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THE NON-LINEAR BENDING OF A CLAMPED CIRCULAR PLATE UNDER UNIFORM NORMAL PRESSURE

by

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SYMBOLS USED

a = radius of the plate

h = thickness of the plate

p = uniform normal pressure

p^* = pa^4/Eh^4 = dimensionless pressure

u = radial displacement of points in the middle surface

w = normal displacement of points in the middle surface

w^* = w/h = dimensionless normal displacement

E = Young's modulus

ν = Poisson's ratio; we choose $\nu = .3$

D = $Eh^3/\ 12(1-\nu^2)$ = flexural rigidity of the plate

ψ = radial stress function

β = tangent to the angle of deflection

σ_r = radial membrane stress

σ_r^* = $\sigma_r a^2/Eh^2$ = dimensionless radial membrane stress

σ_θ = circumferential membrane stress

σ_θ^* = $\sigma_\theta a^2/Eh^2$ = dimensionless circumferential membrane stress

$\sigma_{r,b}$ = radial bending stress

$\sigma_{r,b}^*$ = $\sigma_{r,b} a^2/Eh^2$ = dimensionless radial bending stress

$\sigma_{\theta,b}$ = circumferential bending stress

$\sigma_{\theta,b}^*$ = $\sigma_{\theta,b} a^2/Eh^2$ = dimensionless circumferential bending stress

I. Introduction

A theoretical analysis is presented for the elastic deflections and stresses of an initially flat complete circular plate with clamped edges under uniform normal pressure. The edge supports are assumed to clamp the plate in such a way that not only is the deflection assumed to be zero at the edge, but rotation around the edge is also disallowed. We also assume the fixed edge condition that no radial displacement is allowed. We ignore the fact that under experimental conditions slippage might occur. The plate under consideration is initially, in its undeformed state, a circular disc of radius a and of thickness $h \ll a$. No prior assumption is made about the ratio a/h although values in the literature commonly range from 100 to 200. We shall see later that when all the quantities are written in dimensionless terms, the results are independent of a/h. In fact, the only role a/h does play is in the question of whether the theory and the plate equations are valid for too small a/h.

The plate is presumed to be constructed of homogeneous isotropic material which is elastic in the sense that for deflections up to a certain range, the plate will return to its initial undeformed shape when the pressure is removed. Furthermore, it is assumed that within the elastic range for the material there is a subrange within which Hooke's Law is valid, i.e. the strains are linear functions of the stresses.

The plane section through the undeformed plate parallel to the circular faces of the plate and midway between them will be called the middle plane. If a cylindrical coordinate system is attached to the plate with origin at the center of and z-axis normal to the middle plane, then, in terms of the coordinates (r,θ,z) , the edge of the plate will consist of the points (a,θ,z) , the upper and lower faces of the plate will be described by $z = \pm h/2$, and the middle plane by $z = 0$.

The theory furthermore assumes that each line segment of length h normal to and bisected by the middle plane of the undeformed plate is mapped by the deformation isometrically onto a line segment of length h in such a way that the image of the middle plane is the locus of the midpoints of the image segments in question. This locus is called the middle surface of the deformed plate. Rigorously, a point $(r,\theta,0)$ is mapped onto a point $(r+u,\theta,w)$, where u is the radial displacement and w is the vertical displacement. In the theory, $w \gg u$, and we may think of w as the displacement normal to the undeformed middle surface. These assumptions are part of what is called the "Kirchhoff Hypotheses" in the literature. Comparisons of theoretical and experimental results for pressures in the so called "linear range" , where stresses and displacements are so small that the linearized equations give solutions negligibly different from the full non-linearized equations, i.e. for dimensionless pressures p^* up to about 2 and dimensionless deflections w^* up to a small fraction of the thickness of the plate, suggest that the above hypotheses are reasonable in the linear range. We shall extend the theory to the non-linear range, and explore the mathematical implications of the theory for dimensionless pressures up to 400 , and for dimensionless deflections up to about 5 thicknesses. We will compare the results so obtained with known experimental and theoretical results which have been collected for values p^* as high as 240, (Levy [5] , Way [11] , Keller and Reiss [4]).

We will find that the method outlined in this paper is not only simpler in principle than the method of power series of Way and the method of finite differences of Keller and Reiss, but the principle employed here can be used to solve many related problems. Equally significant, the solutions obtained by the methods of this paper can be decomposed into what we will call "modes" (see section III) , in such a way that we can easily show how the different modes contribute to the deflection and stress results. This is a generalization of the method of superposition of solutions or Fourier type expansions of mathematical physics. Furthermore, the method outlined here permits us to impose all the boundary conditions at once, and, even the most advanced solution that we attempted did not involve more complicated systems than ten equations in ten unknowns. This is an obvious improvement over the two methods mentioned above. Finally, all the results referred to in this paper can be obtained with but a few minutes of computer time, (see section **IV**).

II. Equations

The fundamental differential equations of Föppl and von Kármán for the bending of a circular plate are given by Vol'mir [10] , p. 180 , in the form:

$$\text{I.} \quad \frac{D}{h}\nabla^2\nabla^2 w = H(w,\Phi) + \frac{p}{h}$$

$$\text{II.} \quad \frac{1}{E}\nabla^2\nabla^2\Phi = -\frac{1}{2}H(w,w)$$

where ∇^2 is the Laplacian, which we will write in polar form, and the operator H is given by:

$$H(w,\Phi) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{\partial^2\Phi}{\partial r^2} -$$

$$- 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)$$

where w is the vertical displacement, Φ is the so called stress function, E is Young's modulus, ν is Poisson's ratio, $D = Eh^3 / 12(1-\nu^2)$, and p is the pressure.

When assuming axisymmetry there is no θ dependance, so that $\frac{\partial w}{\partial\theta} = \frac{\partial\Phi}{\partial\theta} = 0$, and the partial derivatives become ordinary derivatives. The operator H then takes the form:

$$H(w,\Phi) = \frac{d^2 w}{dr^2}\left(\frac{1}{r}\frac{d\Phi}{dr}\right) + \left(\frac{1}{r}\frac{dw}{dr}\right)\frac{d^2\Phi}{dr^2} \quad .$$

If we insert H in this form into Equations I and II and write the Laplacian ∇^2 in polar form, then, if we also choose p to be constant as we assume in this paper, Equations I and II can each be integrated once. We obtain:

$$\text{III.} \quad Dr\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right) = h\frac{dw}{dr}\frac{d\Phi}{dr} + \frac{pr^2}{2} + C_1$$

$$\text{IV.} \quad r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)\right) = -\frac{E}{2}\left(\frac{dw}{dr}\right)^2 + C_2$$

where C_1 and C_2 are constants of integration. Thus we obtain a system of third order ordinary differential equations. The fact that the plate is complete, i.e. includes the origin $r = 0$, permits us to assume the existence of $\frac{dw}{dr}$, the tangent to the angle of deflection, and $\frac{d\Phi}{dr}$, the radial stress function, at $r = 0$. The axisymmetry implies further that $\frac{dw}{dr}$ and $\frac{d\Phi}{dr}$ must both be zero at $r = 0$. Thus, $C_1 = C_2 = 0$.

If we now introduce the variables $\beta(r) = -\frac{dw}{dr}$ and $\psi(r) = h\frac{d\Phi}{dr}$, Equations III and IV take the form:

$$\text{V.} \quad D\left(\beta'' + \frac{1}{r}\beta' - \frac{1}{r^2}\beta\right) = \frac{1}{r}\beta\psi - \frac{1}{2}pr$$

$$\text{VI.} \quad \psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi = -\frac{Eh}{2r}\beta^2$$

This pair of second order differential equations is now required to satisfy four boundary conditions which we impose as follows. For our case of a complete axisymmetric plate we may assume that the tangent plane to the deformed surface will be horizontal at the center of the plate. Thus we have:

$$(1) \quad \beta(0) = 0 \ .$$

If the radial membrane stress at the center of the plate, $\sigma_r(0)$, is finite, the radial stress function, $\psi(r)$, defined to be the radial membrane stress multiplied by the radius, see (21), will have value zero at the center of the plate, i.e. at $r = 0$. Thus we will have:

(2) $\psi(0) = 0$.

In the case that we are dealing with, namely the plate clamped at its edge, i.e. at $r = a$, we have the further condition that the angle of deflection at the edge must be zero, and thus its tangent β will be zero too. Thus, at $r = a$:

(3) $\beta(a) = 0$.

Finally, we impose the condition that for a fixed edge, the radial displacement, $u(a)$, must be zero. The radial displacement is given by Vol'mir [10] , p. 177, as follows:

$$u(r) = r\,E\,\left(\psi'(r) - \frac{\nu}{r}\psi(r)\right) / h \quad .$$

We can thus impose the fourth boundary condition as follows:

(4) $\psi'(a) - \frac{\nu}{a}\psi(a) = 0$.

Equations V and VI can also be derived from the work of Reissner [9] , by neglecting terms of order higher than the third and appropriate change of variables, and indeed, the notation in them is his.

III. Method of Solution

The advantage of writing Equations I and II in the form V and VI , is now readily apparent. Appearing on the left sides of both Equations V and VI is the Bessel operator of index one. Thus, if we write:

$$L(X) = \frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} - \frac{1}{r^2} X$$

for the Bessel operator of index one, we find that the Equations V and VI can be written in the form:

$$\text{VII.} \qquad D\,L(\beta) - \frac{1}{r}\beta\psi + \frac{1}{2}pr = 0$$

$$\text{VIII.} \qquad L(\psi) + \frac{Eh}{2r}\beta^2 = 0$$

From the theory of Bessel functions (cf. Churchill [1]) we know that if $X = J_1(\lambda r)$ then $L(X) = -\lambda^2 X$, where λ^2 can be called the formal eigenvalue associated with the formal eigenfunction X .

The boundary condition (3) of section II suggests as a tentative choice:

$$(5) \qquad \beta = AJ_1(\lambda r)$$

where

$$(6) \qquad J_1(\lambda a) = 0 \ .$$

We could then also choose to set:

$$(7) \qquad \psi = BJ_1(\lambda r) \ .$$

It is known that there is a sequence of positive real numbers $0 < k_1 < k_2 < \cdots < k_n < \cdots$ tending to infinity, such that $\lambda = k_n$, $n = 1,\ldots$, satisfies (6) . It is furthermore known that $J_1(k_n r)$, $n = 1, \ldots$, form a complete orthonormal set on the interval $[0, a]$ with respect to the weight function r .

It is clear that (5) and (7) cannot be made to satisfy the differential equations VII and VIII . However, the completeness of the above set implies that the solution functions β and ψ , which are assumed to exist, can each be expanded in infinite series of these Bessel functions with undetermined coefficients, namely:

$$\beta = \sum_{n=1}^{\infty} A_n J_1(k_n r)$$

and

$$\psi = \sum_{n=1}^{\infty} B_n J_1(k_n r)$$

where the coefficients are determined in the Fourier-Bessel manner, namely:

$$A_n = \frac{2}{a^2 (J_2(k_n a))^2} \int_0^a r J_1(k_n r) \beta \, dr$$

and

$$B_n = \frac{2}{a^2 (J_2(k_n a))^2} \int_0^a r J_1(k_n r) \psi \, dr \quad .$$

We reason as follows. If Equations VII and VIII have solutions, then these solutions could certainly be expanded in an infinite series of Bessel functions, since the set of Bessel functions is complete. We could assume therefore, that β and ψ are of the form given above, where the A_n and B_n are undetermined, and will be determined by the differential equations and the boundary conditions. Purely formally, one could find the A_n and the B_n as follows. Taking the above expansions for β and ψ and inserting them into Equations VII and VIII , we can then expand every term appearing in the resulting equations in a series of these same functions. We obtain two infinite sets of equations which we proceed to set equal to zero. Alternatively, from the Bubnov-Galerkin point of view, we insert these expansions for β and ψ into Equations VII and VIII , multiply each term of the resulting equations by $rJ_1(k_m r)$ and integrate from 0 to a , for each n , $n = 1, \ldots$

Either way, we arrive at the same doubly infinite set of coupled quadratic equations for the A_n and the B_n . We note that Equation VIII allows us to eliminate the B_n and we can thus obtain a singly infinite set of cubic equations in the A_n .

