

# Distributed Motion Coordination of Swarms of Mobile Agents

*by*

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# Abstract

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The natural phenomena of swarms, characterized by grouping of a large number of entities, can be observed in many living beings such as flocks of birds and schools of fish. The inspiring aspect of these phenomena is that although the intelligence of the individual members of the swarm is limited, a sophisticated and efficient group behavior is still achieved.

On the engineering side, in the last few years, distributed coordination control of a large scale multi-agent dynamical system has invoked increasing interest in the control community. A large group of mobile agents (e.g., mobile robots or mobile sensor nodes), geared with computing, sensing and communication devices can serve as a platform for a variety of civilian and military applications such as environmental monitoring, space exploration and battle-field mobile surveillance.

In this thesis, we use algebraic graphs to model the topologies of the swarm that embody the neighborhood, communication or the sensing relations among the members. We consider the general situation that the swarm's topology dynamically changes as the spacing among agents evolves with time. We study the underlying mechanisms in natural swarms, present a systematic methodology to analyze the behavior of a large group of mobile agents, and develop a unified framework for the controller design of a general range of coordinated motions of the swarm. By exploiting the presented framework as a guideline, we investigate and design decentralized controllers for two application scenarios of the coordinated motion of the swarm,

namely, flocking in homogeneous environments and virtual leader tracking control. In addition, we examine the performance of the presented controllers under unknown but bounded disturbances. We show that with the presented controllers, the swarm can achieve the desired collective motion with bounded errors even in the presence of unknown disturbances.

*To Na*

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# Nomenclature

$m_i$	The mass of the $i$ th agent
$n$	The dimension of the space in which the swarm moves
$N$	The number of agents in the swarm
$\mathbb{N}_i$	The set of the $i$ th agent's neighbors
$r_i$	The position vector of the $i$ th agent relative to a ground reference origin
$r_{ij}$	The relative position vector from agent $j$ to agent $i$
$\ r_{ij}\ $	The relative distance between two agents ( $i$ and $j$ )
$u_i$	The motion controller of the $i$ th agent
$v_i$	The velocity vector of the $i$ th agent

# Chapter 1

## Introduction

In nature, there are a great number of fascinating phenomena that people will give a big ‘waw’ to at his first sight. Many scientists and engineers have gained motivations and inspirations from these phenomena in history. Self-organized group behavior of swarms, characterized by grouping of a large number of entities with or without formation of patterns, is one of such phenomena in nature. It can be broadly observed in the life spans of various living beings. For instance, within the fishes, over 50% of fish species school at some point in their lives (Shaw, 1970). The biological groups that form a self-organized collective behavior range from microscopic cellular populations and bacterial colonies (Czirok et al., 1996; Grünbaum, 1998; Lutscher, 2002), to macroscopic fish schools and bird flocks (Shaw, 1970; Berthold et al., 2001), and to enormous-size locust swarms (Edelstein-Keshet et al., 1998; Edelstein-Keshet, 1999).

After millions of years of evolution, these species perform a collective group behavior with certain reasons. The group behaviors are generally believed to have certain advantages over individual ones, such as increasing the chance to find food, enhancing the reproductive success of members in the group, and maintaining biologically sufficient number to reproduce next generations. In addition, the survivorship of individuals under danger from predators is believed to be greater when moving in a

---

large group (Shaw, 1970; Hamilton, 1971). For example, a predator facing a large number of fishes that quickly move together has difficulty in singling out and tracking individual fish in the group. The increasing size of fish group also decreases the success of the predator's attack per encounter with a prey. If a single prey is targeted, a greater success rate per attack will be obtained (Parrish and Edelstein-keshet, 1999; Zheng et al., 2005; Beecham and Farnsworth, 1999).

The inspiring aspect of these swarming phenomena is that although the intelligence of the individual member is limited, the efficient and sophisticated group behavior can still be achieved without a global coordinator.

On the other hand, in engineering fields, with the development of RFID and advanced hardware technology in the last few decades, compact and inexpensive sensors, actuators and wireless communication devices are becoming more pervasive. A group of small mobile agents, such as mobile sensor nodes or robots, equipped with inexpensive and compact sensors, actuators, and communication modules can serve as a platform for a wide range of emerging engineering applications. It may also provide an efficient and cost-effective solution for certain challenging missions that are demanding for a single and expensive unit packed with many advanced equipments. For this reason, in the last few years, the coordinated and cooperative control of large scale multi-agent dynamical systems has attracted rapidly growing interests in the control systems community (Olfati-Saber et al., 2007; Ren et al., 2007; Martínez et al., 2007c; Murray, 2007). The coordinated motion control of a large group of mobile agents is one of the important topics due to its potential applications in a variety of civilian and military areas, such as storm monitoring and tracking, disaster search and rescue, aircraft formation flight and large-area mobile surveillance (Ren et al., 2007; Martínez et al., 2007c; Murray, 2007).

Besides the coordinated motion control, there are many other issues that need to be addressed for the successful implementation of a distributed coordination system,

such as design of a power-efficient communication protocol, a distributed data processing and information fusion algorithm. In this thesis, our study is focused on the distributed coordination of the motions of the swarm.

The motivation of this thesis is to study the biological swarms from the control point of view, discover the general underlying mechanisms that govern these phenomena, develop a unified framework for the distributed motion coordination of a swarm of mobile agents, and apply the derived principle to design scalable control strategies for certain engineering applications of the coordinated motions of multi-agent dynamical systems.

In this chapter, we first review some important works on swarms in several related research areas, and then provide a brief exposition of the contribution and the outline of this thesis.

## 1.1 Literature Reviews

The phenomena of swarms have invoked extensive research interests in mathematical biology, physics, control systems and robotics communities. We examine some important literature in these areas respectively.

### 1.1.1 Biology and Physics Literature

Biologists have observed and studied the swarming behaviors of different species for decades. Some interesting phenomena were first observed and recorded by them. For example, Miller *et al.* found that a swarm of sandhill cranes can flock together in a same speed with a constant spacing (about 5.8ft) (Miller and Stephen, 1966), and tuna shoals are observed to school together with a separation of 0.16-0.25 body length in shapes of 1D “Solider”, 2D “surface”, and 3D “ball” (Newlands, 2002).

A large number of works have been reported by biologists for a better understand-

ing of why and how these creatures form a collective group behavior in different environments. In most studied biological swarms, the group behavior is generally assumed to be mediated by two major types of social interactions: the internal interactions among swarm members, and the external interactions between individuals and the environment (see Grünbaum and Okubo, 1994; Mogilner and Edelstein-keshet, 1999; Mogilner et al., 2003; Niwa, 1994, 1996; Holmes et al., 1994; Couzin et al., 2002; Parrish et al., 2002; Adioui et al., 2003, and references therein).

Explicitly or implicitly based on this principle, a variety of models and algorithms have been presented in the mathematical biology literature. Most of the models and approaches can be categorized into two main branches: in reference to the Lagrangian and Eulerian descriptions of fluid motion, they are referred to as Eulerian and Lagrangian (Mirabet et al., 2007; Morale et al., 2005).

The Eulerian approach is based on a continuum model that applies partial differential equations to describe the evolving population density of the swam (Holmes et al., 1994; Edelstein-Keshet et al., 1998; Mogilner and Edelstein-keshet, 1999). The swarm members are not considered individually. Eulerian approach is commonly used to explain the social behavior of organisms such as bacteria (Grünbaum, 1998; Czirok et al., 1996). A typical Euclidean model is an *advection-diffusion-reaction* equation of swarm population density  $\rho(x, t)$  (Holmes et al., 1994; Edelstein-Keshet et al., 1998; Mogilner and Edelstein-keshet, 1999)

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) - \frac{\partial}{\partial x} (\rho V(\rho)) + B(\rho)$$

where the reaction term  $B(\rho)$  is due to the population density dynamics, the diffusion term  $\frac{\partial}{\partial x} (D(\rho) \frac{\partial \rho}{\partial x})$  and the advection term  $\frac{\partial}{\partial x} (\rho V(\rho))$  are the joint outcome of individual behaviors and environmental influences.

The Lagrangian approach uses certain individual-based behavior rules or the classical Newtonian mechanics law to study the motion of each member (Grünbaum and Okubo, 1994; Mogilner and Edelstein-keshet, 1999; Mogilner et al., 2003; Niwa, 1996).

The typical individual-based internal mutual interaction rules used in most models, for example, of fish schools, include short-distance repulsion, long-distance attraction and middle-range alignment (also called "parallel orientation"). It is commonly believed that individuals can sense and adjust its motion according to certain neighbors through the Attraction/Alignment/Repulsion (A/A/R) interactions. Many different functions have been presented by mathematic biologists for the mutual interactions (Mogilner and Edelstein-keshet, 1999; Niwa, 1996; Viscido et al., 2004; Couzin et al., 2002). Also many efforts have been made to propose a more reasonable or "realistic" set of functions for A/A/R interactions by comparing the simulation results with the real fishes (Viscido et al., 2004; Kunz and Hemelrijk, 2003).

Both Eulerian and Lagrangian methods have their own advantages and disadvantages. The fact that the Eulerian approach studies the swarm at a macroscopic level leads to ease in dealing with the emerging properties of large and dense populations, while the microscopic view of Lagrangian methods offers the advantage of directly relating the studies to field observations of individuals in a real biological system (Mirabet et al., 2007; Morale et al., 2005).

In this thesis we also apply the Lagrangian approach to study the underlying rules by which the individual motions can lead to a collective behavior of the swarm in a self-organized manner.

In parallel to deterministic considerations, both Eulerian and Lagrangian models can be stochastic or statistic (Niwa, 1994, 1996; Adiou et al., 2003; Eftimie et al., 2007; Parrish et al., 2002). A statistical mechanics approach, complemented with synergetics is explored to analyze transients and fluctuations in behavior of individuals in a school of fish (Haken, 1983). In (Niwa, 1994, 1996), a stochastic differential equation model with locomotory, arrayal, and grouping forces has been proposed and analyzed. A discrete stochastic model is applied in (Gueron et al., 1996) to investigate the dynamics of migrating herds. A hierarchical decision scheme and neutral zone

(seen in some mammals) are explored to analyze the behavior of herd members with homogeneous and heterogeneous speeds (Gueron et al., 1996).

The swarming phenomena in nature have many remarkable features that have invoked intense research interests (Parrish et al., 2002; Parrish and Edelstein-keshet, 1999). One of them is the collective motion of the swarm, in which the agents have a common linear velocity or a common angular velocity. The former and the later are called *parallel type* and *circular type* collective motions, respectively (Couzin et al., 2002; Paley et al., 2007). When the entire group moves in parallel collectively, all swarm members have the same velocity. In this thesis, the presented control framework can provide a solid and thorough explanation for these types of collective motions in the natural swarms.

Besides mathematical biologists, physicists have also produced plenty research results on self-organization of swarms (see Vicsek et al., 1995; Gregoire and Chate, 2004; Aldana et al., 2007, and references therein).

The study that physicists generally undertake is to model the individual swarm member as a particle, which is usually called self-driven or self-propelled particle, and analyze or simulate the collective behavior of the swarm. They also consider the mutual interactions among particles and apply either Eulerian or Lagrangian methods to perform the analysis or simulations.

Vicsek et al. presented a widely referred statistical simulation model for a group of self-propelled particles to align their headings (Vicsek et al., 1995). In Vicsek's model, each particle is assumed to move at a constant speed and only needs to change its heading at each time step. The particles change their headings based on nearest neighborhood law, i.e., each particle considers only certain swarm-mates within a certain neighborhood. In the presented model, each particle's heading is updated by the average of the headings of its nearest neighbors and itself with some random noises added. Simulations in (Vicsek et al., 1995) show that when the density of

particles is large and the noise is small, the headings of all particles converge to a common value. This is a surprising result that indicates that a collective motion can be achieved by a simple rule, and is followed by a series of papers in the physics and the control communities (Vicsek et al., 1995; Czirok and Vicsek, 2000; Gregoire and Chate, 2004; Aldana et al., 2007; Jadbabaie et al., 2003).

Vicsek et al. also showed that a high noise, with/without low particle density, leads to a non-transport phase where the average velocity is zero; whereas in a low noise, the swarm moves in a particular direction that depends on the initial conditions. They call such transition from a stationary state to a mobile state *kinetic phase* (Vicsek et al., 1995; Czirok and Vicsek, 2000)

In (Mikhailov and Zanette, 1999), the authors consider a dynamic model of swarm of self-propelled particles with attractive long-range interactions. They show that the system can be found in either coherent travelling state or incoherent oscillatory state and that the increase in noise intensity leads to a transition from a coherent to oscillatory state. Similarly, in (Shimoyama et al., 1996), the authors propose a model that represents several sorts of cluster motion observed in nature including collective rotation, chaos, and wandering. The article (Levine and Rappel, 2001) describes a model that exhibits coherent localized solutions in one and two dimensions. The solution is of finite extent and the density drops sharply to zero at the edges of the swarm as in biological swarms. Moreover, they develop a continuum version of their discrete model and show that both models are equivalent.

Similar to Vicsek's model, a behavior based simulation model was proposed earlier by Reynolds to animate the coordinated motion of a group of agents (Reynolds, 1987). He named the simulated entity as *boids*, and his work started a new topic called *artificial life* in computer graphics. Reynolds' animation model consists of three simple steering behaviors to maneuver individual boids, which are based on the position and velocities of nearby swarm-mates:

- **Separation:** steer to avoid crowding local swarm-mates within certain range
- **Alignment:** steer towards the average heading of local swarm-mates
- **Cohesion:** steer to move toward the average position of local swarm-mates

Reynolds' model also applies the neighborhood law, in which each agent has direct access to the whole scene's geometric description, but reacts only to swarm-mates within a certain small neighborhood around itself. Swarm-mates out of this local neighborhood are ignored. Linear superposition of these three rules results in all agents moving in an aggregation while avoiding collisions. Although they are developed independently, it turns out that Vicsek's model is a special case of boids (Tanner et al., 2003a,b).

In the mathematical biology and physics literature, the primary methodology exploited by the Lagrangian models involves proposing certain mathematical equations or behavior rules for individual motions, then using computer simulations to investigate the emerging group behavior of the swarm.

To a certain extent, this simulation based methodology, which may be behavior-based or equation-based, proceeds as scientists usually do: to understand and explain certain phenomena in nature, certain hypotheses and mechanisms are proposed, then specific models are introduced; based on the proposed models, intensive simulations are performed; by comparing the simulation results with observational or experimental data, the models, or certain parameters of the models, are modified to produce more realistic simulations; and finally the models are accepted or rejected on the basis of how well the observed phenomena can be reproduced.

Although some questions can be answered by intensive simulations, the simulation based methodology can not always offer a clear and thorough understanding of the underlying principles. Literally, the analysis based on simulation results is limited to only a certain subset of the entire parameter space, regardless of how compre-

hensively the simulations are performed. In addition, an ultimate simulation that considers every relevant aspect or detail may not be possible. However, an analytically mathematical approach may provide the key to a general and thorough study of the self-organized swarming phenomena.

The lack of a general analytical study in Lagrangian based models also creates confusion in some existing results in the literature. An illustrative example (Eftimie et al., 2009) is Vicsek's model. In Vicsek's model, each particle moves with a constant speed and updates its heading according to the average of the headings of its neighbors, with some random noises added. Simulations show that when the density of particles is large and the noise is small, the headings of all particles converge to a common value. Vicsek's model was initially thought to exhibit a continuous transition from disordered to ordered motion (implying a supercritical bifurcation) (Vicsek et al., 1995), but a later work (Gregoire and Chate, 2004) shows that this transition is discontinuous (implying a subcritical bifurcation). Another result (Aldana et al., 2007) reports that the transition can be either continuous or discontinuous, depending on the way in which the noise is introduced. Nevertheless, the applicability of the result presented in (Aldana et al., 2007) to Vicsek's model continues to be in debate (Chaté et al., 2007).

In recent years, some researchers in mathematical biology and physics have started to analytically study the swarming phenomena, and some efforts have been made on the topics of alignment and aggregation of swarms. Li et al. (2008c) used perturbation theory and linearization based stability analysis to analytically study the local conditions for "perfect" schools in 1D and 2D spaces. The same analytical technique is used for circular formation of swarms in (Lukeman et al., 2009). However, the analysis in (Li et al., 2008c; Lukeman et al., 2009) is only for swarms in 1D and 2D spaces, and the swarm studied there must have a special structure: the swarm members sequentially line up along a straight line or around a closed circle, and each

member only interacts with at maximum two immediate neighbors (front and back).

### 1.1.2 Control Literature

Although there are intensive works on swarms of biological beings or self-propelled particles, it is only most recently that the distributed coordination control of multi-agent dynamical systems has drawn research interest in the control systems community. In a certain extent, the emerging researches on swarms in the control systems community are contributed to certain related engineering applications, such as distributed robotic systems (Murray, 2007, Parker et al., 98). However, the latest research on distributed robotic systems consider only a finite number of robots (Parker et al., 98), not as a biological swarm that consists of an arbitrarily large of elements. In fact, this is another motivation of this thesis. We aim to expand the size of the distributed robotic system, and design a scalable control strategy for a group comprising an arbitrary number of robots.

Depending on the coordination tasks, the agents may need to mobilize and be deployed to certain locations in a certain environment; maintain a particular pattern; move in a collective manner; track a leader or meet at a common position. The corresponding topics are called *mobilization* and *deployment*, *formation control*, *flocking*, *leader tracking* and *rendezvous*, respectively. A few controllers have been presented for these topics in the literature (Martínez et al., 2007c; Olfati-Saber et al., 2007; Ren et al., 2007; Murray, 2007).

In one of the pioneering works on swarms (Jadbabaie et al., 2003) by control engineers, Jadbabaie et al. presented a discrete kinematic model and a decentralized arithmetic averaging rule to achieve the heading *alignment* as in Vicsek's model (Vicsek et al., 1995). Some concepts from algebraic graph theory and matrix analysis are utilized to prove the convergence of all agents' headings. In other words, (Jadbabaie et al., 2003) analytically proves Vicsek's model. As a further theoretical extensions

of (Jadbabaie et al., 2003), Moreau (2005) shows that consensus is achieved asymptotically if the union of the “information exchange graph” is connected.

Gazi and Passino (2003) applied a continuous first-order kinematic model and presented a decentralized controller to study the *aggregation* of swarms in  $n$ -dimensional space. An explicit bound of the swarm size is also derived in (Gazi and Passino, 2003). The results in (Gazi and Passino, 2003) are extended to a class of virtual force functions in (Gazi and Passino, 2002). Their later work (Gazi and Passino, 2004) demonstrates the collective behavior of swarms moving in different environments. Liu and Passino (2004) used a second-order dynamic model and a different control law to study the stable *foraging* of swarms in certain noisy environments. However, all the controllers proposed in (Gazi and Passino, 2003, 2004; Liu and Passino, 2004) require every agent to have the knowledge of the global states of all other swarm members, which is apparently not the case in biological populations.

In (Olfati-Saber, 2006), Saber presented a classic *flocking* algorithm for a group of agents modelled by double integrators. The algorithm is partially motivated by Reynolds’ boid model (Reynolds, 1987). In (Reynolds, 1987), three flocking rules are presented to generate computer animations of collective motion of a group of points. The purposes of the three flocking rules are collision avoidance, velocity matching, and flock centering (Reynolds, 1987). Similarly, Saber’s controller (Olfati-Saber, 2006) has three components: one term uses the gradient of a collective potential to guarantee flock centering and collision avoidance among members, the second term applies velocity consensus to make agents’ velocities matched, and the third term incorporates navigational feedback to follow a leader. By considering an obstacle as a special type of agent, the algorithm is extended to have obstacle-avoidance capability. The algorithm is validated for flocking with undirected communication topologies.

A similar flocking controller is presented for swarms with fixed and dynamic topologies (Tanner et al., 2003a,b, 2007). The presented controller includes a ve-

locity consensus term to align the velocity of the agents, and another term to avoid collision and keep cohesion in the group. With the assumption of an always connected graph, the controller is proved to lead the velocities of all members converge to a same value for dynamic topology. However, the controllers (Jadbabaie et al., 2003; Tanner et al., 2003a,b, 2007) do not explicitly consider the environmental effects.

Some other flocking controllers that consider different scenarios or different conditions have been presented. In (Tanner, 2004), a flocking controller is presented for a team of mobile vehicle with nonholonomic constraint. In (Moshtagh and Jadbabaie, 2007; Moshtagh et al., 2009), a geodesic control law is presented for a group of kinematic nonholonomic agents to flock along a 2D circle (planar motion) or 3D sphere. It is demonstrated that flocking is possible even when the swarm topology changes over time if the joint connectivity of the topological graph is preserved. In (Lee and Spong, 2007), the authors considered the mass of the agents and studied the flocking of swarm that evolves on a slow-switching balanced communication graph.

One may notice that most of the flocking controllers presented in the literature have a common feature: the use of consensus terms. *Consensus* process is to reach an agreement regarding a certain information or state of interest of different agents.

For a group of mobile agents to flock together, each member needs to achieve velocity agreement with respect to its neighbors. Due to this requirement, it is intuitive to think of applying consensus components in the flocking controller. For the same reason, flocking is considered as a special application of consensus (Olfati-Saber et al., 2007; Ren et al., 2007).

*Consensus* is another research topic related to distributed coordination of multi-agent dynamical systems, which has attracted eruptive attentions in the last few years (Ren et al., 2007; Olfati-Saber et al., 2007). In a typical consensus problem, the states of the agents are in a general space, not necessarily related to a motion in Euclidean space such as position and velocity. Thus, the heading alignment in Vicsek's model

is a consensus (Vicsek et al., 1995; Jadbabaie et al., 2003; Olfati-Saber et al., 2007).

A classic consensus algorithm is presented in (Saber et al., 2003; Olfati-Saber and Murray, 2004) for a group of agents with fixed and switching topology and communication delays. Using a simple single integrator model for each agent, the authors discussed average consensus problems for directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time delay and fixed topology. Moreover, a disagreement function is introduced for disagreement dynamics of directed networks with switching topology. The undirected networks case is discussed in (Olfati-Saber and Murray, 2003).

Started from (Saber et al., 2003; Olfati-Saber and Murray, 2004), more and more results are produced on consensus problem by using either the first or the second order integrator model in continuous or discrete domain (Ren and Beard, 2005; Ren, 2007, 2008; Olfati-Saber et al., 2007; Ren et al., 2007; Hatano and Mesbahi, 2005; Bauso et al., 2007). Two recent surveys (Olfati-Saber et al., 2007; Ren et al., 2007) reports the major works on consensus in the past few years. In most of those results, the agents' states generally converge to an equilibrium with all equal components asymptotically or in finite time.

However, a close examination shows that the consensus terms may not be necessary for flocking. For example, in Saber's controller (Olfati-Saber, 2006), by selecting proper design parameters, the leader-following feedback term may guarantee each agent to stably track the leader; and since the leader is unique, then agent's velocity is converged to a common value (the leader's). Thus, the velocity consensus term in (Olfati-Saber, 2006) that is explicitly deployed to achieve velocity match may not be necessary.

Few works on consensus have considered unknown disturbances on the agent's state (Xiao et al., 2007; Bauso et al., 2007; Huang and Manton, 2007). Due to the presence of unknown disturbances, asymptotic convergence to equilibria with all

equal components is generally impossible. In (Xiao et al., 2007), a least mean square optimization method is used to choose the constant coefficients in the classic linear consensus rule. The long term consensus error is then minimized. Two other papers (Bauso et al., 2007; Huang and Manton, 2007) deal with the noise in a similar way by introducing a different weighting factor for the information in consensus algorithm. But they consider different conditions on the measurement noise, namely, unknown but bounded and stochastic (Bauso et al., 2007; Huang and Manton, 2007).

Even though the literature on consensus and flocking is now becoming expansive, and some authors did start to examine the consensus algorithm under disturbances, but to the best of our knowledge, there is still no much work studying the distributed motion coordination of swarms under disturbances. Due to the unknown disturbances, the relative distance between two paired neighbors may arbitrarily change in a way such that, in the worst case, some of the agents may lose sensing or communication links with any other swarm member at a certain step. Therefore, the condition of an always connected graph that is generally assumed for the swarm's topology in the literature may not hold. In this thesis, we are motivated to develop a different and robust control algorithm for the coordinated motion of the swarm under unknown disturbances.

Developing models and controllers for distributed motions is important not only because we can apply them in certain engineering applications, but also we may use the designed algorithms in “controlling” natural swarms (herds, schools, flocks) of animals. An interesting example for this is (Vaughan et al., 2000), where the authors develop a mobile robot that gathers a flock of ducks and maneuvers them safely to a specified goal position. They use a potential field model for flocking behavior and test different methods for generalized swarm patterns by simulations. Then they use a robot to control a group of real ducks and show that the behavior of the ducks in real world is similar to the one in simulations.

### 1.1.3 Other Literature

In optimization area, it is a challenging work to efficiently find a global or “best” local optimal solution for some highly complex questions that have a large number of variables or constraints. A few algorithms inspired by the natural swarms are proposed to solve these problems (Passino, 2002; Dorigo et al., 1999; Clerc and Kennedy, 2002; Kennedy and Eberhart, 2001).

The recently popular “ant algorithm” (Kennedy and Eberhart, 2001) is an optimization algorithm based on mimicking the social swarming behavior of ant colonies . The focus is on biomimicry for the solution of combinatorial optimization algorithms (e.g., shortest path algorithms). In (Passino, 2002), the author shows that chemotactic behaviors of *E. coli* coupled with evolutionary and “elimination/dispersal” events provide a non-gradient distributed and parallel optimization procedure that can be used for adaptive control and cooperative control problems. Also, the author used a similar characterization of an “attractant/repellent” profile to study the swarm behaviors as a distributed optimization method.

Another emerging evolutionary optimization algorithm, called “particle swarm optimization”, is described in (Clerc and Kennedy, 2002; Kennedy and Eberhart, 2001). The basic idea of such algorithm is based on the interaction of individuals in a population of particles that represent local searching agents. The aggregation behavior of the agents can lead to an agreement or a consensus among the mutual interactions of local optimum searching. The book (Kennedy and Eberhart, 2001) also discusses several different systems such as the operation of a brain that can be modelled as a swarm of interacting agents.

## 1.2 Contributions and Outline of the Thesis

To successfully implement a distributed coordination system in practice, many issues need to be addressed, such as designing a scalable control algorithm for the distributed motions of the agents, designing a reliable and power-efficient communication protocol and a computational-efficient information fusion algorithm. In this thesis, our study is focused on the study of the motion coordination of the swarm.

For different motion coordination tasks, different controllers are needed. In the literature, a few controllers have been presented for different topics of distributed motion coordinations, namely, deployment, formation control, flocking, rendezvous, and leader tracking. Two recent surveys report most of the up-to-date results (see Martínez et al., 2007c; Murray, 2007, and references therein). However, a systematic methodology or framework that can provide a general guideline for the controller design for a wide range of motion coordination problem is lacked.

In this thesis, we consider the general situation in that the swarm's topology is time-varying. To explicitly consider the effect of swarm's dynamic topology on the swarm's distributed motion, we use algebraic graphs to model the topologies of the swarm that manifest the neighborhood and the information-exchanging (communication or sensing) relations among members. We abstract the fundamental characterizations of the coordinated motions of both biological and engineering swarms, and present a systematic and generalized methodology to study the distributed motion of the swarm. We develop a novel and unified framework that can serve as a guideline for the control strategy design for different distributed motion coordinations of the swarm. We also review some typical control terms and their functionalities, which can be selected as components of a feasible and provably correct controller for a specific motion coordination task. To demonstrate the presented framework, we apply it to investigate and design control rules for two application scenarios of the distributed motion of the swarm, namely flocking of swarms in homogenous environments with

obstacle avoidance, and virtual leader tracking control of swarms. We also study the collective motion of swarm under unknown but bounded disturbances. Without any assumption on the graph connectivity, we show that the presented controllers can lead the swarm to have a collective motion with bounded error.

The contributions of the thesis include four aspects. First, we present a systematic and general methodology to study the distributed motions of both biological and engineering swarms. This methodology allows us to analyze the distributed motion of the swarm from two different levels: group-level and inter-individual-level, regardless of how complicated the motions of the agents are.

Second, based on the two-level view methodology, we develop a unified framework for the coordinated motion of the swarm, which can be used as a guideline for designing controller for a wide range of motion coordination tasks.

Third, by applying the presented methodology and framework, we design a novel consensus-term free controller for the leader-tracking control of swarm. Compared to the pre-existing controllers in the literature, the presented controller does not use any consensus term to achieve the collective motion of the swarm. Therefore, the condition of always connected graph, which is generally assumed in the literature, is relieved.

Fourth, we examine the group motion of the swarm under unknown uncertainties, which is not well studied in the literature. We show that in the presence of unknown and bounded disturbances, the desired motion coordination of the swarm can still be achieved.

The rest body of this thesis is organized as follows. We first briefly discuss the non-smooth analysis and the nonsmooth Lyapunov stability theories in Chapter 2, which play the role of the primary mathematical tool we use to analyze the swarm's coordinated motions. In Chapter 3, we present a simplified model for individual agents, and define several proximity graphs to represent the topologies of the neighborhood

and the information-exchanging (sensing or communication) relations among swarm members. Then, we illustrate the systematic two-level view methodology and present the unified framework for general coordinated motions of swarm. Some typical control terms and their functionalities are also reviewed in Chapter 3. In Chapter 4, we apply the presented framework as a guideline to design a decentralize controller that needs only local information to make the swarm asymptotically flock in homogeneous environments. By the presented controller, we also prove the correctness of the A/A/R models that are broadly used in the mathematical biology literature. In Chapter 5, we present a consensus-free controller for virtual leader tracking control of swarms. We consider two cases where the virtual leader has a constant and a time-varying velocities. In Chapter 6, we study the performance of the controllers in the presence of unknown but bounded disturbances. Without any assumption on the graph connectivity, we show that the presented controllers can lead the swarm to form a collective motion with bounded errors.

## Chapter 2

# Mathematical Tools for Analysis of Swarms

In a setting of distributed coordination and cooperative control of a large group of mobile agents, each agent normally needs to interact or exchange information with certain other members in order to achieve the designated group tasks. For example, for a team of two mobile agents equipped with different types of sensors, if both come close to a same scene, for a better understanding of the surrounding, they may need to share and fuse the sensing information acquired by individual onboard sensors. After the information exchanging and update, each agent may take different actions for the next step.

In general, such information-exchanging relation primarily depends on the relative distance between two agents. As the agents are in motion, the spacing of the swarm dynamically changes, so do the relations. These changes lead to discontinuity and nonsmoothness in the motion controller of the agents. Through this thesis, we use the nonsmooth analysis and the nonsmooth stability theories as the main mathematical tool to analyze the swarm's motion. In this chapter, we describe some fundamental concepts and results of the nonsmooth analysis and the nonsmooth stability theories.

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Even though the mainstream research on systems and control theory have been built in the realm of the smooth world, nonsmooth dynamical systems exist in a variety of science and engineering applications. For example, in the area of mechanics, the evolution of rigid bodies is often subject to velocity jumps and force discontinuities as a result of Coulomb friction or contact with some other objects in the environment.

Moreover, some challenging issues arising in smooth systems call for the investigation of nonsmooth theory. This is the case, for instance, in the stabilization of close-loop control systems, the well-known Brockett's theorem (Brockett, 1983) implies that many systems, even drift-less systems, cannot be stabilized by means of continuous state feedbacks. Nonsmooth feedback control may provide a better solution for this challenge. In fact, one nonsmooth feedback control approach has been widely used by the control engineers, namely, sliding mode control. By sliding mode control, one first defines a sliding surface on which the system dynamics restricted to this surface can be easily stabilized, then feedback controllers are developed to drive the system states toward the sliding surface. Note that the dynamics of the close-loop system is nonsmooth on the sliding surface.

A dynamical system is discontinuous (also nonsmooth) if in its dynamic model

$$\dot{x}(t) = f[x(t)], \tag{2.1}$$

$f(x)$  is discontinuous.

In control engineering, the reasons that lead a close-loop system to be nonsmooth may originate from the plant, the sensor or observer, or the control inputs. In this thesis, we consider that the discontinuity of the system is only from the controller.

For an autonomous dynamical system with a control input  $u(t)$ , its close-loop dynamics is

$$\dot{x}(t) = f(x, u), \tag{2.2}$$

in which  $u \in \mathbb{R}^m$  is the set of allowable control inputs. If the function of the controller

$$u = u[x(t)] \tag{2.3}$$

is discontinuous, then the close-loop system (2.2) is nonsmooth.

To analyze and design controller for nonsmooth control systems, the first question we need to address is to define a solution for the discontinuous ordinary differential equation as in (2.2).

## 2.1 Filippov Solution to Discontinuous ODE

For an ordinary differential equation with discontinuous right-hand side, a general specification of its solution is presented by Filippov (Filippov, 1988).

**Definition 2.1** (Filippov, 1988) Consider a finite dimensional differential equation of the form

$$\dot{x}(t) = f(x) \tag{2.4}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a measurable and essentially locally bounded, possibly discontinuous vector field. A vector function  $x(\cdot)$  is called a *Filippov solution* of (2.4) on  $[t_0, t_1]$  if  $x(\cdot)$  is absolutely continuous on  $[t_0, t_1]$  and for almost all  $t \in [t_0, t_1]$

$$\dot{x} \in K[f](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\Omega)=0} \overline{\text{co}}\{f(B(x, \delta)) - \Omega\} \tag{2.5}$$

where  $\mu$  denotes the usual Lebesgue measure in  $\mathbb{R}^n$ .

Alternatively, one can show (Paden and Sastry, 1987) that there exists a set  $\Omega_x$  of measure zero such that

$$K[f](x) \equiv \overline{\text{co}}\{\lim f(x_i, t) \mid x_i \rightarrow x, x_i \notin \Omega \cup \Omega_x\} \tag{2.6}$$

where  $\Omega$  is any set of measure zero.

**Proposition 2.1** (Filippov, 1988; Yakubovich et al., 2004) If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable and essentially locally bounded, that is, bounded on a bounded neighborhood of every point, excluding sets with zero measure, then for all  $x(0) \in \mathbb{R}^n$ , there exists a Filippov solution to (2.4) with the initial condition  $x(0)$ .

Moreover, if  $f$  is upper semicontinuous with nonempty, compact, convex values and locally bound, the existence of Filippov solution of (2.4) is guaranteed (Filippov, 1988).

**Proposition 2.2** (Filippov, 1988; Bacciotti and Ceragioli, 1999) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable and essentially locally bounded. If for all  $x \in \mathbb{R}^n$ , there exists a  $\varepsilon > 0$  such that  $f$  is essentially one-sided Lipschitz in  $B(x, \varepsilon)$ , then for all  $x(0) \in \mathbb{R}^n$ , there exists a unique Filippov solution to (2.4) with the initial condition  $x(0)$ .

See the definition of one-sided Lipschitz function in Appendix A.1.

Recall that the uniqueness condition of the solution of continuous ODE is that the vector field  $f(x)$  is locally Lipschitz at every  $x$ . Note that locally Lipschitz functions always satisfy the one-sided Lipschitz condition in a neighborhood of  $x$ , but the reverse is not true (Bacciotti and Ceragioli, 1999; Cortés, 2008).

## 2.2 Differential Inclusion

A differential inclusion (Aubin and Cellina, 1984; Dontchev and Lempio, 1992; Smirnov, 2002; Cellina, 2005) is a generalization of ordinary differential equation. It specifies a multi-valued (also called *set-valued*) map, rather than a single value as the classic ordinary differential equation (Aubin and Cellina, 1984).

Let  $F : \mathbb{R}^n \rightarrow B(\mathbb{R}^n)$  be a set-valued map. The differential inclusion associated with  $F$  is

$$\dot{x}(t) \in F[x(t)]. \tag{2.7}$$

$x_e$  is the equilibrium of the differential inclusion if  $0 \in F(x_e)$  for all  $t \in [0, \infty)$ .

**Definition 2.2** (Dontchev and Lempio, 1992; Cortés, 2008) A solution of the differential inclusion in (2.7), also called *Caratheodory solution*, is a map  $x : [t_0, t_1] \rightarrow \mathbb{R}^n$  such that  $x(\cdot)$  is absolutely continuous on the interval  $[t_0, t_1] \subset [0, \infty)$  and  $\dot{x}(t) \in F[x(t)]$  for almost all  $t \in [t_0, t_1]$ .

Differential inclusions can be used to understand and suitably interpret discontinuous ordinary differential equations. By replacing the differential equation

$$\dot{x}(t) = f(x)$$

as in (2.4) with a differential inclusion

$$\dot{x}(t) \in K[f](x),$$

in which  $K[f](x)$  is determined by (2.5), one can see that the Filippov solution of a discontinuous differential equation is nothing but a Caratheodory solution of the corresponding differential inclusion.

