24}$$

$$\Delta^2 E_2^B(R_D)>0 \tag{4.25}$$

In this section. we prove that the expected base backorders decrease as R_D increases. It is also **convex** with respect to R_D.

4.5 CONVEXITY OF THE EXPECTED BASE ON-HAND INVENTORY WITH RESPECT TO R_B

4.5.1 When $R_B \geq 0$

$$\begin{aligned}\Delta E_1^B(R_B) &= \frac{1-P_{out}^D}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B}[1-P(R_B+Q_B+1-x,\mu_B)] \\ &+\frac{1-P_{out}^D}{Q_B}\sum_{x=0}^{R_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\ &+\frac{1}{Q_B}\sum_{x=0}^{R_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^*} p_{so}^B(m)[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\ &+\frac{1}{Q_B}\sum_{x=0}^{R_B+Q_B} p_{so}^B([\frac{R_B-x}{Q_B}+1]^*)[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^*Q_B-x,\mu_B)]>0\end{aligned} \tag{4.26}$$

$$\begin{aligned}\Delta^2 E_1^B(R_B) &= \frac{1-P_{out}^D}{Q_B}[P(R_B+2,\mu_B)-P(R_B+Q_B+2,\mu_B)] \\ &+\frac{1}{Q_B}\sum_{m=0}^{[\frac{R_B+1}{Q_B}]^*} p_{so}^B(m)[P(R_B+2-mQ_B,\mu_B)-P(R_B+Q_B+2-mQ_B,\mu_B)] \\ &+\frac{1}{Q_B}p_{so}^B([\frac{R_B+1}{Q_B}+1]^*)[1-P(R_B+Q_B+2-[\frac{R_B+1}{Q_B}+1]^*Q_B,\mu_B)]>0\end{aligned} \tag{4.27}$$

4.5.2 When $R_B<0$

$$\Delta E_1^B(R_B)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{Q_B+R_B}[1-P(Q_B+R_B+1-x,\mu_B)]>0 \tag{4.28}$$

$$\begin{aligned}\Delta^2 E_1^B(R)=&\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{R_B+Q_B+1}[1-P(Q_B+R_B+2-x,\mu_B)]\\&-\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{Q_B+R_B}[1-P(Q_B+R_B+1-x,\mu_B)]\\&=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(Q_B+R_B+2,\mu_B)]>0\end{aligned} \tag{4.29}$$

The on-hand inventory at bases increases as the R_B increases. It is a convex function of R_B.

4.6 CONVEXITY OF THE EXPECTED BASE BACKORDERS WITH RESPECT TO R_B

4.6.1 When $R_B\geq 0$

$$\begin{aligned}\Delta E_2^B(R_B)=&-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=1}^{\infty}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]\\&-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}bin(y,m,\frac{1}{N_B})\\&[\sum_{z=(1,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\&+\sum_{z=(1,mQ_B-R_B)}^{\infty}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]<0\end{aligned} \tag{4.30}$$

$$\Delta^2 E_2^B(R_B)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}[P(R_B+2,\mu_B)-P(R_B+Q_B+2,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}bin(y,m,\frac{1}{N_B}) \tag{4.31}$$
$$[\sum_{z=(1,(m-1)Q_B-R_B)}^{(2,(m-1)Q_B-R_B)}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(1,mQ_B-R_B}^{z=(2,mQ_B-R_B)}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]>0$$

4.6.2 When $R_B<0$

$$\Delta E_2^B(R_B)=-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=1}^{-R_B-1}[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=-R_B}^{\infty}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}bin(y,m,\frac{1}{N_B}) \tag{4.32}$$
$$[\sum_{z=(1,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=mQ_B-R_B}^{\infty}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]<0$$

$$\Delta^2 E_2^B(R_B)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}[1-P(R_B+Q_B+2,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}bin(y,m,\frac{1}{N_B}) \tag{4.33}$$
$$\sum_{z=(1,(m-1)Q_B-R_B)}^{(2,(m-1)Q_B-R_B)}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]>0$$

Section 4.6 shows that the expected base backorders decrease as the R_B increases. The expected base backorders is also a convex function of R_B.

4.7 NUMERICAL RESULTS OF THE CONVEXITY ANALYSIS

We use a two-echelon (Q, R) inventory model with following system parameters to study the convexity of the inventory system with respect to Q_B, Q_D, R_B, R_D. In the inventory model, λ_B= 0.25/week, N_B=10, L_B=2 weeks, L_D=3 weeks, Q_B=6 units, Q_D=10 base orders, R_B =-3 and R_D=-2 base orders.

4.7.1 Convexity with Respect to Q_B

We study the effect of Q_B on the expected base on-hand inventory, the expected base backorders and the expected depot on-hand inventory. When Q_B changes, following assumptions are kept: 1. For R_B<0, Q_B>- R_B, 2. Q_B>$\lambda_B L_B$, 3. Q_D>$\lambda_B L_D N_B / Q_B$.

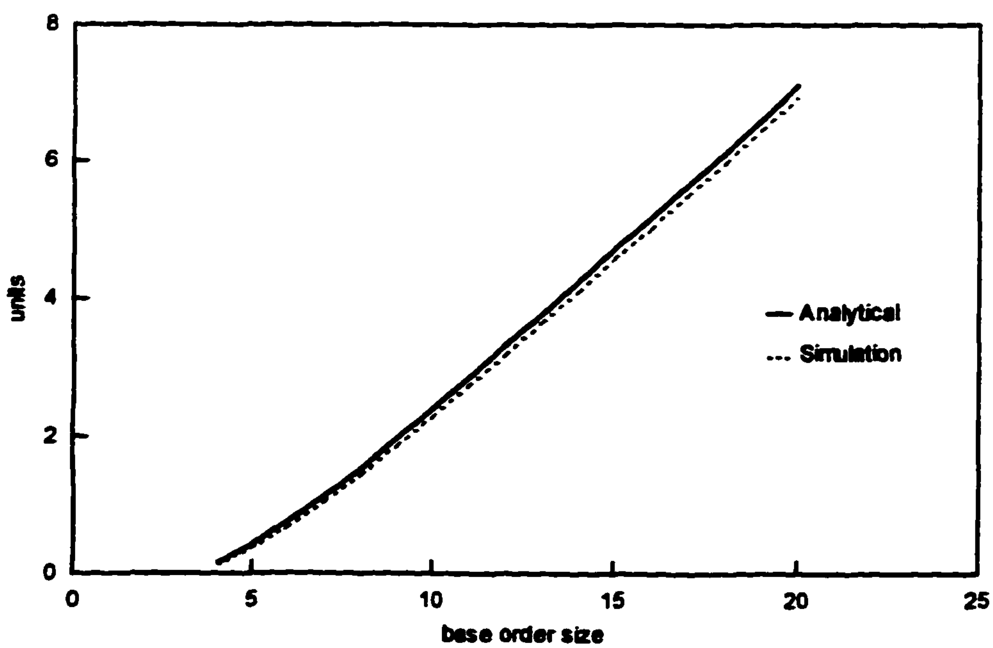

Figure 4.1 The Effect of Q_B on the Base On-hand Inventory

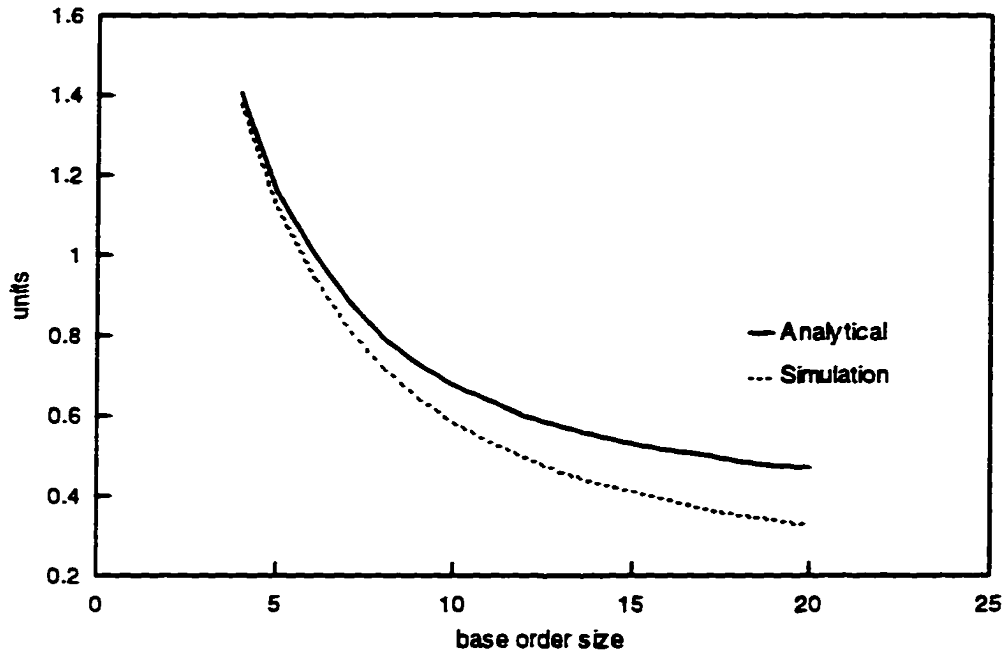

Figure 4.2 The Effect of Q_B on the Base Backorders

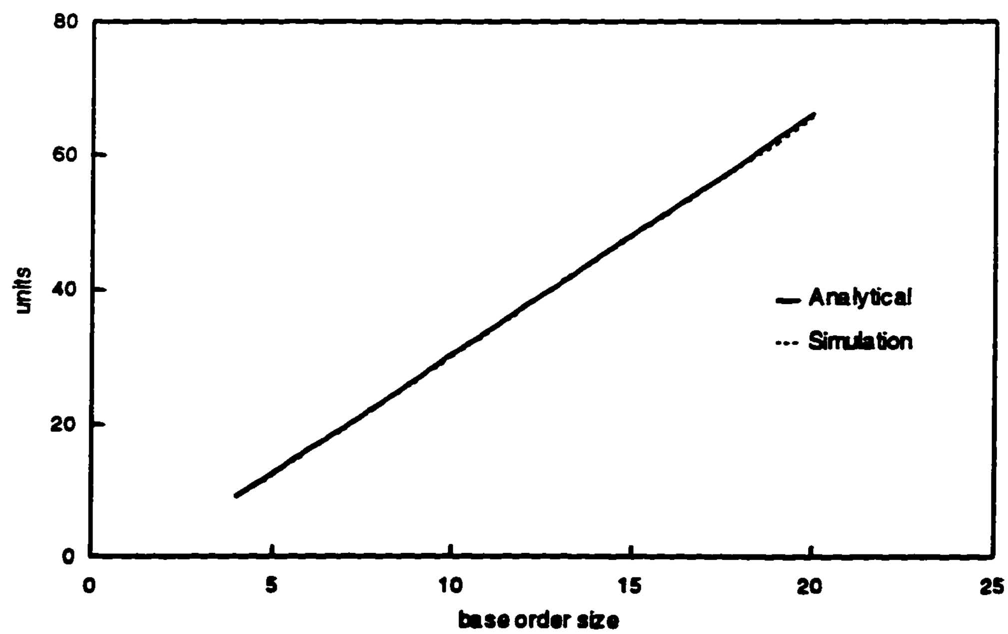

Figure 4.3 The Effect of Q_B on the Depot On-hand Inventory

Figure 4.1, **Figure 4.2** and **Figure 4.3** show that the expected base on-hand inventory and backorders and the expected depot on-hand inventory are **convex** with respect to Q_B. Therefore, the total cost of the inventory system is a **convex** function of Q_B.

4.7.2 Convexity with Respect to Q_D

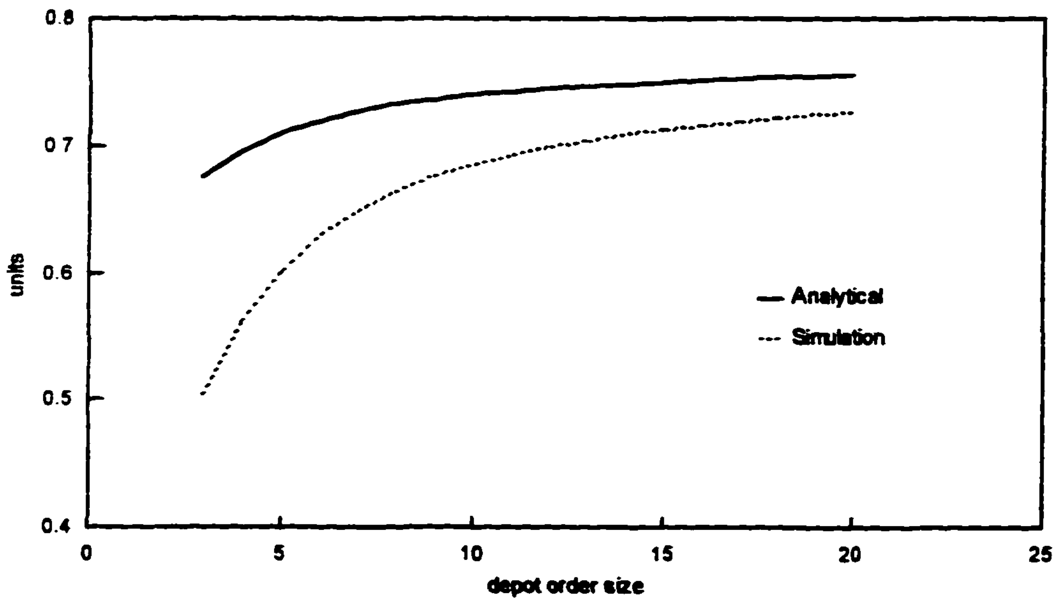

Figure 4.4 The Effect of Q_D on the Base On-hand Inventory

In **Figure 4.4**, the expected base on-hand inventory is not a convex function with respect to Q_D. With the increase of Q_D, the depot has more inventory to supply base orders. The expected base on-hand inventory will increase. However, the expected base on-hand inventory will not increase forever as Q_D increases. The upper limit of the expected base on-hand inventory is the expected on-hand inventory of a single echelon model.

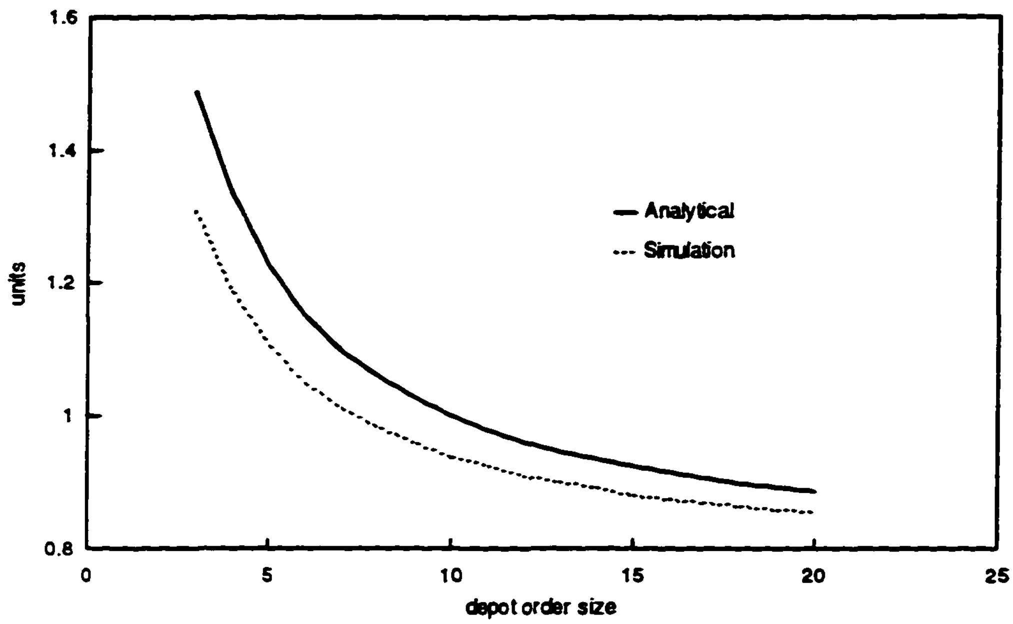

Figure 4.5 The Effect of Q_D on the Base Backorders

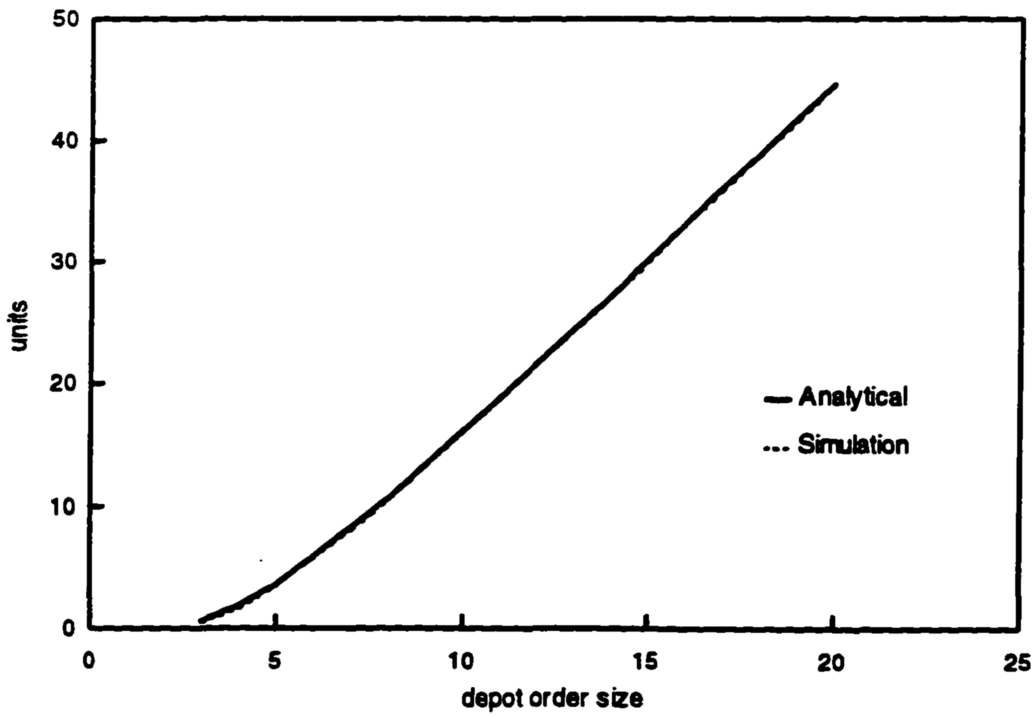

Figure 4.6 The Effect of Q_D on the Depot On-hand Inventory

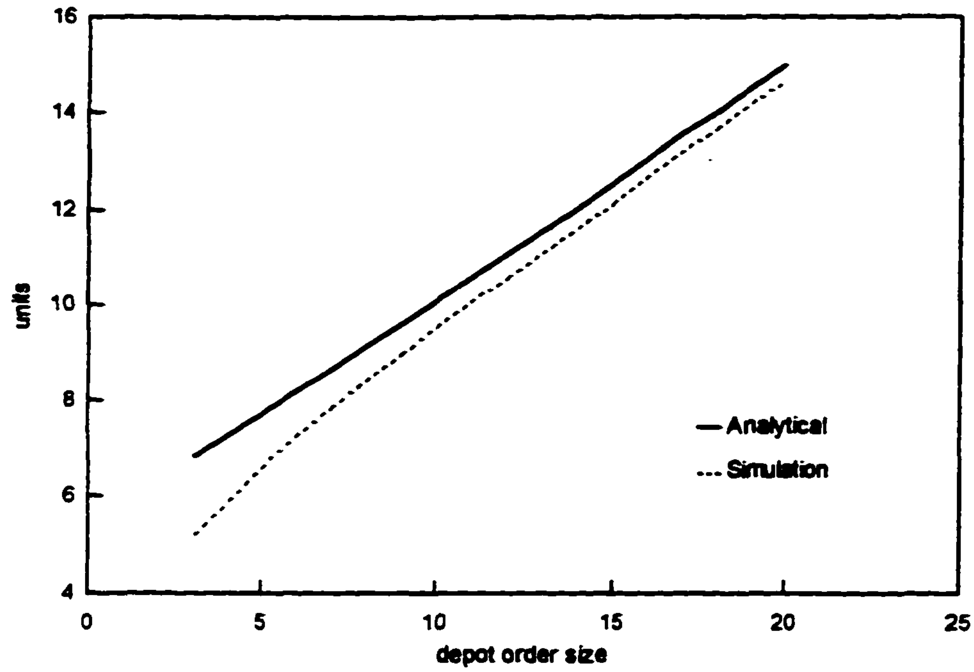

Figure 4.7 The Effect of Q_D on the Total On-hand Inventory

In **Figure 4.5** and **Figure 4.6**, the expected base backorders and the expected depot on-hand inventory are convex with respect to Q_D. Although **Figure 4.4** shows that the expected base on-hand inventory is **not** convex with respect to Q_D, the total expected on-hand inventory of the system shown at **Figure 4.7** is **convex** with respect to Q_D. In calculating the expected total on-hand inventory, we treat one base order at the depot as $(1+R_B/Q_B)$ units at a base. With such a conversion, the expected total on-hand inventory of the system is a convex function with respect to Q_D.

In our convexity analysis, we do not prove the convexity of the system total cost with respect to Q_B and Q_D. **Part 4.7.1** and **Part 4.7.2** show the convexity of the system total cost with respect to Q_B and Q_D. In Chapter 6, we use the assumption of the total cost convexity to develop an algorithm to search the optimal decision variables value and the system minimum total cost.

4.7.3 Convexity with Respect to R_B

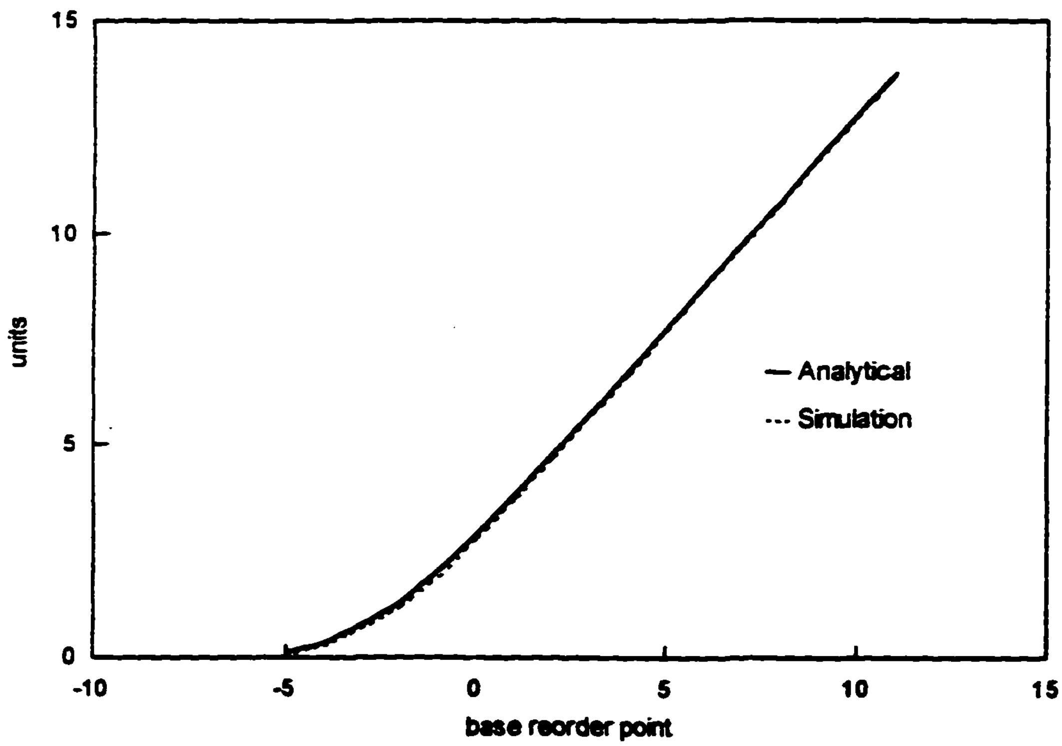

Figure 4.8 The Effect of R_B on the Base On-hand Inventory

The expected base on-hand inventory and backorders are **convex** with respect to R_B. The base reorder point R_B does not affect the depot performance. Therefore, the expected depot on-hand inventory will not change with R_B. The total cost of the inventory system is a **convex** function with respect to R_B.

