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SOME EXTENSIONS AND APPLICATIONS OF WEYL'S IDENTITY

by

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Chapter 1. Introduction

In his paper on convex surfaces [10], Herman Weyl proved an identity which relates the Gaussian curvature to the mean curvature by using differential operators. He used the identity to obtain an a priori bound for the mean curvature. Wintner [12] pointed out that Weyl's identity has other interesting applications. Recently Martin Katzen [5] found and corrected an error in Weyl's statement of the identity which affected some earlier applications. Here we are concerned with certain extensions and applications of the Weyl-Katzen identity.

Weyl indicated that he was dissatisfied with his proof and asked for a shorter one. With the exception of S. S. Chern's result discussed below, no attempt to satisfy Weyl's request has been carried out. In Chapter 2 we give a concise proof of the Weyl-Katzen identity which, unlike Chern's work, uses the notation of Weyl rather than the exterior algebra of Cartan.

Weyl was concerned with surfaces of positive Gaussian curvature but, as Wintner pointed out, the identity is meaningful and correct for surfaces of negative Gaussian curvature; however the identity is definitely not meaningful when the Gaussian curvature is zero. In view of the large number of important and difficult unsolved problems involving surfaces whose Gaussian curvature changes sign, it is desirable to extend the Weyl-Katzen identity to this case. We make this extension in Chapter 3, and also, using an idea due to Wintner, we modify the identity to include umbillics. We conclude this chapter by showing some properties of surfaces for which the Gaussian curvature is identically zero can be derived from the extended formula.

In S.S. Chern's paper [2], he derives some formulas which he claims are essentially equivalent to the Weyl identity, although they are expressed in different formalism. In view of the fact that M. Katzen proved the identity incorrect, we feel it was desirable to check Chern's work. In Chapter 4 we verify that Chern's results correspond to the correct form of the Weyl-Katzen identity. However, perhaps due to a typographical error, the formula in [2] which corresponds most closely to the Weyl identity is not the one claimed by Chern.

In Chapter 5 we attempt to extend the theorem, which asserts that on a surface of positive Gaussian curvature the mean curvature cannot have a local maximum where the Gaussian curvature has a local minimum, to the case where curvature can vanish. We are not completely successful; however we can prove a result which suggests that such an extension is possible.

Chapter 2. Preliminaries

For purposes of the following work we need some fundamental formulas from the theory of manifolds and surfaces.

1. Fundamentals from the theory of manifolds

Let M be an n-dimensional manifold of class $C^r (r \geq 4)$ with Riemannian structure $(\quad , \quad)$. We will denote by T and T^* the tangent space and cotangent space of M respectively. Λ^k will be the space of k forms for $k = 0,1,2,\ldots,n$, Γ will be the vector fields on M and d the exterior derivative. Corresponding to the local coordinates $(u_1,\ldots,u_n)$ we will let $(\partial/\partial u_1,\ldots,\partial/\partial u_n)$ denote the basis in the tangent space and let $(du_1,\ldots,du_n)$ be the dual basis in the cotangent space. For each point $x \in M$, the metric induces an isomorphism between the tangent and cotangent spaces and hence leads us to define the operator

$$g\colon \quad \Gamma \to \Lambda^1$$

as follows:

For v and $w \in \Gamma$

$$(g(v))(w) = (v,w). \tag{2.1.1}$$

Let $f \in C^1(M,R^1)$. Then the gradient of f is given by

(2.1.2) $$\operatorname{grad} f = g^{-1} df.$$

We now introduce an operator

$$m\colon \Lambda^{n-1} \to \Gamma$$

which is closely related to the Hodge star operator

$$*\colon \Lambda^{n-1} \to \Lambda^{1}.$$

Let ω be a volume element, that is an element of Λ^n that does not vanish. The Hodge star operator depends on the metric while m depends only on the choice of ω. This property of m facilitates our later calculations. To define m we identify T with its double dual T^{**}. For $\alpha \in \Lambda^{n-1}$

$$m(\alpha)\colon \Lambda^{1} \to C^0(M,R^1)$$

where for $\beta \in \Lambda^1$

(2.1.3) $$(m(\alpha))(\beta) = s \quad \text{where} \quad \alpha \wedge \beta = s\omega$$

if ω is taken to be the volume element naturally associated with the metric. We note that m is related to $*$ by the equation $m = g^{-1}*$.

Finally we define the divergence as the operator

$$\operatorname{div}\colon \Gamma \to C^0(M,R^1)$$

by the formula

$$(2.1.4) \qquad \operatorname{div}(X)\omega = (-1)^{n-1} dm^{-1}(X)$$

for a vector field X.

Clearly the divergence depends only on the volume element and not on the metric; however in all of our applications it should be understood that ω is taken to be the volume element associated with the metric. (Equivalently one can also view $\operatorname{div} X$ in terms of the Lie derivative of X, namely $(\operatorname{div} X)\omega = L_X\omega$, cf. [1])

We state two n-dimensional versions of commonly used three deminsional formulas whose proofs we sketch below.

Let $\phi, \theta \in C^1(M,R^1)$ and $f \in C^2(M,R^1)$. Then

$$(2.1.5) \qquad \operatorname{grad}(\phi\theta) = \phi \operatorname{grad} \theta + \theta \operatorname{grad} \phi$$

$$(2.1.6) \qquad \operatorname{div}(\phi \operatorname{grad} f) = \phi \operatorname{div} \operatorname{grad} f + (\operatorname{grad} \phi, \operatorname{grad} f)$$

The proof of (2.1.5) follows immediately from the definitions of g and grad in (2.1.1) and (2.1.2), the derivation property of d, and the linearity of g^{-1}.

To prove (2.1.6) we have by the definition of divergence in (2.1.4)

$$\operatorname{div}(\phi \operatorname{grad} f)\omega = (-1)^{n-1} dm^{-1}(\phi \operatorname{grad} f).$$

By the linearity of m^{-1} we replace $m^{-1}(\phi \operatorname{grad} f)$ by

$\phi m^{-1}(\text{grad } f)$ in the above equation, and we expand, using the derivation property of d, to get

$$(2.1.7)\quad \text{div}(\phi \text{ grad } f)\omega = \phi(\text{div grad } f)\omega + \qquad m^{-1}(\text{grad } f) \wedge d\phi.$$

We will show that for $\beta \varepsilon \Gamma$

$$(2.1.8)\quad m^{-1}(\beta) \wedge d\phi = (d\phi(\beta))\omega = (\text{grad } \phi, \beta)\omega$$

where the second equality is immediate from the definition (2.1.2) of grad ϕ. It suffices to prove the first equality in (2.1.8) for a basis (e_i) $i = 1,2,\ldots,n$ of Γ. We observe from the definition of m in (2.1.3) that for

$\omega = e_1^* \wedge e_2^* \wedge \ldots \wedge e_n^*$, (e_i^*) the dual basis,

$$m^{-1}(e_i) = (-1)^{n-i} \quad e_1^* \wedge e_2^* \wedge \ldots \wedge \hat{e}_i^* \wedge \ldots \wedge e_n^*$$

where the superscript $\wedge$ indicates that the term is omitted. Then for $i = 1,2,\ldots,n$

$$\begin{aligned} m^{-1}(e_i)\wedge d\phi &= (-1)^{n-i}(e_1^*\wedge\ldots\wedge\hat{e}_i^*\wedge\ldots e_n^*) \wedge (\sum_{k=1}^{n} d\phi(e_k)e_k^*) \\ &= d\phi(e_i)(-1)^{n-i}(e_1^*\wedge\ldots\wedge\hat{e}_i^*\wedge\ldots\wedge e_n^*) \wedge e_i^* \\ &= d\phi(e_i)\,\omega. \end{aligned}$$

The result (2.1.6) follows from (2.1.7) and (2.1.8) with $\beta = \text{grad } f$.

2. Expressions involving the gradient for degenerate metrics

The Weyl identity has not been interpreted at points where the Gaussian curvature is zero. In this case the determinant of the second fundamental form is zero and the difficulty stems from the fact that certain quantities in the identity involve the gradient with respect to the second fundamental form. In this section we develope an operator S which handles the undefined quantities and which permits us in Chapter 3, Section 3 to generalize the identity for all values of K.

Let h_{ij} be the inner product of the vectors $\partial/\partial u_i$ and $\partial/\partial u_j$ with respect to the given metric. In later sections we will have occasion to interprete relations regarding the gradient of a function with respect to a "metric" when $\det(h_{ij})$ is non-positive. As Wintner [12] observed, our formula for the gradient is still well-defined when $\det(h_{ij})$ is negative. It turns out that, in our work leading to the Weyl-Katzen identity (*) of Chapter 3, Section 2, the gradient is multiplied by $\det(h_{ij})$. This leads us to define the operator

$$S:\ C^1(M,R^1) \to \Gamma \qquad \text{as}$$

(2.2.1) $\quad S(f) = \det(h_{ij}) \text{ grad } f \quad \text{for} \quad \det(h_{ij}) > 0.$

It is easy to see that the expressions for grad f and $S(f)$ are given by

$$(2.2.2) \quad \operatorname{grad} f = (1/(\det(h_{pq}))) \sum_{i,k} (\partial f/\partial u_i)\overline{h}_{ki}(\partial/\partial u_k)$$

and

$$(2.2.3) \qquad S(f) = \sum_{i,k} (\partial f/\partial u_i)\overline{h}_{ki}(\partial/\partial u_k)$$

where $\overline{h_{pq}}$ is the cofactor of h_{pq}. Since (2.2.2) is well-defined when $\det(h_{ij}) < 0$ and (2.2.3) is well-defined for all values of $\det(h_{ij})$, we extend the definitions of grad f and S(f) by the above equations. We remark that (2.2.1) is now valid for $\det(h_{ij}) \neq 0$.

