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WREATH PRODUCTS OF LIE ALGEBRAS

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WREATH PRODUCTS OF LIE ALGEBRAS

by

FRANCES E. SULLIVAN

A dissertation submitted to the  
Graduate Faculty in Mathematics in  
partial fulfillment of the requirements  
for the degree of Doctor of Philosophy,  
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1980

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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DEDICATION

"Praise God from whom  
all blessings flow..."

To my mother, Mrs. Ruthie C. Sullivan, and my father, the late Mr. M. N. Sullivan, Sr., two of the many unsung heroes and champions of quality education for Black youth in South Carolina; they did much to direct me to this place.

To my dear son Jackie, the next generation -- one strong brick in the foundation of the Black Nation, one cornerstone in the foundation of the New World of mankind.

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## INTRODUCTION

The main thrust of this thesis is to investigate the decomposability of certain algebras into wreath products, direct products and free products.

In 1964 A. I Šmel'kin [12] introduced the notion of a verbal wreath product of groups. In 1973 Šmel'kin [11] introduced the analogous product for Lie algebras. In

particular, he defined the verbal  $\mathcal{B}$ -wreath product  $W$  of the Lie algebras  $A$  and  $T$  (denoted  $W = A \text{ wr }_{\mathcal{B}} T$ ) where

$A$  is contained in a variety  $\mathcal{B}$  of Lie algebras over a commutative ring  $\Lambda$  with 1. This wreath product is the semidirect product of  $K$  and  $T$ , where  $K$  is the ideal generated by  $A$ .

In general, it is not an easy task to decipher this wreath product. However, Šmel'kin remarks that this wreath product has a fairly simple description if  $\mathcal{B}$  is the variety of abelian algebras,  $\mathcal{A}$ , and the underlying coefficient ring  $\Lambda$  is a field. In this case, this verbal  $\mathcal{A}$ -wreath product is the semidirect product of  $B$  and  $T$ , where  $B$  is a free  $T$ -module whose rank is equal to the dimension of  $A$ . In this case  $W$  is simply denoted  $A \text{ wr } T$ .

This thesis will be mainly concerned with such wreath products. However we will adopt a rather different viewpoint towards wreath products. Thus we will define, in

general, quite different wreath products from those introduced by A. I. Šmelkin [17]. In the spirit of group theory, we will define wreath products in terms of generators and defining relations (see Chapter 1, Section 1). Indeed, we are motivated throughout by group theory. In fact, this thesis may be viewed as an attempt to carry over to Lie algebras some theorems in group theory.

In Chapter 0 we record some of the basic definitions and theorems that will be used throughout this thesis.

We define in Chapter 1 the wreath product of Lie algebras and discuss its structure. In Section 3 of Chapter 1 we show that the center of a wreath product of Lie algebras is trivial. This theorem may be viewed as the analogue for Lie algebras of the corresponding theorem for groups. We recall some of the details. In 1959, Gilbert Baumslag [2] proved that if  $W$  is the wreath product of a non-trivial group  $A$  by an infinite group  $B$ , then  $W$  has trivial center. Peter M. Neumann [10] in 1964 generalized this result of Gilbert Baumslag's and proved that if  $A$  is not trivial the center of  $W = A \text{ wr } B$  is the center of  $D$ , where  $D$  is the subgroup of the base group of  $W$  consisting of all of the constant functions.

In Section 4, Chapter 1, we prove that the wreath product of a non-trivial abelian Lie algebra by an arbitrary Lie algebra is directly indecomposable. Peter M. Neumann [10] proved the corresponding theorem for groups, specifically, that the wreath product of a non-trivial

group by an infinite group is directly indecomposable. In fact he goes further and states conditions on the groups for the wreath product to be directly decomposable.

We investigate associative algebras in Chapter 2 for two primary reasons. First, the universal enveloping algebra of a Lie algebra is an associative algebra. Secondly, most of the Lie algebra computations are actually performed in its universal enveloping algebra. In particular, we prove that the free product of associative algebras is directly indecomposable.

In Chapter 3 we prove a most useful result, namely, that the universal enveloping algebra of a free product of Lie algebras is the free product of the respective universal enveloping algebras. Hence, working inside the universal enveloping algebra of a free product of Lie algebras reduces to working inside of the free product of associative algebras. Chapter 3 concentrates on the indecomposability of free products of Lie algebras. Two main theorems are proved, namely:

Theorem A:       A free product of Lie algebras is never decomposable into a wreath product of Lie algebras.

Theorem B:       The free product of Lie algebras is directly indecomposable.

Chapter 4 concludes the thesis with several miscellaneous results on the finite dimensionality of some Lie algebras and certain ideals of Lie algebras. We also

prove that an extension of a finitely presented Lie algebra by a finitely presented Lie algebra is finitely presented.

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CHAPTER 0

## PRELIMINARIES

We first present some preliminary definitions, conventions and notation.

Section 1.

If  $F$  is a field, a left  $F$ -vector space  $A$ , equipped with a multiplication

$$\mu : A \times A \rightarrow A$$

is termed a bilinear algebra if

$$(i) \quad (r_1 a_1 + r_2 a_2, b) \mu = r_1 (a_1, b) \mu + r_2 (a_2, b) \mu$$

$$(ii) \quad (a, r_1 b_1 + r_2 b_2) \mu = r_1 (a, b_1) \mu + r_2 (a, b_2) \mu$$

where  $r_1, r_2 \in F, a_1, a_2, b, b_1, b_2 \in A$ .

$A$  is said to be associative if its multiplication is associative, that is, if

$$((a_1, a_2) \mu, a_3) \mu = (a_1, (a_2, a_3) \mu) \mu \quad (a_i \in A)$$

We will usually write  $ab$  in place of  $(a, b) \mu$  if  $A$  is an associative algebra.

Definition 0.1. A bilinear algebra  $A$  over a field  $F$  is termed a Lie algebra if

$$(i) \quad (a, a) \mu = 0 \text{ for all } a \in A$$

$$(ii) \quad [a, b, c] + [b, c, a] + [c, a, b] = 0$$

for all  $a, b, c \in A$  where  $[a, b, c] = ((a, b) \mu, c) \mu$ .

The second condition, (ii), is called the Jacobi Identity.

We will usually denote the multiplication in a Lie algebra

by  $[\cdot, \cdot]$ , that is, we will write  $[a, b]$  in place of  $(a, b)_{\mathcal{L}}$ . It is clear from the preceding definitions that  $[b, a] = -[a, b]$ .

The concepts of subalgebras, ideals, factor algebras and the isomorphism theorems for both associative and Lie algebras will be taken for granted.

The derived algebra of a Lie algebra  $L$  is denoted  $L'$  or  $[L, L]$  where

$$[L, L] = \left\{ \sum [x_i, y_i] \mid x_i, y_i \in L \right\}.$$

We define the center  $C$  of a Lie algebra  $L$  to be the following set

$$C = \left\{ c \in L \mid [a, c] = 0 \text{ for all } a \in L \right\}.$$

A Lie algebra  $L$  is called abelian if  $L = C$ .

Let  $\underline{U}$  be an associative algebra over some field  $F$ .

We define a new operation on  $\underline{U}$  by

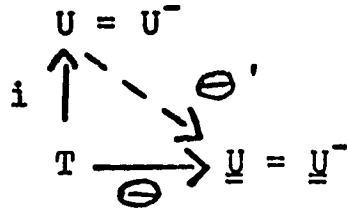
$$[x, y] = xy - yx$$

for all  $x, y \in \underline{U}$ , where the operations on the right side are the usual ones in  $\underline{U}$ .  $\underline{U}$ , with the given multiplication replaced by  $[\cdot, \cdot]$ , becomes a Lie algebra, as is easily verified, called the Lie algebra of the associative algebra  $\underline{U}$ . This Lie algebra is denoted by  $\underline{U}^-$ .

Definition 0.2. A pair  $(U, i)$  is said to be a universal enveloping algebra of the Lie algebra  $T$  if  $U$  is an associative algebra (with 1) and  $i$  is a homomorphism of  $T$  into  $U^-$  and the following hold:

if  $\underline{U}$  is any associative algebra and  $\Theta$  is a homomorphism of  $T$  into  $\underline{U}$ , then there exists a unique homomorphism  $\Theta'$  of  $U$  into  $\underline{U}$  such that

$\Theta = i\Theta'$ , that is, the diagram



is commutative. As usual, the pair  $(U, i)$  turns out to be unique (up to isomorphism) and so we shall simply refer to it as the universal enveloping algebra of  $T$ .

Let  $L$  be a Lie algebra over a field  $F$  generated by a set  $X$ .  $L$  is said to be free on  $X$  if, given any mapping  $\phi$  of  $X$  into a Lie algebra  $M$  there exists a unique homomorphism  $\psi: L \rightarrow M$  extending  $\phi$ .

Definition 0.3. A presentation is a pair  $(X; R)$  where

$X$  is a set and  $R$  is a set of  $X$ -words, that is,  $R$  is a subset of the free (associative or Lie, as is appropriate) algebra on  $X$ .

Definition 0.4. If  $L$  is a Lie (or associative) algebra and  $(X; R)$  is a presentation then we shall write

$$L = (X; R)$$

if there exists a map  $\Theta: X \rightarrow L$  (set map) such that the extension  $\varphi$  of  $\Theta$  to  $F = F(X)$  = the free Lie (or associative) algebra on  $X$  satisfies:

- (i)  $\varphi$  is onto;
- (ii)  $\ker \varphi = \text{id}_F(R)$ .

Notice also that this implies  $L = F/\text{id}_F(R)$  where

$\text{id}_F(R)$  is the least ideal of  $F$  generated by  $R$ .

Recall that  $\Theta$  is sometimes referred to as the "presentation map" and that it is often injective. Usually

$\Theta$  is implicit.

Definition 0.5. Let  $X$  and  $Y$  be Lie subalgebras of a Lie algebra  $W$  (denoted:  $X \leq W, Y \leq W$ ). Then  $W$  is said to be the direct product of its subalgebras  $X$  and  $Y$  (denoted:  $X \times Y$ ) if

(i)  $X$  and  $Y$  are ideals of  $W$  (denoted:  $X \trianglelefteq W$  and  $Y \trianglelefteq W$ );

(ii)  $W$  is the direct sum of  $X$  and  $Y$  (denoted:  $W = X \oplus Y$ ), that is,  $X \cap Y = 0$  and  $W = X + Y$  (in other words, every  $w \in W$  can be written as  $w = x + y, x \in X, y \in Y$ ).

Notice that if  $W = X \times Y$  then  $[x, y] = 0$  for all  $x \in X$  and  $y \in Y$ .

Definition 0.6. Let  $W$  be a Lie algebra. Let  $B \trianglelefteq W, T \leq W$ . Then  $W$  is the split extension of  $B$  and  $T$  (denoted:  $W = B \rtimes T$ ) if  $W = B \oplus T$ .

It is clear that

$$W/B = B \oplus T/B \cong T/B \cap T \cong T.$$

Also in  $B \rtimes T$  the multiplication is

$$[b+t, b'+t'] = ([b, b'] + [t, b'] + [b, t']) + [t, t'].$$

Notice that if  $B$  is abelian then

$$[b+t, b'+t'] = ([t, b'] + [b, t']) + [t, t'].$$

Definition 0.7. A derivation  $\mathfrak{S}$  of a bilinear algebra

$U$  is a linear map

$$\mathfrak{S} : U \longrightarrow U$$

satisfying the product rule

$$(ab)\delta = a(b)\delta + (a)\delta b .$$

So if  $U$  is a Lie algebra this becomes

$$[a,b]\delta = [a,b\delta] + [a\delta,b] .$$

The set of all derivations of  $U$  is denoted as  $\text{Der } U$  .

If  $\text{End } U$  is as usual the set of linear transformations from  $U$  to  $U$  , then clearly  $\text{Der } U$  is a vector subspace of  $\text{End } U$  . It is easily checked that the bracket

$[\delta, \delta']$  of two derivations is again a derivation.

This turns  $\text{Der } U$  into a Lie algebra (indeed a Lie subalgebra of  $(\text{End } U)^-$  ).

## Section 2.

The following are some theorems that we will make use of throughout this paper.

The first of these is the so-called Poincaré - Birkhoff-Witt Theorem. In order to explain, let  $V$  be a vector space over a field  $F$  . Then the tensor algebra

$TV$  of  $V$  is the direct sum of vector spaces

$$T^n V = V \otimes_F V \otimes_F \dots \otimes_F V \quad (n=0,1, \dots)$$

Here  $T^0 V$  is by definition simply  $F$  . The multiplication in  $TV$  is defined via distributivity by

$$(a_1 \otimes \dots \otimes a_p) \cdot (b_1 \otimes \dots \otimes b_m) = a_1 \otimes \dots \otimes a_p \otimes b_1 \otimes \dots \otimes b_m .$$

This of course turns  $TV$  into an associative algebra over  $F$  .

Now suppose  $L$  is a Lie algebra (over  $F$  ). On forgetting the multiplication in  $L$  , the tensor algebra

TL on L can be formed. Let now I be the ideal of TL generated by the elements of the form

$$a \otimes b - b \otimes a - [a, b]$$

where  $a, b \in L$  and, of course,  $[a, b]$  is the product in L itself. Now put

$$U = TL/I$$

and denote the coset of  $a_1 \otimes \dots \otimes a_n$  in U by

$$a_1 a_2 \dots a_n :$$

$$a_1 \otimes a_2 \otimes \dots \otimes a_n + I = a_1 a_2 \dots a_n \quad (a_i \in L).$$

Then, adopting the notation above, we now state the following theorem.

Theorem 0.8 (Poincaré-Birkhoff-Witt [4])

If  $i$  is the homomorphism of L into  $U^-$  defined by

$$i: a \longmapsto a + I \quad (a \in L)$$

then  $(U, i)$  is the universal enveloping algebra of L. Furthermore if

$$x_1, x_2, \dots$$

is any ordered basis of L then 1 together with the elements

$$x(i,1)x(i,2) \dots x(i,m) \quad ((i,1) \leq \dots \leq (i,m))$$

where we will write  $x(i,j)$  for  $x_{i_j}$  and  $(i,j)$  for  $i_j$ , form a basis for U.

Notice that this contains the information that  $i$  is actually a monomorphism. L can therefore be identified with its image in  $U^-$ . Moreover, if L is now thought of as a subset of U and if  $a, b \in L$ , then

$$ab - ba = [a, b]$$

where the right hand side can be viewed either as the given product in the Lie algebra  $L$  itself or as the left hand side, that is, the difference of two products in the associative algebra  $U$ .

We will need the following analogues of two well-known theorems from group theory.

Theorem 0.9. (von Dyck's Theorem [6])

Suppose  $A$  and  $B$  are Lie algebras where  $A$  and  $B$  can be presented as follows:

$$A = (X; R) \quad \text{and} \quad B = (T; R \cup S).$$

Then the map that sends  $x$  qua element of  $A$  to  $x$  qua element of  $B$  defines a homomorphism of  $A$  onto  $B$ .

Theorem 0.10. (B. H. Neumann [6])

If

$$L = (x_1, \dots, x_n; R) \quad (n < \infty)$$

is a finitely presented Lie algebra, then there exists a finite subset  $S \subseteq R$  such that

$$L = (x_1, \dots, x_n; S).$$

The proofs of these theorems are analogous to the corresponding theorems for groups and will be omitted here.

All of the Lie algebras discussed in this thesis will have as the underlying ground field, the commutative field  $F$ . If  $G$  is a group and  $x, y \in G$  then  $[x, y] = x^{-1}y^{-1}xy$  will denote the commutator of  $x$  and  $y$  in  $G$ .  $x^y = y^{-1}xy$  will denote the conjugate of  $x$  by  $y$ . If  $L$  is a Lie algebra,  $x, y \in L$  then  $[x, y]$  is the Lie product of  $x$  and  $y$  in  $L$ .

## CHAPTER 1

### WREATH PRODUCTS

#### Section 1. Introduction

Recall that if  $A$  and  $T$  are two groups where

$$A = (X;R) \text{ and } T = (Y;S)$$

then a presentation for the wreath product of  $A$  and  $T$  (denoted:  $A \text{ wr } T$ ) is:

$$A \text{ wr } T = (X \cup Y; R \cup S \cup \{ [x, x', w(y)] \mid x, x' \in X \\ \text{and } w(y) \text{ is any word in } Y \neq 1 \text{ in } T \} ).$$

By analogy we make the following

Definition 1.1. If  $A$  and  $T$  are two Lie algebras where

$$A = (X;R) \text{ and } T = (Y;S)$$

then the wreath product of  $A$  and  $T$ , denoted  $A \text{ wr } T$ , is defined by

$$A \text{ wr } T = (X \cup Y; R \cup S \cup \{ [ [w(x), v(y)], [w'(x), v'(y)]] \} \\ \cup \{ [w(x), [w'(x), v'(y)]] \} ) .$$

Lemma 1.2.  $P = A \times T$ , the direct product of Lie algebras  $A$  and  $T$ , is a quotient of  $W$ .

