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New results concerning stability of multidimensional digital filters

Žilović, Mihailo Slobodan, Ph.D.

City University of New York, 1993

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New Results Concerning Stability of Multidimensional Digital Filters

by

Mihailo Žilović

A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1993

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Abstract

New Results Concerning Stability of Multidimensional Digital Filters

by

Mihailo Žilović

Adviser: Professor Leonid Roytman

This thesis introduces three new fundamental results concerning stability of digital filters.

It is well known that in order for a 2-D digital filter to be BIBO stable, it is necessary that the denominator of the real rational filter transfer function is a discrete scattering Hurwitz polynomial (DSHP). A very simple concept for a test procedure for 2-D DSHP is presented first. The concept is based on the definition of the stability threshold (stability margin) and of the geometrical properties of the root loci of the polynomial under the test. Polynomials with a finite number of simple zeros on T^2 are considered first. Necessary and sufficient conditions for such polynomials to be DSHP are presented. Also, for the case when a polynomial possesses multiple zeros on T^2 the necessary condition for a polynomial to be a DSHP is derived. All the conditions are related to the values of a polynomial and its partial derivatives at the set of points on T^2.

The second result deals with a very special class of m-D (m>2) first and higher order digital filters, which are proven to be asymptotically stable with the existence of the transfer function polar singularities in the closed unit polydisk. It is also

proven that every 2-D digital filter with DSHP in the denominator of its rational transfer function is always asymptotically stable.

Finally, a new contribution to the very old, but still vital problem of the determination of the upper bound for polynomial zeros is given. The newly introduced bound is obtained through the application of Cauchy's bound theorem to a new polynomial, which is derived from the original one by way of an exponential transform.

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Hackensack, New Jersey

January 1993

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Chapter 1

Introduction

In the processing of multidimensional signals, many operations might be performed on multidimensional sequences which are also performed on one dimensional ones. These operations can be sampling, filtering or transform computation. Generally speaking there is not a straightforward extension of the rules of one dimensional signal processing to the multidimensional one. Thus, for example, in the sampling of the one dimensional signal, the rate at which a bandlimited signal is sampled is adjusted. In the multidimensional case, not only the sampling rate, but also the geometric arrangement of the samples can be adjusted.

The main concern of this thesis is the mathematics for handling multidimensional digital filters (systems), which is far less complete [1]-[4], than the one used for the analysis of one dimensional systems [5],[6]. Many problems arise due to the incompleteness of mathematical tools.

Consideration of the single-input-single-output digital filter with rational transfer function, involves the application of theory of scalar polynomials. The fundamental theorem of algebra states that any one dimensional (univariate) polynomial can be factored as a product of lower degree polynomials. It allows easy determination of the pole locations and the simple pole zero cancelation. On the other hand, there is no fundamental theorem for multivariate polynomials, and in general the problem of their factorization is still open. There are several consequences of the structure of multivariate polynomials related to filter stability. First of all, it is extremely difficult to determine the pole locations, because poles are not isolated points, but rather multidimensional surfaces or manifolds. Furthermore, due to the nonfactorability of multivariate polynomials, it is very hard to do the pole zero cancelation in the multidimensional transfer function. Finally, the multivariate polynomials can be mutually prime and still have finitely many isolated common zeros. Such common zeros of the relatively prime numerator and denominator polynomials of a rational transfer function are called nonessential singularities of the second kind. Therefore, the numerator polynomial affects stability of the multidimensional filter.

The stability region for a one-dimensional filter is well known. It is said that the linear shift invariant causal filter is stable in any sense, if and only if the transfer function is devoid of poles in the entire unit disk. A two-dimensional filter is bounded-input-bounded-output (BIBO) and, hence, asymptotically stable if its transfer function has no poles in the closed unit bidisk and no nonessential singularities of the second kind in the closed unit bidisk, except, possibly on the distinguished boundary of the unit bidisk. For the three and higher dimensional filters, it is said that the filter is BIBO, and therefore asymptotically stable if the transfer function has no poles in the closed unit polydisk and no nonessential singularities of the second kind in the open unit polydisk.

Three different contributions toward the completeness of the multidimensional mathematical tools are given in this thesis. After a review of the current results regarding stability of two-dimensional digital filters in Chapter 2, a simple procedure for evaluating the necessary condition for BIBO stability of two-dimensional filters is given in Chapter 3. This condition states that the denominator of the filter transfer has to be a discrete scattering Hurwitz polynomial. A simple procedure for testing whether the given polynomial is a discrete scattering Hurwitz one is presented. Chapter 4 presents the proof of existence of asymptotically stable filters in the case when the transfer function has singularities in the open unit polydisk. Using the fact that the impulse response of the multidimensional dimensional first order filter can be represented as a term of the multinomial expansion, it is shown that a three-dimensional filter transfer function can have poles in the open unit tridisk, and the filter is still asymptotically stable. The result is generalized to the higher dimensional case, showing that the stability margin increases as the dimension of the filter increases. Finally, Chapter 5 gives a new result on the very old, but continuously vital problem of quick assessment of stability, or stability margins. The new upper bound for the polynomial zeros is introduced. It is based on the exponential polynomial transformation, which gives the possibility for further improvement of the bound.

Chapter 2

An Overview of Stability Issues of 2–D Linear Shift-Invariant Digital Filters

In this chapter certain stability properties of 2–D linear shift-invariant digital filters are discussed. The class of quarter plane filters is considered. A difference equation which describes the input-output relation of the filters from such a class is:

$$\begin{aligned} y(n_1, n_2) = \sum_{k_1=0}^{K_1^p} \sum_{k_2=0}^{K_2^p} p(k_1, k_2) x(n_1 - k_1, n_2 - k_2) \\ - \sum_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{K_1^q} \sum_{k_2=0}^{K_2^q} q(k_1, k_2) y(n_1 - k_1, n_2 - k_2) \end{aligned} \tag{2.1}$$

where $x(n_1, n_2)$ and $y(n_1, n_2)$ represent the input and the output sequences respectively. Equation (2.1) describes the first quadrant filters, which are also termed causal. A feature of such filters is related to the fact that the value of a given point $y(n_1, n_2)$ of the sequence depends only on the values of those points $x(k_1, k_2)$ of the input sequence for which both $k_1 \leq n_1$ and $k_2 \leq n_2$.

The two dimensional z-transform of (2.1) leads to the 2–D rational transfer function

$$G(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)} \tag{2.2}$$

where $P(z_1, z_2)$ and $Q(z_1, z_2)$ are mutually prime polynomials in two variables (P and Q have no irreducible factors in common). Assuming that $Q(z_1, z_2) \neq 0$ at the origin, by continuity argument it can be derived that $Q(z_1, z_2) \neq 0$ in some neighborhood of the origin. Thus $G(z_1, z_2)$ can be expanded into power series in that neighborhood as:

$$G(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} g(n_1, n_2) z_1^{n_1} z_2^{n_2} \tag{2.3}$$

where $g(n_1, n_2)$ is the impulse response of $G(z_1, z_2)$.

2.1 Various Stability Definitions

It is well known that a digital filter is bounded-input-bounded-output (BIBO) stable if and only if

$$g(n_1,n_2) \in l_1 \quad i.e. \quad \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |g(n_1,n_2)| < \infty \tag{2.4}$$

It is said that the impulse response is square summable if:

$$g(n_1,n_2) \in l_2 \quad i.e. \quad \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |g(n_1,n_2)|^2 < \infty \tag{2.5}$$

and that the impulse response is bounded if for some finite M:

$$|g(n_1,n_2)| < M \quad \forall\, n_1,n_2 \tag{2.6}$$

Let us define

$$U^2 = \{(z_1,z_2) : |z_i| < 1,\ i = 1,2\} \tag{2.7}$$

to be the open unit bidisk, and

$$\overline{U}^2 = \{(z_1,z_2) : |z_i| \leq 1,\ i = 1,2\} \tag{2.8}$$

to be the closed unit bidisk, and finally

$$T^2 = \{(z_1,z_2) : |z_1| = |z_2| = 1\} \tag{2.9}$$

to beto be the distinguished boundary of the unit bidisk. Note that $\overline{U}^2 - T^2 \neq U^2$.

Consider now the 2–D rational transfer function of equation (2.2). A 2–D tuple (point) (z_1,z_2) such that $Q(z_1,z_2) = 0$, but $P(z_1,z_2) \neq 0$ is called a pole or a nonessential singularity of the first kind (such a point is analogous to a pole in the one dimensional case). A 2–D tuple (point) (z_1,z_2) such that $Q(z_1,z_2) = P(z_1,z_2) = 0$ is called a nonessential singularity of the second kind (such a point has no one dimensional analogs). Obviously if (z_1,z_2) is a pole, then $G(z_1,z_2) = \infty$. If (z_1,z_2) is a nonessential singularity of the second kind, the value of $G(z_1,z_2)$ is undefined.

Chronologically, the first important stability theorem for a 2–D filter is due to Shanks et. al. [7], who stated that $G(z_1, z_2)$ is BIBO stable if and only if

$$Q(z_1, z_2) \neq 0 \quad for\ all \quad \{(z_1, z_2) : |z_1| \leq 1,\ |z_2| \leq 1\} \tag{2.10}$$

According to Shanks' theorem, filter is BIBO stable if and only if the rational transfer function (2.2) possesses neither poles, nor nonessential singularities of the second kind in $\overline{U}^2$. The efficient procedures for testing whether a given polynomial satisfies (2.10), were given very soon after Shanks' result was published [8], [9]

Five years upon the appearance of [7] (1977), Goodman [10], showed that the necessity condition of Shanks' theorem does not hold. After very lengthy calculations, Goodman proved that a filter with transfer function

$$G_1(z_1, z_2) = \frac{(1 - z_1)^8 (1 - z_2)^8}{2 - z_1 - z_2} \tag{2.11}$$

is BIBO stable, inspite of the fact that the transfer function (2.11) possesses nonessential singularity of the second kind at the point $z_1 = z_2 = 1$, Thus, he showed by example that Shanks' theorem (2.10) is only a sufficient condition for BIBO stability. The main theorem of [10], introduced also in the comprehensive overview on multidimensional system stability [11], stated the following:
If $G(z_1, z_2)$ (2.2) represents a BIBO stable filter, then $G(z_1, z_2)$ has no poles in $\overline{U}^2$, and no nonessential singularities of the second kind in $\overline{U}^2$, except possibly on T^2. Various other results on 2–D stability problems were given in [10], and they can be summarized in the following relationships:

$$
\begin{array}{llll}
a)\ BIBO\ \ stability & \Longleftrightarrow & g(n_1,n_2)\in l_1 & \\
b)\ Q(z_1,z_2)\neq 0\ in\ \overline{U}^2 & \substack{\not\Leftarrow \\ \Rightarrow} & BIBO\ \ stability & \\
c)\ Q(z_1,z_2)\neq 0\ in\ \overline{U}^2 - T^2 & \substack{\Leftarrow \\ \not\Rightarrow} & BIBO\ \ stability & \\
d)\ g(n_1,n_2)\in l_2 & \substack{\Leftarrow \\ \not\Rightarrow} & BIBO\ \ stability & \\
e)\ \lim\limits_{n_1,n_2\to\infty} g(n_1,n_2)=0 & \substack{\Leftarrow \\ \not\Rightarrow} & g(n_1,n_2)\in l_1\ or\ g(n_1,n_2)\in l_2 & (2.12) \\
f)\ Q(z_1,z_2)\neq 0\ in\ U^2 & \substack{\Leftarrow \\ \not\Rightarrow} & |g(n_1,n_2)|\leq M<\infty,\quad \forall n_1,n_2 & \\
g)\ |G(z_1,z_2)|\leq N<\infty\ in\ U^2 & \Longrightarrow & g(n_1,n_2)\in l_2 & \\
h)\ Q(z_1,0)\neq 0\ in\ \overline{U} & \Longrightarrow & \sum\limits_{n_1=0}^{\infty}|g(n_1,n_2)|<\infty,\quad \forall n_2 &
\end{array}
$$

The relationship (h) from (2.12) is also known from [12]. It was defined as a condition which ensures practical stability (Agathoklis and Bruton [12] actually extended the result to m-D case).

After the appearance of [10], a lot of effort was made toward the result which should give the general solution to the open problem of stability in the presence of nonessential singularities of the second kind [13]–[17], [2]. Finally, Roytman et. al. [18] (1987) ended various discussions on BIBO stability of 2–D digital filters. They utilized numerous mathematical results concerning 2–D complex functions analysis [19]–[24], and established an extremely simple procedure for testing l_1 and l_2 stability, as well as boundedness of 2–D digital with a transfer function possessing nonessential singularities of the second kind on T^2.

Starting from the fact that if the transfer function $G(z_1,z_2)$ (2.2) does not have any singularity in $\overline{U}^2$, Parseval's integral can be used in order to evaluate the sum of squares of the impulse response coefficients

$$S=\left(\frac{1}{2\pi j}\right)^2 \oint\limits_{|z_1|=1}\oint\limits_{|z_2|=1} G(z_1,z_2)G\left(\frac{1}{z_1},\frac{1}{z_2}\right)\frac{dz_1dz_2}{z_1z_2} \tag{2.13}$$

where

$$S = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} g^2(n_1, n_2) \tag{2.14}$$

Assuming that $G(z_1, z_2)$ has a nonessential singularity of the second kind on T^2, Parseval's integral (2.13) is not applicable for the derivation of (2.14). Considering a linear transformation $G(z_1, z_2) \to G(kz_1, kz_2)$, it can be seen that for k<1, $G(kz_1, kz_2)$ does not have any singularity in $\overline{U}^2$. Thus, Parseval's integral can be applied to compute the sum of squares of the impulse response of $G(kz_1, kz_2)$:

$$S(k) = \left(\frac{1}{2\pi j}\right)^2 \oint_{|z_1|=1} \oint_{|z_2|=1} G(kz_1, kz_2) G\left(\frac{k}{z_1}, \frac{k}{z_2}\right) \frac{dz_1 dz_2}{z_1 z_2} \tag{2.15}$$

Using the argument similar to those in [10], it was shown in [18] that

$$S = \lim_{k \to 1} S(k) \tag{2.16}$$

Rewriting (2.15) as follows:

$$S(k) = \left(\frac{1}{2\pi j}\right)^2 \oint_{|z_1|=1} \oint_{|z_2|=1} \frac{P_k(z_1, z_2) P_k\left(\frac{1}{z_1}, \frac{1}{z_2}\right)}{Q_k(z_1, z_2) Q_k\left(\frac{1}{z_1}, \frac{1}{z_2}\right)} \frac{dz_1 dz_2}{z_1 z_2} \tag{2.17}$$

where $P_k(z_1, z_2) = P(kz_1, kz_2)$

and $Q_k(z_1, z_2) = Q(kz_1, kz_2)$

and expressing P_k and Q_k in terms of its paraconjugates $\widetilde{P_k}$ and $\widetilde{Q_k}$ (Fetweiss and Basu [25]-[27])

$$P_k\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \frac{1}{z_1^{m_1}} \frac{1}{z_2^{n_1}} \widetilde{P_k}(z_1, z_2) \tag{2.18}$$

and

$$Q_k\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \frac{1}{z_1^{m_2}} \frac{1}{z_2^{n_2}} \widetilde{Q_k}(z_1, z_2) \tag{2.19}$$

where m_1 and n_1, m_2 and n_2 are the maximum degrees of z_1 and z_2 in $P_k(z_1, z_2)$ and $Q_k(z_1, z_2)$ respectively, Roytman et. al. arrived at the expression:

$$S(k) = \left(\frac{1}{2\pi j}\right)^2 \oint_{|z_1|=1} \oint_{|z_2|=1} \frac{P_k(z_1, z_2) \widetilde{P_k}(z_1, z_2)}{Q_k(z_1, z_2) \widetilde{Q_k}(z_1, z_2)} z_1^m z_2^n dz_1 dz_2 \tag{2.20}$$

where m and n are integers.

Instead of computing $S(k)$ as it was suggested in [23] and [24], it was simply investigated in [18] whether $S(k)$ has a finite limit for $k \to 1$. That led to the following theorems on l_2 stability:

Theorem 2.1 [18]: Let the real transfer function $G(z_1, z_2) = P(z_1, z_2)/Q(z_1, z_2)$ has no polar singularities in $\overline{U}^2$ and no singularities of the second kind in $\overline{U}^2$ except for simple ones at (α, β) and, hence, at $(1/\alpha, 1/\beta)$ on T^2. Further, let P and Q be relatively prime [28], [29]. Then $G(z_1, z_2)$ is l_2 stable if and only if

$$m_\alpha\left(R_{z_2}\left[\tilde{Q}, Q\right]\right) \leq 2m_\alpha(R_{z_2}[P, Q]) \tag{2.21}$$

or, alternatively

$$m_\beta\left(R_{z_1}\left[\tilde{Q}, Q\right]\right) \leq 2m_\beta(R_{z_1}[P, Q]) \tag{2.22}$$

where m_α denotes the multiplicity of the factor $(z_1 - \alpha)$ in the z_2 resultant R_{z_2}, and m_β denotes the multiplicity of the factor $(z_2 - \beta)$ in the z_1 resultant R_{z_1} (definitions and the necessary details about resultants are given in the Appendix).