3. Some formulas from the theory of surfaces

Let M be a surface in Euclidean 3-space of class $C^r (r \geq 4)$ given locally by a one-to-one parameterization of the form $X(u,v) = (x(u,v), y(u,v), z(u,v))$ where the pair (u,v) ranges over a sufficiently small open domain G of the Cartesian (u,v)-plane, $X(u,v)$ is a function of class $C^r (r \geq 4)$ and the vectors $X_1 = (\partial x/\partial u, \partial y/\partial u, \partial z/\partial u)$ and $X_2 = (\partial x/\partial v, \partial y/\partial v, \partial z/\partial v)$ are linearly independent at each point of G.

The functions $g_{ij} = X_i \cdot X_j$ and $\ell_{ij} = X_{ij} \cdot N$ are of class C^{r-1} and C^{r-2} respectively where N is the unit normal to the surface so chosen that $\ell_{ii} > 0$, $i = 1, 2$

if $\det(\ell_{ij}) > 0$. Thus we have a 2-dimensional Riemannian manifold where the metric induced by the imbedding X corresponds to the matrix $I = (g_{ij})$.

In the discussion below we will be using a special parameterization, lines of curvature coordinates. We note the following theorem in [6, p. 56] which permits such a parameterization:

> Let P_0 be a point on a surface (of class C^r, $r \geq 5$) which is neither an umbillic nor a flat point. Then there exists a neighborhood of P_0 on the surface and a parameterization (u,v) of that neighborhood such that the parametric curves are the lines of curvature.

We remark that there is a loss of differentiability involved in changing to lines of curvature coordinates. (See [11, p. 860] for a discussion of this point.) However because of the invariant nature of the Weyl-Katzen identity, the formula (*) in Chapter 3, Section 2 which we will derive remains valid for surfaces of class C^4 by a standard approximation argument.

We also note that the parametric curves are the lines of curvature if and only if $g_{12} = \ell_{12} = 0$ and in this case $\ell_{ii} = k_i g_{ii}$, $i = 1,2$ where k_1 and k_2 are the principal curvatures.

In lines of curvature coordinates the Gaussian curvature, K, is given by

(2.3.1) $K = \det II/\det I = k_1 k_2$ where $II = (\ell_{ij})$

and the mean curvature by

(2.3.2) $$\begin{aligned} H &= (1/2)((\ell_{11}/g_{11}) + (\ell_{22}/g_{22})) \\ &= (1/2)(k_1 + k_2). \end{aligned}$$

We also define the quantity

(2.3.3) $$J = (1/2)(k_1 - k_2)$$

and note that

(2.3.4) $$J^2 = H^2 - K.$$

The Mainardi-Codazzi equations in lines of curvature coordinates become as in [6, p. 66]

(2.3.5) $$\begin{aligned} (\ell_{11})_v &= (g_{11})_v H \\ (\ell_{22})_u &= (g_{22})_u H \end{aligned}$$

and Theorem Egregium takes the form (in any orthogonal coordinates):

$$(2.3.6)\qquad K = -(1/(2(g_{11}g_{22})^{\frac{1}{2}}))(a + b)$$

$$a = ((g_{22})_u/(g_{11}g_{22})^{\frac{1}{2}})_u$$

$$b = ((g_{11})_v/(g_{11}g_{22})^{\frac{1}{2}})_v$$

Beltrami Parameters

In what follows let us refer to our coordinates (u,v) of M as (u^1,u^2). Let θ and ϕ be elements of $C^r(M,R^1)$ for $r \geq 2$. Using the metric corresponding to $I = (g_{ij})$ and letting (g^{ij}) be the inverse of I, we denote by $\nabla'(\phi,\theta)$ the inner product of grad ϕ with grad θ which equals

$$\sum_{\alpha,\beta} (\partial\phi/\partial u^{\alpha})g^{\alpha\beta}(\partial\theta/\partial u^{\beta}).$$

For $\theta = \phi$, $\nabla'(\phi,\phi)$ is more briefly denoted by $\nabla'\phi$. ∇' is called the first differential parameter of Beltrami with respect to the first fundamental form.

We denote by $\nabla''(\phi,\theta)$ the expression we get by replacing $g^{\alpha\beta}$ by $\ell^{\alpha\beta}$ in the above equation where the matrix $(\ell^{\alpha\beta})$ is the inverse to $II = (\ell_{\alpha\beta})$. ∇'' is called the first Beltrami parameter with respect to the second fundamental form.

It is easy to check that (cf. [8, p. 227])

$$\text{div grad } \phi = (1/(\det I)^{\frac{1}{2}}) \sum_{\alpha\beta} \partial((\det I)^{\frac{1}{2}} g^{\alpha\beta}(\partial\phi/\partial u^{\beta}))/\partial u^{\alpha}.$$

This expression is denoted by $\Delta'\phi$ and Δ' is called the second differential parameter of Beltrami with respect to the first fundamental form. For $K > 0$ replacing I by II and $g^{\alpha\beta}$ by $\ell^{\alpha\beta}$ in the right hand side of the above equation gives an expression denoted by $\Delta''\phi$ and Δ'' is called the second differential parameter of Beltrami with respect to the second fundamental form.

4. Some remarks on notation

In much of our future work we will have occasion to consider two distinct metrics on our manifold M. Corresponding to the metric given by the matrix I, we will denote the inner product of two vector fields v and w by $(v,w)_I$, the operator between Γ and Λ^1 given in (2.1.1) by g_I and the gradient of a function f by $\text{grad}_I f$. We will denote our volume element by ω_I and choose it to be

$$\omega_I = (\det I)^{\frac{1}{2}} du_1 \wedge \dots \wedge du_n. \tag{2.4.1}$$

We denote the operator between Λ^{n-1} and Γ defined in (2.1.3) by m_I. Finally the divergence defined by the volume element ω_I of a vector field X will be denoted by $\text{div}_I X$.

Chapter 3. The Weyl Identity

The proof of the Weyl identity that appears in Katzen's thesis is accomplished essentially by following the prescription set down by Weyl in [10], but using lines of curvature coordinates to simplify the computations. It remains a lengthy procedure in which one starts with the formulas for the mean and Gaussian curvatures in terms of the first and second fundamental forms, the Theorem Egregium, the Mainardi-Codazzi equations and five formulas gotten by differentiating some of the above. From these ten equations, the second derivatives of the coefficients of the second fundamental form are eliminated. The remaining equations are manipulated by a tedious algebraic procedure to yield the identity (*) in Chapter 3, Section 2. Because of the lengthyness of these computations Weyl omitted the proof and requested that a more direct proof be found. The proof in this chapter fulfills his request.

In Section 2 we assume that our surface is free of umbillics and, for $K \neq 0$, we prove the correct form of the Weyl identity (*) as established in Katzen's thesis. In the course of this work we will interpret certain surface invariants when $K < 0$.

In Section 3 we generalize the result obtained in Section 2 by eliminating the restriction $K \neq 0$.

In Section 4 we employ an idea of Wintner's to get a Weyl-type identity for the case where our surface has umbillics.

1. Preliminaries to the concise proof

Before we go on to the concise proof of the Weyl identity we establish two lemmas.

Lemma 1: *Given a Riemannian n-dimensional manifold* M *with two Riemannian metrics corresponding to the matrices* I *and* II. *Then for any vector field* v

$$(3.1.1)\qquad \mathrm{div}_I(v)\omega_I = \mathrm{div}_{II}((\det I/\det II)^{\frac{1}{2}}v)\omega_{II}$$

Proof. We first claim that

$$(3.1.2)\qquad m_I^{-1} = (\det I/\det II)^{\frac{1}{2}}m_{II}^{-1}$$

By definition (2.1.3) if $\alpha \varepsilon \Lambda^{n-1}$ and $\beta \varepsilon \Lambda^1$ then

$$(3.1.3)\qquad m_I(\alpha)(\beta) = s \quad \text{where} \quad \alpha \wedge \beta = s\omega_I$$

$$m_{II}(\alpha)(\beta) = t \quad \text{where} \quad \alpha \wedge \beta = t\omega_{II}.$$

Since $\omega_I = (\det I/\det II)^{\frac{1}{2}}\omega_{II}$ using (3.1.3)

$$\alpha \wedge \beta = s(\det I/\det II)^{\frac{1}{2}}\omega_{II}$$

or

$$t = s(\det I/\det II)^{\frac{1}{2}}$$

Thus $m_{II} = (\det I/\det II)^{\frac{1}{2}}m_I$ which gives (3.1.2).

