

RUSSIAN DOLL POROELASTICITY

by

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Abstract

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Advisor: Professor Stephen Cowin

In this thesis I solve three interconnected problems related to the understanding of cortical bone poroelasticity. The first and most basic problem addressed is the unconfined compression of a cylindrical annular disk of fluid-filled, porous, elastic solid between smooth, impermeable plates. The theory of anisotropic poroelastic materials is used to develop an analytical solution for this axisymmetric problem in which the plane of transverse isotropy is perpendicular to the cylindrical axis. A general solution is obtained from which the stress relaxation response is obtained. The solution to this boundary value problem is determined for three different boundary conditions on the pore pressure. These boundary conditions are (i) free fluid passage across the outer and inner cylindrical boundaries of the disk; (ii) free fluid passage across the outer cylindrical boundary and no flow across the inner boundary (i.e. zero pressure gradient); (iii) free fluid passage across the inner cylindrical boundary and no flow across the outer boundary. These solutions appeared in Gailani and Cowin (2008).

The second problem is to detail a design and execute an experimental method to measure the permeability of the lacunar-canalicular porosity (PLC) of bone tissue. The analytical solution of problem #1 is used as a theoretical basis for outlining a step-by-step

procedure to experimentally determine the permeability of the disk. In addition, procedures will be described for measuring the permeability of the inner walls (Haversian canal) of an osteon. This is achieved by experimentally employing any of the three different solutions obtained for the basic problem described above, particularly the one that is described as (iii) above. An iterative optimization technique, the least squares method, is used to obtain the material properties including the permeability which yield the best curve fit of the analytical solution to the experimental data.

The two problems described are a prelude to constructing a more detailed model of the porous structure of osteonal cortical bone. This new porous structure model is said to be “nested” in the sense that smaller pore size porosity, the PLC, only interchanges interstitial bone fluid with the larger pore size porosity, the vascular porosity (PV). In the construction of the nested porosity model, called Russian doll poroelasticity, the analytical solution of problem #1 can be used twice, once at the PLC level (the osteon) and again at the PV level (the whole bone cross-section). The significance of the result is basic to the understanding of interstitial flow in bone tissue which, in turn, is basic to understanding nutrient transport from the vasculature to the bone cells buried in the bone tissue and to mechanotransduction in bone tissue. In addition it gives an insight into the elastic response of soft tissues versus hard tissues and saturated swampy soils versus saturated rocks and as an analytical tool to analyze unconfined compression experiments performed on these materials.

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This thesis arose in part out of two years of research that has been done since I joined Professor Cowin and his research group. By that time, I have worked with a great number of people whose contribution to the research and the making of the thesis deserved special mention. It is a pleasure to convey my gratitude to them all in my humble acknowledgement.

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TO
My Mother,
Father,
My wife,
My children
AND
Jaylani &
Ali

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Nomenclature

$\hat{A}_i$	Biot effective stress coefficient, $i = (r, \theta, z)$
a	ratio of the inner radius to the outer radius, c/b
BC1	first drainage condition – flow across the outer boundary only
BC2	second drainage condition – flow across the inner boundary only
BC3	third drainage condition – flow across both boundaries
b	outer radius
c	inner radius
$\hat{\mathbf{C}}^i$	($i = m, d, u$) elasticity or stiffness matrix for the matrix material, drained elastic constants and undrained elastic constants; this is a symmetric second order tensor in six dimensions that is totally equivalent to the customary representation of elasticity as a fourth order tensor in three dimensions
$\hat{\mathbf{E}}$	strain vector in six dimensions; this vector is completely equivalent to the customary representation of strain as a second order tensor in three dimensions, denoted by $\mathbf{E}$
E^i	($i = m, d, u$) modulus of elasticity
$H_n^{(p)}(x)$	Hankel functions
$I_n(x)$	modified Bessel functions of the first kind
i	imaginary number ($i = \sqrt{-1}$)
$J_n(x)$	Bessel functions of the first kind
K^i	($i = m, d, f$) bulk modulus
$\hat{\mathbf{K}}$	permeability vector in six dimensions; this vector is completely equivalent to the customary representation of anisotropic permeability as a second order tensor in three dimensions, denoted by $\mathbf{K}$
$K_n(x)$	modified Bessel functions of the second kind
k_{rr}	radial permeability
$\hat{\mathbf{O}}$	double gradient operator as vector in six dimensions; this vector is completely equivalent to the customary representation of the double or open gradient product, a second order tensor in three dimensions, denoted by $\nabla \otimes \nabla$, $\hat{\mathbf{O}} = [\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \frac{\partial^2}{\partial x_3^2}, \sqrt{2} \frac{\partial^2}{\partial x_2 \partial x_3}, \sqrt{2} \frac{\partial^2}{\partial x_1 \partial x_3}, \sqrt{2} \frac{\partial^2}{\partial x_1 \partial x_2}]^T$
p	pore fluid pressure
P	total applied force
r	radius ($c \leq r \leq b$)
s	Laplace transform variable, inverse of time
$\hat{\mathbf{S}}^i$	($i = m, d, u$) compliance matrix for the matrix material, drained elastic constants and drained elastic constants; this is a symmetric second rank tensor in six dimensions that is totally equivalent to the customary