Theorem 2.2 [18]: Let $G(z_1, z_2) = P(z_1, z_2)/[Q(z_1, z_2)]^n$, where the function $P(z_1, z_2)/Q(z_1, z_2)$ is defined as in Theorem 2.1 and n is a positive integer. Then G is l_2 stable if and only if

$$(2n-1)m_\alpha\left(R_{z_2}\left[\tilde{Q}, Q\right]\right) \leq 2m_\alpha(R_{z_2}[P, Q]) \tag{2.23}$$

or, alternatively

$$(2n-1)m_\beta\left(R_{z_1}\left[\tilde{Q}, Q\right]\right) \leq 2m_\beta(R_{z_1}[P, Q]) \tag{2.24}$$

Corollary: Let

$$G(z_1, z_2) = \left[\frac{P(z_1, z_2)}{Q(z_1, z_2)}\right]^n \tag{2.25}$$

where P/Q is defined as in Theorem 2.1 and n is a positive integer. Then G is l_2 stable if and only if

$$(2n-1)m_\alpha\Big(R_{z_2}\big[\tilde{Q},Q\big]\Big) \le 2nm_\alpha(R_{z_2}[P,Q]) \tag{2.26}$$

or, alternatively

$$(2n-1)m_\beta\Big(R_{z_1}\big[\tilde{Q},Q\big]\Big) \le 2nm_\beta(R_{z_1}[P,Q]) \tag{2.27}$$

This corollary follows from Theorem 2.2 and the fact that $m_\alpha(R_{z_2}[P^n,Q]) = nm_\alpha(R_{z_2}[P,Q])$

The process of defining the condition for l_1 stability, Roytman et. al. [18] started with the assumption that

$$\frac{\partial^2}{\partial z_1 \partial z_2}(z_1 z_2 G^*) \in l_2 \tag{2.28}$$

where G^* denotes the radial limit of G [19]. It is well known that the Fourier coefficients of G^* are the same as those of G, if $G \in H^\infty$. Since it is only these Fourier coefficients that matter for stability, it can be worked with G^*. From (2.28), it follows that

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(n_1+1)^2(n_2+1)^2 g^2(n_1,n_2) < \infty \tag{2.29}$$

Applying Cauchy-Schwartz's inequality, it can be seen that

$$\begin{aligned}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}|g(n_1,n_2)| &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left|\frac{(n_1+1)(n_2+1)}{(n_1+1)(n_2+1)}\right||g(n_1,n_2)| \\ &\le \left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{1}{(n_1+1)^2(n_2+1)^2}\right]^{1/2} \\ &\cdot\left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(n_1+1)^2(n_2+1)^2|g(n_1,n_2)|^2\right]^{1/2} < \infty\end{aligned} \tag{2.30}$$

Thus, a sufficient condition for l_1 stability can be formulated as follows:

$$if \quad \frac{\partial^2}{\partial z_1 \partial z_2}(z_1 z_2 G^*) \in l_2, \quad then \quad G(z_1,z_2) \in l_1 \tag{2.31}$$

To test for the l_2 of $\partial^2/\partial z_1\partial z_2(z_1z_2G^*)$, only the term

$$\frac{z_1z_2P}{Q^3}\left(\frac{\partial^2 Q}{\partial z_1\partial z_2}\right) \tag{2.32}$$

which exhibits the most singular behavior around the singularity has to be considered. By applying Theorem 2.2 to (2.32), it can be seen that $\partial^2/\partial z_1\partial z_2(z_1z_2G^*) \in l_2$, if and only if

$$5m_\alpha\left(R_{z_2}\left[\tilde{Q}, Q\right]\right) \leq 2m_\alpha\left(R_{z_2}\left[z_1, z_2P\frac{\partial^2 Q}{\partial z_1\partial z_2}, Q\right]\right) \tag{2.33}$$

or

$$5m_\alpha\left(R_{z_2}\left[\tilde{Q}, Q\right]\right) \leq 2m_\alpha(R_{z_2}[P, Q]) \tag{2.34}$$

since the singularity at (α, β) is simple. In order to find a less strict condition than (2.34), it can be checked for the existence of a positive integer n, such that $[G(z_1, z_2)]^n \in l_1$. By using (2.31), and the result of Theorem 2.2, it can be shown that $G^n \in l_1$ for some n, if and only if

$$m_\alpha\left(R_{z_2}\left[\tilde{Q}, Q\right]\right) < m_\alpha(R_{z_2}[P, Q]) \tag{2.35}$$

The results on l_1 can be summarized in the following theorem:

Theorem 2.3 [18]: Let $G(z_1, z_2) = P(z_1, z_2)/[Q(z_1, z_2)]^n$, where the function $P(z_1, z_2)/Q(z_1, z_2)$ is defined as in Theorem 2.1 and n is a positive integer. Then G is l_1 stable if and only if

$$m_\alpha\left(R_{z_2}\left[\tilde{Q}, Q\right]\right) < m_\alpha(R_{z_2}[P, Q]) \tag{2.36}$$

or, alternatively

$$m_\beta\left(R_{z_1}\left[\tilde{Q}, Q\right]\right) < m_\beta(R_{z_1}[P, Q]) \tag{2.37}$$

Assuming that G, as defined in Theorem 2.1 is bounded in U^2, it directly follows that $G \in H^\infty(U^2)$. Therefore $G^n \in H^\infty(U^2)$ for any positive integer n, meaning that

$G \in l_2$ for any n. From the corollary of the Theorem 2.2, it can be seen that the equation (2.26) is satisfied for all n if and only if

$$m_\alpha\left(R_{z_2}\left[\widetilde{Q}, Q\right]\right) \leq m_\alpha(R_{z_2}[P, Q]) \tag{2.38}$$

Necessary condition for boundedness (2.38), was proved [18], to be the sufficient as well. The complete results on boundedness can be summarized in the following theorem:

Theorem 2.4 [18]: Let $G(z_1, z_2) = P(z_1, z_2)/[Q(z_1, z_2)]^n$, where the function $P(z_1, z_2)/Q(z_1, z_2)$ is defined as in Theorem 2.1 and n is a positive integer. Then G is bounded in U^2 if and only if

$$nm_\alpha\left(R_{z_2}\left[\widetilde{Q}, Q\right]\right) \leq m_\alpha(R_{z_2}[P, Q]) \tag{2.39}$$

or, alternatively

$$nm_\beta\left(R_{z_1}\left[\widetilde{Q}, Q\right]\right) \leq m_\beta(R_{z_1}[P, Q]) \tag{2.40}$$

It should be pointed out that the results of Theorem 2.1 - Theorem 2.4 hold if the real transfer function G (2.2) possesses finitely many zeros (α_i, β_i), $i = 1, 2, \ldots, n$ and, hence $(1/\alpha_i, 1/\beta_i)$ on T^2. In that case conditions of Theorem 2.1 - Theorem 2.4 have to be checked for each i, $i = 1, 2, \ldots, n$.

The results of [18] were extended [30], and then generalized by the same authors in [31], where the case when the transfer function G (2.2) has multiple singularities on T^2 is analyzed. Studies of BIBO-stability of inverse 2-D digital filters [32], [33] used results of [18] as the starting point in any discussion. Finally, [18], together with [34] serves as a good foundation for the still generally unsolved problem of BIBO stability of three and higher dimensional filters [35], [36].

At the end of this chapter, it has to be noted that in any stability analysis, the most important issue is the effect of the numerator polynomial on stability of the transfer function G (2.2). Also, the nature of $Q(z_1, z_2)$ has to be analyzed in more detail. Let us assume that Q and its paraconjugate $\widetilde{Q}$ (2.19) are not relatively prime. Then Q and $\widetilde{Q}$ have a common factor of one of the following types:

a) $F(z_1)$

b) $F(z_2)$

c) $F(z_1, z_2)$

In the case a), Q and $\widetilde{Q}$ have a common factor which is independent of z_2. In such a case $F(z_1)$ has a zero in $|z_1| \leq 1$ irrespective of the value of z_2, showing that G has a polar singularity in $\overline{U}^2 - T^2$. That ensures instability of G. Similar reasoning can be used for part b). In the case c), we have

$$R_{z_1}\left(\widetilde{Q}, Q\right) \equiv 0 \quad and \quad R_{z_2}\left(\widetilde{Q}, Q\right) \equiv 0 \tag{2.41}$$

Since Q and $\widetilde{Q}$ have reciprocal roots, and $Q \neq 0$ in $\overline{U}^2$ except at (α, β) and $(1/\alpha, 1/\beta)$ on T^2, it is evident that the common factor $F(z_1, z_2)$ is zero at (α, β) and $(1/\alpha, 1/\beta)$ only. In such a case, it is clear that (2.20) is not finite. Thus G is not stable. Therefore, it can be concluded that a necessary condition for boundedness, and for l_2, or l_1 stability of $G(z_1, z_2)$ is that $Q(z_1, z_2)$ be a discrete scattering Hurwitz polynomial (DSHP).

2.2 On Hurwitz Polynomials

The definition of a Hurwitz polynomial goes back to late 1800's, when A. Hurwitz and E. J. Ruth independently published a method of investigating the stability of 1–D linear time-invariant systems [37], [38]. Their method checks for the existence of the roots of a characteristic polynomial in the right half of the complex s-plane.

Thus, a new class of polynomials, known as Hurwitz polynomials was defined. Polynomials of such a class are free of roots in the right hand side of a complex plane.

Comprehensive work of Fettweis and Basu [25]-[27] introduced various definitions and properties of multidimensional Hurwitz and discrete Hurwitz polynomials. Considering 2-D filter (system), it is said that the polynomial $R(s_1, s_2)$ is a Hurwitz polynomial in the widest sense if

$$R(s_1, s_2) \neq 0 \quad for \quad Re\ s_i > 0 \quad i = 1, 2 \tag{2.42}$$

A polynomial $R(s_1, s_2)$ is said to be a scattering Hurwitz polynomial if (2.42) is satisfied and $R(s_1, s_2)$ and its paraconjugate $\widetilde{R}(s_1, s_2)$ (2.18), (2.19) are mutually prime.
Using the transformation

$$z_i = \frac{1 - s_i}{1 + s_i} \qquad s_i = \frac{1 - z_i}{1 + z_i} \tag{2.43}$$

which satisfies

$$\begin{aligned} Re\ s_i > 0 \quad &\Longleftrightarrow \quad |z_i| < 1 \\ Re\ s_i < 0 \quad &\Longleftrightarrow \quad |z_i| > 1 \\ Re\ s_i = 0 \quad &\Longleftrightarrow \quad |z_i| = 1 \end{aligned} \tag{2.44}$$

for $i = 1, 2$, we can define a polynomial $Q(z_1, z_2)$, also called the associated polynomial of $R(s_1, s_2)$, such that

$$Q(z_1, z_2) = R\left(\frac{1 - z_1}{1 + z_1}, \frac{1 - z_2}{1 + z_2}\right)(1 + z_1)^{r_1}(1 + z_2)^{r_2} \tag{2.45}$$

where r_1 and r_2 are maximal degrees of s_1 and s_2 in $R(s_1, s_2)$.
The original polynomial $R(s_1, s_2)$ can be reconstructed using the following relation:

$$R(s_1, s_2) = Q\left(\frac{1 - s_1}{1 + s_1}, \frac{1 - s_2}{1 + s_2}\right)\left(\frac{1 + s_1}{2}\right)^{r_1}\left(\frac{1 + s_2}{2}\right)^{r_2} \tag{2.46}$$

Similarly, if $Q(z_1, z_2)$ is given, with maximal degrees of z_1 and z_2 being q_1 and q_2 respectively, the associated polynomial $R(s_1, s_2)$ can be made according to (2.43), by means of

$$R(s_1, s_2) = Q\left(\frac{1 - s_1}{1 + s_1}, \frac{1 - s_2}{1 + s_2}\right)(1 + s_1)^{q_1}(1 + s_2)^{q_2} \tag{2.47}$$

and the original polynomial $Q(z_1, z_2)$ can be reconstructed using

$$Q(z_1, z_2) = R\left(\frac{1-z_1}{1+z_1}, \frac{1-z_2}{1+z_2}\right)\left(\frac{1+z_1}{2}\right)^{q_1}\left(\frac{1+z_2}{2}\right)^{q_2} \tag{2.48}$$

Therefore, the following definitions and properties hold for discrete Hurwitz polynomials [27]:

Definition 2.1: A polynomial $Q(z_1, z_2)$ is said to be a discrete Hurwitz polynomial in the widest sense if

$$Q(z_1, z_2) \neq 0 \quad for \quad |z_i| < 1 \quad i = 1, 2 \tag{2.49}$$

Definition 2.2: A polynomial $Q(z_1, z_2)$ is said to be a discrete scattering Hurwitz polynomial (DSHP) if (2.49) is satisfied and Q its paraconjugate $\widetilde{Q}$ (2.18), (2.19) are mutually prime.

Definition 2.3: A polynomial $Q(z_1, z_2)$ is said to be a discrete Hurwitz polynomial in the strictest sense if

$$Q(z_1, z_2) \neq 0 \quad for \quad |z_i| \leq 1 \quad i = 1, 2 \tag{2.50}$$

Property 2.1: A discrete scattering Hurwitz polynomial $Q(z)$ in a single variable z has no zero for $|z| \leq 1$.

Property 2.2: If $Q(z_1, z_2)$ is a DSHP in two variables z_1 and z_2, then:

a) $Q(z_1, z_2) \neq 0$ as well for $|z_1| = 1$ and $|z_2| < 1$

as for $|z_1| < 1$ and $|z_2| = 1$

b) $Q(z_1, z_2) = 0$ can hold for at most finite number of points on T^2.

Property 2.3: If $Q(z_1, z_2)$ is a DSHP, any factor of $Q(z_1, z_2)$ is also a DSHP.

Property 2.4: A polynomial $Q(z_1, z_2)$ is a DSHP if and only if every proper factor of $Q(z_1, z_2)$ has no zero for $|z_i| < 1$, $i = 1, 2$, and at least one zero for a (z_1, z_2) with $1 < |z_i| < \infty$, $i = 1, 2$

Note: Properties 2.3 and 2.4 are very important in the consideration of a polynomial with a finite number of multiple zeros on T^2.

The most significant of the above definitions is the definition of DSHP, since as it was pointed out at the and of the previous section, DSHP plays an important role in stability considerations. Therefore, it is important to have a simple test procedure for 2–D DSHP. Fettweis and Basu [39] presented a very complex test procedure. A relatively simpler test was given by Rajan and Reddy [40]. They considered $Q(z_1, z_2)$ as a 1–D polynomial in z_2 with coefficients being polynomials in z_1:

$$Q(z_1, z_2) = \sum_{n=0}^{N} a_n(z_1) z_2^n \tag{2.51}$$

Defining Schur-Cohn matrix $C_2(z_1) = [c_{ij}(z_1)]$ associated with $Q(z_1, z_2)$, where

$$\begin{aligned} &for\ \ i \leq j \quad c_{ij}(z_1) = \sum_{k=1}^{i} a_{N-i+k}(z_1)\left[a_{N-j+k}(z_1)\right]^* - [a_{i-k}(z_1)]^* a_{j-k}(z_1) \\ &for\ \ i > j \quad c_{ij}(z_1) = [c_{ji}(z_1)]^* \end{aligned} \tag{2.52}$$

Rajan and Reddy [40] presented the following theorem, which can serve as a test for a 2–D DSHP:

Theorem 2.5: A polynomial $Q(z_1, z_2)$ is a 2–D DSHP if and only if

a) $Q(z_1, b) \neq 0$ for all z_1 such that $|z_1| \leq 1$ and for some b such that $|b| < 1$ (b can be advantageously chosen as zero)

b) $C_2(e^{j\theta})$ is negative semidefinite for all θ in the interval [0, 2π]

c) det $\left[C_2(e^{j\theta})\right] \not\equiv 0$

The above result is equivalent to the test procedure presented by the same authors in [41], where they used the continuity property of the eigenvalues of the Schur-Cohn matrix, as was done by Šiljak [42].

As can be seen, the presented test procedure is very inconvenient in the case when the degree of a polynomial is higher then three.

Chapter 3

Threshold of Stability and Test Procedure for 2–D Discrete Scattering Hurwitz Polynomials

In this chapter a simple concept for the test procedure for 2–D DSHP is introduced. The concept is based on the definition of stability threshold (stability margin) and geometrical properties of the root loci of the polynomial under the test. Polynomials with a finite number of simple zeros on T^2 are considered first. Necessary and sufficient conditions for such polynomials to be DSHP are presented. Also, in case when a polynomial possesses multiple zeros on T^2, the necessary condition for a polynomial to be DSHP is derived.

3.1 2–D Threshold of Stability and Conditions for a Polynomial to be a DSHP

Stability thresholds (margins) of 2–D digital filters are defined [43] in terms of singularities of the transfer function. They hold a close relationship to the settling time of the 2–D impulse response and can, therefore, serve as a measure of stability of 2–D digital filters.

Consider the class of quarter-plane causal filters with the first quadrant as their region of support and the rational transfer function

$$G(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)} \tag{3.1}$$

where P and Q are mutually prime polynomials in two variables. Assume that a sufficient condition for BIBO stability is satisfied (filter is structurally stable)

$$Q(z_1, z_2) \neq 0 \qquad \text{in} \qquad \{(z_1, z_2) : |z_1| \leq 1,\ |z_2| \leq 1\} \tag{3.2}$$

For such a stable filter, the stability margins σ_1 and σ_2 can be defined [44]–[46], as the largest bidisk for which

$$Q(z_1, z_2) \neq 0 \qquad \text{in} \qquad \{(z_1, z_2) : |z_1| < 1 + \sigma_1,\ |z_2| \leq 1\} \tag{3.3}$$

$$Q(z_1, z_2) \neq 0 \qquad \text{in} \qquad \{(z_1, z_2) : |z_1| \leq 1,\ |z_2| < 1 + \sigma_2\} \tag{3.4}$$

Assuming stability (3.2), z_1–stability threshold or z_1–stability margin σ_1 as defined in (3.3) is the minimal value satisfying

$$Q(z_1, z_2) = 0 \tag{3.5}$$

for some z_1 and z_2, such that

$$|z_1| = 1 + \sigma_1 \qquad |z_2| = 1 \tag{3.6}$$

To compute σ_1, as defined in (3.3) one can vary z_2 along the unit disk in the z_2–plane ($z_2 = e^{j\theta}$), and find the loci of the roots of

$$Q\left(z_{1k}, e^{j\theta}\right) \equiv 0 \tag{3.7}$$

where z_{1k} is one of the roots of (3.5) as a function of θ. A possible locus of one of the roots z_{1k} given by (3.7) is shown on Figure 1. Considering Figure 1, it can be observed that

$$\sigma_1 = |z_{1k}|_{\min} - 1 \tag{3.8}$$

where $|z_{1k}|_{\min}$ is the absolute value corresponding to the point of the curve z_{1k} closest to the origin of the z_1–plane. Geometrical observation of Figure 1 leads to the conclusion that at the closest point of the curve z_{1k} to the origin of the z_1–plane (A), it is necessary that the derivative $dz_{1k}/d\theta$ is perpendicular to the radius vector z_{1k} [46]. Since the zeros of the one dimensional polynomial $Q\left(z_1, e^{j\theta_0}\right)$ at a point $z_2 = e^{j\theta_0}$ are continuous functions of θ_0, the derivative $dz_{1k}/d\theta$ is meaningful. The case when more than one branch of (3.7) coincide at a point $z_2 = e^{j\theta_0}$ corresponds to a multiple zero of the one-dimensional polynomial $Q(z_1, e^{j\theta_0})$, and therefore $\partial Q/\partial z_1 = 0$ at that point. That case is commented later in this chapter. Hence,

$$\arg\left(j\frac{dz_{1k}}{d\theta}\right) = \arg(z_{1k}) + \mu\pi, \quad \mu = 0, 1 \tag{3.9}$$

at the point (A) defining σ_1. For further manipulations it is more convenient to write (3.9) in the form

$$j\frac{dz_{1k}}{d\theta} \times z_{1k} = 0 \tag{3.10}$$

where x denotes a vector product between two vectors in the complex plane[1].