Since there is no way that we can deal with the full infinite set of equations at once, we shall modify our approach and adopt a slightly altered point of view. We will attempt to find approximate solutions for β and ψ in the form of finite series of Bessel functions, instead of the infinite series mentioned above, and obtain finite sets of equations in a finite number of variables, equations which may be viewed as truncations of the infinite set. A paper by Professor Rauch [8] , dealing with a similar problem, suggests just such a solution. We will then attempt to solve these equations by numerical methods. It is our hope, ultimately borne out, that these functions β and ψ in the form of these truncated series will yield approximations to the solution of the problem.

We note that the β and ψ so chosen, with finitely many terms, will immediately satisfy the boundary conditions (1) , (2) , and (3) of section II , since each term of the series does. However, (4) will not be satisfied for the above choice of ψ , since $\psi'(a)$ is not zero. Fortunately however, there is yet another eigenfunction that satisfies Bessel's equation of index one, corresponding to the eigenvalue zero, namely the function $X = r$. Therefore, if we amend ψ and write ψ as the sum:

$$(8) \quad \psi = B_0 r + \sum_{n=1}^{N} B_n J_1(k_n r)$$

and take for β :

$$(9) \quad \beta = \sum_{n=1}^{N} A_n J_1(k_n r)$$

then, not only will these β and ψ satisfy the boundary conditions (1) , (2) , and (3) , but they will also satisfy (4) if we require $B_0 + \sum_{n=1}^{N} B_n J_1'(k_n a) - \nu B_0$ to be zero, where the prime denotes derivative with respect to r . We therefore see that by choosing:

$$(10) \quad B_0 = -\sum_{n=1}^{N} B_n J_1'(k_n a) \,/\, (1 - \nu)$$

we will have met all the above requirements.

Let us therefore take β and ψ as in (8) and (9) above, and insert them into the left sides of Equations VII and VIII .

We obtain:

$$D \sum_{n=1}^{N} (-k_n^2)\, A_n J_1(k_n r)$$
$$- \left(\sum_{n=1}^{N} A_n J_1(k_n r)\right)\left(B_0 r + \sum_{n=1}^{N} B_n J_1(k_n r)\right) / r$$
$$+ pr / 2$$

and

$$\sum_{n=1}^{N} (-k_n^2)\, B_n J_1(k_n r) + Eh \left(\sum_{n=1}^{N} A_n J_1(k_n r)\right)^2 / 2r \quad .$$

Now, instead of attempting to set these expressions equal to zero in the above form, we will multiply each term of the above expressions by $rJ_1(k_m r)$, integrate from 0 to a , and set the resulting expressions equal to zero. This is the method of Bubnov-Galerkin [10] . We obtain:

IX.
$$D \int_0^a rJ_1(k_m r) \sum_{n=1}^{N} (-k_n^2)\, A_n J_1(k_n r)\, dr$$
$$- B_0 \int_0^a rJ_1(k_m r) \sum_{n=1}^{N} A_n J_1(k_n r)\, dr$$
$$- \int_0^a J_1(k_m r) \sum_{n=1}^{N} B_n J_1(k_n r) \sum_{n=1}^{N} A_n J_1(k_n r)\, dr$$
$$+ \frac{p}{2} \int_0^a r^2 J_1(k_m r)\, dr$$
$$= 0$$

and

X.
$$\int_0^a rJ_1(k_m r) \sum_{n=1}^{N} (-k_n^2)\, B_n J_1(k_n r)\, dr$$
$$+ \frac{Eh}{2} \int_0^a J_1(k_m r) \left(\sum_{n=1}^{N} A_n J_1(k_n r)\right)^2 dr$$
$$= 0$$

We observe that since (cf. Menzel [6] p. 59) :

$$\frac{d}{dx}\left(x^n J_n(\alpha x)\right) = \alpha x^n J_{n-1}(\alpha x)$$

we will have

$$\alpha \int_0^r x^n J_{n-1}(\alpha x)\, dx = r^n J_n(\alpha r)$$

and, if we choose $x = k_1 r$ and $\alpha = k_n/k_1$, we find

$$(11) \quad \int_0^a r^2 J_1(k_n r)\, dr = a^2 J_2(k_n a) / k_n \quad .$$

Furthermore, from the orthogonality property of Bessel functions, (cf. Churchill [1] p. 184) , we have:

$$(12) \quad \int_0^a r J_1(k_m r) J_1(k_n r)\, dr = \delta_{mn}\, a^2 \left(J_2(k_n a) \right)^2 / 2$$

where δ_{mn} is the so called Kronecker delta and is defined by

$$\delta_{mn} = \begin{cases} 0 \text{ if } & m \neq n \\ 1 \text{ if } & m = n \end{cases} \quad .$$

We shall adopt the following notation:

$$(13) \quad j_{lmn} = \int_0^a J_1(k_l r) J_1(k_m r) J_1(k_n r)\, dr \quad .$$

These integrals can be calculated to any desired degree of accuracy by any of several numerical methods which will be discussed in section IV .

It should be noted that when N is greater than 1 , the integrals of the form (12) in which $n \neq m$ will all be zero. We will therefore find that Equations IX and X take the form

$$\text{XI.} \quad \begin{aligned} &- D\, (k_n a)^2 (J_2(k_n a))^2 A_n / 2 \\ &- B_0\, a^2 (J_2(k_n a))^2 A_n / 2 \\ &- \sum_{m=1}^{N} \sum_{l=1}^{N} B_l A_m j_{lmn} \\ &+ p\, a^2 J_2(k_n a) / 2k_n \\ &= 0 \end{aligned}$$

and

XII. $$- B_n (k_n a)^2 (J_2(k_n a))^2 / 2 + Eh \sum_{m=1}^{N} \sum_{l=1}^{N} A_l A_m j_{lmn} / 2 = 0$$

N pairs of such equations in all, one pair for each n, $n = 1,..,N$.

We note that Equation XII permits us to solve for each of the B_n in terms of the A_n . In fact, for each n,

$$(14) \quad B_n = Eh \sum_{m=1}^{N} \sum_{l=1}^{N} A_l A_m j_{lmn} / (k_n a)^2 (J_2(k_n a))^2 .$$

Furthermore, given the facts (cf. Churchill [1]) that

$$J_1'(k_n r) = k_n \left\{ J_0(k_n r) - J_1(k_n r) / k_n r \right\}$$

and

$$J_0(k_n a) = - J_2(k_n a)$$

and, $J_1(k_n a) = 0$, we will be able to write (10) as follows:

$$(15) \quad B_0 = \sum_{n=1}^{N} B_n k_n J_2(k_n a) / (1 - \nu)$$

$$= \frac{Eh}{(k_1 a)(1-\nu)} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{l=1}^{N} A_l A_m j^*_{lmn} / Q$$

where we have introduced

$$(13a) \quad j^*_{lmn} = k_1 j_{lmn} = \int_0^{k_1 a} J_1(\frac{k_l}{k_1} x) J_1(\frac{k_m}{k_1} x) J_1(\frac{k_n}{k_1} x) \, dx$$

where

$$(16) \quad x = k_1 r$$

and where $Q = (k_n a)(J_2(k_n a))$.

If we now introduce the new variables

$$(17) \quad A_n^* = \frac{a}{h} A_n \quad , \quad n = 1,..,N$$

and substitute (14) and (15) into Equation XI , replace the flexural rigidity D by $Eh^3 / 12(1 - \nu^2)$, and then multiply every term of the resulting expression by $2a^2k_n / Eh^4(J_2(k_na))$, we obtain a single set of cubic equations in the variables A_n^* , $n = 1,..,N$, of the following form:

$$\text{XIII.} \quad a_nA_n^* + b_n \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j_{lik}^* A_i^* A_k^* A_n^* / c_l$$
$$+ d_n \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j_{lmn}^* j_{lik}^* A_m^* A_i^* A_k^* / e_l$$
$$= p^*$$

one such equation for each n , $n = 1,..,N$, and where,

$$a_n = (k_na)^3(J_2(k_na)) / (12 (1 - \nu^2))$$

$$b_n = (k_na)(J_2(k_na)) / ((k_1a)(1 - \nu))$$

$$c_l = (k_la)(J_2(k_la))$$

$$d_n = 2(k_na) / ((J_2(k_na))(k_1a)^2)$$

$$e_l = (k_la)^2(J_2(k_la))^2$$

and

$$(18) \qquad p^* = pa^4/Eh^4 \ .$$

It should be noted that XIII is written in terms of the variables A_n^* and the ratio a/h does not appear as a parameter, so that in fact, the solutions of XIII will not depend on the thickness of the plate.

There remains only to solve the system XIII of N equations in the N variables A_n^* . The computer method employed for the solution of this system will be outlined in section IV .

Assuming that we have computed the coefficients A_n and B_n , for $n = 1,..,N$, where N has been fixed previously, we can write down our approximate solutions (8) and (9) . We can furthermore proceed to calculate the approximate deflection of the plate at any point of radial distance r from the center of the plate, as well as the approximate stresses, both membrane and bending, for any fixed pressure p^* . We will make use of the following information. Since the deflection at the edge of the plate is zero, in our case of the clamped edge, i.e. $w(a) = 0$, we may write:

$$\begin{aligned} w(r) &= w(r) - w(a) \\ &= \int_a^r \frac{dw}{dr}\,dr \\ &= \int_a^r (-\beta)\,dr \\ &= \int_r^a \beta\,dr \end{aligned}$$

Thus we have:

$$\begin{aligned} (19) \qquad w(r) &= \int_r^a \sum_{n=1}^N A_n J_1(k_n r)\,dr \\ &= \sum_{n=1}^N \frac{A_n}{k_1} \int_{k_1 r}^{k_1 a} J_1\left(\frac{k_n}{k_1} x\right) dx \\ &= -\sum_{n=1}^N \frac{A_n}{k_1} \frac{k_1}{k_n} J_0\left(\frac{k_n}{k_1} x\right) \Bigg|_{k_1 r}^{k_1 a} \\ &= \sum_{n=1}^N \frac{A_n}{k_n} \left(J_0(k_n r) - J_0(k_n a) \right) \end{aligned}$$

where we have used (9) and (16) , and the last relationship was taken from Menzel [6] , p. 59.

In particular then, the central deflection, i.e. at $r = 0$, is given by:

(20) $$w(0) = \sum_{n=1}^{N} \frac{A_n}{k_n} (1 - J_0(k_n a) .$$

The membrane stresses, radial σ_r , and circumferential, σ_θ , are given by the following, (cf. Vol'mir [10] p. 173) :

(21) $$\sigma_r(r) = \Psi(r) / hr$$
$$= \frac{B_0}{h} + \sum_{n=1}^{N} \frac{B_n}{h} \frac{J_1(k_n r)}{r}$$

(22) $$\sigma_\theta(r) = \frac{d\Psi(r)}{dr} \frac{1}{h}$$

$$= \frac{B_0}{h} + \sum_{n=1}^{N} \frac{B_n}{h} J_1'(k_n r)$$

$$= \frac{B_0}{h} + \sum_{n=1}^{N} \frac{B_n k_n}{h} (J_0(k_n r) - J_1(k_n r)/k_n r)$$

where we have used (8) , the derivative J_1' is taken with respect to r , and the last relationship was taken from Menzel [6] , p. 59 .

