Dynamics of Certain Families of Transcendental Meromorphic Functions

by

Shenglan Yuan

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Linda Keen

Date — Chair of Examining Committee

Józef Dodzuik

Date — Executive Officer

Linda Keen

Frederick Gardiner

Jun Hu

Nikola Lakic

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Dynamics of a certain class of transcendental meromorphic functions

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Shenglan Yuan

Advisor: Professor Linda Keen

The theory of iterated transcendental functions has been extensively studied in the past two decades. We are interested in some "slices" of parameter spaces of certain classes of meromorphic functions with two asymptotic values, $\mathcal{T}_{a,\lambda}$, $\mathcal{S}_{\lambda}$. We study the properties of the dynamic plane of functions in the families. We also study parametric representation of the families. We study the relationships between $\mathcal{S}_{\lambda}$ and the tangent family, between $\mathcal{S}_{\lambda}$ and the exponential family, between $\mathcal{T}_{a,\lambda}$ and the tangent family and between $\mathcal{T}_{a,\lambda}$ and the exponential family.

The functions $T_{a,\lambda}$ have two asymptotic values, one is $-\lambda$ and the other one is $a\lambda$. Under conjugation, the family can be written as $\{T_{a,\lambda}(z) = a\lambda\frac{\exp(z)-\exp(-z)}{\exp(z)+a\exp(-z)}, a \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{0\}\}$. We can see that as a approaches ∞, the asymptotic value $a\lambda$ escapes to ∞, and that each function $T_{a,\lambda}(z)$, on any compact subset, will uniformly converge to the exponential function $\lambda\exp(2z) - \lambda$. We will show that there is dynamic convergence as $a \to \infty$, and we will study the relationship of the hyperbolic components of the two families, $\mathcal{T}_{a,\lambda}$ and $\lambda\exp(2z) - \lambda$.

In the family $\mathcal{S}_{\lambda}$ each function S_{λ} has two asymptotic values, 0 and

λ, and 0 is also a pole. We will show that each component of the Fatou set of S_λ is simply connected, and that there is at most one completely invariant domain of the Fatou set. We will also prove that these results can be generalized to functions with finitely many singular values and certain restrictions.

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To Michael and Altai Abrams

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

In recent years, there has been increasing interest in the study of the transcendental entire and meromorphic functions (see, e.g., [1,2, 20, 21,31,32]). In particular, there are many studies about the exponential functions $\lambda \exp(z)$ (see, [6,8,11]), one of the simplest entire functions, which has only one asymptotic value and no critical values. There are also several studies about the tangent family $\lambda \tan(z)$, (see, e.g., [18,22,23,24]) which is the simplest one parameter family of meromorphic functions with two symmetrically located asymptotic values. There has been a veritable explosion in studies of the general entire function and meromorphic functions and much more is known about the functions with finitely many singular values.

We are interested in various "slices" of parameter spaces of certain classes of meromorphic functions with two asymptotic values, $T_{a,\lambda}$, S_λ. We study the properties of dynamic planes and parameter planes of $T_{a,\lambda}$ and S_λ. We study the relationship between S_λ and the tangent functions $\lambda\tan(z)$, as well as S_λ and the exponential functions $E_\lambda(z)$. The relationship between $T_{a,\lambda}$ and $T_\lambda(z)$, as well as $T_{a,\lambda}$ and $E_\lambda(z)$ are also studied.

1.2 Notations

Let us introduce the notations will be used in this thesis.

- $\mathcal{E}_\lambda$ stands for the exponential family $\{E_\lambda(z) = \lambda\exp(2z) - \lambda, \lambda \in \mathbb{C} \setminus \{0\}\}$.

- $\mathcal{S}_\lambda$ stands for the family with two asymptotic values. One of the asymptotic values is also a pole, whose normal form is $\{S_\lambda = \lambda\frac{\exp(z)}{\exp(z)-\exp(-z)}, \lambda \in \mathbb{C} \setminus \{0\}\}$.

- $\tilde{F}$ stands for the family of meromorphic functions with finitely many singular values such that one of the asymptotic values is a pole. In addition, we exclude those functions with exactly one pole that is also an omitted value. That is, we assume that any $f \in \tilde{F}$, f has either at least two poles or f has exactly one pole which is not an omitted value.

- $\mathcal{T}_\lambda$ stands for the tangent family $\{T_\lambda = \lambda\tan(z), \lambda \in \mathbb{C} \setminus \{0\}\}$.

- $\mathcal{T}_{a,1}$ stands for the family with two asymptotic values with the normal form $\{T_{a,1} = a\frac{\exp(z)-\exp(-z)}{\exp(z)+a\exp(-z)}, \lambda \in \mathbb{C} \setminus \{0\}\}$.

- $\mathcal{T}_{a,\lambda}$ stands for the family with two asymptotic values with the normal form $\{T_{a,\lambda} = a\lambda\frac{\exp(z)-\exp(-z)}{\exp(z)+a\exp(-z)}, \lambda \in \mathbb{C} \setminus \{0\}\}$.

1.3 Dynamic Plane of the Functions S_λ

Asymptotic values in the transcendental meromorphic functions play the same role as the critical values in the rational functions. The function S_λ has two asymptotic values 0 and λ, no critical values, and 0 is also a pole. That is, 0 is the preimage of ∞, which is an essential singularity. S_λ behaves more like the function in the exponential family than the function in the tangent family because of the following:

- For each $\lambda \neq 0$, the Julia set of S_λ is connected. There is no Julia set which is homeomorphic to a Cantor set. For the functions in the tangent family, when $|\lambda| < 1, \lambda \neq 0$, the Julia set of $\lambda \tan(z)$ is a Cantor set.

- Each component of the Fatou set of S_λ is simply connected. Each component of the Fatou set of the function in the exponential family is simply connected. When $|\lambda| > 1$, each component of the Fatou set of the function in the tangent family is simply connected.

- There is no isolated Jordan curve in the Julia set of S_λ. For a function in the tangent family, when $\lambda > 1$, the Julia set is the real line. For a function in the exponential family, there is no isolated Jordan curve in the Julia set.

- The Julia set of S_λ contains Cantor bouquets. The Julia set of the exponential function E_λ contains Cantor bouquets.

- For a function S_λ, there is at most one completely invariant component of the Fatou set. For a function in the tangent family when $\lambda > 1$, there are two completely invariant Fatou components, the 'Upper half plane' and the 'Lower half plane'. For a function in the exponential family, there is at most one completely invariant Fatou component.

We prove that all the above properties can be generalized for functions in the family of transcendental meromorphic functions with finitely many singular values with the property that one of the asymptotic values is also a pole and some other conditions.

1.4 Parameter Plane of $\mathcal{S}_\lambda$

We describe the parameter plane of $\mathcal{S}_\lambda$, which has a similar structure to that of the tangent family $\mathcal{T}_\lambda$.

- There exists a positive number $R > 0$, such that when $\Re\lambda > R$, then S_λ has an attracting fixed point and thus $\mathcal{S}_\lambda$ has an unbounded hyperbolic component Ω_1 in the right half plane. When

$\lambda \in \Omega_1$, the Julia set of S_λ contains a Cantor bouquet. The tangent family $\mathcal{T}_\lambda$ has an unbounded hyperbolic component Ω_1 in the right half plane where there are two attracting fixed points. When $\lambda \in \Omega_1$, the Julia set of T_λ is an isolated Jordan curve homeomorphic to the real line R, and there are two completely invariant components of the Fatou set of T_λ. The exponential family $\mathcal{E}_\lambda$ has a bounded component Ω_1 which has a fixed attracting point, and the Julia set of E_λ contains a Cantor bouquet.

- Each hyperbolic component of $\mathcal{E}_\lambda$, except the component Ω_1, is unbounded. All hyperbolic components of $\mathcal{T}_\lambda$, except the component Ω_1 in the right half plane and a component Ω_2 in the left half plane, are bounded. For the family $\mathcal{S}_\lambda$, there are unbounded components and bounded components. We give some criteria for the hyperbolic component to be bounded or unbounded. In the right half plane there is the unbounded component Ω_1 and in the left half plane it seems that there are infinitely many unbounded hyperbolic components. Whether there are unbounded hyperbolic components with periodic cycles of all orders remains unresolved.
- There are bounded hyperbolic components of $\mathcal{S}_\lambda$ with all orders $k > 1$. Like the tangent family, we can describe all the bounded hyperbolic components by the combinatorial deployment of the virtual center. For the family S_λ, there is a hyperbolic component pair $(\Omega_p,\, \Omega'_p)$ attached at each prepole p_n of order $p-1$, where

$p' = p + 2$. S_λ has an attracting cycle of period p for any $\lambda \in \Omega_p$ and S_λ has an attracting cycle of period $p + 2$ for any $\lambda \in \Omega'_p$. For the tangent family $\mathcal{T}_\lambda$, there is a hyperbolic component pair Ω_p, Ω'_p attached at each prepole p_n of order $p - 1$, where $p' = 2p$. T_λ has two attracting cycles of period p for any $\lambda \in \Omega_p$ and T_λ has an attracting cycle of period $2p$ for any $\lambda \in \Omega'_p$.

- For S_λ there is periodical doubling along the internal ray of Ω_2 when the virtual center is the even pole $2n\pi$, and there is periodical doubling along the internal ray of Ω'_2 when the virtual center is the odd pole $(2n + 1)\pi i$.

1.5 Dynamic Planes of $T_{a,\lambda}$ and $E_\lambda(z)$

We are interested in the dynamic planes of $T_{a,1}$ and $E(z)$. These are the real slices of the families $T_{a,\lambda}$ and E_λ when $\lambda = 1$. We are interested in the relation between the functions in two families. We have the following:

- When $a \to \infty$, the functions $T_a(z)$ converge uniformly on any compact subset to the exponential function $E(z)$.

- For the function $T_a(z)$, when $a > 1$, the Julia set is a curve bounded by the vertical lines l_1 and l_2, where l_1 is the line $x = 0$ and l_2 is the line $x = \frac{1}{2}\log(a)$. The Julia set passes through the points $z = n\pi i$ on l_1 and the points $z = \frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ on

l_2, for all n. The Fatou set consists of two completely invariant components, each of which contains one of the asymptotic values.

When $a = 1$, the function $T_a(z) = \tanh(z)$ is conjugate to the tangent function $\tan(z)$ and the Julia set is the imaginary axis $x = 0$.

When $0 < a < 1$, the Fatou set consists of one completely invariant component containing both asymptotic values, and the Julia set of $T_a(z)$ is a Cantor set.

- We prove that the Julia set of $T_{a,1}$ converges to the Julia set of the exponential function $E(z)$, which is a Cantor bonquet.

For any fixed real λ, we study the relationship of the dynamic planes of $T_{a,\lambda}$ and $E_\lambda(z)$, and we also describe the Symbolic Dynamics for $T_{a,\lambda}$ and E_λ, and prove that $T_{a,\lambda}$ converges to E_λ in the combinatorial sense.

1.6 The Parameter Planes of $\mathcal{T}_{a,\lambda}$ and $\mathcal{E}_\lambda$

We study the parameter plane of $\mathcal{T}_{a,\lambda}$, when a is real, and discuss the relation of $\mathcal{T}_{a,\lambda}$ to the tangent family $\mathcal{T}_\lambda$ when a is near 1. We also study the relation of $\mathcal{T}_{a,\lambda}$ to the exponential family $\mathcal{E}_\lambda$ when a is near ∞. When $a \to \infty$, the hyperbolic components of $\mathcal{T}_{a,\lambda}$ approach the hyperbolic components of $\lambda \exp(z)$. That is, for a λ, such that $\exp_\lambda(z)$ has a periodic cycle of period p, there exists a sequence λ_i, $i = 1, 2, \cdots, n, \cdots$, such that $T_{a_{\lambda_i},\lambda_i}(z)$ has a periodic cycle of period p, satisfying $\lambda_i \to \lambda$ and $a_{\lambda_i} \to \infty$ as $i \to \infty$.

1.7 Outline

In Chapter One we will introduce the meromorphic families that we are interested in. We will list our results and discuss and compare our results with known properties of the exponential family and the tangent family.

In Chapter Two we will include some basic prelimaries that will be needed in later chapters regarding the theory of the Julia set and the Fatou set.

In Chapter Three we will study the dynamic plane of functions in the family $\mathcal{S}_\lambda = \lambda\frac{\exp(z)}{\exp(z)-\exp(-z)}, \lambda \in \mathbf{C} \setminus \{0\}$. We also will study the family $\tilde{F}$ of the transcendental meromorphic functions with finitely many singular values, one of whose asymptotic values is a pole. In addition, we assume that functions in $\tilde{F}$ have either a minimum of two poles or one pole which is not an omitted value. We prove that for a function in either of these families there is at most one completely invariant domain. We also discuss the connectivity of the component of the Fatou set.

In Chapter Four we will study the parameter plane of the family S_λ. We introduce the notion of virtual center and show how the deployment of bounded components are related to virtual centers.

In Chapter Five we will study the dynamic plane of functions in the family $\mathcal{T}_{a,\lambda}$ when both a and λ are real and prove the dynamic convergence to the exponential function as a approaches ∞.

In Chapter Six we will discuss the parameter plane of the family $\mathcal{T}_{a,\lambda}$ for a fixed a which is real. We will also discuss the relationship between the parameter planes of the family $\mathcal{T}_{a,\lambda}$ and the exponential family $\mathcal{E}_{\lambda}$.

In Chapter Seven we will discuss open questions and future interests.

Chapter 2

Preliminaries

2.1 Basic Definitions

2.1.1 Julia Sets and Fatou Sets

Let us recall some definitions and basic facts about the dynamics of meromorphic functions in general. For a complex analytic function f, the Julia set carries the interesting dynamical information (See, [5,7,8,26]).

Definition 1 *The Fatou set of f is defined to be: $\{z : z \in \hat{\mathbb{C}}$ and there exists a neighborhood U of z such that all f^n, $n \in \mathbb{N}$, is defined in U and form a normal family in $U\}$. The Julia set $J(f)$ is the complement of the Fatou set.*

For a function a meromorphic function f the Julia set $J(f)$ can also be described by the following equivalent conditions:

- $J(f)$ is the closure of the set of repelling periodic points of f.

- There are points whose forward orbit lands on a pole and are thus preimages of ∞. Such points are called prepoles and $J(f)$ is the closure of the set of prepoles of f.

2.1.2 Cantor Sets and Cantor Bouquets

The Julia set of some rational functions and transcendental meromorphic functions is homeomorphic to the standard Cantor set. Topologically, a Cantor set is non-empty, perfect, compact, totally disconnected, and metrizable. A perfect set is a closed set in which every point is an accumulation point.

A Cantor bouquet appears in the Julia set of certain entire functions. Roughly speaking, a Cantor bouquet can be thought of as a collection of cantor sets of curves. Topologically, it can be defined as follows (See [8,14]) :

Definition 2 *A Cantor bouquet is a subset of C which is homeomorphic to a straight brush. By a straight brush, we mean that a subset* $\mathbf{S}$ *of* $[0,\infty) \times (\mathbb{R} - \mathbb{Q})$ *has the following properties:*

- *Hairiness. For every $s \in \mathbb{R} - \mathbb{Q}$, there is a $t_s \in [0,\infty)$ such that all points $(t,s) \in \mathbf{S}$ where $t \geqslant t_s$. The hair h_s is defined by $h_s = [t_s,\infty) \times \{s\}$. The point $e_s = (t_s, s)$ is called the endpoint of the hair h_s.*

- *Density. The set* $\{\alpha|(y,\alpha) \in \mathbf{S}$ *for some* $y \in [0,\infty)\}$ *is dense in* $\mathbb{R} - \mathbb{Q}$*. Moreover, each endpoint of a hair is the limit from above and from below of other endpoints of hairs.*
- *Closed.* $\mathbf{S}$ *is closed in* $\mathbb{R}^2$*.*

2.1.3 Singular Values, Critical Values, and Asymptotic Values

As we know, studying the orbit of the critical points or values gives us a lot of information about the dynamics of the rational functions. Similarly, by studying what happens to the singular values of transcendental entire or meromorphic functions, we garner much information about the dynamics of transcendental functions.

Definition 3 *Suppose that* $f : D \to \hat{\mathbb{C}}$ *is a nowhere locally constant holomorphic function where* D *is open and* $D \subset \hat{\mathbb{C}}$*. A point* $u \in \hat{\mathbb{C}}$ *is said to have a regular covering if it has a neighborhood* U *such that each component* $f^{-1}(U)$ *is mapped holomorphically onto* U *by* f*. A singular value is a value at which* f *is not a regular covering.*

Definition 4 *A point* z *is called a critical value if there exists a point* $w \in f^{-1}(z)$*, a neighborhood* U *of* z*, and a neighborhood* W *of* w *such that* $f : U \setminus \{z\} \to W \setminus \{w\}$ *is a degree-d covering for some* $d > 1$*.* w *is called a critical point.*

Definition 5 *A point* v *is called an asymptotic value if there exists a path* $\alpha : [0,\infty) \to D$ *such that* $\alpha(t) \to \partial D$ *and* $f \circ \alpha(t) \to v$ *as* $t \to \infty$*.*

Notice that if v is isolated in the singular set then we can choose a small neighborhood V of v whose closure contains no other singular values. Let U be the component of $f^{-1}(V)$ containing the tail of α. Then $f : U \to V \setminus \{v\}$ is a universal covering.

Definition 6 *If we can associate a given asymptotic value v with an asymptotic tract, that is, a simply connected unbounded domain A such that $f(A)$ is a punctured neighborhood of v, then v is called a logarithmic singularity.*

Definition 7 *A set U is called completely invariant under f if $z \in U$ implies $f(z) \in U$ when $f(z)$ is defined. Also, for all w such that $f(w) = z$, $w \in U$. That is, $U \subset f(U)$ and $f^{-1}(U) \subset U$.*

2.2 Fatou and Julia Sets

The following background material about Fatou and Julia sets can be found in the books (see, [4], [7], [26]) or in the paper (see, [5]).