We now prove the lemma.

$$
\begin{aligned}
(\mathrm{div}_I v)\omega_I &= (-1)^{n-1} d m_I^{-1}(v) \\
&= (-1)^{n-1} d((\det I/\det II)^{\frac{1}{2}} m_{II}^{-1}(v)) \quad \text{by (3.1.2)} \\
&= (-1)^{n-1} d m_{II}^{-1}((\det I/\det II)^{\frac{1}{2}} v) \quad \text{by the linearity of } m_{II}^{-1} \\
&= \mathrm{div}_{II}((\det I/\det II)^{\frac{1}{2}} v)\omega_{II}.
\end{aligned}
$$

Lemma 2: *Let M be a two-dimensional Riemannian manifold. Let e_1, e_2 be an orthonormal basis in the tangent space. Then the Gaussian curvature K becomes*

$$K\omega = \mathrm{div}((\mathrm{div}\ e_1)e_1 + (\mathrm{div}\ e_2)e_2)\omega$$

where the volume element ω is chosen in accordance with the metric as in Chapter 3, Section 4.

We note here that Weatherburn first proved this result for surfaces in [9] by direct computation.

Proof: Choosing orthogonal parametric curves let $I = (g_{ij})$ where $g_{12} = g_{21} = 0$. In the notation of Section 4, Chapter 2, $\omega_I = (g_{11}g_{22})^{\frac{1}{2}} du \wedge dv$ and we will prove the lemma in the form

$$K\omega_I = \mathrm{div}_I((\mathrm{div}_I e_1)e_1 + (\mathrm{div}_I e_2)e_2)\omega_I. \tag{3.1.4}$$

In orthogonal coordinates Theorem Egregium becomes as in (2.3.6)

$$K = -(1/(2(g_{11}g_{22})^{\frac{1}{2}}))(a + b)$$

$$a = ((g_{22})_u/(g_{11}g_{22})^{\frac{1}{2}})_u$$

$$b = ((g_{11})_v/(g_{11}g_{22})^{\frac{1}{2}})_v$$

If $E = (\delta_{ij})$, the usual Euclidian metric, we have for the vector field $X = c\partial/\partial u + d\partial/\partial v$

$$\mathrm{div}_E(X) = \partial c/\partial u + \partial d/\partial v$$

Thus we can write Theorem Egregium as

$$K(g_{11}g_{22})^{\frac{1}{2}} = -(1/2)\mathrm{div}_E(X)$$

where $c = (g_{22})_u/(g_{11}g_{22})^{\frac{1}{2}}$

and $d = (g_{11})_v/(g_{11}g_{22})^{\frac{1}{2}}$

Since $\omega_I = (g_{11}g_{22})^{\frac{1}{2}}du \wedge dv$ we have

$$K\omega_I = -\mathrm{div}_E(X/2)\omega_E .$$

Writing the right hand side in terms of the metric I, by Lemma 1 we have

$$K\omega_I = -\mathrm{div}_I(X/(2(g_{11}g_{22})^{\frac{1}{2}}))\omega_I . \tag{3.1.5}$$

We now examine the vector field $X/(2(g_{11}g_{22})^{\frac{1}{2}})$. After replacing $\partial/\partial u$ and $\partial/\partial v$ by $(g_{ii})^{\frac{1}{2}}e_i$ $i = 1,2$ respectively and by observing

$$((g_{11})^{\frac{1}{2}})_u = (g_{11})_u/(2(g_{11})^{\frac{1}{2}})$$

and

$$((g_{22})^{\frac{1}{2}})_v = (g_{22})_v/(2(g_{22})^{\frac{1}{2}})$$

we have

$$(3.1.6)\quad X/(2(g_{11}g_{22})^{\frac{1}{2}}) = (((g_{22})^{\frac{1}{2}})_u/(g_{11}g_{22})^{\frac{1}{2}})e_1 + (((g_{11})^{\frac{1}{2}})_v/(g_{11}g_{22})^{\frac{1}{2}})e_2$$

We will show that in (3.1.6) above the coefficient of e_i is $\mathrm{div}_I e_i$ for $i = 1,2$.

$$\begin{aligned}(\mathrm{div}_I e_1)\omega_I &= \mathrm{div}_I((1/(g_{11})^{\frac{1}{2}})(\partial/\partial u)\omega_I \\ &= \mathrm{div}_E((g_{22})^{\frac{1}{2}})(\partial/\partial u)\omega_E \\ &= \mathrm{div}_E((g_{22})^{\frac{1}{2}})_u\omega_E.\end{aligned}$$

Therefore

$$(3.1.7)\qquad \mathrm{div}_I e_1 = ((g_{22})^{\frac{1}{2}})_u/(g_{22}g_{11})^{\frac{1}{2}}$$

since $\omega_E = (1/(g_{11}g_{22})^{\frac{1}{2}})\omega_I.$

The same argument gives

$$(3.1.8)\qquad \mathrm{div}_I e_2 = ((g_{22})^{\frac{1}{2}})_v/(g_{11}g_{22})^{\frac{1}{2}}$$

And so finally we have from (3.1.6), (3.1.7), (3.1.8)

$$(3.1.9) \quad X/(2(g_{11}g_{22})^{\frac{1}{2}}) = (\mathrm{div}_I e_1)e_1 + (\mathrm{div}_I e_2)e_2.$$

Thus from equations (3.1.5) and (3.1.9) we obtain the desired result (3.1.4).

In the rest of this section we use the notation introduced in Chapter 2, Section 3 to consider a two dimensional surface imbedded in three space with $K > 0$ which is free from umbillics. The restrictions of $K > 0$ and $J \neq 0$ will be dealt with in later sections.

Let (u,v) be lines of curvature coordinates in what follows and let the coefficients of the first and second fundamental forms be represented by $I = (g_{ij})$ $g_{12} = g_{21} = 0$ and $II = (\ell_{ij})$ $\ell_{12} = \ell_{21} = 0$ respectively and let $\ell_{ii} = k_i g_{ii}$ $i = 1,2$ where k_i are the principal curvatures. Let (e_1,e_2) be an orthonormal basis in the directions of the u and v parametric curves respectively and let (e_1^*, e_2^*) be its dual basis in the cotangent space. Then we have according to our convention in Chapter 2, Section 4

$$(3.1.10) \quad \text{a)} \quad g_I(e_i) = e_i^* \qquad i = 1,2$$

$$g_I^{-1}(e_i^*) = e_i$$

$$\text{b)} \quad g_{II}(e_i) = k_i e_i^* \qquad i = 1,2$$

$$g_{II}^{-1}(e_i^*) = e_i/k_i$$

The first statement in (b) follows from

$$(g_{II}(e_1))(e_1) = (e_1,e_1)_{II} = k_1 = (k_1 e_1^*)(e_1) \quad \text{and}$$
$$(g_{II}(e_1))(e_2) = (e_1,e_2)_{II} = 0.$$

Thus $g_{II}(e_1) = k_1 e_1^*$.

We will later make use of the observation that given two metrics $I = (g_{ij})$ and $II = (g_{ij}{}')$ then for any $\tau \varepsilon \Gamma$ and $f \varepsilon C^1(M,R^1)$

$$(3.1.11) \qquad (\text{grad}_I f,\tau)_I = (\text{grad}_{II} f,\tau)_{II} = df(\tau)$$

We may sometimes write $\text{grad}_I f \cdot \tau$ for any of the expressions in (3.1.11).

The Mainardi-Codazzi equations in lines of curvature coordinates as given earlier are

$$(2.3.5) \qquad \begin{aligned} (\ell_{11})_v &= (g_{11})_v H \\ (\ell_{22})_u &= (g_{22})_u H \end{aligned}$$

We also have from (2.3.2) and (2.3.3)

$$(3.1.12) \qquad \begin{aligned} k_1 &= H + J \\ k_2 &= H - J \end{aligned}$$

We will rewrite the Mainardi-Codazzi equations (2.3.5) replacing the coefficients of the second fundamental form by expressions involving H and J. We substitute $\ell_{11} = k_1 g_{11}$ in the first Mainardi-Codazzi equation and carry out the differentiation to obtain

$$(k_1)_v g_{11} + k_1 (g_{11})_v = (g_{11})_v H.$$

Differentiating the first equation in (3.1.12) with respect to v, substituting in the above and rearranging terms gives

$$(3.1.13) \quad (g_{11})_v / g_{11} = (k_1)_v / (-J) = (H_v + J_v)/(-J).$$

A similar procedure on the second Mainardi-Codazzi equation yields

$$(3.1.14) \quad (g_{22})_u / g_{22} = (k_2)_u / J = (H_u - J_u)/J.$$

Section 2. Concise proof of the Weyl identity for a surface which is free from umbillics.