	representation of elasticity as a fourth order tensor in three dimensions and it has the advantage of being the matrix inverse of $\hat{\mathbf{C}}^i$ above
S	Skempton parameter
t	time
t_o	period of the ramp loading
$\hat{\mathbf{T}}$	stress vector in six dimensions; this vector is completely equivalent to the customary representation of stress as a second order tensor in three dimensions, denoted by $\mathbf{T}$
$(\sim)$	tilde over a function indicates the Laplace transform of the function
u	displacement
$Y_n(x)$	Bessel functions of the second kind
$\hat{\mathbf{U}}$	six-dimensional vector representation of a 3D unit tensor ([1,1,1,0,0,0])

Greek Symbols

ε	strain
ϕ	porosity
λ	ratio of r/b (non-dimensional)
μ	viscosity
ν	Poisson's ratio
$\mathfrak{S}^{-1}$	inverse of Laplace transform
τ	dimensionless time
ζ	variation of the fluid volume per unit volume of the porous material

Subscripts and Superscripts

d	drainage condition
f	fluid
m	matrix material
R_{eff}	Reuss lower bound
V_{eff}	Voigt upper bound
L	PLC porosity
V	PV porosity

Introduction

1.1 Theory of poroelasticity

The study of the mechanical loading of the skeletal system and its effects on biological tissues is a major element of the field of biomechanics. The first applications of mathematical and physical principles to the study of biological phenomena are attributed to Galileo's use of the pendulum to measure the human pulse rate, and Borelli's studies on locomotion, in the seventeenth century. The usefulness of mathematical and mechanical modeling in elucidating biological phenomena has become evident ever since, as advances were made in physiology as well as mechanics. A variety of orthopedic biomechanical problems arise from the normal and pathological behavior of the skeletal system. The concept of bone adaptation in response to mechanical environment was described by Wolff in 1892 and is known as Wolff's law.

Bone is a living tissue and has an arterial blood supply and a venous return. Nutrients are transferred from the blood across blood vessel walls to the interstitial fluid within a bone and the nutrients are then, by the time varying mechanical loads applied to the bone, convected by differential fluid pressure, to the bone cells buried in the stiff bone matrix as much as 150 microns away from the blood supply. The wastes from the bone cells are then convected back to the interstitial fluid in pericellular space surrounding the blood vessels, and from there to the bone lymphatic system which convects the waste materials in interstitial fluid from the bone. The venous return of blood is often aided by the contraction of the muscles surrounding the bone. The structure of this fluid supply and waste disposal is very similar to a water and sewer systems for a city in many formal details with the exception that the city

systems are man made and man understands them while the bone's system is nature's design and man does not understand how it works to the degree that we can make reliable transport computations.

The problem addressed in this thesis is the fluid movement from the perivascular regions to the bone cells in the bone matrix and vice versa. The perivascular regions in bone are the domains exterior to the blood vessels but interior to the blood vessel tunnels in bone tissue (Haversian or osteonal and Volkmann canals). An understanding of this fluid movement is key to understand bone health and its mechanosensory system. There is a connection between fluid movement and mechanosensation in bone. The strain levels in bone are generally less than 0.2% due to the stiffness of bone while the minimum strain level bone cells can sense is about 0.5%. In bone, there is a fluid movement-activated strain magnification system that amplifies the stiff bone tissue strain levels to those a cell can sense. This occurs when the interstitial fluid passes over the bone cell's dendritic surface in the approximate 400 nm diameter canaliculi; the dendritic structures, or cytoplasmic processes, are about 200 nm in diameter. Understanding bone mechanotransduction is fundamental to the understanding of how to treat osteoporosis, how to cope with microgravity and how to design prostheses that are implanted in bone tissue.

The theory of deformation of a porous elastic solid containing a compressible fluid was established by Biot in several papers. In Biot (1941) the isotropic case was considered, and in Biot (1955) the theory is generalized to anisotropic materials. General solutions to the elastic equations were established in Biot (1956-a). The theory of wave propagation in such systems was examined in Biot (1956-b).