2.2.1 Basic Properties of Fatou and Julia sets

Lemma 2.2.1 *The Fatou set and Julia set are completely invariant.*

Lemma 2.2.2 *Either the Julia set is $\hat{\mathbb{C}}$ or the Julia set has an empty interior.*

Definition 3 *If the backward orbit of z_0 is finite, then we say that z_0 is an exceptional point.*

Lemma 2.2.4 *If z_0 is not an exceptional point in the Julia set, then the Julia set is the closure of the backward orbit of z_0.*

Proposition 2.2.5 *If f is meromorphic then the Julia set $J(f)$ is a perfect set.*

2.2.2 Periodic Points and Periodic Components

Periodic points play an important role in iteration theory. By definition, periodic points consist of points such that $f^n(z) = z$ for some $n \geqslant 1$ and $f^k(z) \neq z$ for any $k < n$. For a periodic point z_0 of period n, the multiplier of z_0 is defined by $(f^n)'(z_0)$. Notice that the multiplier is independent of the choice of conjugation of the function. (When $z_0 \to \infty$, the multiplier is defined as $(g^n)'(0)$; where $g(z) = 1/f(1/z)$.) A periodic point is called attracting, indifferent, or repelling when the modulus of the multiplier is less than 1, equal to 1, or greater than 1. A periodic point of period 1 is called a fixed point.

By definition, the Fatou set F is open. By convention, a component U of the Fatou set F always means a connected component of F. For a component U, $f^n(U)$ is contained in some component, which we label U_n, of F. A component U is called preperiodic if there exist $m > n \geqslant 0$ such that $U_m = U_n$. A component U is called periodic component of period n if we have $U_n = U$ for some $n > 0$. U is called invariant if $f(U) \subset U$. A component U is called a wandering domain if U is not preperiodic.

2.2.3 The Classification of Periodic Components

Let U be a periodic component of period p, then we have one of the following:

- U is attractive: each U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$.

- U is parabolic: the boundary of U contains a periodic point z_0 of period p. The multiplier of z equals $\exp(2\pi i p/q), (p,q) = 1$, and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$.

- U is a Siegel disk: there exists an holomorphic homeomorphism $\phi : U \to D$ where D is the unit disc such that $\phi(f^p(\phi^{-1}(z))) = \exp(2\pi i\theta)z$ for some irrational number θ .

- U is a Herman ring: there exists an holomorphic homeomorphism $\phi : U \to A$ where A is an annulus, $A = \{z : 1 < |z| < r\}, r > 1$ such that $\phi(f^p(\phi^{-1}(z))) = \exp(2\pi i\theta)z$ for some irrational number θ .

- U is an essentially parabolic or Baker domain: the boundary of U contains a point z_0, such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined.

Chapter 3

Dynamic Planes of S_λ and $\tilde{F}$

3.1 Basic Properties of S_λ

In this section we describe the basic properties of the dynamic plane of the family S_λ (see, [DK]), defined as those meromorphic functions which have two asymptotic values 0, λ, one of which is a pole, and no critical values. By Nevalinna theory (see, [Nevalinna]), these functions are characterized by the fact that they have constant Schwarzian derivative.

Proposition 3.1.1 *A meromorphic function f with two asymptotic values 0 and λ, and no critical values, has Schwarzian derivative equal to $2k$, and if, in addition, 0 is also a pole of f, then f has the following normal form:*

$$f(z) = S_\lambda = \lambda \frac{\exp(z)}{\exp(z) - \exp(-z)}.$$

Proof. From the fundamental theorem of Nevanlinna, (see, [DK],[Nevalinna].) a function with constant Schwarzian derivative $2k$ has the following form:

$$f(z) = \frac{Ae^{2kz} + B}{Ce^{2kz} + D}, \ A, B, C, D \in \mathbb{C}, \ AD - BC \neq 0$$

Its two asymptotic values are A/C and B/D. If one is 0, then either $A/C = 0$ or $B/D = 0$; we also have $C + D = 0$. Since 0 is to be a pole, f has either the form $f = \frac{\lambda}{e^{2z}-1}$ or $f_1 = \frac{\lambda e^{2z}}{e^{2z}-1}$. Note that f and f_1 are conjugate to each other by the map $z \to -z$.

□

For the next three propositions of this section, we assume that λ is real. The following results can be found in [DK].

Proposition 3.1.2 *When $\lambda > 0$, S_λ has the following mapping properties:*

- *S_λ preserves the real line, $S_\lambda(\mathbb{R}^+) = \mathbb{R}^+$ and $S_\lambda(\mathbb{R}^-) = \mathbb{R}^-$.*
- *For each integer k, S_λ maps the horizontal line $z = \frac{2k+1}{2}i$ onto the interval $(0, \lambda)$ in $\mathbb{R}$.*

Proposition 3.1.3 *S_λ has two fixed points, $x_1 < 0 < x_2$, on the real line. If $\lambda > 0$, then x_2 is an attracting fixed point. Moreover, if $\Re z > 0$, then $S_\lambda^n(z) \to x_2$ as $n \to \infty$. It follows that the Julia set $J(S_\lambda)$ is contained in the left half plane $\Re z \leqslant 0$.*

Proof. Let $f_\lambda(x) = S_\lambda(x) - x$, $x \in \mathbb{R}$. When $\lambda > 0$, we have $f_\lambda(x) \to +\infty$ as x approaches 0 from the right side, and $f_\lambda(x) \to -\infty$ as x approaches $+\infty$. Thus, there is a fixed point $x_2 > 0$ on the positive x axis. By a similar argument, we can show that there is a fixed point $x_1 < 0$ on the negative real axis. We can also show that $S'_\lambda(x_2) < 1$ and $|S'_\lambda(x_1)| > 1$ by a straightforward computation. In addition, S_λ maps each vertical line with $\Re z > 0$ to a circle which is contained in the plane $\Re z > \frac{\lambda}{2}$. Thus, by Schwarz lemma, $S^n_\lambda(z) \to z_2$ as $n \to \infty$.

□

Proposition 3.1.4 *The Julia set $J(S_\lambda)$ contains a Cantor bouquet.*

Proof. First we show that $J(S_\lambda)$ contains $\mathbb{R}^- \cup \{0\}$. Let $x < x_1$. It can be shown that $|(S^2_\lambda)'(x)| > 1$ by a straightforward computation. It can also be shown that $\{S^n_\lambda\}$ is not a normal family at x. Thus $J(S_\lambda)$ contains $(-\infty, x_1)$, and one branch of $S^{-1}_\lambda(-\infty, x_1]$ is $[x_1, 0)$, so $J(S_\lambda)$ contains $\mathbb{R}^- \cup \{0\}$ and all its preimages. Since $S_\lambda(t + n\pi i) = \mathbb{R}^- \cup \{0\}$, the preimages of $\mathbb{R}^- \cup \{0\}$ are countably many horizontal lines, $\{l_n = t + n\pi i, n \in \mathbb{Z}\}$. The preimages of each line, l_n, for a fixed n are countably many curves. Each curve is contained in the strip $H_{k_1} = \{s + k_1\pi i, s + (k_1 + 1)\pi i, s < 0\}$ and approaches ∞. We denoted the curve as α_{k_1}. For a fixed k_1, the preimages of the curve α_{k_1} are again countably many curves. Each curve lies in the strip H_{k_2} and approaches ∞. We denoted the curve as α_{k_1,k_2}. This process can

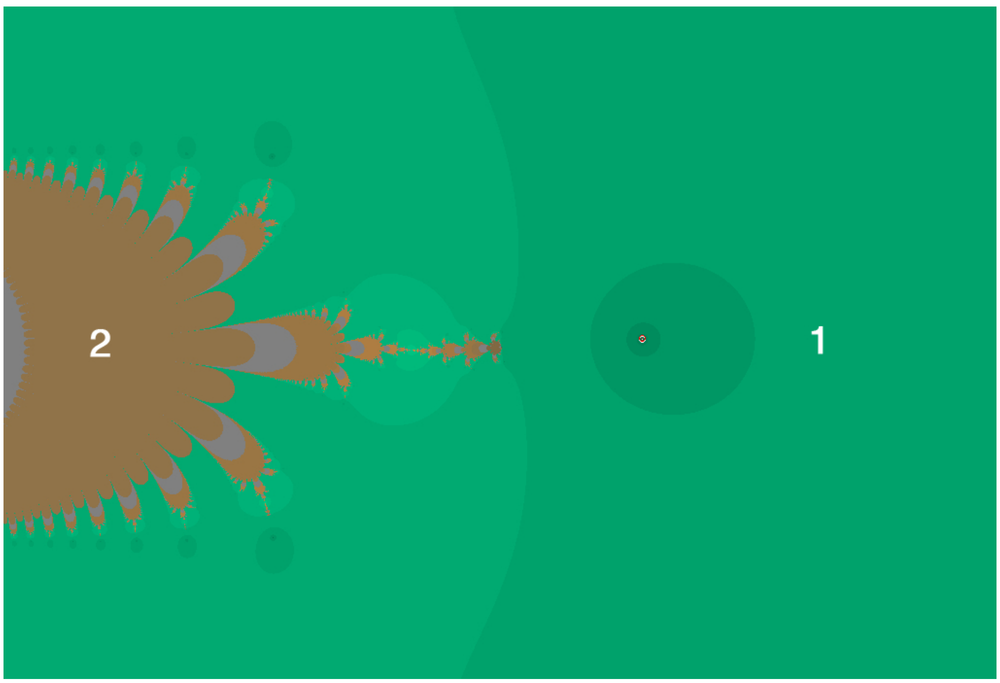

Figure 3.1 The Julia set of $S_\lambda(z)$ when $\lambda = 0.3828125 + 0.03125i$
The green region (1) is the Fatou set, which contains an attracting fixed point. The Julia set consists of the Cantor set of curves in the dark brown region (2) (Due to its complexity, a computational error makes it appear larger than it is).

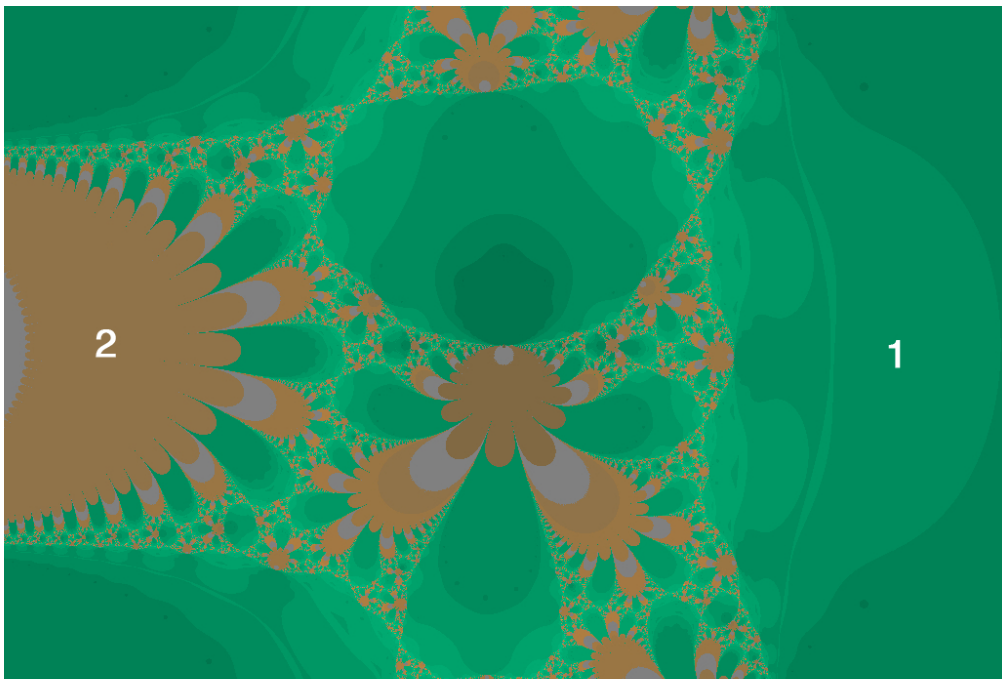

Figure 3.2 The Julia set of $S_\lambda(z)$ when $\lambda = 3.5i$
The green region (1) is the Fatou set, which contains an attracting periodic cycle of period 2. The Julia set consists of the Cantor set of curves in the dark brown region (2) (Due to its complexity, a computational error makes it appear larger than it is).

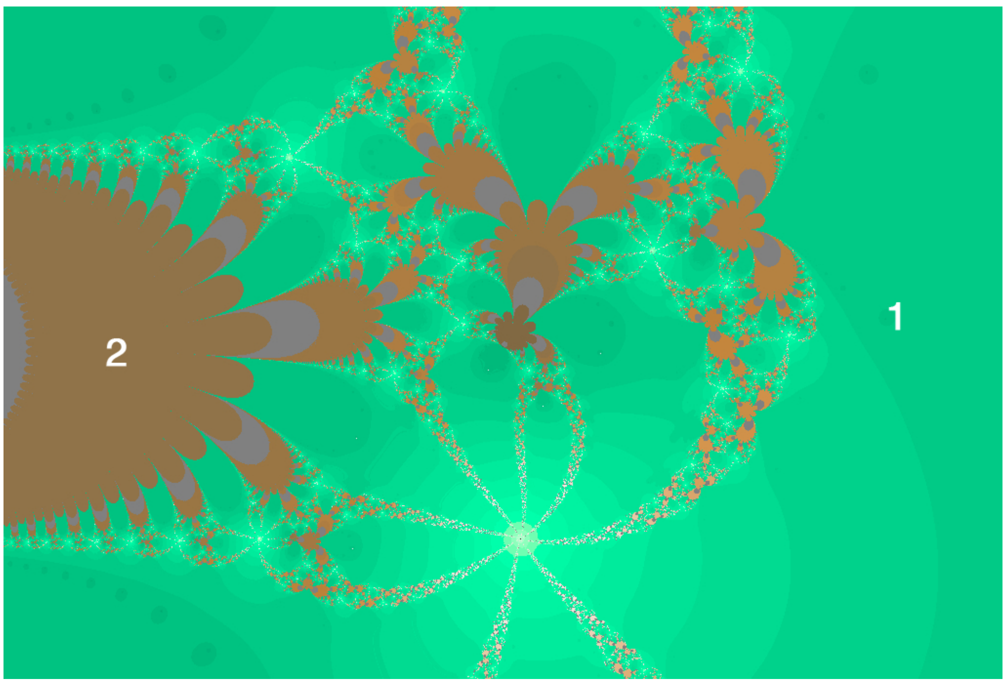

Figure 3.3 The Julia set of $S_\lambda(z)$ when $\lambda = -0.6 - 0.8071875i$
The green region (1) is the Fatou set, which contains an attracting periodic cycle of period 8. The Julia set consists of the Cantor set of curves in the dark brown region (2) (Due to its complexity, a computational error makes it appear larger than it is).

be continued for each $(k_1, k_2, \cdots)$. We can find a curve $\alpha_{k_1,k_2,\cdots}$. Thus, we find a Cantor set collection of curves in the Julia set which have limit on ∞. These curves form a Cantor bouquet.

□

The following propositions hold for any $\lambda \in \mathbb{C} \setminus \{0\}$:

Proposition 3.1.5 *The Fatou set of S_λ has at most one completely invariant domain. When $\lambda > 0$, the Fatou set of S_λ has one completely invariant domain.*

Proof. Let D be a completely invariant domain of the Fatou set of S_λ. Then D contains a fixed point in its interior or on its boundary that attracts a critical point or an asymptotic value of S_λ. Note that S_λ has only two asymptotic values, 0 and λ, and it has no critical values. Since 0 is also a pole, it is in the Julia set $J(S_\lambda)$. Thus only λ can belong the Fatou set of S_λ, and S_λ has at most one completely invariant domain.It can be derived directly from the above proposition that when $\lambda > 0$, the Fatou set of S_λ has a completely invariant domain, and hence exactly one.

□

Proposition 3.1.6 *Any neighborhood of* 0 *contains countably many nondegenerate continua in the Julia set $J(S_\lambda)$.*

Proof. If S_λ has an attracting fixed point, by the above proposition, we know that the Julia set contains all the preimages α_i of $\mathbb{R}^- \cup \{0\}$. We can pull back each α_i to 0. Since 0 is a pole, each of these pullbacks is in $J(S_\lambda)$ and is a continuum limiting on 0.

Suppose S_λ has an attracting periodic cycle with period $p > 1$. Let $U_0, U_1, \cdots, U_{p-1}$ be the immediate basin of the attracting cycle $z_0, z_1, \cdots, z_{p-1}$. Assume that U_0 contains the asymptotic value λ and that U_{p-1} is contained in the asymptotic tract of λ.That is, U_{p-1} is an unbounded component, and there is a continuum α in the Julia set approaching ∞. Pulling back α to 0, we find a continuum β_0 on the boundary of U_0 in the Julia set limiting on the pole 0. There are also countably many preimages of β_0, $\beta_i, i = 1, 2, \cdots$ which are contained in the Julia set and are inside the asymptotic tract of 0. Pulling back each β_i, we find countably many continua γ_i in the Julia set limiting on the pole 0.

□

Corollary 3.1.7 *No invariant component N of the Fatou set of S_λ can contain a curve γ whose winding number satisfies $n(\gamma, 0) \neq 0$ and lies in $\{z : |z| > s > 0\}$.*

Proof. This corollary follows directly from the fact that any neighborhood of 0 contains countably many nondegenerate continua in the Julia set. Thus, if γ is any curve in N that lies in $\{z : |z| > s\}$ for some

$s > 0$, the bounded component of its complement contains a continuum $\alpha \subset J(S_\lambda)$ limiting at 0. If we choose a repelling periodic point z_0 of period k on α and a subset $\alpha' \subset \alpha$ containing 0, then $S_\lambda^{2k}(\alpha')$ contains a continuum from z_0 to ∞ and must thus intersect γ. γ can not be contained in N.

□

Proposition 3.1.8 *There is no isolated Jordan arc in the Julia set of S_λ.*

By definition, a closed Jordan arc is called isolated in the Julia set if there exists an open set which contains the arc but no other points in the Julia set except for the end points of the arc.

Proof. Suppose that such an arc exists and is parametrized by $\gamma : [0,1] \to \mathbb{C}$. The repelling periodic points are dense in $J(S_\lambda)$. We may therefore assume that there is a repelling fixed point in the arc $z_1 = \gamma(t_1)$. In [DK] it was shown that the prepoles are dense in this arc too. Let us assume that p_0 is a prepole in this arc. Then there exists a k such that $S_\lambda^k(p_0)$ is a pole and $S_\lambda^{k+1}(p_0) = \infty$. Let U be an open set containing this arc but no other points in the Julia set. Then $S_\lambda^{k+1}(U)$ is contained in a neighborhood O of ∞ and one of the preimages of O is a neighborhood of 0. By the previous proposition, there are countably many continua limiting on 0 in the Julia set. Therefore there are countably many continua in the Julia set approaching ∞. Pulling back these continua to p_0, we find countably many continua in the Julia set limiting on p_0 and lying in U. This contradicts the fact that U contains no other points in the Julia set.

