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by

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Abstract

TECHNIQUES OF FEEDBACK SUB-OPTIMAL CONTROL

by

Chanbin Park

Advisors: Prof. George M. Kranc and Prof. Frederick E.Thau

A design procedure is developed for obtaining sub-optimal feedback control laws.

The approach taken is based on the results of the L-problem in functional analysis. The new design procedure requires the solution of a set of linear-algebraic equations. An estimate of the error resulting from use of the sub-optimal control law is also derived. This design approach differs from preveous sub-optimal control techniques in that there is no need to solve a Riccati matrix equation.

Four examples (1) two integrator problem(minimizing maximum amplitude of control) (2) Bushaw's problem (3) three integrator problem (minimum time problem with hard constraint on control) are presented to demonstrate the effectiveness of the design procedure. Analytic sub-optimal control laws are derived and the performance of the resulting closed-loop systems is simulated using a digital computer program.

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I. INTRODUCTION

1.1 Motivation and Summary of Results

During the period of the last 15 years, several general mathematical methods[3,5,27] for optimal control problems have been presented. Except for very simple cases all these methods lead to complicated computational procedures. Also in many cases the solution is obtained as an open-loop solution, that is, it is valid only for a single specified initial condition and the presence of deviations from a pre-calculated trajectory is often ignored. One could of course precalculate optimal solutions for all possible initial conditions and thus theoretically speaking, implement the feedback structure for the optimal control problems. This procedure would lead, however, to prohibitive requirements on the storage space of the control computer and it is therefore unrealistic.

For this reason much research[6,9,10,14,15,16,26] in recent years has been devoted towards the development of practical feedback control solutions often called suboptimal or quasi-optimal, where the objective is to obtain a relatively simple, cheap feedback control law which yields a performance criterion "close" to the optimal one and which can be easily implemented.

These techniques have certain common characteristics: first, the proposed suboptimal feedback solutions are assumed to be operating in the presence of "small deviations"; second their theoretical development is based on the Pontryagin Maximum Principle[27] and therefore, it is not surprising that the resulting computational problem is that of a two-point boundary-value problem and involves solution of a Riccati matrix differential equation; third either some form of the quadratic criterion is used[6,14,15,16] for a nonlinear plant or when the admissible control has a magnitude bound, then a solution can be obtained for all practical purposes for a linear plant only. (Either one has to linearize the nonlinear plant around the nominal optimal trajectory or one has to be able to approximate a given nonlinear plant by a simplified nonlinear model[9,10] whose feedback control law is already known.)

The objective of the research presented here is to develop suboptimal control schemes based on the methods of functioal analysis (The L-problem[1,18,19,22,23,24]). As in other methods the feedback suboptimal control system is assumed to be operating in the vicinity of an open-loop nominal optimal trajectory. The admissible control may have a magnitude constraint and in general is assumed to be an element of a "suitable" L_p space[29], $1 \leq p \leq \infty$. The motivation for the use of the L-problem is that it eliminates the need for solving a two-point boundary-value problem. However, it introduces a different type of a compuational problem, a problem of approximation in L_p space. The seeming disadvantage of the L-problem method is that it can only be applied to a linear system. However, this is not a serious objection, since nonlinear processes can be linearized around the nominal optimal trajectory which in any case is assumed to be known.

The results of the dissertation may be summarized briefly as follows:

(1) It is shown that the optimal control law of the "minimum norm problem" can be obtained as the solution of a set of nonlinear algebraic equations (Theorem 1 in Chapter II)

(2) The direction of the "optimum lambda vector" used in calculating the optimum control law is shown to be constant on the optimum trajectory (Theorem 2 in Chapter II).

(3) It is shown in Chapter III, theorem 3, that the optimum control law of the " minimum time problem" can also be obtained as the solution of a set of nonlinear algebraic equations.

(4) A new design procedure for both problems is developed in Chapters II,III, and IV.

(5) A bound is obtained on the size of the allowed deviation from the nominal optimal trajectory corresponding to a given terminal error (Chapter II,III, and IV).

(6) The method developed for linear system is extended to nonlinear systems in Chapter V by linearizing the given nonlinear system around the nominal optimal control and its corresponding trajectory.

(7) It is shown in appendix B that the "optimum lambda vector" used in the functional analysis approach turns out to be the costate vector of the Maximum principle[27] evaluated at the terminal time within a

scalar multiplicity.

1.2 Mathematical Formulation of the Problem

We are given a dynamic system which is to be controlled and which can be described by

$$\frac{d\underline{x}(t)}{dt} = A(t)\underline{x}(t) + B(t)\underline{u}(t) \tag{1-1}$$

$$\underline{y}(t) = C(t)\underline{x}(t) \tag{1-2}$$

where $\underline{x}(t)$, $\underline{u}(t)$ and $\underline{y}(t)$ are respectively the system state variable, control input, and output vectors; $A(t)$, $B(t)$, $C(t)$ are respectively (nxn), (nxr), (mxn) matrices whose elements are known continuous, real-valued functions of time t, where $t \geqslant 0$. The solution of the vector differential equation for a given initial state $\underline{x}_o$ at time t_i is [3.31]

$$\underline{x}(t) = \Phi(t,t_i)\underline{x}(t_i) + \int_{t_i}^{t} \Phi(t,s)B(s)\underline{u}(s)ds \tag{1-3}$$

where $\Phi(t,s)$ is the system transition matrix [3.31]. Therefore, the output vector at terminal time t_f is

$$\underline{y}(t_f) = C(t_f)\Phi(t_f,t_i)\underline{x}(t_i) + \int_{t_i}^{t_f} C(t_f)\Phi(t_f,s)B(s)\underline{u}(s)ds \tag{1-4}$$

Define

$$\underline{e}(t_f,\underline{x}(t_i)) \triangleq \underline{y}(t_f)-C(t_f)\Phi(t_f,t_i)\underline{x}(t_i) \tag{1-5}$$

and

$$H(t_f,s) \triangleq C(t_f)\Phi(t_f,s)B(s) \tag{1-6}$$

This yields the relatively simple description

$$\underline{e}(t_f,\underline{x}(t_i)) = \int_{t_i}^{t_f} H(t_f,s)\underline{u}(s)\, ds \tag{1-7}$$

Now let us state the problems to be solved in this thesis:

1.2.1 Problem 1

Given an initial state $\underline{x}_0$ at time t_i and a desired final output $\underline{y}^d$ at a fixed time t_f, find a feedback "suboptimal" control system which forces the dynamic system after an elapsed time $T=t_f-t_i$, to $\underline{y}(t_f)=\underline{y}^d$ while the performance index $J=\|\underline{u}\|_p$, $1\leq p$, where $\|\underline{u}\|_p$ [28] is a suitable norm in L_p space, is minimized.

1.2.2 Problem 2

Given an initial state $\underline{x}_0$ at time t_i and a desired output $\underline{y}^d$, we wish to find a feedback "suboptimal" control system which enables the dynamic system governed by (1-1) and (1-2) initially in state $\underline{x}(t_i)=\underline{x}_0$ to satisfy $\underline{y}(t_f^*)=\underline{y}^d$ for the least elapsed time $T^*=t_f^*-t_i$ subject to a constraint on the norm of $\underline{u}$ that is $\|\underline{u}\|_p \leqslant L_o$, where L_o is a given positive number.

For both problems it is assumed that the dynamic system is completely output controllable[21] and further assumed that the process operates in the presence of "small perturbations" and that the actual trajectory $\underline{x}(t)$ is not "far" from the nominal optimal trajectory $\underline{x}_o^*(t)$, the nominal solution trajectory of the problems. It is also assumed that all the states are accessible for measurement.

Remark 1-1

A dynamic system is called completely output controllable [21] if and only if there exists an input $\underline{u}(t)$ which enables the system with an arbitrary set of initial states to achieve any desired output in a finite interval t_i to t_f. One can show [20,21] that a dynamic system is completely output controllable if for every nonzero vector $\underline{\lambda}$, the inner product $\langle \underline{\lambda}, \underline{h}_j(t_f, s) \rangle$ is not identically zero over any subinterval of t_i, t_f for every j, which corresponds to a normal system[25]; $\underline{h}_j$ is the j-th column of matrix $H(t_f, s)$. *****

Remark 1-2

"suitable norm" in L_p space which can incorporate numerous practical constraints are defined [28] as follows:

Case 1 Systems with one input

In this case define

$$\|u\|_p \triangleq \left[\int_{t_i}^{t_f} |u(s)|^p \, ds \right]^{1/p}, \quad p \geq 1 \tag{1-8}$$

The set of real-valued time functions u(t) which is defined on an interval t_i to t_f and which has a finite norm indicated by (1-8) is called a Banach space[8,29] denoted by L_p.

Case 2 Systems with multiple inputs and an overall Constraint on all inputs

In this case define

$$\|\underline{u}\|_p \triangleq \left(\int_{t_i}^{t_f} \sum_{i=1}^{r} |u_i(s)|^p ds \right)^{1/p}, \quad p \geq 1 \tag{1-9}$$

Case 3 Systems with multiple inputs and different types of constraints on each input

In this case define

$$\|\underline{u}\|_p \triangleq \left(\sum_{i=1}^{r} \left(\int_{t_i}^{t_f} |u_i(t)|^{p_i} \, dt \right)^{p/p_i} \right)^{1/p} \tag{1-10}$$

The resulting Banach space will be called a product L_p

[17] space. *****

Remark 1-3

Suppose that one wishes to minimize the energy, area or the maximum amplitude of the single control input, one can incorporate these physical constraints with norm defined in (1-8)[17,18,19,22] with p=2, p=1, or p=∞. Energy, area, and the maximum amplitude for each input are applied in the same manner. *****

1.3 Historical Background

As has been mentioned, solutions to optimal control problems are not always easily obtained. In addtion even when it is possible to obtain the desired optimal solution it may be impossible to implement it, that is, to construct a controller capable of duplicating the exact mathmatical results. For example, determination of the feedback control law for a particular problem may require a forbidding amount of computer memory space to carry out a complex set of computations in real time. Hence, determination of the optimal feedback control law would require a large economic effort. This is not always justified since it may be possible to use a relatively simple and cheap control law which yields a performance criterion close to the optimal one. We have thus traded simplicity and lower cost for the suboptimal performance.

In order to solve the suboptimal problem, it is necessary to know the open-loop nominal optimal trajectory when the technique in this thesis is used; one can obtain this using the well-established method of the L-problem [1,18,19,22,23,24] which is outlined in appendix A. It is shown there how to obtain an open-loop nominal optimal control $\underline{u}^*(t)$ both for the case of the minimum-norm optimal problem (see problem 1) and the time-optimal one (see problem 2). Once the nominal optimal control $\underline{u}^*(t)$ is formulated, the nominal optimal trajectory $\underline{x}_o^*(t)$ can be obtained.

Of course other methods are available to obtain $\underline{x}_o^*(t)$ such as Pontryagin Maximum Principle[27], Dynamic Programming [5], and various Steepest Descent methods[7,14,15]. These methods have the advantage over the L-problem approach in that they are not limited to linear systems. As has been mentioned, the open-loop solution of the optimal problems has a limited usefulness. It can at best serve as a guide as to what is the best under idealized condtions.

A number of investigators [2,4,6,9,16,26] attacked the practical aspect of the optimal control problem. A point of view of some of them which bears a relation to the presented research depends on the assumption(not unreasonable in many cases) that the system operates in the presence of small perturbations, which continuously or otherwise cause "small" deviation from the nominal optimal trajectory monitored at the output of the system will be $\underline{x}_o^*(t) + \Delta\underline{x}$ and since $\underline{x}_o^*(t)$ is known, $\Delta\underline{x}$ can be measured. A suitable correction on $\underline{u}^*(t)$, $\Delta\underline{u}$ is generated to make the actual output $\underline{u}^*(t) + \Delta\underline{u}$. When $\Delta\underline{x}$ is sufficiently small, a good approximation to $\Delta\underline{u}(t)$ can be made presumably by setting $\Delta\underline{u} = M \Delta\underline{x}$, where M is a matrix which has to be precalculated.

The elements of the matrix M using the second variation method[6,14,15,16] can be found only in situations where there is no hard constraint on the control input. Essentially, only quadratic types of constraints can be handled by this method. In reference [15] the second variation method is extended to the case of the magnitude bound by the use of penalty function. This increases the dimensionality of the problem and the amount of required computations.

An alternative approach is described in reference 9.10. Instead of trying to find directly a correction $\Delta \underline{u}$, an attempt is made to calculate $\Delta \underline{p}$, the change in the costate vector $\underline{p}$ as a result of small changes $\Delta \underline{x}$ from the nominal optimal trajectory. Although this approach can handle magnitude constraints and others, the technique requires in the general case of a nonlinear problem that an approximate, simplified nonlinear model must be determined. For this, however, there is no general technique available and authors discuss only a few special cases.

The manner in which the approach of this dissertation differs from these previous techniques can be summarized in the following table:

	Breakwell Speyer, Bryson	Friedland Sarachick, Thau	Proposed Technique
Value of p in performance index	$p = 2$	$1 < p < \infty$	$1 < p \leq \infty$
Control constraints for minimum time problem	none	yes	yes
Error analysis	none	none	yes
Computational problem	Solution of two-point boundary-value problem	Solution of matrix Riccati equation	Solution of linear algebraic equations

Table 1-1

<u>Comparison of Suboptimal Control Techniques</u>

In the case of magnitude constraint on control, the proposed technique can handle situations where deviations may occur after the first switching instant on the nominal optimal trajectory. The other techniques cannot account for such deviations. Hence, the contribution of this dissertation is that a feedback suboptimal control design procedure is presented together with an error analysis to estimate its effectiveness.

II SOLUTION OF PROBLEM 1 THE LAMBDA METHOD

2.1 Introduction

A solution of the minimum-norm suboptimal control problem presentrd in this thesis is based on the assumption that the dynamic system operates in the presence of small perturbations during the actual operation and hence, in this restricted problem, a suboptimal controller which yields a performance close to the optimal one and which can be readily implemented is formulated.

The objective of this Chapter is to explore the feasibility of designing a suboptimal control law for problem 1 based on the results of the L-problem. It is for simplicity assumed that in this Chapter that the control is scalar and that suitable norm is defined by (1-8).

2.2 An Alternative Method on Obtaining Λ^*

As has been reviewed in Appendix A, one can obtain the open-loop optimal control of the problem 1*(see appendix A), problem of finding control u(t) which makes

$$\|u\|_p = \min \qquad (2.2\text{-}1)$$

while keeping

$$\int_{t_i}^{t_f} \underline{h}(t_f,s)u(s)ds = \underline{e}(t_f,\underline{x}(t_i)) \qquad (2.2\text{-}2)$$

where t_f is fixed.

One can write optimal control $u^*(t)$, the solution of Problem 1* as

$$u^*(t) = u^*(t_f,\lambda^*,t) \qquad (2.2\text{-}3)$$

where λ^* is the minimand in the expression (A-12) in appendix A with

$$\lambda^{*T}\, \underline{e}(t_f,\underline{x}(t_i)) = 1 \qquad (2.2\text{-}4)$$

Remark 2-1

One can of course obtain λ^* as the maximand in the expression (A-11). In this case we see that the optimum lambda vector λ^* is maximand within non-zero scalar multiplicity. Thus the direction of the vector λ^* is sufficient to determine the optimum lambda vector λ^*. Therefore, the maximum value of (A-11) is independent of the number $\sum_{i=1}^{m}\lambda_i^* e_i(t_f,\underline{x}(t_i))$ so that the quantity of $\sum_{i=1}^{m}\lambda_i^* e_i(t_f,\underline{x}(t_i))$ may be set to 1 as has already been done in appendix A. As an advantage for this approach, one can reduce the dimension of λ^* in the maximization by one, i.e., m to m-1. Once we set $\sum_{i=1}^{m}\lambda_i^* e_i(t_f,\underline{x}(t_i)) = 1$, the magnitude as well as the direction of the optimum lambda vector are fixed for this particular $\sum_{i=1}^{m} \lambda_i^* e_i(t_f, \underline{x}(t_i)) = 1$. *****

The optimum control $u^*(t)$, (2.2-3) which is the solution of (2.2-1) and (2.2-2) is completely specified by the optimum lambda vector λ^*. The optimum control $u^*(t)$ must satisfy (2.2-2). Substituting (2.2-3), that is, (A-17) in Appendix A, into (2.2-2) gives

$$\frac{\int_{t_i}^{t_f} \underline{h}(t_f,s)\left|h(t_f,\underline{\lambda}^*,s)\right|^{q-1} \operatorname{sgn}(h(t_f,\lambda^*,s)\, ds}{(\|h^*\|_q)^q} = \underline{e}(t_f,\underline{x}(t_i)) \tag{2.2-5}$$

where

$$h(t_f,\lambda^*,t) \triangleq \sum_{i=1}^{m} \lambda_i^* h_i(t_f,t) \triangleq h^*$$

Now premultiplying the both sides of (2.2-5) by λ^{*T} (transpose of λ^*), we see that the side constraint (2.2-4) is automatically satisfied. Therefore, one may obtain the optimum lambda vector λ^* as the solution of nonlinear algebraic equations (2.2-5) instead of finding λ^* as the maximand in the expression (A-11) in Appendix A for given initial state $\underline{x}(t_i)$ at time t_i, terminal time t_f and q. Thus, one can summarize this as Theorem 1:

Theorem 1

λ^*, optimum lambda vector which is maximand in the expression (A-11) in Appendix A, can be obtained as the solution of (2.2-5).

2.2.1 λ^* for Single Output System

To clarify Theorem 1 consider the dynamic system with a single input and a single output. In this special case the optimum lambda vector becomes a scalar and its value can be obtained directly from (2.2-4) or (2.2-5)

$$\lambda^* = 1/\, e(t_f, \underline{x}(t_i)) \tag{2.2-6}$$

Since the optimum lambda vector can be explicitly obtained for all initial states $\underline{x}(t_i)$, the optimum control can be obtained as a function of $\underline{x}(t_i)$ and can be found by substituting (2.2-6) into (A-17) in Appendix A as follows:

$$u^*(t) = \frac{e(t_f, \underline{x}(t_i)) \left| h(t_f, t) \right|^{q-1} \operatorname{sgn}(h(t_f, t))}{(\|h\|_q)^q} \tag{2.2-7}$$

where $1 \leq q$. This agrees with the result given in reference 18.

Remark 2-2

We obtained an open-loop type of optimal control which is function of initial state $\underline{x}(t_i)$ and initial time t_i. If we take the initial time t_i to be the current time t and the initial state $\underline{x}(t_i)$ to be the current state $\underline{x}(t)$, we obtain feedback optimal control. *****

2.2 λ^* for Multiple Output Systems

<u>Case 1</u> Energy constraint on control input; p=2:q=2

In this case also, using the Theorem 1 one can obtain λ^* explicitly. From (2.2-5) we obtain

$$\lambda^* = (\|h^*\|_2)^2 W(t_f,t_i)\, \underline{e}(t_f,\underline{x}(t_i)) \qquad (2.2\text{-}8)$$

where

$$W(t_f,t) = \left(\int_{t_i}^{t_f} \underline{h}(t_f,s)\underline{h}(t_f,s)\, ds \right)^{-1} \text{ and } \underline{h}^T(t_f,t) \text{ is}$$

transpose of $\underline{h}(t_f,t)$. Now premultiplying both sides of (2.2-8) by $\underline{e}^T(t_f,\underline{x}(t_i))$, the transpose of $\underline{e}(t_f,\underline{x}(t_i))$ and recalling that $\underline{e}^T(t_f,\underline{x}(t_i))\lambda^* = 1$, we can obtain

$$(\|h^*\|_2)^2 = \frac{1}{\underline{e}^T(t_f,\underline{x}(t_i))W(t_f,t_i)\underline{e}(t_f,\underline{x}(t_i))} \qquad (2.2\text{-}9)$$

Complete output controllability of the system guarantees that $W(t_f,t_i)^{-1}$ is positive definite and therefore $W(t_f,t_i)$ exists. Thus the optimum lambda vector λ^* can be obtained by inserting (2.2-9) into (2.2-8) as follows:

$$\underline{\lambda}^* = \frac{W(t_f,t_i)\ \underline{e}(t_f,\underline{x}(t_i))}{\underline{e}^T(t_f,\underline{x}(t_i))W(t_f,t_i)\underline{e}(t_f,\underline{x}(t_i))} \tag{2.2-10}$$

Therefore, optimal control $u^*(t)$ for this case can be obtained explicitly for any initial state $\underline{x}(t_i)$; Inserting (2.2-10) into (A-17) in appendix A gives

$$u^*(t) = \underline{h}^T(t_f,t)\ W(t_f,t_i)\ \underline{e}(t_f,\underline{x}(t_i)) \tag{2.2-11}$$

As it has been pointed out in Remark 2-2, feedback optimal control can be obtained if we take initial time t_i to be the current time t and the initial state $\underline{x}(t_i)$ to be the current state $\underline{x}(t)$:

$$u^*(t) = \underline{h}^T(t_f,t)\ W(t_f,t)\ \underline{e}(t_f,\underline{x}(t)) \tag{2.2-12}$$

Case 2 Magnitude constraint on control input; $p=\infty$

When we observe (2.2-5) for the case of $q=1$, it is seen that the immediate difficulty is raised in calculating the optimum lambda vector $\underline{\lambda}^*$ for all $\underline{x}(t_i)$ as the number of outputs is increased. However, for the dynamic system whose impulse response belongs to a Tchebycheff system[22,23,24] certain simplifications are possible and the simplifications depends on[23,24] the terminal vector $\underline{e}(t_f,\underline{x}(t_i))$:

(i) $e_1 \neq 0$, $e_j = 0$, $j=2,3,\ldots,m$

In this case a closed form relation can be obtained.

(ii) $e_1, e_2, \ldots, e_s \neq 0$, $e_{s+1} = e_{s+2} = \cdots = e_m = 0$

where $1 < s \leq m$

For the case of (ii) the problem is reduced to the Geronimus problem[13], the problem of finding s-coefficients which maximizes a linear form subject to an integral constraint.

As far as the feedback control system is concerned the case (i) becomes the case of s=m in case (ii) during the process of the system. Thus, to implement the continuous feedback control system it is necessary to solve the Geronimus problem on-line. Therefore, it requires in general, a forbidding amount of computer storage space to generate λ^* for all initial states. Thus, one must trade simplicity and lower cost for allowable approximation of the optimum lambda vector which yields acceptable performance of the dynamic system. One way of approximating λ^* will be presented in the coming section on the assumption that the dynamic system operates in the presence of small perturbations.

2.3 A Structure of Feedback Suboptimal Control System

It has been shown in section 2.2 that λ^* has been explicitly obtained for p=2 and for the single input - single output systems($1 < p \leq \infty$). In this section one obtains an approximate expression for λ^* for the single input - multiple output systems ($1 < p \leq \infty$) on the assumption that the system operates in the presence of small perturbations. External unknown forces can cause perturbations during operation; for example a gust of wind or change in temperature; another example of engineering interest is the regulator problem for which the possible range of initial states is large. Yet another type of perturbation is that due to imperfect knowledge of the system.

However, in this section one focuses on the perturbations essentially due to the external force and assumes that all the system parameters, reference input and sensing devices are precisely known.

Now let us develop the feedback suboptimal control system. The basic idea used in obtaining the suboptimal control is the utilization of the general form of the open-loop optimal control input which is obtained in appendix A using the method of functional analysis. This solution can be written as

$$u^*(t) = u^*(t_f, \lambda_x^*, t) \qquad (2.3\text{-}1)$$

where t_f is the fixed terminal time and λ_x^* is the optimum lambda vector whose precomputed elements depend on the desired final output $\underline{y}^d$ and the initial state $\underline{x}(t)$ of the system, different in general from $\underline{x}_o$, the nominal initial state.

Remark 2-6

Note that $\lambda^* = \lambda_x^*$ when computation of λ^* is performed for the initial state $\underline{x}(t_i)=\underline{x}(t)$ and the desired final output $\underline{y}^d$. *****

It is seen in (2.3-1) that the optimal control $u^*(t)$ depends on parameter λ_x^*. However, as has been mentioned, the on-line computation of λ_x^* requires prohibitive amount of computer memory space for the case of single input - multiple output systems. Therefore, one must trade simplicity and lower cost for the approximate value of λ_x^*. One way of approximating λ_x^* will be presented in this section. The approximate calculation of λ_x^* is based on the assumption that the system operates in the presence of small perturbations. Before developing a method of finding approximate λ_x^*, it will be helpful to define the following items (refer to fig.2-1):

(1) $\underline{x}_o^*(t)$, the optimum nominal trajectory is the solution of equation (1-1) with u replaced by $u^*(t_f,\lambda^*,t)$ and $\underline{x}(t_i)= \underline{x}_o$.

(2) $\underline{x}^*(t)$, the optimum trajectory is the solution of equation (1-1) with u replaced by $u^*(t_f,\lambda_x^*,t)$ and $\underline{x}(t_i)=\underline{x}(t)$.

(3) $\underline{x}(t)$ is the actual trajectory not far from the nominal optimal one.

(4) $\Delta\underline{x}(t) = \underline{x}(t) - \underline{x}_o^*(t)$, deviation measured by sensing devices.

(5) $\bar{u}^*(t)$ is suboptimal control.

(6) λ_t^* is the value of the λ^* at every instant of time along the nominal optimal trajectory $\underline{x}_o^*(t)$.

In the proposed scheme λ_x^* can be approximated by first obtaining $\Delta\lambda$, small change in λ_t^*, as a result of small change $\Delta\underline{x}$ in $\underline{x}_o^*$, i.e.,

$$\lambda_x^* = \lambda_t^* + \Delta\lambda \qquad (2.3\text{-}2)$$

When the deviation $\Delta\underline{x}$ is a small quantity, it is shown in

Graphical Illustration Of Notation

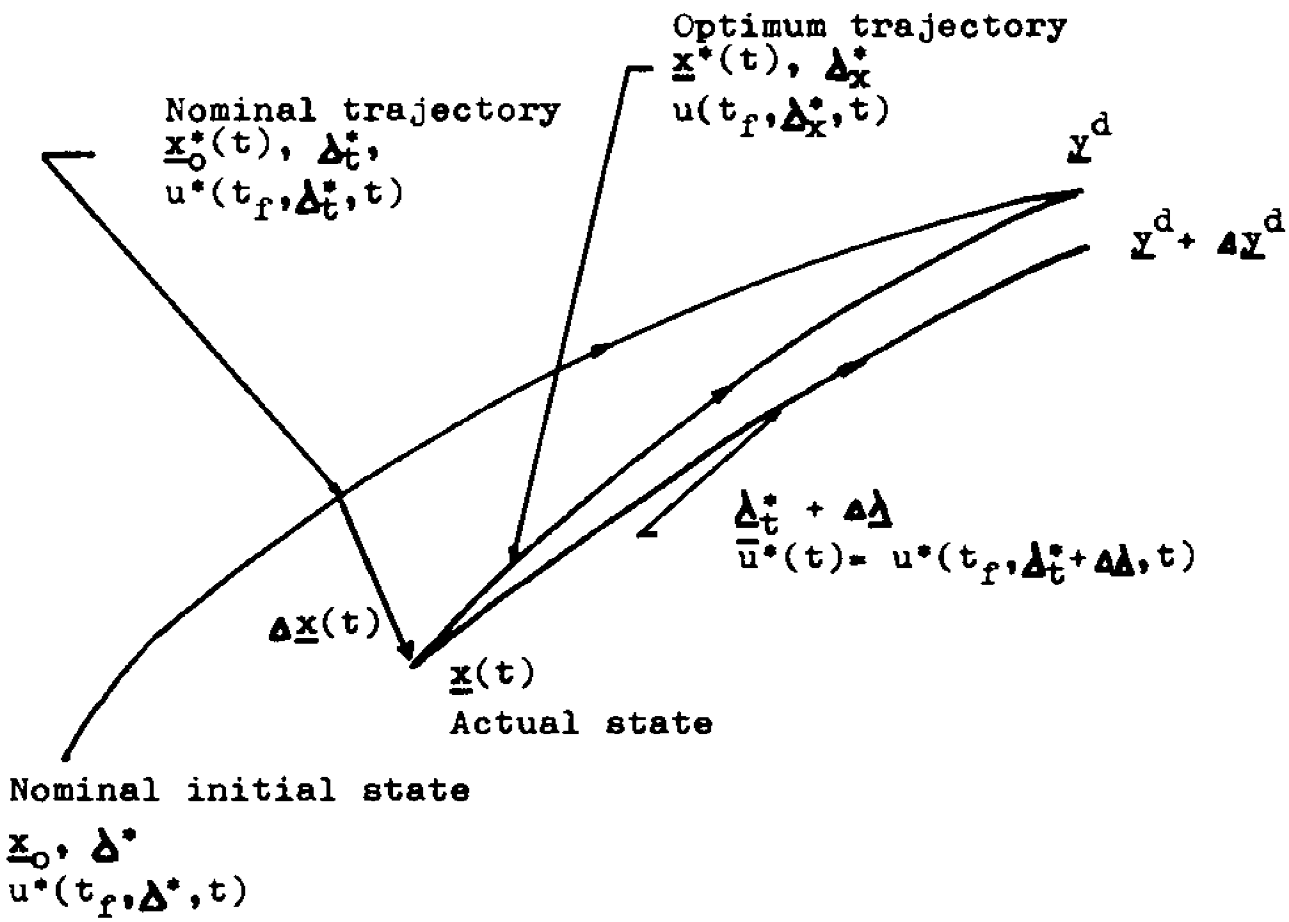


Fig. 2-1

a later section that $\Delta \underline{x}$ can be approximately related to $\Delta \lambda$ by a linear relation. Thus, the deviation $\Delta \underline{x}$ must be small enough to be linearly related to $\Delta \lambda$. Suppose that the linear transformation which maps $\Delta \underline{x}$ into $\Delta \lambda$ is found, i.e.,

$$\Delta \lambda = \left[M_{nq} \right] \Delta \underline{x} \ , \quad 1 \leqslant q < \infty \tag{2.3-3}$$

Then the feedback suboptimal control system proposed in this thesis is shown in fig.2-2. In this figure the box marked "suboptimal controller" realizes eq.(2.3-1), that is, (A-17) in appendix A with t_i, λ^* replaced by t and λ_x^*, respectively. To find λ_x^* in addition to $\Delta \lambda$, it is necessary to know λ_t^*. It turns out that the determination of λ_t^* can be achieved without the necessity of repeating at every instant of time the computational process indicated by (A-11) in appendix A. The determination of λ_t^* is based on the following theorem in which we show that the direction of λ_t^* is constant on the optimum trajectory for all t, $t_i \leqslant t \leqslant t_f$.

<u>Theorem 2</u>

The direction of λ_t^* which defines the optimum control input $u_t^*(t)$ at time t when starting from initial state $\underline{x}_o^*(t)$

SUBOPTIMAL CONTROL SCHEME

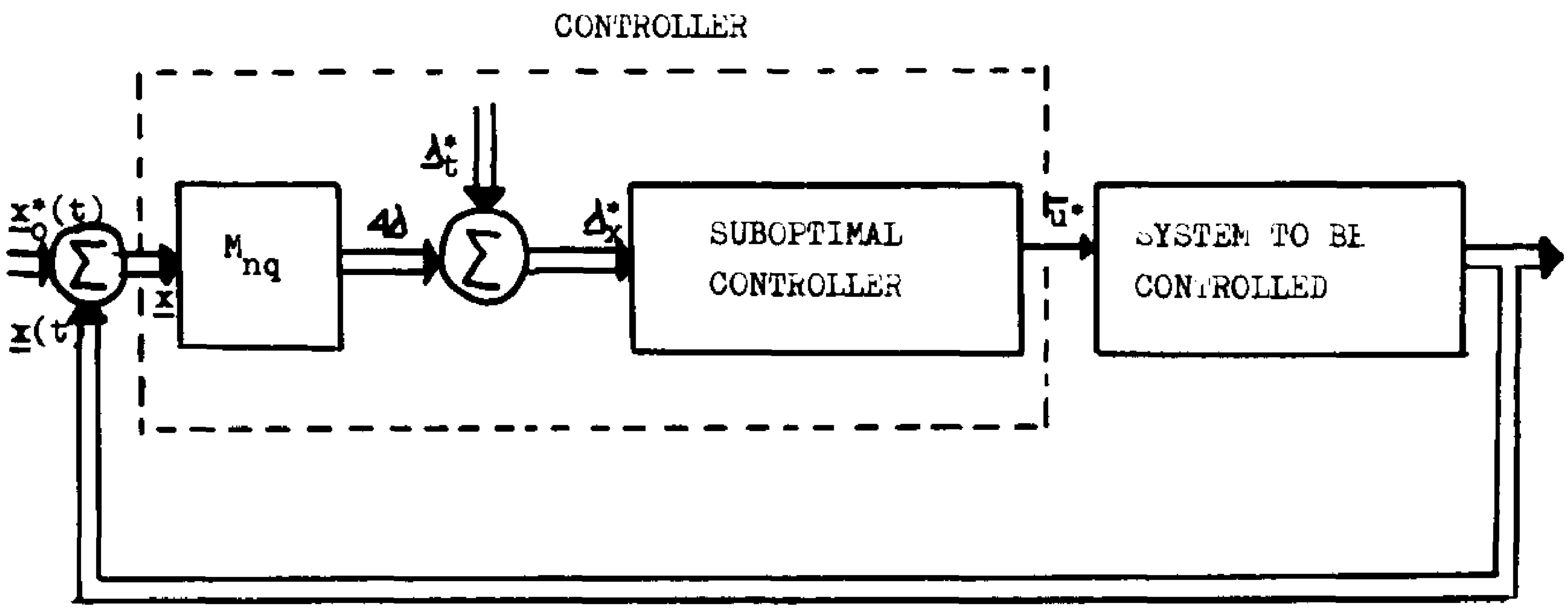


Fig. 2-2

is invariant for all t, $t_i \leq t \leq t_f$, where initial state $\underline{x}_o$ is defined at time t_i.