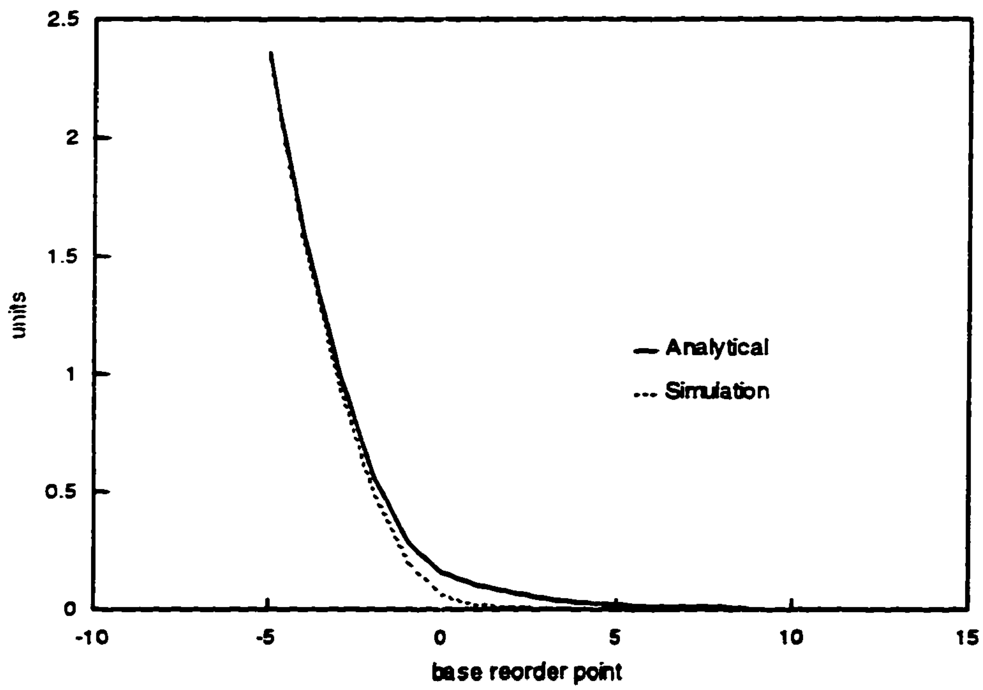

Figure 4.9 The Effect of R_B on the Base Backorders

4.7.4 Convexity with Respect to R_D

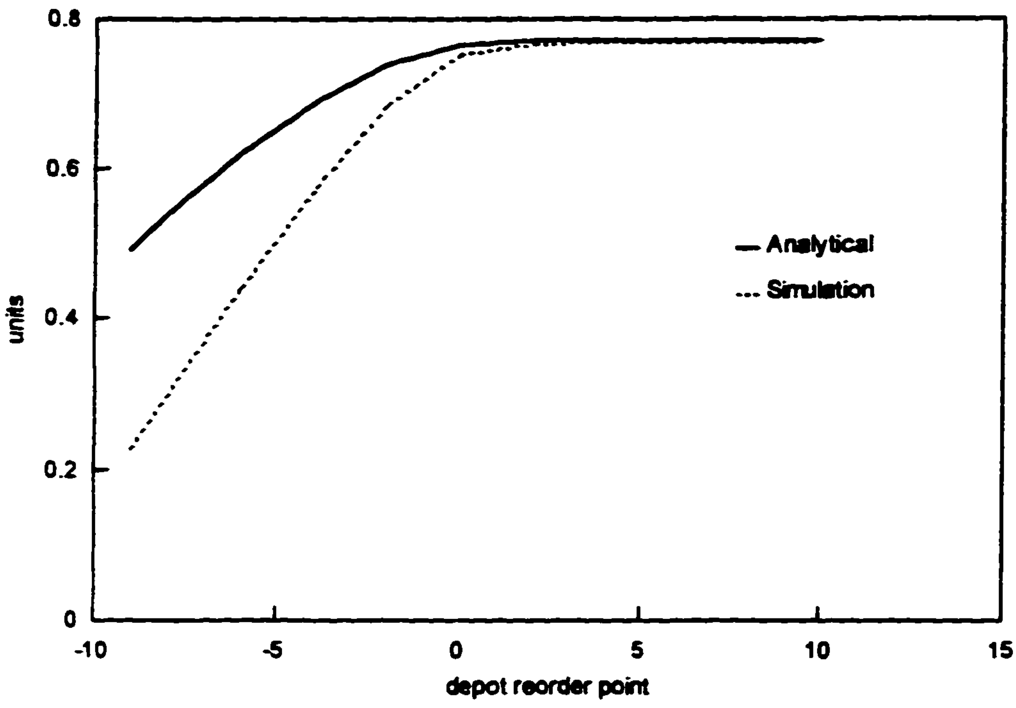

Figure 4.10 The Effect of R_D on the Base On-hand Inventory

The expected base on-hand inventory is not a convex function with respect to R_D. We have a similar case in the convexity analysis of the expected base on-hand inventory with respect to Q_D.

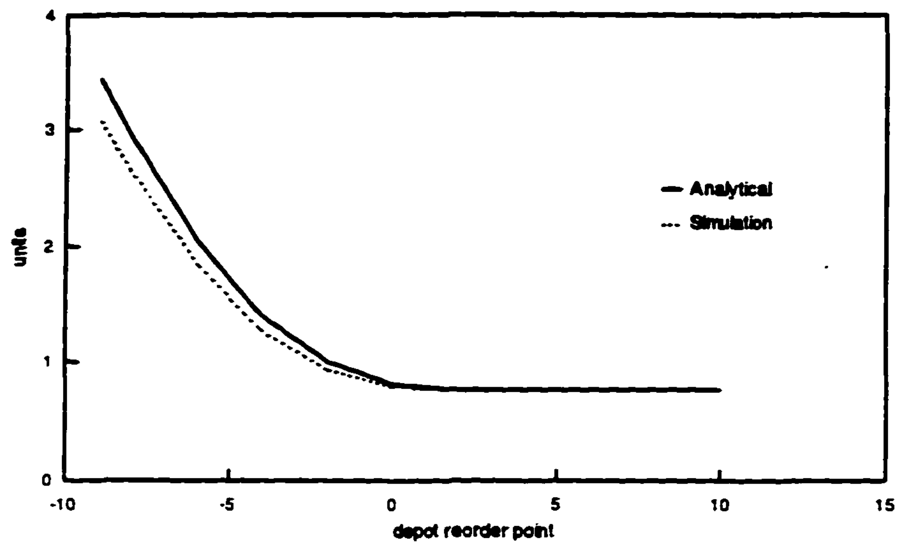

Figure 4.11 The Effect of R_D on the Base Backorders

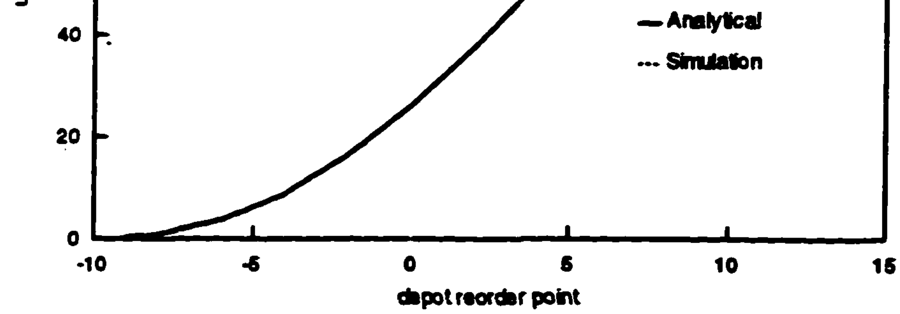

Figure 4.12 The Effect of R_D on the Depot On-hand Inventory

The expected base backorders and depot on-hand inventory are convex with respect to R_D.

When we get the total on-hand inventory, we make a similar conversion as we did with Q_D. With the conversion, one base order at the depot is equivalent to $(1+R_B/Q_B)$ units at a base. The expected total on-hand inventory of the system is a **convex** function with respect to R_D.

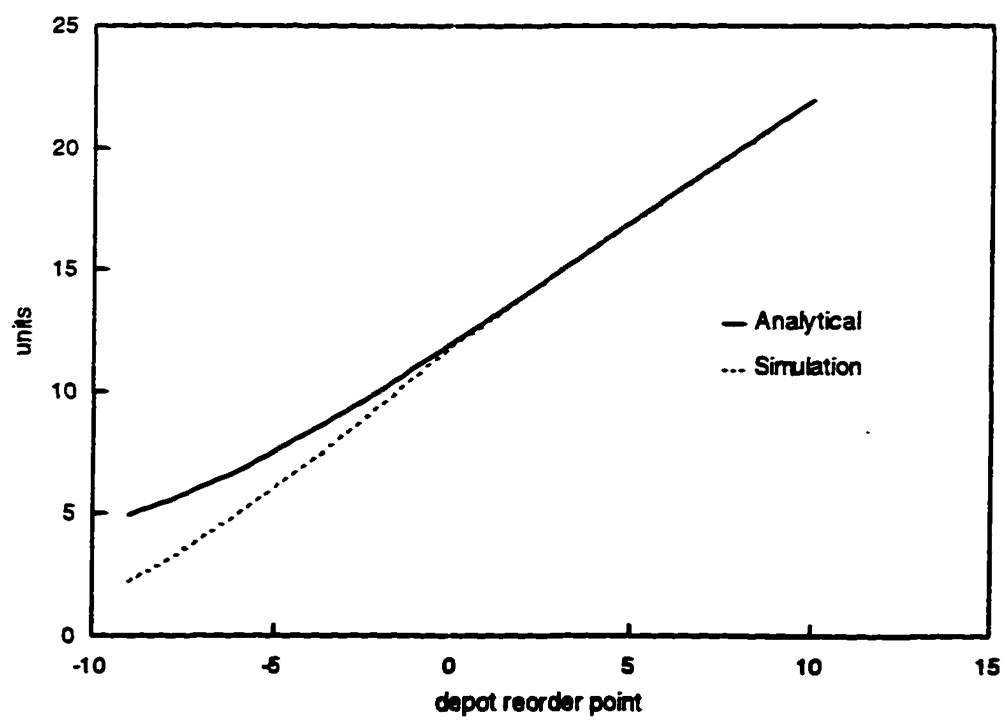

Figure 4.13 The Effect of R_D on the Total On-hand Inventory

Our numerical example confirms our convexity analysis. In this chapter, we theoretically prove that:

1. The depot expected on-hand inventory is a **convex** function with respect to R_D and Q_D. It is not affected by R_B.

2. The base expected on-hand inventory is a **convex** function with respect to R_B. However, the base expected on-hand inventory is **not** a convex function with respect to R_D. With the assumption that the holding cost per base order at the depot is as large as the holding cost of $(1+R_B/Q_B)$ units at a base, the total system expected total holding cost is a **convex** function with respect to R_D.

3. The expected backorders at bases is a **convex** function with respect to R_B and R_D.

We get several conclusions from our numerical examples:

4. The depot expected on-hand inventory is a **convex** function with respect to Q_B.

5. The base expected on-hand inventory is a **convex** function with respect to Q_B. However, the base expected on-hand inventory is **not** a convex function with respect to Q_D. With the assumption that the holding cost per base order at the

depot is as large as the holding cost of $(1+R_B/Q_B)$ units at a base, the total system expected total holding cost is a convex function with respect to Q_D.

6. The expected backorders at the bases is a convex function with respect to Q_B and Q_D.

7. Simulations of the same system have similar results.

CHAPTER 5

SIMULATIONS AND OUTPUT ANALYSIS

5.1 DESCRIPTION OF THE SIMULATION MODEL

Simulations of two-echelon (Q, R) models were initially programmed with Simscript II.5, a simulation language developed by CACI. Simscript II.5 can be used in discrete and continuous simulations. The simulation models are later programmed in C. The analytical model of the two-echelon (Q, R) inventory system is programmed with C.

Computer simulations are widely used in inventory studies. **Figure 1.2** can be used to show the simulation. **Table 5.1** lists input and output of our simulation and goodness of fit tests. In our two-echelon inventory model simulations, the depot receives base orders and supplies these base orders from its warehouse. When the depot inventory level reaches its reorder point, the depot requests Q_D base orders from the outside supplier and receives its supplies in the leadtime L_D. We assume that the outside supplier never runs out of stock.

Table 5.1 Input, Process and Output of the Simulation and Tests

INPUT	PROCESS	OUTPUT
Simulation Parameters:	Simulation	System Characteristics:
. number of random seeds		. Average and Std. Deviation of
. warm-up time	Correlation Test	base on-hand inventory
. number of customers		base backorders
	Independence Test	depot on-hand inventory
System Parameters:		number of base orders
. B_B, H_B, H_D, N_B, L_B, L_D,	Goodness of Fit	number of depot orders
S_B, S_D, λ_D	Tests	total system cost
Decision Variable Values:		Goodness of Fit Test Results
. Q_B, Q_D, R_B, R_D		. Average and Std. Deviation of
		χ^2 test
		A-D tests
		K-S tests
		runs-up test
		correlation test

If the depot has inventory when a base requests an order, the base will receive its order in the leadtime L_B. If the depot runs out of stock, a base order is backlogged at the depot until the depot receives its supplies from the outside supplier. When there are several outstanding base orders in the depot, the depot supplies these base orders according to the sequence that these orders have arrived at the depot. It is the FCFS policy.

In our analytical study and computer simulation, we assume that base orders have the same number of items. The reorder point at all bases is identical. Customers' arrival process in each base is an independent Poisson process with the same arrival rate. When a base runs out of stock, all customers who arrive at the base will wait. There is no loss of sales because of stockout.

In our simulation model, each base is simulated with an independent Poisson arrival process. We record the interval of base orders arrived at the depot in our Simscript programmed simulation. Because we approximate the demand at the depot by a Poisson process in our analytical modeling, we use several goodness of fit tests to examine the accuracy of the approximation. In computer simulations, the initial stage and stopping mechanism affect the accuracy of measurements. In our simulation model, we design a routine to start and stop the accounting process of the base orders at the depot. We use several random seeds in the simulation to produce independent replications in measuring the performance of the model.

With Simscript, the simulation does not stop until all customers arriving at bases are served. When we run a simulation, we want the simulation to stop even though there are customers waiting at the bases or the depot. We also want the measurement of the system performance, such as on-hand inventory in bases and the depot and backorders at the bases, to start after an initial simulation period. In order to do this, we use C to program the inventory simulation model when we measure the system performance, such as the on-hand inventory and backorders.

In our C programmed simulation models, we have a runtime routine to control the start and stop of the system measurements. Comparing the results of Simscript programmed simulation and C programmed simulation, we conclude that the C programmed simulation is more stable and closer to our analytical model. The C programmed simulations are also running much faster.

Since we design a routine in the Simscript model to control the start and stop of the base order accounting process at the depot, the results of goodness of fit tests are not affected by the Simscript own start and stop mechanisms. **Figure 5.1** to **Figure 5.6** are flowcharts of the simulation.