In 1923, Karl Terzaghi developed what is called the Terzaghi consolidation model. The model is a special lumped two-parameter model, shown in Figure 1, which is used in biomechanics and soil mechanics. It combines a spring element and a Darcy permeability element. When the load is initially placed on the piston, the spring supporting the piston initially takes none of the load because it cannot deflect because the piston can't move because of due to the incompressible fluid that is filling the chamber. However, once the water has a chance to exit the cylinder through the orifice in the piston, it is possible for the piston to begin moving downward under the action of the applied compressive loading. The process will continue until the spring deflects an amount that is sufficient to create a force equal to the applied loading. The relationship between the force applied to the piston and the deflection of the piston is given by the same constitutive equation that characterizes the Voigt model.

1.2 Porosity of Cortical Bone

The pore sizes in cortical bone are approximately three discrete sizes; the largest pore size (approximately 50 μm diameter) is associated with the vascular porosity (PV), the second largest pore size (approximately 0.3 μm diameter) with the canaliculi in the lacunar-canalicular porosity (PLC) and the smallest pore size (approximately 10 nm diameter) is in collagen-apatite porosity (PCA), see Cowin (1999). The typical pore size of the vascular porosity channels is not the blood vessel pore size; rather it is the size of the tubular tunnels (Haversian systems or osteons and Volkmann canals) containing the blood vessels, the arterioles and the venules, and considerable extra pore fluid space. The interstitial pore fluid in the collagen-apatite porosity (PCA) is considered to be bound to the solid structure in this study and is not of interest in the present considerations. Thus, the two pore size structures of

interest in the flow of bone pore fluid are the vascular porosity (PV), and the lacunar-canalicular porosity (PLC); the ratio of the pore sizes is approximately 167. The vascular porosity has an estimated intrinsic permeability of $k = 6.35 \times 10^{-14} \text{ m}^2$, Rouhana et al., (1981), while there are two reported estimates of the permeability of the lacunar-canalicular porosity.

A theory-based estimate of the intrinsic permeability of the lacunar-canalicular porosity was provided by Zhang et al. (1998) as $k = 1.47 \times 10^{-20} \text{ m}^2$ and an experimentally based estimate was provided later by Smit et al. (2002), $k = 2.2 \times 10^{-22} \text{ m}^2$. The latter estimate is two orders of magnitude smaller than the previous one for the permeability of the lacunar-canalicular porosity. Thus two estimates for the ratio of the permeabilities of the vascular porosity to those of the lacunar-canalicular porosity are 4.32×10^7 and 8.82×10^{10} . These ratio estimates and the approximate ratio of the diameters as 167 are measures of the significant size difference, pore pressure difference and relaxation times in these two distinct pore size porosities.

The pore fluid pressure in these two pore size porosities is also quite different. Under physiologically possible rapid rise-time loadings of bone, the pore fluid pressure may rise considerably in the PLC (Zhang et al., 1998). The decay time for this pore pressure rise is much larger in the PLC than it is in the PV. The PV is a low-pore-fluid-pressure domain because the PV permeability is sufficiently large to remit a rapid decay of a pressure pulse. This must be the case because the PV contains thin-walled blood vessels carrying blood with a pressure of 40 to 60 mmHg; a pore-fluid-pressure significantly greater than 40 to 60 mmHg will collapse these blood vessels and a prolonged increase in the a pore-fluid-pressure significantly above 40 to 60 mmHg would deprive the tissue of oxygen and nutrients.