These expressions (21) and (22) are valid except at $r = 0$, where $J_1(0) = 0$, and we define $J_1(0)/0$ by means of (25) .

The bending stresses, radial $\sigma_{r,b}$, and circumferential, $\sigma_{\theta,b}$, are calculated at the face of the plate where these stresses are maximum, i.e. at $z = h/2$. These can be found in Vol'mir ([10] , pp. 173 and 189) in the following form:

(23) $$\sigma_{r,b} = \frac{6D}{h^2} (\frac{d\beta}{dr} + \nu\frac{\beta}{r})$$

$$= \frac{6D}{h^2} (\sum_{n=1}^{N} A_n J_1'(k_n r) + \frac{\nu}{r} \sum_{n=1}^{N} A_n J_1(k_n r))$$

$$= \frac{6k_1 D}{h^2} \left(\sum_{n=1}^{N} A_n \frac{k_n}{k_1} J_0\left(\frac{k_n}{k_1} x\right) - \right.$$

$$\left. - (1 - \nu) \sum_{n=1}^{N} A_n J_1\left(\frac{k_n}{k_1} x\right) / x \right)$$

We also have:

$$(24) \qquad \sigma_{\theta, b} = \frac{6 D}{h^2} \left(\frac{\beta}{r} + \nu \frac{d\beta}{dr} \right)$$

$$= \frac{6k_1 D}{h^2} \left(\nu \sum_{n=1}^{N} A_n \frac{k_n}{k_1} J_0\left(\frac{k_n}{k_1} x\right) + \right.$$

$$\left. + (1 - \nu) \sum_{n=1}^{N} A_n J_1\left(\frac{k_n}{k_1} x\right) / x \right)$$

where we have used (9) and (16) , the derivative of J_1 is found as in (22) , and these expressions are valid for all values of r between 0 and a except at $x = k_1 r = 0$, namely at $r = 0$, the center of the plate, where $J_1(0) / 0$ must be defined by means of (25) .

As noted above, the stress functions are all undefined at the center of the plate because the expression $J_1(x) / x$, which appears in each of the stress functions (21) through (24) , approaches the limit zero over zero as x tends to 0 . In order to evaluate this limit, we examine the power series representation of Bessel's function which is given (cf, Menzel [6] p. 56) in the form:

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 2!} + \frac{x^5}{2^5 2! 3!} - + \ldots$$

If we divide this power series, term by term, by x , and take the limit of the resulting expression as x tends to 0 , we obtain the following:

$$(25)\quad \lim_{x\to 0} \frac{J_1(\frac{k_n}{k_1}x)}{x} = \lim_{x\to 0}\left(\frac{k_n}{2k_1} - \frac{k_n^3\, x^2}{k_1^3\, 2^3\, 2!} + \ldots \right)$$

$$= \frac{1}{2}\frac{k_n}{k_1} \quad .$$

Returning to (19) and (21) , we may now define the following dimensionless quantities:

$$(26)\quad w^*(r) = w(r) / h$$

$$= \sum_{n=1}^{N} \frac{1}{(k_n a)} A_n^* \left(J_0(k_n r) - J_0(k_n a) \right)$$

and

$$(27)\quad \sigma_r{}^*(r) = \sigma_r(r)\, a^2 / Eh^2$$

$$= \frac{1}{(1-\nu)(k_1 a)} \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{k=1}^{N} j_{mlk}^* \, A_l^* \, A_k^* / Q$$

$$+ \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{k=1}^{N} j_{mlk}^* \, J_1(k_m r)\, A_l^* A_k^* / (k_1 r) Q^2$$

where $Q = (k_m a)(J_2(k_m a))$.

Using (22) , we may also define:

$$(28)\quad \sigma_\theta{}^*(r) = \sigma_\theta(r)\, a^2 / Eh^2$$

$$= \frac{1}{(1-\nu)(k_1 a)} \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{k=1}^{N} j_{mlk}^* \, A_l^* \, A_k^* / Q$$

$$+ \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{k=1}^{N} \left\{ \frac{k_m}{k_1} J_0(k_m r) - \frac{J_1(k_m r)}{k_1 r} \right\} j_{mlk}^* \, A_l^* A_k^* / Q^2$$

where again $Q = (k_m a)(J_2(k_m a)$.

Using (23) , we may define:

$$(29) \quad \sigma^*_{r,b}(r) = \sigma_{r,b}(r)\, a^2 / Eh^2$$

$$= \frac{k_1 a}{2(1-\nu^2)} \left\{ \sum_{m=1}^{N} A_m^* \frac{k_m a}{k_1 a} J_0(k_m r) - \right.$$

$$\left. - (1-\nu) \sum_{m=1}^{N} A_m^* J_1(k_m r) / k_1 r \right\} .$$

Finally, using (24) , we may define:

$$(30) \quad \sigma^*_{\theta,b}(r) = \sigma_{\theta,b}(r)\, a^2 / Eh^2$$

$$= \frac{k_1 a}{2(1-\nu^2)} \left\{ \nu \sum_{m=1}^{N} A_m^* \frac{k_m a}{k_1 a} J_0(k_m r) + \right.$$

$$\left. + (1-\nu) \sum_{m=1}^{N} A_m^* J_1(k_m r)/(k_1 r) \right\} .$$

We can utilize (25) to write (29) and (30) in the following simple form for the special case $r = 0$, (i.e. at the center of the plate) :

$$(31) \quad \sigma^*_{r,b}(0) = \frac{1}{4(1-\nu)} \sum_{m=1}^{N} A_m^* \, (k_m a)$$

$$(32) \quad \sigma^*_{\theta,b}(0) = \frac{1}{4(1-\nu)} \sum_{m=1}^{N} A_m^* \, (k_m a)$$

We furthermore note the fact that at the center of the plate we have the following simple relationships between the stresses:

$$(33) \quad \sigma_r^*(0) = \sigma_\theta^*(0)$$

and

$$(34) \quad \sigma^*_{r,b}(0) = \sigma^*_{\theta,b}(0) \quad .$$

IV. Calculation

Having set up the previous equations for the general case using N in (8) and (9) , we will now demonstrate how one may carry out the computations. As an illustration, let us work out the details for the particular case $N = 1$. From (15) we obtain:

$$B_0 = Eh\ A_1^2\ j^*_{111} \ / \ (1 - \nu)(k_1 a)^2 (\ J_2(k_1 a)\ )^2 \ .$$

From (14) we obtain:

$$B_1 = Eh\ A_1^2\ j^*_{111} \ / \ (k_1 a)^2 (\ J_2(k_1 a)\ )^2 \ .$$

We utilize these to write down, by means of (8) and (9) , $\beta = A_1 J_1(k_1 r)$ and $\psi = B_0 r + B_1 J_1(k_1 r)$, where B_0 and B_1 are as above. We call this our "one mode solution" , where the definition of an "n-mode solution" is taken to be a solution of (8) and (9) where $N = n$. We note that the one mode solution is only a first approximation to the solution of Equations V and VI of section II . To refine the approximation to the solution will involve taking a larger N . We note immediately a great advantage of this method, namely that we can refine our solutions as often as we wish, the only limitation being the consideration of time. We will find furthermore, that for a fixed p^* , the A_i^* of an n-mode solution will be reasonable guesses for the first n coefficients A_i^* of the (n+1)-mode solution, and thus, it is a fairly easy matter to go from an n-mode solution to an (n+1)-mode solution, (see Table vi) .

It remains now to solve the system XIII of N equations in the N variables A_n^* , $n = 1,..,N$, for any fixed integer N . It soon becomes apparent that to deal with such a system, even with the aid of a desk calculator, is already a massive task for $N = 3$, and becomes prohibitively complicated for larger N . We are compelled to seek assistance from computer methods, which can, through the use of DO loops, very conveniently deal with summations of the type which we encounter in the system XIII .

When $N = 1$, the "system" XIII will consist of just one equation in the one unknown A_1^* , which can be solved in several ways, one of which, the Delta Method, is outlined below. When N is larger than 1 , there are again several methods for solving the system XIII , although some methods will have drawbacks. For example, if we rewrite the system XIII , solving for the linear term in each equation, we obtain:

$$(35) \qquad A_n^{*(q+1)} = p^*/a_n - b_n S_{likn}/a_n - d_n T_{mlik}/a_n$$

for each n , $n = 1,..,N$, where a_n , b_n , and d_n are as in XIII , and

$$S_{likn} = \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j_{lik}^{*} A_i^{*(q)} A_k^{*(q)} A_n^{*(q)} / (k_1 a)(J_2(k_1 a))$$

and

$$T_{mlik} = \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j_{lmn}^{*} j_{lik}^{*} A_m^{*(q)} A_i^{*(q)} A_k^{*(q)} / Q^2$$

where $Q^2 = (k_1 a)^2 (J_2(k_1 a))^2$, and where we have adopted the notation $A_n^{*(q)}$ to mean the q-th iterate of A_n^* . Then, for a fixed value p^* , we may choose initial values $A_n^{*(0)}$, $n = 1,..,N$, which we insert into the right hand sides of

(35) . We will obtain, on the left hand sides of (35) , $A_n^{*(1)}$, $n = 1,..,N$, the first iterates of the A_n^* . We may proceed to put these $A_n^{*(1)}$ back into the right hand sides of (35) , to find the $A_n^{*(2)}$, $n = 1,..,N$. We could continue to iterate in this manner until we have met the convergence criterion, $\sum_{n=1}^{N} \left| A_n^{*(i)} - A_n^{*(i-1)} \right| < \varepsilon$, where ε is chosen small. It was found that this linear iteration scheme actually works for $p^* < 5$, but fails to work reliably for larger p^* .

Another linear iteration scheme that was tried had the form

$$A_n^{*(q+1)} = (p^* - a_n A_n^{*(q)} - d_n T_{mlik}) / b_n U_{lik}$$

for every n , $n = 1,..,N$, where a_n , b_n , d_n , and T_{mlik} are as in (35) , and

$$U_{lik} = \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j_{lik}^* A_i^{*(q)} A_k^{*(q)} / (k_1 a)(J_2(k_1 a)) .$$

This iteration scheme also met with but limited success.

A number of other iteration techniques were also attempted, and although some met with considerable success, none seemed to work for all values of the parameter p^* , and in particular, there was some difficulty in finding the A_n^* when p^* had the intermediate values $9 \leq p^* \leq 26$. This was judged a serious setback, and we therefore turned to the investigation of what is called the Newton-Jacobi Secant Method in the literature, [7] , which we refer to as the Delta Method. To describe this method we introduce the following notation. We shall write $\bar{A}_N^{(q)}$ to mean the q-th iterate of the vector with the N components $A_n^{*(q)}$, $n = 1,..,N$. Then, for the system of N equations, which we shall write as $f_n(\overline{A}_N^{(q)}) = 0$, $n = 1,.,N$, in the N unknowns $A_n^{*(q)}$, $n = 1,..,N$, we may apply Newton's method [2] to each equation separately where we consider the n-th equation as an equation in the n-th unknown, $A_n^{*(q)}$, i.e. $f_i(\overline{A}_N^{(q)}) = 0$ is an equation in the one variable $A_i^{*(q)}$, $i = 1,.,N$. Furthermore, the secant variation uses difference quotients instead of derivatives. We observe that we do not use the full N-dimensional generalization of Newton's method which would involve the computation and inversion either of the Jacobian matrix or the finite difference approximation to it.