We are now in a position to prove the following form of the Weyl identity (*) in the neighborhood of a point P which is not an umbillic and at which $K \neq 0$. We will interpret this formula for $K < 0$.

$$(*) \quad 2KJ^2 = K\Delta'' H - K\nabla''(J^2, H)/J^2 + \nabla''(K, H)/2 - \Delta' K/2 + \nabla'(J^2, K)/(2J^2).$$

By Lemma 2

$$K\omega_I = \mathrm{div}_I((\mathrm{div}_I e_1) e_1 + (\mathrm{div}_I e_2) e_2)\omega_I$$

We proceed to write the vector $(\mathrm{div}_I e_1)e_1 + (\mathrm{div}_I e_2)e_2$ in terms of H, J and K. From (3.1.7) in the proof of Lemma 2 we have

$$\mathrm{div}_I e_1 = ((g_{22})^{\frac{1}{2}})_u/(g_{11}g_{22})^{\frac{1}{2}}$$

$$= (1/(2(g_{11})^{\frac{1}{2}})((g_{22})_u/g_{22})$$

which becomes by (3.1.14)

(3.2.1) $$\mathrm{div}_I e_1 = (1/(2(g_{11})^{\frac{1}{2}})(H_u - J_u)/J)$$

Similarly from (3.1.8) and (3.1.13) we have

(3.2.2) $$\mathrm{div}_I e_2 = (1/(2(g_{22})^{\frac{1}{2}}))((g_{11})_v/g_{11})$$

$$= (1/(2(g_{22})^{\frac{1}{2}}))(-H_v - J_v)/J$$

From $J^2 = H^2 - K$ we have

(3.2.3) $$J_u = (J^2)_u/(2J)$$

$$= (2HH_u - K_u)/2J$$

and an analagous equation with u replaced by v. By substituting (3.2.3) in (3.2.1) and the analogue to (3.2.3) in (3.2.2) and using (3.1.12) we write $\mathrm{div}_I e_1$ and $\mathrm{div}_I e_2$ in terms of H, J K, k_1 and k_2 as follows:

(3.2.4) $$\mathrm{div}_I e_1 = (1/(2(g_{11})^{\frac{1}{2}}))(-k_2H_u/J^2 + K_u/2J^2)$$

$$\mathrm{div}_I e_2 = (1/(2(g_{22})^{\frac{1}{2}})(-k_1H_v/J^2 + K_v/2J^2)$$

By writing $k_i = K/k_j$ for $i = 1,2$ $i \neq j$ and rearranging terms we have

(3.2.5) $$(\mathrm{div}_I e_1)e_1 + (\mathrm{div}_I e_2)e_2 = A + B \quad \text{where}$$

$$A = (1/4J^2)((K_u/(g_{11})^{\frac{1}{2}})e_1 + (K_v/(g_{22})^{\frac{1}{2}})e_2)$$

and

$$B = -(K/2J^2)(H_u/(g_{11})^{\frac{1}{2}})(e_1/k_1) + (H_v/((g_{22})^{\frac{1}{2}})(e_2/k_2))$$

It is easily seen that (3.2.5) can be expressed by

(3.2.6) $$A = (1/4J^2)(dK(e_1)e_1 + dK(e_2)e_2)$$

$$B = -(K/2J^2)(dH(e_1)(e_1/k_1) + dH(e_2)(e_2/k_2))$$

We note here that the discussion up to this point holds for $K > 0$ and $K < 0$. We now assume that $K > 0$.

Using equations (3.1.10) a) and b) and the linearity of g_I and g_{II} we get

(3.2.7) $$A = (1/4J^2)(g_I^{-1}(dK(e_1)e_1^* + dK(e_2)e_2^*))$$

$$B = -(K/2J^2)(g_{II}^{-1}(dH(e_1)e_1^* + dH(e_2)e_2^*))$$

which is simply

$$(3.2.8)\qquad \begin{aligned} A &= (1/4J^2)g_I^{-1}(dK) \\ &= (1/4J^2)\mathrm{grad}_I\, K \qquad\qquad \text{and} \\ B &= -(K/2J^2)g_{II}^{-1}(dH) \\ &= -(K/2J^2)\mathrm{grad}_{II}H. \end{aligned}$$

Thus from (3.2.5) and (3.2.8) we write in terms of H, J, K the vector

$$(3.2.9)\qquad (\mathrm{div}_I e_1)e_1+(\mathrm{div}_I e_2)e_2 = (1/4J^2)\mathrm{grad}_I K-(K/2J^2)\mathrm{grad}_{II}H$$

So by Lemma 2 we have

$$(3.2.10)\qquad K\omega_I = \mathrm{div}_I((K/2J^2)\mathrm{grad}_{II}H - (1/4J^2)\mathrm{grad}_I K)\omega_I.$$

Using Lemma 1 on the first factor of the right hand side of (3.2.10) and noting that $K = \det II/\det I$ we have

$$(3.2.11)\qquad \begin{aligned} K\omega_I &= \mathrm{div}_{II}(((K)^{\frac{1}{2}}/2J^2)\mathrm{grad}_{II}H)\omega_{II} \\ &\quad - \mathrm{div}_I((1/4J^2)\mathrm{grad}_I K)\omega_I. \end{aligned}$$

Observing $\omega_{II} = (K)^{\frac{1}{2}}\omega_I$, equating the coefficients of ω_I in (3.2.11), using the expansion properties (2.1.5) and (2.1.6), substituting $\mathrm{grad}_A(1/J^2) = -(\mathrm{grad}_A(J^2))/J^4$ for $A = I$ and II and finally multiplying by J^2 we get the Weyl identity (*).

Suppose now that $K < 0$. The use of a metric corresponding to a negative definite matrix does not affect the formal definition of g and thus the proof is valid through (3.2.10). The proof appears to break down at equation (3.2.11) since the volume element ω_{II} is equal to $(\det II)^{\frac{1}{2}} du \wedge dv$ where $\det II$ is negative and also we see a factor of $(K)^{\frac{1}{2}}$. We overcome this difficulty by considering the complex tangent space as generated by the vectors $\partial/\partial u$ and $\partial/\partial v$ over the complex field C. We denote this space by T_c. Accordingly we replace R^1, Λ^1, Γ by C, Λ_c, Γ_c. After making these replacements m, d and div are defined by exactly the same formulas as in Chapter 2, Section 2. The expansion formulas (2.1.5) and (2.1.6) and Lemma 1 for real vector fields v remain valid. That Lemma 1 is still valid for real v follows directly by observing that for $\omega_A = (\det A)^{\frac{1}{2}} du \wedge dv$, $m_A{}^{-1}(\partial/\partial u) = -(\det A)^{\frac{1}{2}} dv$ and $m_A{}^{-1}(\partial/\partial v) = (\det A)^{\frac{1}{2}} du$, for $A = I$ and II. The entire proof for $K < 0$ follows verbatim from the proof above which was done in the real Grassman manifold. If $K < 0$ the identity $(*)$ is still a real valued identity since div_{II} of a real vector field is real.

3. The generalized Weyl identity

We now examine a form of the Weyl identity on a surface which is free from umbillics but where K is permitted to be zero. The Beltrami parameters of the first and second kind with respect to the second fundamental form, ∇'' and Δ'', which appear in $(*)$ in Section 2 of this chapter are not well defined if the Gaussian curvature is zero. Furthermore some of the quantities in question appear in the terms $K\Delta''H$ and $K\,\mathrm{grad}_{II}H$ which take the form $0/0$ when $K = 0$. We alter the statement of the Weyl identity to obtain a form which is valid for all values of K.

$$2KJ^2 = L_2H - L_1K \tag{3.3.1}$$

where the L_i $i = 1,2$ are differential operators for which L_1 is elliptic and L_2 is hyperbolic, parabolic or elliptic at a point according as K at that point is less than zero, equal to zero or greater than zero respectively. Specifically

$$L_2H = \mathrm{div}_I(S_{II}(H)/\det I) - (1/J^2)(S_{II}(H)/\det I) \cdot \mathrm{grad}_I J^2$$

$$L_1K = (1/2)\Delta''K - (1/2J^2)\nabla'(K,J^2)$$

where $S_{II}(H)$ is defined in equation (2.2.3) for $(h_{ij}) = II$ and where the inner product in L_2H is given as in (3.1.11.)

We remark that for $\det II \neq 0$, $S_{II}(H)/\det I = K \operatorname{grad}_{II} H$ and by using the expansion formulas (2.1.5) and (2.1.6) we obtain (*) again.

The proof of the Weyl identity in lines of curvature coordinates is valid up to equation (3.2.4) when $K = 0$. Using (3.2.4) we obtain

$$(3.3.2) \quad (\operatorname{div}_I e_1)e_1 + (\operatorname{div}_I e_2)e_2 = (1/(4J^2))\operatorname{grad}_I K + -(1/(2J^2))((k_2H_u/(g_{11})^{\frac{1}{2}})e_1+(k_1H_v/(g_{22})^{\frac{1}{2}})e_2).$$

We note that the second vector on the right hand side of (3.3.2) is precisely $S_{II}(H)/\det I$. Thus from (3.3.2), the above remark and Lemma 2 we have

$$(3.3.3) \quad K = \operatorname{div}_I((1/2J^2)(S_{II}(H)/\det I) - (1/4J^2)\operatorname{grad}_I K$$

Again using the expansion formulas (2.1.5) and (2.1.6) on (3.3.3), letting $(\operatorname{grad}_I J^2)/J^4$ replace $-\operatorname{grad}_I(1/J^2)$ and multiplying by $2J^2$ we have equation (3.3.1) which is valid at nonumbillics for all values of K.