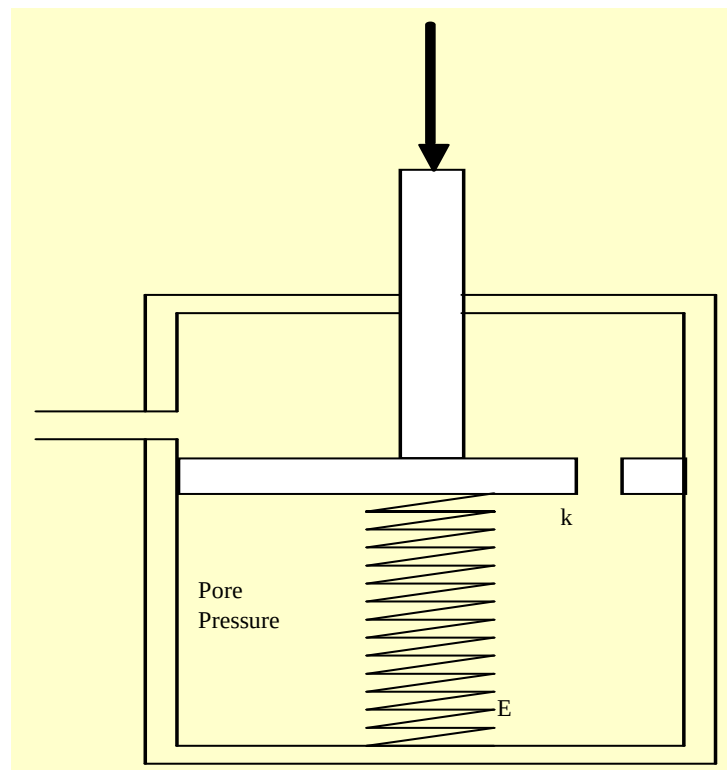


Figure 1: The consolidation of the Terzaghi element, (Cowin and Doty, 2007).

1.3 The model of the PLC in an osteon

As noted above, all the space in the lacunae and the canaliculi is called the lacunar-canalicular porosity (PLC). A sketch of a partial cross-section of an osteon (Figure 2) shows some of the small-scale features of the PLC and related structures of interest. Osteons are considered here to be hollow, roughly cylindrical structures of 100 to 150 μm in outer radii and 40 to 60 μm in inner radii that lie very roughly ($\pm 15^\circ$) along the long axes of a bone and contain at their center a roughly cylindrical canal (see Cowin (1999) and Huja et al. (2006)). The osteonal canal (Figure 2), the hollow at the center of the osteon, is part of the PV and will be discussed later. The walls of the osteonal canals are covered with cells and behind the cells are the entrances to the canaliculi. The canaliculi are passageways that run between lacunae or from the lacunae to the osteonal canal (Figure 1). Weinbaum *et al.* (1994) and Cowin *et al.* (1995) present a detailed theoretical model of the contents of the PLC. The PLC is the porosity associated with the osteocytes; the osteocyte is the prime candidate for the mechanosensory cell in bone. Weinbaum *et al.* (1994) and Cowin *et al.* (1995) suggested that the region between the cell membrane of the osteocyte (and the osteocytic process) on one hand and the walls of the canaliculi and lacunae on the other, is filled with the cell glycocalyx (also called the surface matrix or capsule). The content of this PLC, the combination of the glycocalyx and the bone fluid, is thought to function as a gel. This model has been further developed in a number of papers, e.g., Han et al., (2004).

Cement lines appear as the boundary of osteons (Figure 2). They represent the outer boundary of our osteon model. It was thought at one time that canaliculi did not cross the cement lines, but evidence that they do is now well established (Haines *et al.*, 1983; Curtis *et*

al., 1985). The number density per unit area of the canaliculi crossing cement lines has not been established, nor has the anatomical distribution of their crossing been investigated. It is possible that canaliculi only cross cement lines when there is an isolated osteocyte on the other side, perhaps residing in a small piece of a former osteon removed by remodeling. This would make the number density per unit area of canaliculi crossing cement lines dependent on anatomical site and therefore a parameter with a very inhomogeneous value. It follows that the permeability of this interface is unknown and could be very inhomogeneous. The other possibility is that bone cells in one osteon need to communicate with those in neighboring osteons in order to complete a whole bone cell network. If this latter case is in fact the situation, then the trespassing of the cement lines by the canaliculi will be a more homogeneous occurrence.

The cross-section of an osteon containing the PLC is an annular region bounded by two roughly circular structures, the cement line on the outside and the wall of the osteonal canal on the inside. These boundaries are the two small concentric circles in Figure 2, which is an illustration of a pie shaped section of this cross-section. The volume of the osteon containing the PLC is an approximately circular cylinder with an annular cross-section. The model of the osteon employed here is this volumetric PLC domain. The boundary of this domain formed by the wall of the osteonal canal is assumed to be porous and the boundary formed by the cement line is assumed to be impermeable. There are three rationales for making this assumption; (1) the number of canaliculi crossing the cement line is unknown, and their observation has been infrequent, (2) an earlier study (Wang et al., 1999) concluded that it made little difference if the boundary was assumed to be impermeable or fully permeable in

the global predictions of physiological significance, (3) it is not a difficult assumption to remove if necessary.