Proof

Principle of Optimality guarantees that $u_t^*(t)$ must be an optimum policy. Hence,

$$u_t^*(t) = u^*(t) \tag{2.3-4}$$

for all t, $t_i \leq t \leq t_f$

where $u^*(t)$ is the optimum control corresponding to the initial state $\underline{x}_o$ and is given by (A-17). And also from (A-17) in appendix A

$$u_t^*(t) = \frac{\left|\sum_{i=1}^{m} \lambda_{ti}^* h_i(t_f,t)\right|^{q-1} \operatorname{sgn}\left(\sum_{i=1}^{m} \lambda_{ti}^* h_i(t_f,t)\right)}{\int_t^{t_f} \left|\sum_{i=1}^{m} \lambda_{ti}^* h_i(t_f,s)\right|^q ds} \tag{2.3-5}$$

where $1 \leq q < \infty$

Comparing (A-17) and (2.3-5), it follows from the condtion (2.3-4) that

a. $\operatorname{sgn}(\langle \underline{\Lambda}_t^*, \underline{h}(t_f,t)\rangle) = \operatorname{sgn}(\langle \underline{\Lambda}^*, \underline{h}(t_f,t)\rangle)$ (2.3-6)

b. $|u^*(t)| = |u_t^*(t)|$ (2.3-7)

To prove the theorem, it is sufficient to show that

$u^*(t) = u_t^*(t)$ when

$$\underline{\lambda}_t^* = m(t_f,t)\underline{\lambda}^* \tag{2.3-8}$$

holds for all t, $t_i \leq t \leq t_f$. This can be demonstrated by observing that condition (2.3-7) is satisfied when

$$m(t_f,t) = \frac{\int_{t_i}^{t_f} \left| \sum_{i=1}^{m} \lambda_i^* \, h_i(t_f,s) \right|^q ds}{\int_{t_i}^{t_f} \left| \sum_{i=1}^{m} \lambda_i^* h_i(t_f,s) \right|^q ds} \tag{2.3-9}$$

and that this automatically satisfies the requirement of condition (2.3-6) since $m(t_f,t) > 0$ for all t, $t_i \leq t \leq t_f$. Hence, the theorem is proven. Alternatively $m(t_f,t)$ can also be derived directly from the side condition, that is, since the side condition must be satisfied on the optimum trajectory

$$\langle \underline{\lambda}_t^*, \underline{e}(t_f, \underline{x}_o^*(t)) \rangle = 1 \tag{2.3-10}$$

Inserting (2.3-9) into (2.3-10) and solving for $m(t_f,t)$,

$$m(t_f,t) = \frac{1}{\langle \underline{\lambda}^*, \underline{e}(t_f, \underline{x}_o^*(t)) \rangle} \tag{2.3-11}$$

Note that (2.3-9) is equivalent to (2.3-11). To see this we observe that (1-7) with t_i, $\underline{x}(t_i)$, u(s) replaced by

t, $\underline{x}_o^*(t)$, $u^*(t_f,\underline{\lambda}^*,t)$ respectively yields

$$\underline{e}(t_f,\underline{x}_o^*) = \frac{\int_t^{t_f} \underline{h}(t_f,s)\left|h(\underline{\lambda}^*)\right|^{q-1} \operatorname{sgn}(h(\underline{\lambda}^*))\, ds}{\int_{t_i}^{t_f} \left|h(\underline{\lambda}^*)\right|^q \, ds} \qquad (2.3\text{-}12)$$

where

$$h(\underline{\lambda}^*) \triangleq \sum_{i=1}^{m} h_i(t_f,s)\lambda_i^*$$

Inserting (2.3-12) into (2.3-11) gives

$$\frac{1}{\langle \underline{\lambda}^*,\underline{e}(t_f,\underline{x}_o^*(t))\rangle} = \frac{\int_{t_i}^{t_f} \left|h(\underline{\lambda}^*)\right|^q \, ds}{\int_{t}^{t_f} \left|h(\underline{\lambda}^*)\right|^q \, ds} \qquad (2.3\text{-}13)$$

Thus, $\underline{\lambda}_t^*$ can be precomputed and $\underline{\lambda}_x^*$ be obtained at every instant of time on line as $\underline{\lambda}_x^* = \underline{\lambda}_t^* + \Delta\underline{\lambda}$ provided that $\Delta\underline{\lambda}$ is obtained on line. As has already been mentioned $\Delta\underline{\lambda}$ is estimated approximately from (2.3-3) where m x n linear transformation matrix $\left[M_{nq}\right]$ is to be precomputed. To determine $\left[M_{nq}\right]$ we first define from (1-5) that

$$\Delta\underline{e} \triangleq \underline{e}(t_f,\underline{x}(t)) - \underline{e}(t_f,\underline{x}_o^*(t))$$

Then we can have

$$\Delta\underline{e} = -C(t_f)\Phi(t_f,t)\Delta\underline{x} \qquad (2.3\text{-}14)$$

since $\underline{y}(t_f) = \underline{y}^*(t_f) = \underline{y}^d$, where $\underline{y}^*(t_f) = C(t_f)\underline{x}_o^*(t_f)$.

It turns out that

$$\left[M_{nq}\right] = -\left[R_{nq}\right] C(t_f)\Phi(t_f,t) \qquad (2.3\text{-}15)$$

where $\left[R_{nq}\right]$ is an m x m linear transformation matrix defined by

$$\Delta\underline{\lambda} = \left[R_{nq}\right]\Delta\underline{e} \qquad (2.3\text{-}16)$$

A method for computing matrix $\left[R_{nq}\right]$ is described in the following section.

2.3.1 Derivation of Linear Transformation Matrix $\left[R_{nq}\right]$

It will be shown in this section that an m x m linear transformation matrix $\left[R_{nq}\right]$ is determined through a Taylor series expansion. To do this we first note in theorem 1 that λ_t^* can be obtained as the solution of (2.2-5) with t_i, $\underline{x}(t_i)$, λ^* replaced by t, $\underline{x}_o^*(t)$, λ_t^*, respectively, that is

$$\underline{e}(t_f,\underline{x}_o^*(t)) = \frac{\int_t^{t_f} \underline{h}(t_f,s)\left|h(\lambda_t^*)\right|^{q-1} \operatorname{sgn}(h(\lambda_t^*))ds}{(\|h(\lambda_t^*)\|_q)^q} \qquad (2.3\text{-}17)$$

Define the right hand side of (2.3-17) as $\underline{z}(\Lambda_t^*, t_f)$. Then, (2.3-17) can be rewritten as

$$\underline{e}(t_f, \underline{x}_o^*(t)) = \underline{z}(\Lambda_t^*, t_f)$$

The above equation must be valid for any state $\underline{x}(t)$ which may or may not be on the optimum trajectory $\underline{x}_o^*(t)$ if there exists a Λ_x^* corresponding to $\underline{x}(t)$(otherwise, the corresponding optimum control input dose not exist);

$$\underline{e}(t_f, \underline{x}^*(t)) = \frac{\int_t^{t_f} \underline{h}(t_f, s) \left| h(\Lambda_x^*) \right|^{q-1} \operatorname{sgn}\left(h(\Lambda_x^*)\right) ds}{\left(\| h(\Lambda_x^*) \|_q\right)^q} \tag{2.3-18}$$

$$= \underline{z}(\Lambda_x^*, t_f)$$

Now the linear transformation matrix R_{nq} which maps $\Delta\underline{e}$ into $\Delta\Lambda$ is obtained through a Taylor series expansion of (2.3-18) around Λ_t^* and $\underline{x}_o^*(t)$(see (2.3-2) and the definition of $\Delta\underline{x}$ in item 4 in pages 23). From the definition of $\Delta\underline{e}$

$$\Delta\underline{e} = \underline{z}(\Lambda_t^* + \Delta\Lambda\ ,\ t_f) - \underline{z}(\Lambda_t^*, t_f) \tag{2.3-19}$$

Substituting a Taylor series expansion for $\underline{z}(\Lambda_t^* + \Delta\Lambda,\ t_f)$ into (2.3-19) gives

$$\Delta \underline{e} = \left[\frac{\partial \underline{z}}{\partial \underline{\Delta}_x^*}\right] \Delta \lambda + R$$

where $\left[\frac{\partial \underline{z}}{\partial \underline{\Delta}_x^*}\right] \triangleq \left[\frac{\partial \underline{z}(\underline{\Delta}_x^*, t_f)}{\partial \underline{\lambda}_x^*}\right]_{\lambda_x^* = \lambda_t^*}$ and R is the remainder.

Now define

$$\left[R_{nq}^{-1}\right] = \left[\frac{\partial \underline{z}}{\partial \underline{\Delta}_x^*}\right] \tag{2.3-20}$$

To evaluate $\left[\frac{\partial \underline{z}}{\partial \underline{\lambda}_x^*}\right]$ one takes the symbolic derivative 31.33 for sufficiently small $\Delta \underline{e}$(that is, for sufficiently small deviation $\Delta \underline{x}$)

$$\left[R_{nq}^{-1}\right] = \left[\frac{\partial \underline{z}}{\partial \underline{\Delta}_x^*}\right] = K_q\Bigg(\frac{(q-1) \int_t^{t_f} \underline{h}(t_f,s)\underline{h}(t_f,s)^T \left|h(\lambda_t^*)\right|^{q-2} ds}{K_q^2} +$$

$$\frac{2\int_t^{t_f} \underline{h}(t_f,s)\underline{h}(t_f,s)^T \left|(h(\lambda_t^*)\right|^{q-1} \delta(h(\lambda_t^*))\, ds}{K_q^2} \Bigg) -$$

$$\frac{\int_t^{t_f} \underline{h}(t_f,s)\left|h(\lambda_t^*)\right|^{q-1} \operatorname{sgn}(h(\lambda_t^*))ds\,(q \int_t^{t_f} \underline{h}(t_f,s)^T\left|h(\lambda_t^*)\right|^{q-1} \operatorname{sgn}(h(\lambda_t^*))ds}{K_q^2}$$

(2.3-21)

where $s \in [t, t_f]$ and $\delta(t)$ is an unit impulse and

$$K_q \triangleq \int_t^{t_f} |h(\Delta_t^*)|^q \, ds$$

For q=1:p=∞ (minimizing the maximum amplitude of control) (2.3-21) becomes

$$\left[R_{n2}^{-1}\right] = \frac{2\int_t^{t_f} \underline{h}\ \underline{h}^T \delta(h(\Delta_t^*))ds}{K_1} - \underline{e}(t_f, \underline{x}_o^*(t))\underline{e}(t_f, \underline{x}_o^*))^T \qquad (2.3\text{-}22)$$

where e^T is the transpose of $\underline{e}$ and $\underline{h}^T$ is transpose of $\underline{h}$,

$$\underline{h} \triangleq \underline{h}(t_f, s)$$

The above equation (2.3-22) can be further simplified upon using the formula[33,31]

$$\delta(h(\Delta_t^*)) = \sum_v \frac{\delta(s-s_v^*)}{\left|\frac{d}{ds}(h(\Delta_t^*))\right|_{s=s_v^*}} \qquad (2.3\text{-}23)$$

where s_v^* is a simple zero, if any, of $h(\Delta_t^*)$, $t \leq s \leq t_f$. Then performing the indicated integration of (2.3-22) by using the formula (2.3-23)

$$\left[R_{n1}^{-1}\right] = \sum_v \frac{2\underline{h}(t_f, s_v^*)\underline{h}(t_f, s_v^*)^T}{K_1\left|\frac{d}{ds}\langle\Delta_t^*, \underline{h}(t_f, s)\rangle\right|_{s=s_v^*}} - \underline{e}(t_f, \underline{x}_o^*)\underline{e}(t_f, \underline{x}_o^*)^T$$

(2.3-24)

where s_v^* must be in the interval t to t_f.

Remark 2-7

Suppose that $v=1,2,\ldots,k$ where k is the number of times that the nominal open loop optimal control, $u^*(t_f,\Delta^*,t)$, switches. Then (2.3-24) suggests that (k+1) suboptimum controller must be constructed so as for the v-th controller to be valid in the interval s_{v-1}^* to s_v^*. For example, for the n-th order system with all real eigen-values, there can be at most n suboptimum controllers since there are at most n-1 switching[3.27]. *****

For $p=2:q=2$(Energy case of control), (2.3-21) becomes

$$\left[R_{n2}^{-1}\right] = \frac{(W(t_f,t))^{-1}}{k_2} - 2\,\underline{e}(t_f,\underline{x}_o^*(t))\underline{e}(t_f,\underline{x}_o^*(t))^T \tag{2.3-25}$$

Now to obtain the linear transformation matrix $\left[R_{nq}\right]$ which transforms $\Delta\underline{x}$ into $\Delta\underline{\lambda}$ and which is defined as as the inversion of the matrix $\left[R_{nq}^{-1}\right]$, one must make the inversion of $\left[R_{nq}^{-1}\right]$ and the existence of this inversion requires that the determinant of $\left[R_{nq}^{-1}\right]$ must be non-zero, that is,

$$\mathrm{Det}\left(\left[R_{nq}^{-1}\right]\right) \neq 0 \tag{2.3-26}$$

In fact the above condition is always satisfied whenever $1 < q < \infty$; when $q=1$($p=\infty$, one is trying to minimize the maximum magnitude of control) the condition (2.3-26) is satisfied when there are at least n-1 switching instants between the current time and the final time. The above statement follows from the uniqueness condition [24.38] for Δ_x^* since, if we consider $\underline{e}$ as a function of Δ_x^* (see (2.3-18)), then a necessary condition for unique invertibility of the function Δ_x^* is precisely the condition (2.3-26).

Thus, to evaluate $\left[R_{nq}\right]$ for $q=1$, $q=2$ it is necessary to obtain the inverse of the matrix given by (2.3-25), (2.3-24), respectively. It follows from the above invertibility condition that $\left[R_{n2}\right]$ can always be found and that for a system with n real eigenvalues for an n x n system matrix A, $\left[R_{n1}\right]$ can be obtained only for the time interval up to the first switching instant s_1^* assuming that the original open-loop optimal control,$u^*(t_f,\Delta^*,t)$ contained (n-1) of these instants. If these conditions are not satisfied, we can attempt to estimate an approximate $\left[R_{n1}\right]$, namely $\left[\bar{R}_{n1}\right]$. This of course may be necessary whenever we have to worry about deviations which occur beyond the first switching time interval(for example, $t \;\; s_v^*$, $v=1,2,\ldots,n-1$ in the case of the system with n real eigen-values).

Remark 2-8

Using a matrix idenity[36]

$$(I + A\,B\,)^{-1} = I - A(\,I + B\,A\,)^{-1}\,B$$

one may obtain R_{n2} which is the inversion of $\left[R_{n2}^{-1}\right]$ given by (2.3-25), as

$$\left[R_{n2}\right] = K_2(I - 2\,\lambda_t^*\,\underline{e}(t_f, \underline{x}_o^*(t))\;W(t_f,t)$$

where

$$W(t_f,t) \triangleq \left(\int_t^{t_f} \underline{h}(t_f,s)\underline{h}(t_f,s)^T\,ds\right)^{-1} \quad \ldots\ldots$$

2.3.2 Estimation of $\left[\bar{R}_{n1}\right]$

(a) Pseudo Inverse Method

When $\left[R_{n1}^{-1}\right]$ is a singular matrix, one cannot obtain $\Delta\lambda$ such that $\Delta\underline{e} = \left[R_{n1}^{-1}\right]\Delta\lambda$. However, one can try to obtain the best $\Delta\lambda$ as the minimand of $\left\|\Delta\underline{e} - \left[R_{n1}^{-1}\right]\Delta\lambda\right\|$, where for simplicity the inner product is taken for the norm. There is no unique solution to this problem. However, one can further stipulate that the choice for the best $\Delta\lambda$ should be made such that $\|\Delta\lambda\|$ has the least value. Under this condition it can be shown that [12,31]

$$\Delta\lambda = B^T (BB^T)^{-1} (A^TA)^{-1}A^T \Delta\underline{e} \tag{2.3-27}$$

hence

$$\left[\bar{R}_{n1}\right] = B^T(B\ B^T)^{-1}(A^TA)^{-1}\ A^T \tag{2.3-28}$$

where $\left[R_{n1}^{-1}\right] = A\ B$ and the rank of $\left[R_{n1}^{-1}\right]$ is $k \leq m$

A is m x k matrix of rank k

B is k x m matrix of rank k

For any fixed instant of time it is always possible to decompose $\left[R_{n1}^{-1}\right]$ into matrices A abd B defined above.

(b) An Alternative Method

Instead of using the Pseudo-inverse method an approximate $\Delta\lambda$ can be obtained by assuming that for small $\Delta\underline{e}$ the direction of vector $\underline{\lambda}_x^*$ essentially remains the same as that of $\underline{\lambda}_t^*$ and therefore, $\underline{\lambda}^*$, i.e.,

$$\Delta\lambda = a(t)\ \underline{\lambda}^* \tag{2.3-29}$$

where a(t) is a scalar function to be determined.

Using the side condition

$$\left\langle \underline{\lambda}_x^*,\ \underline{e}(t_f, \underline{x}^*(t)\right\rangle = 1 \tag{2.3-30}$$

it follows from (2.3-29)and (2.3-30) that

$$\Delta\lambda = \frac{-\left\langle \underline{\lambda}^*, \Delta\underline{e}\right\rangle}{1/m(t_f,t) + \left\langle \underline{\lambda}^*, \Delta\underline{e}\right\rangle}\ m(t_f,t)\ \underline{\lambda}^* \tag{2.3-31}$$

Thus, $\Delta\lambda$ can be obtained on line since λ^*, $m(t_f,t)$ can be precomputed

Remark 2-9

If we assume that $\lambda_x^* = a(t)\,\lambda^*$ through the process (from the start to the end of operation), we can make use of (2.3-31) in estimating $\Delta\lambda$ through the process. It is worth noting that an estimation of $\Delta\lambda$ through (2.3-3) is based on the assumption of small changes in magnitude as well as the direction of of λ_t^* as a result of $\Delta\underline{x}$. *****

2.3.3 Suboptimal Control for Single Input and Single Output Systems

Consider the dynamic system which consists of single control input and single control output. For this system, as usual, $\left[M_{nq}\right]$ must be determined to develop the suboptimal control system. To determine $\left[M_{nq}\right]$, $\left[R_{nq}\right]$ must be calculated(see (2.3-15). To evaluate the $\left[R_{nq}\right]$ we first observe that $\langle\lambda_t^*, \underline{h}(t_f,t)\rangle$ are scalar quantities, respectively and from (2.3-10) that $\lambda_t^* = 1/e(t_f,\underline{x}_o^*(t))$. Therefore, the first term in $\left[R_{nq}^{-1}\right]$ (2.3-21) becomes $(q-1)e(t_f,\underline{x}_o^*(t))^2$ and the second term in $\left[R_{nq}^{-1}\right]$ vanishes for all q, $1 \leqslant q$ when recalling formula (2.3-23) and the last term results

$q\ e(t_f,\underline{x}_o^*)^2$. Thus we obtain

$$\left[R_{nq}\right] = -1/e(t_f,\underline{x}_o^*(t)) \qquad (2.3\text{-}32)$$

and hence from (2.3-15)

$$\left[M_{nq}\right] = -1/e(t_f,\underline{x}_o^*(t))C(t_f)\Phi(t_f,t) \qquad (2.3\text{-}33)$$

2.4 Simplified Suboptimum Controller

Optimum control (2.3-1) can be rewritten here for the sake of convenience

$$u^*(t) = K_x^*\left|\Delta_x^{*T}\underline{h}(t_f,t)\right|^{q-1} \operatorname{sgn}\left(\Delta_x^{*T}\ \underline{h}(t_f,t)\right) \qquad (2.4\text{-}1)$$

where a scalar K_x^* is evaluated to satisfy (1-8) and obtains

$$K_x^* = \frac{\Delta_x^{*T}\underline{e}(t_f,\underline{x}_o^*(t))}{(\|h(\Delta_x^*)\|_q)^q} \qquad (2.4\text{-}2)$$

Note in (2.4-2) that on-line integration routine is required to evaluate K_x^*. This problem also arises in realizing the suboptimal control law (2.3-5), where Δ_t^* is replaced by Δ_x^*. To overcome this difficulty, an approximate solution of K_x^* is presented in this section.

When we recall the side condition, one can rewrite (2.4-2) as

$$K_x^*(\|\Delta_x^{*T}\underline{h}(t_f,s)\|_q)^q = 1 \qquad (2.4\text{-}3)$$

with the side condition

$$\underline{\lambda}_x^{*T}\ \underline{e}(t_f,\underline{x}^*(t)) = 1 \qquad (2.4\text{-}4)$$

Note in (2.4-3),(2.4-4) that $K_x^* = K^*$ in case when $\underline{x}_o^*(t) = \underline{x}_o$ at time $t=t_i$. Define K_t^* as the value of K_x^* on the optimal nominal trajectory $\underline{x}_o^*(t)$. It is shown in reference 1 that a small change of $\underline{x}_o^*$, $\Delta\underline{x}$, results in a small change in the minimum norm, and therefore, a small change in K_t^*, ΔK. Thus, one can write

$$K_x^* = K_t^* + \Delta K \qquad (2.4\text{-}5)$$

as a result of small deviation $\Delta\underline{x}$, and a corresponding small change $\Delta\underline{\lambda}$ such that

$$\underline{\lambda}_x^* = \underline{\lambda}_t^* + \Delta\underline{\lambda} \qquad (2.4\text{-}6)$$

Inserting (2.4-5),(2.4-6) into (2.4-3) gives

$$(K_t^* + \Delta K)(\|\underline{\lambda}_t^{*T} + \Delta\underline{\lambda}^T)\underline{h}(t_f,s)\|_q)^q = 1 \qquad (2.4\text{-}7)$$

Taking a Taylor series expansion in the left hand side of (2.4-7) gives

$$K_t^*(\|\underline{\lambda}_t^{*T}\underline{h}\|_q)^q + K(\|\underline{\lambda}_t^{*T}\underline{h}\|_q)^q + qK_t^*\int_t^{t_f}\underline{h}^T(t_f,s)\left|\underline{\lambda}_t^{*T}\underline{h}\right|^{q-1}$$

$$\operatorname{sgn}(\underline{\lambda}_t^{*T}\underline{h}(t_f,s)\ ds\ \Delta\underline{\lambda} + R = 1 \qquad (2.4\text{-}8)$$

SIMPLIFIED SUBOPTIMAL CONTROL SCHEME

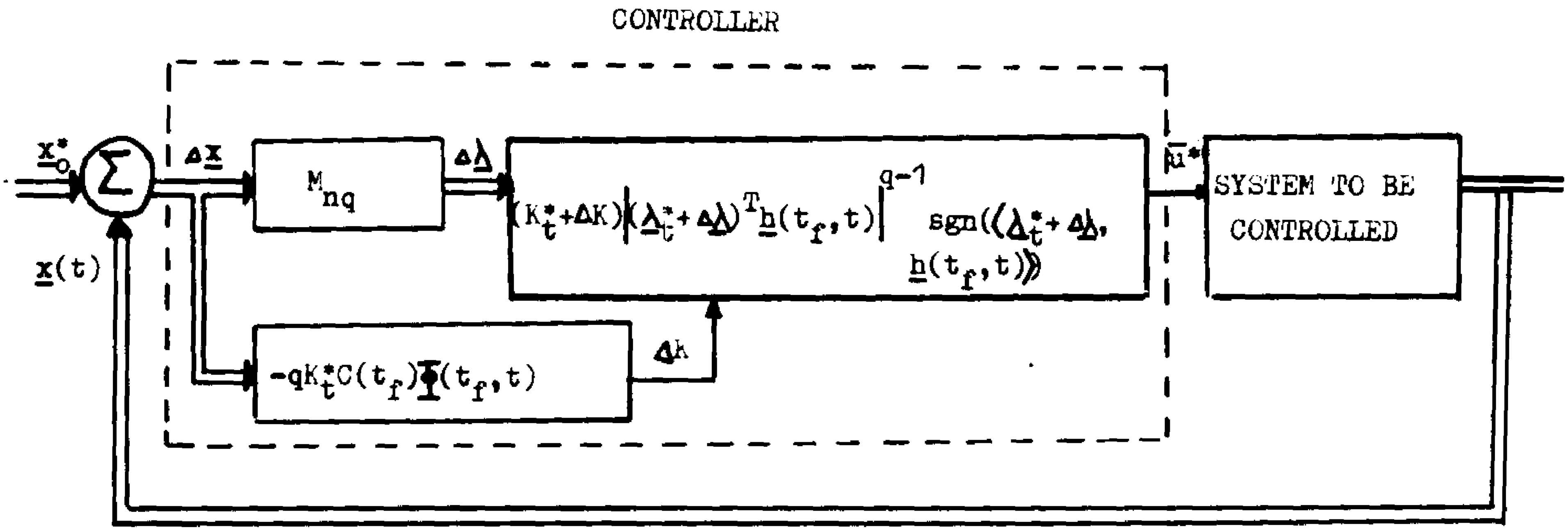


Fig. 2-3

where R is the remainder.

Note that the third term in the left hand side of (2.4-8) becomes $q\ \underline{e}^T(t_f,\underline{x}_o^*(t))\Delta\underline{\lambda}$ and the zero order terms of ΔK and $\Delta\underline{\lambda}$ in both sides of equation (2.4-8) vanish.

Collecting the first order terms of ΔK, $\Delta\underline{\lambda}$ gives

$$\Delta K = -\ q\ K_t^*\ \underline{e}(t_f,\underline{x}_o^*(t))^T\ \Delta\underline{\lambda} \qquad (2.4\text{-}9)$$

In similar way one obtains from (2.4-4)

$$\underline{e}(t_f,\underline{x}_o^*(t))\Delta\underline{\lambda} = -\Delta\underline{e}\ \lambda_t^* \qquad (2.4\text{-}10)$$

where $\Delta\underline{e}$ was specified in (2.3-14)

Inserting (2.4-10) into (2.4-9) gives

$$K = -\ q\ K_t^*\ (\ C(t_f)\underline{\Phi}(t_f,t)\Delta\underline{x}\ )^T\ \underline{\lambda}_t^* \qquad (2.4\text{-}11)$$

Therefore, one can obtain the suboptimal controller $\overline{u}^*(t)$ as

$$\overline{u}^*(t) = (K_t^*+\Delta K)\left|(\underline{\lambda}_t^* + \Delta\underline{\lambda})^T\underline{h}(t_f,t)\right|^{q-1} \operatorname{sgn}((\underline{\lambda}_t^* + \Delta\underline{\lambda})^T\underline{h}(t_f,t)) \qquad (2.4\text{-}12)$$

where $\Delta\underline{\lambda}$, ΔK are obtained from (2.3-3), (2.4-11), respectively. The resulting feedback suboptimal control system can be shown in fig. 2-3 .

2.5 Error Analysis

Let $\underline{y}(t) = \underline{x}(t)$, i.e., $C(t) = I$ for the sake of simplicity and let $\Delta \underline{x}^d$ be the terminal error deviated from the desired state $\underline{x}^d$ at the terminal time t_f as a result of small deviation $\Delta \underline{x}$. In this section we deal with the following problem: For a given positive number r_o such that

$$\| \Delta \underline{x}^d \| \leqslant r_o \tag{2.5-1}$$

find a number $r_n(t, r_o)$ which satisfies

$$\| \Delta \underline{x}(t) \| \leqslant r_n(t, r_o) \tag{2.5-2}$$

while employing the suboptimal control scheme proposed. One may give an explanation of the phisical meaning of the problem as that of finding a bound on the size of the allowed deviation from the nominal trajectory corresponding to a given terminal error.

To solve the problem, recall that

$$\lambda_x^{*T} \underline{e}(t_f, \underline{x}^*(t)) = 1 \tag{2.5-3}$$

where $\underline{e}(t_f, \underline{x}^*(t)) = \underline{x}^d - \Phi(t_f, t)\, \underline{x}^*(t)$

Note that (2.5-3) can be obtained by premultiplying (2.3-18) by λ_x^* and that equality in (2.5-3) must hold for the optimum lambda vector λ_x^* corresponding to $\underline{x}^*(t)$. However, in the suboptimal control scheme, λ_x^* has been approximately estimated as

$$\lambda_x^* = \lambda_t^* + \left[M_{nq}\right]\Delta \underline{x} \qquad (2.5\text{-}4)$$

When (2.5-4) is inserted into (2.5-3), the equality in (2.5-3) dose not hold and one may expect the terminal error $\Delta \underline{x}^d$ deviated from the desired state $\underline{x}^d$ due to the approximate value of λ_x^*. Therefore, one may rewrite (2.5-3) as

$$(\lambda_t^* + \left[M_{nq}\right]\Delta \underline{x})^T\left(\underline{x}^d + \Delta \underline{x}^d - \Phi(t_f,t)(\underline{x}_o^*(t) + \Delta \underline{x}(t))\right) = 1 \qquad (2.5\text{-}5)$$

Note that the matrix $\left[M_{nq}\right] = -\left[R_{nq}\right]C(t_f)\Phi(t_f,t)$ has been chosen so that the zero-order and the first-order terms in $\underline{x}$ vanish, i.e.,

$$\lambda_t^{*T}(\underline{x}^d - \Phi(t_f,t)\underline{x}_o^*(t)) = 1 \text{ and}$$

$$-\lambda_t^{*T}\Phi(t_f,t)\Delta \underline{x} + (\underline{x}^d - \Phi(t_f,t)\,\underline{x}_o^*(t))^T\left[M_{nq}\right]\Delta \underline{x} = 0 \qquad (2.5\text{-}6)$$

To see the reason why (2.5-6) dose hold, recall first that

$$\Delta \underline{e} = -\Phi(t_f,t)\Delta \underline{x}(t); \quad \underline{e}(t_f,\underline{x}_o^*(t)) = \underline{x}^d - \Phi(t_f,t)\underline{x}_o^*(t); \Delta \underline{e} = \left[R_{nq}^{-1}\right]\Delta\lambda$$

and therefore, one may rewrite the left hand side of (2.5-6) as $(\lambda_t^{*T}\left[R_{nq}^{-1}\right] + \underline{e}(t_f,\underline{x}_o^*(t))^T\Delta\lambda$ which is identically zero since premultiplying $\left[R_{nq}^{-1}\right]$ given in (2.3-21), by λ_t^{*T} gives $\underline{e}(t_f,\underline{x}_o^*(t))^T$. Therefore, (2.5-5) is reduced to

$$(\lambda_t^* + \left[M_{nq}\right]\underline{x})^T\Delta \underline{x}^d = (\left[M_{nq}\right]\underline{x})^T\Phi(t_f,t)\Delta \underline{x} \qquad (2.5\text{-}7)$$

Now applying Schwarz's inequality in the left hand side of (2.5-7) and solving the inequality equation for $\Delta \underline{x}^d$ gives

$$\left\| \Delta \underline{x}^d \right\| \geqslant \frac{\left| \Delta \underline{x}^T \left[M_{nq} \right]^T \Phi(t_f, t) \Delta \underline{x} \right|}{\left\| \Delta_t^* \right\| + \left\| \left[M_{nq} \right] \Delta \underline{x} \right\|} \qquad (2.5\text{-}8)$$

where the norm is taken as the square root of the inner product of vectors. Recall that our objective is to obtain $r_n(t, r_o)$ such that $\|\Delta \underline{x}\| \leqslant r_n(t, r_o)$, where $r_n(t, r_o)$ is the upper bound on $\|\Delta \underline{x}\|$. To do this one expresses the right hand side of inequality (2.5-8) in terms of $\|\Delta \underline{x}\|$: For this purpose, make use of a relation [31]

$$\sqrt{\rho_{min}(A^T A)} \leqslant \frac{\| A \underline{x} \|}{\| \underline{x} \|} \leqslant \sqrt{\rho_{max}(A^T A)} \qquad (2.5\text{-}9)$$

where $\rho_{min}(A^T A)$ and $\rho_{max}(A^T A)$ are the minimum and maximum eigenvalues of the matrix $(A^T A)$ respectively.

In order to obtain the least upper bound on $\|\Delta \underline{x}\|$ one takes the coefficient of the first order and the second order of $\|\Delta \underline{x}\|$ as $\sqrt{\rho_{min}\left\{ \left[M_{nq}^T \right] \left[M_{nq} \right] \right\}}$ and $\sqrt{\rho_{max}\left\{ \left[M_{nq}^T \right] \left[M_{nq} \right] \right\} \rho_{max}(\Phi^T \Phi)}$ respectively. Thus, one can find the following inequality

$$\frac{\sqrt{\rho_{max}\left\{ \left[M_{nq}^T \right] \left[M_{nq} \right] \right\} \rho_{max}(\Phi^T \Phi)} \, \|\Delta \underline{x}\|^2}{\| \Delta_t^* \| + \sqrt{\rho_{min}\left\{ \left[M_{nq}^T \right] \left[M_{nq} \right] \right\}} \, \|\Delta \underline{x}\|} \geqslant \frac{\left\| \Delta \underline{x}^T \left[M_{nq}^T \right] \Phi(t_f, t) \Delta \underline{x} \right\|}{\| \Delta_t^* \| + \left\| \left[M_{nq} \right] \Delta \underline{x} \right\|}$$

(2.5-10)

Now solving

$$r_o \geqslant \frac{\sqrt{\rho_{max}\{[M_{nq}^T][M_{nq}]\}\,\rho_{max}(\Phi^T\Phi)}\,\|\Delta\underline{x}\|^2}{\|\lambda_t^*\| + \rho_{min}(\{[M_{nq}^T][M_{nq}]\})^{1/2}\|\Delta\underline{x}\|}$$

for $\underline{x}$ and noting $\underline{x}$ 0 gives

$$\|\Delta\underline{x}\| \leqslant \frac{r_o a_2 + \sqrt{(r_o^2\ a_2^2 + 4r_o a_1\|\lambda_t^*\|)}}{2a_1} \qquad (2.5\text{-}11)$$

where $a_1 \triangleq \sqrt{\rho_{max}\{[M_{nq}^T][M_{nq}]\}\,\rho_{max}(\Phi^T\Phi)}$ and

$a_2 \triangleq \sqrt{\rho_{min}\{[M_{nq}^T][M_{nq}]\}}$

Therefore, one determines $r_n(t,r_o)$ from (2.5-11) as

$$r_n(t,r_o) = \frac{r_o a_2 + \sqrt{r_o^2\ a_2^2 + 4r_o a_1\|\lambda_t^*\|}}{2a_1} \qquad (2.5\text{-}12)$$

2.6 Design Procedure

A design procedure for the problem 1 developed in chapter II can be summarized as follows:

<u>Step 1</u> Calculate the state transition matrix $\Phi(t_f,t)$, the optimum lambda vector λ^* and λ_t^* defined by (2.3-8). Calculate the nominal optimal control $u^*(t)$ and its corresponding trajectory $\underline{x}_o^*(t)$.

<u>Step 2</u> Calculate the appropriate linear transformation matrix

$\left[M_{nq}\right] = -\left[R_{nq}\right] C(t_f)\Phi(t_f,t)$, where the matrix $\left[R_{nq}\right]$ is the inversion of $\left[R_{nq}^{-1}\right]$ and the matrix $\left[R_{nq}^{-1}\right]$ is defined by (2.3-21)

Step 3 Calculate a scalar function K_t^* and K, where

$$K_t^* = \frac{1}{(\|\langle \underline{\lambda}_t^*, \underline{h}(t_f,t)\rangle\|_q)^q}$$

and ΔK is defined by (2.4-11)

Step 4 On-line computation on the deviation $\Delta \underline{x}$ and the suboptimum controller $\overline{u}^*(t)$ defined by (2.4-12).

2.7 Illustrative Example

To clarify the application of the theory developed in this chapter, an illustrative example is presented in this section.