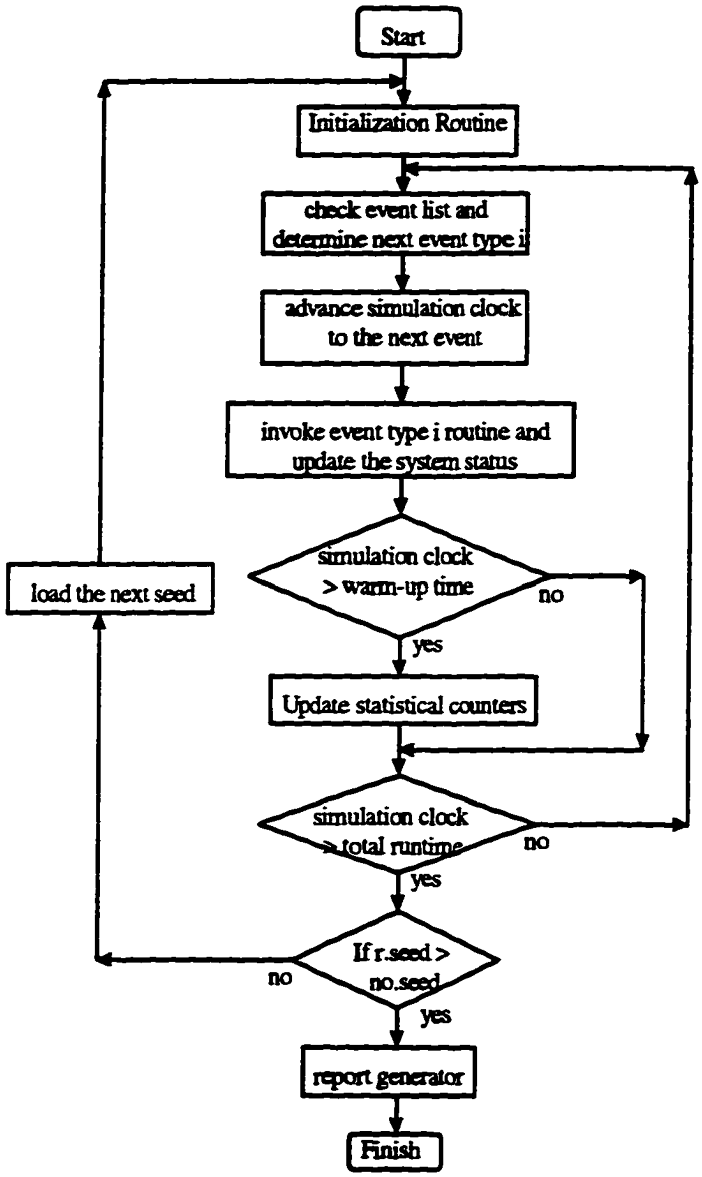

Figure 5.1 Simulation Flowchart

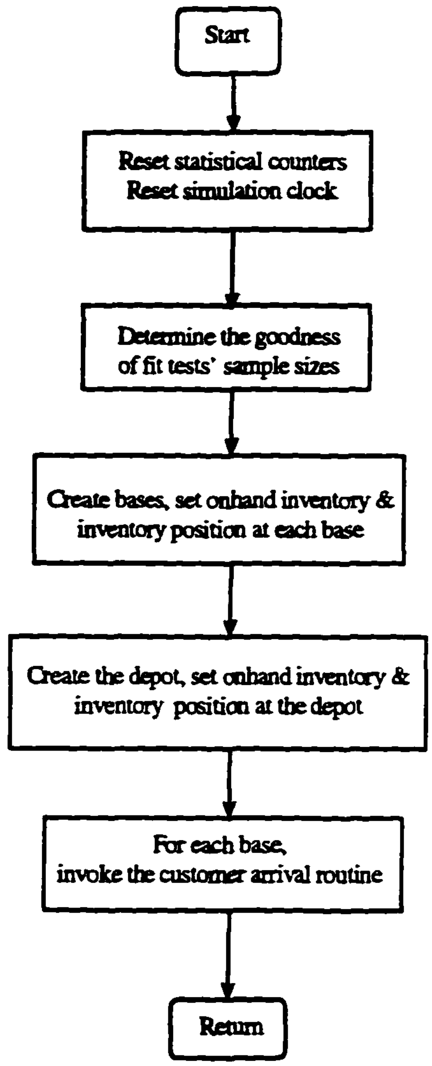

Figure 5.2 Initialization Routine

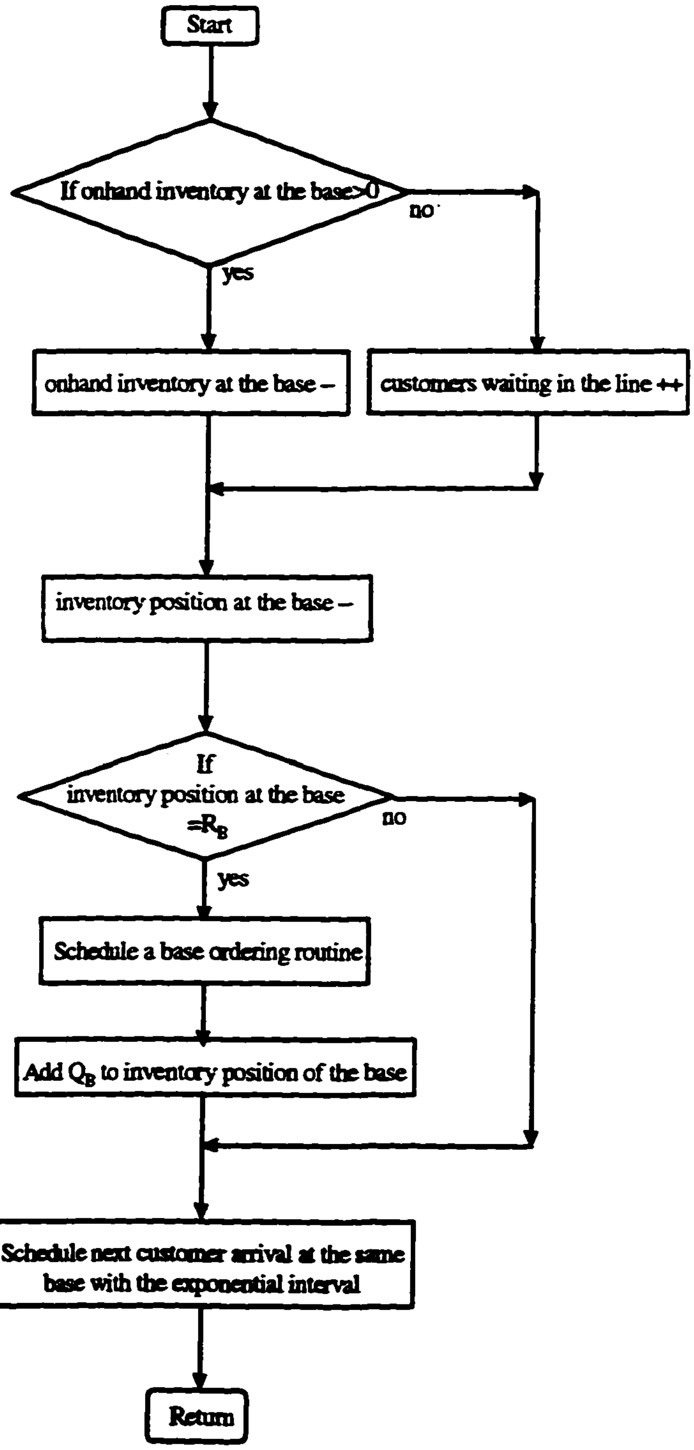

Figure 5.3 Base Customer Arrival Routine

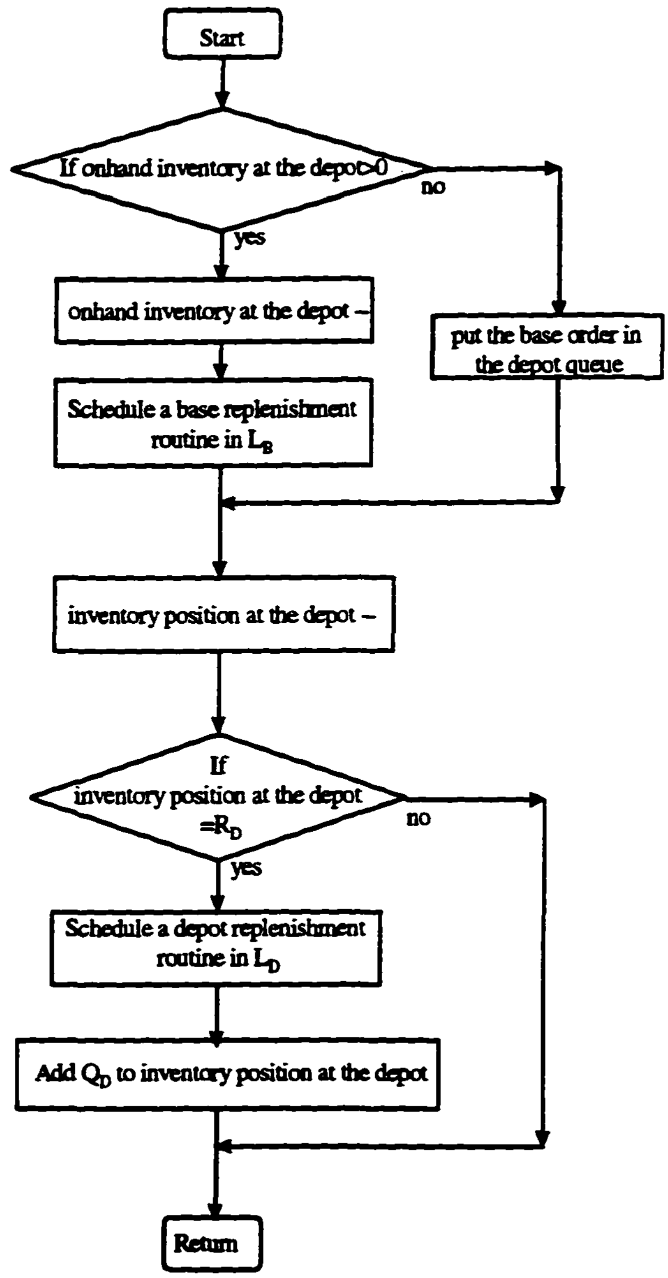

Figure 5.4 Base Ordering Routine

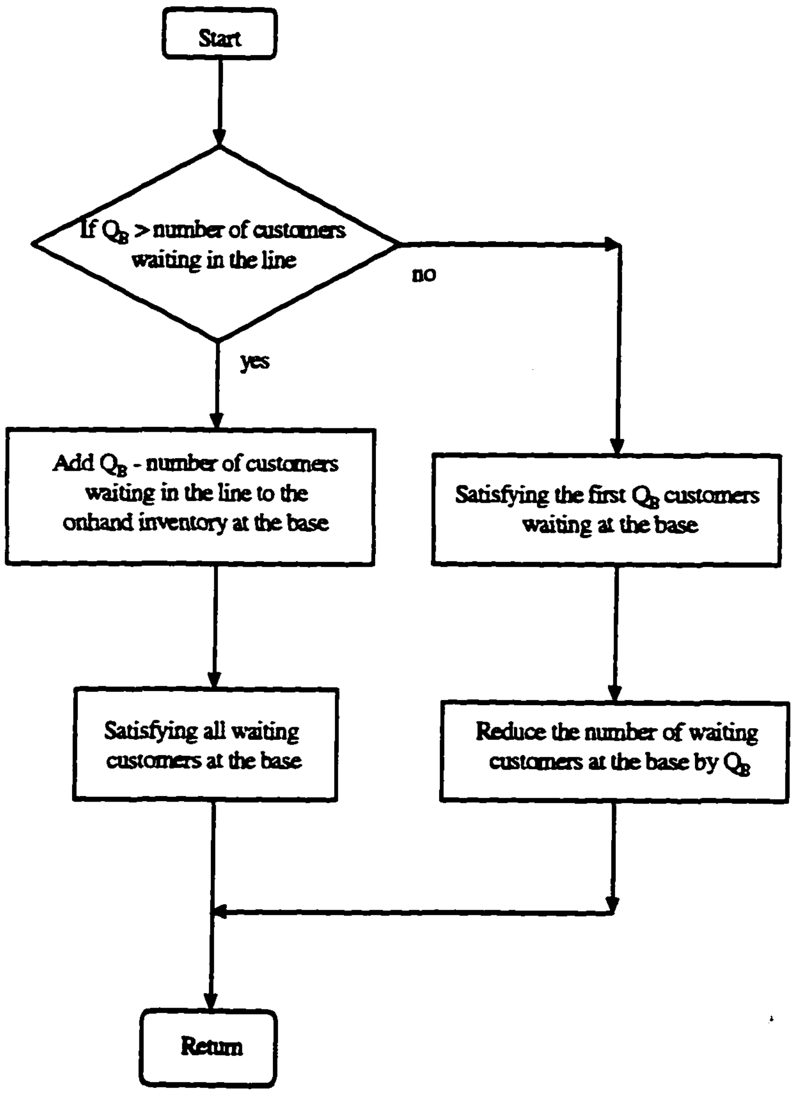

Figure 5.5 Base Replenishment Routine

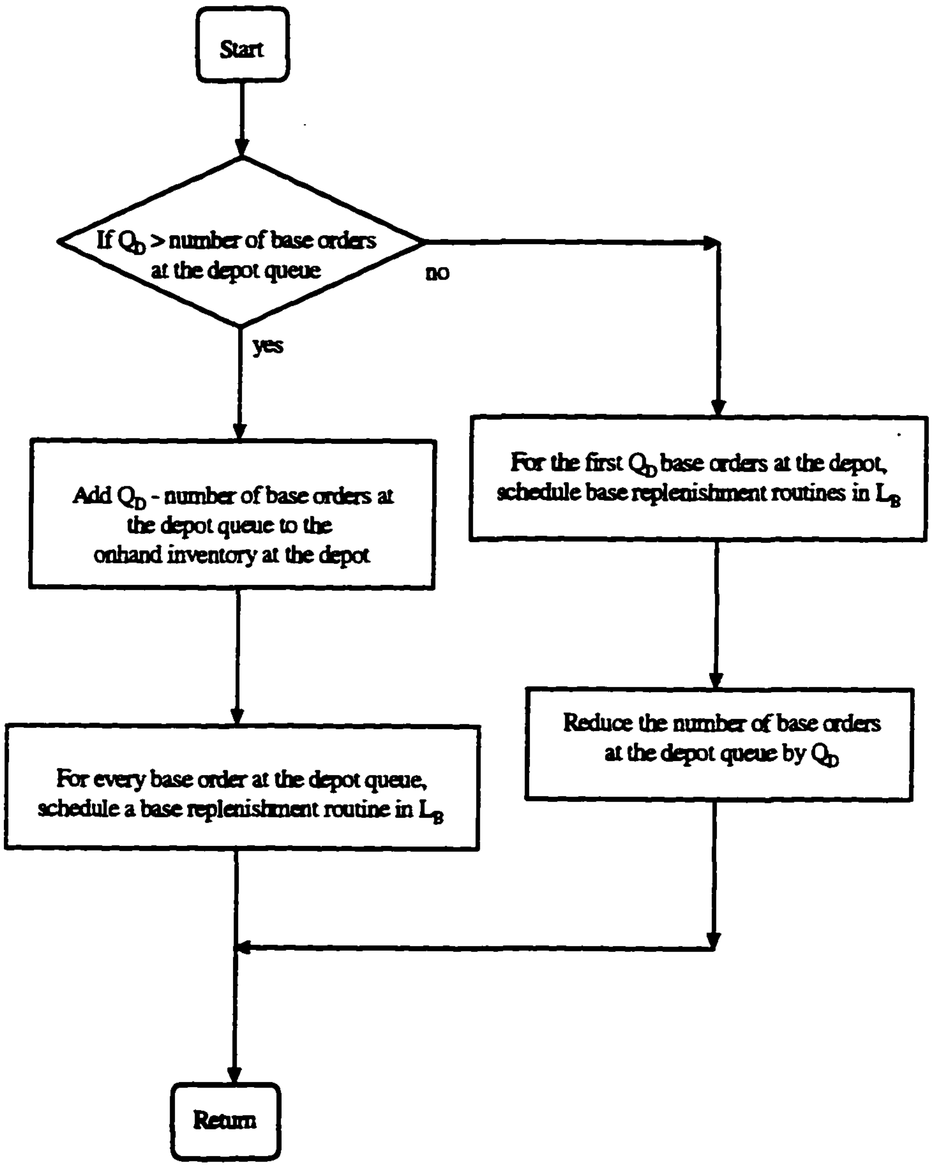

Figure 5.6 Depot Replenishment Routine

5.2 GOODNESS OF FIT TESTS

In our study, the demand process at the depot is approximated by a Poisson process, when the bases use a (Q, R) policy. We use the simulation results to test the Poisson approximation. We also test the feasibility of the approximation of the depot demand process by a normal distribution. The Kolmogrov-Smirnov test and the Anderson-Darling test are used in the Poisson approximation and the normal approximation. In addition to the above tests, we also run a χ^2 test for the Poisson approximation. We study the sensitivity of approximations to the number of bases and base order size. The number of bases and base order size affect the accuracy of the depot demand approximation. When bases have the one-for-one policy, the demand process of the depot is a Poisson process. As base batch size increases, the approximation becomes poor. On the other hand, the approximation of the depot demand process by a Poisson or a normal distribution is improved as the number of bases increases.

5.3 THE χ^2 TEST

The χ^2 test is a commonly used test of goodness of fit. It can be used with continuous variables and discrete variables. In a χ^2 test, data need to be grouped into cells. In order to get an unbiased test, the cells in a test should be equal-probable cells. With equal-probable cells, the expected number of events in each cell is the same.

$$\chi^2 = \sum_{i=1}^{M} \frac{(N_i - np_i)^2}{np_i} \tag{5.1}$$

The number of cells in a χ^2 test affects the accuracy of the test. A good χ^2 test should have roughly twenty cells. When there are a few events in the test, Watson (1957) developed a formula to do a χ^2 test with ten or fewer cells. Moor (1986) gives a convenient formula for the number of equal-probable cells in a χ^2 test:

$$M = 2n^{\frac{2}{5}} \textit{ where n is the number of events} \in \textit{the test}$$

When there are a very large number of events in a χ^2 test, the result is also not good. We use the demand process at each base to check the χ^2 test itself. When we simulate the inventory model, the arrival interval of demand at each base is generated by an exponential process. The χ^2 test value varies greatly with different bases and the simulation time. We conclude that the χ^2 test is volatile and should not be used alone for goodness-of-fit test.

In our study, we use the χ^2 test for an approximation of the depot demand process by a Poisson process. As the intervals between two demands in a Poisson process are exponential random variables, we test these intervals to find out whether they can be approximated by exponentially distributed random variables.

With M equal-probable cells in an exponential distribution test, the boundary points of each cell i are:

$$\begin{aligned} &start\ point=-\frac{1}{\mu_D}\ln(1-\frac{i-1}{M}) \\ &end\ point=-\frac{1}{\mu_D}\ln(1-\frac{i}{M}) \quad where\ i=1,..,M-1 \\ &start\ point=\frac{1}{\mu_D}\ln M,\ end\ point=\infty \quad where\ i=M \end{aligned} \tag{5.2}$$

Although the χ^2 test is a widely used goodness-of-fit test, its power is very low. In χ^2 tests, data are grouped into cells. This practice reduces its test power. When data are grouped into cells, events in an equal-probable cell are represented by the mean cell value in the χ^2 test. Moor(1986) indicates that χ^2 test should not be used when the full ungrouped sample of data is available. Both the Kolmogrov-Smirnov test and the Anderson-Darling test have more test power.

We also conducted a χ^2 test for approximating the depot demand process by a normal distribution. There are two issues in such an approximation. The first issue is the point process vs. continuous process. The superposition of base demand processes is a point process, which takes only integer values. The normal distribution is a continuous process. We also do not know the distribution of intervals corresponding to a normal distributed demand process. We have to test the demand process itself. With the χ^2 test requirement of equal-probable cells, the boundary points of some cells in the normal approximation test may be between two successive integers. As we know, the probability that the number of demands falls within such cells in any simulation is zero. The equal-probable cells assumption is violated. The second issue is how to simulate the demand process by a normal distribution. The normal random variable has the range between [-∞, ∞]. When we use a normal distribution variable as the interval, we may get a negative interval value. There is no interval distribution with a corresponding normal demand process.

5.4 THE KOLMOGROV-SMIRNOV TEST (K-S TEST)

The Kolmogrov-Smirnov (K-S) test compares the simulation output with the hypothesized distribution. In the simulation process, our model counts the number of base orders in each interval. The K-S test has more test power than the χ^2 test because the K-S test does not group data.

In a K-S test, the empirical distribution $F_n(x)$ is compared with the hypothetical distribution F(x). D_n is largest distance between $F_n(x)$ and F(x) for all values of x.

$$D_n^+ = \max[\frac{i}{n} - F(x_{(i)})],\ D_n^- = \max[F(x_{(i)} - \frac{i-1}{n}]$$
$$D_n = \max[D_n^+, D_n^-] \quad (5.3)$$

5.4.1 The K-S Test for the Poisson Approximation

When the arrival process is a Poisson distribution, the arrival intervals are exponentially distributed. For the Poisson process' goodness-of-fit test, we test the distribution of depot demand intervals as an exponential distribution. There are two steps. First, our model calculates the hypothetical cumulative probability with the depot average arrival rate μ_D of the sample and the sample size n.

$$
\begin{gathered}
F(x_i)=1-e^{-\mu_D x_i} \\
D_n^+=\max\left(\left|\frac{i}{n}-F(x_i)\right|\right), \quad D_n^-=\max\left(\left|F(x_i)-\frac{i-1}{n}\right|\right) \\
D_n=\max(D_n^+,D_n^-)
\end{gathered}
\qquad (5.4)
$$

Then, the simulation calculates the K-S test statistics value K^2 for the exponential distribution

$$
K^2=\left(D_n-\frac{0.2}{n}\right)\left(\sqrt{n}+0.26+\frac{0.5}{\sqrt{n}}\right) \qquad (5.5)
$$

5.4.2 The K-S Test for the Normal Distribution

To test the depot arrival process against a normal distribution, the simulation decides the cell size and the number of cells in each sample. The cell size is defined as the number of base orders in a cell at the K-S test. Based on the customer arrival rate at each base, the number of bases, and the base order size, our simulation model gives the interval for each cell. At the end of simulation, our model tests the number of base orders in each cell to see if they can be approximated by a normal distribution. To test the normal distributed demands at the depot, we standardize the number of base orders x_i by y_i, with the sample mean number of base orders at the depot μ and the sample standard deviation of the number of base orders s.

$$y_i=\frac{x_i-\mu}{s} \tag{5.6}$$

Then, the cumulative normal probability $F(x_i)$ and D_n are calculated as:

$$\begin{aligned} F(x_i)&=1-\frac{1}{2}(1+C_1y_i+C_2y_i^2+C_3y_i^3+C_4y_i^4)^{-4} \quad when:\ y_i\geq 0 \\ &=\frac{1}{2}(1-C_1y_i+C_2y_i^2-C_3y_i^3+C_4y_i^4)^{-4} \quad when:\ y_i<0 \\ C_1&=0.196854,\ C_2=0.115194,\ C_3=0.000344,\ C_4=0.019527 \end{aligned} \tag{5.7}$$

$$\begin{gathered} D_n^+=\max(|\frac{i}{n}-F(x_i)|),\quad D_n^-=\max(|F(x_i)-\frac{i-1}{n}|) \\ D_n=\max(D_n^+,D_n^-) \end{gathered} \tag{5.8}$$

The K-S test statistics value K^2 for the normal distribution is:

$$K^2=(\sqrt{n}-0.01+\frac{0.85}{\sqrt{n}})D_n \tag{5.9}$$

Modified critical values for adjusted K-S test statistics (Law, 1991)

	1-α				
Distribution	0.850	0.900	0.950	0.975	0.990
Exponent. Distr.	0.926	0.990	1.094	1.190	1.308
Normal Distr.	0.775	0.819	0.895	0.955	1.035

The simulation model collects several samples' K^2 values with each random seed. The simulation model gives the mean and the standard deviation of K^2. After we run the simulation with different random seeds, we get the average, the maximum, the minimum and standard deviation of K^2 values with different random seeds. We compare the average simulation value of the K^2 against the modified critical values for adjusted K-S test statistics. If the K^2 is greater than the critical value of certain confidence level, we reject the hypothesis that the arrival of base orders at the depot can be approximated by such a distribution.

5.5 THE ANDERSON-DARLING (A-D) TEST

Law(1991) indicates that the K-S test gives the same weights to the difference between the empirical distribution and the hypothetical distribution for every x_i. The Anderson-Darling test calculates the discrepancies in the tails. As a result, the A-D test has a higher test power than the K-S test does. For a valid A-D test, the sample size must be equal to or greater than 8. (D'Agostino, 1986) In our inventory models, we study the base order arrivals at the depot. With each random seed, our model divides the simulation run into several samples. In each sample, our simulation collects the interval times between arrivals of the base orders. We have about 40 base orders at the depot in each sample when we test the arrival process against the exponential distribution. That is, we use a sample size of 40 in our study.

When we test the arrival process of the base orders at the depot against the normal process, we still use the sample size of 40 in our data collection. Since there is no distribution of arrival interval corresponding to a normal demand, we program the simulation model to set the intervals. In each interval, the simulation model collects the number of the base orders at the depot. Our simulation model makes sure that the expected number of base orders in each interval is 20. There are forty such intervals in each sample. The simulation model calculates A^2 value for each sample. With each random seed, there are several samples. That is, there are several A^2 values. The simulation model is programmed to calculate the average A^2 value for each random seed.

As we run the simulation model with several random seeds, we receive the mean and standard deviation of A^2 of the simulation. Using the mean value of A^2, we test the hypothesis that the number of base orders in each sample can be approximated by a normal distribution. The standard deviation of A^2 gives the effect of different random seeds in the simulation.

The Anderson-Darling statistics A^2 is defined as:

$$A^2 = -\frac{\sum_{i=1}^{n}(2i-1)[\ln Z_i + \ln(1-Z_{n+1-i})]}{n} - n \quad \textit{where: } Z_i = F(x_i) \tag{5.10}$$

5.5.1 The Anderson-Darling Test for the Poisson Process

Similar to the K-S test, we use the A-D test for the approximation of the distribution of intervals between arrivals at the depot by an exponential process. There are two steps. First, our simulation model calculates the hypothetical cumulative probability of z_i and A^2, where μ_D is the mean depot arrival rate and n is the sample size.

$$z_i = 1 - e^{-\mu_D x_i}$$

$$A^2 = -\frac{\sum_{i=1}^{n}(2i-1)[\ln z_i + \ln(1-z_{n+1-i})]}{n} - n \tag{5.11}$$

Then, the model calculates the modified A-D test statistics value A^* for the exponential distribution.

$$A^{*}=A^{2}(1.0+\frac{0.75}{n}+\frac{2.25}{n^{2}}) \qquad (5.12)$$

5.5.2 The Anderson-Darling Test for the Normal Distribution

First, our simulation model standardizes the depot demand x_i by y_i, where μ is the mean depot demand and s is the standard deviation of the sample depot demand.

$$y_i=\frac{x_i-\mu}{s}$$

Then, the model calculates the cumulative normal probability $F(x_i)$ and A^2:

$$\begin{aligned} F(x_i)&=1-\frac{1}{2}(1+C_1y_i+C_2y_i^2+C_3y_i^3+C_4y_i^4)^{-4} \quad \textit{when: } y_i\geq 0 \\ &=\frac{1}{2}(1-C_1y_i+C_2y_i^2-C_3y_i^3+C_4y_i^4)^{-4} \quad \textit{when: } y_i<0 \\ C_1&=0.196854,\ C_2=0.115194,\ C_3=0.000344,\ C_4=0.019527 \end{aligned} \qquad (5.13)$$

$$A^{2}=-\frac{\sum_{i=1}^{n}(2i-1)[\ln z_i+\ln(1-z_{n+1-i})]}{n}-n \quad \textit{where } z_i=F(x_i) \qquad (5.14)$$

We get the modified A-D test statistics value A* for the normal distribution:

$$A^{*}=A^{2}(1.0+\frac{0.75}{n}+\frac{2.25}{n^{2}}) \qquad (5.15)$$

Modified critical values for adjusted A-D test statistics (D'Agostino 1986)

1-α				
0.850	0.900	0.950	0.975	0.990
0.631	0.752	0.873	1.035	1.159

We compare the mean simulation value of the A^2 against the modified critical values for the adjusted Anderson-Darling test statistics. If A^2 is greater than the critical value of certain confidence level, we will reject the hypothesis that the arrival of base orders at the depot can be approximated by a normal distribution.

5.6 THE INDEPENDENCE AND CORRELATION TESTS

The depot demand process is assumed to be an independent process when we analyze the inventory system. With our simulation results, we test the independence and correlation of successive base orders at the depot. We program the runs-ups test and the correlation test into our simulation.

5.6.1 The Runs-up Test

We use the runs-up test to check the independence of demand arrival process at the depot. In the runs-up test, we use runs-up break points. A runs-up break point is defined as any point at which the present demand interval x_i is smaller than the predecessor x_{i-1}. The number of intervals r between successive break points is a runs-up of length of r.

Ucount(r) represents the number of runs-up length of r. In the runs-up test, we count statistics UV, the number of runs-up of length of 1, .., 6. For the runs-up with length greater than 6, we count them as length of 6.

$$UV = \frac{1}{n}\sum_{i=1}^{6}\sum_{j=1}^{6}(ucount(i)-nb_i)(ucount(j)-nb_j)a_{ij}$$

$$[b_1,\ b_2,\ \ldots\ b_6] = [\frac{1}{6}, \frac{5}{24}, \frac{11}{120}, \frac{19}{720}, \frac{29}{5040}, \frac{1}{840}] \qquad (5.16)$$

$$\begin{vmatrix} a_{1.1} & a_{1.2} & \cdots & a_{1.6} \\ a_{2.1} & a_{2.2} & \cdots & .. \\ .. & .. & & .. \\ & & & \\ & & & \\ a_{6.1} & & & a_{6.6} \end{vmatrix} = \begin{vmatrix} 4529.4 & 9044.9 & 13568 & 18091 & 22615 & 27892 \\ & 18097 & 27139 & 36187 & 45234 & 55789 \\ & & 40721 & 54281 & 67852 & 83685 \\ & & & 72414 & 90470 & 111580 \\ & & & & 113262 & 139476 \\ & & & & & 172860 \end{vmatrix}$$

UV is distributed as a χ^2 distribution with 6 degrees of freedom. Grafton(1981) suggests that 4,000 is the minimum sample size for a proper runs-up test.

5.6.2 The Correlation Test

We conduct a correlation test to check the relationship between two successive depot demand intervals. The correlation test also examines the independence of depot demand intervals.

The covariance between two successive base order intervals in a sample with mean of μ and standard deviation of σ is defined as

$$C_{i,i-1}=E[(x_i-\mu)(x_{i-1}-\mu)] \qquad (5.17)$$

Although $C_{i,i-1}$ shows the relationship between two successive intervals, it is difficult to interpret. (Law, 1991) Covariance is similar to the sum of squares, which is the square of measurement. We use correlation $\rho_{i,i-1}$ to measure the independence of intervals of base orders.

$$\rho_{i,i-1}=\frac{C_{i,i-1}}{\sigma^2} \qquad (5.18)$$

$\rho_{i,i-1}$ value is between [-1,1]. When $\rho_{i,i-1}$ is close to -1 or 1, the intervals are highly negatively or positively correlated. That is, the intervals are not independent. When $\rho_{i,i-1}$ is close to 0, the intervals are independent. In our simulation output, we find that $\rho_{i,i-1}$ is close to 0. That is, the base order intervals at the depot can be considered as independent random variables. We get a similar conclusion with the runs-up test.

5.7 GOODNESS OF FIT TEST RESULTS OF DEPOT DEMAND

We use the Poisson process to approximate the demand process at the depot. Since the size of base orders and the number of bases affect the approximation, we run our simulations with different numbers of bases in and base order sizes. We also test the independence and correlation of demand intervals at the depot with different numbers of bases and base order sizes.

5.7.1 The Effect of Number of Bases and Q_B on the Poisson Approximation

In Franken's article, he proved that as the number of bases in the system increases, the Poisson approximation is better. We use several goodness-for-fit tests to examine the depot demand process against the Poisson process and the normal process. The following tables are the goodness-of-fit test results with Poisson distribution.

Table 5.2 χ^2 Test Results of the Poisson Approximation

# of bases / Q_B	4	5	6	8	10	15	20
2	30.668	27.117	26.169	24.212	22.247	21.768	21.152
3	39.785	32.274	27.674	25.557	23.255	21.553	21.057
4	43.090	31.762	31.550	26.399	23.109	21.242	21.592
5	46.405	35.450	30.413	24.406	23.643	21.417	20.533
6	47.720	36.744	28.699	25.588	23.091	21.665	20.490
7	48.332	38.761	30.145	25.755	23.416	21.771	20.056
8	52.190	36.871	30.958	24.897	23.422	22.493	21.469

Critical point χ^2 value for the χ^2 distribution with 19 d.f.

1-α	0.250	0.500	0.750	0.900	0.950	0.975	0.990
χ^2	14.562	18.338	22.718	27.204	30.144	32.852	36.191

Table 5.3 K-S Test Results of the Poisson Approximation

# of bases Q_B	**4**	**5**	**6**	**8**	**10**	**15**	**20**
2	0.8240	0.782	0.7763	0.7543	0.7327	0.7248	0.7267
3	0.8925	0.8268	0.7683	0.7657	0.7374	0.7178	0.7110
4	0.8819	0.8332	0.7871	0.7625	0.7295	0.7307	0.7304
5	0.9065	0.8397	0.7917	0.7723	0.7463	0.7361	0.7197
6	0.9156	0.8366	0.7881	0.7535	0.7399	0.7279	0.7160
7	0.9176	0.8508	0.8077	0.7577	0.7559	0.7146	0.7203
8	0.9470	0.8596	0.7917	0.7270	0.7561	0.7347	0.7206

Modified critical values for adjusted K-S test statistics (Law, 1991)

1-α	0.850	0.900	0.950	0.975	0.990
Expon. Distr.	0.926	0.990	1.094	1.190	1.308

Table 5.4 A-D Test Results of the Poisson Approximation

# of bases / Q_B	4	5	6	8	10	15	20
2	0.7700	0.6876	0.6675	0.6500	0.6261	0.6023	0.6237
3	0.8713	0.7692	0.6809	0.6677	0.6393	0.6015	0.6600
4	0.8706	0.7705	0.7091	0.6544	0.6081	0.6045	0.6243
5	0.9310	0.7914	0.7051	0.6920	0.6504	0.5907	0.6157
6	0.9692	0.7846	0.7085	0.6599	0.6209	0.6080	0.5852
7	0.9410	0.8123	0.7313	0.6603	0.6543	0.6004	0.6195
8	1.0164	0.8182	0.7279	0.6415	0.6398	0.6219	0.6158

Modified critical values for adjusted A-D test statistics (D'Agostino 1986)

$1-\alpha$	0.85	0.9	0.95	0.975	0.99
A^2	0.631	0.752	0.873	1.035	1.159

From these tables, we reach the following conclusions. With 90% of confidence, the Poisson approximation of the depot demand process can not be rejected with eight or more bases. The accuracy of the Poisson approximation increases with more bases in the system. As the batch size increases, the Poisson approximation is slowly deteriorating. This conclusion is consistent with the Franken's article. From these tables, we can also find out that the Anderson-Darling test has more test power than the K-S and χ^2 tests.

5.7.2 The Effect of the Number of Bases and Q_B on the Normal Approximation:

According to the central limit theory, the demand process at the depot can be approximated by a normal process when the system has many bases. In our study, we compare the Poisson approximation with the normal approximation. We use the Komogrov-Smirnov test and the Anderson-Darling test to conduct goodness-of-fit tests of a normal distribution. We indicated at the beginning of this chapter that the χ^2 test is not a good test for our study because a χ^2 test has low test power. The following two tables are the test results for a normal distribution approximation.

Table 5.5 K-S Test Results of the Normal Approximation

# of bases / Q_B	4	5	6	8	10	15	20
2	0.7037	0.7112	0.7270	0.7544	0.7195	0.7445	0.7149
3	0.7765	0.7512	0.7458	0.7926	0.7776	0.7762	0.8109
4	0.7761	0.7643	0.7963	0.7525	0.8248	0.8047	0.8202
5	0.8851	0.7332	0.7785	0.7926	0.7690	0.7850	0.7928
6	0.9024	0.8445	0.8765	0.8750	0.8254	0.7804	0.8318
7	0.8760	0.8252	0.8772	0.7699	0.9065	0.8284	0.8427
8	0.8579	0.9083	0.9098	0.9007	0.9685	0.8503	0.8058

Modified critical values for adjusted K-S test statistics (Law, 1991)

1-α	0.850	0.900	0.950	0.975	0.990
Normal Distr.	0.775	0.819	0.895	0.955	1.035

Table 5.6 A-D Test Results of the Normal Approximation

# of bases Q_B	4	5	6	8	10	15	20
2	0.6425	0.6387	0.6253	0.6248	0.6239	0.6127	0.5988
3	0.7416	0.7369	0.7032	0.7389	0.6800	0.6340	0.6366
4	0.8334	0.7856	0.7837	0.7619	0.7559	0.7161	0.6765
5	0.9287	0.9217	0.9139	0.7713	0.8426	0.6448	0.6396
6	1.0134	0.9196	0.9216	0.8501	0.8195	0.7327	0.6796
7	1.0187	0.9751	0.9378	0.9597	0.9099	0.7425	0.7304
8	1.1822	1.1308	1.0997	0.9076	0.7647	0.6079	0.7132

Modified critical values for adjusted A-D test statistics (D'Agostino 1986)

1-α	0.85	0.9	0.95	0.975	0.99
A^2	0.631	0.752	0.873	1.035	1.159

From these tables, we reach the following conclusions: With 90% of confidence, the normal approximation of depot process can not be rejected with more than ten bases. The accuracy of the normal approximation increases with more bases in the inventory system. The approximation of depot demand with a normal distribution deteriorates more rapidly than that with a Poisson process as Q_B increases. This finding is another reason that the Poisson approximation of the demand process at the depot is more desirable than the normal approximation. The Anderson-Darling test also shows that it has more test power than the K-S test.

We conclude that the Poisson approximation is better than the normal approximation with fewer than ten bases in the inventory system.