## 2.3 Clarke's Nonsmooth Analysis

**Definition 2.3** (Clarke, 1990) For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  which is Lipschitz near given  $x$ , the *generalized directional derivative* of  $f$  at  $x$  in the direction of  $w \in \mathbb{R}^n$ , denoted by  $f^\circ(x; w)$ , is defined as

$$f^\circ(x; w) \triangleq \limsup_{y \rightarrow x, h \downarrow 0} \frac{f(y + hw) - f(y)}{h}. \quad (2.8)$$

Note that the classic one-sided directional derivative (right directional derivative)  $f'(x; w)$  may not always exist, whereas the limit in the definition of generalized directional derivative  $f^\circ(x; w)$  is always well-defined. See the definition of the right directional derivative of a function in Appendix A.2.

If the right directional derivative of a function exists, it may be different from its generalized directional derivative. If they are equal, we call the function *regular* as defines in the following.

**Definition 2.4** (Clarke, 1990) A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be *regular* if for all  $w \in \mathbb{R}^n$ , the usual right directional derivative  $f'(x, w)$  exists, and  $f^\circ(x, w) = f'(x, w)$ .

Examples of regular functions include continuously differentiable functions and functions which can be written as the point-wise maximum of a set of smooth functions (Clarke, 1990). A locally Lipschitz function at  $x$  which is convex is also regular (Bacciotti and Ceragioli, 1999; Cortés, 2008).

From Rademacher's Theorem (Clarke, 1990), we know that a locally Lipschitz function is continuously differentiable almost everywhere in the sense of Lebesgue measure.

**Definition 2.5** (Clarke, 1990) Let function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz near given  $x$ , and  $X$  is a real Banach space. The *generalized gradient* of  $f$  at  $x$ , denoted by  $\partial f(x)$ , is the subset of  $X^*$  defined by

$$\partial f(x) \triangleq \{\xi \in X^* \mid f^\circ(x; w) \geq \langle \xi, w \rangle, \forall w \in X\}, \quad (2.9)$$

in which  $X^*$  is the dual space of continuous linear functionals on  $X$ .

For the finite dimensional case, we have the following convenient characteristics of the generalized gradient.

**Theorem 2.1** (Clarke, 1990; Bacciotti and Ceragioli, 1999; Yakubovich et al., 2004) Let function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be locally Lipschitz near  $x$ . Let  $\Omega_f$  be the set of points in  $\mathbb{R}^n$  at which  $f$  is not differentiable, and  $\Omega \in \mathbb{R}^n$  be any subset of measure zero. Then

$$\partial f(x) = \overline{\text{co}}\{\lim_{i \rightarrow \infty} \nabla f(x_i) : x_i \rightarrow x, x_i \notin \Omega \cup \Omega_f\}. \quad (2.10)$$

Note that the set  $\Omega$  in the above theorem can be arbitrarily chosen for a simple computation of  $\partial f$  as far as it has zero measure. The resulting  $\partial f$  is independent of the choice of  $\Omega$  (Cortés, 2008).

According to Theorem 2.1, we can see that the generalized gradient of  $f$  at  $x$  consists of all convex combinations of the possible limits of the gradient at all the neighboring points  $x_i$  where  $f$  is differentiable.

Note that the generalized gradient is generally a set of vectors. For a differentiable

function, its generalized gradient is a singleton that is equal to the classical gradient,  $\partial f(x) = \{\nabla f(x)\}$ .

Calculus associated with generalized gradient usually involves set inclusion operations. When function are regular, the inclusions turn to be equalities. One can refer to (Clarke, 1990; Cortés, 2008) for more calculus laws on general gradient.

For locally Lipschitz function, the generalized gradient and the generalized directional derivative are related by

$$f^\circ(x; w) = \max\{\xi \cdot w, \xi \in \partial f(x)\}. \quad (2.11)$$

## 2.4 Lyapunov Stability Theory for Nonsmooth Systems

In order to analyze the stability of discontinuous dynamical systems, similar to continuous case, one can also resort to a Lyapunov function. Before proceeding to present the main Lyapunov stability theorem for nonsmooth systems, we first introduce the nonsmooth case of the chain rule that can be used to derive the set-valued Lie derivative of a scalar function with respect to a discontinuous vector field.

**Theorem 2.2** (Shevitz and Paden, 1994) Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a Lipschitz and regular function. Let  $x(\cdot)$  be a Filippov solution to (2.4) on an interval containing  $t$ . Then  $V(x(t))$  is absolutely continuous,  $\frac{d}{dt}V(x(t))$  exists almost everywhere, and

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \check{V}(x), \quad (2.12)$$

where

$$\check{V}(x) = \bigcap_{\xi \in \partial V(x)} \xi^T \cdot K[f](x). \quad (2.13)$$

It can easily be shown that the global Lipschitz continuity requirement for  $V(x)$  in the above theorem can be relaxed to be local (Bacciotti and Ceragioli, 1999; Cortés, 2008).

In order to relate the above chain rule to the usual Lie derivative,  $\dot{\tilde{V}}(x)$  is alternatively denoted by  $\tilde{L}_f V(x)$  (Cortés, 2005), which represents the set-valued Lie derivative of scalar function  $V(x)$  along discontinuous vector field  $f(\cdot)$ . And using the locally Lipschitz condition, the chain rule is re-stated as follows.

**Theorem 2.3** (Cortés, 2005, 2008) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be measurable and essentially locally bounded, and  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a locally Lipschitz. Let  $x : [t_0, t_1] \rightarrow \mathbb{R}^n$  be a Filippov solution to (2.4) such that  $V(x(t))$  is regular for almost all  $t \in [t_0, t_1]$ . Then  $\frac{d}{dt}V(x(t))$  exists almost all  $t \in [t_0, t_1]$ , and

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \tilde{L}_f V(x), \quad (2.14)$$

where

$$\tilde{L}_f V(x) \triangleq \{a \in \mathbb{R} \mid \exists v \in K[f](x), \text{ such that } \xi^T \cdot v = a, \forall \xi \in \partial V(x)\}. \quad (2.15)$$

For  $\forall x \in \mathbb{R}^n$ ,  $\tilde{L}_f V(x)$  is a closed and bounded interval in  $\mathbb{R}$ , and possibly empty. If  $V(x)$  is continuously differentiable at  $x$ , then  $\tilde{L}_f V(x) = \{dV \cdot v \mid \forall v \in K[f](x)\}$ . If, in addition, the vector field  $f$  is smooth, then  $\tilde{L}_f V(x)$  turns to be the singleton  $\{L_f V(x)\}$ , which is the usual Lie derivative of  $V$  along the direction of  $f$ .

By using the above chain rule, the Lyapunov stability theory has been extended for discontinuous dynamical systems (Shevitz and Paden, 1994; Bacciotti and Ceragioli, 1999).

**Theorem 2.4** (Bacciotti and Ceragioli, 1999; Cortés, 2008) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a measurable and essentially locally bounded set-valued map. Let  $x_e$  be an equilibrium of the differential inclusion of

$$\dot{x} \in K[f]. \quad (2.16)$$

Let  $D$  be an open and connected set and  $D \subseteq \mathbb{R}^n$  with  $x_e \in D$ . Consider a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ . If  $V(x)$  is a positive definite, locally Lipschitz and regular on  $D$ , and for all  $x \in D$ ,

1. if  $\max \tilde{L}_f V(x) \leq 0$ , then  $x_e$  is a strongly stable equilibrium;
2. if  $\max \tilde{L}_f V(x) < 0$  for all  $x \in D \setminus \{x_e\}$ , then  $x_e$  is a strongly asymptotically stable equilibrium.

Similar to the smooth case, we have the nonsmooth version of LaSalle's invariance principle as follows.

**Theorem 2.5** (Shevitz and Paden, 1994) Let  $\Omega$  be a compact set such that every Filippov solution to the autonomous system  $\dot{x} = f(x)$ ,  $x(0) = x(t_0)$  starting in  $\Omega$  is unique and remains in  $\Omega$  for all  $t \geq t_0$ . Let  $V : \Omega \rightarrow \mathbb{R}$  be a time independent locally Lipschitz and regular function such that  $v \leq 0$  for all  $v \in \tilde{V}$  (if  $\tilde{V}$  is a empty set then this is trivially satisfied). Define  $S = \{x \in \Omega \mid 0 \in \tilde{V}\}$ . Then every trajectory in  $\Omega$  converges to the largest invariant set in the closure of  $S$ .

Bacciotti and Ceragioli extended the invariance principle to a more general result in which the requirement of the uniqueness of the Filippov solution is relieved (Bacciotti and Ceragioli, 1999). To save space, we do not include the discussion of it here. One can refer to (Bacciotti and Ceragioli, 1999) for details.

## Chapter 3

# A Unified Framework for Distributed Motion Coordination

A large group of mobile agents (e.g., mobile robots, mobile sensor nodes or unmanned aerial vehicles (UAVs)), which have computing, sensing, communication and mobility capabilities, can be of great use in a variety of application areas, such as environmental monitoring, battle-field surveillance and space exploration. In these applications, to achieve a coordinated motion is one of the fundamental and important missions for the team. Depending on the specific application, the agents may need to mobilize and be deployed to certain locations, maintain a particular pattern, move in a collective manner, track a leader or meet at a common location. Correspondingly, these coordinated motions are called *mobilization* and *deployment, formation control, flocking, leader tracking* and *rendezvous*, respectively.

The main challenge in designing control algorithms to achieve these collaborated motions is to obtain a decentralized and scalable solution to a large scale coordination problem with limited or constrained resources.

In the literature, many different controllers have been presented to address different distributed motion coordinations. Two recent surveys report most of the up-

to-date results (see Martínez et al., 2007c; Murray, 2007, and references therein). However, a systematic methodology or framework that can provide a general guideline for the controller design for a wide range of motion coordination problem is lacked.

In this chapter, we abstract the general characterizations of most of the coordinated motions of the swarm, present a systematic and generalized methodology for the control strategy design, and develop a novel and unified control framework for a general range of motion coordination tasks of the swarm.

### 3.1 Modelling of Individual Swarm Members

Before we proceed to present the framework, we first introduce two generalized models for the individual agents and the entire group in this and the next sections, respectively.

Consider the individuals in a swarm consisting of  $N$  mobile agents that move in an Euclidean space. Let  $r_i$  represent the position vector of the  $i$ th ( $i = 1, 2, \dots, N$ ) agent relative to a ground reference origin ( $r_i \in \mathbb{R}^n$ ,  $n = 1, 2$ , or  $3$ ). For each agent in the swarm, it can be modelled by

$$\begin{cases} \dot{r}_i = v_i, \\ \dot{v}_i = \frac{1}{m_i} F_i, \end{cases} \quad (3.1)$$

in which  $v_i$  is the velocity vector,  $m_i$  is the mass, and  $F_i$  is the driving force. For simplicity, we do not consider the dimension of each agent and treat it as a point mass through this thesis. We also assume that the mass of each agent is known. Note that in  $2D$  or  $3D$  space,  $r_i$ ,  $v_i$  and  $u_i$  are all defined as vectors.

If the mass is given, without loss of generality, we can assume a unit mass for each

agent. Then, the above model can be written as

$$\begin{cases} \dot{r}_i = v_i, \\ \dot{v}_i = u_i, \end{cases} \quad (3.2)$$

in which  $u_i$  is the control input. If  $u_i$  is available, with a given  $m_i$ , the driving force of each agent is

$$F_i = m_i u_i. \quad (3.3)$$

Define

$$r_{ij} = r_i - r_j \quad (3.4)$$

to represent the relative position vector from  $r_i$  to  $r_j$ . Let  $\|r_{ij}\|$  be the relative Euclidean distance between agents  $i$  and  $j$ .

Due to the advantages of scalability, robustness and fault tolerance, decentralized control strategies, rather than centralized ones, are generally deployed for the distributed cooperative control of a large scale system. Since the control input  $u_i$  is for individual agents, by designing a control law for  $u_i$ , one can obtain a decentralized solution for the distributed motion coordination of the swarm.

For different coordination tasks, different decentralized controllers with different forms are needed to achieve the desired coordinated motion of the entire group. In this thesis, we abstract the underlying nature of the coordinated motion of the swarm, develop a unified control framework for the general collaborated motions of swarm, and then apply it into certain motion coordination tasks.

**Remark 3.1** *For some coordinated motions of the swarm, it may be sufficient to use a kinematic model rather than the above dynamic model for each agent. For example, in the deployment of a mobile sensor network, each sensor node needs to be deployed at certain location for a certain objective such as obtaining the maximum coverage. Since only the position states of the nodes are concerned, and if only a velocity-level*

input is involved for each node, then one can use the following first-order model

$$\dot{r}_i = u_i. \tag{3.5}$$

For generality, in this thesis, we use the second-order model for individual agents (3.2). Without much modification, the proposed framework can be applied for agents that have a first-order model.

## 3.2 Modelling of Swarms with Dynamic Topology

When the group of mobile agents move together in a coordinated fashion, they need to interact or exchange state information with each other. Each member may have interaction and information-exchanging relations with a fixed or dynamic set of swarm mates. In this thesis, we study the general situation in which each agent interacts and exchanges information with a dynamic, not fixed, set of neighboring members depending on the relative distance between them. As the spacing among agents evolves with time, the neighborhood relations may dynamically change, so do the interaction or the information-exchanging relations. To explicitly consider the effect of such dynamic change on the group behavior, we present a mathematical model for the topologies of the relations among swarm members.

For a swarm of mobile agents moving together, we can consider the entire group as an interaction and information-exchanging network. As graph theory has been widely applied to model and analyze network systems (Biggs, 1993; Godsil and Royle, 2001), the topology of the swarm can also be represented by an algebraic graph.

Without considering any effect due to the dimension of individual agents, we can simply model each swarm member as the vertex point of a graph, while the relations among them can be modelled by the edges of the graph. Depending on the metrics used to study the relations, the graph edges can be weighted or unweighed. Also,

depending on the acting direction of the relations, the edges are either directed or undirected, accordingly, the graph is directed (called *digraph*) or undirected (Biggs, 1993; Godsil and Royle, 2001).

Through this thesis, we consider that the agents interact and share information with certain other members to achieve the desired coordinated motion. Thus, the graph edges embody the interaction and the information-exchanging relations within the swarm. Moreover, unweighed edges are used to describe the relations among swarm members.

Moreover, as discussed in previous chapters, we assume that each agent interacts and exchanges information with its neighbors that are located in a pre-defined neighborhood range. Other members beyond this neighborhood range will not be considered to have relations with the agent. In other words, the interaction and the information-exchanging relations among agents are based on their neighborhood relations. Thus, the graph edges also represent the neighborhood relations of the swarm.

To be clarified, we consider that the information-exchanging relation between two agents is different from the interaction relation. The later represents the relation other than information-exchanging, e.g., mutual interaction force (attraction and repulsion), or role assignment relation.

Note that both the neighborhood relation and the interaction relation between two agents, from their essence, are reciprocal, i.e., bi-directional. For example, two neighboring agents are the neighbors of each other. Hence, if the graph edges represent the neighborhood or the interaction relations of the swarm, then the graph is undirected.

For the information-exchanging relations among agents, for simplicity, we assume that they are bi-directional. Thus, if the graph depicts the information-exchanging topology of the swarm, then it is undirected. However, note that in some application

scenarios, because of certain physical constraints, the information exchange flows might be directional. For example, in a group of autonomous helicopters, if the helicopters use vision sensors to detect each other's states, because the vision sensors cannot be mounted toward to the up, then it may occur that some helicopters in a higher altitude can see the ones in a lower altitude, but not in the reverse way.

Based on the above discussion, we define the following topological graphs to represent the topologies of the neighborhood, interaction and information-exchanging relations among the agents.

**Definition 3.1 (Swarm's Neighborhood Topology Graph)** *The topological graph of the neighborhood relations among the members of a swarm is an undirected graph, denoted by  $\mathcal{G}_N = \{\mathcal{V}, \mathcal{E}\}$ , consisting of:*

1. a set of vertices,  $\mathcal{V} = \{1, \dots, N\}$ , indexed by the agents in the swarm;
2. a set of edges,  $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : \|r_{ij}\| \leq d_1\}$ , in which the undirected edges represent the neighborhood relations between unordered pairs of vertices.

It is clear to see that in the above definition,  $d_1$  determines the neighborhood range for every agent. We name it as *neighborhood defining distance*.

**Definition 3.2 (Swarm's Interaction Topology Graph)** *The topological graph of the interaction relations among the members of a swarm is an undirected graph, denoted by  $\mathcal{G}_I = \{\mathcal{V}, \mathcal{E}\}$ , consisting of:*

1. a set of vertices,  $\mathcal{V} = \{1, \dots, N\}$ , indexed by the agents in the swarm;
2. a set of edges,  $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : \|r_{ij}\| \leq d_1\}$ , in which the undirected edges represent the interaction relations between unordered pairs of vertices.

Note that in the above definition, the maximum range of interaction is the same as the *neighborhood defining distance*  $d_1$ . This is because that each agent is assumed to

interact only with its neighbors, and all of the agents beyond the neighborhood range will not be considered to have interaction with. Therefore, the interaction relations and the neighborhood relations of the swarm share a same topology. By using the same  $d_1$  for the neighborhood defining distance and the maximum range of interaction, the neighborhood graph  $\mathcal{G}_N$  and the interaction graph  $\mathcal{G}_I$  become equivalent.

**Definition 3.3 (Swarm’s Information-exchanging Topology Graph)** *The topological graph of the information-exchanging relations among the members of a swarm is an undirected graph, denoted by  $\mathcal{G}_{IE} = \{\mathcal{V}, \mathcal{E}\}$ , in which  $\mathcal{V} = \{1, \dots, N\}$  is the set of vertices indexed by the agents in the swarm;  $\mathcal{E} = \{\mathcal{E}_{ij}\}$  is the edge set and  $\mathcal{E}_{ij}$  exists between two different vertices  $i$  and  $j$  only if there exists information-exchanging relation between them.*

Note that the agents in a swarm can use either communication channels or sensing devices to get each other’s states. According to the mechanism on how the state information is exchanged among members, the edges of the information-exchanging graph  $\mathcal{G}_{IE}$  embody either communication or sensing relations of the agents. Thereupon, we further define the sensing and the communication topology graphs as follows.

**Definition 3.4 (Swarm’s Sensing Topology Graph)** *The topological graph of the sensing relations among the members of a swarm is an undirected graph, denoted by  $\mathcal{G}_S = \{\mathcal{V}, \mathcal{E}\}$ , consisting of:*

1. a set of vertices,  $\mathcal{V} = \{1, \dots, N\}$ , indexed by the agents in the swarm;
2. a set of edges,  $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : \|r_{ij}\| \leq d_s\}$ , in which the undirected edges represent the sensing relations between unordered pairs of vertices.

In the above definition,  $d_s$  is the maximum sensing range that determines whether two agents can sense each other.

**Definition 3.5 (Swarm’s Communication Topology Graph)** *The topological graph of the communication relations among the members of a swarm is an undirected graph, denoted by  $\mathcal{G}_C = \{\mathcal{V}, \mathcal{E}\}$ , consisting of:*

1. *a set of vertices,  $\mathcal{V} = \{1, \dots, N\}$ , indexed by the agents in the swarm;*
2. *a set of edges,  $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : \|r_{ij}\| \leq d_c\}$ , in which the undirected edges represent the communication relations between unordered pairs of vertices.*

In the above definition,  $d_c$  is the maximum communication range of the agents.

Note that in the preceding Definition 3.4 and 3.5, we use a same value for the maximum sensing/communication range of all agents. In practice, different agents may have different types of sensors or different communication powers. Thus, for each agent, the maximum range of sensing or communication link may vary. However, for simplicity, we assume that all the agents in a swarm have the same maximum sensing/communication range.

In addition, considering that the working ranges of most sensors, in particular the ranging sensors such as infrared and sonar sensors, are not so long as most communication radios, thus, it is usual that  $d_s \leq d_c$ .

Note that as observed in some real fish schools in the nature (Parrish et al., 2002), some fishes may be able to sense some other fishes far beyond, but they only interact with certain nearby neighbors. That is, the agents may not interact with all the agents that they can sense or communicated with. Therefore, in general, we consider that  $d_s$  and  $d_c$  are larger than  $d_1$ , thus, we have

$$\mathcal{G}_N = \mathcal{G}_I \subseteq \mathcal{G}_S \subseteq \mathcal{G}_C. \tag{3.6}$$

Note that in real applications, the agents may use only one of the mechanisms, communication channels or sensors, to exchange the state information. Thus, the aforementioned sensing and communication graphs may not co-exist for a swarm.

For simplicity, define

$$\mathcal{G} \triangleq \mathcal{G}_N \cup \mathcal{G}_I \cup \mathcal{G}_S \cup \mathcal{G}_C \quad (3.7)$$

to denote the topological graph of a swarm in a general sense. For a swarm with dynamic topology,  $\mathcal{G}$ ,  $\mathcal{G}_N$ ,  $\mathcal{G}_I$ ,  $\mathcal{G}_S$  and  $\mathcal{G}_C$  are all time-varying.

Note that all of the graphs defined above are proximity graphs (Biggs, 1993; Godsil and Royle, 2001; Olfati-Saber et al., 2007; Martínez et al., 2007c). Proximity graph is a graph in that the vertices represent different entities in an Euclidean space, and the edges depend on the relative Euclidean distance between the vertices. There are several proximity graphs defined in graph theory and computational geometry, such as *Gabriel Graph* (GG), *Nearest Neighbor Graph* (NNG), *k-nearest neighbor graph* (k-NNG), and *farthest neighbor graph* (FNG). However, the topological graphs defined in the thesis are distinct from them. To make it clarified, we provide brief definitions of some of the proximity graphs commonly used in graph theory as follows.

- **Nearest Neighbor graph (NNG):** The nearest neighbor graph of a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is that all of its edges  $(p, q) \in \mathcal{E}$  exist such that there is no vertex  $w \in \mathcal{V}$  where  $\|pw\| < \|pq\|$  and  $\|qw\| < \|pq\|$ .

Note that the nearest neighbor graph is not symmetric, i.e., if  $p$  is the nearest neighbor of  $q$ , but  $q$  is not necessary the nearest neighbor of  $p$ .

- **k-nearest Neighbor Graph (k-NNG):** The k-nearest neighbor graph is a graph in which two vertices  $p$  and  $q$  are connected by an edge, if  $\|pq\|$  is among the  $k$ -th smallest distances from  $p$  to all other vertices in  $\mathcal{V}$ .

Clearly, the *NNG* is a special case of the *k-NNG*, namely it is the *1-NNG*.

- **Farthest Neighbor Graph (FNG):** For a graph with  $N$  vertices, a  $(N-1)$  nearest neighbor graph is called the farthest neighbor graph.

- **Gabriel Graph (GG):** A Gabriel graph of a set of vertices  $\mathcal{E}$  has an edge between vertices  $p$  and  $q$  if and only if  $\|pq\| \leq \|pw\| + \|qw\|$  for  $\forall w \in \mathcal{E}$ .

It is clear that by the defined  $\mathcal{G}_N$  in Definition 3.1, each agent can have neighborhood relations with multiple agents, rather than just a single nearest agent as in the nearest neighbor graph (NNG). Moreover, it is not hard to see that  $NNG \subseteq GG \subseteq \mathcal{G}_N$ .

Define

$$\mathbb{N}_i \triangleq \{j : (i, j) \in \mathcal{E}, \mathcal{E} \subset \mathcal{G}_N\} \quad (3.8)$$

to represent the set of the  $i$ th agent's neighbors. According to Definition 3.1, it is equivalent to

$$\mathbb{N}_i = \{j : \|r_{ij}\| \leq d_1, j \neq i, j = 1, \dots, N\}. \quad (3.9)$$

Let  $r = [r_1^T, r_2^T, \dots, r_N^T]^T$  and  $v = [v_1^T, v_2^T, \dots, v_N^T]^T$  represent the position and the velocity vectors of the entire swarm, respectively.

Define

$$\bar{r} = \frac{1}{N} \sum_{i=1}^N r_i \quad (3.10)$$

to represent the position of the geometric center of the swarm, and define

$$\bar{v} = \frac{1}{N} \sum_{i=1}^N v_i \quad (3.11)$$

to represent the velocity of the geometric center. Clearly,  $\bar{v}$  is also the average velocity of all swarm members.

The swarming phenomena in the nature have many remarkable features that have invoked intense research interests (Parrish et al., 2002; Parrish and Edelstein-kesket, 1999). One of them is the collective motion of the swarm, in which all the agents have a common linear or angular velocity. The former and the later are called *parallel-type*

and *circular-type* collective motions, respectively (Couzin et al., 2002; Paley et al., 2007). When the entire swarm moves in parallel collectively, all swarm members have the same velocity. Denote this same velocity by  $v_0$ . It is easy to see that  $\bar{v} = v_0$  as the parallel-type of collective motion forms. In this thesis, the presented control framework can provide a solid and thorough explanation for the parallel-type of collective motions in natural swarms.

In the engineering side, for specific coordination tasks, the collaborated motion of the swarm normally requires both the states of individual agents ( $r_i$  and  $v_i$ ), and of the entire group ( $r$ ,  $v$ ,  $\bar{r}$  or  $\bar{v}$ ) to achieve certain desired values. In this thesis, we develop a unified framework for the control input  $u_i$  of general distributed motions of the swarm.

### 3.3 A Framework for Distributed Motion Coordination

In this section, we abstract and generalize the underlying characteristics of most coordinated motions of swarm, and present a systematic methodology to develop a unified framework, which can serve as a guideline to design decentralizes controllers for different coordination tasks of the swarm.

In a real application scenario, according to the specific task, the swarm members may need to be deployed to certain locations (*depolyment*), maintain a desired pattern (*formation control*), move in a collective manner (*flocking*), track a leader (*leader tracking control*), or rendezvous at a certain place (*rendezvous*). In the literature, specific controllers are proposed for the corresponding coordination tasks (see Martínez et al., 2007c; Murray, 2007, and references therein). Some of the approaches to design decentralized controllers are also reviewed in the literature (Martínez et al., 2007c;

Bullo et al., 2009). However, a systematic methodology and a general framework for the control strategy design are missed.

By a close examine of most of the coordinated motions of swarms in both the nature and the engineering applications, we can find an abstract and general feature of the system that allows us to develop a unified motion framework for the swarm.

During the steady stage of the coordinated motions of the swarm, either a biological group or an engineering team, no matter what the specific motions are, we can always view the behavior of the entire system from two different stand points: group-level and inter-individual-level. When a coordination motion is formed, the entire swarm always has a certain group-level behavior, and at the same time, in the inter-individual level, certain motion relations may exist among the members. For example, in the nature, when the migrating birds fly together as a group to a common destination, in the inter-individual level, they may also keep a certain pattern and occasionally switch the role of leading for saving energy (Berthold et al., 2001; Cutts and Speakman, 1994). In the engineering side, for instance, in the formation tracking control of swarms, the entire group shows the behavior of tracking a certain reference trajectory in the final phase, while the individual agents keep a certain relative distance from each other to maintain a desired formation.

Note that the aforementioned methodology of two-level views of the swarm is not the same as any approach that only studies the motions of the group and individual agents for the controller design. To illustrate the difference between these two methods, let us consider the rendezvous of swarm. In a distributed rendezvous problem, all of the agents are required to meet at a same location from their initial positions. If the rendezvous mission is accomplished, the states of individual agents are that  $r_i = r^*$  and  $v_i = 0$ , where  $r^*$  denotes the meeting location. If only studying the motions of the individuals and the group, one may have no clue to design the controller because the individual motions are the same as the group's. However, in this case, we can still

apply the two-level view methodology, and examine the final stage of the entire system from the group and the inter-individual levels: in the group-level, all the agents have the states that  $r_i = r^*$  and  $v_i = 0$ ; and in the inter-individual-level, we can consider it as that the relative distance between any two agents is zero ( $\|r_{ij}\| = 0$ ), or their positions are equal ( $r_i = r_j$ ). After revealing this feature of the system, we can design a decentralized controller ( $u_i$ ) that includes two components to achieve the rendezvous task: one term uses the traditional velocity negative feedback term to drive the agent's velocity to be zero ( $v_i \rightarrow 0$ ); and the other term can use a distance negative feedback term to make  $\|r_{ij}\| \rightarrow 0$ , or use a position consensus term to make the positions of agents become the same. By exploring these investigation, it is not hard to come up with a decentralized controller as  $u_i = -k_1 r_i - k_2 v_i - \sum_{j \in \mathbb{N}_i} (r_i - r_j)$ . In fact this is one of the classical rendezvous controller presented in the literature (Ren et al., 2007; Martínez et al., 2007c; Cortés et al., 2004a).

This two-level view methodology of the swarm is the core *clavis* that enables us to develop a unified framework for the controller design of the swarm. By studying the behaviors of the entire system from these two sub-levels, we can design a decentralized control law that includes several different components for the behaviors of the system in different levels, subsequently, make the swarm achieve the desired coordination objectives.

For a better description of the essence of the two-level view methodology, by referring to the field of economics theory, we also use the words of ‘macro-level’ and ‘micro-level’ to denote the group-level and inter-individual-level, respectively.

Based on the above discussion, we propose a unified decentralized motion framework for the swarm as follows

$$u_i = \underbrace{\Phi_1^i}_{\text{macro-level term}} + \underbrace{\Phi_2^i}_{\text{micro-level term}}, \quad (3.12)$$

in which the macro-level term  $\Phi_1^i$  is primarily targeted to the group-level behavior of the swarm, and the micro-level term  $\Phi_2^i$  mainly exerts the effect on the states of

the agents in the inter-individual-level. By applying the two-level view methodology and this framework, one can design specific terms for  $\Phi_1^i$  and  $\Phi_2^i$  to achieve a desired coordinated motion of the swarm.

Note that the two terms  $\Phi_1^i$  and  $\Phi_2^i$  in the framework (3.12) are only a generalized representation of the components of the final controller for the coordination task. It can serve as a guideline to design decentralized controllers for different coordinated motions of the swarm. In the final controller, it may include a few more or less specific terms for the desired task. Moreover, in some situations, in particular when the group-level task is nothing more than to achieve a certain objective in the individual-level of the swarm, one may need only one term in the controller. For example, in the classical velocity or position consensus control problem, since the team's group-level target is to only achieve state consensus between the agents, then a single component as  $\Phi_2^i$ , e.g., a consensus term, is sufficient (Jadbabaie et al., 2003; Olfati-Saber and Murray, 2004; Ren and Beard, 2005).

Note that in most of the cooperation control of swarm, the individual behaviors and the group's behavior are tightly associated. In fact, by decentralized control approaches, the group-level motion of the swarm is generally the result of the motions of individual agents. Hence, in some cases, the individual motions may be the same as the group's. As shown in the above rendezvous example, this does not prevent us from using the presented methodology and framework to design control algorithms.

Note that to obtain a decentralized control strategy, the macro-level control term  $\Phi_1^i$  should not be centralized. This requires that  $\Phi_1^i$  should not use any set of global states of all of the agents.

In fact, the presented two-level view methodology has been implicitly exercised by many engineers to design controllers for distributed coordinations of swarms. From the aforementioned rendezvous example, one can see that in a certain extent, some controllers presented in the literature are just special cases of the proposed frame-

work. To save space, we do not discuss them individually here. To demonstrate the effectiveness of the presented framework, we apply it into two motion coordination problems in the following chapters, namely, flocking of swarms in homogeneous environments and virtual-leader tracking control of swarms.

### 3.4 Optional Terms for the Framework

As mentioned in the preceding section, the presented framework can serve as a guideline to design a decentralized controller for the desired coordination task of the swarm. Different tasks may need different forms of the macro-level  $\Phi_1^i$  and micro-level  $\Phi_2^i$ . To gain an insight on how to select proper components for  $\Phi_1^i$  and  $\Phi_2^i$ , we review some commonly used control terms and their functionalities for different coordination tasks studied in the literature. We also propose a new term with certain effects on the swarm, which can be used for certain tasks.

As one may notice, by the presented framework, the desired coordinated motion of the swarm can be described by two sub-tasks in the group and the inter-individual levels. With the control tasks handy, we can design different terms for  $\Phi_1^i$  and  $\Phi_2^i$  and achieve the requested goal.

In control engineering, people may propose different controllers with different mathematical terms for a same control task. From the physical sense, each term in the controller may have its own underlying effect on the system. Bearing in mind of the functionalities of different terms, one can efficiently design proper terms to attain a feasible and provably correct controller in an objective-oriented way.

By reviewing the controllers presented for different coordination tasks in the literature, we list some typical micro-level and macro-level terms and their functionalities in the following Table 3.1 and 3.2.

For the group-level terms  $\Phi_1^i$ , one of the common approaches used in the literature is that to encode the desired group-level motion of the swarm as an optimization

Table 3.1: Optional terms of  $\Phi_1^i$ 

Terms	Functionality
$-kr_i; -kv_i$	To stabilize the states
$-k(r_i - r_d); -k(v_i - v_d)$	To track a reference signal
$-k(r_i - r_d) - k_2(v_i - v_d) + \dot{v}_d$	To track a reference velocity
$-\nabla_{r_i} J(r); -\nabla_{v_i} J(v)$	To achieve an objective function $J(\cdot)$

objective function  $J(\cdot)$  with a maximum or minimum corresponding to the equilibrium state of the desired coordinated motion, then use the descent gradient term ( $-\nabla_{r_i} J(r)$  or  $-\nabla_{v_i} J(v)$ ) to achieve the optimum. As an example, let us keep reviewing some classic works on the rendezvous problem. In view of that rendezvous implies the relative distances among the agents become zero, an intuitive choice of the objective function is

$$J(r) = \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \|r_i - r_j\|. \quad (3.13)$$

Apparently,  $J(r)$  has a global minimum if and only if  $r_i = r_j$ , which stands for the final stage when the rendezvous mission is accomplished. Hence, one just need to use the term of  $-\nabla_{r_i} J(r)$  in the controller, then the group-level goal can be achieved. This example demonstrates that in a certain sense, one can formulate the distributed motion coordination problem as a distributed optimization problem. However, it is well known that solving a distributed optimization is computationally demanding and not a trivial work at all.

When defining the objective function for the distributed motion coordination problem, one may easily fall in a pitfall that is to use the global states of all agents. This will cause the controller to be centralized. For the above rendezvous task, a better candidate of the objective function will be

$$J(r) = \min\{\|r_i - r_j\|, i \neq j, i, j \in 1, \dots, N\}. \quad (3.14)$$

Another problem of using optimization formulation is that local optimum may cause the system not to achieve the desired critical states.

In the two distributed coordination tasks studied in the following chapters, we do not use the optimization based control terms for the macro-level  $\Phi_1^i$ .

Table 3.2: Optional terms of  $\Phi_2^i$

Terms	Functionality
$-\sum_{j \in \mathbb{N}_i} (r_i - r_j)$	Position consensus
$-\sum_{j \in \mathbb{N}_i} (v_i - v_j)$	Velocity consensus
$\nabla_{r_i} U_i = \sum_{j \in \mathbb{N}_i} \nabla_{r_i} U_{ij}$	Gradient force from mutual or structural potential functions
$\sum_{j \in \mathbb{N}_i} (f_r - f_a) \frac{r_{ij}}{\ r_{ij}\ }$	Mutual virtual attraction and repulsion (A/R) forces *

\* Proposed terms in this thesis.

In Table 3.2, we list some of the control terms that can be used for the inter-individual level motions of the agents. The effect of each term is well self-defined by its mathematical form and briefly mentioned in the table. To save space, we do not discuss them individually.

In Table 3.2, an term worthy to be pointed out is the gradient force from a structural potential function of the swarm. Structural potential functions are broadly used in the literature for formation control of multi-vehicle and multi-robot systems (Olfati-Saber and Murray, 2002; Ögren et al., 2004; Olfati-Saber, 2006; Do, 2007, 2008). The structural potential function of a swarm that maintains a certain formation is a positive definite function determined by the set of algebraic structural constraints of the geometrical pattern of the group (Olfati-Saber and Murray, 2002). The formation of the swarm can be defined by a unique structural potential function. In a certain point, the structural potential function of a group of agents can be viewed

as an optimization objective function that has a global minimum corresponding to the depicted formation. Therefore, by using the negative gradient of the structural potential function, one can obtain the desired formation of the group (Olfati-Saber and Murray, 2002; Ögren et al., 2004; Olfati-Saber, 2006; Do, 2007, 2008).

Note that in Table 3.2, we proposed a new term  $(\sum_{j \in \mathbb{N}_i} (f_r - f_a) \frac{r_{ij}}{\|r_{ij}\|})$ , which will be used in the following chapters. By this term, we assume that in the inter-individual level, each agent interacts with its neighbors through certain attraction and repulsion forces. The attraction/repulsion (A/R) forces drag or push agents towards or away from each other, and subsequently affect individual velocity and position. Since these forces may not really exist in physical systems, we name them *virtual forces*.