Consider the inertia plant which can be represented as the dynamic system which is governed by (1-1) and (1-2) where the matrices A, B, C in this case are given respectively as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (2.7\text{-}1)$$

Given an initial state $\underline{x}_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ at time $t = t_i = 0$ and the desired output $\underline{y}(t_f) = \underline{y}^d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at time $t_f=2$, we wish to solve the problem 1 for the case of $p = \infty$ (minimizing the maximum amplitude of the control u(t))

Step 1 Calculate the state transition matrix, the optimum lambda vector λ^* and λ_t^*. Calculate the nominal optimum control and its corresponding nominal trajectory $\underline{x}_o^*$:

(a) The state transition matrix for this system can be obtained [3.31] as

$$\Phi(t_f,t) = \begin{bmatrix} 1 & t_f-t \\ 0 & 1 \end{bmatrix}$$

(b) Making use of a method of functional analysis reviewed in appendix A, one can obtain the optimum lambda vector

$$\lambda^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad (2.7\text{-}2)$$

and $\lambda_t^* = m(t_f,t)\,\lambda^*$, where $m(t_f,t)$ is obtained from (2.3-11) with t_f,λ^* replaced by 2, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively, i.e.,

$$m(2,t) = 1/(-x_{o1}^*(t) - (1-t)x_{o2}^*(t))$$

where x^*_{o1} and x_{o2}^* indicate elements of $\underline{x}_o^*(t)$

(c) The nominal optimal control $u^*(t)$ becomes

$$u^*(t) = \text{sgn}(1-t) \tag{2.7-3}$$

(d) The nominal optimal trajectory $\underline{x}_o^*(t)$ valid for $t \leqslant 1$ is given by

$$\underline{x}_o^*(t) = \begin{bmatrix} -1 + t^2/2 \\ 2 \end{bmatrix} \tag{2.7-4}$$

and for $t \geqslant 1$ is given by

$$\underline{x}_o^*(t) = \begin{bmatrix} -2 + 2t - 1/2t^2 \\ 2-t \end{bmatrix} \tag{2.7-5}$$

Step 2 Calculate the appropriate linear transformation matrix $\left[M_{n1}\right]$

(a) $0 \leqslant t < 1$ (during the first switching interval)

To obtain the matrix $\left[M_{n1}\right]$ (see (2.3-15)) one first obtains the inversion of $\left[R_{n1}^{-1}\right]$, where $\left[R_{n1}^{-1}\right]$ can be obtained from (2.3-24) with t_f, k_1, s_1^*, $\underline{h}(t_f, s)$ replaced by 2, 1, 1, $\begin{bmatrix} 2-s \\ 1 \end{bmatrix}$ respectively.

Performing the inversion of the matrix $\left[R_{n1}^{-1}\right]$ yields

$$\left[R_{n1}\right] = -m(2,t)^3/2 \begin{bmatrix} 2/m(2,t) - e_2^{*2} & -2/m(2,t) + e_1^* e_2^* \\ -2/m(2,t) + e_1^* e_2^* & 2/m(2,t) - e_1^{*2} \end{bmatrix}$$

where e*'s are the elements of the vector $\underline{e}(t_f, \underline{x}_o^*(t))$

which is obtained from (1-5) as

$$\underline{e}(2,\underline{x}_o^*(t)) = -\begin{bmatrix}1 & 2-t\\ 0 & 1\end{bmatrix}\underline{x}_o^*(t)$$

(b) $1 \leq t < 2$

In this interval the matrix R_{n1}^{-1} becomes

$$\left[R_{n1}^{-1}\right] = -\begin{bmatrix}e_1^{*2} & e_1^* e_2^*\\ e_1^* e_2^* & e_2^{*2}\end{bmatrix}$$

Since the determinant of $\left[R_{n1}^{-1}\right]$ is zero, one makes use of the pseudo-inverse method described in section (2.3.2) to obtain the approximate inversion. To do this decompose $\left[R_{n1}^{-1}\right]$ appropriately as

$$A = -\begin{bmatrix}1\\ e_2^*/e_1^*\end{bmatrix}, \quad B = \left[e_1^{*2} \quad e_1^* e_2^*\right]$$

and performing the indicated calculation of (2.3-28) gives

$$\left[\overline{R}_{n1}\right] = -1/(e_1^{*2} + e_2^{*2})^2 \begin{bmatrix}e_1^{*2} & e_1^* e_2^*\\ e_1^* e_2^* & e_2^{*2}\end{bmatrix} \qquad (2.7\text{-}7)$$

Therefore, $\left[M_{n1}\right]$ is obtained from (2.3-15) as

$$\left[M_{n1}\right] = -\left[R_{n1}\right]\begin{bmatrix}1 & 2-t\\ 0 & 1\end{bmatrix} \qquad (2.7\text{-}8)$$

where $[R_{n1}]$ is specified in (2.7-6) and (2.7-7) for the interval $[0,1)$ and $[1,2)$ respectively.

<u>Step 3</u> Precomputation of K_t^* and ΔK

(a) K_t^* is obtained from (2.4-2) with λ_x^*, $\underline{x}^*(t)$, t_f, q replaced by λ_t^*, $\underline{x}_o^*(t)$, 2, 1, respectively:

$$K_t^* = \frac{1}{\| h(\lambda_t^*)\|_1} \qquad (2.7\text{-}9)$$

(b) ΔK is obtained from (2.4-11), where q=1, $C(t_f)=I$, the transition matrix $\Phi(2,t)$ and the optimum lambda vector λ_t^* are found in step 1.

<u>Step 4</u> On-line computation of the suboptimal controller

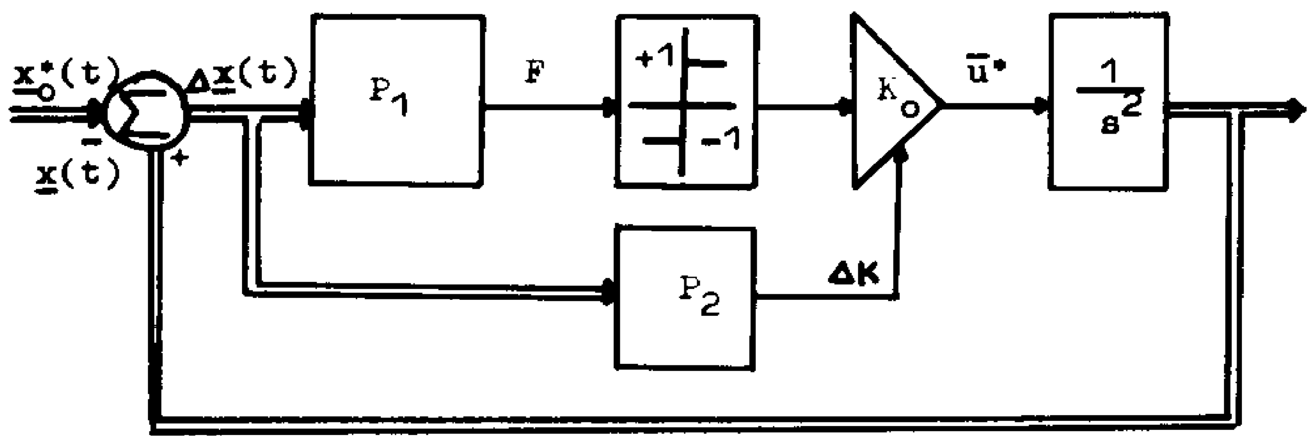
The suboptimal controller can be simply programmed from (2.4-12) as

$$\bar{u}^*(t)=(K_t^*+ K)\mathrm{sgn}((1+\Delta\lambda_1/m(2,t))(2-t)+(-1+\Delta\lambda_2/m(2,t))$$

where $\Delta\lambda$'s are the elements of the vector $\Delta\lambda$ and are obtained through (2.3-3) and $[M_{n1}]$ is specified by (2.7-6).

To demonstrate the effectiveness of the technique developed in this Chapter, the example was simulated using a digital computer for the initial state $x_{o1}=-1.0$, $x_{o2}= 0.0$ at time t = 0. In this simulation for practical purposes

SUBOPTIMAL CONTROL SYSTEM OF THE EXAMPLE



where

K_o, P_1, P_2 are realized so that

$$K_o = 1 + \Delta K$$

$$F = 1-t -0.5\Delta x_1 +(0.75+0.5t)\Delta x_2$$

$$\Delta K = (\Delta x_1 + (1-t)\Delta x_2)m(2,t), \quad m(2,t)=1/(x_{o1}^*-(1-t)x_{o2}^*)$$

and

$$x_{o1}^* = -1 + t^2/2$$

$$x_{o2}^* = t \qquad \text{for} \quad 0 \leqslant t \leqslant 1 \; ;$$

$$x_{o1}^* = -2 +2t-0.5t^2$$

$$x_{o2}^* = 2-t \qquad \text{for } 1 \leqslant t \leqslant 2.$$

Fig.2-4

SUMMARY OF PERFORMANCE OF SUBOPTIMAL CONTROL SYSTEM

INITIAL STATE	SUBOPTIMAL SYSTEM		OPEN-LOOP SYSTEM USING NOMINAL CONTROL	
	MAGNITUDE OF CONTROL	TERMINAL ERROR	MAGNITUDE OF CONTROL	TERMINAL ERROR
$x_1(o) = -0.90$ $x_2(o) = -0.10$	1.0	$x_1(2)=0.060$ $x_2(2)=0.060$	1.0	$x_1(2)=-0.1$ $x_2(2)=-0.1$
$x_1(0) = -0.90$ $x_2(0) = 0.10$	0.8	$x_1(2)=0.067$ $x_2(2)=0.067$	1.0	$x_1(2)=0.3$ $x_2(2)=0.1$
$x_1(0) = -1.10$ $x_2(0) = -0.10$	1.2	$x_1(2)=0.053$ $x_2(2)=0.053$	1.0	$x_1(2) = -0.3$ $x_2(2) = -0.1$
$x_1(0) = -1.10$ $x_2(0) = 0.10$	1.0	$x_1(2)=0.059$ $x_2(2)=0.059$	1.0	$x_1(2)= 0.1$ $x_2(2) = 0.1$

Table 2-1

PHASE PORTRAIT AND CONTROL OF THE EXAMPLE

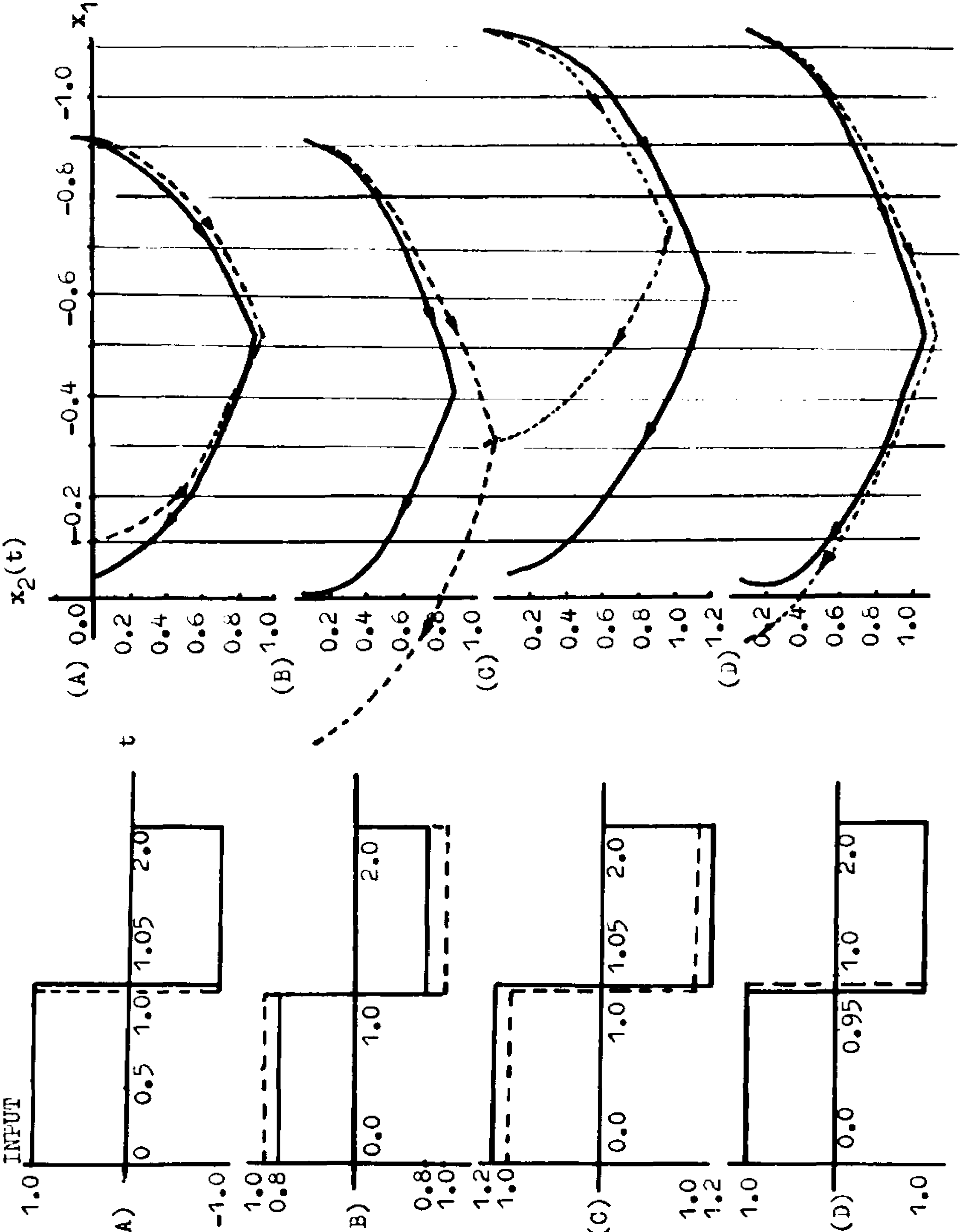


Fig.2-5

$[M_{n1}]$ was evaluated at time t=1 second, i.e., the matrix $[M_{n1}]$ which transforms $\Delta\underline{x}$ into $\Delta\underline{\lambda}$ becomes constant. This simplification is not unreasonable since deviations are most effective around the switching instant t=1 second. The suboptimal control system based on this simplification is obtained as shown in fig. (2-4) and some parameters of controller to be realized are given there. In this simulation, the controller measured the output and corrected the preprogrammed control law at every 0.05 seconds. As the process time approaches the terminal time 2 seconds, the maginitude of m(2,t) becomes large and hence one cannot realize this. In this simulation the function m(2,t) was allowed to saturate at 1000. Under these limitations the performances between the open loop nominal control without correction regardless of deviation $\Delta\underline{x}$ and the suboptimal one are compared and are summarized in table 2-1.

According to the error analysis, one could estimate a bound on the size of deviation $\Delta\underline{x}$: For a given positive number r_o=0.1 such that $\|\Delta\underline{x}^d\| \leqslant r_o$, one wishes to find $r_n(0,r_o)$. Using (2.5- 2) one obtains $r_n(0,0.1)$ = 0.189 which is allowable bound on initial deviation that will result in the allowed terminal error.

The phase portrait in x_1-x_2 plane for each case A,B,C,D is given in fig.2-5, where the actual trajectory for each case, indicated by the solid line is compared with the trajectory due to the open loop nominal control without correction, indicated by the dot line. And also in fig 2-5 the suboptimal control for each case is compared with the open loop nominal one.

2.8 Summary

In this chapter it was shown that a design procedure of a feedback suboptimum control law for the minimum norm control problem was developed. The assumptions on which the technique is based are that the dynamic system operates in the presence of small deviations during the actual operation and that the deviations are that due to the unknown external forces and all the system parameters, reference output and sensing devices are idealized.

The limitation of the approach is that one cannot apply this technique to the system which is not normal.

III SOLUTION OF PROBLEM 2 THE LAMBDA METHOD

3.1 Introduction

It has been shown in chapter II that problem 1 can be reduced to the problem of obtaining a suboptimal lambda vector in terms of the initial state for the given terminal time t_f and also that the problem of estimating the suboptimal lambda vector can be reduced for certain cases to a relatively simpler problem, that of finding an inversion of an m x m linear transformation matrix $[R_{nq}]$.

In this chapter we explore the feasibility of deriving a suboptimal control law for problem 2 based on an idea and scheme similar to that used in obtaining the suboptimal lambda vector and an approximate expression for the terminal time of the given problem. Also it turns out that the problem of estimating the suboptimal lambda vector and the terminal time can be reduced to the problem of finding an inversion of a suitable linear transformation matrix.

3.2 A Theorem on Obtaining λ^* and t_f^*

To develop a Theorem on obtaining an optimum pair optimum lambda vecor λ^* and the least terminal time t_f^*, we first recall that according to theorem 1 one could find the optimum lambda vector of the problem in terms of initial state for the given terminal time t_f, i.e., it is obtained as a solution of

$$\underline{e}(t_f,\underline{x}(t_i)) = \frac{\int_{t_i}^{t_f} \underline{h}(t_f,s) \left| h(\underline{\lambda}^*) \right|^{q-1} \operatorname{sgn}(h(\underline{\lambda}^*))ds}{\| h(\underline{\lambda}^*) \|_q^{\,q}} \tag{3.2-1}$$

where t_f is fixed in the minimum norm control problem. However, in the minimum time control problem, the terminal time t_f is not fixed but must be determined as a part of the problem. To determine the least terminal time t_f^*, one recalls that the norm of control u(t) is bounded by L_o, $\|u\|_p \leq L_o$. In fact, as has been reviewed in appendix A, one can determine t_f^* as the least value of t_f for which (A-23) in appendix A is satisfied with the equality sign namely

$$\frac{1}{\| h(\underline{\lambda}^*)\|_q} = L_o \tag{3.2-2}$$

subject to

$$\langle \underline{\lambda}^*, \underline{e}(t_f,\underline{x}(t_i))\rangle = 1 \tag{3.2-3}$$

where $\underline{\lambda}^*$ is the optimum lambda vector obtained as a solution of (2.2-5), namely, (3.2-1).

Inserting (3.2-2) into (3.2-1) with t_f replaced by the least terminal time t_f^* gives

$$\underline{e}(t_f^*,\underline{x}(t_i)) = L_o^q \int_{t_i}^{t_f^*} \underline{h}(t_f^*,s)\left|\langle\underline{\lambda}^*,\underline{h}(t_f^*,s)\rangle\right|^{q-1} \operatorname{sgn}\langle\underline{\lambda}^*,\underline{h}(t_f^*,s)\rangle \, ds \tag{3.2-4}$$

Therefore, the optimum lambda vector $\underline{\lambda}^*$ and the least terminal time t_f^* must satisfy (3.2-1) to (3.2-3) or equivalently (3.2-3) and (3.2-4). Thus one can state a theorem as a summary:

<u>Theorem 3</u>

If there exists a time optimal control for the problem 2 with an initial state $\underline{x}(t_i)$, an optimum pair $\underline{\lambda}^*$ and t_f^* corresponding to the initial state $\underline{x}(t_i)$ can be obtained as a solution of (3.2-3) and (3.2-4) simultaneously.

Thus, (m+1) undetermined control parameters $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*, t_f^*$ may be determined from the solution of (m+1) nonlinear algebraic equations given by (3.2-3) and (3.2-4) for a given initial state $\underline{x}(t_i)$ and for all q, $1 \leq q < \infty$. For $q = \infty$: $p=1$ the theorem 3 is still valid. However, in this case the limiting process $q \to \infty$ is required and depends on the property of $\langle \lambda^*, \underline{h}(t_f^*, s) \rangle$.

3.2.1 λ^* and t_f^* for Single Input and Single Output Systems

As the first application of theorem 3, consider the dynamic systems which consist of single input and single output. In this case the optimum lambda vector λ^* becomes a scalar. Thus, from (3.2-3) and (3.2-4) we have two nonlinear algebraic equations for two unknowns λ^* and t_f^*

$$\lambda^* = 1/e(t_f^*, \underline{x}(t_i)) \tag{3.2-5}$$

where t_f^* is the least value of t_f for which

$$e(t_f, \underline{x}(t_i)) = L_o^q \int_{t_i}^{t_f} \left| h(t_f, s) \right|^q ds \tag{3.2-6}$$

is satisfied for a given q, $1 \leq q < \infty$.

3.2.2 λ^* and t_f^* for Single Input and Multiple Outputs Systems

<u>Case 1</u> Energy constraint on control input(p=2:q=2)

As the second application of theorem 3, consider the dynamic systems which consist of a single input and multiple outputs, where the control input is bounded in energy,i.e., $\|u\|_2 \leq L_o$. In this case also, using theorem 3 gives

$$\Delta^* = 1/L_o^2 \, W(t_f^*,t_i) \, \underline{e}(t_f^*,\underline{x}(t_i)) \qquad (3.2\text{-}7)$$

where

$$W(t_f^*,t_i) \; = \left[\int_{t_i}^{t_f^*} \underline{h}(t_f^*,s)\underline{h}(t_f^*,s)^T \, ds\right]^{-1}$$

and t_f^* is the least value of t_f for which

$$\underline{e}(t_f,\underline{x}(t_i))W(t_f,t_i)\underline{e}(t_f,\underline{x}(t_i)) = L_o^2 \qquad (3.2\text{-}8)$$

must be satisfied.

As has been pointed out in Remark 2-4, the completely controllability of the system guarantees that $W(t_f^*,t_i)$ is positive definite. To compute the least terminal time t_f^* one solves an nonlinear algebraic equation which may not be always easy for every initial state $\underline{x}(t_i)$. However, when this problem is attempted by the method of the Maximum Principle [27], one must solve a two-point boundary-value problem which may not be well suited for computer methods.

Case 2 Magnitude bound on control input($p=\infty$: $q=1$)

It is seen from (3.2-3) and (3.2-4) that an immediate difficulty is raised in computing the optimum pair Δ^* and t_f^* for every initial state as the dimension of output vector is increased. Therefore, one must seek a method for computing an approximate optimum pair. A sub-optimal computation of the optimum pair will be discussed in section 3.3.

3.3 A Development of Minimum Time Suboptimum Control Systems

As has been mentioned, the computation of λ^* and t_f^* for every $\underline{x}(t_i)$ are in general difficult and so is the computation of the corresponding optimal control. However, a suboptimal control may be obtained by first approximating λ^* and t_f^* for every $\underline{x}(t_i)$. For this purpose one assumes that the dynamic system operates in the presence of small perturbations. Before presenting a method for obtaining approximate λ^* and t_f^* for every initial state, it may be helpful to define the following items:

(1) $u^*(t)$ is optimal control obtained in appendix A and can be written as

$$u^*(t) = u^*(T_x^*, \lambda_x^*, t) \qquad (3.3\text{-}1)$$

for an initial state $\underline{x}^*(t)$, where T_x^* is the computed le st elapsed time and λ_x^* is the optimum lambda vector which is the minimand of (A-12) with $\langle \lambda_x^*, \underline{e}(t_f^*, \underline{x}^*(t)) \rangle = 1$.

(2) $\underline{x}^*(t)$, the optimum trajectory, is the solution of

(1-1) with u(t) replaced by $u^*(T_x^*, \lambda_x^*, t)$. $\underline{x}(t_i) = \underline{x}^*(t)$.

(3) $\underline{x}_o^*(t)$, an nominal trajectory, is the solution of (1-1) with u(t) replaced by $u^*(T^*, \lambda^*, t)$ and $\underline{x}(t_i) = \underline{x}_o$, where T^* is the least elapsed time corresponding to initial state $\underline{x}_o$.

(4) $\underline{x}(t)$ is the actual trajectory not far from the nominal one.

(5) $\bar{u}^*(t)$ is the suboptimal control.

(6) $\underline{x}(t) = \underline{x}(t) - \underline{x}_o^*(t)$ and the deviation $\Delta\underline{x}$ is assumed to be measured continuously.

(7) λ_t^* is defined as the vector λ^* at every instant of time on the nominal trajectory $\underline{x}_o^*(t)$.

In the proposed scheme λ_x^* and T_x^* may be obtained by first finding $\Delta\lambda$(small change in λ_t^*) and Δt_f(small change in t_f^*) as a result of small change in $\underline{x}_o^*(t)$, $\Delta\underline{x}$, namely,

$$\lambda_x^* = \lambda_t^* + \Delta\lambda \qquad (3.3\text{-}2)$$

$$t_{fx}^* = t_f^* + \Delta t_f \qquad (3.3\text{-}3)$$

as a result of $\Delta\underline{x}$. When the deviation $\Delta\underline{x}$ is a small quantity, $\Delta\lambda$ and Δt_f can be approximately related to $\Delta\underline{x}$ by a linear transformation matrix. Suppose that the linear transformation matrix which maps $\Delta\underline{x}$ into $\Delta\lambda$ and Δt_f

is found(see section 3.3.1), i.e., suppose

$$\left[\frac{\Delta\lambda}{\Delta t_f}\right] = \left[\frac{M_{tq}}{g_q}\right] \Delta \underline{x} \ , \ 1 \leqslant q < \infty \qquad (3.3\text{-}4)$$

is known, where $\left[\frac{\Delta\lambda}{\Delta t_f}\right]$ is defined as the transpose of composite vector $\Delta\lambda_1 \cdot \Delta\lambda_2, \ldots, \Delta\lambda_n, \Delta t_f$. Then the feedback suboptimum control scheme can be as shown in fig 3-1. In this figure the box marked "suboptimum controller" realizes (3.3-1) with T_x^*, λ_x^* replaced by $T^* + \Delta T$, $\lambda_t^* + \Delta\lambda$, respectively, where $\Delta\lambda$ and Δt_f is given by (3.3-4):

$$\bar{u}^*(t) = \bar{L}_o(t)^q \left| \langle \lambda_t^* + M_{tq}\Delta\underline{x},\ \underline{h}(t_f^* + g_q\Delta\underline{x}, t)\rangle \right|^{q-1} \operatorname{sgn}(\langle \lambda_t^* + M_{tq}\Delta\underline{x},\ \underline{h}(t_f^* + g_q\ \Delta\underline{x},\ t)\rangle) \qquad (3.3\text{-}5)$$

where $\bar{L}_o(t)$ is the time varing bound; when $p = \infty$, $\bar{L}_o(t) = L_o$ since the magnitude bound is independent of time. However, this is not true for $1 < p < \infty$ since in this case the amount of "energy" associated with $\|u\|_p$ which can be utilized depends on the amount equal to the given L_o minus the energy expended during the interval t_i to t. Hence,

$$\bar{L}_o(t)^p = L_o^p - \int_{t_i}^{t} |u(s)|^p \, ds \ \text{for} \ 1 < p < \infty \qquad (3.3\text{-}6)$$

Note that $\bar{L}_o(t_i) = L_o$.

SUBOPTIMAL CONTROL SCHEME

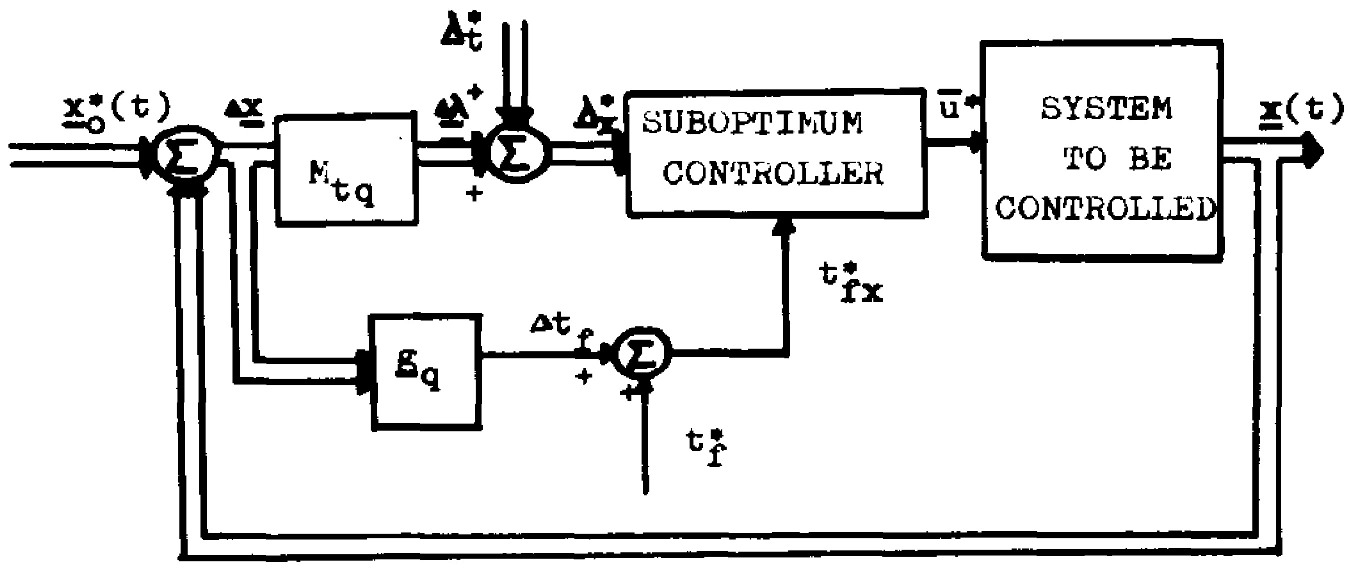


Fig. 3-1

To find Δ_x^* it is necessary to know Δ_t^* in addition to $\Delta\lambda$. As has been already shown in chapter II, one can determine Δ_t^* as

$$\Delta_t^* = m(t_f^*,t)\Delta\underline{\lambda}^*$$

where $m(t_f^*,t)$ is a scalar function specified by (2.3-8).

3.3.1 Derivation of Linear Transformation Matrix

In this section we derive an expression for the matrix $\left[\frac{M_{tq}}{g_q}\right]$ so that approximately

$$\left[\frac{\Delta\lambda}{\Delta t_f}\right] = \left[\frac{M_{tq}}{g_q}\right]\Delta\underline{x}$$

is satisfied for the sufficiently small deviations $\Delta\underline{x}$.

To do this we first note in appendix A that t_{fx}^* is the least value of t_f for which

$$\left[\int_t^{t_f}\left|\langle\Delta_x^*,\underline{h}(t_f,s)\rangle\right|^q ds\right]^{1/q} = 1/L_o(t) \qquad (3.3\text{-}7)$$

is satisfied with

$$\langle\underline{\Delta}_x^*,\ \underline{e}(t_f,\underline{x}^*(t))\rangle = 1 \qquad (3.3\text{-}8)$$

where $L_o(t)$ is the time varing bound and $L_o(t)^p$ is defined by (3.3-6).

According to the theorem 3, one can obtain Δ_x^* and t_{fx}^*

as s simultaneous solution of

$$\underline{e}(t^*_{fx},\underline{x}^*(t)) = L_o(t)^q \int_t^{t^*_{fx}} \underline{h}(t^*_{fx},s) \left| \langle \lambda^*_x, \underline{h}(t^*_{fx},s) \rangle \right|^{q-1}$$

$$\operatorname{sgn}\left(\langle \lambda^*_x, \underline{h}(t^*_{fx},s)\rangle\right) ds \triangleq \underline{r}(\lambda^*_x, t^*_{fx}) \qquad (3.3\text{-}9)$$

and

$$\langle \lambda^*_x, \underline{e}(t^*_{fx},\underline{x}^*(t)) \rangle = 1 \qquad (3.3\text{-}10)$$

As a simultaneous solution of (3.3-9) and (3.3-10) for a given initial state $\underline{x}^*(t)$, one can write

$\lambda^*_x = \lambda^*_x(\underline{x}^*(t),t)$ and $t^*_{fx} = t^*_{fx}(\underline{x}^*(t),t)$; note that when $\underline{x}^*(t) = \underline{x}_o$, then $\lambda^*_x = \lambda^*$ and $t^*_{fx} = t^*_f$; when $\underline{x}^*(t) = \underline{x}^*_o(t)$, then $\lambda^*_x = \lambda^*_t$ and $t^*_{fx} = t^*_f$.

Remark 3-1

For p=2:q=2 the simultaneous solution of (3.3-9) and (3.3-10) for λ^*_x and t^*_{fx} gives

$$\lambda^*_x = L_o(t)^{-2} W(t^*_{fx},t)\, \underline{e}(t^*_{fx},\underline{x}^*(t)) \qquad (3.3\text{-}11)$$

where t^*_{fx} is the least value of t_f for which

$$\underline{e}(t_f,\underline{x}^*(t))^T W(t_f,t)\underline{e}(t_f,\underline{x}^*(t)) = L_o(t)^2 \qquad (3.3\text{-}12)$$

is satisfied. *****

Now an approximate linear relation between $\Delta\underline{x}$ and $\Delta\lambda$, Δt_f is obtained through a Taylor series expansion of (3.3-9) and (3.3-10) around $\underline{x}_o^*(t)$ and Δ_t^* at time t.

To do this let $\Delta\underline{e}$ be defined as

$$\Delta\underline{e} \triangleq \underline{e}(t_f^*+\Delta t_f,\ \underline{x}_o^*(t)+\Delta\underline{x}(t))-\underline{e}(t_f^*,\underline{x}_o^*(t)) \qquad (3.3\text{-}13)$$

From the definition of $\underline{e}(t_f,\underline{x}(t))$(see (1-5)), one can write (3.3-13) as

$$\Delta\underline{e}=-(C(t_f^*+\Delta t_f)\Phi(t_f^*+\Delta t_f,t)(\underline{x}_o^*(t)+\Delta\underline{x}(t))-C(t_f^*)\Phi(t_f^*,t)\underline{x}_o^*(t)) \qquad (3.3\text{-}14)$$

since $\underline{y}_x(t_{fx}^*) = \underline{y}_o(t_f^*) = \underline{y}^d$, where $\underline{y}_x(t_{fx}^*)=C(t_{fx}^*)\underline{x}^*(t_{fx}^*)$ and $\underline{y}_o(t_f^*)=C(t_f^*)\underline{x}_o^*(t_f^*)$.

From (3.3-13), (3.3-9) one can express $\underline{e}$ as

$$\Delta\underline{e} = \underline{r}(\Delta_t^* + \Delta\lambda,\ t_f^* + \Delta t_f) - \underline{r}(\Delta_t^*,t_f^*) \qquad (3.3\text{-}15)$$

Substituting a Taylor series expansion for $\underline{r}(\Delta_t^*+ \Delta\lambda,t_f^*+\Delta t_f)$ into (3.3-15) gives

$$\Delta\underline{e} = \left[\frac{\partial \underline{r}}{\partial\Delta_t^*}\right]\Delta\lambda + \left[\frac{\partial \underline{r}}{\partial t_f^*}\right]\Delta t_f + R \qquad (3.3\text{-}16)$$

where R is the remainder and

$$\left[\frac{\partial \underline{r}}{\partial\Delta_t^*}\right]=\left[\frac{\partial\underline{r}(\Delta_x^*,t_f^*)}{\partial\Delta_x^*}\right]_{\Delta_x^*=\Delta_t^*} \quad \text{and} \quad \left[\frac{\partial \underline{r}}{\partial t_f^*}\right]=\left[\frac{\partial\underline{r}(\Delta_t^*,t_{fx}^*)}{\partial t_{fx}^*}\right]_{t_{fx}^*=t_f^*}$$

To evaluate $\left[\frac{\partial \underline{r}}{\partial\Delta_t^*}\right]$ and $\left[\frac{\partial \underline{r}}{\partial t_f^*}\right]$ one takes the symbolic derivative [31,33] for sufficiently small $\Delta\underline{e}$(that is, for sufficiently small $\Delta\underline{x}$),i.e.,

$$\left[\frac{\partial \Gamma}{\partial \Delta_t^*}\right] = \Gamma_o(t)^q \left(\int_t^{t_f^*} \underline{h}(t_f^*,s)\underline{h}(t_f^*,s)^T \left|\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right|^{q-2} ds(q-1) \right.$$

$$\left. + 2\int_t^{t_f^*} \underline{h}(t_f^*,s)\underline{h}(t_f^*,s)^T \left|\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right|^{q-1} \delta\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle \right) ds \tag{3.3-17}$$

where $\delta(\ )$ is a delta function [31,33].