It is also known that for the value of z_{1k} and θ at the point (A),

$$Q(z_1, z_2)|_{z_2=e^{j\theta}} = 0 \tag{3.11}$$

and, since (A) is the point on the curve $z_{1k} = f(\theta)$ defined by (3.7)

$$\frac{dQ}{d\theta} = \left(\frac{\partial Q}{\partial z_1}\frac{dz_1}{d\theta} + \frac{\partial Q}{\partial z_2}\frac{dz_2}{d\theta}\right)\bigg|_{z_2=e^{j\theta}} = 0 \tag{3.12}$$

Assuming that $\partial Q/\partial z_1 \neq 0$, it directly follows from (3.12) that

$$\frac{dz_{1k}}{d\theta}\bigg|_{(\alpha,\beta)} = -\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}\frac{dz_2}{d\theta}\bigg|_{(\alpha,\beta)} = -\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}jz_2\bigg|_{(\alpha,\beta)} \tag{3.13}$$

Substituting (3.13) into (3.10)

$$\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}z_2 \times z_1 = 0 \tag{3.14}$$

or

$$\frac{\partial Q}{\partial z_1}z_1 \times \frac{\partial Q}{\partial z_2}z_2\bigg|_{z_2=e^{j\theta}} = 0 \tag{3.15}$$

Note that if $\partial Q/\partial z_1 = 0$ at the point (A), (3.15) is again satisfied.

Equations (3.11) and (3.15) form a set of two equations in two variables θ and z_1 [46], [47]. The solution with the minimal absolute value of $|z_1|$ renders σ_1 (3.8). Note

[1] The expression for the vector product between two vectors a and b in the complex plane is given by: $a \times b = Re(a)Im(b) - Re(b)Im(a)$

that (3.11) and (3.15) can readily be put in the form of three polynomial equations in three variables $e^{j\theta}$, $e^{j\theta_1}$ and $|z_1|$, where $z_1 = |z_1|e^{j\theta_1}$. They can be solved for $|z_1|$, using, for instance, the method proposed in [48].

The same arguments are relevant for the computation of σ_2 as defined in (3.4)

$$\sigma_2 = |z_{2k}|_{\min} - 1 \tag{3.16}$$

where z_2 is the solution with the minimal absolute value $|z_2|_{\min}$ of the set of equations:

$$Q(z_1, z_2)|_{z_1=e^{j\theta}} = 0 \tag{3.17}$$

and

$$\left.\frac{\partial Q}{\partial z_1} z_1 \times \frac{\partial Q}{\partial z_2} z_2\right|_{z_1=e^{j\theta}} = 0 \tag{3.18}$$

Therefore, from the Property 2 of DSHP given in the previous chapter (section 2.2), and the above discussions, it follows that the calculations of stability margins σ_1 and σ_2 can lead to the test procedure for a polynomial to be DSHP. If the polynomial has a zero on T^2, it should be simply required that both σ_1 and σ_2 have to be equal to zero. Also, it has to be pointed out that neither σ_1, nor σ_2 can be identically equal to zero. If, for instance $\sigma_1 \equiv 0$, then the root locus $z_{1k} = f(\theta)$ is a circle of radius one in the complex z_1–plane. If that is the case, $Q(z_1, z_2)$ possesses infinitely many zeros on T^2. That contradicts the Property 2.2 of DSHP (section 2.2). In that case, according to Definition 2.1 $Q(z_1, z_2)$ is a discrete Hurwitz polynomial in the widest sense. The typical example of what was just stated, is the consideration of the polynomial $Q(z_1, z_2) = 1 - z_1 z_2$. Note also that if $\sigma_1 \equiv 0$, then the z_2–resultant $R_{z_2}\left[\tilde{Q}, Q\right]$ is identically equal to zero as well.

Furthermore, if the following condition:

$$\sigma_1 > 0, \qquad \sigma_2 > 0 \tag{3.19}$$

is satisfied, then $Q(z_1, z_2)$ does not have any zeros on T^2. From Definition 2.3, it can be concluded that it is a discrete Hurwitz polynomial in the strictest sense, also termed as discrete very strict Hurwitz polynomial (DVSHP). Finally, if either σ_1 or σ_2 is less than zero, it can be concluded the $Q(z_1, z_2)$ is not a Hurwitz polynomial.

Direct implementation of the stability threshold unfortunately leads to very tedious calculations, solving two sets of equations (3.11), (3.15) and (3.17), (3.18). Starting from the geometrical observation that the root loci of $Q(z_{1k}, e^{j\theta}) \equiv 0$ and $Q(e^{j\theta}, z_{k2}) \equiv 0$ have to be tangential to the unit disk in the corresponding complex planes (see Figure 2), the simple procedure for testing if a given polynomial is a DSHP is presented in the next section. The test is based on evaluations of the values of the polynomial and its partial derivatives at the points on T^2 where $Q(z_1, z_2) = 0$.

3.2 A New Test Procedure for 2-D DSHP in Case that a Polynomial has a Finite Set of Simple Zeros on T^2

Consider a 2-D real polynomial $Q(z_1, z_2)$ which possesses finitely many zeros on T^2,

$$Q(\alpha_i, \beta_i) = 0, \qquad i = 1, 2, \ldots, n \tag{3.20}$$

where $|\alpha_i| = |\beta_i| = 1$.

Since $Q(z_1, z_2)$ has a set of zeros (α, β) on T^2, it directly follows from the discussion given in the previous section that σ_1 and σ_2 are either zero or negative. Clearly, $Q(z_1, z_2)$ cannot be DVSHP. But, if both σ_1 and σ_2 are equal to zero, then $Q(z_1, z_2)$ is DSHP. From the geometry of the root loci, that means that none of the root loci of $Q(z_{1k}, e^{j\theta}) \equiv 0$ and $Q(e^{j\theta}, z_{k2}) \equiv 0$ cross unit disks $|z_1| = 1$ and $|z_2| = 1$, but only touch them at the points where $Q(z_1, z_2) = 0$ from the outside of the unit disks $|z_1| = 1$ and $|z_2| = 1$. Thus, the set of necessary conditions, which is sufficient for $Q(z_1, z_2)$ (3.21) to be a DSHP can be given as follows:

a) For the polynomial $Q(z_1, z_2)$ to be a DSHP it is necessary that all the different loci $z_{1k} = f_k(e^{j\theta})$, where $Q(z_{1k}, e^{j\theta}) \equiv 0$ are located outside of the unit disk in the z_1-plane and are tangential to it at the set of points α_i, $i = 1, 2, \ldots, n$.

b) For the polynomial $Q(z_1, z_2)$ to be a DSHP it is necessary that all the different loci $z_{2k} = g_k(e^{j\theta})$, where $Q(e^{j\theta}, z_{2k}) \equiv 0$ are located outside of the unit disk in the z_2-plane and are tangential to it at the set of points β_i, $i = 1, 2, \ldots, n$.

Conditions (a) and (b) are sufficient for $Q(z_1, z_2)$ to be a DSHP.

Assume that $Q(z_1, z_2)$ has a simple zero at (α, β) on T^2, that is

$$\left.\frac{\partial Q}{\partial z_1}\right|_{(\alpha,\beta)} \neq 0, \qquad \left.\frac{\partial Q}{\partial z_2}\right|_{(\alpha,\beta)} \neq 0 \tag{3.21}$$

Thus, there is only one branch $z_{2k} = g_k(e^{j\theta})$ of the equation $Q(e^{j\theta}, z_{2k}) \equiv 0$ which may cross or touch the unit disk in the z_2-plane at the point β, when $z_1 = \alpha$, that is $\beta = g_k(\alpha)$. According to (3.18) $g_k(e^{j\theta})$ is tangential to the unit disk $|z_2| = 1$ if

$$\left.\left(\frac{\partial Q}{\partial z_1} z_1 \times \frac{\partial Q}{\partial z_2} z_2\right)\right|_{(\alpha,\beta)} = 0 \tag{3.22}$$

or

$$\arg\left.\left(\frac{\partial Q/\partial z_1}{\partial Q/\partial z_2}\right)\right|_{(\alpha,\beta)} = \arg\left.\left(\frac{z_2}{z_1}\right)\right|_{(\alpha,\beta)} + \mu\pi, \qquad \mu = 0, 1 \tag{3.23}$$

Setting $z_{1k} = f_k(e^{j\theta})$, and recalling (3.14) it can be shown that $f_k(e^{j\theta})$ is tangential to the unit disk $|z_1| = 1$ if

$$\left.\left(\frac{\partial Q}{\partial z_2} z_2 \times \frac{\partial Q}{\partial z_1} z_1\right)\right|_{(\alpha,\beta)} = 0 \tag{3.24}$$

or

$$\arg\left.\left(\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}\right)\right|_{(\alpha,\beta)} = \arg\left.\left(\frac{z_1}{z_2}\right)\right|_{(\alpha,\beta)} + \mu\pi, \qquad \mu = 0, 1 \tag{3.25}$$

which is equivalent to (3.22) and (3.23). However, if (3.24) or (3.25) is satisfied, it is not sufficient for the root loci $f_k(e^{j\theta})$ and $g_k(e^{j\theta})$ to be the outside tangents to the unit disk in the corresponding complex planes. They can be the inside tangents, or can cross the unit disk if $f_k(e^{j\theta})$ or $g_k(e^{j\theta})$ has an inflection point at (α, β).

The sufficient condition for the root loci $f_k(e^{j\theta})$ and $g_k(e^{j\theta})$ to be the outside tangents to the unit disk in the corresponding complex plane, and therefore for $Q(z_1, z_2)$ to be a DSHP is given by the set of necessary conditions in the following theorem:

Theorem 3.1:

If a complex polynomial with real coefficients $Q(z_1, z_2) = 0$ at a finite set of points (α_i, β_i) on T^2, then $Q(z_1, z_2)$ is a DSHP iff all the following conditions are satisfied for $\forall i,\ i = 1, \ldots, n$

1)

$$\left(\frac{\partial Q}{\partial z_1} z_1 \times \frac{\partial Q}{\partial z_2} z_2\right)\Bigg|_{(\alpha_i, \beta_i)} = 0 \tag{3.26}$$

2)

$$\left(\left|\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}\right|^2 + Re\left(\frac{d^2 z_1}{d\theta^2}\bar{z}_1\right)\right)\Bigg|_{(\alpha_i, \beta_i)} > 0 \tag{3.27}$$

where

$$\frac{d^2 z_1}{d\theta^2} = -\frac{\left(\frac{\partial^2 Q}{\partial z_1^2}\left(\frac{dz_1}{d\theta}\right)^2 + 2\frac{\partial^2 Q}{\partial z_1 \partial z_2}\frac{dz_1}{d\theta}\frac{dz_2}{d\theta} + \frac{\partial^2 Q}{\partial z_2^2}\left(\frac{dz_2}{d\theta}\right)^2 + \frac{\partial Q}{\partial z_2}\frac{d^2 z_2}{d\theta^2}\right)}{\frac{\partial Q}{\partial z_1}} \tag{3.28}$$

$$\frac{dz_1}{d\theta} = -\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1} j z_2, \qquad z_2 = e^{j\theta}$$

3)

$$\left(\left|\frac{\partial Q/\partial z_1}{\partial Q/\partial z_2}\right|^2 + Re\left(\frac{d^2 z_2}{d\theta^2}\bar{z}_2\right)\right)\Bigg|_{(\alpha_i, \beta_i)} > 0 \tag{3.29}$$

where

$$\frac{d^2 z_2}{d\theta^2} = -\frac{\left(\frac{\partial^2 Q}{\partial z_2^2}\left(\frac{dz_2}{d\theta}\right)^2 + 2\frac{\partial^2 Q}{\partial z_1 \partial z_2}\frac{dz_1}{d\theta}\frac{dz_2}{d\theta} + \frac{\partial^2 Q}{\partial z_1^2}\left(\frac{dz_1}{d\theta}\right)^2 + \frac{\partial Q}{\partial z_1}\frac{d^2 z_1}{d\theta^2}\right)}{\frac{\partial Q}{\partial z_2}} \tag{3.30}$$

$$\frac{dz_2}{d\theta} = -\frac{\partial Q/\partial z_1}{\partial Q/\partial z_2} j z_1, \qquad z_1 = e^{j\theta}$$

Proof: Setting $z_2 = e^{j\theta}$, and assuming that $Q(z_1, z_2) = 0$, one can find $z_1 = f(e^{j\theta}) = |z_1|e^{j\theta_1(\theta)}$. Since the root locus $z_1 = f(e^{j\theta})$ is located in the z_1-plane, in order to find $|z_1|_{\min}$ one should find $d|z_1|/d\theta_1$, equate it to zero, and for such θ_1 satisfying $d|z_1|/d\theta_1 = 0$, check for $d^2|z_1|/d\theta_1^2 > 0$. Obviously, $d|z_1|/d\theta_1$ cannot be found directly, but the following relation holds:

$$\frac{d|z_1|}{d\theta_1} = \frac{d|z_1|}{d\theta}\frac{d\theta}{d\theta_1} \tag{3.31}$$

Theoretically, it is possible to find $d|z_1|/d\theta$ from the fact that $z_1 = f(e^{j\theta})$, but instead of looking for the explicit expression of $|z_1|$, the following relation can be used:

$$\frac{d|z_1|^2}{d\theta} = 2|z_1|\frac{d|z_1|}{d\theta} \tag{3.32}$$

Hence, for $z_1 \neq 0$, $d|z_1|/d\theta = 0$ and $d|z_1|/d\theta_1 = 0$, when $d|z_1|^2/d\theta = 0$. From the fact that Q is a complex polynomial with the real coefficients and that $Q(z_1, e^{j\theta}) \equiv 0$, it follows that

$$\frac{d|z_1|^2}{d\theta} = \frac{d}{d\theta}(z_1\bar{z}_1) = 2Re\left(\frac{dz_1}{d\theta}\bar{z}_1\right) = 2Im\left(\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}\bar{z}_1 z_2\right) \tag{3.33}$$

Using the property, that the imaginary part of the product of two complex numbers a and b can be expressed in terms of the vector product as:

$$Im(ab) = \bar{a} \times b \tag{3.34}$$

it can be seen that

$$\frac{d|z_1|^2}{d\theta} = 2\ z_1 \times \left(\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}z_2\right) = 2\left|\frac{\partial Q}{\partial z_1}\right|^2\left(\frac{\partial Q}{\partial z_1}z_1 \times \frac{\partial Q}{\partial z_2}z_2\right) \tag{3.35}$$

Under the initial assumption that Q has simple zeros at the set of points (α_i, β_i) on T^2, it is clear that $d|z_1|^2/d\theta = 0$ if and only if

$$\frac{\partial Q}{\partial z_1}z_1 \times \frac{\partial Q}{\partial z_2}z_2 = 0 \tag{3.36}$$

at these points. Therefore, the necessary condition for tangentiality of the root loci $z_1 = f(e^{j\theta})$ to the unit disk in the $|z_1|$-plane at the points (α_i) is derived. Setting $z_1 = e^{j\theta}$, and looking for $d|z_2|^2/d\theta = 0$, one would end up with the condition as in (3.36). Hence, the proof for Part 1 of Theorem 1. In order to prove Part 2, $d^2|z_1|/d\theta_1^2$, when $d|z_1|/d\theta_1 = 0$ has to be derived first. It can be observed that:

$$\frac{d^2|z_1|^2}{d\theta_1^2} = \frac{d^2|z_1|^2}{d\theta^2}\left(\frac{d\theta}{d\theta_1}\right)^2 + \frac{d|z_1|^2}{d\theta}\frac{d^2\theta}{d\theta_1^2} = \frac{d^2|z_1|^2}{d\theta^2}\left(\frac{d\theta}{d\theta_1}\right)^2 \tag{3.37}$$

since $d|z_1|^2/d\theta = 0$, when $d|z_1|/d\theta_1 = 0$. On the other hand,

$$\frac{d^2|z_1|^2}{d\theta^2} = 2\left(\frac{d|z_1|}{d\theta}\right)^2 + 2|z_1|\frac{d^2|z_1|}{d\theta^2} = 2|z_1|\frac{d^2|z_1|}{d\theta^2} \tag{3.38}$$

since $d|z_1|/d\theta = 0$, when $d|z_1|/d\theta_1 = 0$. Furthermore,

$$\frac{d^2|z_1|^2}{d\theta_1^2} = 2\left(\frac{d|z_1|}{d\theta_1}\right)^2 + 2|z_1|\frac{d^2|z_1|}{d\theta_1^2} = 2|z_1|\frac{d^2|z_1|}{d\theta_1^2} \tag{3.39}$$

when $d|z_1|/d\theta_1 = 0$. From (3.37), (3.38) and (3.39), it directly follows that $d^2|z_1|/d\theta_1^2 > 0$, if and only if $d^2|z_1|^2/d\theta^2 > 0$, and $d\theta/d\theta_1 \neq 0$. Now, it has to be proved that $d\theta/d\theta_1 \neq 0$, whenever $d|z_1|/d\theta_1 = 0$. As stated earlier, $z_1 = f(e^{j\theta}) = |z_1|(\theta)e^{j\theta_1(\theta)}$. Therefore,

$$\frac{dz_1}{d\theta} = \frac{d|z_1|}{d\theta}e^{j\theta_1} + jz_1\frac{d\theta_1}{d\theta} \tag{3.40}$$

Since θ_1 is a real function of the real argument θ and $Q(z_1, e^{j\theta}) \equiv 0$, it follows that:

$$\frac{d\theta_1}{d\theta} = Im\left(\frac{\frac{dz_1}{d\theta}}{z_1}\right) \tag{3.41}$$

Applying (3.34), it is easy to see, that for $|z_1| = 1$

$$\frac{d\theta_1}{d\theta} = \left|\frac{\partial Q}{\partial z_1}\right|^2\left(j\frac{dQ}{dz_1}z_1 \times \frac{dQ}{dz_2}z_2\right) \tag{3.42}$$

From the definition of a vector product, if there are two vectors a and b, such that $|a| \neq 0$ and $|b| \neq 0$, then if $a \times b = 0$, it immediately follows that $(ja) \times b \neq 0$. Hence, whenever $\left(\frac{\partial Q}{\partial z_1}z_1 \times \frac{\partial Q}{\partial z_2}z_2\right)\Big|_{(\alpha,\beta)} = 0$, $d\theta/d\theta_1 \neq 0$ at the same point.