In fact, the hypothesis of virtual forces in biological swarms (e.g., fish schools and bird flocks) has been widely adopted in the mathematical biology community for decades (see Grünbaum and Okubo, 1994; Mogilner and Edelstein-keshet, 1999; Mogilner et al., 2003; Niwa, 1994, 1996; Holmes et al., 1994; Couzin et al., 2002; Parrish et al., 2002; Adiou et al., 2003, and references therein). It is generally assumed that the short-distance repulsion force prevents collision among members and the long-distance attraction keeps the group cohesive.

Note that many different functions have been presented in the literature for the virtual forces (Mogilner and Edelstein-keshet, 1999; Niwa, 1996; Viscido et al., 2004; Couzin et al., 2002). Additionally, many efforts have been made to propose a more “reasonable” or realistic set of functions for the A/R interaction forces by comparing the simulation results with experimental or observational data in real animal groups (Viscido et al., 2004; Huth and Wissel, 1994; Kunz and Hemelrijk, 2003). In this thesis, we show that the presented motion framework works for a general range of A/R forces, thus, it avoids the problem of looking for a set “better” of functions. In other words, we prove that the A/R models commonly used in the mathematical biology literature can lead to collective behaviors of the swarm. We will illustrate

the use of the virtual A/R forces for the micro-level term  $\Phi_2^i$  of the framework in the next chapters.

**Remark 3.2** *Note that because of the difference of information availability, driving forces or communication powers among members, the agents in a swarm may not be homogeneous. For example, for a leader-tracking swarm, due to limited communication range and information routing protocol, the leader's states may not be available for some members, particularly when the spatial scale of the group is large. In such scenario, a heterogeneous controller needs to be designed for different individuals.*

**Remark 3.3** *In this thesis, our study is focused on the coordinated motion of the swarm. Hence, the states  $r_i$  and  $v_i$  in the model of individual agents (3.2) are the metrics of the Euclidean space, i.e., position and velocity.*

*As mentioned in the first chapter (Introduction), there are, in fact, many other types of distributed coordination problems in science and engineering fields, such as distributed data fusion or sensor fusion. By considering the system's states in a general state space, rather than in Euclidean space, one can extend the use of the presented framework into some other distributed coordination topics.*

## Chapter 4

# Flocking of Swarms in Homogeneous Environments

In the last chapter, we present a systematic methodology to study the behavior of a swarm of mobile agents, and develop a unified framework for the controller design of the distributed and coordinated motion of the swarm. To demonstrate its effectiveness, we apply the framework to design scalable controllers for two different application scenarios in the rest of this thesis.

One of the important coordinated motions of swarms that have invoked intense interests in control engineers is flocking. *Flocking*, originally, refers to the behavior exhibited when a group of birds, called a flock, are in flight or foraging. It is paralleled with the schooling behavior of fish and the swarming behavior of insects in the nature. Through this thesis, *flocking* of swarms means that a large number of agents move together in a collective fashion. It is also interchangeable with *swarming* and *schooling*.

Depending on the specific application, the collective motion of the agents could be with formation constraints (Ögren et al., 2004; Olfati-Saber, 2006; Tanner et al., 2007), or without formation constraints (Jadbabaie et al., 2003; Moshtagh and Jad-

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babaie, 2007). The former is also called *formation maneuvering* or *formation tracking* (Ren et al., 2007).

In the literature, some controllers have been presented for the flocking control of swarms. Jadbabaie et al. (2003) used a discrete kinematic model and an arithmetic averaging rule to make agents move in a common direction. However, the agents studied there are assumed to have a constant and same velocity. In their later work (Tanner et al., 2003a, 2007), a continuous dynamic model and a decentralized controller are presented for flocking of swarms with dynamic topology. The controllers in (Tanner et al., 2003a, 2007) include a velocity consensus term to align the velocity of the agents, and another term to avoid collision and keep cohesion in the group. With the assumption of connected graph, the controller is proved to make the velocities of all members converge to a same value. However, the environmental effect is not explicitly considered in (Jadbabaie et al., 2003; Tanner et al., 2003a, 2007). Liu and Passino (2004) used a second-order dynamic model and present a controller for the collective foraging of swarms in certain noisy environments. However, the controller requires each member to know the global states of all other swarm mates (Liu and Passino, 2004).

In (Olfati-Saber, 2006), the author presented a decentralized flocking algorithm for a group of agents to track a virtual leader. The controller (Olfati-Saber, 2006) has three components: one term uses the gradient of a collective potential to guarantee flock centering and collision avoidance among members, the second term applies velocity consensus to make agents' velocities matched, and the third term incorporates navigational feedback to follow a leader. In a certain extent, Saber's controller (Olfati-Saber, 2006) is an mathematical implementation of Reynolds' boid model (Reynolds, 1987). In the next chapter, we will study virtual leader tracking control of swarms and present a novel consensus-term free controller.

In this chapter, we study the flocking of swarms in homogeneous environments

without any formation requirement. The environmental potential is assumed to have homogeneous effects on all agents. We first apply the two-level view methodology presented in the last chapter to develop a new decentralized controller for the agents. The controller uses virtual attraction/repulsion (A/R) forces as the micro-level term of the framework. We assume that during the motion of the swarm, each member senses and interacts with its neighbors via virtual repulsion and attraction (A/R) forces, while perceiving and following the gradient force of the environment. The controller needs only local information to achieve the collective motion of the group. By the mathematical tool of nonsmooth stability analysis, we show that the controller enables that the swarm achieves the collective motion of flocking and no collision occurs among swarm members.

Moreover, by considering virtual attraction/alignment/repulsion (A/A/R) forces, we expand the analysis to the A/A/R models that are commonly used in the mathematical biology literature. In addition, by incorporating obstacles within the environmental potentials, we show that the presented controller leads the swarm to flock together while avoiding obstacles in the environment. Several sets of simulation results are included to demonstrate the controller.

## 4.1 A Virtual Forces Based Controller

In this section, we apply the framework presented in the last chapter to develop a new decentralized controller for swarms to flock in a homogeneous environment. The controller is based on virtual attraction/repulsion forces. We illustrate the use of the virtual A/R forces for the inter-individual-level motion of the swarm. We prove that the presented controller enables that the velocities of all agents asymptotically converge to a common value (their average), and the spacings among neighboring agents remain unchanging.

When a swarm of mobile agents flocks in a certain environment, in the inter-

individual-level, each agent needs to keep a certain distance from its neighbors to make the group cohesive and collision free among agents. We assume that certain virtual attraction/repulsion forces exist between paired agents to play such role. The virtual attraction and repulsion forces drag or push agents towards or away from each other, and subsequently affect individual velocity and position.

The hypothesis of virtual forces in social creatures (e.g., fish schools and bird flocks) has been widely accepted in the mathematical biology community for decades (see Grünbaum and Okubo, 1994; Mogilner and Edelstein-keshet, 1999; Mogilner et al., 2003; Niwa, 1994, 1996; Holmes et al., 1994; Couzin et al., 2002; Parrish et al., 2002; Adiou et al., 2003, and references therein). In the literature, two different means are usually used to define the individual motions: one uses descriptive behavior rules and the other uses mathematical equations. In the behavior-based simulation studies (Reynolds, 1987; Grünbaum and Okubo, 1994; Kunz and Hemelrijk, 2003; Huth and Wissel, 1994; Viscido et al., 2004), an agent is pre-assigned certain behaviors and assumed to adopt different actions (attraction, repulsion, or alignment) according to the range in which the perceived neighbors are positioned. Similarly, in mathematical equation based models (Mogilner and Edelstein-keshet, 1999; Couzin et al., 2002; Flierl et al., 1999), the agent's motion (velocity and position) are defined to be affected by virtual forces from certain other members according to dynamical equations guided by the classic Newton mechanics law.

Despite the different approaches used to describe the individual motions in the mathematical biology literature, it is generally assumed that the swarm members sense and adjust their behaviors or motions through certain virtual forces with their neighbors. The attraction forces keep the group cohesive while the repulsion forces can prevent collision happening among agents.

Fig.4.1 shows the diagram of two neighboring agents ( $i$  and  $j$ ) and the virtual forces between them, in which  $G_{ij}$  is the combined force of the virtual attraction and

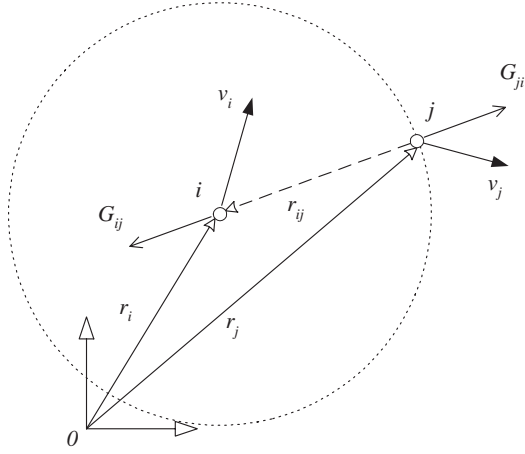


Figure 4.1: Two neighboring agents ( $i$  and  $j$ ) and their mutual interaction.

repulsion forces from agent  $j$  to  $i$ , defined by

$$G_{ij} = F_r(\|r_{ij}\|) - F_a(\|r_{ij}\|), \quad (4.1)$$

where  $F_a(\|r_{ij}\|)$  and  $F_r(\|r_{ij}\|)$  are the virtual attraction and repulsion forces from  $j$  to  $i$ , respectively. Note that the virtual forces  $F_a$ ,  $F_r$ , and the combined force  $G_{ij}$  are all along the direction of the vector  $r_{ij}$ . Hence, we denote

$$F_a(\|r_{ij}\|) = f_a(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}, \quad (4.2)$$

and

$$F_r(\|r_{ij}\|) = f_r(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}, \quad (4.3)$$

where  $f_a(\|r_{ij}\|)$  and  $f_r(\|r_{ij}\|)$  represent the amplitudes of the virtual attraction and repulsion forces. Then,  $G_{ij}$  can be written as

$$G_{ij} = (f_r(\|r_{ij}\|) - f_a(\|r_{ij}\|)) \frac{r_{ij}}{\|r_{ij}\|}. \quad (4.4)$$

Note that  $f_a(\|r_{ij}\|)$  and  $f_r(\|r_{ij}\|)$  are scalar functions that only depend on the relative distance  $\|r_{ij}\|$ .

In the literature many different functions have been presented for the virtual forces  $f_a$  and  $f_r$  (Mogilner and Edelstein-keshet, 1999; Niwa, 1996; Viscido et al., 2004; Couzin et al., 2002). Also, many efforts have been made to propose a more “reasonable” or realistic set of functions for the virtual forces by comparing the simulation results with experimental or observational data in real animal groups (Viscido et al., 2004; Huth and Wissel, 1994; Kunz and Hemelrijk, 2003). In this thesis, the presented motion rule works for a general range of virtual forces, thus, it avoids the problem of looking for a “better” set of functions for the virtual forces. We show that regardless of which specific functions the virtual forces take, the collective group behavior of the swarm can be achieved.

Moreover, since the perception range of any creature is limited, we consider that individual agents can only interact with certain neighbors. The motion of each agent is affected by virtual forces from its neighbors within a certain range, and all other agents beyond this range will not be considered to have interactions.

Note that, for the same reason of the limited sensing or communication range in both nature and engineering, it is hard for each agent to know the states of all other swarm-mates as assumed by some controllers in the literature (Gazi and Passino, 2003, 2004; Liu and Passino, 2004). To make it more feasible, we present an decentralized motion controller that only requires local information to achieve the collective motion of the group.

After illustrating the virtual attraction/repulsion forces, we apply it as the micro-level term  $\Phi_2^i$  in the motion framework and design a novel controller for the agents to flock in a homogeneous environment.

In this section, we study swarms flocking in a dynamic environment, such as schools of fish move along ocean currents (Muss et al., 2001; Armsworth, 2001; Humston et al., 2000). Assume that the swarm moves in an environment with a potential

function  $J(r)$ . The gradient of  $J(r)$  at  $r_i$  is denoted by  $\nabla_{r_i} J(r)$ . Each agent hardly has the full knowledge of the global environmental potential  $J(r)$ , but it is assumed to be able to sense the local information around its current position,  $\nabla_{r_i} J(r)$ . This assumption can be justified by observations in some biological systems. For example, European robins and homing pigeons can sense the magnetic field of the earth at their current location to determine their heading directions (Wiltschko and Wiltschko, 1996), and some reef fish can perceive and school along the ocean currents surrounding them (Muss et al., 2001; Armsworth, 2001; Humston et al., 2000).

Moreover, we assume that the environment has homogeneous effects on all agents (Flierl et al., 1999; Beecham and Farnsworth, 1999), i.e.,  $\nabla_{r_i} J(r)$  is the same for  $i = 1, \dots, N$ . We also assume  $\nabla_{r_i} J(r)$  is bounded.

When the swarm flocks in a certain environment, in the steady stage, the entire swarm exhibits a group-level motion of following the path ‘clue’ of the environmental potential, and at the same time in the inter-individual-level, each agent keeps a certain spacing from each other to make the group cohesive and collision free. The group-level motion can be obtained by the macro-level term  $\Phi_1^i$  of the framework, and the inter-individual-level behavior can be achieved by the virtual A/R forces. In viewing of this feature of the swarm in the steady stage, we apply the framework presented in the last chapter and design a decentralized controller for each agent as

$$u_i = \underbrace{-k_p[v_i - \nabla_{r_i} J(r)]}_{\Phi_1^i} + \underbrace{\sum_{j \in \mathbb{N}_i} (f_r - f_a) \frac{r_{ij}}{\|r_{ij}\|}}_{\Phi_2^i}, \quad (4.5)$$

where  $k_p > 0$  is a design parameter,  $f_a$  and  $f_r$  are the amplitudes of the virtual attraction and repulsion forces from agent  $j$  to  $i$ . The implication of this controller is that each agent perceives and follows the gradient force of the environment ( $\nabla_{r_i} J(r)$ ), and at the same time interacts with its neighbors via virtual attraction/repulsion forces to adjust its motion.

**Remark 4.1** *Note that the measurements involved in the controller (4.5) are just*

relative distances between pairs of agents, i.e.,  $x_i - x_j$ , rather than global positions  $x_i$  or  $x_j$ . This means that the GPS sensor devices can be saved and relative range sensors are sufficient.

In nature, different biological groups may show different types of collective motions in different environments (Parrish et al., 2002). Our study is focused on the parallel type, rather than the circular type (also called “milling”, Lukeman et al., 2009), of collective motion of the swarm (Couzin et al., 2002; Paley et al., 2007). When the entire group moves in parallel collectively, all members share a common velocity and have constant spacings among them. In other words, if the speeds and the headings of all members become the same, the collective motion of the swarm is formed.

For simplicity, denote

$$g(\|r_{ij}\|) = f_r(\|r_{ij}\|) - f_a(\|r_{ij}\|). \quad (4.6)$$

to represent the amplitude of the combined virtual A/R forces.

For swarms with dynamic topology, in order to prevent collision among agents and meanwhile keep the whole group cohesive, the combined virtual A/R forces need to be

$$g(\|r_{ij}\|) = \begin{cases} > 0 & 0 \leq \|r_{ij}\| < d_0, \\ = 0 & \|r_{ij}\| = d_0, \\ < 0 & d_0 < \|r_{ij}\| \leq d_1, \\ = 0 & \|r_{ij}\| > d_1. \end{cases} \quad (4.7)$$

Furthermore, let  $g(\|r_{ij}\|)$  be locally Lipschitz. Clearly,  $g(\|r_{ij}\|)$  is discontinues at  $d_1$ . For simplicity of analysis, we assume that  $g(\|r_{ij}\|)$  is continuous for  $\|r_{ij}\| < d_1$ , i.e., it is discontinuous only at  $d_1$ .

As one can see, the conditions in (4.7) is very general. It implies nothing but that the virtual forces between two agents need to be selected such that when two agents

come too close ( $\|r_{ij}\| < d_0$ ), the repulsion force should dominate; and if  $\|r_{ij}\| > d_0$ , attraction force play the role. Note that if the agent  $j$  is not in the neighborhood range of  $i$ , ( $\|r_{ij}\| > d_1$ ), then no any virtual force between them is considered, i.e.,  $g(\|r_{ij}\|) = 0$ .

We will show that for any set of virtual forces, as long as the condition (4.7) is satisfied, the decentralized controller (4.5) enables that all agents asymptotically have a same velocity ( $\bar{v}$ ), thereby the parallel type of collective motion is formed.

Define error states

$$e_{v_i} = v_i - \bar{v}. \quad (4.8)$$

It is straight to have

$$\begin{aligned} \dot{v}_i - \dot{\bar{v}} &= -k_p(v_i - \bar{v}) + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + k_p[\nabla_{r_i} J(r) \\ &\quad - \frac{1}{N} \sum_{i=1}^N \nabla_{r_i} J(r)] - \frac{1}{N} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}. \end{aligned} \quad (4.9)$$

Since  $\|r_{ij}\| = \|r_{ji}\|$  and  $g(\|r_{ij}\|) = g(\|r_{ji}\|)$ , then

$$g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = -g(\|r_{ji}\|) \frac{r_{ji}}{\|r_{ji}\|} \quad (4.10)$$

By using the adjacent matrix  $A = [a_{ij}]_{N \times N}$  defined by (Godsil and Royle, 2001; Biggs, 1993)

$$a_{ij} = \begin{cases} 0, & j = i \text{ or } j \notin \mathbb{N}_i, \\ 1, & j \in \mathbb{N}_i, \end{cases} \quad (4.11)$$

one can have

$$\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = \sum_{i=1}^N \sum_{j=1}^N a_{ij} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}. \quad (4.12)$$

Note that according to the definition of the neighbor set  $\mathbb{N}_i$  in the Section 3.2, the adjacent matrix  $A$  (4.11) is always symmetric regardless of whether the graph  $\mathcal{G}$  is

connected. Hence, by using (4.10), it yields to

$$\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = \sum_{i=1}^N \sum_{j=1}^N a_{ij} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = 0. \quad (4.13)$$

Thus, (4.9) becomes

$$\dot{v}_i - \dot{\bar{v}} = -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + k_p [\nabla_{r_i} J - \frac{1}{N} \sum_{i=1}^N \nabla_{r_i} J(r)] \quad (4.14)$$

We assume all agents move in a same environment which has identical effect on each one, i.e.,  $\nabla_{r_i} J(r) = \nabla_{r_j} J(r), \forall i \neq j$ . For example (Gazi and Passino, 2004),  $J(r) = \sum_{i=1}^N J(r_i) = \sum_{i=1}^N a^T \cdot r_i + b$ , where  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ . Then

$$\nabla_{r_i} J(r) - \frac{1}{N} \sum_{i=1}^N \nabla_{r_i} J(r) = 0. \quad (4.15)$$

Thus, we have

$$\dot{v}_i - \dot{\bar{v}} = -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}. \quad (4.16)$$

Because along the boundary of neighborhood range,  $g(\|r_{ij}\|)$  is discontinuous, i.e., not differentiable, then the derivative  $\dot{e}_{v_i}$  may not exist at a certain subset. Thus, one can only have differential inclusion rather than differential equation for  $\dot{e}_{v_i}$  (Dontchev and Lempio, 1992; Filippov, 1988).

$$\dot{e}_{v_i} \in^{a.e.} K[e_{v_i}] = \left\{ -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \right\} \quad (4.17)$$

Define

$$E_{ij}(\|r_{ij}\|) = \int_{\|r_{ij}\|}^{d_0} g(\tau) d\tau. \quad (4.18)$$

Then for  $\|r_{ij}\| < d_1$ :

$$E_{ij} = \int_{\|r_{ij}\|}^{d_0} g(\tau) d\tau,$$

for  $\|r_{ij}\| \geq d_1$ :

$$\begin{aligned} E_{ij} &= - \int_{d_0}^{\|r_{ij}\|} g(\tau) d\tau = - \left\{ \int_{d_0}^{d_1} g(\tau) d\tau + \int_{d_1}^{\|r_{ij}\|} g(\tau) d\tau \right\} \\ &= - \int_{d_0}^{d_1} g(\tau) d\tau = \text{const.} \end{aligned}$$

i.e.,

$$E_{ij} = \begin{cases} \|r_{ij}\| < d_1 : & = \int_{\|r_{ij}\|}^{d_0} g(\tau) d\tau, \\ \|r_{ij}\| \geq d_1 : & = \text{const.} \end{cases} \quad (4.19)$$

Clearly  $E_{ij}(\|r_{ij}\|)$  is continuous everywhere. But because  $g(d_1) \neq 0$ , it is not differentiable at  $\|r_{ij}\| = d_1$ .

Furthermore, we have its general gradient (Clarke, 1983; Clarke et al., 1998)

$$\partial E_{ij} = \begin{cases} \|r_{ij}\| < d_1 : & = -g(\|r_{ij}\|), \\ \|r_{ij}\| > d_1 : & = 0, \\ \|r_{ij}\| = d_1 : & = \overline{\text{co}}[-g(d_1), 0]. \end{cases} \quad (4.20)$$

in which  $\overline{\text{co}}[\cdot]$  is the closed convex hull.

Use the candidate Lyapunov function

$$V_t = \frac{1}{2} \sum_{i=1}^N e_{v_i}^T e_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N E_{ij}(\|r_{ij}\|). \quad (4.21)$$

Clearly  $V_t$  is a function of  $e_{v_i}$  and  $r_{ij}$ , and smooth about  $e_{v_i}$ . But because  $E_{ij}$  is nonsmooth about  $\|r_{ij}\|$ , so is  $V_t$ . From (4.19) we know  $V_t \geq 0$ .

Note that if the swarm's topology is fixed, i.e., if the neighborhood relations among members do not change, one can use a different candidate Lyapunov function as

$$V_2 = \frac{1}{2} \sum_{i=1}^N e_{v_i}^T e_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} E_{ij}(\|r_{ij}\|). \quad (4.22)$$

However,  $V_2$  is not suitable for swarms with dynamic topology. This is because by using the adjacent matrix  $A = [a_{ij}]_{N \times N}$  in (4.11),  $V_2$  can be written as

$$V_2 = \frac{1}{2} \sum_{i=1}^N e_{v_i}^T e_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} E_{ij}(\|r_{ij}\|).$$

Then its derivative with respect to time is

$$\frac{d}{dt}V_2 = \frac{1}{2} \sum_{i=1}^N e_{v_i}^T \dot{e}_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \left( \frac{da_{ij}}{dt} E_{ij} + a_{ij} \frac{dE_{ij}}{dt} \right).$$

If the swarm's topology is not fixed, i.e.,  $a_{ij}$  is not constant, then  $\frac{d}{dt}a_{ij}$  is not zero. But because  $a_{ij} = 0$  or  $1$ , it is impossible to evaluate  $\frac{d}{dt}a_{ij}$ . Therefore,  $V_2$  cannot be used for the analysis of swarms with dynamic topology. From this, one can see the difficulty brought by the swarm's dynamic topology to the analysis of the group's collective motion.

We will evaluate how the Lyapunov function  $V_t$  (6.22) evolves with time by studying its derivative  $\frac{d}{dt}V_t$ , and show that the velocities of all agents asymptotically converge to  $\bar{v}$  and the entire swarm forms a parallel type of collective motion. Before proceeding to derive  $\frac{d}{dt}V_t$ , we need to prove that  $V_t$  is differentiable at least almost everywhere in the sense of usual Lebesgue measure. Since  $V_t$  is a function of  $e_{v_i}$  and  $\|r_{ij}\|$ , we need to show it is differentiable almost everywhere in the entire domain of  $(e_{v_i}, \|r_{ij}\|)$ .

**Lemma 4.1** *The function  $V_t$  (6.22) is differentiable almost everywhere in its domain.*

*Proof:* From its definition in (6.22) we know  $V_t$  is a function of  $e_{v_i}$  and  $\|r_{ij}\|$ , and continuously differentiable about  $e_{v_i}$ . But because  $E_{ij}$  is not continuously differentiable about  $\|r_{ij}\|$ , so is  $V_t$ . Since  $E_{ij}(\|r_{ij}\|)$  is differentiable except at  $d_1$ , then it is locally Lipschitz on its domain except  $d_1$ . Moreover, because  $g(\cdot)$  is locally Lipschitz, then the general gradient of  $E_{ij}$  exists (4.20). Hence,  $E_{ij}(\|r_{ij}\|)$  is also locally Lipschitz at  $d_1$ . Furthermore, according to (6.22),  $V_t$  is also locally Lipschitz. Therefore, from Rademacher's Theorem (Clarke, 1990),  $V_t$  is differentiable almost everywhere in its entire domain. □

In order to use the chain rule in the Section 2.4 to derive the set-valued Lie derivative of  $V_t$ , we need to show it is regular everywhere (see the definition of *regular function* in Definition 2.4 ).

**Lemma 4.2** *The function  $V_t$  is a regular function everywhere in its domain.*

*Proof:* Because  $e_{v_i}^T e_{v_i}$  is convex function, we just need to prove  $E_{ij}$  is regular in order to show  $V_t$  is regular everywhere (Bacciotti and Ceragioli, 1999; Cortés, 2008). And since  $E_{ij}$  is smooth everywhere except at  $d_1$ , we only need to prove it is regular at  $d_1$ . To show its regularity at  $d_1$ , we need to prove the equality between its generalized directional derivative and classical one-sided directional derivative, i.e.,  $E_{ij}^\circ(d_1, w) = E'_{ij}(d_1, w)$ , where  $E'_{ij}(d_1, w) = \lim_{h \downarrow 0} \frac{E_{ij}(d_1+hw) - E_{ij}(d_1)}{h}$  and  $E_{ij}^\circ(d_1, w) = \lim_{y \rightarrow d_1} \sup_{h \downarrow 0} \frac{E_{ij}(y+hw) - E_{ij}(y)}{h}$ .

If  $w < 0$ :  $E'_{ij}(d_1, w) = \lim_{h \downarrow 0} \frac{E_{ij}(d_1+hw) - E_{ij}(d_1)}{h} = \lim_{h \downarrow 0} \frac{hw E'_{ij}(d_1^-)}{h} = w E'_{ij}(d_1^-)$ , and  $E_{ij}^\circ(d_1, w) = \lim_{y \rightarrow d_1} \sup_{h \downarrow 0} \frac{E_{ij}(y+hw) - E_{ij}(y)}{h} = \lim_{h \downarrow 0} \frac{E_{ij}(d_1+hw) - E_{ij}(d_1)}{h} = w E'_{ij}(d_1^-)$ ,  
 If  $w \geq 0$ :  $E'_{ij}(d_1, w) = \lim_{h \downarrow 0} \frac{E_{ij}(d_1+hw) - E_{ij}(d_1)}{h} = \lim_{h \downarrow 0} \frac{hw E'_{ij}(d_1^+)}{h} = 0$ , and

$$\begin{aligned} E_{ij}^\circ(d_1, w) &= \lim_{y \rightarrow d_1} \sup_{h \downarrow 0} \frac{E_{ij}(y+hw) - E_{ij}(y)}{h} = \lim_{z \rightarrow d_1} \sup_{h \downarrow 0} \frac{E_{ij}(z) - E_{ij}(z-hw)}{h} \\ &\leq \lim_{z \rightarrow d_1} \sup_{h \downarrow 0} \frac{E_{ij}(d_1) - E_{ij}(z-hw)}{h} = - \lim_{z \rightarrow d_1} \inf_{h \downarrow 0} \frac{E_{ij}(z+h(-w)) - E_{ij}(d_1)}{h} \\ &= - \lim_{h \downarrow 0} \frac{E_{ij}(d_1) - E_{ij}(d_1)}{h} = 0. \end{aligned}$$

So, in both cases  $E_{ij}^\circ(d_1, w) = E'_{ij}(d_1, w)$ . □

**Theorem 4.1** *Consider a swarm of  $N$  mobile agents moving in an homogeneous environment. With any set of virtual forces that satisfies the condition in (4.7), the decentralized controller (4.5) makes the velocity vectors of all agents asymptotically converge to a common value (the average  $\bar{v}$ ).*

*Proof:* Since  $V_t(e_{ij}, \|r_{ij}\|)$  is locally Lipschitz, we have its generalized gradient as

$$\partial V_t = \overline{\text{co}}\{\lim \nabla V_t(e_{v_i}, \|r_{ij}\|), \|r_{ij}\| \notin \Omega_V, i, j = 1, \dots, N\},$$

in which  $\Omega_V$  is the set of measure zero where the gradient of  $V_t$  is not defined. Specifically,

$$\partial V_t = [e_{v_1}^T, \dots, e_{v_N}^T, \frac{1}{2}\partial E_{11}, \dots, \frac{1}{2}\partial E_{ij}, \dots, \frac{1}{2}\partial E_{NN}]^T. \quad (4.23)$$

For simplicity, denote  $\zeta_{ij} = \frac{1}{2}\partial E_{ij}$ , then

$$\partial V_t = [e_{v_1}^T, \dots, e_{v_N}^T, \zeta_{11}, \dots, \zeta_{ij}, \dots, \zeta_{NN}]^T. \quad (4.24)$$

From Theorem 3.2, we know

$$\frac{dV_t}{dt} \in^{a.e.} \dot{\tilde{V}}_t, \quad (4.25)$$

where

$$\dot{\tilde{V}}_t = \bigcap_{\xi \in \partial V_t} \xi^T \cdot \left\{ K[e_{v_1}], \dots, K[e_{v_N}], \frac{d\|r_{11}\|}{dt}, \dots, \frac{d\|r_{ij}\|}{dt}, \dots, \frac{d\|r_{NN}\|}{dt} \right\}^T.$$

From (6.23) we have

$$\dot{\tilde{V}}_t = \bigcap_{\xi \in \partial V_t} \left\{ \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\}. \quad (4.26)$$

To find out  $\dot{\tilde{V}}_t$  on the whole domain of  $\|r_{ij}\|$ , we discuss it piece-wisely.

If for  $\forall i, \|r_{ij}\| > d_1$  where  $j \in \{1, \dots, N\} \setminus \{i\}$ , then  $\zeta_{ij} = 0$ ,  $K[e_{v_i}] = -k_p e_{v_i}$ , then

$$\begin{aligned} & \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^N e_{v_i}^T \cdot (-k_p e_{v_i}) \\ & + \sum_{i=1}^N \sum_{j=1}^N 0 \cdot \frac{d\|r_{ij}\|}{dt} = -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i}. \end{aligned} \quad (4.27)$$

If for  $\forall i, \|r_{ij}\| < d_1$  where  $j \in \mathbb{N}_i$ , then  $\zeta_{ij} = -\frac{1}{2}g(\|r_{ij}\|)$ . Then, we have

$$\begin{aligned} \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} &= \sum_{i=1}^N e_{v_i}^T \cdot \{-k_p e_{v_i} + \\ &\sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}\} - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt}. \end{aligned} \quad (4.28)$$

Since for  $j \notin \mathbb{N}_i$ ,  $g(\|r_{ij}\|) = 0$ , then

$$\sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt}.$$

Because

$$\begin{aligned} \frac{d\|r_{ij}\|}{dt} &= \left[\frac{d}{dt} r_{ij}\right]^T \cdot \nabla_{r_{ij}} \|r_{ij}\| = (v_i - v_j)^T \cdot \nabla_{r_{ij}} \|r_{ij}\| = \\ &= v_i^T \cdot \nabla_{r_{ij}} \|r_{ij}\| + v_j^T \cdot \nabla_{r_{ji}} \|r_{ji}\|, \end{aligned} \quad (4.29)$$

where we use  $\nabla_{r_{ji}} \|r_{ij}\| = \frac{r_{ij}}{\|r_{ij}\|}$ , then equation (4.28) becomes

$$\begin{aligned} \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} &= -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} (v_i \\ &- \bar{v})^T g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} - \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \left\{ \left(\frac{d}{dt} r_{ij}\right)^T \cdot \nabla_{r_{ij}} \|r_{ij}\| \right\} \\ &= -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} - \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \bar{v}^T g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \\ &\left( v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} \right) - \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) (v_i^T \cdot \nabla_{r_{ij}} \|r_{ij}\|) \\ &= -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i}. \end{aligned} \quad (4.30)$$

If for  $\forall i, \|r_{ij}\| = d_1$  where  $j \in \mathbb{N}_i$ , then  $\zeta_{ij} \in \overline{c0}[-\frac{1}{2}g(d_1), 0]$ , and  $K[e_{v_i}] = -k_p e_{v_i} +$

$\bar{c}\bar{o}[g(d_1), 0] \sum_{j \in \mathbb{N}_i} \frac{r_{ij}}{\|r_{ij}\|}$ . Then we have:

$$\begin{aligned}
 \tilde{V}_t \big|_{\|r_{ij}\|=d_1} &= \bigcap_{\xi \in \partial V_i} \left\{ \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\} \\
 &= \bigcap_{\zeta_{ij} \in \bar{c}\bar{o}[-\frac{1}{2}g(d_1), 0]} \left\{ \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\} \\
 &= \bigcap_{\zeta_{ij} \in \bar{c}\bar{o}[-\frac{1}{2}g(d_1), 0]} \left\{ \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \bar{c}\bar{o}[g(d_1), 0] \right. \\
 &\quad \left. (v_i - \bar{v})^T \cdot \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \zeta_{ij} \left[ \frac{dr_{ij}}{dt} \right]^T \cdot \nabla_{r_{ij}} \|r_{ij}\| \right\} \\
 &= \bigcap_{\zeta_{ij} \in \bar{c}\bar{o}[-\frac{1}{2}g(d_1), 0]} \left\{ \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} + \bar{c}\bar{o}[g(d_1), 0] \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \right. \\
 &\quad \left. v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} + 2\zeta_{ij} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} v_i^T \cdot \nabla_{r_{ij}} \|r_{ij}\| \right\} \\
 &= \bigcap_{\zeta_{ij} \in \bar{c}\bar{o}[-\frac{1}{2}g(d_1), 0]} \left\{ \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} + (\bar{c}\bar{o}[g(d_1), 0] + 2\zeta_{ij}) \right. \\
 &\quad \left. \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} \right\} \subseteq \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} + \bigcap_{\zeta_{ij} \in \bar{c}\bar{o}[-\frac{1}{2}g(d_1), 0]} \\
 &\quad \left\{ (\bar{c}\bar{o}[g(d_1), 0] + 2\zeta_{ij}) \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} v_i^T \cdot \frac{r_{ij}}{\|r_{ij}\|} \right\}. \tag{4.31}
 \end{aligned}$$

Since

$$\bigcap_{\zeta_{ij} \in \bar{c}\bar{o}[-\frac{1}{2}g(d_1), 0]} \bar{c}\bar{o}[g(d_1), 0] + 2\zeta_{ij} = \{0\}, \tag{4.32}$$

then

$$\dot{\tilde{V}}_t \big|_{\|r_{ij}\|=d_1} \subseteq \left\{ \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} \right\}. \quad (4.33)$$

Thus, on the entire domain, we have

$$\dot{\tilde{V}}_t \subseteq \left\{ \alpha \mid \alpha = \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} \leq 0 \right\}. \quad (4.34)$$

Therefore, we have

$$\frac{dV_t}{dt} \in^{a.e.} \left\{ \alpha : \alpha = \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} \leq 0 \right\}. \quad (4.35)$$

Thus, all  $\frac{d}{dt}V_t \leq 0$ . This means  $e_{v_i}$  is stable for any agent.

Furthermore, Note that the only option to make  $\frac{d}{dt}V_t = 0$  in (4.35) is  $e_{v_i} = 0$ . Thus, from the nonsmooth LaSalle's invariance principle (Bacciotti and Ceragioli, 1999; Cortés, 2008), we know that the states  $(e_{v_i}, \|r_{ij}\|)$  will reach the largest invariant set in

$$\begin{aligned} \bar{S} &= cl(\{(e_{v_i}, \|r_{ij}\|) : 0 \in \dot{\tilde{V}}_t, i, j = 1, \dots, N\}) \\ &= cl(\{(0, \|r_{ij}\|) : 0 \in \dot{\tilde{V}}_t, i, j = 1, \dots, N\}). \end{aligned} \quad (4.36)$$

where  $cl(\cdot)$  is the closure of a set. Note that in the invariant set  $\bar{S}$ ,  $e_{v_i} = 0$ . This means that the velocity vectors of all agents asymptotically converge to a common value (the average  $\bar{v}$ ).  $\square$

We now proceed to discuss the spacings among agents. For simplicity, let  $d_{ij} = \|r_{ij}\|$ , then

$$d_{ij}^2 = (r_i - r_j)^T (r_i - r_j), \quad (4.37)$$

and

$$\frac{d}{dt}d_{ij}^2 = 2(r_i - r_j)^T (v_i - v_j). \quad (4.38)$$

Note that  $\frac{d}{dt}(r_i - r_j) = v_i - v_j = e_{v_i} - e_{v_j}$ . And since  $e_{v_i} = 0$  as  $t \rightarrow \infty$ , then  $e_{v_i} - e_{v_j} = 0$ , i.e.,  $\frac{d}{dt}(r_i - r_j) = 0$  as  $t \rightarrow \infty$ . Thus,  $r_i - r_j$  is not strictly increasing and  $r_i - r_j \neq \infty$ .