One also obtains

$$\left[\frac{\partial \Gamma}{\partial t_f^*}\right] = \Gamma_o(t)^q \left(\underline{h}(t_f^*,t_f^*) \left|\langle \Delta_t^*, \underline{h}(t_f^*,t_f^*)\rangle\right|^{q-1} \operatorname{sgn}\left(\langle \Delta_t^*, \underline{h}(t_f^*,t_f^*)\rangle\right) \right.$$

$$+ \int_t^{t_f^*} \dot{\underline{h}}(t_f^*,s) \left|\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right|^{q-1} \operatorname{sgn}\left(\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right) ds +$$

$$\int_t^{t_f^*} \underline{h}(t_f^*,s)\langle \Delta_t^*, \dot{\underline{h}}(t_f^*,s)\rangle \left|\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right|^{q-2} \operatorname{sgn}\left(\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right) ds(q-1)$$

$$\left. + 2\int_t^{t_f^*} \underline{h}(t_f^*,s) \left|\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle\right|^{q-1} \langle \Delta_t^*, \dot{\underline{h}}(t_f^*,s)\rangle \delta\langle \Delta_t^*, \underline{h}(t_f^*,s)\rangle \right) ds \tag{3.3-18}$$

where $\dot{\underline{h}}(t_f^*,s) = \left.\left|\frac{\partial \underline{h}(t_{fx}^*,s)}{\partial t_{fx}^*}\right|\right|_{t_{fx}^* = t_f^*}$

The above equation (3.3-17) and (3.3-18) can be simplified using the formula indicated in (2.3-23),namely, after evaluating the integrals involing delta functions in

(3.3-17), (3.3-18) by the use of (2.3-23) and employing the properties of the transition matrix [3]

$$\dot{\Phi}(t_f^*, s) = A(t_f^*)\Phi(t_f^*, s)$$

then one can simplify (3.3-16) for the following cases:

(i) Magnitude constraint on control ($p=\infty$: $q=1$)

Let $C = I$ for simplicity.

$$\Delta \underline{e} = \Big[L_o(\underline{h}(t_f^*, t_f^*)\,\mathrm{sgn}(\langle \lambda_t^*, \underline{h}(t_f^*, t_f^*)\rangle) + 2\sum_v \frac{\underline{h}(t_f^*, s_v^*)\underline{h}(t_f^*, s_v^*)^T}{\left|\frac{d}{ds}(\langle \lambda_t^*, \underline{h}(t_f^*, s)\rangle)\right|_{s_v^*}} A(t_f^*)^T \lambda_t^*$$

$$+ A(t_f^*)\underline{e}(t_f^*, \underline{x}_o^*(t)\Big]\Delta t_f + 2L_o \sum_v \frac{\underline{h}(t_f^*, s_v^*)\underline{h}(t_f^*, s_v^*)}{\left|\frac{d}{ds}\langle \lambda_t^*, \underline{h}(t_f^*, s)\rangle\right|_{s_v^*}} \Delta\lambda \tag{3.3-19}$$

(ii) Energy constraint on control($p=2$:$q=2$)

$$\Delta \underline{e} = \Big[L_o(t)^2 (\underline{h}(t_f^*, t_f^*)\langle \lambda_t^*, \underline{h}(t_f^*, t_f^*)\rangle + \int_t^{t_f^*} \underline{h}(t_f^*, s)\underline{h}(t_f^*, s)^T \cdot$$

$$\mathrm{sgn}(\langle \lambda_t^*, \underline{h}(t_f^*, s)\rangle)\; A(t_f^*)^T \lambda_t^*\; ds) + A(t_f^*)\underline{e}(t_f^*, \underline{x}_o^*(t))\Big]\Delta t_f$$

$$+ L_o(t)^2 \int_t^{t_f^*} \underline{h}(t_f^*, s)\underline{h}(t_f^*, s)^T\, ds\, \Delta\lambda \tag{3.3-20}$$

where $C(t) = I$ for the sake of simplicity

As indicated in (3.3-16), in particular, (3.3-19) and (3.3-20) for the case of $p=\infty$ and $p = 2$ respectively, only

m equations are given for (m + 1) unknowns $\Delta\underline{\lambda}$, Δt_f.
An additional equation may be obtained from (3.3-10) in a similar way that we have obtained (3.3-16) through a Taylor series expansion:

$$\underline{\lambda}_t^{*T}\Delta\underline{e} = -\Delta\underline{\lambda}^T\underline{e}(t_f^*,\underline{x}_o^*(t)) \qquad (3.3\text{-}21)$$

To find $\begin{bmatrix} M_{tq} \\ \hline \underline{g}_q \end{bmatrix}$ which maps $\Delta\underline{x}$ into $\Delta\underline{\lambda}$, one notes from (3.3-14) in case of C(t) = I that

$$\Delta\underline{e} = -(A(t_f^*)\Phi(t_f^*,t)\underline{x}_o^*(t)\Delta t_f - \Phi(t_f^*,t)\Delta\underline{x}) \qquad (3.3\text{-}22)$$

Thus, the linear transformation matrix can be obtained approximately from (3.3-21) and (3.3-16) after discarding the remainder terms in (3.3-16), i.e.,

$$\begin{bmatrix} M_{tq} \\ \hline \underline{g}_q \end{bmatrix} = \begin{bmatrix} W_q \\ \hline \underline{z}_q \end{bmatrix}^{-1} \begin{bmatrix} -\Phi(t_f^*,t) \\ \hline -\underline{\lambda}_t^{*T}\Phi(t_f^*,t) \end{bmatrix} \qquad (3.3\text{-}23)$$

where the submatrices W_q and $\underline{z}_q$ can be properly defined from (3.3-16), (3.3-21).

The inversion of the matrix $\begin{bmatrix} W_q \\ \hline \underline{z}_q \end{bmatrix}$ depends on the determinant of $\begin{bmatrix} W_q \\ \hline \underline{z}_q \end{bmatrix}$. In case when the determinant is zero, we employ the paeudo-inverse technique as has been already developed in the chapter II for the minimum norm control problem.

3.3.2 Pseudo-inverse Technique

When the determinant of $\begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix}$ becomes zero, one cannot find $\begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix}^{-1}$ such that

$$\begin{bmatrix} \Delta\lambda \\ \Delta t_f \end{bmatrix} = \begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix}^{-1} \begin{bmatrix} \Phi(t_f^*, t) \\ \lambda_t^{*T} \Phi(t_f^*, t) \end{bmatrix} \Delta \underline{x} \qquad (3.3\text{-}24)$$

In this case one can try to obtain the best $\begin{bmatrix} \Delta\lambda \\ \Delta t_f \end{bmatrix}, \begin{bmatrix} \Delta\bar{\lambda} \\ \Delta t_f \end{bmatrix}$ as the minimand of

$$\left\| \begin{bmatrix} \Phi(t_f^*, t) \\ \lambda_t^{*T} \Phi(t_f^*, t) \end{bmatrix} \Delta \underline{x} - \begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix} \begin{bmatrix} \Delta\lambda \\ \Delta t_f \end{bmatrix} \right\|,$$ where for simplicity

the inner product norm is taken. There may be no unique solution to this problem or there may be no consistent solution to this problem. However, one can further reasonably stipulate that the choice for the best $\Delta\lambda$ and Δt_f should be made such that

$$\left\| \begin{matrix} \Delta\lambda \\ \Delta t_f \end{matrix} \right\| = \text{minimum}$$

Under this condition one can show [12,31] that

$$\begin{bmatrix} \Delta\lambda \\ \Delta t_f \end{bmatrix} = B^T(BB^T)^{-1}(A^TA)^{-1}A^T \begin{bmatrix} \Phi(t_f^*, t) \\ \lambda_t^* \Phi(t_f^*, t) \end{bmatrix} \Delta \underline{x} \qquad (3.3\text{-}25)$$

Hence,

$$\begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix}^{-1} = B^T(BB^T)^{-1}(A^TA)^{-1}A^T \qquad (3.3\text{-}26)$$

where

$$\begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix} = A\ B \text{ and } \begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix} \text{ is rank } k, \quad k \leqslant m$$

A = m x k matrix of rank k

B = k x m matrix of rank k

For any fixed instant of time t, it is always possible to decompose $\begin{bmatrix} W_q \\ \underline{z}_q \end{bmatrix}$ into the matrices A and B indicated above.

3.4 Approximate Error Analysis

In this section one deals with the same problem posed in section (2.5) in chapter II: Given a positive number r_o such that

$$\| \Delta\underline{x}^d \| \leqslant r_o \qquad (3.4\text{-}1)$$

find a number r_t which satisfies

$$\| \Delta\underline{x}(t) \| \leqslant r_t \qquad (3.4\text{-}2)$$

while providing the suboptimal control input into the syst m to be controlled.

To solve the problem one makes use of (3.3-10) which is rewritten here for the sake of convenience

$$\langle \underline{\lambda}_x^*, \underline{e}(t_{fx}^*, \underline{x}^*(t)) \rangle = 1 \qquad (3.4\text{-}3)$$

Note that (3.4-3) can be obtained by premultiplying (3.3-9) by $\underline{\lambda}_x^{*T}$ and that the equality in (3.4-3) must hold for any $\underline{x}(t)$ such that $\underline{x}(t_{fx}^*) = \underline{x}^d$ and for its corresponding $\underline{\lambda}_x^*$. However, in the suboptimal control scheme $\underline{\lambda}_x^*$ and the terminal time t_{fx}^* has been approximately estimated as

$$\underline{\lambda}_x^* = \underline{\lambda}_t^* + [M_{tq}]\Delta\underline{x} \qquad (3.4\text{-}4)$$

and

$$t_{fx}^* = t_f^* + \underline{g}_q \Delta\underline{x} \qquad (3.4\text{-}5)$$

Thus, one expects the terminal state error $\Delta\underline{x}^d$ deviated from the desired state $\underline{x}^d$ due to the approximate value of $\underline{\lambda}_x^*$ and t_{fx}^*. Hence, one must include the error term $\Delta\underline{x}^d$ in (3.4-3) with $\underline{\lambda}_x^*$, t_{fx}^*, and $\underline{x}(t)$ replaced by $\underline{\lambda}_t^* + [M_{tq}]\Delta\underline{x}$, $t_f^* + \underline{g}_q\Delta\underline{x}$ and $\underline{x}_o^*(t) + \Delta\underline{x}(t)$, respectively so that the equality in (3.4-3) dose hold, i.e.,

$$\langle \underline{\lambda}_t^* + [M_{tq}]\Delta\underline{x},\ \underline{x}^d + \Delta\underline{x}^d - \underline{\Phi}(t_f^* + \underline{g}_q\Delta\underline{x}, t)(\underline{x}_o^*(t) + \Delta\underline{x}) \rangle = 1 \qquad (3.4\text{-}6)$$

Recalling that the matrices $[M_{tq}]$ and $\underline{g}_q$ have been chosen so that the zero order and the first order terms of $\Delta\underline{x}$ vanish in a Taylor series expansion of (3.4-6) and assuming that $\Delta\underline{x}$ is small enough to neglect the third order and the higher order terms of $\Delta \underline{x}$, and making use of some properties of the transition matrix [31] gives

$$\langle \Delta_t^* + [M_{tq}]\Delta\underline{x}, \Delta\underline{x}^d \rangle = \langle \Delta_t^*, \dot{\Phi}\Delta\underline{x} \rangle \varepsilon_q \Delta\underline{x} + \langle [M_{tq}]\Delta\underline{x}, \Phi\Delta\underline{x} \rangle + \langle [M_{tq}]\Delta\underline{x}, \quad \underline{x}_o^* \rangle \varepsilon_q \Delta\underline{x}$$
$$+ (1/2) \langle \Delta_t^*, \ddot{\Phi}\, \underline{x}_o^*(t) \rangle (\varepsilon_q \Delta\underline{x} \ )^2 \qquad (3.4\text{-}7)$$

where $\Phi = \Phi(t_f^*, t)$, $\dot{\Phi} = A(t_f^*)\Phi(t_f^*, t)$ and

$$\ddot{\Phi} = \left. (\partial A(t_{fx}^*)/\partial t_{fx}^* \ )\Phi(t_f^*, t) + A(t_f^*)^2 \Phi(t_f^*, t) \right|_{t_{fx}^* = t_f^*}$$

Now applying Schwarz's inequality to both sides of (3.4-7) and solving the inequality equation for $\underline{x}^d$ gives

$$\|\Delta\underline{x}^d\| \geq \| \langle \Delta_t^*, \dot{\Phi}\Delta\underline{x} \rangle \varepsilon_q \Delta\underline{x} + \langle [M_{tq}]\Delta\underline{x}, \Phi\Delta\underline{x} \rangle + \langle [M_{tq}]\Delta\underline{x}, \dot{\Phi}\, \underline{x}_o^*(t) \rangle \ \varepsilon_q \Delta\underline{x}$$
$$+ (1/2) \langle \Delta_t^*, \ddot{\Phi}\, \underline{x}_o^*(t) \rangle (\varepsilon_q \Delta\underline{x})^2 \quad / (\|\Delta_t^*\| + \|M_{tq}\Delta\underline{x}\|) \quad (3.4\text{-}8)$$

where the norm is taken as the square root of the inner product of vectors. Recall that our objective is to obtain r_t such that $\|\Delta\underline{x}\| \leq r_t$, where r_t is the least upper bound on $\|\Delta\underline{x}\|$. To do this one makes use of a relation (2.5-9) in (3.4-8) to express the inequality (3.4-8) in terms of $\|\Delta\underline{x}\|$. This yields an equality of quadratic of $\|\Delta\underline{x}\|$. However, in order to find the least upper bound on $\|\Delta\underline{x}\|$ one takes the coefficient of the first order of $\|\Delta\underline{x}\|$ as small as possible and the coefficient of the second order of $\|\Delta\underline{x}\|$ as large as possible:

$$r_o \geq \frac{(Q_1 + Q_2 + Q_3 + Q_4)\,\|\Delta \underline{x}\|^2}{\|\Delta_t^*\| + \sqrt{\rho_{min}\{[M_{tq}]^T[M_{tq}]\}}\,\|\Delta \underline{x}\|} \qquad (3.4\text{-}9)$$

where

$$Q_1 \triangleq \|g_q\|\|\Delta_t^*\|\sqrt{\rho_{max}(\dot{\Phi}^T\dot{\Phi})}$$

$$Q_2 \triangleq \sqrt{\rho_{max}(M_{tq}^T\, M_{tq})\,\rho_{max}(\Phi^T\Phi)}$$

$$Q_3 \triangleq \sqrt{\rho_{max}(M_{tq}^T\, M_{tq})\,\rho_{max}(\dot{\Phi}^T\Phi)}$$

$$Q_4 \triangleq \quad (1/2)\|\langle\Delta_t^*, \ddot{\Phi}\underline{x}_o^*(t)\rangle\|\,\|g_q\|^2$$

Solving the inequlity (3.4-9) for $\|\Delta \underline{x}\|$ and noting $\|\Delta \underline{x}\| \geq 0$ gives

$$\|\Delta \underline{x}\| \leq \frac{r_o\, Q_5 + \sqrt{(r_o^2\, Q_5^2 + 4\, r_o\, Q\,\|\Delta_t^*\|)}}{2\,Q} \qquad (3.4\text{-}10)$$

where $Q \triangleq Q_1 + Q_2 + Q_3 + Q_4$ and

$$Q_5 \triangleq \sqrt{\rho_{min}(M_{tq}^T\, M_{tq})}$$

Therefore, one determines an approximate bound on $\|\Delta \underline{x}(t)\|$, at time t as

$$r_t = \frac{r_o\, Q_5 + \sqrt{r_o^2\, Q_5^2 + 4\, r_o\|\Delta_t^*\|}}{2\,Q} \qquad (3.4\text{-}11)$$

3.5 Design Procedure

A design procedure for the problem 1 developed in this chapter can be summarized as follows:

Step 1 Calculate the state transition matrix $\Phi(t_f, t)$, the optimum lambda vector λ^* and $\lambda_t^* = m(t_f^*, t)\lambda^*$ as defined by (2.3-8), where t_f^* is the least terminal time computed. Calculate the nominal optimal control(the open-loop optimal control) $u^*(t)$ and its corresponding trajectory $\underline{x}_o^*(t)$.

Step 2 Calculate appropriate linear transformation matrix or matrices

$$\begin{bmatrix} M_{tq} \\ \hline g_q \end{bmatrix} = \begin{bmatrix} W_q \\ \hline \underline{z}_q \end{bmatrix}^{-1} \begin{bmatrix} \Phi(t_f^*, t) \\ \hline \lambda_t^* \Phi(t_f^*, t) \end{bmatrix}$$

where $\begin{bmatrix} W_q \\ \hline \underline{z}_q \end{bmatrix}^{-1}$ is the inversion of $\begin{bmatrix} W_q \\ \hline \underline{z}_q \end{bmatrix}$, which is properly defined from (3.3-16), (3.3-21) as indicated in (3.3-23).

Step 3 Continuous measurements on

$$\Delta\underline{x} = \underline{x}(t) - \underline{x}_o^*(t) \text{ and } L_o(t)$$

Step 4 On-line computation of the suboptimal control $\bar{u}^*(t)$:

$$\bar{u}^*(t) = L_o(t)^q \left| \langle \lambda_t^* + M_{tq}\Delta\underline{x}, \underline{h}(t_f^* + g_q\Delta\underline{x}, t)\rangle \right|^{q-1} \operatorname{sgn}(\langle \lambda_t^* + M_{tq}\Delta\underline{x}, \underline{h}(t_f^* + g_q\Delta\underline{x}, t)\rangle)$$

3.6 Illustrative Example (Bushaw's Problem)

To clarify the application of the theory developed in this chapter, one illustrative example is presented below:

The system to be controlled is assumed to consist of two integrators and therefore, its differential equation can be written as

$$\frac{d\underline{x}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (3.6\text{-}1)$$

$$\underline{y}(t) = \underline{x}(t) \qquad (3.6\text{-}2)$$

Given an initial state $\underline{x}(0) = \underline{x}_o = \begin{bmatrix} -100 \\ 0 \end{bmatrix}$ and desired final output $x_1^d = 0$, $x_2^d = 0$ find the time suboptimal control system for $p=\infty$ and $L_o=1$, namely,

$$|u(t)| \leqslant 1$$

<u>Step 1</u> Obtain the state transition matrix and the Nominal optimal control and its corresponding trajectory.

(a) $$\underline{\lambda}^* = 1/100 \begin{bmatrix} 1 \\ -100 \end{bmatrix} \qquad (3.6\text{-}3)$$

(b) Nominal optimal control becomes

$$u^*(T^*, \underline{\lambda}^*, t) = \text{sgn } (10-t) \qquad (3.6\text{-}4)$$

where the least elapsed time $T^* = 20$ and hence the least terminal time $t_f^* = 20$.

(c) Nominal trajectory

$$\underline{x}_o^*(t) = \begin{bmatrix} -100 + 0.5t^2 \\ t \end{bmatrix} \quad \text{for } t \leqslant 10$$

$$\underline{x}_o^*(t) = \begin{bmatrix} -0.5t^2 + 20t - 200 \\ 20 - t \end{bmatrix} \quad \text{for } t \geqslant 10 \qquad (3.5\text{-}5)$$

(d) State transition matrix

$$\underline{\Phi}(t_f, t) = \begin{bmatrix} 1 & t_f - t \\ 0 & 1 \end{bmatrix}$$

(e) $\underline{\lambda}_t^* = m(t_f^*, t)\ \underline{\lambda}^*$

where $m(t_f^*, t) = 100/(-x_{o1}^* - (10-t)x_{o2}^*)$

Step 3 Linear Transformation Matrices

To obtain $\begin{bmatrix} M_{t1} \\ \hline \underline{g}_1 \end{bmatrix}$ it is required to know $\begin{bmatrix} W_1 \\ \hline \underline{z}_1 \end{bmatrix}^{-1}$:

(a) $0 \leqslant t < 10$

Making use of (3.3-19), (3.3-21), (3.3-22), one can obtain

$$\begin{bmatrix} W_1 \\ \hline \underline{z}_1 \end{bmatrix} = \begin{bmatrix} (2/m)10^4 & (2/m)10^3 & 20 \\ (2/m)10^3 & (2/m)10^2 & 1 \\ -e_1^* & -e_2^* & (-m/100)e_2^* \end{bmatrix}$$

where $m = m(20, t)$; $e_1^* = e_1(20, \underline{x}_o^*(t))$; $e_2^* = e_2(20, \underline{x}_o^*)$

Performing the inversion of this matrix and calculating the matrix indicated by (3.3-23) gives

$$\left[\frac{M_{t1}}{g_1}\right] = \begin{bmatrix} \frac{m^2}{2\times10^5}(e_2^* +20) & \frac{m^2}{2\times10^5}(20e_2^*-te_2^*+200-20t) \\ \frac{m^2}{2\times10^5}(e_1^*-20e_2^*-200) & \frac{m^2}{2\times10^5}(-te_1^*+e_2^*(20t-200) -2000+200t) \\ -1/10 & -(10-t)/10 \end{bmatrix}$$

(3.6-6)

(b) $10 \leqslant t$

For this region $\left[\frac{W_1}{g_1}\right]$ becomes from (3.3-19),(3.3-21)

$$\left[\frac{W_1}{g_1}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -e_1^* & -e_2^* & -e_2^*m/100 \end{bmatrix}$$

This is a singular matrix. Thus, in this region one can estimate the best $\left[\frac{M_1}{g_1}\right]$ by first performing the pseudo-inverse of the matrix. For this purpose one follows the indicated partition of $\left[\frac{W_1}{g_1}\right]$ as

$$\left[\frac{W_1}{g_1}\right] = A\ B$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 0 & -1 \\ -e_1^* & -e_2^* & -e_2^*m/100 \end{bmatrix}$$

And performing the pseudo-inverse indicated by (3.3-26) gives

$$\left[\frac{\overline{W}_1}{\underline{z}_1}\right]^{-1} = -1/(e_1^*+e_2^*) \begin{bmatrix} 0 & e_1^*e_2^*m/100 & -e_1^* \\ 0 & e_2^{*2}m/100 & -e_2^* \\ 0 & -e_1^{*2}-e_2^* & 0 \end{bmatrix} \quad (3.6\text{-}7)$$

Substituting (3.6-7) into (3.3-25) gives

$$\left[\frac{\overline{M}_{t1}}{\underline{g}_1}\right] = -1/(e_1^{*2}+e_2^{*2}) \begin{bmatrix} -e_1^*m/100 & e_1^*e_2^*m/100 & -(10-t)me_1^*/100 \\ -e_2^*m/100 & e_2^{*2}m/100 & -e_2^*(10-t)m/100 \\ 0 & -e_1^{*2}-e_2^{*2} & \end{bmatrix}$$

(3.6-8)

<u>Step 3</u> Continuous measurements on

$\Delta\underline{x} = \underline{x}(t) - \underline{x}_0^*(t), \; \overline{L}_0(t) = 1$

where $\underline{x}_0^*(t)$ is precomputed by (3.6-5) and must be stored.

<u>Step 4</u> On-line computation of the suboptimal control $\overline{u}^*(t)$

The suboptimum controller is programmed by use of (3.3-5), namely,

$$\overline{u}^*(t) = \text{sgn}(10-t+\underline{g}_1\Delta\underline{x} + (100/m)(20+\underline{g}_1\Delta\underline{x}-t)\cdot (m_{t11}\Delta x_1 + m_{t12}\Delta x_2) + (100/m) (m_{t21}\Delta x_1 + m_{t22}\Delta x_2)) \quad (3.6\text{-}9)$$

where m_{tij} indicates the i-th row and the j-th column of the matrix M_{t1} specified by (3.6-6) for $0\leq t\leq 10$ and by (3.6-8) for $t\geq 10$ and $\Delta x_1, \Delta x_2$ are the elements of the vector $\Delta \underline{x}$.

To demonstrate the effectiveness of this technique the example was simulated using a digital computer for the initial deviations from the nominal initial state $x_1(0)=-100$ and $x_2(0)=0.0$. In this simulation the suboptimum controller measured the output $\underline{x}(t)$ and corrected the nominal control at every 0.05 seconds. The summary of results is tabulated in table 3-1, where the performances between the optimal control system and the suboptimal one, the performances between the suboptimal system and the system with open-loop control without correction are compared. In case no correction is made on control input regardless of deviations, the terminal state error is more than 15 times greater than those obtained using the suboptimal control scheme.

The phase portrait for each case A, B, C, D is given in fig 3-2 where an actual trajectory indicated by the solid line (—— —— ——) is compared with the trajectory due to the open loop control without correction, indicated by the dot line (-------) and also is compared with the nominal trajectory indicated by (——x——x——). In fig 3-2 the suboptimal control indicated by the solid line is also compared with the nominal one indicated by

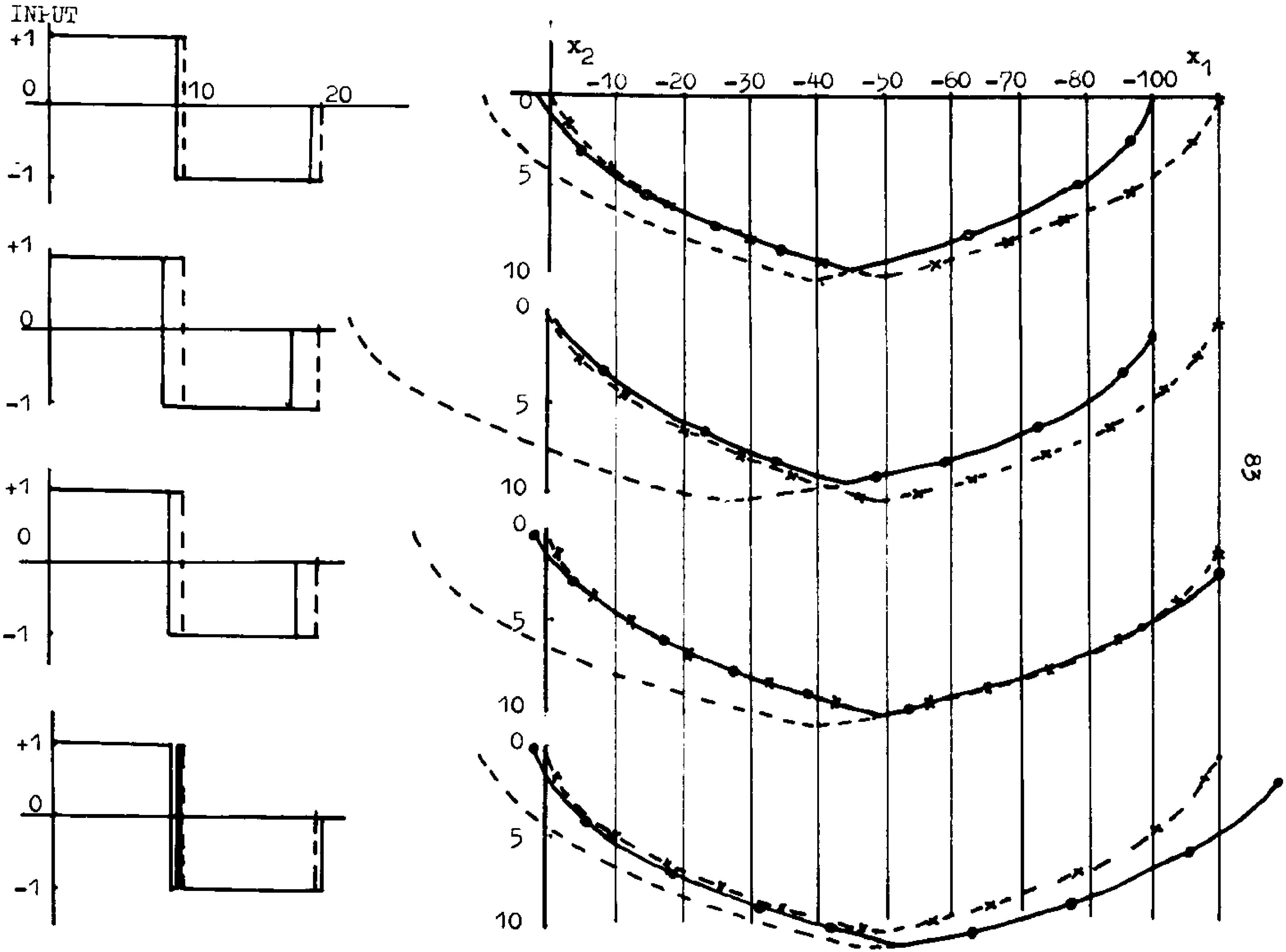


Fig. 3-2

SUMMARY OF RESULTS

	INITIAL STATE	OPTIMUM SYSTEM		SUBOPTIMUM SYSTEM		SYSTEM WITHOUT CORRECTION	
	$x_1(0), x_2(0)$	TERMINAL TIME t^*_{fx}	TERMINAL STATE $x^*_1(t^*_{fx})$, $x^*_2(t^*_{fx})$	TERMINAL TIME $20.0 + \Delta t_f$	TERMINAL STATE x_1 ; x_2	TERMINAL TIME t^*_f	TERMINAL STATE x_1 ; x_2
(A)	-90.0, 0.0	18.97	0.00 0.00	18.99	0.25; 0.00	20.00	10.00;0.00
(B)	-90.0, 1.0	18.03	0.00 0.00	17.89	-1.19; 0.00	20.00	30.00;1.00
(C)	-100.0,1.0	19.05	0.00 0.00	19.09	0.50; 0.00	20.00	20.00;1.00
(D)	-110.0,1.0	20.02	0.00 0.00	20.09	0.78; 0.00	20.00	10.00;1.00

Disturbances are made on initial state $x_1 = -100.0$, $x_2 = 0.0$

Table 3-1

DEMONSTRATION OF ERROR BOUND

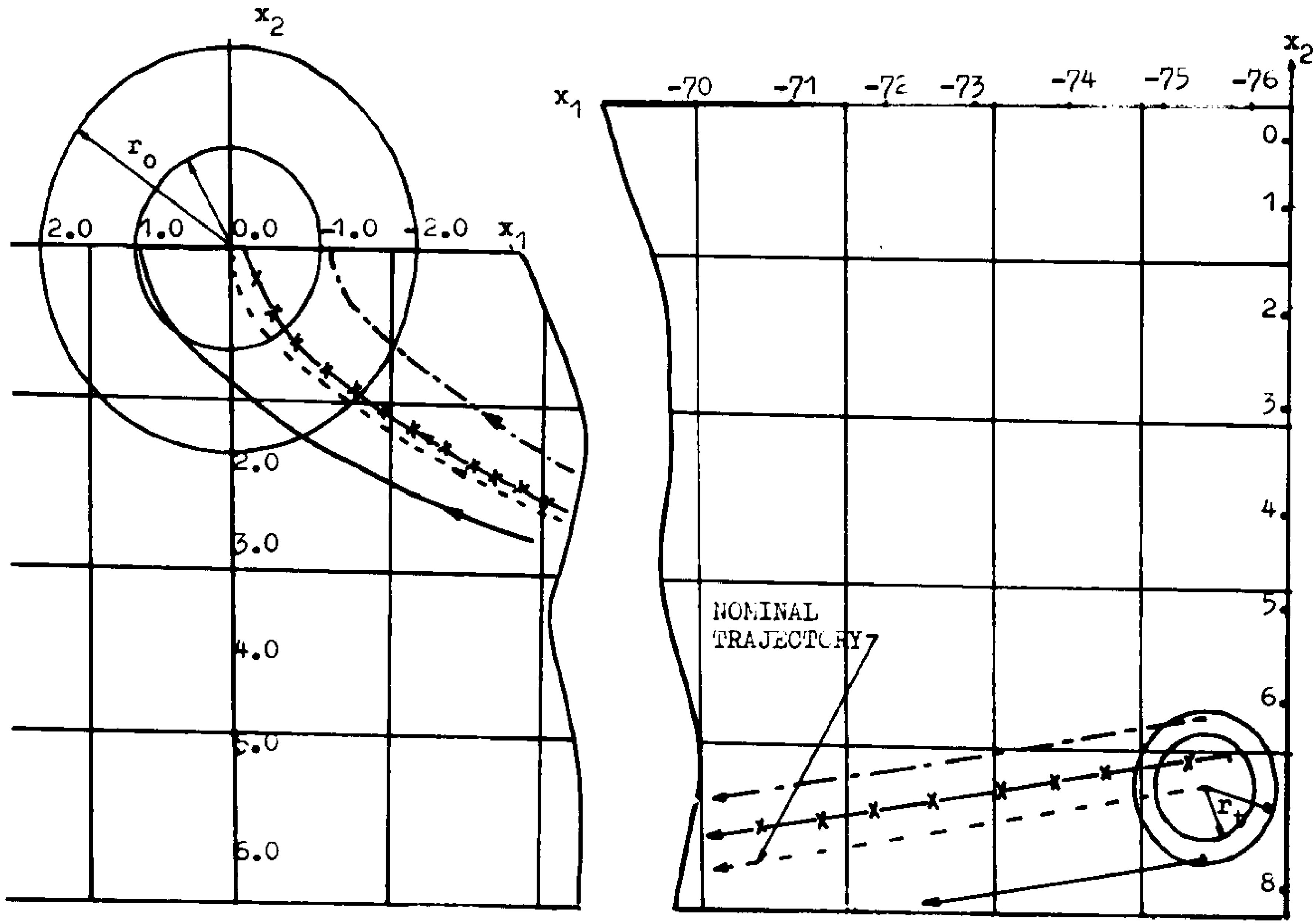


Fig.3-3

dot line. To illustrate the error analysis, let the radius of the terminal error be 1 or 2 units. Then, making use of (3.4-11) at time t= 7 seconds, one can precompute the error bound r_t, 0.55 and 0.78 corresponding to r_o=1 and r_o=2, respectively. To demonstrate the error bound precomputed for given r_o, one made deviations from $\underline{x}_o^*$ at time 7 seconds as indicated in fig.3-3 and the suboptimal control ended up with a terminal error which was within the error bound predicted for each corresponding deviation.

3.7 Summary

In this chapter it was shown that a design procedure of a feedback suboptimum control law for the minimum time control problem was developed.

The proposed technique is based on the assumption that the dynamic system operates in the presence of small perturbations during the actual operation and that all the system parameters, reference output and sensing devices are idealized.

The limitation of the approach is that one cannot apply this technique to the system which is not normal.

The illustrative example showed that in case no correction is made on the nominal control regardless of perturbations, the terminal error is more than 15 times greater than those obtained using the suboptimal control scheme.