Thus, let us rewrite the system XIII in the form:

$$(36) \qquad f_n(\overline{A}_N^{(q)}) = T1_n + T2_n + T3_n - p^*$$

for each n , $n = 1,..,N$, where

$$T1_n = a_n A_n^{*(q)}$$

$$T2_n = b_n \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j^*_{lik} \, A_i^{*(q)} A_k^{*(q)} A_n^{*(q)} \, / \, c_l$$

$$T3_n = d_n \sum_{m=1}^{N} \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} j^*_{lmn} \, j^*_{lik} \, A_m^{*(q)} A_i^{*(q)} A_k^{*(q)} \, / \, e_l$$

where a_n , b_n , c_l , d_n , and e_l are given in XIII . Let us now hold all the components of the vector $\overline{A}_N^{(q)}$ fixed except for the n-th component which we shall increment by a small number Δx . We shall designate this new vector by $\overline{A}_N^{(q)} + \Delta\overline{A}_n$, where $\Delta\overline{A}_n$ represents the increment of the n-th component by Δx . We may now calculate, again using (36) , $f_n(\bar{A}_N^{(q)} + \Delta\bar{A}_n)$. We can now form the quotient:

$$(f_n(\bar{A}_N^{(q)} + \Delta\bar{A}_n) - f_n(\bar{A}_N^{(q)})) / \Delta x ,$$

which we will designate by $\Delta f_n/\Delta x$. The Delta Method then calculates the next iterates $A_n^{*(q+1)}$ from the $A_n^{*(q)}$ for each n , $n = 1,..,N$, by the following iterative scheme:

$$(37) \qquad A_n^{*(q+1)} = A_n^{*(q)} - f_n(\bar{A}_N^{(q)}) / (\Delta f_n/\Delta x)$$

We continue to calculate the $A_n^{*(q)}$ cyclically until we have met our convergence criterion, $\sum_{n=1}^{N} \left| A_n^{*(q)} - A_n^{*(q-1)} \right| < \varepsilon$. By choosing Δx small, on the order of 10^{-5} , it was found that the Delta Method was successful in finding the A_n^* , $n = 1,..,N$, for all integer values of the parameter p^* in the range $0 \leq p^* \leq 400$, and in fact the Delta Method seems to work for an unlimited range of the parameter p^* . However, as mentioned earlier, for loads p^* in the upper half of this range, the bending equations as set down in section II may no longer be valid, and there seems little point in pursuing the results for p^* larger than 400 . We may observe however, that the deflection results, w^* , and the stress results, σ_r^* and σ_θ^* , for p^* up to about 240 , are in excellent agreement with experimental results when the A_n^* are calculated by the method outlined above, (see section V) . A copy of the Fortran program used to find the A_n^* by the Delta Method is included after this discussion.

Several computational details are worthy of mention. Firstly, it will be noticed that all computation was carried out to double precision in Fortran, which, although most accurate, is slow and wasteful of computer time, and unnecessary once the Delta Method has been established as successful. For all practical purposes, no results need to be calculated to greater accuracy than three or four decimal places, and as such, single precision Fortran is sufficient.

A useful trick to speed convergence will be found on line 45 of the Fortran main program used to calculate the A_n^* . It was found that cutting the size of Δx in half every time before reentering the iterative loop, made the program dramatically more efficient, and thus cut the computing time and cost significantly. This is consistent with the Newton-Jacobi Method as described in the literature.

It should furthermore be mentioned that a number of standard acceleration techniques of numerical analysis, such as Aitken's Δ^2 process [2] , were attempted to speed the convergence of the Delta Method, but they did not meet with success.

With regard to the computation of the cubic integrals j_{lmn}^* , a comparison of the methods of Simpson's Rule and Gaussian Quadrature showed the latter method in every way superior. We calculated the values of these integrals using Simpson's Rule with 10 , 20 , 50 , and 100 thousand subdivisions and found very good agreement, that is, the 100,000 subdivisions answer agreed to eight decimal places or better with the 50,000 subdivisions answer, and they in turn were also in close agreement with the 10 and 20 thousand subdivisions results. However, for a fraction of the computer time and cost, using the built in Gaussian Quadrature Fortran subprogram DQG32 , the best version of Gaussian Quadrature available on the IBM 370 at the present time, we were able to duplicate the results obtained by means of Simpson's Rule with 100,000 subdivisions to ten decimal places. Thus, whereas one of the integrals j^*_{1mn} computed by means of Simpson's Rule with 100,000 subdivisions required about one minute of computer time on the IBM 370 , it could be computed in about 0.01 minutes using DQG32 . In fact, using this method, it is possible to compute the 220 such integrals required for solutions up to ten modes in about two minutes. Also, for any one fixed value of p^* , it is possible to calculate, by means of the program supplied in this paper, deflections and stresses, using a nine or ten mode solution, in about 15 seconds. These estimates are based on the use of double precision arithmetic in the Fortran programming. If single precision arithmetic is employed, the computation time should be more than halved.

Finally, we wish to point out, in order to evaluate $J_n(x)$, $n = 0,1$, for any argument x , a new computer subroutine was written, as the package program to compute Bessel functions found in the IBM 370 was found to be inadequate with regard to precision. In fact, there was no double precision Bessel function in the machine at all. These new subprograms are called BJOK1(x) and BJ1K1(x) respectively.

These programs are accurate to at least ten decimal places, (compare with the tables in Gray et. al. [3]) . These programs calculate the value of $J_n(x)$ by means of its power series representation for x smaller than 17 , and by means of an asymptotic expression (cf. Gray et. al. [3]) , for x larger than 17 .

V. Discussion of Results

In lieu of a convergence proof for the method described in the previous sections, we will examine the consistency of the results found by the application of these methods. We may note that the different central deflection curves, corresponding to solutions with different number of modes, exhibit the following phenomenon. For all values p^* , except for the values $9 \leq p^* \leq 26$, the one mode solution yields the highest deflection value among the solutions examined, and the two mode solution the lowest. The higher mode solutions all lie between these first two solutions and tend upwards towards the one mode solution, in the manner exhibited in Table i , so that we begin to suspect that the one mode solution is a good approximation to, or at least an upper bound for, the central deflection. For the entire range $0 \leq p^* \leq 240$ the one mode solution differs from the nine mode solution in no case by more than $2\frac{1}{2}\%$. For $9 \leq p^* \leq 26$, some of the higher mode solutions yield slightly larger central deflection results than the one mode solution, but nowhere did they differ from the one mode solution by more than about two tenths of one percent. As far as the graphical sketching is concerned, the five, seven, and higher mode solutions are almost indistinguishable from each other for the entire range $0 \leq p^* \leq 240$, and lie just below the graph of the one mode solution.

As far as the deflection curves given for a fixed p^* , we may note the following. Although the one mode solution is a good approximation to the central deflection it is a poor approximation at any other point of the plate. However, a quite reasonable approximation to the deflection at any point of the plate is given by the five mode solution, which seems to differ from the nine mode solution, the best solution that we calculated, by less than 1% for any p^* in the region of primary interest, $0 \leq p^* \leq 240$.

The fact that the higher mode solutions seem to converge in this sense, points out another major advantage of using this method. As the evidence in the tables at the end of this section suggests, a nine mode solution is sufficient to predict deflection and membrane stress results for $0 \leq p^* \leq 240$, and a nine mode solution does not require an inordinate amount of work. In fact, as noted earlier, such solutions may be obtained with but several minutes of computer work.

Unfortunately, the bending stress results were not quite so definitive, and even a nine mode solution did not exhibit settled behavior for p^* even as small as 25 . However, there is every indication that the addition of more modes will yield more satisfactory results. For very small p^* , such as $p^* = 5$, the nine mode solution did in fact yield quite acceptable results even for the bending stresses. In any case, there is no serious setback, since the interested reader can certainly pursue the solutions (8) and (9) to as many modes as necessary, as mentioned earlier.

Comparisons have been made between our results and the earlier theoretical work and results of Way [11] , who based his solutions on the method of power series, as well as with the work of Keller and Reiss [4] , whose solutions are based on the method of finite differences. Comparisons were also made with the experimental results of Levy [5] . Unfortunately, all the results of the above mentioned sources are tabulated graphically, which tends to make comparisons more difficult, but, the following general remarks can be made. If we examine the theoretical results, the clearest results for small p^* , $0 \leq p^* \leq 12$, can be found in the paper by Way. His results for this range of pressures p^* seem to be indistinguishable from those obtained by the methods outlined in this paper. The theoretical results given by Keller and Reiss, are given for a larger range of pressures p^* , but again are only tabulated graphically, and in fact, the given scale for the graphs is different from that of Way and Levy, which tends to doubly complicate comparisons. However, they claim that their results are in "excellent agreement" with the results given by Way, as well as the results found by the extension of Way's method. As near as we could judge, our results are also in close agreement with theirs.

The experimental results given by Levy are given for pressures p^* in the range $0 \leq p^* \leq 240$, and are again tabulated graphically. We may conclude from his results, as near as we could judge, that although his deflection results appear to be consistently larger than the results obtained by the methods of theis paper, for no value of the parameter p^* in the range mentioned above, do his results seem to differ from our nine mode solution by more than $2\frac{1}{2}\%$. In fact, this is very encouraging, as there are various possible sources of error in the experimental results which could account for these discrepancies, as well as a number of possible improvements that we can still make on our method. The most fundamental source of error that would account for these discrepancies is the fact that the plate equations as set down in the theory and outlined in section II , may be inadequate for large p^* where we begin to notice the above mentioned discrepancies. We note that all results, theoretical and experimental, are in very close agreement for small p^* , and it is only as p^* increases that we begin to note differences in the results.

Our own results may be improved by taking a larger N in (8) and (9) . The higher mode solutions will undoubtedly be very consistent, although there is no reason to expect that the higher mode solutions will be any more consistent with the earlier results, than were the solutions that we calculated.

More likely sources of discrepancy between theoretical and experimental results may lie in the fact that the experimental plate may actually not behave in the manner dictated by the theory due to slippage and yielding of the plate under experimental conditions. Thus, it is possible that the plate which is assumed to be clamped and fixed at its edge, may actually behave like a simply supported plate when p^* becomes large enough. We suggest that if the methods of this paper would be pursued for the case of the simply supported edge, one would find, for the larger p^* , closer agreement with the experimental results given by Levy for what he assumes to be a clamped fixed edge.

Further sources of discrepancy between our results and experimental results could arise from small errors in measurement caused by inaccuracies in the gauges or from the fact that the material being tested is in fact not as homogeneous as the experimenter might desire.

Some interesting observations emerge from the analysis of the methods of this paper. We note, for example, that the difference in deflections, as predicted by the solutions of different modes, varies with the parameter p^* , and, as noted earlier, the smaller p^* , the smaller these differences. This is particularly encouraging, as it is precisely the smaller p^* for which we expect the theory to hold and the equations of section II to be valid. The fact that very good agreement with experimental results was obtained for the values of p^* in the range $0 \leq p^* \leq 240$ with such a small number of modes as nine or ten, and with so little effort, clearly demonstrates the advantage of this method.

An interesting oddity that emerged from the study of this problem, was the fact that the problem seemed to favor solutions with an odd number of modes, i.e. N odd. We find for example, that the nine mode solution yields deflection and stress results which are demonstrably better than the more refined ten mode solution. The reason for this phenomenon is not quite clear, although its occurrence is readily apparent in the tables and graphs that follow this discussion.