To see that L_1 is an elliptic operator on K and that L_2 is an operator on H whose type is determined by the sign of K we compute the coefficients of the higher derivatives and we obtain

$$L_1K = (g_{22}K_{uu} - 2g_{12}K_{uv} + g_{11}K_{vv})/(2(g_{11}g_{22}-g_{12}{}^2)$$

$$+ K_u(\ldots) + K_v(\ldots) \qquad \text{and}$$

$$L_2H = (\ell_{22}H_{uu} - 2\ell_{12}H_{uv} + \ell_{11}H_{vv})/(g_{11}g_{22}-g_{12}{}^2)$$

$$+ H_u(\ldots) + H_v(\ldots).$$

We see that L_1 is elliptic since the matrix $(g_{ij}/(2(g_{11}g_{22}-g_{12}{}^2)))$ has determinant equal to $1/(4(g_{11}g_{22}-g_{12}{}^2))$ which is strictly greater than zero. The matrix $(\ell_{ij}/(g_{11}g_{22}-g_{12}{}^2))$ corresponding to L_2 has determinant $(\ell_{11}\ell_{22}-\ell_{12}{}^2)/(g_{11}g_{22}-g_{12}{}^2)^2$ which is equal to $K/(g_{11}g_{22}-g_{12}{}^2)$ and hence L_2 is a hyperbolic, parabolic or elliptic operator at a point P according to whether K at P is negative, zero or positive respectively.

4. Umbillics

Following the method of Wintner [12], we will show that (3.4.1) below which is obtained from (3.3.1) by multiplying by $2J^2$ and which has been proven for nonumbillics is valid at an umbillic P.

$$(3.4.1) \quad 4KJ^4 = J^2(2\mathrm{div}_I(S_{II}(H)/\det I) - \Delta'K) +$$

$$-2(S_{II}(H)/\det I) \cdot \mathrm{grad}_I J^2 + \nabla'(K,J^2).$$

Either the umbillic P is a limit of nonumbillics or P has a neighborhood consisting entirely of umbillics. In the former case using the continuity of the quantities involved (M was assumed to be of class C^r, $r \geq 4$), by taking limits we obtain (3.4.1) at P. In the latter case, this neighborhood of P lies on a sphere or is part of a plane. In each instance, J is identically zero causing (3.4.1) to be trivial and true.

5. Some consequences when K is identically zero.

We want to illustrate how the extended form of the Weyl-Katzen identity can be used to obtain some known results on surfaces where K is identically zero.

From the Weyl identity (3.3.1), $2KJ^2 = L_2H - L_1K$, we have for K identically zero

$$L_2H \equiv 0. \tag{3.5.1}$$

If $k_2 \equiv 0$ then $H \equiv J \equiv 2k_1$ and (3.5.1) becomes

$$H^2 \mathrm{div}_I(S_{II}(H)/(H^2 \det I)) \equiv 0 \tag{3.5.2}$$

Upon using definition (2.2.5) for $S_{II}(H)$, (3.5.2) becomes

$$H^2 \mathrm{div}_I((2H_v/(g_{22}H))X_1) \equiv 0. \tag{3.5.3}$$

(3.5.4) Therefore $2H^2((g_{11}/g_{22})^{\frac{1}{2}}(\ln H)_v)_v \equiv 0.$

In a neighborhood of a nonflat point (u_0,v_0) we have from (3.5.4)

(3.5.5) $((\ln H)_v(g_{11}/g_{22})^{\frac{1}{2}})_v \equiv 0$ and integrating we obtain

(3.5.6) $(\ln H)_v = c(g_{22}/g_{11})^{\frac{1}{2}}$ for a constant c.

We now restrict our discussion to the line of curvature $u = u_0$. Integrating (3.5.6) from v_0 to v we get

(3.5.7) $H(u_0,v) = H(u_0,v_0)\exp\int_{v_0}^{v} c(g_{22}(u_0,v)/g_{11}(u_0,v)^{\frac{1}{2}}dv.$

From (3.5.7) if $H(u_0,v_0) \neq 0$ then $H(u_0,v) \neq 0$ along the line of curvature lying in any compact neighborhood of (u_0,v_0)

Furthermore if $H_v(u_0,v_0) = 0$, then H is identically a constant on the line of curvature $u = u_0$. For from (3.5.6) the constant c is zero hence (3.5.7) gives $H(u_0,v) = H(u_0,v_0)$.

Chapter 4. Reconciliation of Chern's work with the Weyl identity.

In his paper [10], Weyl describes the computations which should lead to his identity as a "langweilige Rechnung." He states "Es ist wahrscheinlich, dass ein geschickterer Rechner die Formel (33) auf veil leichteren Wege wird ermitteln konnen, als hier angeduetet wurde." In 1945 S.S. Chern published the paper "Some new characterizations of the Euclidean sphere" [2] in which he derives an equation (32) which he states is essentially the equation in Weyl's paper for $K > 0$. Actually this remark of Chern's was probably a misprint since this equation turns out to be Theorem Egregium in terms of the principal curvatures. Equation (38) in Chern's paper more closely resembles the Weyl identity. However, as will be seen, it will take a lengthy calculation to get from Chern's formulas (32) and (38) to the Beltrami parameters which appear in the Weyl identity.

The following computations show that Chern's work implies the identity $(*)$ in Chapter 3, Section 2 and not the incorrect form of the Weyl identity which appears in Weyl's paper [10]. Chern uses the notation of E. Cartan to arrive at his formula (32). First we will write Chern's equation (32) in the notation of the previous sections and then we will reconcile this formula with the identity $(*)$.

Chern takes a differentiable, closed, orientable surface S and attaches right handed rectangular trihedrals $Pe_1e_2e_3$ at nonumbillic points P of S as follows. At P there are two principal curvatures r_i $i = 1,2$ and corresponding directions d_i $i = 1,2$ which are fixed by the convention $r_1 > r_2$. The trihedral $Pe_1e_2e_3$ is attached to P so that (1) e_3 is the unit vector along the outward normal and (2) e_1, e_2 are the unit vectors along d_1, d_2 respectively. In what follows the reference to equation (X) in Chern's paper will be denoted by C:(X). Between these trihedrals he has the vectorial differential equation C:(5). Concerning these formulas we claim

$$(4.1) \quad \omega_1 = (g_{11})^{\frac{1}{2}}du$$

$$(4.2) \quad \omega_2 = (g_{22})^{\frac{1}{2}}dv$$

$$(4.3) \quad \omega_{12} = (1/(2(g_{11}g_{22})^{\frac{1}{2}}))(-(g_{11})_v du + (g_{22})_u dv)$$

$$(4.4) \quad d\omega_1 = -\omega_2 \wedge \omega_{12}$$

$$(4.5) \quad d\omega_2 = \omega_1 \wedge \omega_{12}$$

To verify (4.1) and (4.2) we need only examine dP. Recalling that e_i is the unit vector $(g_{ii})^{-\frac{1}{2}}P_i$ for

$i = 1,2$

$$\begin{aligned} dP &= P_1 du + P_2 dv \\ &= (g_{11})^{\frac{1}{2}}(du)e_1 + (g_{22})^{\frac{1}{2}}(dv)e_2 \\ &= \omega_1 e_1 + \omega_2 e_2. \end{aligned}$$

Relations (4.4) and (4.5) come easily after examining $d(dP) = 0$. To acquire (4.3) let

$$(4.6) \qquad \omega_{12} = b\,du + a\,dv$$

From (4.1)

$$d\omega_1 = ((g_{11})^{\frac{1}{2}})_v dv \wedge du \qquad \text{and from (4.4)}$$

$$d\omega_1 = -(g_{22})^{\frac{1}{2}}(b)dv \wedge du. \qquad \text{Thus}$$

$$(4.7) \quad b = -(g_{11})_v/(2(g_{11}g_{22})^{\frac{1}{2}})$$

A similar manipulation gives

$$(4.8) \quad a = (g_{22})_u/(2(g_{11}g_{22})^{\frac{1}{2}})$$

Thus (4.6), (4.7) and (4.8) give (4.3).

In Chern's equations C:(23) he defines r_{ij}, $i = 1,2$. We find the r_{ij} in terms of the principal curvatures r_1 and r_2 and the metric coefficients to be:

$$r_{11} = (r_1)_u/(g_{11})^{\frac{1}{2}} \tag{4.9}$$

$$r_{12} = (r_1)_v/(g_{22})^{\frac{1}{2}} \tag{4.10}$$

$$r_{21} = (r_2)_u/(g_{11})^{\frac{1}{2}} \tag{4.11}$$

$$r_{22} = (r_2)_v/(g_{22})^{\frac{1}{2}} \tag{4.12}$$

We show the evaluation of r_{1j}, $j = 1,2$ below. r_{2j}, $j = 1,2$ is done similarly.

$$dr_1 = (r_1)_u du + (r_1)_v dv$$

$$= ((r_1)_u/(g_{11})^{\frac{1}{2}}\omega_1 + ((r_1)_v/(g_{22})^{\frac{1}{2}})\omega_2 \tag{4.13}$$

Thus from C:(23) by using (4.1), (4.2) and (4.13) we obtain (4.9) and (4.10).

