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Dual Resonance Models and their Currents

by

Edward A. Johnson

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Myriam P. Sarachik

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ABSTRACT

Dual Resonance Models and their Currents
by
Edward A. Johnson

Advisor: Professor Michio Kaku

Work continues in the exciting area of dual resonance theory. Inspite of their present inadequacies, dual models promise a qualitative leap in our understanding of the fundamental processes. The Veneziano model had its origin as a solution of the finite energy sum rules which tie together the high energy and low energy sectors of nuclear reactions. In a brave way, it advanced the concept of duality in hadronic interactions. We show how dual resonance models were re-derived from the concept of a string tracing out a surface in space-time. Thus, interacting strings reproduce the dual amplitudes. A scheme for tackling the unitarity problem began to develop. As a consistent theory of hadronic processes began to be built, workers at the same time were naturally led to expect that leptons could be included with hadrons in a unified dual theory. Thus, there is a search for dual amplitudes which would describe interactions between hadrons and currents (for example, electrons) as well as interactions involving only hadrons. Such amplitudes, it is believed,

will be the correct ones, describing the real world. Such amplitudes will provide valuable information concerning such things as hadronic form factors. We describe the great difficulties in building current-amplitudes with the required properties of proper factorization on a good spectrum, duality, current algebra, and proper asymptotic behavior.

Dual models at the present time require for consistency, an intercept value of $\alpha_0 = 1$ and a dimension value of d= 26 (or d=10). There have been speculations that the unphysical dimension may be made physical by associating the 'extra dimensions' with certain internal degrees of freedom. However, we would like the theory itself, to force the dimension d=4. It is quite possible that the dimension problem and the intercept problem are tied together and that resolving either problem will resolve the other.

In this work, we construct order by order, a new dual current that is manifestly factorizable and which appears to be valid for arbitrary space-time dimension. The fact that this current is not bound at d=26, leads to interesting speculations on the nature of dual currents.

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1. How Dual Theory Originated.

The goal of dual theory has been to construct amplitudes describing purely hadronic interactions, i.e. strong interactions. As such, dual theory despite its shortomcings, has provided a crude picture which experiments show to be surprisingly close to the truth. Real accuracy will involve taking into account other factors; an important one being the requirement that a correct theory be capable of also describing interactions between hadrons and currents. Such a requirement will go a long way towards restricting the choice of dual models and finding the most truthful representation of the real world.

Researchers developed dual theory starting from the fact that Lagrangian techniques used in quantum electrodynamics pose problems when applied to strong interactions because here the coupling constants are not small. Further it was noted that cross sections of interacting nucleons show features indicating that these reactions can proceed by way of intermediate states called resonances.(1) For example the process

$$a_1 + a_2 \longrightarrow a_3 + a_4$$

(where a_i are nucleons) at specific energies, appears to occur in two steps; thus $a_1 + a_2 \longrightarrow R \longrightarrow a_3 + a_4$

in which the intermediate particle R has a very short lifetime. The mass m_R of R is just the value of incoming energy at the bump that appears on a plot of cross section versus incoming energy. A bump indicates a region of rapid increase in cross section as a function of incoming energy. Its width is proportional to the lifetime of the resonance. Many such bumps have been noted, the number of bumps or resonances increasing in a regular way with incoming energy. The existence of a spectrum of resonances, each of which despite its short lifetime is accepted on an equal footing with neutrons or protons, makes the list of hadrons a very long one. To use the field theory approach here would be extremely difficult as it seems to involve postulating a distinct field for each strongly interacting particle.

Some researchers therefore rejected the field theory approach as a useful one in the strong interaction domain, and focused attention on constructing an S-matrix based on general postulates of quantum relativistic theory. With this minimal set of postulates, researchers then proceeded to hunt for an additional property of the S-matrix structure that would single out specifically strong interactions.

The S-matrix postulates[2] are the following:

(1) Poincare invariance. Consider a reaction in which the total number of initial particles plus the total number of final particles is N. Consider the following partition of the set of N particles:

$$\left(P_1^A, P_2^A, \ldots, P_n^A\right), \left(P_1^B P_2^B, \ldots. P_m^B\right) \quad \text{where } n+m = N \qquad (1.1)$$

A partition into two sets is called a channel. Let us call the particular channel represented in 1.1 the K^{th} channel. Then

$$S_K \equiv -\left(\sum_i P_i^A\right)^2 = -\left(\sum_i P_i^B\right)^2$$

follows from momentum conservation and S_K, a Lorentz invariant, is called the Mandelstam variable for the K^{th}, channel. If there are M channels then the amplitude for the reaction is described by $\left(S_1, S_2, \ldots\ldots, S_M\right)$

(2) Crossing. Consider the reaction $i \rightarrow f$, in which the total number of initial particles plus the total number of final particles is N. Let their momenta be $P_1, P_2, \ldots, P_N$. A particle that is incoming is assigned energy $P_i^0 > 0$. A particle that is outgoing is assigned energy $P_i^0 < 0$. The Mandelstam variables $(S_1 S_2, \ldots\ldots, S_M)$ describing the reaction $i \rightarrow f$ will belong to region R of S_K- space. Suppose now, keeping the same N particles, that the signs of some of the P_i^0 are changed. Thus a different process $i' \rightarrow f'$ is indicated, in which the variables $(S_1, S_2, \ldots, S_M)$ now belong to a different region say R' of S_K-space. Crossing symmetry now says that the corresponding amplitudes $\langle f|S|i\rangle$ and $\langle f'|S|i'\rangle$ are both obtainable by analytic continuation from a single function called the N-point function.

(3) Unitarity is the principle that probability is conserved. This requires that the S-matrix satisfy $S^+S = SS^+ = I$

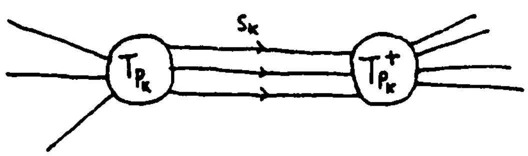
or if T is introduced by $S = I + iT$ we have

$$T - T^+ = iT^+T$$

The equation is now taken between states $|i\rangle$ and $|f\rangle$, and a complete set of intermediate states $|n\rangle\langle n|$ is inserted. This gives

$$\langle f|T_{P_K}|i\rangle - \langle f|T^{+}_{P_K}|i\rangle = i\sum_{n}\langle f|T^{+}_{P_K}|n\rangle\langle n|T_{P_K}|i\rangle \qquad (1.2)$$

Here P_K is the momentum associated with $|i\rangle$ and with $|f\rangle$ and with each state $|n\rangle$ that contributes to the sum.



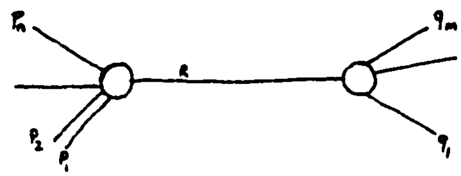
This diagram illustrates the right hand side of eq 1.2 showing a K^{th} channel reaction.

Let the Mandelstam variables involved be $S_1, \cdots, S_K, \cdots, S_M$. Then it can be shown that the left hand side of eq 1.2 is proportional to $\mathrm{Im}\, A(s_1, s_2, \cdot s_K \cdots, s_M)$ (Here, we have omitted complications due to internal quantum numbers.) Thus $\mathrm{Im}\, A$ must be non-zero for the discrete point $S_K = M_K^2$, if M_K is the mass of a possible single-particle state $|n\rangle$ with the right quantum numbers.

In fact, $\mathrm{Im}\, A$ must be non-zero at any valve of S_k corresponding to a possible process in channel K or particles exchanged in channel K. Neglecting for the moment the requirements of crossing symmetry, Im A can be zero at all other valves of S_K.

(4) Analyticity. The principle of causality is that an effect cannot precede its cause in time; and that signals cannot travel faster than c. Researchers have been led by this principle to

postulate amplitudes that, in all Mandelstam variables, are totally analytic except for those singularities demanded by the existence of intermediate particles, unitarity, and crossing symmetry. Using eq 1.2 one can then show that, for example, a single-particle state of mass M_K exchanged in channel K corresponds to a simple pole at $S_K = M_K^2$ in the amplitude $A(S_1 S_2, \ldots S_K, \ldots S_M)$
Thus in the single exchange reaction shown below



the amplitude for $P_1 + P_2 + \cdots + P_n \rightarrow q_1 + q_2 + \cdots + q_m$ can be shown to be given by

$$A(n,m) \sim \frac{A(n,R)\,A(m,R)}{S - M_R^2} \qquad (1.3)$$

where A(n,m), A(n,R), and A(mR) are respectively the n+m point function, the n+1 point function, and the m+1 point function. Notice the factorization of the residue in equation 1.3 .

The above postulates 1 through 4 are the important basics for any S-matrix theory. To this list researchers have added the following postulates to pinpoint strong interactions.

(5a) That all strongly interacting particles lie on Regge trajectories that are approximately linear with a universal slope $\alpha' \sim 1\,(GeV)^{-2}$. Thus if for a Regge trajectory $\alpha(s)$, we have $\alpha(m_R^2) = \sigma$ where σ is the spin of a single particle of mass M_R exchanged in a certain channel, then since $\alpha(s) = \alpha_0 + \alpha' s$, the contribution of M_R to the amplitude will be given by

$$\sum_{\ell} \frac{C_\ell}{\alpha(s) - \ell} \quad \underset{s \sim m_R^2}{\sim} \quad \frac{C_\sigma}{\alpha'(s - m_R^2)}$$

In other words, the imaginary part of $\alpha(s)$ will be practically zero and this, of course, will mean the resonance width will be infinitesimally small. There is experimental evidence supporting, in an indirect way, the existence of very narrow resonance widths. Technically, zero resonance widths are a violation of unitarity. However, researchers feel this to be a valid approximation as long as the residues of the poles factor, (as in eq 1.3) without ghosts, and so give the correct coupling constants.

(5b) That neglecting all but single-particle type exchanges can give a reasonably good Born approximation. It can be shown that an infinite number of single particle exchanges are then necessary in order to ensure consistency with crossing and good behavior at high energy.

2. How a Dual Amplitude is Constructed.

Consider the 4-point function. Assuming that in one of the 3 channels involved there are no resonances and that resonances only get exchanged in channel 1 $(1+2 \rightarrow 3+4)$ and in channel 2 $(\bar{4}+1 \rightarrow \bar{2}+3)$ we have for fixed S_2 :

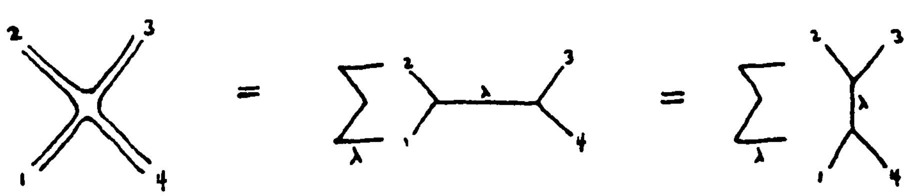
$$A_4(s_1, s_2) = \sum_{\lambda} \frac{G_\lambda(s_2)}{-k_1^2 - m_{1\lambda}^2}$$

where $M_{1\lambda}$ is the λ^{th} resonance that can be exchanged in channel 1, K_1 is the momentum flowing through channel 1. In the same way regarding S_1 as fixed and varying S_2 in the physical region of channel 2 we have:

$$A_4(s_1, s_2) = \sum_{\lambda} \frac{\bar{G}_\lambda(s_1)}{-k_2^2 - m_{2\lambda}^2}$$

These equations are just applications of the defining postulates. Lets assume here that $G = \bar{G}$. Now the duality statement is

$$A_4(s_1, s_2) = \sum_{\lambda} \frac{G_\lambda(s_2)}{-k_1^2 - m_{1\lambda}^2} = \sum_{\lambda} \frac{G_\lambda(s_1)}{-k_2^2 - m_{2\lambda}^2}$$

or 

The residues must factorize so we have

$$\sum_\lambda \frac{G_\lambda(s_2)}{-K_1^2 - m_{1\lambda}^2} = \sum_\lambda g_{12\lambda} \frac{1}{-K_1^2 - m_{1\lambda}^2} g_{34\lambda}$$

Lets introduce resonance states $|m_{n\lambda}, K_n\rangle$ where n denotes channel, K_n the momentum flowing through the n^{th} channel, $M_{n\lambda}$ the mass of the λ^{th} resonance available in the n^{th} channel.

Lets also introduce a momentum operator P and mass operator M^2 and vertex operator V given by

$$P|m_{n\lambda}, K_n\rangle = K_n|m_{n\lambda}, K_n\rangle$$

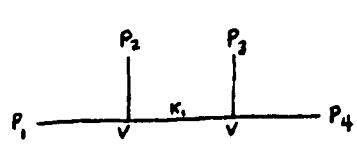
$$M^2|m_{n\lambda}, K_n\rangle = m_{n\lambda}^2|m_{n\lambda}, K_n\rangle$$

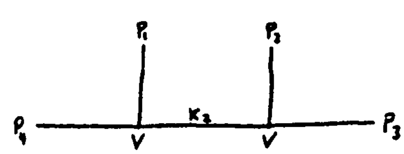
$$\langle 12|V|K_n, m_{n\lambda}\rangle = g_{12\lambda}$$

Then

$$\sum_\lambda \frac{G_\lambda(s_2)}{-K_1^2 - m_{1\lambda}^2} = \sum_\lambda \langle 12|V|m_{1\lambda} K_1\rangle\langle m_{1\lambda} K_1| \frac{1}{-P^2 - M^2}|m_{1\lambda} K_1\rangle\langle m_{1\lambda} K_1|V|34\rangle$$

$$= \langle 12|V \frac{1}{-P^2 - M^2} V|34\rangle = \langle 41|V \frac{1}{-P^2 - M^2} V|23\rangle \qquad (2.1)$$

or $A_4 =$ 

$=$ 

Thus the amplitude has a pole when the channel energy is equal to the resonance energy.

In studying the N-point function we will only consider planar channels.[3] For the N interacting particles ordered 1,2,3,..., N-1,N a planar channel is defined as a partition of the form

$(b+1, b+2, \ldots, n, 1, 2, \ldots, a)$, $(a+1, a+2, \ldots, b-1, b)$

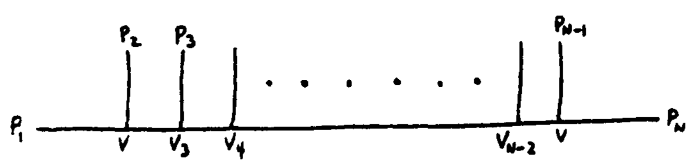
A function $A_N^{(123\cdots N)}$ is constructed satisfying dual theory requirements and having singularities only in its planar channels. The complete N-point function is then taken to be a sum over all possible permutations of $(1, 2, \ldots N-1, N)$. This procedure, which, of course, assumes that resonances do not get exchanged in every channel, but only in planar channels, seems to lead to results in good agreement with experiment.

It can be shown that for any particular ordering (eg. $1, 2, \ldots, N-1, N$) there are exactly N-3 non-overlapping planar channels. Two channels are said to overlap if they have at least one line in common and neither is a subset of the other. In order for dual theory to be consistent with Feynmann diagram analysis, we must require that resonances get exchanged simultaneously in non-overlapping channels only. That is resonances are not exchanged simultaneously in overlapping channels.

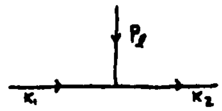
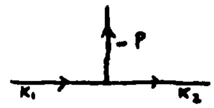
Using the same method used to get eg2.1 we find for A_N (dropping the superscript):

$$A_N = \left\langle 1\,2 \,\middle|\, V \frac{1}{-p^2-M^2} V_3 \frac{1}{-p^2-M^2} V_4 \frac{1}{-p^2-M^2} \cdots\cdots V_{N-2} \frac{1}{-p^2-M^2} V \,\middle|\, N-1, N \right\rangle$$

or $A_N =$



where we define the V_ℓ vertex by

$\langle m_{1\lambda}\, K_1 | V_\ell | m_{2\lambda'}\, K_2 \rangle = g_{K_1+P_\ell \to K_2} =$  $=$ 

Putting $\langle 12 | V | m_{n\lambda}\, K_n \rangle = g_{12\lambda} = \langle 1 | V_2 | m_{n\lambda}\, K_n \rangle$

and $\langle m_{n\lambda}\, K_n | V | N-1, N \rangle = g_{\lambda\, N-1, N} = \langle m_{n\lambda}\, K_n | V_{N-1} | N \rangle$

we have $A_N = \langle 1 | V_2 \frac{1}{-P^2-M^2} V_3 \frac{1}{-P^2-M^2} V_4 \cdots V_{N-2} \frac{1}{-P^2-M^2} V_{N-1} | N \rangle$

and by duality $A_N = \langle N | V_1 \frac{1}{-P^2-M^2} V_2 \frac{1}{-P^2-M^2} V_3 \cdots V_{N-3} \frac{1}{-P^2-M^2} V_{N-2} | N-1 \rangle$

Now put $P^2 + M^2 = L_0 - 1$ and note that $\frac{1}{L_0 - 1} = \int_0^1 \frac{dx}{x^2} x^{L_0}$

$$\therefore\ A_N = \int_0^1 \cdots \int_0^1 \frac{dx_2}{x_2^2} \cdots \frac{dx_{N-2}}{x_{N-2}^2} \langle 1 | V_2\, x_2^{L_0} V_3\, x_3^{L_0} \cdots V_{N-2}\, x_{N-2}^{L_0}\, V_{N-1} | N \rangle$$

If now we define a new vertex $V(P_n, z) = z^{L_0} V_n z^{-L_0}$ then

$$A_N = \int_0^1 \cdots \int_0^1 \frac{dx_2}{x_2^2} \cdots \frac{dx_{N-2}}{x_{N-2}^2}$$

$$\times\ \langle 1 | V(P_2, 1) V(P_3, x_2) V(P_4, x_2 x_3) \cdots V(P_{N-2}, x_2 x_3 \cdots x_{N-3}) V(P_{N-1}, x_2 x_3 \cdots x_{N-2}) (x_2 x_3 \cdots x_{N-2})^{L_0} | N \rangle$$

Since the external particle $|N\rangle$ must lie on its mass shell we have $(L_0 - 1)|N\rangle = 0$. If now we make the following change of variables:

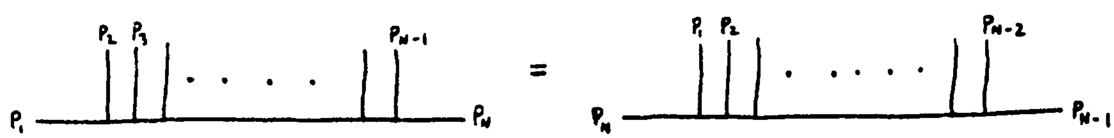
$$\left.\begin{array}{ll} x_2 & = y_3 \\ x_2 x_3 & = y_4 \\ \vdots & \\ x_2 x_3 \cdots x_{N-2} & = y_{N-1} \end{array}\right\} \quad \text{we find that} \quad \prod_{i=2}^{N-2} \frac{dx_i}{x_i} = \prod_{i=3}^{N-1} \frac{dy_i}{y_i}$$

$$\therefore\ A_N = \underset{1 = y_2 > y_3 > \cdots > y_{N-1} > 0}{\int \cdots \int} \frac{dy_3}{y_3} \cdots \frac{dy_{N-1}}{y_{N-1}} \langle 1 | V(p_2, y_2) V(p_3, y_3) \cdots V(p_{N-2}, y_{N-2}) V(p_{N-1}, y_{N-1}) | N \rangle$$

and by duality we also have

$$A_N = \underset{1 = y_1' > y_2' > \cdots > y_{N-2}' > 0}{\int \cdots \int} \frac{dy_2'}{y_2'} \cdots \frac{dy_{N-2}'}{y_{N-2}'} \langle 1 | V(p_1, y_1') V(p_2, y_2') \cdots V(p_{N-3}, y_{N-3}') V(p_{N-2}, y_{N-2}') | N-1 \rangle$$

or $A_N =$



Thus the duality condition is seen to amount to just a cyclic re-ordering of variables as shown in the diagram. This symmetry can be expressed by the group $SU(1,1)$. To see this, let it be required that

$$\lim_{z \to 0} \frac{V(p_n, z)}{z} |0,0\rangle = |n\rangle \qquad \text{and} \qquad \lim_{z \to \infty} \langle 0,0 | z V(p_n, z) = \langle n | \tag{2.2}$$

Then the duality condition can be cast in the form

$$\left.\begin{aligned} A_N &= \int\cdots\int \frac{y_1}{y_N}\frac{dy_3}{y_3}\cdots\frac{dy_{N-1}}{y_{N-1}} \langle 0,0|V(P_1,y_1)V(P_2,y_2)\cdots V(P_{N-1},y_{N-1})V(P_N,y_N)|0,0\rangle \\ &\quad (y_1=\infty,\ y_2=1,\ y_N=0\ ;\ \ y_2>y_3>\cdots>y_N) \\ &= \int\cdots\int \frac{y_N'}{y_{N-1}'}\frac{dy_2'}{y_2'}\cdots\frac{dy_{N-2}'}{y_{N-2}'} \langle 0,0|V(P_N,y_N')V(P_1,y_1')V(P_2,y_2')\cdots V(P_{N-1},y_{N-1}')|0,0\rangle \\ &\quad (y_N'=\infty,\ y_1'=1,\ y_{N-1}'=0\ ;\ \ y_1'>y_2'>\cdots>y_{N-1}') \end{aligned}\right\}(2.3)$$

Consider the Mobius transformations:

$$y' = Ay = \frac{a_1 y + a_2}{a_3 y + a_4} \qquad \text{where } a_1a_4 - a_2a_3 = 1$$

It is easily shown that given numbers n_1, n_2, n_3 and n_1', n_2', n_3' that a unique A can be found such that $n_i' = A n_i$;

Also, given numbers n_1, n_2, n_3 , we can find a particular A such that $An_1 = n_2$, $An_2 = n_3$, $An_3 = n_1$

Given A let it be required that a unitary operator U(A) exist such that

$$U(A)\frac{V(P,y)}{y}U^{-1}(A) = \frac{1}{(a_3y+a_4)^2}\frac{V(P,Ay)}{(Ay)} \qquad (2.4)$$

and $\quad U(A)|0,0\rangle = 0 \qquad (2.5)$

then since $dy = (a_3y+a_4)^2 d(Ay)$ it can be shown that the duality constraint (2.3) will be satisfied.