#### Proof

Clearly  $P$  and  $W$  are both generated by  $X \cup Y$ .

Since  $P$  is the direct product of  $A$  and  $T$

$$[ [w(x), v(y)], [w'(x), v'(y)]] = 0$$

and

$$[w(x), [w'(x), v(y)]] = 0 .$$

Since  $A$  and  $T$  are embedded in  $A \times T$ , both  $R$  and  $S$  are satisfied in  $A \times T$ . Hence all the relations of  $W$  are satisfied in  $P$  and  $P$  is a quotient of  $W$  by von Dyck's Theorem(0.9).

Lemma 1.3.  $A \hookrightarrow W$ , that is,  $A$  is embedded in  $W$ .

Proof

In  $W = A \text{ wr } T$ , set  $\tilde{A} = L_a(X)$ , that is, the Lie subalgebra generated by  $X$ , and  $\tilde{T} = L_a(Y)$ . Clearly,  $\tilde{A} \leq W$  and  $\tilde{T} \leq W$ . Let

$$\lambda: W \twoheadrightarrow A \times T$$

be the homomorphism (actually the epimorphism) of Lemma 1.2.

Then

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{i} & W & \xrightarrow{\lambda} & A \times T \\ & & & & \downarrow \alpha \\ & & & & A \end{array}$$

$\searrow \nu$

Where  $\alpha: a \mapsto a$ ,  $t \mapsto 0$ . Here  $\nu$  is the composition of  $\alpha$ ,  $\lambda$  and  $i$ , that is,

$$\nu = i \lambda \alpha .$$

It is clear that  $\nu$  is the map that sends  $x$  qua element of  $\tilde{A}$  to  $x$  qua element of  $A$ .

Now  $\tilde{A}$  clearly satisfies the relations  $R$  of  $A$ .

Hence by von Dyck (Theorem 0.9), we have the epimorphism

$$\rho: A \twoheadrightarrow \tilde{A}$$

which sends  $x$  qua element of  $A$  to  $x$  qua element of  $\tilde{A}$ .

It is clear that  $\rho \nu = 1$  and  $\nu \rho = 1$ , so

$$\tilde{A} \cong A$$

that is

$$A \longleftrightarrow W$$

and the lemma is proved.

By a similar argument, we also have

Lemma 1.4.  $T \longleftrightarrow W$ , that is,  $T$  embeds into  $W$ .

So we may now identify  $A$  with  $\tilde{A}$  and  $T$  with  $\tilde{T}$ .

Definition 1.5. Let  $W = A \text{ wr } T$ . Then the bottom  $B$  of  $W$  is defined to be

$$B = \text{id}_W(A).$$

Notice that if  $M$  is an abelian ideal of a Lie algebra  $L$  then  $M$  can be turned into an  $L/M$  - (Lie algebra) module by setting

$$m(a + M) = [m, a] \quad (m \in M, a \in L).$$

This "action" of  $L/M$  on  $M$  can be extended to an action of  $U(L/M)$  on  $M$ . This then turns  $M$  into a right  $U(L/M)$  - module in the usual sense.

Theorem 1.6. If  $A$  is an abelian Lie algebra, then the bottom  $B$  of  $W = A \text{ wr } T$  is also abelian. If  $B$  is viewed as a right  $U = U(T)$ -module then  $B$  is actually a free  $U$ -module freely generated by any chosen basis of the vector space  $A$ .

The proof of this theorem will be divided up into four lemmas, which depend in part on the following construction. The hypothesis of Theorem 1.6 will be adopted throughout.

Let  $\bar{A}$  and  $\bar{T}$  be copies of  $A$  and  $T$  respectively. Let  $\bar{B}$  be the free right  $U(\bar{T})$ -module on a basis for  $\bar{A}$ .

We can form the semidirect product

$$\bar{W} = \bar{B} \rtimes \bar{T}$$

of  $\bar{B}$  and  $\bar{T}$  using this action of  $U(\bar{T})$  on  $\bar{B}$ .

We remind the reader of the details. First the vector space structure of  $\bar{W}$  is defined by putting

$$\bar{W} = \bar{T} \oplus \bar{B} .$$

Then  $\bar{W}$  is turned into a Lie algebra by defining

$$(1) \quad [y_1+x_1, y_2+x_2] = [y_1, y_2] + (x_1y_2 - x_2y_1)$$

where  $x_i \in \bar{B}$ ,  $y_i \in \bar{T}$  for  $i = 1, 2$ . Clearly  $xy$  is the product given in  $\bar{B}$  as a  $U(\bar{T})$  module.

Notice that

$$(2) \quad \left\{ \begin{array}{l} (i) \quad [\bar{b}, \bar{t}] = \bar{b}\bar{t} \text{ for } \bar{t} \in \bar{T}, \bar{b} \in \bar{B} \\ (ii) \quad \bar{B} \trianglelefteq \bar{W} \\ (iii) \quad \bar{W}/\bar{B} \cong \bar{T} . \end{array} \right.$$

Finally observe that

$$\bar{W} = L_a(\bar{Y} \cup \bar{X}) .$$

The preceding remarks are made use of to prove the following lemma.

Lemma 1.7. There exists a canonical homomorphism

$$\Theta : W \longrightarrow \bar{W}$$

which maps  $x \longmapsto \bar{x}$  and  $y \longmapsto \bar{y}$  where  $x \in X, y \in Y$ .

Notice that  $\Theta$ , if it exists, is simply the extension of the map

$$x \longmapsto \bar{x}, y \longmapsto \bar{y} \quad (x \in X, y \in Y) .$$

But the latter map does indeed extend to  $W$  because the defining relations of  $W$  go over into relations in  $\bar{W}$ .

Observe next that

Lemma 1.8.  $\text{id}_{\bar{W}}(\bar{A}) = \bar{B}$ .

Proof

To see this observe first that since  $\bar{A} \leq \bar{B}$

$$\text{id}_{\bar{W}}(\bar{A}) \leq \bar{B}.$$

In order to prove the reverse inclusion notice that if

$\bar{b} \in \bar{B}$  then  $\bar{b}$  can be expressed in the form

$$\bar{b} = \sum \bar{a}\bar{u}.$$

Here the  $\bar{a}$  range over part of a basis for  $\bar{A}$  while the

$\bar{u} \in U(\bar{T})$ . Now each  $\bar{u}$  occurring can be written in the form

$$\bar{u} = \sum \gamma \bar{t}(j,1)\bar{t}(j,2) \dots \bar{t}(j,r)$$

where  $\gamma \in F$ ,  $(j,1), \dots, (j,r) \in J$ ,  $(j,1) \leq (j,2) \leq \dots \leq (j,r)$ , and  $\{\bar{t}_j | j \in J\}$  is an ordered basis for  $\bar{T}$ . So

$$\bar{b} = \sum \bar{a}\bar{u} = \sum \sum \gamma \bar{a}\bar{t}(j,1) \dots \bar{t}(j,r).$$

But it follows, on repeatedly invoking (2(ii)), that

$$\gamma \bar{a}\bar{t}(j,1) \dots \bar{t}(j,r) = \gamma [ [ \dots [ [ \bar{a}, \bar{t}(j,1) ] , \bar{t}(j,2) ] \dots ] , \bar{t}(j,r) ] .$$

So

$$\gamma \bar{a}\bar{t}(j,1) \dots \bar{t}(j,r) \in \text{id}_{\bar{W}}(\bar{A}).$$

Hence  $\bar{b} \in \text{id}_{\bar{W}}(\bar{A})$ . This completes the proof of the lemma.

Now since  $\Theta : A \longrightarrow \bar{A}$  then it follows that

$\Theta : B \longrightarrow \bar{B}$ . Indeed, we claim the following:

Lemma 1.9. The restriction of  $\theta$  to  $B$ ,

$$\theta|_B \longrightarrow \bar{B}, \text{ is an isomorphism.}$$

Proof

It suffices to prove that there exists a  $\rho$  such that  $\rho: \bar{B} \longrightarrow B$  where  $\rho(\theta|_B) = 1$  and  $(\theta|_B)\rho = 1$ .

Recall that if  $W$  is a Lie algebra,  $B$  an abelian ideal of  $W$ , then  $B$  can be viewed as a  $U(W/B)$ -module by setting

$$a(w+B) = [a, w].$$

Since  $W/B \cong T$ ,  $B$  can therefore be thought of as a  $U(T)$ -module. But since  $T \cong \bar{T}$ ,  $B$  can also be thought of as a  $U(\bar{T})$ -module. Hence since both  $B$  and  $\bar{B}$  are  $U(\bar{T})$ -modules, we can use the "freeness" of  $\bar{B}$  to show that  $\rho$  exists.

In other words, if  $B$  is any  $U(\bar{T})$ -module and  $\rho': \bar{X} \longrightarrow B$  (where here  $\rho'$  sends  $\bar{x}$  to  $x$ ), then there exists an extension homomorphism  $\rho$  of  $\rho'$  such that  $\rho: \bar{B} \longrightarrow B$ . Clearly  $\rho$  is such that  $\rho(\theta|_B) = 1$  and  $(\theta|_B)\rho = 1$  as desired.

Moreover, it is clear from (1) that  $B$  is abelian.

The proof of this lemma completes the proof of Theorem 1.6, but we can additionally observe that

Lemma 1.10.  $\theta: \dot{W} \longrightarrow \bar{W}$  is an isomorphism.

Proof

Since  $W = L_a(A \cup T)$  it follows that

$$W = B + T.$$

So if  $w \in W$ , then

$$w = b + t \quad (b \in B, t \in T).$$

Suppose that  $w \in \ker \Theta$ . Thus

$$0 = w\Theta = b\Theta + t\Theta.$$

Notice that  $b\Theta \in \bar{B}$  and  $t\Theta \in \bar{T}$ . Since

$$\bar{W} = \bar{B} \oplus \bar{T}$$

it follows that

$$b\Theta = 0 = t\Theta.$$

But  $\Theta$  is an isomorphism when restricted to  $B$  and  $T$ .

Hence  $b=0$  and  $t=0$ , that is  $w=0$  and the proof of

Lemma 1.10 is complete.

It is clear that we have

Corollary 1.10.1.  $W = A \text{ wr } T = B \oplus T$ .

This is easily observed. Since  $W \cong \bar{W}$  and  $\bar{W} = \bar{T} \oplus \bar{B}$ , as a vector space,  $\Theta^{-1}$  maps  $\bar{T}$  onto  $T$  and  $\bar{B}$  onto  $B$ . So  $W = T \oplus B = B \oplus T$ .

## Section 2. A universal mapping theorem.

Proposition 1.11. Let  $A$ ,  $A^*$  and  $B$  be arbitrary Lie algebras. Let  $\alpha : A \rightarrow A^*$  be an epimorphism. Then there exists an epimorphism

$$\mu : A \text{ wr } B \rightarrow A^* \text{ wr } B.$$

### Proof

Consider the following presentations:

$$A = (X; R) \text{ and } B = (Y; S).$$

Then

$$W = A \text{ wr } B = (X \cup Y; Z)$$

where

$$Z = R \cup S \cup \{ [w(x), [w'(x), v'(y)]] \} \cup \{ [[w(x), v(y)], [w'(x), v'(y)]] \}$$

Notice, since  $\varphi : A \rightarrow A^*$  is onto,  $X$  generates  $A^*$ . Also  $A^* \cong A/\ker \varphi \cong F(X)/Q$ , where the latter isomorphism is induced by the isomorphism between  $A$  and  $F(x)/id_F(R)$ . Notice that  $R \subseteq Q$  and  $A^* = (X; Q)$ .

Now

$$W^* = A^* \text{ wr } B = (X \cup Y; Z^*)$$

where

$$Z^* = Q \cup S \cup \{ [w(x), [w'(x), v'(y)]] \} \cup \{ [[w(x), v(y)], [w'(x), v'(y)]] \}$$

Since  $Z^* \supset Z$  it follows immediately (from von Dyck's Theorem (0.9)) that the canonical map from  $W$  to  $W^*$  is a homomorphism.

Section 3. The center of a wreath product of Lie algebras is trivial.

Let  $A$  be an abelian Lie algebra,  $T$  an arbitrary Lie algebra. Recall that

$$A \text{ wr } T = T \oplus B = \{ t+b \mid t \in T, b \in B \}$$

where  $B$  is the free right  $U = U(T)$ -module on a basis for  $A$ , and the Lie product in  $T \oplus B$  is defined as follows:

$$[t_1+b_1, t_2+b_2] = [t_1, t_2] + (b_1 t_2 - b_2 t_1)$$

where  $b_i \in B, t_i \in T$  (see (1)).

Consider the set:

$$N = \left\{ \begin{pmatrix} u & 0 \\ b & 0 \end{pmatrix} \mid u \in U(T), b \in B \right\}$$

These matrices can be multiplied naturally, that is

$$\begin{pmatrix} u & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} u' & 0 \\ b' & 0 \end{pmatrix} = \begin{pmatrix} uu' & 0 \\ bu' & 0 \end{pmatrix}$$

and as such form an associative algebra. For any  $s, t \in N$  define  $[s, t] = st - ts$  where the operations on the right are the usual ones in  $N$  as an associative algebra.

$(N, [ \ ])$  is a Lie algebra. Consider

$$M = \left\{ \begin{pmatrix} t & 0 \\ b & 0 \end{pmatrix} \mid t \in T, b \in B \right\}.$$

Clearly  $M \subseteq N$ . Define a vector space map  $\varphi$  such that

$$\varphi : W = T \oplus B \longrightarrow M$$

where

$$\varphi : t + b \longmapsto \begin{pmatrix} t & 0 \\ b & 0 \end{pmatrix}.$$

It is easy to prove now that

Proposition 1.12.  $\varphi$  is a Lie algebra homomorphism. In fact,  $\varphi$  is an isomorphism.

We now prove

Proposition 1.13. The center  $Z(M)$  of  $M$  is zero.

Proof

If  $\begin{pmatrix} t' & 0 \\ b' & 0 \end{pmatrix} \in Z(M)$  then  $\left[ \begin{pmatrix} t' & 0 \\ b' & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \right] = 0$  for all  $b \in B$ , by definition. But

$$\left[ \begin{pmatrix} t' & 0 \\ b' & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ -bt' & 0 \end{pmatrix}$$

for all  $b \in B$ . Hence we must have  $-bt' = 0$  for all

$b \in B$ . In particular, for  $b = a_i$ , where  $a_i$  is one of the generators for  $A$ ,  $bt' = a_i t' = 0$ . But since the  $a_i$  are all basis elements, we must have  $t' = 0$ .

So if

$$\begin{pmatrix} t' & 0 \\ b' & 0 \end{pmatrix} \in Z(M)$$

then  $t'=0$  . Now

$$\begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} \in Z(M)$$

implies that

$$\left[ \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix}, \begin{pmatrix} t & 0 \\ b & 0 \end{pmatrix} \right] = 0$$

for all  $t \in T, b \in B$  . Hence

$$\begin{pmatrix} 0 & 0 \\ b't & 0 \end{pmatrix} = 0$$

for all  $t \in T$  implies that  $b't=0$  for all  $t \in T$  . Now

$b' \in B$  implies that  $b' = \sum_i a_i u_i$  ,  $u_i \in U(T)$  . If

$t \neq 0$  , then

$$0 = b't = \sum_i a_i (u_i t)$$

Hence  $u_i t = 0$  (all  $i$  ) implies  $u_i = 0$  (all  $i$  ) since  $t \neq 0$  and

$U$  is an integral domain. So  $b'=0$  and  $Z(M)=0$  .

Proposition 1.14.  $Z(W)=0$ .

Proof

$Z(M)=0$  and  $W \cong M$  . So  $Z(W)=0$ .

#### Section 4. Direct decompositions of wreath products of Lie algebras.

A. In this section we will show that the wreath product of a (non-trivial) abelian Lie algebra  $A$  by a (non-trivial) Lie algebra  $T$  is directly indecomposable. We do not know what happens in the general case because the structure of the wreath product of arbitrary Lie algebras is unclear.

In order to prove this we will first show that if a Lie algebra is directly decomposable then the dimension of the centralizers of all of its elements must be greater than or equal to two. We then choose an element from the wreath product and show that the dimension of its centralizer is less than two. Hence, a wreath product of Lie algebras can not be a direct product of Lie algebras.

B. Let  $W = A \text{ wr } T$  where  $A \neq 0$  is an abelian Lie algebra and  $T \neq 0$  is an arbitrary Lie algebra. Recall that  $W = A \text{ wr } T = B \oplus T$  where  $B$  is the free right  $U$ -module on a basis for  $A$  and  $U$  is the universal enveloping algebra of  $T$ .

We begin by proving (see Definition 0.5)

Lemma 1.15.  $W \neq B \times T$ .