One general remark about the graphs that follow is in order. We have chosen to represent the radial distance along the plate in terms of the variable $x = k_1 r$, where x varies from 0 to the first zero of Bessel's function of index one, namely $k_1 a \approx 3.8317$. Thus, the scaling is such that $0 \leq x \leq k_1 a$, where a point at the edge of the plate has radial distance $x = k_1 a$ from the center of the plate.

Table i

The following are the central deflections as predicted by the solutions of modes one through ten for some representative pressures p* .

mode	p*/ 5	100	200	400
1	0.685	2.884	3.716	4.747
2	0.650	2.488	3.116	3.889
3	0.679	2.829	3.628	4.623
4	0.674	2.755	3.494	4.398
5	0.679	2.839	3.637	4.630
6	0.678	2.817	3.594	4.550
7	0.679	2.847	3.648	4.643
8	0.679	2.837	3.630	4.608
9	0.679	2.850	3.654	4.652
10	0.679	2.846	3.645	4.635

Figure i: Central Deflection versus Pressure

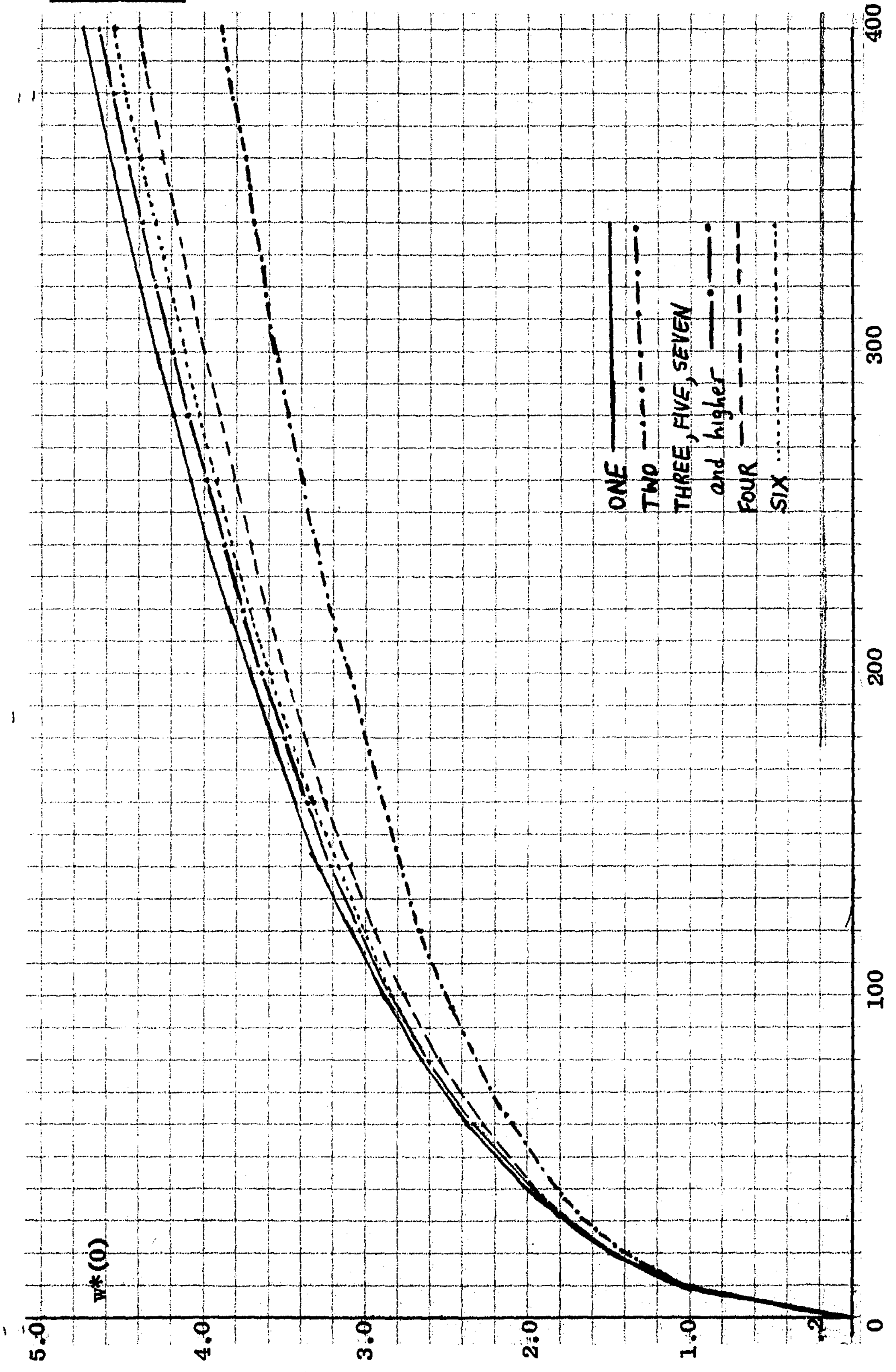

Table ii

The following are some values of the deflection $w^*(x)$ for $p^* = 5$, at some representative points $x = k_1 r$ along the radius of the plate, as predicted by the solutions of different modes. The central deflection, $w^*(0)$, can be found in Table i ; the deflection at the edge of the plate, $w^*(k_1 a)$, was always found to be less than 10^{-10} in absolute value, which is certainly in agreement with the clamping condition. As for scaling, see the remark on page 33 .

mode	x / 0.5	1.5	2.5	3.5
1	0.655	0.446	0.173	0.011
2	0.634	0.491	0.221	0.015
3	0.655	0.491	0.233	0.017
4	0.655	0.494	0.235	0.018
5	0.657	0.496	0.235	0.019
6	0.657	0.496	0.235	0.019
7	0.657	0.497	0.236	0.020
8	0.658	0.497	0.236	0.020
9	0.658	0.497	0.236	0.020
10	0.658	0.497	0.236	0.020

Figure ii: Shape of Deflected Plate for Pressure $p^* = 5$

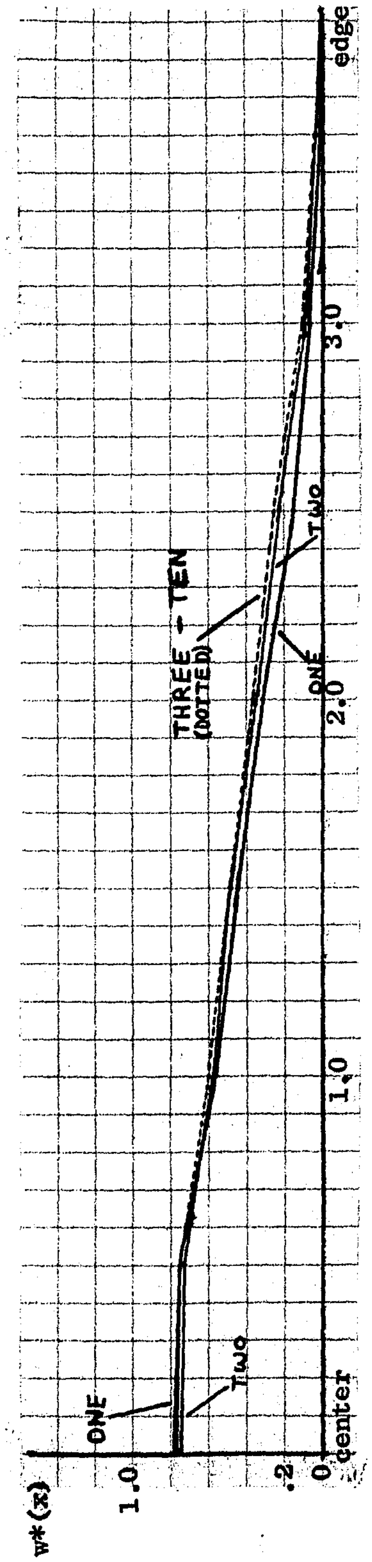

Table iii

The description here is the same as that for Table ii, except that here the pressure $p^* = 200$. For scaling, see the remark on page 33.

mode	x / 0.5	1.5	2.5	3.5
1	3.553	2.423	0.939	0.060
2	3.139	2.904	1.527	0.114
3	3.491	2.854	1.757	0.153
4	3.481	2.934	1.818	0.182
5	3.530	2.994	1.824	0.203
6	3.556	2.999	1.828	0.218
7	3.559	3.008	1.839	0.228
8	3.577	3.023	1.851	0.236
9	3.576	3.026	1.858	0.241
10	3.585	3.028	1.860	0.245

Figure iii : Shape of Deflected Plate for Pressure $p^* = 200$

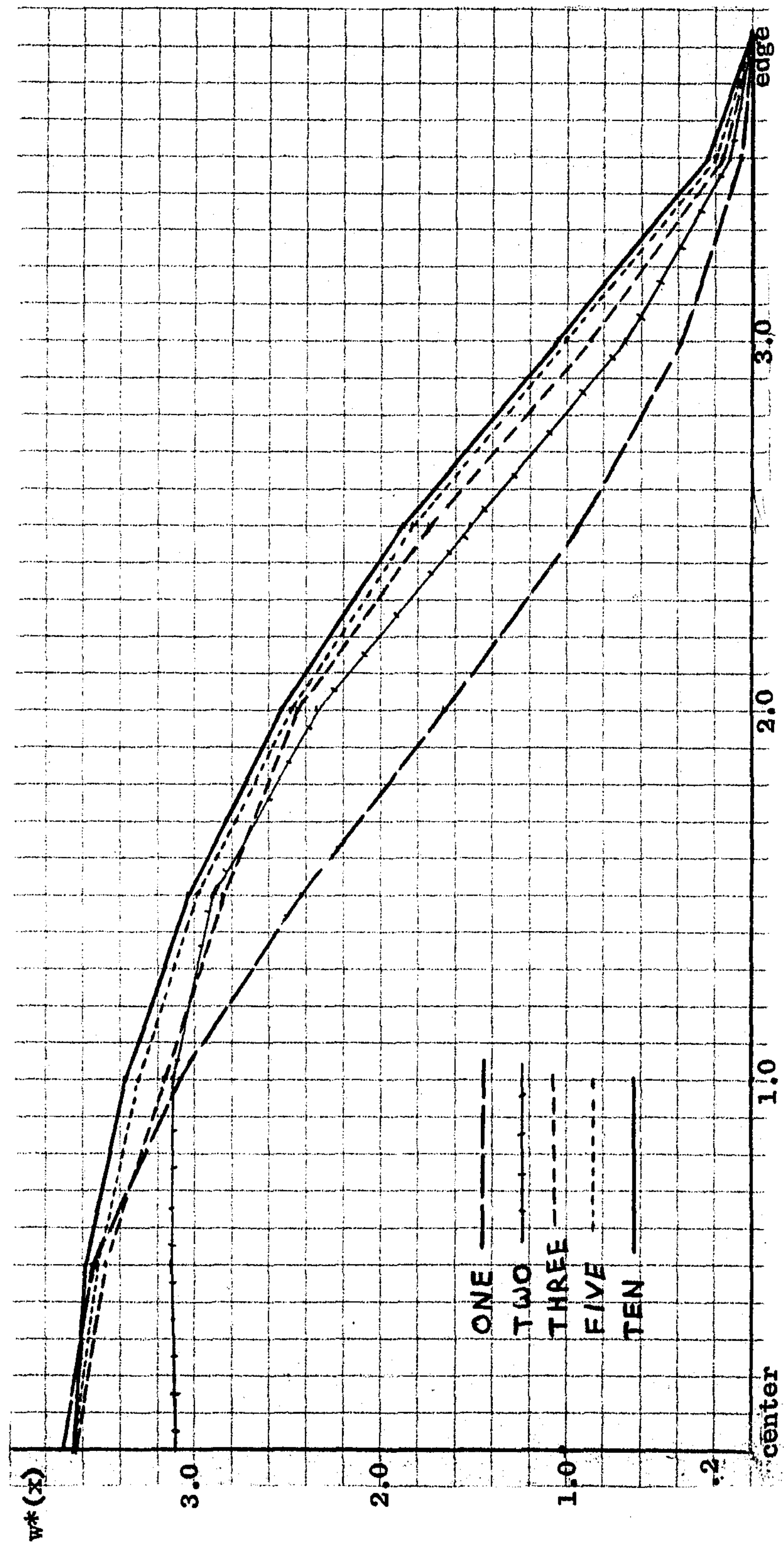

Table iv

The following is a list of the values of the membrane stresses, σ_r^* and σ_θ^*, for some representative pressures, p^*, at the center and at the edge of the plate.