The expression for $d^2|z_1|^2/d\theta^2$ is given in the following form:

$$\frac{d^2|z_1|^2}{d\theta^2} = \frac{d^2}{d\theta^2}(z_1\bar{z}_1) = 2\left|\frac{dz_1}{d\theta}\right| + 2Re\left(\frac{d^2z_1}{d\theta^2}\bar{z}_1\right) \tag{3.43}$$

From the fact that $Q(z_1, e^{j\theta}) \equiv 0$, it follows that

$$\frac{dQ}{d\theta} = \frac{d^2Q}{d\theta^2} \equiv 0 \tag{3.44}$$

Thus, as we previously used

$$\frac{dz_1}{d\theta} = -\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1}jz_2 \tag{3.45}$$

and finally

$$\frac{d^2z_1}{d\theta^2} = -\frac{\left(\frac{\partial^2 Q}{\partial z_1^2}\left(\frac{dz_1}{d\theta}\right)^2 + 2\frac{\partial^2 Q}{\partial z_1\partial z_2}\frac{dz_1}{d\theta}\frac{dz_2}{d\theta} + \frac{\partial^2 Q}{\partial z_2^2}\left(\frac{dz_2}{d\theta}\right)^2 + \frac{\partial Q}{\partial z_2}\frac{d^2z_2}{d\theta^2}\right)}{\frac{\partial Q}{\partial z_1}} \tag{3.46}$$

Hence, the end of the proof of Part 2. In order to prove Part 3, one can set $z_1 = e^{j\theta}$, define $z_2 = g(e^{j\theta}) = |z_2|(\theta)e^{j\theta_2(\theta)}$ and follow the same steps as in the proof of Part 2. ■

Theorem 3.1 gives a very simple procedure for testing whether a given polynomial is a DSHP, assuming that the set of polynomial zeros on the unit bidisk is known. The test procedure is even simpler if we use the following corollaries:

Corollary 3.1: If $Q(z_1, z_2)$ has real simple zeros at $(\pm 1, \pm 1)$, the condition of Part 1 is always satisfied.

Corollary 3.2: If $Q(z_1, z_2) = Q(z_2, z_1)$ and if $\alpha_i = \beta_i$, then the condition of Part 2 is the same as the condition of Part 3.

Corollary 3.3: For the complex conjugate zeros, the condition of Part 1 has to be checked for only one zero from the each pair.

Note that if the left hand side of (3.27) or (3.29) is less than zero, then the root loci $f_k(e^{j\theta})$ or $g_k = (e^{j\theta})$ are the inside tangents to the unit disk in the corresponding complex plane. If the left hand side of (3.27) or (3.29) is equal to zero, and if

$$Re\left(\frac{d^3 z_i}{d\theta^3}\bar{z}_i\right) + 3Re\left(\frac{d^2 z_i}{d\theta^2}\frac{d\bar{z}_i}{d\theta}\right) \neq 0 \qquad i = 1,2 \tag{3.47}$$

then the root loci $f_k(e^{j\theta})$ or $g_k = (e^{j\theta})$ cross the unit disk in the corresponding complex plane (see Figure 3)[2]. In that case, the function $|z_1|(\theta)$ or $|z_2|(\theta)$ has the inflection point at (α_i, β_i). According to the initial assumption that $Q(z_1, z_2)$ has a set of simple zeros on T^2, the root loci $f_k(e^{j\theta})$ or $g_k = (e^{j\theta})$ cannot have any cusps and wedges due to the fact that $\frac{\partial Q}{\partial z_1}\Big|_{(\alpha_i,\beta_i)} \neq 0$ and $\frac{\partial Q}{\partial z_2}\Big|_{(\alpha_i,\beta_i)} \neq 0$, and hence, $f_k(e^{j\theta})$ and $g_k(e^{j\theta})$ are continuous functions of θ with the finite derivatives with respect to θ.

The obtained results can be illustrated through the following examples:

Example 3.1: Let

$$Q(z_1, z_2) = z_2^2 + z_1 z_2 + 2z_2 + 4z_1 - 8 \tag{3.48}$$

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2 Expressions for $\frac{dz_i}{d\theta}$ and $\frac{d^2 z_i}{d\theta^2}$ are given by (3.29) and (3.31). $\frac{d^3 z_i}{d\theta^3}$ can be derived from $\frac{d^3 Q}{d\theta^3} \equiv 0$

possessing only one simple zero on T^2 at a point (1,1). Since we have a real zero, the condition of Part 1 of Theorem 3.1 is satisfied according to corollary 3.1. Partial derivatives of Q at the point (1,1) have the following values:

$$\begin{aligned} \frac{dQ}{dz_1} &= \frac{dQ}{dz_2} = 5 \\ \frac{d^2Q}{dz_1^2} &= 0 \\ \frac{d^2Q}{dz_2^2} &= 2 \\ \frac{d^2Q}{dz_1 dz_2} &= 1 \end{aligned} \tag{3.49}$$

For $\frac{dz_1}{d\theta} = -\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1} j z_2 = -j$ and $z_2 = e^{j\theta} = 1$, (3.27) becomes

$$1^2 + Re\left(-\frac{0 + 2(-j)j + 2(j)^2 + 5(j)^2}{5} 1 \right) = 2 > 0 \tag{3.50}$$

and the condition of Part 2 of Theorem 3.1 is satisfied.

For $\frac{dz_2}{d\theta} = -\frac{\partial Q/\partial z_1}{\partial Q/\partial z_2} j z_1 = -j$ and $z_1 = e^{j\theta} = 1$, the left hand side of (3.29) becomes

$$1^2 + Re\left(-\frac{2(-j)^2 + 2(-j)j + 0 + 5(j)^2}{5} 1 \right) = 2 > 0 \tag{3.51}$$

and the condition of the Part 3 of Theorem 3.1 is satisfied as well. Hence, Q is a DSHP.

The same example was given in [40], where the test procedure goes in the following direction: first of all $Q(z_1, z_2)$ (3.48) has to be written as a polynomial in z_2 with coefficients being polynomials in z_1

$$Q(z_1, z_2) = z_2^2 + (z_1 + 2)z_2 + 4(z_1 - 2) \tag{3.52}$$

Then, according to Theorem 2.5 stated in the section 2.2, it has to be checked for

$$Q(z_1, 0) = 4z_1 - 8 \neq 0 \quad for \quad |z_1| \leq 1 \tag{3.53}$$

Using (2.52), Schur-Cohn matrix is obtained in the following form:

$$C_2\left(e^{j\theta}\right) = \begin{bmatrix} -79 + 64\cos\theta & 14 - 7e^{j\theta} + 8e^{-j\theta} \\ 14 - 7e^{j\theta} + 8e^{-j\theta} & -79 + 64\cos\theta \end{bmatrix} \tag{3.54}$$

Then,

$$\det\left[C_2\left(e^{j\theta}\right)\right] = 5,932 - 10,250\cos\theta + 4,320\cos^2\theta$$
$$= (5,932 - 4,320\cos\theta)(1 - \cos\theta) \tag{3.55}$$

Following the second requirement of Theorem 2.5, it can be seen that det $\left[C_2\left(e^{j\theta}\right)\right]$ is not identically zero, but becomes zero at θ=0. Choosing θ=π, cosθ=-1, and the Schur-Cohn matrix becomes

$$C_2\left(e^{j\pi}\right) = \begin{bmatrix} -143 & 13 \\ 13 & -143 \end{bmatrix} \tag{3.56}$$

Finally,

$$\det\left[\lambda I - C_2\left(e^{j\pi}\right)\right] = (\lambda + 143)^2 - 13^2$$
$$= (\lambda + 156)(\lambda + 130) \tag{3.57}$$

showing that the Schur-Cohn matrix is negative definite, and, hence $Q(z_1, z_2)$ is DSHP.

From what was just shown, it is easy to observe that a new proposed procedure given by Theorem 3.1 is much simpler than the one given in [40]. As was mentioned before, the test procedure for 2–D DSHP given in [40] was found to be simpler then the one proposed in [39].

Example 3.2: Let

$$Q(z_1, z_2) = 1 + 0.5z_1 + 0.5z_2 - 2z_1z_2 \tag{3.58}$$

The only zero of given Q is again at the point (1,1). Therefore, the condition of Part 1 of Theorem 3.1 is satisfied according to Corollary 3.1. Partial derivatives of Q have the following values at the point (1,1):

$$\frac{dQ}{dz_1} = \frac{dQ}{dz_2} = -1.5$$
$$\frac{d^2Q}{dz_1^2} = \frac{d^2Q}{dz_2^2} = 0 \tag{3.59}$$
$$\frac{d^2Q}{dz_1dz_2} = -2$$

According to Corollary 3.2, we have to check only the condition of Part 2 of the Theorem 3.1, where for $\frac{dz_1}{d\theta} = -\frac{\partial Q/\partial z_2}{\partial Q/\partial z_1} jz_2 = -j$ and $z_2 = e^{j\theta} = 1$, (3.27) becomes

$$1^2 + Re\left(-\frac{0 + 2(-2)(-j)(j) + 0 + (-1.5)(j)^2}{-1.5}1\right) = -\frac{2}{3} < 0 \tag{3.60}$$

Thus, it can be concluded that Q (3.58) is not a DSHP.

Example 3.3: Let

$$Q(z_1, z_2) = 1.5 - z_1 - z_2 \tag{3.61}$$

It is easy to see that the given Q has a pair of complex conjugate simple roots at

$$(\alpha_1, \beta_1) = \left(0.75 + j\sqrt{0.4375}, 0.75 - j\sqrt{0.4375}\right) \tag{3.62}$$

and

$$(\alpha_2, \beta_2) = \left(0.75 - j\sqrt{0.4375}, 0.75 + j\sqrt{0.4375}\right) \tag{3.63}$$

Checking the condition of Part 1 of Theorem 3.1, it can be derived that:

$$\left(\frac{\partial Q}{\partial z_1} z_1 \times \frac{\partial Q}{\partial z_2} z_2\right)\bigg|_{(\alpha_1, \beta_1)} = -1.5\sqrt{0.4375} \tag{3.64}$$

Clearly, Q (3.61) is not a DSHP.

3.3 Necessary Condition for a 2-D Polynomial with a Finite Set of Multiple Zeros on T^2 to be a DSHP

Let us first consider the case when Q has only one zero at (α, β) but of order[3], say r. In that case, Q can be represented as

$$Q(z_1, z_2) = A(z_1, z_2) \prod_{i=1}^{r} Q_i(z_1, z_2) \tag{3.65}$$

where $A(\alpha, \beta) \neq 0$, and each of the Q_i's (i=1, ..., r) has a simple zero at (α, β). From (3.65), it follows that the r-th mixed derivative of Q at (α, β) has the form:

$$\frac{\partial^r Q(z_1, z_2)}{\partial z_1^t \partial z_2^{r-t}}\bigg|_{(\alpha,\beta)} = A(\alpha, \beta) \sum_s \left(\frac{\partial Q_{i_1}}{\partial z_1} \cdots \frac{\partial Q_{i_t}}{\partial z_1} . \frac{\partial Q_{j_1}}{\partial z_2} \cdots \frac{\partial Q_{j_{r-t}}}{\partial z_2}\right)\bigg|_{(\alpha,\beta)} \tag{3.66}$$

[3] The order can be computed by taking the derivatives $\partial^k Q/\partial z_1^k$ and $\partial^k Q/\partial z_2^k$, and determining the lowest value of k for which both of them are non-zero at the point (α, β). Incidentally, if $(\partial^p Q/\partial z_1^p)|_{(\alpha,\beta)} = 0$ and $(\partial^q Q/\partial z_1^q)|_{(\alpha,\beta)} \neq 0$ (or vice-versa) for p,q<k, then it follows from [14] that Q is not a DSHP.

where

$$s = \{1, 2, \ldots, r\}$$
$$s_i = \{i_1, i_2, \ldots, i_n\} \in s \qquad (3.67)$$
$$s_j = \{j_1, j_2, \ldots, j_n\} = s - s_i$$

and $0 \le t \le r$. Setting t = r in (3.66), the following expression can be obtained:

$$\left.\frac{\partial^r Q}{\partial z_1^r}\right|_{(\alpha,\beta)} = r! A(\alpha,\beta) \left.\frac{\partial Q_1}{\partial z_1} \cdots \frac{\partial Q_r}{\partial z_1}\right|_{(\alpha,\beta)} \qquad (3.68)$$

From (3.66) and (3.68) we obtain

$$\left.\frac{D_{z_1}^t D_{z_2}^{r-t} Q}{D_{z_1}^r Q}\right|_{(\alpha,\beta)} = \frac{1}{r!} \sum_{s_j} \left.\frac{D_{z_2} Q_{j_1}}{D_{z_1} Q_{j_1}} \cdots \frac{D_{z_2} Q_{j_{r-t}}}{D_{z_1} Q_{j_{r-t}}}\right|_{(\alpha,\beta)} \qquad (3.69)$$

where

$$\frac{\partial}{\partial z_2} = D_{z_2}$$
$$\frac{\partial}{\partial z_1} = D_{z_1} \qquad (3.70)$$
$$\frac{\partial^r}{\partial z_1^t \partial z_2^{r-t}} = D_{z1}^t D_{z_2}^{r-t}$$

Let us now define the following r-variables, X_i, i = 1, ..., r:

$$X_i = \left.\frac{D_{z_2} Q_i}{D_{z_1} Q_i}\right|_{(\alpha,\beta)}, \qquad i = 1, \ldots, r \qquad (3.71)$$

Then (3.69) can be written as:

$$\left.\frac{D_{z_1}^t D_{z_2}^{r-t} Q}{D_{z_1}^r Q}\right|_{(\alpha,\beta)} = \frac{(r-t)!}{r!} \sum_{v_1} X_{m_1} \ldots X_{m_{r-t}} \Big|_{(\alpha,\beta)} \qquad (3.72)$$

where v_i represents all possible combinations of (r-t) elements of the set $(m_1, m_2, \ldots, m_r)$. From (3.72) it can be seen that

$$\sum_{v_i} X_{m_1} X_{m_2} \ldots X_{m_{r-t}} \Big|_{(\alpha,\beta)} = C_{r-t}^r t! \left.\frac{D_{z_1}^t D_{z_2}^{r-t} Q}{D_{z_1}^r Q}\right|_{(\alpha,\beta)} \qquad (3.73)$$

From (3.73), it follows that $X_1, X_2, \ldots, X_r$ can be represented as the zeros of one r-th order polynomial $P_r(X)$ whose coefficients are homogeneous expansions of the type (3.73):

$$P_r(X) = X^r + a_1 X^{r-1} + a_2 X^{r-2} + \ldots + a_i X^{r-i} + a_r \qquad (3.74)$$

where

$$a_i = C_i^r (r-i)!(-1)^i \frac{D_{z_1}^{r-i} D_{z_2}^i Q}{D_{z_1}^r Q}\bigg|_{(\alpha,\beta)} \tag{3.75}$$

Let us now assume that for any polynomial Q_i, (i=1, ..., r), the loci are tangential at (α, β). Invoking (3.26), and combining it with (3.71) we have

$$\arg X_i = \arg\left(\frac{\alpha}{\beta}\right) + \mu\pi, \qquad \mu = 0, 1 \tag{3.76}$$

Thus, it follows from (3.76) that for tangentiality of the loci for all polynomials Q_i, all the zeros of the polynomial $P_r(X)$ have to have the same arguments. It can be easily observed that by transforming $P_r(X)$ into $\overline{P}_r(X)$, given by

$$\overline{P}_r(X) = X^r + b_1 X^{r-1} + b_2 X^{r-2} + \ldots + b_i X^{r-i} + b_r \tag{3.77}$$

where $b_i = a_i(\beta/\alpha)^{r-i}$, one can check the necessary tangentiality conditions for all Q_i (i=1, ..., r), which are also necessary for $Q(z_1, z_2)$ to be a DSHP, by verifying that all the roots of $\overline{P}_r(X)$ are real. The beauty of this result is the verification of real roots of $\overline{P}_r(X)$ without actually computing the roots, but simply finding the "Sturm functions" [49], [50]. Note that in case when $(\alpha, \beta) = (\pm 1, \pm 1)$, the necessary tangentiality condition for all Q_i $i = 1, \ldots, r$ is always satisfied.

If $Q(z_1, z_2)$ has a finite set of multiple zeros (α_j, β_j), $j = 1, \ldots, n$ on T^2, the condition (3.76) has to be checked for each j to ascertain whether all the different loci are tangential to the unit disks at the points (α_j, β_j). However, in case of complex conjugate zeros, (α_j, β_j) and $(\alpha_{j+1}, \beta_{j+1})$, $(\alpha_j, \beta_j) = (1/\alpha_{j+1}, 1/\beta_{j+1})$ holds, and hence, it is necessary to check for only one of the zeros, say (α_j, β_j).

Example 3.4: Consider the polynomial

$$Q(z_1, z_2) = 2z_2^4 - 5z_1 z_2^2 + 3z_1^2 + 9z_2^2 - 11z_1 + 10 \tag{3.78}$$

possessing a pair of complex conjugate roots $(\alpha_1, \beta_1) = (1, j)$ and $(\alpha_2, \beta_2) = (1, -j)$ on T^2. It can be easily check that these roots are of multiplicity two. Since the roots are complex conjugate, it is sufficient to check for only one of them, say (α_j, β_j). By (3.77), the following polynomial can be formed:

$$\overline{P}_2(X) = X^2 + b_1 X + b_2 \tag{3.79}$$

where $b_1 = ja_1$ and $b_2 = a_2$. Applying (3.75),

$$\begin{aligned} a_1 &= C_1^2(1)!(-1)^1 \left(\frac{\frac{\partial^2 Q}{\partial z_1 \partial z_2}}{\frac{\partial^2 Q}{\partial z_1^2}} \right) \Bigg|_{(1,j)} = \frac{10}{3} j \\ a_2 &= C_2^2(0)!(-1)^2 \left(\frac{\frac{\partial^2 Q}{\partial z_2^2}}{\frac{\partial^2 Q}{\partial z_1^2}} \right) \Bigg|_{(1,j)} = -\frac{8}{3} \end{aligned} \tag{3.80}$$

Hence,

$$\overline{P}_2(X) = X^2 - \frac{10}{3} X - \frac{8}{3} \tag{3.81}$$

It is obvious that both roots of $\overline{P}_2(X)$ are real and thus the tangentiality condition holds at the point (1, j) and also at (-1, j). Hence, Q given by (3.78) satisfies a necessary condition to be a DSHP. It should be noted that Q (3.78) can be expressed as $Q = Q_1 Q_2$, where

$$Q_1(z_1, z_2) = z_2^2 - z_1 + 2 \tag{3.82}$$

and

$$Q_2(z_1, z_2) = 2z_2^2 - 3z_1 + 5 \tag{3.83}$$

Using the test procedure proposed by Theorem 3.1, it can be easily checked that both Q_1 and Q_2 are DSHP. Therefore, according to Property 2.3 and 2.4 of DSHP (section 2.2), Q (7.78) is indeed a DSHP.

Example 3.5: Consider the polynomial

$$Q(z_1, z_2) = 2z_1^2 z_2 + 2z_1 z_2^2 - 0.5z_1^2 - 0.5z_2^2 - 5z_1 z_2 + 2 \tag{3.84}$$

Since Q has a real root $(\alpha, \beta) = (1, 1)$ of multiplicity two on T^2, it can be concluded that it satisfies the necessary condition to be a DSHP. Observing that $Q = Q_1 Q_2$, where

$$Q_1(z_1, z_2) = 2 - z_1 - z_2 \tag{3.85}$$

and

$$Q_2(z_1, z_2) = 1 + 0.5z_1 + 0.5z_2 - 2z_1z_2 \tag{3.86}$$

it is obvious that Q (3.84) is not a DSHP, due to the fact that its factor Q_2 is not a DSHP as shown by example 3.2.