To see this we need only point out that  $T$  is not an ideal of  $W$ . If  $T$  were an ideal of  $W$ , it would have to contain all products of the form  $[a_i, t]$ , where the  $a_i$  are basis elements of  $A$  and  $t \in T$ ,  $t \neq 0$ . But the products  $[a_i, t] = a_i t \neq 0$ . Hence  $B \cap T \neq 0$ . This contradicts the definition of direct product.

Lemma 1.16.  $W$  is not the (non-trivial) direct product of  $B$  and  $T'$ , where  $T' \neq 0$ ,  $T' \trianglelefteq W$ .

Proof

Suppose

$$W = B \times T' \quad (T' \neq 0)$$

Let  $t \in T$ ,  $t \neq 0$ , then  $t = b + t'$ ,  $b \in B$ ,  $t' \in T'$ .

Choose some  $a \in B$  where  $a$  is one of the basis elements

of A . Then

$$\begin{aligned} 0 &= [a, t'] && \text{since } B \cap T' = 0 , \\ &= [a, t-b] \\ &= [a, t] - [a, b] \\ &= [a, t] && \text{since } B \text{ is abelian.} \end{aligned}$$

So  $0 = at$  in  $B$  . But  $a$  is part of a basis, so  $a \neq 0$  . Hence it must be that  $t=0$  . But this is a contradiction. Therefore

$$W \neq B \times T' .$$

Definition 1.17. If  $L$  is a Lie algebra and  $w \in L$  , the centralizer of  $w$  ,  $cr(w)$  , is defined as follows:

$$cr(w) = \{x \in L \mid [w, x] = 0\} .$$

Lemma 1.18. Let  $L$  be any Lie algebra such that

$L = C \times D$  where  $C \leq L$  ,  $D \leq L$  ,  $C \neq 0 \neq D$  . If  $\ell \in L$  then the  $\dim(cr(\ell)) \geq 2$  , where  $\dim$  denotes dimension.

Proof

Let  $\ell \in L$  . Without loss of generality, we may assume  $\ell \neq 0$  . Then  $\ell = c + d$  ,  $c \in C$  ,  $d \in D$  . Now

$$[c+d, d] = [c, d] + [d, d] = 0 .$$

So  $d \in cr(\ell)$ . Also

$$[c+d, c] = [c, c] + [d, c] = 0 .$$

Hence  $c \in cr(\ell)$  .

If  $c \neq 0 \neq d$  , then  $c$  and  $d$  are linearly independent since they occur in different factors. So clearly,  $\dim(cr(\ell)) \geq 2$  .

Suppose one of  $c$  and  $d$  is zero, say  $c$  . Then  $\ell = c$  . So all of  $D$  centralizes  $\ell$  . Hence,  $cr(\ell) \supseteq \{c, D\}$  . Therefore, in all cases,  $\dim(cr(\ell)) \geq 2$  .

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c. Choose  $w=a+t$  where  $a$  is some basis element of  $A$  and  $t \in T$ ,  $t \neq 0$ . If  $c \in \text{cr}(a+t)$  then

$$0 = [a+t, c] = [a, c] + [t, c].$$

This implies that

$$(3) \quad 0 = [a, c] + [t, c].$$

Proposition 1.19. A non-zero element  $c \in \text{cr}(a+t)$  is neither an element of  $B$  nor an element of  $T$ .

Proof

(i) Suppose  $c \in B$ . Then from (3) we obtain

$$0 = [a, c] + [t, c] = -ct.$$

So  $c=0$ , a contradiction.

(ii) Suppose  $c \in T$ . Then from (3) we obtain

$$0 = [a, c] + [t, c]$$

and so

$$(4) \quad [a, c] = [c, t].$$

But the left hand side (lhs) of (4) is in  $B$  and the right hand side (rhs) of (4) is in  $T$ , so that we must have

$$[a, c] = 0 = [c, t]$$

since  $B \cap T = 0$ . This implies that  $c=0$  since  $a$  is some basis element of  $A$ .

Hence  $c$  is neither an element of  $T$  nor of  $B$ .

This allows me to prove

Theorem 1.20. The centralizer  $C$  of  $w=a+t$  is one-dimensional.

We shall divide this theorem into two lemmas (1.25 and 1.26) and one corollary (1.26.1). But first we recall the following, and state some definitions.

If  $\{u_j | j \in J\}$  is an ordered basis for  $T$ , then the elements

$u(i,1)u(i,2)\dots u(i,r)$  ( $(i,1) \leq (i,2) \leq \dots \leq (i,r)$ ) and 1 form a basis for  $U$  (by the Poincaré-Birkhoff-Witt Theorem (0.8)).

Definition 1.21. An element of  $U$  is termed a monomial if it is written in the form

$$\alpha u(i,1)u(i,2)\dots u(i,r) \quad (\alpha \in F, \alpha \neq 0)$$

where the  $u$ 's are basis elements of  $T$ . We allow  $r=0$  which is interpreted simply as  $\alpha$ .

Definition 1.22. An element of  $U$  is termed a straight monomial if it is written in the form

$$u(i,1)u(i,2)\dots u(i,r) \quad (\alpha \in F, \alpha \neq 0)$$

and the  $u(i,j)$  are basis elements of  $T$ , where

$$(i,1) \leq (i,2) \leq \dots \leq (i,r)$$

Definition 1.23. The weight of a straight monomial  $m$ , where

$$m = \alpha u(i,1)u(i,2)\dots u(i,r)$$

$(i,1) \leq \dots \leq (i,r)$  is defined to be  $r$ . We will write:

$$\text{wt}(m) = r$$

Notice that the definitions exclude 0 from the straight monomials.

Definition 1.24. An element  $u \in U$  is termed homogeneous of weight  $r$  if it can be written as a sum of straight monomials of weight  $r$ .

Notice that if  $u \in U$  ( $u \neq 0$ ), then it can be written uniquely in the form

$$(5) \quad u = u(0) + u(1) + \dots + u(p)$$

where  $u(r)$  is homogeneous of weight  $r$ . Notice that  $u(p)$  is the term of highest weight.

Lemma 1.25. If

$$\alpha u(i,1)u(i,2)\dots u(i,r)$$

is a straight monomial and if  $u_k$  is any basis element of  $T$ , then

$$\begin{aligned} & \alpha u(i,1)u(i,2)\dots u(i,r) \cdot u_k = \\ & \sigma + \alpha u(i,1)\dots u(i,k)u_k u(i,j+1)\dots u(i,r) \end{aligned}$$

where  $\sigma$  is either zero or a sum of straight monomials of weight less than  $r+1$  and

$$(i,1) \leq (i,2) \leq \dots \leq (i,j) \leq k \leq (i,j+1) \leq \dots \leq (i,r)$$

Proof

The proof will be by induction on  $r$ , the weight of the monomial.

Suppose  $r=1$ , then for

$$m = \alpha u(i,1)$$

$$m \cdot u_k = \alpha u(i,1)u_k$$

If  $(i,1) \leq k$ , then we are finished, that is,  $u(i,1)u_k$  is a straight monomial as it stands. If  $(i,1) > k$ , then

$$\begin{aligned} u(i,1)u_k &= u_k u(i,1) - u_k u(i,1) + u(i,1)u_k \\ &= u_k u(i,1) + [u(i,1), u_k], \quad k < (i,1) \end{aligned}$$

So  $u(i,1)u_k = u_k u(i,1) + \sum \alpha_j u_j$  ,  $\alpha_j \in F$   
 since  $[u(i,1), u_k] \in T$  . Notice that  $\text{wt}(\alpha_j u_j) =$   
 $1 < r+1 = 2$  , for all  $r$  as required.

Now suppose the theorem is true for some  $r > 1$  ,  
 that is

$$u(i,1)u(i,2)\dots u(i,r) \cdot u_k =$$

$$\sigma + u(i,1)\dots u(i,j)u_k u(i,j+1)\dots u(i,r)$$

where  $\sigma$  is a sum of straight monomials of weight  
 less than  $r+1$  and

$$(i,1) \leq (i,2) \leq \dots \leq (i,j) \leq k \leq (i,j+1) \leq \dots \leq (i,r) .$$

Consider  $u(i,1)u(i,2)\dots u(i,r)u(i,r+1)$  , a straight  
 monomial of weight  $r+1$  .

Case 1. Suppose  $k \geq (i,r+1)$  . Then  $u(i,1)\dots u(i,r+1) \cdot u_k$   
 is a straight monomial and we are done.

Case 2. Suppose  $k < (i,r+1)$  . Then

$$u(i,1)\dots u(i,r)u(i,r+1) \cdot u_k =$$

$$u(i,1)\dots u(i,r)(u_k u(i,r+1) + u(i,r+1)u_k - u_k u(i,r+1)) =$$

$$u(i,1)\dots u(i,r)(u_k u(i,r+1) + [u(i,r+1), u_k]) =$$

$$u(i,1)\dots u(i,r)u_k u(i,r+1) + u(i,1)\dots u(i,r) \sum \alpha_j u_j =$$

$$(u(i,1)\dots u(i,j)u_k u(i,j+1)\dots u(i,r) + \sigma)u(i,r+1)$$

$$+ \sum \alpha_j u(i,1)\dots u(i,r)u_j =$$

$$(6) \left\{ \begin{array}{l} u(i,1)\dots u(i,j)u_k u(i,j+1)\dots u(i,r)u(i,r+1) \\ + \sigma u(i,r+1) + \sum \alpha_j u(i,1)\dots u(i,r)u_j \end{array} \right.$$

where

$$(i,1) \leq \dots \leq (i,j) \leq k \leq (i,j+1) \leq \dots \leq (i,r) \leq (i,r+1) .$$

Since by the induction hypothesis  $\sigma$  is either  
 zero or is a sum of straight monomials of weight less  
 than  $r+1$  ,  $\sigma u(i,r+1)$  can be written as a sum of

straight monomials of weight less than  $r+2$  . Similarly we can "straighten" the monomials involved in

$$\sum_j \alpha_j u(i,1) \dots u(i,r) \cdot u_j$$

and obtain a sum of straight monomials of weight less than  $r+2$  , or zero. Rewriting (6) we are led to the equation

$$(7) \begin{cases} u(i,1) \dots u(i,r) u(i,r+1) \cdot u_k = \\ u(i,1) \dots u(i,j) u_k u(i,j+1) \dots u(i,r) u(i,r+1) + \sigma' \end{cases} .$$

where

$$(i,1) \leq \dots \leq (i,j) \leq k \leq (i,j+1) \leq \dots \leq (i,r) \leq (i,r+1)$$

and  $\sigma'$  is a sum of straight monomials of weight less than  $r+2$  , or  $\sigma'$  is zero. This concludes the proof of Lemma 1.25.

It follows from the proof of Lemma 1.25 that we have proved

Lemma 1.26. If  $u = u(0) + \dots + u(p)$  , uniquely,

( $u \neq 0$ ) and  $u_k$  is a basis element of  $T$  , then

$$u \cdot u_k = v(0) + v(1) + \dots + v(p+1)$$

where  $v(i)$  is either zero or homogeneous of weight  $i$  for  $i=0, \dots, p$  and  $v(p+1)$  is homogeneous of degree  $p+1$  ,  $p+1 \geq 0$  .

As a consequence, we have the following corollary.

Corollary 1.26.1. If  $u = u(0) + \dots + u(p)$  , ( $u \neq 0$ ) ,

uniquely, and  $t = \sum_j \alpha_j u_j \in T$  ,  $t \neq 0$ , then

$$ut = v(0) + v(1) + \dots + v(p+1) , \quad v(p+1) \neq 0$$

where  $v(i)$  is the homogeneous component of weight  $i$  or zero, and  $p+1 > 0$  .

D. We now make use of the following calculations.  
 Suppose  $c \in C = cr(a+t)$  . If  $c=d+s$  ,  $d \neq 0 \neq s$  ,  $d \in B$  ,  
 $s \in T$  , then

$$[a+t, d+s] = [a, d] + [a, s] + [t, d] + [t, s] = 0$$

which implies

$$as - dt = 0 = [s, t]$$

since  $B \cap T = 0$  . So  $as = dt$  and  $[s, t] = 0$  .

Now if  $d = \sum a_i u_i$  ,  $u_i \in U$  , the universal enveloping algebra of  $T$  , then  $dt = \sum a_i (u_i t)$  . Since  $as = dt =$

$\sum a_i (u_i t)$  then  $dt = a_k (u_k t)$  (for some  $k$ ) and all other

$u_i t = 0$  . So  $u_i = 0$  since  $U$  is an integral domain. Here

$a_k = a$  . Simplifying notation we write  $u$  for  $u_k$  . So

$$(8) \quad d = au .$$

Therefore

$$(9) \quad as = a(ut) \text{ implies } s = ut .$$

We shall examine this equation (9), which involves only elements in  $T$  .

Recall that  $s=ut$  where  $s, t \in T$  ,  $s \neq 0 \neq t$  ,  $u \in U$  and  $u \neq 0$  . So we may write  $s=s(1)$  . Write  $u$  in the form

$$u = u(0) + \dots + u(p) , (u(p) \neq 0, p \geq 0) .$$

So

$$ut = v(0) + \dots + v(p+1)$$

by Corollary 1.26.1 . Hence, we obtain

$$s(1) = v(0) + \dots + v(p+1) .$$

Hence  $p=0$  is the only possibility, that is,  $u \in F$  .

So rewriting (8) we have the equation

$$d = \alpha a \quad (\alpha \in F) .$$

Now  $d = \alpha a$  and  $s = \alpha t$ , since  $d = au$  and  $s = ut$  by (8) and (9), where  $d+s \in \text{cr}(a+t)$ . So  $d+s = \alpha a + \alpha t = \alpha(a+t)$ . Therefore the centralizer of  $w = a+t$  is one dimensional as desired. This completes the proof of Theorem 1.20.

## CHAPTER 2

### ASSOCIATIVE ALGEBRAS

#### Section 1. Introduction

We first define a free product of associative algebras ( $F$ , the underlying commutative field).

Definition 2.1.  $C$  is said to be the free product of a family of associative algebras  $\{C_j \mid j \in J\}$  (with  $1$  for each  $j$ ) iff

- (i)  $C = \text{alg}(C_j \mid j \in J)$ ;
- (ii) for every associative algebra  $C'$  and every family of algebra homomorphisms  $\alpha_j: C_j \rightarrow C'$ , there exists an algebra homomorphism  $\alpha$  such that  $\alpha: C \rightarrow C'$  extends  $\alpha_j$  ( $j \in J$ ).

Let  $A$  and  $B$  be associative algebras with  $1$ . Let  $X = \{1\} \cup \{e_i^A \mid i \in I\}$  and  $Y = \{1\} \cup \{e_i^B \mid i \in I\}$  be bases for  $A$  and  $B$  respectively. Consider  $\{1\}$  and the set of monomials

$$(1) \quad a = e^{\gamma_1(i,1)} e^{\gamma_2(i,2)} \dots e^{\gamma_r(i,r)}$$

where  $\gamma_i \in \{A, B\}$  and  $\gamma_i \neq \gamma_{i+1}$ . It is easily seen that this set constitutes a basis for  $P = A * B$ , the free product of associative algebras  $A$  and  $B$ . We also term  $\alpha a$  ( $\alpha \in F$ ,  $\alpha \neq 0$ ), where  $a$  is given by (1), a monomial of weight  $r$ . Notice that every element  $u \in P$ ,  $u \neq 0$ , can be written uniquely in the form

$$(2) \quad u = u(0) + u(1) + \dots + u(n)$$

where  $u(0) \in F$  and each  $u(i)$  is either a sum of monomials of weight  $i$  or zero. We term  $u(i)$  the homogeneous component of  $u$  of weight  $i$ . Notice that the form (2) depends on the choice of bases for  $A$  and  $B$ . Since this choice is made at the outset we shall have no occasion to mention it further. We term  $u$  of weight  $n$  if  $u(n) \neq 0$ .

Section 2. A free product of associative algebras  $A$  and  $B$  is directly indecomposable if  $A$  and  $B$  are integral domains.

A. In this first part of Section 2 we prove Theorem 2.2.

Let  $A$  and  $B$  be associative algebras. If  $A$  and  $B$  are integral domains, then there exists an element  $p \in P$  ( $p \neq 0$ ),  $P = A * B$ , such that  $cr(p) = F[p]$ .

Proof

Choose  $p \in P$  ( $p \neq 0$ ) such that  $p = ef$ , where  $1 \neq e$  is an element of  $X$  and  $1 \neq f$  is an element of  $Y$ . We will show that  $C = cr(ef) = F[ef]$ . Let  $u \in C$ ,  $u \neq 0$ . We will prove that  $u \in F[ef]$  by induction on the weight of  $u$ .