The following is one method of implementing conditions (2.2,2.4,2.5), namely the Veneziano model.[4]

One introduces harmonic oscillator operators a_m^μ where $m = 0, \pm1, \pm2, \ldots\ldots$ and $\mu = 0, 1, 2, \ldots, d-1$

where d is the dimension of space-time (for this model d need not be 4).

These operators satisfy the commutation relations

$$\left[a_m^\mu , a_n^\nu\right] = m g^{\mu\nu} \delta_{m,-n} \qquad a_{-n}^\mu \equiv a_n^{\mu +} \tag{2.6}$$

Here $g^{\mu\nu}$ is needed for manifest covariance of the model.

Introduce also the position operator q^μ canonical to $p^\mu \equiv a_0^\mu$

$$\left[q^\mu , p^\nu\right] = i g^{\mu\nu}$$

The resonance states of the system are then taken to be

$$\prod_{r=1}^{R(\lambda)} a_{-m(\lambda)_r}^{\mu(\lambda)_r} \left|0, k_n\right\rangle \equiv \left|m_{n\lambda}, k_n\right\rangle$$

which are excited states consisting of components of various angular momenta. On the other hand the state describing one of the external particles is expressed as a ground state:

$$\left|n\right\rangle \equiv \left|0, p_n\right\rangle \equiv e^{i p_n \cdot q} \left|0, 0\right\rangle$$

The vertex operator satisfying (2.2), (2.4), (2.5) is found to be

$$V(p_n, z) = z^{L_0} V_n z^{-L_0}$$

$$\text{where} \quad V_n = \left(\exp \sum_{r=1}^{\infty} \frac{p_n \cdot a_{-r}}{r}\right) e^{i p_n \cdot q} \left(\exp - \sum_{s=1}^{\infty} \frac{p_n \cdot a_s}{s}\right) \tag{2.7}$$

The required unitary operators are found to be expressible as $U(\Lambda) = e^{i\alpha \cdot L}$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are real numbers and $L = (L_{-1}, L_0, L_1)$ are operators that form a subalgebra of a more general algebra discovered by Virasoro.[5] The general L_m operators are

$$\left.\begin{aligned} L_0 &= P^2 + \sum_{n=1}^{\infty} a_{-n}\cdot a_n \\ L_m &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty} a_{m-n}\cdot a_n \qquad m=1,2,3,\ldots \\ L_{-m} &= L_m^{+} \end{aligned}\right\} \tag{2.8}$$

The Virasoro algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\,\delta_{m,-n} \tag{2.9}$$

It is easily shown that

$$[L_n, V(P_i, y)] = y^n\left(y\frac{d}{dy} + nP_i^2\right)V(P_i, y) \tag{2.10}$$

Because of eq 2.10

we find that in order to implement eq 2.4 that the N interacting particles of mometa $P_1, P_2, \ldots, P_N$ must each be taken to be scalars with $P_i^2 = 1$. A useful fact deriveable from the algebraic properties of the harmonic oscillators is the relation:

$$\langle 0,0|\,\frac{V(P_1,Y_1)}{Y_1}\cdots\frac{V(P_N,Y_N)}{Y_N}\,|0,0\rangle = \prod_{i<j}(Y_i - Y_j)^{P_i\cdot P_j} \tag{2.11 A}$$

which is the Koba-Nielson formula. Thus the amplitude A can be cast in the form:

$$A_N = \frac{(Y_a-Y_b)(Y_a-Y_c)(Y_b-Y_c)}{dY_a\,dY_b\,dY_c}\int\prod_{i=1}^{N}dY_i\,\theta(Y_i-Y_{i+1})\prod_{i<j}(Y_i-Y_j)^{P_i\cdot P_j} \tag{2.11 B}$$

Here the three variables Y_a, Y_b, Y_c which are held fixed in the integration can be chosen to be any three out of the N variables Y_i .

This aribtrariness is due to the fact that by a Mobius transformation we can send any three Y_i variables onto any other set of three. Furthermore, one can make the mapping $(Y_a, Y_b, Y_c) \rightarrow (Y_1, Y_2, Y_N) \rightarrow (\infty, 1, 0)$ via a Mobius transformation.

Using $\alpha(s) = \alpha_0 + \alpha' s$ with $\alpha_0 = 1$ we have

$\alpha(s) = 1 - (P_1 + P_2)^2$, $\alpha(t) = 1 - (P_2 + P_3)^2$

we can, after some algebra, rewrite the Koba-Nielson form for A_4 and get

$$A_4 = \int_0^1 dx\, x^{-\alpha_s - 1} (1-x)^{-\alpha_t - 1} \equiv \beta(-\alpha_s, -\alpha_t)$$

B is the beta function

$$\beta(-\alpha_s, -\alpha_t) = \frac{\Gamma(-\alpha_s)\,\Gamma(-\alpha_t)}{\Gamma(-\alpha_s - \alpha_t)}$$

This is the original Veneziano amplitude. Notice its explicit s-t symmetry. This amplitude contains an infinite series of poles in both s and t arising respectively from the poles of the gamma functions $\Gamma(-\alpha_s)$ and $\Gamma(-\alpha_t)$ which occur whenever the appropriate α passes through a positive integer. Thus the S-channel poles are exhibited by rewriting A_4 as

$$A_4 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(-\alpha_t)}{\Gamma(-\alpha_t - n)} \frac{1}{n - \alpha_s} \qquad (\alpha_t < 0)$$

We see that the residue of the n^{th} s-channel pole is a polynomial in t

of degree n, and thus can be expanded in terms of Legendre polynomials $P_j(\cos\theta)$ with $j = 0, 1, 2, \ldots, n$. The amplitude therefore describes a leading trajectory together with an infinite series of equally spaced daughter trajectories.

Letting s become very large while keeping t fixed we find the Regge asymptotic behavior

$$A_4 \sim \Gamma(-\alpha_t)\,(-\alpha_s)^{\alpha_t}$$

for $s \to \infty$ in any direction in the complex plane except on the real axis where the poles lie.

It can be shown that A $(N > 4)$ can be written as a multiple beta function.

3. The States of the Veneziano Model

Physical states are states $|\phi\rangle$ which satisfy two conditions:

$$\left.\begin{array}{llr} \text{(i)} & (L_0 - 1)|\phi\rangle = 0 & \text{(the mass-shell condition)} \\ \text{(ii)} & L_m|\phi\rangle = 0 \quad m = 1,2,3\cdots & \text{(the subsidiary conditions)} \end{array}\right\} 3.1$$

Mass-shell states that have zero coupling to any physical state are called spurious. Lets introduce the states $|\psi_n\rangle$ such that $(L_0 + n - 1)|\psi_n\rangle = 0$ where n is an integer greater than zero. An example of such a state $|\psi_n\rangle$ is the state $|0,k\rangle$ with $k^2 \neq 1$. Consider the state $L_{-n}|\psi_n\rangle$ introduced by Virosoro. Using the commutation relations we see this state must necessarily be a mass-shell state, ie $(L_0 - 1)(L_{-n}|\psi_n\rangle) = 0$

The amplitude for one particle of type $L_{-n}|\psi_n\rangle$ and N-1 scalar particles is

$$A_N = \langle\psi_n|L_n V(P_2)\frac{1}{L_0 - 1}V(P_3)\cdots\frac{1}{L_0 - 1}V(P_{N-1})|0,P_N\rangle$$

$=$ 

$$= \langle\psi_n|(L_n - L_0 - n + 1)V(P_2)\frac{1}{L_0 - 1}V(P_3)\cdots\frac{1}{L_0 - 1}V(P_{N-1})|0,P_N\rangle$$

(here $V(P) \equiv V(P,1)$)

But using eq 2.10 we see that $[L_n - L_0, V(P)] = n\,V(P)$

and from eq 2.9 we see that $(L_n - L_0 + 1)\frac{1}{L_0 - 1} = \frac{1}{L_0 + n - 1}(L_n - L_0 - n + 1)$

Thus the operator $(L_n - L_0 - n + 1)$ can be successively commuted to the right until we have a factor $(L_n - L_0 + 1)$ acting on the state $|0,P_N\rangle$. But since $L_n|0,P\rangle = 0$ and $(L_0 - 1)|0,P\rangle = 0$ we find that $A_N = 0$. Thus $L_{-n}|\psi_n\rangle$ decouples from all physical "tree states" and is spurious. It can be shown that any spurious state is of the form $L_{-n}|\psi_n\rangle$.

Ghost states are states with negative norm. Such states are produced by the time component of each harmonic oscillator. For example the state $a^0_{-n}|0\rangle$ has norm $\langle 0|a^0_n a^0_{-n}|0\rangle = g^{00} n \langle 0|0\rangle = -n$. In a remarkable proof, it has been shown that the requirement of eq 3.1 is sufficient to eliminate all ghosts provided that $d \leq 26$. The problem of ghosts arises because quantization in a covariant manner introduces superfluous degrees of freedom in the system. It is then required to check whether the physical states all have positive norm. For this purpose we introduce the transverse space[6]

$$T = \{ \Psi_t \}$$

where $$\Psi_t = \prod_j A^{i_j\,+}_{n_j}(K_0)\,|0, p\rangle \qquad (3.2)$$

where $\alpha' p^2 = 1$, $K_0^2 = 0$, $K_0 \cdot p = 1$

(this choice of p and the parameter K_0 amounts to a restriction of Lorentz frame).

and the transverse excitation operators $A^i_n(K_0)$ constructed by Del Giudice, Di Vecchia and Fubini[6] satisfy

$$\left[L_m , A^{i+}_n(K_0) \right] = 0 \qquad (3.3)$$

$$\left[A^i_m(K_0) , A^{j+}_n(K_0) \right] = n\,\delta_{nm}\,\delta^{ij} \qquad (3.4)$$

Equation 3.3 implies that the A_n^{i+} operators transform physical states into physical states; in particular T is a subspace of the space $\{|\phi\rangle\}$ of physical states. Equation 3.4 then proves that T is a space with positive definite norm. For d=26, $\alpha_0 = 1$ it can be shown that T is equivalent to $\{|\phi\rangle\}$ i.e. that for any physical state ϕ we have a ψ_t such that

$$\phi = \psi_t + \eta \qquad (3.5)$$

where η is a state of zero norm, decoupled from all states. While (3.5) only holds for d=26, $\alpha_0 = 1$, it can be proven that for $d < 26$, $\alpha_0 = 1$ that the physical states are then a subspace of those for d=26. Thus, the physical states for $d < 26$ are also of positive norm.

The first few levels of the generalized Veneziano model have the following:

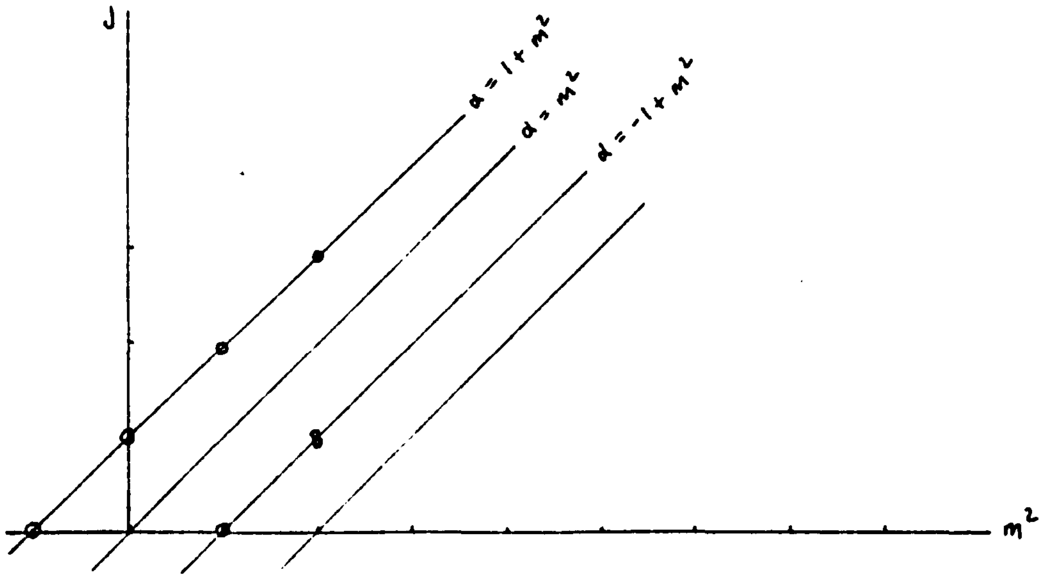
(1) $|0,k\rangle$ has $J=0$, $m^2 = -1$ (a tachyon)

(2) $a_{-1}^{\mu}|0,k\rangle$ has $J=1$, $m^2 = 0$ (a massless meson)

(3) $a_{-1}^{\mu}a_{-1}^{\nu}|0,k\rangle$ has $J=2$, $m^2 = 1$

There is no $J=1$, $m^2 = 1$ state. In fact for all n-point functions the first daughter is absent at all levels.

The corresponding physical trajectories are shown below



The leading trajectory has intercept $\alpha_0 = 1$ and consequently carries a tachyon (particle with mass $m^2 < 0$). This is the worst feature. Among other bad features of the spectrum, is that there is no light pseudoscalar to represent the pion in the model. Unrealistic spectrums are unfortunately typical of dual models at the present time. Continued search for better dual models seem very much warranted, however.

4. The String Concept

It has been demonstrated that dual amplitudes can be derived and understood in terms of the breaking and joining of strings[7] evolving in space. Unfortunately the unphysical dimension d=26 is seen to be required for consistency of the theory. The appearance of such a critical dimension for space-time is of quantum mechanical origin and is not understood at present.

Just as a point particle traces a world line in space-time, a string traces a world sheet $X^\mu(\sigma,\tau)$.

The string is defined by

$$X^\mu = X^\mu(\sigma,\tau) \tag{4.1a}$$

$$\left(\frac{\partial X}{\partial \tau}\right)^2 \leq 0 \quad , \quad \left(\frac{\partial X}{\partial \sigma}\right)^2 > 0 \tag{4.1b}$$

Here the metric is taken as $g^{ii} = 1 = -g^{00}$

The conditions 4.1b ensure that a Lorentz frame exists in which the equations 4.1a of the string reduce to

$$X^i = X^i(\sigma,\tau) \qquad i = 1,2,3$$

$$X^0 = \tau$$

i.e. the parameter τ is just the time in a particuar frame.

Note that for σ fixed we have $(c=1)$:

$$ds^2 = \left(\frac{\partial X}{\partial \tau}\right)^2 d\tau^2 = \left(-1 + \frac{\partial X^i}{\partial \tau}\frac{\partial X_i}{\partial \tau}\right) d\tau^2 \leq (-1 + c^2)\, d\tau^2 = 0 \tag{4.3}$$

i.e. we have a time-like or light-like interval;

while for τ fixed we have:

$$ds^2 = \left(\frac{\partial x}{\partial \sigma}\right)^2 d\sigma^2 = \frac{\partial X^i}{\partial \sigma}\frac{\partial X_i}{\partial \sigma}\, d\sigma^2 \quad > 0 \qquad (4.4)$$

i.e. we have a space-like interval.

The parameter σ is taken to have range $0 \leq \sigma \leq \pi$

By analogy with the point particle case (where the dynamics derives from demanding that the invariant length of the world line be stationary) here we demand that the area of the world sheet be minimized. Thus we define the action

$$S = \int_{\tau_i}^{\tau_f}\int_0^{\pi} d\tau\, d\sigma\, \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{-\dot{X}^2 X'^2 + (\dot{X}X')^2} \qquad (4.5)$$

where $\dot{X}_\mu \equiv \frac{\partial X_\mu}{\partial \tau}$, $X'_\mu \equiv \frac{\partial X_\mu}{\partial \sigma}$

and α' is a fundamental constant with dimensions of $(\text{length})^2$.

Then $\delta S = 0$ and $\delta X^\mu\big|_{\tau=\tau_i} = \delta X^\mu\big|_{\tau=\tau_f} = 0$ are found to imply the following:

$$\frac{\partial}{\partial \tau}\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} + \frac{\partial}{\partial \sigma}\frac{\partial \mathcal{L}}{\partial X'^\mu} = 0 \qquad (4.6a)$$

$$\frac{\partial \mathcal{L}}{\partial X'^\mu} = 0 \quad \text{at } \sigma = 0, \pi \qquad (4.6b)$$

The conserved total momentum is found to be

$$P^{\mu} \equiv \int_{c} \cdot \left(K^{\mu} d\sigma + R^{\mu} d\tau \right)$$

where

$$K^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}_{\mu}} = \frac{1}{(2\pi\alpha')^2} \frac{1}{\mathcal{L}} \left[X'^{\mu}(\dot{X} X') - \dot{X}^{\mu} X'^2 \right] \tag{4.7}$$

$$R^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial X'_{\mu}} = \frac{1}{(2\pi\alpha')^2} \frac{1}{\mathcal{L}} \left[\dot{X}^{\mu}(\dot{X} X') - X'^{\mu} \dot{X}^2 \right] \tag{4.8}$$

squaring 4.8 gives $(2\pi\alpha')^2 R^2 + \dot{X}^2 = 0$

thus 4.6b implies that $\dot{X}^2 = 0$ at $\sigma = 0, \pi$ ie that the end points of the string move with the speed of light.

The action (eg 4.5) is left invariant by the transformations

$$X^{\mu}(\sigma,\tau) \longrightarrow X^{\mu}(\bar{\sigma}, \bar{\tau})$$

where

$$\bar{\sigma} = \bar{\sigma}(\sigma, \tau)$$
$$\bar{\tau} = \bar{\tau}(\sigma, \tau) \tag{4.9}$$

The transformations 4.9 amount to just a reparametrization of the surface of evolution of the string. The transformations 4.9 form a continuous group, called the gauge group of the string. Infinitesimal generators of the group are the transformations

$$X(\sigma,\tau) \rightarrow \left[1 - \epsilon f(\sigma,\tau) \frac{\partial}{\partial\sigma} - \epsilon g(\sigma,\tau) \frac{\partial}{\partial\tau} \right] X(\sigma,\tau) \tag{4.10}$$

which corresponds to the infinite change of parameters

$$\sigma \rightarrow \sigma + \epsilon f(\sigma,\tau) \quad , \quad \tau \rightarrow \tau + \epsilon g(\sigma,\tau) \tag{4.11}$$

From eg4.7 we find that

$$X' \cdot K = 0 \tag{4.12a}$$

$$(2\pi\alpha')^2 K^2 + X'^2 = 0 \tag{4.12b}$$

Equations 4.12 are constraint equations. They are incompatible with canonical poisson brackets where all X's and K's are treated as independent dynamical variables. We will therefore disregard equations 4.12 and regard X and K as independent when establishing poisson brackets. Afterwards, we will impose egs 4.12 as initial conditions on the classical variables. Thus the canonical poisson brackets are taken to be

$$\left\{ X^\mu , X^\nu \right\} = \left\{ K^\mu , K^\nu \right\} = 0 \tag{4.13}$$

$$\left\{ X^\mu(\sigma) , K^\nu(\sigma') \right\} = g^{\mu\nu} \, \delta(\sigma - \sigma') \tag{4.14}$$

It then follows that

$$\left\{ X'^\mu(\sigma) , K^\nu(\sigma') \right\} = g^{\mu\nu} \frac{\partial}{\partial\sigma} \delta(\sigma - \sigma') \tag{4.15}$$

If we extend the range of $X'(\sigma)$ and $K(\sigma)$ to $-\pi \leq \sigma \leq \pi$ by letting $X'(-\sigma) = -X'(\sigma)$, $K(-\sigma) = K(\sigma)$

and if we define functionals

$$L_f = \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma\, f(\sigma) \left[\sqrt{2\alpha'}\,\pi\, K(\sigma) + \frac{X'(\sigma)}{\sqrt{2\alpha'}}\right]^2 \qquad (4.16)$$

then the constraint eqs (4.12) can be written equivalently as

$$L_f = 0 \qquad \text{[for any function } f(\sigma)\text{]} \qquad (4.17)$$

One can easily verify that the functionals L_f form a closed algebra ie

$$\left\{ L_f , L_g \right\} = L_{f\otimes g} \qquad (4.18)$$

where

$$f\otimes g \equiv f g' - f' g$$

The fact that $\mathcal{L}$ is invariant under the gauge group of the string manifests itself in the impossibility of inverting $K = K(x', \dot{x})$ to get $\dot{x} = \dot{x}(K, x')$. In fact, it is easy to verify using eq4.7 that $\int d\sigma \left[K(\sigma)\dot{X}(\sigma) - \mathcal{L}\right] \equiv 0$. However once an effective hamiltonian, H, is defined we can use $\dot{x} = \{x, H\}$. Thus a choice of H is equivalent to a specification of gauge. Now

$$\left\{L_h, X^\mu(\sigma)\right\} = -\frac{1}{2}\left[h(\sigma) + h(-\sigma)\right] 2\pi\alpha'\, K^\mu(\sigma) \quad - \frac{1}{2}\left[h(\sigma) - h(-\sigma)\right] X'^\mu(\sigma) \qquad (4.19)$$

We see from (4.19) that the even part of L_h transforms X into K and thus generates a dynamical evolution of the system; while the odd part of L_h corresponds to a reparametrization of the string at fixed T. Thus we can fix the gauge by taking $h(\sigma)$ to be an even function of σ ; in particular we will choose $h = 1$ and put $H \equiv L_{(1)}$

Then $$H = \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma \left[\sqrt{2\alpha'}\,\pi K(\sigma) + \frac{X'(\sigma)}{\sqrt{2\alpha'}}\right]^2$$

$$= \frac{1}{8\pi\alpha'}\int_{-\pi}^{\pi} d\sigma \left[(2\pi\alpha')^2 K^2 + X'^2\right] \qquad (4.20)$$

With H we find the equations of motion:

$$\dot{X} = 2\pi\alpha' K \qquad (4.21\,a)$$

$$\dot{K} = \frac{X''}{2\pi\alpha'} \qquad (4.21\,b)$$

and therefore $\ddot{X} - X'' = 0$ (4.22)

Inserting (4.21a) into (4.8) and using (4.12) we get

$$2\pi\alpha' R^\mu + X'^\mu = 0$$

The boundary conditions (4.6b) thus become

$$X'^\mu(0) = X'^\mu(\pi) = 0 \qquad (4.23)$$

Thus the equations of motion (4.6) assume the linear form 4.22, 4.23 in the particular chosen guage.