IV ALTERNATIVE SOLUTIONS OF PROBLEM 1 AND PROBLEM 2 FOR A SPECIAL CASE

4.1 Introduction

In this chapter an alternative solution to problem 1 and problem 2 in the case of $p=\infty$ (minimizing the maximum magnitude of control in problem 1 and the minimum time problem with hard constraint on control) is developed by correcting the switching instants $s_1^*, s_2^*, s_3^*, \ldots, s_v^*$ instead of correcting the optimum lambda vector $\boldsymbol{\lambda}^*$ as a result of a deviation from a nominal trajectory.

An advantage of this approach is that one can obtain relatively simple controller structures for certain cases. However, one cannot apply this technique to general cases.

It is assumed in this technique that there is a known bound on the total number of switching instants for the optimal trajectory $\underline{x}^*(t)$. Also assume for simplicity that $\underline{y}(t) = \underline{x}(t)$ and the control is scalar.

4.2 Alternative Solution of Problem 1 ($p = \infty$)

In the case when $p = \infty$ (minimizing the maximum amplitude of the control) in problem 1, one can rewrite the optimum control $u^*(t_f, \underline{\lambda}_x^*, t)$ given by (2.3-1) in chapter II as

$$u^*(t) = UK_x^* \prod_{k=1}^{m} \operatorname{sgn}(t - s_{xk}^*) \qquad (4.2\text{-}1)$$

where U is +1 or -1 determined by the nominal optimal control $u^*(t_f, \underline{\lambda}^*, t)$ and $s_{x1}^*, s_{x2}^*, \ldots, s_{xm}^*$ are the optimum switching instants not far from $s_1^*, s_2^*, \ldots, s_m^*$ as a result of small deviation $\Delta \underline{x}$ and for each $s_{xk}^*, k=1,2,\ldots,m$

$$\langle \underline{\lambda}_x^*, \underline{h}(t_f, s_{xk}^*) \rangle = 0 \qquad (4.2\text{-}2)$$

is satisfied; K_x^* denotes the minimum norm and $\underline{\lambda}_x^*$ is defined in (2.3-1).

Note that the vector $\underline{S}_x^*$, the transpose of $(s_{x1}^*, \ldots, s_{xm}^*)$ and K_x^* depend on the initial state $\underline{x}(t)$ and the final desired output $\underline{y}^d$. Also note that $K_x^* = K^*$ and $\underline{S}_x^* = \underline{S}^*$, where $\underline{S}^*$ is the vector of switching instants determined from $u^*(t_f, \underline{\lambda}^*, t)$, when the computations of $\underline{K}^*$ and $\underline{S}^*$ are performed for the nominal initial state $\underline{x}_o$ and the desired final output $\underline{y}^d$.

Consider a time-invariant system in which the eigen-values of the system matrix A are all real and non-positive. Pontryagin showed[27] that the number of switching instants m dose not exceed n-1 for the system under consideration. More generally the technique can be applied when there is a known bound on the total number of switching instants for the optimal trajectory (see section 5.2). Let us assume here that m=n-1 without loss of generality and that

$$t \leqslant s^*_{x1} \leqslant s^*_{x2} \leqslant s^*_{x3} \quad \cdots \quad s_{xn-1} \leqslant t_f$$

Inserting the optimal control (4.2-1) into (2.2-2) and rewriting the latter in terms of switching instants gives

$$\underline{e}(t_f,\underline{x}(t))=(-1)^{n-1}UK^*_x\Big(\int_t^{s^*_{x1}} \underline{h}(t_f-s)ds - \int_{s^*_{x1}}^{s^*_{x2}} \underline{h}(t_f-s)ds + \cdots$$

$$\cdots + (-1)^{n-1} \int_{s^*_{xn-1}}^{t_f} \underline{h}(t_f-s)\, ds\Big) \qquad (4.2\text{-}3)$$

One can observe that the only terms in (4.2-3) which are not fixed by the system description are K^*_x and $\underline{S}^*_x$ for the initial state $\underline{x}(t)$ at time t. Therefore, one can express (4.2-3) as

$$\begin{bmatrix} \underline{S}_x^* \\ \hline K_x^* \end{bmatrix} = \underline{Q}_n^{-1}(t,\underline{x}(t)) \tag{4.2-4}$$

where $\underline{Q}_n^{-1}(t,\underline{x}(t))$ is inverse mapping of $\underline{Q}_n(\underline{S}_x^*,K_x^*)$ and $\underline{Q}_n(\underline{S}_x^*,K_x^*)$ is defined from (4.2-3) as

$$\underline{e}(t_f,\underline{x}(t) = \underline{Q}_n(\underline{S}_x^*,K_x^*) \tag{4.2-5}$$

If one could determine the inverse mapping of $\underline{Q}_n(\underline{S}_x^*,K_x^*)$ which satisfies (4.2-3) for a given initial state $\underline{x}(t)$ at time t, the optimal control which transfers $\underline{x}(t)$ to $\underline{y}^d$ with the minimum norm of control will then be completely specified. Thus, the problem of obtaining the optimal control which minimizes the maximum amplitude of control is equivalent to obtaining the $\underline{Q}_n^{-1}(t,\underline{x}(t))$. However, it may be in general difficult to obtain the inverse mapping for all $\underline{x}(t)$. Hence an approximate solution of $\underline{S}_x^*$ and K_x^* is attempted to obtain on the assumption of small perturbations during the actaul process. As has been already assumed in the previous chapters, all the system parameters reference output, and sensing devices are idealized and the perturbations are essentially due to the external forces.

Before developing a design procedure, it may be helpful to define the following terms :

(1) $\underline{S}^{*(0)}$ is a vector of switching instants using nominal control $u^*(t_f,\Delta^*,t)$ and $K^{*(0)}$ is a minimum norm of the nominal control. $\underline{S}^{*(0)}$, $K^{*(0)}$ can be calculated as a solution of (4.2-3) when $\underline{x}(t) = \underline{x}_o$ (nominal initial state) at time $t = t_i$.

(2) $\underline{S}^{*(k)}$, $K^{*(k)}$ are a nominal switching instant vector, the nominal minimum norm, respectively and are corrected k-times from $\underline{S}^{*(0)}$, $K^{*(0)}$, respectively.

(3) $\Delta\underline{S}^{(k)}$ is a vector, a small change in $\underline{S}^{*(k)}$ as a result of deviation $\Delta\underline{x}$, where

$$\Delta\underline{x}(t) = \underline{x}(t) - \underline{x}^{(k)}(t) \qquad (4.2\text{-}6)$$

(4) $\underline{x}^{(k)}(t)$ is a nominal trajectory corresponding to $\underline{S}^{*(k)}$ and $K^{*(k)}$. Note that $\underline{x}^{(0)}(t)$ is a nominal trajectory corresponding to $S^{*(0)}$, $K^{*(0)}$, namely $u^*(t_f,\Delta^*,t)$.

In the proposed scheme $\underline{S}^*_x$ and K^*_x can be approximated by first obtaining $\Delta\underline{S}^{(k)}$, a small change in $\underline{S}^{*(k)}$, and $\Delta K^{(k)}$ a small change in $K^{*(k)}$ as a result of a small change in $\underline{x}^{(k)}(t)$, $\Delta\underline{x}(t)$, i.e.,

$$\underline{S}^*_x = \underline{S}^{*(k)} + \Delta\underline{S}^{(k)} \qquad (4.2\text{-}7)$$

$$K^*_x = K^{*(k)} + \Delta K^{(k)} \qquad (4.2\text{-}8)$$

where k is the number of correction made on the nominal trajectory $\underline{x}^{(0)}(t)$ and $k \geqslant 0$.

When the deviation $\Delta\underline{x}$ is a small quantity, it is shown later that $\Delta\underline{x}$ can be approximately related to $\Delta\underline{S}^{(k)}$, $\Delta K^{(k)}$ by a linear relation. Thus, the deviation $\Delta\underline{x}$ must be small enough to be linearly related to $\Delta\underline{S}^{(k)}$, $\Delta K^{(k)}$. Suppose that the linear relation is found, i.e.,

$$\begin{bmatrix} \Delta\underline{S}^{(k)} \\ \hline \Delta K^{(k)} \end{bmatrix} = \begin{bmatrix} Q_n^{(k)} \end{bmatrix} \Delta\underline{x} \qquad (4.2\text{-}9)$$

where $\begin{bmatrix} Q_n^{(k)} \end{bmatrix}$ is an n x n matrix whose elements depend on $\underline{S}^{*(k)}$, $K^{*(k)}$, $\underline{x}^{(k)}(t)$. The derivation of this matrix is developed in the next section.

To find $\underline{S}^*_x, K^*_x$ it is necessary to know the nominal quantities $\underline{S}^{*(k)}, K^{*(k)}, \underline{x}^{(k)}(t)$ in addition to $\Delta\underline{S}^{(k)}$, $\Delta K^{(k)}$. The nominal quantities depend on the number k, the nunmber of corrections made on the nominal quantities $\underline{S}^{*(0)}$, $K^{*(0)}$, $\underline{x}^{(k)}(t)$. When k = 0, the nominal quantities become $\underline{S}^{*(0)}$ $K^{*(0)}, \underline{x}^{(0)}(t)$ and are employed through the process; when k becomes large enough to obtain $\underline{S}^*_x$, K^*_x for each $\underline{x}(t)$, this procedure would implement the feedback structure but may lead to prohibitive requirements on the storage space of computer. Let k be the number of original nominal switching instants, the dimension of the vector $\underline{S}^{*(0)}$ and let the dimension of the vector $\underline{S}^{*(0)}$ be n-1. Moreover, let the corrections on the nominal quantities be made at each nominal switching instant, namely

$$\underline{S}^{*(k)} = \underline{S}^{*(k-1)} + \Delta\underline{S}^{(k-1)} \qquad (4.2\text{-}10)$$

$$K^{*(k)} = K^{*(k-1)} + \Delta K^{(k-1)}, \quad k \geqslant 1 \qquad (4.2\text{-}11)$$

where $\underline{S}^{(k-1)}$ and $K^{(k-1)}$ are constant evaluated at time $t = s_k^{*(k)}$ and $s_v^{*(k)}$'s are elements of the vector $\underline{S}^{*(k)}$.

The estimation of the nominal trajectory $\underline{x}^{(k)}(t)$ can be accomplished by solving (1-1) with $u^*(t)$ (4.2-1), where $\underline{S}_x^*$, K_x^*, m are replaced by $\underline{S}^{*(k)}$, $K^{*(k)}$, n-1, respectively and with initial state $\underline{x}(s_k^{*(k)})$ at initial time $t = s_k^{*(k)}$, namely,

$$\underline{x}^{(k)}(t) = \Phi(t_f - s_k^{*(k)})\underline{x}(s_k^{*(k)}) + U\, K^{*(k)}(-1)^{n-k}(-\underline{r}(t, s_k^{*(k)}) + \underline{r}(t,t)) \qquad (4.2\text{-}12)$$

where $s_k^{*(k)} \leqslant t \leqslant s_{k+1}^{*(k+1)}$; $r(t,s) \triangleq \int \underline{h}(t-s)\, ds$

Note that when the running time t is in the interval $s_k^{*(k)}$ to $s_{k+1}^{*(k+1)}$, one can set

$$s_1^{*(k)} = s_2^{*(k)} = \dots = s_{k-1}^{*(k)} = s_k^{*(k)} = t$$

and $s_{k+1}^{*(k)}$, $s_{k+2}^{*(k)}$, ..., $s_{n-1}^{*(k)}$ are constant switching instants which are evaluated at $t = s_k^{*(k)}$ in the previous interval $s_{k-1}^{*(k-1)}$ to $s_k^{*(k)}$. Thus, one can have

$$\underline{S}^{*(k)} = \left[t, t, \ldots, t, s^{*(k)}_{k+1}, s^{*(k)}_{k+2}, \ldots, s^{*(k)}_{n-1}\right]^T \qquad (4.2\text{-}13)$$

Suppose that the linear transformation matrix $\left[Q_n^{(k)}\right]$, the nominal quantities $\underline{S}^{*(k)}$, $K^{*(k)}$, $\underline{x}^{(k)}$ for given k are precomputed. Then, the suboptimal control $\bar{u}_k^*(t)$ can be obtained as

$$\bar{u}_k^*(t) = (K^{*(k)} + \Delta K^{(k)}) \left(\prod_{v=1}^{n-1} \operatorname{sgn}(t - s_v^{*(k)} - \Delta s_v^{(k)}) \right) U \qquad (4.2\text{-}14)$$

for the interval $s_k^{*(k)}$ to $s_{k+1}^{*(k+1)}$.

4.2.1 Linear Transformation Matrix $\left[Q_n^{(k)}\right]$

The transformation matrix $\left[Q_n^{(k)}\right]$ which maps $\Delta\underline{x}$ into $\Delta\underline{S}^{(k)}$, $\Delta K^{(k)}$ is obtained through a Taylor series expansion of (4.2-5) around the nominal quantities $\underline{S}^{*(k)}$, $K^{*(k)}$, $\underline{x}^{(k)}(t)$. To do this define

$$\Delta\underline{e} = \underline{e}(t_f, \underline{x}(t)) - \underline{e}(t_f, \underline{x}^{(k)}(t)) \qquad (4.2\text{-}15)$$

then

$$\Delta\underline{e} = \underline{Q}_n(\underline{S}^{*(k)} + \Delta\underline{S}^{(k)}, K^{*(k)} + \Delta K^{(k)}) - \underline{Q}_n(\underline{S}^{*(k)}, K^{*(k)}) \qquad (4.2\text{-}16)$$

Substituting a Taylor series expansion for $\underline{Q}_n(\underline{S}^{*(k)} + \Delta\underline{S}^{(k)}, K^{*(k)} + \Delta K^{(k)})$ into (4.2-16) gives

$$\Delta\underline{e} = \left[\frac{\partial \underline{Q}_n(\underline{S}_x^*, K^{*(k)})}{\partial \underline{S}_x^*}\right] \Delta\underline{S}^{(k)} + \left[\frac{\partial \underline{Q}_n(\underline{S}^{*(k)}, K_x^*)}{\partial K_x^*}\right] \Delta K^{(k)} + R$$

where the partial derivatives are evaluated at

$\underline{S}^*_x = \underline{S}^{*(k)}$, $K^*_x = K^{*(k)}$ and R is the remainder.

Performing the indicated partial derivatives in (4.2-17) gives

$$\Delta\underline{e} = (-1)^{n-1} 2UK^{*(k)} \sum_{v=1}^{n-1} \underline{h}(t_f - s_v^{*(k)}) \Delta s_v^{(k)} (-1)^{v+1}$$

$$+ (1/k^{*(k)}) \underline{e}(t_f, \underline{x}^{(k)}(t)) \Delta K^{(k)} \qquad (4.2\text{-}18)$$

where note that $s_1^{*(k)} = s_2^{*(k)} = \ldots = s_k^{*(k)} = t$

Recalling the definition of $\Delta\underline{e}$ (see (4.2-15)) and associating it with (4.2-18), one can find $\left[Q_n^{(k)}\right]$ as

$$\left[Q_n^{(k)}\right] = -(Q_{nS}^{(k)} \;\; Q_{nK}^{(k)})^{-1} \Phi(t_f - t) \qquad (4.2\text{-}19)$$

where $(Q_{nS}^{(k)} \; Q_{nK}^{(k)})$ is an augumented matrix whose submatrices $\left[Q_{nS}^{(k)}\right]$ and $\left[Q_{nK}^{(k)}\right]$ are properly defined from (4.2-18) and (4.2-15).

Remark 4-1

In the case when $C \neq I$, i.e., the dimension of the output is m which is not equal to n, the dimension of the vector $\underline{S}^*_x$ is at most n-1(the bound on the switching instants are still n-1). Therefore, (4.2-18) may not be enough to determine $\Delta\underline{S}^{(k)}$ and $\Delta K^{(k)}$ and additional n-m equations are required: Consider the n-1 equations generated from the switching condition, (4.2-2)

$$\langle \underline{\Delta}_x^*, \underline{h}(t_f - s_{xv}^*) \rangle = 0 \qquad (4.2\text{-}20)$$

where v= 1,2,...,n-1 and the dimension of the vector $\underline{\Delta}_x^*$ is m.

Consider also the constraint

$$\langle \underline{\Delta}_x^*, \underline{e}(t_f, \underline{x}(t)) \rangle = 1 \qquad (4.2\text{-}21)$$

One can easily derive the n-m equations in terms of $\underline{S}_x^*$, $\underline{x}(t)$ by eliminating the m vector $\underline{\Delta}_x^*$ from (4.2-20) and (4.2-21). Define these n-m equations as

$$v_i(\underline{S}_x^*, K_x^*) = 0 \ , i=1,2,\ldots,n-m \qquad (4.2\text{-}22)$$

From (4.2-18),(4.2-22) an approximate linear relation between $\Delta\underline{S}^{(k)}$, $\Delta K^{(k)}$ and $\Delta\underline{x}$ can be obtained through a Taylor series expansion as usual. *****

To estimate $\left[Q_n^{(k)}\right]$ in (4.2-19) one must perform the inversion of the matrix $\left[Q_{nS}^{(k)} \ \ Q_{nK}^{(k)}\right]$. The inversion of this matrix depends on the determinant of $\left[Q_{nS}^{(k)} \ Q_{nK}^{(k)}\right]$. In cases where the determinant is zero, one can employ the pseudo-inverse technique developed in chapter II and III. In this case one may try to obtain the best $\left[\frac{\Delta\underline{S}^{(k)}}{\Delta K^{(k)}}\right]$ as the minimand of

$$\left\| \Delta \underline{x} - \left[Q_{nS}^{(k)} \; Q_{nK}^{(k)} \right] \begin{bmatrix} \Delta \underline{S}^{(k)} \\ \hdashline \Delta K^{(k)} \end{bmatrix} \right\| \tag{4.2-23}$$

where the norm is taken as the inner product.

To this problem there may be no unique solution or even no consistent solution. However, the choice for the best $\Delta \underline{S}^{(k)}$, $\Delta K^{(k)}$ can be made so that

$$\begin{bmatrix} \Delta \underline{S}^{(k)} \\ \hdashline \Delta K^{(k)} \end{bmatrix} = \text{minimum}$$

Under this condtion one can have that

$$\begin{bmatrix} \Delta \underline{S}^{(k)} \\ \hdashline \Delta K^{(k)} \end{bmatrix} = -B^T(BB^T)^{-1} \; (A^TA)^{-1} \; A^T \Phi(t_f - t) \Delta \underline{x} \tag{4.2-24}$$

where the matrices A, B are n x r, r x n, respectively and its rank is r ;

$$\left[Q_{nS}^{(k)} \; Q_{nK}^{(k)} \right] = A \; B$$

Remark 4-2

In practice as a further simplification, the nominal quantities $\underline{x}^{(0)}(t)$, $\underline{S}^{*(0)}$, $K^{*(0)}$ are used through the whole process without further correcting the nominal quantities, namely, for the small deviation $\Delta\underline{x}(t) = \underline{x}(t) - \underline{x}^{(0)}(t)$ one may estimate the approximate $\underline{S}^*_x, K^*_x$ as

$$\underline{S}^*_x = \underline{S}^{*(0)} + \Delta \underline{S}^{(0)}$$

and

$$K_x^* = K^{*(0)} + \Delta K^{(0)}$$

where $\Delta \underline{S}^{(0)}$ and $\Delta K^{(0)}$ can be estimated as

$$\left[\frac{\underline{S}^{(0)}}{K^{(0)}}\right] = \left[Q_n^{(0)}\right] \Delta \underline{x}$$

and the linear transformation matrix $\left[Q_n^{(0)}\right]$ is found from (4.2-19) with k =0. Note that the matrix $\left[Q_n^{(0)}\right]$ depends on $\underline{S}^{*(0)}$, $K^{*(0)}$, $\underline{x}^{(0)}(t)$.

The resulting suboptimal control becomes in this case

$$\tilde{u}_j^*(t) = U(K^{*(0)} + \Delta K^{(0)}) \prod_{k=1}^{n-1} \operatorname{sgn}\,(t - s_k^{*(0)} - \Delta s_k^{(0)}).$$

The pseudo-inverse technique may be applied in the inversion of the matrix appropriately. *****

4.3 Alternative Solution of Problem 2 for a Special Case

As has been mentioned in appendix A, the minimum time control problem is closely related to the minimum-norm control problem. Hence, one may obtain the suboptimal control law for problem 2 when p $=\infty$ in a similar way to that obtained in the previous section as an alternative solution of the problem 1 ($p=\infty$).

Note that in the least time control problem the magnitude of the control is bounded by L_o and the terminal time t_f is not fixed but must be determined as a part of the problem. In references [18,19,22,23] it was shown that the least terminal time t^*_{fx} can be determined as the least value of t_f for which

$$K^*_x = L_o \tag{4.3-1}$$

where K^*_x is the minimum norm defined in the previous section and computed for the initial state $\underline{x}(t)$ and the desired state $\underline{y}^d$. Thus, one can rewrite the optimal control of the problem 2 ($p=\infty$) directly from (4.2-1) as

$$u^*(t) = U\, L_o \prod_{k=1}^{m} \operatorname{sgn}\,(t-s^*_{xk}) \tag{4.2-1}$$

where U is +1 or -1 determined by the nominal optimal control $u^*(t^*_f,\lambda\,,t)$ and $s^*_{x1},s^*_{x2},\ldots,s^*_{xm}$ are the optimum switching instants not far from $s^*_1,s^*_2,\ldots,s^*_m$ as a result of small deviation $\Delta\underline{x}$ and for each s^*_{xk}, $k=1,2,\ldots,m$

$$\langle \Delta^*_x, \underline{h}(t^*_{fx}, s^*_{xk}) \rangle = 0 \tag{4.2-2}$$

is satisfied; t^*_{fx} denotes the least terminal time for which (4.3-1) is satisfied.

Assume that a bound on the total number of switching instants for the optimal trajectory is given.

In this section also we consider a time-invariant system in which the eigen-values of the system matrix A are all real. Then, one can assume without loss of generality that $m = n-1$ and that

$$t \leqslant s^*_{x1} \leqslant s^*_{x2} \leqslant s^*_{x3} \quad \cdots \quad s^*_{n-1} \leqslant t^*_{fx}$$

Inserting the optimal control (4.3-2) into (2.2-2) with $t_i, t_f, \underline{x}(t_i)$ replaced by $t, t^*_{fx}, \underline{x}(t)$, respectively gives

$$\underline{e}(t^*_{fx}, \underline{x}(t)) = (-1)^{n-1}\, UL_o\Big(\int_t^{s^*_{x1}} \underline{h}(t^*_{fx}-s)ds - \int_{s^*_{x1}}^{s^*_{x2}} \underline{h}(t^*_{fx}-s)ds$$

$$+ \cdots + (-1)^{n-1} \int_{s^*_{xn-1}}^{t^*_{fx}} \underline{h}(t^*_{fx}-s)\, ds \Big) \qquad (4.3\text{-}4)$$

One can observe that the only terms in (4.3-4) which are not fixed by the system description are t^*_{fx} and $\underline{S}^*_x$ for the initial state $\underline{x}(t)$ at time t. Therefore, one can express (4.3-4) as

$$\begin{bmatrix} \underline{S}^*_x \\ \hline t^*_{fx} \end{bmatrix} = \underline{Q}_t^{-1}(t, \underline{x}(t)) \qquad (4.3\text{-}5)$$

where $\underline{Q}_n^{-1}(t, \underline{x}(t))$ is the inverse mapping of $\underline{Q}_t(\underline{S}^*_x, t^*_{fx})$ and $\underline{Q}_t(\underline{S}^*_x, t^*_{fx})$ is defined from (4.3-4) as

$$\underline{x}(t) = \underline{Q}_t(\underline{S}^*_x, t^*_{fx}) \qquad (4.3\text{-}6)$$

If one could obtain the inverse mapping of $\underline{Q}_t(\underline{S}^*_x, t^*_{fx})$ which satisfies (4.3-4) for each initial state $\underline{x}(t)$ and the desired output $\underline{y}^d$, the optimal control which transfers $\underline{x}(t)$ to $\underline{y}^d$ in the least elapsed time T^*_x, $t^*_{fx}-t$, will then be completely specified. However, it may be in general difficult to obatin the inverse mapping for each $\underline{x}(t)$, hence an approximate solution of $\underline{S}^*_x, t^*_{fx}$ is attempted to obtain on the assumption of small perturbations during the actual process. As has been already assumed in the previous section,all the system parameters, reference output, and sensing devices are idealized and the perturbations are essentially due to the external unknown forces.

A design procedure of obtaining the suboptimal control for the problem 2 ($p=\infty$) is analogous to the one developed in the previous section. The terms $\underline{S}^{*(0)}$, $\underline{S}^{*(k)}$ $\Delta\underline{S}^{(k)}$, $\Delta\underline{x}(t)$, and $\underline{x}^{(k)}(t)$ defined in the previous section with t_f replaced by t^*_f will be used in this section; Define $t^{*(k)}_f$ as the nominal terminal time corrected k-times from $t^{*(0)}_f$; $t^{*(0)}_f = t^*_f$.

In analogy to the suboptimal control $\overline{u}^*(t)$ of problem 1, one can obtain the suboptimal control of problem

2 ($p=\infty$) as

$$\bar{u}_k^*(t)=UL_o \prod_{v=1}^{n-1} \text{sgn}\,(t-s_v^{*(k)}-\ s_v^{(k)}) \qquad (4.3\text{-}7)$$

where the least terminal time t_{fx}^* may be determined as

$$t_{fx}^* = t_f^{*(k)} + \Delta t_f^{(k)} \qquad (4.3\text{-}8)$$

and $\Delta t_f^{(k)}$, $\Delta \underline{S}^{(k)}$ are small change in $t_f^{*(k)}$, $\underline{S}^{*(k)}$, respectively as a result of small deviation $\Delta\underline{x}$, a small change in $\underline{x}^{(k)}(t)$. The linear transformation matrix $\left[Q_t^{(k)}\right]$ which transform $\Delta\underline{x}$ into $\Delta\underline{S}^{(k)}$, $\Delta t_f^{(k)}$

$$\begin{bmatrix} \Delta\underline{S}^{(k)} \\ \hline \Delta t_f^{(k)} \end{bmatrix} = \left[Q_t^{(k)}\right]\Delta\underline{x} \qquad (4.3\text{-}9)$$

is derived in the next section.

4.3.1 Derivation of the Linear Transformation Matrix $\left[Q_t^{(k)}\right]$

In analogy to obtaining the matrix $\left[Q_n^{(k)}\right]$ in the previous section, one can determine without difficulty $\left[Q_t^{(k)}\right]$ which satisfies (4.3-9). To do this recall (4.3-6), which can be obtained by premultiplying both sides of (4.3-4) by $\Phi(t-t_{fx}^*)$, namely, $\underline{Q}_t(\underline{S}_x^*, t_{fx}^*)$ can be identified as

$$\underline{x}(t) = \Phi(t-t_{fx}^*)\underline{x}(t_{fx}^*)-(-1)^{n-1}UL_o\Big(\int_t^{s_{x1}^*} \underline{h}(t-s)\,ds - \int_{s_{x1}^*}^{s_{x2}^*} \underline{h}(t-s)ds$$

$$+\dots\dots+ (-1)^{n-1} \int_{s^*_{xn-1}}^{t^*_{fx}} \underline{h}(t-s)\, ds \Big) = \underline{Q}_t(\underline{S}^*_x, t^*_{fx}) \qquad (4.3\text{-}10)$$

The matrix $\left[Q_t{}^{(k)}\right]$ which satisfies (4.3-9) is obtained through a Taylor series expansion of (4.3-10) around the nominal quantities $\underline{x}^{(k)}(t)$, $\underline{S}^{*(k)}, t_f^{*(k)}$ in a similar to that the matrix $\left[Q_n{}^{(k)}\right]$ is obtained in the previous section: From (4.3-10)

$$\Delta \underline{x}(t) = \underline{Q}_t(\underline{S}^{*(k)} + \Delta\underline{S}^{(k)}, t_f^{*(k)} + \Delta t_f{}^{(k)}) - \underline{Q}_t(\underline{S}^{*(k)}, t_f^{*(k)}) \qquad (4.3\text{-}11)$$

where

$$\Delta \underline{x}(t) = \underline{x}(t) - \underline{x}^{(k)}(t) \qquad (4.3\text{-}12)$$

Substituting a Taylor series expansion for $\underline{Q}_t(\underline{S}^{*(k)} + \Delta\underline{S}^{(k)}, t_f^{*(k)} + \Delta t_f{}^{(k)})$ into (4.3-11) gives

$$\Delta \underline{x} = R + \frac{\partial\, \underline{Q}_t(\underline{S}^*_x, t_f^{*(k)})}{\partial\, \underline{S}^*_x} \Delta\underline{S}^{(k)} + \frac{\partial \underline{Q}_t(\underline{S}^{*(k)}, t^*_{fx})}{\partial\, t^*_{fx}} \Delta t_f{}^{(K)}$$

where the partial derivatives are evaluated at $\underline{S}^*_x = \underline{S}^{*(k)}$, $t^*_{fx} = t_f^{*(k)}$ and R is the remainder.

Performing the indicated partial derivatives and noting that $\underline{x}(t^*_{fx}) = \underline{y}^d$ and

$$\frac{\partial \underline{\Phi}(t-t_f)}{\partial t_f} = -\underline{\Phi}(t-t_f)\, A$$

gives [31]

$$\Delta \underline{x} = \Phi(t-t_f^{*(k)})A\underline{y}^d \Delta t_f^{(k)} - (-1)^{n-1} UL_o (2 \sum_{v=1}^{n-1} \underline{h}(t-s_v^*(k))(-1)^{v-1}$$

$$\Delta s_v^{(k)} + (-1)^{n-1} \underline{h}(t-t_f^{*(k)})\Delta t_f^{(k)} \qquad (4.3\text{-}13)$$

where $s_1^{*(k)} = s_2^{*(k)} = \ldots s_k^{*(k)} = t$

Thus, one can obtain the matrix which satisfies (4.3-9) as

$$\left[Q_t^{(k)}\right] = \left[Q_{tS}^{(k)} \quad Q_{t_f}^{(k)}\right]^{-1} \qquad (4.3\text{-}14)$$

where $\left[Q_{tS}^{(k)} \; Q_{t_f}^{(k)}\right]$ is the augumented matrix whose submatrices $\left[Q_{tS}^{(k)}\right]$, $\left[Q_{t_f}^{(k)}\right]$ are properly defined from (4.3-13).

To measure deviations $\Delta\underline{x}$ defined by (4.3-12), one must determine $\underline{x}^{(k)}(t)$. This can be accomplished by solving (1-1) with u(t) replaced by $\bar{u}_k^*(t)$ indicated by (4.3-7), namely and an initial state $\underline{x}(s_k^{*(k)})$. This gives

$$\underline{x}^{(k)}(t) = \Phi(t-s_k^{*(k)})\underline{x}(s_k^{*(k)}) - UL_o(-1)^{n-k}(\underline{r}(t, s_k^{*(k)}) - \underline{r}(t,t)) \qquad (4.3\text{-}15)$$

where $s_k^{*(k)} \leq t \leq s_{k+1}^{*(k+1)}$; $\quad \underline{r}(t,s) \triangleq \int \underline{h}(t,s)\, ds$

Thus, one can obtain the suboptimal control scheme as shown in Figure 4-1. In this Figure one notes that the nominal trajectory $\underline{x}^{(k)}(t)$ and the matrix $\left[Q_t^{(k)}\right]$

SUBOPTIMAL CONTROL SCHEME

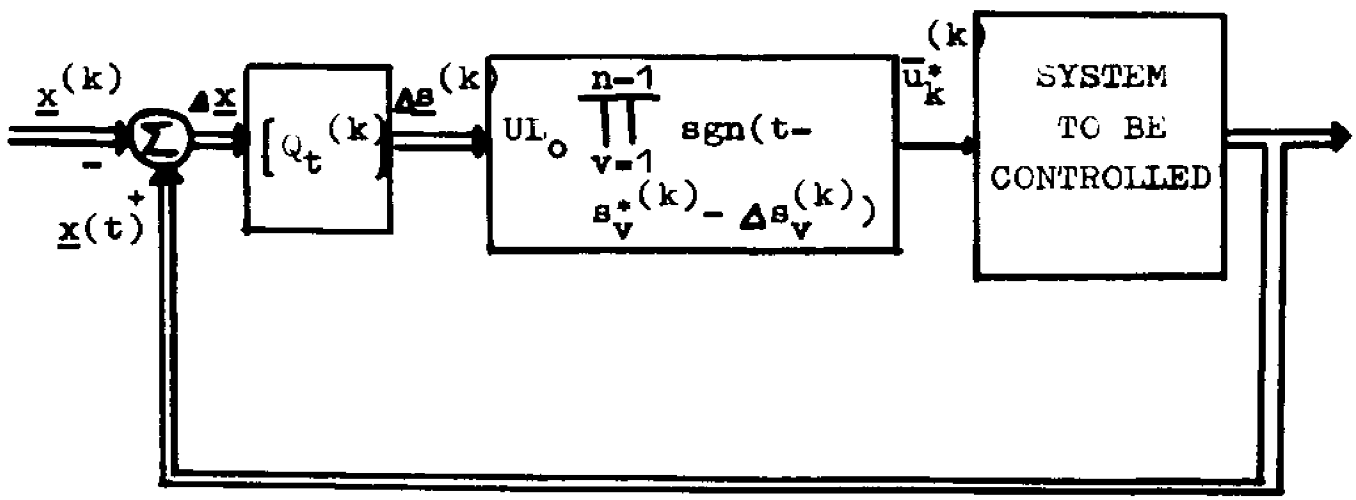


Fig. 4-1

may be computed by simple substitutions at time t,

$t = s_k^{*(k)}$. In practice as a further simplification, one

may assume k = 0, i.e.,

$$\underline{S}_x^* = \underline{S}^{*(0)} + \Delta\underline{S}^{(0)} \qquad (4.3\text{-}16)$$

$$t_{fx} = t_f^{(0)} + \Delta t_f^{(0)} \qquad (4.3\text{-}17)$$

where $\underline{S}$, t_f are obtained as

$$\begin{bmatrix} \underline{S}^{(0)} \\ t_f^{(0)} \end{bmatrix} = \left[Q_t^{(0)}\right] \Delta\underline{x} \qquad (4.3\text{-}18)$$

and $\Delta\underline{x} = \underline{x}(t) - \underline{x}^{(0)}(t)$. In this simplified case, the suboptimal control $\bar{u}^*(t)$ yields

$$\bar{u}_0^*(t) = UL_0 \prod_{v=1}^{n-1} \text{sgn}\,(t - s_v^{*(0)} - s_v^{(0)}) \qquad (4.3\text{-}19)$$

where $\Delta\underline{S}^{(0)}$ and $\Delta t_f^{(0)}$ are found from (4.3-18).

4.4 Approximate Error Analysis

In this section consider following problem: Given a positive number r_0 such that

$$\| \Delta\underline{y}^d \| \leqslant r_0 \qquad (4.4\text{-}1)$$

find a number r_s such that

$$\| \Delta\underline{x} \| \leqslant r_s \qquad (4.4\text{-}2)$$

while providing the suboptimal control into the system to be controlled.