Let  $u \in C$ ,  $wt(u) \leq 1$ . Writing  $u$  in its homogeneous form, we have

$$u = u(0) + u(1)$$

Since  $u \in C = cr(ef)$ , we have

(3)  $u(ef) = (ef)u$

If  $u(1) \neq 0$  then  $u(1) = \sum \alpha x + \sum \beta y$  ( $x \in X, y \in Y$ ).

Consider the term of highest weight on the left hand side of (3) (lhs (3)) and the right hand side of (3) (rhs (3)):

$$\text{lhs}(3) = \sum \beta yef \text{ and } \text{rhs}(3) = \sum \alpha efx .$$

So

$$(4) \quad \sum \beta yef = \sum \alpha efx$$

which is not possible since lhs(4) ends in  $f \in Y$  and rhs(4) ends in  $x \in X$  . So we must have  $\beta = 0 = \alpha$  for all  $\alpha$  and  $\beta$  .

Therefore  $u = u(0) \in F$  and  $u \in C$  is trivially a polynomial in  $ef$  . So the theorem is proved for  $\text{wt}(u) \leq 1$  .

Let us also consider the case when  $u \in C$  has weight two. That is

$$u = u(0) + u(1) + u(2) \quad (u(2) \neq 0) .$$

Since  $u \in C$  we have

$$(5) \quad u(ef) = (ef)u .$$

Consider  $u(2)$  , the term of highest weight in  $u$  :

$$(6) \quad u(2) = \sum \alpha we + \sum \alpha' w'e' + \sum \beta vf + \sum \beta' v'f'$$

where  $w, w', v, v'$  are all of weight one, that is, are equal to  $e, f$  or one of the elements  $e'$  in  $X - \{e\}$  or  $f'$  in  $Y - \{f\}$  . It is clear that we have taken all the terms in  $u(2)$  and grouped them as follows: first, those terms ending in  $e$  ; second, those terms ending in the elements of  $X - \{e\}$  , denoted  $e'$  ; third, those terms ending in  $f$  ; and last, those terms ending in elements in  $Y - \{f\}$  , denoted  $f'$  . All of the

monomials are straight (see Definition 1.22). So

$v$  and  $v'$  are equal to  $e$  or one of the elements  $e'$  and  $w$  and  $w'$  are equal to  $f$  or one of the elements  $f'$ .

Consider the terms of highest weight, that is of weight four, in (5):

$$(7) \quad \sum \beta v f e f + \sum \beta' v' f' e f = \sum \beta e f v f + \sum \beta' e f v' f'$$

Since no terms of the lhs(7) end in any  $f'$  ( $f \neq f'$ ), it must be that

$$\sum \beta' e f v' f' = 0$$

Hence

$$(8) \quad \beta' = 0$$

Rewriting (7) we have

$$\sum \beta v f e f = \sum \beta e f v f$$

which implies that for some  $i$  and  $j$

$$\beta_i v_i f e f = \beta_j e f v_j f$$

Hence

$$\beta_i = \beta_j \quad \text{and} \quad v_i = e = v_j$$

for all  $i$  and  $j$ . So (6) may now be rewritten as

$$(9) \quad u(2) = \sum \alpha w e + \sum \alpha' w' e' + \beta e f$$

Now

$$(10) \quad u(1) = \gamma e + \delta f + \sum \gamma' e' + \sum \delta' f'$$

We now consider the terms of weight three in (5):

$$(11) \quad \left\{ \begin{array}{l} \sum \alpha w e f + \sum \alpha' e' f + \delta f e f + \sum \delta' f' e f \\ \sum \alpha e f w e + \sum \alpha' e f w' e' + \gamma e f e + \sum \gamma' e f e' \end{array} \right. =$$

where " $\sqcup$ " indicates that an amalgamation takes place between these factors since they occur in the same basis.

Since every term on lhs(11) ends in  $f$ , we must have  $lhs(11) = 0 = rhs(11)$ . Considering rhs(11) and the fact

that  $e \neq e'$ , we obtain

$$\sum \alpha f w e + \gamma e f e = 0 .$$

So

$$e f (\sum \alpha w + \gamma \cdot 1) e = 0 .$$

Since  $P$  is an integral domain

$$\sum \alpha w + \gamma \cdot 1 = 0 .$$

But no linear combination of basis elements is zero unless all the coefficients are zero. Hence

$$(12) \quad \alpha = 0 = \gamma$$

for all  $\alpha$ . Also since

$$\sum \alpha' e f w' e' + \sum \gamma' e f e' = 0$$

then

$$\sum \alpha' e f w' e' + \sum \gamma' e f e' = 0$$

and

$$e f (\sum \alpha' w' e' + \sum \gamma' e') = 0$$

so

$$(13) \quad \sum \underbrace{\alpha' w' e'}_{(i)} + \sum \underbrace{\gamma' e'}_{(ii)} = 0 .$$

But (13(i)) begins with an element of  $Y$  and (13(ii)) begins with an element of  $X$  so the only way (13) can be true is if

$$(14) \quad \alpha' = 0 = \gamma'$$

So (12) and (14) give us that (9) may be written as

$$(15) \quad u(2) = \beta(e f)$$

Considering lhs(11), it may now be rewritten as

$$(16) \quad \sum \zeta f e f + \sum \zeta' f' e f = 0 .$$

But  $f \neq f'$  so  $\zeta = 0 = \zeta'$  for all  $\zeta$  and  $\zeta'$ .

Hence (12), (14) and (16) give us that  $u(1) = 0$ .

Hence  $u = u(0) + \beta(ef)$  which is clearly in  $F[ef]$ .

Hence the theorem is proved for  $wt(u) \leq 2$ .

Next we show that if  $u \in cr(ef)$  and  $wt(u) = m = 2n - 1$  then  $u(2n - 1) = 0$  (that is,  $u = u(0)$ ). Let

$$u = u(0) + \dots + u(2n - 1).$$

Clearly we have  $u(1) = 0$  for  $n = 1$  from the first case considered (page 28). Consider  $u(2n-1)$ ,  $n \geq 2$ . Write  $u(2n-1)$  as follows:

$$(17) \left\{ \begin{aligned} u(2n-1) = & \sum \alpha x w_x e + \sum \bar{\alpha} y w_y e + \sum \alpha' x' w_x' e' + \\ & \sum \bar{\alpha}' y' w_y' e' + \sum \beta x'' v_x'' f + \sum \bar{\beta} y'' v_y'' f + \\ & \sum \beta' x''' v_x''' f' + \sum \bar{\beta}' y''' v_y''' f' \end{aligned} \right.$$

where  $x, x', x'', x''' \in X$  and  $y, y', y'', y''' \in Y$ .

The monomial  $w_x$  is the remaining (straight) part of the straight monomial of  $u(2n-1)$  that ends in  $e$  and begins in  $x \in X$  (the other monomials in (17) are similarly defined).

Again we have

$$(18) \quad u(ef) = (ef)u.$$

Consider the terms of highest weight (that is, weight  $2n+1$ ) in (18):

$$(19) \left\{ \begin{aligned} & \sum \beta x'' v_x'' f e f + \sum \bar{\beta} y'' v_y'' f e f + \sum \beta' x''' v_x''' f' e f + \\ & \sum \bar{\beta}' y''' v_y''' f' e f = \\ & \sum \alpha e f x w_x e + \sum \alpha' e f x' w_x' e' + \sum \beta e f x'' v_x'' f + \\ & \sum \beta' e f x''' v_x''' f' \end{aligned} \right.$$

Since no term on lhs(19) ends in  $f'$  ( $f \neq f'$ ) we must have

$$\sum \beta' e f x''' v_x''' f' = 0.$$

So

$$\beta' = 0 .$$

Similarly

$$\sum \alpha e f x w_x e = 0 = \sum \alpha' e f x' w_x' e' .$$

So

$$(21) \quad \alpha = 0 = \alpha'$$

for all  $\alpha$  and  $\alpha'$  . So (19) may be rewritten as

$$(22) \quad \left\{ \begin{array}{l} \sum \beta x'' v_x'' f e f + \sum \bar{\beta} y'' v_y'' f e f + \sum \bar{\beta}' y''' v_y''' f' e f \\ \sum \beta e f x'' v_x'' f \end{array} \right. =$$

Since every term on rhs(22) begins with  $e f$  , we must have that

$$(23) \quad \sum \bar{\beta} y'' v_y'' f e f + \sum \bar{\beta}' y''' v_y''' f' e f = 0 .$$

But since the first summation ends in  $f e f$  and the second ends in  $f' e f$  ( $f \neq f'$ ), we must have

$$(24) \quad \bar{\beta} = 0 = \bar{\beta}'$$

for all  $\bar{\beta}$  and  $\bar{\beta}'$  .

Finally (22) may be written as

$$(25) \quad \sum \beta x'' v_x'' f e f = \sum \beta e f x'' v_x'' f .$$

So

$$(\sum \beta x'' v_x'' f e - \sum \beta e f x'' v_x'') f = 0 .$$

Since  $P$  is an integral domain we must have

$$\sum \beta x'' v_x'' f e - \sum \beta e f x'' v_x'' = 0$$

which implies that

$$\sum \beta x'' v_x'' f e = \sum \beta e f x'' v_x'' .$$

So all  $x'' = e$  and we may write

$$(26) \quad \sum \beta e v_x'' f e = \sum \beta e f e v_x'' .$$

We obtain

$$\sum \beta v_x'' f e = \sum \beta f e v_x''$$

which implies that all  $v_x''$  must begin with  $fe$  and end with  $fe$ . Now  $wt(v_x'') = 2n-3$  is odd. For  $n = 4$  then  $wt(v_x'') = 5$  so

$$v_x'' = fe \cdot R \cdot fe$$

where  $R$  is the remaining factor of  $v_x''$ ,  $wt(R) = 1$ .

Considering  $v_x''$  in (26) we have for some  $i$  and  $j$

$$\begin{aligned} \beta_i fe \cdot R_i \cdot fe \cdot fe &= \\ \beta_j fe \cdot fe \cdot R_j \cdot fe & \end{aligned}$$

so

$$\beta_i R_i \cdot fe = \beta_j fe \cdot R_j$$

But  $wt(R_i) = 1 = wt(R_j)$  implies that  $f = e$ , a contradiction. So

$$\sum \beta ev_x'' fe = \sum \beta ef ev_x''$$

only if

$$\sum \beta ev_x'' fe = 0 = \sum \beta ef ev_x''$$

which implies that

$$(27) \quad \beta = 0$$

So (17) may be rewritten as

$$u(2n-1) = \sum \bar{\alpha} y w_y e + \sum \bar{\alpha}' y' w_y' e'$$

Write  $u(2n-2)$  as follows:

$$\begin{aligned} u(2n-2) = & \sum \alpha^* x^* w_x^* e + \sum \bar{\alpha}^* y^* w_y^* e + \sum \alpha'^* x'^* w_x'^* e' + \\ & \sum \bar{\alpha}'^* y'^* w_y'^* e' + \sum \beta^* x'' v_x'' f + \sum \bar{\beta}^* y'' v_y'' f + \\ & \sum \beta'^* x''' v_x''' f' + \sum \bar{\beta}'^* y''' v_y''' f' \end{aligned}$$

where  $x^*, x'^*, x'', x'''' \in X$ ,  $y^*, y'^*, y'', y'''' \in Y$

and all monomials are as in (17) except that their weight

is  $2n-4$ .

In (19) we considered the terms of weight  $2n+1$  .

Now consider the terms of weight  $2n$  in (18):

$$(28) \left\{ \begin{array}{l} \sum \alpha x w_x e f + \sum \bar{\alpha} y w_y e f + \sum \alpha' x' w_x' e' e f + \sum \bar{\alpha}' y' w_y' e' e f + \\ \sum \beta x'' v_x'' f e f + \sum \bar{\beta} y'' v_y'' f e f + \sum \beta' x''' v_x''' f' e f \\ \quad + \sum \bar{\beta}' y''' v_y''' f' e f = \\ \sum \bar{\alpha} e f y w_y e + \sum \bar{\alpha}' e f y' w_y' e' + \sum \bar{\beta} e f y'' v_y'' f + \sum \bar{\beta}' e f y''' v_y''' f' + \\ \sum \alpha^* e f x^* w_x^* e + \sum \alpha'^* e f x'^* w_x'^* e' + \sum \beta^* e f x'' v_x'' f \\ \quad + \sum \beta'^* e f x''' v_x''' f' \end{array} \right. .$$

Notice that all terms on lhs(28) end in  $f$  and since  $f \neq f'$  (also  $e \neq e'$ ) we must have

$$(i) \quad \sum \bar{\alpha} e f y w_y e + \sum \alpha^* e f x^* w_x^* e = 0 .$$

So

$$e(\sum \bar{\alpha} f y w_y + \sum \alpha^* f x^* w_x^*)e = 0 .$$

Since  $P$  is an integral domain,

$$\sum \bar{\alpha} f y w_y + \sum \alpha^* f x^* w_x^* = 0$$

which implies that

$$f(\sum \bar{\alpha} y w_y + \sum \alpha^* x^* w_x^*) = 0$$

so

$$(29) \quad \sum \bar{\alpha} y w_y + \sum \alpha^* x^* w_x^* = 0 .$$

Since  $wt(y w_y) = 2n-2$  and  $wt(x^* w_x^*) = 2n-3$  and since the first summation begins in  $y \in Y$  and the second in  $x^* \in X$ , it must be that

$$(30) \quad \bar{\alpha} = 0 = \alpha^* .$$

$$(ii) \quad \sum \bar{\alpha}' e f y' w_y' e' + \sum \alpha'^* e f x'^* w_x'^* e' = 0$$

so

$$e f(\sum \bar{\alpha}' y' w_y' e' + \sum \alpha'^* x'^* w_x'^* e') = 0 .$$

Since  $P$  is an integral domain we have

$$(31) \quad \sum \bar{\alpha} y' w_y' e' + \sum \alpha' x' w_x' e' = 0 .$$

But clearly this can only be so if

$$(32) \quad \bar{\alpha} = 0 = \alpha' *$$

since the first summation in (31) begins with  $y'$  and the second summation begins with  $x' *$  .

So finally (30) and (32) give us that (17) is zero, that is

$$u(2n-1) = 0$$

for all  $n$  .

Since we have shown that if  $u$  is of odd weight then  $u = u(0)$  , we now prove that if  $u$  is any centralizer of  $ef$  of even weight then the theorem is true for it.

Consider

$$u = u(0) + \dots + u(2n) \quad (n \geq 2) .$$

Write  $u(2n)$  as follows:

$$(33) \quad \left\{ \begin{array}{l} u(2n) = \sum \alpha x w_x e + \sum \bar{\alpha} y w_y e + \sum \alpha' x' w_x' e' \quad + \\ \sum \bar{\alpha}' y' w_y' e' + \sum \beta x'' v_x'' f + \sum \bar{\beta} y'' v_y'' f \quad + \\ \sum \beta' x''' v_x''' f' + \sum \bar{\beta}' y''' v_y''' f' \end{array} \right.$$

where the  $x$ 's,  $y$ 's and monomials are as in (17), except the monomials have weight  $2n-2$  .

Again we have

$$(34) \quad u(ef) = (ef)u .$$

Consider the terms of highest weight (that is, weight =  $2n+2$ ) in (34):

$$(35) \quad \left\{ \begin{array}{l} \sum \beta x'' v_x'' f e f + \sum \bar{\beta} y'' v_y'' f e f + \sum \beta' x''' v_x''' f' e f \quad + \\ \sum \bar{\beta}' y''' v_y''' f' e f \end{array} \right. =$$

$$(35) \left\{ \begin{aligned} \sum \alpha e f x w_x e + \sum \alpha' e f x' w_x' e' + \sum \beta e f x'' v_x'' f \\ + \sum \beta' e f x''' v_x''' f' \end{aligned} \right. .$$

Since no term on lhs(35) ends in  $f'$  ( $f \neq f'$ ) we must have

$$\sum \beta' e f x''' v_x''' f' = 0 .$$

So

$$\beta' = 0 .$$

Similarly

$$\sum \alpha e f x w_x e = 0 = \sum \alpha' e f x' w_x' e' .$$

So

$$(36) \quad \alpha = 0 = \alpha'$$

for all  $\alpha$  and  $\alpha'$  . So (35) may be rewritten as

$$(37) \left\{ \begin{aligned} \sum \beta x'' v_x'' f e f + \sum \bar{\beta} y'' v_y'' f e f + \sum \bar{\beta}' y''' v_y''' f' e f \\ = \sum \beta e f x'' v_x'' f \end{aligned} \right. .$$

Since every term on rhs(37) begins with  $e f$  , we must have that

$$(38) \quad \sum \bar{\beta} y'' v_y'' f e f + \sum \bar{\beta}' y''' v_y''' f' e f = 0 .$$

But since the first summation ends in  $f e f$  and the second ends in  $f' e f$  ( $f \neq f'$ ), we must have that

$$(39) \quad \bar{\beta} = 0 = \bar{\beta}'$$

for all  $\bar{\beta}$  and  $\bar{\beta}'$  .