5.7.3 The Runs-up Test and the Correlation Test

Table 5.7 Correlation Test Results of the Depot Demand Intervals

# of bases Q_B	4	5	6	8	10	15	20
2	-0.0939	-0.0868	-0.0898	-0.0714	-0.0572	-0.0470	-0.0416
3	-0.1682	-0.1412	-0.1282	-0.1059	-0.0873	-0.0596	-0.0446
4	-0.2044	-0.1770	-0.1504	-0.1257	-0.0898	-0.0670	-0.0582
5	-0.2317	-0.1955	-0.1705	-0.1300	-0.1101	-0.0715	-0.0443
6	-0.2529	-0.2062	-0.1728	-0.1344	-0.1153	-0.0701	-0.0546
7	-0.2624	-0.2117	-0.1839	-0.1431	-0.1134	-0.0692	-0.0544
8	-0.2811	-0.2217	-0.1904	-0.1264	-0.1067	-0.0768	-0.0552

Table 5.8 Runs-up Test Results of the Depot Demand Intervals

# of bases Q_B	4	5	6	8	10	15	20
2	6.6201	7.3389	7.7694	8.5185	9.0897	9.5026	6.5454
3	15.810	8.0239	6.6795	7.9053	5.0337	7.4207	7.9223
4	16.969	8.7519	6.6634	7.8703	7.0369	6.6622	7.0373
5	16.464	10.381	8.0742	8.9703	5.6794	7.0132	6.3494
6	15.533	11.871	6.8676	7.6324	8.4125	6.2157	7.6711
7	16.692	10.993	6.0735	6.2101	6.8267	9.6196	5.9982
8	21.083	10.870	8.4669	5.6432	6.7885	6.8120	8.1648

Critical point χ^2 value for the chi-square distribution with 6 d.f.

$1-\alpha$	0.250	0.500	0.750	0.900	0.950	0.975	0.990
χ^2	3.455	5.348	7.841	10.645	12.592	14.449	16.812

The runs-up test value of the base order intervals has a χ^2 value with six degrees of freedom. The runs-up test results show that base orders at the depot are independent random variables when there are more than five bases in the two-echelon inventory system.

With the correlation test results, we conclude that the base order intervals at the depot are independent random variables and the correlation between successive demand intervals is very weak. **Table 5.7** and **Table 5.8** show that the correlation test provides more information about the base orders at the depot than the runs-up test does.

We realize that the demand interval has a negative correlationship and the absolute value of the correlation increases as Q_B increases. The negative correlationship results are reasonable. When a base requests an order from the depot, it is unlikely that the same base will request another order from the depot soon. On the other hand, if a base has not requested an order for a while, the inventory position at that base will be close to the reorder point. It is likely that the base will request an order from the depot soon. As Q_B increases, it is unlikely that the next depot demand comes from the same base. That is why the depot demand interval has a negative correlation and the absolute value of the correlation increases as Q_B increases.

As we know, in the regular inventory model, the order sizes, the reorder points and the total cost are not very sensitive to the demand process. We compare the simulation model of the two-echelon inventory system with the analytical model in Chapter 6.

CHAPTER 6

THE ALGORITHM OF CALCULATING THE MINIMUM TOTAL COST

6.1 DESCRIPTION OF THE ALGORITHM

In Chapter 4, we define the total cost function as:

$$TC=N_B\left(\frac{\lambda_B}{Q_B}S_B+H_BE_1^{\ B}+B_BE_2^{\ B}\right)+\frac{\lambda_D}{Q_D}S_D+Q_BH_DE_1^{\ D}$$

Figure 1.2 can be used to shows our analytical algorithm. **Table 6.1** lists the input and the output of our algorithm. The total cost is a function of variable B_B, H_B, H_D, N_B, Q_B, Q_D, R_D, R_B, S_B, S_D, λ_B, λ_D. In the total cost function, B_B, H_B, H_D, N_B, S_B, S_D, λ_B are inventory system parameters. Q_B, Q_D, R_B, R_D are decision variables and λ_D is a function of variable N_B, Q_B, λ_B.

In our study, we develop an algorithm to find the values of decision variable Q_B, Q_D, R_D, R_B to minimize the total cost of the two-echelon inventory system. In Chapter 5, we have proven the sufficient conditions of the convexity of the total cost function with respect to the reorder points at the bases and the depot. Although we did not prove the sufficient conditions of the total cost function convexity with respect to the ordering quantities at the bases and the depot, we applied our algorithm with more than 300 sets of wide-range system parameters and initial starting points. We always arrived at the same minimum total cost. We are sure that our algorithm will give the optimal values of the decision variables and the minimum total cost with the optimal decision variable values.

Table 6.1 Input, Process and Output of the Analytical Algorithm

INPUT	PROCESS	OUTPUT
System Parameters: . B_B, H_B, H_D, N_B, L_B, L_D, S_B, S_D, λ_D Initial Values of Decision Variable: . Q_B^0, Q_D^0, R_B^0, R_D^0	Analytical Algorithm	Optimal Values of Decision Variables: . Q_B^*, Q_D^*, R_B^*, R_D^* System Characteristics: . expected base on-hand inventory . expected base backorders . expected depot on-hand inventory . expected total system cost

In the algorithm, we search for the minimum total cost by changing one decision variable at a time. There are four decision variables Q_B, Q_D, R_B, R_D in the total cost function. The first step is to calculate the total cost TC^0 with initial values Q_B^0, Q_D^0, R_B^0, R_D^0. N is the number of the unchanged decision variables in each iteration. The second step is to change the Q_B value by one unit and calculate the total cost TC^1 with the new Q_B value. If the total cost TC^1 is lower than TC^0, Q_B^0 will be changed and the number of unchanged variables will be reduced. In the same way, the algorithm will check the total cost by changing Q_D, R_B, R_D values. At the end of each iteration, the number of unchanged variables is checked. When the number of unchanged variables is four, the total cost is at its minimum and the search for further reducing total cost stops. If the number of unchanged variables is less than four, the algorithm will start a new iteration to minimize the total cost.

As we know, decision variable Q_B, Q_D, R_B, R_D are related and some system parameters limit the ranges of decision variables. For example, the average demand during the base leadtime limits the minimum base order size Q_B. The average base orders at the depot during the depot leadtime limits the minimum depot order size Q_D^0. These requirements are necessary with a stable inventory system. If the average demand during the leadtime is equal or greater than the base order size, the number of customers waiting for the items increases indefinitely as the inventory system runs. We have a similar requirement in the queuing theory. The arrival rate in a signal queuing system must be smaller than the service rate of the system.

The reorder point at each base R_B limits the ordering quantity at each base Q_B. In our inventory system, the reorder points at each base and the depot can be negative. When the reorder point at a base is negative, the base will only place an order when customers are backlogged. If the reorder point at a base is negative, the ordering quantity at the base is greater than the absolute value of the reorder point at the base. After receiving its supplies from the depot, the base will have at least one unit of on-hand inventory at its facility if there is no demand during the base leadtime.

We program the algorithm in four parts. The dotted lines in **Figure 6.1** show these four parts. Part A is to see the effect of changing the base order size on the total cost. It is shown in detail in **Figure 6.2**. Part B is to see the effect of changing the depot order size on the total cost. It is shown in **Figure 6.3**. Part C is to see the effect of changing the base reorder point on the total cost. It is shown in **Figure 6.4**. Part D is to see the effect of changing the depot reorder point on the total cost. It is shown in **Figure 6.5**.

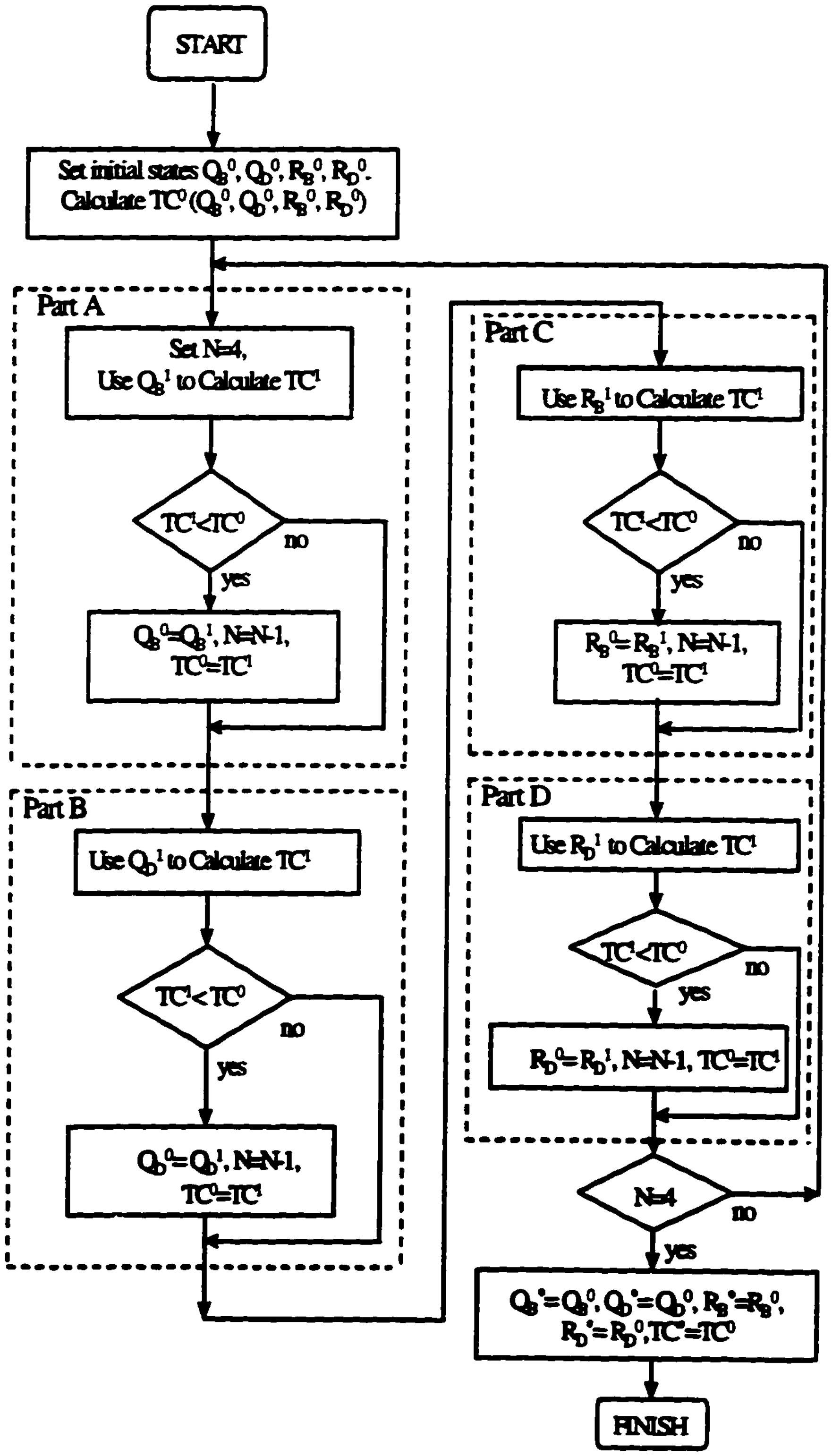

Figure 6.1 Flowchart of Minimizing Total Cost

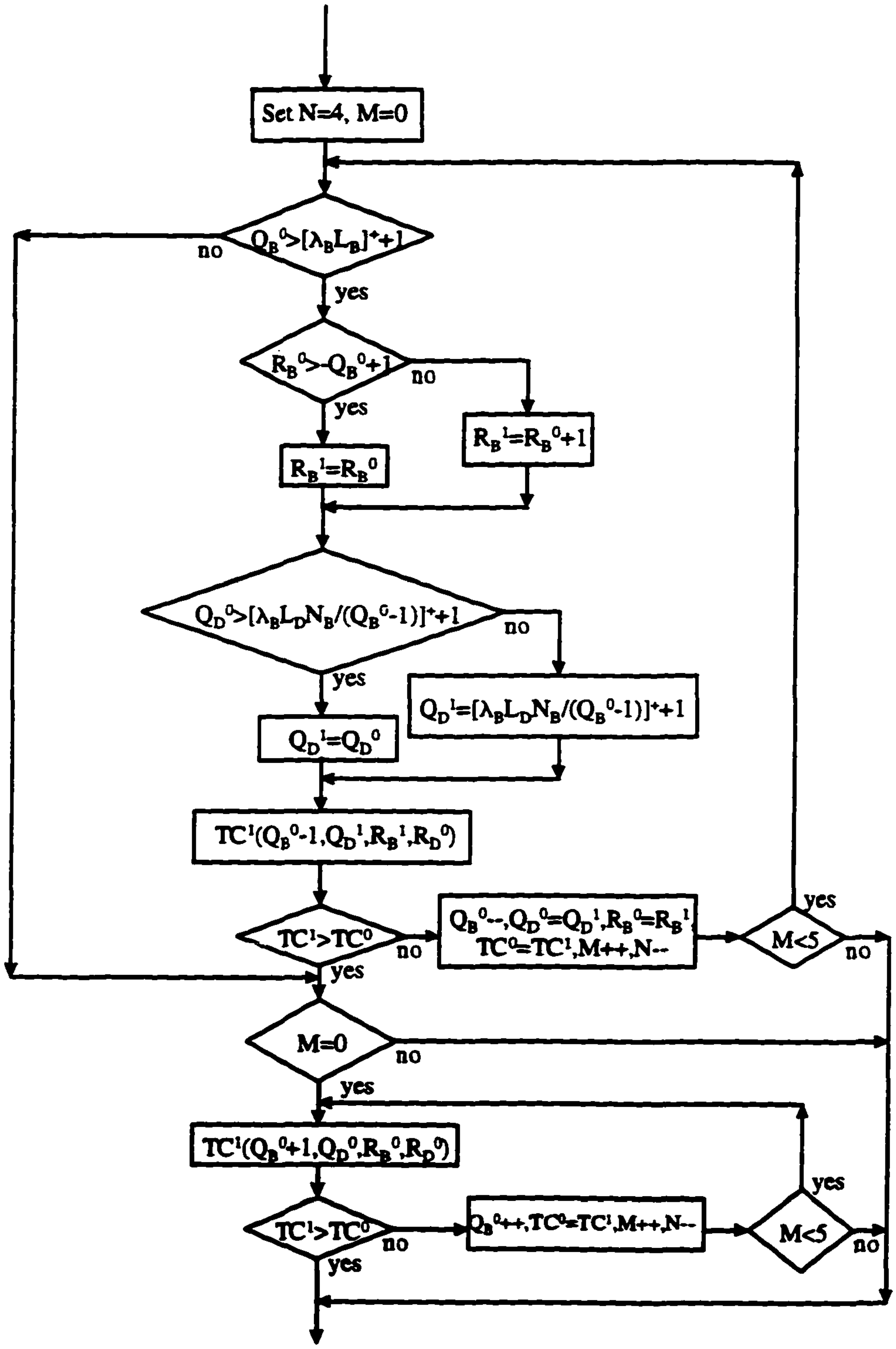

Figure 6.2 Flowchart of Changing the Base Order Size
(Part A of Figure 6.1)

Figure 6.2 is the flowchart of calculating the total cost when the ordering quantity at the base changes. M is the number of steps in the search of the Q_B value. We limit the M value to be five. If changing Q_B value can lower the total cost, the algorithm will continue its search to minimize the total cost by changing the Q_B value in the same direction up to five times before it switches to another decision variable.

If the M value is very large, the total cost's reduction decreases because of the diminishing reduction of the total cost in respect to the change of the decision variable. If the M value is one, the algorithm will check the total cost with respect to all other decision variables before it returns to the decision variable that reduces the total cost. We realize that when M is five, the efficiency of the algorithm is very high.

In **Figure 6.2**, the algorithm starts by calculating the total cost when Q_B decreases. It checks the feasibility of decreasing Q_B. As it is said, the ordering quantity at each base must be greater than the average leadtime demand at each base. If it is not feasible to decrease the Q_B value, the algorithm starts to calculate the total cost when Q_B increases.

When it decreases Q_B value, the algorithm checks the reorder point at each base. If the reorder point at each base is negative and the ordering quantity at each base is not enough to get a positive on-hand inventory after the replenishment, the base reorder point will be increased. The next step is to check the Q_D value. After the reduction of the ordering quantity at each base, the average number of orders at the depot during the depot leadtime may increase. The algorithm will make the necessary change in the ordering quantity at the depot to make sure that the ordering quantity at the depot is enough to cover the average number of orders at the depot during the depot leadtime.

After all these necessary adjustments to the decision variables, the algorithm calculates the total cost when the ordering quantity at the base is reduced by one base order. If the total cost is reduced, the algorithm will continue its process to reduce the ordering quantity at the base and calculate the total cost with the new ordering quantity. This process continues for at most five times before the algorithm starts with a different decision variable.

If the ordering quantity at each base cannot be reduced or if the total cost increases with the ordering quantity reduction, the algorithm starts its search for the total cost reduction by increasing the ordering quantity at each base. If the total cost is reduced with the increasing ordering quantity at the base, the algorithm will continue the process for at most five times before it turns to search other decision variables for the total cost reduction. When the total cost of the inventory system decreases with the increasing or the decreasing of the ordering quantity at the base, the number of unchanged decision variables N is reduced. If any increase or decrease of the ordering quantity of the base will not reduce the total cost, the algorithm starts to check the effect of changing the ordering quantity of the depot on the total cost in **Figure 6.3**. Then, N value is still four.

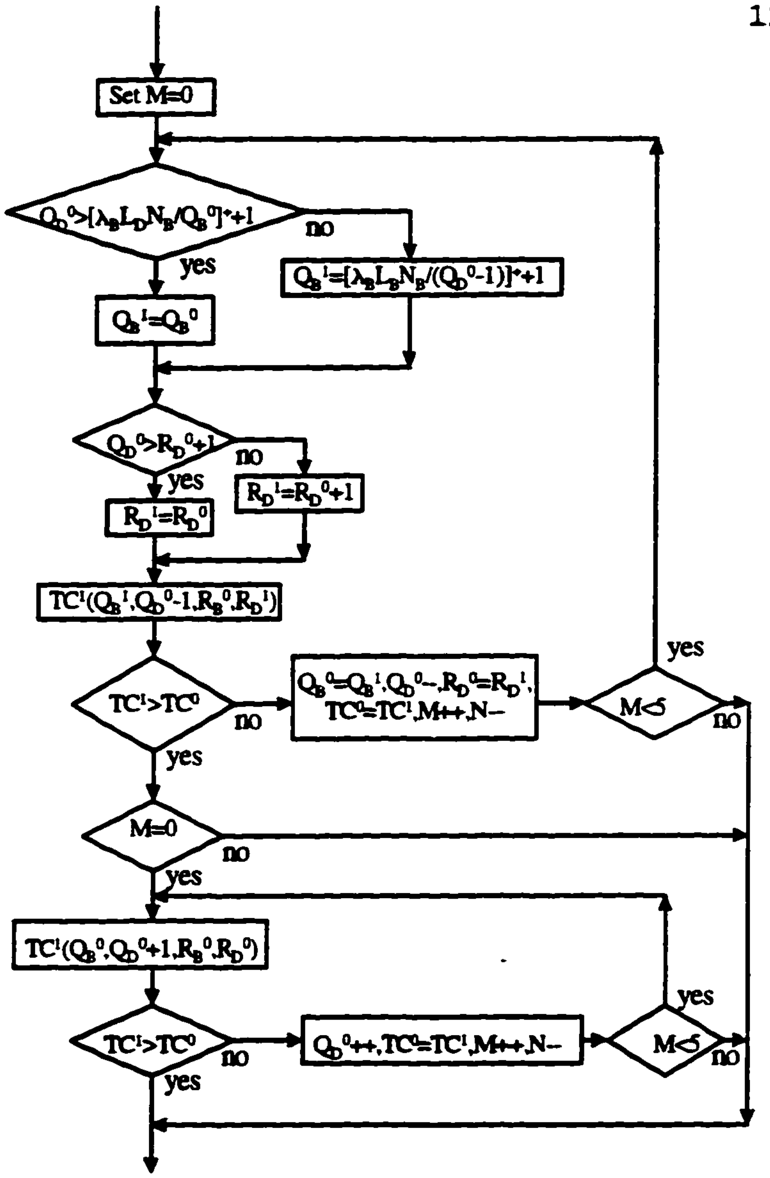

Figure 6.3 Flowchart of Changing the Depot Order Size (Part B of Figure 6.1)

In **Figure 6.3**, the algorithm uses the total cost from **Figure 6.2**. It checks the feasibility of decreasing Q_D. As has been stated, the ordering quantity at the depot must be greater than the average orders during the depot leadtime from bases. Since increasing the ordering quantity at each base will decrease the number of base orders at the depot, the algorithm will increase the ordering quantity of each base to decrease the number of base orders at the depot. By doing that, the algorithm decreases the depot order size.

With the necessary increase of the base order size, the algorithm checks the reorder point at the depot. If the reorder point at the depot is negative and the ordering quantity at the depot is not enough to have on-hand inventory after the replenishment, the reorder point at the depot will be increased.

After all these necessary adjustments to the decision variables, the algorithm calculates the total cost when the ordering quantity at the depot is reduced by one base order. If the total cost is reduced, the algorithm will continue its reduction of the ordering quantity at the depot and calculate the total cost with the new ordering quantity. This process continues for at most five times before the algorithm starts with the decision variable R_B.

If the total cost is increasing following the depot ordering quantity reduction, our algorithm starts its search of total cost reduction by increasing the depot ordering quantity. If the total cost is reduced following the increasing ordering quantity at the depot, the algorithm continues the process for at most five times before it turns to search other decision variables for the total cost reduction. When the total cost of the inventory system is reduced with the increasing or decreasing the ordering quantity at the depot, the number of unchanged decision variables N is reduced.

The algorithm that checks the effect of changing the reorder point at the base on the total cost is shown in **Figure 6.4**.

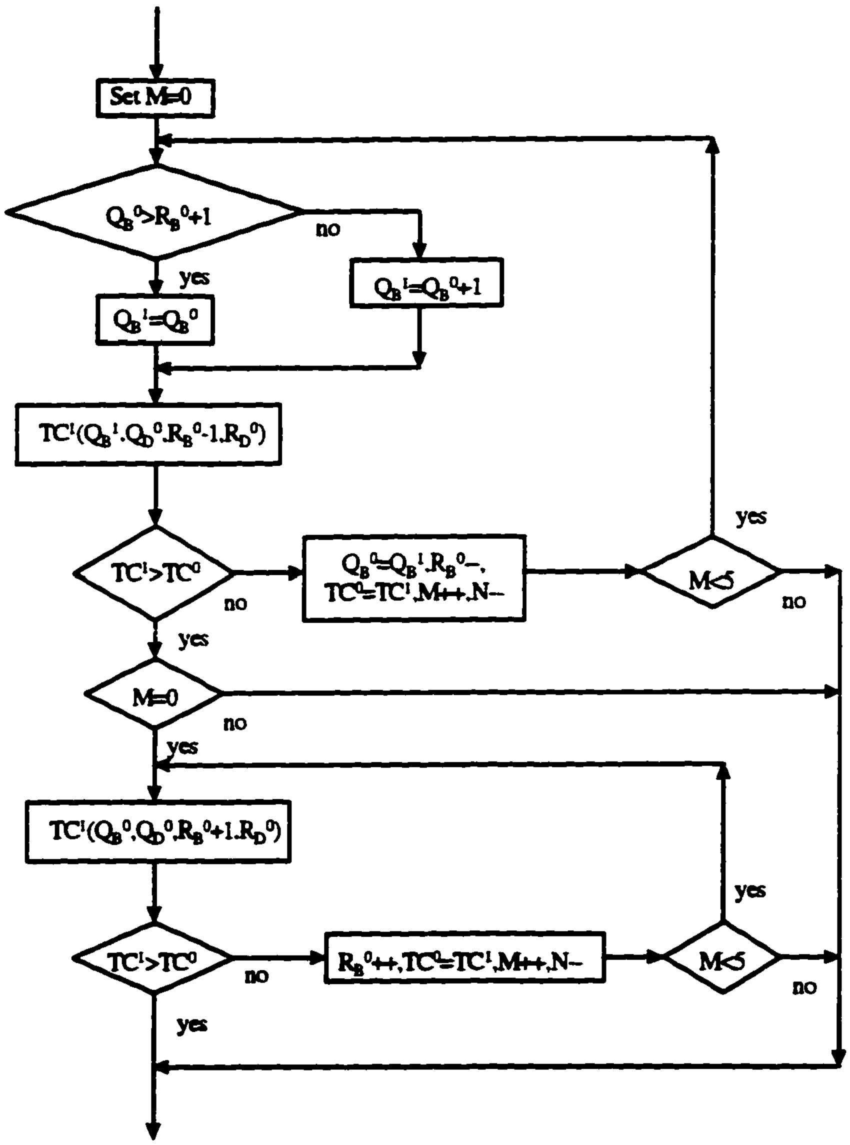

Figure 6.4 Flowchart of Changing the Base Reorder Point
(Part C of Figure 6.1)

First, our algorithm checks the values of the reorder point and ordering quantity at the base. When the reorder point at each base is negative and the absolute value of the base reorder point is not smaller than the ordering quantity at the base by two units, the algorithm will increase the ordering quantity at the base by one unit. In that way, the algorithm can calculate the total cost with the reorder point at the base is reduced by one unit. By making such an adjustment, the required relationship between the reorder point and the ordering quantity of the base is met.

If the total cost with the reduction of the reorder point at the base is smaller than the previous one, the algorithm will continue to reduce the reorder point at the base by making a similar adjustment to the ordering quantity at the base and substitute the total cost with the reduced one. This process may last at most five times before the algorithm starts with the depot reorder point in **Figure 6.5**. If the total cost with the reduction of the base reorder point is not smaller than before, the algorithm checks the total cost with the increased base reorder point. Then, the ordering quantity of the base does not change. Any reduction of the total cost with increasing the base reorder point will keep the process for at most five times before the algorithm starts with the depot reorder point. The number of unchanged decision variables is reduced when the base reorder point changes.

If any change in the reorder point of the base does not bring down the total cost, the number of unchanged decision variables remains unchanged and the algorithm checks the effect of the depot reorder point at the depot on the total cost in **Figure 6.5**.