□

Proposition 3.1.9 *The Julia set of S_λ is connected and all the Fatou components of S_λ are simply connected.*

Proof. Assume that U is a Fatou component of S_λ which is multiply connected. It was shown in (see, e.g., [GK]) when f has finitely many singular values there is no wandering domain. Thus, U is a preperiodic component of the Fatou set. Let γ be an S_λ^n invariant, non-homotopically trivial curve in U, that is, $S_\lambda^n(\gamma) \subset U$. Since U is multiply connected, γ contains a preimage of a pole in the bounded component of its complement, D_γ. That is, some iterate $S_\lambda^k(\gamma)$ contains some pole p_k in $S_\lambda^k(D_\gamma)$. Assume that γ has been chosen to pass very close to the prepole such that the bounded component of the complement of $S_\lambda^{k+1}(\gamma)$, $D_{S_\lambda^{k+1}(\gamma)}$, contains both asymptotic values 0 and λ. By the previous proposition there are countably many continua α_i in the Julia set attached at 0. The image of each α_i must intersect $S_\lambda^{k+1}(\gamma)$, which contradicts the fact that γ is in the Fatou set which is invariant.

□

Corollary 3.1.10 *There is no totally disconnected invariant subset in $J(S_\lambda)$.*

Proof. That there is no totally disconnected invariant subset in $J(S_\lambda)$ is direct result of the above proposition.

□

3.2 The Dynamic Planes of the function $f \in \tilde{F}$

We let P be the family of transcendental meromorphic functions with exactly one pole, which is also an omitted value. It can be shown that if $f \in P$, then f can assume the form $f(z) = \frac{\exp(g(z))}{z^m}$ for some positive m and some entire function $g(z)$. Here, we may consider analytic self-maps of $\mathbb{C}^*$ without requiring 0 as a pole of a function (see, e.g.,[K1],[K2]).

We let $\tilde{F}$ be the family of meromorphic functions with finitely many singular values such that one of the asymptotic values is a pole. In addition, we exclude those functions with exactly one pole that is also an omitted value. That is, we assume that any $f \in \tilde{F}$, f has either at least two poles or f has only one pole which is not an omitted value. That is equivalent to saying that the backward orbit of ∞ is an infinite set. Note that the prepoles are dense in the Julia set of any function $f \in \tilde{F}$ (see,[5]). In this section we study the dynamics and the invariance and connectivity properties of the components of the Fatou set for these functions.

3.2.1 Completely Invariant Domains

In the paper [BKY] it was shown that there are at most two completely invariant domains for transcendental meromorphic functions with finitely many singular values. In [KK1] it was shown that when λ is real and $\lambda \geqslant 1$ for the family $\lambda \tan(z)$, there are two completely invariant domains, the upper half plane and the lower half plane. In the papers [Baker1] and [EL], it was shown that there is at most one completely invariant component of the Fatou set for any transcendental entire function. We will show here that for $f \in \tilde{F}$ there is also at most one completely invariant domain.

The following lemmas can be found in [BKY]:

Lemma 3.2.1 *If a transcendental meromorphic function, f, has finitely many singular values, and f has either at least two poles or has only one pole which is not an omitted value, and if N is a completely invariant component of the Fatou set of f, then $\partial N = J(f)$.*

Proof. Let z_0 be a point in $J(f)$ and let U be any neighborhood of z_0. The prepoles are dense in the Julia set of f (See, e.g.,[KK1], [BKY]). Therefore there exists a point $z_1 \in U$ and an integer n such that $f^n(z_1)$ is a pole of f and $f^{(n+1)}(U)$ contains a full neighborhood of ∞. By the complete invariance, $N \subset f^{-1}(N)$, and by Picard's theorem, N is unbounded. Therefore N meets $f^{n+1}(U)$ and we deduce $z_0 \in \partial N$.

□

Lemma 3.2.2 *Let f be a function as in the previous lemma. If there are two or more completely invariant components N_0, N_1, $\cdots$ of the Fatou set of f, then each of them is simply connected.*

Proof. Let N_0 and N_1 be completely invariant components of the Fatou set of f. By the previous lemma, $\partial N_0 = \partial N_1 = J(f)$. Suppose that N_0 is not simply connected. Then there exists a non-homotopically trivial Jordan curve γ in N_0. Let B_0 be the bounded component of the complement of γ. B_0 must contain a component α_1 of ∂N_0, and the complement of B_0 contains another component α_2 of ∂N_1. Since $\partial N_0 = \partial N_1$, $\alpha_1 \subset \partial N_1$ as well. We can find an open set $U \subset N_1$ containing points z_i with z_i converging to some point z^* in α_1. Similarly, we can find an open set, $V \subset N_1$, containing points z_i' with z_i' converging to some point $z^{*'}$ in α_2. Since N_1 is arcwise connected, we can find a curve δ in N_1 joining points z_i and z_j'. Then δ intersects γ, which implies that N_0 and N_1 are not distinct.

□

Lemma 3.2.3 *Suppose that f is a transcendental meromorphic function with finitely many singular values, and N is a simply connected completely invariant component of f. Then ∞ is an accessible point of ∂N.*

Proof. Since f has finitely many singular values, N contains a finite subset, S_0, of the singular values of f. Let D be a Jordan domain, bounded by the curve γ, such that $\overline{D} \subset N$ and $S_0 \subset D$. Let K be any component of $f^{-1}(N \setminus \overline{D})$. Then $f : K \to N \setminus \overline{D}$ is a covering. Since N is a completely invariant component of f, we have $K \subset N$. Since $N \setminus \overline{D}$ is doubly connected, K is either doubly connected or simply connected.

We show that K is simply connected. If K is doubly connected, both $\pi(N \setminus \overline{D})$ and $\pi(K)$ are infinite cyclic, and $f : K \to N \setminus \overline{D}$ is a finite to one covering map. One boundary component γ_1 of K is an inverse branch of the curve γ. If D contains an asymptotic value, then $\gamma_1 = f^{-1}(\gamma) \to \gamma$ is infinite to one map. Thus $f : K \to N \setminus \overline{D}$ can not be a finite to one covering map, and hence K can not be doubly connected. If there are no asymptotic values in D, then each component of $f^{-1}(\gamma)$ is bounded, since f is transcendental, there are infinitely many components. There are points belonging to the Julia set in each component of $f^{-1}(N \setminus \overline{D})$. Thus $N \setminus \overline{D}$ is infinitely connected. This contradicts the assumption that N is simply connected. Thus K cannot be doubly connected.

Now K is simply connected, and $\forall z \in N \setminus \overline{D}$, there is preimage of z in $K \subset N$. Let γ_0 be a simple closed curve in $N \setminus \overline{D}$ which separates the two boundary components of $N \setminus \overline{D}$. Let z_0 be a point in the curve γ_0, and let $\gamma_{z_1 z_2}$ be a preimage of γ_0, which starts at z_1 and ends at some point z_2 such that $f(z_1) = f(z_2) = z_0$. Lift γ_0 again to z_2 and get $\gamma_{z_2 z_3}$, which starts at z_2 and ends at some point z_3 such that $f(z_2) = f(z_3) = z_0$. Continuing this process, we get $\gamma_{z_n z_{n+1}}$, which begins at z_n and ends at z_{n+1}, with $f(z_n) = f(z_{n+1}) = z_0$. We claim that the curve $\Gamma = \gamma_{z_1 z_2}\gamma_{z_2 z_3}\cdots\gamma_{z_n z_{n+1}}\cdots$ in N joins z_1 to ∞. Since all the z_n are different, $\lim_{n\to\infty} z_n = \infty$. If not, let z_* be a limit point of z_n. Then for any neighborhood U of z_*, there are infinitely many z_n such that $f(z_n) = f(z_*) = z_0$. This in turn implies that f is a constant map in U, which is a contradiction. For a disc D_x with center $x \in \gamma_0$, such that $D_x \subset N \setminus \overline{D}$, all branches of f^{-1} are holomorphic in D_x and take disjoint values at x. The branches of f^{-1} form a normal family in D_x since they omit all values in $\overline{D}$. The subsequence which we write as f_n^{-1} such that $f_n^{-1}(x) \in \gamma_{z_n z_{n+1}}$, converges pointwise to ∞ and therefore uniformly to ∞ in the spherical metric. We can cover γ_0 with a finite number of discs D_x. For each we have $\gamma_{z_n z_{n+1}} \to \infty$, and the spherical length of $\gamma_{z_n z_{n+1}} \to 0$. Thus ∞ is accessible along γ.

□

For the rest of this section, we assume that f is a meromorphic transcendental function and that $f \in \tilde{F}$.

Theorem 3.2.4 *The Fatou set of $f \in \tilde{F}$ has at most one completely invariant domain.*

Proof. Suppose that the Fatou set of f has two completely invariant components N_1 and N_2. By the previous lemma, there is a curve Γ_1 in N_1 with $\Gamma_1 \to \infty$. Let p be an asymptotic value which is also a pole, $p \in \partial N_1$ and $p \in \partial N_2$. Then there is a preimage $\Gamma_1' = f^{-1}(\Gamma_1)$ which approaches p and lies in N_1. That is, p is accessible in N_1 along Γ_1.

Joining Γ_1 and Γ_1' in N_1, we get a curve γ_1 joining p to ∞. Similarly we can find a curve γ_2 in N_2 which joins p to ∞. $\Gamma = \gamma_1 \cup \gamma_2$ forms a Jordan curve which separates $\hat{\mathbb{C}}$ into two regions, D_1 and D_2. We choose points $r_1 \neq p$ and $r_1 \neq \infty$ in γ_1 and $r_2 \neq p$ and $r_2 \neq \infty$ in γ_2 and a curve in the region D_1 which connects r_1 and r_2. It must meet the Julia set. Thus there is a continuum $\beta_1 \subset J(f) \cap D_1$ which has one end at p and the other at ∞. Similarly, there is a continuum $\beta_2 \subset J(f) \cap D_2$ which has one end at p and the other at ∞.

Since p is an asymptotic value of f, let U be a neighborhood of p and $V = f^{-1}(U)$ be the asymptotic tract of p. Let $\beta_1^0 = \beta_1 \cap U$, then the preimages of β_1^0 in V are infinitely many continua, $\alpha_1, \alpha_2, \cdots$, extending to ∞ and belonging to the Julia set. They lie either in D_1 or D_2. Assume that D_1 contains infinitely many continua α_{i_1}, α_{i_2}, $\cdots$ in the Julia set approaching to ∞. We choose R to be a large enough positive number, with $D_R = \{z||z| > R\}$, such that each region A_j surrounded by curves α_j, α_{j+1} and D_R has a pre-image B_j inside U. Since $\partial N_1 = \partial N_2 = J(f)$, for every $j, j = 1, 2, \cdots$, the domains A_j and B_j belong alternately to N_1 and N_2. Assume that the domains A_1 and B_1 belong to N_1, then the domains A_2 and B_2 belong N_2 and the domains A_3 and B_3 belong to N_1, and so on. The domains A_j share the same boundary point at ∞, and the domains B_j share the boundary point at p. This contradicts the fact that N_1 and N_2 are simply connected.

□

3.2.2 The Connectivity of the Components of Fatou Sets

In this section, we prove that each component of the Fatou set of $f \in \tilde{F}$ is simply connected.

Proposition 3.2.5 *Let $f \in \tilde{F}$, and let p_0 be an asymptotic value of f which is also a pole, then any neighborhood of p_0 contains countably many nondegenerate continua in the Julia set.*

Proof. First we claim that if there is one nondegenerate continuum α belonging to the Julia set in a neighborhood U of p_0, then there are countably many continua β_i belonging to the Julia set in U. Since p_0 is an asymptotic value, the pullbacks of α are countably many continua α_i inside the asymptotic tract of p_0 approaching ∞. And since p_0 is also a pole, for each α_i, there is one pullback β_i belonging to the Julia set and limiting on p_0. Thus there are countably many nondegenerate continua belonging to the Julia set in U. There is no Baker domain for any function $f \in \tilde{F}$.

Case 1: If the Fatou set of f has only one component N, and then either N contains an attracting fixed point z_0 or ∂N contains a parabolic fixed point, N can not be a Baker domain since f has finitely many singular values (see,e.g. [4]). Suppose that there is no nondegenerate continuum belonging to the Julia set in a neighborhood U of p_0. Then there exists a non-homotopically trivial curve γ belonging to the Fatou set in U, and we may assume that γ passes very close to p_0 so that there exists $\gamma_1 \subset \gamma$ such that $f(\gamma_1)$ is inside the asymptotic tract of

p_0, considering the forward orbits of γ_1, $f^2(\gamma_1)$ and we can choose the sequence $\gamma^{(n)}$, the part of γ, such that $f^{2n+1}(\gamma^{(n)})$ is inside the asymptotic tract of p_0. Thus, there exists a limit point $z*$ on the γ such that there forward orbit of $z*$ is accumulating at ∞. This contradicts the fact that the forward orbit of γ approaches the attracting fix point z_0.

Case 2: If there are more than one components in the Fatou set. Thus there is a nondegenerate continuum α in the Julia set, and since the prepoles are dense in the Julia set, there is a prepole p in the α, and $f^n(p)$ is a pole. Thus $f^n(\alpha)$ is a continuum β approaching ∞, and there is a preimage of β, α', which is a nondegenerate continuum in the Julia set, limiting on the pole p_0.

□

Corollary 3.2.6 *If a meromorphic function $f \in \tilde{F}$, and p is an asymptotic value of f which is also a pole, then there is no invariant component N of the Fatou set of f that can contain a curve γ whose winding number satisfies $n(\gamma, p) \neq 0$, and which lies in $\{z : |z - p| > s > 0\}$.*

Proof. This follows directly from the previous proposition that any neighborhood of p contains countably many nondegenerate continua in the Julia set. Thus if γ is any curve in N that lies in $\{z : |z - p| > s\}$ for some $s > 0$, the bounded component of its complement contains a continuum α limiting at p in the Julia set. Since the repelling periodic points are dense in the Julia set, there exists some periodic point $z_0 \in \alpha$

with $f^n(z_0) = z_0$. Thus $f^{2n}(\alpha)$ contains the point z_0 and extends to ∞. Therefore it must intersect γ, γ can not be contained in N.

□

Corollary 3.2.7 *There is no isolated Jordan curve in the Julia set of f.*

Proof. Suppose that such an arc exists and is parametrized by $\gamma : [0,1] \to \mathbb{C}$. Since the repelling periodic points are dense in this arc (see, [4], [26]). We may therefore assume that there is a repelling fixed point in the arc $z_1 = \gamma(t_1)$. The prepoles are dense in this arc too (see, [4], [12],[13]). Let us assume that p is a prepole in this arc. Then there exists a k, $S_\lambda^k(p)$ is a pole and $S_\lambda^{k+1}(p) = \infty$. Let U be an open set such that it contains this arc but no other points in the Julia set. Then $S_\lambda^{k+1}(U)$ is contained in a neighborhood O of ∞ and one of the preimages of O is in a neighborhood of p_0. By the previous proposition, there are countably many continua limiting on p_0 in the Julia set. Therefore there are countably many continua in the Julia set approaching ∞. Pulling back these continua, we get countably many continua in the Julia set limiting on p and lying in U. This contradicts the fact that U contains no other points in the Julia set.

□

Theorem 3.2.8 *The Julia set of $J(f)$ is connected, and all Fatou components of f is simply connected.*

Proof. Assume that U is a Fatou component of f which is multiply connected. As we know, when f has finitely many singular values, there is no wandering domain. Thus, U is preperiodic component of Fatou set $F(f)$. Let γ be a f^n invariant non-homotopically trivial curve in U, that is, $f^n(\gamma) \subset U$. Since U is multiply connected, γ contains a preimage of a pole in the bounded component of its complement, D_γ. That is, some iterate $f^k(\gamma)$ contains some pole p_k in $f^k(D_\gamma)$. Assume that γ has been chosen to be very close to the prepole such that the bounded component of its complement $D_{f^{k+1}(\gamma)}$ of the $f^{k+1}(\gamma)$ contains all asymptotic values. Thus $f^{k+1}(\gamma)$ must have non-zero winding number with respect to the origin. Let p_0 be the asymptotic value which is also a pole. Let O_{p_0} be a small neighborhood of p_0 and $O_{p_0} \subset D_{f^{k+1}(\gamma)}$, and let α be a continuum attached to p_0 which is in the Julia set and let p_1 be a repelling fixed point in α. Let α_0 be the part of α which connects p_0 and p_1. The image of α_0 is a continuum has one end at p_1 and the other end at ∞. Thus it must intersect $f^{k+1}(\gamma)$, which contradicts the Fatou set is invariant.

□

Corollary 3.2.9 *If $f \in \tilde{F}$, then there is no totally disconnected invariant subset in $J(f)$.*

Chapter 4

Parameter Space of $\mathcal{S}_\lambda(z)$

In this chapter, we will describe the parameter space of $\mathcal{S}_\lambda$. Our description is analogous to the combinatorial description of the hyperbolic components of the tangent family. Like the tangent family (See, [KK1], [KK2], [KY]), functions in $\mathcal{S}_\lambda$ have no critical point, thus the hyperbolic components do not have centers at which the periodic cycle is superattracting. We will introduce a notion of the virtual center, a unique common point on the boundary of a pair of hyperbolic components. We describe how the deployment of hyperbolic components are related to the virtual centers.

4.1 The Tangent Family

In this section, we will summarize the properties of the hyperbolic components of the tangent family $\mathcal{T}_\lambda = \{\lambda \tan(z), \lambda \in \mathbb{C} \setminus \{0\}\}$. Each function in $\mathcal{T}_\lambda$ has two symmetric asymptotic values λ and $-\lambda$. There are two unbounded hyperbolic components, Ω_2 in the left half plane and Ω_1 in the right half plane. There are two attracting fixed points for any function in $\mathcal{T}_\lambda$ with $\lambda \in \Omega_1$; there is an attracting periodic cycle of period 2 for any function in $\mathcal{T}_\lambda$ with $\lambda \in \Omega_2$. There is a hyperbolic component which is the punctured unit disk $\Delta^* = \{\lambda, |\lambda| < 1\} \setminus \{0\}$, so that when $\lambda \in \Delta^*$, the function $\lambda \tan(z)$ has an attracting fixed point at 0 which attracts both asymptotic values, and its Julia set is a Cantor set.

4.1.1 The Hyperbolic Components of $\mathcal{T}_\lambda$

All hyperbolic components of $\mathcal{T}_\lambda$ are bounded except for Ω_1 and Ω_2. All hyperbolic components, except Δ^*, appear in pairs. Each component pair has a unique common point, which we call the virtual center of the component pair. The virtual center of the pair of unbounded components Ω_1 and Ω_2 is the point at infinity (See,[KK1] and [KK2]).