Finally (37) may be written as

$$\sum \beta x'' v_x'' f e f = \sum \beta e f x'' v_x'' f .$$

So

$$(\sum \beta x'' v_x'' f e - \sum \beta e f x'' v_x'') f = 0 .$$

Since  $P$  is an integral domain we must have

$$\sum \beta x'' v_x'' f e - \sum \beta e f x'' v_x'' = 0$$

which implies that

$$\sum \beta x'' v_x'' fe = \sum \beta efx'' v_x''$$

So all  $x'' = e$  and we may write

$$(40) \quad \sum \beta e v_x'' fe = \sum \beta e f e v_x''$$

We obtain

$$(41) \quad \sum \beta v_x'' fe = \sum \beta f e v_x''$$

But this says that  $v_x''$  must begin and end with  $fe$ , that is

$$(42) \quad v_x'' = fe \cdot R_1 \cdot fe \quad (\text{wt}(v_x'') = 2n-2)$$

where  $R_1$  represents the remaining  $2n-6$  factors of  $v_x''$ .

Suppose  $R_1 = f' \cdot R_2$  then for  $n = 4$ , we would have

$$(43) \quad v_x'' = fe \cdot f' \cdot R_2 \cdot fe \quad (\text{wt}(R_2)=1)$$

Replacing (43) in (41) we obtain, for some  $i$  and  $j$ ,

$$(44) \quad \beta_i \underbrace{fe \cdot f' \cdot R_2 \cdot fe}_{v_x'' = v_i''} \cdot fe = \beta_j f e v_j''$$

so

$$(45) \quad \beta_i f' \cdot R_2 \cdot fe \cdot fe = \beta_j v_j''$$

This says that some  $v_x'' = v_j''$  begins with  $f'$ . But this can not be so since all  $v_x''$  begin with  $f$  and  $f \neq f'$  for any  $f'$ . So clearly the only choice for the first factor in  $R_1$  is  $f$ . In fact, from (45) we know that the first factor must be  $fe$ . So  $R_1 = fe$  since  $\text{wt}(R_1) = 2$ . Also from (45) we have that  $\beta_i = \beta_j$  and  $v_i'' = f e f e f e = v_j''$ . So continuing this process inductively for all  $n$  we see that there is only one term, that is

$$(46) \quad \sum \beta \underbrace{x'' v_x'' f}_{(2n)} = \beta e (fe)^{n-1} f = \beta (ef)^n$$

Now

$$(47) \quad u(2n) = \sum \bar{\alpha} y w_y e + \sum \bar{\alpha}' y' w_y' e' + \beta (ef)^n .$$

In (35) we considered the terms of weight  $2n+2$  .

We now consider the terms of weight  $2n+1$  in (34). But first we write  $u(2n-1)$  as follows:

$$(48) \quad \left\{ \begin{aligned} u(2n-1) = & \sum \alpha * x * w_x^* e + \sum \bar{\alpha} * y * w_y^* e + \sum \alpha' * x' * w_x' e' + \\ & \sum \bar{\alpha}' * y' * w_y' e' + \sum \beta * x'' * v_x'' * f + \sum \bar{\beta} * y'' * v_y'' * f \\ & + \sum \beta' * x''' * v_x''' * f' + \sum \bar{\beta}' * y''' * v_y''' * f' \end{aligned} \right.$$

where the  $x$ 's ,  $y$ 's and the monomials are as in (17)

except that each monomial has weight  $2n-3$ . So the terms of weight  $2n+1$  in (34) are:

$$(49) \quad \left\{ \begin{aligned} & \sum \bar{\alpha} y w_y e e f + \sum \bar{\alpha}' y' w_y' e' e f + \sum \beta * x'' * v_x'' * f e f + \sum \bar{\beta} * y'' * v_y'' * f e f + \\ & \sum \beta' * x''' * v_x''' * f' e f + \sum \bar{\beta}' * y''' * v_y''' * f' e f = \\ & \sum \bar{\alpha} e f y w_y e + \sum \bar{\alpha}' e f y' w_y' e' + \sum \alpha * e f x * w_x^* e + \sum \alpha' * e f x' * w_x' e' + \\ & \sum \beta * e f x'' * v_x'' * f + \sum \beta' * e f x''' * v_x''' * f' . \end{aligned} \right.$$

Since all terms on lhs(49) end in  $f$  then rhs(49) =

$0 = \text{lhs}(49)$  . Now  $e \neq e'$  and rhs(49) = 0 imply that

$$(i) \quad \sum \bar{\alpha} e f y w_y e + \sum \alpha * e f x * w_x^* e = 0$$

so

$$e f (\sum \bar{\alpha} y w_y + \sum \alpha * x * w_x^*) e = 0 .$$

Since  $P$  is an integral domain,

$$(50) \quad \sum \bar{\alpha} y w_y + \sum \alpha * x * w_x^* = 0 .$$

But this can only be so if

$$(51) \quad \bar{\alpha} = 0 = \alpha *$$

since the terms in the first summation in (50) begin with

$y \in Y$  and the terms in the second summation in (50) begin with  $x^* \in X$  . Also (from (49))

$$(ii) \sum \bar{\alpha} e f y' w'_y e' + \sum \alpha' e f x' w'_x e' = 0$$

so

$$e f (\sum \bar{\alpha} y' w'_y e' + \sum \alpha' x' w'_x e') = 0 .$$

But  $P$  an integral domain, implies that

$$(52) \quad \sum \bar{\alpha} y' w'_y e' + \sum \alpha' x' w'_x e' = 0 .$$

Clearly (52) can only be true if

$$(53) \quad \bar{\alpha} = 0 = \alpha' .$$

since the terms in the first summation in (52) begin with  $y' \in Y$  and the terms in the second summation begin with  $x' \in X$  .

So from (51) and (53) we now may write (47) as follows:

$$(54) \quad u(2n) = \beta(ef)^n .$$

Clearly  $\beta(ef)^n \in \text{cr}(ef)$  since  $u \in \text{cr}(ef)$  , so

$$z = u - u(2n) \in \text{cr}(ef) .$$

But  $\text{wt}(z) < 2n$  hence the theorem is true for  $z$  , that is  $z \in F[ef]$  . Since  $\beta(ef)^n$  is clearly an element of  $F[ef]$  , we have that  $u = z + u(2n) \in F[ef]$  , and the theorem is proved.

B. Suppose for  $P = A * B$  , the free product of associative algebras  $A$  and  $B$  ,  $P = C \times D$  , that is,  $P$  can be decomposed into a direct product ( $C \neq 0 \neq D$ ). The next two lemmas are central in completing the proof that if  $A$  and  $B$  , associative algebras, are integral domains, then the free product of associative algebras  $A$  and  $B$  is directly indecomposable.

Lemma 2.4. If  $p \in P$ ,  $p \neq 0$ ,  $p = c + d$ ,  $c \in C$ ,  $d \in D$ , then  $c$  and  $d$  are elements of  $\text{cr}(p)$ .

Proof

If  $p \in P = C \times D$  ( $p \neq 0$ ,  $C \neq 0 \neq D$ ), then  $p = c + d$  uniquely. Thus

$$\begin{aligned} pc - cp &= (c + d)c - c(c + d) \\ &= c^2 + dc - c^2 - cd \\ &= 0 \end{aligned}$$

since all  $c, d$  products are zero ( $C \cap D = 0$ ). Similarly,  $pd - dp = 0$ . So  $c$  and  $d$  are both elements of  $\text{cr}(p)$ .

Lemma 2.5. Let  $p \in P$ ,  $p \neq 0$ ,  $P = A *a B = C \times D$ ,  $C \neq 0 \neq D$ . Then the centralizer of  $p$ ,  $\text{cr}(p)$ , is not a polynomial ring in one variable.

Proof

By Lemma 2.4  $c$  and  $d$  are both elements of the centralizer of  $p = c + d$ . Suppose  $\text{cr}(p)$  is a polynomial ring in one variable, say  $F[q]$ . Then  $c = f(q)$  and  $d = h(q)$ ,  $f(q), h(q) \in F[q]$ . If  $c \neq 0 \neq d$  then  $0 = cd = f(q)h(q)$  since  $c$  and  $d$  occur in different direct factors. But this contradicts  $F[q]$  an integral domain. So  $\text{cr}(p)$  is not a polynomial ring in one variable.

But clearly Lemma 2.5 contradicts Theorem 2.2. So the stated decomposition of  $P = A *a B$  is not possible, that is,  $P = A *a B$  is directly indecomposable.

## CHAPTER 3

### FREE PRODUCTS

#### Section 1. Introduction

We will first present the following comments on free products of Lie algebras.

Suppose  $A$  and  $B$  are Lie algebras. Let  $X$  and  $Y$  be the ordered bases for  $A$  and  $B$  respectively. Let  $U = U(A)$  and  $V = U(B)$  be universal enveloping algebras of  $A$  and  $B$  respectively. Form  $Q = U *_{\mathfrak{a}} V$ , the free product of  $U$  and  $V$ . We denote the free product of Lie algebras  $A$  and  $B$  by:  $A *_L B$ .

Definition 3.1.  $C$  is said to be the free product of a family of Lie algebras  $\{C_j | j \in J\}$  iff

$$(i) \quad C = L_{\mathfrak{a}}(C_j | j \in J);$$

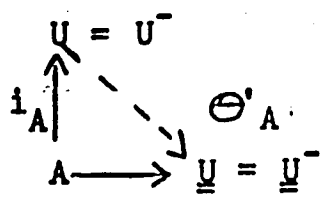
(ii) for every Lie algebra  $C'$  and every family of algebra homomorphisms  $\alpha_j: C_j \longrightarrow C'$ , there exists an algebra homomorphism  $\alpha$  such that

$$\alpha: C \longrightarrow C' \text{ extends the } \alpha_j \text{ (} j \in J \text{)}.$$

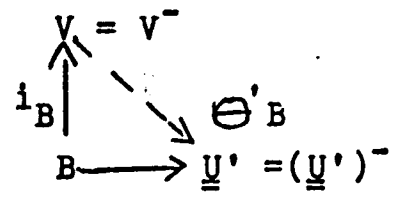
Lemma 3.2.  $L_{\mathfrak{a}}(A \cup B) = A *_L B$ .

#### Proof

We will investigate  $L_{\mathfrak{a}}(A \cup B)$ , the Lie algebra generated by  $A \cup B$ , in  $Q = U *_{\mathfrak{a}} V$ . From the definition of the universal enveloping algebra, we have that the following diagrams are commutative:



and



where  $\underline{U}$  and  $\underline{U}'$  are arbitrary associative algebras and  $\Theta_A$  and  $\Theta_B$  are homomorphisms such that

$$\Theta_A : A \longrightarrow \underline{U}' \quad \text{and} \quad \Theta_B : B \longrightarrow (\underline{U}')^- .$$

In order for the definition of  $P = L_a(A \cup B)$  to coincide with the definition of other free products, we need to show the existence of an algebra homomorphism  $\gamma$  where  $\gamma$  is as follows:

- (1)  $\left\{ \begin{array}{l} \text{if } L \text{ is any Lie algebra and if } \alpha : A \longrightarrow L \\ \text{and } \beta : B \longrightarrow L \text{ then there exists a homomor-} \\ \text{phism } \gamma : P \longrightarrow L \text{ which extends } \alpha \text{ and } \beta . \end{array} \right.$

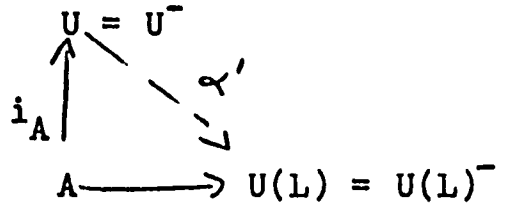
Consider  $U(L)$ , the universal enveloping algebra of  $L$ , where  $L$  is a Lie algebra such that there exists homomorphisms  $\alpha$  and  $\beta$  where

$$\alpha : A \longrightarrow L \quad \text{and} \quad \beta : B \longrightarrow L .$$

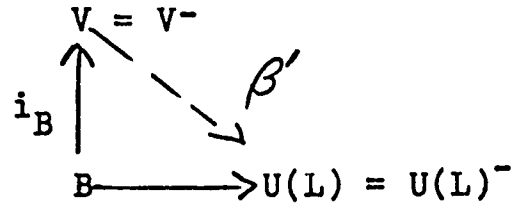
Then, since  $L \leq U(L)^-$ ,  $\alpha$  can be extended to  $U(L)^-$ .

By an abuse of language, we call this extension  $\alpha$ , that is,  $\alpha : A \longrightarrow U(L)^-$ . Similarly,  $\beta : B \longrightarrow U(L)^-$ .

So, there exist  $\alpha'$ , such that  $\alpha' : U \longrightarrow U(L)$  and  $\beta'$  such that  $\beta' : V \longrightarrow U(L)$  where



and



are commutative.

Since we have the existence of homomorphisms

$\alpha'$  and  $\beta'$  such that

$$\alpha': U \longrightarrow U(L) \quad \text{and} \quad \beta': V \longrightarrow U(L) ,$$

the definition of the free product  $U *a V$  gives us the existence of a homomorphism  $\gamma'$  which extends  $\alpha'$  and  $\beta'$  such that

$$\gamma': U *a V \longrightarrow U(L)$$

(the universal mapping property for free products). Now let the desired  $\gamma$  of (1) be the restriction of  $\gamma'$  to  $L_a(A \cup B)$  in  $U *a V$ . So  $L_a(A \cup B)$  has the universal mapping property, and we now write

$$L_a(A \cup B) = A *_L B = P .$$

Lemma 3.3. If  $A$  and  $B$  are Lie algebras, then

$$U(A *_L B) = U(A) *a U(B) .$$

Proof

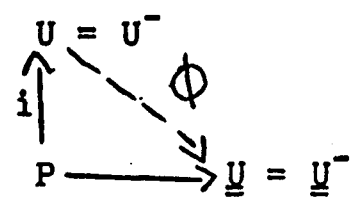
We first recall the definition of the universal enveloping algebra of  $P = A *_L B$ .

A universal enveloping algebra of  $P = A *_L B$  is a pair  $(U, i)$  where  $U$  is an associative algebra with 1, over  $F$ ,  $i: P \longrightarrow U$  is a linear map satisfying

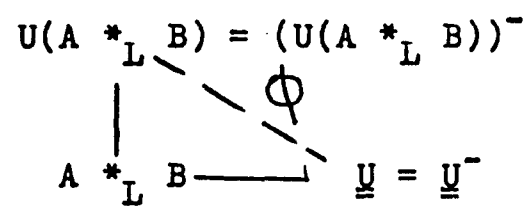
$$(2) \quad i([x, y]) = i(x)i(y) - i(y)i(x)$$

for  $x, y \in P$ , and the following hold:

for any associative  $F$ -algebra  $\underline{U}$  with 1, and any linear map  $j: P \longrightarrow \underline{U}$  satisfying (2), there exists a unique algebra homomorphism  $\phi: U \longrightarrow \underline{U}$  (sending  $1_U$  to  $1_{\underline{U}}$ ) such that  $\phi \circ i = j$ , that is



is commutative. Equivalently,



is commutative.

By the definition of the universal enveloping algebras of  $A$  and  $B$ , there exists injections  $i_A$  and  $i_B$  such that

$$i_A : A \longrightarrow U(A)^- \quad \text{and} \quad i_B : B \longrightarrow U(B)^- .$$

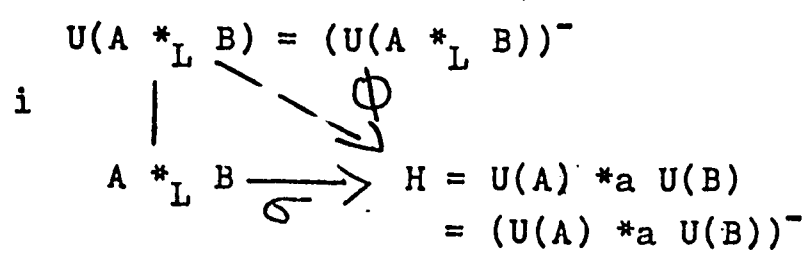
Now form  $U(A) *_{\mathfrak{a}} U(B) = H$ . Clearly,  $U(A)^-$  is a subalgebra of  $H^-$  and  $U(B)^-$  is a subalgebra of  $H^-$ , so  $i_A$  can be extended to an injection from  $A$  into  $H^-$ . Similarly,  $i_B$  can be extended to an injection from  $B$  into  $H^-$ . By an abuse of language, call these extensions  $i_A$  and  $i_B$  respectively.

So  $i_A : A \longrightarrow H^-$  and  $i_B : B \longrightarrow H^-$  can be extended to an injection  $\sigma$  such that

$$\sigma : A *_L B \longrightarrow H^-$$

by the universal mapping property of free products.