All results are taken from the nine mode solution. For scaling, see the remark on page 33.

	σ_r^*		σ_θ^*	
p^*	center	edge	center	edge
5	0.446	0.227	0.446	0.068
25	2.581	1.486	2.581	0.446
100	7.758	4.994	7.758	1.498
200	12.878	8.646	12.878	2.594
400	21.117	14.643	21.117	4.393

Figure iv : Membrane Stresses

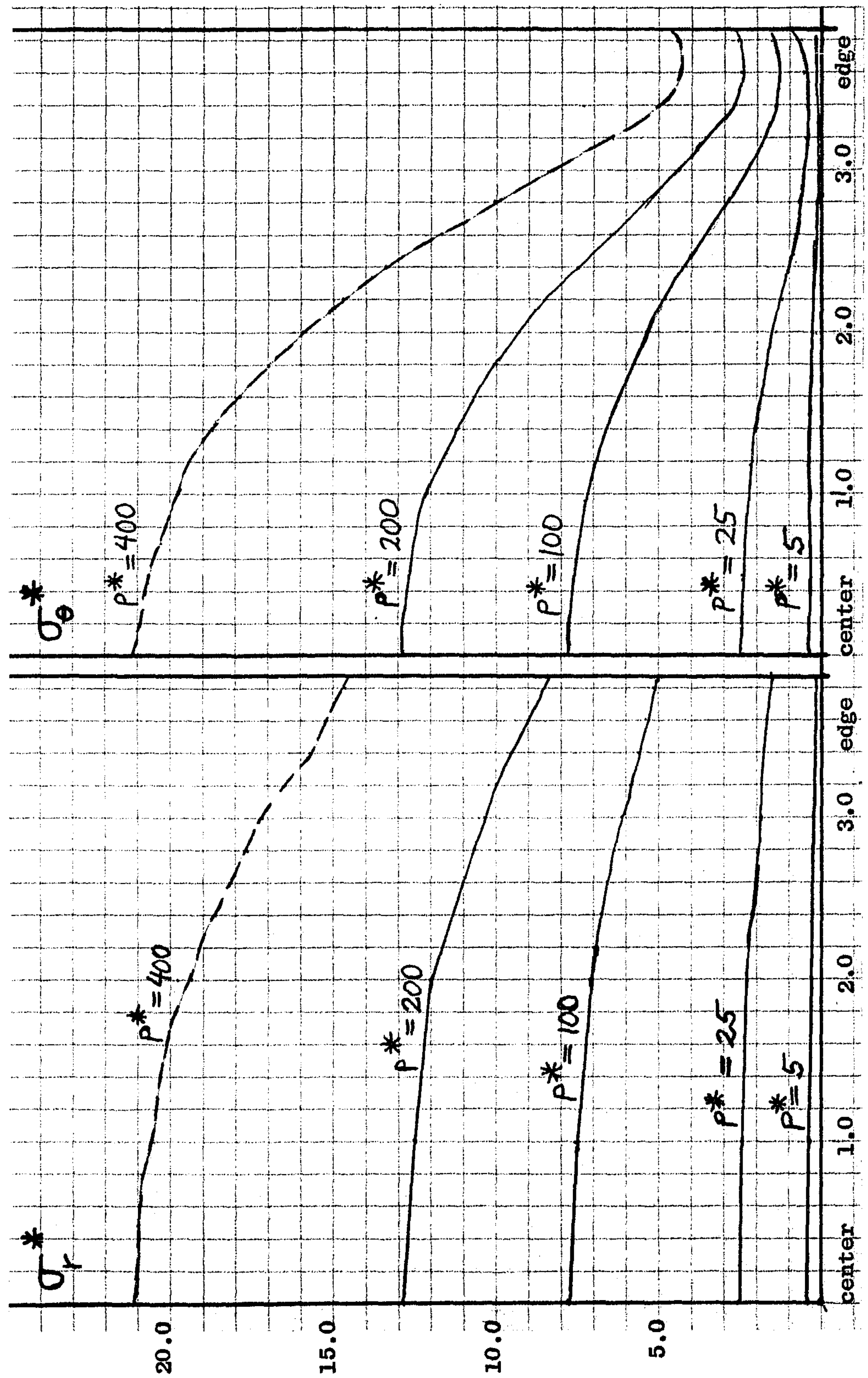

Table V

The following is a list of the maximum bending stress values taken at the center and at the edge of the plate for some representative pressures p^*. All results are taken from the nine mode solution. We may note that the point of zero stress moves towards the edge of the plate as p^* increases. The oscillation in the graphs is due to the inadequacy of a nine mode solution, and would probably smooth out for higher mode solutions. For scaling, see the remark on page 33.

	$\sigma^*_{r,b}$		$\sigma^*_{\theta,b}$	
p^*	center	edge	center	edge
5	1.877	-3.016	1.877	-0.905
25	3.796	-10.080	3.796	-3.024
100	6.099	-25.041	6.099	-7.512
200	8.242	-38.286	8.242	-11.486
400	11.696	-57.328	11.696	-17.198

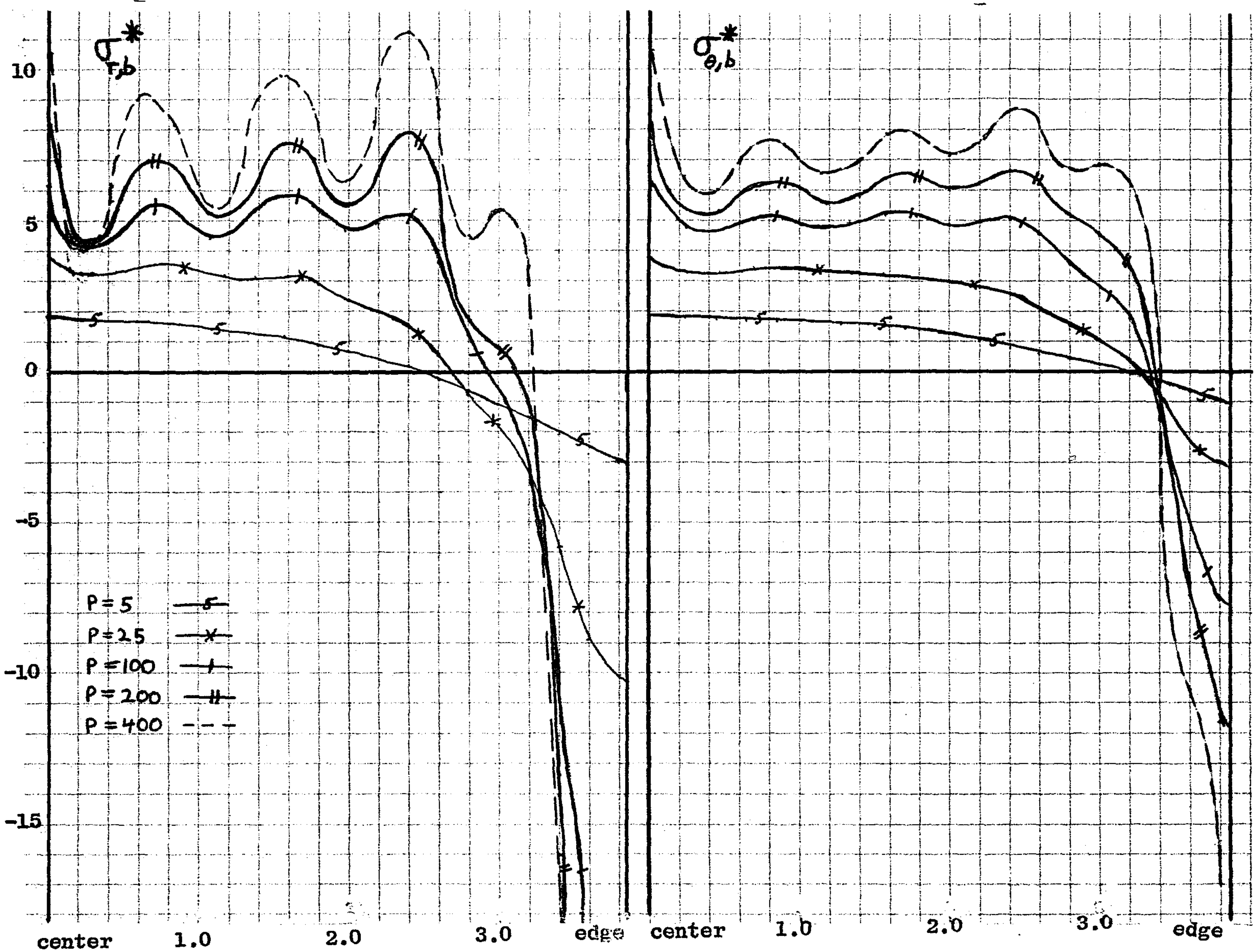

Figure V : Bending Stresses

Figure vi :