Example 3.6: Let

$$Q(z_1, z_2) = z_2^5 - z_1z_2^3 + 3z_2^3 - z_1z_2^2 + z_1^2 + z_2^2 - z_1z_2 - 3z_1 + 2z_2 + 2 \tag{3.87}$$

It can be easily checked that Q has a pair of complex conjugate roots $(\alpha_1, \beta_1) = (1, j)$ and $(\alpha_2, \beta_2) = (1, -j)$ on T^2, and that these roots are of multiplicity two. Since the roots are complex conjugate, it is sufficient to check the necessary condition for a DSHP only for (α_1, β_1). By (3.77):

$$\overline{P}_2(X) = X^2 + b_1X + b_2 \tag{3.88}$$

where $b_1 = ja_1$ and $b_2 = a_2$. Applying (3.75),

$$\begin{aligned} a_1 &= C_1^2(1)!(-1)^1 \left(\frac{\frac{\partial^2 Q}{\partial z_1 \partial z_2}}{\frac{\partial^2 Q}{\partial z_1^2}} \right) \Bigg|_{(1,j)} = 2j - 2 \\ a_2 &= C_2^2(0)!(-1)^2 \left(\frac{\frac{\partial^2 Q}{\partial z_2^2}}{\frac{\partial^2 Q}{\partial z_1^2}} \right) \Bigg|_{(1,j)} = -4j \end{aligned} \tag{3.89}$$

Hence,

$$\overline{P}_2(X) = X^2 + 2(1-j)X - 4j \tag{3.90}$$

Since the roots of (3.91) are $X_1 = 2$ and $X_2 = -2j$, it can be concluded that Q (3.87) is not a DSHP.

Figure 1: Locus of the root of $z_{1k} = f_k(\theta)$

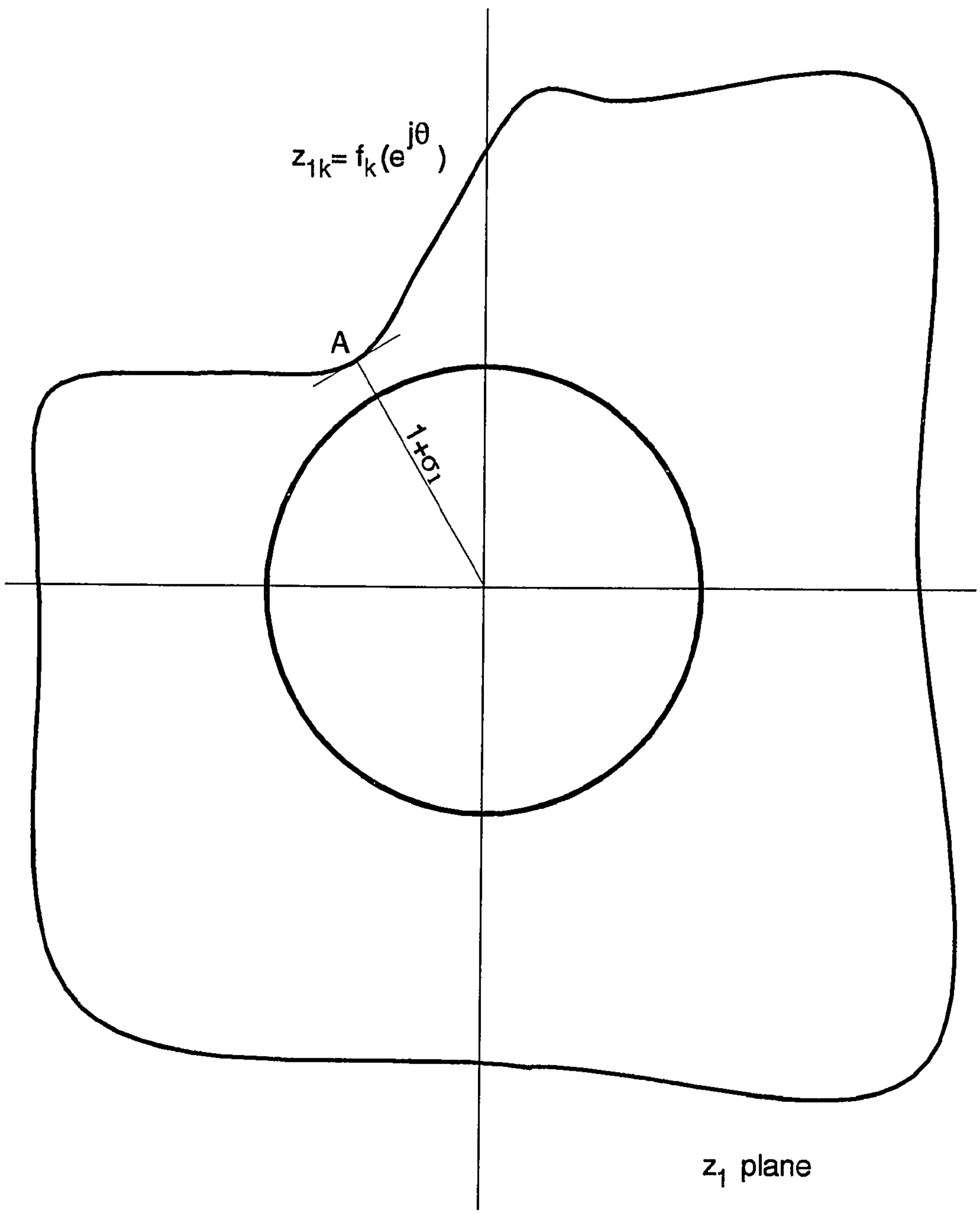


Figure 2: Locus of the root of $z_{1k} = f_k(\theta)$ when $Q(z_1, z_2)$ is a DSHP

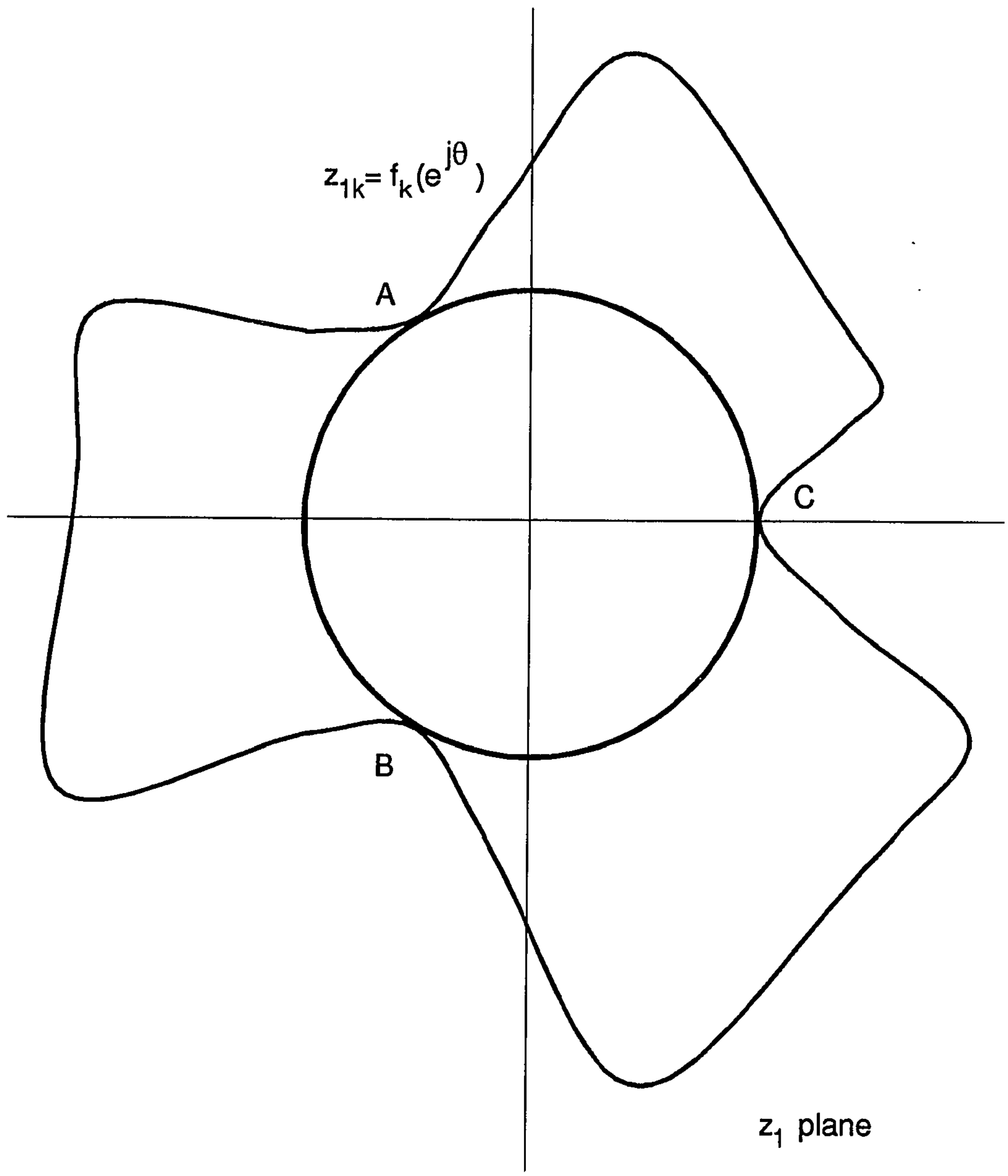
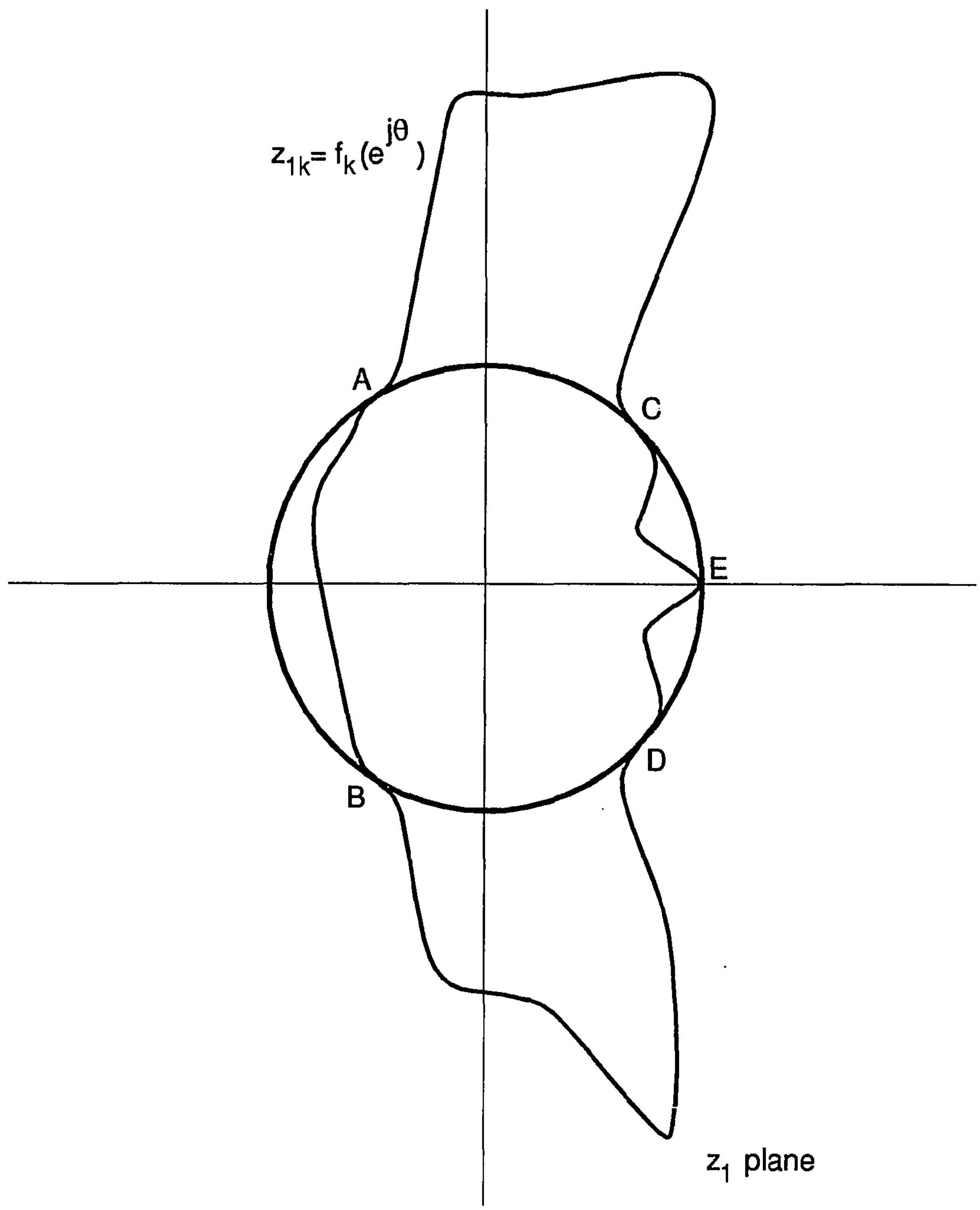


Figure 3: Possible locus of the root of $z_{1k} = f_k(\theta)$ when the necessary condition for $Q(z_1, z_2)$ to be a DSHP is satisfied



Chapter 4

Asymptotic Stability of m-D Digital Filters with the Presence of Poles in the Closed Unit Polydisk

In this chapter, a very simple proof is given that every 2-D digital filter with the real rational function $G(z_1, z_2) = P(z_1, z_2)/Q(z_1, z_2)$ is asymptotically stable if Q is a DSHP. Also, for the very special class of m-D (m>2) first and higher order filters, it is shown that the filter transfer function can possess nonessential singularities of the first kind (poles) in the closed unit polydisk, and that the filter can still be asymptotically stable.

4.1 m-D (m>2) Stability Definitions

Consider the class of hyperquadrant causal filters whose impulse responses have the first hyperquadrant as their region of support. A m-D filter of such class is characterized by its rational transfer function

$$G(z_1, \ldots, z_m) = \frac{P(z_1, \ldots, z_m)}{Q(z_1, \ldots, z_m)} \tag{4.1}$$

where P and Q are mutually prime polynomials in m variables. If there is a point $(z_1^a, \ldots, z_m^a)$, where

$$P(z_1^a, \ldots, z_m^a) = Q(z_1^a, \ldots, z_m^a) = 0 \tag{4.2}$$

then that point is called the nonessential singularity of the second kind (NSSK). Assuming that $Q(z_1, \ldots, z_m) \neq 0$ at the origin, by continuity argument it can be derived that $Q(z_1, \ldots, z_m) \neq 0$ in some neighborhood of the origin. Thus G can be expanded into power series in this neighborhood as:

$$G(z_1, \ldots, z_m) = \sum_{n_1=0} \cdots \sum_{n_m=0} g(n_1, \ldots, n_m) z_1^{n_1} \ldots z_m^{n_m} \tag{4.3}$$

where $g(n_1,\ldots,n_m)$ is the impulse response of $G(z_1,\ldots,z_m)$. We define

$$U^m = \{(z_1,\ldots,z_m) : |z_i| < 1,\ i = 1,\ldots,m\} \tag{4.4}$$

to be the open unit polydisk,

$$\overline{U}^m = \{(z_1,\ldots,z_m) : |z_i| \le 1,\ i = 1,\ldots,m\} \tag{4.5}$$

to be the closed unit polydisk, and

$$T^m = \{(z_1,\ldots,z_m) : |z_i| = 1,\ i = 1,\ldots,m\} \tag{4.6}$$

to be distinguished boundary of the unit polydisk.

The filter is said to be l_1 (BIBO) stable if and only if

$$g(n_1,\ldots,n_m) \in l_1 \Leftrightarrow \sum_{n_1=0}\cdots\sum_{n_m=0}|g(n_1,\ldots,n_m)| < \infty \tag{4.7}$$

It is l_2 stable if and only if

$$g(n_1,\ldots,n_m) \in l_2 \Leftrightarrow \sum_{n_1=0}\cdots\sum_{n_m=0}|g(n_1,\ldots,n_m)|^2 < \infty \tag{4.8}$$

and finally, a m-D digital filter is asymptotically stable if and only if

$$\lim_{n_1,\ldots,n_m\to\infty} g(n_1,\ldots,n_m) = 0 \tag{4.9}$$

Using the definition given in [12], a filter is practical-BIBO stable if and only if the following m inequalities are satisfied

$$\sum_{n_1=0}^{N_1}\cdots\sum_{n_k=0}^{N_k}\cdots\sum_{n_m=0}^{N_m}|g(n_1,\ldots n_m)| < \infty \quad \text{for } k = 1,\ldots,m \tag{4.10}$$

where $N_1,\ldots,N_{k-1},N_{k+1},\ldots,N_m$ are finite integers. In terms of filter transfer function (4.1), the latest condition (4.10) can be expressed as:

$$Q(0,\ldots,0,z_k,0,\ldots,0) \neq 0 \quad \text{for } z_k \in \overline{U} \qquad k = 1,\ldots,m \tag{4.11}$$

According to (4.10), a 2–D filter is practical-BIBO stable if $g(n_1, n_2)$ is row- and column-absolute summable [12]. In terms of (4.11), it is equivalent to:

$$
\begin{aligned}
Q(z_1, 0) &\neq 0 \quad \text{for } z_1 \in \overline{U} \\
Q(0, z_2) &\neq 0 \quad \text{for } z_2 \in \overline{U}
\end{aligned}
\tag{4.12}
$$

Finally, following the definition given in [51], a m-D first hyperquadrant causal digital filter is said to be asymptotically stable under all finitely extended bounded input signals $x(n_1, \ldots, n_m)$ where

$$
\begin{aligned}
|x(n_1, \ldots, n_m)| &\leq M \qquad && \text{for } n_1 + \ldots + n_m \leq R \\
x(n_1, \ldots, n_m) &= 0 && \text{for } n_1 + \ldots + n_m > R \\
n_i &\geq 0 && i = 1, \ldots, m
\end{aligned}
\tag{4.13}
$$

M being a bounded real number, R being some positive integer, if all the outputs of m-D digital filter $y(n_1, \ldots, n_m)$ expressed by m-dimensional convolution sum

$$
y(n_1, \ldots, n_m) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_m=0}^{n_m} g(n_1 - i_1, \ldots, n_m - i_m) \times x(i_1, \ldots, i_m) \tag{4.14}
$$

asymptotically reaches zero for $(n_1 + \ldots + n_m) \to 0$.