For each hyperbolic component Ω_p, we can define the eigenvalue map $m : \Omega_p \to \Delta^*$ by $\lambda \mapsto m_\lambda$, where m_λ is the multiplier of an attracting or neutral periodic cycle of T_λ containing the point z_λ, the attracting periodic point of the function $T_\lambda(z)$. For each $\alpha \in \mathbb{R}$, the *internal ray* $R(\alpha)$ is defined by $R(\alpha) = m^{-1}(re^{2\pi i\alpha}), 0 < r < 1$. The following results can be found in [KK1].

Proposition 4.1.1 *The eigenvalue map $m : \Delta^* \to \Delta^*$ is the identity. For each component Ω_p, except Δ^*, the eigenvalue map is an infinite degree regular covering map.*

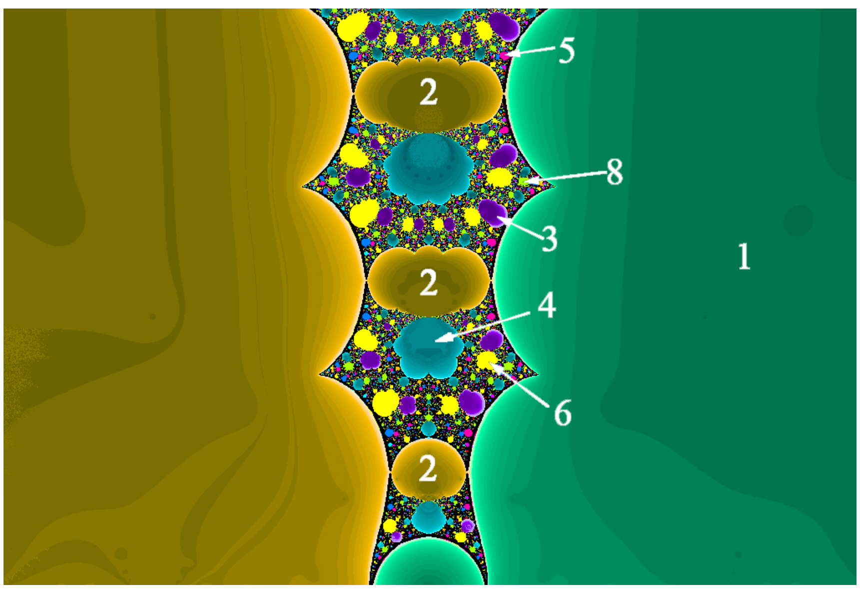

Figure 4.1 The parameter space of $\lambda \tan(z)$
Each color represents a component of parameters of the function that has an attracting periodic cycle of a certain order (as delineated by the numbers).

The eigenvalue map lifts to a conformal isomorphism from Ω_p to the upper half plane $\mathbb{H}$, $\tilde{m} : \Omega_p \to \mathbb{H}$ by

$$\tilde{m}(\lambda) \mapsto 2\pi \arg m(\lambda) - i \log |m(\lambda)|$$

where the branch of the logarithm is chosen so that the internal ray $R(0)$ is mapped to $-i\log(r)$.

Definition 2 *The boundary point corresponding to the point at infinity of the component Ω_p is called its virtual center. Recall that the hyperbolic components have no center since the functions in $\mathcal{T}_\lambda$ have no critical points.*

Proposition 4.1.3 *For any hyperbolic component Ω, there is a conformal homeomorphism $M : \Omega \to \mathbb{H}$ between Ω and the upper half plane $\mathbb{H}$. Moreover, M extends continuously to the boundary $\partial\Omega$ such that $i\infty$ is the unique boundary point in the upper half plane $\mathbb{H}$ corresponding to the virtual center λ^*. Thus, the virtual center is unique.*

Proposition 4.1.4 *Let Ω_p be a hyperbolic component which is not Δ^*. Then for $\lambda \in \Omega_p$, T_λ has either*

- *one attracting periodic cycle; both asymptotic values are in the immediate basin of the cycle and p is even, or*
- *two attracting periodic cycles of period p, symmetric with respect to the origin, both cycles have the same multiplier, and each has a basin that contains one of the asymptotic values.*

We use the notation Ω_p for the hyperbolic components with two distinct attracting cycles, and Ω'_p for the components with a single cycle of period $2p$.

Proposition 4.1.5 *For any bounded hyperbolic component Ω_p or Ω'_p, with $p > 1$, the virtual center λ^* is finite and $T_{\lambda^*}^{\circ p-1}(\lambda^* i) = \infty$, that is, $\lambda^* i$ is a prepole of order $p-1$.*

The following theorem from [KK1] gives a combinatorial description of the component pairs using the itineraries of the virtual centers.

Theorem 4.1.6 *Let $\lambda_0 i$ be a prepole of order $p-1$ with $T_\lambda^{\circ p-2}(\lambda_0 i) = s_n$, where $s_n = \frac{2n+1}{2}\pi$ is a pole of T_λ. Then λ_0 is the virtual center of a component pair (Ω_p, Ω'_p) and $\lambda_0 \in \partial\Omega_p \cap \partial\Omega'_p$, where T_λ with $\lambda \in \Omega_p$ has two attracting cycles of period p and T_λ with $\lambda \in \Omega'_p$ has one attracting cycle of period $2p$.*

4.1.2 Periodic Doubling

In this subsection we summarize the period doubling phenomena for the tangent family $\mathcal{T}_\lambda$. When λ is on the imaginary axis any function $T_\lambda = \lambda \tan(z)$ maps the imaginary axis to the real axis, and T_λ^2 maps the imaginary axis to the imaginary axis. Along the imaginary axis, there exists a period doubling phenomena analogous to the phenomena on the real axis for the quadratic family. Standard period doubling occurs at a parabolic point when an attracting cycle bifurcates to a cycle of double the period. Non-standard period doubling occurs at a parabolic

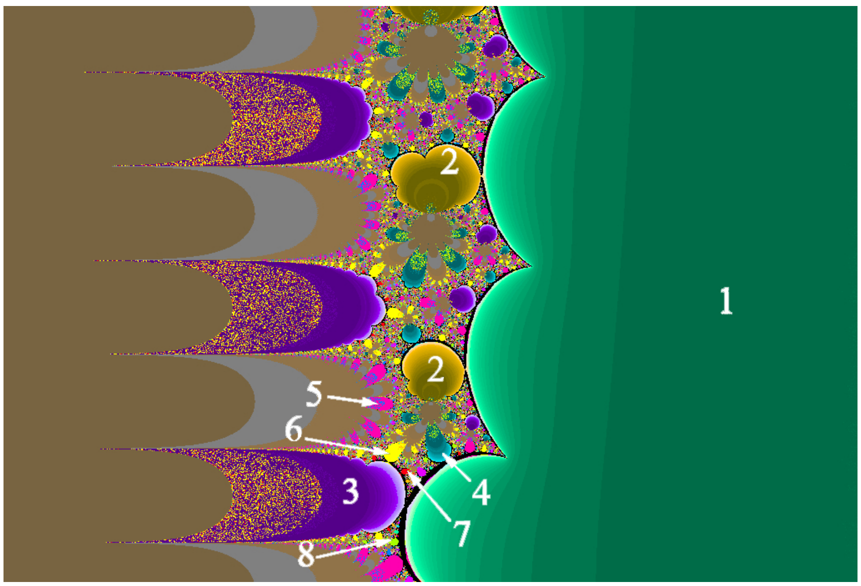

Figure 4.2 The parameter space of $S_\lambda(z)$
Each color represents a component of parameters of the function that has an attracting periodic cycle of a certain order (as delineated by the numbers).

point when an attracting periodic cycle bifurcates to two cycles of the same period. The following result can be found in [KK] and [KY].

Theorem 4.1.7 *Both standard and non-standard period doubling occur in the tangent family.*

4.2 The family $\mathcal{S}_\lambda$

In this section, We give the basic description of the parameter space of $\mathcal{S}_\lambda(z), \lambda \in \mathbb{C} \setminus \{0\}$. As for the tangent family, we let Ω_p be a hyperbolic component such that when $\lambda \in \Omega_p$, the function S_λ has an attracting periodic cycle of period p. There are bounded and unbounded hyperbolic components. There is an unbounded hyperbolic component Ω_1 in the right half plane such that when $\lambda \in \Omega_1$, the function S_λ has an attracting fixed point. There are unbounded components in the left half plane. As for the tangent family, we will give a description of bounded components using the virtual centers.

Definition 1 *Let m_λ denote the multiplier of an attracting or neutral periodic cycle of S_λ containing the point z_λ. If Ω_p is an arbitrary hyperbolic component of $\mathcal{H}$ and Δ^* is the unit disk punctured at the origin, the eigenvalue map $m : \Omega_p \to \Delta^*$ is defined by $\lambda \to m_\lambda$. For each $\alpha \in \mathbb{R}$ the internal ray $R(\alpha)$ is defined by $R(\alpha) = m^{-1}(re^{2\pi i\alpha}), 0 < r < 1$*

Definition 2 *The corresponding point $m^{-1}(re^{2\pi i\alpha})$ as $r \to 0$ in the Ω_p is called the virtual center.*

Proposition 4.2.3 *Given a hyperbolic component Ω_p. For any $\lambda \in \Omega_p$, let $z_i(\lambda)$ be an attracting periodic point of period p of the function S_λ. If there exists an $\epsilon > 0$ such that $|z_i(\lambda) - k\pi i| > \epsilon$ for all $\lambda \in \Omega_p, i = 0, 1, \cdots, p-1$, and all $k \in Z$, then Ω_p is an unbounded component.*

Proof. By the hypothesis, there exists an $R > 0$ such that $exp(z_i(\lambda)) < R$, for all $i = 0, 1, \cdots, p-1$. If not, there is an i and a sequence of $\lambda_j \in \Omega_p$ such that $\exp(z_i(\lambda_j)) \to \infty$ as $j \to \infty$. Thus, there exists a k_0, $z_{i-1}(\lambda_j) \to k_0\pi i$, which contradicts the assumption that there exists an $\epsilon > 0$ such that $|z_i(\lambda) - k\pi i| > \epsilon$ for all $\lambda \in \Omega_p, i = 0, 1, \cdots, p-1,$ and all $k \in Z$. Also $z_i > \epsilon$ which follows directly from the assumption when $k = 0$. Thus we have that $\frac{\exp(z_i)}{z_i}$ is bounded for every i.

Since $S'_\lambda(z) = \lambda\frac{-2}{(\exp(z)-\exp(-z))^2}$ and $\lambda \cdot \frac{\exp(z_{i-1})}{z_i} = \exp(z_{i-1}) - \exp(-z_{i-1})$, we find that the multiplier m_λ is:

$$m_\lambda = [S_\lambda^{\circ p}(z_0(\lambda))]' = \prod_{i=1}^{p} S'_\lambda[S_\lambda^{i-1}(z_0(\lambda))] = \prod_{i=1}^{p} \frac{-2}{\lambda\frac{exp(2z_{i-1})}{z_i^2}}.$$

Since $\frac{\exp(z_i)}{z_i}$ is bounded, the only way that $m_\lambda \to 0$ is that $\lambda \to \infty$; that is, Ω_p is unbounded.

□

Proposition 4.2.4 *For any bounded hyperbolic component Ω_p with $p > 1$, the virtual center λ^* is finite and $S_{\lambda^*}^{(p-1)}(\lambda^*) = \infty$; that is, λ is a prepole of order $p-1$.*

Proof. For $\lambda \in \Omega_p$, let $z_0 = z_0(\lambda)$ be the attracting periodic point of S_λ of period p belonging to the component N_0 of the Fatou set that contains the asymptotic tract and is such that both λ and z_1 are in

$N_1 = S_\lambda(N_0)$ and S_λ has an attracting cycle of period p. Denote the preimage of z_0 in the periodic cycle by z_{p-1} and the preimage of N_0 by N_{p-1}; then the map $S_\lambda : N_{p-1} \to N_0$ is bijective. Since N_0 contains the asymptotic tract and the map is bijective, there must be a unique pole, p_n, on ∂N_{p-1}. To see this, note that there is a preasymptotic tract at p_n in N_{p-1} containing a preimage of either $z = x$ or $z = -x$ for large $x > 0$. So if ∂N_{p-1} contained any other pole, there would be a preasymptotic tract in N_{p-1} of the same segment and $S|_{N_{p-1}}$ would not be injective.

Since $S'_\lambda = \lambda \frac{-2}{(\exp(z)-\exp(-z))^2}$ and $\lambda = z_i \cdot \frac{(\exp(z_{i-1})-\exp(-z_{i-1}))}{\exp(z_{i-1})}$, the multiplier m_λ can be written as

$$m_\lambda = [S_\lambda^{\circ p}(z_0(\lambda))]' = \prod_{i=1}^{p} S'_\lambda[S_\lambda^{i-1}(z_0(\lambda))] = \prod_{i=1}^{p} \frac{-2z_i}{\exp(2z_{i-1}) - 1}.$$

The only way some factor may tend to 0 as $\lambda \to \lambda^*$ is if for some i, $\exp(2z_i) - 1 \to \infty$, that is, for some i, $\Re z_i \to \infty$. Since z_0 is in the asymptotic tract, we conclude that $\Re z_0 \to \infty$. By hypothesis $p > 1$ and $\lambda^* \neq \infty$ so $z_{p-2} \neq z_{p-1}$. We have

$$\lim_{\lambda \to \lambda^*} \lambda \frac{\exp(z_{p-1}(\lambda))}{\exp(z_{p-1}(\lambda)) - \exp(-z_{p-1}(\lambda))} = \lim_{\lambda \to \lambda^*} z_0 = \infty.$$

We conclude that $\lim_{\lambda \to \lambda^*} z_{p-1}(\lambda) = p_n$.

□

The following propositions are standard adaptations of the arguments for the tangent family (See, [Jiang], [KK1]).

Proposition 4.2.5 *Let $q > 1$ and $\lambda \in \partial\Omega_p$ be such that the multiplier $m(\lambda)$ is a primitive q^{-th} root of unity. Then there is a "bud" component Ω_{pq} with an attracting cycle of order pq tangent to the component Ω_p at λ.*

Theorem 4.2.6 *For each pole $n\pi i$ and $n \neq 0$, there exists a λ close to $n\pi i$, such that S_λ has an attracting cycle of period 2 and there exists a λ' close to $n\pi i$, such that $S_{\lambda'}$ has an attracting cycle of period 4.*

Proof. Assume that ϵ is a small positive number and $\epsilon = |\lambda - n\pi i|$. Let us consider the ball $B_\lambda = B(\lambda, \epsilon)$. Choose R_λ so large that $\mathcal{A} = \{\Re z > R_\lambda > |\frac{1}{2}\log(\epsilon)|\}$ is an asymptotic tract of λ such that $n\pi i \in S_\lambda(\mathcal{A})$. Let $C = S^{-1}_{n,\lambda}(\mathcal{A})$ be a triangular region with a vertex at $n\pi i$. For $\lambda \in C$, set $D = \cup_{k\in\mathbb{Z}} S^{-1}_{k,\lambda}(C)$. To claim that $S^2_\lambda(D) \subset D$, we need the condition

$$\Re S_\lambda(\lambda) > R_\lambda.$$

Let $x_0 = \frac{1}{2}\log\epsilon$, then

$$\Re S^{-1}_\lambda(x_0) = \frac{1}{2}|\log(\frac{x_0}{x_0 - \lambda})| \approx \frac{\lambda}{2x_0} = \frac{\lambda}{|\log(\epsilon)|}.$$

Thus we have

$$|S^{-1}_\lambda(x_0) - n\pi i| \approx \frac{\lambda}{|\log(\epsilon)|},$$

and we can choose ϵ small enough such that $\frac{\lambda}{|\log(\epsilon)|} \geqslant \epsilon$. Therefore,

$$\epsilon = |\lambda - n\pi i| \leqslant |S_\lambda^{-1}(x_0) - n\pi i|$$

so that we have

$$\Re S_\lambda(\lambda) > x_0.$$

On the other hand, there are preimages $w_{k,\lambda} = S_{k,\lambda}^{-1}(n\pi i)$, with

$$\Re w_{k,\lambda} = \Re S_{k,\lambda}^{-1}(n\pi i) = \Re[\frac{1}{2}\log(\frac{\lambda+\epsilon}{\epsilon})].$$

Thus $\Re w_{k,\lambda} \approx \frac{1}{2}\log|\epsilon|$ Therefore we have

$$\Re S_\lambda(\lambda) > \Re w_{k,\lambda}.$$

Which proves that there exists λ, such that S_λ has an attracting cycle of period 2.

Let $\mathcal{A}^- = \{\Re z < -\frac{1}{2}\log|\epsilon|\}$ be an asymptotic tract of 0, and let $\mathcal{B}^+$ be an preasymptotic tract $S_{n,\lambda}^{-1}(\mathcal{A}^-)$, that is, an open set containing the pole $n\pi i$ on its boundary. Then

- $S_\lambda^{-1}(\mathcal{B}^+)$ is an asymptotic tract $\mathcal{A} = \{\Re z > r'\}$.
- $S_\lambda(\mathcal{A}^-)$ is an open set containing 0.
- $S_\lambda^2(\mathcal{A}^-)$ is an asymptotic tract $\mathcal{A} = \{\Re z > r"\}$

Let $D = S_{n,\lambda}^{-1}(\mathcal{B}^+)$. We need to show that $S_\lambda^4(D) \subset D$, that is, $\Re S_\lambda^3(\lambda) > \Re S_\lambda^{-1}(\lambda)$. As above, we have

$$\Re S_\lambda(\lambda) < \Re S_{n,\lambda}^{-1}(z)$$

and also we have

$$S_\lambda(z) > S_{n,\lambda}^{-1}(\lambda).$$

This proves that there exists λ', such that S'_λ has an attracting cycle of period 4.

□

We have the following immediate corollary.

Corollary 4.2.7 *For each pole $n\pi i$, $n \neq 0$, there is a hyperbolic component pair (Ω_2, Ω'_2). Moreover Ω_2 and Ω'_2 are tangent at the pole $n\pi i$.*

Furthermore, we have the following:

Proposition 4.2.8 *Choose a hyperbolic component pair (Ω_2, Ω'_2) attached at the pole $2n\pi i$, $n \neq 0$. For any $\lambda \in \Omega_2$, if the multiplier $m(\lambda)$ is real, then $m(\lambda) < 0$ and for any $\lambda' \in \Omega'_2$, if the multiplier $m(\lambda')$ is real, then $m(\lambda') > 0$. Therefore there exists a $\lambda_0 \in \partial\Omega_2$ with $m(\lambda_0) = -1$, such that there is a hyperbolic component Ω_4 tangent to Ω_2 at λ_0 so that for $\lambda \in \Omega_4$, S_λ has an attracting cycle of order 4.*

Choose a hyperbolic component pair (Ω_2, Ω'_2) attached at the pole $(2n+1)\pi i$, $n \neq 0$. For any $\lambda \in \Omega_2$, if the multiplier $m(\lambda)$ is real, then

$m(\lambda) > 0$ and for any $\lambda' \in \Omega_2'$, if the multiplier $m(\lambda')$ is real, then $m(\lambda') < 0$. Therefore there exists a $\lambda_0' \in \partial\Omega_2'$ with $m(\lambda_0') = -1$, such that there is a hyperbolic component Ω_8 tangent to Ω_2' at λ_0' so that for $\lambda \in \Omega_8$, S_λ has an attracting cycle of order 8.