Notice that if we substitute (4.21a) into 4.12) we get

$$\left.\begin{aligned} X' \cdot \dot{X} &= 0 \\ \dot{X}^2 + X'^2 &= 0 \end{aligned}\right\} \quad (4.24)$$

The particular choice of gauge in which equations (4.24) holds is called the orthonormal gauge. Using the methods of differential geometry it can be shown that equations (4.24) are precisely the conditions for the equations of a minimal surface to reduce to d'Alembert equations.

$X(\sigma)$ and $K(\sigma)$ can be expanded in normal modes in a way that will satisfy equations 4.21 4.22 4.23. Thus one gets

$$X(\sigma,\tau) = X_0 + 2\alpha' P\tau + \sum_{n=1}^{\infty}\sqrt{\frac{2\alpha'}{n}}\cos n\sigma\left(a_n e^{-in\tau} + \bar{a}_n e^{in\tau}\right) \tag{4.25}$$

$$K(\sigma,\tau) = \frac{P}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\sqrt{\frac{n}{2\alpha'}}\cos n\sigma\left(-i a_n e^{-in\tau} + i\bar{a}_n e^{in\tau}\right) \tag{4.26}$$

where because of (4.13)(4.14) the poisson brackets of a, $\bar{a}$ are:

$$\left\{a_n^{\mu}, a_n^{\nu}\right\} = 0 \quad , \quad \left\{a_m^{\mu}, \bar{a}_n^{\nu}\right\} = i g^{\mu\nu}\delta_{m,n}$$

P and X_0 are constants. P is the total momentum of the string. X_0 is the barycentric coordinate.

The initial values of the dynamic variables are subject to the constraint $L_f = 0$. We choose the following discrete set for the functionals :

$$L_n \equiv L(e^{in\sigma}) \qquad n = 0, \pm 1, \pm 2, \ldots$$

Then expanding into normal modes we find

$$= -i\sqrt{2n\alpha'}\ p\cdot a_n + \sum_{m=1}^{\infty}\sqrt{(m+n)m}\ \bar{a}_m\cdot a_{m+n} - \frac{1}{2}\sum_{m=1}^{n-1}\sqrt{(n-m)m}\ a_m\cdot a_{n-m} \qquad n>0$$

$$= H = \alpha' p^2 + \sum_{n=1}^{\infty} n\,\bar{a}_n\cdot a_n$$

$$_{-n} = \bar{L}_n$$

The initial conditions are then $L_n = 0$, $n = 0, \pm 1, \pm 2, \ldots\ldots$

To quantize the system:

Replace the c-numbers a_n , $\bar{a}_n$, X_0 , P with operators satisfying

$$\left[a_m^{\mu}, a_n^{\nu +}\right] = g^{\mu\nu}\delta_{m,n}$$

$$\left[X_0^{\mu}, P^{\nu}\right] = i g^{\mu\nu}$$

$$a_m^{\mu}|0\rangle = P^{\mu}|0\rangle = 0$$

$$|K^{\mu}\rangle = e^{iK\cdot X_0}|0\rangle$$

$$\therefore\ P^{\mu}|K^{\mu}\rangle = K^{\mu}|K^{\mu}\rangle$$

A basis of the Fock space is given by

$$|\lambda, K\rangle = \prod_r \left(a_r^{\mu_r +}\right)^{\lambda_r} e^{iK\cdot X_0}|0\rangle$$

In the definition of the operator L_0 one encounters an ambiguity from the ordering of the operators. (No such ambiguity occurs in the definition of the L_n with $n \neq 0$).

We fix this ambiguity by defining

$$L_0 = :L_0: = \alpha' p^2 + \sum_n n\,a_n^{+} a_n$$

We must allow, however, for the presence of an undetermined additive constant in the quantum equation corresponding to $L_0 = 0$. The wave functions $|\phi\rangle$ of the physical states must satisfy

$$\langle \phi | L_n - \alpha_0 \delta_{n,0} | \phi \rangle = 0$$

or

$$\begin{cases} (L_0 - \alpha_0)\,|\phi\rangle = 0 & & (4.36) \\ L_n\,|\phi\rangle = 0 & n > 0 & \\ \langle\phi|\,L_{-n} = 0 & n > 0 & (4.37) \end{cases}$$

Conditions (4.36)(4.37) replace the operator eqs $L_n = 0$ which are incompatible with the commutation relations (4.30).

Eq4.36 is a spectral condition. It gives

$$(-\alpha' p^2 + \alpha_0)\,|\phi\rangle = (\alpha' s + \alpha_0)\,|\phi\rangle = \alpha(s)\,|\phi\rangle = \sum_n n\, a_n^+ a_n\,|\phi\rangle$$

5. Interactions of the Strings[1]

There are several ways to introduce interactions of the string. First we will consider interactions of the string with an electromagnetic field.

The most general dual amplitude is

$$A_N = \langle k_N \lambda_N | V(k_{N-1}) \frac{1}{L_0 - 1} V(k_{N-2}) \cdots \frac{1}{L_0 - 1} V(k_2) | k_1 \lambda_1 \rangle \tag{5.1}$$

where $|k_1 \lambda_1\rangle$ and $|k_N \lambda_N\rangle$ are arbitrary states of excitation which satisfy the Virasoro conditions but which need not be in the ground state. Thus the dual amplitude can be conceived as the amplitude for the transition between two states, $|k_1 \lambda_1\rangle$ and $|k_N \lambda_N\rangle$ of the string. This transition is induced by the emission of quanta of some external field with momenta $k_2, k_3, \ldots, k_{N-1}$. In order to introduce an interaction between the string and the external field that will reproduce the dual amplitude one proceeds as follows: Add a Maxwell interaction term S_1 to the free action S of eq4.5.

$$S_1 = -\frac{e}{\pi} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \; F_{\mu\nu}(x) \, \dot{X}^{\mu} X'^{\nu} \equiv \int d\tau \, d\sigma \, \mathcal{L}_1 \tag{5.2}$$

$$\text{where } F_{\mu\nu}(x) = \frac{\partial A_\nu}{\partial X_\mu} - \frac{\partial A_\mu}{\partial X_\nu}$$

After integration by parts we find this interaction reduces to

$$S_1 = -\frac{e}{\pi} \int_{\tau_i}^{\tau_f} d\tau \left\{ \dot{X}_\mu(\tau, \pi) A^\mu\big(X(\tau,\pi)\big) \quad \dot{X}_\mu(\tau, 0) A^\mu\big(X(\tau, 0)\big) \right\} \tag{5.3}$$

Notice that since for a closed string we have $X_\mu(\tau,\pi) = X_\mu(\tau,0)$, it follows that a closed string does not interact with a Maxwell field.

On the other hand for the open string one derives the equations of motion, solves them and then quantizes the system. Then the interaction hamiltonian $H_1(\tau) = -\int d\sigma\, \mathcal{L}_1$ is used to calculate the transition $|k_1\,\lambda_1\rangle$ to $|k_N\,\lambda_N\rangle$.

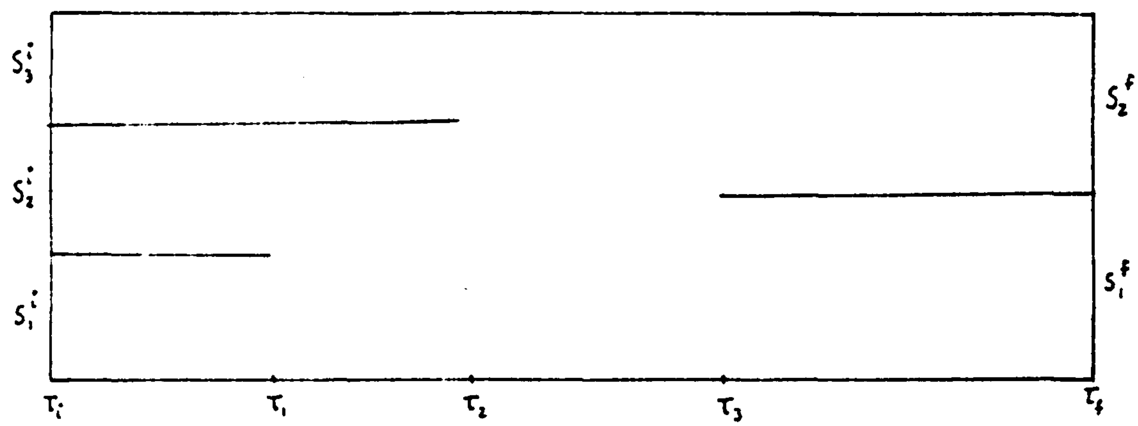
$$\text{ie.}\quad (\lambda_1|\lambda_N) = \int_{-\infty}^{\infty} d\tau_{N-1} \int_{-\infty}^{\tau_{N-1}} d\tau_{N-2} \cdots \int_{-\infty}^{\tau_4} d\tau_3 \int_{-\infty}^{\tau_3} d\tau_2$$

$$\times \langle k_N\lambda_N | H_1(k_{N-1},\tau_{N-1}) \cdots H_1(k_2,\tau_2) | k_1\lambda_1\rangle \tag{5.4}$$

The right hand side of eg5.4 is found to duplicate the dual amplitude (5.1). Furthermore, the quantity $H_1(k,0)$ is found to be just the emission vertex $V(k)$ of the dual amplitude. There is one problem that makes the above derivation non-trivial however. It is expected that every time we consider the interaction of strings with some external field, that the interaction, as in the case of the Maxwell field, will take place at the boundary. This will mean that the interaction hamiltonian will contain the expressions $\exp\{ik\cdot X(\tau,0)\}$ and $\exp\{ik\cdot X(\tau,\pi)\}$. But the operator

$\exp(ik\cdot x)$ is not defined for $k^2 \neq 0$ and so a non-trivial problem of normal ordering is involved. The precise way to obtain the vertices of the dual model is by considering the self interaction of strings.

In the interacting-string picture, interactions take place by the joining and splitting of strings as shown, for example, in the following diagram where three strings join and then split and end up as two strings.



The surface of evolution of the strings at a given moment is discontinuous across the solid lines so that there are three separate strings when $\tau < \tau_1$. At time τ_1 two of them join; at τ_2 the string so formed joins the third. At τ_3 the strings split into two, which go to $\tau = \tau_f$.

In general for an arbitrary number of strings one proceeds as follows: S^i and S^f are respectively the initial and final configuration of the strings. Σ is a typical surface of evolution of the system, existing during the change from S^i to S^f .

Σ is parametrized by assigning the coordinates of its points as $\chi^\mu = \chi^\mu(\sigma,\tau)$

where σ , τ does not necessarily have the same range as in the case of the free string.

A is the area of Σ

$S = \frac{1}{2\pi\alpha'} A$ is the action associated with Σ

Consider the functional

$$I(S^i, S^f) = \int (D\chi)\, e^{iS} \qquad (5.5)$$

which is a sum over all possible surfaces of evolution. This is a functional integral over functions χ . Then Feynmann's path integral formulation of quantum mechanics gives the transition amplitude as

$$T = \lim_{\substack{\tau_i \to -\infty \\ \tau_f \to \infty}} \int \prod_h DS_h^i \prod_k DS_k^f \prod_h \Psi_{\alpha_h}(S_h^i, \tau_i) \prod_k \Psi_{\alpha_k}^*(S_k^f, \tau_f) \times I(S_h^i, \tau_i ; S_k^f, \tau_f) \qquad (5.6)$$

where Ψ_{α_h} and Ψ_{α_k} are the wave functions of the initial and final free strings.

When we do the functional integration I, we recall that S is given by $S = -\frac{1}{2\pi\alpha'}\int d\tau\, d\sigma \sqrt{-\dot{\chi}^2\chi'^2 + (\dot{\chi}\chi')^2}$. The presence of the square root makes the calculation of I almost impossible. Moreover an integration over all functions $\chi(\sigma,\tau)$ would be redundant, because many different functions, related by a change of parametrization, can represent the same surface of evolution. Thus in order to make the functional integration well defined, we must impose a constraint on the functional integral, in the form of a guage requirement that specifies a parametrization. By choosing a suitable gauge we can in fact change the form of the action from square root to quadratic. The only suitable gauge found so far is the transverse gauge described as follows:

The transverse gauge.

First introduce for any vector u^{α} the light-like components defined by

$$u^{\pm} = (u^{0} \pm u^{d-1})/\sqrt{2}$$

$$u_{\perp} = (u^{1}, \ldots, u^{d-2}) = \{u^{i}\}$$

Choose the τ parameter proportional to the time coordinate in some given Lorentz frame:

$$\eta \cdot \chi = \sqrt{\alpha'}\,\tau \qquad (5.7)$$

where is the light-like vector $\eta^{-} = 1\,,\ \eta^{+} = \eta^{i} = 0$

Label the σ coordinate points so that the density of momentum in the η direction is a constant:

$$\eta \cdot K = \frac{1}{2\pi\sqrt{\alpha'}} \qquad (5.8)$$

Thus $P^{+} = \int_{0}^{\sigma_{max}} d\sigma\, K^{+}(\sigma) = \frac{\sigma_{max}}{2\pi\sqrt{\alpha'}}$ is the total (+) momentum.

Equations (5.7) and (5.8) specify the transverse gauge. The transverse gauge can be shown to be a further specification of gauge which does not spoil the orthonormal guage. The independent dynamical variables turn out to be

$\chi_\perp,\ K_\perp,\ P^+,\ x_0^-$

while the variables χ^-, K^- turn out to be dependent variables, ie functions of $\chi_\perp$ and $K_\perp$. Solving to find the classical equations of motion and then quantizing the string one proceeds in the same way as in the covariant quantization. The normal mode operators $\{a_n^i\}$ now used in quantization, differ from the covariant operators $\{a_n^\mu\}$ since the a_n^i act wholly within the physical positive normed $\{a_n^\mu\}$ space. In fact the a_n^i are related to the A_n^i operators introduced earlier. Of course, the use of the transverse gauge is a non-covariant method of quantization. Upon checking one finds that quantization of the string in the transverse gauge leads to a covariant quantum system only when d=26 and $\alpha_0 = 1$.

Lets return now to the calculation of I in (5.5). For a general surface of evolution (not satisfying the classical equations of motion) $P^+(\tau)$ will not be a constant. In summing over paths in (5.5) we should include all possible surfaces of evolution. But since we know that $P^+(\tau)$ will finally be conserved we assume that we can restrict the summation over paths to these surfaces of evolution that conserve P^+. This amounts to a functional integration over the transverse variables $x^i(\sigma,\tau)$ only. The functional integration (5.5) can be reduced to an integral over $x_\perp$ only, by inserting in the integrand the function

$$\delta\left[(\dot{x} - x')^2\right] \quad \delta\left[x^+ - f(\sigma,\tau)\right]$$

Notice that $(\dot{x}-x')^2 = 0$ implies that

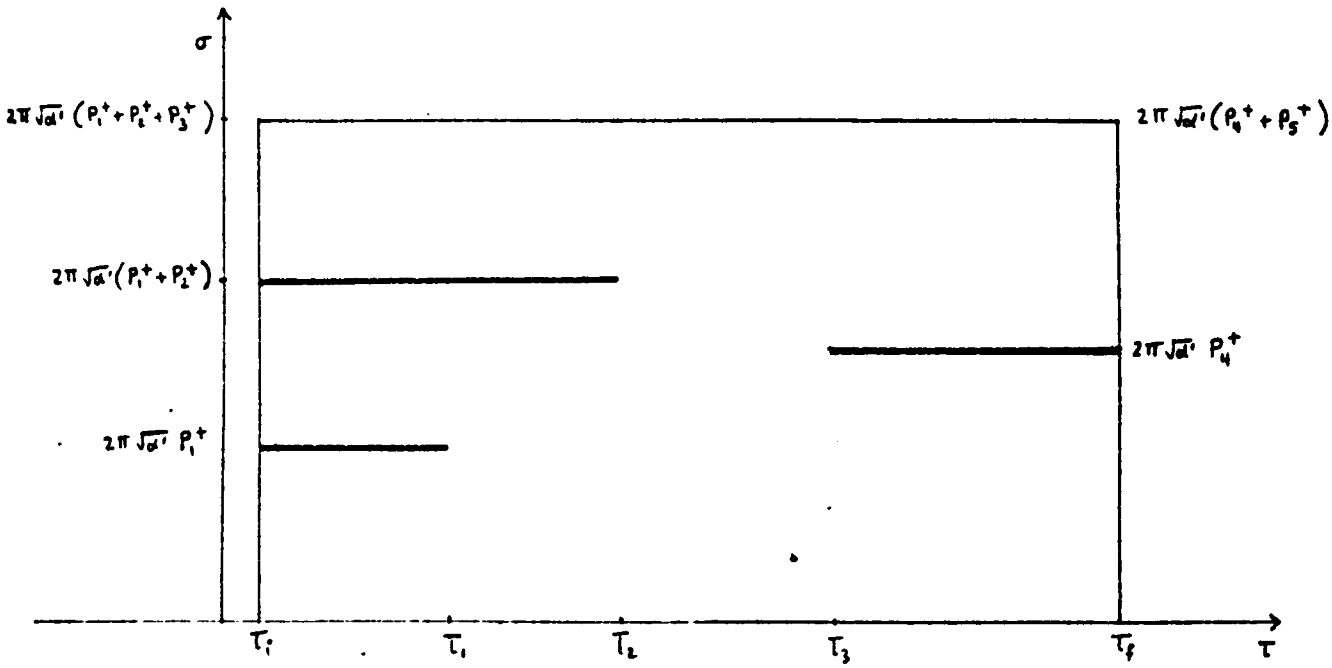
$$\sqrt{-\dot{\chi}^2 \chi'^2 + (\dot{\chi}\chi')^2} \quad = \quad \tfrac{1}{2}\left(\chi'^2 - \dot{\chi}^2\right)$$

The transition amplitude can then be cast in the form

$$\begin{aligned} T = \lim_{\substack{\tau_i \to -\infty \\ \tau_f \to \infty}} \int \prod_n d\tau_n \; V(\tau_1 \cdots \tau_N, \tau_i \tau_f) \quad & \int D\chi^i(\sigma,\tau) \\ \times \prod_h \psi_{\alpha_h}\left(\chi_\perp^h(\sigma,\tau_i)\right) \quad & \prod_k \psi^*_{\alpha_k}\left(\chi_\perp^k(\sigma,\tau_f)\right) \\ \times \exp\left\{ \frac{-i}{4\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_{max}} d\sigma \left[\dot{\chi}_\perp^2(\sigma,\tau) - \chi_\perp'^2(\sigma,\tau)\right] \right. & \left. \vphantom{\int} \right\} \end{aligned} \qquad (5.10)$$

Here $V(\tau)$ is an appropriate weight factor.

The domain of σ, τ as the system of strings evolves from S^i to S^f for the case of the 5-point function is shown below.



It is convenient to perform a wick rotation in (5.10) replacing τ with $i\tau$. Afterwards a complex variable Z is introduced: $Z = \tau + i\sigma$ The singularity structure in the Z plane is as shown in the diagram; ie the heavy horizontal lines are the cuts. The actual calculation of (5.10) is quite involved and will not be reproduced here.(8) Suffice it to say that the dual N-point function can be exactly rederived. However, transition amplitudes derived by use of the transverse gauge can be shown to be covariant and reproduce the dual amplitudes only for d=26.

6. Neveu-Schwarz and Ramon Models(9)

Although this paper deals primarily with the Veneziano model we give a brief description of the Neveu-Schwarz (also called the dual pion model) and the Ramon model.

In the N-S and R models we have the operators

$X^\mu(\sigma)$, $K^\mu(\sigma)$

of the conventional Veneziano model. In addition we have the anticommuting operators $\Gamma^\mu(\sigma)$.

Both the N-S and the R model can be derived from a string picture in which the string has, superimposed on it, a continuum spin structure. $X^\mu(\sigma)$ and $K^\mu(\sigma)$ are expanded in terms of a_n^μ and $a_n^{\mu +}$ where n is an integer and these operators satisfy $[a_n^\mu, a_m^{\nu +}] = g^{\mu\nu}\delta_{nm}$ as shown earlier.

On the other hand Γ^μ can be expanded in two different ways corresponding to the choices

either $\qquad \Gamma^\mu(-\pi) = -\Gamma^\mu(\pi)$

or $\qquad \Gamma^\mu(-\pi) = +\Gamma^\mu(\pi)$

The first choice gives

$$\Gamma^\mu(\sigma) = i\sqrt{2} \sum_{r = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots} b_r^\mu \exp ir\sigma \qquad (b_{-r} \equiv b_r^+)$$

this is the choice made in constructing the N-S model.

The second choice gives

$$\Gamma^{\mu}(\sigma) = \sum_{m=0,\pm 1,\pm 2,\ldots} d_m^{\mu} \exp im\sigma \qquad (d_{-m} = d_m^{+})$$

which is the choice made in constructing the R model.

For the N-S model we have

$$\{ b_r^{\mu}, b_{-s}^{\nu} \} = g^{\mu\nu} \delta_{rs}$$

and the Fock space $\prod_{ij} b_{r_i}^{\mu_i +} a_{n_j}^{\nu_j +} |0\rangle$ consists of boson states.

For the R model we have

$$\{ d_m^{\mu}, d_{-n}^{\nu} \} = -2 g^{\mu\nu} \delta_{mn}$$

and the Fock space $\prod_{ij} d_{m_i}^{\mu_i +} a_{n_j}^{\nu_j +} |u\rangle$ consists of fermion states.

Here $|u\rangle$ is an ordinary Dirac spinor.