Moreover, because  $v_i - v_j = e_{v_i} - e_{v_j} = 0$  as  $t \rightarrow \infty$ , then from (4.38) we know  $\frac{d}{dt}d_{ij}^2 = 0$  as  $t \rightarrow \infty$ . Hence,  $d_{ij}^2$  will be a constant. Since it is always that  $d_{ij} \geq 0$ , then  $d_{ij}$  will be a non-negative constant. This means that as the collective motion of the swarm is achieved, the relative distance between two neighboring agents ( $d_{ij}$ ) will remain unchanging.

Furthermore, from the final evaluation of  $\frac{d}{dt}V_t$  (4.35) and the LaSalle's invariance principle, we know that before the velocity convergence occurs, the Lyapunov function  $V_t$  will be strictly decreasing and heading to its global minimum until the states  $(e_{v_i}, \|r_{ij}\|)$  reach the invariant set  $\bar{S}$  (4.35). Once  $\bar{S}$  is approached, the states  $(e_{v_i}, \|r_{ij}\|)$  will stay inside  $\bar{S}$ , despite whether or not the minimum of  $V_t$  is achieved. From the definition of  $V_t$  in (6.22) one can easily see that  $V_t$  has a global minimum ( $V_t = 0$ ) if and only if  $e_{v_i} = 0$  and  $\|r_{ij}\| = d_0$ . However, it should be noted that the invariant set  $\bar{S}$  only has  $e_{v_i} = 0$ , thus, the state set by which  $V_t = 0$  is only a subset of  $\bar{S}$ . This means that after the invariant set  $\bar{S}$  is reached, it will always be that  $e_{v_i} = 0$ , and the global minimum of  $V_t$  may or may not be achieved, i.e.,  $\|r_{ij}\|$  may or may not equal to  $d_0$ .

In other words, the preceding analysis indicates that after the parallel type of collective motion is established (i.e.,  $e_{v_i} = 0$ ), the relative distance between two neighboring agents  $\|r_{ij}\|$  will remain unchanging, and may belong to a ball around  $d_0$  where the attraction and repulsion forces are balanced.

**Remark 4.2** *Note that all information needed by the controller (4.5) can be locally sensed, thus, the swarm's topological graph refers to sensing graph rather than commonly used communication graph in the literature (Leonard and Fiorelli, 2001; Moreau, 2005; Olfati-Saber, 2006; Tanner et al., 2007). The advantage of this config-*

uration is that by the proposed controller, range sensors are sufficient and the agents do not need to communicate their states with each other, which means communication modules are not needed. Subsequently all the assumptions and issues about communication setup (such as time delay and noise) are relieved.

## 4.2 Swarming Under Virtual A/A/R Forces

In the last section, we show that by the proposed controller, a pair of virtual attraction and repulsion forces lead the swarm to form a collective parallel motion. In this section, we extend the analysis to the common A/A/R models presented by mathematical biologists, which use Attraction/Alignment/Repulsion (A/A/R) forces. We show that the A/A/R models also enable that the swarm forms a collective parallel motion in a self-organized manner.

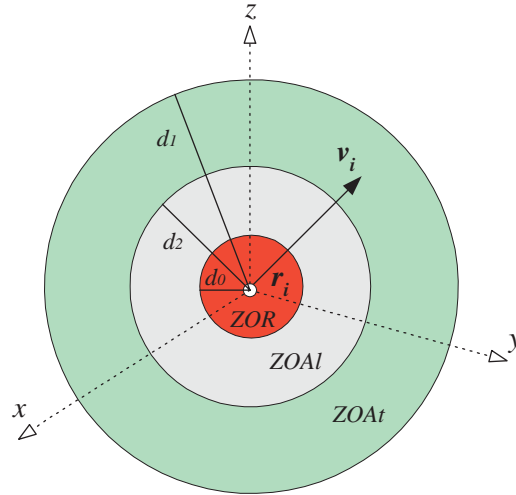


Figure 4.2: Interaction zones associated with agent  $i$ : zone of repulsion (ZOR), zone of alignment (ZOAI) and zone of attraction (ZOAt).

In the mathematical biology literature, many individual-based models use three

types of virtual forces for the mutual interactions among agents: short-distance repulsion force, long-distance attraction force and middle-range alignment force (also called “parallel orientation”) (Couzin et al., 2002; Parrish et al., 2002). Compared to the models using only A/R forces, the additional alignment force ensures that individuals maintain a motion alignment with other swarm-mates.

Fig.4.2 shows the three non-overlapping interaction zones associated with an agent, in which  $d_0$ ,  $d_2$  and  $d_1$  are the respective radii. If a neighbor comes too close to agent  $i$  and locate in the repulsion zone (ZOR), the combined effect of the virtual forces will be repulsive, while in attraction zone (ZOAt) the dominated attraction force will try to bring them closer. In alignment zone (ZOAl) (also called *neutral zone*), there will be no virtual attraction and repulsion forces to push or drag the neighbors. The agents just align their orientation angles based on the alignment force or the environmental force (Couzin et al., 2002; Kunz and Hemelrijk, 2003). Note that the maximum range of attraction zone is the same as the neighborhood defining distance  $d_1$ .

For most of the A/A/R models in the literature, regardless of which functions the virtual A/A/R forces use, they can be generalized as the controller in (4.5):

$$u_i = -k_p[v_i - \nabla_{r_i} J(r)] + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|},$$

with the combined virtual forces satisfying

$$g(\|r_{ij}\|) = \begin{cases} > 0 & 0 \leq \|r_{ij}\| < d_0, \\ = 0 & d_0 \leq \|r_{ij}\| \leq d_2, \\ < 0 & d_2 < \|r_{ij}\| \leq d_1, \\ = 0 & \|r_{ij}\| > d_1. \end{cases} \quad (4.39)$$

Moreover, to satisfy the Lipschitz condition for analyzing the collective motion of the swarm, let

$$g(d_0^-), g(d_2^+) \text{ and } g(d_1) \neq \infty, \quad (4.40)$$

in which  $d_0^-$  and  $d_1^+$  mean approaching to  $d_0$  from the lower side and to  $d_1$  from the upper side, respectively.

As one can see, the conditions in (4.39)-(4.40) are very general. It implies that the virtual forces between two agents need to be selected such that the amplitude of combined virtual forces  $g(\|r_{ij}\|)$  has three effect zones with limit ranges as in Fig. 4.2. Note that  $d_0$ ,  $d_1$  and  $d_2$  in (4.39) are the radii of the corresponding zones in Fig. 4.2.

In general cases,  $g(\|r_{ij}\|)$  is not continuous along the boundaries of the aforementioned three zones. For example,  $g(\|r_{ij}\|)$  is zero in ZA1, but not zero in ZOAt, thus,  $g(\|r_{ij}\|)$  may not be continuous on the boundary between ZOAl and ZOAt. However, for simplicity, we assume that it is continuous inside each zone.

Note that compared to the virtual A/R forces in (4.7), the only difference to use A/A/R forces is that the range of balanced force in which  $g(\|r_{ij}\|) = 0$  changes from a point  $d_0$  in (4.7), to an interval  $[d_0, d_2]$  in (4.39). Thus, by exploring a similar analysis as in the proof of Theorem 4.1, it is not hard to show that the A/A/R forces also lead to a collective flocking motion of the swarm.

**Corollary 4.1** *Using the same motion controller as (4.5), if  $g(\|r_{ij}\|)$  is selected to have Attraction/Alignment/Repulsion (A/A/R) zones and satisfy the conditions (4.39)-(4.40), then the swarm achieves the parallel type of collective motion.*

*Proof:* Use the same Lyapunov function as (6.22):

$$V_t = \frac{1}{2} \sum_{i=1}^N e_{v_i}^T e_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N E_{ij}(\|r_{ij}\|).$$

Similar to the proof of Theorem 4.1, we can show  $V_t$  is differentiable and regular almost everywhere.

According to (4.39),  $g(\|r_{ij}\|) = 0$  on  $[d_0, d_2]$ ; then in the proof part of Theorem 4.1, the evaluation of  $\dot{\tilde{V}}_t$  on  $d_0$  needs to be expanded to an interval of  $d_0 \leq \|r_{ij}\| \leq d_2$

(i.e., the alignment zone). Note that for  $d_0 < \|r_{ij}\| \leq d_2$ , the general gradient  $\partial E_{ij}$  and the error dynamics  $K[e_{v_i}]$  are the same as when  $\|r_{ij}\| = d_0$ . Thus, expanding the subset  $d_0$  to  $[d_0, d_2]$  does not affect the evaluation of  $\dot{V}_t$ . Therefore, one can have the same final result of  $\frac{d}{dt}V_t$  as in (4.35).  $\square$

It is noticed that Corollary 4.1 suggests that using A/R forces in the controller (4.5) is a special case of using A/A/R forces.

Combining the results stated in Theorem 4.1 and Corollary 4.1, we can reach the following finding. By the individual motion framework (4.5), if the virtual forces between two swarm members have finite ranges, and the attraction/repulsion forces are balanced (i.e.,  $g(\|r_{ij}\|) = 0$ ) at a certain distance (i.e., A/R model), or in a certain range (i.e., A/A/R model), then the entire swarm can form a parallel type of collective motion in a self-organized manner.

**Remark 4.3** *Note that the presented motion rule (4.5) works for a general range of virtual forces. In other words, we analytically prove that the A/A/R models commonly used in the mathematical biology literature can lead to collective behaviors of the swarm, in spite of specific functions the A/A/R forces take.*

### 4.3 Flocking with Obstacle Avoidance

In the last two sections, we show that the presented controller makes a swarm to form a collective motion in an open environment. In this section, we examine a swarm moving in a constrained environment with obstacles avoidance.

In robotics, obstacle avoidance is a fundamental issue and has been intensely studied in the last decades. A popular approach used in the robotics literature is the potential field method (Rimon and Koditschek, 1992; Hwang and Ahuja, 1992; Koren

and Borenstein, 1991; Borenstein et al., 1996; Ge and Cui, 2000). The basic concept of the potential field method is to treat the robot's configuration as a point, and fill the robot's motion space with an artificial potential field in which the robot is attracted to its goal position and is repulsed away from the obstacles. The resulting trajectory is the path for the robot. The potential field method is particularly attractive because of its mathematical elegance and simplicity (Rimon and Koditschek, 1992; Hwang and Ahuja, 1992; Koren and Borenstein, 1991; Ge and Cui, 2000).

Note that in the presented controller (4.5), environmental potential is used to guide the swarm to achieve a collective motion. In view of this feature of the controller, we can incorporate an additional artificial potential field, which is generated by the obstacles, with the environmental potential in the controller. Accordingly, the swarm can avoid the obstacles while flocking in the environment.

To demonstrate the obstacle avoidance capability of the controller, we present several sets of simulations results of swarms moving in certain homogeneous environments with existence of different obstacles in the next section.

Note that the potential field method for obstacle avoidance has some inherent limitations (Koren and Borenstein, 1991). A common problem is the trap situation caused by the local minima of the potential field. The robot may stop at a local minima of the potential field and not keep moving forward to its final goal. To eliminate the local minima, harmonic potential functions are used (Kim and Khosla, 1992; Ge and Cui, 2000). See Appendix A.3 for the description of harmonic function.

## 4.4 Simulation Study

In this section, we present a comprehensive simulation study on swarms flocking in different environments. The environment is assumed to have homogeneous effects on all agents. In the following figures, the stars and circles represent agents' initial and final positions, respectively.

We first simulate a swarm flocking in an open space by using virtual attraction/repulsion (A/R) forces. The the amplitude of the combined virtual A/R forces is selected to be that for  $\|r_{ij}\| \leq d_1$  :  $g(\|r_{ij}\|) = -\log(\|r_{ij}\|/d_0)/(\|r_{ij}\|^2 + 1)$ , and for  $\|r_{ij}\| > d_1$  :  $g(\|r_{ij}\|) = 0$ , in which  $d_0 = 4$  and The neighborhood defining distance  $d_1 = 12$ . The design constants  $k_p = 5$ .

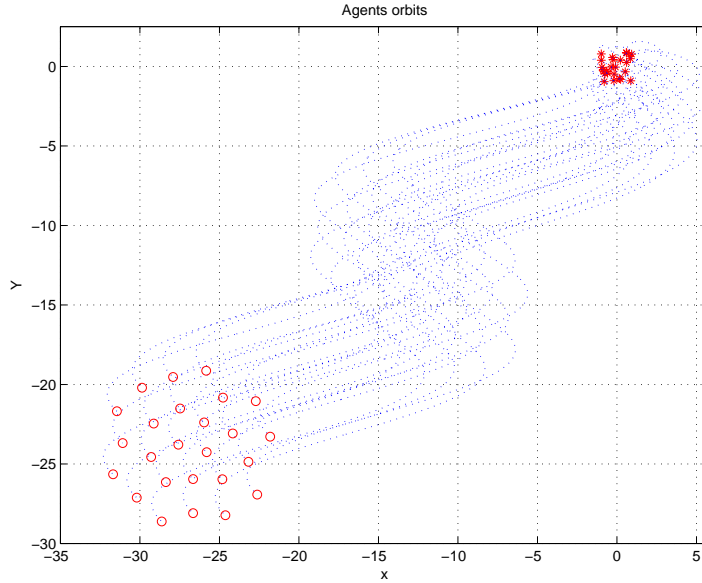


Figure 4.3: Agents’ trajectories on  $x - y$  plane when the swarm moves in a 2D environment that has a sinusoid-like gradient. ( $N = 25$ ).

Fig.4.3 and 4.4 show a swarm ( $N = 25$ ) moving in a 2D open environment that has a sinusoid-like gradient. At the initial stage, all agents are randomly located in a  $2 \times 2$  square centered at  $[0, 0]$ . The agents’ velocities are randomly given. Fig.4.3 shows the trajectories of the agents, and Fig.4.4 shows the velocity convergence. It is clear that all swarm members eventually move together to follow the sinusoid “wave” of the environment, as a group of fish schooling along the ocean currents.

Then we simulate a swarm flocking in an open space by using virtual attraction/alignment/repulsion (A/A/R) forces. The neighborhood defining distance  $d_1 = 24$ . We select the amplitude of the combined virtual forces to be piece-wise linear as:

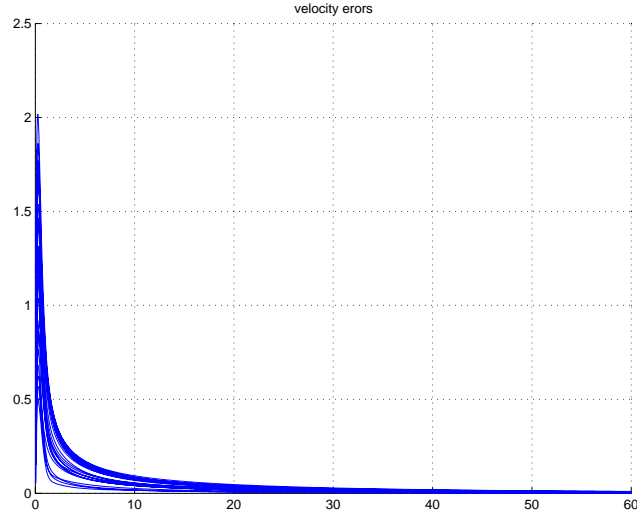


Figure 4.4: Velocity convergence of the agents as shown in Fig. 4.3 ( $N = 25$ ).

for  $\|r_{ij}\| < d_0$ :  $g(\|r_{ij}\|) = -30\|r_{ij}\| + 320$ ; and for  $d_1 \geq \|r_{ij}\| > d_2$ :  $g(\|r_{ij}\|) = -30\|r_{ij}\| + 400$ , in which  $d_0 = 10$  and  $d_2 = 14$ . The alignment zone lies between  $d_0 \sim d_2$ . Clearly  $g(\|r_{ij}\|)$  is not continuous at  $d_0$ ,  $d_1$  and  $d_2$ . The design parameter  $k_p = 5$ .

Fig.4.5(a)–4.5(d) show a swarm ( $N = 50$ ) moving in a 2D environment that has a sinusoid wave like gradient. Initially all agents are randomly located in a  $10 \times 10$  square center at  $(0, 0)$ . Their velocities are randomly given. Fig.4.5(a) shows the trajectories of agents and Fig.4.5(b) shows the velocity convergence. The steady motion of the swarm is shown in Fig.4.5(d). From Fig.4.5(a) it is clear that all swarm members asymptotically flock together to follow the sinusoid-like “wave” of the environment gradient. As one can see, using either A/R or A/A/R forces makes the swarm form the collective motion.

Finally, we simulate a swarm flocking in an environment in presence of different obstacles. To examine the behavior of the swarm under the influence of obstacle, we simulate three situations in that three different sizes of round-shape obstacles exist in the environment.

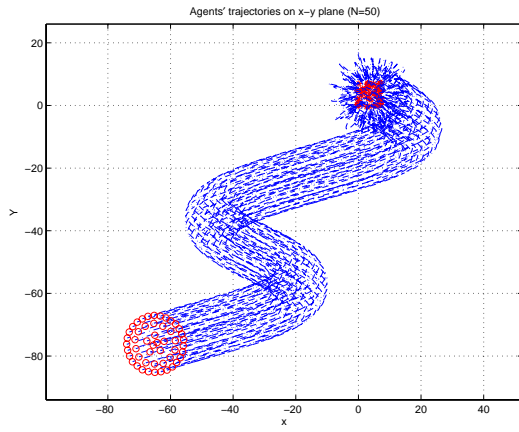
In the following simulations, we use the virtual A/R forces and select the combined effect of virtual forces to be: for  $\|r_{ij}\| \leq d_1$ :  $g(\|r_{ij}\|) = -30\|r_{ij}\| + 30d_0$ , in which  $d_0 = 10$  and  $d_1 = 12$ . The design parameter  $k_p = 5$ . The environmental potential has a linear gradient  $[-1, -1]^T$ . Initially all the agents are located in a  $25 \times 25$  square center at  $(0, 0)$ . Their velocities are randomly given. The center of the obstacle is at  $(-35, -35)$ .

Fig. 4.6 shows the motion of a swarm of  $N = 15$  agents when a small size obstacle is in the path of the swarm. The radius of the round-shape obstacle is 2.5. Swarm's trajectory and topology at different time steps are shown in Fig.4.7. The dot-line circles in Fig.4.6(d) is the half of neighborhood range of each agent. If the circles of two agents are overlapped, then these two are neighbors. Note that compared to the neighborhood defining distance  $d_1 = 12$ , the obstacle with radius of 2.5 is quite small. When the swarm comes to the obstacle, certain agents avoid it, but the topology of the entire group is not affected much. Thus, the entire group exhibits a surrounding motion around the obstacle.

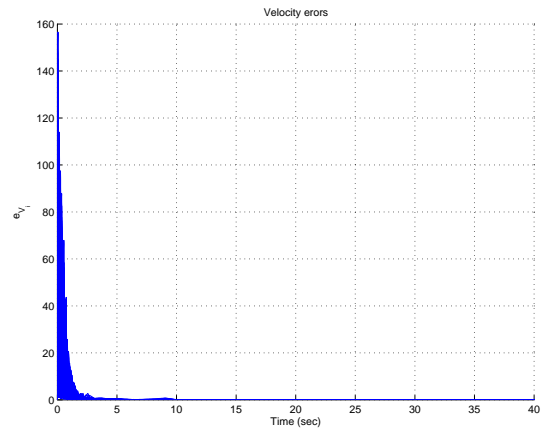
Fig. 4.8 shows the motion of a swarm of  $N = 20$  agents when a median-size obstacle is in the path of the swarm. The radius of the round-shape obstacle is 5. Swarm's trajectory and topology at different time steps are shown in Fig.4.9a and 4.9b. Note that due to the size and the location of the obstacle, the swarm splits after avoiding the obstacle. This verifies that the presented controller can lead to the collective motion of the swarm without any assumption on the connectivity of swarm's topological graph.

Fig. 4.10 shows the motion of a swarm of  $N = 20$  agents when a large size obstacle is in the path of the swarm. The radius of the round-shape obstacle is 9. Note that compared to the neighborhood defining distance  $d_1 = 12$ , the obstacle is quite big. Swarm's trajectory and topology at different time steps are shown in 4.11a and 4.11b. Note that due to the size and the location of the obstacle, the entire group moves

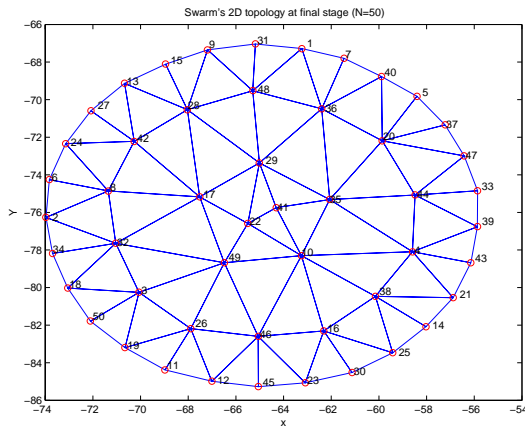
aside from the obstacle.



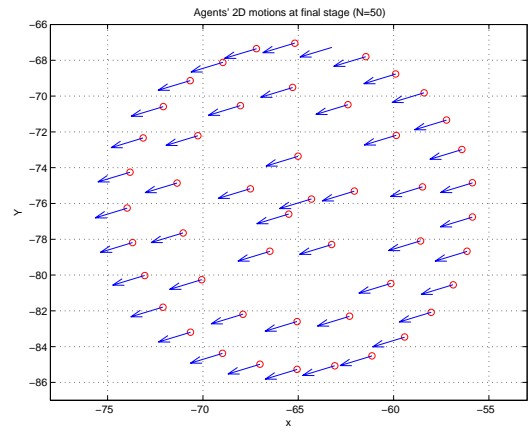
(a) Swarm's trajectory



(b) velocity convergence



(c) Swarm's steady topology



(d) Agent's final motion

Figure 4.5: A swarm moves in an open 2D environment that has a sinusoid-like gradient (N=50).

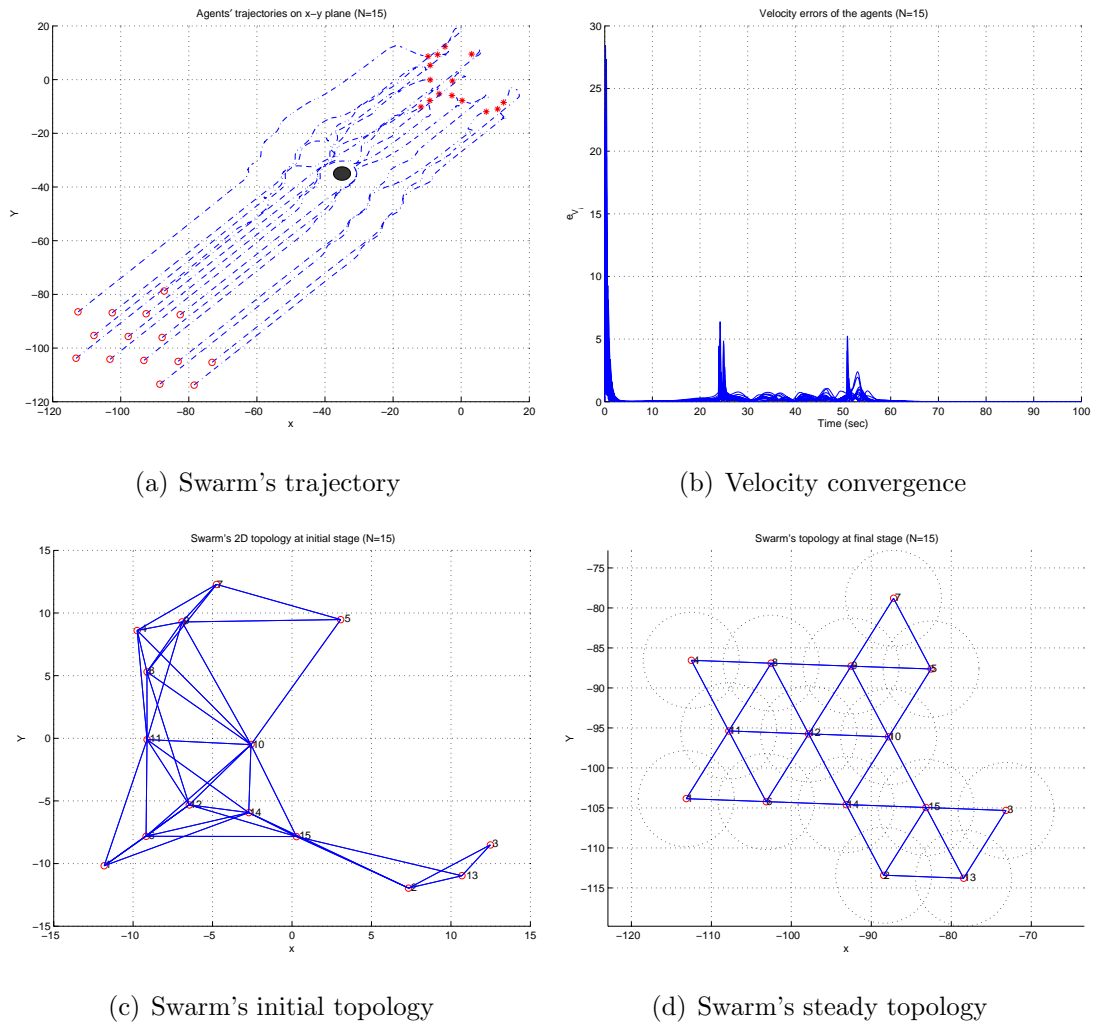


Figure 4.6: A swarm flocks around a small obstacle in a linear environment (N=15).

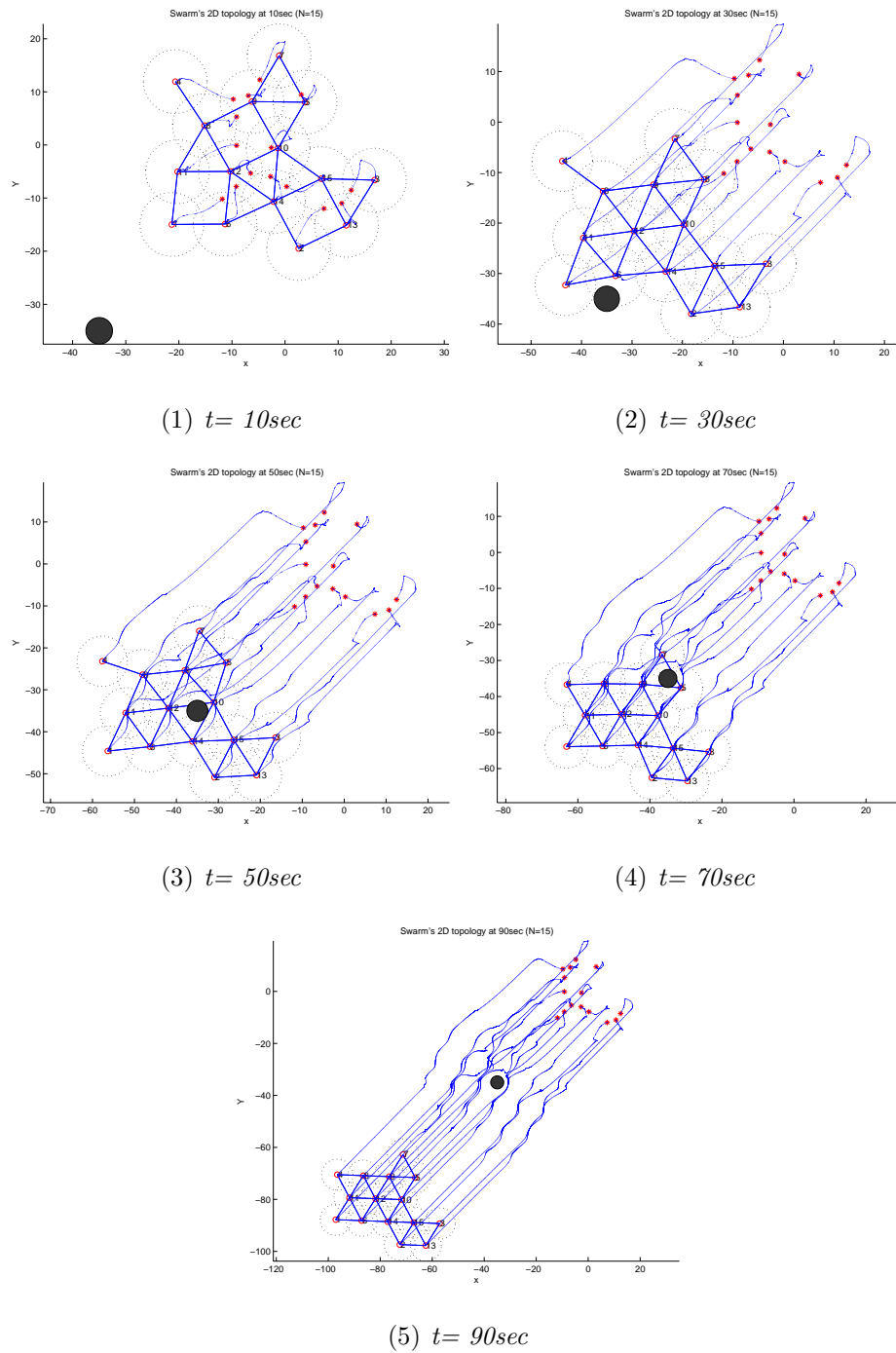


Figure 4.7: Swarm's trajectory and topology at different time steps as shown in Fig. 4.6(a).

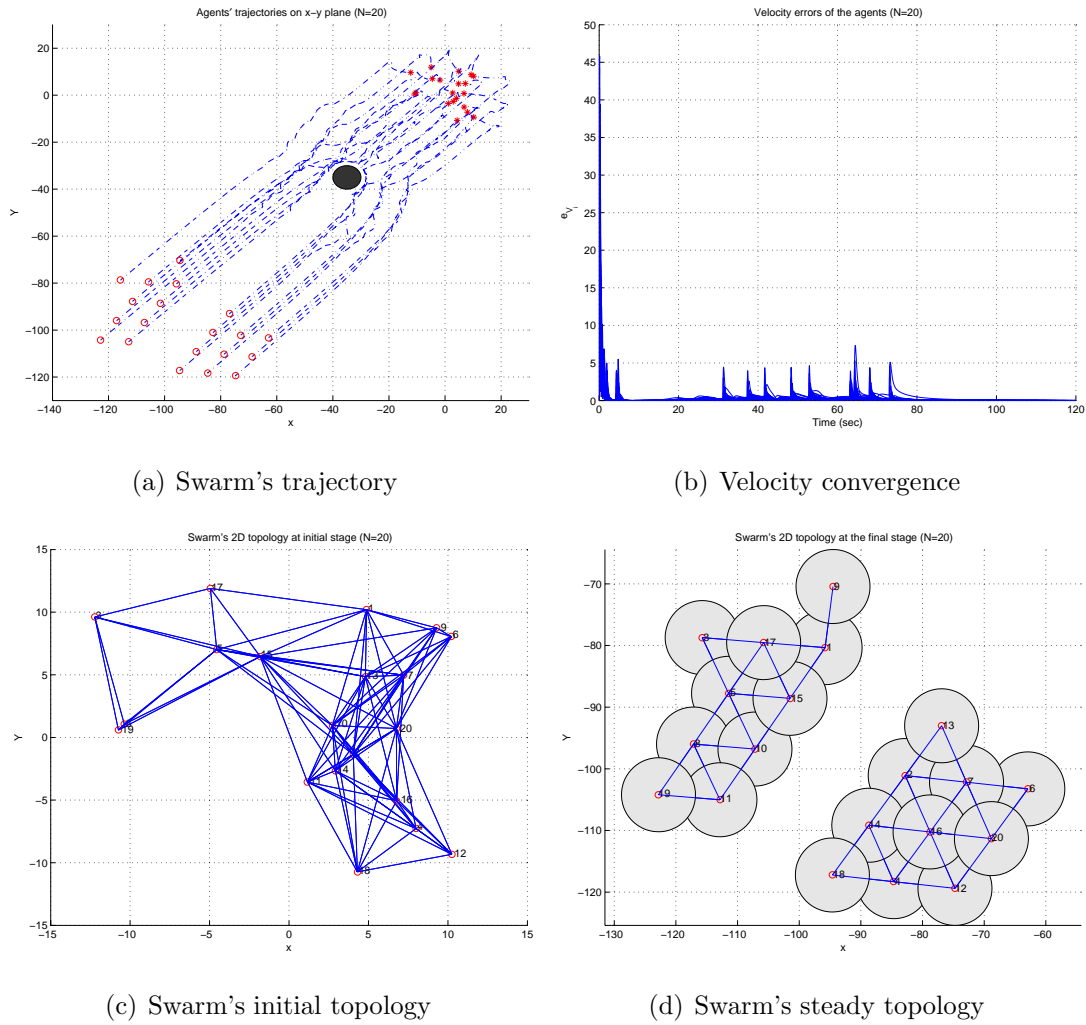


Figure 4.8: A swarm splits after avoiding a median-size obstacle in a linear environment (N=20).

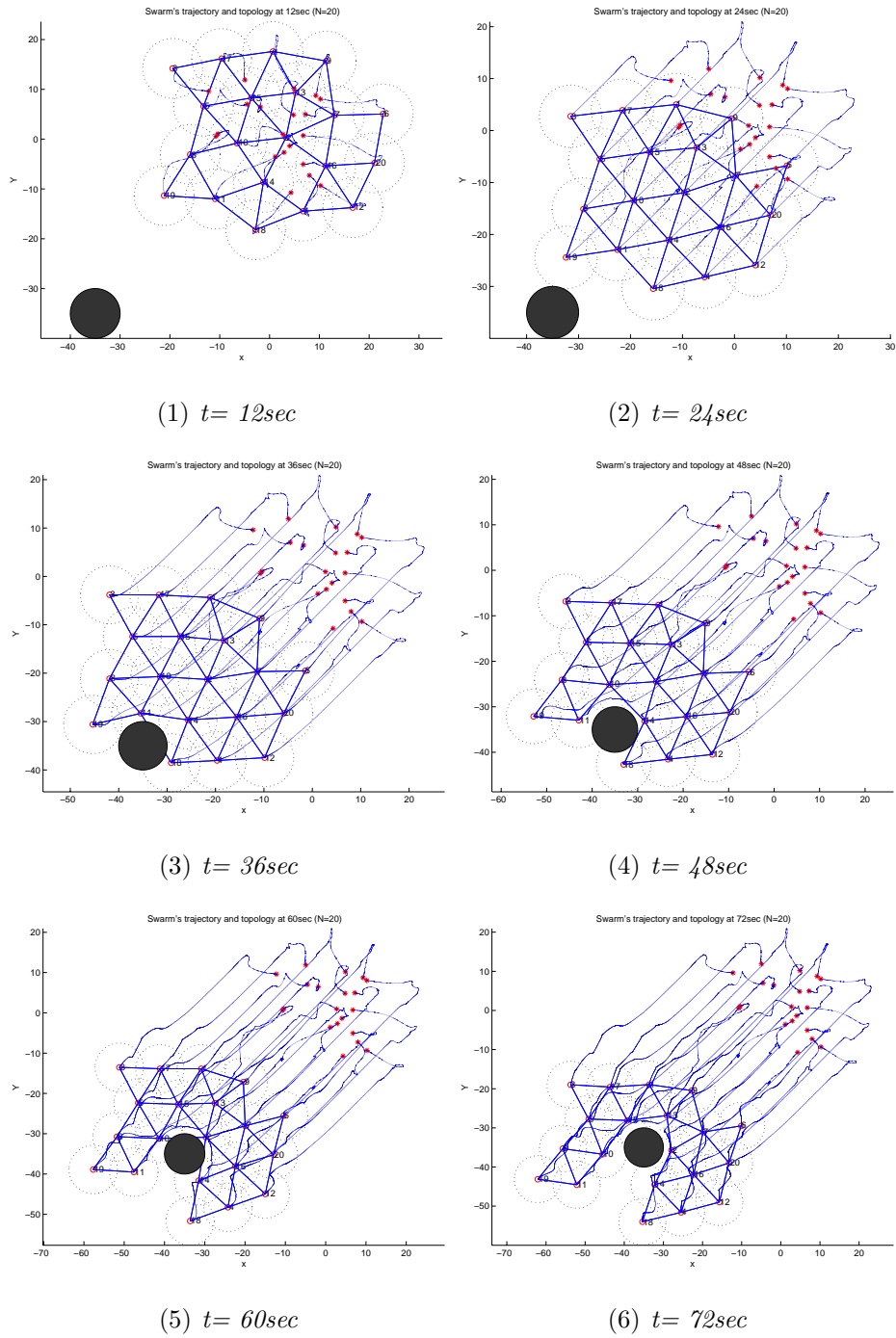


Figure 4.9a: Swarm's trajectory and topology at different time steps as shown in Fig. 4.8(a).