Proof. Let L_λ be the set such that $m(\lambda)$ is real for any $\lambda \in L_\lambda$. Let $\lambda_1 = x_1 + y_1 i \in L_\lambda$ be a parameter which is close to the pole $n\pi i$, the common point of $\partial\Omega_2$ and $\partial\Omega_2'$. Thus x_1 is close to 0 and y_1 is close to $n\pi$. Let z_0 and z_1 be the periodic cycle of period 2 corresponding to S_{λ_1}. Then we have

$$m(\lambda_1) = \prod_{i=0}^{i=1} \frac{-2\lambda}{(\exp(z_i) - \exp(-z_i))^2} = \frac{(x_1^2 - y_1^2 + 2x_1 y_1 i)(c + di)}{c^2 + d^2}$$

where $c = \Re(\prod_{i=0}^{i=1}(\exp(z_i) - \exp(-z_i))^2)$ and $d = \Im(\prod_{i=0}^{i=1}(\exp(z_i) - \exp(-z_i))^2)$. Since $\Im m(\lambda_1) = 0$, we have

$$\frac{c}{d} = \frac{x_1^2 - y_1^2}{2x_1 y_1}.$$

Therefore,

$$m(\lambda_1) = \frac{(x_1^2 - y_1^2 - \frac{4x_1^2 y_1^2}{x_1^1 - y_1^2})c}{c^2 + d^2}.$$

So that $sign(m(\lambda_1)) = sign(-c)$.

Now we prove that when the virtual center is the pole $p_n = 2n\pi i$, we have $c > 0$, that is, we need to show that

$$\Re(\exp(z_0) - \exp(-z_0))^2 \cdot (\exp(z_1) - \exp(-z_1))^2) > 0.$$

Since z_0 is close to the pole and $\Re z_1$ is a large positive number, we have

$$\Re(\exp(z) - \exp(-z))^2 = \Re(\exp(2z) + \exp(-2z) - 2)$$

$$= \exp(2\Re z)\cos(2\Im z) + \exp(-2\Re z)\cos(2\Im z) - 2.$$

Since $\Re z_0$ is close to 0 and $\Im z_0$ is close to $2n\pi$, we have $\Re(\exp(z_0) - \exp(-z_0)^2) > 0$, and since $\Re z_1$ is a large positive number and $\Im z_1$ is close to $\frac{1}{2}\Im z_0$, we have $\Re(\exp(z_1) - \exp(-z_1)^2) > 0$. Thus $m(\lambda_1) < 0$. Since for all $\lambda \in L_\lambda$, $m(\lambda) \neq 0$, it follows that for all $\lambda \in L_\lambda$, $m(\lambda)$ have the same sign.

Using the same argument, we can show that when the virtual center $p_n = (2n+1)\pi i$, we have $c < 0$. Therefore, $m(\lambda) > 0$ when $\lambda \in L_{\lambda'}$. Using a similar method we can show that when $\lambda' \in L_{\lambda'}$ and the virtual center $p_n = 2n\pi i$, we have $m(\lambda') > 0$ and when the virtual center $p_n = (2n+1)\pi i$, we have $m(\lambda') < 0$.

□

In general, we have the following:

Theorem 4.2.9 *Let λ_0 be a prepole of order $p-1$ with $S^{\circ(p-2)}(\lambda_0) = p_n = n\pi i$. Then λ_0 is the virtual center of a hyperbolic component pair (Ω_p, Ω_{p+2}). That is, $\lambda_0 \in \partial\Omega_p \cap \partial\Omega_{p+2}$, and for any $\lambda \in \Omega_p$, S_λ has an attracting cycle of period p and for any $\lambda \in \Omega_{p+2}$, S_λ has an attracting cycle of period $p+2$.*

Proof. The proof is similar to the proof in [KK1] for the tangent family. We'll show that there exists a λ arbitrarily close to λ_0 such that S_λ has an attracting periodic cycle of period p. To construct the attracting periodic cycle, we want to find a λ and a domain D in the dynamic plane such that $S_\lambda^p(D) \subset D$.

Let $\mathcal{A}_\lambda = \{z|\Re(z) > R\}$ be an asymptotic tract of λ and $\mathcal{A}_0 = \{z|\Re(z) < -R\}$ be an asymptotic tract of 0, let $U_{\lambda_0} = S_{\lambda_0}(\mathcal{A}_\lambda)$, that is a neighborhood of λ_0 and let $U_0 = S_\lambda(\mathcal{A}_0)$, that is a neighborhood of 0. For $\lambda \in U_{\lambda_0}$, let $\mathcal{B}_n^+$ be the common preasymptotic tract:

$$\mathcal{B}_n^+ = \bigcap_{\lambda \in U_{\lambda_0}} S_{n,\lambda}^{-1}(\mathcal{A}_\lambda)$$

attached to the pole p_n. Let $\mathcal{B}_0^-$ be the common preasymptotic tract

$$\mathcal{B}_0^- = \bigcap_{\lambda \in U_{\lambda_0}} S_{0,\lambda}^{-1}(\mathcal{A}_0)$$

attached to the pole 0. For any $\lambda \in U_{\lambda_0}$, define the map $f : f(\lambda) = S_\lambda^{p-2}(\lambda)$. Then $f(U_{\lambda_0})$ is an open set containing the pole p_n and there exists open set $U_{\lambda_0}^+ \subset U_{\lambda_0}$ such that $U_{\lambda_0}^+ = f^{-1}(\mathcal{B}_n^+)$. For any $\lambda \in U_{\lambda_0}^+$, S_λ^{p-1} belongs to an asymptotic tract $\mathcal{A}' = \{\Re z > r'\}$ where possibly $r' < r$. Moreover, for inverse branches such that

$$S_{\lambda_0,\mathbf{n}_{p-2}}^{-(p-2)}(p_n) = \lambda_0$$

We have that $z_\lambda = S^{-(p-2)}_{\lambda,\mathbf{n}_{p-2}}(p_n) \neq \lambda_0$, so that there are preimages of z_λ in each strip $w_{k,\lambda}$ which depends on λ continuous and if $\lambda \to \lambda_0$, then $\Re w_{\lambda,k} \to \infty$ Let $\kappa_\lambda = |\lambda - z_\lambda|$ and consider the ball $B_\lambda = B(z_\lambda, \kappa_\lambda)$. Then $S^{p-2}_\lambda(B_\lambda)$ is an open set containing the pole p_n, Using the principal part of S_λ we see that $S^{p-1}_\lambda(B_\lambda) = \hat{\mathbb{C}} - B(0, R_\lambda)$ where R_λ is some positive real number.

Set $D = \cup_{k\in\mathbb{Z}} S^{-1}_{k,\lambda}(B^+_n)$, To claim that $S^p_\lambda(D) \subset D$, we need to show that $\Re S^{p-1}_\lambda(\lambda) > \Re w_{k,\lambda}$. Let

$$K = \max_{\lambda\in\overline{B_\lambda}, z\in\overline{B_\lambda}|(S^{p-2}_\lambda)'(z)|}$$

We have

$$\Re w_{k,\lambda} = \Re S^{-1}_\lambda(z_\lambda) \leqslant \Re S^{-1}_\lambda(\lambda + \kappa_\lambda) = \Re[\frac{1}{2}\log\frac{\lambda+\kappa_\lambda}{\kappa_\lambda}]$$

Thus, $\Re w_{k,\lambda} \approx \frac{1}{2}|\log|\kappa_\lambda||$.

$$|S^{p-2}_\lambda(\lambda) - p_n| \leqslant |\lambda - z_\lambda| K = K\kappa_\lambda$$

on the other hand, we have

$$\Re S^{p-1}_{n,\lambda}(\lambda) > \frac{1}{2}|\log|\kappa_\lambda||$$

Thus $\Re S^{p-1}_\lambda(\lambda) > \Re w_{k,\lambda} \approx \frac{1}{2}|\log|\kappa_\lambda||$. Now we can construct the suitable domain D such that $S^p_\lambda(D) \subset D$. Using a similar argument, we can construct a domain D' with $S^{p+2}_\lambda(D') \subset D'$.

□

The following proposition is similar result as in [KK1] for the tangent family.

Proposition 4.2.10 *Let λ_n be a virtual center such that $S_{\lambda_n}^{p-2}(\lambda_n) = n\pi i$. Then there is a sequence of component pairs $(\Omega_{p,k}\Omega_{p+2,k}), k \in \mathbb{Z}$, with centers λ_k where $S_{\lambda_k}^{p-1}(\lambda_k) = k\pi i$ and $\lambda_k \to \lambda_n$ as $k \to \infty$*

Proof. Let $D(\lambda_n, \epsilon)$ be an neighborhood of the asymptotic value λ of S_λ in the dynamic plane. Let $D'(\lambda_n, \epsilon)$ be the corresponding neighborhood in the parameter plane. Thus we have the map $g : D'(\lambda_n, \epsilon) \to \hat{C}$ with $g(\lambda) = S_\lambda^{(p-1)}(\lambda)$. Then $g(D'(\lambda_n, \epsilon))$ is a neighborhood of ∞, therefore for any large k, there exists λ_k such that $g(\lambda_k) = k\pi i$. Thus, by the pervious theorem λ_k is the virtual center of $(\Omega_{p,k}\Omega_{p+2,k})$; moreover, $\lambda_k \to \lambda_n$ as $k \to \infty$.

□

Chapter 5

The Dynamical Plane of $T_{a,\lambda}$

5.1 The Dynamical plane of $T_a(z)$ and $E(z)$

In this section we discuss the dynamical plane for the family $T_a(z)$ and for $E(z)$. These are the real slices of the families $T_{a,\lambda}$ and E_λ when $\lambda = 1$. Furthermore, we'll study the relation between the two families. First let us describe the Julia set of the family $T_a(z)$

Lemma 5.1.1 *For the family $T_a(z)$, when $0 < a < 1$, the function $T_a(z)$ has an attracting fixed point at $z = 0$ and two repelling fixed points on the real axis.*

When $a > 1$, there is a repelling fixed point at $z = 0$ and two attracting fixed points on the real axis with one in $a > Re(z) > \frac{1}{2}\log(a)$ and the other in $-1 < Re(z) < -\frac{1}{64}$.

When $a = 1$, there is a parabolic fixed point at $z = 0$.

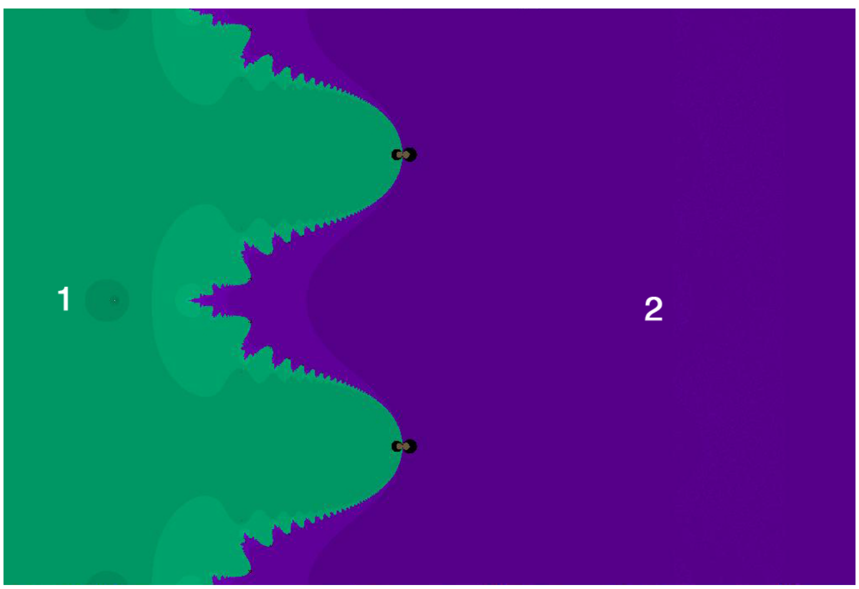

Figure 5.1 The Julia set of $T_{100,1}(z)$
The Julia set is the boundary curve of the green region (1) and the purple region (2).

Proof. First we observe that $z = 0$ is a fixed point of $T_a(z)$, for all a. By direct calculation, we see that when $0 < a < 1$,

$$|T_a'(0)| = \frac{2a}{a+1} < 1.$$

That is, $z = 0$ is an attracting fixed point. When $a > 1$,

$$|T_a'(0)| = \frac{2a}{a+1} > 1.$$

That is, $z = 0$ is a repelling fixed point.

When $a > 1$,

$$T_a(\frac{1}{2}\log(a)) - \frac{1}{2}\log(a) = \frac{a - 1 - \log(a)}{2} > 0$$

and

$$T_a(a) - a = \frac{a\exp(2a) - a}{\exp(2a) + a} - a = \frac{-a - a^2}{\exp(2a) + a} < 0.$$

Therefore, when $a > z > \frac{1}{2}\log(a)$, there exists a zero for the function $T_a(z) - z$, that is the fixed point z_0 of $T_a(z)$. Notice

$$|T_a'(z_0)| = |\frac{2a(1+a)}{(\exp(z_0) + a\exp(-z_0))^2}|$$

$$= |\frac{2a(a+1){z_0}^2}{a^2(\exp(z_0) - \exp(-z_0))^2}| = |\frac{2(a+1){z_0}^2}{(\exp(2z_0) + \exp(-2z_0) - 2)a}|$$

$$< |\frac{2(a+1){z_0}^2}{a(4{z_0}^2)}| < \frac{a+1}{2a} < 1.$$

Here we used the fact that $\exp(z_0)+a\exp(-z_0) = \frac{a(\exp(z_0)-\exp(-z_0))}{z_0}$ and $\exp(2z)+\exp(-2z) = 1+2z+\frac{(2z)^2}{2!}+\cdots+1-2z+\frac{(-2z)^2}{2!}+\cdots > 2+(2z)^2$. Thus, z_0 is an attracting fixed point. Using a similar argument, and by direct calculation, we have $T_a(-\frac{1}{64}) + \frac{1}{64} < 0$ and $T_a(-1)+1 > 0$. Therefore, when $-\frac{1}{64} > z > -1$, there exists a zero for the function $T_a(z) - z$, that is, a fixed point of $T_a(z)$. We can also estimate that $|T_a'(z)| < 1$, that is, it is an attracting fixed point.
When $a = 1$, we have

$$T_a'(0) = 1.$$

That is, 0 is a parabolic fixed point.

□

Proposition 5.1.2 *For the family $T_a(z)$, when $a > 1$, the Julia set is a curve bounded by the vertical lines l_1 and l_2, where l_1 is the line $x = 0$ and l_2 is the line $x = \frac{1}{2}\log(a)$. It passes through the points $z = n\pi i$ on l_1 and the points $z = \frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ on l_2, for all n. The Fatou set consists of two completely invariant components each of which contains one of the asymptotic values.*

When $a = 1$, the function $T_a(z) = \tanh(z)$ is conjugate to the tangent function $\tan(z)$ and the Julia set is the imaginary axis $x = 0$.

When $0 < a < 1$, the Fatou set consists of one completely invariant component containing both asymptotic values and the Julia set of $T_a(z)$ is Cantor set.

Proof. When $a > 1$, the function $T_a(z)$ maps the line l_1 into the left half plane H_L and maps H_L to a region to the left of $T_a(l_1)$. By Lemma 3.1, there is one attracting fixed point P_L in the left half plane and the other attracting fixed point P_R in the right half plane. By the Schwarz lemma, under iteration, the left half plane is attracted to P_L. The line l_2 is mapped to the line $x = \frac{a-1}{2}$ which is, since $a > 1$, at the right of the line $x = \frac{1}{2}\log(a)$. Again by the Schwarz lemma , under iteration, the region $Re(z) > \frac{1}{2}\log(a)$ is attracted to the attracting fixed point P_R. The other two cases follow the arguments in the paper [KK1].

□

It is well known that the Julia set of the exponential function $E(z)$ is a Cantor bouquet. We give a brief explanation of this fact. The left half plane $x < 0$ is mapped to the disk $|z + 1| < 1$ which eventually will be attracted to the attracting fixed point. The preimages of the left half plane form infinitely many finger-like regions. These infinitely many fingers are along the lines $x+\frac{1}{2}n\pi i$ for all n, one in each horizontal strip of width 2π. There are infinitely many narrower fingers which are mapped to each finger along the line $x + \frac{1}{2}n\pi i$. Thus the left half plane together with all the fingers are in the Fatou set. Their boundaries form a set of infinitely many curves all connected to ∞ in the Julia set.

We observe the following fact. When $a \to \infty$, the family $T_a(z)$ also converges uniformly on any compact subset to the exponential $E(z)$. Specifically, we have

Theorem 5.1.3 *Given any compact set $K \subset \mathbb{C}$, and any $\varepsilon > 0$, $\exists R > 0$ such that $\forall a > R$ and $z \in K$,*

$$|T_a(z) - (\exp(2z - 1))| < \varepsilon.$$

Proof. A simple calculation shows that for all z, we have,

$$|T_a(z) - (exp(2z) - 1)| = |\frac{\exp(2z) - \exp(4z)}{\exp(2z) + a}|$$

$$\leq \frac{|\exp(4z)| + |\exp(2z)|}{|\exp(2z) + a|}$$

We can find a large enough, such that

$$K \subset \{z | Rez < \frac{1}{5}\log(a)\}.$$

For $z \in K$, we have,

$$|T_a(z) - (\exp(2z) - 1)| \leq \frac{|\exp(2z)| + |\exp(4z)|}{a - |\exp(2z)|}$$

$$= \frac{\exp(Re(2z)) + \exp(Re(4z))}{a - \exp(Re(2z))} < \frac{a^{\frac{2}{5}} + a^{\frac{4}{5}}}{a - a^{\frac{2}{5}}}.$$

The right hand side approaches 0 as $a \to \infty$. Therefore $\forall \varepsilon > 0$, we can find $R > 0$ such that when $a > R$, $\frac{a^{\frac{2}{5}} + a^{\frac{4}{5}}}{a - a^{\frac{2}{5}}} < \varepsilon$.