Any attempts to understand and model the biomechanical properties of bone must be firmly rooted in the knowledge of the bone structure at both the ultrastructural and microstructural levels of organization. Mechanical properties of bone are determined by complex relationships between material and structural properties such as tissue mineralization, size and composition of mineral crystals, anisotropy, porosity, permeability ...etc. The osteon is composed of a central Haversian canal housing a blood vessel and is surrounded by alternating mineralized collagen lamellae. As early as 1959 Ascenzi and Fabri (1959) worked with partially isolated secondary osteons. Their technique of separation has improved in Ascenzi and Bonucci (1967), Ascenzi et al. (1985), and Ascenzi et al. (1994). Frasca et al. (1976) showed that an osteon can be separated by propagating a fracture through the natural boundary of the osteon. Neil Dong et al. (2005) designed and assembled a precision and computer controlled osteon pushout micro-testing system. The isolated single osteon is idealized in this study as a porous annular cylinder fully saturated and subjected to unconfined compression in a stress relaxation test. In Gailani and Cowin (2007) we reported our analysis of the unconfined compression of a thin annular, circular disk of a poroelastic material when squeezed between two perfectly smooth, impermeable plates. Under these conditions, the compression applied along the z-axis, the longitudinal axis of the circular disk, causes a radial expansion. This radial expansion is unconstrained and free fluid exudation and/or imbibing can occur across the cylindrical boundary surface. Since contact with the plates compressing the disks is assumed to be totally frictionless, the deformation is expected to be independent of

(z). From these simple intuitive arguments, we see that interstitial fluid must flow in the radial direction.

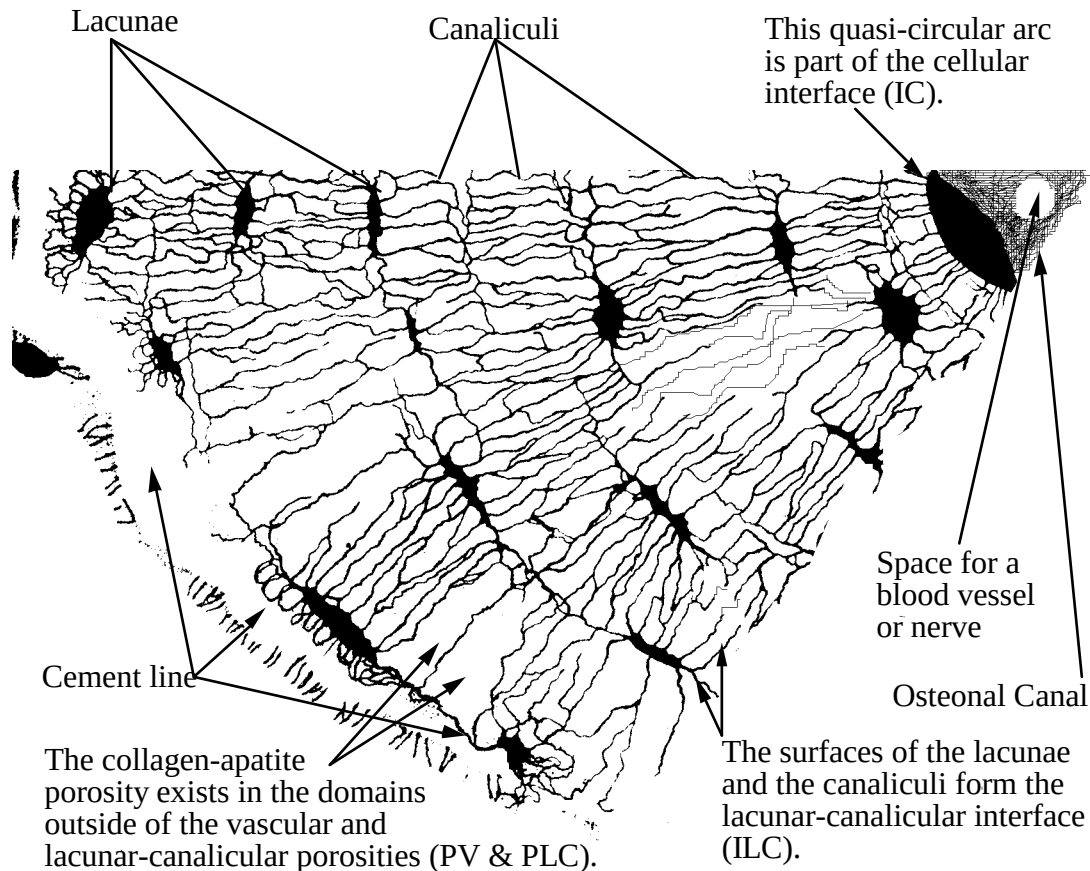


Figure 2: A pie-shaped section of an osteon. The osteonal canal is on the upper right, the cement line to the left. The osteonal canal is part of the vascular porosity (PV), the lacunae and the canaliculi are part of the lacunar-canalicular porosity (PLC) and the material in the space that is neither PV or PLC contains the collagen-apatite porosity (PCA). The three interfaces, the cement line, the cellular interface (IC) and the lacunar-canalicular interface are each indicated. The radius of an osteon is usually about 100 μm , and the long axis of a lacuna is about 15 μm . Using this information it should be possible to establish the approximate scale of the printed version of this illustration. Cowin (1999)

2.