The equations of constraint can be expressed in terms of gauge operators L_n, G_r, F_n (where n is an integer, and r is a half integer); these operators being expressed as integrals of X, K and Γ. The algebra of these gauges is

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{8} d (m^3 - m) \delta_{m,-n}$$

$$[L_m, G_r] = \left(\tfrac{1}{2} m - r\right) G_{m+r}$$

$$[L_m, F_n] = \left(\tfrac{1}{2} m - n\right) F_{m+n}$$

$$\{G_r, G_s\} = 2L_{r+s} + \tfrac{1}{2}d\left(r^2 - \tfrac{1}{4}\right)\delta_{r,-s}$$

$$\{F_m, F_n\} = 2L_{m+n} + \tfrac{1}{2}d\left(m^2 - \tfrac{1}{4}\right)\delta_{m,-n}$$

The classical equations of constraint are given quantum mechanically as restrictions on the physical states.

For the boson states we have

$$L_n|\phi\rangle_b = G_r|\phi\rangle_b = 0 \qquad n, r > 0$$

$$(L_0 - \alpha_0)|\phi\rangle_b = 0$$ (the mass shell condition)

For the fermion states we have

$$L_n|\phi\rangle_f = F_m|\phi\rangle_f = 0 \qquad n, m > 0$$

$$(F_0 - i\sqrt{\alpha'}M)|\phi\rangle_f = 0$$ (the mass shell condition)

By arguments analogous to those used for the Veneziano model, consistency of the spinning-string theory requires:

(i) That the boson leading trajectory intercept α_0 be fixed at $\alpha_0 = 1$.

(ii) That the mass of the fermion ground state be $M^2 = 0$

(iii) That the mass of the boson ground state be $\alpha' M^2 = -\frac{1}{2}$

(iv) That the dimension of spac -time be $d = 10$.

The transverse states form a complete set of physical states only when (i) -(iv) holds. Similar to the case of the Veneziano model, there is a critical dimension beyond which ghosts appear. Its value is d=10. The amplitudes for the interaction of a system of strings with spin can be calculated using functional integration generalizing the method used in the spinless case.

7. Off-Mass-Shell Amplitudes

There is great interest in the problem of defining an amplitude describing a process in which some of the external legs are currents (eg leptons). Defining a current leg is believed to involve extrapolating the leg to off-mass-shell. Thus such an extrapolation will enable weak and electromagnetic currents to be incorporated in dual models. Many dynamical properties of hadrons can be tested via electromagnetic or weak currents (eg chiral symmetry, current algebra, and scaling properties). Researchers believe a solution of the currents problem will go a long way towards finding the most correct dual model picture of the real world. Also, it seems reasonable to see the problem of constructing currents as connected ultimately with the search for the best Lagrangian formulation of dual models.

Requirements of Off-Mass-Shell Amplitudes.

Researchers now agree on the following restrictions for currents: (i) Currents should be formulated within the original Hibert space of physical particle states, which was defined from the analysis of purely hadronic dual amplitudes involving only on-shell external states. (ii) The poles of the off-shell amplitude in all channels should correspond to the physical states of the on-shell theory. (iii) the residues of the poles should factorize without ghosts.

The phenomenological observation that the high energy behavior and the low energy form of the hadron-hadron scattering amplitudes are related, is what led to the duality approach to the strong interactions. This relation was formulated on theoretical grounds as the finite energy sum rules.[10] The duality concept as a solution to the finite energy sum rule equations provides a relation between the low and high-energy experimental quantities. If the hadrons show duality properties in their mutual interactions, one may ask whether these properties play a role in the interactions of hadrons with leptons too. Recent sophisticated laboratory equipment has made possible the launching of ever higher energy lepton-hadron experiments. Thus these questions become more important.

There are two factors which make the generalization of duality ideas to currents nontrivial. First, the currents carry four momenta q_i which are not restricted to any mass shell. So, q_i^2 appear as additional variables in which the amplitudes have resonances and other singularities. The question is whether the singularities arising in lepton-hadron scattering obey the same kind of duality restrictions as in purely hadron-hadron scattering. Secondly, the presence of the new variables q_i^2 which can grow to infinity with energy give rise to a variety of high energy limits, e.g. the Bjorken limit. In the search for the correct duality

duality phenomena we have to find a limit playing the role analogous to that of the Regge limit in the hadron-hadron scattering.

Early application of the duality ideas to currents was to express the electroproduction structure functions as sums of resonances with a view to describing the observed scaling. Experimental support for duality came after. The scaling in the Bjorken limit was found to take place at unexpectedly low values of $-q^2$. Furthermore, it was observed that the simple asymptotic scaling interpolates the nucleon resonance region. These experimental results indicate that in the inclusive lepton-nucleon scattering there is again a relation between the asymptotic behavior of the amplitudes and their values in some nonasymptotic region, in much the same way as in hadron-hadron scattering. The Bjorken limit is suggested to be the suitable high energy limit.

8. Deep-Inelastic Lepton-Hadron Processes

For the study of hadron structure, reactions of the following type are of special interest:

inelastic lepton-nucleon scattering, namely,

$$l + N \longrightarrow l' + \boxed{n} \qquad (8.1)$$

and annihilation reactions of the kind

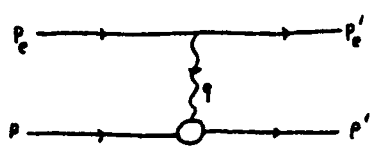
$$e^+ + e^- \longrightarrow \boxed{n} \qquad (8.2)$$

Here $\boxed{n}$ is any possible hadronic system produced.

In (8.1) only the outgoing lepton is detected and the sum is taken over all the hadrons. In (8.2) those events which correspond only to hadronic final states are isolated and summed. In the limit when q^2 (the squared mass of the corresponding hadronic current) is very large in magnitude and $\boxed{n}$ carries a very high mass, these reactions (8.1) and (8.2) fall under the general category of deep inelastic lepton-hadron processes.[11]

Consider the following elastic scattering with single photon exchange:

$$e + p \rightarrow e' + p' \qquad \text{or}$$



The scattering amplitude is

$$S = \frac{1}{(2\pi)^3} \frac{m_e e^2}{\sqrt{P_{e0}' P_{e0}}} \frac{\bar{U}_{e'} \gamma^\mu U_e}{q^2 + i\epsilon} \langle p', s' | J_\mu^{EM} | p, s \rangle \, i(2\pi)^4 \, \delta^4(p' + p_e' - p - p_e)$$

Where $\partial^\mu J_\mu^{EM} = 0$

$$\langle p', s' | J_\mu^{EM} | p, s \rangle = \frac{M}{\sqrt{P_0 P_0'}} \bar{U}_{s'}(p') \left\{ F_1(q^2) \gamma_\mu + i \frac{\kappa}{2M} F_2(q^2) \sigma_{\mu\nu} q^\nu \right\} U_s(p)$$

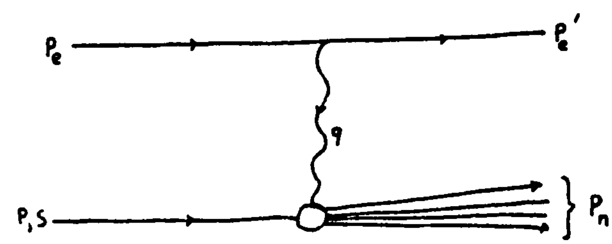
with the form factors $F_1(q^2)$, $F_2(q^2)$ real , M = proton mass

and $F_1(0) = F_2(0) = 1$, κ is anomalous magnetic moment of proton.

It is found experimentally that $F_1(q^2)$ and $F_2(q^2)$ fall of rapidly for large increasing $|q^2|$.

Consider now the inelastic reaction:

$e + p \rightarrow e' + \boxed{n}$



The scattering amplitude in this case is,

$$S = \frac{1}{(2\pi)^3} \frac{m_e e^2}{\sqrt{P_{e0} P'_{e0}}} \frac{\bar{U}_{e'} \gamma^\mu U_e}{q^2 + i\epsilon} \langle n | J_\mu^{EM} | P, S \rangle \, i(2\pi)^4 \, \delta^4(P + q - P_n)$$

$$\text{Let} \quad W_{\mu\nu}^e \equiv \frac{P_0}{M} \sum_n \overline{\sum_s} \langle P, S | J_\mu^{EM} | n \rangle \langle n | J_\mu^{EM} | P, S \rangle (2\pi)^6 \, \delta^4(P + q - P_n)$$

P- and T- invariance imply $W_{\mu\nu}^e = W_{\nu\mu}^e$ (8.5)

Current conservation requires $q^\mu W_{\mu\nu}^e = 0$ (8.6)

From (8.5) and (8.6) we find that $W_{\mu\nu}^e$ must have the form:

$$W_{\mu\nu}^e = \left(-g^{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1^e + \frac{1}{M^2}\left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P \cdot q}{q^2} q_\nu\right) W_2^e$$

where $W_i^e \equiv W_i^e(q^2, \nu)$ are real (called inelastic ep structure functions) can show that

$$W_i^e(q^2, \nu) = - W_i^e(q^2, -\nu)$$

$$\nu \equiv \frac{p \cdot q}{M}$$

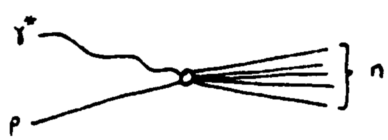
$$P_n^2 = (p+q)^2 = M^2 + 2M\nu + q^2$$

The deep inelastic region is where q^2 and P_n^2 are large compared to M.

$$\frac{d^2\sigma}{dq^2\, d\nu} = \frac{4\pi\alpha^2}{q^4} \frac{P_{e_0}'}{P_{e_0}} \left(2W_1^e \sin^2\frac{\theta_e}{2} + W_2^e \cos^2\frac{\theta_e}{2} \right)$$

The dependence of W_1^e and W_2^e on q^2, ν completely characterizes deep inelastic electron scattering. In a sense W_1^e and W_2^e represent our lack of knowledge of the bottom vertex, or describe the process $\gamma^* + p \to \boxed{n}$ where γ^* is a virtual pheton. So, alternately, we can think of e-N deep inelastic scattering as actually studying the process virtual photon-nucleon scattering. The cross section depends on both the photon energy and "mass."

W =

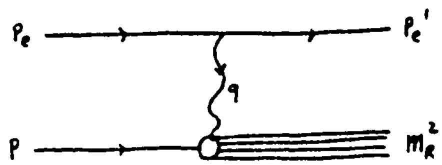


The virtual photon has scalar and transverse cross sections ie $\sigma_{\gamma^* p} = \sigma_T + \sigma_L$ (σ_T refers to photons polarized perpendicular to $\vec{q}$ and σ_L refers to photons polarized parallel to $\vec{q}$). We can show that

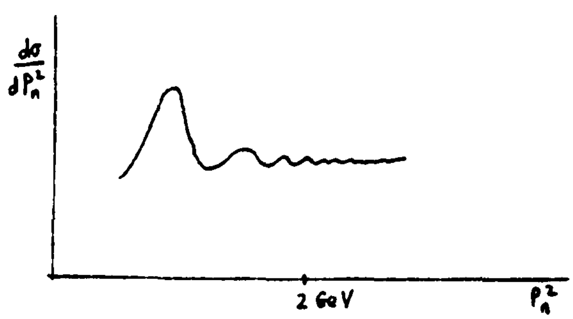
$$W_1^e = \frac{\sqrt{\nu^2 - q^2}}{4\pi^2 \alpha}\, \sigma_T$$

$$W_2^e = \frac{\sqrt{\nu^2 - q^2}}{4\pi^2 \alpha}\, \frac{-q^2}{\nu^2 - q^2}\left(\sigma_T + \sigma_L\right)$$

In leptons-induced resonance production P_n^2 becomes m_R^2 - the mass of the resonance. eg $e + P \rightarrow e' + N^*$

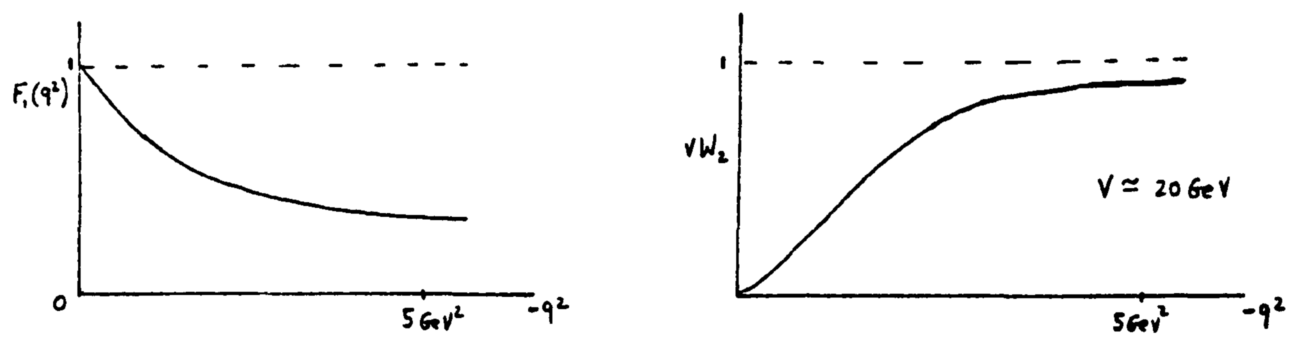


(Electro-resonance production by single-photon exchange)



This curve shows a differential cross section with $-q^2 \gg 1\,GeV^2$. It shows bumps in the resonance region It is then smoothed out. This smoothing marks the onset of the deep inelastic domain.

There is a striking difference in the behavior of the elastic form factors and the deep inelastic structure functions at large $-q^2$. Take ep scattering for instance. The elastic form factors fall off very fast as $-q^2$ increases. Thus $F_1(q^2)$ experimentally tends to zero at least as fast as $1/q^4$ for large $-q^2$. This is regarded as an effect of the structure of the proton due to strong interactions; if the electron saw the proton as a point particle, F_1 should have remained at unity for all q^2. On the other hand, for the inelastic case the situation is that experimentally νW_2 is fairly large for large $-q^2$ and is nearly constant there. It is exhibiting, so to speak, a point like structure in the behavior of the proton in deep inelastic scattering.



Bjorken limit:[12] Defined $-q^2 \to \infty$, $\nu \to \infty$, $\omega \equiv \dfrac{2M\nu}{-q^2}$ fixed.
(Bj)

In the B_J limit the structure functions have the following scaling behavior:

$$\lim_{B_J} W_1(q^2, \nu) = \frac{1}{M} F_1(\omega)$$

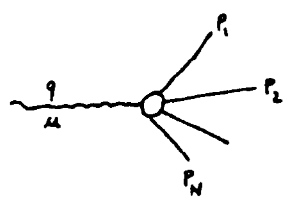
$$\lim_{B_J} \nu W_2(q^2, \nu) = F_2(\omega)$$

The Parton Model

The above scaling rule was originally proposed on theoretical grounds by Bjorken, and experimentally verified. Shortly afterwards the parton model[13] was proposed to provide a framework for understanding Bjorken scaling. According to the parton model, a hadron is a composite of two or more partons. The partons are taken to be point particles with fixed internal quantrum numbers. The partons are bound together by forces at least as strong as the strong interactions. According to the model, the interaction among hadrons can be resolved into the interaction among partons. It is assumed that when a virtual photon ~~parton~~ of sufficiently large q^2 is absorbed by a hadron, the photon is absorbed incoherently i.e. absorbed by only one of the partons. The parton that absorbed the virtual photon ~~hadron~~ is then excited with reference to the remainder of the hadron. The parton cannot escape. Finally the initial hadron breaks up into two or more hadrons. To study the scattering one goes to the Lorentz frame in which the z-component of the initial hadron momentum is infinite. If one then calculates the contributions to W_1 and W_2 from the individual scattering of the parton it it is found that the Bjorken scaling rule is re-derived.

9. Can currents be included in Veneziano models ?

In attempting to include currents in a narrow resonance world, one hopes to gain some insight into the detailed behavior of weak and electromagnetic form factors, electroproduction structure functions, and more generally into the behavior of the matrix elements of the weak and electromagnetic currents between arbitrary hadronic states, as a funtion of momentum transfer. An early attempt to construct a one current amplitude is the following: Consider the amplitude $V_\mu(q, p_i)$ for one current and N spinless particles.

$V_\mu(q, p_i) =$ 

Suppose we impose the following seemingly reasonable conditions on Vu:

(a) $q_u \ V^u = 0$

(b) Regge behavior in all $S_{ik} = (P_i+P_{i+1}+\ldots+P_k)^2$

(c) Meromorphy in q^2 and in the Sik with simple poles for positive real values of these variables.

(d) The residues of poles in Sik are polynomials of finite order in the variables of the overlapping channels.

The residues of poles in q^2 are products of a vector meson scattering amplitude and the strength of the current-vector coupling.

e) The dispersion relations in q^2 and in Sik have no subtractions.

f) Factorization holds, so that the residue of any pole in Sik is a product of some V^μ and a purely hadronic scattering amplitude.

The constraints on analytic behavior in the Sik, contained in (b) - (e), are precisely the same as those discussed for N-point functions. The restrictions on q^2 behavior are the minimal ones required for guage invariance, and factorization. Given a pure hadronic narrow resonance amplitude for a vector meson coupling to N spinless particles, it is straight forward to write down a function satisfying (b) - (f). Let us write the amplitude for N spinless hadrons (momenta P1 P2...PN) and one vector meson (mass M , momentum) as $A_N^{\mu,r}(q, P_i)$

This is just an extention of the N+1 point function for N+1 Scalor hadrons given by 2.11. Then a function satisfying (b) - (f) is given by

$$V^\mu(q, P_i) = \sum_r g_r(q^2) \frac{m_r^2}{m_r^2 - q^2} \left(g^\mu{}_\nu - \frac{q^\mu q_\nu}{q^2} \right) A_N^{\nu, r}(q, P_i) \qquad (9.1)$$

where $\sum_r g_r(o) = 1$ and the $g_r(q^2)$ are as yet undetermined entire functions. We still need to satisfy the guage invariance condition (a), at $q^2=0$. We can do this by requiring that

$q_u A^u n = 0$ (9.2)

Equations 9.1 and 9.2 introduced by Brower and Weis[14] was one of the early methods of satisfying requirements (a) - (f). Unfortunately there is no apparent method of finding the g_r function and hence the q^2 dependence. For this reason the ansatz 9.1 gave no non-trivial information concerning current form factors. Of course, the problem of the bad spectrum that plagues purely hadronic narrow resonance models is carried over into the Brower-Weis ansatz.

10. The Drummond-Rebbi model

The approach followed by many workers was to construct an amplitude for currents, which would reduce to the standard An (for example, 2.11) when the current momenta were placed on-shell. We illustrate by showing the procedure followed by Drummond and Rebbi.[15]

The amplitude for N scalar photons of momenta q_1, q_2.. q_N with arbitrary q_i^2 (i.e. not necessarily on-shell) is taken to be

$$A_N = \int \frac{1}{dR} \prod_{i=1}^{N} \frac{d^2 z_i}{\rho(z_i)^2} F_N(z, q)$$

$$\text{where } F_N(z,q) \equiv \prod_{i=1}^{N} \rho(z_i)^{q_i^2} \prod_{1 \le i < j \le N} |z_i - z_j^*|^{2 q_i \cdot q_j}$$

$$\text{and } \rho(z) \equiv \left| \tfrac{1}{2}(z - z^*) \right|$$

10(1)

$\int dR$ gives an infinite volume that has to be factored out. The integration region is the entire upper half of the complex Zi planes.

Note that under a real Mobius transformation, A, we have:

$$z \to Az = \frac{a_1 z + a_2}{a_3 z + a_4} \quad \text{and} \quad z^* \longrightarrow A z^*$$

$$\frac{d^2 z}{\rho(z)^2} \longrightarrow \frac{d^2 z}{\rho(z)^2}$$

$$F_N(z,q) \to F_N(Az,q) = \prod_{i=1}^{N} \left| \frac{a_1 a_4 - a_2 a_3}{(a_3 z_i + a_4)^2} \right|^{q \cdot q_i} F_N(z,q)$$

where $q = \sum_{i=1}^{N} q_i$ When momentum is conserved $q = 0$ and the integrand is invariant.

Suppose one or more points, say $Z_1 \ldots Z_j$ approach a common point on the real axis. Thus let

$Z_1 = X_1 + i\epsilon$, $Z_i = (X_1 + \epsilon X_i') + i\epsilon Y_i'$ $\quad i = 2, \ldots, j$

Then for small ϵ the integrand of 10(1) behaves as

$$\epsilon^{\left(\sum_{i=1}^{j} q_i\right)^2 - 2}$$

Thus A_N has poles at

$$\left(\sum_{i=1}^{j} q_i\right)^2 = 1, 0, -1, \ldots\ldots$$

If we want to put all the external lines on their mass shell we must let all the complex points Zi approach the real axis, ie put $Z_i = X_i + i\epsilon Y_i$ $i = 1, 2, \ldots, N$ then $d^2 Z_i = -2i\epsilon\, dx_i\, dy_i$

Integrating and afterwards setting $\epsilon = 0$ we find:

$$A_N \sim \prod_{i=1}^{N} \frac{1}{q_i^2 - 1} \int \frac{1}{dR} \prod_{i=1}^{N} dx_i \prod_{1 \le i < j \le N} |x_i - x_j|^{2 q_i q_j} \qquad 10(4)$$

This is just the standard Koba-Nielsen amplitude. Since we must distinguish among different orderings of the points on the real axis, 10(4) actually gets replaced by 1/2 (N-1)! different non cyclic permutations of the external lines.

Three of the integration variables in 10(1) must be eliminated. Here is one way to do this. Choose three one-dimensional arcs on the complex plane:

$Z = f(\lambda_i; t)$ i= 1.2.3, (λ_i : constant) $\qquad$ 10(5)

Thus for example, $f(\lambda_i, t)$ may be three concentric circles or three vertical lines, etc.

Then use a Mobius transformation to map the complex points Z1, Z2, Z3, onto the three points, $f(\lambda_1,t_1)$, $f(\lambda_2,t_2)$, $f(\lambda_3,t_3)$.

Thus $Z_i = M f(\lambda_i\ t_i) \equiv \dfrac{m_1 f(\lambda_i t_i) + m_2}{m_3 f(\lambda_i t_i) + m_4}$, $m_1 m_4 - m_2 m_3 = 1$, m_i real

On calculating a jacobian, it can be shown that

$$\prod_{i=1}^{3} \frac{d^2 Z_i}{\rho(Z_i)^2} = \prod dm_i \prod_{i=1}^{3} \frac{dt_i}{\rho(t_i)^2} \mu(\lambda_1,\lambda_2,\lambda_3 ; t_1, t_2, t_3)$$

Finally, the infinite integration $\prod dm_i \equiv dR$ is divided out.