Finally we have that there exists  $\phi$  where



and this diagram is commutative.

On the other hand the inclusion of  $A$  into  $P$ ,

$$A \hookrightarrow P = A *_L B$$

yields by the Poincaré-Birkhoff-Witt Theorem an inclusion  $f$  of  $U(A)$  into  $U(P) = U(A *_L B)$ ,

$$f: U(A) \hookrightarrow U(P) = U(A *_L B)$$

and similarly an inclusion  $g$  of  $U(B)$  into  $U(P)$

$$g: U(B) \hookrightarrow U(P) = U(A *_L B) .$$

Hence by the universal mapping property,  $f$  and  $g$  extend to a homomorphism  $\varphi$  of  $Q$  into  $U(P)$ ,

$$\varphi: Q \longrightarrow U(P) = U(A *_L B) .$$

Notice that  $\theta\varphi$  and  $\varphi\theta$  are the identity on  $A$  and  $B$  and hence on  $P$  and  $Q$  respectively. Hence

$$U(P) \cong Q$$

as required.

Let

$$X = \{e_i^A \mid i \in I\} \quad \text{and} \quad Y = \{e_i^B \mid i \in I\}$$

be bases for  $A$  and  $B$  respectively. Let  $\bar{X}$  and  $\bar{Y}$  be the respective bases for  $U(A)$  and  $U(B)$  (see Poincaré-Birkhoff-Witt Theorem 0.8). The basis for  $Q = U(A) *_L U(B)$  is given in Chapter 2, Section 1. If  $0 \neq p \in P = A *_L B$  then it is clear that  $p$  may be thought of as an

element of  $Q$  (actually  $Q^-$ ). Writing  $p$  in its homogeneous weight form (see (2), Chapter 2, Section 1) we have

$$p = p(0) + p(1) + \dots + p(n) \quad (\text{wt}(p)=n)$$

where  $p(i)$  is a sum of straight monomials of weight  $i$  or zero and  $p(0) \in F$ . For each monomial

$$u = w_1 w_2 \dots w_d \quad (\text{wt}(u)=d)$$

in  $p(d)$  where each  $w \in \{\bar{X}, \bar{Y}\}$ , each  $w$  has the form

$$w = e_{j_1}^A \dots e_{j_s}^A \quad \text{or} \quad w = e_{i_1}^B \dots e_{i_t}^B$$

where  $e^A$ 's  $\in X$  and  $e^B$ 's  $\in Y$ , respective bases for  $A$  and  $B$ . So  $u = w_1 w_2 \dots w_d$  may be viewed as

$$(3) \quad u = e_{i_1}^A \dots e_{i_{r_1}}^A \cdot e_{j_1}^B \dots e_{j_{r_2}}^B \dots e_{\ell_1}^B \dots e_{\ell_{r_d}}^B$$

where weight  $u = d$ . From (3) we define the microweight of a monomial  $u$  to be  $r = \sum_{i=1}^d r_i$ . Notice that every element  $p \in P$  ( $p \neq 0$ ) can be expressed uniquely in the form

$$(4) \quad p = p_0 + p_1 + \dots + p_m$$

where  $p_0 \in F$  and each  $p_i$  is either a sum of monomials of microweight  $i$  or zero. We term  $p_i$  the homogeneous component of  $p$  of microweight  $i$ . If  $p_m \neq 0$  then  $m$  is said to be the microweight of  $p$  (denoted  $\text{microwt}(p)=m$ ).

Notice that

$$p = p(0) + p(1) + \dots + p(n) \quad (\text{wt}(p)=n)$$

(homogeneous weight form)

$$= p_0 + p_1 + \dots + p_m \quad (\text{microwt}(p)=m)$$

(homogeneous microweight form)

and  $m \geq n$  . Clearly then,

$$\text{wt}(p) = n \text{ does not imply } \text{microwt}(p) = n$$

(in general).

Also if  $p$  and  $q \in A *_L B \leq Q^-$  , and  $\text{microwt}(p) = m$  ( $m > 0$ ) and  $\text{microwt}(q) = n$  ( $n > 0$ ) , then

$$\text{microwt}(pq) = m + n$$

where this product,  $pq$  , takes place in  $Q$  .

Section 2. The wreath product of Lie algebras is freely indecomposable.

Let  $L = A \text{ wr } T$  be the wreath product of an abelian Lie algebra  $A$  by an arbitrary Lie algebra  $T$  . Suppose it were possible to decompose  $L$  into a (non-trivial) free product of Lie algebras, that is suppose

$$A \text{ wr } T = C *_L D$$

where  $*_L$  indicates that this is a free product of Lie algebras. Let  $X$  be an ordered basis for  $C$  ,  $Y$  an ordered basis for  $D$  . Let  $V$  be the associative subalgebra of  $U(L)$  generated (in  $U(L)$ ) by  $C$  and  $W$  be the associative subalgebra of  $U(L)$  generated by  $D$  .

Then

Lemma 3.4.  $U(L) = V *_a W$  .

Lemma 3.5. Suppose that the Lie algebra  $L$  is a non-trivial free product, that is

$$L = C *_L D \quad (C \neq 0 \neq D) .$$

Then every abelian ideal of  $L$  is zero.

Proof

Let  $B$  be a non-trivial abelian ideal of  $L$ .

Notice that if either  $x \in C$  or  $y \in D$  then

$$(5) \quad [b, [b, x]] = 0 \quad \text{and} \quad [b, [b, y]] = 0$$

for every  $b \in B$ . In particular for  $x \in X$ ,  $b \neq 0$ ,

$$\begin{aligned} 0 &= [b, [b, x]] \\ &= [b, (bx - xb)] \\ &= b(bx - xb) - (bx - xb)b \\ &= b^2x - 2bxb + xb^2 \\ &= b^2x + (xb - 2bx)b \end{aligned}$$

Hence

$$(6) \quad b^2x = (2bx - xb)b$$

Since  $U(L) = U(C) * U(D)$ , we can avail ourselves of the usual description of the elements of  $U(L)$  in terms of a basis of  $U(C)$  and  $U(D)$ . We now write

$$b = \sum \alpha w_1 \dots w_r$$

as the following sum:

$$(7) \quad b = cx + \sum \delta dx' + \sum \epsilon ey$$

where

- (i) we take all terms in  $v$  ending in  $x$ , factor  $x$  out of each term and call the sum of what remains,  $c$ ;
- (ii) then we consider the sum of all terms ending in other elements  $x' \in X - \{x\}$ , factor  $x'$  out of the term and call what remains  $d$ , where  $\delta$  is the coefficient associated with each of the terms; and finally

(iii) we consider all terms in  $b$  ending in the  $y$ 's and do as in (ii), ( $y \in Y$ ).

We assume that  $\sum_{e \in Y} ey \neq 0$ . Notice that if  $b$  had no terms ending in  $y$ , then all terms must end in one of the elements  $x'$  (or  $x$ ). So instead of considering  $[b, [b, x]]$  we would have examined  $[b, [b, y]]$ , and the argument would proceed in the same manner.

Substituting (7) in rhs(6) we may now write rhs(6) as follows:

$$\begin{aligned} (2bx-xb)(cx + \sum \delta dx' + \sum_{e \in Y} ey) &= \\ (2bx-xb)(cx) + (2bx-xb)(\sum \delta dx') + (2bx-xb)(\sum_{e \in Y} ey) &. \end{aligned}$$

Consider the term

$$(8) \quad (2bx-xb)(\sum_{e \in Y} ey) .$$

Clearly all of these terms will end in some  $y$ . But there are no terms ending in  $y$  on lhs(6), so

$$(2bx-xb)(\sum_{e \in Y} ey) = 0 .$$

So, since  $U(L)$  is an integral domain, one of  $(2bx-xb)$  and  $(\sum_{e \in Y} ey)$  must be zero. But  $\sum_{e \in Y} ey \neq 0$  by hypothesis. If  $2bx-xb = 0$  then  $\text{rhs}(6) = 0$ . This would imply that  $\text{lhs}(6) = 0$ , that is,  $b^2x = 0$ . Since  $x \neq 0$ , it follows that  $b = 0$  as desired. Hence  $B = 0$ .

This yields in particular the

Theorem 3.6.

If

$$L = A \text{ wr } T \quad (A \neq 0 \neq T)$$

is the wreath product of the abelian Lie algebra  $A$  with

$T$  , then  $L$  is freely indecomposable.

Section 3. The free product of Lie algebras is directly indecomposable.

Let  $P = A * B \leq (U(A) * U(B))^- = (U(P))^-$  .

Let  $p = [e, f] \in P$  where  $e \in X$  ,  $f \in Y$  , and  $X$  and  $Y$  are the respective ordered bases of  $A$  and  $B$  . (So  $[e, f] \neq 0$ ). Recall that by the definition of the universal enveloping algebra,  $P$  sits inside of  $U(P)^-$  and we can identify  $P$  with its image in  $U(P)^-$  . In fact

$$[e, f] = ef - fe$$

in  $U(P)^-$  .

Lemma 3.7.  $C = \text{cr}(ef - fe) = F[(ef - fe)]$  .

Proof

Let  $u \in C = \text{cr}(ef - fe)$  ( $u \neq 0$ ) . We will prove that  $u \in F[(ef - fe)]$  by induction on the microweight of  $u$  .

Let  $u \in C$  and  $\text{microweight}(u) \leq 1$  . Then by (4), Section 1,  $u$  may be written in its homogeneous microweight form as follows:

$$u = u_0 + u_1 .$$

Since  $u$  is a centralizer of  $(ef - fe)$  , ( $u \neq 0$ ) , then

$$(9) \quad u(ef - fe) = (ef - fe)u .$$

If  $u_1 \neq 0$  then

$$u_1 = \sum \alpha x + \sum \beta y$$

(where  $x \in X$  ,  $y \in Y$  ,  $X$  and  $Y$  are the respective ordered bases of  $A$  and  $B$  ). Consider the terms of

highest microweight on the lhs(9) and the rhs(9):

$$\text{lhs}(9) = \sum \alpha xef - \sum \alpha xfe + \sum \beta yef - \sum \beta yfe$$

$$\text{rhs}(9) = \sum \alpha efx - \sum \alpha fex + \sum \beta efy - \sum \beta fey$$

( $\alpha, \beta \in F$ ). It is clear then that we must have

$$-\sum \alpha xfe = \sum \alpha efx \quad .$$

That is, there exists  $\alpha_1$  and  $\alpha_2$  such that

$$-\alpha_1 x_1 fe = \alpha_2 efx_2$$

and so

$$-\alpha_1 = \alpha_2 \quad \text{and} \quad x_1 = e = x_2 \quad .$$

So

$$(10) \quad x_i = e \quad \text{and} \quad -\alpha_i = \alpha_i$$

for all  $i$  and (10) gives us that all  $\alpha = 0$  .

Similarly  $\sum \beta yef = -\sum \beta fey$  . leads us to the conclusion that all  $\beta$  are zero.

Therefore  $u = u_0 \in F$  and  $u \in C$  is trivially a polynomial in  $F[(ef - fe)]$  . So the lemma is proved for  $\text{microwt}(u) \leq 1$  .

Let us also consider the case when  $u \in C$  has microweight two. That is

$$u = u_0 + u_1 + u_2 \quad , \quad (u_2 \neq 0) \quad .$$

We want to show that  $u$  is a polynomial in  $(ef - fe)$  , and, in fact,  $u_2 = \gamma(ef - fe)$  ( $\gamma \in F$ ) . Since

$u \in C$  then

$$(11) \quad u(ef - fe) = (ef - fe)u \quad .$$

Consider  $u_2$  , the term of highest microweight in  $u$  :

$$(12) \quad u_2 = \sum \alpha w e + \sum \alpha' w' e' + \sum \beta v f + \sum \beta' v' f'$$

where  $w, w', v, v'$  are all of microweight one, that

is, are equal to  $e$ ,  $f$  or one of the elements  $e'$  in  $X - \{e\}$  or  $f'$  in  $Y - \{f\}$ . It is clear that we have taken all of the terms in  $u_2$  and separated them into four groups: first, those terms ending in  $e$ ; second, those terms ending in the elements of  $X - \{e\}$ , denoted  $e'$ ; third, those terms ending in  $f$ ; and last, those terms ending in elements from  $Y - \{f\}$ , denoted  $f'$ . It should be pointed out that the monomials were straight (see Definition 1.22) before  $e$ ,  $e'$ ,  $f$  and  $f'$  were factored out. Hence the parts remaining are straight, that is,  $v$ ,  $v'$ ,  $w$  and  $w'$  are straight monomials.

Consider the terms of highest microweight in (11):

$$(13) \left\{ \begin{array}{l} \tau_{\alpha} w e e f - \tau_{\alpha} w e f e + \tau_{\alpha'} w' e' e f - \tau_{\alpha'} w' e' f e + \\ \tau_{\beta} v f e f - \tau_{\beta} v f f e + \tau_{\beta'} v' f' e f - \tau_{\beta'} v' f' f e = \\ \tau_{\alpha} e f w e - \tau_{\alpha} f e w e + \tau_{\alpha'} e f w' e' - \tau_{\alpha'} f e w' e' + \\ \tau_{\beta} e f v f - \tau_{\beta} f e v f + \tau_{\beta'} e f v' f' - \tau_{\beta'} f e v' f' \end{array} \right. .$$

So

$$\tau_{\alpha'} e f w' e' = 0 = -\tau_{\alpha'} f e w' e'$$

since no terms on the lhs(13) end in  $e'$  ( $e \neq e'$ ). (By a similar argument we find that no terms end in  $f'$ ). Hence

$$(14) \quad \alpha' = 0 = \beta'$$

for all  $\alpha'$  and  $\beta'$ .

We may now rewrite (13) as follows:

$$(15) \left\{ \begin{array}{l} \tau_{\alpha} w e e f \underset{(i)}{-} \tau_{\alpha} w e f e \underset{(ii)}{+} \tau_{\beta} v f e f \underset{(iii)}{-} \tau_{\beta} v f f e \underset{(iv)}{=} \\ \tau_{\alpha} e f w e \underset{(v)}{-} \tau_{\alpha} f e w e \underset{(vi)}{+} \tau_{\beta} e f v f \underset{(vii)}{-} \tau_{\beta} f e v f \underset{(viii)}{=} \end{array} \right. .$$

Also we have from (14) that

$$(16) \quad u_2 = \sum \alpha w e + \sum \beta v f .$$

Suppose that there exists some  $w_1$  such that

$w_1 = e$  ( $\text{microwt}(w_1) = 1$ ). Then from (15(i)) we would have

$$\alpha_1 w_1 e e f = \alpha_1 e e e f .$$

But from a quick observation of the rhs(15) it is immediate this term has no match. Hence it must be that  $\alpha_1 = 0$ , that is, all  $w$  must equal  $f$ . (Clearly  $e'$  and  $f'$  are ruled out as choices since no terms on the rhs(15) can possibly begin with  $e'$  or  $f'$ .)

Suppose there exists some  $v_1$  such that  $v_1 = f$ . Then from (15(iii)) we have that

$$\beta_1 v_1 f e f = \beta_1 f f e f .$$

But it is at once clear that no terms on the rhs(15) can possibly begin with  $ff$ . Therefore all  $v$  must equal  $e$ . So we may write  $u_2$  as

$$(17) \quad u_2 = \sum \alpha w e + \sum \beta v f = \alpha f e + \beta e f .$$

Rewriting (15) we are led to the equation

$$(18) \quad \left\{ \begin{array}{l} \alpha f e e f - \alpha f e f e + \beta e f e f - \beta e f f e = \\ \alpha e f f e - \alpha f e f e + \beta e f e f - \beta f e e f . \end{array} \right.$$

Clearly we have

$$\alpha f e e f = -\beta f e e f$$

as our only choice. Hence

$$(19) \quad \alpha = -\beta$$

and  $u_2 = \beta(e f - f e)$  as desired. Notice that

$$(\beta(e f - f e))(e f - f e) = (e f - f e)(\beta(e f - f e)) .$$

So  $u_2 \in \text{cr}(ef - fe)$  . Since  $u \in \text{cr}(ef - fe)$  then

$$z = u - u_2 \in \text{cr}(ef - fe) \text{ . But } z = u - u_2 = u_0 + u_1 \text{ .}$$

So  $\text{microwt}(z) \leq 1$  . But the lemma has been proved for centralizers of  $\text{microwt} \leq 1$  , and in fact, if  $\text{microwt}$

$(z) \leq 1$  then  $z = u_0 \in F$  . So  $u = u_0 + \beta(ef - fe)$  ,

$u_0 \in F$  . Therefore if  $u \in \text{cr}(ef - fe)$  and  $\text{microwt}(u) \leq 2$  , then  $u$  is a polynomial in  $(ef - fe)$  .

We now suppose that  $\text{microwt}(u) = n > 2$  , and that if  $v \in C$  ,  $\text{microwt}(v) < n$  then  $v$  is a polynomial in  $F[(ef - fe)]$  .