The Mainardi-Codazzi equations in lines of curvature coordinates become for $J = (r_1 - r_2)/2$

$$(g_{11})_v = -(r_1)_v g_{11}/J \tag{4.14}$$

$$(g_{22})_u = (r_2)_u g_{22}/J \tag{4.15}$$

In C:(31) Chern defines r_{ijk}'s. We will show below that

$$r_{122} = W + X \qquad \text{where} \tag{4.16}$$

(4.17) $W = (1/(g_{22})^{\frac{1}{2}})((r_1)_v/(g_{22})^{\frac{1}{2}})_v$ and

(4.18) $X = ((r_1)_v - 2(r_2)_u)((g_{22})_u/(2g_{11}g_{22}))$ and that

(4.19) $r_{211} = Y + Z$ where

(4.20) $Y = (1/(g_{11})^{\frac{1}{2}})((r_2)_u/(g_{11})^{\frac{1}{2}})_u$ and

(4.21) $Z = -(2(r_1)_v - (r_2)_v)((g_{11})_v/(2g_{11}g_{22}))$.

In the first equation of C: (31) we replace ω_{12} by (4.3) and, collecting terms with respect to du and dv, we observe that the dv term on the left hand side is $(W^1 + X^1)dv$ where

$W^1 = (r_{12})_v$ which by (4.10) is

$= ((r_1)_v/(g_{22})^{\frac{1}{2}})_v$ and by (4.3)

$X^1 = (r_{11} - 2r_{12})((g_{22})_u/(2(g_{11}g_{22})^{\frac{1}{2}}))$

which becomes using (4.9) and (4.11)

$$X^1 = ((r_1)_u/(g_{11})^{\frac{1}{2}} - 2(r_2)_u/(g_{11})^{\frac{1}{2}})((g_{22})_u/(2(g_{11}g_{22})^{\frac{1}{2}}).$$

Hence from the first equation in C: (31) we have

$$r_{122}\omega_2 = (W^1 + X^1)dv \qquad \text{or by (4.2)}$$

$$r_{122} = (W^1 + X^1)/(g_{22})^{\frac{1}{2}}.$$

Finally we obtain (4.16) by observing $W = W^1/(g_{22})^{\frac{1}{2}}$ and $X = X^1/(g_{22})^{\frac{1}{2}}$. In a similar manner using (4.11), (4.3), (4.10), (4.12) and (4.1) we can prove (4.19).

We now show equation C: (32) of Chern's is Theorem Egregium in lines of curvature coordinates with terms involving $(g_{11})_v$ and $(g_{22})_u$ replaced by the Mainardi-Codazzi equations (4.14) and (4.15). Substituting in C: (32), (4.14), (4.15) and $r_1 - r_2 = 2J$ we have

$$2KJ = W + X - Y - Z. \tag{4.22}$$

Carrying out the differentiation in (4.17) and (4.20) W and $-Y$ become

$$W = (r_1)_{vv}/g_{22} - (r_1)_v(g_{22})_v/(2(g_{22})^2) \quad \text{and}$$

$$-Y = -(r_2)_{uu}/g_{11} - (r_2)_u(g_{11})_u/(2(g_{11})^2).$$

Substituting (4.15) in the expression (4.18) for X gives

$$X = (r_1)_u(r_2)_u/(2g_{11}J) - (r_2)_u(r_2)_u/(g_{11}J).$$

Substituting (4.14) in the expression (4.21) for $-Z$ gives

$$-Z = -(r_1)_v(r_1)_v/(g_{22}J) + (r_1)_v(r_2)_v/(2g_{22}J).$$

Thus (4.22) becomes after replacing X, Y, Z, W from the above and dividing by 2J

$$(4.23)\quad K = (r_1)_{vv}/(2Jg_{22})-(r_2)_{uu}/(2Jg_{11})-((r_1)_v)^2/(2J^2g_{22}) + -((r_2)_u)^2/(2J^2g_{11})+(r_1)_u(r_2)_u/(4J^2g_{11}) + -(r_1)_v(g_{22})_v/(4J(g_{22})^2)-(r_2)_u(g_{11})_u/(4J(g_{11})^2) + +(r_1)_v(r_2)_v/(4J^2g_{22}).$$

(4.23) is an expression for the Gaussian curvature K in terms of the principal curvatures r_1 and r_2 and the coefficients of the first and second fundamental forms using lines of curvature coordinates as our parameterization.

To reconcile our form of the Weyl identity with Chern's work we use Chern's equation C: (38) in addition to C: (32) which we examined in the previous paragraph. In C: (38), $\ell(K)$ is the second differential Beltrami parameter of K with respect to the first fundamental form which Chern writes in terms of the principal curvatures. With regard to C: (38) all the ingredients have been calculated in (4.16), (4.19) and (4.9) - (4.12). It remains for us to obtain r_{111} and and r_{222} in terms of the principal curvatures and the metric coefficients.

Chern defines r_{111} and r_{222} by successivly applying the exterior differential operator d to $K = r_1r_2$ in his equations C: (33) - C: (37). From C: (33) and (4.1) and (4.2) it is easy to observe

$$K_1 = Ku/(g_{11})^{\frac{1}{2}} \tag{4.24}$$

$$K_2 = Kv/(g_{22})^{\frac{1}{2}} \tag{4.25}$$

To calculate K_{11} in terms of the r_i's and the g_{ij}'s we examine the du term on the left hand side of the first equation in C: (34) which becomes from (4.3)

$$((r_1)_u + r_2(g_{11})_v/(2(g_{11}g_{22})^{\frac{1}{2}}))du.$$

Thus using (4.1) we obtain from the first equation in

C: (34)

$$K_{11} = (K_1)_u/(g_{11})^{\frac{1}{2}} + K_2(g_{11})_v/(2g_{11}(g_{22})^{\frac{1}{2}})$$

Hence by (4.24) and (4.25)

$$(4.26)\quad K_{11} = (1/g_{11})^{\frac{1}{2}}(K_u/(g_{11})^{\frac{1}{2}})_u + K_v(g_{11})_v/(2g_{11}g_{22}).$$

A similar examination of the dv term in the second equation in C: (34) and use of (4.3), (4.2), (4.24) and (4.25) give

$$(4.27)\quad K_{22} = (1/g_{22})^{\frac{1}{2}}(K_v/(g_{22})^{\frac{1}{2}})_v + K_u(g_{22})_u/(2g_{11}g_{22}).$$

We will show that

$$(4.28)\quad r_{111} = (1/g_{11})^{\frac{1}{2}}((r_1)_u/(g_{11})^{\frac{1}{2}})_u+((g_{11})_v/(g_{11}g_{22}))(r_1/r_2)(r_1)_v$$
$$+(g_{11})_v(r_1)_v/(2g_{11}g_{22}) \qquad \text{and}$$

$$(4.29)\quad r_{222} = (1/g_{22})^{\frac{1}{2}}((r_2)_v/(g_{22})^{\frac{1}{2}})_v+((g_{22})_v/(g_{11}g_{22}))(r_2/r_1)(r_2)_u$$
$$+(g_{22})_u(r_2)_u/(2g_{11}g_{22}).$$

In (4.26) we substitute for K_u, K_v, K_{uu} and K_{vv} the expressions

$$K_u = r_1(r_2)_u + r_2(r_1)_u$$
$$K_{uu} = r_1(r_2)_{uu} + 2(r_1)_u(r_2)_u + r_2(r_1)_{uu}$$

and the corresponding expressions with u replaced by v. Thus we get

$$K_{11} = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{where}$$

$$\alpha_1 = r_1(1/g_{11})^{\frac{1}{2}}((r_2)_u/(g_{11})^{\frac{1}{2}})_u$$

$$\alpha_2 = 2(r_2)_u(r_1)_u/g_{11} + r_2(1/g_{11})^{\frac{1}{2}}((r_1)_u/(g_{11})^{\frac{1}{2}})_u$$

and $$\alpha_3 = ((g_{11})_v/(2g_{11}g_{22}))(r_2(r_1)_v + r_1(r_2)_v).$$

Using the notation (4.19) we have

$$\alpha_1 = r_1Y$$
$$= r_1r_{211} + r_1(r_1)_v/(g_{11}g_{22}) - r_1(r_2)_v(g_{11})_v/(2g_{11}g_{22})$$

which, modifying the middle term by multiplying by r_2/r_2, can be rewritten as

$$\alpha_1 = 2r_{11}r_{21} + r_2(1/g_{11})^{\frac{1}{2}}((r_1)_u/(g_{11})^{\frac{1}{2}})_u.$$

Thus $\alpha_1 + \alpha_2 + \alpha_3$ takes the form of the right hand side of the third equation in C: (37) and, if we collect the coefficients of r_2, we have the expression for r_{111} given in (4.28). Similarly from the fourth equation of C: (37) we examine K_{22} and using (4.10), (4.12), (4.16) and (4.27) and, collecting the coefficient of r_1, we get (4.29) as the expression for r_{222}.