To solve the problem observe (4.3-10): When approximate value of $\underline{S}_x^*$, t_{fx}^* for a given initial state $\underline{x}(t)$ and the desired final output $\underline{y}^d$, i.e.,

$$\underline{S}_x^* = \underline{S}^{*(k)} + \Delta \underline{S}^{(k)}$$

$$t_{fx}^* = t_f^{*(k)} + \Delta t_f^{(k)}$$

are inserted into (4.3-10), the equality in (4.3-10) dose not hold. Therefore, one may expect an error in the desired output $\underline{y}^d$, i.e., $\underline{y}^d + \Delta \underline{y}^d$.

Let

$$\Delta \underline{S}^{(k)} = \left[Q_1^{(k)}\right] \Delta \underline{x} \qquad (4.4\text{-}3)$$

and

$$\Delta t_f^{(k)} = \left[\underline{Q}_2^{(k)}\right] \Delta \underline{x} \qquad (4.4\text{-}4)$$

Let $\underline{Q}_{1v}^{(k)}$ be the v-th row vector of $\left[Q_1^{(k)}\right]$

Inserting an approximate values of $\underline{S}_x^*$, t_{fx}^* obtained through (4.2-7),(4.3-8), respectively into (4.3-10) gives

$$\underline{x}^{*(k)}(t) + \Delta \underline{x} = \Phi(t - t_f^{*(k)} - \underline{Q}_2^{(k)} \Delta \underline{x})(\underline{y}^d + \Delta \underline{y}^d) - (-1)^{n-1}\, UL_o$$

$$2 \sum_{v=1}^{n-1} \underline{r}(t, s_v^{*(k)} + Q_{1v}^{(k)} \Delta \underline{x})(-1)^{v-1} - \underline{r}(t,t)$$

$$+ (-1)^{n-1} \underline{r}(t, t_f^{*(k)})\, \underline{Q}_2^{(k)} \Delta \underline{x}) \qquad (4.4\text{-}5)$$

where $\underline{r}(t,s) \triangleq \int \underline{h}(t-s)\, ds$

Since $Q_1^{(k)}$, $\underline{Q}_2^{(k)}$ are chosen so that the zero order terms and the first order terms of $\Delta\underline{x}$ vanished, one can express (4.4-5) as

$$0 = \Phi(t-t_f^{*(k)}-\underline{Q}_2^{(k)}\Delta\underline{x})\Delta\underline{y}^d + (1/2)\Phi(t-t_f^{*(k)})A^2\underline{y}^d(\underline{Q}_2^{(k)}\Delta\underline{x})^2$$

$$-(-1)^{n-1}UL_o\sum_{v=1}^{n-1}(-1)^{v-1}\Phi(t-s_v^{*(k)})\ AB(\underline{Q}_{1v}^{(k)}\Delta\underline{x})^2$$

$$+(1/2)(-1)^{n-1}\Phi(t-t_f^{*(k)})AB(\underline{Q}_2^{(k)}\Delta\underline{x})^2\ ) + R \qquad (4.4\text{-}6)$$

where R is the remainder.

Assume that $\underline{x}$ is small enough to neglect the remainder terms. One may solve $\Delta\underline{y}^d$ by premultiplying (4.4-6) by $\Phi(t_f^{*(k)} + \Delta t_f^{(k)} - t\ )$ to give

$$\Delta\underline{y}^d = -\frac{1}{2}\ A^2B\ \underline{y}^d(\underline{Q}_2^{(k)}\Delta\underline{x})^2 + (-1)^{n-1}UL_o\sum_{v=1}^{n-1}\Phi(t_f^{*(k)}-s_v^{*(k)})$$

$$AB(\underline{Q}_{1v}^{(k)}\Delta\underline{x})^2(-1)^{v-1} + \frac{1}{2}(-1)^{n-1}\Phi(0)AB(\underline{Q}_2^{(k)}\Delta\underline{x})^2 \qquad (4.4\text{-}7)$$

From (4.4-7) one may not precompute $\Delta\underline{y}^d$ for a given deviation $\Delta\underline{x}$ since the computation of the matrix $[Q_t^{(k)}]$ is required on-line. However, one may resolve this problem by seeking a further approximate equality relation in (4.4-7): Note from (4.2-10),(4.3-8) that

$$t_f^{*(k)} = t_f^{*(0)} + \Delta t_f^{(1)} + \ldots\ldots + \Delta t_f^{(k)} \qquad (4.4\text{-}8)$$

$$s_v^{*(k)} = s_v^{*(0)} + \Delta s_v^{(1)} + \ldots + \Delta s_v^{(k)} \qquad (4.4\text{-}9)$$

and from (4.3-14) and from the definition of $\underline{Q}_{1v}^{(k)}$, $\underline{Q}_2^{(k)}$ that

$$\underline{Q}_{1v}^{(k)} = \underline{Q}_{1v}^{(k)}(t_f^{*(k)}, \underline{S}^{*(k)}) \qquad (4.4\text{-}10)$$

where $v=1,2,\ldots,n-1$

$$\underline{Q}_2^{(k)} = \underline{Q}_2^{(k)}(t_f^{*(k)}, \underline{S}^{*(k)}) \qquad (4.4\text{-}11)$$

Inserting (4.4-10),(4.4-11) with $t_f^{*(k)}$, $\underline{S}^{*(k)}$ replaced by the right hand sides of (4.4-8), (4.4-9), respectively, into (4.4-7) and neglecting the third order and the higher order terms of $\Delta\underline{x}$ in the resulting equation gives

$$\Delta\underline{Y}^d = -\frac{1}{2}A^2\underline{Y}^d(\underline{Q}_2^{(k)}(t_f^{*(0)}, \underline{S}^{*(0)})\Delta\underline{x})^2 + (-1)^{n-1}UL_o\sum_{v=1}^{n-1} \Phi(t_f^{*(0)} - s_v^{*(0)})AB(-1)^{v-1}(\underline{Q}_{1v}^{(k)}(t_f^{*(0)}, \underline{S}^{*(0)})\Delta\underline{x})^2 + \frac{1}{2}(-1)^{n-1}AB(\underline{Q}_2^{(k)}(t_f^{*(0)}, \underline{S}^{*(0)})\Delta\underline{x})^2 \qquad (4.4\text{-}12)$$

Now applying the Schwarz inequality in (4.4-12) gives

$$\|\Delta\underline{Y}^d\| \leq \Big(\Big\|\frac{1}{2}A^2\underline{Y}^d + \frac{1}{2}UL_oAB\Big\|\|\underline{Q}_2^{(k)}(t_f^{*(0)}, \underline{S}^{*(0)})\|^2 + L_o\Big\|\sum_{v=1}^{n-1}(-1)^{v-1}\Phi(t_f^{*(0)}, s_v^{*(0)})AB\Big\|\Big\|\sum_{v=1}^{n-1}\underline{Q}_{1v}^{(k)}(t_f^{*(0)}, \underline{S}^{*(0)})(-1)^{v-1}\Big\|^2\Big)\|\Delta\underline{x}\|^2 \qquad (4.4\text{-}13)$$

where the norm is taken as the square root of the inner product norm.

Since the objective is to find a number r_s such that $\|\Delta \underline{x}\| \leq r_s$ for a given positive number r_o which satisfies (4.4-2), one solves (4.4-13) for $\|\Delta \underline{x}\|$ in case of $\|\Delta \underline{y}^d\| \leq r_o$ and obtains r_s as

$$r_s = \frac{\sqrt{2\, r_o}}{\left\| A^2 \underline{y}^d + UL_o AB \right\|^{1/2} \left\| Q_2^{(k)} \right\| + \left\| \sum_{v=1}^{n-1} \underline{X}(t_f^{*(0)}, s_v^{*(0)}) AB \right\|^{1/2} \sqrt{L_o}}$$

$$\overline{\sum_{v=1}^{n-1} (-1)^{v-1} Q_{1v}(t_f^{*(0)}, \underline{S}^{*(0)})} \qquad (4.4\text{-}14)$$

4.5 Design Procedure

A design procedure for the problem 2 (in case of $p = \infty$) developed in chapter IV can be summarized as follows:

Assume that the upper bound on the number of optimal switching instants are known. In the following design procedure consider the dynamic system all of whose eigen-values are real and its number of switchings is n-1. Let k, the number of corrections be n-1 and the corrections be made on the nominal switching instant.

Step 1 Calculate the state transition matrix $\Phi(t,s)$, the nominal switching instants $s_1^{*(0)}, s^{*(0)}, \ldots, s_{n-1}^{*(0)}$, the least terminal time $t_f^{*(0)}$, and the nominal trajectory $\underline{x}^{(0)}(t)$. Determine U from the nominal optimum control $u^*(t_f^{*(0)}, \lambda^*, t)$.

Step 2 Calculate the nominal trajectory $\underline{x}^{(k)}(t)$ at the nominal instant time $s_k^{*(k)}$ and the nominal switching instants corrected k-times: Use (4.3-15), (4.2-10), respectively.

Step 3 Calculate the matrix $\left[Q_t^{(k)}\right]$ at time $t = s_k^{*(k)}$ for each k, $k \geqslant 0$. Make use of (4.3-14) to obtain $\left[Q_t^{(k)}\right]$.

Step 4 On-line computation of the suboptimal control defined by (4.3-7).

4.6 Illustrative Examples

Consider Bushaw's problem presented in section 3.6 in Chapter III.

Note in this problem that the system matrix A has two real roots and both are zero. Hence, there is at most only one switching instant s_{x1}^*. In this example the suboptimal control indicated by (4.3-7) yields

$$\bar{u}_k^*(t) = U \operatorname{sgn}\left(t - s_1^{*(k)} - \Delta s_1^{(k)}\right) \tag{4.6-1}$$

Let k, the number of corrections be 1 and the correction be made on the nominal switching instant.

Step 1 Calculate the state transition matrix, the 0-nominal switching instant $s_1^{*(0)}$, the 0-least terminal time $t_f^{*(0)}$, the 0-nominal trajectory $\underline{x}^{(0)}(t)$, i.e.,

$$s_1^{*(0)} = 10; \quad t_f^{*(0)} = 20 ;$$

$$\underline{x}^{(0)}(t) = \begin{bmatrix} -100+0.5\, t^2 \\ t \end{bmatrix} \quad \text{for } t \leqslant 10$$

$$\underline{x}^{(0)}(t) = \begin{bmatrix} -0.5t^2 + 20t - 200 \\ 200 - t \end{bmatrix} \quad \text{for } 10 \leqslant t \leqslant 20 \qquad (4.6\text{-}2)$$

U = -1 determined from the nominal optimal control.

Step 2 Calculate $\underline{x}^{(1)}(t)$ by making use of (4.3-15):

$$\underline{x}^{(1)}(t) = \Phi(t - s_1^{*(1)})\underline{x}(s_1^{*(1)}) + \begin{bmatrix} ts_1^{*(1)} - 0.5(s_1^{*(1)})^2 - 0.5t^2 \\ s_1^{*(1)} - t \end{bmatrix} \qquad (4.6\text{-}3)$$

where $s_1^{*(1)} \leqslant t \leqslant t_f^{*(1)}$ and

$$s_1^{*(1)} = s_1^{*(0)} + \Delta s_1^{(0)}(s_1^{*(1)}) \qquad (4.6\text{-}4)$$

$$t_1^{*(1)} = t_f^{*(0)} + \Delta t_f^{(0)}(s_1^{*(1)}) \qquad (4.6\text{-}5)$$

Step 3 Calculate the matrices $\left[Q_t^{(k)}\right]$ for k=0, 1:

Make use of (4.3-14) to obtain $\left[Q_t^{(k)}\right]$.

In the ase when k =0,

$$\left[Q_{tS}^{(0)} \vdots Q_{t_f}^{(0)}\right] = -\begin{bmatrix} 2(t-10) & -(t-20) \\ 2 & -1 \end{bmatrix}$$

Performing the inversion of the matrix indicated above gives

$$\left[Q_t^{(0)}\right] = -\frac{1}{20}\begin{bmatrix} 1 & (20-t) \\ 1 & (10-t) \end{bmatrix} \qquad (4.6\text{-}6)$$

In the case when k = 1, one can obtain $\left[Q_t^{(1)}\right]$ in a similar way to that $\left[Q_t^{(0)}\right]$:

$$Q_t^{(1)} = \frac{1}{2(t-t_f^{*(1)})}\begin{bmatrix} 1 & (t_f^{*(1)} - t) \\ 2 & 0 \end{bmatrix} \qquad (4.6\text{-}7)$$

<u>Step 4</u> The on-line computation of the suboptimal control

(i) $0 \leqslant t \leqslant s_1^{*(1)}$

$$\bar{u}_0^*(t) = -\operatorname{sgn}(t-10-\Delta s_1^{(0)}) \qquad (4.6\text{-}8)$$

where $\Delta s_1^{(0)}$ is determined from (4.6-6) and (4.3-9) with k = 0 and $\Delta \underline{x}$ is the difference between the $\underline{x}(t)$ and $\underline{x}^{(0)}(t)$, where $\underline{x}^{(0)}(t)$ is defined by (4.6-2)

(ii) $s_1^{*(1)} \leqslant t \leqslant t_f^{*(1)}$

$$\bar{u}_1^*(t) = -\operatorname{sgn}(-\Delta s_1^{(1)}) \qquad (4.6\text{-}9)$$

where $\Delta s_1^{(1)}$ is determined from (4.6-7) and (4.3-9) with k = 1 ; $\Delta \underline{x}$ is the difference between the $\underline{x}(t)$ and $\underline{x}^{(1)}(t)$, where $\underline{x}^{(1)}(t)$ is determined from (4.6-3). Note that $s_1^{*(1)}$ is measured at time t so that the running time t may satisfy

$$10 - \frac{1}{20} (\Delta x_1(t) + (20-t)\Delta x_2(t)) - t = 0$$

and that the process is stopped at time t so that the running time t may satify

$$20 + \Delta t_f^{(0)} + \Delta t_f^{(1)}(t) - t = 0$$

Remark 4-1

When k=0, as has been mentioned earlier, a further simplification may be possible, i.e., it is not required to estimate the nominal quantities $\underline{S}^{*(1)}, \underline{x}^{(1)}, t_f^{*(1)}$; use all the formulas with $\underline{S}^{*(1)}$, $\underline{x}^{(1)}$, $t_f^{*(1)}$ replaced by $\underline{S}^*(0)$, $\underline{x}^{(0)}(t), t_f^{*(0)}$, respectively. *****

It is interesting to note that

$$s_1^{*(1)} = \frac{1}{2(t-t_f^{*(1)})} (\Delta x_1 + (t_f^{*(1)}-t)\Delta x_2) = 0$$

may represent an approximate switching line: To demonstrate this some deviations from the nominal trajectory $\underline{x}^{(0)}(t)$ are made at 17 seconds, 18 seconds, 19 seconds as indicated in Figure 4-2. The suboptimal control (4.6-9) is calculated

ACTION OF SUBOPTIMAL CONTROLLER AROUND NOMINAL TRAJECTORY

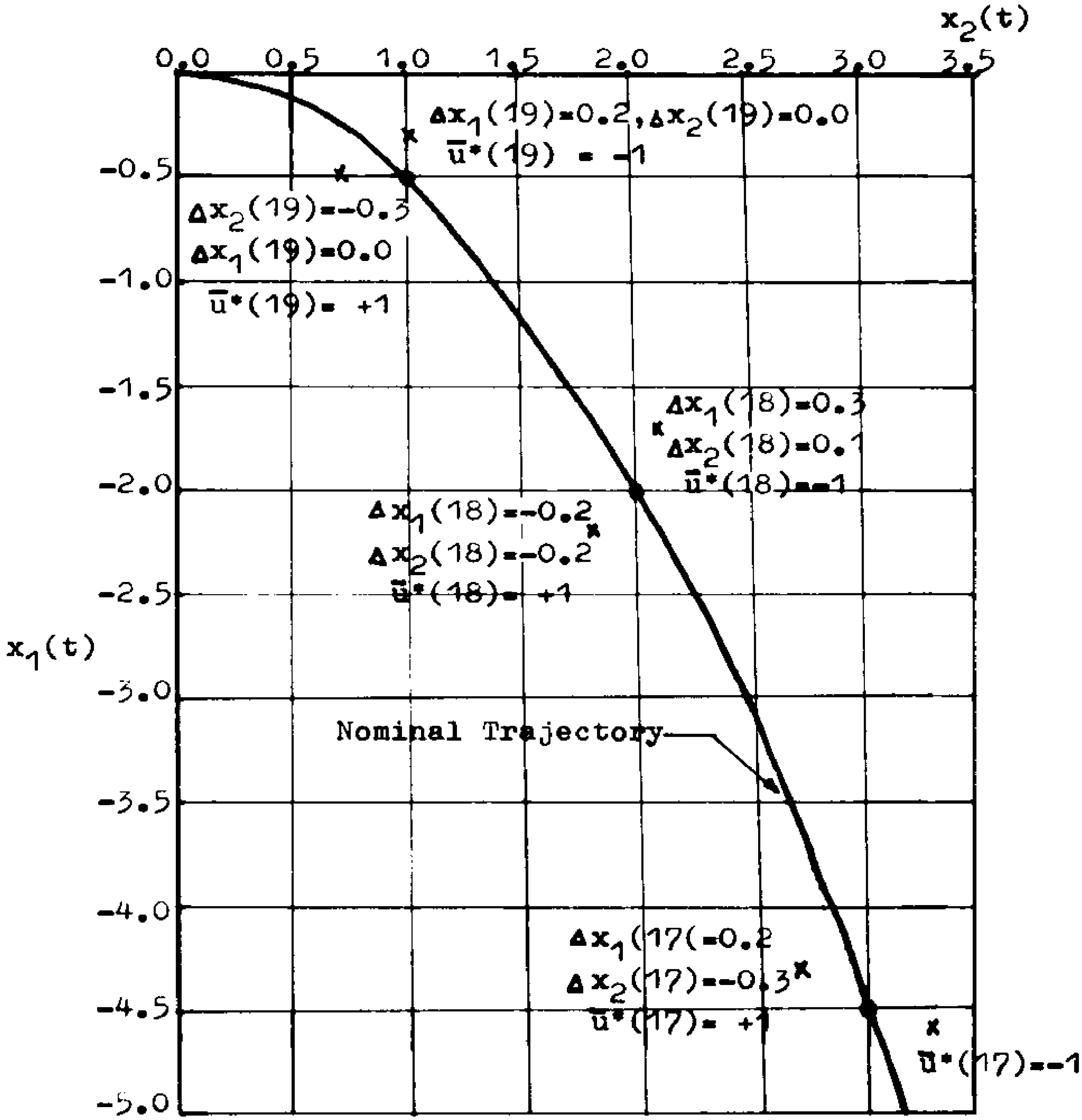


Fig. 4-2

for each deviation and is as shown in Figure 4-2. Thus, the Figure shows that $\Delta s_1^{(1)} = 0$ can represent the suboptimal switching line.

As a second problem, consider a dynamic system described by the first order differential equations

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (4.6\text{-}10)$$

$$\underline{y}(t) = \underline{x}(t) \qquad (4.6\text{-}11)$$

Given an initial state $\underline{x}_o = [-1 \quad 0 \quad 0]^T$ at an initial time $t_i = 0$ and a desired final output $x_1^d = x_2^d = x_3^d = 0$; we wish to find the suboptimal control which transfers the initial state $\underline{x}_o$ to $\underline{x}^d$ in the least time when the control is bounded by 0.5, i.e., $|u(t)| \leq 0.5$.

It is easy to see in this problem that there are at most two switching instants s^*_{x1}, s^*_{x2} . In this example the suboptimal control indicated by (4.3-7) yields

$$\bar{u}^*_k(t) = 0.5 \text{ sgn } (t - s_1^{*(k)} - \Delta s_1^{(k)})(t - s_2^{*(k)} - \Delta s_2^{(k)}) \qquad (4.6\text{-}12)$$

Let k, the number of corrections be 2 and the corrections be made at the nominal switching instant.

Step 1 Calculate the state transition matrix, the 0-nominal quantities $\underline{S}^{*(0)}$, $t_f^{*(0)}$, $\underline{x}^{(0)}(t)$:

(a) transition matrix

$$\Phi(t-s) = \begin{bmatrix} 1 & t-s & 1/(2(t-s)^2) \\ 0 & 1 & t-s \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $s_1^*(0) = 1$; $s_2^{*(0)} = 3$; $t_f^{*(0)} = 4$

(c)

i) $0 \leq t \leq 1$

$$x_1^{(0)}(t) = \frac{1}{12} t^3 - 1$$

$$x_2^{(0)}(t) = \frac{1}{4} t^2$$

$$x_3^{(0)}(t) = 0.5\, t$$

ii) $1 \leq t \leq 3$

$$x_1^{(0)}(t) = -\frac{1}{12} t^3 + 0.5t^2 - 0.5t - \frac{5}{6}$$

$$x_2^{(0)}(t) = -0.25t^2 + t - 0.5$$

$$x_3^{(0)}(t) = -0.5t + 1$$

iii) $3 \leq t \leq 4$

$$x_1^{(0)}(t) = -\frac{1}{12} t^3 - t^2 + 4t - \frac{16}{3}$$

$$x_2^{(0)}(t) = 0.25\, t^2 - 2t + 4$$

$$x_3^{(0)}(t) = 0.5t - 2$$

(d) $U = +1$

<u>Step 2</u> Calculate $\underline{x}^{(k)}(t)$ by making use of (4.3-15)

$$\underline{x}^{(k)}(t) = \underline{\Phi}(t-s_k^{*(k)})\underline{x}(s_k^{*(k)}) - \int_{s_k^{*(k)}}^{t} 0.5\,(t-s)B\,ds \tag{4.6-17}$$

where k= 1, 2 and

$$s_1^{*(1)} = s_1^{*(0)} + \Delta s_1^{(0)}(s_1^{*(1)}) \tag{4.6-18}$$

$$s_2^{*(2)} = s_2^{*(1)} + \Delta s_2^{(1)}(s^{*(2)}) \tag{4.6-19}$$

<u>Step 3</u> Calculate the matrices $\left[Q_t^{(k)}\right]$ for k=0,1,2

Make use of (4.3-14) to obtain $\left[Q_t^{(k)}\right]$:

(i) k = 0

$$Q_t^{(0)} = \frac{1}{3}\begin{bmatrix} -1 & (t-3.5) & -0.5(t-3)(t-4) \\ -3 & 3t-7.5 & -1.5(t-1)(t-4) \\ -4 & 4t-8 & -(t-1)(t-3)2 \end{bmatrix}$$

(ii) k = 1

$$Q_t^{(1)} = \frac{4}{(t-3)(t-4)}\begin{bmatrix} -0.5 & -(3.5-t) & -0.5(t-3)(t-4) \\ 0.5(t-4) & -0.25(t-4)^2 & 0 \\ (t-3) & -0.5(t-3)^2 & 0 \end{bmatrix}$$

where $s_2^{*(1)} = 3$; $t_f^{*(1)} = 4$ for simplicity.

(iii) k=2

In this case one cannot make inversion of the matrix $\left[Q_{tS}^{(2)} \vdots Q_{t_f}^{(2)}\right]$. To employ the Pseudo-inverse method, one decomposes the matrix as indicated in section 4.2 , Chapter IV

$$\left[Q_{tS}^{(2)} \vdots Q_{t_f}^{(2)}\right] = A\ B$$

where

$$A = \begin{bmatrix} 0.5(t-4)^2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 0 & 0.5(t-4) \\ 1 & -1 & 0.5 \end{bmatrix}$$

Performing the indicated Pseudo-inversion in section 4.2 gives

$$\left[\bar{Q}_t^{(2)}\right] = \begin{bmatrix} \frac{-(t-4)}{(t-4)^4+4} & \frac{-2}{(t-4)(4+(t-4)^4)} & 0.5 \\ \frac{(t-4)}{(t-4)^4+4} & \frac{2}{(t-4)(4+(t-4)^4)} & -0.5 \\ \frac{4(t-4)}{(t-4)^4+4} & \frac{8}{(t-4)(4+(t-4)^4)} & 0 \end{bmatrix} \tag{4.6-22}$$

where $t_f^{*(2)} = 4$ for the sake of simplicity.

Step 4 The on-line computation of the suboptimal control:

(i) $0 \leq t \leq s_1^{*(1)}$

$$\bar{u}_0^*(t) = 0.5\,\mathrm{sgn}(t-1-\Delta s_1^{(0)})(t-3-\Delta s_2^{(0)}) \tag{4.6-23}$$

where $\Delta s_1^{(0)}$; $\Delta s_2^{(0)}$ are determined from (4.6-20) and (4.3-9) with k = 0 and $\Delta \underline{x}$ is the difference between

$\underline{x}(t)$ the actual output and $\underline{x}^{(0)}(t)$ defined by (4.6-14) to (4.6-16).

(ii) $s_1^{*(1)} < t \leq s_2^{*(2)}$

$$\bar{u}_1^*(t) = 0.5 \text{ sgn } (\Delta s_1^{(1)}) \text{ sgn } (t - s_2^{*(1)} - \Delta s_2^{(1)})$$

where $\Delta s_1^{(1)}$, $\Delta s_2^{(1)}$ are determined from (4.6-21) and (4.3-9) with $k = 1$ and $\Delta \underline{x}(t)$ is the difference between the actual output $\underline{x}(t)$ and $\underline{x}^{(1)}(t)$ defined by (4.6-17) with $k=1$; $s_2^{*(1)} = 3 + \Delta s_2^{(0)} + \Delta s_2^{(1)}(s_1^{*(1)})$.

(iii) $s_2^{*(1)} \leq t < t_f^{*(2)}$

$$\bar{u}_2^*(t) = 0.5 \text{ sgn}(\Delta s_1^{(2)}) \text{ sgn } (\Delta s_2^{(2)})$$

where $\Delta s_1^{(2)}$, $\Delta s_2^{(2)}$ are determined from (4.6-22) and (4.3-9) with $k = 2$ and $\Delta \underline{x}$ is the difference between $\underline{x}(t)$ and $\underline{x}^{(2)}(t)$ defined by (4.6-17) with $k = 2$.

To demonstrate the effectiveness of the technique the suboptimal control system was simulated using a digital computer for the initial deviation from the nominal initial state $x_{o1} = -1$, $x_{o2} = x_{o3} = 0$. An error in position was taken as the initial deviation. The reason for this deviation is that one can obtain the optimal control in references [22.23.24] so that one may compare the performance between the optimal control and the suboptimal one. In this

SUMMARY OF RESULTS

			A	B	C	D
INITIAL POSITION			-1.100	-1.300	-0.800	-0.700
NOMINAL CONTROL	SWITCHING INSTANTS	$s_1^{*(0)}$	1.000	—	—	—
		$s_2^{*(0)}$	3.000	—	—	—
		$t_f^{*(0)}$	4.000	—	—	—
	TERMINAL ERROR	Δx_1	-0.100	-0.300	0.200	0.300
		Δx_2	0.000	—	—	—
		Δx_3	0.000	—	—	—
SUBOPTIMAL CONTROL	SWITCHING INSTANTS	$s_1^{*(1)}$	1.030	1.090	0.930	0.900
		$s_2^{*(2)}$	3.100	3.300	2.790	2.690
		$t_f^{*(3)}$	4.140	4.400	3.740	3.610
	TERMINAL ERROR	Δx_1	-0.013	-0.023	0.004	0.033
		Δx_2	-0.010	-0.033	0.000	0.009
		Δx_3	0.000	-0.010	0.010	0.015
OPTIMAL CONTROL	SWITCHING INSTANTS	s_{x1}^*	1.033	1.092	0.928	0.893
		s_{x2}^*	3.099	3.276	2.784	2.679
		t_{fx}^*	4.133	4.370	3.713	3.572
	TERMINAL ERROR	$\Delta \underline{x}$	$\underline{0}$	—	—	—

Table 4-1

PHASE PORTRAIT OF THE EXAMPLE

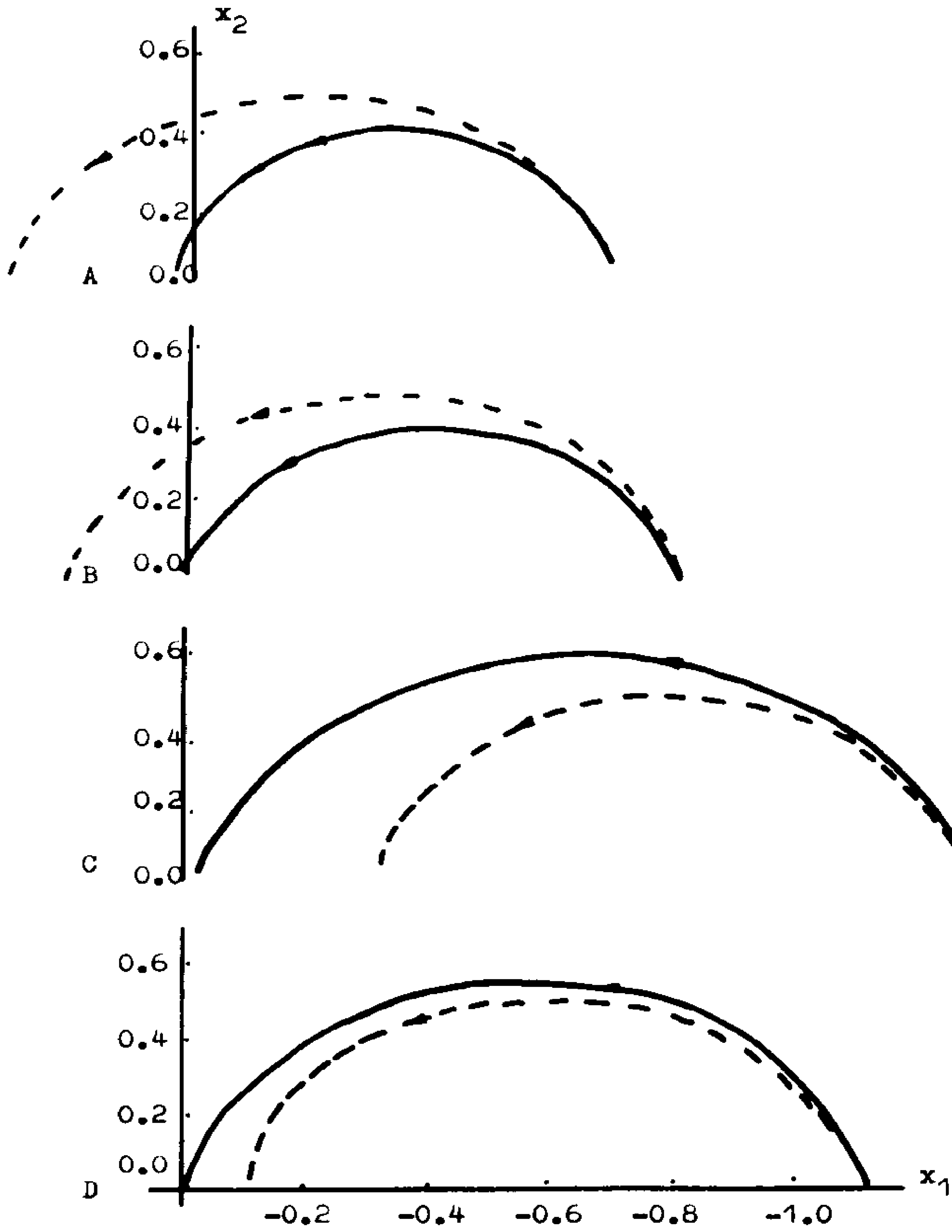


Fig. 4-3

CONTROL INPUT OF THE EXAMPLE

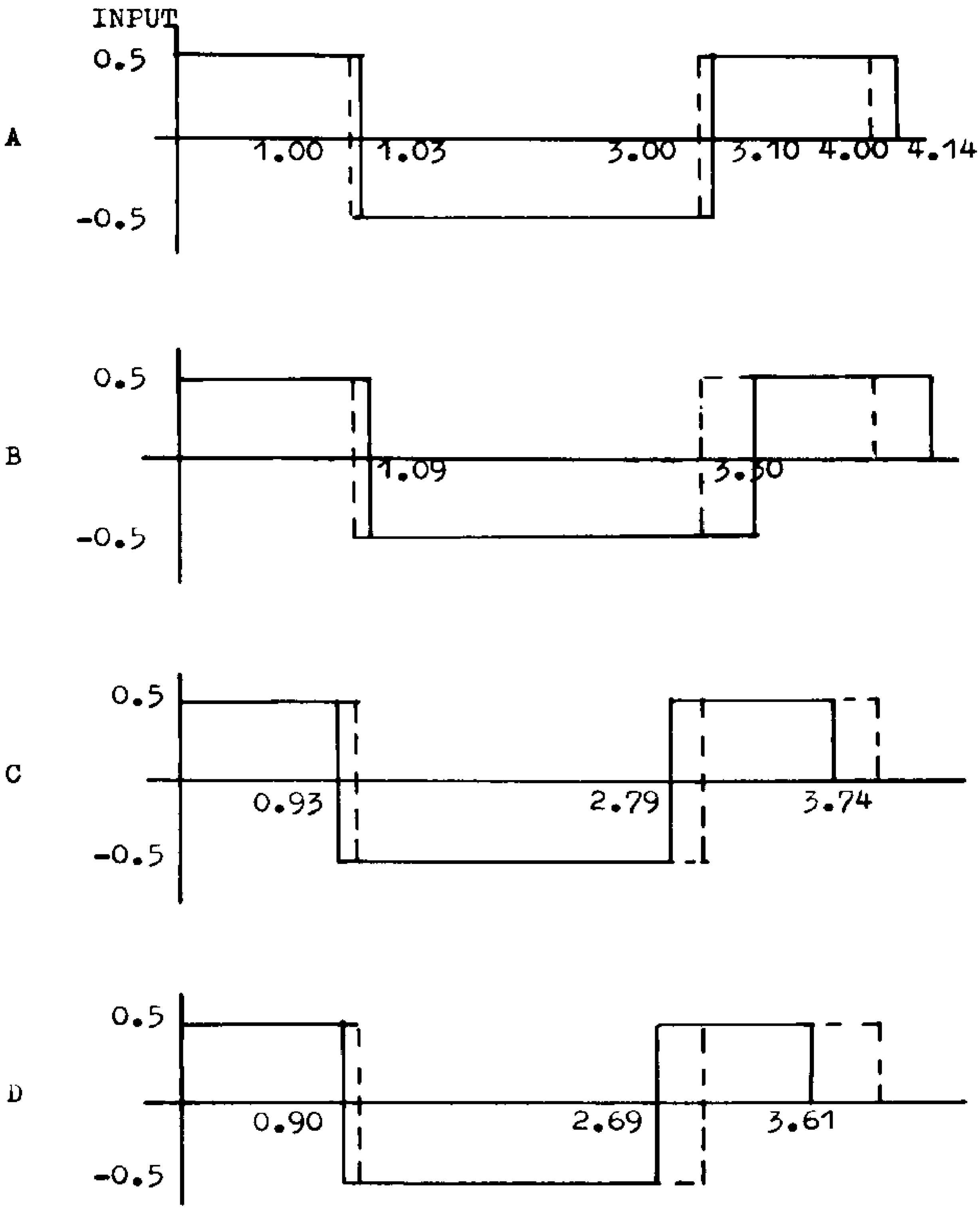


Fig.4-4

simulation the suboptimal controller measures the output at every 0.01 seconds and corrects the preprogrammed suboptimal control. The summary of the results is tabulated in Table 4-1. The Table 4-1 shows that for 30% deviation from the initial position the suboptimal control results within a circle of radius 0.04 from the desired output while the nominal control without correction results within a radius 7.5 times that of the suboptimal one. In Figure 4-3 the suboptimal control and the nominal one are compared. In Figure 4-4 a state trajectory is plotted for each case A, B, C, D in x_1-x_2 plane, where the solid line, dot line indicate the state trajectory due to the suboptimal control, and due to the nominal control without correction, respectively.