To calculate an emplitude for a process involving N particles, ℓ of which are on shell, we take Z1, Z2, ..Zℓ as real and put $q_1^2 = q_2^2 : .. = q_\ell^2 = -1$

To demonstrate factorization.

The amplitude for m+n photons with momenta $(q_1 \ldots q_n)$ and $(K_1 \ldots K_m)$ can be written as:

$$A_{n+m} = \int \frac{1}{dR} \prod_{ij} \frac{d^2 z_i}{\rho(z_i)^2} \frac{d^2 t_j}{\rho(t_j)^2} F_n(z, q)\ F_m(t, K) \prod_{ij} \left| z_i - t_j^* \right|^{2 q_i K_j} \qquad 10(7)$$

We have seen that the poles in the variable $q^2 = \left(\sum_{i=1}^{n} q_i\right)^2$ occur when all the Z_i come together on the real axis. We will now show that these poles have factorizing residues.

First, recall that a real Mobius transformation, A, can be written as

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix} \qquad \det A = 1$$

and $$Az = \frac{a_1 z + a_2}{a_3 z + a_4}$$

We change the integration variables in 10 (7) by the following parametrization:

$$Z_i = M\bar{Z}_i \qquad t_i = N\bar{t}_i$$

where M and N are two real Mobius transformations. $\bar{Z}_i$ are a basic set with three of them confined to arcs as shown in 10 (5). $\bar{t}_i$ are similarly defined. Then 10 (7) can be cast in the form:

$$A_{n+m} = \int \prod_{i,j} \frac{d^2 z_i}{\rho(z_i)^2} \frac{d^2 t_j}{\rho(t_j)^2} \left(\frac{dM\,dN}{dR}\right) F_n(Mz, q)\, F_m(Nt, k) \prod_{i,j} \left| Mz_i - Nt_j^* \right|^{2 q_i k_j} \qquad 10(9)$$

Where $$dM \equiv \prod_{i=1}^{4} dm_i\, \delta(\det M - 1)$$

$$dN \equiv \prod_{i=1}^{4} dn_i\, \delta(\det N - 1)$$

In order to be able to remove dR we next change variables from (M,N) to (L,P) where L=N and $P = N^{-1}M$

then since $$\frac{\partial(\ell_1\, \ell_2\, \ell_3\, \ell_4\, p_1\, p_2\, p_3\, p_4)}{\partial(m_1\, m_2\, m_3\, m_4\, n_1\, n_2\, n_3\, n_4)} = 1$$

we have dMdN = dLdP where $dP = \prod_{i=1}^{4} dP_i\, \delta(\det P - 1)$

Now let $P = e^{\alpha}(B_2^{+})^{-1} \Lambda B_1$ where α is a redundant integration variable which will be dropped, and

$$B_1 = \begin{bmatrix} 1-x_1 & 0 \\ 1 & -x_1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \qquad B_2^+ = \begin{bmatrix} 1-x_2 & -1 \\ 0 & -x_2 \end{bmatrix}$$

We find that det $P = e^{2\alpha} \lambda x_1 x_2 (1-x_1)(1-x_2)$

we put det $P = 1$.

Then we find that

$$\frac{\partial(p_1\, p_2\, p_3\, p_4)}{\partial(\alpha\, \lambda\, x_1\, x_2)} = \frac{1}{\lambda^2 \, |x_1 x_2 (1-x_1)(1-x_2)|}$$

$$\therefore \quad dP = dB_2 \, d\Lambda \, dB_1$$

where $dB_i = \dfrac{dx_i}{|x_i(x_i - 1)|} \qquad d\Lambda = \dfrac{d\lambda}{\lambda^2}$

The above choice of parametrization turns out to be particularly useful in demonstrating factorization.

If now we put

$$N = R B_2^+ \quad , \quad M = e^{\alpha} R \Lambda B_1$$

we see that since

$$\frac{\partial(n_1\, n_2\, n_3\, n_4\, m_1\, m_2\, m_3\, m_4)}{\partial(r_1\, r_2\, r_3\, r_4\, \alpha\, \lambda\, x_1\, x_2)} = 1$$

we have $dMdN = dR\,(dB_2 d\Lambda dB_1) = dRdP$

Thus dR is removed from the integration. We then set $R = 1$ and $\alpha = 0$

Now 10 (9) can be cast in the form

$$A_{n+m} = \int \prod_{ij} \frac{d^2 z_i}{\rho(z_i)^2} \frac{d^2 t_j}{\rho(t_j)^2} \, dB_2 \, \frac{d\lambda}{\lambda^2} \, dB_1 \, F_n(\Lambda B_1 z, q) \, F_m(B_2^+ t, \kappa) \prod_{ij} \left| B_2^+ t_j^* - \Lambda B_1 z_i \right|^{2 q_i \kappa_j} \tag{10(10)}$$

Now from 10 (2) we see that

$$F_n(\Lambda B_1 z, q) = |\lambda|^{q^2} F_n(B_1 z, q)$$

Also from 10(2) we see that

$$F_m(B_2^+ t, \kappa) = \prod_{j=1}^{m} \left| B_2^+ t_j \right|^{-2 q \cdot \kappa_j} F_m\left(\frac{-1}{B_2^+ t}, \kappa\right) \qquad q = -\kappa$$

Now $\dfrac{-1}{B_2^+ t} = B_2\left(\dfrac{-1}{t}\right)$ where $B_2 = \begin{bmatrix} -x_2 & 0 \\ 1 & 1-x_2 \end{bmatrix}$

and $\prod_j \dfrac{d^2 t_j}{\rho(t_j)^2}$ is invariant under the replacement

$$t_i \longrightarrow \frac{-1}{t_i}$$

Thus we can rewrite 10 (10)

$$A_{n+m} = \int \prod_{ij} \frac{d^2 z_i}{\rho(z_i)^2} \frac{d^2 t_j}{\rho(z_j)^2} \, dB_2 \, dB_1 \int_{-\infty}^{\infty} d\lambda \, |\lambda|^{-2+q^2} F_n(B_1 z, q) \, F_m(B_2 t, \kappa)$$

$$\times \prod_{ij} \left| 1 + \lambda (B_1 z_i)(B_2 t_j^*) \right|^{2 q_i \kappa_j}$$

Using the fact that $\log(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$, $|x| < 1$

We have, for small λ :

$$\left[1 + \lambda(B_1 z_i)(B_2 t_j^*)\right]^{q_i k_j} = \exp\left[-\sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n} q_i k_j \left(B_1 z_i\, B_2\, t_j^*\right)^n\right]$$

$$\therefore I \equiv \prod_{ij} \left|1 + \lambda(B_1 z_i)(B_2 t_j^*)\right|^{2 q_i k_j}$$

$$= \exp\left\{-\sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n}\left[u_n V_n^* + u_n^* V_n\right]\right\}$$

where $u_n = \sum_{i=1}^{n} q_i (B_1 z_i)^n$, $V_n = \sum_{j=1}^{m} k_j (B_2 t_j)^n$

If now we introduce two sets of harmonic oscillators

$$[a_{n\mu}, a^+_{m\nu}] = g_{\mu\nu}\,\delta_{mn} \qquad [b_{n\mu}, b^+_{m\nu}] = g_{\mu\nu}\,\delta_{mn}$$

$$[a, a] = [b, b] = [a, b] = 0$$

$$\text{and} \quad H = \sum_{n=1}^{\infty} n\left(a_n^+ \cdot a_n + b_n^+ b_n\right)$$

then using the properties of these oscillators ie that

$$(-\lambda)^{n a_n^+ \cdot a_n} \exp\left(-\frac{1}{\sqrt{n}} k \cdot a_n^+\right)|0\rangle = \exp\left(-\frac{(-\lambda)^n}{\sqrt{n}} k \cdot a_n^+\right)|0\rangle$$

$$\langle 0| e^{xa} e^{ya^+} |0\rangle = e^{x^* y}$$

We find that I can be rewritten as

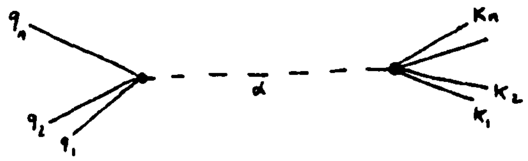
$$I = \langle 0 | \exp\left[\sum_n \tfrac{1}{\sqrt{n}} \left(a_n \cdot u_n + b_n \cdot u_n^* \right) \right] \quad (-\lambda)^H \quad \exp\left[- \sum_n \tfrac{1}{\sqrt{n}} \left(a_n^+ \cdot v_n^* + b_n^+ \cdot v_n \right) \right] | 0 \rangle$$

$$\therefore \quad A_{n+m} \sim \langle U | \frac{1 + (-1)^H}{H + q^2 - 1} | V \rangle \qquad 10(15)$$

$$\text{where } | V \rangle \equiv \int \prod_j \frac{d^2 t_j}{\rho(t_j)^2} \, dB_2 \; F_m(B_2 t, k) \exp\left[- \sum_n \tfrac{1}{\sqrt{n}} \left(a_n^+ \cdot v_n^* + b_n^+ \cdot v_n \right) \right] | 0 \rangle$$

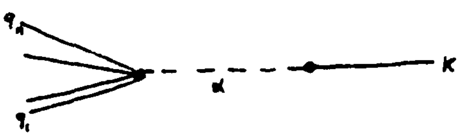
The residues of the poles have the factorizing form $\sum_\alpha \langle U | \alpha \rangle \langle \alpha | V \rangle$ where $|\alpha\rangle$ is an occupation number state involving both a^+ and b^+.

A_{n+m} =



Because two sets of oscillators are required for factorization and $\sum_n n a_n^+ a_n |V\rangle \ne \sum_n n b_n^+ b_n |V\rangle$ the states on the leading trajectory are increasingly degenerate with increasing mass. Factorization in the channel corresponding to an external leg can be shown to have the same form as 10 (15).

ie. A_{n+1} =

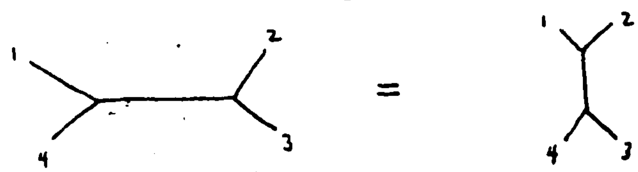


However, for the theory to be completely consistent the hadron S-matrix has to be enlarged, the original one (of Koba-Nielson which factorized with only one set of osciallators) becoming simply a self-consistent sector of the bigger one. What this means is that if we treat the off-shell leg as a photon, then states can be excited electromagnetically which cannot be reached through strong interaction processes. This is physically unrealistic. Furthermore the model has certain problems with ghosts. Some of the problems of the Drummond-Rebbi model were resolved by the subsequent work of other researchers as we shall see.

11. A program for unitarity in narrow resonnance models

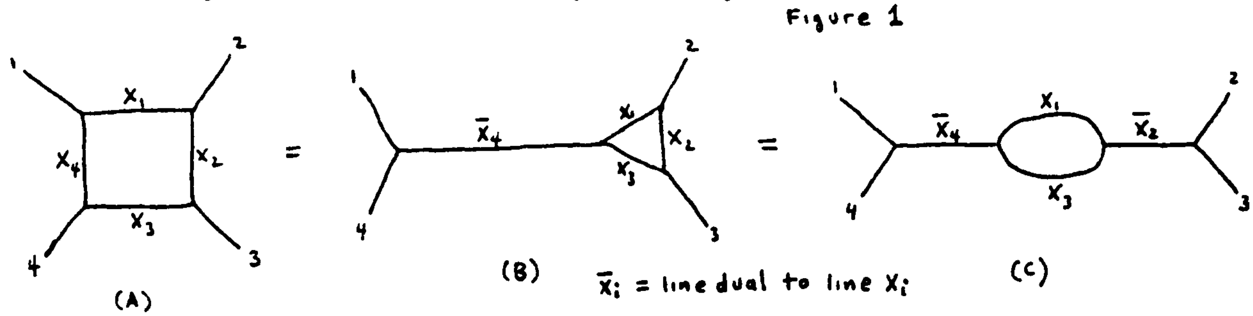
Before continuing our study of current-amplitudes, we will first look more closely at the problem of unitarity in narrow resonance models. We believe such an examination to be highly relevant to the study of current-amplitudes and to the search for more realistic dual resonance models of hadron scattering.

The program initiated by Kikkawa, Sakita, and Virasora[16] is to build a perturbative series in which the Veneziano representation plays the role of a Born approximation. In higher-order terms, contributions of many-particle intermediate states are included in a way similar to usual perturbation theory, ie Feymann-like diagrams with closed loops are included. This approach stems from the fact that zero-width resonances obviously cause Veneziano type amplitudes to violate unitarity. Now recall that in the Veneziano model we have the property of duality which in the case of the 4-point function means that the amplitude can be expressed equivalently, either as a sum over s-channel poles, or as a sum over t channel poles. This is expressed in the following diagram.



The program initiated by K.S.V. is to extend the definition of duality to apply to internal lines. Thus the following Feymann-like diagrams will be equal to each other by duality.

Figure 1



$\bar{X}_i$ = line dual to line X_i

Here duality applied to each internal line means we replace lines in one channel by the corresponding lines in the crossed channel. The Feymann approach to get an exact amplitude would be to take a sum

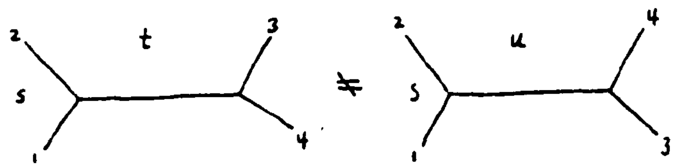
A + B + C + + ...

where each term in the sum is different.

But instead, here we want the amplitude for each diagram to be given by a single expression ie.

A = B = C = ...

In what follows use will be made of the twist operator. The need for this operator can be seen by considering the obvious fact shown in the following diagram.

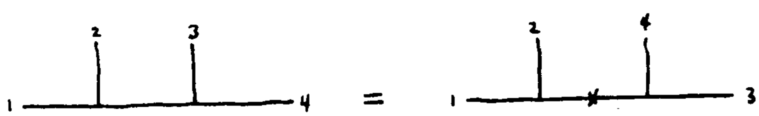


(A) (B)

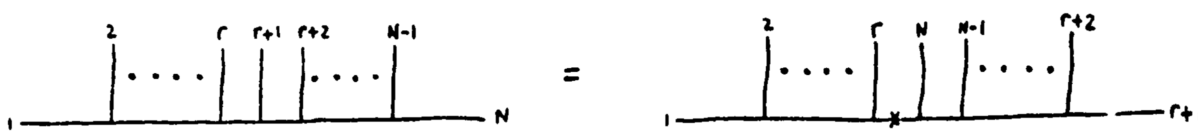
Figure 2.

In figure 2A there are poles in the t channel but in 2B there are no poles in the t channel.

We therefore introduce a twisted line as follows:



And in general for N-point amplitudes we have



Or in the operator language we have

$$V(k_{r+1})\,D\,V(k_{r+2})\cdots D\,V(k_{N-1})\,|0,k_N\rangle = \Omega\,V(k_N)\,D\,V(k_{N-1})\cdots D\,V(k_{r+2})\,|0,k_{r+1}\rangle \tag{11.1}$$

Where Ω is the twist operator.

It can be shown that the operator we require, ie that satisfies eg (11.1) is given by

$$\Omega = (-1)^R\, e^{-L_+}$$

Continuing in the K.S.V. approach to unitarity, we start by calculating amplitudes for single loop diagrams of the following kind:

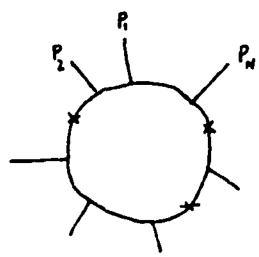


Figure 3.

The N-point loop diagram shown in figure 3 is a generalization of figure 1A. An arbitrary number of its internal lines are twisted. Before calculating the loop amplitude we must first gather together certain operators.

$$L_0(P) = -\tfrac{1}{2}P^2 + \sum_{n=1}^{\infty} n\, a_n^+ \cdot a_n$$

$$L_+(P) = P\cdot a_1^+ + \sum_{n=1}^{\infty} \sqrt{n(n+1)}\; a_{n+1}^+ \cdot a_n$$

$$L_-(-P) = P\cdot a_1 + \sum_{n=1}^{\infty} \sqrt{n(n+1)}\; a_n^+ \cdot a_{n+1}$$

The three generators Lo(p), $L_\pm(P)$ generate the algebra SU (11).

Define A (p) = Lo (p) - L_-(p)
and A^+(-p) = Lo (P) - L_+(P)

A^+(P) has the property that it creates when applied to any state in the Hilbert space, spurious states that do not couple to any number of external scalar particles. The projection operator is

$$\mathcal{P} = I - \left[A^+(-P) - \alpha_0\right] \left\{ A(P)\left[A^+(-P) - \alpha_0\right] \right\}^{-1} A(P) \qquad \text{(II.2)}$$

The projected propagator is $\bar{D}(P) = \mathcal{P}^+(-P)\, D(P)\, \mathcal{P}(P)$

where $D(P) = \int_0^1 dx\,(1-x)^{\alpha_0 - 1}\, x^{R-\alpha(P^2)-1}$

The projected propagator $\bar{D}$ (P) is hermetian and contains states which have lineraly independent couplings to external scalar particles. The operator A^+ is necessary since otherwise one must enlarge the Hilbert space by introducing additional scalar modes in order to achieve factorization of the twisted verticies.

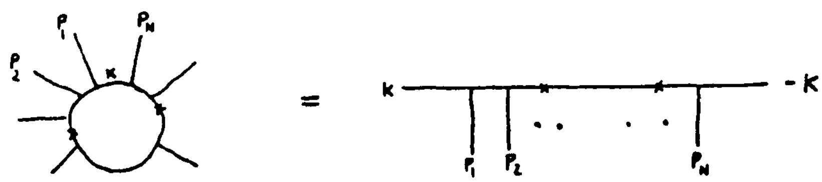
The twist operator we found is

$$\Omega = (-1)^R e^{-L_+}$$

but this choice is not completely unique. We are free to choose Ω so that can define θ in the following way

$$\theta(z) \equiv \Omega(1-z)^A = (1-z)^{-A}\Omega$$

To evaluate the N-point graph in which an arbitrary number of internal lines are twisted, one evaluates the following integral:



$$= \int d^4K \, Tr\left\{ V(P_1)\, D'(K+P_1)\, V(P_2)\, D'(K+P_1+P_2) \cdots V(P_N)\, D'(K) \right\}$$

$$\text{where} \quad D'(P) = \begin{cases} \bar{D}(P)\,\theta(P) & \text{for a twisted line} \\ \bar{D}(P) & \text{for untwisted line} \end{cases}$$

Using the properties of P and Ω and of coherent states, it can be shown that the loop integral with M adjacent twisted internal lines and N adjacent untwisted internal lines (figure 4) is given by :

$$F_{N,M} = \int d^4k \int \prod_{i=1}^{M} du_i \; u_i^{-\alpha(k_i^2)-1} \prod_{j=1}^{N} dv_j \; v_j^{-\alpha(k_j^2)-1} \qquad \left[1-(-1)^M \omega\right]^2$$

$$\times \left[(1-u_M v_1 v_2 \cdots v_N u_1) \cdots (1-u_r v_\ell v_{\ell+1} \cdots v_{\ell+m} u_{r+1}) \cdots \prod_{i=1}^{M-1} (1-u_i u_{i+1}) \right]^{\alpha_0 - 1}$$

$$\times \left[\prod_{j=1}^{N} (1-v_j) \right]^{\alpha_0 - 1}$$

$$\times \; \mathrm{Tr} \left\{ (-u_1)^R V(p_1) (-u_2)^R V(p_2) \cdots (-u_M)^R V(p_M) \quad v_1^R \, V(p_{M+1}) \cdots v_N^R \, V(p_{M+N}) \right\}$$

$$\text{where} \quad \omega = \prod_{j=1}^{N} \prod_{i=1}^{M} u_i v_j \qquad (1.7)$$

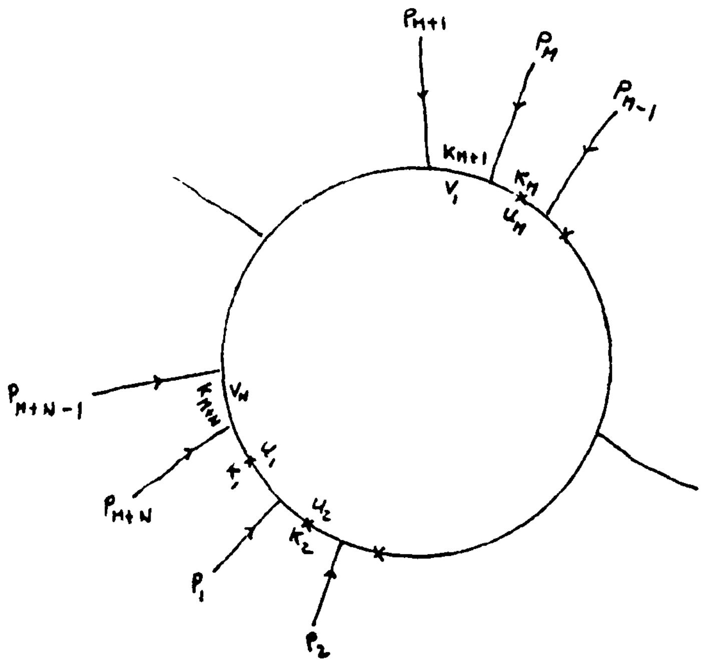


Figure 4.