Central Deflection versus Stresses for the 9-mode Solution

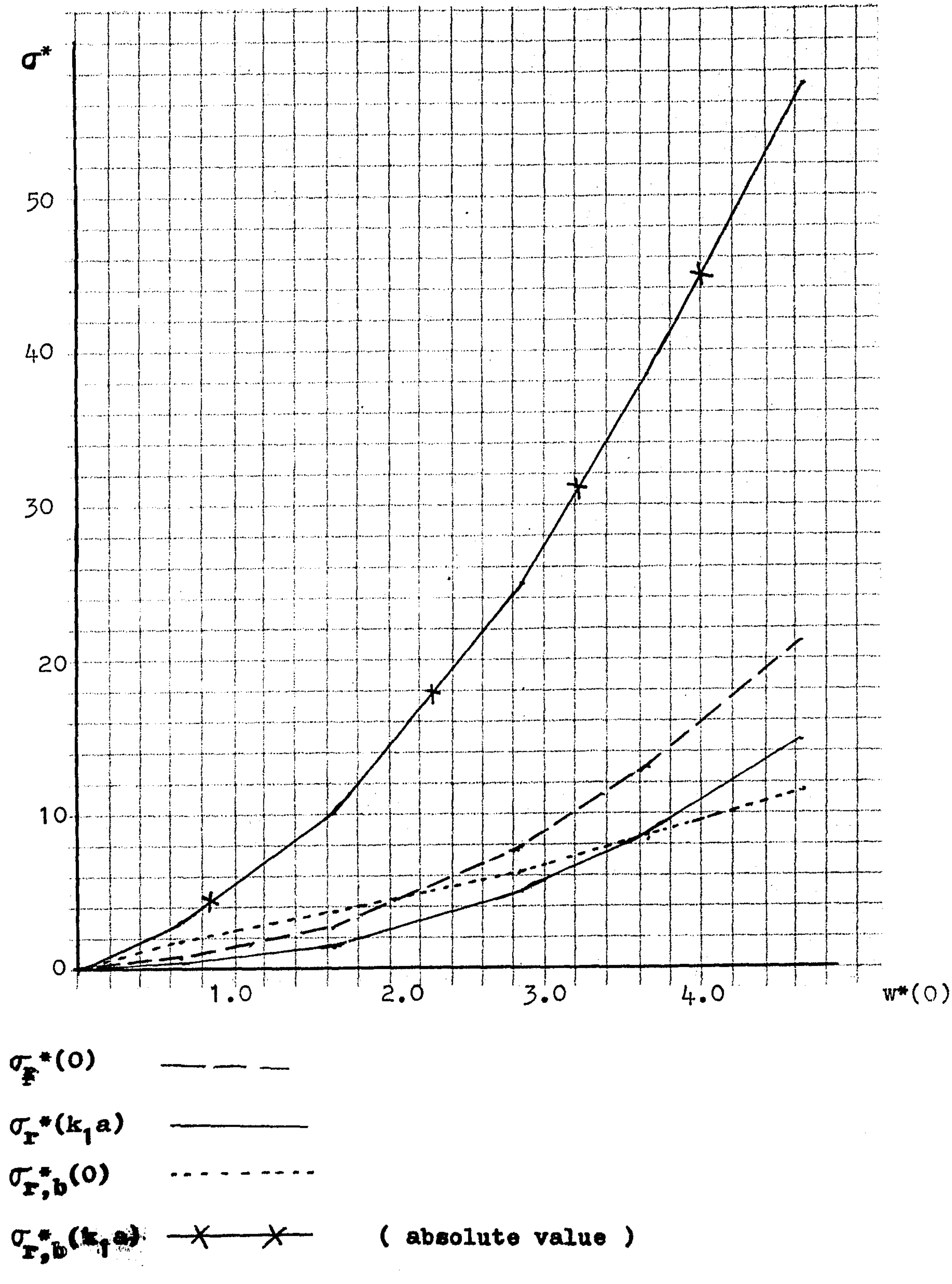

Table vi

The following is a table of the coefficients A_i^* , obtained by the Delta Method of section IV , for $p^* = 5$ and $p^* = 200$, for the 9-mode and 10-mode solutions.

	$p^* = 5$		$p^* = 200$	
	9-mode	10-mode	9-mode	10-mode
A_1^*	2.19476	1.93335	10.48802	10.49449
A_2^*	-0.61341	-0.51625	-6.89167	-6.89639
A_3^*	0.24512	0.20510	4.38212	4.38484
A_4^*	-0.12544	-0.10481	-2.82794	-2.82944
A_5^*	0.07405	0.06183	1.89204	1.89298
A_6^*	-0.04794	-0.04000	-1.31808	-1.31881
A_7^*	0.03310	0.02761	0.95305	0.95388
A_8^*	-0.02397	-0.01999	-0.71117	-0.71258
A_9^*	0.01800	0.01502	0.54260	0.54673
A_{10}^*	-	-0.01162	-	-0.42743

The following is a list of the values of the integrals j^*_{lmn} . We have given only the first 35 of these integrals which are sufficient to carry the reader through the five mode solution which was judged adequate for deflection and membrane stress results. The 6-mode solution requires 21 more such integrals; the 7-mode solution requires 28 more than the 6-mode solution, and the 10-mode solution requires a total of 220 such integrals. The program that follows, on page 48 , permits the calculation of any of these integrals that one may desire. Note that l,m,n must be supplied on line 8 of the main program. All results are given to ten decimal places, since we obtained the same results for j^*_{lmn} when computed by means of Simpson's Rule as by means of Gaussian Quadrature, (see section IV)

l,m,n	j^*_{lmn}
1,1,1	0.3146252900
1,1,2	0.0939580066
1,1,3	-0.0063748586
1,1,4	0.0020834662
1,1,5	0.0652953781
1,2,2	0.1847634203
1,2,3	0.0614188788
1,2,4	-0.0047322899
1,2,5	0.0016924525
1,3,3	0.1292775419
1,3,4	0.0453735092

l,m,n	j^*_{lmn}
1,3,5	-0.0037014462
1,4,4	0.0992433367
1,4,5	0.0359318829
1,5,5	0.0804885087
2,2,2	0.1411398543
2,2,3	0.1278776157
2,2,4	0.0428271836
2,2,5	-0.0036718097
2,3,3	0.1069686181
2,3,4	0.0956151982
2,3,5	0.0327593871
2,4,4	0.0842932976
2,4,5	0.0760750084
2,5,5	0.0691869329
3,3,3	0.1076462086
3,3,4	0.0872908932
3,3,5	0.0741028972
3,4,4	0.0871778406
3,4,5	0.0715907000
3,5,5	0.0723941245
4,4,4	0.0773087587
4,4,5	0.0736948958
4,5,5	0.0662928802
5,5,5	0.0652953782

```
FORTRAN IV G LEVEL  21                MAIN              DATE = 75147           13/11/19                 PAGE 0001

             C      EVALUATING INTEGRALS BY GAUSSIAN QUADRATURE
 0001               IMPLICIT REAL*8(A-H,O-Z)
 0002               DIMENSION G(30),H(30),E(30)
 0003               COMMON/ROOTS/A1K,A2K,A3K,A4K,A5K,A6K,A7K,A8K,A9K,ATK
 0004               COMMON/PWRS/K1PWR,K2PWR,K3PWR,K4PWR,K5PWR,K6PWR,K7PWR,K8PWR,K9PWR,
                   1KTPWR
 0005               EXTERNAL SIMINT
 0006               XL=0.0D00
 0007               XU=A1K
 0008             8 READ(5,892) K1PWR,K2PWR,K3PWR,K4PWR,K5PWR,K6PWR,K7PWR,K8PWR,K9PWR,
                   1KTPWR
 0009           892 FORMAT(10I3)
 0010               IF(K8PWR.EQ.5) STOP
 0011               CALL DQG32(XL,XU,SIMINT,VALUE)
 0012               WRITE(6,991) K1PWR,K2PWR,K3PWR,K4PWR,K5PWR,K6PWR,K7PWR,K8PWR,K9PWR
                   1,KTPWR
 0013           991 FORMAT('0FOR K1PWR=',I3,2X,'K2PWR=',I3,2X,'K3PWR=',I3,2X,
                   1'K4PWR=',I3,2X,'K5PWR=',I3,'K6PWR=',I3,'K7PWR=',I3,'K8PWR=',I3,2X,
                   2'K9PWR=',I3,2X,'K10PWR=',I3)
 0014               WRITE(6,990) VALUE
 0015           990 FORMAT(' VALUE OF INTEGRAL BY GUASSIAN QUADRATURE =',D23.16)
 0016               GO TO 8
 0017               END
```

```
FORTRAN IV G LEVEL  21                BLK DATA          DATE = 75147           13/11/19                 PAGE 0001

 0001               BLOCK DATA
 0002               IMPLICIT REAL*8(A-H,O-Z)
 0003               COMMON/ROOTS/A1K,A2K,A3K,A4K,A5K,A6K,A7K,A8K,A9K,ATK
 0004               DATA A1K,A2K,A3K/.38317059702D01,.70155866698D01,1.01734681351D01/
 0005               DATA A4K,A5K,A6K/13.3236919363D00, 16.4706300509D00,19.6158585D00/
 0006               DATA A7K,A8K,A9K/22.7600843806D00,25.9036720876D00,29.04682853D00/
 0007               DATA ATK/32.189679910974D00/
 0008               END
```

FORTRAN IV G LEVEL 21 SIMINT DATE = 75147 13/11/19 PAGE 0001

```
      FUNCTION SIMINT(X)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/ROOTS/A1K,A2K,A3K,A4K,A5K,A6K,A7K,A8K,A9K,ATK
      COMMON/PWRS/K1PWR,K2PWR,K3PWR,K4PWR,K5PWR,K6PWR,K7PWR,K8PWR,K9PWR,
     1KTPWR
      Y=A2K/A1K
      Z=A3K/A1K
      ZZ=A4K/A1K
      ZZZ=A5K/A1K
      ZZZZ=A6K/A1K
      YY=A7K/A1K
      YYY=A8K/A1K
      YYYY=A9K/A1K
      YTEN=ATK/A1K
      IF(K1PWR.EQ.0) GO TO 55
      B1=BJ1K1(X)**K1PWR
      GO TO 60
   55 B1=1.0D00
   60 IF(K2PWR.EQ.0) GO TO 65
      B2=BJ1K1(X*Y)**K2PWR
      GO TO 70
   65 B2=1.0D00
   70 IF(K3PWR.EQ.0) GO TO 75
      B3=BJ1K1(X*Z)**K3PWR
      GO TO 80
   75 B3=1.0D00
   80 IF(K4PWR.EQ.0) GO TO 85
      B4=BJ1K1(X*ZZ)**K4PWR
      GO TO 86
   85 B4=1.0D00
   86 IF(K5PWR.EQ.0) GO TO 88
      B5=BJ1K1(X*ZZZ)**K5PWR
      GO TO 90
   88 B5=1.0D00
   90 IF(K6PWR.EQ.0)GO TO 92
      B6=BJ1K1(X*ZZZZ)**K6PWR
      GO TO 93
   92 B6=1.0D00
   93 IF(K7PWR.EQ.0) GO TO 94
      B7=BJ1K1(X*YY)**K7PWR
      GO TO 95
   94 B7=1.0D00
   95 IF(K8PWR.EQ.0) GO TO 96
      B8=BJ1K1(X*YYY)**K8PWR
      GO TO 97
   96 B8=1.0D00
   97 IF(K9PWR.EQ.0) GO TO 98
      B9=BJ1K1(X*YYYY)**K9PWR
      GO TO 199
   98 B9=1.0D00
  199 IF(KTPWR.EQ.0) GO TO 99
      B10=BJ1K1(X*YTEN)**KTPWR
      GO TO 299
   99 B10=1.0D00
  299 SIMINT=B1*B2*B3*B4*B5*B6*B7*B8*B9*B10
      RETURN
      END
```

```
      FUNCTION BJ1K1(X)
C     COMPUTING BESSEL FNS AT PTS X(N)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION G(30),H(30),E(30)
      IF(X.GT.17.0) GO TO 601
      T=X/2.0D00
      V=T
      DO 102 M=1,50
      EM=M
      T=T*(-1.0D00*X*X)/(EM*(EM+1.0D00)*4.0D00)
      V=V+T
      TT=DABS(T)
      IF(TT.LT.1.0D-12) GO TO 101
  102 CONTINUE
      GO TO 121
  101 BJ1K1=V
      RETURN
  601 PI=3.1415926535D00
      M=10
      DO 100 N=1,M
  100 G(N)=4-(2*N-1)**2
      H(1)=G(1)
      H(2)=G(1)*G(2)
      DO 120 N=3,M
  120 H(N)=H(N-2)*G(N)*G(N-1)
      G(1)=1
      DO 130 N=2,M
  130 G(N)=G(N-1)*N
  135 E(1)=8.*X
      DO 140 N=2,M
  140 E(N)=E(N-1)*8.*X
      B=1
      C=0
      DO 150 N=2,M,2
      B=B+(-1)**(N/2)*H(N)/(G(N)*E(N))
  150 C=C+(-1)**(N/2+1)*H(N-1)/(G(N-1)*E(N-1))
      B=B*DCOS(X-.75D00*PI)
      C=C*DSIN(X-.75D00*PI)
      BJ1K1=DSQRT(2./(PI*X))*(B-C)
      RETURN
  121 WRITE (6,995) X
  995 FORMAT('0FOR X=',F20.16,'BESSEL SERIES FAILED TO CONVERGE WITHIN
     250 STEPS')
      STOP
      END
```

The program that follows can be used to calculate the deflections and stresses in which we are interested. We note that the Delta Method is an essential part of this process, as are the Bessel function subprograms. The following should be noted:

The Bessel function subprogram BJ1K1(x) must be supplied when running this program. A copy of this subprogram appears on page 50 of this discussion.