4.2 Asymptotic Stability of 2-D Digital Filters

The research on asymptotic stability of 2-D digital filters can be summarized in the following theorem:

Theorem 4.1: Let

$$
G(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)} \tag{4.15}
$$

represent a 2–D rational transfer function and let G has no polar singularities in $\overline{U}^2$ and no singularities of the second kind anywhere except for the simple ones at (α, β) and, hence, at $(1/\alpha, 1/\beta)$ on T^2. Further, let P and Q, and Q and its paraconjugate $\widetilde{Q}$ be relatively prime. Then $G(z_1, z_2)$ is asymptotically stable.

Proof: Referring to [18], if $H(z_1, z_2) = P_1(z_1, z_2)/Q(z_1, z_2)$ has no polar singularities in $\overline{U}^2$ and no singularities of the second kind anywhere except for the simple ones at (α, β) on T^2, and P_1 and Q, and Q and its paraconjugate $\widetilde{Q}$ are relatively prime, then a sufficient condition for G to be l_1 stable is that

$$m_\alpha(R_{z_2}[\underset{\sim}{Q}, Q]) < m_\alpha(R_{z_2}[P_1, Q]) \tag{4.16}$$

where m_α denotes the multiplicity of the factor $(z_1 - \alpha)$ in the z_2-resultant R_{z_2}. If Q and its paraconjugate $\widetilde{Q}$ are relatively prime, or in other words Q is a DSHP, then the left hand side of (4.16) is always finite. In case that $R_{z_2}[\underset{\sim}{Q}, Q] \equiv 0$ then the value of infinity is considered for the left hand side of (4.16) (section 2.2). Obviously,$H_k(z_1, z_2) = 1/Q(z_1, z_2)$ is not l_1 stable, but we can always find $P_1(z_1, z_2) = (z_1 - \alpha)^k$ to ensure l_1 stability of the 2–D digital filter described by its transfer function $H = P_1/Q$. Since Q is DSHP, $Q(z_1, 0) \neq 0$ in $\overline{U}$ and $Q(0, z_2) \neq 0$ in $\overline{U}$, according to [12], H represents practical-BIBO stable filter. Therefore, for $P_1(z_1, z_2) = (z_1 - \alpha)^k$, the sum of absolute values along row n_2 is finite,

$$\sum_{n_1=0}^{\infty} |h(n_1, n_2)| < \infty \tag{4.17}$$

for any positive integer k. Furthermore if H is l_1 stable, then $\lim_{n_1, n_2 \to \infty} h(n_1, n_2) = 0$.

Transfer function H can be represented as $H(z_1, z_2) = (z_1 - \alpha)H_1(z_1, z_2)$, where $H_1(z_1, z_2) = (z_1 - \alpha)^{k-1}/Q(z_1, z_2)$. Therefore, the impulse response $h(n_1, n_2)$, can be written as:

$$h(n_1, n_2) = h_1(n_1 - 1, n_2) - \alpha h_1(n_1, n_2) \tag{4.18}$$

Defining a positive integer N such that $|h(n_1, n_2)| \leq \Delta$ for $n_1 > N - 1$, where Δ is infinitesimally small positive number, from (4.18) it can be derived that

$$|h_1(n_1 - 1, n_2)| - |h_1(n_1, n_2)| \leq |h(n_1, n_2)| \leq \Delta \tag{4.19}$$

for $n_1 > N - 1$. Thus,

$$|h_1(N + p, n_2)| \geq |h_1(N, n_2)| - p\Delta \tag{4.20}$$

Therefore, the minimum value of the sum of absolute values of $h_1(n_1, n_2)$ along row n_2 can be expressed as:.

$$\sum_{n_1=0}^{\infty} |h_1(n_1, n_2)| = \frac{|h_1(n_1, n_2)|^2}{2\Delta} \tag{4.21}$$

showing that for sufficiently small Δ, the left hand side of (4.21) tends to infinity. According to (4.17), this contradicts the fact that $\sum_{n_1=0}^{\infty} |h_1(n_1, n_2)| < \infty$. The only way to resolve that contradiction is to conclude that $h_1(n_1, n_2) \to 0$. Keeping the same procedure, it can be shown that $H_2(z_1, z_2) = (z_1 - \alpha)^{k-2}/Q(z_1, z_2)$, H_3..., and finally $H_k = 1/Q$ are asymptotically stable.

Furthermore, since H_k is asymptotically stable, G is also asymptotically stable, due to fact that with

$$P(z_1, z_2) = \sum_{j_1=0}^{M_1} \sum_{j_2=0}^{M_2} b_{j_1,j_2} z_1^{n_1} z_2^{n_2} \tag{4.22}$$

from the expression of G (4.15) and the shift theorem

$$g(n_1, n_2) = \sum_{j_1=0}^{M_1} \sum_{j_2=0}^{M_2} b_{j_1,j_2} h_k(n_1 - j_1, n_2 - j_2) \\ u(n_1 - j_1, n_2 - j_2) \tag{4.23}$$

where $u(n_1, n_2)$ is m-D unit step, defined as

$$u(n_1, n_2) = \begin{cases} 1, & \text{for } \; n_i \geq 0, \quad i = 1, 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.24}$$

■

4.3 Asymptotic Stability of the First-Order m-D (m>2) Digital Filters with Poles in the Closed Unit Polydisk

The transfer function of a purely first order m-D digital filter is given by

$$G(z_1, \ldots, z_m) = \frac{P(z_1, \ldots, z_m)}{1 - a_1 z_1 - \ldots - a_m z_m} \tag{4.25}$$

where $P(z_1, \ldots, z_m)$ is a real polynomial in $(z_1, \ldots, z_m)$ given as

$$P(z_1, \ldots, z_m) = \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} b_{j_1,\ldots,j_m} z_1^{n_1} \ldots z_m^{n_m} \tag{4.26}$$

It has been shown in [52], that for structural stability (no singularities in $\overline{U}^m$), it is necessary and sufficient that

$$\sum_{i=1}^{m} |a_i| < 1 \tag{4.27}$$

By the same author, it was proven in [51], that the following condition:

$$\sum_{i=1}^{m} |a_i| \leq 1 \tag{4.28}$$

corresponds to the fact that there are no singularities in $\overline{U}^m$, except possibly on T^m. That means that all purely first order linear shift-invariant m-D (m≥2) digital filters with no singularities in $\overline{U}^m$, except possibly on T^m are asymptotically stable in the finite input sense.

In the following analysis, it is going to be investigated, whether is possible to have an asymptotically stable filter defined by the transfer function (4.25), with the condition

$$\sum_{i=1}^{m} |a_i| > 1 \tag{4.29}$$

If that filter is asymptotically stable, it has to be shown that $|g(n_1, \ldots, n_m)| \to 0$ when $(n_1 + \ldots + n_m) \to \infty$.

Let us start from the fact that the impulse response $h(n_1, \ldots, n_m)$ of the first order filter with the transfer function

$$H(z_1, \ldots, z_m) = \frac{1}{1 - a_1 z_1 - \ldots - a_m z_m} \tag{4.30}$$

is defined by

$$h(n_1, \ldots, n_m) = \frac{(n_1 + \ldots + n_m)!}{n_1! \ldots n_m!} a_1^{n_1} \ldots a_m^{n_m} u(n_1, \ldots, n_m) \tag{4.31}$$

where $u(n_1, \cdots, n_m)$ is m-D unit step, defined by

$$u(n_1,\ldots,n_m) = \begin{cases} 1, & \text{for} \quad n_i \geq 0, \qquad i = 1,\ldots,m \\ 0 & \text{otherwise} \end{cases} \tag{4.32}$$

Setting $n = (n_1 + \ldots + n_m)$, it follows that:

$$|h(n_1,\ldots,n_m,n)| = \frac{n!}{n_1!\ldots n_m!}|a_1|^{n_1}\ldots|a_m|^{n_m}u(n_1,\ldots,n_m) \tag{4.33}$$

from where it can be concluded that the absolute value of the impulse response

$$\frac{(n_1+\ldots+n_m)!}{n_1!\ldots n_m!}|a_1|^{n_1}\ldots|a_m|^{n_m} \tag{4.34}$$

is a term of the multinomial expansion:

$$(|a_1|+\ldots+|a_m|)^n = \sum_{n_1+\ldots+n_m=n} \frac{n!}{n_1!\ldots n_m!}|a_1|^{n_1}\ldots|a_m|^{n_m} \tag{4.35}$$

If the filter is asymptotically stable each of the terms of the multinomial expansion (4.35) has to tend to zero, namely

$$\frac{(n_1+\ldots+n_m)!}{n_1!\ldots n_m!}|a_1|^{n_1}\ldots|a_m|^{n_m} \to 0 \tag{4.36}$$

when $(n_1 + \ldots + n_m) = n \to \infty$.

Assuming that $|a_1| = |a_2| = \ldots = |a_m| = |a|/m$, the following binomial expansion has to be analyzed for m=2:

$$(|a_1|+|a_2|)^n = \left(\frac{|a|}{2}\right)^n \sum_{n_1,\, n_1+n_2=n} \frac{n!}{n_1!n_2!} \tag{4.37}$$

Since the binomial coefficients have the following properties for $n_1 + n_2 = n$:

$$\begin{aligned} &\frac{n!}{n_1!n_2!} \leq \frac{n!}{\left(\frac{n}{2}!\right)^2} && \text{for k even} \\ &\frac{n!}{n_1!n_2!} \leq \frac{n!}{\frac{(n+1)}{2}!\frac{(n-1)}{2}!} \leq \frac{(n+1)!}{\left(\frac{(n+1)}{2}!\right)^2} && \text{for k odd} \end{aligned} \tag{4.38}$$

it can be concluded that for $n \to \infty$

$$|h(n_1,n_2)| \leq \frac{(2n)!}{(n!)^2}\left(\frac{|a|}{2}\right)^{2n} \tag{4.39}$$

Representing n! [53] as:

$$n! = \sqrt{2\pi n}\ n^n e^{-n} e^{\Omega(n)/(12n+12)} \tag{4.40}$$

where $0 < \Omega(n) < 1 \quad \forall n$, it is obvious that

$$\lim_{n\to\infty} \frac{(2n)!}{(n!)^2}\left(\frac{|a|}{2}\right)^{2n} = \lim_{n\to\infty} |a|^{2n} \frac{1}{\sqrt{\pi n}}\ e^{\frac{\Omega(2n)}{24n+12} - \frac{\Omega(n)}{6n+6}} \tag{4.41}$$

Hence, under the condition that $|a_1| = |a_2| = |a|/2$, the first order filter with the transfer function (4.30) is asymptotically stable if and only if $|a| \leq 1$, which goes along the line of the condition (4.28). This result was basically obtained in [10] where the stability of the filter with the transfer function

$$G_4(z_1, z_2) = \frac{1}{1 - \frac{1}{2}z_1 - \frac{1}{2}z_2} \tag{4.42}$$

was analyzed. It was proven that the filter (4.42) is BIBO unstable but that it is asymptotically stable. It was also pointed that along the diagonal ($n_1 = n_2$), $g_4(n_1, n_2)$ approaches zero as $1/\sqrt{n}$, when n tends to infinity. This behavior contrasts with that in the one dimensional case where, if the impulse response approaches zero, it must do so geometrically.

Moving to the 3–D case, keeping the assumption that the absolute values of all the first order filter coefficients are equal ($|a_1| = |a_2| = |a_3| = |a|/3$), we are involved in the analysis of the following trinomial expansion:

$$(|a_1| + |a_2| + |a_3|)^n = \left(\frac{|a|}{3}\right)^n \sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!n_2!n_3!} \tag{4.43}$$

where

$$\sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!n_2!n_3!} = \sum_{n_1,\, n_1+n_2=n} \frac{n!}{n_1!n_2!} 1^{n_1}(1+1)^{n_2} \tag{4.44}$$

As can be seen, each term of the binomial expansion in the form of the right hand side of (4.44) is a product of the binomial coefficient and the term of the binomial expansion $(1+1)^{n_2}$. Using the properties of the binomial coefficients (4.38), it can be observed that when $n \to \infty$

$$\frac{n!}{n_1!n_2!}\frac{n_2!}{n_3!n_4!} \le \frac{n!}{n_1!n_2!}\frac{n_2!}{\left((\frac{n_2}{2})!\right)^2} \le \frac{n!}{\left((\frac{n}{2})!\right)^2}\frac{(\frac{n}{2})!}{\left((\frac{n}{4})!\right)^2!} \tag{4.45}$$

Therefore, the maximal term of the trinomial expansion on the left hand side of (4.44), and, hence the maximal coefficient of the trinomial expansion (4.43) is the product of the maximal term of the binomial expansion

$$(1+1)^{n_2} = \sum_{n_3,\ n_3+n_4=n_2} \frac{n_2!}{n_3!n_4!} \tag{4.46}$$

and the maximum of the binomial coefficients $n!/(n_1!n_2!)$ where $n_1+n_2 = n$. Obviously, the maximum of $n_2!/(n_3!n_4!)$, where $n_3 + n_4 = n_2$ depends of the value of n_2. Thus, using the properties of the binomial coefficients (4.38), it can be concluded that for $|a_1| = |a_2| = |a_3| = |a|/3$, each term in the trinomial expansion (4.43), and therefore $|h(n_1, n_2, n_3)|$ (4.33), satisfies the following relation when $n \to \infty$:

$$|h(n_1, n_2, n_3)| \le \frac{(4n)!}{(2n)!(n!)^2}\left(\frac{|a|}{3}\right)^{4n} \tag{4.47}$$

Using the expression for n! (4.40), it can be derived that

$$\begin{aligned} &\lim_{n\to\infty} \frac{(4n)!}{(2n)!(n!)^2}\left(\frac{|a|}{3}\right)^{4n} = \\ &\lim_{n\to\infty}\left(\frac{|a|^4 4^4}{2^2 3^4}\right)^n \frac{1}{\pi n\sqrt{2}}\, e^{\frac{\Omega(4n)}{48n+12}-\frac{\Omega(2n)}{24n+12}-\frac{\Omega(n)}{6n+6}} \end{aligned} \tag{4.48}$$

which shows that $|h(n_1, n_2, n_3)|$ in (4.33) tends to zero when $n = (n_1 + n_2 + n_3) \to \infty$, if and only if

$$\frac{|a|^4 4^4}{2^2 3^4} \le 1 \quad \Rightarrow \quad |a| \le \left(\frac{81}{64}\right)^{1/4} \approx 1.06 \tag{4.49}$$

proving that the condition (4.29) can be satisfied, and still have asymptotically stable purely-first order digital. That means that the condition (4.28) is only a sufficient condition for asymptotic stability, and that for m=3, there is a possibility for the existence of the transfer function singularities inside the unit tridisk. Thus, for example, setting

$a_1 = a_2 = a_3 = 1.05/3$, a 3–D first order digital filter is asymptotically stable even with the fact that its transfer function given by

$$H(z_1, z_2, z_3) = \frac{1}{1 - \frac{1.05}{3}z_1 - \frac{1.05}{3}z_2 - \frac{1.05}{3}z_3} \tag{4.50}$$

possesses singularity at the point $z_1 = z_2 = z_3 = 1/1.05$ in $\overline{U}^3$.

The general result on asymptotic stability for the special class of m-D (m>2) digital filters can be stated in the following theorem:

Theorem 4.2: All purely first order linear shift-invariant m-D (m>2) digital filters with the transfer function

$$G(z_1, \ldots, z_m) = \frac{P(z_1, \ldots, z_m)}{1 - a_1 z_1 - \ldots - a_m z_m} \tag{4.51}$$

where $P(z_1, \ldots, z_m)$ is real polynomial in $z_1, \ldots, z_m$, having the property that $|a_1| = \cdots = |a_m| = |a|/m$ are asymptotically stable in the finite input sense if and only if

$$|a| \leq \frac{m}{t_1}\left(\prod_{i=2}^{m-1} t_i^{t_i}\right)^{1/t_1} \tag{4.52}$$

where $t_i = 2^{m-i}$, thus allowing singularities in the closed unit polydisk $\overline{U}^m$.

Proof: Expressing $P(z_1, \ldots, z_m)$ as in (4.26), from the expression of G in (4.51) and the shift theorem, it can be concluded that G is asymptotically stable if

$$H(z_1, \ldots, z_m) = \frac{1}{1 - a_1 z_1 - \ldots - a_m z_m} \tag{4.53}$$

is asymptotically stable. For m=3, the asymptotic stability of (4.53) was previously proved. Applying the same procedure in order to extend the result to the higher order case (m>3), it can be observed that each coefficient of the multinomial expansion

$$\left(\frac{|a|}{m} + \cdots + \frac{|a|}{m}\right)^n = \left(\frac{|a|}{m}\right)^n (1 + \cdots + 1)^n \tag{4.54}$$

is the product of m-1 binomial coefficients

$$\left.\frac{n!}{n_1! n_2!}\right|_{n_1+n_2=n} \quad \left.\frac{n_2!}{n_3! n_4!}\right|_{n_3+n_4=n_2} \quad \cdots \quad \left.\frac{n_{2m-4}!}{n_{2m-3}! n_{2m-2}!}\right|_{n_{2m-3}+n_{2m-2}=n_{2m-4}} \tag{4.55}$$

where each of the product terms, except the very first one, depends on the preceding one. Thus, maximizing the first term in (4.55), and then all consecutive ones, we end up with the fact that each term of the multinomial expansion (4.54) is less or equal to

$$\frac{(t_1 n)!}{n! \prod_{i=1}^{m-2} (t_i n)!} \left(\frac{|a|}{m} \right)^{t_1 n} \tag{4.56}$$

Using the expression for the n! (4.40), it can be seen that the limit of (4.56) tends to zero when n goes to infinity, if the condition (4.52) of the Theorem 4.2 is satisfied. The allowance of singularities in the closed unit polydisk $\overline{U}^m$ follows directly from the fact that $\sum_{i=1}^{m} |a_i| > 1$ ■

From the result of Theorem 4.2, it can be seen that the stability margin (SM) of the filter increases as m increases. Thus, for m=2, we have zero stability margin, but for m=3, SM≈0.06, for m=4, SM≈0.189, for m=5, SM≈0.36, and so on. The increase in the stability margin of the filter follows naturally according to the fact that the number of terms in the multinomial expansion of the type (4.35) enormously increases as m increases. Thus, for example, there are n+1 terms in the binomial expansion, while the number of terms in the trinomial expansion equals to (n+1)(n+2)/2.