We are now in a position to obtain the Weyl identity $(*)$ in the notation of Beltrami parameters from equations C: (32) and C: (38). We write

$$2J^2 = (J/K)(2K/J) \quad \text{which by C:(32)}$$

$$= (J/K)(r_{122} - r_{211}) \quad \text{which by}$$

letting $r_1 - r_2 = 2J$ becomes

$$2J^2 = ((r_1 - r_2)/(2K))(r_{122} - r_{211}) \tag{4.30}$$

Adding C: (38) and (4.30) and letting $K = r_1 r_2$ we have

$$(1/2K)\Delta' K + 2J^2 = (r_{222}+r_{122})/(2r_2)+(r_{111}+r_{211})/(2r_1) + {} \tag{4.31}$$
$$+r_{11}r_{22}/(r_1 r_2) + r_{12}r_{22}/(r_1 r_2).$$

Substituting from equations (4.16), (4.19), (4.28), (4.29) we have the right hand side of (4.31) is equal to $A + B + C + D + E + F$ where

$$A = ((r_2)_v/(g_{22})^{\frac{1}{2}})_v/(2(g_{22})^{\frac{1}{2}}r_2)+((r_1)_v/(g_{22})^{\frac{1}{2}})_v/(2(g_{11})^{\frac{1}{2}}r_2) \tag{4.32}$$

$$D = ((r_1)_u/(g_{11})^{\frac{1}{2}})_u/(2(g_{11})^{\frac{1}{2}}r_1)+((r_2)_u/(g_{11})^{\frac{1}{2}})_u/(2(g_{11})^{\frac{1}{2}}r_1) \tag{4.33}$$

$$B = (g_{11})_v(r_1)_v/(2g_{11}g_{22}r_2)-(g_{11})_v(r_1)_v/(4g_{11}g_{22}r_1) + {} \tag{4.34}$$
$$+(g_{11})_v(r_2)_v/(4g_{11}g_{22}r_1)$$

(4.35) $E = (g_{22})_u(r_2)_u/(2g_{11}g_{22}r_1)-(g_{22})_u(r_2)_u/(4g_{11}g_{22}r_2) +$

$+(g_{11})_u(r_1)_u/(4g_{11}g_{22}r_2)$

(4.36) $C = r_{12}r_{22}/(r_1r_2)$

(4.37) $F = r_{11}r_{21}/(r_1r_2)$.

We observe the following symmetry in the equations above. If we replace g_{22} by g_{11}, v by u and r_2 by r_1 then A becomes D, B becomes E and C becomes F using (4.9) - (4.12). In the computations which will follow we write only the derivatives with respect to v and will denote the v components of the Weyl identity with the superscript v. Thus we are examining (4.31) in the form

(4.38) $$((1/2K)\Delta' K + 2J^2)^v = A + B + C$$

We first reexamine A. Recalling $2H = r_1 + r_2$ and $g_{ii} = \ell_{ii}/r_i$ for $i = 1,2$, we have

$$A = (\ell_{11}/r_2)^{\frac{1}{2}}(1/\ell_{11}\ell_{22})^{\frac{1}{2}}(H_v(\ell_{11}/\ell_{22})^{\frac{1}{2}}(r_2/\ell_{11})^{\frac{1}{2}})_v.$$

Using the product rule for differentiation on the last equation and keeping the first two factors together we arrive at

(4.39) $$A = A_1 + A_2 \quad \text{where}$$

$$(4.40)\quad A_1 = (1/\ell_{11}\ell_{22})^{\frac{1}{2}}(H_v(\ell_{11}/\ell_{22})^{\frac{1}{2}})_v \qquad \text{and}$$

$$(4.41)\quad A_2 = ((r_2/\ell_{11})^{\frac{1}{2}})_v(H_v(\ell_{11})^{\frac{1}{2}}/(\ell_{22}(r_2)^{\frac{1}{2}})).$$

We note that A_1 is precisely the v component of the second Beltrami differential parameter with respect to the second fundamental form which we denote by $(\Delta''H)^v$. Thus (4.38) becomes

$$(4.42)\quad ((1/(2K)\Delta'K)^v + (2J^2)^v = (\Delta''H)^v + A_2 + B + C$$

Using (4.41), $\ell_{ii} = r_i g_{ii}$ $i = 1,2$ and $2H_v = (r_1)_v + (r_2)_v$:

$$(4.43)\quad A_2 = st \qquad \text{where}$$

$$s = (1/(4r_2))((r_1)_v/g_{22} + (r_2)_v/g_{22})$$

$$\text{and}\quad t = ((r_2)_v/r_2 - (r_1)_v/r_1 - (g_{11})_v/g_{11}),$$

and using (4.9) - (4.12) and rearranging terms (4.43) becomes:

$$(4.44)\quad A_2 = (r_{12})^2((r_1+r_2)/(4r_1r_2(r_1-r_2))+(r_{22})^2(1/(4(r_2)^2) +$$
$$+r_{12}r_{22}(1/(4(r_2)^2)+(r_1+r_2)/(4r_1r_2(r_1-r_2)).$$

Using (4.14) and then (4.10) we have

$$(g_{11})_v/(g_{11}(g_{22})^{\frac{1}{2}}) = -(r_1)_v/((g_{22})^{\frac{1}{2}}J)$$
$$= -r_{12}/J$$

Thus examining B in (4.34) we have

$$(4.45)\quad \begin{aligned} B = {} & (r_{12})^2(-1/(r_2(r_1-r_2)) + 1/(2r_1(r_1-r_2))) + \\ & +r_{12}r_{22}(-1/(2r_1(r_1-r_2)). \end{aligned}$$

So from (4.44), (4.45) and (4.36) we have

$$(4.46)\quad \begin{aligned} A_2+B+C = {} & (r_{12})^2(3r_2-3r_1)/(4r_1r_2(r_1-r_2)) + \\ & +r_{12}r_{22}((5r_1-5r_2)/(4r_1r_2(r_1-r_2))-1/(4(r_2)^2)) + \\ & +(r_{22})^2(1/(4(r_2)^2)). \end{aligned}$$

Finally from (4.42) we have

$$(4.47)\quad (\Delta''H)^V - ((1/2K)\Delta'K)^V - (2J^2)^V = -(A_2 + B + C).$$

We claim that

$$-(A_2+B+C) = 2(\nabla''(H,J)/J)^V-(\nabla'(K,J)/KJ)^V-(\nabla''(K,H)/(2K))^V.$$

Using the techniques and equations above we have

$$(4.48)\quad 2(\nabla''(H,J)/J)^V = ((r_{12})^2-(r_{22})^2)/(r_2(r_1-r_2))$$

$$(4.49)\quad \begin{aligned} -(\nabla'(K,J)/(KJ))^V = {} & ((r_{22})^2-(r_{22}r_{12}))/(r_2(r_1-r_2)) + \\ & +(r_{12}r_{22}-(r_{12})^2)/(r_1(r_1-r_2)). \end{aligned}$$

$$(4.50)\quad \begin{aligned} (\nabla''(K,H)/(2K))^V = {} & -((r_{12})^2+r_{12}r_{22})/(4r_1r_2) + \\ & -(r_{12}r_{22}+(r_{22})^2)/(4(r_2)^2). \end{aligned}$$

Adding (4.48), (4.49), (4.50) and collecting terms in r_{ij} we have precisely the negative of equation (4.46). Thus (4.47) becomes

$$(4.51)\quad (\Delta''H-(1/2K)\Delta'K-2J^2)^V = (2\nabla''(H,J)/J+\nabla'(K,J)/(KJ)-\nabla''(K,H)/(2K))^V$$

Hence by symmetry the Weyl identity (*) follows.

Chapter 5. An application of the Weyl identity

Let S be a surface which is free from umbillics so that a Weyl identity of the form

$$(3.31) \qquad L_2H - L_1K = 2KJ^2$$

holds on S. If $K > 0$ then L_1 and L_2 are both elliptic operators. Therefore we have the known result that H cannot have a maximum where K has a minimum, for then the right hand side of (3.3.1) is strictly positive while the left hand side is non-positive. There is reason to believe that the requirement of $K > 0$ can be relaxed to $K \geq 0$ to yield the following result: If H has a positive maximum at the same point P where K takes on a minimum of zero, then H assumes its maximum value at every point along a line of curvature through P. Furthermore if we add the assumption that the set of points at which K is minimized is a subset of the set of points at which H is maximized then this line of curvature is an asymptotic curve of S.

One reason for believing that the conjecture is true is that the conjecture seems to be related to the known result in the same way that Hartman and Sacksteder's result on elliptic and parabolic operators in [3] is related to E. Hopf's result on elliptic operators in [4].

One might guess that it would be possible to prove the conjecture by imitating the technique of E. Hopf, that is by perturbing H and $-K$ as functions on S and using $L_2 H + L_1(-K) \geq 0$ in a region. A more thorough examination of the problem indicates that it is more probable that the surface itself must be perturbed. We state and prove the following theorem whose corollary suggests that the conjecture is true.