□

In general, we have the following,

Theorem 5.1.4 *Given any compact set $K \subset \mathbb{C}$, given any $\lambda \neq 0$ and any $\varepsilon > 0$, $\exists R(\lambda) > 0$ such that $\forall a > R$ and $z \in K$,*

$$|T_{a,\lambda}(z) - E_\lambda(z)| < \varepsilon.$$

Remark 5 *In Section 5.2, we study the convergence of the family $T_a(z)$ to the function $E(z)$ in the dynamic plane. In the Section 5.3, we study the general case $T_{a,\lambda}$ and E_λ.*

5.2 Symbolic Dynamics for $T_{a,\lambda}$ and E_λ

5.2.1 Symbolic Dynamics for $T_{a,\lambda}$

Now let us study the inverse branches of the family $T_{a,\lambda}$. For any fixed real number a, the function $T_{a,\lambda}(z)$ is periodic with period π and so is an infinite to one cover of $\hat{C} - \{-\lambda, a\lambda\}$. The origin is the fixed point and the points $z = n\pi i$ are the pre-images of the origin. The poles are $\frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$. The image of any vertical line segment between $\frac{1}{2}\log(a) + \frac{2n-1}{2}\pi i$ and $\frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ is the vertical line $x = \frac{a-1}{2}$. We denote by l_n the horizontal line $l_n = x + \frac{2n+1}{2}\pi i, x \in R, n \in Z$. We denote by L_n the horizontal strip between the lines l_{n-1} and l_n. Then the function $T_{a,\lambda}$ maps each horizontal strip L_n onto $\hat{C} - \{-\lambda, a\lambda\}$.

The inverse of $T_{a,\lambda}$ is given by the following multivalued formula:

$$T_{a,\lambda}^{-1}(z) = \frac{1}{2}\log(\frac{a\lambda + az}{a\lambda - z})$$

Let $\lambda = \lambda_x + i\lambda_y$, $z = x + iy$; then

$$Re(T_{a,\lambda}^{-1}(z)) = \frac{1}{2}\log|\frac{a\lambda + az}{a\lambda - z}| =$$

$$\frac{1}{4}\log(\frac{(a^2|\lambda|^2 - a|z|^2 + (a^2-a)(\lambda_x x + \lambda_y y))^2 + (a^2+a)^2(y\lambda_x - x\lambda_y)^2}{((a\lambda_x - x)^2 + (a\lambda_y - y)^2)^2}),$$

$$Im(T_{a,\lambda}^{-1}(z)) = \frac{1}{2}\arctan(\frac{(a^2+a)(y\lambda_x - x\lambda_y)}{a^2|\lambda|^2 - a|z|^2 + (a^2-a)(\lambda_x x + \lambda_y y)}).$$

In the second formula we must specify which branch of the arctan we use. We therefore denote by $T_{n,a,\lambda}^{-1}$ the branch of the inverse whose image is in the strip L_n. For a given $p \in \mathbb{N}$ and any sequence $\mathbf{n}_p = (n_1, n_2, n_3, \cdots, n_p)$, we define a branch of $T_{a,\lambda}^{-p}$ by

$$T_{a,\mathbf{n}_p,\lambda}^{-p} = T_{a,n_p,\lambda}^{-1} \circ T_{a,n_{p-1}\lambda}^{-1} \circ \cdots \circ T_{a,n_1,\lambda}^{-1}.$$

Definition 1 *We call the sequence $\mathbf{n}_p$ the itinerary of the map $T_{a,\lambda}^{-p}$. We say an infinite sequence $\mathbf{n} = \{n_1, n_2, \cdots\}$ has bounded itinerary if there exists N, such that $|n_i| < N$ for all i.*

When $\lambda > \frac{a+1}{2a}$, we have that 0 is a repelling fixed point of $T_{a,\lambda}$. There are two attracting fixed points P_L in the left half plane and P_R in the right half plane. Using the same argument as in proposition 5.1.2, we know that the left half plane, $x < 0$, lies in the basin of P_L, and the right half plane, $x > \frac{1}{2}\log|a|$, lies in the basin of P_R. Now let us define the rectangles

$$R_n = \{z \in L_n | 0 \leq Re(z) < \frac{1}{2}\log(a)\},$$

and set

$$\mathbf{R}_N = \cup R_i, |i| < N.$$

For any n, the inverse map $T^{-1}_{a,\lambda,n}$ is well-defined and analytic on any compact subset of R_n and takes values strictly inside the strip R_n. Hence $T^{-1}_{a,n,\lambda}$ is a strict contraction in the Poincare metric on R_n. As a consequence, for any infinite bounded sequence $\mathbf{n}$ with $|n_i| < N$, we can define the following:

$$T^{-1}_{a,\mathbf{n},\lambda} = \cdots \circ T^{-1}_{a,n_p,\lambda} \circ T^{-1}_{a,n_{p-1},\lambda} \circ \cdots \circ T^{-1}_{a,n_1,\lambda}(z).$$

For any $z \in \mathbf{R}_N$, $T^{-1}_{a,\mathbf{n},\lambda}$ is well-defined on $\mathbf{R}_N$.

Proposition 5.2.2 *Let* $\mathbf{n} = n_1 n_2 \cdots$ *be any bounded infinite sequence with* $|n_i| < N$. *Then there are some limit points* $z_{a,\lambda}(\mathbf{n})$ *in* $\mathbf{R}_N$ *whose itinerary under* $T_{a,\lambda}$ *is* $\mathbf{n}$. *These points lie in the Julia set of* $T_{a,\lambda}$, *where* $\lambda > \frac{a+1}{2a}$. *Moreover, if* $\mathbf{n}$ *is a repeating sequence, then* $z_{a,\lambda}(\mathbf{n})$ *is a repelling periodic point.*

Proof. From the above discussion we know for any k and any sequence $\mathbf{n}_k = n_1 n_2 \cdots n_k$, we have that the inverse map $F_k = T^{-1}_{a,n_k,\lambda} \circ T^{-1}_{a,n_{p-1}\lambda} \circ \cdots \circ T^{-1}_{a,n_1,\lambda}(z)$ is a contraction map in the region $\mathbf{R}_N$. The maps F_k are uniformly bounded and thus form normal family in the region $\mathbf{R}_N$. Therefore there exist limit functions for any subsequence F_{k_i}. Any limit function must be a constant by the Schwarz Lemma. When $n_1 n_2 \cdots n_{n_0} n_1 n_2 \cdots n_{n_0}, \cdots$ is a repeating sequence, then the maps of the subsequence $F_{n_0}, F_{2n_0}, \cdots, F_{kn_0}, \cdots$ are contractions in the region R_{n_0}. Thus this subsequence converges to an attracting fixed point z_0 of $T_{a,\lambda}^{-n_0}$ in the region R_{n_0}. Thus z_0 is a repelling fixed point of $T_{a,\lambda}^{n_0}$ and a repelling periodic point of $T_{a,\lambda}$.

□

When $\lambda < \frac{a+1}{2a}$, 0 is the attracting fixed point of $T_{a,\lambda}$, and the real line is mapped to the interval $(-\lambda, a\lambda)$, which is attracted to the point 0. Since both asymptotic values are attracted to 0, the whole Fatou set is the basin of 0. The lines $x + n\pi i$ for all n are also mapped to the interval $(-\lambda, a\lambda)$. Thus these lines $l_n = x + n\pi i$ together with open strips L_n about them are in the basin of 0, that is, in the Fatou set. The far left half plane $H_L = \{z | Re(z) < -R\}$ is mapped to a small disk about $-\lambda$, so this half plane H_L lies in the basin of 0 as well. We can choose R large enough so that $T_{a,\lambda}(H_L) \subset L_0$. The far right half plane $H_R = \{z | Re(z) > R'\}$ is mapped to a small disk around the asymptotic value $a\lambda$. So this half plane also lies in the Fatou set. We can also choose R' large enough so that $T_{a,\lambda}(H_R) \subset L_0$. Let $\mathcal{O}$ be the union of these pieces of the basin. That is, $\mathcal{O} = \cup H_L \cup H_R \cup l_n$. So $T_{a,\lambda}(\mathcal{O}) \subset \mathcal{O}$. $\mathcal{O}$ is connected. Thus the complement of $\mathcal{O}$ consists of infinitely many closed, simply connected regions R_i. We have that $T_{a,\lambda}$

maps each R_i one to one onto $\hat{C} - T_{a,\lambda}(\mathcal{O})$, so $T_{a,\lambda}(R_i) \supset R_j \cup \{\infty\}$ for each j. There is a pole in each R_j. Thus for any given $\lambda \neq 0$ and $z \in J(T_{a,\lambda})$, we have that the orbit of z is contained in the $\cup R_i \cup \{\infty\}$ for all $i \in Z$. For any itinerary $\mathbf{n} = (n_1, n_2, \cdots)$, there exists at least one point z corresponding to the sequence $(n_1, n_2, \cdots)$. Thus we may use symbolic dynamics to associate to each $z \in J(T_{a,\lambda})$ an itinerary.

Definition 3 *For any $z \in J(T_{a,\lambda})$, we have one of the following two forms for the itinerary of z:*

$$n(z) = n_0 n_1 n_2 \cdots \ \textit{or} \ n(z) = n_0 n_1 n_2 \cdots n_{n-1} \infty.$$

Here each $n_i \in Z$ and $n_i = k$ if and only if $T^i_{a,\lambda}(z) \in R_k$. If z is a prepole, $T^n_{a,\lambda}(z) = \infty$ and $T^{n-1}_{a,\lambda}(z)$ is a pole; we associate the finite sequence $n_0 n_1 n_2 \cdots n_{n-1}\infty$ to z.

Thus for any given λ and any point $z \in J(T_{a,\lambda})$, let Λ denote the set of all possible such itineraries, consisting of $(n_0, n_1, n_2, \cdots)$ where $n_j \in \mathbb{Z}$ and all finite sequences of the form $(n_0, n_1, n_2, \cdots n_j, \infty)$. There is a topology on Λ originally defined by Moser [Mo], by taking the usual neighborhood basis about an infinite itinerary. For a finite itinerary $\mathbf{n} = (n_1, n_2, \cdot, n_j, \infty)$, we associate a neighborhood basis of $\mathbf{n}$ to all (finite or infinite) sequences $n_0, n_1, \cdots, n_j \tau \cdots$ where $|\tau| \geq K$ for some $K \in \mathbb{Z}^+$. In this topology, Λ is homeomorphic to a Cantor set. There is a natural map called the shift automorphism $\sigma : \Lambda \to \Lambda$ defined by $\sigma(n_0, n_1, n_2, \cdots) = (n_1, n_2, \cdots)$. Note that $\sigma(\infty)$ is not defined.

Let $\Lambda_{N,a,\lambda}$ be the set of all itineraries which remain in the region $\mathbf{R}_N$. Let Σ_N consist of all bounded itineraries $\mathbf{n} = (n_0, n_1, n_2, \cdots)$ with $|n_i| < N$. Then we have the same result for all maps in our family as for tangent and exponential maps, which were proven in [KK] and [Devaney] respectively. We also claim any point z in the Julia set is uniquely determined by its itinerary. Using an argument from [McMullen], we deduce that $z \in J(T_{a,\lambda})$ and ${T'_{a,\lambda}}^n(z) \to \infty$ if z is not a pole. At the poles, we have that $T'_{a,\lambda}(z) = \infty$. Therefore $T_{a,\lambda}$ is expanding on its Julia set. Since two points corresponding to the same itinerary must remain a bounded distance apart, it follows that two points in the Julia set cannot have the same itinerary. Thus, we have the following Theorem.

Theorem 5.2.4 *Suppose $N > 0$. For each $\lambda \neq 0 \in \mathbf{C}$, $\Lambda_{N,a,\lambda}$ is homeomorphic to Σ_N and $T_{a,\lambda}|\Lambda_{N,a,\lambda}$ is conjugate to the shift map on Σ_N.*

Thus for any given itinerary $\mathbf{n} \in \Sigma_N$, we can define $z_{\lambda,a}(\mathbf{n})$ be the unique point in $\Lambda_{N,a,\lambda}$ under this homemorphism.

5.2.2 Symbolic Dynamics for E_λ

Now we set up the combinatorics for E_λ.

For the exponential family E_λ (see, [Devaney]), we can define horizontal strips $R(k) = R_\lambda(k)$ by

$$R(k) = \{z \in \mathbb{C} | (k - \frac{1}{2})\pi < Imz < (k + \frac{1}{2})\pi\}.$$

For any given $\lambda \neq 0$, we may use symbolic dynamics to associate to each $z \in J(E_\lambda)$ an itinerary of the following form:

$$\mathbf{n}(z) = n_0, n_1, n_2, \cdots$$

where $n_i = k$ if and only if $E_\lambda^i(z) \in R(k)$.

Now let us define the rectangle

$$R_0^b(n_i) = \{z \in R_{n_i} | 0 \leqslant Rez \leqslant b\} \text{ and } R_{0,K}^b = \cup_{|n_i| \leqslant K} R_0^b(n_i).$$

Throughout this section, we fix b such that

$$|\lambda|(\exp(2b) - 1) > K + |\lambda|.$$

Our choice of b guarantees that when $|n_i| < K$, $E_\lambda(R_0^b(n_i))$ covers each $R_0^b(n_j)$ for each $|n_j| < K$. Let $\Lambda_{K,\lambda}$ be the set of points whose orbits remain in R_0^b.

Definition 5 *Σ_K is the set of itineraries $\{\mathbf{n} = (n_0, n_1, \cdots,)\}$ such that $|n_i| < K$ for all $i \in Z$.*

The following theorem is from [Devaney]

Theorem 5.2.6 *Suppose $K > 0$. For each $\lambda \in \mathbf{C}$, Σ_K is homeomorphic to $\Lambda_{K,\lambda}$ and $E_\lambda | \Lambda_{K,\lambda}$ is conjugate to the shift map on Σ_K.*

Thus for any given itinerary $\mathbf{n} \in \Sigma_K$, we can define $z_\lambda(\mathbf{n})$ to be the unique point in $\Lambda_{K,\lambda}$ under this homemorphism.

We have the following

Theorem 5.2.7 *For any given itinerary* $\mathbf{n} \in \Sigma_K$*, as* $a \to \infty$ *the corresponding points* $z_{\lambda,a}(\mathbf{n}) \to z_\lambda(\mathbf{n})$.

Proof. Let $\mathbf{n} = (n_0, n_1, n_2, \cdots)$ be any given bounded itinerary, and choose a $z \in R_T \cap R_b$. First, we claim that as $a \to \infty$, $T^{-1}_{a,\lambda,n_1}(z) \to E^{-1}_{\lambda,n_1}(z)$. This is clear since when $a \to \infty$, $T_{a,\lambda}(z)$ is uniformly convergent in the compact set to $E_\lambda(z)$. By induction, assume that when $i = k$, we have

$$T^{-1}_{a,\lambda,n_i} \circ \cdots \circ T^{-1}_{a,\lambda,n_1} \circ T^{-1}_{a,\lambda,n_0}(z) \to E^{-1}_{a,\lambda,n_i} \circ \cdots \circ E^{-1}_{\lambda,n_1} \circ E^{-1}_{\lambda,n_0}(z).$$

For any $\epsilon > 0$, there exists a R such that when $a > R$, $|z_{a,\lambda}(\mathbf{n_k}) - z_\lambda(\mathbf{n_k})| < \frac{1}{2}\epsilon$. When $i = k+1$, we have $T^{-1}_{a,\lambda,n_{k+1}} \to E^{-1}_{\lambda,n_{k+1}}$ and $|T^{-1}_{a,\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k})) - E^{-1}_{\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| < \frac{1}{2}\epsilon$ we can choose a large enough if necessary. Therefore $|T^{-1}_{a,\lambda,n_{k+1}}(z_{a,\lambda}(\mathbf{n_k})) - E^{-1}_{\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| < |T^{-1}_{a,\lambda,n_{k+1}}(z_{a,\lambda}(\mathbf{n_k})) - T^{-1}_{a,\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| + |T^{-1}_{a,\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k})) - E^{-1}_{\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| < \epsilon$.

Thus we have

$$\cdots \circ T^{-1}_{a,\lambda,n_2} \circ T^{-1}_{a,\lambda,n_1} \circ T^{-1}_{a,\lambda,n_0}(z) \to \cdots \circ E^{-1}_{a,\lambda,n_2} \circ E^{-1}_{\lambda,n_1} \circ E^{-1}_{\lambda,n_0}(z)$$

□

5.3 The Dynamic Planes of $T_{a,\lambda}$ and E_λ

Next we'll discuss the relation between Julia sets of the function $T_{a,\lambda}$ when $\lambda > \frac{1}{2}$ and the exponential function E_λ. As we stated before, the Julia set can be characterized in two ways: the closure of the pre-poles and the closure of the repelling fixed points. First, when for any given $\lambda > \frac{1}{2}$, there exists a large k, such that when $a > k$, $z = 0$ is the repelling fixed point of $T_{a,\lambda}(z)$. The imaginary axis contains the pre-images of $z = 0$. Now let us consider the pre-images of the imaginary axis. By direct calculation, the first pre-image is,

$$\exp(4Re(z)) + (a-1)\exp(2Re(z))\cos(2Im(z)) - a = 0.$$

The curve passes through the points $z = n\pi i$ for all $n \in Z$ and the poles $\frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ for all $n \in Z$. Let us denote this curve C_{T_1} and let $C_{T_1}^{i_1}$ be that part of the curve in R_i. That is,

$$C_{T_1}^{i_1} = \{\forall z \in C_{T_1}^{i_1} | z \in C_{T_1}, z \in R_{i_1}\}$$

Let $C_{T_2}^{i_1,i_2}$ be that part of the pre-image of the curve $C_{T_1}^{i_1}$ that lies in R_{i_2}. and let C_{T_2} be the pre-images of C_{T_1}, that is, $C_{T_2} = \cup C_{T_2}^{i_1,i_2}$ In general, we can follow this procedure; for each itinerary $\mathbf{i}_p = i_1, i_2, \cdots, i_p$, we can define

$$C_{T_p}^{i_1,i_2,\cdots,i_p} = T_{a,i_p}^{-1} \circ T_{a,i_{p-1}}^{-1} \circ \cdots \circ T_{a,i_1}^{-1}(z),$$

where $Re(z) = 0$ and $C_{T_p}^{i_1,i_2,\cdots,i_p} \in R_{i_p}$ and C_{T_p} is the pre-image of $C_{T_{p-1}}$.