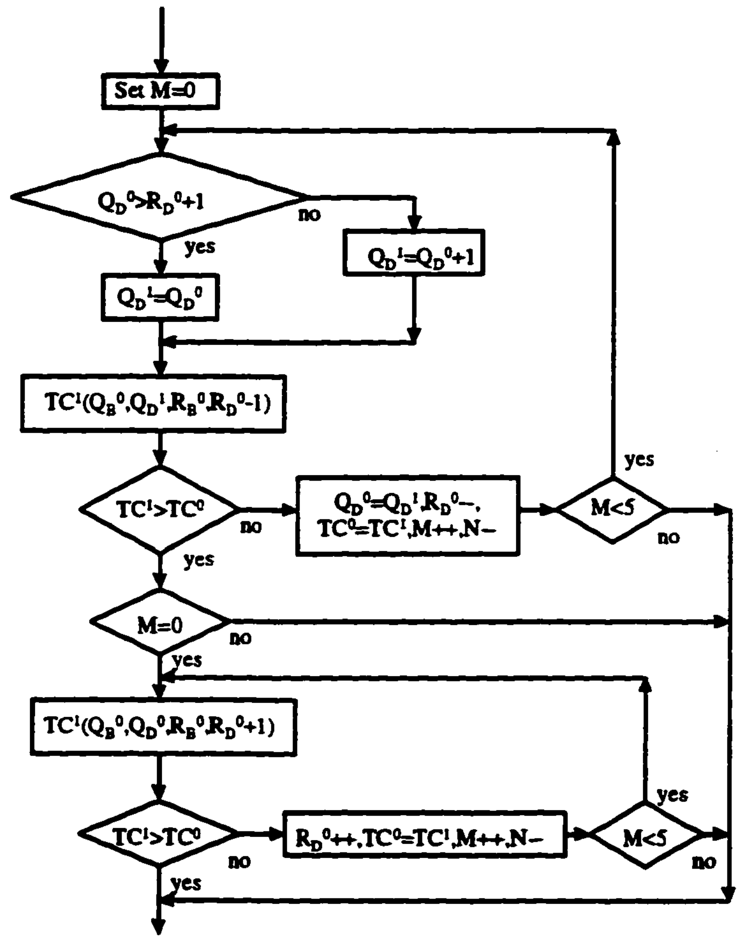

Figure 6.5 Flowchart of Changing the Depot Reorder Point (Part D of Figure 6.1)

Figure 6.5 shows the flowchart of changing the depot reorder point. It is the last part of each iteration which is very similar to the flowchart of **Figure 6.4**. After the algorithm goes through the iteration from **Figure 6.2** to **Figure 6.5**, it checks the number of unchanged decision variables.

If the number of unchanged decision variables is four, the total cost of the inventory system can not be reduced by changing any decision variable Q_B, Q_D, R_D, R_B. The total cost reaches its minimum value, and the decision variables are at their optimal values.

If the number of unchanged decision variables is less than four, at least one decision variable has changed its value, and the total cost is reduced in the previous iteration. The algorithm will continue its search until the number of unchanged decision variables after an iteration is four.

Summary of the Algorithm

We use this algorithm for more than three hundred sets of systems parameters. For each set of inventory system parameters, we use several different initial starting points. The algorithm always ends in the same minimum total cost and optimal decision variable values. That means that our algorithm did not stop at any local optimal values.

When we use different initial starting points with the same set of inventory system parameters, the algorithm searches different routes. Some decision variables change their search directions in the process. For example, the value of the base reorder point may initially increase and later decrease in the search process. All other decision variables may experience similar search direction changes.

In our algorithm, we consider the relationships between the decision variables and parameters, we are certain that our algorithm checks every possible change of these decision variables. It may be the reason our algorithm always ends in one optimal solution. The total costs at initial starting points are normally very high and are reduced dramatically at their optimal levels.

6.2 THE EFFECT OF INVENTORY SYSTEM PARAMETERS ON THE TOTAL COST AND THE DECISION VARIABLES

In our study, as in most inventory model studies, the objective is to discover the minimum total inventory cost and the values of the system decision variables with the minimum total cost. Our algorithm uses inventory system parameters to determine the minimum total cost and the ordering quantities and the reorder points at bases and the depot. To see the effect of the system parameters on the total cost and the decision variables shown below, we change one parameter and keep all other parameters unchanged in the search for minimum total cost.

Because of the complex relationship among the system parameters and the decision variables, we ran more than 300 sets of scenarios. Some results may not give clear relationship among certain system parameters and decision variables. We present several tables from what we got. These tables show relationships clearly. For all the rest tables, certain relationships cannot be easily seen and sometimes one or more decision variables change slightly or remain the same when a system parameter changes. However, the results of those tables do not contradict any observation or conclusion we get from the following tables. We also ran simulations with system parameters and the optimal decision variable values. By doing this, we can compare the total costs of our analytical model with those of simulations.

6.2.1 The Effect of B_B on the Total Cost and Decision Variables

The following system parameters: H_B=\$1.0/unit, H_D=\$0.5/week, S_B=\$25, S_D=\$10, L_B=3 weeks, L_D=2 weeks, N_B=10, λ_B=0.25 customers/week are used in the study of unit base backorders cost on the system performance.

Table 6.2 Effect of B_B on the Total Cost and Decision Variables

B_B	Q_B	Q_D	R_B	R_D	Simulation	TC
0.2	7	7	-6	-5	19.68970	19.20687
0.5	6	3	-4	-1	24.98041	24.98314
1.0	5	4	-2	-1	30.17663	31.01598
2.0	4	4	-1	0	35.35956	36.08229
4.0	4	4	-1	1	40.15273	40.53383
6.0	4	3	-1	2	44.39243	44.45568
8.0	3	5	0	2	46.40051	46.65807
10.0	4	3	0	2	46.65772	47.143337
13.0	4	4	0	2	48.07777	48.542332
16.0	4	4	0	2	49.09657	49.87827
22.0	4	3	0	3	52.26281	52.33252

The reorder points at the depot and the bases increases as the unit base backorders cost increases. The effect of the increasing unit base backorders cost on the ordering quantities of the depot and bases is not obvious from this table. Generally speaking, the base ordering quantity decreases and the depot ordering quantity increase as the unit base backorders cost increases. From **Table 6.2**, the effect of change decision variables on the total cost can be seen. Reorder points at the depot and bases have more effect on the total cost than the ordering quantities at the depot and the bases have.

When the unit base backorders cost changes from $8.0/week to $10.0/week, ordering quantities at the depot and bases change in different direction. The optimal decision variable values do not always change in one direction. It makes a general formula for the optimal decision variable values impossible.

6.2.2 The Effect of H_D on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$1/unit, H_B=\$0.05/week, S_B=\$15, S_D=\$15, L_B=3 weeks, L_D=1 week, N_B=16, λ_B=0.5 customers/week,

Table 6.3 Effect of H_D on the Total Cost and Decision Variables

H_D	Q_B	Q_D	R_B	R_D	Simulation	TC
0.001	19	26	0	1	14.6140	14.6253
0.005	19	12	0	0	15.2123	15.28959
0.01	18	9	0	0	15.8861	15.78354
0.03	18	4	0	0	17.0848	17.0305
0.08	18	4	1	-1	17.6737	17.8928
0.1	17	4	1	-1	18.0258	19.20698
0.15	18	3	1	-1	19.4409	20.01609
0.2	17	3	1	-1	19.9018	20.64735
0.25	16	3	1	-1	22.6817	21.25313
0.5	17	2	1	-1	24.0709	25.07547
1	13	3	3	-2	24.2272	25.07547

As the holding cost per unit at the depot increases, the depot reorder point decreases, because the depot holding cost increases. With the same reason, the depot ordering quantity is reduced. With the lower reorder point at the depot, the average on-hand inventory at the depot will be smaller. At the same time, the probability that a base order is backlogged at the depot is greater. To compensate for the higher probability of stockout at the depot, the base reorder point increases.

The reorder point and ordering quantity at the depot are measured by ordering quantity at the base. They are not measured by the unit of items customers order at bases. The reductions of ordering quantity and reorder point at the depot are also performed by decreasing the ordering quantity at each base.

When our algorithm is searching the minimum total cost, it considers costs both at the depot and the base levels. When the holding cost at the depot increases from $0.25/week to $0.50/week, the ordering quantity at the base increases and the base reorder point remains the same. When the base reorder point remains the same and the probability that the depot runs out of stock increases, the ordering quantity of the base increases to reduce the increasing average stockout cost.

When the holding cost at the depot increases from $0.50/week to $1.00 /week, the ordering quantity at the depot increases. That seems to be contrary to what we discussed above. In (Q, R) inventory systems, the change per unit in reorder point has more impact on the average on-hand inventory and average backorders than that in the ordering quantity does. That means that decreasing one unit in the reordering point and increasing one unit in the ordering quantity of the depot at the same time will bring down the average on-hand inventory at the depot.

When the holding cost increases from $0.50/week to $1.00/week, our algorithm searches the reduction of the total cost by decreasing the reorder point at the depot.

However, the ordering quantity at the depot is 2 and the reorder point at the depot is -1, when the holding cost at the depot is $0.50/week. With our assumption that $Q_D \geq -R_D+1$, the reordering point cannot be reduced without increasing the ordering quantity at the same time. The algorithm calculates the total cost with decreasing the reordering point and increasing the ordering quantity at the depot simultaneously. **Figure 6.5** shows the flowchart of doing the search. As we mentioned before, our algorithm searches every possible change of decision variables in reducing total cost.

6.2.3 The Effect of H_B on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$1/unit, H_D=\$0.005/week, S_B=\$15, S_D=\$15, L_B=3 weeks, L_D=1 week, N_B=16, λ_B=0.5 customers/week.

Table 6.4 Effect of H_B on the Total Cost and Decision Variables

H_B	Q_B	Q_D	R_B	R_D	Simulation	TC
0.01	39	6	1	0	7.5293	7.587416
0.02	28	8	1	0	10.2266	10.19677
0.03	23	10	1	0	12.1500	12.20822
0.04	21	11	0	0	13.9098	13.8694
0.05	19	12	0	0	15.2240	15.28951
0.1	18	13	0	0	21.6713	20.71778
0.2	10	22	-1	1	27.7917	27.79039
0.3	9	25	-1	1	32.5269	32.54421
0.4	7	32	-1	1	36.2750	36.27716
0.5	7	32	-1	1	39.2740	39.27601
0.6	6	38	-1	1	41.9005	41.92485
1.00	6	38	-2	1	48.4344	48.43144
2.00	5	44	-3	2	58.7968	58.03149

As the base holding cost per unit increases, the base reorder point decreases. For the same reason, the ordering quantity at each base is reduced. With the lower reorder point at each base, the average on-hand inventory at each depot will be smaller. At the same time, the probability that a customer is backlogged at each base is greater. To compensate for the higher probability of stockout at the base, the depot reordering point and the ordering quantity at the depot increase.

There are two reasons that the depot reorder point and ordering quantity increase. First, the reorder point and ordering quantity at the depot are measured by base orders, and not by the unit of the product customers order at bases. To keep the same amount of goods at the depot when the ordering quantity of each base is decreasing, the reordering point and ordering quantity at the depot increase. Second, to compensate for the reduced reorder point and ordering quantity at each base, we also need to increase the reorder point and ordering quantity at the depot. By doing that, the probability that the depot runs out of stock decreases. Of course, the total cost of the inventory system increases as the holding cost at each base increases.

6.2.4 The Effect of S_B on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$8/unit, H_B=\$0.1/week, H_D=\$0.05/week, S_D=\$20, L_B=5 weeks, L_D=4 week, N_B=2, λ_B=1.0 customers/week.

Table 6.5 Effect of S_B on the Total Cost and Decision Variables

S_B	Q_B	Q_D	R_B	R_D	Simulation	TC
0.1	6	7	8	2	3.96774	4.050511
0.5	6	7	8	2	4.16080	4.183844
1	6	7	8	2	4.32756	4.350511
2	6	7	8	2	4.56402	4.683842
5	9	5	8	1	5.66765	5.54982
8	10	5	8	1	5.88445	6.180018
10	12	4	8	1	6.22626	6.542497
12	12	4	8	1	6.55974	6.87583
15	13	4	8	1	7.16520	7.34902
20	16	3	7	1	7.84938	7.962154
25	17	3	7	1	8.27700	8.560926
30	18	3	7	1	9.06855	9.12185
45.0	23	2	7	1	10.19982	10.45712
50.0	25	2	6	1	10.62870	10.86145
60.0	26	2	6	1	11.42978	11.643013

The base setup cost directly affects the base ordering cost. With the increase of the setup cost at each base, the ordering quantity at each base increases. With the increase in ordering quantity at each base, the average on-hand inventory at each base will increase and the average backorders at each base will decreases. To counteract the effect of the increasing base ordering quantity, the reorder point at each base decreases.

With the increasing number of units in a base order at the depot, the number of items in each depot order to the outside supplier and the number of items at the reorder point of the depot increase. At the depot level, the ordering quantity and the reordering point in term of base orders are decreasing to reduce the impact of ordering quantity increase at the base level.

6.2.5 The Effect of S_D on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$0.50/unit, H_B=\$1.0/week, H_D=\$0.2/week, S_B=\$25, L_B=2 weeks, L_D=3 week, N_B=10, λ_B=0.25 customers/week.

Table 6.6 Effect of S_D on the Total Cost and Decision Variables

S_D	Q_B	Q_D	R_B	R_D	Simulation	TC
0.1	6	2	-4	0	21.7874	22.3755
1	6	2	-4	0	22.9749	22.563
2	6	4	-4	-1	23.3697	23.1079
3	6	4	-4	-1	23.47398	23.212
5	6	4	-4	-1	23.6824	23.4204
7	6	4	-4	-1	23.89085	23.6287
10	5	5	-3	-1	24.0930	24.4013
12	5	5	-3	-1	24.1931	24.6013
15	5	6	-3	-1	24.3090	24.8986
20	5	6	-3	-1	24.9258	25.3152
25	5	6	-3	-1	25.8425	25.7319
30	5	8	-3	-2	26.3877	26.1307
40	5	8	-3	-2	26.9002	26.7557
50.0	5	9	-3	-2	27.65605	27.32386
60.0	5	11	-3	-3	28.1377	27.92380

The setup cost at the depot directly affects the ordering cost at the depot. With the increase of the setup cost at the depot, the ordering quantity of the depot increases. The increasing of the depot order size causes the increasing of the average on-hand inventory at the depot. At the same time, the probability that a base order is backlogged at the depot is reduced. To counteract the effect of the increasing ordering quantity, the reorder point at the depot decreases.

As the ordering cost at the depot increases, the ordering cost at each base becomes relatively less expensive. With a cheaper ordering cost at each base, the base ordering quantity is reduced. To compensate for the decrease in ordering quantity of each base, the reordering point at each base increases slightly. The reorder points at the base and the depot are not very sensitive to the changes of the ordering cost. The ordering quantity of each base also changes slightly with the changes of the ordering cost of the depot. The major effect of the depot ordering cost changes is on the depot ordering quantity.

6.2.6 The Effect of λ_B on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$0.05/unit, H_B=\$1.0/week, H_D=\$0.005/week, S_B=\$15, S_D=\$15, L_B=3 weeks, L_D=1 week, N_B=16.

Table 6.7 Effect of λ_B on the Total Cost and Decision Variables

λ_B	Q_B	Q_D	R_B	R_D	Simulation	TC
0.01	3	11	-1	-1	1.89748	1.90334
0.02	4	12	-1	-1	2.85489	2.83726
0.05	6	12	-1	0	4.68919	4.67179
0.1	9	11	-1	0	6.77772	6.78247
0.2	11	13	0	0	9.65805	9.68079
0.3	14	13	0	0	11.7800	11.80283
0.5	18	13	0	0	15.2183	15.69002
0.7	21	13	1	0	18.0020	18.11838
1	25	13	2	1	21.6600	21.70212
1.2	28	13	2	1	23.6720	23.73225
1.5	31	13	3	1	26.4480	26.54998
1.8	34	13	4	1	29.0061	29.11227
2	37	13	4	1	30.5762	30.68401
2.5	40	13	6	1	34.1460	34.3626
3	45	13	7	1	37.3604	37.65782
4	51	13	10	2	43.3680	43.49781

When customer arrival rate increases, the total demand of the inventory system increases. The customer arrival rate affects all decision variables of the inventory system.

With the increase of the customer arrival rate, the ordering quantity and reordering point of the base increases to satisfy the increasing customer demand.

With more demand at each base, the probability that a base order is backlogged at the depot increases. The reorder point of the depot increases with the increasing customer demand. By making that, the probability that a base order backlogged at the depot is reduced. Although the ordering quantity of the depot to the outside supplier increases slightly, the actual number of units ordered each time from the depot increases because of the increasing size of base order to the depot.

When the customer arrival rate increases from 0.05 customer/week to 0.2 customer/week, the ordering quantity of the depot decreases and then increases. The number of units the depot orders from its outside supplier during this time increases from 72 to 143. We conclude that one particular decision variable may not change in one direction when a system parameter makes consistent changes and decision variables work together to minimize the total cost. The ordering quantity of the depot changes slightly with the changes of the customer arrival rate at each base. The major effect of customer arrival rate is on the ordering quantity of each base.

6.2.7 The Effect of N_B on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$8.0/unit, H_B=\$0.1/week, H_D=\$0.05/week, S_B=\$10, S_D=\$20, L_B=5 weeks, L_D=4 weeks, λ_B=1 customer/week.

Table 6.8 Effect of N_B on the Total Cost and Decision Variables

N_B	Q_B	Q_D	R_B	R_D	Simulation	TC
2	12	4	8	1	6.31614	6.5425
3	11	5	8	2	8.85900	9.07921
4	13	5	7	2	10.8885	11.3551
6	13	6	7	3	15.3771	15.854
8	13	7	7	4	19.7480	20.2196
10	14	8	7	4	23.5812	24.4825
12	14	8	7	5	27.8835	28.6252
14	14	9	7	6	32.2467	32.7368
16	14	9	7	7	36.2820	36.8834
18	13	10	7	8	40.3650	41.0234
20	14	10	7	8	44.0221	44.9333
22	14	11	7	9	48.2496	48.9647
24	15	11	7	9	52.0180	52.969
26	15	12	7	9	55.6547	56.9912
28	15	12	7	10	59.7431	60.8854
30	14	13	7	11	63.5133	64.8292

The number of bases in the two-echelon inventory system is a very important system parameter in the study. As we know, the number of bases in the system affects the Poisson approximation of the demand process at the depot. When the number of bases increases, our approximation of the depot demand by a Poisson process is improved. At the same time, the demand at the depot will increase.

To satisfying the increasing demand from bases, the depot's ordering quantity and reorder point increase. Consequently, the depot average holding cost increases. The increasing of the ordering quantity from each base causes further raising of the holding cost at the depot. To compensate the holding cost increase at the depot, the reordering point at each base decreases. By making such a change in the base reorder point, the total holding cost of the inventory system will not increase as fast as one without the reduction of the base reorder point.

6.2.8 The Effect of L_B on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$1.0/week, H_B=\$0.5/week, H_D=\$0.005/week, N_B=16, S_B=\$15, S_D=\$15, L_D=1 week, λ_B=0.5 customer/week.

Table 6.9 Effect of L_B on the Total Cost and Decision Variables

L_B	Q_B	Q_D	R_B	R_D	Simulation	TC
0.2	17	14	-1	0	14.5860	14.65441
0.5	18	13	-1	0	14.6370	14.70411
1	18	13	-1	0	14.7585	14.82679
1.5	18	13	-1	0	15.0380	15.00777
2	18	13	0	0	15.0485	15.10619
2.5	18	13	0	0	15.0974	15.17055
3	19	12	0	0	15.2154	15.28959
3.5	18	13	1	0	15.3295	15.40455
4	18	13	1	0	15.3699	15.44644
4.5	19	12	1	0	15.4570	15.53265
5	19	12	1	0	15.5750	15.62898
5.5	18	13	2	0	15.6371	15.69028
6	19	12	2	0	15.6735	15.75328
6.5	19	12	2	0	15.7720	15.85358
7	18	13	3	0	15.8405	15.91467
8	19	12	3	0	16.0572	16.03839
9	19	12	4	0	16.0777	16.15546

With the increasing leadtime at each base, the replenishment of base orders takes a longer time. If the base runs out of stock, customers have to wait a longer time. The reorder point at each base increases to satisfy more customers at each base when the base sends its requests of orders to the depot.

A large ordering quantity at each base has a similar effect on the increasing leadtime at each base as the reordering point at each base does. An interesting observation is that increasing ordering quantity at each base by one unit and decreasing the ordering quantity at the depot by one base order will compensate for the increasing of the base leadtime.

We ran several different sets of data. We always ended with the unchanged reorder point of the depot when the leadtime at each base changes. Our conclusion is that the effect of the base leadtime on the depot reorder point is very small.

6.2.9 The Effect of L_D on the Total Cost and Decision Variables

We use the following system parameters: B_B=\$8.0/week, H_B=\$0.1/week, H_D=\$0.05/week, N_B=2, S_B=\$10, S_D=\$20, L_B=5 week, λ_B=1.0 customer/week.

Table 6.10 Effect of L_D on the Total Cost and Decision Variables

L_D	Q_B	Q_D	R_B	R_D	Simulation	TC
0.2	15	3	7	-1	5.19700	5.56768
0.5	13	4	8	-1	5.63512	5.94617
1	12	4	7	0	5.89180	5.99939
1.5	12	4	8	0	5.97554	6.08861
2	12	4	8	0	6.08320	6.22014
2.5	11	4	7	1	6.31220	6.42101
3	11	4	8	1	6.32830	6.43676
3.5	11	4	8	1	6.4329	6.47778
4	12	4	8	1	6.52155	6.5425
4.5	12	4	8	1	6.71623	6.62985
5	10	4	8	2	6.75301	6.67431
5.5	11	4	8	2	6.82827	6.78119
6	11	4	8	2	6.88612	6.80957
7	12	4	8	2	6.89564	6.92758
8	11	5	8	2	6.99447	7.08663
9	10	5	8	3	7.01246	7.10644

When the leadtime of the depot increases, the replenishment of the depot stock takes a longer time. If the depot runs out of stock, bases have to wait longer time before they can receive their orders. It will increase the expected base backorders cost.

Contrary to the change of the base leadtime, the depot leadtime has a major effect on the reordering point at the depot. However, the effect of the depot leadtime on the ordering quantity of the depot is very small. The increasing of the ordering quantity of the depot with the increasing of the depot leadtime can be observed with different data sets. Again, the combination of the reorder point and the ordering quantity of each base deserves a detailed study. The sequence of the changes in the ordering quantity and reorder point at each base is that the reorder point at each base increases before the increase of the ordering quantity at each base. That is, the base reorder point changes have more effect on the backorders cost than base ordering quantity does.

6.3 CONCLUSION

With these tables, we can determine the effect of system parameters on the minimum total cost and the optimal decision variable values. As we showed early, the changes in a particular decision variable may not be in the same direction when a system parameter changes because of interactions among decision variables. These tables show that the total cost with the analytical model and that with the simulation are very close. It means that the Poisson process approximation of the base orders process at the depot is reasonable. Our analytical model can accurately represent the two-echelon (Q, R) inventory system. **Table 6.11** shows the general effect of system parameters on the decision variables and total cost of the two-echelon inventory model. Because decision variables only take integer values and interactions among decision variables, the direction of changes for some decision variables may be switched when a system parameter changes its value.

Table 6.11 Effect of System Parameters on the Total Cost and Decision Variables

	Q_B	Q_D	R_B	R_D	TC
B_B ⇑	⇓	⇑	⇑	⇑	⇑
H_B ⇑	⇓	⇑	⇓	⇑	⇑
H_D ⇑	⇓	⇓	⇑	⇓	⇑
L_B ⇑	⇑	⇓	⇑	unchanged	⇑
L_D ⇑	⇓	⇑	⇑	⇑	⇑
N_B ⇑	⇑	⇑	⇓	⇑	⇑
S_B ⇑	⇑	⇓	⇓	⇓	⇑
S_D ⇑	⇓	⇑	⇑	⇓	⇑
λ_B ⇑	⇑	⇑	⇑	⇑	⇑

Certain results in **Table 6.11** are intuitive. For example, when the unit holding cost at the depot H_D increases, the inventory holding cost at the depot increases. As a result, the depot should keep fewer inventories than before. The depot order size Q_D and reorder point R_D decrease. To compensate for the decreasing depot on-hand inventory, we would assume that bases raise their order size and reorder point. However, the base order size actually decreases. The reason for decreasing Q_B is that Q_D is measured in terms of Q_B. When Q_B decreases, the depot will have fewer inventories than that with unchanged Q_B. To reduce the effect of the decreasing in Q_B and Q_D on bases, the base reorder point R_B increases.

CHAPTER 7 CONCLUSION AND FUTURE RESEARCH

In our two-echelon inventory system with (Q, R) policy, the depot demand process is approximated by a Poisson process. With such an approximation, the analysis of the depot demand process is performed without considering the inventory level at each base. Our analysis of the depot performance is independent of its bases. It dramatically reduces the complexity of the two-echelon inventory model. When we study each base performance, we consider the effect of the depot inventory level on the bases.

· We theoretically prove the sufficient condition of the convexity of the total holding cost with respect to R_D. The sufficient condition is that a base order holding cost at the depot is as large as the holding cost of $(1+R_B/Q_B)$ units at a base. In most inventory systems, the reorder point R is usually smaller than the order size Q. As soon as the holding cost of a base order at the depot is as large as the holding cost of two units at a base, the total cost of the inventory system is convex with respect to D_D. We also prove the convexity of other parts of total cost function with respect to R_B and R_D. We show that the expected base on-hand inventory is not a convex function of the reorder point at the depot. We also explain the reason of the non-convexity of the expected on-hand inventory at bases with respect to the ordering quantity and the reorder point at the depot. With our numerical examples, we show that the same condition is sufficient to establish the convexity of the total holding cost function with respect to Q_D.