The Unconfined Compression of a Poroelastic Annular Cylindrical Disk

2.1 Introduction

The problem of confined and unconfined compression tests of porous disks compressed between two parallel rigid and impermeable smooth plates has been under investigation by many authors in the last 30 years. The impermeable frictionless plates would allow flow only in the radial direction in the unconfined compression experiment as shown in Figure 3. The theory of anisotropic poroelastic materials was used to develop an analytical solution for this axisymmetric problem in which the plane of transverse isotropy is perpendicular to the cylindrical axis. A general solution was obtained from which both the stress relaxation response and creep response to imposed displacement or load step ramp displacements may be obtained. The algebraic complexity of the problem stems from the need to have the solution to the problem for three different sets of boundary drainage conditions on fluid flow or no flow through the inner and outer boundaries of the annular disk. The analytical solution demonstrates the effects of the assumptions of isotropy vs. transverse isotropy and compressibility vs. incompressibility of the poroelastic constituents. The advantages of this solution are that they provide insight into the nature of the elastic response and fluid flow of soft tissues versus hard tissues and saturated swampy soils versus saturated rocks and as an analytical tool to analyze unconfined compression experiments performed on these materials.

The experimental technique of subjecting circular disks of soft tissue to unconfined compression in order to determine the soft tissue's permeability and some of its elastic constants has become widely used in the last few decades (Armstrong et al., 1984; Cheng and Bilston, 2007; Cohen et al., 1998; Fortin et al., 2000; Korhonen et al., 2002; LeRoux and Setton et al., 2002; Wu and Herzog, 2006; Yin and Elliott, 2004, and many others). The original study of unconfined compression appeared in a paper by Mandel (1953). The analysis of the experimental data from such a test is based on an analytical solution to an infinitesimally deformed saturated poroelastic disk loaded in a similar manner. For the analysis of soft tissues the assumption that the fluid and solid constituents of the poroelastic medium are both incompressible is reasonable. However in the case of hard tissues the constituent incompressibility assumption is no longer reasonable because the effective bulk modulus of the poroelastic solid constituent is almost an order of magnitude stiffer than that of the poroelastic fluid constituent, thus the solid constituent shields the fluid constituent from stress. The pressures in the solid and fluid phases of hard tissue may be of significantly different magnitudes whereas these pressures are reasonably assumed to be the same in soft tissues; thus the constituent incompressibility assumption fits for soft tissues but is inappropriate for hard tissues. The analytical solution to a saturated poroelastic disk subjected to unconfined compression was given in Cowin and Mehrabadi (2007). In this paper that solution is extended to the situation in which the disk is annular rather than circular in order to provide an analytical solution for the basis of a yet-to-be-developed unconfined compression experimental technique for determining the lacunar-canalicular permeability of bone tissue and some of the elastic constants of a similarly shaped osteonal bone specimen.

An unconfined compression experiment is illustrated schematically in Fig. 3. In this

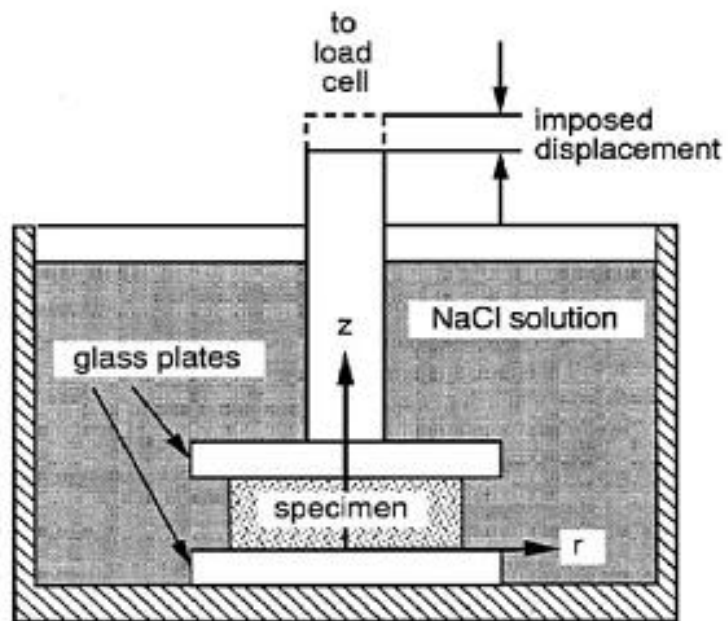


Figure 3: Schematic of unconfined compression test, Cohen et al., (1998).