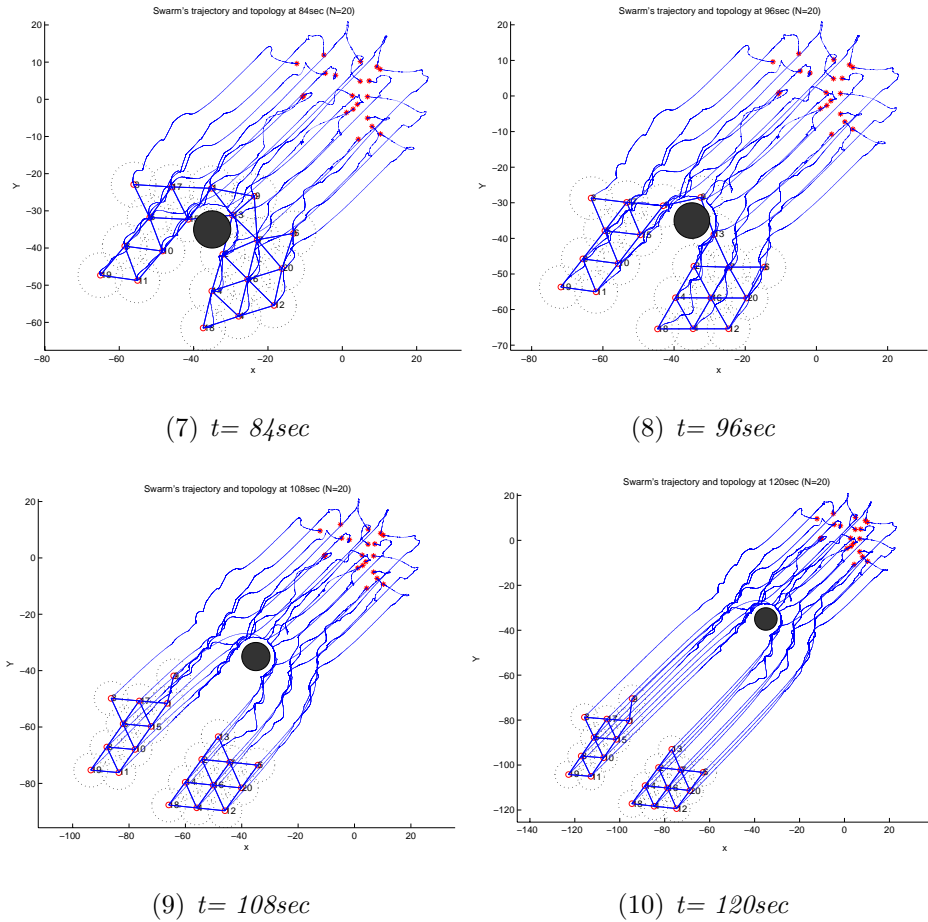


Figure 4.9b: Swarm's trajectory and topology at different time steps as shown in Fig. 4.8(a) (Continued).

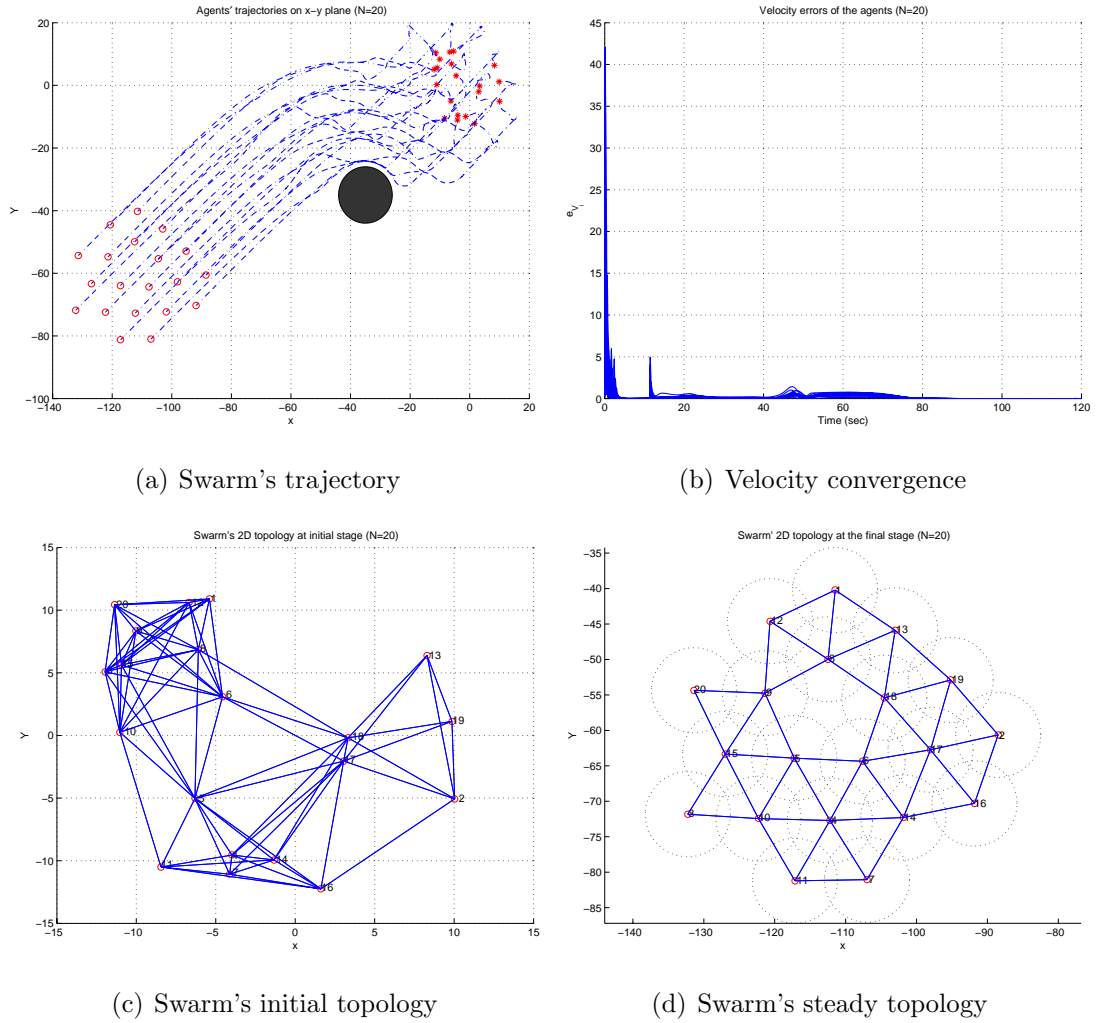


Figure 4.10: A swarm avoids a large obstacle while maintaining a formation in a linear environment ( $N=20$ ).

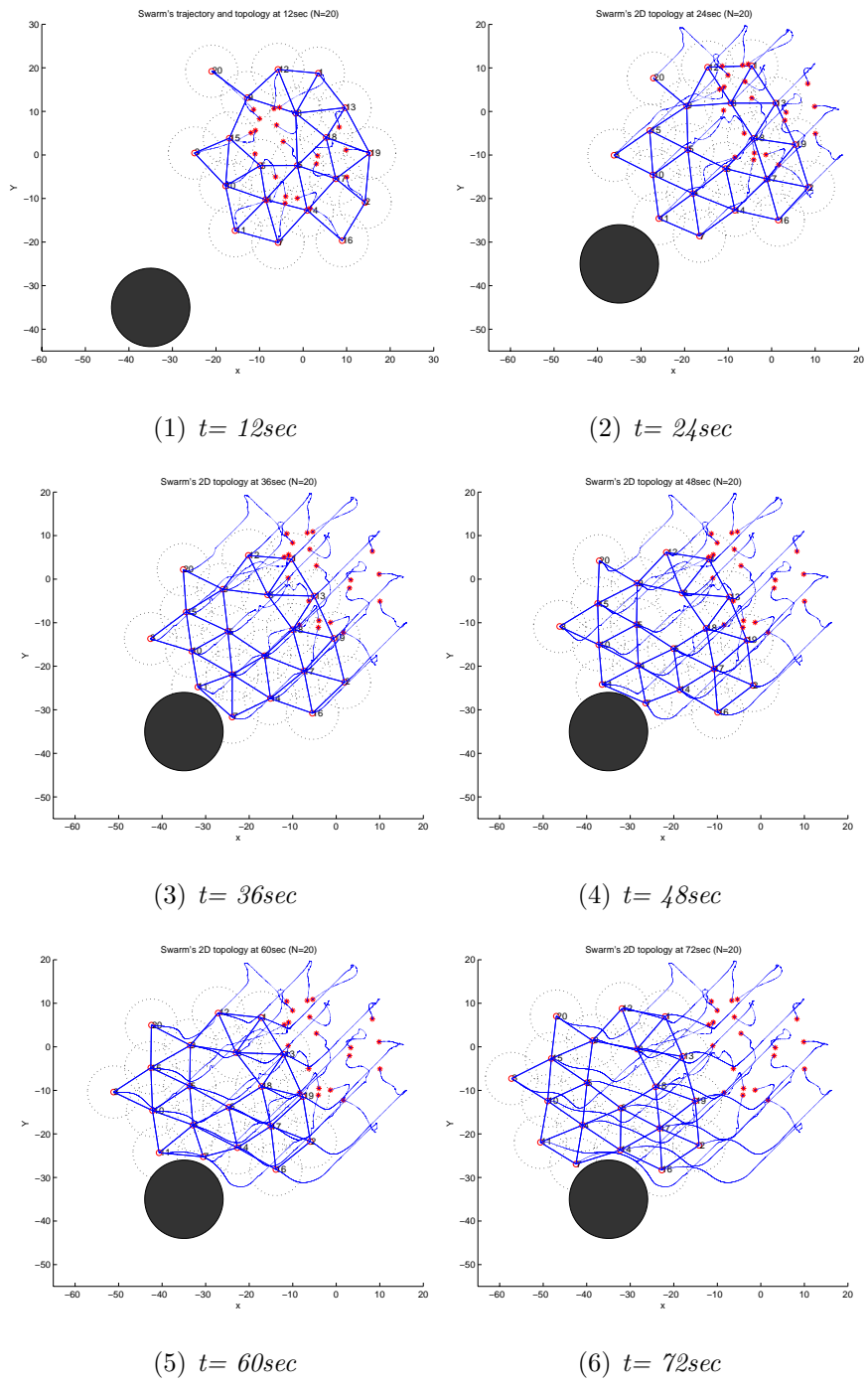
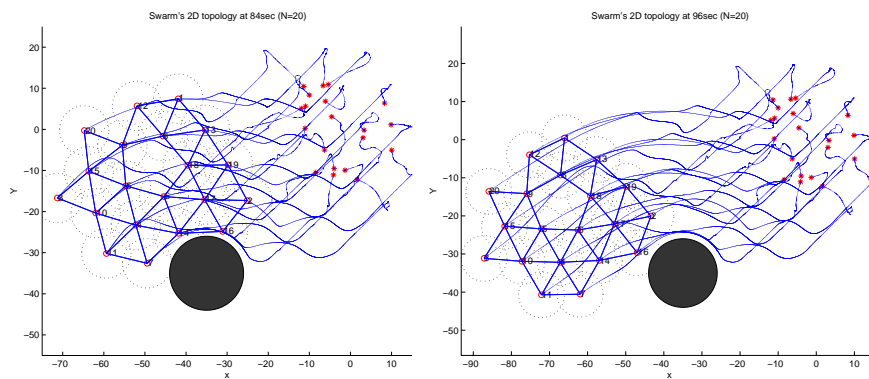
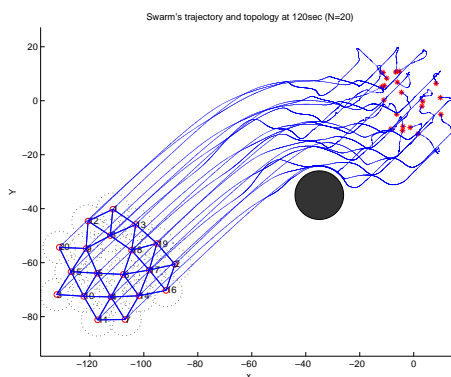


Figure 4.11a: Swarm's trajectory and topology at different time steps as shown in Fig. 4.10(a).



(7)  $t = 84sec$

(8)  $t = 96sec$



(9)  $t = 120sec$

Figure 4.11b: Swarm's trajectory and topology at different time steps as shown in Fig. 4.10(a) (*Continued*).

## Chapter 5

# Virtual Leader Tracking Swarms

In the last chapter, we study the flocking of swarms in a homogeneous environment. As shown in the analysis and simulations, the entire group can move together at the desired speed; moreover, by sensing the local gradient force of the potential field of the environment, the swarm can navigate through an adversary surrounding presented with obstacles. If considering the gradient force of the environment potential as a reference velocity signal in the controller, one can view the issue discussed in the last chapter as the velocity tracking control of swarms.

In certain areas, however, the swarm may need to not only track a reference velocity signal, but also a reference position trajectory. For example, in the environmental monitoring and the surveillance of a large battle field, it is more efficient to deploy multiple unmanned aerial vehicles (UAVs) equipped with different sensors, rather than just a single UAV packed with many sensors. The team of UAVs may need to fly together and patrol along a certain routine-like trajectory. This type of mission demands the swarm not only to move along a desired velocity, but also along a certain desired trajectory. By doing so, the entire group exhibits a behavior that seems to track a leader. Since the leader does not really exist, this type of coordination task is commonly called *virtual leader tracking control* or *virtual leader following control* of

swarms. If the leader is one of the agents, we call it *leader tracking control* or *leader following control* of swarms. The major difference between these two topics is that in the later, the leader agent has interactions with follower agents, thus, its motion needs to take account of both the reference signal and other swarm members.

Since a real leader has two states (velocity and position), *virtual leader tracking control* implied that two reference signals are available for tracking: velocity and position. Depending on the specific application, the swarm may or may not need to maintain a formation while tracking the virtual leader. For the former, it is usually called *formation tracking control*.

In this chapter, we investigate the position and velocity tracking control of swarms, with and without formation requirements. The design task is to control the swarm tracking any continuous reference signal while the geometric center of the group following the virtual leader's trajectory. We first review the classic Saber's controller in the literature, then apply the unified framework presented in Chapter 3 to design a novel consensus-term free decentralized controller. We show the swarm tracking a virtual leader with a constant and a time-varying velocities, respectively. Several sets of simulation are presented to demonstrate the controllers.

## 5.1 Saber's Controller for Virtual Leader Tracking

In (Olfati-Saber, 2006), the author presented a classic flocking algorithm for a group of agents modelled by double integrators. The algorithm is partially motivated by Reynolds' boid model (Reynolds, 1987). In (Reynolds, 1987) Reynolds presented three flocking rules to generate computer animations of collective motion of a group of points (Reynolds, 1987). Similarly, Saber's controller (Olfati-Saber, 2006) has three components: one term uses the gradient of a collective potential to guarantee flock centering and collision avoidance among members, the second term applies veloc-

ity consensus to make agents' velocities matched, and the third term incorporates navigational feedback to follow a leader. By considering an obstacle as a special type of agent, the algorithm is extended to have obstacle-avoidance capabilities. The algorithm is validated for flocking with undirected communication topologies (Olfati-Saber, 2006).

A similar flocking controller is presented for swarms with fixed and dynamic topologies (Tanner et al., 2003a,b, 2007). Some other flocking controllers that consider different scenarios or different conditions have also been presented (Lee and Spong, 2007; Tanner, 2004; Moshtagh and Jadbabaie, 2007; Moshtagh et al., 2009).

It is noticed that most of the flocking controllers presented in the literature have a common feature: the use of consensus components. For a group of mobile agents to flock together, each member need to achieve velocity agreement with respect to its neighbors. Due to this fact, it is intuitive to think of applying consensus components in the flocking controller. For the same reason, flocking is considered as a special application of consensus by some researchers (Olfati-Saber et al., 2007; Ren et al., 2007).

However, a close examination shows that the consensus terms may not be necessary for flocking. For example, in Saber's controller (Olfati-Saber, 2006), by properly choose the design parameters, the leader-following feedback term may guarantee each agent to stably track the leader; and since the leader is unique, then agent's velocity is converged to a common value (the leader's). With the term of gradient force from the collective potential, collision is avoided and cohesion is maintained among agents. By using these two terms, the swarm may already obtain the collective motion of flocking. Thus, the velocity consensus term in (Olfati-Saber, 2006) that is explicitly included to achieve velocity match may not be necessary.

Another justification for this assertion that consensus terms are not necessary can be found in some biological systems. For example, the European robins and homing

pigeons sense the magnetic field of the earth while flocking together (Wiltschko and Wiltschko, 1996), and some tropical reef fish perceive and school along the ocean currents in a massive group (Muss et al., 2001; Armsworth, 2001). Apparently a consensus-like process that requires velocity comparison and information communication (neighbor’s velocity) is hardly involved in these living beings. Indeed, the swarming phenomena in many different natural species, like fish schools and bird flocks, have been intensively studied by mathematical biologists for decades. It is generally assumed that each swarm member can perceive and follow a certain environmental guidance, meanwhile, sensing and interacting with its neighbors (see Grünbaum and Okubo, 1994; Niwa, 1996; Couzin et al., 2002; Parrish et al., 2002; Mogilner et al., 2003, and references therein).

## 5.2 A Consensus-term Free Controller

In this section, we present a consensus-term free controller for the swarm to track a virtual leader that has a constant velocity, while making the geometric center of the group to asymptotically follow the leader’s trajectory.

By applying the framework in Chapter 3, we propose a consensus-term free controller for each agent as

$$u_i = \underbrace{-k_1(r_i - r_0) - k_2(v_i - v_0)}_{\Phi_1^i} + \underbrace{\sum_{j \in \mathbb{N}_i} (f_r - f_a) \frac{r_{ij}}{\|r_{ij}\|}}_{\Phi_2^i} \quad (5.1)$$

in which  $k_1$  and  $k_2$  are positive design parameters. The implication of this controller is that each agent tracks the position ( $r_0$ ) and the velocity ( $v_0$ ) reference signals of the virtual leader, while interacting with its neighbors via virtual repulsion and attraction forces.

If the group-level task imposes that the geometric center of the swarm tracks a desired trajectory with a constant relative position, denoted by  $\tilde{h}$ , we can just change  $\Phi_1^i$  in the controller as

$$\Phi_1^i = -k_1[(r_i - r_0) - \tilde{h}] - k_2(v_i - v_0). \quad (5.2)$$

By rewriting it as

$$\Phi_1^i = -k_1[r_i - (r_0 + \tilde{h})] - k_2(v_i - v_0), \quad (5.3)$$

one can consider it as shifting the desired position signal from  $r_0$  to  $r_0 + \tilde{h}$ . Therefore, for simplicity, in this section we consider the reference position signal is along the trajectory of the virtual leader.

Note that since the swarm is not required to maintain a particular pattern, we use virtual forces, rather than a structural potential function, to generate the internal interactions  $\Phi_2^i$  in the controller (5.1). If a certain pattern needs to be preserved during the tracking motion, one can just replace the  $\sum_{j \in \mathbb{N}_i} (f_r - f_a) \frac{r_{ij}}{\|r_{ij}\|}$  with  $\sum_{j \in \mathbb{N}_i} \nabla_{r_i} U_{ij}$  in (5.1), where the structural potential functions  $U_{ij}$ 's are designated for the desired pattern.

For simplicity, denote

$$g(\|r_{ij}\|) = f_r(\|r_{ij}\|) - f_a(\|r_{ij}\|), \quad (5.4)$$

then the controller (5.1) can be written as

$$u_i = -k_1(r_i - r_0) - k_2(v_i - v_0) + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \quad (5.5)$$

In order to prevent collision among agents and keep the whole group cohesive, the combined effect of virtual forces,  $g(\|r_{ij}\|)$ , should satisfy

$$g(\|r_{ij}\|) = \begin{cases} > 0 & 0 \leq \|r_{ij}\| < d_0, \\ = 0 & \|r_{ij}\| = d_0, \\ < 0 & d_0 < \|r_{ij}\| \leq d_1, \\ = 0 & \|r_{ij}\| > d_1. \end{cases} \quad (5.6)$$

Similarly to the Section 4.1, we also assume that  $g(\|r_{ij}\|)$  is locally Lipschitz, and continuous at  $\|r_{ij}\| < d_1$ . That is, its discontinuity occurs only at  $d_1$ .

With the controller handy, we need to prove that it makes the velocities of all agents converge to a common velocity, which also approaches to the virtual leader's velocity, and the geometric center of the swarm asymptotically follows the virtual leader's trajectory.

Before proceeding, we define an error state as

$$e_{r_i} = r_i - \bar{r}, \quad (5.7)$$

in which

$$\bar{r} = \frac{1}{N} \sum_{i=1}^N r_i \quad (5.8)$$

is the geometric center of the swarm.

Then we have the error dynamics as

$$\dot{e}_{r_i} = e_{v_i}, \quad (5.9)$$

$$\dot{e}_{v_i} = \dot{v}_i - \dot{\bar{v}} \quad (5.10)$$

Using (5.1), it yields to

$$\dot{v}_i - \dot{\bar{v}} = -k_1(r_i - \bar{r}) - k_2(v_i - \bar{v}) + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} - \frac{1}{N} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}. \quad (5.11)$$

Since  $g(\|r_{ij}\|) = g(\|r_{ji}\|)$ , then  $g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = -g(\|r_{ji}\|) \frac{r_{ji}}{\|r_{ji}\|}$ . In addition, because  $\mathbb{N}_i$  is always symmetric regardless that the swarm's topological graph  $\mathcal{G}$  is connected or not, then we have

$$\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = 0. \quad (5.12)$$

Therefore, similarly to in Section 4.1, we have the following differential inclusion for the error dynamics:

$$\dot{e}_{v_i} \in^{a.e.} K[e_{v_i}] = \left\{ -k_1 e_{r_i} - k_2 e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \right\}. \quad (5.13)$$

Use a candidate Lyapunov function as

$$V_t = \frac{1}{2}k_1 \sum_{i=1}^N e_{r_i}^T e_{r_i} + \frac{1}{2} \sum_{i=1}^N e_{v_i}^T e_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N E_{ij}(\|r_{ij}\|), \quad (5.14)$$

where  $E_{ij}(\|r_{ij}\|)$  is defined same as in Section 4.1:

$$E_{ij}(\|r_{ij}\|) = \int_{\|r_{ij}\|}^{d_0} g(\tau) d\tau, \quad (5.15)$$

with its general gradient being

$$\partial E_{ij} = \begin{cases} \|r_{ij}\| < d_1 : & = -g(\|r_{ij}\|), \\ \|r_{ij}\| > d_1 : & = 0, \\ \|r_{ij}\| = d_1 : & = \overline{\text{co}}[-g(d_1), 0]. \end{cases} \quad (5.16)$$

in which  $\overline{\text{co}}[\cdot]$  is the closed convex hull.

Clearly,  $V_t$  is a nonnegative function of  $e_{r_i}$ ,  $e_{v_i}$  and  $\|r_{ij}\|$ , and continuously differentiable with respect to  $e_{r_i}$  and  $e_{v_i}$ . But because  $E_{ij}$  is not continuously differentiable about  $\|r_{ij}\|$ , so is  $V_t$ . As it is shown in Lemma. 4.1,  $E_{ij}(\|r_{ij}\|)$  is locally Lipschitz in its domain. Thus,  $V_t$  is also locally Lipschitz. Therefore, from Rademacher's Theorem (Clarke, 1990),  $V_t$  is differentiable almost everywhere in its entire domain.

Compared to the Lyapunov function used in the Section 4.1,  $V_t$  defined here (5.14) has an additional term of  $\sum_{i=1}^N e_{r_i}^T e_{r_i}$ . Because  $\sum_{i=1}^N e_{r_i}^T e_{r_i}$  is convex,  $V_t$  is regular (see the proof of Lemma. 4.2).

**Theorem 5.1** *For a virtual leader with bounded  $v_0$  and  $r_0$ , if the combined effect of the virtual repulsion and attraction forces are selected such that  $g(\|r_{ij}\|)$  satisfies (5.6), then the decentralized controller (5.1) enables that the velocities of all agents asymptotically converge to their average velocity.*

*Proof:* Since  $V_t$  is a function of  $e_{r_i}$ ,  $e_{v_i}$  and  $\|r_{ij}\|$ , we denote it by  $V_t(e_{r_i}, e_{v_i}, \|r_{ij}\|)$ .

Because  $V_t$  is locally Lipschitz and differentiable almost everywhere, we have its generalized gradient (Clarke, 1990) as

$$\partial V_t = \overline{\text{co}}\{\lim \nabla V_t(e_{r_i}, e_{v_i}, \|r_{ij}\|), \|r_{ij}\| \notin \Omega_V, i, j = 1, \dots, N\},$$

in which  $\Omega_V$  is a set with zero measure where the gradient of  $V_t$  is not defined. Specifically,

$$\partial V_t = [k_1 e_{r_1}^T, \dots, k_1 e_{r_N}^T, e_{v_1}^T, \dots, e_{v_N}^T, \frac{1}{2} \partial E_{11}, \dots, \frac{1}{2} \partial E_{ij}, \dots, \frac{1}{2} \partial E_{NN}]^T. \quad (5.17)$$

For simplicity, denote  $\zeta_{ij} = \frac{1}{2} \partial E_{ij}$ , then  $\partial V_t$  can be written as

$$\partial V_t = [k_1 e_{r_1}^T, \dots, k_1 e_{r_N}^T, e_{v_1}^T, \dots, e_{v_N}^T, \zeta_{11}, \dots, \zeta_{ij}, \dots, \zeta_{NN}]^T. \quad (5.18)$$

Using the chain rule of set-valued Lie derivative and the error dynamics (5.13), we have

$$\frac{dV_t}{dt} \in^{a.e.} \dot{\tilde{V}}_t, \quad (5.19)$$

where

$$\dot{\tilde{V}}_t = \bigcap_{\xi \in \partial V_t} \xi^T \cdot \{e_{v_1}^T, \dots, e_{v_N}^T, K[e_{v_1}], \dots, K[e_{v_N}], \frac{d\|r_{11}\|}{dt}, \dots, \frac{d\|r_{ij}\|}{dt}, \dots, \frac{d\|r_{NN}\|}{dt}\}^T \quad (5.20)$$

with  $K[e_{v_i}]$  is determined in (5.13).

Using the notation defined in (5.18), it yields to

$$\dot{\tilde{V}}_t = \bigcap_{\xi \in \partial V_t} \left\{ \sum_{i=1}^N k_1 e_{r_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\}. \quad (5.21)$$

For simplicity, let

$$\Gamma = \sum_{i=1}^N k_1 e_{r_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt}. \quad (5.22)$$

Note that  $V_t$  is continuous and differentiable about  $e_{r_i}$  and  $e_{v_i}$ , but not  $\|r_{ij}\|$ . Also note that due to the discontinuity of  $g(\|r_{ij}\|)$  at  $\|r_{ij}\| = d_1$ ,  $K[e_{v_i}]$  and  $\zeta_{ij}$  share the

same nonsmooth subset. Thus, similarly to the Section 4.1, we discuss  $\tilde{V}_t$  piece-wisely on the sub-domain of  $\|r_{ij}\|$  in order to evaluate  $\frac{dV_i}{dt}$  in its entire domain.

If for  $\forall i, \|r_{ij}\| > d_1$  where  $j \in \{1, \dots, N\} \setminus \{i\}$ , then  $\zeta_{ij} = 0$  and  $K[e_{v_i}] = -k_1 e_{r_i} - k_2 e_{v_i}$ . Hence,

$$\begin{aligned} \Gamma &= \sum_{i=1}^N k_1 e_{r_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \cdot (-k_1 e_{r_i} - k_2 e_{v_i}) + \sum_{i=1}^N \sum_{j=1}^N 0 \cdot \frac{d\|r_{ij}\|}{dt} \\ &= -k_2 \sum_{i=1}^N e_{v_i}^T e_{v_i}. \end{aligned} \quad (5.23)$$

If for  $\forall i, \|r_{ij}\| < d_1$  where  $j \in \mathbb{N}_i$ , then  $\zeta_{ij} = -\frac{1}{2}g(\|r_{ij}\|)$ . Then we have

$$\begin{aligned} \Gamma &= \sum_{i=1}^N k_1 e_{r_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \cdot \left\{ -k_1 e_{r_i} - k_2 e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \right\} - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \\ \frac{d\|r_{ij}\|}{dt} &= -k_2 \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} e_{v_i}^T g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \left\{ -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt} \right\}. \end{aligned} \quad (5.24)$$

By applying the same technique as in (4.30) in the proof of Theorem 4.1, we have

$$\Gamma = -k_2 \sum_{i=1}^N e_{v_i}^T e_{v_i}. \quad (5.25)$$

If for  $\forall i, \|r_{ij}\| = d_1$  where  $j \in \mathbb{N}_i$ , then  $\zeta_{ij} \in \overline{\text{co}}[-\frac{1}{2}g(d_1), 0]$ , and  $K[e_{v_i}] = -k_1 e_{r_i} - k_2 e_{v_i} + \overline{\text{co}}[g(d_1), 0] \sum_{j \in \mathbb{N}_i} \frac{r_{ij}}{\|r_{ij}\|}$ . Then

$$\begin{aligned} \tilde{V}_t |_{\|r_{ij}\|=d_1} &= \bigcap_{\xi \in \partial V_t} \left\{ \sum_{i=1}^N k_1 e_{r_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\} \\ &= \bigcap_{\zeta_{ij} \in \overline{\text{co}}[-\frac{1}{2}g(d_1), 0]} \left\{ \sum_{i=1}^N k_1 e_{r_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\}. \end{aligned} \quad (5.26)$$

By using  $K[e_{v_i}]$ , we have

$$\begin{aligned} \dot{\tilde{V}}_t \big|_{\|r_{ij}\|=d_1} = & \bigcap_{\zeta_{ij} \in \overline{\text{co}}[g(d_1), 0]} \left\{ \sum_{i=1}^N -k_2 e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} e_{v_i}^T \cdot \overline{\text{co}}\left[-\frac{1}{2}g(d_1), 0\right] \frac{r_{ij}}{\|r_{ij}\|} \right. \\ & \left. + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \zeta_{ij} \frac{d\|r_{ij}\|}{dt} \right\}. \end{aligned} \quad (5.27)$$

By exploring the same evaluation as in (4.33), we can have

$$\dot{\tilde{V}}_t \big|_{\|r_{ij}\|=d_1} \subseteq \left\{ \sum_{i=1}^N -k_2 e_{v_i}^T e_{v_i} \right\}. \quad (5.28)$$

Therefore, on the entire domain of  $\|r_{ij}\| \geq 0$ , we have

$$\dot{\tilde{V}}_t \subseteq \left\{ \alpha : \alpha = \sum_{i=1}^N -k_2 e_{v_i}^T e_{v_i} \right\}. \quad (5.29)$$

Using (5.19), it leads to

$$\frac{dV_t}{dt} \in^{a.e.} \left\{ \alpha : \alpha = \sum_{i=1}^N -k_2 e_{v_i}^T e_{v_i} \leq 0 \right\}. \quad (5.30)$$

Thus, all  $\frac{d}{dt}V_t \leq 0$ . This means that for any agent,  $e_{v_i}$  is stable, and  $(e_{r_i}, \|r_{ij}\|)$  are bounded.

From (5.30), one can see that the only option for  $\alpha = 0$  is  $e_{v_i} = 0$ . Thus, from the nonsmooth version of LaSalle's invariance principle, we know that the states  $(e_{v_i}, e_{r_i}, \|r_{ij}\|)$  approach to the largest invariant set in

$$\begin{aligned} \bar{S} &= cl(\{(e_{v_i}, e_{r_i}, \|r_{ij}\|) \mid 0 \in \dot{\tilde{V}}_t, i, j = 1, \dots, N\}) \\ &= cl(\{(0, e_{r_i}, \|r_{ij}\|) \mid 0 \in \dot{\tilde{V}}_t, i, j = 1, \dots, N\}). \end{aligned} \quad (5.31)$$

where  $cl(\cdot)$  is the closure of a set. This means that the velocity vectors of all agents asymptotically converge to the average  $\bar{v}$ .  $\square$

Furthermore, we show that if the virtual leader has a constant velocity, the average velocity of the swarm  $\bar{v}$  will asymptotically approach to it.

**Theorem 5.2** *If the velocity of the virtual leader satisfies that  $\dot{v}_0 = 0$ , then the average velocity of the swarm asymptotically approaches to  $v_0$  and the geometric center of the group asymptotically tracks the virtual leader's trajectory.*

*Proof:* Consider  $\bar{r} = \frac{1}{N} \sum_{i=1}^N r_i$  and  $\bar{v} = \frac{1}{N} \sum_{i=1}^N v_i$  as two virtual states of the system and evaluate their dynamics as follows. It is straight to have

$$\dot{\bar{v}} = \frac{1}{N} \sum_{i=1}^N \dot{v}_i = \frac{1}{N} \sum_{i=1}^N \left\{ -k_1(r_i - r_0) - k_2(v_i - v_0) + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \right\}$$

Since it is always that  $\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = 0$  in spite of that the swarm's topological graph is connected, then

$$\dot{\bar{v}} = -k_1(\bar{r} - r_0) - k_2(\bar{v} - v_0).$$

Thus, we have

$$\begin{cases} \dot{\bar{r}} = \bar{v}, \\ \dot{\bar{v}} = -k_1(\bar{r} - r_0) - k_2(\bar{v} - v_0). \end{cases} \quad (5.32)$$

This is actually the dynamic equation of the geometric center of the swarm. If we do not consider the internal structure of the swarm, the virtual leader tracking control can be considered, in a certain extent, just as a tracking control problem of the above  $(\bar{r}, \bar{v})$  system.

Define two error states as

$$e_{\bar{r}} = \bar{r} - r_0; \quad e_{\bar{v}} = \bar{v} - v_0. \quad (5.33)$$

Then,

$$\begin{aligned} \dot{e}_{\bar{r}} &= e_{\bar{v}}, \\ \dot{e}_{\bar{v}} &= -k_1 e_{\bar{r}} - k_2 e_{\bar{v}} - \dot{v}_0. \end{aligned}$$

Write it into matrix form as

$$\begin{bmatrix} \dot{e}_{\bar{r}} \\ \dot{e}_{\bar{v}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} e_{\bar{r}} \\ e_{\bar{v}} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{v}_0. \quad (5.34)$$

Since  $\dot{v}_0 = 0$ , then we have the error dynamics as

$$\begin{bmatrix} \dot{e}_{\bar{r}} \\ \dot{e}_{\bar{v}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}}_P \begin{bmatrix} e_{\bar{r}} \\ e_{\bar{v}} \end{bmatrix}. \quad (5.35)$$

Considering the preceding linear model, it is not hard to know that for any positive  $k_1$  and  $k_2$ ,  $P$  is always Hurwitz. Thus,  $(e_{\bar{r}}, e_{\bar{v}})$  have an asymptotically stable equilibrium point at  $(0,0)$ , i.e.,  $e_{\bar{r}}$  and  $e_{\bar{v}}$  asymptotically goes to zero.

Therefore, the entire swarm asymptotically obtain a common velocity that is equal to the virtual leader's velocity, and the geometric center asymptotically tracks the virtual leader's trajectory.  $\square$

### 5.3 Tracking a Virtual Leader with Time-varying Velocity

In the last section, we present a consensus-term free controller for a swarm to track a virtual leader. We show that if the virtual leader's velocity is constant, then the velocities of all agents asymptotically reach the leader's velocity and the geometric center of the swarm asymptotically track the trajectory of the virtual leader.

However, in a more general situation, the swarm may need to follow a virtual leader with a time-varying or even arbitrary velocity, for example, along a circle or an ellipse.

One might be tempted to use the same controller as in (5.1). However, it cannot achieve the desired group-level task. As an example, in the Fig. 5.1, we use the controller (5.1) and show the motion of a group of 10 agents tracking a virtual leader along an ellipse. Apparently, when the virtual leader moves along an ellipse, its velocity is not constant. Fig.5.1(a) shows the collective motion of the group following

the virtual leader, in which the red line is the leader's trajectory. As one can see, Fig.5.1(b) shows that the velocities of all agent converge to the average velocity, but the average velocity does not converge to the virtual leader's velocity (see Fig.5.1(c)).

The reason for this behavior is that if the virtual leader's velocity is not a constant, i.e.,  $\dot{v}_0 = 0$ , then according to the error dynamics in (5.34),  $e_{\bar{r}}$  and  $e_{\bar{v}}$  cannot asymptotically reach the equilibrium  $(0, 0)$ .