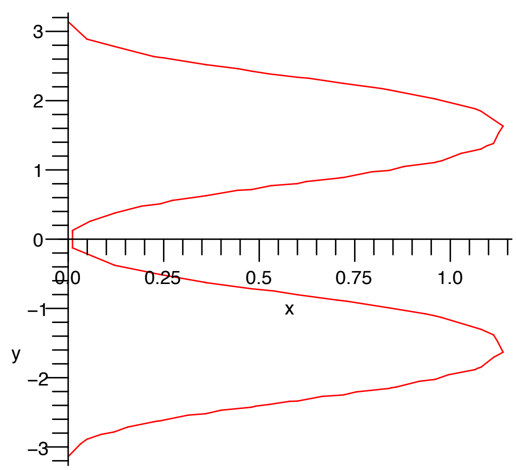

Figure 5.2 The inverse image of $T_{a,\lambda}$ when $a = 10$

By induction, we can show that the curve $C_{T_{k+1}}$ contains all of the pre-poles P_i up to $i = k$, where P_i is the pole of $T_a^i(z)$, and contains all of the pre-images of 0 under i^{th} iteration up to $i = k+1$. When $k \to \infty$, we claim that the curve $C_{T_k} = \cup C_{T_k}^{i_1,\cdots,i_k}$ converges to the Julia set of T_a. Clearly from the above discussion the limit curve C_{T_∞} contains the Julia set $J(T_a)$. On the other hand, assume that there is a point z is in the limit curve which is not a pre-pole. Then there exists a k and $z \in C_{T_k}$ with $T_a^k(z)$ on the imaginary axis. If $T_a^k(z)$ is not pre-image of 0, then $T_a^{k+1}(z)$ lies in the left half plane, and thus eventually maps to the attracting fixed point in the left half plane. Therefore, z is not in the curve $C_{T_{k+1}}$. This implies that z is not in the limit curve. Which is a contradiction. So, z must be in the Julia set.

We can use the same method to describe the Julia set of the exponential function $E(z)$. As above, we know that $z = 0$ is a repelling fixed point of $E(z)$ and the points $z = n\pi i$ are pre-images of $z = 0$. The pre-images of the imaginary axis will be the union of curves, denoted by $\cup_{i_1=-\infty}^{\infty} C_1^{i_1}$ where $C_1^{i_1}$ is in the region

$$L_{i_1} = \{\forall z \in L_{i_1} | \frac{2i_1 - 1}{2} < Im(z) < \frac{2i_1 + 1}{2}\}.$$

In general, we denote

$$C_p^{i_1,i_2,\cdots,i_p} = E_{i_p}^{-1} \circ E_{i_{p-1}}^{-1} \circ \cdots \circ E_{i_1}^{-1}(z),$$

where $Re(z) = 0$ and $C_p^{i_1,i_2,\cdots,i_p} \in L_{i_p}$. Again as above, we can show that the limit curves will be in the Julia set of $E(z)$, which we saw in previous section is Cantor bouquet.

Theorem 5.3.1 *When $a > \frac{1}{2}$, 0 is a repelling fixed point of $T_{a,1}(z)$. For any given itinerary $\mathbf{i}_k = i_1, i_2, \cdots, i_k$, as $a \to \infty$ the curve $C_{T_k}^{i_1,\cdots,i_k}$ of $T_a(z)$ converges to the curve $C_k^{i_1,\cdots,i_k}$, where $C_{T_k}^{i_1,\cdots,i_k}$ and $C_k^{i_1,\cdots,i_k}$ are defined as above.*

Proof. We prove this by induction. When $n = 1$, from the calculation as we stated above, the curve $C_{T_1}^{i_1}$ is

$$\exp(4Re(z)) + (a-1)\exp(2Re(z))\cos(2Im(z)) - a = 0$$

restricted in the region $\frac{2i_1-1}{2} < Im(z) < \frac{2i_1+1}{2}$, and the curve $C_1^{i_1}$ is

$$\exp(2Re(z))\cos(2Im(z)) - 1 = 0.$$

We have the curve

$$\exp(4Re(z)) + (a-1)\exp(2Re(z))\cos(2Im(z)) - a = 0.$$

Therefore

$$\cos(2Im(z)) = \frac{1 - \frac{1}{a}\exp(4Re(z))}{(1-\frac{1}{a})\exp(2Re(z))}$$

$$\to \frac{1}{\exp(2Re(z))} \text{ as } a \to \infty.$$

This shows $C_{T_1}^{i_1}$ goes to $C_1^{i_1}$

Assume that when $n = k$, the curve $C_{T_k}^{i_1,\cdots,i_k}$ approaches the curve $C_k^{i_1,\cdots,i_k}$. When $n = k+1$, $\forall z \in C_{T_{k+1}}^{i_1,\cdots,i_{k+1}}$, we have

$$T_a(z) = \zeta \in C_{T_k}^{i_1,\cdots,i_k}, \zeta = a\frac{\exp(2z) - 1}{\exp(2z) + a}.$$

We also have

$$z = \frac{1}{2}\log(\frac{a\zeta + a}{a - \zeta})$$

where log takes the branch of $\frac{2i_{k+1}-1}{2} < Im(z) < \frac{2i_{k+1}+1}{2}$.

We know as $a \to \infty$,

$$Re(T_a^{-1}(z)) = \frac{1}{2}\log|\frac{a + az}{a - z}| = \frac{1}{4}\log(\frac{(a^2 - a|z|^2 + (a^2 - a)x)^2 + (a^2 + a)^2y^2}{((a - x)^2 + y^2)^2})$$

$$= \frac{1}{4}\log(\frac{(1 - \frac{1}{a}|z|^2 + (1 - \frac{1}{a})x)^2 + (1 + \frac{1}{a})^2y^2}{((1 - \frac{1}{a}x)^2 + \frac{1}{a}y^2)^2})$$

$$\to \frac{1}{4}\log((1 + x)^2 + 1)^2 + y^2);$$

$$Im(T_a^{-1}(z)) = \frac{1}{2}\arctan(\frac{(a^2 + a)y}{a^2 - a|z|^2 + (a^2 - a)x})$$

$$\frac{1}{2}\arctan(\frac{(1 + \frac{1}{a})y}{1 - \frac{1}{a}|z|^2 + (1 - \frac{1}{a})x})$$

$$\to \frac{1}{2}\arctan(\frac{y}{1 + x}).$$

Now we observe the inverse of $E^{-1}(z) = \frac{1}{2}\log(1 + z)$. $Re(E^{-1}(z)) =$

$\frac{1}{4}\log((x^2+1)^2+y^2)$ and $Im(E^{-1}(z)) = \frac{1}{2}\arctan(\frac{y}{1+x})$. Therefore we have $\frac{1}{2}\log(\frac{a\zeta+a}{a-\zeta}) \to \frac{1}{2}\log(\zeta+1)$. By induction $\zeta \in C_{T_k}^{i_1,\cdots,i_k} \to$ some point $\zeta' \in C_k^{i_1,\cdots,i_k}$, and therefore, $\frac{1}{2}\log(\zeta+1)$ approaches $\frac{1}{2}\log(\zeta'+1)$ which is in the curve $C_{k+1}^{i_1,\cdots,i_{k+1}}$.

□

From the above theorem, we have the following as an immediate corollary:

Theorem 5.3.2 *As $a \to \infty$, Julia set of $T_a(z)$ converges pointwise to the Julia set of $E(z)$.*

For any $z \in J(T_{a,\lambda})$ which is not prepole, there is a corresponding unique infinite itinerary and as $a \to \infty$, $z \to z_0$ which is the the end point of $J(E_\lambda)$.

Chapter 6

Parameter Space of $\mathcal{T}_{a,\lambda}$

In this chapter, we will discuss the parameter plane of the family $\mathcal{T}_{a,\lambda}$ for a fixed real a. When a approaches ∞, we will describe how the parameter plane of $\mathcal{T}_{a,\lambda}$ relates to that of the exponential family. Set

$$\mathcal{E}_\lambda = \{E_\lambda(z) = \lambda \exp(2z) - \lambda, \lambda \in \mathbb{C} \setminus \{0\}\}.$$

Note that the exponential function $E_\lambda = \lambda \exp(2z) - \lambda$ we study here is conjugate to the exponential function $\lambda \exp(z)$. We have

$$L_{a,b}^{-1} \circ E_\lambda \circ L_{a,b} = \lambda' \exp(z)$$

where $L_{a,b} = \frac{1}{2}z - \lambda$ and $\lambda' = 2\lambda \exp(-2\lambda)$. Note that the map from λ to λ' is not one to one. The λ plane we have here is an ∞ to 1 covering of the λ' plane. See the pictures below. In λ' plane, there is a bounded heart-shaped component which represents parameters of the functions having an attracting fixed point, there is an unbounded light brown component which represents parameters of functions having an attracting periodic cycle of period 2 in the left half plane, there are countably many purple finger-like components which represent parameters of functions having an attracting periodic cycle of period 3, there are countably many finger-like components which represent parameters of functions having an attracting periodic cycle of period 4 between every two purple components, and so on. The bounded heart-shaped component in λ' plane corresponds to the circle and the unbounded region in the right-half plane in the λ plane. The unbounded component in the left half plane in the λ' space corresponds to the countably many finger like shaped components in the λ space. Each finger-like component in the λ' plane corresponds to countably many narrower finger-like components in the λ plane.

6.1 Parameter Plane of $\mathcal{E}_\lambda$

In this section, we will give an outline of the properties of the hyperbolic components of the family $\mathcal{E}_\lambda$, which is the direct corollary of the properties of the family $\{\lambda \exp(z)\}$.

Proposition 6.1.1 *When $|\lambda| < \frac{1}{2}$, the function E_λ has an attracting fixed point at 0. When λ is real and $\lambda > \frac{1}{2}$, the function E_λ has an attracting fixed point at some negative real number. When λ is real and $\lambda < -\frac{1}{2}$, the function E_λ has an attracting periodic cycle of period 2.*

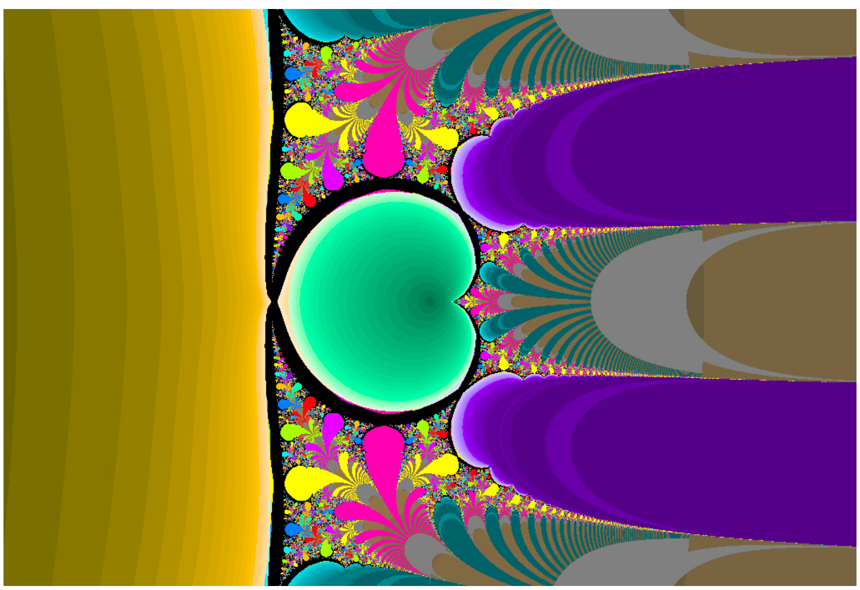

Figure 6.1 The parameter space of $E_\lambda = \lambda \exp(z)$

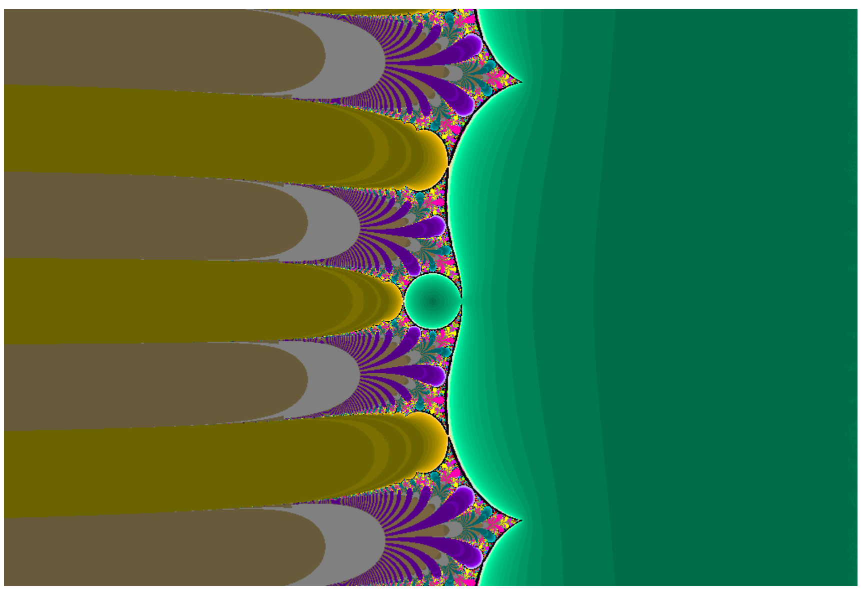

Figure 6.2 The parameter space of $\mathcal{E}_\lambda = \lambda \exp(2z) - \lambda$

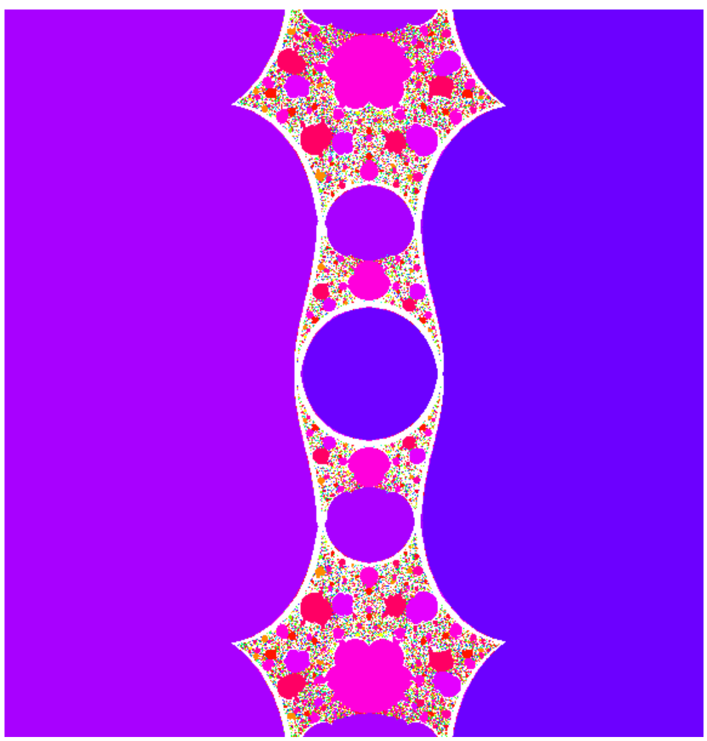

Figure 6.3 The parameter space of $T_{a,\lambda}$ when $a = 1.01$

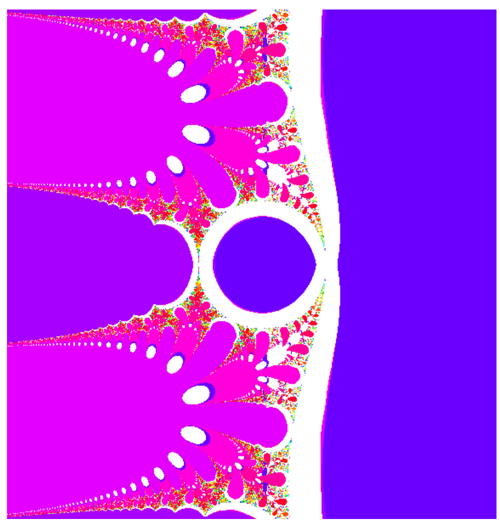

Figure 6.4 The parameter space of $T_{a,\lambda}$ when $a = 10000$

Proof. It is clear that 0 is a fixed point of E_λ and $E'_\lambda(0) = 2\lambda$. When $|\lambda| < \frac{1}{2}$, $|E'_\lambda(0)| < 1$. When $|\lambda| > \frac{1}{2}$, $|E'_\lambda(0)| > 1$. When $\lambda > \frac{1}{2}$, the function $g(x) = E_\lambda(x) - x \to -\infty$ as $x \to \infty$; also $g(-\lambda) = \lambda \exp(-2\lambda) > 0$. When $x < 0$ and x is very close to 0, $g(x) < 0$, $g(\frac{1}{2}-\lambda) = \lambda \exp(1-2\lambda) - \frac{1}{2} < 0$. Thus $g(x)$ has a fixed point x_0 between $\frac{1}{2} - \lambda$ and $-\lambda$ and $E'_\lambda(x_0) = 2\lambda \exp(2x_0) = 2(x_0 + \lambda) < 1$. That is, x_0 is an attracting fixed point. When $\lambda < -\frac{1}{2}$, we have $E^2_\lambda(\lambda) - \lambda < 0$ and $E^2_\lambda(-\frac{1}{2}\ln(\frac{|\lambda|}{n})) = \lambda \exp(-2n - 2\lambda) - \lambda > 0$, where we choose $n < |\lambda|$ and n is so close to λ such that $\lambda \exp(-2n - 2\lambda) - \lambda + \frac{1}{2}\ln(\frac{\lambda}{n}) > 0$ Thus there exists an attracting periodic cycle of period 2 between $-\frac{1}{2}\ln(\frac{|\lambda|}{n})$ and λ, and we can calculate that the multiplier of the periodic cycle less than 1.

□

It follows from the above proposition that there is a hyperbolic component $\Omega_1 = \{\lambda, |\lambda| < \frac{1}{2}\} \setminus \{0\}$ such that when $\lambda \in \Omega_1$, the function E_λ has an attracting fixed point at 0. There is a component Ω'_1 which lies in the right half plane $\Re z > \frac{1}{2}$ such that when $\lambda \in \Omega'_1$, the function E_λ has a repelling fixed point at 0 and a single attracting fixed point. There is a component Ω_2 which lies in the left half plane $\Re z < -\frac{1}{2}$ such that when $\lambda \in \Omega_2$, the function E_λ has a repelling fixed point at 0 and an attracting periodic cycle of period 2.