Our algorithm searches every possible change of decision variables to discover the minimum total cost and the optimal decision variables for any set of system parameters and compare our analytical minimum total costs to those from the simulations. The differences between these total costs are less than 1%, which is a very good result.

We used a 486 DX 66 personal computer with 16 M RAM to run the analytical model and simulation. It usually takes less than two minutes to get the optimal solution with our algorithm. With the analytical model, the computer run time depends on the optimal values of the decision variable. The computer run time for simulations depends on the number of customers in the simulation. The source codes of our analytical model and simulations are available from the author upon request.

With computer simulation results, several goodness of fit tests are used to test the Poisson approximation and normal approximation of the depot demand process. We compare these test results and show the test power of each goodness of fit test. To show the low test power of the χ^2 test, the customer arrival processes at bases are used to conduct the χ^2 test, whose inter-arrivals are generated by exponential random variables. We conclude that the χ^2 test accuracy depends on the number of events in the test. If there are too many events or just a few events in the test, the χ^2 test result is not reliable. With the simulation results, the effects of number of bases and base orders size on the Poisson approximation of the depot demand are discussed. With eight or more bases, the Poisson approximation of the demand process at the depot is very good. With fewer than eight bases, the Poisson approximation of the demand process at the depot is not satisfied. However, the results from our analytical model are still very close to those obtained by simulation. The reason is that the sensitivity of the optimal solution of an inventory system is very low. That makes our approach of modelling two-echelon inventory systems a reasonable one.

In future studies of two-echelon inventory models, two depots in the system may be studied. In such a model, the leadtimes between one depot and certain bases is shorter than that of another depot. We can also consider the lateral transshipments among bases when a base runs out of stock. In our study, we assume that customer demand is backlogged when bases run out of stock. A future study may consider lost sales or partial backorders. Our model can also be easily adapted to the assumption of different base leadtimes and base reorder points.

Appendix 1

THE PROOF OF UNIFORM DISTRIBUTED INVENTORY POSITION IN THE DEPOT

When $R_D \geq 0$:

We present the relationship between inventory position and on-hand inventory or backorders. The inventory position is R_D+x, when there are R_D+x, R_D-Q_D+x ,...., $R_D+x-[(R_D+x)/Q_D]^{-}Q_D$ on-hand inventories or $[(R_D+x)/Q_D+1]^{-}Q_D-R_D-x$,...., backorders. That is:

$$\begin{aligned}\rho(R_D+x) &= \psi_1^D(R_D+x)+\psi_1^D(R_D+x-Q_D)+\ldots+\psi_1^D(R_D+x-[\frac{R_D+x}{Q_D}]^{-}Q_D)\\ &\quad+\psi_2^D([\frac{R_D+x}{Q_D}+1]^{-}Q_D-R_D-x)+\ldots\\ &=\psi_1^D(R_D+x)+\sum_{i=1}^{[\frac{R_D+x}{Q_D}]^{-}}\psi_1^D(R_D+x-iQ)+\sum_{i=[\frac{R_D+x}{Q_D}]^{-}+1}^{\infty}\psi_2^D(iQ_D-R_D-x)\end{aligned} \tag{1}$$

As we know:

$$\begin{aligned}\psi_1^D(x) &= \frac{1}{Q_D}[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \quad 0\leq x<R_D+1\\ &= \frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \qquad R_D+1\leq x\leq R_D+Q_D\end{aligned} \tag{2}$$

$$\psi_2^D(y)=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \quad y\geq 0 \tag{3}$$

$$\begin{aligned}
&\psi_1^D(R_D+x)+\sum_{i=1}^{[\frac{R_D+x}{Q_D}]^*}\psi_1^D(R_D+x-iQ)=\frac{1}{Q_D}[1-P(R_D+Q_D+1-(R_D+x),\mu_D)]\\
&+\frac{1}{Q_D}\sum_{i=1}^{[\frac{R_D+x}{Q_D}]^*}[P[R_D+1-(R_D+x-iQ_D),\mu_D]-P[R_D+Q_D+1-(R_D+x-iQ_D),\mu_D]]\\
&=\frac{1}{Q_d}[1-P(Q_D+1-x,\mu_D)]+\frac{1}{Q_D}\sum_{i=1}^{[\frac{R_D+x}{Q_D}]^*}[P(iQ_D+1-x,\mu_D)-P(Q_D+iQ_D+1-x,\mu_D)]\\
&=\frac{1}{Q_D}[1-P([\frac{R_D+x}{Q_D}+1]^*Q_D+1-x,\mu_D)]
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\sum_{i=[\frac{R_D+x}{Q_D}]^*+1}^{\infty}\psi_2^D(iQ_D-R_D-x)\\
&=\frac{1}{Q_D}\sum_{i=[\frac{R_D+x}{Q_D}]^*+1}^{\infty}[P(1+iQ_D-x,\mu_D)-P(Q_D+1+iQ_D-x,\mu_D)]\\
&=\frac{1}{Q_D}P(1+[\frac{R_D+x}{Q_D}+1]^*Q_D-x,\mu_D)
\end{aligned} \tag{5}$$

That is: $$\begin{aligned}
&\rho(R_D+x)=\psi_1^D(R_D+x)+\sum_{i=1}^{[\frac{R_D+x}{Q_D}]^*}\psi_1^D(R_D+x-iQ)\\
&+\sum_{i=[\frac{R_D+x}{Q_D}]^*+1}\psi_2^D(iQ_D-R_D-x)=\frac{1}{Q_D}
\end{aligned} \tag{6}$$

When $R_D<0$:

We present the relationship between inventory position and on-hand inventory or backorders.

1. When $R_D+x\geq 0$

The inventory position is R_D+x when there are R_D+x on-hand inventory or Q_D-R_D-x,...., backorders. That is:

$$\begin{aligned} &\textit{When } R_D<0, R_D+x\geq 0\\ &\rho(R_D+x)=\psi_1^D(R_D+x)+\psi_2^D(Q_D-R_D-x)+\ldots\\ &=\psi_1^D(R_D+x)+\sum_{i=1}^{\infty}\psi_2^D(iQ_D-R_D-x), \end{aligned} \quad (7)$$

As we know:

$$\psi_1^D(x)=\frac{1}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad 0\leq x\leq R_D+Q_D \quad (8)$$

$$\psi_2^D(y)=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \quad y\geq -R_D-1 \quad (9)$$

$$\psi_1^D(R_D+x)=\frac{1}{Q_D}[1-P(R_D+Q_D+1-R_D-x,\mu_D)]=\frac{1}{Q_D}[1-P(Q_D+1-x,\mu_D)] \quad (10)$$

$$\sum_{i=1}^{\infty} \psi_2^D(iQ_D-R_D-x)$$
$$=\frac{1}{Q_D}\sum_{i=1}^{\infty} [P(iQ_D-R_D-x+R_D+1,\mu_D)-P(iQ_D-R_D-x+R_D+Q_D+1,\mu_D)]$$
$$=\frac{1}{Q_D}\sum_{i=1}^{\infty} [P(iQ_D+1-x,\mu_D)-P(iQ_D+Q_D+1-x,\mu_D)]=\frac{1}{Q_D}P(Q_D+1-x,\mu_D) \quad (11)$$

When $R_D+x<0$

$$\textit{That is: } \rho(R_D+x)=\psi_1^D(R_D+x)+\sum_{i=1}^{\infty} \psi_2^D(iQ_D-R_D-x)$$
$$=\frac{1}{Q_D}[P(Q_D+1-x,\mu_D)+1-P(Q_D+1-x,\mu_D)]=\frac{1}{Q_D} \quad (12)$$

The inventory position is R_D+x, when there are R_D-x, Q_D-R_D-x,...., backorders.

That is:

$$\textit{When } R_D<0, R_D+x<0$$
$$\rho(R_D+x)=\psi_2^D(-R_D-x)+\psi_2^D(Q_D-R_D-x)+\dots$$
$$=\sum_{i=0}^{\infty} \psi_2^D(iQ_D-R_D-x) \quad (13)$$

As we know:

$$\psi_2^D(x)=\frac{1}{Q_D}[1-P(R_D+Q_D+1+x,\mu_D)] \quad 0\le x\le -R_D-1 \quad (14)$$

$$\psi_2^D(x)=\frac{1}{Q_D}[P(x+R_D+1,\mu_D)-P(x+R_D+Q_D+1,\mu_D)] \quad y\ge -R_D-1 \quad (15)$$

$$\psi_2^D(-R_D-x)=\frac{1}{Q_D}[1-P(R_D+Q_D+1-R_D-x,\mu_D)]=\frac{1}{Q_D}[1-P(Q_D+1-x,\mu_D)] \tag{16}$$

$$\begin{aligned}&\sum_{i=1}^{\infty}\psi_2^D(iQ_D-R_D-x)\\ &=\frac{1}{Q_D}\sum_{i=1}^{\infty}[P(iQ_D-R_D-x+R_D+1,\mu_D)-P(iQ_D-R_D-x+R_D+Q_D+1,\mu_D)]\\ &=\frac{1}{Q_D}P(Q_D-x+1,\mu_D)\end{aligned} \tag{17}$$

That is: $$\begin{aligned}&\rho(R_D+x)=\psi_2^D(-R_D-x)+\sum_{i=1}^{\infty}\psi_2^D(iQ_D-R_D-x)\\ &=\frac{1}{Q_D}[P(Q_D+1-x,\mu_D)+1-P(Q_D+1-x,\mu_D)]=\frac{1}{Q_D}\end{aligned} \tag{18}$$

Conclusion:

The inventory position in the depot is uniformly distributed among $[R_D+1, R_D+Q_D]$ with $1/Q_D$ as the state probability no matter whether $R_D \geq 0$.

Appendix 2

THE PROOF OF UNIFORM DISTRIBUTED INVENTORY POSITION IN BASES

When the Reorder Point at Bases $R_B \geq 0$:

Similar to the depot case, the inventory position in bases is R_B+x when there are R_B+x, R_B-Q_B+x ,...., $R_B+x-[(R_B+x)/Q_B]^{-}Q_B$ on-hand inventories or $[(R_B+x)/Q_B+1]^{-}Q_B-R_B-x$,...., backorders. That is:

$$
\begin{aligned}
\rho(R_B+x) &= \psi_1^B(R_B+x)+\psi_1^B(R_B+x-Q_B)+\ldots+\psi_1^B(R_B+x-[\frac{R_B+x}{Q_B}]^{-}Q_B) \\
&\quad +\psi_2^B([\frac{R_B+x}{Q_B}+1]^{-}Q_B-R_B-x)+\ldots \\
&= \psi_1^B(R_B+x)+\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^{-}} \psi_1^B(R_B+x-iQ)+\sum_{i=[\frac{R_B+x}{Q_B}]^{-}+1}^{\infty} \psi_2^B(iQ_B-R_B-x)
\end{aligned}
\qquad (1)
$$

As we know:

$$
\psi_1^B(x)=\frac{1-P_{out}^D+p_{so}^B(0)}{Q_B}[1-P(R_B+Q_B+1-x,\mu_B)] \quad where \quad R_B+1 \leq x \leq R_B+Q_B \qquad (2)
$$

$$\psi_1^B(x)=\frac{1-P_{out}^D}{Q_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^-} p_{SO}^B(m)[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \tag{3}$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B-x}{Q_B}+1]^-)[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^-Q_B-x,\mu_B)] \quad 1\le x\le R_B$$

$$\psi_1^B(R_B+x-iQ_B)=\frac{1-P_{out}^D}{Q_B}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^-} p_{so}^B(m)[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)] \tag{4}$$
$$+\frac{1}{Q_B}p_{so}^B([\frac{iQ_B-x}{Q_B}+1]^-)[1-P(iQ_B+Q_B+1-[\frac{iQ_B-x}{Q_B}+1]^-Q_B-x,\mu_B)]$$

$$\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^-} \psi_1^B(R_B+x-iQ_B)$$
$$=\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^-} \frac{1-P_{out}^D}{Q_B}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^-}\sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^-} p_{so}^B(m)[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)] \tag{5}$$
$$+\frac{1}{Q_B}\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^-} p_{so}^B([\frac{iQ_B-x}{Q_B}+1]^-)[1-P(iQ_B+Q_B+1-[\frac{iQ_B-x}{Q_B}+1]^-Q_B-x,\mu_B)]$$

$$\text{That is: } \sum_{i=1}^{[\frac{R_B+x}{Q_B}]^{*}} \psi_1^B(R_B+x-iQ_B)$$

$$= \sum_{i=1}^{[\frac{R_B+x}{Q_B}]^{*}} \frac{1-P_{out}^D}{Q_B}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]$$

$$+\frac{1}{Q_B}\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^{*}} \sum_{m=0}^{i-1} p_{so}^B(m)[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)] \tag{6}$$

$$+\frac{1}{Q_B}\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^{*}} p_{so}^B(i)[1-P(Q_B+1-x,\mu_B)]$$

$$\psi_2^B(z)=\frac{1-P_{out}^D}{Q_B}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$

$$+\frac{1}{Q_B}\sum_{m=0}^{[\frac{z-R_B}{Q_B}]^{*}} p_{so}^B(m)[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \tag{7}$$

$$+\frac{1}{Q_B}p_{so}^B([\frac{R_B+z}{Q_B}+1]^{*})[1-P(z+R_B+Q_B+1-[\frac{R_B+z}{Q_B}+1]^{*}Q_B,\mu_B)]$$

$$\psi_1^B(R_B+x)=\frac{1-P_{out}^D+p_{so}^B(0)}{Q_B}*[1-P(Q_B+1-x,\mu_B)] \quad 1\le x\le Q_B \tag{8}$$

$$
\begin{aligned}
&\sum_{i=[\frac{R_B+x}{Q_B}]^{+}+1}^{\infty} \psi_2^{B}(iQ_B-R_B-x) \\
&=\frac{1-P_{out}^{D}}{Q_B}\sum_{i=[\frac{R_B+x}{Q^{B}}]^{+}+1}^{\infty} [P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)] \\
&\quad+\frac{1}{Q_B}\sum_{i=[\frac{R_B+x}{Q_B}]^{+}+1}^{\infty} \sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^{+}} p_{so}^{B}(m) \\
&\quad[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)] \\
&\quad+\frac{1}{Q_B}\sum_{i=[\frac{R_B+x}{Q_B}]^{+}+1}^{\infty} p_{so}^{B}([\frac{iQ_B-x}{Q_B}+1]^{+})[1-P(iQ_B+Q_B+1-[\frac{iQ_B-x}{Q_B}+1]^{+}Q_B-x,\mu_B)] \qquad (9) \\
&=\frac{1-P_{out}^{D}}{Q_B}\sum_{i=[\frac{R_B+x}{Q_B}]^{+}+1}^{\infty} [P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)] \\
&\quad+\frac{1}{Q_B}\sum_{i=[\frac{R_B+x}{Q_B}]^{+}+1}^{\infty} \sum_{m=0}^{i-1} p_{so}^{B}(m)[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)] \\
&\quad+\frac{1}{Q_B}\sum_{i=[\frac{iQ_B-x}{Q_B}]^{+}+1}^{\infty} p_{so}^{B}(i)[1-P(Q_B+1-x,\mu_B)]
\end{aligned}
$$

$$\sum_{i=1}^{[\frac{R_B-x}{Q_B}]^*} \psi_1^B(R_B+x-iQ_B)+\sum_{i=[\frac{R_B-x}{Q_B}]^*+1}^{\infty} \psi_2^B(iQ_B-R_B-x)$$

$$=\frac{1-P_{out}^D}{Q_B}\sum_{i=1}^{\infty}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]$$

$$+\frac{1}{Q_B}\sum_{i=1}^{\infty}\sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^*} p_{so}^B(m)[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)] \quad \textbf{(10)}$$

$$+\frac{1-P(Q_B+1-x,\mu_B)}{Q_B}\sum_{i=1}^{\infty} p_{so}^B(i)$$

$$=\frac{1-P_{out}^D+\sum_{m=0}^{\infty} p_{so}^B(m)}{Q_B}P(Q_B+1-x,\mu_B)+\frac{1-P(Q_B+1-x,\mu_B)}{Q_B}\sum_{i=1}^{\infty} p_{so}^B(i)$$

$$=\frac{P(Q_B+1-x,\mu_B)}{Q_B}+\frac{1-P(Q_B+1-x,\mu_B)}{Q_B}[P_{out}^D-p_{so}^B(0)]$$

$$\psi_1^B(R_B+x)+\sum_{i=1}^{[\frac{R_B+x}{Q_B}]^*} \psi_1^B(R_B+x-iQ_B)+\sum_{i=[\frac{R_B+x}{Q_B}]^*+1}^{\infty} \psi_2^B(iQ_B-R_B-x)$$

$$=\frac{P(Q_B+1-x,\mu_B)}{Q_B}+\frac{1-P(Q_B+1-x,\mu_B)}{Q_B}[P_{out}^D-p_{so}^B(0)] \quad \textbf{(11)}$$

$$+\frac{1-P_{out}^D+p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]=\frac{1}{Q_B}$$

That is: $\rho(R_B+x)=\frac{1}{Q_B}$ *where* $1\le x\le Q_B, R_B\ge 0$ *[end]*

When The Base Reorder Point $R_B<0$:

We present the relationship between inventory position and on-hand inventory or backorders.

When $R_B+x\ge 0$

The inventory position is R_B+x when there are R_B+x on-hand inventory or Q_B-R_B-x,...., backorders. That is:

$$\begin{aligned} &\textit{When } R_B<0, R_B+x\ge 0 \\ &\rho(R_B+x)=\psi_1^B(R_B+x)+\psi_2^B(Q_B-R_B-x)+\ldots \\ &=\psi_1^B(R_B+x)+\sum_{i=1}^{\infty}\psi_2^B(iQ_B-R_B-x) \end{aligned} \tag{12}$$

As we know:

$$\psi_1^B(x)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(Q_B+R_B+1-x,\mu_B)] \quad \textit{where: } 0\le x\le Q_B+R_B \tag{13}$$

$$\psi_1^B(R_B+x)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)] \tag{14}$$

$$\psi_2^B(z)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(z+R_B+Q_B+1,\mu_B)] \quad where\ 0\le z\le -R_B-1 \tag{15}$$

$$\begin{aligned}\psi_2^B(z)=&\frac{1-P_{out}^D}{Q_B}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]\\&+\frac{1}{Q_B}\sum_{m=0}^{[\frac{z+R_B}{Q_B}]^{+}} p_{so}^B(m)[P(z+R_B+1-mQ_B,\mu_B)-P(z+Q_B+R_B+1-mQ_B,\mu_B)]\\&+\frac{1}{Q_B}p_{so}^B([\frac{z+R_B}{Q_B}+1]^{+})[1-P(z+R_B+1-[\frac{z+R_B}{Q_B}]^{+}Q_B,\mu_B)] \quad where\ z\ge -R_B\end{aligned} \tag{16}$$

$$\begin{aligned}\psi_2^B(iQ_B-R_B-x)=&\frac{1-P_{out}^D}{Q_B}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]\\&+\frac{1}{Q_B}\sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^{+}} p_{so}^B(m)[P(iQ_B+1-x-mQ_B,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)]\\&+\frac{1}{Q_B}p_{so}^B([\frac{iQ_B-x}{Q_B}+1]^{+})[1-P(iQ_B+1-x-[\frac{iQ_B-x}{Q_B}]^{+}Q_B,\mu_B)]\end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\psi_2^B(iQ_B-R_B-x)=&\frac{1-P_{out}^D}{Q_B}\sum_{i=1}^{\infty}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]\\&+\frac{1}{Q_B}\sum_{i=1}^{\infty}\sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^{+}} p_{so}^B(m)[P(iQ_B+1-x-mQ_B,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)]\\&+\frac{1}{Q_B}\sum_{i=1}^{\infty}p_{so}^B([\frac{iQ_B-x}{Q_B}+1]^{+})[1-P(iQ_B+1-x-[\frac{iQ_B-x}{Q_B}]^{+}Q_B,\mu_B)]\end{aligned} \tag{18}$$

$$\sum_{i=1}^{\infty} \psi_2^B(iQ_B-R_B-x)=\frac{1-P_{out}^D}{Q_B}P(Q_B+1-x,\mu_B)$$
$$+\frac{1}{Q_B}\sum_{m=0}^{\infty} p_{so}^B(m)P(Q_B+1-x,\mu_B)+\frac{1}{Q_B}\sum_{i=1}^{\infty} p_{so}^B(i)[1-P(Q_B+1-x,\mu_B)] \tag{19}$$
$$=\frac{1}{Q_B}P(Q_B+1-x,\mu_B)+\frac{P_{out}^D-p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]$$

That is: $\rho(R_B+1)=\psi_1^B(R_B+x)+\sum_{i=1}^{\infty} \psi_2^B(iQ_B-R_B-x)$

$$=\frac{1}{Q_B}P(Q_B+1-x,\mu_B)+\frac{P_{out}^D-p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]$$
$$+\frac{1-P_{out}^D+p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]=\frac{1}{Q_B}\quad [end] \tag{20}$$

When $R_D+x<0$

The inventory position is R_B+x, when there are $-R_B-x$, Q_B-R_B-x,...., backorders. That is:

When $R_B<0, R_B+x<0$

$$\rho(R_B+x)=\psi_2^B(-R_B-x)+\psi_2^B(Q_B-R_B-x)+\ldots$$
$$=\sum_{i=0}^{\infty} \psi_2^B(iQ_B-R_B-x) \tag{21}$$

As we know:

$$\psi_2^B(z)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(z+R_B+Q_B+1,\mu_B)] \quad \textit{where } 0\le z\le -R_B-1 \tag{22}$$

$$\psi_2^B(-R_B-x)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)] \tag{23}$$

$$\begin{aligned}\psi_2^B(z)=&\frac{1-P_{out}^D}{Q_B}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]\\&+\frac{1}{Q_B}\sum_{m=0}^{[\frac{z+R_B}{Q_B}]^*}p_{so}^B(m)[P(z+R_B+1-mQ_B,\mu_B)-P(z+Q_B+R_B+1-mQ_B,\mu_B)]\\&+\frac{1}{Q_B}p_{so}^B([\frac{z+R_B}{Q_B}+1]^*)[1-P(z+R_B+1-[\frac{z+R_B}{Q_B}]^*Q_B,\mu_B)]\quad where\ z\geq -R_B\end{aligned} \tag{24}$$

$$\begin{aligned}\psi_2^B(iQ_B-R_B-x)=&\frac{1-P_{out}^D}{Q_B}[P(iQ_B+1-x,\mu_B)-P(iQ_B+Q_B+1-x,\mu_B)]\\&+\frac{1}{Q_B}\sum_{m=0}^{[\frac{iQ_B-x}{Q_B}]^*}p_{so}^B(m)[P(iQ_B+1-mQ_B-x,\mu_B)-P(iQ_B+Q_B+1-mQ_B-x,\mu_B)]\\&+\frac{1}{Q_B}p_{so}^B([\frac{iQ_B-x}{Q_B}+1]^*)[1-P(iQ_B+1-[\frac{iQ_B-x}{Q_B}]^*Q_B-x,\mu_B)]\end{aligned} \tag{25}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\psi_2^B(iQ_B-R_B-x)=&\frac{1}{Q_B}P(Q_B+1-x,\mu_B)\\&+\frac{P_{out}^D-p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]\end{aligned} \tag{26}$$

$$\textit{That is: } \rho(R_B+1)=\sum_{i=0}^{\infty}\psi_2^B(iQ_B-R_B-x)$$
$$=\frac{1-P_{out}^D+p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]+\frac{1}{Q_B}P(Q_B+1-x,\mu_B)$$
$$+\frac{P_{out}^D-p_{so}^B(0)}{Q_B}[1-P(Q_B+1-x,\mu_B)]=\frac{1}{Q_B} \quad [end] \qquad (27)$$

Conclusion:

The inventory position in the bases is uniformly distributed among $[R_B+1, R_B+Q_B]$ with $1/Q_B$ as the state probability no matter whether $R_B \geq 0$.

Appendix 3

COMPLETE PROOF OF THE CONVEXITY OF THE TOTAL COST FUNCTION

PART 1. OVERVIEW

In Appendix 3, we prove the convexity of the total cost function by proving the convexity of each part of the total cost function whenever it is possible.

2.1 Depot on-hand inventory convexity with respect to R_D.

2.2 Depot on-hand inventory convexity with respect to Q_D.

3.1 Base on-hand inventory convexity with respect to R_B.

3.2 Base on-hand inventory convexity with respect to R_D. The base on-hand inventory is not a convex function of R_D. Our study finds the sufficient condition under which the total holding cost of on-hand inventory of the system is a convex function with respect to R_D.

4.1 Base backorders convexity with respect to R_B.

4.2 Base backorders convexity with respect to R_D.

In Appendix 3, we use the symbol of $\propto$ in the proof of convexity of certain parts of system total cost function with respect to a decision variable. Since the total cost function is very long, we only consider the effect of the decision variable on the total cost. If a part of the total cost is not affected by the decision variable, we simplify the cost function by eliminating the unchanged part or by proportionally increasing the value of the cost function. We use $\propto$ to indicate that we perform such a simplification in the proof of the convexity.