4.4 Asymptotic Stability of the Higher Order Filters

Extending the result of Theorem 4.2 to the higher order filters, the variable transformation introduced in [51] is considered:

Let us define

$$\begin{aligned} z_1' &= z_1^{L_{11}} \cdots z_m^{L_{1m}} \\ &\vdots \\ z_m' &= z_1^{L_{m1}} \cdots z_m^{L_{mm}} \end{aligned} \tag{4.57}$$

and

$$\det(L) = \begin{vmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & & \vdots \\ L_{1m} & \cdots & L_{mm} \end{vmatrix} \tag{4.58}$$

where L_{ij} are nonnegative integers. Therefore, the resulting transfer function $H(z_1, \ldots, z_m)$, given by

$$H(z_1, \ldots, z_m) = \frac{1}{1 - a_1 z_1^{L_{11}} \cdots z_m^{L_{1m}} - \cdots - a_m z_1^{L_{m1}} \cdots z_m^{L_{mm}}} \tag{4.59}$$

has the same impulse response as:

$$H(z_1', \ldots, z_m') = \frac{1}{1 - a_1 z_1' - \ldots - a_m z_m'} \tag{4.60}$$

if $h(n_1, \ldots, n_m)$ is described in a new coordinate system with unit base vectors:

$$\begin{aligned} e_1 &= (L_{11} \cdots L_{1m})^T \\ &\vdots \\ e_m &= (L_{m1} \cdots L_{mm})^T \end{aligned} \tag{4.61}$$

where e_i, $i = 1, \ldots, m$ describes the base vectors of the system $(n_1, \ldots, n_m)$ in the new coordinate system $(n_1', \ldots, n_m')$.

For the class of the higher order filters defined above, together with the condition that $|a_1| = \cdots = |a_m| = |a|/m$, we have a new class of the asymptotically stable higher order digital filters with the singularities in $\overline{U}^m$, i.e. $\sum_{i=1}^{m} |a_i| > 1$.

Example 4.1: Consider the 4–D transfer function

$$G(z_1, z_2, z_3, z_4) = \frac{(z_1 z_2 z_3 z_4 - 1)(z_2^2 - 1)}{1 - b(z_1^2 z_2 - z_2^2 z_3 - z_3^4 z_4 - z_1^5 z_4^7)} \tag{4.62}$$

Since

$$\det(L) = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 5 & 0 & 0 & 7 \end{vmatrix} \neq 0 \tag{4.63}$$

it was shown in [51] that the G of (4.62) is asymptotically stable for b=1/4. By Theorem 4.2, asymptotic stability is guaranteed for $b \leq 2^{-7/4}$.

4.5 Implementation of the New Result on Asymptotic Stability of m-D (m>2) Direct Realization Filters Under the Influence of Nonlinearities

Multidimensional digital filters are realized with a fixed point arithmetics (finite word length). Such realization introduces various types of nonlinearities like magnitude truncation quantization, saturation overflow etc.. The effect of nonlinearities on the stability of m-D has been investigated in [54], [55], [56], [57], and more extensively in the recent publication of Bauer and Bauer and Jury [51], [52], [58] and [59].

Considering direct form realization, the output of the filter can be described by the following nonlinear difference equation:

$$\begin{aligned} y(n_1, \cdots, n_m) = &\sum_{\substack{i_1=0 \\ (i_1,\cdots,i_m)\neq(0,\cdots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} Q_{i_1\cdots i_m}(a_{i_1\cdots i_m} y(n_1 - i_1, \cdots, n_m - i_m)) \\ &+ \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} Q_{j_1\cdots j_m}(b_{j_1\cdots j_m} x(n_1 - j_1, \cdots, n_m - j_m)) \end{aligned} \tag{4.64}$$

where all of the nonlinearities satisfy the sector condition

$$-1 \leq \frac{Q_{i_1\cdots i_m}(x)}{x} \leq 1, \qquad x \neq 0 \tag{4.65}$$

and

$$Q_{i_1\cdots i_m}(0) = 0 \tag{4.66}$$

or equivalently,

$$Q_{i_1\cdots i_m}(x) = k_{i_1\cdots i_m} x \tag{4.67}$$

where $k_{i_1\cdots i_m} \in [-1, 1]$. A single nonlinearity $Q_{i_1,\cdots i_m}$ in (4.64) can represent the concatenation of multiple nonlinearities, where each of them satisfy conditions (4.65) and (4.66). Asymptotic stability of (4.64) can be checked through asymptotic stability of a positive coefficient linear shift-invariant filter.

Theorem 4.3 [51]: The nonlinear difference equation (4.64) is asymptotically stable under all finitely extended input signals (4.13), if the linear shift-invariant filter, whose output is described by the linear difference equation:

$$\begin{aligned} y_{lin}(n_1,\cdots,n_m) &= \sum_{\substack{i_1=0 \\ (i_1,\cdots,i_m)\neq(0,\cdots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} |a_{i_1\cdots i_m}| y_{lin}(n_1-i_1,\cdots,n_m-i_m) \\ &+ \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} |b_{j_1\cdots j_m}| x_{lin}(n_1-j_1,\cdots,n_m-j_m) \end{aligned} \tag{4.68}$$

is asymptotically stable under all finitely extended input signals.

Proof: Using the comparison principle [51], [59], [56], [57], and assuming the following:

a) zero initial conditions for the nonlinear filter, namely

$$y(n_1-i_1,\cdots,n_m-i_m)=0 \tag{4.69}$$

if $i_k > n_k$, for some $k = 1,\ldots,m$.

b) the initial states of the linear reference filter such that

$$|y(n_1-i_1,\cdots,n_m-i_m)| \le y_l(n_1-i_1,\cdots,n_m-i_m) \tag{4.70}$$

c) zero initial condition of the linear filter

$$|y(n_1-i_1,\cdots,n_m-i_m)| = y_{lin}(n_1-i_1,\cdots,n_m-i_m)=0 \tag{4.71}$$

d) the input signal to linear filter such that

$$x_{lin}(n_1-i_1,\cdots,n_m-i_m) = |x(n_1-i_1,\cdots,n_m-i_m)| \tag{4.72}$$

for all $n_k \geq 0$, $k = 1, \ldots, m$, it can be determined that for the first recursion:

$$
\begin{aligned}
|y(n_1, \cdots, n_m)| &= \Bigg| \sum_{\substack{i_1=0 \\ (i_1,\cdots,i_m)\neq(0,\cdots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} k_{i_1\cdots i_m} a_{i_1\cdots i_m} y(n_1 - i_1, \cdots, n_m - i_m) \\
&+ \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} k_{j_1\cdots j_m} b_{j_1\cdots j_m} x(n_1 - j_1, \cdots, n_m - j_m) \Bigg| \\
&\leq \sum_{\substack{i_1=0 \\ (i_1,\cdots,i_m)\neq(0,\cdots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} |a_{i_1\cdots i_m}| |y(n_1 - i_1, \cdots, n_m - i_m)| \\
&+ \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} |b_{j_1\cdots j_m}| |x(n_1 - j_1, \cdots, n_m - j_m)| \\
&\leq \sum_{\substack{i_1=0 \\ (i_1,\cdots,i_m)\neq(0,\cdots,0)}}^{N_1} \cdots \sum_{i_m=0}^{N_m} |a_{i_1\cdots i_m}| y_{lin}(n_1 - i_1, \cdots, n_m - i_m) \\
&+ \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} |b_{j_1\cdots j_m}| x_{lin}(n_1 - j_1, \cdots, n_m - j_m) \\
&= y_{lin}(n_1 - i_1, \cdots, n_m - i_m) = |y_{lin}(n_1 - i_1, \cdots, n_m - i_m)|
\end{aligned}
\tag{4.73}
$$

The last equation yields the result

$$
y_{lin}(n_1 - i_1, \cdots, n_m - i_m) \geq |y(n_1 - i_1, \cdots, n_m - i_m)| \tag{4.74}
$$

From (4.70) and (4.73) it follows that the condition (4.74) is satisfied for all further iterations assuming the proper choice of the input to the linear reference filter (4.72). Therefore, if the linear filter is asymptotically stable under all finitely extended bounded input signals, then according to (4.73), the nonlinear filter must also be stable in the same sense. The output of the linear positive coefficient filter (4.68) provides an upper bound for the output of the nonlinear filter (4.64).

Recalling the results of Theorem 4.2 and the discussions made in the previous section, the following corollaries from Theorem 4.3 can be obtained:

Corollary 4.1: All purely first order m–D (m>2) nonlinear digital filters of the form

$$\begin{aligned} y(n_1,\cdots,n_m) &= Q_1(a_1 y(n_1-1,n_2,\cdots,n_m)) \\ &+ Q_2(a_2 y(n_1,n_2-1,n_3,\cdots,n_m)) \\ &\vdots \\ &+ Q_m(a_m y(n_1,\cdots,n_{m-1},n_m-1)) \\ &+ \sum_{j_1=0}^{M_1}\cdots\sum_{j_m=0}^{M_m} Q_{j_1\cdots j_m}(b_{j_1\cdots j_m} x(n_1-j_1,\cdots,n_m-j_m)) \end{aligned} \tag{4.75}$$

with the nonlinearities satisfying the sector condition (4.67) are asymptotically stable in the finite extend input case if $|a_1| = \cdots = |a_m| = |a|/m$ and

$$|a| \le \frac{m}{t_1}\left(\prod_{i=2}^{m-1} t_i^{t_i}\right)^{1/t_1} \tag{4.76}$$

where $t_i = 2^{m-i}$.

Corollary 4.2: All m-D (m>2) nonlinear digital filters of the form

$$\begin{aligned} y(n_1,\cdots,n_m) &= Q_1(a_1 y(n_1-L_{11},\cdots,n_m-L_{1m})) \\ &+ Q_2(a_2 y(n_1-L_{21},\cdots,n_m-L_{2m})) \\ &\vdots \\ &+ Q_m(a_m y(n_1-L_{m1},\cdots,n_m-L_{mm})) \\ &+ \sum_{j_1=0}^{M_1}\cdots\sum_{j_m=0}^{M_m} Q_{j_1\cdots j_m}(b_{j_1\cdots j_m} x(n_1-j_1,\cdots,n_m-j_m)) \end{aligned} \tag{4.77}$$

$$\det(L) = \begin{vmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & & \\ L_{m1} & \cdots & L_{mm} \end{vmatrix}$$

with the nonlinearities satisfying the sector condition (4.67) are asymptotically stable in the finite extend input case if $|a_1| = \cdots = |a_m| = |a|/m$ and (4.74) is satisfied.

Corollary 4.3: All m-D (m>2) nonlinear digital filters of the form (4. 75) and (4.77), where $|a_1| = \cdots = |a_m| = 1/m$ are asymptotically stable in the finite extend input case with the nonlinearities satisfying the sector condition

$$Q_{i_1\cdots i_m}(x) = k_{i_1\cdots i_m} x \tag{4.78}$$

where

$$k_{i_1 \cdots i_m} \in \left[-\frac{m}{t_1} \left(\prod_{i=2}^{m-1} t_i^{t_i} \right)^{1/t_1}, \frac{m}{t_1} \left(\prod_{i=2}^{m-1} t_i^{t_i} \right)^{1/t_1} \right] \tag{4.79}$$

Chapter 5

Exponential Transformation of a Polynomial and a New Bound for the Polynomial Zeros

In this chapter, the classical problem of determination of the upper bound for the polynomial zeros is treated. After a brief review of the existing results, the idea of exponential polynomial transformation, which leads to a new upper bound for the polynomial zeros is presented.

5.1 Bounds on Zeros of a Polynomial

The problem of determining bounds on polynomial zeros has been the subject of intensive research for a long time. It goes back to 1829, when Cauchy [60] gave a very simple expression for the bound in terms of polynomial coefficients. His result states that all the zeroes of the complex polynomial

$$P(z) = z^n + \sum_{i=0}^{n-1} a_i z^i, \qquad a_i \in \mathbb{C} \tag{5.1}$$

lie in the circle

$$|z| < 1 + A \tag{5.2}$$

where

$$A = \max_i \{|a_i|\}, \qquad i = 0, \cdots, n-1 \tag{5.3}$$

Numerous extensions of this result have been given. Chapter VII of the comprehensive book by Marden [54] presents several bounds for the zeros of $P(z)$ as functions of its coefficients. Thus, according to [54, theorem 27.3], the zero z_1 of the largest modulus of $P(z)$, satisfies the inequality

$$r \geq |z_1| \geq \left(2^{1/n} - 1\right) r \tag{5.4}$$

where r is the positive root of the equation

$$|a_0| + |a_1| z + \cdots + |a_{n-1}| z^{n-1} - z^n = 0 \tag{5.5}$$

By [54, equation 27.19], $P(z)$ (5.1) has all its zeros in the circle

$$|z| < \left\{ 1 + \sum_{i=0}^{n-1} |a_i|^2 \right\}^{1/2} \tag{5.6}$$

by [54, equation 27.25] in the circle

$$|z| \leq \left\{ 1 + |a_0|^2 + |a_1 - a_0|^2 + \cdots + |a_{n-1} - a_{n-2}|^2 + |1 - a_{n-1}|^2 \right\}^{1/2} \tag{5.7}$$

and by [54, equation 27.26], in the circle

$$|z| \leq \sum_{i=1}^{n} |a_{n-i}|^{1/i} \tag{5.8}$$

A. Joyal, G. Labelle and Q. I. Rahman in [61] defined a smaller circle than Cauchy's which contains all the zeros of $P(z)$. [61, theorem 1] claims the following: if $B = \max_{0 \leq i \leq n-1} |a_i|$ then all the zeros of $P(z)$ are contained in the circle

$$|z| \leq \frac{1}{2} \left\{ 1 + |a_{n-1}| + \sqrt{(1 - |a_{n-1}|)^2 + 4B} \right\} \tag{5.9}$$

According to [61, corollary 2 of theorem 1], the polynomial $P(z)$ (5.1) has all its zeros in the circle

$$|z| \leq \frac{1}{2} \left(1 + \sqrt{1 + 4B'} \right) \tag{10}$$

where

$$B' = \max_{0 \leq i \leq n-1} |a_{n-1} a_i - a_{i-1}|, \qquad (a_{-1} = 0) \tag{5.11}$$

By [61, corollary 3 of theorem 2], the polynomial $P(z)$ (5.1) has all its zeros in the circle

$$|z| \leq 1 + \sqrt{B''} \tag{5.12}$$

where

$$B'' = \max_{0 \leq i \leq n-1} |(1 - a_{n-1})a_i + a_{i-1}|, \qquad (a_{-1} = 0) \tag{5.13}$$

B. Datt and N. K. Govil in [62], presented two theorems, defining ring-shaped regions which contain all the polynomial zeros. [62, theorem 1] states that $P(z)$ (5.1) has all its zeros in the region

$$\frac{|a_0|}{2(1+A)^{n-1}(nA+1)} \leq |z| \leq 1 + \lambda_0 A \tag{5.14}$$

where $A = \max_{0 \leq i \leq n-1} |a_i|$ and λ_0 is the unique root of the equation $x = 1 - 1/(1 + Ax)^n$ in the interval (0,1). By [62, theorem 1], $P(z)$ (5.1) has all its zeros in the ring-shaped region given by

$$\frac{|a_0|}{2(1+A)^{n-1}(nA+1)} \leq |z| \leq 1 + \left(1 - \frac{1}{(1+A)^n}\right)A \tag{5.15}$$

where $A = \max_{0 \leq i \leq n-1} |a_i|$

F. G. Boese and W. J. Luther [63] gave an estimate for the smallest disk $|z| < R$ containing all the zeros of $P(z)$ (5.1). In their theorem R is defined as

$$R = \begin{cases} \left[\dfrac{A(1-nA)}{1-(nA)^{1/n}}\right]^{1/n}, & A \leq 1/n \\ \min\left\{(1+A)\left(1 - \dfrac{A}{(1+A)^{n+1} - nA}\right),\right. & \\ \left. 1 + \dfrac{2(nA-1)}{n+1}\right\}, & A \geq 1/n \end{cases} \tag{5.16}$$

Recently, Zeheb [64] gave a new extension of Cauchy's result. He introduced polynomial transformations, treating separately cases when an original polynomial is real and complex. Minimizing Cauchy's bound of transformed pair first, Zeheb defined two circular bounds of the original polynomial. Analyzing the real case ($a_i \in \mathbb{R}$), Zeheb transformed $P(z)$ (5.1) into the polynomial

$$Q(z) = (z+\alpha)P(z) = z^{n+1} + \sum_{k=0}^{n} (\alpha a_k + a_{k-1}) z^k \tag{5.17}$$

since all the zeros of $P(z)$ are included in the set of all the zeros of $Q(z)$, he determined the value of α which minimizes Cauchy's bound for the zeros of $Q(z)$. Thus, at the same time he defined a new circular bound for the zeros of the original polynomial $P(z)$ (5.1). The result is stated in [64, theorem 1] as:

$$|z| < 1 + \max_i \{A_{ij}\} \tag{5.18}$$

where

$$A_{ij} = \frac{|a_i a_{j-1} - a_j a_{i-1}|}{|a_i| + |a_j|}, \qquad i,j = 0,\cdots,n; \;\; j > i \qquad a_n = 1, \quad a_{-1} = 0 \tag{5.19}$$

For the case when $P(z)$ (5.1) is a complex polynomial, in [64, theorem 2], he stated that all its zeros lie in the circle

$$|z| < \rho_0 + |a_{i_l}| \rho_0^{i_l - n + 1} \tag{5.20}$$

where ρ_0 is defined as follows:

$$\rho_1 = \min_{1 \le i \le n-1} \left\{ \left(\frac{|a_0|}{|a_i|} \right)^{1/i} \right\} \tag{5.21}$$

and suppose ρ_1 is obtained for $i = i_1$. If

$$g_{i_1} = 1 + |a_{i_1}|(i_1 - n + 1)\rho_1^{i_1 - n} \ge 0 \tag{5.22}$$

then

$$\rho_0 = \min \left\{ \rho_1, [|a_0|(n-1)]^{1/n} \right\} \tag{5.23}$$

If $g_{i_1} < 0$, define

$$\rho_2 = \min_{\substack{1 \le i \le n-1 \\ i \ne i_1}} \left\{ \left(\frac{|a_{i_1}|}{|a_i|} \right)^{1/i - i_1} \right\} \tag{5.24}$$

and suppose ρ_2 is obtained for $i = i_2$. If

$$g_{i_2} = 1 + |a_{i_2}|(i_2 - n + 1)\rho_2^{i_2 - n} \geq 0 \tag{5.25}$$

then

$$\rho_0 = \min\left\{\rho_2, [|a_{i_1}|(n - i_1 - 1)]^{1/n - i_1}\right\} \tag{5.26}$$

If $g_{i_2} < 0$, continue with this process by defining

$$\rho_3 = \min_{\substack{1 \leq i \leq n-1 \\ i \neq i_1, i \neq i_2}} \left\{\left(\frac{|a_{i_2}|}{|a_i|}\right)^{1/i - i_2}\right\} \tag{5.27}$$

and so on. The process terminates when $g_i > 0$ or when all values of $i = 1, \cdots, n-1$are exhausted. Thus, we can say that g_0 is defined as

$$\rho_0 = \min\left\{\rho_{l+1}, [|a_{i_l}|(n - i_l - 1)]^{1/n - i_l}\right\} \tag{5.28}$$

5.2 A New Upper Bound for the Zeros Of a Polynomial

Using the idea of a polynomial transformation in the process of determination of the upper bound for the polynomial zeros, the following concept is proposed:
Let us first define a polynomial

$$R(z) = P^m(z) \tag{5.29}$$

where $P(z)$ is defined by (5.1). Change of variable $z_1 = z^m$ in $R(z)$, and the initial assumption that $P(z) = 0$ lead to the definition of a polynomial

$$P_1(z) = z_1^n + \sum_{k=0}^{n-1} a_k^{(1)} z_1^k \tag{5.30}$$

By Cauchy's theorem [60], all the zeros of $P_1(z)$ lie in the circle

$$|z_1| < 1 + \max_{0 \leq k \leq n-1} \left|a_k^{(1)}\right| \tag{5.31}$$

Hence, all the zeros of a polynomial $P(z)$ lie in the circle

$$|z| < \left(1 + \max_{0 \leq k \leq n-1} \left|a_k^{(1)}\right|\right)^{1/m} \tag{5.32}$$

This result conceptually should give a bound improvement. Unfortunately, it is very difficult to find the general expression for $a_k^{(1)}$'s. In order to simplify that problem, expression of $P(z)$ (5.1) as a sum of even and odd degree polynomials $P(z) = P_{even} + P_{odd}$ is suggested.