4.7 Summary

In this Chapter it was shown that a design procedure of a feedback suboptimum control law for the minimum time control problem ($p=\infty$) was developed.

The proposed technique is based on the assumption that there is a known bound on the number of the optimum switching instants and that the dynamic system operates in the presence of small perturbations during the actual operation and that the perturbations are essentially due to the external unknown forces.

The limitation of the approach is that one cannot apply this technique to the system which is not normal.

The illustrative example showed that in case no correction is made on the nominal control regardless of the perturbations, the terminal error is more than 7.5 times greater than those obtained using the suboptimal control scheme.

V SOLUTION FOR THE NON-LINEAR SYSTEMS

In this Chapter consider the dynamic system to be controlled is described by a non-linear differential equation

$$\frac{d\,\underline{x}_N(t)}{d\,t} = \underline{f}(\underline{x}_N(t), u_N(t), t) \qquad (5\text{-}1)$$

where $\underline{x}_N(t)$ is n-state vector, $u_N(t)$ is for simplicity a scalar control input. $\underline{f}$ is a continuous real-valued n-known vector function of time t, $\underline{x}_N(t)$, and $u_N(t)$ for $t \geqslant 0$ and is continuously differentialable with respect to t, $\underline{x}_N(t)$, and $u_N(t)$.

One cannot apply the technique developed for the linear system to the non-linear one since the technique is based on the results of the L-problem and the L-problem was restricted to the linear system only. However, one may extend the technique to the non-linear systems by linearizing the non-linear system around the nominal optimal control $u_N^*(t)$ and its corresponding trajectory $\underline{x}_N^*(t)$.

5.1 Linearized System

Assuming that the system operates in the presence of small deviations from the nominal trajectory, one can write

$$\underline{x}_N(t) = \underline{x}_N^*(t) + \Delta\underline{x} \qquad (5.1\text{-}1)$$

$$u_N(t) = u_N^*(t) + \Delta u \qquad (5.1\text{-}2)$$

Linearize the non-linear system described by (5-1) around $\underline{x}_N^*(t)$, $u_N^*(t)$ and define $\underline{x}_N$ and u_N so that

$$\frac{d\underline{x}_N(t)}{d\,t} = \underline{f}_{x_N}\ \underline{x}_N + \underline{f}_{u_N}\ u_N \qquad (5.1\text{-}3)$$

is satisfied, where

$$\underline{f}_{\underline{x}_N} \triangleq \left|\frac{\partial\underline{f}(\underline{x}_N, u_N^*, t)}{\partial\underline{x}_N}\right|_{\underline{x}_N^*(t)} ; \quad \underline{f}_{u_N} \triangleq \left|\frac{\partial\underline{f}(\underline{x}_N^*, u_N, t)}{\partial u_N}\right|_{u_N^*(t)}$$

Let $\underline{x}_L(t)$ and $u_L(t)$ be defined as

$$\underline{x}_L(t) = \underline{x}_N^*(t) + \underline{x}_N(t) \tag{5.1-4}$$

$$u_L(t) = u_L^*(t) + u_N(t) \tag{5.1-5}$$

Inserting (5.1-4) and (5.1-5) into (5.1-3) gives

$$\frac{d\underline{x}(t)}{dt} = \underline{f}_{\underline{x}_N}\underline{x}_L(t)+\underline{f}_{u_N}u_L(t)+\underline{f}(\underline{x}_N^*,u_N^*,t)-\underline{f}_{\underline{x}_N}\underline{x}_N^*-\underline{f}_{u_N}u_N^* \tag{5.1-6}$$

where it is assumed that the initial state $\underline{x}_L(t_i)=\underline{x}_N^*(t_i)=\underline{x}_o$. Thus, the above differential equation (5.1-6) describes a linear system derived from what was originally a nonlinear system. Then the suboptimal control systems developed for the linear systems can be applied provided the linearized system meets the requirements of Chapter I.

The linearized system has following properties: Let the optimum control $u_L^*(t)$ be the solution of the problem 1* (see Appendix A) for the linearized system.

<u>Property 1</u>

Solving (5.1-6) with $u_L(t)$ replaced by $u_N^*(t)$ gives

$$\underline{x}_L(t) = \underline{x}_N^*(t)$$

<u>Property 2</u>

The optimum control for the linearized system has the same form as that of $u_N^*(t)$.

Property 3

Let the costate vector [27] for the non-linear system be $\underline{p}_N^*(t)$. And let the optimum lambda vector for the linearized system be λ^*. Then,

$$\lambda^* = K_o \, p_N^*(t_f) \qquad (5.1\text{-}7)$$

where t_f is the fixed terminal time; K_o is scalar.

To show Property 1 one must show that $\underline{x}_L(t) = \underline{x}_N^*(t)$ when $u_L(t) = u_N^*(t)$. When $u_L(t) = u_N^*(t)$, (5.1-6) yields

$$\dot{\underline{x}}_L(t) = \underline{f}(\underline{x}_N^*, u_N^*, t) + \underline{f}_{\underline{x}_N}(\underline{x}_L(t) - \underline{x}_N^*(t))$$

and solving [3.32] the above differential equation

$$\underline{x}_L(t) = \Phi(t,t_i)\underline{x}_o + \int_{t_i}^{t} \Phi(t,s)(\underline{f}(\underline{x}_N^*, \underline{x}_N^*, t) - \underline{f}_{\underline{x}_N}\underline{x}_N^*(s) \, ds \qquad (5.1\text{-}8)$$

where $\Phi(t,s)$ is the transition matrix of $\underline{f}_{\underline{x}_N}$

Making use of Property of the transition matrix

$$\frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t,s) \, \underline{f}_{\underline{x}_N}$$

in (5.1-8) and integrating by parts the integral in (5.1-8) one obtains

$$\underline{x}_L(t) = \Phi(t,t_i)\underline{x}_o + \Big| \Phi(t,s)\underline{x}_N^*(s) \Big|_{t_i}^{t} = \underline{x}_N^*(t)$$

since $\underline{x}_N^*(t_i) \; \underline{x}_o$.

To show Property 2, one makes use of the Maximum Priciple [27]. According to the Maximum Principle, The optimum control $u_N^*(t)$, its corresponding trajectory $\underline{x}_N^*(t)$, and the costate vector $\underline{p}_N^*(t)$ must satisfy

$$\dot{\underline{x}}_N^*(t) = \underline{f}(\underline{x}_N^*(t), u_N^*, t) \qquad (5.1\text{-}9)$$

$$\dot{\underline{p}}_N^*(t) = -(\underline{f}_{\underline{x}_N})^T \underline{p}_N^*(t) \qquad (5.1\text{-}10)$$

with boundary conditions

$$\underline{x}_N^*(t_i) = \underline{x}_o \ ; \ \underline{x}_N^*(t_f) = \underline{x}^d; \ t_f \text{ is fixed}$$

and

$$u_N^*(t) = K_1 \left| (\underline{p}_N^{*T} \underline{f}_{u_N}) \right|^{1/(p-1)} \operatorname{sgn}(\underline{p}_N^{*T} \underline{f}_{u_N}) \qquad (5.1\text{-}11)$$

where K_1 is a constant scalar.

Also applying the Maximum Principle to the linearized system, one may obtain a candidate for the optimum control

$$u_L^*(t) = K_2 \left| \underline{p}_L^{*T}(t) \underline{f}_{u_N}) \right|^{1/(p-1)} \operatorname{sgn}(\underline{p}_L^{*T}(t) \underline{f}_{u_N}) \qquad (5.1\text{-}12)$$

where K_2 is a constant scalar; $\underline{p}_L^*(t)$ is the costate vector corresponding to the linearized system. Moreover, $u_L^*(t)$, $\underline{x}_L^*(t)$, $\underline{p}_L^*(t)$ must satisfy

$$\dot{\underline{x}}_L^*(t) = \underline{f}(\underline{x}_N^*(t), u_N^*(t), t) + \underline{f}_{\underline{x}_N} (\underline{x}_L^*(t) - \underline{x}_N^*(t)) + \underline{f}_{u_N} (u_L^*(t) - u_N^*(t)) \qquad (5.1\text{-}14)$$

$$\dot{\underline{p}}^*_L(t) = -(\underline{f}_{\underline{x}_N})^T \underline{p}^*_L(t) \qquad (5.1\text{-}14)$$

with boundar conditions $\underline{x}^*_L(t_i) = \underline{x}_o$; $\underline{x}^*_L(t_f) = \underline{x}^d$.

According to Property 1, $u^*_N(t)$, a candidate for $u^*_L(t)$ satisfies (5.1-13),(5.1-14). Thus, $u^*_N(t)$ is a candidate for the optimum control. However, since we assumed that the linearized system is normal, $u^*_L(t)$ is unique and $u^*_L(t) = u^*_N(t)$.

To show Property 3, recall that the direction of the optimumlambda vector is the same as that of the costate vector evaluated at the terminal time t_f. According to Property 2, $u^*_L(t) = u^*_N(t)$. Moreover, $u^*_L(t) = u^*_N(t)$ if and only if $\underline{p}^*_N(t) = \underline{p}^*_L(t)$ within a scalar multiplicity. Therefore, Property 3 is asserted.

Hence it has been shown that a solution of a two-point boundary value problem for the nonlinear system is also a solution of the two point boundary value problem for the linearized system. Hence, the nonlinear control law is a candidate for being the optimum control of the linear system. However, what is assumed in the following suboptimum design is that a suboptimum control for the linear system is "close" to the optimum control law for the non-linear one.

No analytic verification for this assumption is available at this time.

Based on the above assumption, one can reduce the original nonlinear problem to a linear one. Then the suboptimum control system developed for the linear system can be applied provided the linearized system meets the requirements of Chapter I. The feedback suboptimal control scheme obtained for the non-linear system is as shown in Figure 5-1. In this Figure, the box marked " suboptimum co ntroller" realizes (2.3-1) (in Chapter II) which is obtained from the linearized system (5.1-6). The seeming difficulty with this approach is that one must estimate the impulse response for the time-varying linearized system. Although there is no general approach to obtain a closed-form expression for the impulse response of the time varying system (5.1-6), there are some specialized techniques [37] for obtaining the impulse response of time-varying linear systems.

In case of problem 2(the minimum time suboptimum control problem) the terminal time is also corrected as a result of a deviation from the nominal trajectory according to the technique developed in Chapter III.

SUBOPTIMAL CONTROL SCHEME

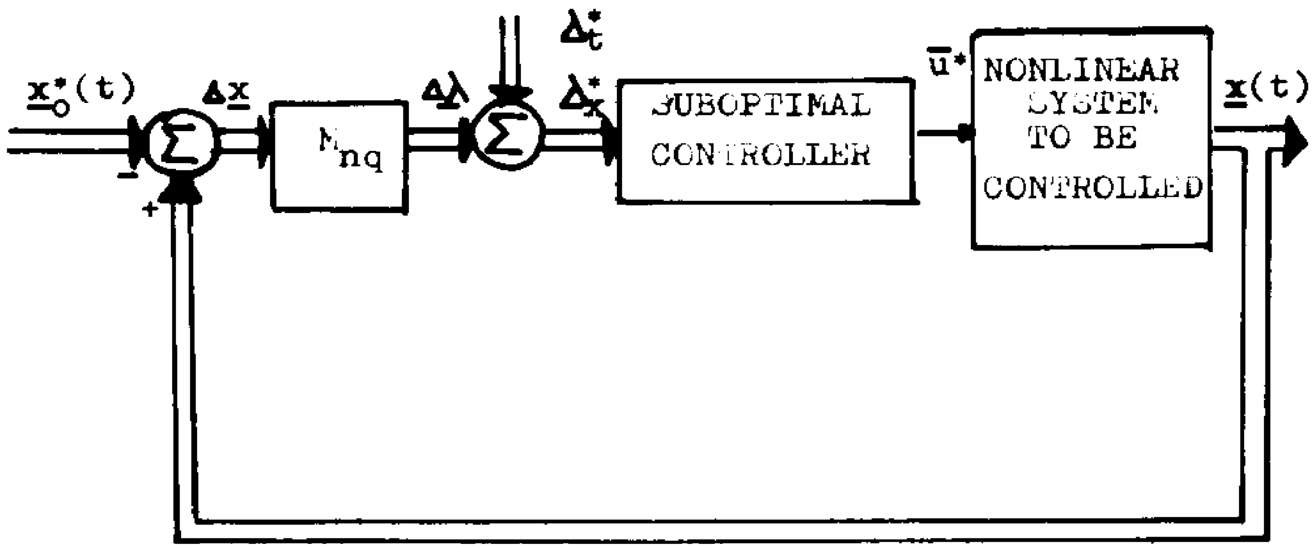


Fig.5-1

In case there is a bound on the number of the switching instants of the optimal control, one can apply the switching method developed in Chapter IV to the corresponding non-linear system(see the illustrative example given in section 5.2).

5.2 Illustrative Example

To clarify the technique developed for the non-linear system, a problem of guiding an air-craft to a terminal take-off area in the least time is presented here: The simplified equations of motion of the air-craft in the horizontal plane are [34]

$$\dot{x}_1 = \cos x_3$$
$$\dot{x}_2 = \sin x_3$$
$$\dot{x}_3 = -u(t) \qquad (5.2\text{-}1)$$

where x_1 and x_2 are the Cartesian coordinates of the air-craft, x_3 is the heading angle measured counter-clockwise from the positive x_1 axis. The problem considered in this example can be stated as follows: Given a dynamic system (5.2-1) with a magnitude constraint on control,i.e., $|u(t)| \leqslant 1$ find a suboptimal control which transfers initial states $x_1(0)= -2$, $x_2(0)= 2$, $x_3(0)= /2$ to the origin in the least time in the presence of perturbations.

It has been shown in reference 34 that a bound on the number of the switching instants is two and that the switching instants for a particular initial state $\underline{x}_0$ are $s_1^{*(0)} = 0.33731$, $s_2^{*(0)} = 4.79701$. And the least terminal time $t_f^{*(0)} = 7.34861$. The corresponding nominal trajectory $\underline{x}^{(0)}(t)$ can be found as

(i) $0 \leq t \leq s_1^{*(0)}$

$$x_1^{(0)}(t) = -2\sin(t_f^{*(0)}+2s_2^{*(0)}-s_1^{*(0)})+\sin(t_f^{*(0)}+2s_2^{*(0)}-2s_1^{*(0)}) + 2\sin(t_f^{*(0)}+s_2^{*(0)})$$

$$x_2^{(0)}(t) = 2\cos(t_f^{*(0)}+2s_2^{*(0)}-s_1^{*(0)})-\cos(t_f^{*(0)}+2s_2^{*(0)}-2s_1^{*(0)}) - 2\cos(t_f^{*(0)}+s_2^{*(0)})$$

$$x_3^{(0)}(t) = t_f^{*(0)}+2(s_2^{*(0)}-s_1^{*(0)}) + t \qquad (5.2\text{-}2)$$

(ii) $s_1^{*(0)} \leq t \leq s_2^{*(0)}$

$$x_1^{(0)}(t) = 2\sin(s_2^{*(0)}-t_f^{*(0)})-2\sin(2s_2^{*(0)}-t_f^{*(0)}-t)$$

$$x_2^{(0)}(t) = 1-2\cos(s_2^{*(0)}-t_f^{*(0)})+\cos(2s_2^{*(0)}-t_f^{*(0)}-t)$$

$$x_3^{(0)}(t) = 2s_2^{*(0)}-t_f^{*(0)}-t \qquad (5.2\text{-}3)$$

(iii) $s_2^{*(0)} \leq t \leq t_f^{*(0)}$

$$x_1^{(0)}(t) = 2\sin(t-t_f^{*(0)})$$

$$x_2^{(0)}(t) = 1-\cos(t-t_f^{*(0)})$$

$$x_3^{(0)}(t) = t - t_f^{*(0)} \tag{5.2-4}$$

To construct the suboptimal control, one first obtains the linearized system: From (5.1-6) one obtains

$$\dot{\underline{x}}_L(t) = \begin{bmatrix} 0 & 0 & -\sin(x_3^{(0)}(t)) \\ 0 & 0 & \cos(x_3^{(0)}(t)) \\ 0 & 0 & 0 \end{bmatrix} \underline{x}_L(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_L(t) + \underline{g}(t) \tag{5.2-5}$$

where

$$\underline{g}(t) = \begin{bmatrix} \cos(x_3^{(0)}(t)) + x_3^{(0)}(t)\sin(x_3^{(0)}(t)) \\ \sin(x_3^{(0)}(t)) - x_3^{(0)}(t)\cos(x_3^{(0)}(t)) \\ 0 \end{bmatrix}$$

<u>Suboptimal Control Law</u>

Since the bound on the number of switching instants of the optimal control is known, one applies the switching method developed in Chapter IV for the sake of simplicity: According to the design procedure presented in Section 4.5, Chapter IV,

<u>Step 1</u> Calculate

(a) State transition matrix $\Phi(t,s)$

In this particular example one can obtain the state transition matrix of $\underline{f}_{\underline{x}_N}$ analytically since $\int_s^t \underline{f}_{\underline{x}_N}(s)ds$ and $\underline{f}_{\underline{x}_N}(t)$ commute [31] for all t. Therefore,

$$\Phi(t,s) = \exp\left(\int_s^t \underline{f}_{\underline{x}_N}(s)\,ds\right)$$

and in this problem one obtains

$$\exp\left(\int_s^t \underline{f}_{\underline{x}_N}(s)ds\right) = I + \int_s^t \underline{f}_{\underline{x}_N}(s)\,ds \qquad (5.2\text{-}6)$$

(b) The nominal trajectory $\underline{x}^{(0)}(t)$, which is calculated by (5.2-2) to (5.2-4)

(c) U = -1

Step 2 Calculate $\underline{x}^{(k)}(t)$

Assume that k = 0 for the sake of simplicity. Thus, employ $\underline{x}^{(0)}(t)$ as the nominal trajectory through the whole process.

Step 3 Calculate $\left[Q_t^{(k)}\right]$

To simplify the the controller structure, a design procedure is taken so that the suboptimal controller measures an initial deviation $\Delta\underline{x}$ and corrects the control input. For this purpose one finds $\left[Q_t^{(0)}\right]$ at initial time t=0 which transforms $\Delta\underline{x}$ into $\Delta\underline{S}^{(0)}, \Delta t_f^{(0)}$. Note that in this case the matrix $\left[Q_t^{(0)}\right]$ becomes constant.

To obtain $\left[Q_t^{(0)}\right]$ indicated by (4.3-14), one first determines $\underline{h}(0,s_v^{*(0)})$, v=1,2. Recall the definition of

$$\underline{h}(t,s) = \Phi(t,s)B$$

where $B = \left[0 \quad 0 \quad -1\right]^T$

By making use of (5.2-6) one obtains

$$\underline{h}(0,s_1^{*(0)}) = \begin{bmatrix} -\sin(s_1^{*(0)}) \\ \cos(s_1^{*(0)}) - 1 \\ -1 \end{bmatrix} \tag{5.2-7}$$

$$\underline{h}(0,s_2^{*(0)}) = \begin{bmatrix} -2\sin(s_1^{*(0)}) - \cos(s_2^{*(0)} - t_f^{*(0)}) \\ -1 - \sin(s_2^{*(0)} - t_f^{*(0)}) + 2\cos(s_1^{*(0)}) \\ -1 \end{bmatrix} \tag{5.2-8}$$

and

$$\underline{h}(0,t_f^{*(0)}) = \begin{bmatrix} -2\sin(s_1^{*(0)}) - 2\cos(s_2^{*(0)} - t_f^{*(0)}) + 1 \\ 2\cos(s_1^{*(0)}) - 2\sin(s_2^{*(0)} - t_f^{*(0)}) - 1 \\ -1 \end{bmatrix} \tag{5.2-9}$$

Now inserting (5.2-7), (5.2-8), (5.2-9) into (4.3-13) with t, k, n, L_o, $\underline{y}^d$ replaced by 0, 0, 3, 1, $\underline{0}$ respectively and making the inversion of the resulting matrix gives

$$\left[Q_t^{(0)}\right] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \qquad (5.2\text{-}10)$$

where $Q_{11}, Q_{12}, \ldots, Q_{33}$ are defined as follows:

Let $\cos(s_1^{*(0)}) \triangleq a$; $\sin(s_1^{*(0)}) \triangleq \bar{a}$; $\sin(s_2^{*(0)} - t_f^{*(0)}) \triangleq \bar{b}$

$\cos(s_2^{*(0)} - t_f^{*(0)}) \triangleq b$; $D \triangleq ab + \bar{a}\,\bar{b}$, then

$Q_{11} = 0.5(a - \bar{b})/D$; $Q_{12} = 0.5(b + \bar{a})/D$; $Q_{13} = -0.5(\bar{a} + b)/D$

$Q_{21} = (0.5a - \bar{b})/D$; $Q_{22} = (0.5\bar{a} + b)/D$

$Q_{23} = (\bar{a}\,\bar{b} - 0.5\bar{a} + ab - b)/D$; $Q_{31} = -\bar{b}/D$; $Q_{32} = b/D$

$Q_{33} = (\bar{a}b + ab - b)/D$ (5.2-11)

<u>Step 4</u> Suboptimum controller

The suboptimum controller can be obtained from (4.3-7) with k = 0 :

$$u_0^*(t) = -\,\mathrm{sgn}(t - s_1^{*(0)} - \Delta s_1^{(0)})(t - s_2^{*(0)} - \Delta s_2^{(0)}) \qquad (5.2\text{-}12)$$

where

$$\Delta s_1^{(0)} = Q_{11}\Delta x_1 + Q_{12}\Delta x_2 + Q_{13}\Delta x_3$$

$$\Delta s_2^{(0)} = Q_{21}\Delta x_1 + Q_{22}\Delta x_2 + Q_{23}\Delta x_3$$

$$\Delta t_f^{(0)} = Q_{31}\Delta x_1 + Q_{32}\Delta x_2 + Q_{33}\Delta x_3$$

and $Q_{11}, Q_{12}, \ldots, Q_{33}$ are defined in (5.2-11)

To demonstrate the effectiveness of the technique, one simulated the simplified suboptimum control system with an initial deviation $\Delta \underline{x}$, a deviation from $x_{o1}=-2.00, x_{o2}= 2.00$ in x_1-x_2 plane and a deviation from $x_{o3} = \pi/2$, a heading angle of the air-craft. The results are tabulated in Table 5-1 where the performances between the suboptimum system and the system with control input without correction are compared. The Table shows that in case no correction is made on control input regardless of disturbances, the terminal state error is more than five times greater than those obtained from using the suboptimum control system. The phase portrait in x_1-x_2 plane for each case A, B, C, D is given in Fig.5-2, 5-3 where an actual trajectory indicated by the solid line is compared with the trajectory corresponding to the nominal control. In Fig.5-4 the suboptimum control for each case is compared with the nominal one.

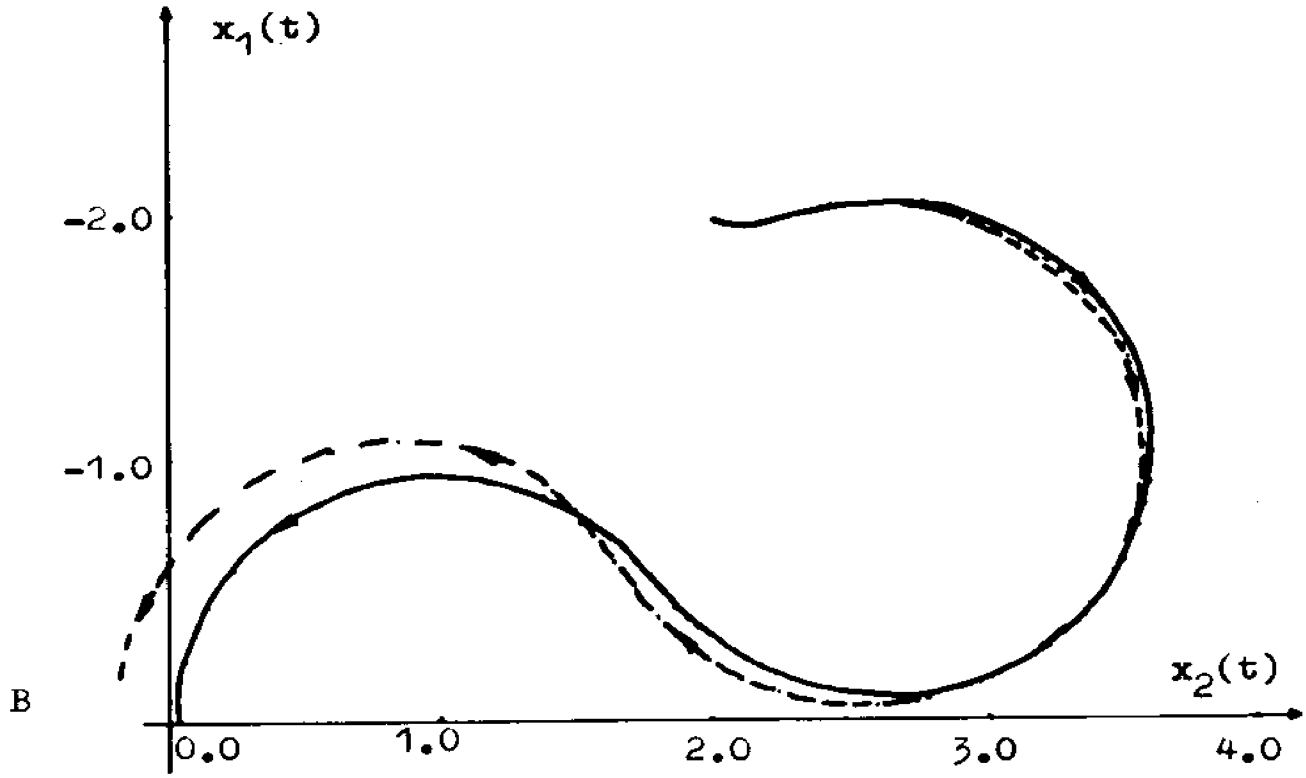
One showed in Fig.5-5 that the mathematical solution of the problem can be simulated using an electronic analogue-computer. In simulation of the air-craft in the horizontal plane, one used three integrators and two oscillators as indicated in the solid line box. One can

SUMMARY OF RESULTS

	INITIAL STATE			SUBOPTIMUM CONTROL SYSTEM				OPEN LOOP CONTROL SYSTEM (NOMINAL CONTROL)			
	x_{o1}	x_{o2}	x_{o3}	$t_f^{*(1)}$	TERMINAL ERROR			$t_f^{*(0)}$	TERMINAL ERROR		
					Δx_1	Δx_2	Δx_3		Δx_1	Δx_2	Δx_3
A	-1.990	2.010	1.700	7.4	-0.010	-0.064	0.020	7.4	0.307	0.300	0.180
B	-2.000	1.990	1.470	7.3	0.039	-0.038	0.010	7.4	-0.157	-0.198	-0.050
C	- .000	2.000	1.620	7.4	0.033	0.044	0.041	—	0.150	0.114	0.1 01
D	-2.000	2.010	1.670	7.4	0.018	0.117	0.030	—	0.240	0.232	0.150

Table 5-1

PHASE PORTRAIT OF EXAMPLE



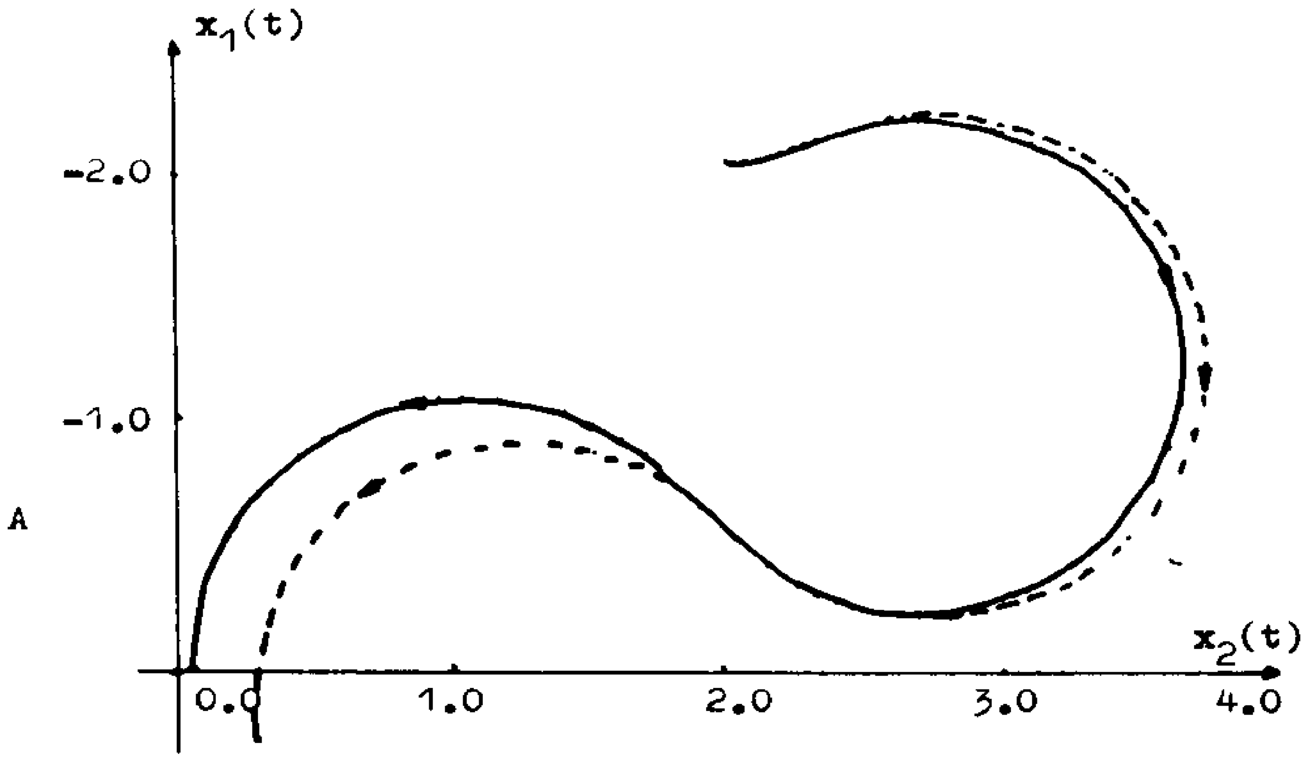
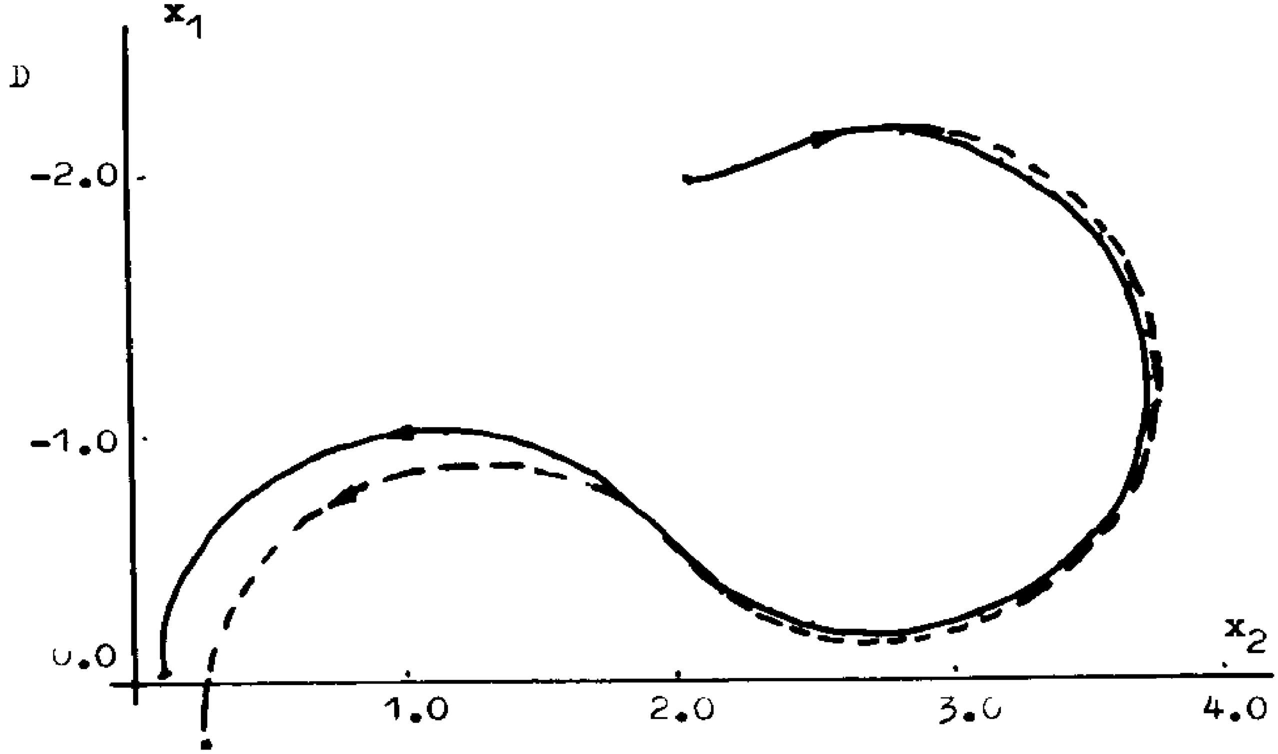