It is interesting to note that although the controller (5.1) is not able to make the swarm accomplish the group-level task (i.e., asymptotically track the virtual leader), the inter-individual-level sub-task can still be achieved. Attributing to the internal interaction term  $\Phi_2^i$  in the controller (5.1), the swarm members stay cohesive and avoid collision among them. This can be seen in the Fig. 5.1 (d) and (e), both of which show the swarm's topologies at the initial and the final stages, respectively.

An optional solution to deal with the non-zero  $\dot{v}_0$  is to use high-gain controllers. By considering  $\dot{v}_0$  as a disturbance in (5.34), one can use a high-gain controller with a large gain  $k_2$  to reject the effect of  $\dot{v}_0$ . However, using high-gain controller may cause the cost of the controller too high to be implemented.

Assuming that the dynamics of the reference velocity is known, we present a decentralized controller to achieve the virtual-leader tracking control of the swarm as follows:

$$u_i = -k_1(r_i - r_0) - k_2(v_i - v_0) + \dot{v}_0 + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \quad (5.36)$$

in which  $k_1$  and  $k_2$  are design parameters,  $\dot{v}_0$  is the dynamics of the reference velocity, and  $g(\|r_{ij}\|)$  has the same condition as in (5.6).

We use the same error states as defined in previous sections:  $e_{r_i} = r_i - \bar{r}$  and  $e_{v_i} = v_i - \bar{v}$ . It is easy to see that

$$\dot{v}_i - \dot{\bar{v}} = -k_1 e_{r_i} - k_2 e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} - \frac{1}{N} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} \quad (5.37)$$

### 5.3. TIME-VARYING REFERENCE TRACKING

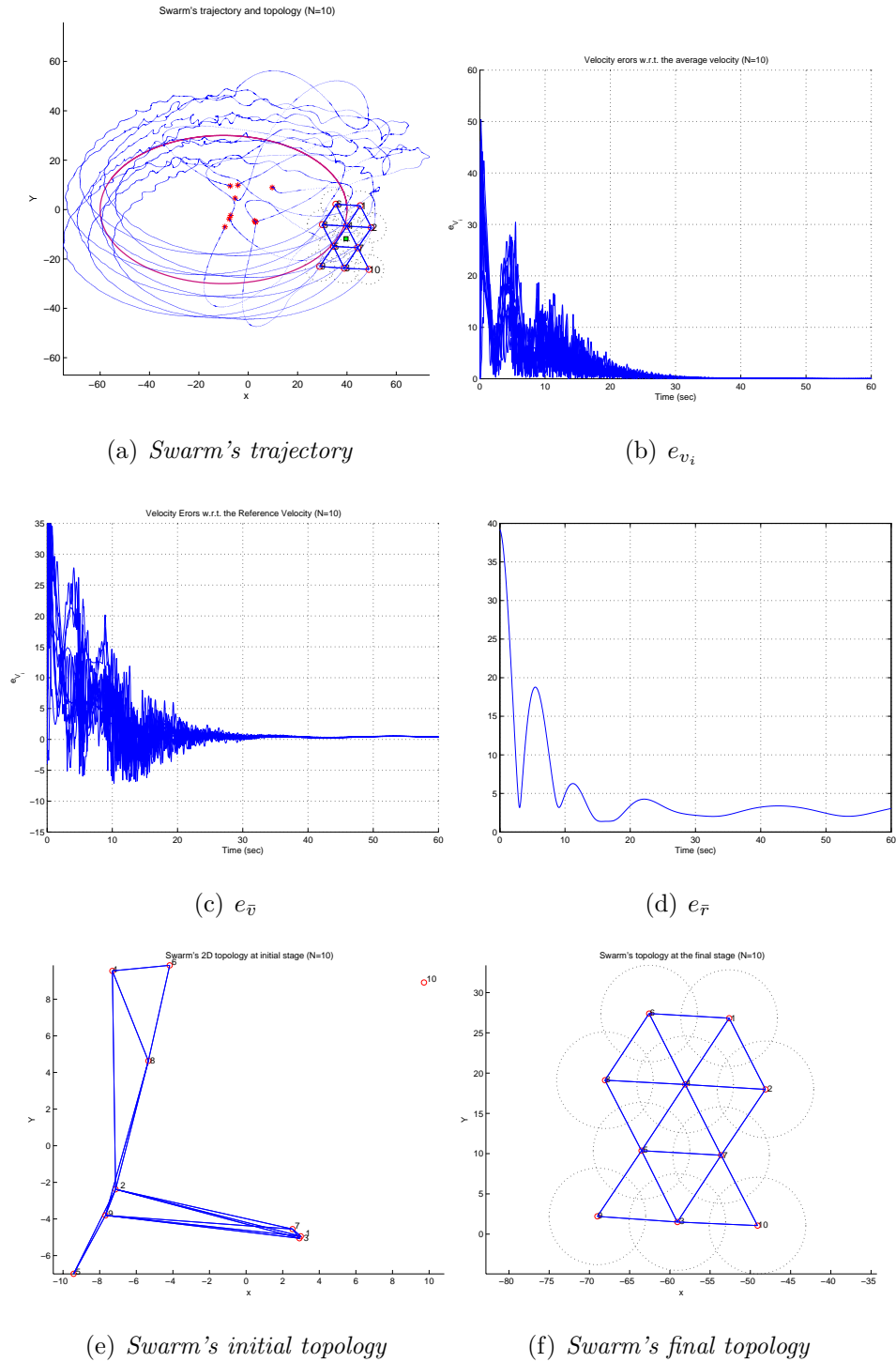


Figure 5.1: A swarm tracks a virtual leader along an ellipse ( $N=10$ ).

As in (5.12), we know

$$\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = 0.$$

Thus, we have the same error dynamics as in (5.13):

$$\dot{e}_{v_i} \in^{a.e.} K[e_{v_i}] = \{-k_1 e_{r_i} - k_2 e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}\}. \quad (5.38)$$

By exploring the same evaluation steps as in the previous section, we obtain the following results for the tracking control of swarms.

**Theorem 5.3** *For a virtual leader with bounded  $r_0$ ,  $v_0$  and  $\dot{v}_0$ , if the combined effect of the virtual repulsion and attraction forces are selected such that  $g(\|r_{ij}\|)$  satisfies (5.6), then the decentralized controller (5.36) enables that the velocities of all agents asymptotically converge to their average velocity.*

*Proof:* Similar to the proof of Theorem 5.1 and skipped here. □

For the group-level behavior of the swarm, we have the following theorem.

**Theorem 5.4** *The average velocity of the swarm asymptotically approaches to virtual leader's velocity and the geometric center of the group asymptotically tracks the virtual leader's trajectory.*

*Proof:* It is straight to have that

$$\dot{\bar{v}} = -k_1(\bar{r} - r_0) - k_2(\bar{v} - v_0) + \dot{v}_0.$$

Using the same error states as defined in (5.33), we have

$$\dot{e}_{\bar{v}} = \dot{\bar{v}} - \dot{v}_0 = -k_1(\bar{r} - r_0) - k_2(\bar{v} - v_0).$$

That is,

$$\begin{bmatrix} \dot{e}_{\bar{r}} \\ \dot{e}_{\bar{v}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} e_{\bar{r}} \\ e_{\bar{v}} \end{bmatrix}. \quad (5.39)$$

Note that this is the same as (5.35). Therefore, by the same analysis as for (5.35), we know that  $e_{\bar{r}} \rightarrow 0$  and  $e_{\bar{v}} \rightarrow 0$  as  $t \rightarrow \infty$ . That is, the velocities of all agents converge to the virtual leader's velocity and the geometric center of the group asymptotically follow the trajectory of the virtual leader.  $\square$

## 5.4 Remarks on the Controllers

**Remark 5.1** Note that in the controllers (5.1) and (5.36), the agents need to know the states ( $r_0$  and  $v_0$ ) of the virtual leader. If applying the same controllers for the problem of leader tracking control, in which one of the agents act as the leader, certain information exchange mechanisms need to be implemented for the follower agents to be aware of the leader's states. For a small size swarm, this can be achieved by certain sensing devices equipped with the agents. For a large swarm, some long-range communication links may be necessary.

Recall that in the controllers for flocking in homogeneous environments (Chapter 4), which can be considered as velocity tracking control, all the information needed by the controller can be locally sensed, and communication modules are not needed. This is the major difference between velocity tracking control and position-velocity tracking control.

**Remark 5.2** Note that for the two controllers (5.1) and (5.36), due to the internal interactions among swarm members, individual agents cannot asymptotically track the reference position trajectory when the geometric center of the swarm tracks.

When applying the framework to design the controller, to accomplish the group-level mission, we explicitly select feedback terms, such as  $-k_1(r_i - r_0)$  and  $-k_2(v_i - v_0)$ , for the intention of driving  $r_i \rightarrow r_0$ . As mentioned above, in the final phase of swarm's motion,  $r_i$  cannot converge to  $r_0$ . Note that although individual agents do not exactly behave as our intention, the group-level task is achieved by exploiting the framework.

From this, one can see the merit of the presented framework as a guideline to design control strategy for the distributed motion coordination of swarms.

**Remark 5.3** Compared to most of the pre-existing works on flocking in the literature (Olfati-Saber, 2006; Tanner et al., 2007; Ren, 2007; Housheng Su and Lin, 2009; Lee and Spong, 2007, etc.), the convergence analysis of the swarm's collective motion in Theorem 5.1 and 5.3 does not require any condition on the connectivity of the swarm's topological graph. This is mainly because, unlike those works in the literature, we do not use local consensus terms in our controllers. If the consensus terms are applied in the controller to gain the velocity or position matching among members, then the swarm's topology graph needs to be always connected. Otherwise, the consensus process cannot spread over the entire group, and some of the agents cannot achieve the convergence.

**Remark 5.4** Note that for the problem of virtual leader tracking control, both the classic Saber's controller and our controllers in (5.1) and (5.36) have an underlying condition that the reference information of the virtual leader needs to be available. However, this may not be the case in real applications.

Moreover, if the proposed controller (5.1) or (5.36) is used for *leader tracking* control, in which the leader is one of the agents, certain explicit or implicit communication channels are required in order to communicate the information from the leader to all follower agents. This may cause a high load of information exchange and large time-delay, particularly when the size of the swarm is large.

A candidate and promising strategy to reduce the communication burden is to add a local consensus term in the micro-level component  $\Phi_i^2$  (Ren, 2007; Housheng Su and Lin, 2009). The follower agents just need to achieve information consensus among local neighbors, and not need to directly communicate with the leader. The leader's states can propagate through the entire group by the local consensus terms. At the stage of this thesis, we are still working on this approach. We do not include the

discussion of it in the thesis.

## 5.5 Simulation Study

To demonstrate the proposed consensus-term free controllers, we present two sets of simulation results of two swarms of mobile agents tracking a virtual leader without any formation requirement. The geometric center of the group is controlled to asymptotically follow the virtual leader's position trajectory.

In the simulations, since the swarm is not required to maintain a specific pattern, we use virtual forces rather than potential field to generate the internal interactions among agents. We select a piece-wise linear function for the virtual forces as: for  $\|r_{ij}\| \leq d_1$ :  $g(\|r_{ij}\|) = -30\|r_{ij}\| + 30d_0$ , in which the critical distance  $d_0 = 10$  and the neighborhood defining distance  $d_1 = 12$ . The design parameters in the controllers are  $k_1 = 5$  and  $k_2 = 5$ .

Fig.5.2(a)-5.5 shows a swarm of 10 agents tracking a virtual leader along a circle. The position trajectory of the virtual leader is defined by  $\|r_0 - [-10, 0]^T\| = 40$ , starting from  $r_0(t = 0) = [30, 0]^T$ . Its velocity is  $[-4 \sin(0.1t), 4 \cos(0.1t)]^T$ . Initially, all the agents are randomly located in a  $20 \times 20$  square centered at  $[0, 0]$ . Their initial velocities are randomly given.

Fig. 5.2(a) shows the trajectories of the agents and the virtual leader. The stars and circles in the figure represent the initial and the final positions of the agents, respectively. The trajectory of the virtual leader in the figure is marked by "V.L.". The small green square in the figure represents the geometric center of the group at the final location.

Fig. 5.2(b) and 5.2(c) show the swarm's topologies at the initial and the final stages, respectively. It is clear that at the final stage, the agents are evenly spaced from each other. Due to the mutual interactions among members, the entire group

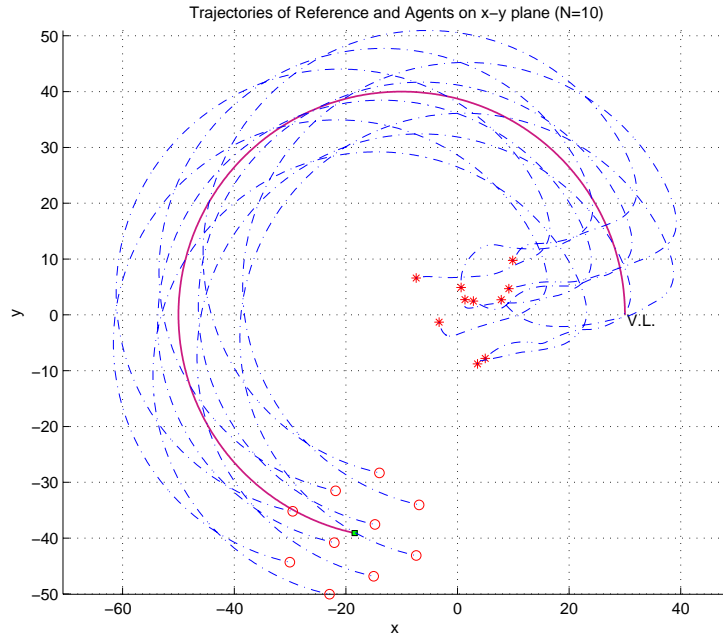
shows a structure of lattice composed by equilateral triangles.

The swarm's trajectory and topology at different time steps are shown in Fig. 5.5. In the figure, the stars represent the initial positions of the agents, the pink circle is the virtual leader's trajectory, and the small green square is the geometric center of the swarm at the corresponding time constant. One can see that the geometric center follows the trajectory of the virtual leader.

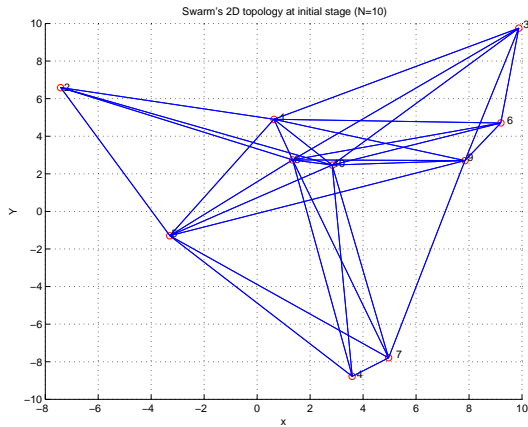
Fig. 5.3(a) and 5.3(b) show the velocity errors of the agents with respect to the average velocity and the virtual leader's velocity, respectively. The tracking error of the geometric center is shown in Fig. 5.4. It is clear that the velocities of all agents converge to the virtual leader's velocity, and the geometric center of the swarm asymptotically track the virtual leader's position trajectory.

Fig.5.6-5.8 shows 12 agents tracking a virtual leader along the same circle as in Fig.5.2(a). The agents are initially located as two separate sub-groups: 5 of the agents are randomly located in a  $20 \times 20$  square centered at  $[25, 70]$ , and all others are randomly positioned around  $[0, 0]$ . Apparently, the swarm's topological graph at the initial period is not connected.

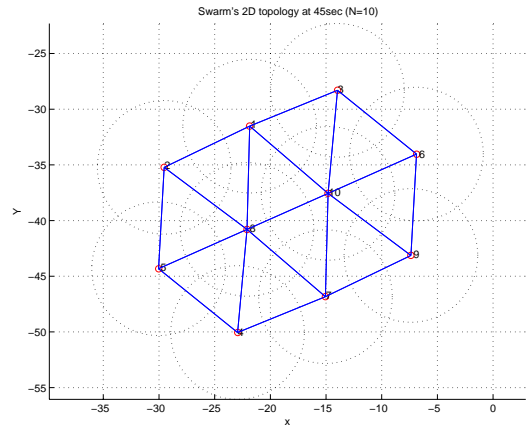
Fig. 5.6(a) shows the trajectories of the agents and the virtual leader. Fig. 5.6(b) and 5.6(c) show the velocity convergence of the group. The swarm's trajectory and topology at different time steps are shown in Fig.5.7. Fig. 5.8 shows the changing of the swarm's topology when the two sub-groups merge. It can be seen that the entire group achieves the tracking task even with unconnected topology.



(a) Trajectories of agents and virtual leader



(b) Swarm's initial topology



(c) Swarm's final topology

Figure 5.2: A swarm tracks a virtual leader along a circle ( $N=10$ ).

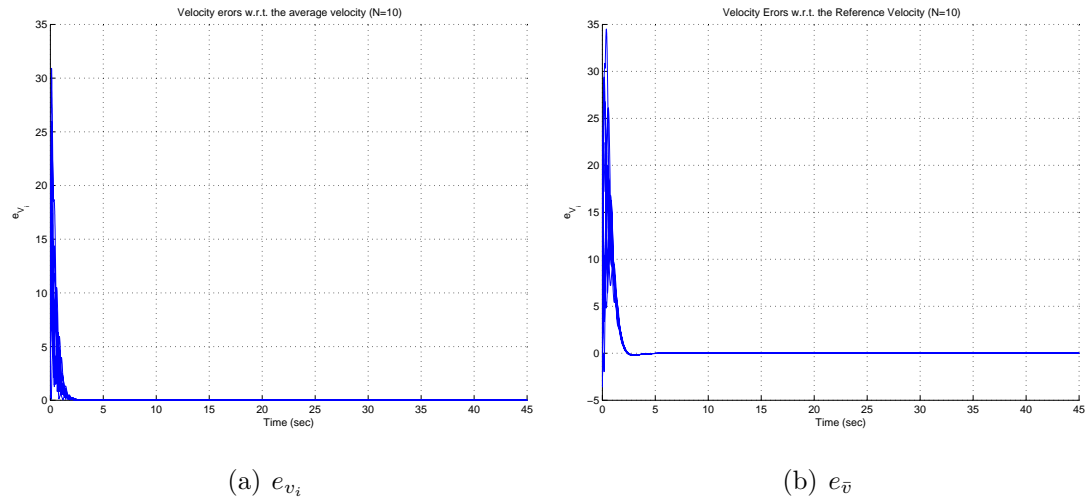


Figure 5.3: Velocity convergence of the swarm as shown in Fig. 5.2(a) .

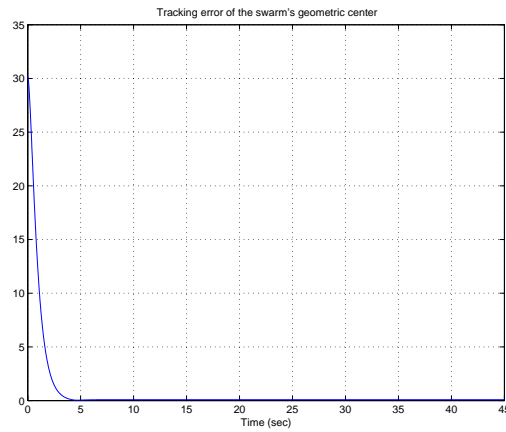


Figure 5.4: Tracking error of the swarm's geometric center w.r.t the virtual leader.

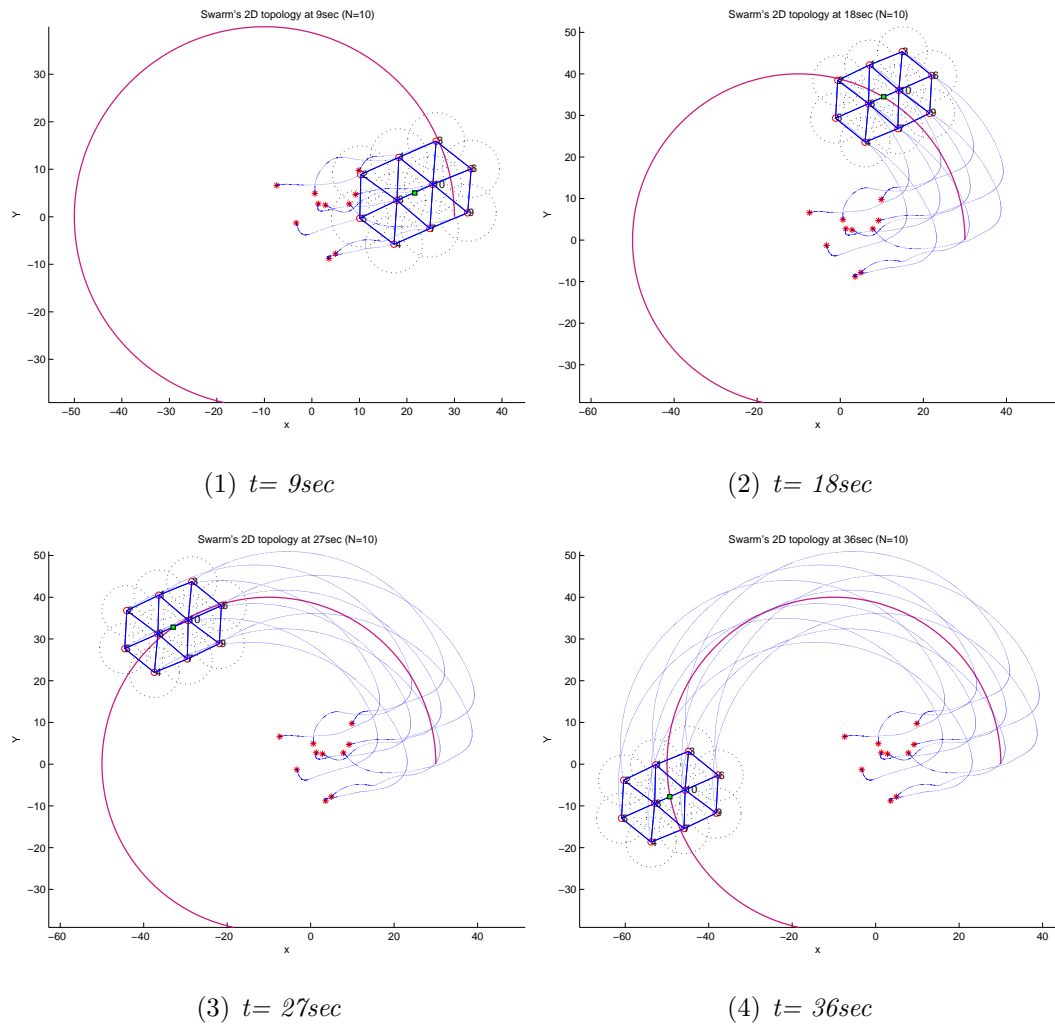
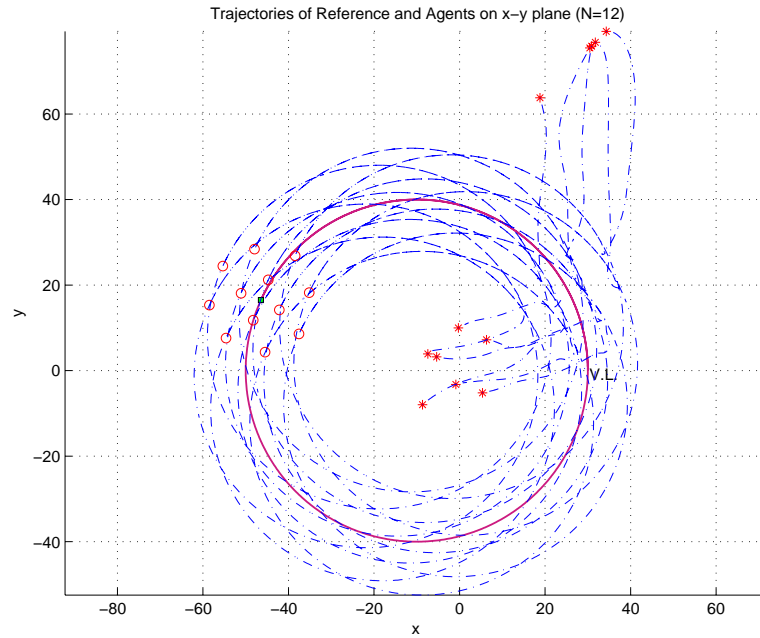


Figure 5.5: Swarm's trajectory and topology at different time steps as shown in Fig. 5.2(a).



(a) Trajectories of agents and virtual leader

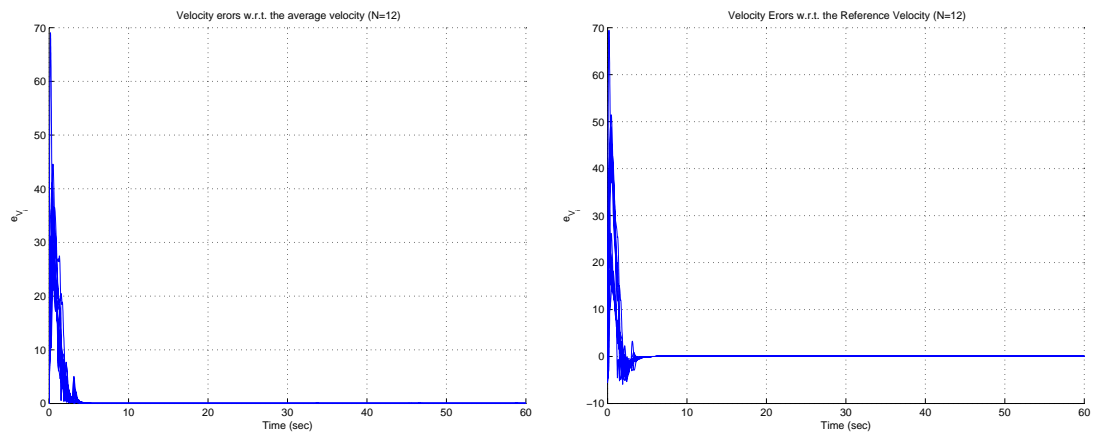
(b)  $e_{v_i}$ (c)  $e_{\bar{v}}$ 

Figure 5.6: A swarm started as two separate sub-groups tracks a virtual leader along a circle ( $N=12$ ).

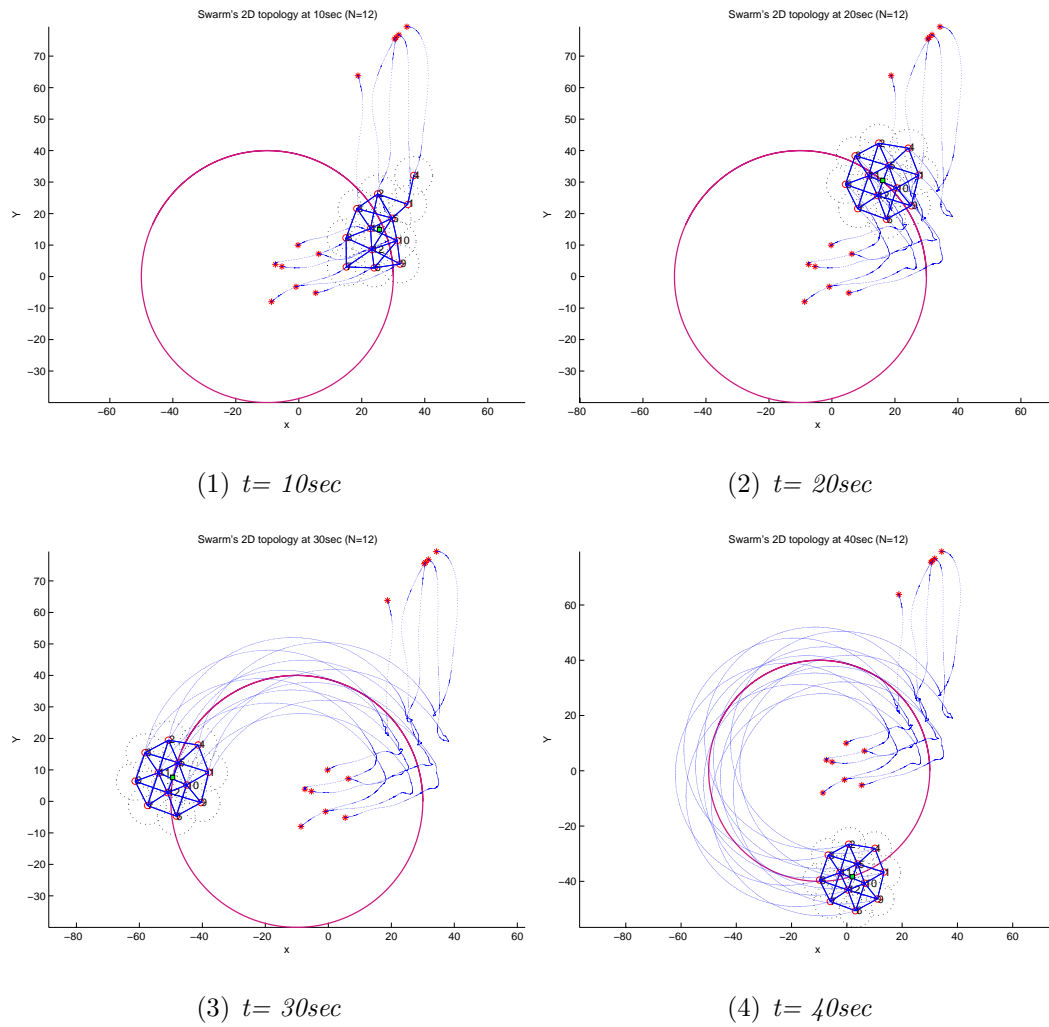


Figure 5.7: Swarm's trajectory and topology at different time steps as shown in Fig. 5.6(a).

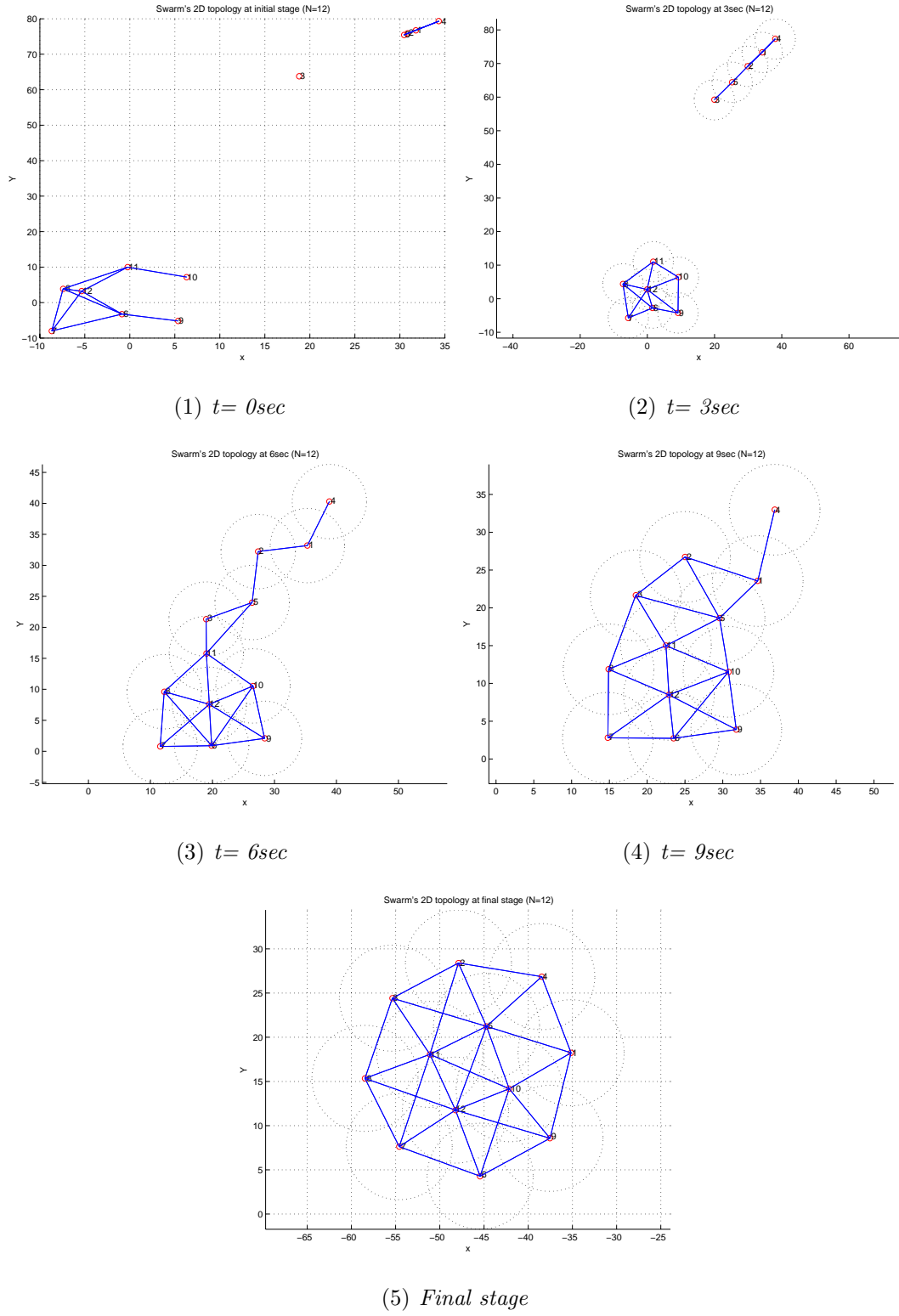


Figure 5.8: Swarm's topology at different time steps as shown in Fig. 5.6(a) (*continued*).  
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# Chapter 6

## Swarms Under Unknown Disturbances

In the last two chapters, we discuss two types of coordinated motions of a group of mobile agents: flocking of swarms in a homogeneous environment and virtual leader tracking with a constant or time-varying reference velocity. One may notice that the swarms considered in those chapters are assumed to be uncertainty free. However, in real application scenarios, there exist a variety of uncertainties in the system, such as disturbance forces, sensing or communication noises, parametric uncertainties. To design a robust and reliable control algorithm for the distributed motion of the swarm, these issues need to be studied.

In this chapter, we consider one of the common uncertainties occurring in engineering applications, disturbances. We examine the group behavior of the swarms with existence of unknown disturbance forces. Because of the presence of unknown disturbances, asymptotical convergence of agents' velocities to equilibria with all equal components is generally impossible (Bauso et al., 2007; Huang and Manton, 2007, 2009).

Moreover, most of the controllers in the literature use consensus approach to ob-

tain the velocity matching among agents (Olfati-Saber, 2006; Tanner et al., 2007; Lee and Spong, 2007; Housheng Su and Lin, 2009; Olfati-Saber et al., 2007; Ren et al., 2007). However, due to the unknown disturbances, the relative distance between two paired neighbors may arbitrarily change in a way that is even beyond what we expect. Consequently, in the worst case, some of the agents may lose sensing or communication connectivity with any other swarm member at a certain time. Therefore, the always connected condition of swarm's topological graph that is generally assumed for consensus terms may not hold anymore.

As discussed in previous chapters, our controllers do not use any consensus term to achieve the velocity matching, thus, not rely on any assumption of the connectivity of the swarm's topological graph. This feature offers the advantage of the presented controllers that allows the group to achieve the desired coordinated motion even with unknown disturbances.

In this chapter, we utilize the mathematical tool of nonsmooth analysis to evaluate the performance of the presented consensus-term free controllers under certain disturbances. The disturbances are unknown but assumed to be bounded by an a priori known bound. We show that the controllers can make the velocities of all agents converge to a common value with bounded errors without any assumption on the graph connectivity. Simulation results are presented to verify the result.

## 6.1 Modelling of Swarms Under Disturbances

Consider a swarm of  $N$  agents with unit mass that move in a  $\mathbb{R}^n$  space with certain disturbances. For the  $i$ th ( $i = 1, 2, \dots, N$ ) agent in the swarm, we have

$$\begin{aligned} \dot{r}_i &= v_i \\ \dot{v}_i &= u_i + \delta_i, \end{aligned} \tag{6.1}$$

where  $r_i$  is its position vector,  $v_i$  is its velocity,  $u_i$  is the control input, and  $\delta_i$  is the disturbance. The disturbance played upon the agent could be caused by the interaction among neighbors or from the environment. We assume that the disturbances are unknown but bounded (UBB).

Fig.6.1 shows two neighboring agents accompanied with certain disturbances. In the figure,  $\Delta_{ij}$  is the disturbance force between agent  $i$  and  $j$ ;  $G_{ij}$  is the internal interaction vector from  $j$  to  $i$ . As discussed in Chapter 3,  $G_{ij}$  could be determined by virtual repulsion and attraction forces or by the gradient force from the structural potential function. Note that in Fig.6.1,  $G_{ij}$  is along the direction of  $r_{ij}$  and  $\Delta_{ij}$  is along an arbitrary direction.