PART 2. DEPOT ON-HAND INVENTORY ANALYSIS

2.1 Depot on-hand inventory convexity with respect to R_D

a. When $R_D \geq 0$:

$$\begin{aligned} E_1^D(R_D) &= \sum_0^{R_D} x\psi_1^D + \sum_{R_D+1}^{Q_D+R_D} x\psi_1^D \\ &= \frac{1}{Q_D}\sum_1^{R_D} x[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \\ &\quad +\frac{1}{Q_D}\sum_{R_D+1}^{R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)] \end{aligned} \tag{1}$$

$$\begin{aligned} E_1^D(R_D+1) &= \sum_0^{R_D+1} x\psi_1^D + \sum_{R_D+2}^{Q_D+R_D+1} x\psi_1^D \\ &= \frac{1}{Q_D}\sum_1^{R_D+1} x[P(R_D+2-x,\mu_D)-P(R_D+Q_D+2-x,\mu_D)] \\ &\quad +\frac{1}{Q_D}\sum_{R_D+2}^{R_D+Q_D+1} x[1-P(R_D+Q_D+2-x,\mu_D)] \\ &= \frac{1}{Q_D}\sum_0^{R_D} (x+1)[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \\ &\quad +\frac{1}{Q_D}\sum_{R_D+1}^{R_D+Q_D} (x+1)[1-P(R_D+Q_D+1-x,\mu_D)] \end{aligned} \tag{2}$$

$$\begin{aligned} \Delta E_1^D(R_D) &= E_1^D(R_D+1)-E_1^D(R_D) \\ &= \frac{1}{Q_D}\sum_0^{R_D} [P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \\ &\quad +\frac{1}{Q_D}\sum_{R_D+1}^{R_D+Q_D} [1-P(R_D+Q_D+1-x,\mu_D)]>0 \end{aligned} \tag{3}$$

$$\begin{aligned}
&\Delta^2 E_1^D(R_D)=\Delta E_1^D(R_D+1)-\Delta E_1^D(R_D)\\
&=\frac{1}{Q_D}\sum_0^{R_D+1}[P(R_D+2-x,\mu_D)-P(R_D+Q_D+2-x,\mu_D)]\\
&+\frac{1}{Q_D}\sum_{R_D+2}^{R_D+Q_D+1}[1-P(R_D+Q_D+2-x,\mu_D)]\\
&-\frac{1}{Q_D}\sum_0^{R_D}[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)]\\
&-\frac{1}{Q_D}\sum_{R_D+1}^{R_D+Q_D}[1-P(R_D+Q_D+1-x,\mu_D)]\\
&=\frac{1}{Q_D}[P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)]>0
\end{aligned} \tag{4}$$

b. When $R_D<0$:

$$E_1^D(R_D)=\sum_0^{Q_D+R_D} x\psi_1^D=\frac{1}{Q_D}\sum_0^{R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)] \tag{5}$$

$$\begin{aligned}
&E_1^D(R_D+1)\\
&=\sum_1^{Q_D+R_D+1} x\psi_1^D=\frac{1}{Q_D}\sum_1^{R_D+Q_D+1} x[1-P(R_D+Q_D+2-x,\mu_D)]\\
&=\frac{1}{Q_D}\sum_0^{R_D+Q_D}(x+1)[1-P(R_D+Q_D+1-x,\mu_D)]
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\Delta E_1^D(R_D)=E_1^D(R_D+1)-E_1^D(R_D)\\
&=\frac{1}{Q_D}\sum_0^{R_D+Q_D}[1-P(R_D+Q_D+1-x,\mu_D)]>0
\end{aligned} \tag{7}$$

$$\Delta^2 E_1^D(R_D)=\Delta E_1^D(R_D+1)-\Delta E_1^D(R_D)$$
$$=\frac{1}{Q_D}\sum_0^{R_D+Q_D-1}[1-P(R_D+Q_D+2-x,\mu_D)]$$
$$-\frac{1}{Q_D}\sum_0^{R_D+Q_D}[1-P(R_D+Q_D+1-x,\mu_D)] \quad (8)$$
$$=\frac{1}{Q_D}[1-P(R_D+Q_D+2,\mu_D)]>0$$

Conclusion: The depot expected on-hand inventory increases as R_D. The depot on-hand inventory is also a **convex** function of R_D.

2.2 Depot on-hand inventory convexity with respect to Q_D

a. When $R_D \geq 0$:

$$E_1^D(Q_D)=\sum_0^{R_D} x\psi_1^D+\sum_{R_D+1}^{Q_D-R_D} x\psi_1^D$$
$$=\frac{1}{Q_D}\sum_1^{R_D} x[P(R_D+1-x,\mu_D)-P(R_D+Q_D+1-x,\mu_D)] \quad (9)$$
$$+\frac{1}{Q_D}\sum_{R_D+1}^{R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)]$$

$$E_1^D(Q_D+1)=\sum_0^{R_D} x\psi_1^D+\sum_{R_D+1}^{Q_D+R_D+1} x\psi_1^D$$
$$=\frac{1}{Q_D+1}\sum_1^{R_D} x[P(R_D+1-x,\mu_D)-P(R_D+Q_D+2-x,\mu_D)] \quad (10)$$
$$+\frac{1}{Q_D+1}\sum_{R_D+1}^{R_D+Q_D+1} x[1-P(R_D+Q_D+2-x,\mu_D)]$$

$$\Delta E_1^D(Q_D) = -\frac{1}{Q_D(Q_D+1)}\sum_0^{R_D} xP(R_D+1-x,\mu_D)$$
$$+\sum_0^{Q_D+R_D}\left(-\frac{x+1}{Q_D+1}+\frac{x}{Q_D}\right)P(R_D+Q_D+1-x,\mu_D)+\frac{1}{2}$$
$$= -\frac{1}{Q_D(Q_D+1)}\sum_0^{R_D} xP(R_D+1-x,\mu_D)$$
$$+\sum_0^{Q_D+R_D}\frac{x-Q_D}{Q_D(Q_D+1)}P(R_D+Q_D+1-x,\mu_D)+\frac{1}{2}$$
$$= -\frac{1}{Q_D(Q_D+1)}\sum_0^{R_D} xP(R_D+1-x,\mu_D) \tag{11}$$
$$+\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D(Q_D+1)}P(R_D+Q_D+1-x,\mu_D)$$
$$+\sum_{Q_D}^{R_D-Q_D}\frac{x-Q_D}{Q_D(Q_D+1)}P(R_D+Q_D+1-x,\mu_D)+\frac{1}{2}$$
$$=\frac{1}{2}+\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D(Q_D+1)}P(R_D+Q_D+1-x,\mu_D)$$
$$>\frac{1}{2}+\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D(Q_D+1)}=\frac{1}{2}+\frac{Q_D-1-2Q_D}{2(Q_D+1)}=0$$

$$\Delta^2E_1^D(Q_D)=\Delta E_1^D(Q_D+1)-\Delta E_1^D(Q_D)$$
$$=\sum_0^{Q_D}\frac{x-Q_D-1}{(Q_D+1)(Q_D+2)}P(R_D+Q_D+2-x,\mu_D)$$
$$-\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D(Q_D+1)}P(R_D+Q_D+1-x,\mu_D)$$
$$\propto -\frac{Q_D+1}{Q_D+2}P(R_D+Q_D+2,\mu_D)+ \tag{12}$$
$$\sum_1^{Q_D}\frac{x-Q_D-1}{Q_D+2}P(R_D+Q_D+2-x,\mu_D)-\sum_0^{Q_D-1}\frac{x-Q_D}{Q_D}P(R_D+Q_D+1-x,\mu_D)$$
$$\propto -(Q_D+1)P(R_D+Q_D+2,\mu_D)-\frac{2}{Q_D}\sum_0^{Q_D-1}(x-Q_D)P(R_D+Q_D+1-x,\mu_D)$$
$$>-(Q_D^2+Q_D)P(R_D+Q_D+2,\mu_D)+P(R_D+Q_D+1,\mu_D)(Q_D^2+Q_D)$$
$$=(Q_D^2+Q_D)[P(R_D+Q_D+1,\mu_D)-P(R_D+Q_D+2,\mu_D)]>0$$

b. When R_D<0

$$E_1^D(Q_D)=\sum_0^{Q_D+R_D} x\psi_1^D=\frac{1}{Q_D}\sum_0^{R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)] \tag{13}$$

$$E_1^D(Q_D+1)=\sum_1^{Q_D+R_D+1} x\psi_1^D=\frac{1}{Q_D+1}\sum_1^{R_D+Q_D+1} x[1-P(R_D+Q_D+2-x,\mu_D)] \tag{14}$$

$$\begin{aligned}
&\Delta E_1^D(Q_D)=E_1^D(Q_D+1)-E_1^D(Q_D)\\
&=\frac{1}{Q_D+1}\sum_1^{Q_D+R_D+1} x[1-P(R_D+Q_D+2-x,\mu_D)]\\
&-\frac{1}{Q_D}\sum_1^{R_D+Q_D} x[1-P(R_D+Q_D+1-x,\mu_D)]\\
&=\frac{1}{Q_D(Q_D+1)}\sum_0^{R_D+Q_D} (Q_D-x)[1-P(R_D+Q_D+1-x,\mu_D)]
\end{aligned} \tag{15}$$

Since $R_D<0$, *that is* $R_D+Q_D<Q_D, \sum_0^{R_D+Q_D} (Q_D-x)>0$

We have $\Delta E_1^D(Q_D)>0$

$$
\begin{aligned}
&\Delta^2 E_1^D(Q_D)=\Delta E_1^D(Q_D+1)-\Delta E_1^D(Q_D)\\
&=\frac{1}{(Q_D+1)(Q_D+2)}\sum_0^{R_D+Q_D+1}(Q_D+1-x)[1-P(R_D+Q_D+2-x,\mu_D)]\\
&-\frac{1}{Q_D(Q_D+1)}\sum_0^{R_D+Q_D}(Q_D-x)[1-P(R_D+Q_D+1-x,\mu_D)]\\
&\propto\sum_0^{R_D+Q_D+1}\frac{Q_D+1-x}{Q_D+2}[1-P(R_D+Q_D+2-x,\mu_D)]\\
&-\sum_0^{R_D+Q_D}\frac{Q_D-x}{Q_D}[1-P(R_D+Q_D+1-x,\mu_D)]\\
&=\frac{Q_D+1}{Q_D+2}[1-P(R_D+Q_D+2,\mu_D)]\\
&-\frac{2}{Q_D(Q_D+2)}\sum_0^{R_D+Q_D}(Q_D-x)[1-P(R_D+Q_D+1-x,\mu_D)]\\
&\propto(Q_D+1)[1-P(R_D+Q_D+2,\mu_D)]\\
&-\frac{2}{Q_D}\sum_0^{R_D+Q_D}(Q_D-x)[1-P(R_D+Q_D+1-x,\mu_D)]\\
&>(Q_D^2+Q_D)[1-P(R_D+Q_D+2,\mu_D)]-2\sum_0^{R_D+Q_D}(Q_D-x)[1-P(1,\mu_D)]\\
&\propto(Q_D^2+Q_D)[1-P(R_D+Q_D+2,\mu_D)]-(Q_D^2+Q_D-R_D^2-R_D)[1-P(1,\mu_D)]\\
&=(Q_D^2+Q_D)[P(1,\mu_D)-P(R_D+Q_D+2,\mu_D)]+(R_D^2+R_D)[1-P(1,\mu_D)]>0
\end{aligned}
\tag{16}
$$

Conclusion: **The expected depot on-hand inventory increases as Q_D. It is also a convex function of Q_D.**

PART 3 BASE ON-HAND INVENTORY ANALYSIS

3.1 Base on-hand inventory convexity with respect to R_B

a. When $R_B \geq 0$:

$$\begin{aligned}
E_1^B(R_B) = {} & \frac{1-P_{out}^D}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
& + \frac{1-P_{out}^D}{Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
& + \frac{1}{Q_B}\sum_{x=0}^{R_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^*} p_{so}^B(m)x[P(R_B+1-mQ_B-x,\mu_B) \\
& \qquad - P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
& + \frac{1}{Q_B}\sum_{x=0}^{R_B+Q_B} p_{so}^B([\frac{R_B-x}{Q_B}+1]^*)x[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^*Q_B-x,\mu_B)]
\end{aligned} \tag{17}$$

$$\begin{aligned}
E_1^B(R_B+1) = {} & \frac{1-P_{out}^D}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B} (x+1)[1-P(R_B+Q_B+1-x,\mu_B)] \\
& + \frac{1-P_{out}^D}{Q_B}\sum_{x=0}^{R_B} (x+1)[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
& + \frac{1}{Q_B}\sum_{x=0}^{R_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^*} p_{SO}^B(m)(x+1) \\
& [P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
& + \frac{1}{Q_B}\sum_{x=0}^{R_B+Q_B} p_{so}^B([\frac{R_B-x}{Q_B}+1]^*) \\
& (x+1)[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^*Q_B-x,\mu_B)]
\end{aligned} \tag{18}$$

$$\Delta E_1^{B}(R_B)=\frac{1-P_{out}^{D}}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B}[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-P_{out}^{D}}{Q_B}\sum_{x=0}^{R_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{x=0}^{R_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^{+}}p_{so}^{B}(m)[P(R_B+1-mQ_B-x,\mu_B) \tag{19}$$
$$-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{x=0}^{R_B+Q_B}p_{so}^{B}([\frac{R_B-x}{Q_B}+1]^{+})$$
$$[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^{+}Q_B-x,\mu_B)]>0$$

$$\Delta E_1^{B}(R_B+1)=\frac{1-P_{out}^{D}}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B}[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-P_{out}^{D}}{Q_B}\sum_{x=-1}^{R_B}[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{x=-1}^{R_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^{+}}p_{so}^{B}(m)[P(R_B+1-mQ_B-x,\mu_B) \tag{20}$$
$$-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{x=-1}^{R_B+Q_B}p_{so}^{B}([\frac{R_B-x}{Q_B}+1]^{+})$$
$$[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^{+}Q_B-x,\mu_B)]$$

$$\begin{aligned}\Delta^2 E_1^B(R_B) &= \frac{1-P_{out}^D}{Q_B}[P(R_B+2,\mu_B)-P(R_B+Q_B+2,\mu_B)] \\ &+\frac{1}{Q_B}\sum_{m=0}^{[\frac{R_B+1}{Q_B}]^*} p_{so}^B(m)[P(R_B+2-mQ_B,\mu_B) \\ &-P(R_B+Q_B+2-mQ_B,\mu_B)] \\ &+\frac{1}{Q_B}p_{so}^B([\frac{R_B+1}{Q_B}+1]^*)[1-P(R_B+Q_B+2-[\frac{R_B+1}{Q_B}+1]^* Q_B,\mu_B)]>0\end{aligned} \tag{21}$$

b. When $R_B<0$:

$$E_1^B(R_B)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{Q_B+R_B} x[1-P(Q_B+R_B+1-x,\mu_B)] \tag{22}$$

$$\begin{aligned}E_1^B(R_B+1) &= \frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_1^{Q_B+R_B+1} x[1-P(Q_B+R_B+2-x,\mu_B)] \\ &= \frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{Q_B+R_B} (x+1)[1-P(Q_B+R_B+1-x,\mu_B)]\end{aligned} \tag{23}$$

$$\Delta E_1^B(R_B)=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{Q_B+R_B} [1-P(Q_B+R_B+1-x,\mu_B)]>0 \tag{24}$$

$$
\begin{aligned}
&\Delta^2 E_1^B(R_B)=\Delta E_1^B(R_B+1)-\Delta E_1^B(R_B) \\
&=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{R_B+Q_B+1}[1-P(Q_B+R_B+2-x,\mu_B)] \\
&-\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}\sum_0^{Q_B+R_B}[1-P(Q_B+R_B+1-x,\mu_B)] \\
&=\frac{1-P_{out}^D+P_{so}^B(0)}{Q_B}[1-P(Q_B+R_B+2,\mu_B)]>0
\end{aligned}
\qquad (25)
$$

Conclusion: The expected base on-hand inventory increases as R_B increases. It is also a convex function of R_B.

3.2 Base on-hand inventory convexity with respect to R_D

a. When $R_B<0$:

$$E_1^B(R_D)=\frac{1-\sum_{y=1}^{\infty}[1-(1-\frac{1}{N_B})^y]\psi_1^D(y)}{Q_B}\sum_{x=0}^{R_B+Q_B}x[1-P(R_B+Q_B+1-x,\mu_B)] \tag{26}$$

1) For $R_D\geq 0$:

$$\psi_2^D(y)=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$

$$E_1^B(R_D)\propto -\frac{\sum_{y=1}^{\infty}[1-(1-\frac{1}{N_B})^y][P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B} *\sum_{x=0}^{R_B+Q_B}x[1-P(R_B+Q_B+1-x,\mu_B)] \tag{27}$$

$$E_1^B(R_D+1)\propto -\frac{\sum_{y=1}^{\infty}[1-(1-\frac{1}{N_B})^{y-1}][P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B} *\sum_{x=0}^{R_B+Q_B}x[1-P(R_B+Q_B+1-x,\mu_B)] \tag{28}$$

$$\Delta E_1^B(R_D)=\frac{\sum_{y=1}^{\infty}(1-\frac{1}{N_B})^{y-1}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_D Q_B} * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]>0 \quad (29)$$

$$\Delta E_1^B(R_D+1)=\frac{\sum_{y=2}^{\infty}(1-\frac{1}{N_B})^{y-2}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_D Q_B} * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \quad (30)$$

$$\Delta^2 E_1^B(R_D)=\frac{\sum_{y=2}^{\infty}(1-\frac{1}{N_B})^{y-2}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B^2 Q_D Q_B} * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] - \frac{P(R_D+2,\mu_D)-P(Q_D+R_D+2,\mu_D)}{N_B Q_B Q_D}\sum_{0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \quad (31)$$

$$\Delta^2 E_1^B(R_D) > -\frac{P(R_D+2,\mu_D)-P(Q_D+R_D+2,\mu_D)}{N_B Q_B Q_D}\sum_{0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] > -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B}) \quad (32)$$

2) For $R_D<0$:

$$\psi_2^D(y)=\frac{1}{Q_D}[1-P(y+R_D+Q_D+1,\mu_D)] \quad where: y\le -R_D-1$$

$$\psi_2^D(y)=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \quad where\ y\ge -R_D-1$$

$$E_1^B(R_D) \propto -\frac{\sum_{y=1}^{-R_D-1}[1-(1-\frac{1}{N_B})^y][1-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} - \frac{\sum_{y=-R_D}^{\infty}[1-(1-\frac{1}{N_B})^y][P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} \quad (33)$$

$$E_1^B(R_D+1) \propto -\frac{\sum_{y=1}^{-R_D-1}[1-(1-\frac{1}{N_B})^{y-1}][1-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} - \frac{\sum_{y=-R_D}^{\infty}[1-(1-\frac{1}{N_B})^{y-1}][P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} \quad (34)$$

$$\Delta E_1^B(R_D) = \frac{\sum_{y=1}^{-R_D-1}(1-\frac{1}{N_B})^{y-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_B Q_D * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} + \frac{\sum_{y=-R_D}^{\infty}(1-\frac{1}{N_B})^{y-1}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} \quad (35)$$

$$\Delta E_1^B(R_D+1)=\frac{\sum_{y=2}^{-R_D-1}(1-\frac{1}{N_B})^{y-2}[1-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_B Q_D * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} + \frac{\sum_{y=-R_D}^{\infty}(1-\frac{1}{N_B})^{y-2}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} \quad (36)$$

$$\Delta^2 E_1^B(R_D)=\frac{1}{N_B Q_B Q_D}[-1+P(R_D+Q_D+2,\mu_D) + \frac{1}{N_B}\sum_{y=2}^{-R_D-1}(1-\frac{1}{N_B})^{y-2}[1-P(y+R_D+Q_D+1,\mu_D)]] * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] + \frac{\sum_{y=-R_D}^{\infty}(1-\frac{1}{N_B})^{y-2}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{N_B^2 Q_D Q_B * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]} \quad (37)$$

$$\Delta^2 E_1^B(R_D) > -\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B} * \sum_{x=0}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] > -\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B}) \quad (38)$$

b. When $R_B \geq 0$:

$$E_1^B = \sum_{x=0}^{R_B+Q_B} x\psi_1^B(x)$$
$$= \frac{1-P_{out}^D}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-P_{out}^D}{Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{x=0}^{R_B}\sum_{m=0}^{[\frac{R_B-x}{Q_B}]^-} p_{so}^B(m)x[P(R_B+1-mQ_B-x,\mu_B) \qquad (39)$$
$$-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{x=0}^{R_B+Q_B} p_{so}^B([\frac{R_B-x}{Q_B}+1]^+)$$
$$x[1-P(R_B+Q_B+1-[\frac{R_B-x}{Q_B}+1]^+Q_B-x,\mu_B)]$$

$$\text{That is: } E_1^B = \sum_{x=0}^{R_B+Q_B} x\psi_1^B(x)$$
$$= \frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \qquad (40)$$
$$\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=0}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$*\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

1) When $R_D \geq 0$:

$$\psi_2^D(y)=\frac{1}{Q_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$

We can prove that:

$$\begin{aligned}
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]\\
&=\sum_{x=Q_B}^{R_B-Q_B-mQ_B} (x-Q_B)[P(R_B+1-(m-1)Q_B-x,\mu_B)-P(R_B+Q_B+1-(m-1)Q_B-x,\mu_B)]\\
&<\sum_{x=0}^{R_B-(m-1)Q_B} x[P(R_B+1-(m-1)Q_B-x,\mu_B)-P(R_B+Q_B+1-(m-1)Q_B-x,\mu_B)]\\
&\quad-\sum_{x=0}^{R_B-mQ_B} Q_B[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}$$

We can also prove that:

$$\begin{aligned}
&\sum_{x=(0,R_B+1-mQ_B)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]\\
&=\sum_{x=(Q_B,R_B+1-(m-1)Q_B)}^{R_B+Q_B-(m-1)Q_B} (x-Q_B)[1-P(R_B+Q_B+1-(m-1)Q_B-x,\mu_B)]\\
&\leq\sum_{x=(0,R_B+1-(m-1)Q_B)}^{R_B+Q_B-(m-1)Q_B} x[1-P(R_B+Q_B+1-(m-1)Q_B-x,\mu_B)]\\
&\quad-\sum_{x=(0,R_B+1-mQ_B)}^{R_B+Q_B-mQ_B} Q_B[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}$$

That is:

$$\begin{aligned}
&\sum_{x=0}^{R_B-(m-1)Q_B} x[P(R_B+1-(m-1)Q_B-x,\mu_B)-P(R_B+Q_B+1-(m-1)Q_B-x,\mu_B)]\\
&>\sum_{x=0}^{R_B-mQ_B} (x+Q_B)[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}$$

$$\begin{aligned}
&\sum_{x=(0,R_B+1-(m-1)Q_B)}^{R_B+Q_B-(m-1)Q_B} x[1-P(R_B+Q_B+1-(m-1)Q_B-x,\mu_B)]\\
&\geq\sum_{x=(0,R_B+1-mQ_B)}^{R_B+Q_B-mQ_B} (x+Q_B)[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}$$

$$E_1^B(R_D)=\frac{1-\frac{1}{Q_D}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B}$$
$$*\sum_{x=R_B+1}^{R_B+Q_B}x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1-\frac{1}{Q_D}\sum_{y=1}^{\infty}[P(Y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B}$$
$$*\sum_{x=0}^{R_B}x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_DQ_B}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \tag{41}$$
$$*\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$\sum_{x=0}^{R_B-mQ_B}x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$+\frac{1}{Q_DQ_B}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$
$$*\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B}x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$part1\ of\ E_1^B(R_D)$$
$$=\frac{1-\frac{1}{Q_D}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B}$$
$$*\sum_{x=R_B+1}^{R_B+Q_B}x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$+\frac{1}{Q_DQ_B}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \tag{42}$$
$$*\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$*\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B}x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$
\begin{aligned}
&part1\ of\ E_1^{B}(R_D+1) \\
&=\frac{1-\frac{1}{Q_D}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B} \\
&\quad *\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&+\frac{1}{Q_DQ_B}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\quad *\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\quad *\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}
\qquad (43)
$$

$$
\begin{aligned}
&part1\ of\ \Delta E_1^{B}(R_D) \\
&=\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_BQ_DQ_B} \\
&\quad [\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&\quad -\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]] \\
&+\frac{1}{Q_DQ_B}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\quad [\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\quad \sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&\quad -\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\quad \sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]
\end{aligned}
\qquad (44)
$$

$$
\begin{aligned}
&\text{part1 of } \Delta E_1^{B}(R_D+1) \\
&= \frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{N_B Q_D Q_B} \\
&\quad \Big[\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&\quad -\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]\Big] \\
&+\frac{1}{Q_D Q_B}\sum_{y=3}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\quad \Big[\sum_{m=0}^{y-2} \binom{y-2}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m-2} \\
&\quad \sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&\quad -\sum_{m=0}^{y-1} \binom{y-1}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m-1} \\
&\quad \sum_{x=(0,R_B-mQ_B+1)}^{R_B-Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]\Big]
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&part1\ of\ \Delta^2 E_1^B(R_D) \\
&> -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B} \\
&[\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&-\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]] \\
&-\frac{1}{Q_D Q_B}[P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)] \\
&[\sum_{m=0}^{1}\binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B-Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{2}\binom{2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{2-m} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B-Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]] \\
&+\frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{N_B Q_D Q_B} \\
&[\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&-\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]]
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&That\ is,\ part1\ of\ \Delta^2 E_1^B(R_D) \\
&> -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B} \\
&[\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&-\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]] \\
&\geq -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B} \\
&\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]
\end{aligned}
\tag{47}
$$

$$part2\ of\ E_1^B(R_D)=$$

$$\frac{1-\frac{1}{Q_D}\sum_{y=1}^{\infty}[P(Y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B}$$

$$\sum_{x=0}^{R_B}x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$

$$+\frac{1}{Q_DQ_B}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \tag{48}$$

$$\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$

$$\sum_{x=0}^{R_B-mQ_B}x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$part2\ of\ E_1^B(R_D+1)=$$

$$\frac{1-\frac{1}{Q_D}\sum_{y=2}^{\infty}[P(Y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B}$$

$$*\sum_{x=0}^{R_B}x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$

$$+\frac{1}{Q_DQ_B}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \tag{49}$$

$$*\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1}$$

$$*\sum_{x=0}^{R_B-mQ_B}x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$
\begin{aligned}
&part2\ of\ \Delta E_1^B(R_D) \\
&= \frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)]}{Q_D Q_B} \\
&* \sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&- \frac{1}{Q_B Q_D}[P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)] \\
&* \sum_{m=0}^{1} \binom{1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{1-m} \\
&* \sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&+ \frac{1}{Q_D Q_B}\sum_{y=2}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&* [\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&* \sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&- \sum_{m=0}^{y} \binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&* \sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&part2\ of\ \Delta E_1^B(R_D+1) \\
&=\frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)]}{N_B Q_D Q_B} \\
&*[\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&-\sum_{x=0}^{R_B-Q_B} x[P(R_B+1-Q_B-x,\mu_B)-P(R_B+1-x,\mu_B)]] \\
&+\frac{1}{Q_D Q_B}\sum_{y=3}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&*[\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
&*\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&*\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]
\end{aligned}
\qquad (51)
$$

$$
\begin{aligned}
& part2\ of\ \Delta^2 E_1^B(R_D) \geq \\
& [\frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{N_B Q_D Q_B} \\
& -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B}] \\
& *[\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
& -\sum_{x=0}^{R_B-Q_B} x[P(R_B+1-Q_B-x,\mu_B)-P(R_B+1-x,\mu_B)]] \\
& -\frac{1}{Q_D Q_B}[P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)] \\
& *[\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m} \\
& *\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
& -\sum_{m=0}^{2} \binom{2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{2-m} \\
& *\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]] \\
& \geq -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_B Q_D} \\
& *\sum_{x=0}^{R_D} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]
\end{aligned} \tag{52}
$$