Statement 5.1: If $P(z) = 0$, then

$$P_{even}^m - (-1)^m P_{odd}^m = 0 \tag{5.33}$$

Proof: Statement follows from the fact that

$$P_{even}^m - (-1)^m P_{odd}^m = (P_{even} + P_{odd}) \sum_{j=0}^{m-1} P_{even}^{m-1-j} (-1)^j P_{odd}^j \tag{5.34}$$

Therefore, instead of deriving $R(z)$ the polynomial

$$R_1(z) = (-1)^{m+n} (P_{even}^m - (-1)^m P_{odd}^m) \tag{5.35}$$

can be derived, from which $P_1(z_1)$ can be defined. Factor $(-1)^{m+n}$ ensures that $a_n^{(1)} = 1$. Still, for $m > 2$ the determination of $P_1(z_1)$ is very complicated. Assuming $m = 3$ and $n = 3$, (5.1) has the form

$$P(z) = z^3 + a_2 z^2 + a_1 z + a_0 \tag{5.36}$$

Using (5.35)

$$\begin{aligned} R_1(z) &= \left(z^3 + a_1 z\right)^3 + \left(a_2 z^2 + a_0\right)^3 \\ &= z^9 + a_2^3 z^6 + a_1^3 z^3 + a_0^3 + 3a_1 z^4 (z^3 + a_1 z) \\ &\qquad + 3a_0 a_2 z^2 (a_2 z^2 + a_0) \end{aligned} \tag{5.37}$$

With the initial assumption that $P(z) = 0$, (5.37) becomes

$$\begin{aligned} R_1(z) &= z^9 + a_2^3 z^6 + a_1^3 z^3 + a_0^3 + 3a_1 z^4 (-a_2 z^2 - a_0) \\ &\qquad + 3a_0 a_2 z^2 (-z^3 - a_1 z) \end{aligned}$$

$$= z^9 + \left(a_2^3 - 3a_1a_2\right)z^6 + \left(a_1^3 - 3a_0a_1a_2\right)z^3 + a_0^3 - 3a_0z^3\left(a_2z^2 + a_1z\right)$$

$$= z^9 + \left(a_2^3 - 3a_1a_2\right)z^6 + \left(a_1^3 - 3a_0a_1a_2\right)z^3 + a_0^3 - 3a_0z^3\left(-z^3 - a_0\right) \tag{5.40}$$

Change of variable $z_1 = z^3$ in (5.40) leads to

$$P_1(z_1) = z_1^3 + a_2^{(1)}z + a_1^{(1)}z + a_0^{(1)} \tag{5.41}$$

where

$$\begin{aligned} a_2^{(1)} &= \left(a_2^3 - 3a_1a_2 + 3a_0\right) \\ a_1^{(1)} &= \left(a_1^3 - 3a_0a_1a_2 + 3a_0^2\right) \\ a_0^{(1)} &= a_0^3 \end{aligned} \tag{5.42}$$

From what was just shown, the only convenient choice for the proposed exponential transformation of a polynomial would be $m = 2$. It can lead to the general expression for the $a_k^{(1)}$'s in (5.30).

Theorem 5.1: All the zeros of a polynomial

$$P(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \tag{5.43}$$

lie in the disk

$$\left\{ z \in \mathbb{C} \mid |z| < \sqrt{1 + A} \right\} \tag{5.44}$$

with

$$A = \max_{0 \le k \le n-1} \left\{ \left| a_k^2 + 2(-1)^k (B - C) \right| \right\} \tag{5.45}$$

where

$$B = \sum_{\substack{0 \le i < j \le [n/2] \\ i+j=k}} a_{2i}a_{2j} \tag{5.46}$$

$$C = \sum_{\substack{0 \le i < j \le [(n-1)/2] \\ i+j=k-1}} a_{2i+1}a_{2j+1} \tag{5.47}$$

$a_n = 1$, $a_{n+1} = 0$ and $[l]$ denotes the integer part of l.

Proof: Expressing $P(z)$ as a sum of even and odd degree polynomials and recalling the statement (5.33) for $m = 2$, by (5.35) we form

$$R_1(z) = (-1)^n (P^2_{even}(z) - P^2_{odd}(z)) = \sum_{k=0}^{n} a_k^{(1)} z^{2k} \tag{5.48}$$

with

$$a_k^{(1)} = (-1)^n \left\{ (-1)^k a_k^2 + 2(B - C) \right\} \tag{5.49}$$

The change of variable $z_1 = z^2$ in $R_1(z)$ leads to the

$$P_1(z_1) = \sum_{k=0}^{n} a_k^{(1)} z_1^k \tag{5.50}$$

From the expression for $a_k^{(1)}$ (5.49), it can be seen that $a_n^{(1)} = 1$ and Cauchy's bound can be applied to $P_1(z_1)$, which gives the following bound for the zeros of $P_1(z_1)$

$$|z_1| < 1 + A \tag{5.51}$$

where $A = \max\limits_{0 \le k \le n-1} \left\{ \left| a_k^{(1)} \right| \right\}$. Therefore, it can be concluded that a bound for the zeros of $P(z)$ is

$$|z| < \sqrt{1 + A} \tag{5.52}$$

Example 5.1: Let

$$P(z) = z^3 + 3z^2 + 2z + 1 \tag{5.53}$$

Using (5.45), $A = \max\{1, 2, 5\}$ and the bound on zeros of $P(z)$ in (5.53) is

$$|z| < \sqrt{6} \approx 2.45 \tag{5.54}$$

This example was given in [64], where Zeheb's bound is $|z| < 2.75$ [64, theorem 1] (5.18), showing the improvement over other existing bounds given by:

Cauchy [60], (5.2) $|z| < 4$,

[54, theorem 27.3], (5.4) $|z| \leq 3.627$,

[54, eq. (27.19)], (5.6) $|z| < 3.873$,

[54, eq. (27.25)], (5.7) $|z| \leq 2.828$,

[54, eq. (27.26)], (5.8) $|z| \leq 5.414$,

Joyal et. al. [61, theorem 1], (5.9) $|z| \leq 4$,

[61, th. 1 (corollary 2)], (5.10) $|z| \leq 3.193$,

[61, th. 1 (corollary 3)], (5.12) $|z| \leq 3$,

Datt and Govil [62, theorem 1], (5.14) $|z| \leq 3.951$,

[62, theorem 2], (5.15) $|z| \leq 3.953$,

Boese and Luther [63], (5.16) $|z| \leq 3.951$

Example 5.2: Let

$$P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1 \tag{5.55}$$

Using (5.45), $A = \max\{1, 2, 5, 10\}$ and the bound on zeros of $P(z)$ in (5.55) is

$$|z| < \sqrt{11} \approx 3.3166 \tag{5.56}$$

The other bounds are:

Zeheb [64, theorem 1], (5.18) $|z| < 3.6$,

Cauchy [60], (5.2) $|z| < 5$,

[54, theorem 27.3], (5.4) $|z| \leq 4.733$,

[54, eq. (27.19)], (5.6) $|z| < 5.568$,

[54, eq. (27.25)], (5.7) $|z| \leq 3.742$,

[54, eq. (27.26)], (5.8) $|z| \leq 7.992$,

Joyal et. al. [61, theorem 1], (5.9) $|z| \leq 5$,

[61, th. 1 (corollary 2)], (5.10) $|z| \leq 4.14$,

[61, th. 1 (corollary 3)], (5.12) $|z| \leq 4$,

Datt and Govil [62, theorem 1], (5.14) $|z| \leq 4.9936$, [62, theorem 2], (5.15) $|z| \leq 4.9936$,

Boese and Luther [63], (5.16) $|z| \leq 4.9936$.

5.3 Bounds Improvement and Recursive Application of the Bound

If the bound (5.45) is applied on the polynomial $P_1(z_1)$ (5.50) instead of $P(z)$ (5.43), looking for A in terms of the $a_k^{(1)}$'s, the bound for all zeros of $P(z)$ can be further improved, bearing in mind that $|z| = \sqrt{|z_1|}$. All other existing bounds can be also improved when applied to $P_1(z_1)$. This is illustrated in the following examples:

Example 5.3: Let

$$P(z) = z^3 + 3z^2 + 2z + 1 \tag{5.57}$$

as in example 1. Using (5.49) and (5.50)

$$P_1(z_1) = z_1^3 - 5z_1^2 - 2z_1 - 1 \tag{5.58}$$

From (5.45) and (5.44) $|z_1| < \sqrt{30}$ and finally

$$|z| < \sqrt{\sqrt{30}} \approx 2.34 < 2.45 \tag{5.59}$$

Example 5.4: Let

$$P(z) = z^3 + 0.1z^2 + 0.1z + 0.04 \tag{5.60}$$

According to [64, theorem 2], (5.20), where the same example was also given, $|z| < 0.65$, but starting from $P(z)$, using (5.49) and (5.50), we end up with

$$P_1(z_1) = z_1^3 + 0.19z_1^2 + 0.002z_1 - 0.0016 \tag{5.61}$$

Referring to the same theorem when determining a bound for the zeros of $P_1(z_1)$ (5.61), and using the fact that $|z| = \sqrt{|z_1|}$, we find that all the zeros of $P(z)$ lie in the disk

$$\{z \in \mathbf{C} |\ |z| < 0.4382\} \tag{5.62}$$

Example 5.5: Let

$$P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1 \tag{5.63}$$

as in Example 5.2.According to (5.49) and (5.50)

$$P_1(z_1) = z_1^4 - 10z_1^3 - 5z_1^2 + 2z_1 + 1 \tag{5.64}$$

Referring to [62, theorem 2], (5.15) $|z_1| \leq 10.999$. Hence,

$$|z| \leq \sqrt{10.999} \approx 3.316 \tag{5.65}$$

which is better then the original bound $|z| \leq 4.99$ for $P(z)$ in (5.63) obtained according to [62, theorem 2], also given in Example 2.

As it can be seen from (5.59), the bound (5.44) on the zeros of the original polynomial $P(z)$ in (5.43) is improved when applied to transformed polynomial $P_1(z_1)$ in (5.50). This procedure can be applied iteratively. $P_1(z_1)$ in (5.50) can be further transformed using (5.49) into $P_2(z_2)$ and $P_2(z_2)$ into $P_3(z_3)$ until the last iteration brings no practical improvement. In practice it turns out that very few iterations are needed. If modulus of one of the polynomial coefficients is much greater than the others, the bound for all zeros of the polynomial $P_r(z_r)$ is approximately equal to $\sqrt{1 + \left|a_k^{(r)}\right|^2_{\max}}$, meaning that the iterative procedure is over when

$$\left|a_k^{(r+1)}\right|_{\max} \approx \left|a_k^{(r)}\right|^2_{\max} \tag{5.66}$$

. Then all zeros of the original polynomial $P(z)$, lie in the disk

$$\left\{ z \in \mathbf{C} \middle| \, |z| < \left(1 + \left|a_k^{(r)}\right|_{\max}\right)^{2^{-r}} \right\} \tag{5.67}$$

If we start the iterative procedure on $P(z)$ given in Example 5.2 (5.55), then

$$\begin{aligned} P_1(z_1) &= z_1^4 - 10z_1^3 - 5z_1^2 + 2z_1 + 1 \\ P_2(z_2) &= z_2^4 - 110z_2^3 + 67z_2^2 - 14z_2 + 1 \\ P_3(z_3) &= z_3^4 - 11966z_3^3 + 1411z_3^2 - 62z_3 + 1 \\ P_4(z_4) &= z_4^4 - 143182334z_4^3 + 507139z_4^2 - 1022z_4 + 1 \end{aligned} \tag{5.68}$$

Obviously $\left|a_3^{(4)}\right| \approx \left|a_3^{(3)}\right|^2$, hence $|z| < (1+11966)^{1/8} \approx 3.2341$. This additional improvement brings the bound very close to the exact one (numerically determined as $|z| \leq 3.2340$). However, the iterative procedure does not necessarily lead to the exact bound. Still, that problem can be solved with the help of [64, theorem 2]. The solution to which the bound converges to can be bypassed toward exact solution as shown in the following example.

Example 5.6: Let

$$P(z) = z^2 - 5z + 6 \tag{5.69}$$

The bound of its zeros $|z| \leq 3$ can be easily found by solving the quadratic equation. On the other hand, all bounds [60]-[64] are very far from the exact one, being in the range from 5.288 [64, theorem 1] to 13.93 [54, eq. (27.25)]. After just two iterations we have

$$P_2(z_2) = z_2^2 - 97z_2 + 1296 \tag{5.70}$$

From this it can be seen that the bound converges to $|z| \leq 6$. But [[64], theorem 2] leads to $|z_2| \leq 97 + 1296/97 = 110.36$. Hence, $|z| \leq (110.36)^{1/4} \approx 3.24$, which is very close to the exact solution. The result can be further improved if we proceed iteratively and then apply [64, theorem 2], converging to the exact bound.

Chapter 6
Summary and Future Work

This thesis presents three new results related to stability of digital filters. Test procedure for a 2–D Scattering Hurwitz polynomial is based on the determination of the values of a polynomial and its derivatives at zeros of a polynomial on T^2. Analysis of the first order m-D (m>2) digital filter shows the possibility for an asymptotically stable filter, even with the presence of poles of the transfer function inside the closed unit polydisk. Exponential transformation of a polynomial leads to a new upper bound for the polynomial zeros.

The results of Chapter 3 and Chapter 4 can be found in [65] and [66]. Results presented in Chapter 5 were recently published [67], [68].

Future work will be directed towards the derivation of necessary and sufficient conditions for BIBO stability of m-D (m>2) digital filter with the presence of nonessential singularities of the second kind on $\overline{U}^m - U^m$. That would be the extension of the results presented in [34], [35], [36] and [17]. Also, the connection of the results of Chapter 5 [67], [68] to the stability threshold of 2–D filters will be investigated.

Appendix

Resultants of Two Two-Variable Polynomials

Let us first consider two one-variable polynomials:

$$f(x) = a_m x^m + \cdots + a_0 \tag{A.1}$$

and

$$g(x) = b_n x^n + \cdots + b_0 \tag{A.2}$$

The product

$$(a_m)^n (b_n)^m \prod_{i,j} (\alpha_i - \beta_j) = (a_m)^n g(\alpha_1) \cdots g(\alpha_m) \tag{A.3}$$

where α_i, $i = 1, \cdots, m$ and β_j, $j = 1, \cdots, n$ are the zeros of $f(x)$ and $g(x)$ respectively, is called a resultant of $f(x)$ and $g(x)$ [20], [19]. Any function which is a constant multiple of (A.3) is also called a resultant of $f(x)$ and $g(x)$. The resultant of $f(x)$ and $g(x)$ is denoted as $R[f(x), g(x)]$. It is important to note that $R[f(x), g(x)]$ is a polynomial in the coefficients of $f(x)$ and $g(x)$. It can be seen from (A.3) that the resultant can be thought of as a product of the various roots of $f(x)$ and $g(x)$.

Let us now consider two two-variable polynomials $f(z_1, z_2)$ and $g(z_1, z_2)$. Assuming that z_1 is a constant, f and g can be considered as one-variable polynomials in z_2. The resultant of these two one-variable polynomials in z_2 is defined as the z_2-resultant of the two two-variable polynomials $f(z_1, z_2)$ and $g(z_1, z_2)$. It is denoted as $R_{z_2}[f(z_1, z_2), g(z_1, z_2)]$. Note that $R_{z_2}[f(z_1, z_2), g(z_1, z_2)]$ is a polynomial in z_1. Similarly $R_{z_1}[f(z_1, z_2), g(z_1, z_2)]$, the z_1-resultant of $f(z_1, z_2)$ and $g(z_1, z_2)$ can be defined as a polynomial in z_2.

In order to evaluate the z_2-resultant of the two two-variable polynomials, let us express $f(z_1, z_2)$ and $g(z_1, z_2)$ as one-variable polynomials in z_2:

$$f(z_1, z_2) = a_m(z_1) z_2^m + \cdots + a_0(z_1) \tag{A.4}$$

and

$$g(z_1, z_2) = b_n(z_1)z_2^n + \cdots + b_0(z_1) \tag{A.5}$$

From [69], it is known that $R_{z_2}[f(z_1,z_2), g(z_1,z_2)]$ is a determinant of the order (m+n), given by

$$\begin{vmatrix} a_m(z_1) & \cdots & a_0(z_1) & 0 & \cdots & & 0 \\ 0 & a_m(z_1) & \cdots & a_0(z_1) & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & & \cdots & a_m(z_1) & \cdots & a_0(z_1) \\ b_n(z_1) & \cdots & b_0(z_1) & 0 & \cdots & & 0 \\ 0 & b_n(z_1) & \cdots & b_0(z_1) & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & & \cdots & b_n(z_1) & \cdots & b_0(z_1) \end{vmatrix} \tag{A.6}$$

Example: Let

$$f(z_1, z_2) = 1 - z_1 - z_2 + z_1 z_2 \tag{A.7}$$

and

$$g(z_1, z_2) = 2 - z_1 - z_2 \tag{A.8}$$

Rearranging f and g as:

$$f(z_1, z_2) = (z_1 - 1)z_2 + 1 - z_1 \tag{A.9}$$

and

$$g(z_1, z_2) = -z_2 + 2 - z_1 \tag{A.10}$$

and applying (A.6)

$$R_{z_2} = \begin{vmatrix} z_1 - 1 & 1 - z_1 \\ -1 & 2 - z_1 \end{vmatrix} = -(z_1 - 1)^2 \tag{A.11}$$

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