Proposition 6.1.2 *For each n there exists an unbounded hyperbolic component $\Omega_{3,n}$ which contains the line $x + \frac{2n+1}{2}\pi i$ for all large negative* *numbers x, such that when $\lambda \in \Omega_{3,n}$, the function E_λ has an attracting periodic cycle of period* 3.

In general, the following theorem regarding the existence of hyperbolic components of the family $\{\lambda \exp(z)\}$ can be found in several papers (see, e.g., [8], [11],[30])

Theorem 6.1.3 *For every $n \geqslant 3$, there exist countably many hyperbolic components in λ-plane in which every exponential function $\lambda \exp(z)$ has an attracting periodic cycle of period n. Moreover, there exist countably many hyperbolic components of period $n+1$ between every two hyperbolic components of period n.*

There exists an analytic curve tending to $-\infty$ such that along the curve the multipliers of the attracting orbit tend to 0.

As we've seen that λ plane of $\{\lambda \exp(z) - \lambda\}$ we have here is an ∞ to 1 covering of λ' plane of $\{\lambda' \exp(z)\}$, we have the following:

Corollary 6.1.4 *For every $n \geqslant 3$, there exist countably many hyperbolic components in λ-plane in which every exponential function $\lambda \exp(2z) - \lambda$ has an attracting periodic cycle of period n. Moreover, there exist countably many hyperbolic components of period $n+1$ between every two hyperbolic components of period n.*

There exists an analytic curve tending to $-\infty$ such that along the curve the multipliers of the attracting orbit tend to 0.

6.2 The Parameter Plane for $T_{a,\lambda}$

In this section we will discuss some properties of the parameter plane for the family $\mathcal{T}_{a,\lambda}$ and their relation to the exponential family E_λ. When $a = 1$, the family $\mathcal{T}_{a,\lambda}$ conjugates to the tangent family $\lambda \tan(z)$. This family has been studied by several authors (see,[Jiang], [KK1], [KK2],[KY]). The stable and unstable sets inherit symmetric properties from the symmetry of the tangent map. That is, $\lambda \tan(z) = -\lambda \tan(-z)$, and asymptotic values of $\lambda \tan(z)$, λ and $-\lambda$, are symmetric with respect to the origin. Either both asymptotic values are attracted to the same cycle, or are attracted to two symmetric cycles respectively. When $a \neq 1$, the family loses this symmetry property. When a is near 1, for each λ, the action of $T_{a,\lambda}$ is similar to the action of $\lambda \tan(z)$; that is, the dynamic and parameter planes of $\mathcal{T}_{a,\lambda}$ and the tangent family exhibit similarities. On the other hand when a is a large real number, the action of $T_{a,\lambda}$ is similar to the action of $\lambda \exp(2z) - \lambda$, and the dynamic and parameter planes of $\mathcal{T}_{a,\lambda}$ and the exponential family $\mathcal{E}_\lambda$ exhibit similarities.

More specifically,

Proposition 6.2.1 *For any given positive real a, and real λ such that $\lambda < \frac{a+1}{2a}$, any function $T_{a,\lambda}$ in the family $\mathcal{T}_{a,\lambda}$ has an attracting fixed point at $z = 0$. When $\lambda > \frac{a+1}{2a}$, the function $T_{a,\lambda}$ has a repelling fixed point at $z = 0$ and two attracting fixed points on the real line. When $\lambda < -\frac{a+1}{2a}$, the function $T_{a,\lambda}$ has a repelling fixed point at $z = 0$ and an* *attracting periodic cycle of period* 2 *on the real line.*

Proof. By direct calculation, we have $T_{a,\lambda}(0) = 0$ and when $\lambda < \frac{a+1}{2a}$ we have

$$|T'_{a,\lambda}(0)| = \frac{2a\lambda}{a+1} < 1.$$

When $\lambda > \frac{a+1}{2a}$ we have

$$|T'_{a,\lambda}(0)| = \frac{2a\lambda}{a+1} > 1,$$

and

$$T_{a,\lambda}(x) - x \to -\infty \text{ as } x \to \infty, T_{a,\lambda}(\frac{1}{2}) - \frac{1}{2} > 0.$$

Thus, $T_{a,\lambda}(z)$ has a fixed point x_1 in the positive real line $x > \frac{1}{2}$ and by direct calculation we can show that $T'_{a,\lambda}(x_1) < 1$. So that x_1 is an attracting fixed point. Using a similar argument, we can show that there exists an attracting fixed point x_2 in the negative real line $x < -\frac{1}{2}$. When $\lambda < -\frac{a+1}{2a}$, there are points x_1 in the positive real line and x_2 in the negative real line that form an attracting periodic cycle of period 2.

□

It can be derived directly from the above proposition that there is a hyperbolic component $\Omega_{1,T} = \{\lambda, |\lambda| < \frac{a+1}{2a} \setminus \{0\}\}$, such that when $\lambda \in \Omega_{1,T}$, the function $T_{a,\lambda}$ has an attracting fixed point at 0. There is a component $\Omega_{1,1,T}$ which lies in the right half plane $\Re z > \frac{a+1}{2a}$, such that when $\lambda \in \Omega_{1,1,T}$, the function $T_{a,\lambda}$ has a repelling fixed point at 0 and two attracting fixed points. There is a component $\Omega_{2,T}$ which lies in the left half plane $\Re z < -\frac{a+1}{2a}$, such that when $\lambda \in \Omega_{2,T}$, the function $T_{a,\lambda}$ has a repelling fixed point at 0 and an attracting periodic cycle of period 2.

Proposition 6.2.2 *Let a be a fixed large positive real number, and r be a real number such that $r > \frac{1}{2}$. For any λ such that $\Re(\lambda) > \max(r\log(a), \sqrt{a})$ and $n\pi - \frac{1}{4}\pi < \Im(\lambda) < n\pi + \frac{1}{4}\pi$ for each $n \in \mathbb{Z}$, the function $T_{a,\lambda}$ has two attracting fixed points.*

Proof. Let $D_{-\lambda,\epsilon}$ be a small disk with center $-\lambda$ and radius ϵ. We choose ϵ so small that $0 < |\epsilon| < \frac{1}{2}|(r-\frac{1}{2})\log(a)|$ and $n\pi - \frac{1}{4}\pi < |\Im(\lambda) - \Im(\epsilon)| < n\pi + \frac{1}{4}\pi$ for some $n \in \mathbb{Z}$ and also $a\exp(-a+|\epsilon|) < |\epsilon|$. We show that when $\Re(\lambda) > \max(r\log(a), \sqrt{a})$ and $n\pi - \frac{1}{4}\pi < \Im(\lambda) < n\pi + \frac{1}{4}\pi$ for some $n \in \mathbb{Z}$, we have

$$T_{a,\lambda}(D_{-\lambda,\epsilon}) \subset D_{-\lambda,\epsilon}.$$

For any $z \in D_{-\lambda,\epsilon}$, we have

$$|T_{a,\lambda}(z) + \lambda| = |\frac{a\lambda\exp(2z) + \lambda\exp(2z)}{\exp(2z) + a}|$$

$$< (\lambda + \frac{a}{\lambda})\Re(\exp(2z)) < 2\lambda\Re(\exp(2z)) < |\epsilon|.$$

Thus, there is an attracting fixed point in $D_{-\lambda,\epsilon}$.

Using a similar argument, we can show that there is an attracting fixed point near the asymptotic value $a\lambda$. Let $D_{a\lambda,\epsilon}$ be a small disk with center $a\lambda$ and radius ϵ, and $0 < \epsilon < \frac{1}{2}|(r - \frac{1}{2})\log(a)|$. We show that when $\Re(\lambda) > \max(r\log(a), \sqrt{a})$ and $n\pi - \frac{1}{4}\pi < \Im(\lambda) < n\pi + \frac{1}{4}\pi$ for each $n \in \mathbb{Z}$,

$$T_{a,\lambda}(D_{a\lambda,\epsilon}) \subset D_{a\lambda,\epsilon}.$$

For any $z \in D_{a\lambda,\epsilon}$, we have $|T_{a,\lambda}(z) - a\lambda| = |\frac{-a\lambda - a^2\lambda}{\exp(2z)+a}| < \epsilon$. Thus, there is an attracting fixed point in $D_{a\lambda,\epsilon}$.

□

Proposition 6.2.3 *When $\Re(\lambda) < -r\log(a)$ where $r < -\frac{1}{2}$, the function $T_{a,\lambda}$ has an attracting periodic cycle of period* 2.

Proof. Using a similar method to the previous theorem, we consider the neighborhoods of $-\lambda$ and $a\lambda$. Let $D_{-\lambda,\epsilon}$ be a neighborhood of the asymptotic value $-\lambda$; we can show that $T^2_{a,\lambda}(D_{-\lambda,\epsilon}) \subset D_{-\lambda,\epsilon}$.

□

Let a be a large enough positive number, and choose $r < \frac{1}{2}$. When $|\Re(\lambda)| < r\log(a)$, the function $T_{a,\lambda}$ behaves similarly to the exponential function E_λ. We have the following:

Proposition 6.2.4 *Let a be a fixed large positive number, and choose $r < \frac{1}{2}$. In addition, assume that $\Re(\lambda) < -r\log(a)$, and that the asymptotic value $-\lambda$ of the function $T_{a,\lambda}$ is attracted to some immediate basin* *of an attracting periodic cycle of period n. Then the other asymptotic value $a\lambda$ is attracted to the same cycle.*

Proof. Assume that the function $T_{a,\lambda}(z)$ has an attracting periodic cycle $z_0, z_1, \cdots, z_{n-1}$ of period n and the asymptotic value $-\lambda$ is attracted to this attractive cycle. Let D_i be the immediate basin containing z_i , $i = 0, 1, \cdots, n-1$ and suppose that D_0 contains $-\lambda$. Then there exists an $\epsilon > 0$ and $k > 0$, such that $T^k_{a,\lambda}(-\lambda)$ lies in the disk $D_{z_0,\epsilon}$ with center z_0 and radius ϵ. Let D_0^{-i}=$T^{-i}_{a,\lambda}(D_{z_0,\epsilon})$ be the inverse lying inside D_i where $i = i(k)$.

On the other hand, we can assume a is so large that

$$T_{a,\lambda}(a\lambda) = a\lambda\frac{\exp(2a\lambda) - 1}{\exp(2a\lambda) + a} \approx -\lambda.$$

In fact, we can choose it large enough such that $T_{a,\lambda}(a\lambda)$ lies in D_0^{-k}. Thus $T^k_{a,\lambda}(a\lambda)$ will be in the disk $D_{z_0,\epsilon}$.

□

Proposition 6.2.5 *When a is a large enough positive number, for each n, there exists a λ which is close to $-\frac{1}{2}\log(a) - \frac{2n+1}{2}\pi i$, such that the function $T_{a,\lambda}$ has an attracting periodic cycle of period 3. There also exists a λ which is close to $-\frac{1}{2}\log(a) - \frac{2n+1}{2}\pi i$, such that the function $T_{a,\lambda}$ has an attracting periodic cycle of period 2.*

Proof. Let $D_{a\lambda,\epsilon_1}$ be a disk with center $a\lambda$ and radius ϵ_1; let $D_{-\lambda,\epsilon_2}$ be a disk with center $-\lambda$ and radius ϵ_2; let $\mathcal{A}_{-\lambda}$ be an asymptotic tract of $-\lambda$; let $\mathcal{A}_{a\lambda}$ be an asymptotic tract of $a\lambda$.

Let $\mathcal{B}_{-\lambda}$ be the pre-asymptotic tract of $\mathcal{A}_{-\lambda}$, $T_{a,\lambda}(\mathcal{B}_{-\lambda}) \subset \mathcal{A}_{-\lambda}$, attached at the pole $p_n = \frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$; let $\mathcal{B}_{a\lambda}$ be the pre-asymptotic tract of $\mathcal{A}_{a\lambda}$, $T_{a,\lambda}(\mathcal{B}_{a\lambda}) \subset \mathcal{A}_{a\lambda}$, attached at the pole $p_n = \frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$. From the previous proposition, we can choose a large enough such that $T_{a,\lambda}(a\lambda)$ is in the disk $D_{-\lambda,\epsilon_2}$, let D_1 be $D_{-\lambda,\epsilon_2} \cap \mathcal{B}_{a\lambda}$. We can show that for a large enough a, we have $T^3_{a,\lambda}(D_1) \subset D_1$. Because we have $T_{a,\lambda}(D_1) \subset \mathcal{A}_{a\lambda}$, we can choose a large enough such that $a\lambda$ is in the asymptotic tract $\mathcal{A}_{-\lambda}$. Thus, $T^{-2}_{a,\lambda}(D_1)$ contains $\mathcal{A}_{a\lambda}$. Therefore, the function $T_{a,\lambda}$ has an attracting periodic cycle of period 3.

Let D_2 be $D_{-\lambda,\epsilon_2} \cap \mathcal{B}_{-\lambda}$. Similarly, we can show that $T^2_{a,\lambda}(D_2) \subset D_2$. Thus the function $T_{a,\lambda}$ has an attracting periodic cycle of period 2.

□

In general, when a is a large positive number, we can show that there exist hyperbolic components of period n for any n.

Theorem 6.2.6 *For a fixed large enough positive number $a = a(n)$, there exists a hyperbolic component pair $(\Omega_{n+2}, \Omega_{n+3})$ near p_n in the λ plane, where p_n is the complex number such that $T^n_{a,p_n}(p_n) = -\frac{1}{2}\log(a) - \frac{2k+1}{2}\pi i$ for some k. When $\lambda \in \Omega_{n+2}$, the function $T_{a,\lambda}$ has an attracting periodic cycle of period $n+2$. When $\lambda \in \Omega_{n+3}$ the function $T_{a,\lambda}$ has an attracting periodic cycle of period $n+3$.*

Proof. Using a similar method to the previous proposition, we choose a neighborhood, D_0, of p_n, and a neighborhood, $T^n_{a,\lambda}(D_0)$, of the pole, $-\frac{1}{2}\log(a) - \frac{2k+1}{2}\pi i$. We can select a suitable subset of D_0 such that $T^{n+1}_{a,\lambda}(D_0)$ is inside an asymptotic tract of the asymptotic value $-\lambda$. For a large enough a, we can choose D', a subset of D_0, such that $T^{n+2}_{a,\lambda}(D') \subset D'$.

We can also choose a subset D'' of D_0 such that $T^{n+1}_{a,\lambda}(D'')$ is inside an asymptotic tract of $a\lambda$ and $T^{n+2}_{a,\lambda}(D'')$ is in a neighborhood of $a\lambda$. We can choose a large if necessary so that a neighborhood of $a\lambda$ is mapped to a neighborhood of $-\lambda$.

□

The following proposition describes the relationship between the hyperbolic component of the family $\mathcal{E}_\lambda$ and the family $\mathcal{T}_{a,\lambda}$ when a is a large positive real number.

Proposition 6.2.7 *For any λ_0 in a hyperbolic component of $\mathcal{E}_\lambda$ such that the function E_{λ_0} has an attracting periodic cycle of period n, there exists a sequence of pair (a_i, λ_{a_i}) with $a_i \to \infty$, $\lambda_{a_i} \to \lambda_0$ such that the function $T_{a_i,\lambda_{a_i}}$ has an attracting periodic cycle of period n.*

Proof. Let λ_0 be a parameter such that the function $E_{\lambda_0} = \lambda_0 \exp(2z) - \lambda_0$ has an attracting periodic cycle of period n; that is, there exists a z_0 such that $E^n_{\lambda_0}(z_0) = z_0$. Let $D_{z_0,\epsilon}$ be a disk with center z_0 and radius ϵ. We can choose ϵ so small that $|E^n_{\lambda_0}(D_{z_0,\epsilon}) - z_0| = \epsilon_1 < \epsilon$.

By theorem 5.1.4, there exists an a_k such that

$$|E^n_{\lambda_0}(z) - T^n_{a_k,\lambda_{a_k}}(z)| < \epsilon_k < |\epsilon - \epsilon_1|.$$

For any $z \in D_{z_0,\epsilon}$, we have

$$|T_{a_k,\lambda_{a_k}}(z) - z_0| \leqslant |T_{a_k,\lambda_{a_k}}(z) - E^n_{\lambda_0}(z)| + |E^n_{\lambda_0}(z) - z_0| \leqslant |\epsilon - \epsilon_1| + \epsilon_1 < \epsilon.$$

This implies that $T_{a_k,\lambda_{z_k}}(D_{z_0,\epsilon}) \subset D_{z_0,\epsilon}$. Therefore, the function $T_{a_k,\lambda_{a_k}}$ has an attracting periodic cycle of period n.

□

Chapter 7

Open Questions

7.1 Open Questions and Future Work

In this chapter we will discuss some open questions and future work.

Unanswered questions regarding the parameter space of S_λ include:

- Are there unbounded hyperbolic components with any given order?
- Is there a curve such that a periodic doubling phenomenon occurs along the curve?
- Can we characterize the boundary of the hyperbolic components of S_λ?

Unanswered questions regarding $T_{a,\lambda}$ include:

In the papers [KK] and [KY], it was shown that for the tangent family $\lambda tanz$, which is conjugate to $T_{1,\lambda}$, there are infinite cascades of non-standard period doubling bifurcation along the imaginary axis.

- A natural question arises: Is there a similar phenomenon for the family $T_{a,\lambda}$ in general? That is, when $a \neq 1$, is there a curve such that a period doubling occurs along the curve?

- For the tangent family $\lambda tanz$, which is conjugate to $T_{1,\lambda}$, both asymptotic values are either attracted to the same cycle or attracted to two symmetric cycles. When $a \neq 1$ for certain a, the asymptotic values λ and $a\lambda$ may be attracted to two cycles with two attracting periodic cycles each with a different period. Richard Oudkerk showed that when one asymptotic value is an attracting fixed point, the other asymptotic value could be attracted to an attracting periodic cycle with any period. We can then ask the following questions:

- Is there a condition for a such that two asymptotic values are attracted to two attracting periodic cycles, each with a different period?

- If one asymptotic value is attracted to an attracting periodic cycle with a certain period, can the other asymptotic value be attracted to an attracting periodic cycle with any period?

- We proved that the family $T_{a,\lambda}$ has dynamical convergence to the exponential family E_λ when $a \to \infty$ along the positive real axis. How does the corresponding dynamic plane change when a is not real and a approaches ∞ along certain curves?

There are also questions regarding more general families of functions. For example:

- We studied transcendental meromorphic functions in the family $\tilde{F}$ with finitely many singular values such that one of the asymptotic values is a pole. We proved that the Fatou set of any function in $\tilde{F}$ is simply connected. What can we say about the dynamic plane of transcendental meromorphic functions with finitely many singular values such that one of the asymptotic values is a prepole or a critical value is a pole?

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