As we know:

$$
\begin{aligned}
& \sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
& *\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
& >\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \\
& *\sum_{0}^{R_{R_B}-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}
$$

As y increases,

$$\sum_{m=0}^{y-1} \binom{y-1}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m-1}$$

$$*\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$-\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}$$

$$*\sum_{0}^{R_{R_B}-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

decreases.

$$\Delta^2 E_1^B(R_D)$$

$$>-\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B}$$

$$*[\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \qquad (53)$$

$$+\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]]$$

$$>-\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}\left(1+\frac{R_B}{Q_B}\right)$$

2) When $R_D<0$:

$$E_1^B(R_D) \propto -\frac{1}{Q_D Q_B}\Big[\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]$$

$$+\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]\Big]$$

$$\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x;\mu_B)]$$

$$-\Big[\frac{\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D}$$

$$+\frac{\sum_{-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D}\Big]$$

$$\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]$$

$$+\frac{1}{Q_B Q_D}\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]$$

$$\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$

$$\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x)] \tag{54}$$

$$+\frac{1}{Q_B Q_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$

$$\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$

$$\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$+\frac{1}{Q_B Q_D}\sum_{y=1}^{-R_D}[1-P(y+R_D+Q_D+1,\mu_D)]$$

$$\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$

$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$+\frac{1}{Q_B Q_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$

$$\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$

$$*\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$
\begin{aligned}
&part1\ of\ E_1^B(R_D) \\
&\propto -\frac{1}{Q_D Q_B}\Big[\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&+\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]\Big] \\
&\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x;\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=1}^{-R_D}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m} \qquad (55) \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}
$$

$$
\begin{aligned}
&part1\ of\ E_1^B(R_D+1) \\
&\propto -\frac{1}{Q_D Q_B}\Big[\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&+\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]\Big] \\
&\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x;\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=2}^{-R_D}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m-1} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m-1} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
&part1\ of\ \Delta E_1^{B}(R_D)= \\
&\frac{1-P(R_D+Q_D+2,\mu_D)}{Q_BQ_D}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&-\frac{1-P(R_D+Q_D+2,\mu_D)}{Q_BQ_D} \\
&[(1-\frac{1}{N_B})\sum_{x=(0,R_B+1)}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&+\frac{1}{N_B}\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]] \\
&+\frac{1}{Q_BQ_D}\sum_{y=2}^{-R_D}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&[\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]] \\
&+\frac{1}{Q_BQ_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&[\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \\
&\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]
\end{aligned}
\qquad (57)
$$

$$part1\ of\ \Delta E_1^B(R_D+1)=\frac{1-P(R_D+Q_D+3,\mu_D)}{N_B Q_B Q_D}$$
$$[\sum_{x=(0,R_B+1)}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$
$$-\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]]$$
$$+\frac{1}{Q_B Q_D}\sum_{y=3}^{-R_D}[1-P(y+R_D+Q_D+1,\mu_D)]$$
$$[\sum_{m=0}^{y-2}\binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2}$$
$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$-\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \tag{58}$$
$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]$$
$$+\frac{1}{Q_B Q_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$
$$[\sum_{m=0}^{y-2}\binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2}$$
$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$
$$-\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$
$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]$$

$$part1\ of\ \Delta^2 E_1^{B}(R_D) \geq \frac{P(R_D+Q_D+2,\mu_D)-P(R_D+Q_D+3,\mu_D)}{N_B Q_B Q_D}$$

$$[\sum_{x=(0,R_B+1)}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$

$$-\sum_{x=(0,R_B-Q_B+1)}^{R_B} x[1-P(R_B+1-x,\mu_B)]]$$

$$-\frac{1}{Q_B Q_D}[1-P(R_D+Q_D+3,\mu_D)]$$

$$[\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m} \tag{59}$$

$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B+Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]$$

$$-\sum_{m=0}^{2} \binom{2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{2-m}$$

$$\sum_{x=(0,R_B-mQ_B+1)}^{R_B-Q_B-mQ_B} x[1-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]$$

$$\geq -\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)]$$

$$
\begin{aligned}
&part2\ of\ E_1^B(R_D) \propto \\
&-\Big[\frac{\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_BQ_D} \\
&+\frac{\sum_{-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B}\Big] \\
&\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&+\frac{1}{Q_BQ_D}\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x)] \\
&+\frac{1}{Q_BQ_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}
\qquad (60)
$$

$$
\begin{aligned}
&part2\ of\ E_1^B(R_D+1) \propto \\
&-\Big[\frac{\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D} \\
&+\frac{\sum_{-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B}\Big] \\
&\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m-1} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}\Big(\frac{1}{N_B}\Big)^m\Big(1-\frac{1}{N_B}\Big)^{y-m-1} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&\textit{part2 of } \Delta E_1^{B}(R_D)= \\
&\frac{1-P(R_D+Q_D+2,\mu_D)}{Q_D Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&[\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x)]] \\
&+\frac{1}{Q_B Q_D}\sum_{y=-R_D}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]. \\
&[\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]
\end{aligned}
\qquad (62)
$$

$$
\begin{aligned}
&part2\ of\ \Delta E_1^{B}(R_D+1)= \\
&\frac{1-P(R_D+Q_D+3,\mu_D)}{Q_D Q_B}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&+\frac{1}{Q_B Q_D}\sum_{y=3}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)] \\
&[\sum_{m=0}^{y-2}\binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]- \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x)]] \\
&+\sum_{y=-R_D}^{\infty}\frac{P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_B Q_D} \\
&[\sum_{m=0}^{y-2}\binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)]]
\end{aligned}
\qquad (63)
$$

$$\begin{aligned}
&part2\ of\ \Delta^2 E_1^B(R_D) \geq \\
&\frac{P(R_D+Q_D+2,\mu_D)-P(R_D+Q_D+3,\mu_D)}{Q_B Q_D} \\
&\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&-\frac{1}{Q_B Q_D}[1-P(R_D+Q_D+3,\mu_D)] \\
&[\sum_{m=0}^{1}\binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x,\mu_B)] \\
&-\sum_{m=0}^{2}\binom{2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{2-m} \\
&\sum_{x=0}^{R_B-mQ_B} x[P(R_B+1-mQ_B-x,\mu_B)-P(R_B+Q_B+1-mQ_B-x)]] \\
&\geq -\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_B Q_D} \\
&\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)]
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\Delta^2 E_1^B(R_D) > \\
&-\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_B Q_D}\sum_{x=R_B+1}^{R_B+Q_B} x[1-P(R_B+Q_B+1-x,\mu_B)] \\
&-\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_B Q_D}\sum_{x=0}^{R_B} x[P(R_B+1-x,\mu_B)-P(R_B+Q_B+1-x,\mu_B)] \\
&> -\frac{1-P(R_D+Q_D+2,\mu_D)}{N_B Q_D}(1+\frac{R_B}{Q_B})
\end{aligned} \tag{65}$$

$$\begin{aligned}
&For\ R_D<0,\ \Delta^2[H_D Q_B E_1^D(R_D)+H_B N_B E_1^B(R_D)] \\
&> \frac{1-P(R_D+Q_D+2,\mu_D)}{Q_D}[H_D Q_B - H_B(1+\frac{R_B}{Q_B})]
\end{aligned} \tag{66}$$

$$\text{For } R_D \geq 0,\ \Delta^2[H_D Q_B E_1^{\,D}(R_D) + H_B N_B E_1^{\,B}(R_D)] > \frac{P(R_D+2,\mu_D) - P(R_D+Q_D+2,\mu_D)}{Q_D}[H_D Q_B - H_B(1+\frac{R_B}{Q_B})] \tag{67}$$

Conclusion: **The expected total holding cost (the holding cost at the depot and bases) increases as R_D increases. The sufficient condition of the convex expected total holding** cost with respect to R_D **is that the depot holding cost per base order is as large as the base** holding cost of $(1+R_B/Q_B)$ **units at bases.**

PART 4. BASE BACKORDERS ANALYSIS

4.1 Base backorders convexity with respect to R_B

a. When $R_B \geq 0$

$$E_2^B(R_B)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \tag{68}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$E_2^B(R_B+1)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=1}^{\infty}(z-1)[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \tag{69}$$
$$\sum_{z=(1,mQ_B-R_B)}^{\infty}(z-1)[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$\sum_{z=(1,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}(z-1)[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$\Delta E_2^B(R_B) = -\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=1}^{\infty}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \tag{70}$$
$$\sum_{z=(1,mQ_B-R_B)}^{\infty}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(1,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]<0$$

$$\Delta E_2^B(R_B+1) = -\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=2}^{\infty}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \tag{71}$$
$$\sum_{z=(2,mQ_B-R_B)}^{\infty}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(2,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$
\begin{aligned}
&\Delta^2 E_2^{\,B}(R_B)= \\
&\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}[P(R_B+2,\mu_B)-P(R_B+Q_B+2,\mu_B)] \\
&+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(1,mQ_B-R_B)}^{z=(2,mQ_B-R_B)}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(1,(m-1)Q_B-R_B)}^{(2,(m-1)Q_B-R_B)}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]>0
\end{aligned}
\tag{72}
$$

b. When $R_B<0$

$$E_2^B(R_B)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=0}^{-R_B-1}z[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=-R_B}^{\infty}z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \quad (73)$$

$$E_2^B(R_B+1)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=1}^{-R_B-1}(z-1)[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=-R_B}^{\infty}(z-1)[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}\sum_{z=mQ_B-R_B}^{\infty}(z-1)[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}\sum_{z=(1,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}(z-1)[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \quad (74)$$

$$\Delta E_2^B(R_B)=-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=1}^{-R_B-1}[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=-R_B}^{\infty}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=mQ_B-R_B}^{\infty}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(1,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]<0 \qquad (75)$$

$$\Delta E_2^B(R_B+1)=-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=2}^{-R_B-1}[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}$$
$$\sum_{z=-R_B}^{\infty}[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=mQ_B-R_B}^{\infty}[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(2,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \qquad (76)$$

$$\Delta^2 E_2^B(R_B)=\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}[1-P(R_B+Q_B+2,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m} \qquad (77)$$
$$\sum_{z=(1,(m-1)Q_B-R_B)}^{(2,(m-1)Q_B-R_B)}[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]>0$$

Conclusion: **The expected base backorders decreases as R_B increases. It is also a convex** function of R_B.

4.2 Base backorders convexity with respect to R_D

a. When $R_B \geq 0$. We assume that $R_B \geq Q_B$.

$$E_2^B = \frac{1-\sum_{y=1}^{\infty} \psi_2^D(y)}{Q_B}$$
$$\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty} \psi_2^D(y)\sum_{m=0}^{y} \binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \tag{78}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{1}{Q_B}\sum_{y=1}^{\infty} \psi_2^D(y)\sum_{m=0}^{y} \binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

1) For $R_D \geq 0$:

We can prove following two inequalities:

$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$=\sum_{z=(-Q_B,(m-1)Q_B-R_B)}^{\infty} (z+Q_B)[P(z+R_B+1-(m-1)Q_B,\mu_B)$$
$$-P(z+R_B+Q_B+1-(m-1)Q_B,\mu_B)]$$
$$\geq\sum_{z=0,(m-1)Q_B-R_B}^{\infty} z[P(z+R_B+1-(m-1)Q_B,\mu_B)$$
$$-P(z+R_B+Q_B+1-(m-1)Q_B,\mu_B)]$$

$$\text{The value of } \sum_{m=0}^{y-1} \binom{y-1}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m-1}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$\text{increases as } y \text{ increases}$$

$$\text{As } R_B \geq Q_B,\ \sum_{m=0}^{1} \binom{1}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{1-m}$$
$$\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]=0$$

$$E_2^B(R_D) \propto -\frac{\sum_{y=1}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_D Q_B}$$
$$\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\frac{\sum_{y=1}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_B Q_D}$$
$$\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \qquad (79)$$
$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\frac{\sum_{y=1}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_D Q_B}$$
$$\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}$$
$$\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$
\begin{aligned}
&E_2^B(R_D+1) \propto \\
&-\frac{\sum_{y=2}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_D Q_B} \\
&\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
&+\frac{\sum_{y=2}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_B Q_D} \\
&\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\frac{\sum_{y=2}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_D Q_B} \\
&\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]
\end{aligned}
\qquad (80)
$$

$$\Delta E_2^B(R_D) = -\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B}$$
$$[\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\sum_{z=0}^{Q_B-R_B-1} z[1-P(z+R_B+1,\mu_B)]]$$
$$+\frac{\sum_{y=2}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_B Q_D}$$
$$[\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)$$
$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \tag{81}$$
$$\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)$$
$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$-\sum_{(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]<0$$

$$
\begin{aligned}
&\Delta E_2^B(R_D+1)= \\
&-\frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{N_B Q_D Q_B} \\
&[\textstyle\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
&+\textstyle\sum_{z=0}^{Q_B-R_B-1} z[1-P(z+R_B+1,\mu_B)]] \\
&+\frac{\sum_{y=3}^{\infty} P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)}{Q_B Q_D} \\
&[\textstyle\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
&\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\textstyle\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
&\textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&-\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&-\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
&-\textstyle\sum_{(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
\Delta^2 E_2^B(R_D) \geq & \frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{N_B Q_D Q_B} \\
& [\textstyle\sum_{z=0} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
& + \textstyle\sum_{z=0}^{Q_B-R_B-1} z[1-P(z+R_B+1,\mu_B)]] \\
& + \frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{Q_B Q_D} \\
& [\textstyle\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m} \\
& \textstyle\sum_{z=(0,mQ_B-R_B)} z[P(z+R_B+1-mQ_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& + \textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B-mQ_B,\mu_B) \\
& -2\textstyle\sum_{m=0}^{2} \binom{2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{2-m} \\
& [\textstyle\sum_{z=(0,mQ_B-R_B)} z[P(z+R_B+1-mQ_B,\mu_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& + \textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& + \textstyle\sum_{m=0}^{3} \binom{3}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{3-m} \\
& [\textstyle\sum_{z=(0,mQ_B-R_B)} z[P(z+R_B+1-mQ_B,\mu_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& + \textstyle\sum_{(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]]>0
\end{aligned}
\qquad (83)
$$

2) For $R_D<0$

$$
\begin{aligned}
&E_2^B(R_D)\propto \\
&-\Big[\frac{\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_BQ_D} \\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B}\Big] \\
&\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
&+\frac{\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B} \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(0,mQ_b-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_BQ_D} \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\frac{\sum_{y=1}^{-R_B-1}[1-P(z+R_D+Q_D+1,\mu_D)]}{Q_BQ_D} \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B} \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&E_2^B(R_D+1) = \\
&-\Bigg[\frac{\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_BQ_D} \\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B}\Bigg] \\
&\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
&+\frac{\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B} \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\Bigg[\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&\qquad -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Bigg] \\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_BQ_D} \\
&\sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
&\Bigg[\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&\qquad -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Bigg]
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
&\Delta E_2^B(R_D)=\frac{[1-P(R_D+Q_D+2,\mu_D)]}{Q_D Q_B}\\
&[\textstyle\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]\\
&-\textstyle\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m}\\
&\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)\\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]\\
&+\frac{\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B}\\
&[\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}\\
&[\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)\\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\textstyle\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]\\
&-\textstyle\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}\\
&[\textstyle\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)\\
&+\textstyle\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]\\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D}\\
&[\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}\\
&[\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)\\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]\\
&-[\textstyle\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}\\
&[\textstyle\sum_{m=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\\
&+\textstyle\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)\\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]<0
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
\Delta E_2^B(R_D+1) = & \frac{1-P(R_D+Q_D+3,\mu_D)}{Q_D Q_B} \\
& [\textstyle\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
& -\textstyle\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m} \\
& \textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& +\frac{\sum_{y=3}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B} \\
& [\textstyle\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
& [\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& -\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
& [\textstyle\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]] \\
& +\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B Q_D} \\
& [\textstyle\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \\
& [\textstyle\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& -[\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \\
& [\textstyle\sum_{m=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]
\end{aligned}
\qquad (87)
$$

$$
\begin{aligned}
\Delta^2 E_2^B(R_D) > & \frac{1-P(R_D+Q_D+2,\mu_D)}{Q_D Q_B} \\
& [-\sum_{z=0}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
& +\sum_{m=0}^{1} \binom{1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{1-m} \\
& \sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& +\frac{1-P(R_D+Q_D+3,\mu_D)}{Q_D Q_B}[\sum_{m=0}^{y-2} \binom{y-2}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-2} \\
& \sum_{z=(0,mQ_B-R_B)}^{mQ_B-R_B-1} z[P(z+R_B+1-mQ_B,\mu_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& -2\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
& [\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& +\sum_{m=0}^{y} \binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
& [\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
& -P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]]>0
\end{aligned}
\tag{88}
$$

since:

$$\sum_{m=0}^{y-1} \binom{y-1}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m-1}$$

$$\Big[\sum_{z=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)$$

$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$-\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Big]$$

$$+\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}$$

$$\Big[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{m=(0,mQ_B-R_B)}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)$$

$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Big]>0$$

The difference decreases

Conclusion: **When $R_D<0$, $R_B \geq Q_B$, the expected base backorders is a convex function of R_D. It is relatively easier to prove the same conclusion when $R_B<Q_B$.**

b. When $R_B<0$

$$
\begin{aligned}
E_2^B &= \frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\
&+\frac{1-\sum_{y=1}^{\infty}\psi_2^D(y)}{Q_B}\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
&+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\frac{1}{Q_B}\sum_{y=1}^{\infty}\psi_2^D(y)\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]
\end{aligned}
\tag{89}
$$

1) For $R_D\geq 0$:

$$
\begin{aligned}
E_2^B(R_D) &= \frac{1-\frac{1}{Q_D}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B} \\
&[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\
&+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]] \\
&+\frac{1}{Q_BQ_D}\sum_{y=1}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B) \\
&-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
E_2^B(R_D+1) = {} & \frac{1-\frac{1}{Q_D}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_B} \\
& \Big[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\
& +\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]\Big] \\
& +\frac{1}{Q_B Q_D}\sum_{y=2}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
& \sum_{m=0}^{y-1}\binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
& \Big[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Big]
\end{aligned}
\qquad (91)
$$

$$\Delta E_2^B(R_D)=\frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{Q_DQ_B}$$

$$[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)]$$

$$+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$

$$-\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m}$$

$$[\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$

$$+\frac{1}{Q_BQ_D}\sum_{y=2}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \tag{92}$$

$$[\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$-[\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$

$$\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]<0$$

$$
\begin{aligned}
\Delta E_2^B(R_D+1) = {} & \frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{Q_D Q_B} \\
& [\textstyle\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\
& +\textstyle\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
& -\textstyle\sum_{m=0}^{1} \binom{1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{1-m} \\
& [\textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]] \\
& +\frac{1}{Q_B Q_D}\textstyle\sum_{y=3}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
& [\textstyle\sum_{m=0}^{y-2} \binom{y-2}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-2} \\
& [\textstyle\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
& -[\textstyle\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
& \textstyle\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
& +\textstyle\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]
\end{aligned}
\tag{93}
$$

$$\Delta^2 E_2^B(R_D) > \frac{P(R_D+2,\mu_D)-P(R_D+Q_D+2,\mu_D)}{Q_D Q_B}$$
$$[-\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$-\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m}$$
$$[\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)$$
$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$
$$+\frac{P(R_D+3,\mu_D)-P(R_D+Q_D+3,\mu_D)}{Q_B Q_D}$$
$$[\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \tag{94}$$
$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)$$
$$-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$-2\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$
$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$
$$+[\sum_{m=0}^{y} \binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$
$$\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$

$$since: \sum_{m=0}^{y-1} \binom{y-1}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m-1}$$

$$\left[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\right.$$

$$\left.+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\right]$$

$$-\left[\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}\right.$$

$$\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$\left.+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\right]>0$$

and the difference decreases as y increases.

$$that\ is:\ \Delta E_2^B(R_B)<0,\ \Delta^2 E_2^B(R_D)>0, \quad when\ R_B<0, R_D\geq 0 \tag{95}$$

2) For $R_D<0$:

$$
\begin{aligned}
&E_2^B(R_D) \propto \\
&-[\frac{\sum_{y=1}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B} \\
&+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_D Q_B} \\
&[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\
&+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]] \\
&+\frac{1}{Q_D Q_B}\sum_{y=1}^{-R_B-1}[1-P(z+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
&+\frac{1}{Q_D Q_B}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
&\sum_{m=0}^{y}\binom{y}{m}\left(\frac{1}{N_B}\right)^m\left(1-\frac{1}{N_B}\right)^{y-m} \\
&[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
&+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]
\end{aligned}
\tag{96}
$$

$$E_2^B(R_D+1) \propto$$
$$-[\frac{\sum_{y=2}^{-R_D-1}[1-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B}$$
$$+\frac{\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]}{Q_DQ_B}]$$
$$[\sum_{z=0}^{-R_B-1}z[1-P(z+R_B+Q_B+1,\mu_B)]$$
$$+\sum_{z=-R_B}^{\infty}z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]]$$
$$+\frac{1}{Q_DQ_B}\sum_{y=2}^{-R_B-1}[1-P(z+R_D+Q_D+1,\mu_D)]$$
$$\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1} \tag{97}$$
$$[\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$
$$+\frac{1}{Q_DQ_B}\sum_{y=-R_D}^{\infty}[P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$
$$\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$
$$[\sum_{z=mQ_B-R_B}^{\infty}z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$
$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1}z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$
\begin{aligned}
\Delta E_2^{B}(R_D) = \frac{1-P(R_D+Q_D+2,\mu_D)}{Q_B Q_D} \\
[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)] \\
+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)] \\
-\sum_{m=0}^{1} \binom{1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{1-m} \\
[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
+\frac{1}{Q_D Q_B}\sum_{y=2}^{-R_B-1} [1-P(z+R_D+Q_D+1,\mu_D)] \\
[\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \\
-\sum_{m=0}^{y} \binom{y}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m} \\
[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,_B)]] \\
+\frac{1}{Q_D Q_B}\sum_{y=-R_D}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)] \\
[\sum_{m=0}^{y-1} \binom{y-1}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m-1} \\
[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B-mQ_B,\mu_B)]] \\
-[\sum_{m=0}^{y} \binom{y}{m}\left(\frac{1}{N_B}\right)^{m}\left(1-\frac{1}{N_B}\right)^{y-m} \\
[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)] \\
+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]
\end{aligned}
\tag{98}
$$

$$\Delta E_2^B(R_D+1)=\frac{1-P(R_D+Q_D+3,\mu_D)}{Q_BQ_D}$$

$$[\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)]$$

$$+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$

$$-\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$+\frac{1}{Q_DQ_B}\sum_{y=3}^{-R_B-1} [1-P(z+R_D+Q_D+1,\mu_D)]$$

$$[\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]] \tag{99}$$

$$-\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$

$$[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$+\frac{1}{Q_DQ_B}\sum_{y=-R_D}^{\infty} [P(y+R_D+1,\mu_D)-P(y+R_D+Q_D+1,\mu_D)]$$

$$[\sum_{m=0}^{y-2} \binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B-mQ_B,\mu_B)]]$$

$$-[\sum_{m=0}^{y-1} \binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$

$$\Delta^2 E_2^B(R_D) > \frac{1-P(R_D+Q_D+2,\mu_D)}{Q_B Q_D}$$

$$[-\sum_{z=0}^{-R_B-1} z[1-P(z+R_B+Q_B+1,\mu_B)]$$

$$+\sum_{z=-R_B}^{\infty} z[P(z+R_B+1,\mu_B)-P(z+R_B+Q_B+1,\mu_B)]$$

$$+\sum_{m=0}^{1} \binom{1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{1-m}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$+\frac{1}{Q_D Q_B}\sum_{y=2}^{-R_B-1}[1-P(z+R_D+Q_D+1,\mu_D)]$$

$$[\sum_{m=0}^{y-2}\binom{y-2}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-2} \tag{100}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$-2\sum_{m=0}^{y-1}\binom{y-1}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m-1}$$

$$[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]$$

$$+\sum_{m=0}^{y}\binom{y}{m}(\frac{1}{N_B})^m(1-\frac{1}{N_B})^{y-m}$$

$$[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]]]$$

$$\textit{since: } \sum_{m=0}^{y-1} \binom{y-1}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m-1}$$

$$\Big[\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$-\sum_{z=(0,(m-1)Q_B-R_B}^{mQ_B-R_B+1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Big]$$

$$+\sum_{m=0}^{y} \binom{y}{m} \left(\frac{1}{N_B}\right)^m \left(1-\frac{1}{N_B}\right)^{y-m}$$

$$\Big[\sum_{m=(0,(m-1)Q_B-R_B)}^{mQ_B-R_B-1} z[1-P(z+R_B+Q_B+1-mQ_B,\mu_B)]$$

$$+\sum_{z=mQ_B-R_B}^{\infty} z[P(z+R_B+1-mQ_B,\mu_B)-P(z+R_B+Q_B+1-mQ_B,\mu_B)]\Big]>0$$

and the difference decreases as y increases, we have

$$\Delta^2 E_2^B(R_D)>0 \tag{101}$$

Conclusion: The expected base backorders is a **convex** function of R_D

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