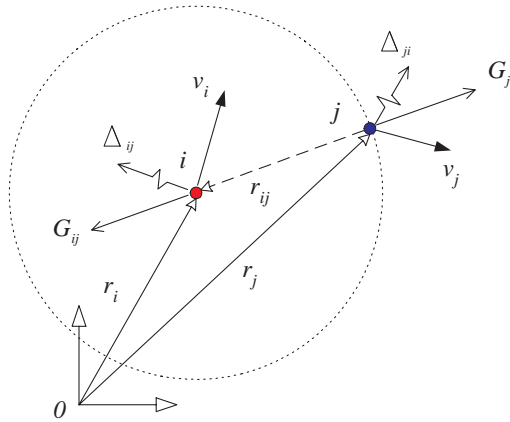


Figure 6.1: Two neighboring agents ( $i$  and  $j$ ) with disturbance.

In this paper, we assume that the disturbance is originated with the interaction between neighboring agents, i.e., the disturbance reciprocally exists between any two paired neighbors. We consider the general condition in which the disturbances between two neighbors are along arbitrary directions, and not necessarily have the same amplitude, i.e.,  $\|\Delta_{ij}\| \neq \|\Delta_{ji}\|$ .

According to Fig.6.1, the total disturbances imposed on agent  $i$  is

$$\delta_i = \sum_{j \in \mathbb{N}_i} \Delta_{ij}, \quad (6.2)$$

where  $\Delta_{ij}$  is unknown but bounded by an a priori known bound

$$\|\Delta_{ij}\| \leq \delta_0. \quad (6.3)$$

Define

$$\bar{r} = \frac{1}{N} \sum_{i=1}^N r_i, \quad (6.4)$$

$$\bar{v} = \frac{1}{N} \sum_{i=1}^N v_i \quad (6.5)$$

to represent the geometric center of the group and the average velocity of all swarm members, respectively.

Let

$$r_{ij} = r_i - r_j, \quad (6.6)$$

and  $\|r_{ij}\| = \|r_{ij}\|_2$  is the relative Euclidean distance between agents  $i$  and  $j$ . Let  $r = [r_1^T, r_2^T, \dots, r_N^T]^T$  and  $v = [v_1^T, v_2^T, \dots, v_N^T]^T$  represent the position and velocity vectors of the entire swarm, respectively.

We use the same graphic representation for the swarm's topology as in previous sections. To save space, we do not repeat the discussion here. See the Section 3.2 for details.

## 6.2 Performance Analysis of the Controllers

For simplicity, in this section, we study only the flocking of swarms in a homogeneous environment and analyze the collective group behavior of the group by the controller

presented in Section 4.1. We show that in homogeneous environments, the controller leads to all agents' velocities converge to the average velocity with bounded errors. For the controller of virtual leader tracking control, one can explore a similar analysis as follows.

As discussed in Section 4.1, the proposed decentralized controller for a swarm to flock in a homogeneous environment is given by

$$u_i = \Phi_1^i + \Phi_2^i = \underbrace{-k_p[v_i - \nabla_{r_i} J(r)]}_{\Phi_1^i} + \sum_{j \in \mathbb{N}_i} \Phi_2^{ij}, \quad (6.7)$$

in which  $\Phi_2^{ij}$ 's are the internal interaction terms among neighboring agents. Depending on whether the group is required to maintain a specific formation,  $\Phi_2^{ij}$  could be determined by virtual attraction and repulsion (A/R) forces, as

$$\Phi_2^{ij} = \{f_r(\|r_{ij}\|) - f_a(\|r_{ij}\|)\} \frac{r_{ij}}{\|r_{ij}\|} \quad (6.8)$$

or by the gradient force from the structural potential function, as

$$\Phi_2^{ij} = \nabla_i U_{ij}. \quad (6.9)$$

For simplicity, we first examine the presented controller for the swarm that does not have any inherent formation constraint as in Section 4.1, which is

$$u_i = -k_p[v_i - \nabla_{r_i} J(r)] + \sum_{j \in \mathbb{N}_i} \{f_r(\|r_{ij}\|) - f_a(\|r_{ij}\|)\} \frac{r_{ij}}{\|r_{ij}\|}, \quad (6.10)$$

where  $k_p > 0$  is a design constant,  $f_r(\cdot)$  and  $f_a(\cdot)$  are the amplitudes of virtual repulsion and attraction forces. The implication of this controller is that each agent perceives and follows the gradient force of the environmental potentials ( $\nabla_{r_i} J(r)$ ), and at the same time interacts with its neighbors through virtual A/R forces. Note that for a homogeneous environment (Flierl et al., 1999),  $\nabla_{r_i} J(r) = \nabla_{r_j} J(r)$  for  $i \neq j$  and  $i, j = 1, \dots, N$ .

Due to the unknown disturbances, the connectivity condition of swarm's topological graph that is commonly assumed in the literature may not hold. As discussed in previous chapters, the controller (6.10) does not include any velocity consensus term, thus, not rely on any condition of the connectivity of the swarm's topology. Attributing to this feature, it is possible that the swarm can still achieve the collective behavior under disturbances.

Denote the combined effect of the virtual attraction and repulsion forces by

$$g(\|r_{ij}\|) = f_r(\|r_{ij}\|) - f_a(\|r_{ij}\|),$$

As in Section 4.1, assume that  $g(\|r_{ij}\|)$  is locally Lipschitz and satisfies

$$g(\|r_{ij}\|) = \begin{cases} > 0 & 0 \leq \|r_{ij}\| < d_0, \\ = 0 & \|r_{ij}\| = d_0, \\ < 0 & d_0 < \|r_{ij}\| \leq d_1, \\ = 0 & \|r_{ij}\| > d_1, \end{cases} \quad (6.11)$$

Moreover, let

$$g(\|r_{ij}\|) = \begin{cases} \rightarrow \infty & \|r_{ij}\| = 0, \\ > \delta_0 & \|r_{ij}\| > d_1. \end{cases} \quad (6.12)$$

For simplicity, we assume that  $g(\|r_{ij}\|)$  is continuous for  $\|r_{ij}\| < d_1$ .

The purpose of imposing the condition (6.12) is to make the repulsion or attraction effects be sufficient enough to overcome the disturbances and keep two neighbors collision and splitting free at  $\|r_{ij}\| = 0$  and  $d_1$ , respectively.

Two of the preceding conditions, (6.11) and (6.12) together, enable that when two agents come too close ( $\|r_{ij}\| < d_0$ ), the virtual repulsion force is big enough to dominate over attraction and disturbance forces, and avoid collision occurring between them; and on the other hand, if they separate too far ( $\|r_{ij}\| > d_0$ ), the virtual attraction force dominates to keep them cohesive.

We will show that if the conditions (6.11) and (6.12) hold, the controller (6.10) makes the velocities of all agents approach to a common value ( $\bar{v}$ ) with bounded errors.

Use the error state

$$e_{v_i} = v_i - \bar{v}. \quad (6.13)$$

From (6.10), it is straight to have

$$\begin{aligned} \dot{v}_i - \dot{\bar{v}} &= -k_p[v_i - \nabla_{r_i} J(r)] + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \delta_i \\ &- \frac{1}{N} \sum_{i=1}^N \left\{ -k_p[v_i - \nabla_{r_i} J(r)] + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \delta_i \right\} \\ &= -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \left( \delta_i - \frac{1}{N} \sum_{i=1}^N \delta_i \right) \\ &+ k_p [\nabla_{r_i} J(r) - \frac{1}{N} \sum_{i=1}^N \nabla_{r_i} J(r)] - \frac{1}{N} \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|}. \end{aligned} \quad (6.14)$$

As in (5.12), we know that  $\mathbb{N}_i$  is always reciprocal for two agents ( $i \neq j$ ), regardless of whether the graph  $\mathcal{G}$  is connected. Thus,

$$\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = \sum_{i=1}^N \sum_{j=1}^N a_{ij} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} = 0. \quad (6.15)$$

Then,

$$\begin{aligned} \dot{v}_i - \dot{\bar{v}} &= -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \\ &+ k_p [\nabla_{r_i} J(r) - \frac{1}{N} \sum_{i=1}^N \nabla_{r_i} J(r)]. \end{aligned} \quad (6.16)$$

Thus, for a homogeneous environment, we have

$$\dot{v}_i - \dot{\bar{v}} = -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j. \quad (6.17)$$

Note that along the boundary of neighborhood range ( $\|r_{ij}\| = d_1$ ),  $g(\|r_{ij}\|)$  is discontinuous, then  $\dot{e}_{v_i}$  may not exist at certain subsets. Since the disturbance  $\Delta_{ij}$  is assumed to originate with the interaction between neighbors,  $\Delta_{ij}$  has the same discontinuity subset as  $g(\|r_{ij}\|)$ . In view of this, we have the following differential inclusion for the error dynamics

$$\dot{e}_{v_i} \in^{a.e.} K[e_{v_i}] = \left\{ -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \right\}. \quad (6.18)$$

Use the same  $E_{ij}$  as in previous chapters

$$E_{ij}(\|r_{ij}\|) = \int_{\|r_{ij}\|}^{d_0} g(\tau) d\tau. \quad (6.19)$$

and its general gradient is

$$\partial E_{ij} = \begin{cases} \|r_{ij}\| < d_1 : & = -g(\|r_{ij}\|), \\ \|r_{ij}\| > d_1 : & = 0, \\ \|r_{ij}\| = d_1 : & = \overline{\text{co}}[-g(d_1), 0]. \end{cases} \quad (6.20)$$

in which  $\overline{\text{co}}[\cdot]$  is the closed convex hull.

Moreover, define

$$\phi_{ij}(\|r_{ij}\|) = \int_0^{\|r_{ij}\|} \delta_0 d\tau. \quad (6.21)$$

Then,  $\partial \phi_{ij} = \delta_0$ .

Select a candidate Lyapunov function as

$$V_t = \frac{1}{2} \sum_{i=1}^N e_{v_i}^T e_{v_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N E_{ij}(\|r_{ij}\|) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \phi_{ij}(\|r_{ij}\|). \quad (6.22)$$

Clearly  $V_t \geq 0$ .  $V_t$  is a function of  $e_{v_i}$  and  $r_{ij}$ , and is differentiable about  $e_{v_i}$ . Due to  $E_{ij}$  is non-differentiable about  $\|r_{ij}\|$ , so is  $V_t$ . Since  $\phi_{ij}(\|r_{ij}\|)$  and  $E_{ij}$  are locally Lipschitz,  $V_t$  is also locally Lipschitz. Then, from Rademacher's Theorem, we know  $V_t$  is differentiable almost everywhere in the sense of regular Lebesgue measure. Moreover, because  $\phi_{ij}$  are convex,  $V_t$  is regular in its domain.

**Theorem 6.1** Consider a swarm of  $N$  mobile agents moving in a homogeneous environment with bounded disturbances as in (6.3). Then with a set of virtual forces that satisfies the condition in (6.11) and (6.12), the decentralized controller (6.10) makes the velocities of all agents converge to the average velocity with bounded errors.

*Proof:* Since  $V_t$  is locally Lipschitz, we have its generalized gradient as

$$\partial V_t = \overline{\text{co}}\{\lim \nabla V_t(e_{v_i}, \|r_{ij}\|), \|r_{ij}\| \notin \Omega_{V_t}, i, j = 1, \dots, N\},$$

in which  $\Omega_{V_t}$  is the set with zero measure, in which the gradient of  $V_t$  is not defined. Specifically,

$$\partial V_t = [e_{v_1}^T, \dots, e_{v_N}^T, \frac{1}{2}\partial E_{11}, \dots, \frac{1}{2}\partial E_{NN}, \frac{1}{2}\partial\phi_{11}, \dots, \frac{1}{2}\partial\phi_{NN}]^T.$$

For simplicity, denote  $\zeta_{ij} = \frac{1}{2}\partial E_{ij}$ ; and since  $\partial\phi_{ij} = \delta_0$ , then

$$\partial V_t = [e_{v_1}^T, \dots, e_{v_N}^T, \zeta_{11}, \dots, \zeta_{ij}, \dots, \zeta_{NN}, \frac{1}{2}\delta_0, \dots, \frac{1}{2}\delta_0]^T. \quad (6.23)$$

From the chain rule of the set-valued Lie derivative of  $V_t$  (Bacciotti and Ceragioli, 1999; Shevitz and Paden, 1994), we know

$$\frac{dV_t}{dt} \in^{a.e.} \dot{\tilde{V}}_t, \quad (6.24)$$

where

$$\dot{\tilde{V}}_t = \bigcap_{\xi \in \partial V_t} \xi^T \cdot \left\{ K[e_{v_1}], \dots, K[e_{v_N}], \frac{d\|r_{11}\|}{dt}, \dots, \frac{d\|r_{NN}\|}{dt}, \frac{d\|r_{11}\|}{dt}, \dots, \frac{d\|r_{NN}\|}{dt} \right\}^T.$$

Using (6.23), we have

$$\dot{\tilde{V}}_t = \bigcap_{\xi \in \partial V_t} \left\{ \sum_{i=1}^N e_{v_i}^T \cdot K[e_{v_i}] + \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} + \sum_{i=1}^N \sum_{j=1}^N \frac{1}{2}\delta_0 \frac{d\|r_{ij}\|}{dt} \right\}. \quad (6.25)$$

For simplicity, let

$$\Gamma_i = e_{v_i}^T \cdot K[e_{v_i}] + \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} + \sum_{j=1}^N \frac{1}{2}\delta_0 \frac{d\|r_{ij}\|}{dt}, \quad (6.26)$$

then

$$\tilde{V}_t = \bigcap_{\xi \in \partial V_t} \left\{ \sum_{i=1}^N \Gamma_i \right\}.$$

Similar to before, we evaluate  $\tilde{V}_t$  on the domain of  $\partial V_t$  piece-wisely. Note that the discontinuity of both error dynamics  $K[e_{v_i}]$  (6.18) and the general gradient  $\partial E_{ij}$  (6.20) are originated from  $g(\|r_{ij}\|)$ , so  $K[e_{v_i}]$  and  $\zeta_{ij}$  share the same nonsmooth domain. Since the disturbance  $\Delta_{ij}$  is assumed to originate with the interaction between neighbors,  $\Delta_{ij}$  has the same discontinuity subset as  $g(\|r_{ij}\|)$ .

If for  $\forall i, \|r_{ij}\| > d_1$  where  $j \in \{1, \dots, N\} \setminus \{i\}$ , then  $\zeta_{ij} = 0$  and  $K[e_{v_i}] = -k_p e_{v_i} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j$ . Then

$$\sum_{i=1}^N \Gamma_i = \sum_{i=1}^N -k_p e_{v_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \left( \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \right) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \delta_0 \frac{d\|r_{ij}\|}{dt}. \quad (6.27)$$

It is not difficult to have

$$\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \delta_0 \frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^N \sum_{j=1}^N \delta_0 e_{v_i}^T \nabla_{r_{ji}} \|r_{ij}\|, \quad (6.28)$$

where we use  $\nabla_{r_{ji}} \|r_{ij}\| = \frac{r_{ij}}{\|r_{ij}\|}$ . Thus, (6.27) becomes

$$\sum_{i=1}^N \Gamma_i = -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \left( \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \right) + \sum_{i=1}^N \sum_{j=1}^N \delta_0 e_{v_i}^T \frac{r_{ij}}{\|r_{ij}\|}. \quad (6.29)$$

If for  $\forall i, \|r_{ij}\| < d_1$  where  $j \in \{1, \dots, N\} \setminus \{i\}$ , then  $\zeta_{ij} = -\frac{1}{2}g(\|r_{ij}\|)$  and  $K[e_{v_i}] = -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{r_{ij}}{\|r_{ij}\|} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j$ . Then,

$$\begin{aligned} \sum_{i=1}^N \Gamma_i &= -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) e_{v_i}^T \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N e_{v_i}^T \\ &\left( \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \right) - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt} + \sum_{i=1}^N \sum_{j=1}^N \frac{\delta_0}{2} \frac{d\|r_{ij}\|}{dt}. \end{aligned}$$

Since  $g(\|r_{ij}\|) = 0$  for  $j \notin \mathbb{N}_i$ , then  $\sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt}$ .

And using similar technique as in (6.28), we can have

$$\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N g(\|r_{ij}\|) \frac{d\|r_{ij}\|}{dt} = \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) v_i^T \frac{r_{ij}}{\|r_{ij}\|}$$

Moreover, since  $\sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) \bar{v}^T \frac{r_{ij}}{\|r_{ij}\|} = 0$ , then,

$$\begin{aligned} \sum_{i=1}^N \Gamma_i &= -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) (v_i - \bar{v})^T \frac{r_{ij}}{\|r_{ij}\|} + \\ &\sum_{i=1}^N e_{v_i}^T (\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j) - \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} g(\|r_{ij}\|) v_i^T \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N \sum_{j=1}^N \\ &\delta_0 e_{v_i}^T \nabla_{r_{ij}} \|r_{ij}\| = -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T (\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j) \\ &+ \sum_{i=1}^N \sum_{j=1}^N \delta_0 e_{v_i}^T \nabla_{r_{ij}} \|r_{ij}\|. \end{aligned} \quad (6.30)$$

If for  $\forall i$ ,  $\|r_{ij}\| = d_1$  where  $j \in \mathbb{N}_i$ , then  $\zeta_{ij} \in \bar{c} \bar{o}[-\frac{1}{2}g(d_1), 0]$  and  $K[e_{v_i}] = -k_p e_{v_i} + \sum_{j \in \mathbb{N}_i} \bar{c} \bar{o}[g(d_1), 0] \frac{r_{ij}}{\|r_{ij}\|} + \delta_i - \sum_{j=1}^N \delta_j$ . Then,

$$\begin{aligned} \dot{\tilde{V}}_t \Big|_{\|r_{ij}\|=d_1} &= \bigcap_{\xi \in \partial V_t} \left\{ \sum_{i=1}^N \Gamma_i \right\} = \bigcap_{\zeta_{ij} \in \bar{c} \bar{o}[-\frac{1}{2}g(d_1), 0]} \left\{ -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} \right. \\ &+ \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \bar{c} \bar{o}[g(d_1), 0] (v_i - \bar{v})^T \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N e_{v_i}^T (\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j) \\ &+ \left. \sum_{i=1}^N \sum_{j=1}^N \zeta_{ij} \frac{d\|r_{ij}\|}{dt} + \sum_{i=1}^N \sum_{j=1}^N \frac{\delta_0}{2} \frac{d\|r_{ij}\|}{dt} \right\} = \bigcap_{\zeta_{ij} \in \bar{c} \bar{o}[-\frac{1}{2}g(d_1), 0]} \left\{ \right. \\ &- k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \bar{c} \bar{o}[g(d_1), 0] v_i^T \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N e_{v_i}^T (\delta_i - \frac{1}{N} \\ &\left. \sum_{j=1}^N \delta_j) + \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} 2\zeta_{ij} v_i^T \nabla_{r_{ij}} \|r_{ij}\| + \sum_{i=1}^N \sum_{j=1}^N \delta_0 e_{v_i}^T \nabla_{r_{ij}} \|r_{ij}\| \right\} \end{aligned}$$

$$\begin{aligned}
 &= \bigcap_{\zeta_{ij} \in \overline{c\bar{o}}[-\frac{1}{2}g(d_1), 0]} \left\{ -k_p \sum_{i=1}^N e_{v_i}^T e_{v_i} + \sum_{i=1}^N e_{v_i}^T \left( \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \right) + \right. \\
 &\quad \left. \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} \left( \overline{c\bar{o}}[g(d_1), 0] + 2\zeta_{ij} \right) v_i^T \frac{r_{ij}}{\|r_{ij}\|} + \sum_{i=1}^N \sum_{j=1}^N \delta_0 e_{v_i}^T \frac{r_{ij}}{\|r_{ij}\|} \right\} \\
 &\subseteq \sum_{i=1}^N e_{v_i}^T \left[ -k_p e_{v_i} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \right] + \\
 &\quad \bigcap_{\zeta_{ij} \in \overline{c\bar{o}}[-\frac{1}{2}g(d_1), 0]} \left\{ \left( \overline{c\bar{o}}[g(d_1), 0] + 2\zeta_{ij} \right) \sum_{i=1}^N \sum_{j \in \mathbb{N}_i} v_i^T \frac{r_{ij}}{\|r_{ij}\|} \right\}.
 \end{aligned}$$

Because

$$\bigcap_{\zeta_{ij} \in \overline{c\bar{o}}[-\frac{1}{2}g(d_1), 0]} \overline{c\bar{o}}[g(d_1), 0] + 2\zeta_{ij} = \{0\}, \quad (6.31)$$

then,

$$\dot{\tilde{V}}_t \big|_{\|r_{ij}\|=d_1} \subseteq \left\{ \sum_{i=1}^N e_{v_i}^T \left[ -k_p e_{v_i} + \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \right] \right\}. \quad (6.32)$$

Therefore, on the entire domain,

$$\dot{\tilde{V}}_t \subseteq \left\{ \alpha : \alpha = \sum_{i=1}^N \left[ -k_p \|e_{v_i}\|^2 + e_{v_i}^T \left( \delta_i - \sum_{j=1}^N \frac{\delta_j}{N} + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \right) \right] \right\}. \quad (6.33)$$

Then from (6.24) we have

$$\begin{aligned}
 \frac{dV_t}{dt} \in^{a.e.} \left\{ \alpha : \alpha = \sum_{i=1}^N \left[ -k_p \|e_{v_i}\|^2 + e_{v_i}^T \left( \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j \right) \right. \right. \\
 \left. \left. + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \right] \right\}. \quad (6.34)
 \end{aligned}$$

Clearly,

$$\alpha \leq \sum_{i=1}^N \left\{ -k_p \|e_{v_i}\|^2 + \|e_{v_i}\| \left\| \delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \right\| \right\}. \quad (6.35)$$

Note that  $\|\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|}\| \leq \|\delta_i\| + \frac{1}{N} \sum_{j=1}^N \|\delta_j\| + \sum_{j=1}^N \delta_0 \|\frac{r_{ij}}{\|r_{ij}\|}\| \leq 3N\delta_0$ , thus,

$$\begin{aligned} \alpha &\leq \sum_{i=1}^N \{-k_p \|e_{v_i}\|^2 + 3N\delta_0 \|e_{v_i}\|\} \\ &= -k_p \sum_{i=1}^N \|e_{v_i}\| \left\{ \|e_{v_i}\| - \frac{3N\delta_0}{k_p} \right\}. \end{aligned} \quad (6.36)$$

From (6.36), it is not hard to see that  $e_{v_i}$  is bounded by  $\frac{3N\delta_0}{k_p}$  and  $V_t$  is bounded. Furthermore, since  $V_t$  is bounded and all the terms  $e_{v_i}^T e_{v_i}$ ,  $E_{ij}$  and  $\phi_{ij}$  in (6.22) are not negative, then  $\phi_{ij}$ 's are also bounded. According to the definition of  $\phi_{ij}$  in (6.21), we know  $\|r_{ij}\|$ 's are bounded. Moreover, since  $V_t \neq \infty$ , then  $E_{ij} \neq \infty$ , hence, using the condition in (6.12), we know that  $\|r_{ij}\| \neq 0$ . This means that collision among agents does not happen even in the presence of unknown disturbances.  $\square$

**Remark 1:** From (6.34), one can see that the actual error's bound depends on the term of  $\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|}$ . Note that  $\frac{r_{ij}}{\|r_{ij}\|}$  is the unit vector along the direction of  $r_{ij}$ . If the agents are well distributed around each other in the final phase,  $\delta_0 \sum_{j=1}^N \frac{r_{ij}}{\|r_{ij}\|} \ll N\delta_0$ , then  $\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} = \sum_{j=1}^N (\Delta_{ij} - \frac{\delta_j}{N}) + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \ll 3N\delta_0$ . This means that the real errors of  $e_{v_i}$  will be much smaller than the ultimate bound. Therefore, the ultimate bound  $\|e_{v_i}\| \leq \frac{3N\delta_0}{k_p}$  in (6.36) is certainly a conservative estimation.  $\square$

**Remark 2:** If the disturbance between two neighboring agents is along the direction of the relative position vector between them, then we can write  $\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} = \sum_{j \in \mathbb{N}_i} \|\Delta_{ij}\| \frac{r_{ij}}{\|r_{ij}\|} - \frac{1}{N} \sum_{j=1}^N \sum_{j \in \mathbb{N}_i} \|\Delta_{ij}\| \frac{r_{ij}}{\|r_{ij}\|} + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|}$ . Furthermore, if the disturbance between two neighbors is reciprocal, i.e.,  $\|\Delta_{ij}\| = \|\Delta_{ji}\|$ , then  $\frac{1}{N} \sum_{j=1}^N \sum_{j \in \mathbb{N}_i} \|\Delta_{ij}\| \frac{r_{ij}}{\|r_{ij}\|} = 0$ , and  $\delta_i - \frac{1}{N} \sum_{j=1}^N \delta_j + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} = \sum_{j \in \mathbb{N}_i} \|\Delta_{ij}\| \frac{r_{ij}}{\|r_{ij}\|} + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|}$ . For a large swarm in which the agents are well distributed around each

other at the final stage,  $\delta_0 \sum_{j=1}^N \frac{r_{ij}}{\|r_{ij}\|} \ll N\delta_0$ . Then  $\sum_{j \in \mathbb{N}_i} \|\Delta_{ij}\| \frac{r_{ij}}{\|r_{ij}\|} + \sum_{j=1}^N \delta_0 \frac{r_{ij}}{\|r_{ij}\|} \ll 2N\delta_0$ . Thus, the final  $e_{v_i}$ 's will be much smaller than the ultimate bound.  $\square$

**Remark 3:** An inherent and very important question arising with unknown disturbances is the preservation of graph connectivity. Unlike most works in the literature, the preceding proof in this paper does not require any assumption that the swarm's topological graph is always connected.  $\square$

Note that the controller (6.10) utilizes the virtual repulsion and attraction forces to generate the sufficient internal interactions among agents. Since the virtual A/R forces are not be designated for a particular structural formation, the controller cannot make the group achieve an arbitrarily desired pattern. As discussed in Chapter 4 and 5, if a certain formation of the swarm is required, it is suggested to replace the virtual forces with gradient forces of a structural potential function. However, due to the existence of unknown disturbances, in general, the relative distance between two paired neighbors may not asymptotically approach to a constant. Thus, a desired pattern may not be asymptotically sustained. This can be shown by the following discussion.

For simplicity, denote  $d_{ij} = \|r_{ij}\|$ . Clearly,

$$d_{ij}^2 = (r_i - r_j)^T (r_i - r_j), \quad (6.37)$$

and

$$\frac{d}{dt} d_{ij}^2 = 2(r_i - r_j)^T (v_i - v_j). \quad (6.38)$$

To preserve a desired pattern,  $d_{ij}$  generally needs to be asymptotically kept as a constant, i.e.,  $d_{ij}^2$  is a constant, then  $\frac{d}{dt} d_{ij}^2 = 0$ . As we show,  $r_i - r_j \neq 0$  and  $v_i - v_j = e_{v_i} - e_{v_j} \neq 0$  as  $t \rightarrow \infty$ , thus,  $\frac{d}{dt} d_{ij}^2 \neq 0$ . Therefore, a fixed pattern of

the group cannot be maintained when unknown disturbances exist. To design a robust controller to achieve and maintain a desired pattern of the entire group in the presence of unknown disturbances needs further investigation.

## 6.3 Simulation Study

In this section, we present the simulation results of two swarms that move in a linear homogeneous environment under UBB disturbances.

We select the neighborhood defining radius  $d_1 = 12$ , and the critical radius  $d_0 = 8$ . The design constant  $k_p = 5$ . The gradient force of the environmental potential is assumed to be homogeneous to each agent. In the following figures, the stars and circles represent agents' initial and final positions, respectively.

Fig.6.2–6.4(5) show a swarm of mobile agents ( $N = 20$ ) moving in a 2D linear environment. The gradient of the environmental potential is  $\nabla_{r_i} J(r) = [-3.0, -3.3]^T$ . The combined effect of virtual forces is selected to be  $g(\|r_{ij}\|) = -\log(\|r_{ij}\|/d_0) - (\|r_{ij}\| - d_0)$  for  $\|r_{ij}\| \leq d_1$ . The disturbance between any pair of agents is along arbitrary direction and bounded by  $\delta_0 = 1$ . Initially, the agents are randomly located in a  $30 \times 30$  square centered at  $(0, 0)$ . Their initial velocities are randomly given. Fig.6.2 shows the group's trajectory on  $X - Y$  plane. The convergence of their velocities is shown in Fig.6.3. The swarm's topologies at different time stages are shown in Fig. 6.7. It is clear that all agents' velocities converge to the average with bounded errors, and in the steady stage, the separation of agents is fairly even.

Fig.6.5– 6.7(2) show a swarm ( $N = 15$ ) moving in the same 2D environment as Fig.6.2. The combined effect of virtual forces is selected to be the same as above. The disturbance is bounded by  $\delta_0 = 2$ . At the initial position, the swarm is not connected, and an agent starts far from the rest agents. See the initial topology of the swarm in Fig.6.7(1). The agents' initial velocities are randomly given. Fig.6.5 shows agents'

trajectories and Fig.6.6 shows the velocity convergence. The swarm's final topology is shown in Fig. 6.7(2). It is clearly shown that all swarm members move together even though the graph is not connected initially.

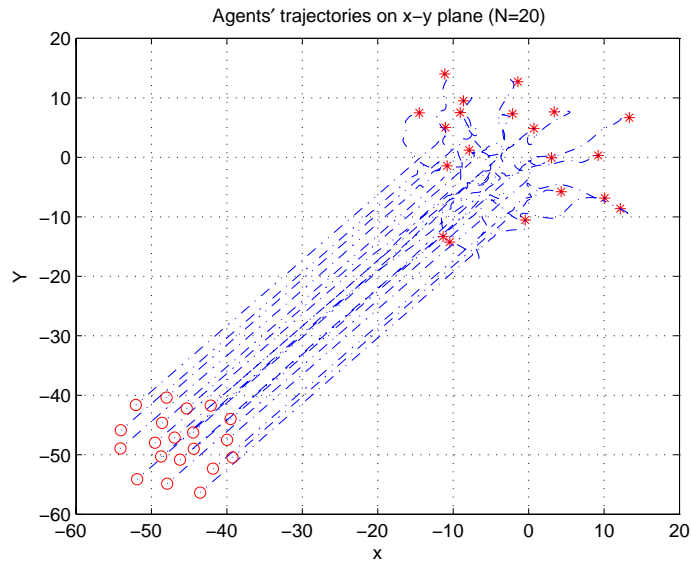


Figure 6.2: Trajectories of the agents when the swarm moves in a 2D environment with linear potential ( $N = 20$ ).

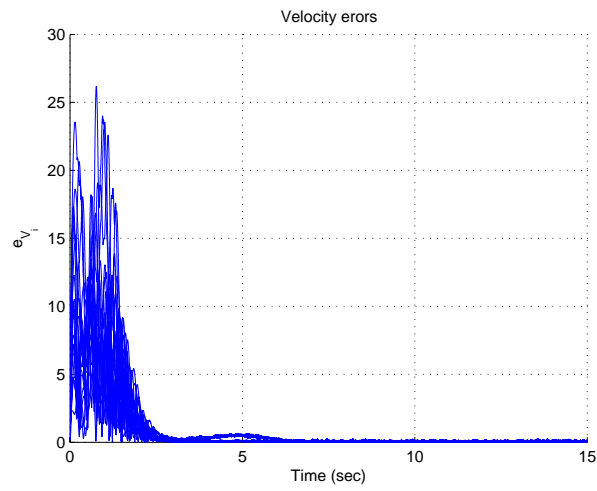


Figure 6.3: Velocity errors of the agents as shown in Fig. 6.2.

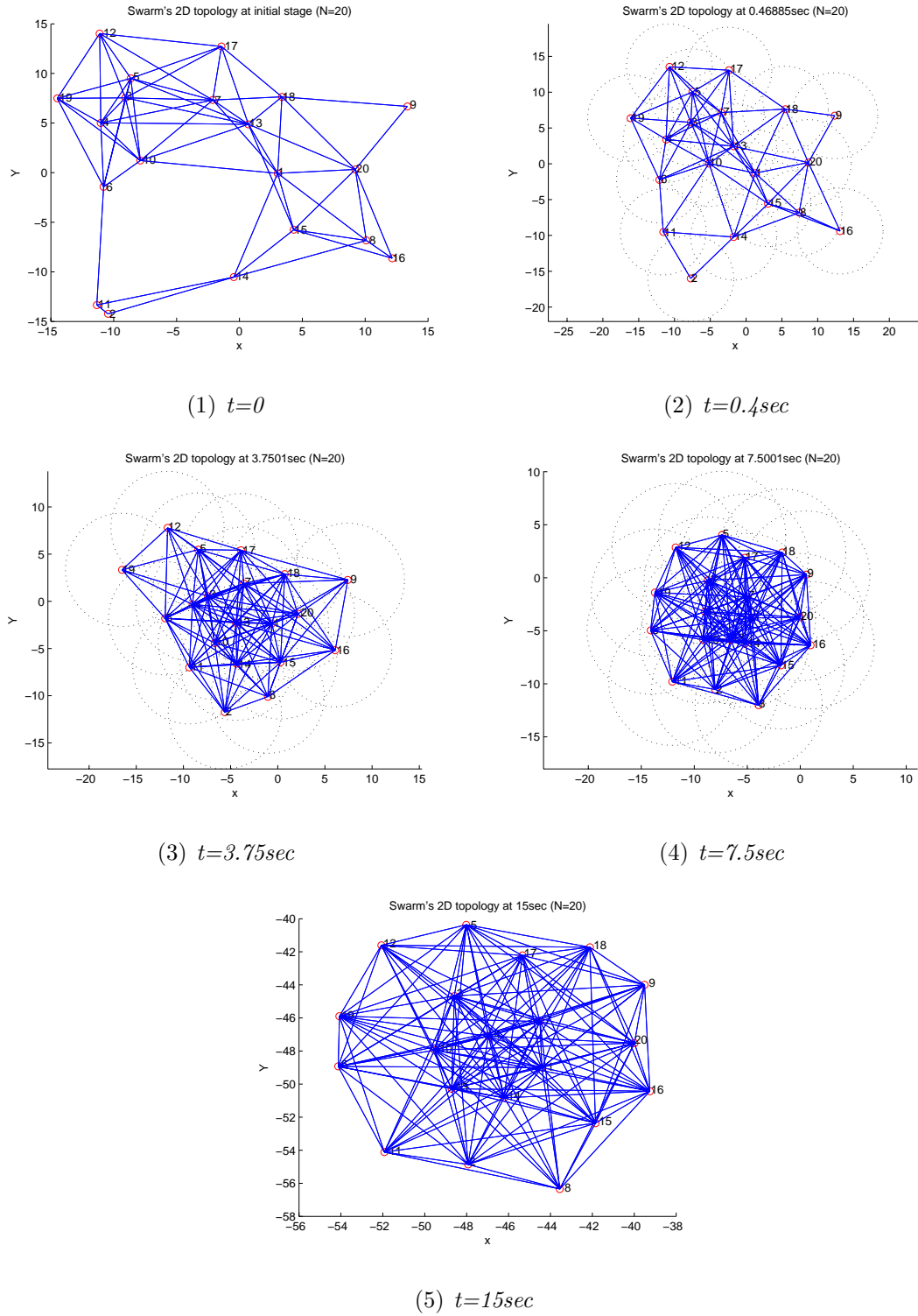


Figure 6.4: Swarm's topologies at different time steps as shown in Fig. 6.2

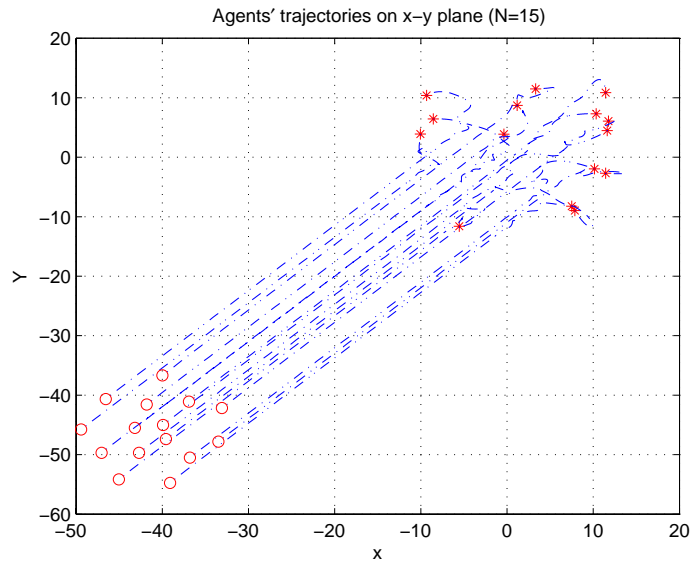


Figure 6.5: Trajectories of the agents when the swarm moves in a 2D linear environment ( $N = 15$ ).