Diagrams with an even number of twists (greater than zero) are called orientable nonplanar diagrams. These have special interest. The trace in eg 11.7 can be evaluated, and then using rather involved mathematics the integral can be evaluated. The general form of the orientable one-loop integral in the case of M (Meven) adjacent twisted lines can be calculated using the techniques of Amati et al and the properties of Jacobi functions. One gets

$$F_{N,M} = 4\pi^2 g^{N+M} \int \prod_{i=1}^{N} dv_i \prod_{j=1}^{M} du_j \, (1-\omega)^2 \frac{f(\omega)^{-4}}{\ln^2 \omega} \quad \omega^{-\alpha_0 - 1}$$

$$\times \left[(1 - u_n v_1 \cdots v_N u_1) \cdots (1 - u_r v_l v_{l+1} \cdots v_{l+m} u_{r+1}) \cdots \prod_{j=1}^{M-1} (1 - u_j u_{j+1}) \right]^{\alpha_0 - 1}$$

$$\times \left[\prod_{i=1}^{N} (1 - v_i) \right]^{\alpha_0 - 1} \prod_{i<j} \psi_{\pm} (c_{ji}, \omega)^{-P_i P_j} \qquad (11.8)$$

ψ_+ is used when an even number of twisted lines are included between Pi and Pj.

ψ_- is used when an odd number of twisted lines are included between Pi and Pj

$\psi_+(x,\omega)$ and $\psi_-(x,\omega)$ are functions of X,W involving Jacobi θ functions.

C_{ji} denotes the product of the U's and V's appearing between Pi and Pj

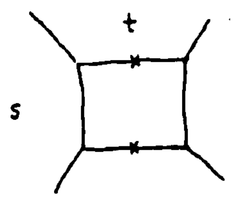
$$\therefore \quad C_{ij} = \frac{\omega}{C_{ji}} \quad i \neq j \quad , \quad C_{ii} = \omega$$

A property of $\Psi_{\pm}$ is that $\Psi_{\pm}(x,\omega) = \Psi_{\pm}\left(\frac{\omega}{x},\omega\right)$ which has as a consequence that it does not matter on which side the U's and V's are chosen in the definition of Cji. This in turn can be shown to imply the cyclic symmetry of $F_{N,M}$ in the external momenta.

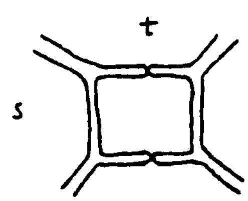
$$f(\omega) \equiv \prod_{n=1}^{\infty}\left(1-\omega^{n}\right)$$

In a typical factor of the form $\left(1 - U_r V_\ell V_{\ell+1} \cdots V_{\ell+m} U_{r+1}\right)$ Ur and Ur+1 refer to two successive twisted lines with M+1 intervening untwisted lines.

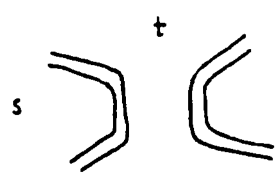
It turns out that all single loop integrals $F_{N,M}$ have divergences. Thus a renormalization procedure, which will not concern us here, is required if unitarity is to be achieved. Orientable non-planar diagrams have striking properties. Unlike other loop integrals, which are renormalizable, orientable nonplanar loop integrals can be shown to have branch point singularities in certain channels. This violates the requirements of perturbative unitarity, which motivated the construction of these diagrams in the first place. The singularities are poles only when $\alpha_0 = 1$ and $d = 26$ ($d = 10$ for the dual pion model.) Surprisingly these singularities only occur in channels with vaccuum quantum numbers. For example consider the following orientable loop diagram:



which can be redrawn the following way with quark lines



thus the diagram can be pulled apart without breaking quark lines in the following way.



Thus there are no quark-antiquarks propagating in the s channel ie the S-channel is the one with quantum numbers of the vacuum. Furthermore the singularities in orientable loop integrals have definite angular momentum properties, lying in fact on Regge trajectories given by $\alpha(x) = \frac{1}{3} + \frac{1}{4}$ where x is the channel (energy).[2] All of this strongly suggests that these singularities be identified with the Pomeranchuck singularity.

12. Operators in the unitarity scheme

We now turn to a better method (17) of reproducing eq. 11.7 which has been useful in constructing currents.

As we know, physical trees of the Veneziano are built up out of propagators $D(p)$ and vertex operators $V(p)$:

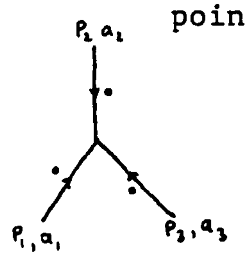
$$D(P) = \int_0^1 dx\, x^{R-\alpha(P^2)-1}(1-x)^{\alpha_0-1} \qquad , \quad \alpha(s) = \alpha_0 + \tfrac{1}{2}s$$

$$R = \sum_{n=1}^{\infty} n\, a^+(n)\cdot a(n)$$

$$V(P) = \exp\left(a^+|P\right)\exp\left(a|P\right)$$

$$(a|P) \equiv \sum_{n=1}^{\infty}\frac{a(n)\cdot P}{\sqrt{n}} = \sum_{m,n=1}^{\infty} a(m)\,\delta_{mn}\frac{P}{\sqrt{n}}$$

We introduce now the symmetric vertex operator, (18) or three-point operator, given by:



$$= V(P_1, P_2, P_3\,;\, a_1, a_2, a_3)$$

$$= g\left\langle 0\right|\exp\left\{(a_1|P_2) + (a_2|P_3) + (a_3|P_1) + (a_1|M_-|a_2) + (a_2|M_-|a_3) + (a_3|M_-|a_1)\right\}$$

Here $(a_i|M_-|a_j) = \sum_{m,n=1}^{\infty} a_i(m)\,(M_-)_{mn}\,a_j(n)$

$$(M_-)_{mn} = \frac{1}{\sqrt{mn}}\frac{1}{B(n,-m)} = (-1)^n\sqrt{\frac{n}{m}}\binom{m}{n}$$

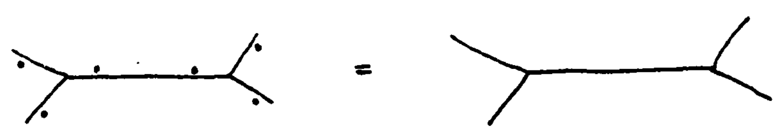
$$(M_+)_{mn} = \frac{1}{\sqrt{mn}}\frac{1}{B(n,m)} = \sqrt{\frac{n}{m}}\binom{m+n-1}{n}$$

Notice that we have different but mutually commuting harmonic operators a_1, a_2, a_3 applicable to each leg in turn. The three-point amplitude for definite states is given by

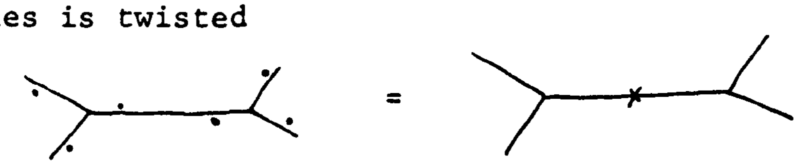
$$V(P_1, P_2, P_3 ; a_1, a_2, a_3)\,|\lambda_1 P_1\rangle\,|\lambda_2 P_2\rangle\,|\lambda_3 P_3\rangle$$

where $|\lambda_i\, P_i\rangle$ is a physical state with occupation number λ and mass P_i^2.

The dots shown in the 3 - point graph and in subsequent diagrams are quite necessary. As will be seen they enable one to tell whether or not an internal line formed by joining two diagrams with a propagator is twisted or not. If the dots match the line is untwisted.



If the dots are on opposite sides of the line, the lines is twisted



Now spurious states are all generated by the operator $A^+(-P)$ acting on any vector of the Hilbert space.

Recall that the spurious states contained in the original Hilbert space can be removed by the projection operator P (P) given by (11.2)

Recall also that

P (P) is constructed so that the projected propagator $\bar{D}(P)$ is free of spurious states.

$$\bar{D}(P) \equiv P^{+}(-P)\,D(P) = D(P)\,P(P)$$

Using the twist operator

$$\Omega(P) = (-1)^{R} \exp\left[-L_{+}(P)\right]$$

We can construct the projected twisted propagator, it is

$$\mathcal{D}(P) = P^{+}(P)\,D(P)\,\Omega(P)\,P(P)$$

This can be rewritten in the following integral form

$$\mathcal{D}(P) = \int_0^\infty dz\, z^{-\alpha(P^2)-1}\,(1+z)^{P^2} \tag{12.2}$$
$$\times \exp\left(a^{+}\middle|\left[\tfrac{z}{1+z}\right]\middle|P\right) N\left(M_-^{T}[-z]\,M_-\right) \exp-\left(a\middle|\left[\tfrac{z}{1+z}\right]\middle|P\right)$$

Here $[x]$ denotes the matrix $\delta_{mn} X^{n}$

Calculations show that 12.2 gives the correct matrix element between arbitrary multi peripheral trees.

The gauge invariance property, namely

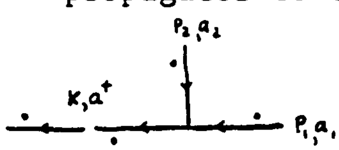
$$\lambda^{A(P)-\alpha_0}\,\mathcal{D}(P) = \mathcal{D}(P) \tag{12.3}$$

can be shown to hold. Indeed 12.3 can be shown to amount to merely a change of variables $z \to \lambda z$ in the integrand.

$$N(B) \equiv :\exp\left\{(a^{+}|B|a) - (a^{+}|a)\right\}:$$

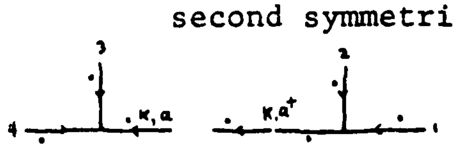
The gauge invariance of $\mathcal{D}(P)$ is the guarantee that the projected twisted propagator is free of spurious states. This is a direct result of the fact that an arbitrary physical tree state of the form

is annihilated by the operator $A-\alpha_0$. The uniqueness theorem of ref (17) is then used to show that 12.2 is a unique form. The projected twist propagator and the symmetric vertex operator can now be used to build up completely symmetric operator expressions for the tree amplitudes describing the scattering of arbitrary levels of the model. In constructing operator tree formulas first attach a projected twist propagator to a symmetric vertex as follows:



$$= \mathcal{D}V_3 = \langle 0|\mathcal{D}(a,K)\,V(-K,P_1,P_2;a^+,a_1,a_2)\,|0,K\rangle \qquad (12.4)$$

The four-point operator is obtained by attaching a second symmetric vertex operator to (12.4) as follows:



$=$

$$\doteq V_3\,\mathcal{D}\,V_3 = \langle 0|\,V(K\,P_3\,P_4;a\,a_3\,a_4)\,\mathcal{D}(a,K)\,V(-K\,P_1\,P_2;a^+a_1\,a_2)\,|0,K\rangle$$

$$= \int_0^1 dx\,\langle 0|\,\mathcal{U}(x;P_1\,P_2\,P_3\,P_4;K)\,|0,K\rangle \qquad (12.5)$$

The cyclically symmetric 4 -point operator V_4 is then gotten by performing a gauge transformation on leg P_3

i.e. $V_4 \equiv V(P_1 P_2 P_3 P_4 \,;\, a_1 a_2 a_3 a_4)$

$$= \int_0^1 dx \langle 0 | (1-X)^{A_3(P_3) - \alpha_0} u(x; P_1 P_2 P_3 P_4 \,;\, k) \, | 0, k \rangle$$

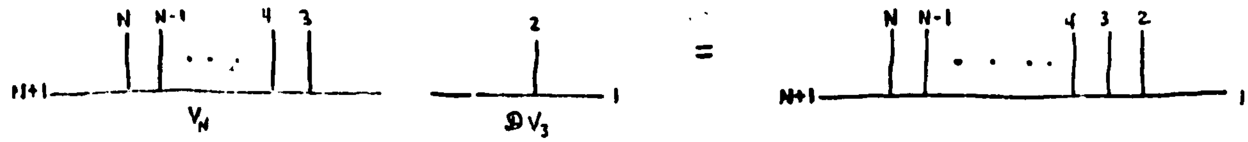
The constructed 4 -point operator is now cyclically symmetry and this guarantees that it possesses s - t duality.

It is important to point out that the internal line in the final 4 - point graph is untwisted.
The operator V4 is a generating functional for the 4 - particle tree graphs of the Veneziano model. To calculate the amplitude for definite states one evaluates the expression.

$$V(P_1 P_2 P_3 P_4 \,;\, a_1 a_2 a_3 a_4) \prod_{i=1}^{4} P^+(P_i) \, | \lambda_i \, P_i \rangle$$

where $| \lambda_i \, P_i \rangle$ is a definite occupation number state with mass Pi^2.

Once the symmetric N-point operator V_N is known, the N+1 point operator can be calculated by attaching a DV_3 operator in the following way.



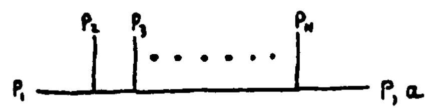
Thus the N+1 point operator can be written down as an integral involving N-2 integration variables. The symmetric N+1 point operator V_{N+1} is gotten by applying the gauge transformation

$$\left(U_{23}\right)^{A_3(P_3)-\alpha_0}$$

to external line 3. Here U_{23} is the integration variable associated with channel (2,3).

For later reference we give the expression for a tree state (which can be derived from V_{N+1}); it is

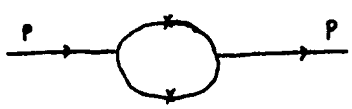
$$|T(P_1 P_2 \cdots P_N ; a)\rangle =$$



$$= g^{N-1}\int \prod_{i=2}^{N} dx_i \prod_{i<j} (x_i - x_j)^{-P_i\cdot P_j} \exp\sum_{i=1}^{N}(a^+|[x_i]|P_i)\,|0\rangle$$

where $P = -\sum_{i=1}^{N} P_i$

We now wish to calculate the non-planar (vaccuum) self-energy operator shown in the following diagram.



This calculation can be performed by attaching two of the DV_3 operators in the following way:

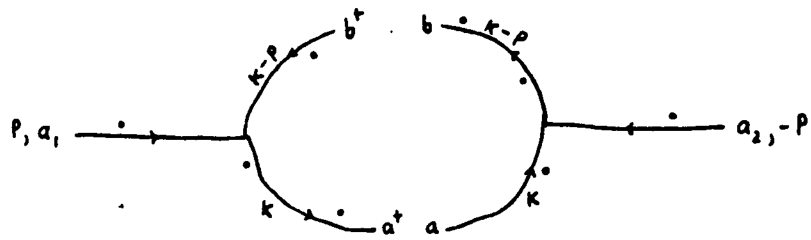


Figure 1.

The right hand side of figure 1 is given by

$$\langle 0 | \mathcal{D}(b^+, k-p)\, V(-p, k, p-k\, ; a_2, a, b)\, |0, k-p\rangle_b |0, k\rangle_a = \int dv\; Y_2(k, a_2, v)$$

The left hand side of figure 1 is given by

$$\langle 0 | \mathcal{D}(a, k)\, V(p, -k, k-p\, ; a_1, a^+, b^+)\, |0, k-p\rangle_b |0, k\rangle_a = \int du\; Y_1(k, a_1, u)$$

The non-planar self-energy operator $\Sigma_T(p)$ is given by

$$\Sigma_T(p) = \int d^4 k \int du\, dv\; Y_2(k, a_2, v)\; Y_1(k, a_1, u) \qquad (12.7)$$

The calculation of 12.7 is a lengthy one involving evaluation of a trace of operators, and a clever change of variables. It turns out that in order to get a finite result, one must not take the UV region to be what one might naively expect. Instead one must integrate over a restricted region. The final expression is

$$\Sigma_T(p) = 4\pi^2 g^2 \int_0^1 \frac{dx\, dy}{\ln^2 \omega}\, \omega^{-d_0 - 1} \left[f(\omega)\right]^{-4} \left[\frac{\Psi_T(x, \omega)}{1-\omega}\right]^{p^2}$$

$$\times \langle 0 | \exp\Big\{ (p|\, F_T(1-\omega) \big[|a_2) - |a_1)\big] + (a_1|(1-\omega) E_T(1-\omega)|a_2)$$

$$+ (a_1|(1-\omega) E(1-\omega)|a_1) + (a_2|(1-\omega) E(1-\omega)|a_2) \Big\}$$

(12.8)

where $W = XY$, $\psi_T \equiv \psi_-$

and F_T , E_T and E are matices expressible in terms of M_- and M_+ .

Our generalized definition of duality now requires there be the following equivalence between graphs.

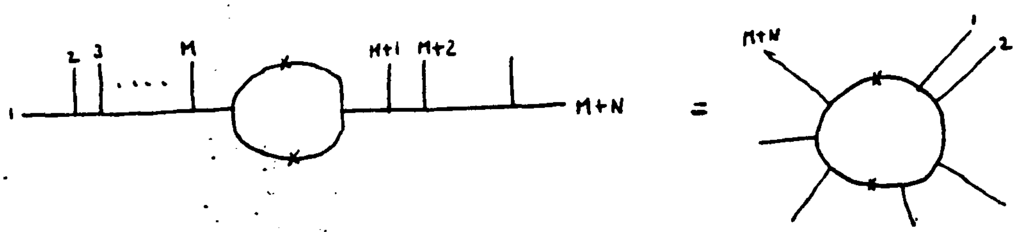


Figure 2.

The properties of the projected twist propagator and the V3 operator can be shown to imply that any single loop diagram with an even non-zero number of twists can be put in a form with only two twists; and these two twists can have any arbitrary positions along the loop. These properties guarantee the K.S.V. duality. The formal demonstration of equivalence shown in Figure 2 involves making nontrivial appropriate changes of variables in the integrands. After careful calculation one finds that indeed the following holds:

$$\langle \tau(p_1 \cdots p_n \,;\, a_1) \,|\, \Sigma_T(p\, a_1\, a_2) \,|\, \tau(p_{n+1} \cdots p_{n+N} \,;\, a_2) \rangle = F_{N,M} \qquad (12.9)$$

where τ is the tree state given by eg 12.6

13. Currents derived from the study of the pomeron

Cremmer and Scherk[19] have shown that for the case $\alpha_0 = 1$ the non-planar graph Fn,M can be refactorized in the following way:

$$F_{N,M} = \langle T(p_1 \cdots p_N ; a) \mid \Sigma_T(p, a) \mid T(p_{N+1} \cdots p_{M+N} ; a) \rangle \qquad (13.1)$$

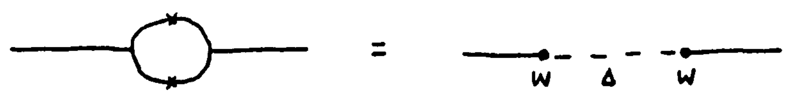
where Σ_T is equivalent to that of eg 3.7 up to the Virasoro guages.

The authors C.S. proved furthermore that (for the case $\alpha_0 = 1$):

$$\Sigma_T(p, a) = \langle 0_{bc} \mid W(a^+, b, c)\, \Delta(p^2, b, c)\, W(a, b^+, c^+) \mid 0_{bc} \rangle \qquad (13.2)$$

where Δ , identified as the pomeron propagator, is defined in a space spanned by two sets of commuting harmonic oscillators b and c.
W is identified as a vertex of transmition between the reggeon space and the pomeron space. All of the singularities of Σ_T are restricted to the pomeron Δ . The factorization exhibited in eg. 13.2 means that the singularities of the pomeron can receive a particle interpretation. When the restriction to d = 26 is made the singularities become Regge poles of half the slope of the reggeons. The degeneracy on the pomeron daughters increases like in the Virasoro-Shapiro model. In fact the pomeron propagator can be shown to be essentially the same as in the

Virasoro-Shapiro model. The equivalence stated in eg 13.2 can be expressed in the following graphical form:



<u>How a current might be obtained.</u>

Neveu and Scherk[20] used the followng procedure: Expand the integrand of $\Sigma_T(P)$ in powers of $q \equiv \exp \frac{2\pi^2}{\ln \omega}$ and keep lowest powers only. Assume $\alpha_0 = 1$ otherwise the integral will not factorize in the manner of eg 13.2 Then it is found that

$$\langle T(k_1 \cdots k_M) |\, \Sigma_T(P)\, | T(k_{M+1} \cdots k_{M+N}) \rangle$$
$$\simeq F_M(k_1 \cdots k_M ; -P)\; P_0(P^2)\; F_N(k_{M+1} \cdots k_{M+N} ; P)$$

where $$F_M(k_1 \cdots k_M ; P) = g^{M-1} \int_0^{\pi} d\theta_M \int_0^{\theta_M} d\theta_{M-1} \cdots \int_0^{\theta_3} d\theta_2 \prod_{1 \le i < j \le M} \left[\sin(\theta_i - \theta_j)\right]^{-k_i k_j}$$

and $$P_0(P^2) = g^2 \left(\frac{\pi}{2}\right)^{P^2} \int_0^1 d\omega \int_0^1 \frac{dx}{x}\, \mu(\omega)\, q^{-\frac{1}{4} - \frac{1}{4}P^2}$$

$$P = \sum k_i = -\sum P_i$$

Notice the form factors. Since it is defined in the reggeon space it can be studies by itself, without reference to the pomeron. The form factor, it turns out, can be factored in the following way:

$$F_M(k_1 \cdots k_M ; P) = \langle 0 | t(P) D(P) | \tau(k_1 \cdots k_M) \rangle$$

where the tadpole operator is

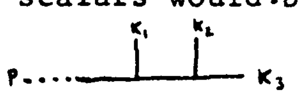
$$t(P) = \pi^{-\alpha(P^2)} \exp\left[\tfrac{1}{2}(a | \tilde{\Delta} | a) + (P | \tilde{F}_T | a) \right]$$

$$\tilde{\Delta} = \sum_{n \neq 0} \left[\frac{1}{n}\right] M_+ \left[-\frac{1}{n}\right]$$

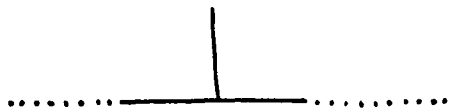
$$\tilde{F}_T = - \sum_{n \neq 0} \left[\frac{1}{n}\right]$$

In addition to obeying factorization in the original Hilbert space of physical particle states, F_M also obeys duality: it being cyclically symmetric in $k_1, k_2 \ldots, k_M$ Furthermore it has acceptable analytic properties as a function of P^2. Thus it was proposed that F_M be taken as a single scalar current amplitude while $\langle 0 | t(P)$ be taken as describing the emission of an off-shell scalar.