Consider  $u = u_0 + u_1 + \dots + u_n$  .  $u \in C$  implies that

$$(20) \quad u(ef - fe) = (ef - fe)u \text{ .}$$

Consider  $u_n$  , the term of  $u$  of highest microweight:

$$u_n = \sum \alpha w e + \sum \alpha' w' e' + \sum \beta v f + \sum \beta' v' f'$$

where  $w, w', v, v'$  are all of microweight  $n-1$  .

Consider the terms of highest microweight in (20):

$$(21) \quad \begin{cases} \sum \alpha w e e f - \sum \alpha w e f e + \sum \alpha' w' e' e f - \sum \alpha' w' e' f e & + \\ \sum \beta v f e f - \sum \beta v f f e + \sum \beta' v' f' e f - \sum \beta' v' f' f e & = \\ \sum \alpha e f w e - \sum \alpha f e w e + \sum \beta e f v f - \sum \beta f e v f & + \\ \sum \beta' e f v' f' - \sum \beta' f e v' f' + \sum \alpha' e f w' e' - \sum \alpha' f e w' e' & . \end{cases}$$

So

$$(22) \quad \sum \alpha' e f w' e' = 0 = - \sum \alpha' f e w' e'$$

since no terms on rhs(21) end in  $e'$  ( $e \neq e'$ ). But this can only be if all  $\alpha' = 0$  . By a similar argument we

also obtain that all  $\beta' = 0$  . Hence we may write

$$(23) \quad u_n = \sum \alpha w e + \sum \beta v f \quad .$$

Also (21) may be rewritten as

$$(24) \quad \left\{ \begin{array}{l} \sum_{(i)} \alpha w e e f - \sum_{(ii)} \alpha w e f e + \sum_{(iii)} \beta v f e f - \sum_{(iv)} \beta v f f e = \\ \sum_{(v)} \alpha e f w e - \sum_{(vi)} \alpha f e w e + \sum_{(vii)} \beta e f v f - \sum_{(viii)} \beta f e v f \end{array} \right. .$$

It is immediate from a spot inspection of (24) that

w and v can not begin (or end) with e' or f' .

Consider the possible endings for the v's and w's .

Case 1. Suppose there exists some  $w_1$  such that  $w_1$  ends in e . Then from (24(v)) we have that

$$\alpha_{1efw_1e} = \alpha_{1ef} \cdot R \cdot ee$$

(where R represents the rest of the  $n-2$  factors in  $w_1$  ).

But no terms on lhs(24) could possible end in ee . So

it must be that  $\alpha_1 = 0$  . Hence all w's must end in f . .

Case 2. Suppose there exists some  $v_1$  such that  $v_1$  ends in f . The from (24(vii)) we have that

$$\beta_{1efv_1f} = \beta_{1ef} \cdot R \cdot ff \quad .$$

Since no terms on the lhs(24) could possibly end in ff ,

it is immediate that  $\beta_1$  must be zero. So all v's end in e .

We observe from (24) that all v's and w's must begin with ef or fe .

It is a straight forward process to observe that in none of the v's or w's can we have a sequence:

$$\underbrace{ff \dots f}_n \quad \text{or} \quad \underbrace{ee \dots e}_n \quad , \quad n > 2 \quad .$$

Suppose such a sequence did exist, say for some  $w_1$ ,  
 $w_1 = fe \cdot R_1 \cdot f$  where  $R_1$  contains a sequence  
 $ff \dots f$ , ( $n$  factors,  $n > 2$ ). Then the  $w_1$ - term  
 in (24(i))

$$\alpha_1 w_1 e e f = \alpha_1 f e \cdot R_1 \cdot f e e f$$

can only have a match in (24(viii)), that is, there  
 exists some  $v_1$  such that

$$\alpha_1 f e \cdot R_1 \cdot f e e f = \beta_1 f e v_1 f$$

which implies

$$v_1 = R_1 \cdot f e e$$

If the  $ff \dots f$  ( $n$  factors) sequence begins  $R_1$   
 then our contradiction is immediate since  $n > 2$ . That  
 is, considering the  $v_1$ -term of any expression on the  
 lhs(24), gives us immediately that there is no possible  
 match on the rhs(24).

If the  $ff \dots f$  ( $n$  factors,  $n > 2$ ) sequence does  
 not begin  $R_1$ , then we continue comparing terms on both  
 sides of (24). Each time a comparison is made, two  
 factors from the beginning of  $R_1$  are cancelled. Since  
 the sequence contains at least three  $f$ 's, we will  
 eventually get to a point where two  $f$ 's will occur at  
 the beginning of some term. But this is not possible  
 since if it is considered in any term on the lhs(24),  
 it will have no match on the rhs(24).

By a similar process of "term-comparison" we also  
 see that each  $w$  and  $v$  must have odd weight.

It should also be noted that the sequence  $ff$   
 and  $ee$  can only occur in  $R_1$  under certain conditions.

Observe the following. Since  $\text{microwt}(w_1) = 2p$  for some  $p$ , then  $\text{microwt}(R_1)$  is odd. Each time a comparison is made,  $R_1$  is "reduced" on the left by two factors. Hence  $ff$  or  $ee$  can not be so situated in  $R_1$  so as to have the first factor in the sequence in the  $(2k+1)$ st position. If so, then on the  $(2k-1)$ st comparison, the  $ff$  or  $ee$  sequence would appear at the beginning of some  $w$  or  $v$ . But this would be a contradiction, since all terms must begin with  $ef$  or  $fe$  by rhs(24).

We finally have that all  $w$ 's are just strings of  $ef$ 's and  $fe$ 's ending in  $f$  (since  $\text{microwt}(w)$  is odd) and similarly all  $v$ 's are strings of  $ef$ 's and  $fe$ 's ending in  $e$  ( $\text{microwt}(v)$  is odd). So the terms of  $u_n$ ,  $w_\alpha$  and  $v_\beta$ , are strings of  $ef$ 's and  $fe$ 's, where all  $\alpha = \pm \gamma$  and all  $\beta = \pm \delta$ , for one particular  $\gamma \in F$ .

By a straight forward listing of all possible  $w$ 's and  $v$ 's that can occur and all terms occurring in the product:  $(ef - fe)^{n/2}$ , we see that they are in a one-to-one correspondence. To see that all terms have coefficients equal to  $\pm \gamma$ , for some  $\gamma \in F$ , we need only go through the comparison of terms in (24).

An arbitrary  $w_\alpha$  ( $\text{microwt}(w_\alpha) = n = 2p$ ) beginning with  $ef$  may be written as:

$$(25) \quad w_\alpha = ef \dots ef \underbrace{fe}_{p-q_1} \dots fe \underbrace{ef}_{p-q_1+q_2} \dots \underbrace{ef}_p$$

That is, we is viewed as consisting of  $p$  pairs of  $ef$ 's and  $fe$ 's, where  $ef$  occurs as the first through the  $(p-q_1-1)$ st factors,  $fe$  occurs as the  $(p-q_1)$ st through  $(p-q_1+q_2-1)$ st factors, and the remaining factors are  $ef$ 's.

To determine the particular term of  $(ef - fe)^{n/2}$  that the term in (25) corresponds to, notice that

$$(26) \quad (ef - fe)^{n/2} = \underset{(1)}{(ef - fe)} \underset{(2)}{(ef - fe)} \dots \underset{(p)}{(ef - fe)} .$$

To match (25), choose, in (26),  $ef$  from factors (1) through  $(p-q_1-1)$ ; next choose  $fe$  from factors  $(p-q_1)$  through  $(p-q_1+q_2-1)$ ; finally, choose  $ef$  from the remaining factors in (26). Form the product of these factors. Clearly this product is equal to (25).

So we finally have that

$$u_n = (ef - fe)^{n/2} \quad (n = 2p) .$$

Clearly  $u_n \in \text{cr}(ef - fe)$  so  $z = u - u_n \in \text{cr}(ef - fe)$ .

But  $\text{microwt}(z) < n$  so the theorem is true for  $z$ ,

that is,  $z \in F[(ef - fe)]$ . Since  $u_n \in F[(ef - fe)]$

we have that

$$u \in F[(ef - fe)]$$

and Lemma 3.7 is proved.

Since it is clear that  $F[(ef - fe)] \subseteq C$  then

$$C = F[(ef - fe)] .$$

Since any centralizer  $u$  of  $[e, f]$  in  $P$  is a centralizer of  $(ef - fe)$  in  $U(P)^-$ , then  $u$  must be an element of  $F[(ef - fe)]$ , that is

$$u = a_0 + a_1(ef-fe) + a_2(ef-fe)^2 + \dots + a_n(ef-fe)^n$$

where  $a_i \in F$ , the underlying field of  $A$  and  $B$ , hence  $U(P)$ . But, clearly since  $P$  sits monomorphically inside of  $U(P)^-$ , and  $u \in P$ , all of the polynomials must have the form

$$u = a_0 + a_1[e, f] \longmapsto a_0 + a_1(ef - fe)$$

Hence we have shown

Lemma 3.8. The centralizer of  $[e, f]$  in  $P$  is one-dimensional.

We now suppose that  $P = A *_L B = C \times D$  ( $C \neq 0 \neq D$ ), that is,  $P$  can be (non-trivially) decomposed into a direct product. By Lemma 1.18,  $\text{cr}(p) \geq 2$  for any

$p \in P$  ( $p \neq 0$ ). But this clearly contradicts Lemma 3.8. Hence, the stated decomposition is not possible, that is, the free product of Lie algebras is directly indecomposable.

CHAPTER 4

MISCELLANEOUS RESULTS

Section 1.

Lemma 4.1. Let  $W$  be a wreath product of Lie algebras

$A$  and  $T$ , where  $A$  is an abelian Lie algebra and

$T$  is arbitrary. We have that  $W = A \text{ wr } T = B \oplus T$ .

Let  $I \subseteq W$ ,  $I \leq B$ ,  $I \neq 0$ . Then  $I$  is not finite dimensional.

Proof

Suppose  $I$  is finite dimensional. Since  $I \neq 0$  there is a basis with non-zero elements in  $I$ , namely

$$k_1, k_2, \dots, k_n.$$

Since  $k_r \in I \leq B$  then  $k_r = \sum a_i u(i,r)$ , for some

$u(i,r) \in U(T)$ , where the  $\{a_i\}$  constitute a basis for  $B$ . Consider  $[k_r, t] \in I$  ( $I \subseteq W$ ), where  $r$  is fixed.

$$\begin{aligned} [k_r, t] &= [\sum a_i u(i,r), t] \\ &= \sum [a_i u(i,r), t]. \end{aligned}$$

So

$$(1) \quad [k_r, t] = \sum a_i (u(i,r)t).$$

Now  $\sum a_i (u(i,r)t) = \sum \alpha(i,r) k_i$ , so we may write

$$\begin{aligned} \sum \alpha(i,r) k_i &= \alpha(1,r) k_1 + \dots + \alpha(n,r) k_n \\ &= \alpha(1,r) (a_1 u(1,1) + \dots + a_n u(n,1)) \\ &\quad + \dots + \end{aligned}$$

$$\alpha(n,r)(a_1 u(1,n) + \dots + a_n u(n,n)) .$$

So

$$(2) \quad \sum \alpha(i,r) k_i = a_1 (\alpha(1,r) u(1,1) + \dots + \alpha(n,r) u(1,n)) \\ + \dots + \\ a_n (\alpha(1,r) u(n,1) + \dots + \alpha(n,r) u(n,n)) .$$

Equating (1) and (2) we obtain

$$u(1,r)t = \alpha(1,r) u(1,1) + \dots + \alpha(n,r) u(1,n)$$

...

...

$$u(n,r)t = \alpha(1,r) u(n,1) + \dots + \alpha(n,r) u(n,n)$$

So  $u(1,r)t$  is an element of the vector space spanned by

$u(1,1), \dots, u(1,n)$  . Hence,  $u(i,r)t$  is an element of the vector space spanned by  $u(1,1), \dots, u(n,n)$

( $1 \leq i \leq n, 1 \leq r \leq n$ , and all  $t \in T$  ). So there exists

some  $(i,r)$  such that  $u(i,r) \neq 0$  since  $k_r \neq 0$  . Con-

sider the following products which are obtained from (1).

They must also be elements of  $I$  (since  $I \trianglelefteq W$  ) .

$$\begin{aligned} [k_r, t, t] &= [[k_r, t], t] \\ &= [\sum a_i (u(i,r)t), t] \\ &= \sum a_i (u(i,r)t)t . \end{aligned}$$

So

$$[k_r, t, t] = \sum a_i (u(i,r)t^2) .$$

By induction it is clear that

$$[k_r, \underbrace{t, t, \dots, t}_q] = \sum a_i (u(i,r)t^q) .$$

It is easily seen that there are an infinite number of these products. They all must be in  $I$  since  $I$  is an ideal of  $W$  . These  $[k_r, t, t, \dots, t]$  products

are linearly independent. To see this, suppose

$$\alpha_1 [k_r, t] + \dots + \alpha_q \underbrace{[k_r, t, t, \dots, t]}_q = 0 .$$

Then

$$\alpha_1 \sum a_i (u(i, r) t) + \dots + \alpha_q \sum a_i (u(i, r) t^q) = 0 .$$

So

$$a_1 \{ \alpha_1 (u(1, r) t) + \dots + \alpha_q (u(1, r) t^q) \} + \dots + a_n \{ \alpha_1 (u(n, r) t) + \dots + \alpha_q (u(n, r) t^q) \} = 0 .$$

But the  $a_i$  are basis elements, hence

$$\sum \alpha_i (u(1, r) t^i) = \dots = \sum \alpha_i (u(n, r) t^i) = 0 .$$

Therefore  $\alpha_i = 0$  for all  $i = 1, 2, \dots, q$  .

But the  $[k_r, t, t, \dots, t]$  products' linear independence contradicts the finite dimensionality of  $I$  . Hence our assumption that  $I$  is finite dimensional must be false.

## Section 2.

Lemma 4.2. Let  $W$  be a wreath product of Lie algebras

$A$  and  $T$  where  $A$  is an abelian Lie algebra and  $T$  is arbitrary ( $W = A \text{ wr } T = B \oplus T$ ) . Suppose  $I \trianglelefteq W$  ,

$I \neq 0$  and  $I$  is not necessarily in  $B$  . Then  $I$  is not finitely generated.

### Proof

Suppose  $I$  is finitely generated. Then there exists a basis for  $I$  with non-zero elements

$$k_1, k_2, \dots, k_n .$$

Let  $r$  be fixed. Then  $k_r \in I \trianglelefteq W = B \oplus T$  implies that

$$k_r = \sum a_i u(i, r) + \sum \gamma_p t_p$$

( $i = 1, \dots, n; p = 1, \dots, m$ ) where  $u(i,r) \in U(T)$ ,

$\gamma_p \in F$  and the  $\{a_i\}$  constitute a basis for  $B$ .

Now let  $t \neq 0$  then

$$\begin{aligned} [k_r, t] &= [\sum a_i u(i,r) + \sum \gamma_p t_p, t] \\ &= [\sum a_i u(i,r), t] + [\sum \gamma_p t_p, t] \\ &= \sum a_i (u(i,r)t) + \sum \gamma_p [t_p, t] \end{aligned}$$

so

$$[k_r, t] = \sum a_i (u(i,r)t) + \sum \gamma_p t'_p$$

( $t'_p \in T$ ). Also,

$$\begin{aligned} [k_r, t, t] &= [\sum a_i (u(i,r)t) + \sum \gamma_p t'_p, t] \\ &= [\sum a_i (u(i,r)t, t) + [\sum \gamma_p t'_p, t]]. \end{aligned}$$

So

$$[k_r, t, t] = \sum a_i (u(i,r)t^2) + \sum \gamma_p t''_p$$

( $t''_p \in T$ ). By induction we obtain

$$[k_r, \underbrace{t, t, \dots, t}_q] = \sum a_i (u(i,r)t^q) + \sum \gamma_p s_p$$

( $s_p \in T$ ).