figure the disk-shaped specimen is compressed between smooth parallel impermeable plates (glass in the illustration) and the pore fluid is forced out of the specimen, across the lateral curved boundaries, in the radial direction. Typical unconfined compression tests are accomplished by imposing either a step in end plate displacement, which would constitute a stress relaxation test, or a step in total applied end force load may be applied, which would constitute a creep test. The specimen contact with the plates is assumed to be frictionless, so the deformation might be expected to be independent of z ; thus this axisymmetric problem becomes a one-dimensional problem in the radius, and fluid flow will occur in the radial direction.

The unconfined compression test of a poroelastic disk with constituents constrained to be incompressible has been used to study the mechanical behavior and load effects of hydrated soft tissues such as articular cartilage. Following Mandel (1953), Armstrong et al. (1984) obtained a theoretical poroelastic solution assuming infinitesimal deformation, frictionless platen interface, isotropic material properties and incompressible constituents and reported that a good agreement between the theoretical response and the experimental data could not be obtained. They attributed this to confinement effects due to surface adhesion at the tissue-platen interface. Cohen et al. (1998) proposed that the high peak stress and rapid relaxation time that are observed experimentally in unconfined compression are due to the confining effect of extracellular matrix against radial displacement producing tensile strains in the transverse plane and that the behavior can be described by a transversely isotropic poroelastic model. Replacing the material isotropy assumption of Armstrong et al. (1984) by an assumption of transverse isotropy, Cohen et al. (1998) compared their theoretical solution to the unconfined tests of the growth plate and the chondroepiphysis, and showed that the

unconfined compression model with incompressible constituents and transverse isotropy provided a good curve fit.

Cowin and Mehrabadi (2007) obtained a theoretical solution for the unconfined compression test of a poroelastic disk using the governing equations of the theory of anisotropic poroelastic materials without the constituent incompressibility constraints. They also demonstrated the effects of changing the porosity on the load intensity and reported that the theoretical approach can be applied to soft materials like cartilaginous tissues or swampy soils with the incompressibility assumptions as well as to hard materials like living bone tissues and porous granites when incompressibility is not assumed.

In this study, the work of Cowin and Mehrabadi (2007) is used to extend their analysis to annular disks. The only difference between the annular disk described here and the one analyzed by Cowin and Mehrabadi (2007) is that the annular disk referred to in this report, has a centered circular hole. The complexity is in the algebraic details, as illustrated by Cowin and Mehrabadi (2007), but complicated in the case of an annular disk by additional boundary conditions on the interior surface and the occurrence of an additional Bessel function in the solutions for the pressure, fluid content and displacement fields.

The solution requires three different sets of boundary drainage conditions because it is possible to perform three different unconfined compression experiments with the same annular disk specimen. The first set of boundary conditions allows free fluid passage from the outer boundary and none across the inner boundary. The second set of boundary conditions is to seal the outer boundary and allow free fluid passage from the inner boundary. The third set of boundary conditions allows fluid passage from both the outer and the inner cylindrical boundaries of the annular disk. The fourth possible set of boundary conditions is the case in

which both curved boundaries are sealed. In this case flow cannot occur and the situation is described as undrained; the stress analysis is trivial.

We have chosen to develop an analytical solution rather than a computational solution because, for our purposes, this is the more desirable choice for several reasons. First, the general character of the solution is indicated by the standard functions appearing in the solution, Bessel functions in this case. Second, in the analytical solution the material parameters do not need to be assigned specific values and may be varied after the solution is obtained to understand their effect on the solution; numerical solutions are restricted in their usefulness because they require parameters to be specified. Third, analytical solutions provide a standard against which numerical solutions may be compared. Most importantly, it is our intention to use the solution presented as the theoretical basis for an experimental procedure and it is highly desirable for us to be able to see the effects of the variation in boundary conditions and material parameters.

2.2 Formulation of the problem

The symmetry of the unconfined compression problem suggests that it is reasonable to assume that the displacement and pressure fields are axisymmetric. The compressive strain in the axial direction is homogeneous, a condition only obtainable if the two impervious plates are perfectly frictionless. A direct result on the kinematics of deformation is that all stresses and strains are independent of (z) . The compressive loading in the z direction and the frictionless plates leads one to conclude that the (r, z) coordinate axes are principal axes. Thus there are no shear strains and hence no shear stresses exist on these coordinate planes. The boundary conditions of zero shear stresses on the ends and the cylindrical outer boundary at $r = b$ are then automatically satisfied. The assumption that the fluid pressure and the strains do

not depend on z implies that there is no relative motion in the (z) direction between the fluid phase and the solid phase.