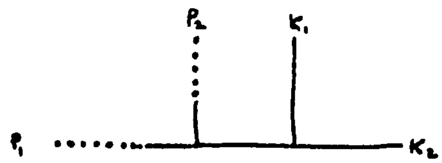
Thus the graph of one off-shell scalar and three on-shell scalars would be the following:

$F_3 =$ 

In fact F_M is similar to the Drummond- Rabbi amplitude but has the improved feature of factoring in the conventional spectrum of states. F_M can be slightly modified to give a planar rather than non planar current. The 2-scalar current amplitude is also decently behaved with correctly located singularities. It is built up by attaching two one-current operators on a symmetric three-point reggeon verte using twisted projected operators. The two-current operator thus has the following graph.



Thus using the two-current operator, one can calculate the amplitude of two off-shell scalars and two on-shell scalars. The graph of this process is the following:



Vector currents have also been constructed, by considering excitations in the pomeron sector.

The origin of the Cremmer-Scherk-Neveu model in dual loop theory, guarantees many of the properties we would like currents to possess. However, the model has a problem with spurious states which has not yet been completely resolved.

14. The Kikawa-Sakita model of vector currents

Nambu[21] presented a scheme for introducing electromagnetic currents in dual resonance models. His methods involved using the conserved currents of a two-dimensional Euclidean space in order to construct conserved currents in Lorentz space. These currents were found to build up a photon-like and a pomeranchukon-like trajectory reminiscent of the Cremmer-Scherk-Neveu theory.

The idea was the following:

Given a field in 2 - space

$\phi_\mu(\vec{r})$, $\mu = 0,1,2,3$, $\vec{r} = (r_1, r_2)$

if, in the 2 - space there exists a conserved field

$j^a(\vec{r})$ $a = 1,2$

$$\frac{\partial j^a}{\partial r^a} = 0$$

then the quantity

$$J_\mu(x) = \int d^2r \, j^a(\vec{r}) \left[\frac{\partial}{\partial r^a} \phi_\mu(\vec{r}) \right] \delta^4 \left[\phi(\vec{r}) - X \right]$$

is conserved. (Here X is the four-coordinate in Lorentz space. Conservation is demonstrated in the following way

$$\frac{\partial J_\mu(x)}{\partial X_\mu} = \int d^2r \, j^a(\vec{r}) \left[\frac{\partial}{\partial r^a} \phi_\mu(\vec{r}) \right] \frac{\partial}{\partial X_\mu} \delta^4 \left[\phi(\vec{r}) - X \right]$$

$$= -\int d^2r \, j^a(\vec{r}) \, \frac{\partial \phi_\mu(\vec{r})}{\partial r^a} \, \frac{\partial}{\partial \phi_\mu} \delta^4 \left[\phi(\vec{r}) - X \right]$$

$$= -\int d^2r \, j^a(\vec{r}) \, \frac{\partial}{\partial r^a} \delta^4 \left[\phi(\vec{r}) - X \right]$$

$$= -\int d^2r \, \frac{\partial}{\partial r^a} \left[j^a(\vec{r}) \, \delta^4(\phi - X) \right]$$

Thus, provided the normal component of the current j^a at the boundary vanishes, we have

$$\frac{\partial J_\mu(x)}{\partial X_\mu} = 0$$

The conserved current in 4 - space can be rewritten as

$$J_\mu(x) = \int V_\mu(\kappa)\, e^{-iK\cdot X}\, d^4K \qquad (14.1)$$

$$\text{where } V_\mu(\kappa) = \int d^2r\, j^a(\vec{r})\left[\frac{\partial}{\partial X^a}\, \phi_\mu(\vec{r})\right] e^{iK\cdot\phi(\vec{r})}$$

$$\text{and } \quad k \cdot V(\kappa) = 0$$

Kikkawa and Sakita[22] generalized Nambu's method to construct a model of vector currents. Their method involved expressing the Drummond-Rebbi amplitude as a functional average over 2-dimensional field quantities. Recall the Drummond-Rebbi amplitude is given by

$$A(\kappa_1 \cdots \kappa_N) = \frac{1}{R}\int \prod_{i=1}^{N} \frac{d^2 z_i}{(r_{2i})^2} \prod_{i,j=1}^{N} |z_i - \bar{z}_j|^{k_i k_j}$$

where $z_j = r_{1j} + i r_{2j}$ rangeof integration is $-\infty \le r_1 \le \infty,\ 0 \le r_2 \le \infty$

The idea now was to introduce a complex scalar field $\phi(r_1 r_2)$ in 2 dimensional Euclidean space.

The Lagrangian density for ϕ is $\mathcal{L} = -\frac{\partial\bar{\phi}}{\partial r^a}\frac{\partial\phi}{\partial r_a}$

It is easily shown that

$$\nabla^2 G_\phi(\vec{r}, 0) = \delta^2(\vec{r})$$

where $G_\phi(\vec{r}, \vec{r}\,') = -\frac{1}{2\pi} \ln |z - \bar{z}\,'|$ is the Green's function.

The functional average is defnined by

$$\left\langle \exp\left[i\int d^2r\, P(\vec{r})\cdot\phi(\vec{r})\right] \right\rangle \equiv \exp\left[-\frac{1}{2}\int d^2r\, d^2r'\, G_\phi(\vec{r},\vec{r}\,')\, P(\vec{r})\cdot P(\vec{r}\,')\right]$$

If now we put $P(\vec{r}) = \sqrt{4\pi}\sum_{j=1}^{N} \delta(\vec{r} - \vec{r}_j)\, K_j$

Then we see that

$$A(K_1 \cdots K_N) = \frac{1}{R}\int \prod_i d^2r_i \left\langle \prod_j V(K_j, \vec{r}_j) \right\rangle$$

where $V(K, \vec{r}) = \frac{1}{r_2^2} \exp\left[i\sqrt{4\pi}\, K\cdot\phi(\vec{r})\right]$

Now let $\phi = H_1 + iH_2$, then we can rewrite $\mathcal{L}$ as

$$\mathcal{L} = -\frac{\partial H_b}{\partial r_a}\frac{\partial H^b}{\partial r^a}$$

$$0 = \delta S = -2\int dr_1 dr_2 \left[\delta H_b \frac{\partial^2 H^b}{\partial r_1^2} + \delta H_b \frac{\partial^2 H^b}{\partial r_2^2}\right] - 2\int dr_1 \left[\frac{\partial H_b}{\partial r_2}\,\delta H^b\right]_{r_2=0}$$

$$\therefore \quad \nabla^2 H_b = 0 \qquad b = 1, 2$$

and boundary conditions are chosen to be

$$\left.\frac{\partial H_1}{\partial r_2}\right|_{r_2=0} = 0 \quad , \quad \left. H_2 \right|_{r_2=0} = 0$$

Using $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, and $z = re^{i\theta}$

we get solutions

$$H^1_\mu(\vec{r}) = \frac{1}{\sqrt{2\pi}}\left\{ Q_\mu - iP_\mu \ln z\bar{z} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}\left[(z^n + \bar{z}^n)a_{n\mu} + (z^{-n} + \bar{z}^{-n})\bar{a}_{n\mu}\right]\right\}$$

$$H^2_\mu(\vec{r}) = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}\left[(z^n - \bar{z}^n)b_{n\mu} + (z^{-n} - \bar{z}^{-n})\bar{b}_{n\mu}\right]$$

Introduce proper time coordinate $\tau = \frac{1}{2}\ln z\bar{z}$

and space variable $\sigma = \frac{1}{2i}\ln\left(z/\bar{z}\right)$

Now let a, $\bar{a}$, b, $\bar{b}$ become operators a, a^+, b, b^+;

and replace the functional average by the vacuum expectation value of the time-ordered product of $V(k_i, \vec{r}_i)$; ie by

$$\left\langle T \prod_i V(k_i, \vec{r}_i) \right\rangle \qquad 14(9)$$

That the quantity $\exp\left[i\sqrt{4\pi}\, k\cdot\phi(\vec{r})\right]$ is well defined can be seen by application of the well known formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

which is valid when [A,B] is a c-number.

In our case

$$[A,B] = 4k^2 \sum_{n=1}^{\infty}\frac{\cos 2n\theta}{n} = 4k^2 \ln|2\sin\theta|$$

$$\therefore \exp\left[i\sqrt{4\pi}\, k\cdot\phi(\vec{r})\right] = 0 \quad \text{for } z \text{ real},\ k^2 > 0 \qquad 14(10)$$

Thus for $K^2 > 0$ the vertex is defined for all values of Z and normal ordering is unnecessary. In contrast, if one uses a real field for Φ, one must use a normal ordered product $:e^{iK\cdot\Phi}:$ to make the vertex well defined. In fact it turns out that the mobius-invariance of $V_\mu(K)$ for arbitrary K^2, is destroyed if a normal ordered product is taken. Hence we use complex ϕ and do not take normal order ng in 14 (9)

Let a conserved current in the internal 2 - space be $J^n_a(\vec{r})$, n is an isopin index, $a = 1, 2$.

Then the isovector current conserved in the external 4 space will be:

$$V^n_\mu(K) = \frac{1}{i\sqrt{4\pi}}\int d^2r \; J^n_a(\vec{r}) \left[\frac{\partial}{\partial r_a} \Phi_\mu(\vec{r})\right] e^{i\sqrt{4\pi}\, K\cdot\Phi} \qquad 14(11)$$

Notice that the isovector current is conserved for $K^2 > 0$ regardless of the boundary property of J^n_a, because of 14 (10).

use for J^n_a the construction:

$J^n_a = \bar{\Psi}\, \sigma_a \tau_n \Psi \qquad a = 1, 2$

also introduce another construction

$\tilde{J}^n_a = i\bar{\Psi}\, \sigma_a \sigma_3 \tau_n \Psi$

and corresponding 4 - current

$$A^n_\mu(K) = \frac{1}{i\sqrt{4\pi}}\int d^2r \; \tilde{J}^n_a(\vec{r})\left[\frac{\partial}{\partial r_a} \, i\Phi_\mu(\vec{r})\right] e^{i\sqrt{4\pi}\, K\cdot\Phi} \qquad 14(12)$$

Both j_a^n and $\tilde{j}_a^n$ are conserved currents. It can be shown that each has the correct transformation properties to guarantee that V_μ^n and A_μ^n each be Mobius invariant. In the Kikkawa-Sakita scheme V_μ^n and A_μ^n are each intended to represent single (fictitious) external photon-emission vertices.

The following two photon emmission vertices were then constructed:

$$e^2 S_{\mu\nu}^{(VV)\,nm}(K_1+K_2) = -2\pi e^2 \delta_{nm} \int d^2r \left(\partial_a \phi_\mu\right)\left(\partial_a \phi_\nu\right) e^{i\sqrt{4\pi}(K_1+K_2)\cdot\phi}$$

$$f^2 S_{\mu\nu}^{(AA)\,nm}(K_1+K_2) = 2\pi f^2 \delta_{nm} \int d^2r \left(\partial_a \phi_\mu\right)\left(\partial_a \phi_\nu\right) e^{i\sqrt{4\pi}(K_1+K_2)\cdot\phi}$$

$$ef\, S_{\mu\nu}^{(VA)\,nm}(K_1+K_2) = 2\pi i ef\, \delta_{nm} \int d^2r\, \epsilon_{ab} \left(\partial_a \phi_\mu\right)\left(\partial_b \phi_\nu\right) e^{i\sqrt{4\pi}(K_1+K_2)\cdot\phi}$$

The conserved-vector-current densities in 4 coordinate Lorentz space will be $V_\mu^n(X)$ and $A_\mu^n(X)$ (given by 14(1))

The covariant Greens function for M V's and N A-s is

$$T^{n_1\cdots\, n_{N+M}}_{\mu_1\cdots\,\mu_{N+M}}(X_1 \cdots X_{N+M}) = \left\langle\!\left\langle 0 \left| T \prod_{i=1}^{M} V_{\mu_i}^{n_i}(X_i) \prod_{j=M+1}^{M+N} A_{\mu_j}^{n_j}(X_j) \right| 0 \right\rangle\!\right\rangle$$

Which can be shown to satisfy the generalized Ward-Takahashi relation:

$$\frac{\partial}{\partial X_{1\mu_1}} T^{n_1\cdots\, n_{N+M}}_{\mu_1\cdots\,\mu_{N+M}} = \sum_i (2\pi)^4 i\, \delta^4(X-X_i)\, \epsilon_{1\alpha_i\beta}\, T^{n_2\cdots\, n_{i-1}\ \beta\ n_{i+1}\cdots\, n_{N+M}}_{\mu_2\cdots\,\mu_{i-1}\ \mu_i\ \mu_{i+1}\cdots\,\mu_{N+M}} \qquad 14(13)$$

We introduce the contraction notation, c, for fields Y_i :

$$c\, Y_1 Y_2 \cdots Y_n = Y_1 Y_2 \cdots Y_n + \sum_{i \neq j} Y_1 Y_2 \cdots \overbrace{Y_i \cdots Y_j} \cdots Y_n$$

+ all doubly contracted products

+ all triply contracted products

+ . . .

In our case we have :

$$\overbrace{V^{n_1}_{\mu_1}(K_1)\, V^{n_2}_{\mu_2}}(K_2) = S^{(VV)\, n_1 n_2}_{\mu_1 \mu_2}(K_1 + K_2)$$

$$\overbrace{A^{n_1}_{\mu_1}(K_1)\, A^{n_2}_{\mu_2}}(K_2) = S^{(AA)\, n_1 n_2}_{\mu_1 \mu_2}(K_1 + K_2)$$

$$\overbrace{V^{n_1}_{\mu_1}(K_1)\, A^{n_2}_{\mu_2}}(K_2) = S^{(VA)\, n_1 n_2}_{\mu_1 \mu_2}(K_1 + K_2)$$

14 (14)

We define the dual-current amplitude for M V's and N A's by

$$T^{n_1 \cdots n_{M+N}}_{\mu_1 \cdots \mu_{M+N}}(K_1 \cdots K_{M+N}) = \frac{1}{R} \left\langle c \prod_{i=1}^{M} V^{n_i}_{\mu_i}(K_i) \prod_{j=M+1}^{M+N} A^{n_j}_{\mu_j}(K_j) \right\rangle \qquad 14(15)$$

The contract quantities 14 (14) are called seagall terms. It can be shown that the proof of eg. 14 (13) requires the existence of seagull terms. Indeed without seagull terms, it is not possible to construct a guage invariant amplitude in most models.

$V_\mu^n(K)$ can be shown to have pole singularities at $-K^2 = 1, 2, 3, \ldots\ldots$

and $A_\mu^n(K)$ can be shown to have pole singularities at $-K^2=0, 1, 2, 3, \ldots\ldots$

There are no singularities in the seagull terms.
The pole at $K^2=0$ in A has some of the properties of the pion.
We can write A in the form

$$A_\mu^n(K) \underset{K^2 \sim 0}{\sim} C_\mu^n + \frac{K_\mu P_n(K)}{K^2} \qquad 14(16) \qquad (C^n \text{ a constant})$$

The righthand side of 14(16) is only conserved in the limit as $K^2 \to 0$.
Thus the right hand side may be viewed as the partially conserved axial current discussed by Nambu.[23]
Thus we can take Pn(K) as the pion-emission vertex defined by

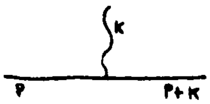
$$K_\mu P_n(K) = \lim_{K^2 \to 0} K^2 A_\mu^n(K)$$

The amplitude of M V-currents and N pions is taken to be

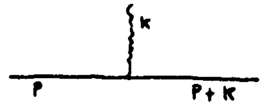
$$T_{\mu_1 \cdots \mu_M}^{n_1 \cdots n_M ; n_{M+1} \cdots n_{M+N}} \left(K_1 \cdots K_M ; K_{M+1} \cdots K_{M+N} \right)$$

$$= \frac{1}{R} \left\langle \prod_{j=M+1}^{M+N} P_{n_j}(K_j)\; c \prod_{i=1}^{M} V_{\mu_i}^{n_i}(K_i) \right\rangle \qquad 14(17)$$

The following associations are made with the constructed operators:



$= V_\mu^n(K)$

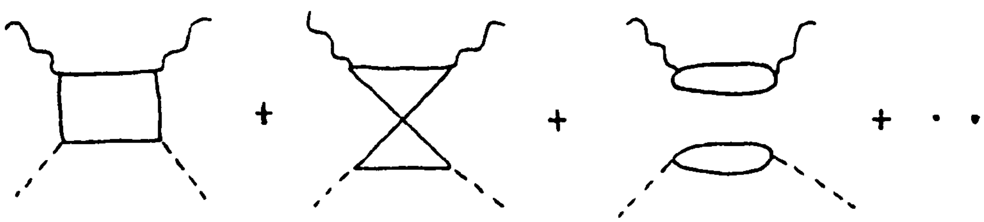


$= A_\mu^n(K)$

$= P_n(K)$

$= V(K_1)V(K_2) + \overline{V(K_1)\,V(K_2)}$

The amplitudes are all in the form of loop graphs. For example compton scattering is given by the following sum of graphs:



Duality, in the sense of the sum of s-channel poles equaling the sum of t-channel poles, is violated because of the seagull terms. However, the Mobius symmetry is of course inherent in the amplitudes. The particle spectrum of the current amplitudes is the same as that of the hadronic amplitudes. Because of the Gaussian K^2 dependence of the pion form factor, the model, like its ancester the Drummond model, is inapplicable in the deep-inelastic limit.

15. Focus on the spectrum problem is constructing currents

The most promising approach, in our opinion, in constructing current-amplitudes, is the procedure initiated by Schwarz[24] for describing off-mass-shell states in dual resonance models. The idea was to construct amplitudes with the requirement, built in from the beginning, that off-mass-shell lines contain the same spectrum as does the corresponding on-mass shell model. This condition, imposed before any other considerations, is necessary if one hopes to describe physical currents. This led to the construction of a dual current requiring a particular choice of space-time dimension D, namely D = 16. This number 16, is disturbing not only because it is unphysical, but also because it does not conform to the critical dimension, namely D=26, of the Veneziano model in which context he worked. Nevertheless, researchers believe that Schwarz' procedure sheds a great deal of light on how, eventually, realistic models will be found.

The procedure starts with the ghost free Veneziano model. In this model it is desired to construct an off-mass-shell scalar state of momentum q. Write the state in the form $S|o,q\rangle$ where S is an operator constructed out of the harmonic oscillator raising operators a^{μ}_{-m}

In order to implement the fundamental spectrum condition, let it be required that the state satisfy the Virasoro conditions:

$$(L_n - L_o + 1 - n)S|o,q\rangle = 0 \qquad n= 1,2,3,.... \tag{15.1}$$

Requirement 15.1 will mean that an arbitrary tree state, of form $V(P_1)\frac{1}{L_0-1}V(P_2)\cdots\frac{1}{L_0-1}V(P_N)\frac{1}{L_0-1}S\,|0,q\rangle$

$=$  $= \Psi$,

will also be annihilated by $(L_n - L_0 + 1 - n)$.
Thus such trees will decouple from spurious states.
Thus all the poles of Ψ both in Pi, and q will correspond to physical states.

In solving eg 15.1 it is sufficient to just consider n=1 and n=2 because L_1, L_2 generate all the additional Virasoro conditions through the algebra. Schwarz showed that a solution to eg. 15.1 is

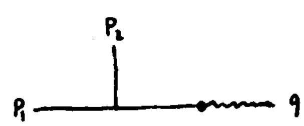
$$S = e^{T} \qquad (15.2)$$

where $T \equiv \frac{1}{2}\sum_{l,m=0}^{\infty} A_{lm}\, a_{-l}\cdot a_{-m}$

and $A_{lm} \equiv \frac{(-1)^{l+m}}{l+m}\binom{-1/2}{l}\binom{-1/2}{m}$

However, the solution (15.2) will only satisfy (15.1 for the case d = 16.

Using this off-shell state $S|0,q\rangle$ Schwarz then proceeded to calculate the ground state form factor

$\langle 0, P_1 |\, V(P_2)\frac{1}{L_0-1}S\,|0,q\rangle$ = 

which unfortunately was found to diverge. To see this use

$V(P_i) = e^{P_i A^{+}}\, e^{i P_i X_0}\, e^{-P_i A}$, $A = \sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a_n}{n}$

$\frac{1}{L_0-1} = \int_0^1 dx\; x^{R-1-\alpha(P^2)}$

$$\sum_{l=1}^{\infty} A_{0l}\, X^{l} = -2 \log\left(\frac{1+\sqrt{1-X}}{2}\right)$$

$$\sum_{l,m=1}^{\infty} A_{lm}\, X^{l} Y^{m} = 2 \log\left[\frac{(1+\sqrt{1-X})(1+\sqrt{1-Y})}{2(\sqrt{1-X}+\sqrt{1-Y})}\right]$$

Then one finds

$$A_2 \equiv \langle 0, P_1 \,|\, V(P_2)\frac{1}{L_0 - 1} S \,|\, 0, q\rangle = \frac{1}{2}\int_0^1 \frac{d\lambda}{1-\lambda}\, \lambda^{-\frac{1}{2}\alpha(-q^2)-1}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n - \frac{1}{2}\alpha(q^2)}$$

Similarly the amplitude for N on-shell ground states and one off-shell state was also found to be divergent. It was then suspected that the divergence problem is related to the fact that $\alpha_0 = 1$ for this model, and that a realistic ghost free model must be found with $\alpha_0 < 1$. In anticipation of modifications that would take place in a theory with $\alpha_0 < 1$, the replacement $(1-\lambda)^1 \to (1-\lambda)^\gamma$ is made,[25] where γ is a parameter to be determined by further research. With this admittedly ad-hoc removal of the divergence, the amplitude for one off-shell state to go into N on-shell ground states is calculated to be

$$A_N^{\gamma} = \langle 0q | S^{+} \frac{1}{L_0 - 1} V(K_1) \cdots \frac{1}{L_0 - 1} V(K_{N-1}) | 0; K_N \rangle$$

$$= \frac{1}{2} \int_0^1 \frac{d\lambda}{(1-\lambda)^{\gamma}} \lambda^{-\frac{1}{2}\alpha(-q^2)-1} \int_0^1 \prod_{i=2}^{N-1} \frac{\theta(x_i - x_{i+1})}{1 - \lambda x_i^2} dx_i \prod_{1 \leq i < j \leq N} \left(\frac{x_i - x_j}{1 - \lambda x_i x_j} \right)^{K_i K_j}$$

where $x_1 = 1$ and $x_N = 0$

The properties of A_n^{γ} (and very encouraging) are the following:

(i) The only singularities are those associated with particle poles.