Theorem: Let a surface S of non-negative Gaussian curvature K and positive mean curvature H be given by $(x,y,z(x,y))$ where $z \in C^r$, $r \geq 4$. Suppose $H(Q)$ is a maximum for H and $K(Q) = 0$ is a minimum for K at a point Q on S. Then for an appropriate reparameterization, the Taylor expansion for $z(x,y)$ is given by

$$z(x,y) = z_{xx}(Q)x^2/2 + z_{xxxx}(Q)x^4/4! + R(x,y)$$

where $z_{xx}(Q) > 0$, $R(x,y) = 0((x^2+y^2)^{5/2})$ and

$$z_{xxxx}(Q) \leq 3(z_{xx}(Q))^3.$$

Remark In the proof of the theorem, using the Weyl identity, we compile the following data: Under the stated hypothesis and reparameterization

$$K_{xx}(Q) = K_{xy}(Q) = K_{yy}(Q) = H_{yy}(Q) = H_{xy}(Q) = 0.$$

Corollary: If $z(x,y)$ is a polynomial of degree ≤ 4, then K is identically zero.

From the corollary and Section 5, Chapter 3 we have that H is equal to its maximum value everywhere on the line of curvature corresponding to the principal curvature zero and thus the conjecture is true for this class of surfaces.

Proof of Theorem Let $\partial z/\partial x = p$, $\partial z/\partial y = q$, $\partial^2 z/\partial x^2 = r$, $\partial^2 z/(\partial x \partial y) = s$ and $\partial^2 z/\partial y^2 = t$. Then for any surface of the form $(x,y,z(x,y))$ we have the following formulas:

$$g_{11} = 1 + p^2 \tag{5.1}$$

$$g_{12} = pq \tag{5.2}$$

$$g_{22} = 1 + q^2 \tag{5.3}$$

$$\ell_{11} = r/(1 + p^2 + q^2)^{\frac{1}{2}} \tag{5.4}$$

$$\ell_{12} = s/(1 + p^2 + q^2)^{\frac{1}{2}} \tag{5.5}$$

$$\ell_{22} = t/(1 + p^2 + q^2)^{\frac{1}{2}} \tag{5.6}$$

$$K = (rt - s^2)/(1 + p^2 + q^2)^2 \tag{5.7}$$

$$H = ((1+q^2)r - 2pqs + (1+p^2)t)/(2(1+p^2+q^2)^{\frac{3}{2}}). \tag{5.8}$$

We reparameterize our surface in the following way.

Let us pick the (x,y) plane so that it is tangent to the surface at Q and so that the origin corresponds to Q. Thus we have

$$z_x(Q) = z_y(Q) = z(Q) = 0. \tag{5.9}$$

Now we rotate the plane so that the x and y axes point in the directions of the lines of curvature corresponding to the positive principal curvature and the zero principal curvature respectively. Thus from (5.4), (5.5), (5.6) and the relationship of the ℓ_{ij}'s to $k_1(Q) = 0$ and $k_2(Q) > 0$ we have

$$\begin{aligned} z_{yy}(Q) &= 0 \\ z_{xx}(Q) &> 0 \\ z_{xy}(Q) &= 0. \end{aligned} \tag{5.10}$$

Differentiating formulas (5.7) and (5.8) and using the conditions (5.9) and (5.10) to evaluate these derivatives at the origin we obtain

$$K_x(Q) = z_{xx}(Q)z_{xyy}(Q) \tag{5.11}$$

$$K_y(Q) = z_{xx}(Q)z_{yyy}(Q) \tag{5.12}$$

$$K_{xx}(Q) = z_{xx}(Q)z_{xxyy}(Q) \tag{5.13}$$

$$K_{xy}(Q) = z_{xx}(Q)z_{xyyy}(Q) \tag{5.14}$$

(5.15) $$K_{yy}(Q) = z_{xx}(Q)z_{yyyy}(Q)$$

and

(5.16) $$H_x(Q) = z_{xyy}(Q) + z_{xxx}(Q)$$

(5.17) $$H_y(Q) = z_{xxy}(Q) + z_{yyy}(Q)$$

(5.18) $$H_{xx}(Q) = z_{xxyy}(Q) + z_{xxxx}(Q) - 3(z_{xx}(Q))^3$$

(5.19) $$H_{xy}(Q) = z_{xyyy}(Q) + z_{xxxy}(Q)$$

(5.20) $$H_{yy}(Q) = z_{xxyy}(Q) + z_{yyyy}(Q).$$

The condition that K is minimized at Q gives, from equations (5.11) and (5.12),

(5.21) $$z_{xyy}(Q) = 0 \quad \text{and}$$

(5.22) $$z_{yyy}(Q) = 0.$$

From the fact that H is extremal at Q we have, from (5.16), (5.17), (5.21) and (5.22) that

(5.23) $$z_{xxx}(Q) = 0 \quad \text{and}$$

(5.24) $$z_{xxy}(Q) = 0$$

Thus from equations (5.21) - (5.24) we see that all the third order derivatives of z at Q are zero.

When H has a maximum at Q and K has a minimum of zero at Q we have

$$(L_2H)(Q) \leq 0 \quad \text{and}$$

$$(L_1K)(Q) \geq 0 \quad \text{and the Weyl identity}$$

for $K \geq 0$ gives from (3.31)

$$L_2H \geq L_1K.$$

Thus $(L_1K)(Q) = 0$ and, since $(L_1K)(Q)$ becomes simply $K_{xx}(Q) + K_{yy}(Q)$, we have

$$K_{xx}(Q) + K_{yy}(Q) = 0. \tag{5.25}$$

(5.25) together with the fact that the quantities $K_{xx}(Q)$ and $K_{yy}(Q)$ are nonnegative give

$$K_{xx}(Q) = 0 \tag{5.26}$$

$$K_{yy}(Q) = 0 \quad \text{and hence} \tag{5.27}$$

$$K_{xy}(Q) = 0 \tag{5.28}$$

From (5.13) - (5.15) in light of (5.26) - (5.28) we have

$$z_{xxyy}(Q) = 0 \tag{5.29}$$

$$z_{xyyy}(Q) = 0 \tag{5.30}$$

(5.31) $$z_{yyyy}(Q) = 0.$$

$(L_2H)(Q) = 0$ also follows from (5.25) but yields no new information.

Since $H_{yy}(Q) = 0$ and H is maximized at Q we have that $H_{xy}(Q) = 0$ and $H_{xx}(Q) \leq 0$. The equality yields from equation (5.19) for $H_{xy}(Q)$ and from (5.30)

(5.32) $$z_{xxxy}(Q) = 0.$$

Hence from (5.29) - (5.32) we see that all the fourth order derivatives of z at Q are zero except possibly z_{xxxx}. The inequality $H_{xx}(Q) \leq 0$, (5.29) and (5.18) give

(5.33) $$z_{xxxx}(Q) \leq 3(z_{xx}(Q))^3.$$

Thus from equations (5.21) - (5.24) and (5.29) - (5.33) the Taylor expansion for z agrees with the one in the statement of the theorem.

If z is a polynomial of degree ≤ 4 its Taylor expansion, by the above theorem, is free from any terms involving y. Thus z_{xy} and z_{yy} are identically zero which give K identically zero from (5.7).

Bibliography

[1] Bishop, R. and R. Crittenden, *Geometry of Manifolds*, Academic Press (1964), New York & London.

[2] Chern, S.S., *Some new characterizations of the euclidean sphere*, "Duke Mathematical Journal," Vol. 12 (1945), pp. 279-290.

[3] Hartman, P. and R. Sacksteder, *On maximal principals for nonhyperbolic partial differential operators*, Rend. del Circ. Mat. di Palermo, (2) Vol. 6 (1957), pp. 218-232.

[4] Hopf, E., *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte der Berliner Akademie der Wissenschaften, Vol. 19 (1927), pp. 147-152.

[5] Katzen, M., *On Weyl's identity*, Dissertation in Mathematics submitted for degree of Doctor of Philosophy, City University of New York (1968).

[6] Laugwitz, D., *Differential and Riemannian Geometry* (1965), Academic Press.

[7] Singer, I.M. and J.A. Thorpe, *Lecture Notes on Elementary Topology and Geometry* (1967), Scott, Foresman and Company.

[8] Weatherburn, C.E., *Differential Geometry of Three Dimensions* (1927), Cambridge University Press.

[9] Weatherburn, C.E., *On isometric systems of curves and*

surfaces, "American Journal of Mathematics," Vol. 49 (1927), pp. 527-534.

[10] Weyl, H., Über de Bestimmung einer geschlossenen Konvexen Flächen durth ihr hinienelement, "Vierteljahrsschritt der Naturforschinden Gesellschaft in Zürich," Vol. 61 (1915), pp. 40-72.

[11] Wintner, A., On indefinite binary Riemannian metrics, "American Journal of Mathematics," Vol. 77 (1955), pp. 853-867.

[12] Wintner, A., On Weyl's identity in differential geometry of surfaces, "Annali di Matematica," (1956), pp. 257-268.

Autobiographical Statement

Eleanor Marian Zeitlin was born in New York City on May 26, 1941. She was graduated from Seward Park High School in 1957, received a Bachelor of Science degree from the College of the City of New York in 1961 and a Master of Science degree from Case Institute of Technology at Cleveland, Ohio in 1963. For the next three years she was a full time instructor of mathematics at Western Reserve University in Cleveland. In September 1966, she resumed graduate studies at the City University of New York where in August 1969 she completed her dissertation. She married Gerald Goldstein in August 1968.