AK(i) refer to the $k_i a$, the zeroes of $J_1(x)$

BJ(i) refer to the values $J_2(k_i a)$

VAL are the values of the j^*_{1mn}

L = i , $1 \leq i \leq 19$ in the output refers to 19 representative points equally spaced along the radius of the plate.

The numbers $k_i a$ and $J_2(k_i a)$, which must be supplied in line 11 of the main program, may be found to sixteen place accuracy in Table IV of Gray et. al. [3] .

Recall that $J_2(k_i a) = - J_0(k_i a)$.

```
          IMPLICIT REAL*8(A-H,O-Z)
          DIMENSION A1(10),A2(10),AA(10),S(10,10,10),AK(10),BJ(10),D(10)
          DIMENSION RR(10),R(19),B(10),SMR(19),SMT(19),SBR(19),SBT(19)
C         N REPRESENTS THE NUMBER OF MODES
          READ(5,880) N
      880 FORMAT(I3)
C         WE HAVE ONLY LEFT ENOUGH ARRAY SPACE FOR N.LE.10
          IF(N.LE.10) GO TO 89
          WRITE(6,901) N
      901 FORMAT('0N=',I3,2X,'IS TOO BIG FOR PRESENT ARRAYS TO HANDLE')
          STOP
       89 CONTINUE
          READ(5,801) (AK(I),BJ(I),I=1,N)
      801 FORMAT(2D20.10)
          DO 702 KARDS=1,165
          READ(5,802) L,M,N,VAL
      802 FORMAT(3I2,D22.16)
          S(L,M,N)=VAL
          S(L,N,M)=VAL
          S(M,L,N)=VAL
          S(M,N,L)=VAL
          S(N,L,M)=VAL
          S(N,M,L)=VAL
      702 CONTINUE
          DO 703 I=1,N
      703 RR(I)=AK(I)/AK(1)
          DO 704 L=1,19
          EL=L
      704 R(L)=AK(1)*EL/19.0
          CONST=1.0 D-10
          CONMAX=1.0 D20
          GNU=.3D00
          DD=AK(1)/(2.0*(1.0-GNU**2))
C       **THIS AREA STARTS THE BALL ROLLING
          P=200.
          A1(1)=.104944934D02
          A1(2)=-.689638863D01
          A1(3)=.438484261D01
          A1(4)=-.282944398D01
          A1(5)=.189297536D01
          A1(6)=-.131880999D01
          A1(7)=.953878968D00
          A1(8)=-.712582100D00
          A1(9)=.546731886D00
          A1(10)=-.427426027D00
C         ITNO STANDS FOR 'ITERATION NUMBER'
          DELTAX=1.D-03
          DO 115 ITNO=1,5000
          DELTAX=.5*DELTAX
          DO 99 L=1,N
          TTT=0.0D00
          DO 32 M=1,N
          T=0.0D00
          DO 31 I=1,N
          DO 30 J=1,N
       30 T=T+S(M,I,J)*A1(I)*A1(J)
```

```
   31 CONTINUE
      TT=T/(AK(M)*BJ(M))
   32 TTT=TTT+TT
      B0=TTT/((1.0D00-GNU)*AK(1))
      UUU=0.0D00
      DO 43 M=1,N
      U=0.0D00
      DO 41 I=1,N
      DO 40 J=1,N
   40 U=U+S(M,I,J)*A1(I)*A1(J)
   41 CONTINUE
      UU=U/(AK(M)**2*BJ(M)**2)
      DO 42 K=1,N
   42 UUU=UUU+UU*S(L,M,K)*A1(K)
   43 CONTINUE
      T1=AK(L)**3*BJ(L)*A1(L)/(12.0D00*(1.0D00-GNU**2))
      T2=AK(L)*BJ(L)*B0*A1(L)
      T3=2.0D00*AK(L)*UUU/(BJ(L)*AK(1)**2)
      BTEMP=P-T1-T2-T3
   13 A2(L)=A1(L)
      A1(L)=A1(L)+DELTAX
      TTT=0.0D00
      DO 37 M=1,N
      T=0.0D00
      DO 36 I=1,N
      DO 35 J=1,N
   35 T=T+S(M,I,J)*A1(I)*A1(J)
   36 CONTINUE
      TT=T/(AK(M)*BJ(M))
   37 TTT=TTT+TT
      B0=TTT/((1.0D00-GNU)*AK(1))
      UUU=0.0D00
      DO 49 M=1,N
      U=0.0D00
      DO 47 I=1,N
      DO 46 J=1,N
   46 U=U+S(M,I,J)*A1(I)*A1(J)
   47 CONTINUE
      UU=U/(AK(M)**2*BJ(M)**2)
      DO 48 K=1,N
   48 UUU=UUU+UU*S(L,M,K)*A1(K)
   49 CONTINUE
      T1=AK(L)**3*BJ(L)*A1(L)/(12.0D00*(1.0D00-GNU**2))
      T2=AK(L)*BJ(L)*B0*A1(L)
      T3=2.0D00*AK(L)*UUU/(BJ(L)*AK(1)**2)
      CTEMP=P-T1-T2-T3
C     DELTA METHOD
      A1(L)=A1(L)-BTEMP*DELTAX/(CTEMP-BTEMP)
      AA(L)=DABS(A1(L)-A2(L))
   99 CONTINUE
      DIF=0.0D00
      DO 105 L=1,N
  105 DIF=DIF+AA(L)
      IF (DIF.LT.CONST) GO TO 124
      GO TO 114
  124 CCTEMP=DABS(CTEMP)
```

```
          IF(CCTEMP.LT.1.D-07) GO TO 125
      114 CONTINUE
          IF (DIF.GT.CONMAX) GO TO 120
      115 CONTINUE
          WRITE(6,905) ITNO
      905 FORMAT(' ITERATION FAILED TO CONVERGE WITHIN',I6,'STEPS')
          STOP
      120 WRITE(6,904) ITNO,DIF
      904 FORMAT(' AFTER',I6,2X,'STEPS ITERATION BLEW UP.DIF=',D23.16)
          STOP
      125 W=0.0D00
          DO 126 L=1,N
      126 W=W+(1.0D00+BJ(L))*A2(L)/AK(L)
          WRITE(6,900)  P,(I,A1(I),I=1,N)
      900 FORMAT('0FOR P=',F8.3,5(1X,'A1(',I1,')=',D16.9)/4(1X,'A1(',I1,')='
         9,D16.9))
          WRITE(6,903) ITNO,W
      903 FORMAT(' REQUIRED',I6,2X,'ITERATIONS.',15X,'CENTRAL DEFLECTION W='
         4,F20.16)
          WRITE(6,386)
      386 FORMAT('0R=0.0 MEANS CENTRAL DEFLECTION   R=3.83 IS DEF AT EDGE')
          DO 203 L=1,N
          BB=0.0
          DO 202 I=1,N
          DO 201 J=1,N
      201 BB=BB+S(L,I,J)*A1(I)*A1(J)
      202 CONTINUE
          B(L)=BB/(AK(L)**2*BJ(L)**2)
      203 CONTINUE
          BB=0.0
          CB=0.0
          DO 211 M=1,N
          BB=BB+RR(M)*B(M)
      211 CB=CB+A1(M)*AK(M)
          SMR(1)=B0+BB*.5D00
          SMT(1)=SMR(1)
          SBR(1)=CB/(4.0*(1.0-GNU))
          SBT(1)=SBR(1)
          DO 213 L=2,19
          BB=0.0
          BC=0.0
          CB=0.0
          CC=0.0
          DEF=0.0D00
          DO 212 M=1,N
          Y=RR(M)*R(L)
          BJ0=BJ0K1(Y)
          BJ1=BJ1K1(Y)
          DERIVJ=Y*BJ0-BJ1
          BB=BB+B(M)*BJ1
          BC=BC+B(M)*DERIVJ
          CB=CB+A1(M)*(DERIVJ+GNU*BJ1)
          CC=CC+A1(M)*(GNU*DERIVJ+BJ1)
          DEF=DEF+A1(M)*(BJ0+BJ(M))/AK(M)
      212 CONTINUE
          WRITE(6,387) R(L),DEF
```

```
0162         387 FORMAT(' FOR R=',F6.2,2X,'DEFLECTION=',D23.16)
0163             SMR(L)=B0+BB/R(L)
0164             SMT(L)=B0+BC/R(L)
0165             SBR(L)=DD*CB/R(L)
0166             SBT(L)=DD*CC/R(L)
0167         213 CONTINUE
0168             WRITE(6,221)
0169         221 FORMAT('0STRESSES S MEMBRANE M BENDING B RADIAL R TANGENTIAL T')
0170             DO 223 L=1,19
0171             WRITE(6,222)L,SMR(L),SMT(L),SBR(L),SBT(L)
0172         222 FORMAT(' L=',I3,2X,'SMR=',D23.16,2X,'SMT=',D23.16,2X,'SBR=',D23.16
                1,2X,'SBT=',D23.16)
0173         223 CONTINUE
0174             STOP
0175             END
```

```
      FUNCTION BJOK1(X)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION G(30),H(30),E(30)
      IF(X.GT.17.0) GO TO 601
      ONE=1.0D00
      T=-1.0D00*X*X/4.0D00
      V=ONE+T
      DO 100 M=1,50
      EM=M
      T=T*(-1.0D00*X*X)/((EM+1.D00)*(EM+1.D00)*4.D00)
      V=V+T
      TT=DABS(T)
      IF(TT.LT.1.0D-12) GO TO 101
  100 CONTINUE
      GO TO 120
  101 BJOK1=V
      RETURN
  120 WRITE (6,995) X
  995 FORMAT('0FOR X=',F20.16,'BESSEL SERIES FAILED TO CONVERGE WITHIN
     250 STEPS')
      STOP
  601 PI=3.141592653589793D00
      M=10
      DO 102 N=1,M
  102 G(N)=-(2*N-1)**2
      H(1)=G(1)
      H(2)=G(1)*G(2)
      DO 121 N=3,M
  121 H(N)=H(N-2)*G(N)*G(N-1)
      G(1)=1
      DO 130 N=2,M
  130 G(N)=G(N-1)*N
  135 E(1)=8.*X
      DO 140 N=2,M
  140 E(N)=E(N-1)*8.*X
      B=1
      C=0
      DO 150 N=2,M,2
      B=B+(-1)**(N/2)*H(N)/(G(N)*E(N))
  150 C=C+(-1)**(N/2+1)*H(N-1)/(G(N-1)*E(N-1))
      B=B*DCOS(X-.25D00*PI)
      C=C*DSIN(X-.25D00*PI)
      BJOK1=DSQRT(2./(PI*X))*(B-C)
      RETURN
      END
```

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THE PROPAGATION OF ELECTROMAGNETIC WAVES

IN AN OVERDENSE PLASMA

by

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