(ii) Poles only occur in chnnels composed of sets of adjacent lines.

(iii) $A_2^{\gamma} = \frac{1}{2} \int_0^1 d\lambda \, \lambda^{-\frac{1}{2}\alpha(-q^2)-1} (1-\lambda)^{-\gamma} \equiv \frac{1}{2} B\left[-\frac{1}{2}\alpha(-q^2) , 1-\gamma\right]$

and $A_2^{\gamma} \underset{q^2 \to \infty}{\sim} \frac{1}{2} \Gamma(1-\gamma) \left[-\frac{1}{2}\alpha(-q^2)\right]^{\gamma-1}$

Thus, since researchers expect dipole behavior of form factors, one might put $\gamma = -1$.

(iv) A_3^{γ} = 

$$= \frac{1}{2}\int_0^1 \frac{d\lambda\, dx}{1-\lambda x^2}\, \lambda^{-\frac{1}{2}\alpha(-q^2)-1} (1-\lambda)^{-\gamma}\, x^{-\alpha(s)-1} \left(\frac{1-x}{1-\lambda x}\right)^{-\alpha(t)-1}$$

where $s = -(k_1+k_2)^2$, $t = -(k_2+k_3)^2$

A_3^{γ} is $s-t$ symmetrical, as can be demonstrated by making the change of variables

$$x = \frac{1-x'}{1-\lambda' x'} \quad , \quad \lambda = \lambda' \qquad \text{in the integrand.}$$

(v) The amplitude when q is put on shell turns out to be

$$A_3^{\gamma}(s,t) \underset{\alpha(-q^2)\sim 0}{\sim} \frac{1}{\alpha(-q^2)}\, B\left[-\alpha(s)\,,\,-\alpha(t)\right]$$

(vi) In the Regge limit, defined by

$|\alpha(s)| \to \infty$ with t, q^2 fixed we get

$$A_3^{\gamma} \sim \frac{1}{2}\begin{cases} \Gamma(-\alpha_t)(-\alpha_s)^{\alpha_t}\, B\left[-\frac{1}{2}\alpha(-q^2),\, \alpha_t-\gamma+1\right] & \text{for } \alpha_t > \gamma-1 \\ \Gamma(1-\gamma)(-\alpha_s)^{\gamma-1}\int_0^1 \frac{dy}{2-y}\, y^{-\gamma}(1-y)^{\gamma-\alpha_t-2} & \text{for } \alpha_t < \gamma-1 \end{cases}$$

i.e. A_3^{γ} contains a fixed pole at $J = \gamma - 1$ in addition to the usual Regge pole. Since fixed poles are generally expected to occur at nonsense values of angular momentum it seems natural to suppose that γ should be 0 or - 1. Thus this model predicts that the fixed pole in electroproduction be correlated with an asymptotic behavior of the form factor.

(vii) In the Bjorken scaling limit taken as

$$|\alpha(s)|\,,\ |\alpha(-q^2)| \rightarrow \infty$$

with t and $\omega = 1 + \frac{s}{q^2}$ held fixed , we get

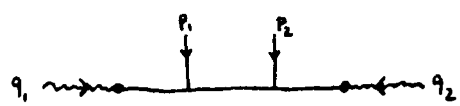
$$A_3^{\gamma} \sim \frac{1}{2}\Gamma(1-\gamma)\left[-\frac{1}{2}\alpha(-q^2)\right]^{\gamma-1}\int_0^{\infty}\frac{dx}{1+2x}\left[1+2(\omega-1)x\right]^{\gamma-1}\left(\frac{x}{1+x}\right)^{-\alpha(t)-1}$$

which is an encouraging result.

Favorable results were also found for the 2 current amplitude.[26]

$$T = \left\langle -q_1 \right| \frac{1}{L_0-1} V(p_1) \frac{1}{L_0-1} V(p_2) \frac{1}{L_0-1} \left| q_2 \right\rangle$$

=



$$|q\rangle \equiv S\,|0,q\rangle$$

A parameter β can be introduced in the calculations of T to remove the divergence in the same way that γ was introduced in the calculations for the one current amplitudes. The calculations are long and tedious, but it is demonstrated that Bjorken scaling holds for the particular choice $d + 4\beta = 8$ (where d is the dimension of space time). This result is significant in that the hadrons of the model contain no point like structures.

This demonstrates that at least some of the parton or light-cone results can be produced by the coherent behavior of an infinite number of hadron states. Further calculations in the Regge limit, demonstrate regge poles and fixed poles of the appropriate type.

16. The dimension problem

We now focus attention on the unphysical dimension d = 16 that Schwarz solution required. In order to arrive at a solution to eg (15.1) which does not require a particular choice of D it is now proposed that we try a solution of the form $S = S(D) = e^T B(D)$ (15.3)

Using the commutation relation between Ln and $\underline{a}_\ell \cdot \underline{a}_m$ the Virasoro conditions (15.1) can then be put in the form:

$$(L_1 - L_0)\, e^T B\, |0,q\rangle = e^T \left[L_1 - L_0 , B \right] |0,q\rangle = 0 \qquad (15.4)$$

$$(L_2 - L_0 - 1)\, e^T B\, |0,q\rangle = e^T \left\{ \left[L_2 + \sum_{r=0}^{\infty} A_{1r}\, \underline{a}_r \cdot a_1 - L_0 , B \right] + \left(\frac{D}{16} - 1 \right) B \right\} |0,q\rangle = 0 \qquad (15.5)$$

We now attempt to construct an operator B which will satisfy 15.4 and 15.5. To convince oneself that such an operator may in fact exist, consider the following:

Let $S(D) = 1 + S_1 + S_2 + S_3 + \cdots\cdots$ (15.6)a

where $S_1 = K_1\, a_0 \cdot a_{-1}$

$$S_2 = K_{21}\, a_0 \cdot a_{-2} + K_{22}\, a_{-1} \cdot a_{-1} + K_{23}\, a_0 \cdot a_{-1}\, a_0 \cdot a_{-1}$$

$$\vdots$$

ie S_n is a linear sum of all products $\prod_{i=1}^{r} a_{-x_i} \cdot a_{-y_i}$ such that $\sum_{i=1}^{r} (x_i + y_i) = n$. (products $a_0 \cdot a_0$ are not included)

One substitutes 15.6 into the equations

$(L_1 - L_0) S|o,q\rangle = 0$ and $(L_2 - L_0 - 1) S|o,q\rangle = 0$,

subject only to the requirement that the coefficients be of the form $K = a + bq^2$ (15.6)b

where a and b are constants.

It turns out that the resulting simultaneous equations can then be uniquely solved coefficient by coefficient; first for the coefficient on the S_1 level, then for those on the S_2 level, the S_3 level, and so on.

Below is shown the results up to the S_6 level (with e^T factored out).

$$
\begin{aligned}
S(D) = e^{T} \Bigg\{ 1 &+ \frac{u}{2^4}\frac{a(02)}{D} - \frac{u}{2^4}\frac{a(11)}{D} - \frac{u}{2^4}\frac{a(0011)}{D(D-1)} - \frac{u}{2^4}\frac{a(12)}{D} - \frac{u}{2^4}\frac{a(0012)}{D(D-1)} \\
&+ \frac{u}{2^4}\frac{a(03)}{D} + \left(\frac{-3u}{2^6} + \frac{uv}{3\cdot 2^8}\right)\frac{a(13)}{D} + \left(\frac{-3u}{2^6} + \frac{uv}{3^2\cdot 2^7}\right)\frac{a(0013)}{D(D-1)} \\
&+ \left(\frac{15u}{2^8} - \frac{uv}{3\cdot 2^{10}}\right)\frac{a(04)}{D} + \left(\frac{-3u}{2^8} - \frac{uv}{2^{10}}\right)\frac{a(22)}{D} + \left(\frac{-3u}{2^8} - \frac{uv}{2^{11}}\right)\frac{a(0022)}{D(D-1)} \\
&+ \frac{uv}{3\cdot 2^9}\frac{a(1102)}{D(D-1)} + \left(\frac{-u}{2^6} - \frac{uv}{3\cdot 2^9}\right)\frac{a(23)}{D} + \left(\frac{-u}{2^6} - \frac{5uv}{3^2\cdot 2^{10}}\right)\frac{a(0023)}{D(D-1)} \\
&+ \left(\frac{-5u}{2^7} + \frac{uv}{2^9}\right)\frac{a(14)}{D} + \left(\frac{-5u}{2^7} + \frac{uv}{3\cdot 2^8}\right)\frac{a(0014)}{D(D-1)} + \left(\frac{7u}{2^7} - \frac{uv}{3\cdot 2^9}\right)\frac{a(05)}{D} \\
&+ \frac{uv}{3\cdot 2^9}\frac{a(1103)}{D(D-1)} - \frac{uv}{3\cdot 2^{10}}\frac{a(2201)}{D(D-1)} + \left(\frac{5\cdot 3\cdot 7u}{2^{11}} - \frac{19uv}{5\cdot 2^{12}} + \frac{uv^2}{5\cdot 3^3\cdot 2^{13}}\right)\frac{a(06)}{D} \\
&+ \left(\frac{-5^2u}{2^{11}} - \frac{11uv}{3\cdot 2^{12}} + \frac{uv^2}{3^2\cdot 2^{13}}\right)\frac{a(24)}{D} + \left(\frac{-5^2u}{2^{11}} - \frac{5uv}{2^{14}} + \frac{uv^2}{3\cdot 2^{16}}\right)\frac{a(0024)}{D(D-1)} \\
&+ \left(\frac{-5\cdot 7u}{2^{10}} + \frac{71uv}{5\cdot 3\cdot 2^{11}} - \frac{uv^2}{5\cdot 3^2\cdot 2^{12}}\right)\frac{a(15)}{D} + \left(\frac{-5\cdot 7u}{2^{10}} + \frac{7\cdot 17uv}{5^2\cdot 3\cdot 2^{10}} - \frac{uv^2}{5^2\cdot 3^2\cdot 2^9}\right)\frac{a(0015)}{D(D-1)} \\
&+ \left(\frac{-5u}{2^{10}} - \frac{uv}{2^{11}} - \frac{uv^2}{3^3\cdot 2^{12}}\right)\frac{a(33)}{D} + \left(\frac{-5u}{2^{10}} - \frac{uv}{3\cdot 2^{11}} - \frac{uv^2}{3^4\cdot 2^{12}}\right)\frac{a(0033)}{D(D-1)} \\
&+ \left(\frac{11\cdot 7uv}{5\cdot 3\cdot 2^{13}} - \frac{uv^2}{5\cdot 2^{15}}\right)\frac{a(1104)}{D(D-1)} + \left(\frac{23uv}{5\cdot 3^2\cdot 2^{11}} + \frac{uv^2}{5\cdot 3\cdot 2^{13}}\right)\frac{a(1203)}{D(D-1)} \\
&+ \left(\frac{-19uv}{3^3\cdot 2^{11}} - \frac{uv^2}{3^4\cdot 2^{13}}\right)\frac{a(2301)}{D(D-1)} + \left(\frac{13uv}{5\cdot 3^3\cdot 2^{10}} - \frac{11uv^2}{5\cdot 3^4\cdot 2^{12}}\right)\frac{a(1302)}{D(D-1)} \\
&+ \left(\frac{-uv}{3\cdot 2^{13}} + \frac{uv^2}{3^2\cdot 2^{15}}\right)\frac{a(1122)}{D(D-1)} + \left(\frac{-uv}{3\cdot 2^{13}} + \frac{uv^2}{3\cdot 2^{16}}\right)\frac{a(001122)}{D(D-1)(D-2)} \\
&+ \cdots \Bigg\}
\end{aligned}
$$

15.7

Where u= D-16, v = D-8

$a(x,y) \equiv a_{x} \cdot a_{-y}$

$$a(x_1 y_1 , x_2 y_2) \equiv 2a(x_1,y_1)a(x_2,y_2) - a(x_1,y_2)a(x_2,y_1) - a(x_1,x_2)\,a(y_1,y_2) \tag{15.9}$$

$$a(0,0\,,1,1\,,2,2) \equiv 2\Big[a(0,0)a(1,1,2,2) - a(0,1)a(0,1,2,2) - a(0,2)a(0,2,1,1)\Big] \tag{15.10}$$

In the calculations (15.7), note the presence of factors 1/D , 1/D-1 , 1/D-2 . These factors may cause S(D) not to be finite. We will presently see how to delete these factors. Note also the presence in calculations (15.7) of groupings of harmonic oscillators $a(x_1 y_1\, x_2 y_2)$ and $a(\,00\,,11\,,22)$.

In general lets define $a(\,x_1 y_1 \,\cdots\cdots\, x_N y_N)$ in the following way:

$$a(x_1 y_1 \cdots x_N y_N\, x_{N+1} y_{N+1})$$
$$= \frac{1}{2}\left[4 - \sum_{j=1}^{N}\left(S_{x_{N+1},x_j} + S_{x_{N+1},y_j} + S_{y_{N+1},x_j} + S_{y_{N+1},y_j}\right)\right]a(x_{N+1},y_{N+1})\,a(x_1 y_1 \cdots x_N,y_N) \tag{15.11}$$

(where $S_{ka}A_{ab} = A_{kb}$, $S_{ab}A_{ab} = A_{ba}$, $S_{ab} = S_{ba}$

ie. $S_{k\ell}$ changes indices.) One can easily show that eqs. 15.9,15.10 agree with this definition. Then by induction one can show the following to be true:

$$a(\,x_1 y_1 \cdots x_j y_j \cdots x_N y_N) = a(\,x_1 y_1 \cdots y_j x_j \cdots x_N y_N) \tag{15.12}$$

$$a(\,x_1 y_1 \cdots x_j y_j \cdots x_k y_k \cdots x_N y_N) = a(\,x_1 y_1 \cdots x_k y_k \cdots x_j y_j \cdots x_N y_N) \tag{15.13}$$

$$a(\,x_1,y_1,x_2 y_2 \cdots x_N,y_N) + a(\,x_1,y_2,x_2,y_1 \cdots x_N y_N) + a(\,x_1,x_2,y_1,y_2 \cdots x_N,y_N) = 0 \tag{15.14}$$

$j,k = 1,2,\ldots\ldots,N$

It follows from 15.12-14 that $a(x_1 y_1 \cdots x_N y_N)$ vanishes if any three indices are equal.

Lets define

$$\nabla a(x_1 y_1 \cdots x_N y_N, x_{N+1}, y_{N+1})$$
$$\equiv \frac{1}{2}\left(1+\sum_{j=1}^{N} S_{x_{N+1},x_j} S_{y_{N+1},y_j}\right)\left[4-\sum_{i=1}^{N}\left(S_{x_{N+1}x_i}+S_{x_{N+1}y_i}+S_{y_{N+1}x_i}+S_{y_{N+1}y_i}\right)\right]$$
$$\times\ \delta_{1,x_{N+1}}\delta_{1,y_{N+1}}\ a(x_1 y_1 \cdots x_N y_N)\,. \tag{15.15}$$

and

$$\nabla a(x,y) \equiv \delta_{1,x}\ \delta_{1,y}$$

The commutation relations turn out to be as follows:

$$\left[L_1, a(x_1 y_1 \cdots x_N y_N)\right] = \sum_{j=1}^{N}\left(x_j S_{x_j,x_j-1} + y_j S_{y_j,y_j-1}\right) a(x_1 y_1 \cdots x_N y_N) \tag{15.16}$$

$$\left[L_2, a(x_1 y_1 \cdots x_N y_N)\right]$$
$$=\left\{\sum_{j=1}^{N}\left[\theta(x_j-2)x_j S_{x_j x_j-2} + \theta(y_j-2)y_j S_{y_j y_j-2} + \delta_{1,x_j} S_{x_j,-1} + \delta_{1,y_j} S_{y_j,-1}\right]\right.$$
$$\left. + (D-N+1)\nabla\right\} a(x_1 y_1 \cdots x_N y_N) \tag{15.17}$$

(Here it is to be understood that the destruction operator a_1, resulting from the application of $S_{x,-1}$ to a product, is to stand to the right of all other operators in the product.)

$$\left[a_{-n}\cdot a_1\ ,\ a(x_1 y_1 \cdots x_N y_N)\right]$$
$$= \sum_{j=1}^{N}\left(\delta_{1,x_j} S_{n,x_j} + \delta_{1,y_j} S_{n,y_j}\right) a(x_1 y_1 \cdots x_N y_N) \tag{15.18}$$

Suppose we now write $S(D) = e^{T}B(D)$ and express B as a linear combination of the operators $a(x_1 y_1 \cdots x_N y_N)$ $N = 1,2,3,\ldots\ldots$ It seems reasonable to take coefficients having a structure similar to that of the operators $a(x_1 y_1 \cdots x_N y_N)$. So lets take coefficients $H(x_1 y_1 \cdots x_N y_N)$ having the following properties:

$$H(x_1 y_1 \cdots x_j y_j \cdots x_k y_k \cdots x_N y_N) = H(x_1 y_1 \cdots x_k y_k \cdots x_j y_j \cdots x_N y_N) = H(x_1 y_1 \cdots y_j x_j \cdots x_k y_k \cdots x_N y_N)$$

$$H(x_1 y_1 x_2 y_2 \cdots x_N y_N) + H(x_1 y_2 x_2 y_1 \cdots x_N y_N) + H(x_1 x_2 y_1 y_2 \cdots x_N y_N) = 0$$

Let $$B(D) = 1 + \sum_{N=1}^{D} \sum_{XY} H(x_1 y_1 \cdots x_N y_N) \frac{a(x_1 y_1 \cdots x_N y_N)}{D(D-1)\cdots(D-N+1)} \tag{15.19}$$

$$\sum_{XY} \equiv \sum_{x_1=0}^{\infty} \sum_{y_1=0}^{\infty} \cdots \cdots \sum_{x_N=0}^{\infty} \sum_{y_N=0}^{\infty}$$

where the D factors were added to accommodate the D term in eq. 15.17. Using (15.16), (15.17) and (15.18) we see that eq. 15.4 is satisfied provided:

$$\sum_{j=1}^{N} \Big[(x_j+1) H(x_1 y_1 \cdots x_j+1, y_j \cdots x_N y_N) + (y_j+1) H(x_1 y_1 \cdots x_j, y_j+1 \cdots x_N y_N) - (x_j+y_j) H(x_1 y_1 \cdots x_j y_j \cdots x_N y_N) \Big] = 0$$

$$N = 1, 2, \ldots, D \qquad x, y \geq 0 \tag{15.20}$$

and (15.5) is satisfied provided that:

$$\left(\frac{D}{16} - 1\right) + H(1,1) = 0 \tag{15.21}$$

and that:

$$\sum_{j=1}^{N} \Big[(x_j+2) H(x_1 y_1 \cdots x_j+2, y_j \cdots x_N y_N) + (y_j+2) H(x_1 y_1 \cdots x_j, y_j+2 \cdots x_N y_N) - (x_j+y_j) H(x_1 y_1 \cdots x_j y_j \cdots x_N y_N) + A_{1,x_j} H(x_1 y_1 \cdots 1, y_j \cdots x_N y_N) + A_{1,y_j} H(x_1 y_1 \cdots 1, x_j \cdots x_N y_N) \Big]$$

$$+ \theta(D-N-1)(N+1)(N+2) H(1,1,x_1,y_1,\cdots x_N y_N)$$

$$+ \left(\frac{D}{16} - 1\right) H(x_1 y_1 \cdots x_N y_N) = 0 \tag{15.22}$$

$$N = 1, 2, \ldots, D$$

$$x, y \geq 0$$

Using (15.21) and the sets of equations corresponding to N=1 and N=2 in (15.20) and (15.22) one can easily calculate all the H coefficients up to the S_6 level and so duplicate the calculations (15.7) exactly.

The explicit expression for $H(x,y)$ can be shown, by direct substitution, to be given by:

$$H(n+k,n) = \frac{-u}{2n+k}\binom{2n+k}{n}\sum_{r=0}^{[\frac{k}{2}]}\sum_{s=0}^{n+r-1}\left(\frac{v}{8}\right)^{s} g(k,r)\, G(n+r-1,\, n+r-1-s)$$

$n, k \geqslant 0$

$$g(k,r) \equiv \begin{cases} (-1)^r \dfrac{1}{k-r}\dbinom{k-r}{r}, & \text{for } \left[\frac{k}{2}\right] \geqslant r \geqslant 0 \\ 0, & \text{otherwise} \end{cases} \quad ; \; g(0,0) = 1$$

$$G(p,\ell) \equiv \frac{1}{8[2(p+1)]!}\left\{1 + \sum_{0\leq k_1 \leq p} F(k_1) + \sum_{0\leq k_1 < k_2 \leq p} F(k_1)F(k_2) + \dots + \sum_{0\leq k_1<\dots<k_\ell \leq p} F(k_1)\cdots F(k_\ell)\right\}$$

$$F(k) \equiv \left(k-\tfrac{1}{2}\right)\left(k+\tfrac{3}{2}\right)$$

We have been unable to find explicit closed expressions for higher coefficients.

For v=0, ie. D=8, we find that B(D) simplifies and is given by:

$$B(8) = 1 + \sum_{xy} \bar{H}(x,y)\left[a(x,y) + \lambda\, a(0,0,x,y)\right]$$

where

$$\bar{H}(n+k,n) = \frac{1}{2n+k}\binom{2n+k}{n}\sum_{r=0}^{[\frac{k}{2}]} g(k,r)\, G(n+r-1,\, n+r-1)$$

$n > 0 \quad , \; k \geqslant 0$

$$\bar{H}(0,0) = 2(7\lambda - 1)$$

where λ is an arbitrary constant.

Conclusion

Given the original assumptions concerning the coefficients in S (namely eq.15.6), we have obtained a unique solution to a dual current in an arbitrary number of space-time dimensions up to the 6^{th} level, and a prescription for carrying the calculations to all orders.

Unfortunately, we have not been able to find a closed analytic expression for such a current (except for the special cases D=8,16). However, the uniqueness of our solution is a strong indication that such an expression exists. Also it is not clear how our solution relates to the theorem of Collins and Friedman.[28] Our solution may be an evasion of their theorem.

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