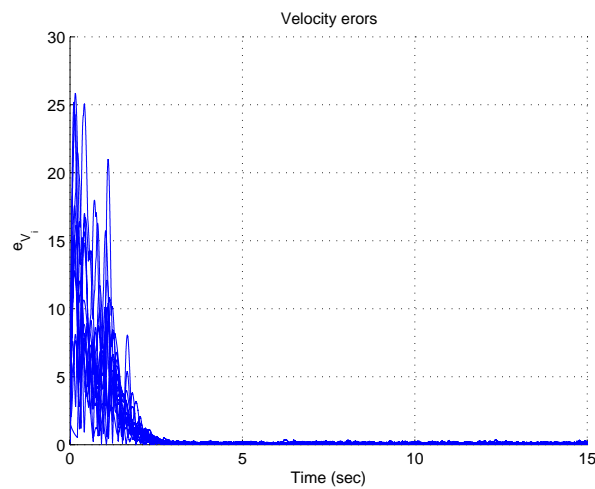


Figure 6.6: Velocity convergence as shown in Fig. 6.5.

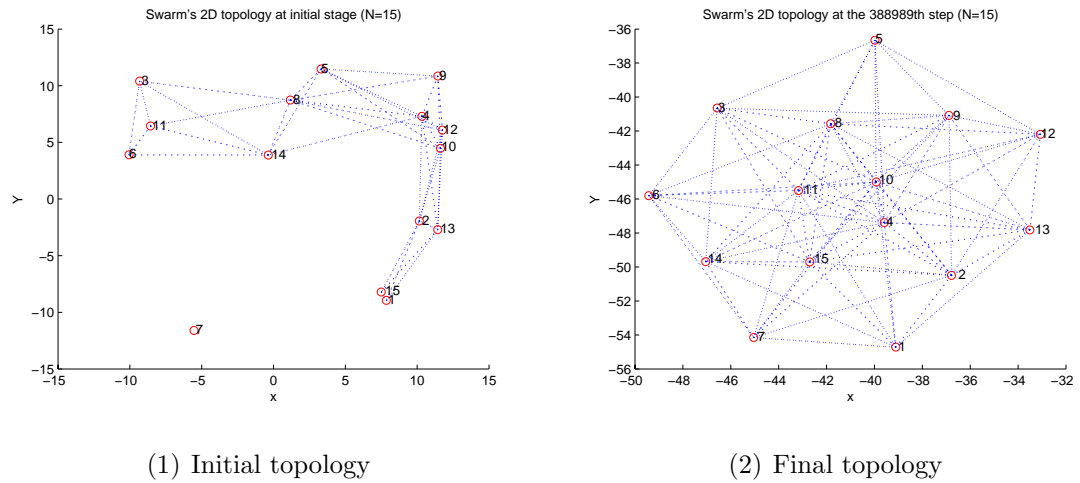


Figure 6.7: Swarm's topologies at the initial and final stages as in Fig. 6.5.

# Chapter 7

## Conclusions and Future Work

In this thesis, we study the distributed motions of swarm of biological and engineering agents and present a systematic methodology to analyze the behavior of a large group of mobile dynamical agents. We develop a novel and unified framework that can serve as a guideline for the control strategy design for different motion coordination tasks of the swarm. We use a simplified dynamics model for individual agents, and utilize algebraic graphs to model the topologies of the swarm that embody the neighborhood, communication or sensing relations among members. We consider the general situation in that the swarm's topology dynamically changes as the spacing among agents evolves. To demonstrate the presented framework, we apply it to design decentralized control algorithms for two application scenarios of coordinated motions of swarms, namely, flocking in homogeneous environments and virtual leader tracking control. By using the theory of nonsmooth analysis and nonsmooth stability analysis, we prove that the presented controllers enable the swarm to achieve the desired coordinated motions. In addition, we examine the performance of the presented controllers under unknown but bounded disturbances. Without any assumption on the graph connectivity, we show that the controllers can lead the swarm to have a collective motion with bounded error even in the presence of unknown disturbances.

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For the future work, there will be two primary directions to extend this thesis. One is to comprehensively study some other inherent issues that arise in practical applications, such as parametric uncertainties, communication delays, sensing or communication noises. The effects of the physical implementation of agents, such as the nonholonomic constraints, the power consumption and the body size of the agents may also need to be considered in certain applications. The other direction to extend the result in this thesis is to apply the proposed framework into some other related research topics on swarms, such as mobilization of swarms, rendezvous, and distributed information fusion. To implement the presented controllers into a real distributed robotic system will be one of our plans for the future work.

# Appendix A

## Mathematical Notes

### A.1 One-sided Lipschitz Function

A function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is one-sided Lipschitz at  $x \in \mathbb{R}^n$  if there exist  $\varepsilon$  and  $L$  both of which are positive, such that

$$(y_1 - y_2)^T [f(y_1) - f(y_2)] \leq L \|y_1 - y_2\|_2^2, \quad (\text{A.1})$$

for all  $y_1, y_2 \in B(x, \varepsilon)$ .

Recall that a locally Lipschitz function is defined as follows.

A function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is locally Lipschitz at  $x \in \mathbb{R}^n$  if there exist  $\varepsilon$  and  $L$  both of which are positive, such that

$$\|f(y_1) - f(y_2)\|_2 \leq L \|y_1 - y_2\|_2, \quad (\text{A.2})$$

for all  $y_1, y_2 \in B(x, \varepsilon)$ .

## A.2 Right Directional Derivative

For a function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ , the right directional derivative of  $f$  at  $x$  in the direction of  $w \in \mathbb{R}^n$ , denoted by  $f'(x, w)$ , is defined as

$$f'(x; w) \triangleq \lim_{h \downarrow 0} \frac{f(x + hw) - f(x)}{h}, \quad (\text{A.3})$$

if the limit exists.

It is usual that one omits the word of ‘right’ and name it as *directional derivative*.

Note that for a continuous function  $f$ , the directional derivative  $f'(x, w)$  may not exist. If the function  $f$  is differentiable at  $x$ , then the directional derivative  $f'(x, w)$  exists along any direction  $w$  and one has  $f'(x, w) = \nabla f(x) \cdot w$ .

## A.3 Harmonic Function

Let a function  $f : D \rightarrow \mathbb{R}$  where  $D$  is an open subset of  $\mathbb{R}^n$ .  $f$  is a *harmonic function* if it is twice continuously differentiable and satisfies the Laplace’s equation, i.e.,

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad (\text{A.4})$$

holds everywhere on  $D$ . This is usually written as

$$\Delta f = 0 \quad (\text{A.5})$$

where  $\Delta$  is the Laplace operator.

Examples of harmonic functions of two variables include  $f(x_1, x_2) = e^{x_1} \sin x_2$ , and  $f(x_1, x_2) = \ln(x_1^2 + x_2^2)$  defined on  $\mathbb{R}^n \setminus \{0\}$  (e.g. the electric potential due to a line charge, and the gravity potential due to a long cylindrical mass). All functions that are constant, linear and affine on  $\mathbb{R}^n$  are harmonic functions.

Harmonic functions are infinitely differentiable. In fact, harmonic functions are also real analytic. If  $f$  is a harmonic function on  $D$ , then all partial derivatives of  $f$

are also harmonic on  $D$ . One of the other important features of harmonic functions is the *maximum principle* stated as follows.

If  $Q$  is any compact subset of  $D$ , then restricted to  $Q$ ,  $f$  attains its maximum and minimum on the boundary of  $Q$ . Thus, if  $D$  is connected, then  $f$  cannot have local maxima or minima, except that  $f$  is constant.

# Bibliography

- Adiou, M., J. P. Treuil, and O. Arino: 2003, Alignment in a fish school: a mixed Lagrangian-Eulerian approach. *Ecological Modelling* 167(1-2), 19–32.
- Aldana, M., V. Dossetti, C. Huepe, V. Kenke, and H. Larralde: 2007, Phase transitions in systems of self-propelled agents and related network models. *Physical Review Letters* 98(9), 095, 702.
- Angeli, D. and P. A. Bliman: 2005, Extension of a result by Moreau on stability of leaderless multi-agent systems. In: *Proceedings of the 46th IEEE Conference on Decision and Control, with European Control Conference*. Seville, Spain, pp. 759–764.
- Armsworth, P. R.: 2001, Directed motion in the sea: Efficient swimming by reef fish larvae. *Journal of Theoretical Biology* 210(1), 81–91.
- Aubin, J.-P. and A. Cellina: 1984, *Differential Inclusions, Set-Valued Maps And Viability Theory*, Vol. 264 of *Grundlehren Der Mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics)*. Berlin: Springer-Verlag.
- Bacciotti, A.: 2004, Some remarks on generalized solutions of discontinuous differential equations. *International Journal of Pure and Applied Mathematics* 10(3), 257–266.
- Bacciotti, A. and F. Ceragioli: 1999, Stability and Stabilization of Discontinuous Systems and Nonsmooth Lyapunov Functions. *ESAIM Journal of Control Optimization and Calculus of Variations* 4, 361–376.

- Bacciotti, A. and F. Ceragioli: 2002,  $L_2$  Gain Stability with respect to Filippov Solutions. In: *the Proceedings of 41th IEEE Conference on Decision and Control*. Las Vegas, NV, pp. 713–716.
- Barbaro, A. B., B. Birnir, and K. Taylor: 2007, Discrete and continuous models of the dynamics of Pelagic fish: orientation-induced swarming and application to the Capelin. Technical report, the eScholarship Repository, University of California. <http://repositories.cdlib.org/cnls/Bio2>.
- Bauso, D., L. Giarre, and R. Pesenti: 2006, Nonlinear protocols for optimal distributed consensus in networks of dynamic agents. *Systems and Control Letters* 55(11), 918–928.
- Bauso, D., L. Giarre, and R. Pesenti: 2007, Lazy consensus for networks with unknown but bounded disturbances. In: *Proceedings of the 46th IEEE Conference on Decision and Control*. New Orleans, LA, pp. 2283–2288.
- Beecham, J. A. and K. D. Farnsworth: 1999, Animal group forces resulting from predator avoidance and competition minimization. *Journal of Theoretical Biology* 198, 533–548.
- Berthold, P., H.-G. Bauer, and V. Westhead: 2001, *Bird Migration: A General Survey*. Oxford: Oxford University Press.
- Bhatia, R.: 1996, *Matrix Analysis*, Vol. 169 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.
- Biggs, N.: 1993, *Algebraic Graph Theory*. Cambridge: Cambridge University Press, 2nd edition.
- Borenstein, J., H. R. Everett, and L. Feng: 1996, *Navigating Mobile Robots: Systems and Techniques*. Wellesley, MA: A. K. Peters, Ltd.
- Borkar, V. and P. Varaiya: 1982, Asymptotic agreement in distributed estimation. *IEEE Transactions on Automatic Control* 27(3), 650–655.

- Brockett, R. W.: 1983, Asymptotic Stability and Feedback Stabilization. In: R. W. Brockett, R. S. Millman, and H. J. Sussmann (eds.): *Geometric Control Theory*. Boston: Birkhauser Verlag, pp. 181–191.
- Bullo, F., J. Cortés, and S. Martínez: 2009, *Distributed Control of Robotic Networks*, Applied Mathematics Series. Princeton University Press. Electronically available at <http://coordinationbook.info>.
- Cellina, A.: 2005, A view on differential inclusions. *Rendiconti del Seminario Matematico Univ. Pol. Torino* 63(3), 197–209.
- Ceragioli, F.: 2000, Discontinuous ordinary differential equations and stabilization. Ph.D. thesis, Università di Firenze, Torino, Italy. Electronically available at <http://calvino.polito.it/~ceragioli>.
- Chaté, H., F. Ginelli, and G. Grégoire: 2007, Comment on “phase transitions in systems of self-propelled agents and related network models”. *Physical Review Letters* 99, 229, 601.
- Clarke, F.: 2004, Lyapunov functions and feedback in nonlinear control. In: M. S. de Queiroz, M. Malisoff, and P. Wolenski (eds.): *Optimal Control, Stabilization and Nonsmooth Analysis*, Vol. 301 of *Lecture notes on control and information science*. New York: Springer, pp. 267–282.
- Clarke, F. H.: 1983, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts. New York: John Wiley & Sons.
- Clarke, F. H.: 1990, *Optimization and Nonsmooth Analysis*, Vol. 5 of *Classics in Applied Mathematics*. Philadelphia, PA: SIAM.
- Clarke, F. H., Y. S. Iedyayev, R. J. Stern, and R. R. Wolenski: 1998, *Nonsmooth Analysis and Control Theory*, Vol. 178 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.
- Cortés, J.: 2005, Achieving coordination tasks in finite time via nonsmooth gradient flows. In: *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*. Seville, Spain, pp. 6376–6381.

- Cortés, J.: 2008, Discontinuous Dynamical Systems: A tutorial on solutions, nonsmooth analysis and stability. *IEEE Control Systems Magazine* 28(3), 36–73.
- Cortés, J.: 2008, Distributed algorithms for reaching consensus on general functions. *Automatica* 44(3), 726–737.
- Cortés, J. and F. Bullo: 2005, Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization* 44(5), 1543–1574.
- Cortés, J. and F. Bullo: 2009, Nonsmooth coordination and geometric optimization via distributed dynamical systems. *SIAM Review* 51(1), 163–189.
- Cortés, J., S. Martínez, and F. Bullo: 2004a, Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Transactions on Automatic Control* 51(8), 1289–1298.
- Cortés, J., S. Martínez, and F. Bullo: 2005a, Spatially-distributed coverage optimization and control with limited-range interactions. *ESAIM: Control, Optimisation and Calculus of Variations* 11, 691–719.
- Cortés, J., S. Martínez, T. Karatas, and F. Bullo: 2004b, Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation* 20(2), 243–255.
- Cortés, J., A. van der Schaft, and P. Crouch: 2005b, Characterization of gradient control systems. *SIAM Journal on Control and Optimization* 44(4), 1192–1214.
- Couzin, I. D., J. Krause, R. James, G. D. Ruxton, and N. R. Franks: 2002, Collective memory and spatial sorting in animal groups. *Journal of Theoretical Biology* 218, 1–11.
- Cutts, C. J. and J. R. Speakman: 1994, Energy savings in formation flight of Pink-footed Geese. *Journal of Experimental Biology* 189, 251–261.
- Czirok, A., E. Ben-Jacob, I. Cohen, and T. Vicsek: 1996, Formation of complex bacterial colonies via self-generated vortices. *Physical Review E* 54, 1791–1801.

- Czirok, A. and T. Vicsek: 2000, Collective behavior of interacting self-propelled particles. *Physica A* 281(1), 17–29.
- de Castro, G. A. and F. Paganini: 2004, Convex synthesis of controllers for consensus. In: *Proceedings of 2004 American Control Conference*. Boston, MA, pp. 4933–4938.
- Do, K. D.: 2007, Bounded controllers for formation stabilization of mobile agents with limited sensing ranges. *IEEE Transactions on Automatic Control* 52(3), 569–576.
- Do, K. D.: 2008, Formation Tracking Control of Unicycle-Type Mobile Robots With Limited Sensing Ranges. *IEEE Transactions on Control Systems Technology* 16(3), 527–538.
- Dontchev, A. L. and F. Lempio: 1992, Difference methods for differential inclusions: A survey. *SIAM Review* 34, 263–294.
- Edelstein-Keshet, L.: 1999, Mathematical models of swarming and social aggregation. *IMA Workshop of “From individual to aggregation”: Modelling animal grouping*. Minneapolis, MN.
- Edelstein-Keshet, L., J. Watmough, and D. Grünbaum: 1998, Do travelling band solutions describe cohesive swarms? An investigation for migratory locusts. *Journal of Mathematical Biology* 36, 515–549.
- Eftimie, R., G. de Vries, and M. A. Lewis: 2009, Weakly nonlinear analysis of a hyperbolic model for animal group formation. *Journal of Mathematical Biology* 59(1), 37–74.
- Eftimie, R., G. de Vries, M. A. Lewis, and F. Lutscher: 2007, Modeling group formation and activity patterns in self-organizing collectives of individuals. *Bulletin of Mathematical Biology* 69, 1537–1565.
- Egerstedt, M. and X. Hu: 2001, Formation control with virtual leaders and reduced communications. *IEEE Transactions on Robotics and Automation* 17(6), 947–951.
- Fax, J. A.: 2001, Optimal and cooperative control of vehicle formations. Ph.D. thesis, Division of Control Dynamical System, California Institute of Technology, Pasadena, CA.

- Fax, J. A. and R. Murray: 2004, Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control* 49(9), 1465–1476.
- Filippov, A. F.: 1988, *Differential Equations with Discontinuous Righthand Sides*, Vol. 18 of *Mathematics and Its Applications (Soviet Series)*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Flierl, G., D. Grünbaum, S. Levin, and D. Olson: 1999, From individual to aggregations: The interplay between behavior and physics. *Journal of Theoretical Biology* 196, 397–454.
- Ganguli, A., J. Cortés, and F. Bullo: 2006, Maximizing visibility in nonconvex polygons: nonsmooth analysis and gradient algorithm design. *SIAM Journal on Control and Optimization* 45(5), 1657–1679.
- Gazi, V. and K. M. Passino: 2002, A Class of Attraction/repulsion Functions for Stable Swarm Aggregations. In: *Proceedings of the 41th IEEE Conference on Decision and Control*. Los Vegas, NV, pp. 2842–2847.
- Gazi, V. and K. M. Passino: 2003, Stability Analysis of Swarms. *IEEE Transactions on Automatic Control* 48(4), 692–697.
- Gazi, V. and K. M. Passino: 2004, Stability Analysis of Social Foraging Swarm. *IEEE Transactions on Systems, Man and Cybernetics - Part B: Cybernetics* 34(1), 539–557.
- Ge, S. S. and Y. J. Cui: 2000, New potential functions for mobile robot path planning. *IEEE Transactions on Robotics and Automation* 16(5), 615–620.
- Godsil, C. and G. Royle: 2001, *Algebraic Graph Theory*, Vol. 207 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.
- Gregoire, G. and H. Chate: 2004, Onset of collective and cohesive motion. *Physical Review Letters* 92(2), 025, 702.
- Grünbaum, D.: 1998, Schooling as a strategy for taxis in a noisy environment. *Evolutionary Ecology* 12(5), 503–522.

- Grünbaum, D. and A. Okubo: 1994, Modelling social animal aggregation. In: S. Levin (ed.): *Frontiers in Mathematical Biology*. New York: Springer, pp. 296–325.
- Gueron, S., S. A. Levin, and D. I. Rubenstein: 1996, The dynamics of herds: from individuals to aggregations. *Journal of Theoretical Biology* 182, 85–98.
- Haken, H.: 1983, *Synergetics: An Introduction: Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry and Biology*. New York, NY: Springer, 3rd edition.
- Hamilton, W. D.: 1971, Geometry for the selfish herd. *Journal of Theoretical Biology* 31, 295–311.
- Hatano, Y. and M. Mesbahi: 2005, Agreement over random networks. *IEEE Transactions on Automatic Control* 50(11), 1867–1872.
- Holmes, E., M. Lewis, and J. Banks: 1994, Partial differential equations in ecology: spatial interactions and population dynamics. *Ecology* 75(1), 17–29.
- Horn, R. A. and C. R. Johnson: 1990, *Matrix Analysis*. Cambridge, U.K.: Cambridge University Press.
- Housheng Su, X. W. and Z. Lin: 2009, Flocking of Multi-agents With a Virtual Leader. *IEEE Transactions on Automatic Control* 54(2), 293–307.
- Huang, M. and J. H. Manton: 2007, Stochasting approximation for consensus seeking: mean square and almost sure convergence. In: *Proceedings of the 46th IEEE Conference on Decision and Control*. New Orleans, LA, pp. 306–311.
- Huang, M. and J. H. Manton: 2009, Coordination and consensus of networked agents with noisy measurements: Stochastic algorithms and asymptotic behavior. *SIAM Journal on Control and Optimization: Special issue on Control and Optimization in Cooperative Networks* 48(1), 134–161.
- Humston, R., J. Ault, M. Lutcavage, and D. Olson: 2000, Schooling and migration of large pelagic fishes relative to environmental cues. *Fisheries Oceanography* 9(2), 136–146.

- Huth, A. and C. Wissel: 1994, The simulation of fish schools in comparison with experimental data. *Ecological Modelling* 75, 135–146.
- Hwang, Y. K. and N. Ahuja: 1992, A potential field approach to path planning. *IEEE Transactions on Robotics and Automation* 8(1), 23–32.
- Jadbabaie, A., J. Lin, and A. S. Morse: 2003, Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules. *IEEE Transactions on Automatic Control* 48(6), 988–1001.
- Kennedy, J. and R. C. Eberhart: 2001, *Swarm Intelligence*. Morgan Kaufmann Publisher.
- Khalil, H. K.: 2002, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall, 3rd edition.
- Kim, J. O. and P. K. Khosla: 1992, Real-time obstacle avoidance using harmonic potential functions. *IEEE Transactions on Robotics and Automation* 8(2), 338–349.
- Kim, Y. and M. Mesbahi: 2006, On maximizing the second smallest eigenvalue of state-dependent graph Laplacian. *IEEE Transactions on Automatic Control* 51(1), 116–120.
- Koren, Y. and J. Borenstein: 1991, Potential Field Methods and Their Inherent Limitations for Mobile Robot Navigation. In: *Proceedings of the IEEE Conference on Robotics and Automation*. Sacramento, CA, pp. 1398–1404.
- Krstić, M., I. Kanellakopoulos, and P. Kokotović: 1995, *Nonlinear and Adaptive Control Design*. New York: Wiley Interscience.
- Kunz, H. and C. K. Hemelrijk: 2003, Artificial fish schools: Collective effects of school size, body size and body form. *Artificial Life* 9, 237–253.
- Kwok, A. and S. Martínez: 2009, Deployment algorithms for a power-constrained mobile sensor network. *International Journal of Robust and Nonlinear Control*. in press.

- Lee, D. and M. W. Spong: 2007, Stable Flocking of Multiple Inertial Agents on Balanced Graphs. *IEEE Transactions on Automatic Control* 52(8), 1469–1475.
- Leonard, N. E. and E. Fiorelli: 2001, Virtual leaders, artificial potentials and coordination control of groups. In: *Proceedings of the 40th IEEE Conference on Decision and Control*. pp. 2968–2973.
- Levine, H. and W. J. Rappel: 2001, Self-organization in systems of self-propelled particles. *Physical Review E* 63(1), 017101–4.
- Li, X., Z. Cai, and J. Xiao: 2005a, Backstepping Based Multiple Robots Formation Control. In: *Proceedings of the 2005 IEEE/RSJ International Conference on Intelligent Robots and Systems*. Albert Canada, pp. 1313–1318.
- Li, X., Z. Cai, and J. Xiao: 2005b, Stable Flocking of Swarms by Local Information. In: *Proceedings of the IEEE International Conference on Systematics, Man and Cybernetics (SMC'05)*. Hawaii, U.S., pp. 3921–3926.
- Li, X., Z. Cai, and J. Xiao: 2007, Biologically Inspired flocking of swarms with dynamic topology in uniform environments. In: *Proceedings of the 46th IEEE Conference on Decision and Control (CDC'07)*. New Orleans, LA, pp. 2252–2257.
- Li, X., Z. Cai, and J. Xiao: 2008a, Stable Swarming by Mutual Interactions of Attraction/Alignment/Repulsion. In: *Proceedings of the 47th IEEE Conference on Decision and Control (CDC'08)*. Cancun, Mexico.
- Li, X., Z. Cai, and J. Xiao: 2008b, Stable Swarming by Mutual Interactions of Attraction/Alignment/Repulsion: Fixed topology. In: *Proceedings of the 2008 IFAC World Congress*. Seoul, Korea, pp. 5143–5148.
- Li, X. and J. Xiao: 2005, Robot Formation Control in Leader-Follower Motion Using Direct Lyapunov Method. *International Journal of Intelligent Control and Systems* 10(3), 244–250.
- Li, X. and J. Xiao: 2006, A Biologically Inspired Controller for Swarms in Dynamic Environment. *International Journal on Intelligent System and Control, Special Issue on Swarm Robotics* 11(3), 154–163.

- Li, Y.-X., R. Lukeman, and L. Edelstein-Keshet: 2008c, Minimal mechanisms for school formation in self-propelled particles. *Physica D: Nonlinear Phenomena* 237(5), 699–720.
- Lin, Z., M. Brouke, and B. Francis: 2004, Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control* 49(4), 622–629.
- Lin, Z., B. Francis, and M. Maggiore: 2005, Necessary and Sufficient Graphical Conditions for Formation Control of Unicycles. *IEEE Transactions on Automatic Control* 50(1), 121–127.
- Liu, Y. and K. M. Passino: 2004, Stable Social Foraging Swarms in a Noisy Environment. *IEEE Transactions on Automatic Control* 49(1), 30–44.
- Lukeman, R., Y.-X. Li, and L. Edelstein-Keshet: 2009, A Conceptual Model for Milling Formations in Biological Aggregates. *Bulletin of Mathematical Biology* 71, 352–382.
- Lutscher, F.: 2002, Modeling alignment and movement of animals and cells. *Journal of Mathematical Biology* 45, 234–260.
- Marshal, J., M. Broucke, and B. Francis: 2004, Formations of vehicles in cyclic pursuit. *IEEE Transactions on Automatic Control* 49(11), 1963–1974.
- Martínez, S., F. Bullo, J. Cortés, and E. Frazzoli: 2007a, On synchronous robotic networks Part I: Models, tasks and complexity notions. *IEEE Transactions on Automatic Control* 52(12), 2199–2213.
- Martínez, S., F. Bullo, J. Cortés, and E. Frazzoli: 2007b, On synchronous robotic networks Part II: Time complexity of rendezvous and deployment algorithms. *IEEE Transactions on Automatic Control* 52(12), 2214–2226.
- Martínez, S., J. Cortés, and F. Bullo: 2007c, Motion coordination with distributed information. *IEEE Control Systems Magazine* 27(4), 75–88.
- Mesbahi, M.: 2005, On state-dependent dynamic graphs and their controllability properties. *IEEE Transactions on Automatic Control* 50(3), 387–392.

- Mikhailov, A. S. and D. H. Zanette: 1999, Noise-induced breakdown of coherent collective motion in swarms. *Physical Review E* 60(4), 4571–4575.
- Miller, R. S. and W. J. D. Stephen: 1966, Spatial relationships in flocks of sandhill cranes. *Ecology* 47(2), 323–327.
- Mirabet, V., P. Auger, and C. Lett: 2007, Spatial structures in simulations of animal grouping. *Ecological Modelling* 201, 468–476.
- Mogilner, A. and L. Edelstein-keshet: 1999, A non-local model for a swarm. *Journal of Mathematical Biology* 38, 534–570.
- Mogilner, A., L. Edelstein-keshet, L. Bent, and A. Spiros: 2003, Mutual interactions, potentials, and individual distance in a social aggregation. *Journal of Mathematical Biology* 47, 353–389.
- Morale, D., V. Capasso, and K. Oelschläger: 2005, An interacting particle system modelling aggregation behavior: from individuals to populations. *Journal of Mathematical Biology* 50, 49–66.
- Moreau, L.: 2004, Stability of continuous-time distributed consensus algorithms. In: *Proceedings of the 43rd IEEE Conference on Decision and Control*. Paradise Island, Bahamas, pp. 3998–4003.
- Moreau, L.: 2005, Stability of Multi-agent Systems with Time-dependent Communication Links. *IEEE Transactions on Automatic Control* 50(2), 169–182.
- Moshtagh, N. and A. Jadbabaie: 2007, Distributed Geodesic Control Laws for Flocking of Nonholonomic Agents. *IEEE Transactions on Automatic Control* 52(4), 681–686.
- Moshtagh, N., N. Michael, A. Jadbabaie, and K. Daniilidis: 2009, Vision-Based, Distributed Control Laws for Motion Coordination of Nonholonomic Robots. *IEEE Transactions on Robotics* 25(4), 851–860.
- Muhammad, A. and M. Egerstedt: 2005, Connectivity graphs as models of local interactions. *Applied Mathematics and Computation*.

- Murray, J. D.: 1993, *Mathematical Biology*. Springer-Verlag, New York.
- Murray, R. M.: 2007, Recent Research in Cooperative Control of MultiVehicle Systems. *Journal of Dynamic Systems, Measurement, and Control* 129(5), 571–583.
- Muss, A., D. R. Robertson, A. Stepien, P. Wirtz, and B. W. Bowen: 2001, Phylogeography of Ophioblennius: The role of ocean currents and geography in reef fish evolution. *Evolution* 55(3), 561–571.
- Newlands, N.: 2002, Shoaling dynamics and abundance estimation: Atlantic Bluefin Tuna (*Thunnus thynnus*). Ph.D. thesis, Fisheries Center, University of British Columbia, Vancouver BC, Canada.
- Niwa, H. S.: 1994, Self-organizing dynamic model of fish schooling. *Journal of Theoretical Biology* 171, 123–136.
- Niwa, H. S.: 1996, Newtonian dynamical approach to fish schooling. *Journal of Theoretical Biology* 181, 47–83.
- Ögren, P., E. Fiorelli, and N. E. Leonard: 2004, Cooperative Control of Mobile Sensor Networks: Adaptive Gradient Climbing in a Distributed Environment. *IEEE Transactions on Automatic Control* 49(8), 1292–1302.
- Olfati-Saber, R.: 2003, Flocking with Obstacle Avoidance. Technical Report Technical Report CIT-CDS 03-006, California Institute of Technology.
- Olfati-Saber, R.: 2006, Flocking for Multi-Agent Dynamic Systems: Algorithms and Theory. *IEEE Transactions on Automatic Control* 51(3), 401–420.
- Olfati-Saber, R., J. A. Fax, and R. M. Murray: 2007, Consensus and Cooperation in Networked Multi-agent Systems. *Proceedings of the IEEE* 95(1), 215–233.
- Olfati-Saber, R. and R. M. Murray: 2002, Distributed Cooperative Control of Multiple Vehicle Formations Using Structural Potential Functions. In: *Proceedings of the 15th IFAC World Congress*. Barcelona, Spain.
- Olfati-Saber, R. and R. M. Murray: 2003, Consensus Protocols for Networks of Dynamic Agents. In: *Proceedings of the 2003 American Control Conference*. pp. 951–956.

- Olfati-Saber, R. and R. M. Murray: 2004, Consensus Problems in Networks of Agents with Switching Topology and Time-delays. *IEEE Transactions on Automatic Control* 49(9), 1520–1533.
- Paden, B. and S. Sastry: 1987, A Calculus for Computing Filippov’s Differential Inclusion with Application to the Variable Structure Control of Robot Manipulators. *IEEE Transactions on Circuits and Systems* 34(1), 73–82.
- Paley, D. A., N. E. Leonard, R. Sepulchre, and I. D. Couzin: 2007, Spatial Models of Bistability in Biological Collectives. In: *Proceedings of the 46th IEEE Conference on Decision and Control*. New Orleans, LA, pp. 4851–4856.
- Parrish, J. K. and L. Edelstein-keshet: 1999, Complexity, Pattern and Evolutionary Trade-offs in Animal Aggregation. *Science* 284, 99–101.
- Parrish, J. K., S. V. Viscido, and D. Grünbaum: 2002, Self-organized fish schools: An examination of emergent properties. *Biological Bulletin* 202, 296–305.
- Regmi, A., R. Sandoval, R. Byrne, H. Tanner, and C. Abdallah: 2005, Experimental implementation of flocking algorithms in wheeled mobile robots. In: *Proceedings of 2005 American Control Conference*. Portland, OR, pp. 4917–4922.
- Ren, W.: 2007, Multi-vehicle consensus with a time-varying reference state. *Systems and Control Letters* 56(7-8), 474–483.
- Ren, W.: 2008, On Consensus Algorithms for Double-Integrator Dynamics. *IEEE Transactions on Automatic Control* 53(6), 1503–1509.
- Ren, W. and R. W. Beard: 2005, Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control* 50(5), 655–661.
- Ren, W., R. W. Beard, and E. M. Atkins: 2007, Information Consensus in Multivehicle Cooperative Control: Collective Group Behavior through Local Interaction. *IEEE Control Systems Magazine* 27(2), 71–82.

- Reynolds, C. W.: 1987, Flocks, herds, and schools: a distributed behavioral model. *Proceedings of the 14th ACM Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'87)* 21(4), 25–34.
- Rimon, E. and D. E. Koditschek: 1992, Exact robot navigation using artificial potential functions. *IEEE Transactions on Robotics and Automation* 8(5), 501–517.
- Ryan, E. P.: 1998, An Integral Invariance Principle for Differential Inclusions with Applications in Adaptive Control. *SIAM Journal on Control and Optimization* 36(3), 960–980.
- Saber, R. O., J. A. Fax, and R. M. Murray: 2003, Consensus protocols for networks of dynamic agents. In: *Proceedings of the 2003 American Control Conference*. Denver, CO, pp. 951–956.
- Sastry, S.: 1999, *Nonlinear Systems: Analysis, Stability and Control*. New York: Springer-Verlag.
- Savkin, A. V.: 2004, Coordinated collective motion of groups of autonomous mobile robots: Analysis of Vicseks model. *IEEE Transactions on Automatic Control* 49(6), 981–982.
- Shaw, E.: 1970, Schooling in fishes: critique and review. In: L. Arson (ed.): *Development and Evolution of Behavior*. San Francisco, CA: Freeman, pp. 452–480.
- Shevitz, D. and B. Paden: 1994, Lyapunov Stability Theory of Nonsmooth Systems. *IEEE Transactions on Automatic Control* 39(9), 1910–1914.
- Shimoyama, N., K. Sugawa, T. Mizuguchi, Y. Hayakawa, and M. Sano: 1996, Collective motion in a system of motile elements. *Physical Review Letters* 76(20), 3870–3873.
- Smirnov, G. V.: 2002, *Introduction to the theory of differential inclusions*, Vol. 41 of *Graduate Studies in Mathematics*. American Mathematical Society.
- Susca, S., S. Martínez, and F. Bullo: 2009, Gradient algorithms for polygonal approximation of convex contours. *Automatica* 45(1), 510–516.

- Tanner, H. G.: 2004, Flocking with Obstacle Avoidance in Switching Networks of Interconnected Vehicles. In: *Proceedings of the 2004 IEEE International Conference on Robotics and Automation (ICRA'04)*, Vol. 3. New Orleans, LA, pp. 3006–3011.
- Tanner, H. G., A. Jababaie, and G. J. Pappas: 2003a, Stable flocking of mobile agent, Part II: Dynamic topology. In: *Proceedings of the 42nd IEEE Conference on Decision and Control*. Maui, HI, pp. 2016–2021.
- Tanner, H. G., A. Jababaie, and G. J. Pappas: 2007, Flocking in Fixed and Switching Networks. *IEEE Transactions on Automatic Control* 52(5), 863–868.
- Tanner, H. G., A. Jadbabaie, and G. J. Pappas: 2003b, Stable flocking of mobile agent, Part I: Fixed topology. In: *Proceedings of the 42nd IEEE Conference on Decision and Control*. Maui, HI, pp. 2010–2015.
- Tanner, H. G., G. J. Pappas, and V. Kumar: 2004, Leader-to-formation stability. *IEEE Transactions on Robotics and Automation* 20(3), 443–455.
- Vaughan, R., N. Sumpter, J. Henderson, A. Frost, and S. Cameron: 2000, Experiments in automatic flock control. *Robotics and Autonomous Systems* 31, 109–117.
- Vicsek, T., A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet: 1995, Novel type of phase transition in a system of self-driven particles. *Physical Review Letters* 75(6), 1226–1229.
- Viscido, S. V., J. K. Parrish, and D. Grünbaum: 2004, Individual behavior and emergent properties of fish schools: a comparison of observation and theory. *Marine Ecology Progress series* 273, 239–249.
- Wiltschko, W. and R. Wiltschko: 1996, Magnetic orientation in birds. *Journal of Experimental Biology* 199, 29–38.
- Xiao, L. and S. Boyd: 2004, Fast linear iterations for distributed averaging. *Systems and Control Letters* 53(1), 65–78.
- Xiao, L., S. Boyd, and S.-J. Kim: 2007, Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing* 67, 33–46.

- Yakubovich, V. A., G. A. Leonov, and A. K. Gelig: 2004, *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, Vol. 14 of *Stability, Vibration and Control of Systems, Series A*. Singapore: World Science Publishing.
- Zheng, M., Y. Kashimoria, O. Hoshinoc, K. Fujitab, and T. Kambaraa: 2005, Behavior pattern (innate action) of individuals in fish schools generating efficient collective evasion from predation. *Journal of Theoretical Biology* 235, 153–167.