Fig.5-2

PHASE PORTRAIT OF EXAMPLE



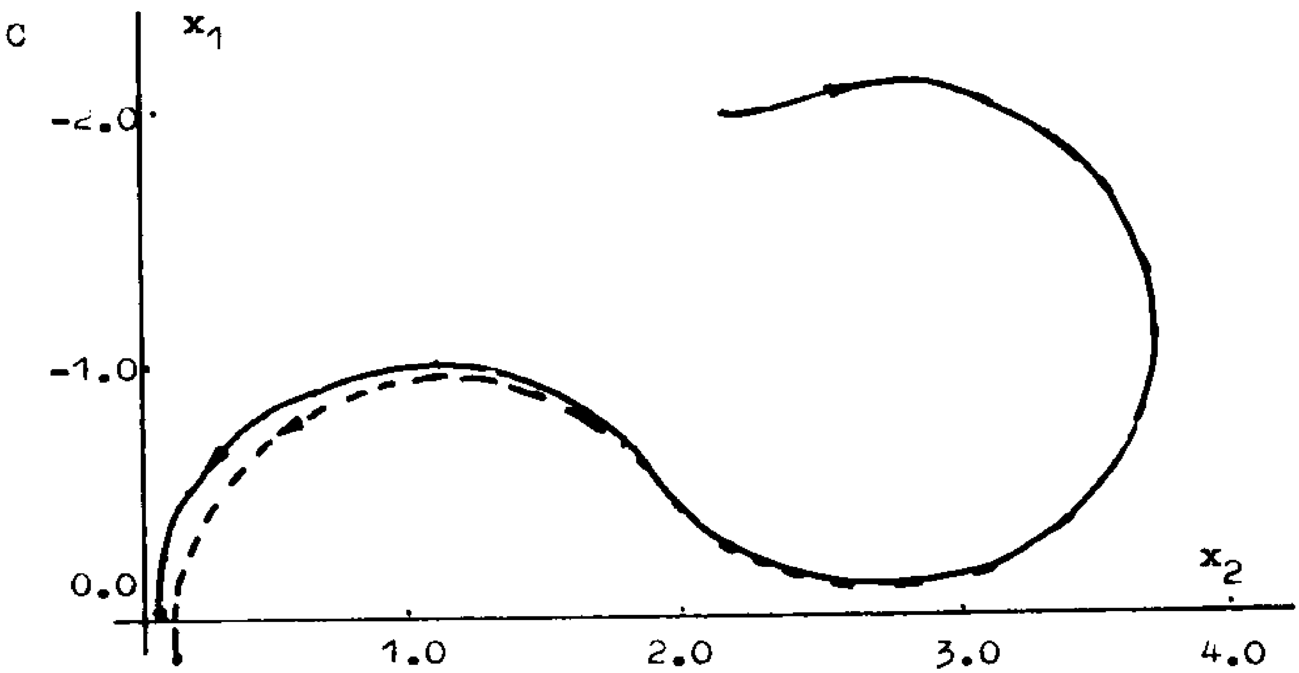


fig.5-3

CONTROL INPUT OF EXAMPLE

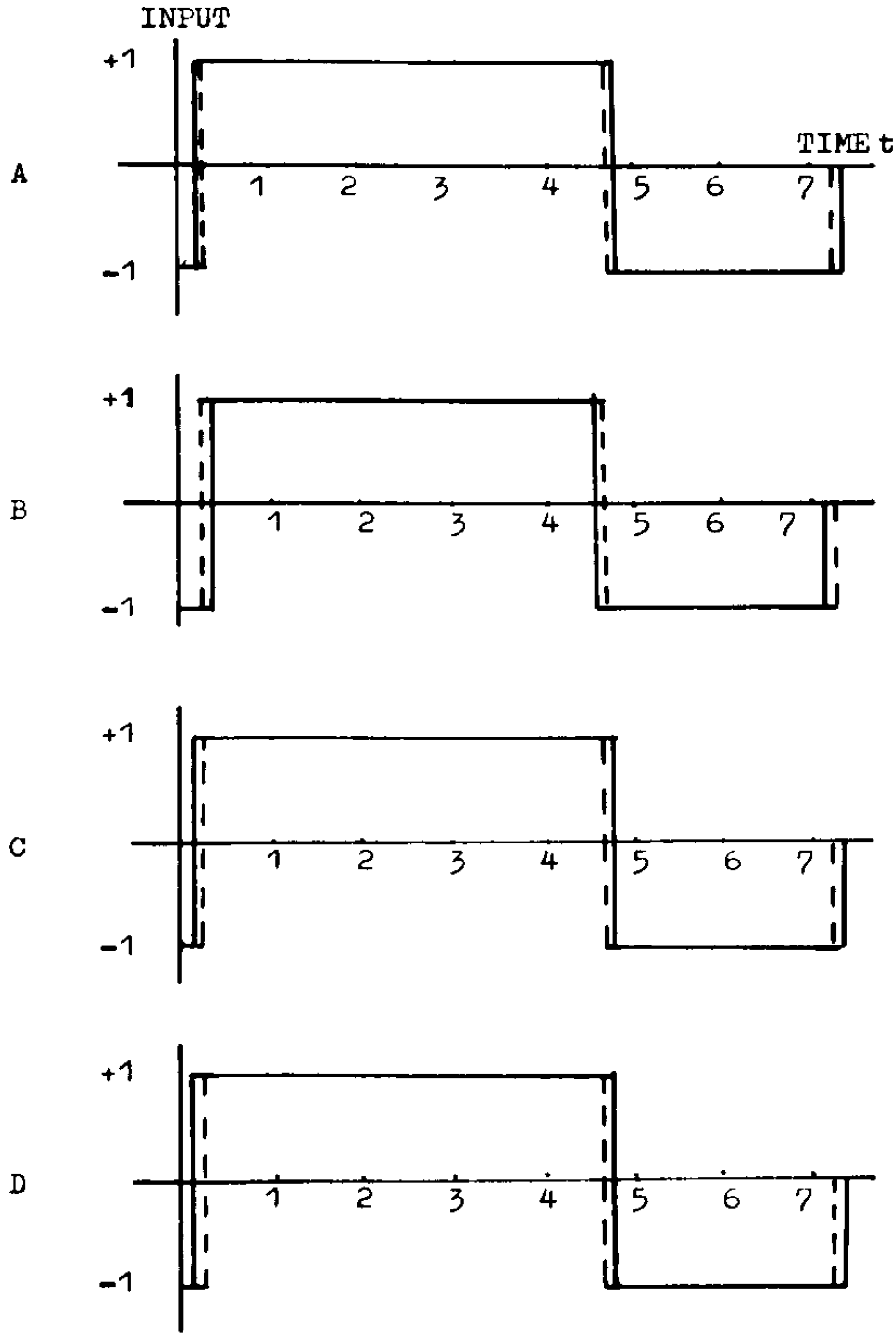


Fig.5-4

SIMULATION OF THE EXAMPLE USING ANALOGUE COMPUTER

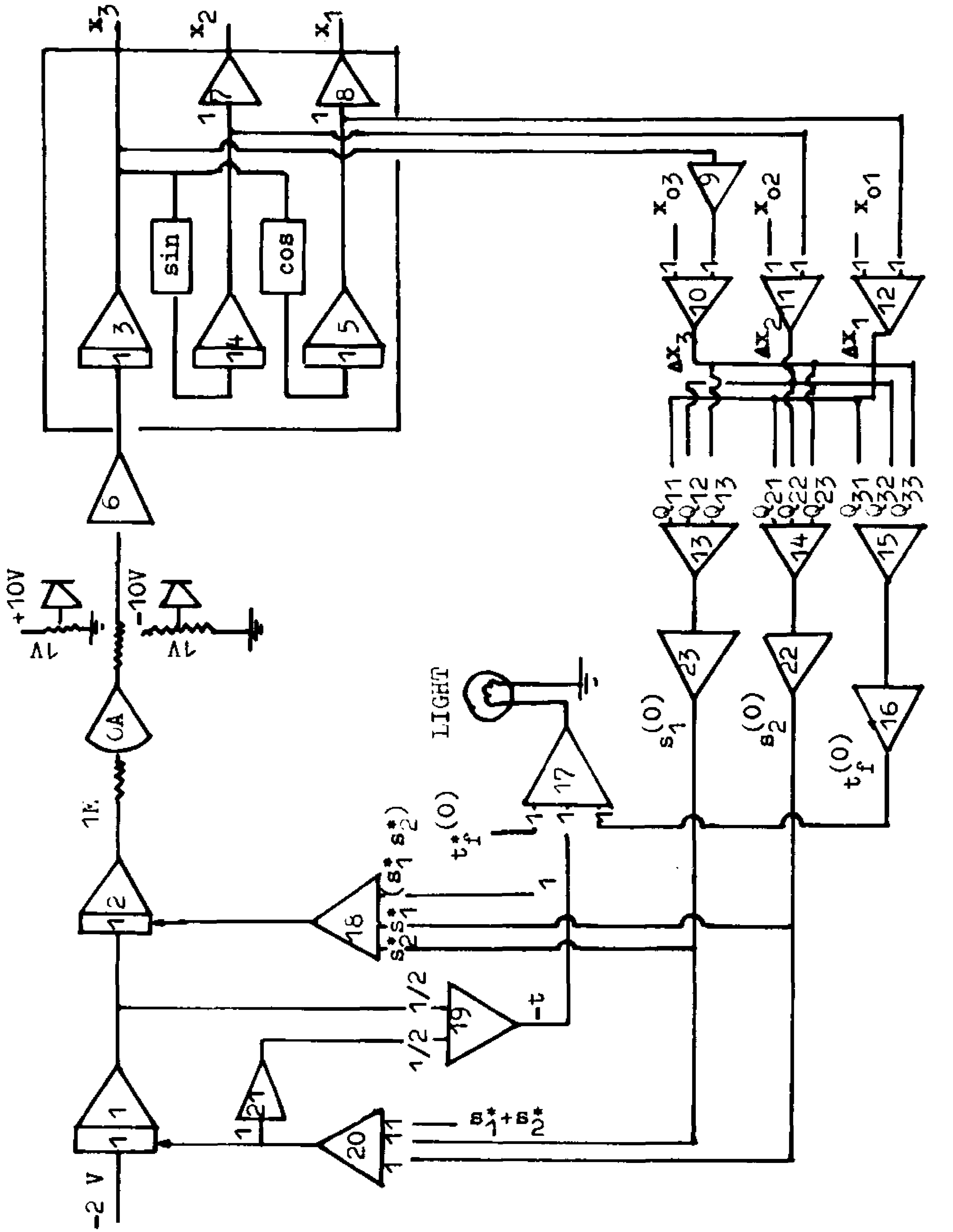


Fig.5-5

measure deviations of the position in x_1-x_2 plane and a heading angle through the summing amplifiers 10,11,12. The corresponding corrections of the switching instants $\Delta s_1^{(0)}, \Delta s_2^{(0)}, \Delta t_f^{(0)}$ are obtained through the summing amplifiers 13,14,15. And the approximate switching instants $s_1^{*(0)}$, $s_2^{*(0)}$ are obtained through 20,18 which are fed into the two integratore 1,2 as an initial condition respectively. The two integrators are for simulating the argument of the signum function given in (5.2-12). One stops operation of the system when the light is off which indicates $t = t_f^{*(0)} + \Delta t_f^{(0)}$.

5.3 Summary

In this Chapter it was shown that a design procedure of a feedback suboptimum control law for the control problems of a non-linear system was developed as an extension of the linear version of the problem.

The proposed technique is based on the assumption that the linearized system meets requirements given in Chapter I and that the suboptimum control for the linearized system is close to the optimum control law for the non-linear system.

The limitation of the approach is that one cannot apply this technique to the non-linear system in which the linearized system is not normal.

The illustrative example showed that in case no correction is made on the nominal control regardless of the perturbations, the terminal error is more than five times greater than those obtained using the suboptimal control scheme.

VI FURTHER COMMENTS

In this chapter the above techniques are extended to the sampled version of problem 2, to the multi-input version of the problem, and to the fuel problem with an additional magnitude constraint on the control input. This chapter also includes some suggestions for further research.

6.1 Sampled version and multi-input version of the problem

6.1.1 Sampled version of problem 2

Consider an m-output and a single-input system whose dynamic behavior is described by the difference equations

$$\underline{x}(n+1) = A(n)\underline{x}(n) + B(n)u(n) \tag{6.1-1}$$

Solving (6.1-1) [28,31] at the N-th step for a given initial state $\underline{x}(n_o)$ at n_o -th step gives

$$\underline{x}(N) = \Phi(N,n_o)\underline{x}(n_o) + \sum_{i=n_o}^{N} \Phi(N,i)B(i)u(i) \tag{6.1-2}$$

where $\Phi(n,n_o)$ is the state transition matrix of matrix A(n).

The sampled version of problem 2* (see appendix A) whose solution is quite analogous to the continuous one can be stated as follows:

Given a completely output controllable [21] system described by (6.2-2); a desired output $\underline{x}^d$ and an initial output $\underline{x}_o$ at the n_o -th step; a constraint on control input

$$\| u(i) \|_p \leqslant L_o \qquad (6.1\text{-}3)$$

Find an optimal control sequence $u(n_o),\ldots,u(N)$ which makes $\underline{x}(N)=\underline{x}^d$ at time N for minimum N. The solution of this problem is quite analogous to the continuous one and is given [36] as

$$u^*(n)=L_o^q \left|\boldsymbol{\lambda}^{*T}\underline{h}(N^*,n)\right|^{q-1} \mathrm{sgn}\,(\boldsymbol{\lambda}^{*T}\underline{h}(N^*,n)) \qquad (6.1\text{-}4)$$

where $\boldsymbol{\lambda}^*$, N^*, $\underline{h}(N^*,n)$ are defined as follows:

(1) $\underline{h}(N,i)=\underline{\Phi}(N,i)\ B(i)$

(2) $\boldsymbol{\lambda}^*$ is the minimand of the expression

$$\min \| K \|_q \qquad (6.1\text{-}5)$$

$$\boldsymbol{\lambda}^T\ \underline{e}(N,\underline{x}(n_o))=1 \qquad (6.1\text{-}6)$$

where $\underline{e}(N,\underline{x}(n_o)) = \underline{x}(N) - \underline{\Phi}(N,n_o)\underline{x}(n_o)$;

$$\| K \|_q = (\sum_{j=1}^{m} (\lambda_j\ h_j(N,i))^q\ )^{1/q}$$

(3) N^* is the least terminal step obtained as the least value of N for which

$$\| K^* \|_q \geq 1/L_o$$

where K^* is the value of K evaluated at $\Delta = \Delta^*$.

Now consider the suboptimal control problem, the problem of finding a suboptimal control sequence $\bar{u}^*(n)$ which makes the actual output approximately equal to the desired one at time N for minimum N in the presence of small disturbances during the process.

In analogy with the continuous version of the problem one can solve the sampled version of the problem:

(i) Suboptimal Controller

From (6.1-4) the suboptimal control can be obtained as

$$\bar{u}^*(n) = \bar{L}_o^{\,q} \left| (\Delta_n^* + \Delta\Delta)^T \underline{h}(N^* + \Delta N, n) \right|^{q-1} \mathrm{sgn}((\Delta_n^* + \Delta\Delta)^T \underline{h}(N^*, n)) \qquad (6.1\text{-}7)$$

where $\bar{L}_o$, Δ_n^*, $\Delta\Delta$, $\Delta\underline{x}$ are defined as follows: (1) $\bar{L}_o$ denotes a time-varying bound on control effort. When $p = \infty$, $\bar{L}_o(n) = L_o$ which is constant since the magnitude bound is independent of time. This, however, is not true for $1 < p < \infty$ since in this case the amount of "energy " associated with $\| u \|_p$ which can be utilized depends on the amount equal to the given L_o minus the " energy " expended during the interval n_o to n. (see (3.3-8) in chapter III).

(2) $\underline{\lambda}_n^*$ is the vector $\underline{\lambda}^*$ computed for $\underline{x}_o^*(n)$, the norminal optimal trajectory and has the same property as in the conti nuous case: the direction of the $\underline{\lambda}_n^*$ is constant and

$$\underline{\lambda}_n^* = m(N^*,n)\,\underline{\lambda}^* \tag{6.1-8}$$

where $m(N^*,n)$ is the sample version of $m(t_f^*,t)$ give in (2.3-9) with the integral operator and t_f, t_i, t replaced by a summation operator and N^*, n_o , n, respectively.

(3) $\Delta\underline{\lambda}$, ΔN are obtained from similar arguments developed in chapter III;

$$\begin{bmatrix} \Delta\underline{\lambda} \\ \Delta N \end{bmatrix} = \begin{bmatrix} M \\ \xi \end{bmatrix} \Delta\underline{x}(n)$$

where

$$\Delta\underline{x}(n) = \underline{x}(n) - \underline{x}_o^*(n)$$

and the linear transformation matrix $\begin{bmatrix} M \\ \xi \end{bmatrix}$ can be obtained in a similar manner to that developed in Chapter III

6.1.2 Multi-input version of the problem

Consider the multi-input version of the problem which is the generalized case [19] of a single control input. Without theoretical difficulty one can apply the suboptimal feedback control scheme developed for a single control input to the multi-version of the problem. Still the direction of the optimum lambda vector is invariant along the optimal trajectory. However, the magnitude of the optimum lambda vector may vary differently for each optimal control input $u_j^*(t)$, i.e., let λ_{tj}^* be the optimum lambda vector in j-th optimal control, then

$$\lambda_{tj}^* = m_j(t_f, t)\, \lambda^*$$

on the optimal trajectory, where $m_j(t_f, t)$ may be determined from (A-25) in appendix A in a similar way to that obtained for single control input. (see (2.3-9)). The suboptimal control scheme is completely analogous to the case of single control input.

6.2 Fuel problem with additional magnitude constraint on Control input (Van Gelder problem)

Consider the case of p=1 in the problem 1. The open loop study has been reported in [18.19.30.36.] . Since the optimal input for p=1 consists of delta functions, it is of interest to study the case when there is an additional magnitude constraint on the input: Given a completely controllable system described by (1-1) in chapter I; a desired output $\underline{x}^d$ at a fixed time t_f and the initial output $\underline{x}_o$ at time t_i; a magnitude constraint on the control input

$$|u(t)| \leqslant 1 \qquad (6.2\text{-}1)$$

Find an optimal control which makes the actual output equal to the desired output while $\|u\|_1$ is minimized.
The solution for this problem in the case of no disturbance is [30.] :

$$u^*(t) = \mathrm{sgn}(\underline{\lambda}^{*T}\underline{h}(t_f,t)) \quad \text{for} \quad 1 < \underline{\lambda}^{*T}\underline{h}(t_f,t) \qquad (6.2\text{-}2)$$

$$u^*(t) = 0 \quad \text{for} \quad 1 > \underline{\lambda}^{*T}\underline{h}(t_f,t) \qquad (6.2\text{-}3)$$

where λ^* are chosen so that the terminal conditions (1-7) with u replaced by u*(t) ((6.2-2) and (6.2-3)) are satisfied, i.e.,

$$\underline{e}(t_f,\underline{x}(t_i)) = \int_{t_i}^{t_f} \underline{h}(t_f,s)u^*(s)\, ds \qquad (6.2\text{-}4)$$

Note that λ^* is a Lagrange multiplier and is slightly different from the optimum lambda vector used in the previous chapters. However, one can see from (6.2-2) and (6.2-3) that the direction of the λ^* is constant on the optimum trajectory.

Now consider the problem of finding, for a given open-loop optimal control u*, the corresponding vector λ^*, and the corresponding optimum trajectory $\underline{x}_o^*$, the suboptimal feedback control law for small deviations from $\underline{x}_o^*$. To solve this problem rewrite (6.2-2) and (6.2-3) as a single form, i.e.,

$$u^*(t)=(1/2)(\operatorname{sgn}(\lambda^{*T}\underline{h}(t_f,t)-1) + \operatorname{sgn}(\lambda^{*T}\underline{h}(t_f,t)+1)) \qquad (6.2\text{-}5)$$

Let $\Delta\lambda$ be small change in λ^* as a result of small deviation from the optimal trajectory $\underline{x}_o^*(t)$. Then, the linear transformation matrix which maps $\Delta \underline{x}$ into $\Delta \underline{\lambda}$ can be found as has been done in chapter II: the linear transformation matrix can be found through a Taylor series expansion of (6.2-4) around λ^* and $\underline{x}_0^*$ with u* replaced by (6.2-5). Suppose that the corresponding matrix M_f is found. Then one can obtain the suboptimum control as

$$\tilde{u}^*(t) = \frac{1}{2}\Big[\operatorname{sgn}((\lambda^* + \Delta\lambda)^T \underline{h}(t_f, t) - 1)$$

$$+ \operatorname{sgn}((\lambda^* + \Delta\lambda)^T \underline{h}(t_f, t) + 1)\Big]$$

where

$$\Delta\lambda = M_f \; \Delta\underline{x}$$

6.3 Suggestions for Further Research

The research reported here also points out that additional work would be worthwhile in the following areas:

(1) Parameter Variation

In this dissertation one has focused on the disturbances essentially due to the external forces and one assumed that all the system parameters are known precisely. Now suppose that the mathematical model of the system is different from the actual system, i.e., the system matrix A, B of the mathematical model are different from the actual one. Then, one may ask how a small change in the system parameters affects the performance of the suboptimal feedback control scheme. In the two integrators problem it was found in the simulation the variation of a_{12}(1st row and 2nd column of matrix A) within 10% results in the terminal error being within a radius .06. However, more general theoretical study is required to obtain a bound on allowed parameter variations corresponding to a given terminal error.

(2) Simplified Linear Transformation Matrix

The linear transformation matrix M_{nq} or M_{tq} are time-varying matrices and therefore, for certain cases, the evaluation and implementation of these matrices may be difficult. Therefore, one may investigate the possibility of whether one can overcome this difficulty by using reasonably simplified matrices: One possibility for this is to use M_{nq} or M_{tq} as constant matrices by evaluating these at switching instants since $\Delta\lambda$ is most effective around switching instants .

(3) Additional State Constraint

It is of interest to study the case of state constraints in addition to control constraints. The open-loop study for this was reported in [17].

Appendix A

In this appendix we wish to review some arguments leading to the solution of problem 1* and 2* (see footnote) stated in chapter I

(1) Solution of the Problem 1*

Consider that the case when there is only a single input (t). Problem 1* can be therefore restated as follows: Find the control function u(t) which makes

$$\| u \|_p = \min \qquad \text{(A-1)}$$

while maintaining

$$f(h_j) = \int_{t_i}^{t_f} h_j(t_f,s)u(s)ds = e_j \qquad \text{(A-2)}$$

where $j=1,2,\dots,m$; t_i and t_f are fixed; e_j is the j-th element of the vector $\underline{e}(t_f,\underline{x}(t_i))$ defined in (1-5); $h_j(t,s)$ is the response observed at time t due to an unit impulse applied at time t = s; $\| u \|_p$ is a norm in L_p space,i.e.,

$$\| u \|_p = \left(\int_{t_i}^{t_f} |u(s)|^p \, ds\right)^{1/p}, \quad p \geqslant 1 \qquad \text{(A-3)}$$

In the problem 1* and 2* we ask for open loop optimal control in the problem 1 and 2 in chapter I respectively.

Consider now a linear functional

$$f(x) = \int_{t_i}^{t_f} x(s)u(s)\, ds \qquad \text{(A-4)}$$

The norm of the functional f of (A-4) is defined as

$$\| f(x) \| = \sup_{x \neq 0} \frac{f(x)}{\| x \|} \qquad \text{(A-5)}$$

One can choose that $\| f \| = \| u \|_p$ when x is in L_q space with norm $\| x \|_q$ given by

$$\| x \|_q = \left(\int_{t_i}^{t_f} \left| \underline{x}(s) \right|^q ds \right)^{1/q} \qquad \text{(A-6)}$$

where q is congugate to p is defined by

$$1/p + 1/q = 1$$

Thus, the variational problem (A-1),(A-2) can be reformulated as follows: Choose the bounded linear functional f which has minimum norm

$$\| f \| = \min \qquad \text{(A-7)}$$

and satisfies

$$f(h_j) = e_j \, , \; j = 1,2,\ldots,m \qquad \text{(A-8)}$$

The problem (A-7),(A-8) is called the L-problem in the Theory of moments [1].

In solving the problem (A-7),(A-8) it is assumed that $h_1, h_2, \ldots, h_m$ are linearly independent. This will certainly be the case if the controlled system is completely output controllable [21]. The case when this condition is not satisfied has been treated in Kreindler's work [20]. Since $h_1, h_2, \ldots, h_m$ are linearly independent, they span a m-dimensional linear space and the linear functional f is completely defined on this space. Let h be defined as

$$h \triangleq \sum_{i=1}^{m} \lambda_i h_i \qquad \text{(A-9)}$$

Then, we have

$$f(h) = f\left(\sum_{i=1}^{m} \lambda_i h_i\right) = \sum_{i=1}^{m} \lambda_i f(h_i)$$

Thus, we obtain

$$f(h) = \sum_{i=1}^{m} \lambda_i e_i \qquad \text{(A-10)}$$

The norm of f may accordingly be computed over this m dimensional space according to the definition (A-5), i.e.,

$$\| f \| = \sup_{\Delta} \frac{\sum_{i=1}^{m} \lambda_i e_i}{\| h \|} \qquad \text{(A-11)}$$

or equivalently [1,17]

$$\| f \| = \frac{1}{\inf_{\lambda} \| h \|} \qquad \text{(A-12)}$$

with

$$\sum_{i=1}^{m} \lambda_i e_i = 1 \qquad \text{(A-13)}$$

where $_i$'s are not all zero; $h = h_q$ and is defined as

$$\| h \|_q = \left(\int_{t_i}^{t_f} \left| \sum_{i=1}^{m} \lambda_i h_i \right|^q ds \right)^{1/q} \qquad \text{(A-14)}$$

where the infimum is to be evaluated over all real values of λ's.

But now, according to the Hahn-Banach Theorem [17] the functional f defined by (A-10) over the linear span h can be extended over the whole L_q space without an increase of the norm. That is, among all the extensions of f, none of which can have a norm smaller than (A-11), there is certainly one which has a norm just equal to that given in (A-11). The norm evaluated in (A-11) or equivalently in (A-12) with (A-13) accordingly is the minimum norm sought in (A-7) consistent with the condition (A-8). This concludes the argument from the Hahn -Banach Theorem, which guarantees the existence of a functional f that solves the problem (A-7),(A-8) and hence of its function component u in L_p space which solves problem (A-1), (A-2).

Now consider finding an expression for the optimum control $u^*(t)$. If $u^*(t)$ is the control which solves the problem 1*, then

$$\| u^* \|_p = \frac{1}{\inf\limits_{\Delta} \| h \|_q}$$

with

$$\sum_{i=1}^{m} \lambda_i e_i = 1$$

Let Δ^* be the minimand of the above expression; then

$$\| u^* \|_p = \frac{1}{\| h^* \|_q} \qquad \text{(A-15)}$$

with $\sum_{i=1}^{m} \lambda_i^* e_i = 1$, where $h^* = \sum_{i=1}^{m} \lambda_i^* h_i$

The existence of solution has been demonstrated by an appeal to Hahn-Banach Theorem [17]. This argument gives no direct information about the form of solution $u^*(t)$. For this we turn to Holder's inequality for functional of the form (A-2) operating on h^* and satisfying the condition (A-8). Using (A-10) this inequality gives

$$\left| f(h^*) \right| = \left| \sum_{i=1}^{m} \lambda_i^* e_i \right| \leq \| u \|_p \, \| h^* \|_q$$

and recalling $\sum_{i=1}^{m} \lambda_i^* e_i = 1$, then we have

$$\| u \|_p \geq \frac{1}{\| h^* \|_q} \quad \text{(A-16)}$$

Comparing (A-15) and (A-16) we see that

$$\| u \|_p = \| u^* \|_p$$

when the Holder's inequality is satisfied with equality condition [17,18,19] ,namely

$$u^*(t) = K \left| \sum_{i=1}^{m} \lambda_i^* \, h_i(t_f,t) \right|^{q-1} \operatorname{sgn}\left(\sum_{i=1}^{m} \lambda_i^* \, h_i(t_f,t) \right) \quad \text{(A-17)}$$

where constant K is evaluated to satisfy (A-6) which implies that

$$f(h^*) = \sum_{i=1}^{m} \lambda_i^* \, e_i \quad \text{(A-18)}$$

Recalling $\sum_{i=1}^{m} \lambda_i^* \, e_i = 1$, we obtain

$$K = \frac{1}{(\| h^* \|_q)^q} \quad \text{(A-19)}$$

Thus, the problem 1* has been reduced to the problem of finding the optimum lambda λ^* which minimizes $\| h \|_q$ subject to $\sum_{i=1}^{m} \lambda_i^* \, e_i = 1$. This is a minimization in m-1 dimensional space and in s me cases can be solved more easily than the two-point boundary-value problem which arises from applying variational methods. Various iterative techniques [35] are available to find the λ^*. And also it has been shown that for certain cases this problem can be reduced [23,24] to a problem in the calculus of approximation [13].

Special Consideration of the Case of p=1

The development leading to the control function (A-17) applies strictly only for $p > 1$ [17], namely, only for $q < \infty$. In the case when p=1, the expression (A-17) must be given a proper interpretation. This can be accomplished by using a limiting process and letting $q \to \infty$ (Kirillova [32]). This leads to

$$u^*(t) = K \sum_{v=1} |k_v| \; \delta(t-t_v) \text{ sgn } (\lambda^* h(t_f,t)) \qquad \text{(A-20)}$$

where $\_^*$ is obtained so that at $\lambda = \lambda^*$

$$\min_{\lambda} \sup_{t_i \leq t \leq t_f} |\lambda^* h(t_f,t)| \qquad \text{(A-21)}$$

has minimum with $\underline{\lambda}^{*T} \underline{e}(t_f, \underline{x}(t_i)) = 1$

Note that $\delta(t-t_v)$ is an unit impulse which occurrs at the instant of t_v at which a supremum occurrs and its amplitude k_v must be determined. Constant K may be determined by letting $\underline{\lambda}^{*T}\underline{e}(t_f,\underline{x}(t_i)) = 1$.

(2) Solution of the Problem 2*

The least time control problem is closely related to the minimum norm control problem (problem 1*). In fact since there is an upper limit on $\|u\|_p$, that is, $\|u\|_p \leqslant L_o$, it is apparent that L_o will be the value of $\|u^*\|_p$ that will enable us to obtain the least elapsed time $T^* = t_f^* - t_i$, if it exists, for which the minimum $\|f\|$ of (A-11) becomes equal to L_o,

$$\|f\| = L_o \tag{A-22}$$

when we recall

$$\|f\| = \frac{\sum_i^m \lambda_i^* e_i}{\|h^*\|_q}$$

the least time condition is

$$\frac{\sum_{i=1}^m \lambda_i^* e_i}{\|h^*\|_q} = L_o \tag{A-23}$$

for which the least value of t_f is obtained as the least terminal time t_f^* if its solution exists.

When the least time condition is satisfied, the control function (A-17) is

$$u^*(t)=L_o^q \left|h^*(t_f^*,t)\right|^{q-1}\operatorname{sgn}(h^*(t_f^*,t)) \qquad \text{(A-24)}$$

In the case when $p=1$, the limiting process letting $q \rightarrow \infty$, can be also applied to the least time problem [17]. It leads as before to impulsive solutions.

Since it can be shown that the product L_p spaces [17] are also Banach spaces, all previous arguments apply to the multi-input cases. The norm on vector $\underline{u}$ whose elements are different inputs of the system is defined as in eq.(1-10), namely

$$\| \underline{u} \|_p = \left(\sum_{j=1}^{r} \left(\int_{t_i}^{t_f} \left|u_j(t)\right|^{p_j} dt \right)^{p/p_j} \right)^{1/p}$$

then the following results can be obtained for the multi-input version of the problem treated in this appendix:

Solution of the problem 1* for multi-input systems [17,18,19].

$$u_j^*(t) = \frac{\left(\int_{t_i}^{t_f} \left|k_j^*(t_f,s)\right|^{q_j} ds \right)^{q/q_j - 1}}{\left(\| \underline{K}^* \|_q \right)^q} \left|K_j^*(t_f,t)\right|^{q_j - 1} \operatorname{sgn}(K_j^*(t_f,t))$$

for $j=1,2,\ldots,r$ (A-25)

where $1/p + 1/q = 1$; $1/p_j + 1/q_j = 1$; $1 \; p \; \infty$; $1 \; p_j \; \infty$

and $\underline{K}(t_f,s) = k_1, k_2, \ldots, k_m$, $k_j = k_j\,(t_f,s) = \boldsymbol{\lambda}^T \underline{h}_j(t_f,s)$

$\underline{h}_j(t_f,s)$ indicates the j-th column of the m x r impulse response matrix of the dynamic system. And

$\underline{K}^*(t_f,s) = k_1^*, k_2^*, \ldots, k_m^*$; $k_j^*(t_f,s) = \boldsymbol{\lambda}^{*T} \underline{h}_j(t_f,s)$

where $\boldsymbol{\lambda}^*$ is maximand of

$$\max_{\underline{\lambda}} \frac{\sum_{i=1}^{m} \lambda i \; e_i}{(\| \underline{K} \|_q)} \qquad \text{(A-26)}$$

Solution of the problem 2* for multi-input systems

In the case of the multi-input version of the problem 2* the least time condition is

$$\frac{\sum_{i=1}^{m} \lambda_i^* \; e_i}{\| \underline{K}^* \|_q} = L_o \qquad \text{(A-27)}$$

for which the least value of t_f is obtained as the least terminal time t_f^* if its solution exists.

When the least time condition is satisfied, the control function (A-25) is

$$u_j^*(t) = L_o^q \left| k_j^*(t_f^*,t) \right|^{q_j - 1} \operatorname{sgn} k_j^*(t_f^*,t) \qquad \text{(A-28)}$$

$$1 \leqslant q < \infty \quad ; \quad 1 \leqslant q_j < \infty$$

In the case when $p=1$, the limiting process letting $q \rightarrow \infty$ can be also applied to the multi-input version of the least time problem. It leads, as before, to impulsive solutions.

Appendix B

In this appendix one shows that a relation between the optimum lambda vector $\underline{\lambda}^*$ and the costate vector $\underline{p}(t)$ [3.27.] is

$$\underline{\lambda}^* = K\ \underline{p}(t_f) \qquad \text{(B-1)}$$

where K is scalar.

To show this, one applies the Maximum Principle [27.] to find the open loop optimal control $u^*(t)$ of the problem 1 in chapter I to obtain

$$u^*(t) = K_1 \left|\underline{p}(t)^T B(t)\right|^{\frac{1}{p-1}} \text{sgn}\ (p(t)^T B(t)) \qquad \text{(B-2)}$$

where K_1 is constant scalar; $\underline{p}(t)$ is the solution of the adjoint system of A(t) and is found to be [3.27.]

$$\underline{p}(t) = \Phi(t_f,t)^T \underline{p}(t_f)$$

where $\Phi(t,s)$ is the transition matrix of A(t).

Now applying the solution technique of the functional analysis to the same problem, one obtains (A-17) in appendix A. Since one deals with the normal system, the optimal control is unique [3.] and therefore, both solutions indicated by (B-2) and (A-17) must be identical. Recall the definition of $\underline{h}(t_f,t)$, $\underline{h}(t_f,t) = \Phi(t_f,t)B(t)$, and note $\frac{1}{p-1} = q-1$ since $1/p + 1/q = 1$, to give (B-1).

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Autobiography

Chanbin Park was born in Okchun-Up in Korea on Octorber 24, 1940. In 1966 he was awarded a B.S. degree in Electrical Engineering from The Seoul National University where he was awarded a Dong-Shin Scholarship from 1964 to 1966. After graduation from The Seoul National University, he joined to a design Group of The Dong-Shin Electrical & Machinery Company, Inc.. He continued his education at The City College of New York, recieving a M.S. degree in Electrical Engineering in 1969 and entered The Graduate School of The City University of New York in 1969, passing his doctoral thesis defense in June, 1973. He was awarded an University Fellowship from 1970 to 1972.

In 1969 Mr. Park was married to SoonZa Shim. They current live in Bronx, New York with their daughter Jeanne, 2.