To see that all the  $[k_r, t, t, \dots, t]$  products are linearly independent, consider the following. Suppose

$$0 = \alpha_1 [k_r, t] + \alpha_2 [k_r, t, t] + \dots + \alpha_q [k_r, \underbrace{t, \dots, t}_q].$$

Then

$$\begin{aligned} \alpha_1 \sum a_i (u(i,r)t) + \dots + \alpha_q \sum a_i (u(i,r)t^q) \\ + \sum \gamma_p t_p = 0. \end{aligned}$$

Therefore

$$\begin{aligned} a_1 \{ \alpha_1 (u(1,r)t) + \dots + \alpha_q (u(1,r)t^q) \} + \dots + \\ a_n \{ \alpha_1 (u(n,r)t) + \dots + \alpha_q (u(n,r)t^q) \} + \sum \gamma_p t_p = 0. \end{aligned}$$

So

$$(3) \left\{ a_1 \left\{ \sum \alpha_j(u(1,r)t^j \right\} + \dots + a_n \left\{ \sum \alpha_j(u(n,r)t^j \right\} \right. \\ \left. = - \sum \gamma_p t^p \right. .$$

Since the lhs of (3) is in  $B$  and the rhs of (3) is in  $T$  then

$$a_1 \left\{ \sum \alpha_j(u(1,r)t^j \right\} + \dots + a_n \left\{ \sum \alpha_j(u(n,r)t^j \right\} = 0$$

(  $B \cap T = 0$  ). Since the  $a_i$  are all basis elements of  $B$  this means that  $\sum \alpha_j(u(i,r)t^j) = \sum \alpha_j(u(i_0,r)t^j)$ ,  $\dots$ ,  $n$ . Consider  $i = i_0$ , then  $\sum \alpha_j(u(i_0,r)t^j) = 0$ . Suppose that  $\alpha_q \neq 0$ , then

$$\alpha_1(u(i_0,r)t) + \dots + \alpha_q(u(i_0,r)t^q) = 0$$

implies that

$$\alpha_q(u(i_0,r)t^q) = - \left\{ \alpha_1(u(i_0,r)t) + \dots + \alpha_{q-1}(u(i_0,r)t^{q-1}) \right\}$$

so

$$u(i_0,r)t^q = -(1/\alpha_q) \left\{ \alpha_1(u(i_0,r)t) + \dots + \alpha_{q-1}(u(i_0,r)t^{q-1}) \right\} .$$

But  $t^q$  can not equal a sum of powers of  $t$  less than

$q$ . Hence  $\alpha_q$  can not be non-zero. So all  $\alpha_i = 0$  and the  $(u(i,r)t^j)$  are linearly independent for all

$i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, q$ .

There are an infinite number of these  $[k_r, t, \dots, t]$  products, all linearly independent, all contained in  $I$ .

But this can not be so since it was assumed  $I$  was finitely generated. Hence the assumption that  $I$  is finitely generated must be false.

Section 3.Theorem 4.3.

Let  $H$  be a finitely generated free Lie algebra and  $A \trianglelefteq H$ . If  $H/A$  is finitely presented then

$A/[A, H]$  is a finite dimensional algebra.

This theorem will be proved in two steps. Following some preliminary statements, we will show in Lemma 4.4 that  $A$  can be realized as a union of specifically constructed vector spaces. Lemma 4.5 will show a similar result for  $[A, H]$ . By observing the factor algebra of these two unions, the finite dimensionality of  $A/[A, H]$  will be clear. But first the preliminary statements.

Let  $H$  be a finitely generated free algebra. Then  $H$  can be presented as

$$H = (x_1, x_2, \dots, x_n) .$$

Since  $A \trianglelefteq H$  then  $[A, H] \subseteq A$ , where

$$[A, H] \stackrel{\text{def.}}{=} L_a([a, h] \mid a \in A, h \in H) .$$

$H/A$  can be presented as follows:

$$H/A = (x_1, x_2, \dots, x_n; A) .$$

Applying Neumann's Theorem (see Theorem 0.10) in the case of Lie algebras, we have that there exists a finite set

$$\{a_1, \dots, a_k\} \subseteq A \text{ such that } A = \text{id}(a_1, \dots, a_k) .$$

Consider the following vector spaces:

$$\begin{aligned} A_1 &= \text{vector space spanned by } \{a_1, \dots, a_k\} \\ &= \text{sp}\{a_1, \dots, a_k\} . \end{aligned}$$

$$A_2 = \text{sp}\{[a_i, h'] \mid i = 1, \dots, k; h' \in H\} .$$

$$A_3 = \text{sp}\{[x, h''] \mid x \in A_2, h'' \in H\} \cup A_2 .$$

Continuing this process, we obtain:

$$A_{N+1} = \text{sp} \{ [y, h] \mid y \in A_N, h \in H \} \cup A_N .$$

Then  $\sum \gamma_N \dots \gamma_1 [ \dots [ [ [a_i, h'] , h'' ] , \dots ] , h^N ]$

is an element of  $A_{N+1}$  .

Lemma 4.4.  $A = \bigcup A_i$  ( $i = 1, \dots, \infty$ ) where the  $A_i$  are defined above.

Proof

Clearly  $\bigcup A_i \subseteq A$  . To see this, suppose that

$b \in \bigcup A_i$  . Then there exists some  $j$  such that

$b \in A_{j+1}$  . So

$$b = \sum \gamma_j \dots \gamma_1 [ \dots [ [a_i, h'] , h'' ] , \dots h^j ] .$$

Now  $[a_i, h'] \in A$  since  $A \supseteq H$  , hence  $[ [a_i, h'] , h'' ] \in$

$A$  , and so on, which implies that  $b \in A$  . That is,

$$\bigcup A_i \subseteq A .$$

To see that  $A \subseteq \bigcup A_i$  , we first observe that

$\bigcup A_i$  is a vector space (by definition) and it contains

$\{a_1, \dots, a_k\}$  . We need only show that  $\bigcup A_i$  is an ideal of  $H$  .

Consider  $[b, h]$  ,  $b \in \bigcup A_i$  ,  $h \in H$  . If

$b \in \bigcup A_i$  then there exists some  $j$  such that

$$b \in A_{j+1} = \text{sp} \{ [y, h] \mid y \in A_j, h \in H \} \cup A_j .$$

So either

$$(i) \quad b = \sum \gamma_j \dots \gamma_1 [ \dots [ [a_i, h'] , h'' ] , \dots ] , h^j ] .$$

If so then

$$[b, h] = \sum \gamma_j \dots \gamma_1 [ [ \dots [a_i, h'] , \dots ] h^j ] , h ] .$$

So

$$[b, h] \in A_{j+2} \subseteq \bigcup A_i ;$$

or

$$(ii) b = \sum \gamma_{j-1} \dots \gamma_1 [[\dots [a_i, h'], \dots], h^{j-1}] .$$

Then

$$[b, h] = \sum \gamma_{j-1} \dots \gamma_1 [[\dots [a_i, h'], \dots], h^{j-1}] .$$

So  $[b, h] \in A_{j+1} \subseteq \cup A_i$  . Hence  $\cup A_i \trianglelefteq H$  and

$$\{a_1, \dots, a_k\} \subseteq \cup A_i \text{ imply that } A \leq \cup A_i .$$

So

$$A = \cup A_i .$$

Lemma 4.5.  $[A, H] = \cup A_i$  .

Proof

First we observe that  $[A, H] \leq \cup A_i$  . If

$b \in \cup A_i$  then there exists some  $j$  such that  $b \in$

$A_{j+1}$  , that is

$$b = \sum \gamma_j \dots \gamma_1 [[\dots [a_i, h'], \dots], h^j] .$$

Clearly,  $b \in L_a \{[a, h] \mid a \in A, h \in H\} = [A, H]$  since

$$[a, h'] \in A , A \trianglelefteq H , \text{ so } [[a, h'], h''] \in [A, H] ,$$

and so on. So  $[A, H] \leq \cup A_i$  .

Conversely, if  $x \in [A, H]$  , then  $x = \sum \beta_j [a, h]_j$  .

Now  $a \in A = \text{id}(a_1, \dots, a_k)$  implies that  $a = \sum \alpha_i a_i$  .

Then

$$\begin{aligned} x &= \sum \beta_j [\sum \alpha_i a_i, h]_j \\ &= \sum \sum \beta_j \alpha_i [a_i, h]_j . \end{aligned}$$

Hence  $x \in A_2 \subseteq \cup A_i$  . So  $[A, H] \leq \cup A_i$  . There-

fore

$$[A, H] = \cup A_i$$

and the lemma is proved.

Now consider  $A/[A, H]$  . It is clear that

$[A, H] \trianglelefteq A$  since  $[[a, h], b] \in [A, H]$  for every  $[a, h] \in [A, H]$  and  $b \in A$  ( $[A, H] \subseteq A$  and  $A \trianglelefteq H$ ). Now

$$A/[A, H] = \bigcup_{i=1}^{\infty} A_i / \bigcup_{i=2}^{\infty} A_i \cong A_1 = \text{sp} \{a_1, \dots, a_k\} .$$

Then  $A_1$  contains a finite basis since it is spanned by a finite number of linearly independent elements. Hence

$A/[A, H]$  is a finite dimensional algebra.

#### Section 4.

Lemma 4.6. An extension of a finitely presented Lie algebra by a finitely presented Lie algebra is finitely presented.

#### Proof

Let  $L$  be a Lie algebra and  $J \trianglelefteq L$ . Suppose  $J$  is finitely presented. Then there exists

$$a_1, a_2, \dots, a_r \in L$$

which generate  $J$  and

$$h_1(a) = h_2(a) = \dots = h_{\rho}(a) = 0$$

defining relations of  $J$ . Suppose also that  $L/J$  is finitely presented. Then there exists

$$\bar{b}_1, \bar{b}_2, \dots, \bar{b}_s \in L/J$$

which generate  $L/J$  and

$$k_1(\bar{b}) = k_2(\bar{b}) = \dots = k_{\sigma}(\bar{b}) = 0$$

defining relations of  $L/J$ .

In each coset  $\bar{b}_i$  choose an element  $b_i$ . Then  $k_{\alpha}(b) = f_{\alpha}(a)$  ( $\alpha = 1, 2, \dots, \sigma$ ), that is  $k_{\alpha}(b)$  can be written as a word in the  $a$ 's. To see this observe the following. If  $b_i \in \bar{b}_i = q_i + J$ ,

$q_i \in L$ , then  $b_i = q_i + j$  for some  $j \in J$ . Now

$$k_\alpha(\bar{b}_i) = k_\alpha(q_i + J) = k_\alpha(q_i) + J = J$$

implies that  $k_\alpha(q_i) \in J$ . So if  $b_i = q_i + j$  then

$$k_\alpha(b_i) = k_\alpha(q_i) + k_\alpha(j). \text{ But } k_\alpha(q_i) \in J \text{ and}$$

$k_\alpha(j) \in J$  imply that  $k_\alpha(b_i) \in J$ . Hence,  $k_\alpha(b_i)$  is a word in the  $a$ 's.

Also,  $[b_i, a_j] = g(i, j)(a)$ , ( $i = 1, \dots, s$ ;

$j = 1, \dots, r$ ), products can be written as words in the  $a$ 's since  $J$  is an ideal.

To see that these  $a$ 's and  $b$ 's generate  $L$ , observe the following: Let  $x \in L$ ,  $J \subseteq L$ . Consider

$$x + J \in L/J \text{ where } L/J \text{ is generated by the } \bar{b}_i \text{'s.}$$

Then

$$x + J = \epsilon_1 b_1 + J + \dots + \epsilon_n b_n + J = \sum \epsilon_i b_i + J.$$

So  $x - (\sum \epsilon_i b_i) \in J$ . Hence

$$x - (\sum \epsilon_i b_i) = \sum \epsilon'_m a_m, \quad a_m \in \{a_1, \dots, a_n\}.$$

Therefore  $x = \sum \epsilon_i b_i + \sum \epsilon'_m a_m$ . So

(4) the  $a$ 's and  $b$ 's generate  $L$ .

In order to see that the  $\rho + \sigma + rs$  relations

$$(5) \begin{cases} h_\beta(a) = 0 & (\beta = 1, 2, \dots, \rho) \\ k_\alpha(b) = f_\alpha(a) & (\alpha = 1, 2, \dots, \sigma) \\ [b_i, a_j] = g(i, j)(a) & (i = 1, \dots, s; j = 1, \dots, r) \end{cases}$$

constitute a set of defining relations for  $L$  it is necessary to show the following:

$$(6) \left\{ \begin{array}{l} \text{If } F = \text{the free Lie algebra generated by} \\ (a_1, \dots, a_s; b_1, \dots, b_r) \text{ and} \\ \varphi : F_a(a_1, \dots, a_s; b_1, \dots, b_r) \longrightarrow L \end{array} \right.$$

((6) continued)  
 (6) is a homomorphism then  $\ker \varphi =$  ideal of  $F$  generated by  
 $(h_{\beta}(a), k_{\alpha}(b) - f_{\alpha}(a), [b_i, a_j] - g(i,j)(a))$  .

Let  $x \in \ker \varphi$  , then  $x\varphi = 0 \in L$  . If  $x$  is an element of  $F_a(a_1, \dots, a_s; b_1, \dots, b_r)$  then

(i)  $x$  is a word in the  $a$ 's which means that  $x\varphi = h_{\beta}(a)$  for some  $\beta$  , so  $x\varphi$  is in the ideal of (6); or

(ii)  $x$  is a word in the  $b$ 's which means that  $x\varphi = k_{\alpha}(b)$  . But  $x\varphi = 0$  and the only words in  $L$  which are zero are words in the  $a$ 's so  $x\varphi = k_{\alpha}(b) = f_{\alpha}(a)$  , and again  $x\varphi$  is in the ideal of (6); or

(iii)  $x$  is a word in the  $a$ 's and  $b$ 's , that is their product is a word in the  $a$ 's and  $b$ 's , so

$$x\varphi = \prod_{i,j} [b_i, a_j] = \prod_{i,j} g(i,j)(a)$$

and again  $x\varphi$  is in the ideal of (6) .

Conversely, let  $x$  be an element of the ideal of (6).

Then  $x$  is some linear combination of the above three generators, that is,

$$(7) \left\{ x = \sum_{p,m,n} \left\{ \begin{aligned} &\gamma_p (h_{\beta}(a))_p + \eta_m (k_{\alpha}(b) - f_{\alpha}(a))_m \\ &+ \xi_n ([b_i, a_j] - g(i,j)(a))_n \end{aligned} \right. \right.$$

where  $\gamma, \eta, \xi$  are elements of the field  $F$  (we will write  $\sum_{p,m,n} \dots$  for (7)).

Consider  $x\varphi$  .

$$\begin{aligned} x\varphi &= \left( \sum_{p,m,n} \dots \right) \varphi \\ &= \sum_{p,m,n} \left( \gamma_p(h_\beta(a))_p \varphi + \eta_m(k_\alpha(b) - f_\alpha(a))_m \varphi \right. \\ &\quad \left. + \sum_n ([b_i, a_j] - g(i,j)(a))_n \varphi \right) . \end{aligned}$$

Since  $\varphi$  is a homomorphism, we have the following:

(i)  $h_\beta(a) = 0$  , hence  $\gamma_p(h_\beta(a))_p \varphi = 0$  ;

(ii)  $k_\alpha(b) - f_\alpha(a) = 0$  , hence

$$\eta_m(k_\alpha(b) - f_\alpha(a)) = 0 ; \text{ and}$$

(iii)  $[b_i, a_j] - g(i,j)(a) = 0$  , hence

$$\sum_n ([b_i, a_j] - g(i,j)(a))_n \varphi = 0 ;$$

for all  $p$  ,  $m$  and  $n$  . Therefore  $x\varphi = 0$  and

$x \in \ker \varphi$  . So finally we have that (6) is true.

Thus, we have exhibited a finite set of generators (4) and a finite set of relations (5) for  $L$  , that is  $L$  is finitely presented.

## BIBLIOGRAPHY

1. Baer, R. and Levi, F., "Freie Produkte und ihre Untergruppen," Comp. Math., 1 (1936), p. 391 - 398.
2. Baumslag, Gilbert, "Wreath products and p-groups," Proc. Cam. Phil. Soc., 55 (1959), p. 224 - 231.
3. Hartley, B., "The residual nilpotence of wreath products," Proc. London Math. Soc. (3) 20 (1970), p. 365 - 392.
4. Humphreys, James E., Introduction to Lie Algebras and Representation Theory, Springer-Verlag, New York Inc., 1972.
5. Jacobson, Nathan, Lie Algebras, Interscience Publishers, John Wiley & Sons, New York, 1962.
6. Kurosh, A. G., The Theory of Groups, I and II, Chelsea Publishing Company, New York, 1956.
7. Magnus, Wilhelm; Karrass, Abraham; and Solitar, Donald, Combinatorial Group Theory, Interscience Publishers, John Wiley & Sons, New York, 1966.
8. Mal'cev, A. I., "Generalized nilpotent algebras," AMS Translations, Series 2, 69, p. 347 - 366.
9. Neumann, Hanna, Varieties of Groups, Ergebnisse der Math. und ihrer Grenzgebiete, Band 37. Springer-Verlag, New York, 1967.
10. Neumann, Peter M., "On the structure of standard wreath products of groups," Math. Z., 84, (1964), p. 343 - 373.

11. Šmel'kin, A. I., "Wreath products of Lie algebras and their applications to the theory of groups," Trans. Moscow Math. Soc., 29 (1973), p. 239 - 252.
12. Šmel'kin, A. I., "Wreath products and varieties of groups," Dokl. Akad. Nauk. SSSR, 157, (1964).