The mathematical formulation begins with assumptions concerning the functional dependence of the pressure, displacements and the fluid content,

$$u_r = u_r(r, t), \quad u_z = z\varepsilon(t), \quad p = p(r, t), \quad \text{and} \quad \zeta = \zeta(r, t), \quad (2-1)$$

where cylindrical coordinates have been employed. The deformation is infinitesimal. The Biot effective stress coefficient vector $\hat{\mathbf{A}}$, so named because it is employed in the definition of the effective stress in equation (13), is related to the stiffness matrix $\hat{\mathbf{C}}^d$ and the compliance matrix $\hat{\mathbf{S}}^m$ as shown by Cowin and Mehrabadi (2007) and in Cowin and Doty (2007) by the relation

$$\hat{\mathbf{A}} = (\hat{\mathbf{1}} - \hat{\mathbf{C}}^d \cdot \hat{\mathbf{S}}^m) \cdot \hat{\mathbf{U}}, \quad (2-2)$$

where $\hat{\mathbf{U}}$ is a six-dimensional vector representation of a 3D unit tensor, and the axis of symmetry of $\hat{\mathbf{S}}^m$ is assumed to be coincident with the transversely isotropic axis of symmetry of $\hat{\mathbf{C}}^d$; this assumption yields

$$\hat{\mathbf{A}} = [\hat{A}_r, \hat{A}_r, \hat{A}_z, 0, 0, 0]^T. \quad (2-3)$$

If $\hat{\mathbf{S}}^m$ and $\hat{\mathbf{C}}^d$ do not have the relationship indicated above, then the solution of the problem under investigation will be more complicated because $\hat{A}_r \neq \hat{A}_\theta$. The *drained* anisotropic elastic constants of $\hat{\mathbf{S}}^d$ and $\hat{\mathbf{C}}^d$ are measured in situations in which the fluid pressure p in the fluid-filled pores is negligible or zero. Draining all the pores before the test or doing the test so slowly achieves the drained condition.

In a porous medium the pores are assumed to be connected to allow the flow of the fluid through them; there are no unconnected pores. The assumed transversely isotropic material symmetry of $\hat{\mathbf{C}}^d$ can be expressed as

$$\hat{\mathbf{C}}^d = \begin{bmatrix} \hat{C}_{rr}^d & \hat{C}_{r\theta}^d & \hat{C}_{rz}^d & 0 & 0 & 0 \\ \hat{C}_{r\theta}^d & \hat{C}_{rr}^d & \hat{C}_{rz}^d & 0 & 0 & 0 \\ \hat{C}_{rz}^d & \hat{C}_{rz}^d & \hat{C}_{zz}^d & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{C}_{44}^d & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{C}_{44}^d & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{C}_{66}^d \end{bmatrix}. \quad (2-4)$$

The relationship between the compliance tensor $\hat{\mathbf{S}}^i$ ($i = m, d, u$) and the effective bulk modulus is simple and important. Hill (1952) showed that $K_{\text{Reff}} = (\hat{\mathbf{U}} \cdot \hat{\mathbf{S}}^i \cdot \hat{\mathbf{U}})^{-1}$, where K_{Reff} is the Reuss lower bound on the effective (isotropic) bulk modulus of the anisotropic elastic material. In the same manner, the Voigt effective bulk modulus of an anisotropic elastic material, $K_{\text{Veff}} = (\hat{\mathbf{U}} \cdot (\hat{\mathbf{S}}^i)^{-1} \cdot \hat{\mathbf{U}})/9$, is the upper bound, thus $K_{\text{Reff}} \leq K_{\text{eff}} \leq K_{\text{Veff}}$. In the case of isotropy the two bounds coincide with the isotropic bulk modulus.

At certain places in the analysis presented, incompressibility will be imposed on the matrix material and the pore fluid. These two constraints require that the pore fluid and the matrix material experience no volume change. The requirement that the fluid be incompressible is implemented by requiring that the reciprocal of the bulk modulus of the fluid tends to zero as the instantaneous density tends to the initial density, or by imposing the condition that the fluid density be a constant. The incompressibility constraint for the matrix material is the requirement that its matrix material compliance satisfy the condition (Cowin and Mehrabadi, 2007)

$$\hat{\mathbf{U}} \cdot \hat{\mathbf{S}}^m = \hat{\mathbf{0}}. \quad (2-5)$$

A direct result from equation (5) is that $\hat{\mathbf{S}}^m$ must be singular in order to satisfy this constraint.

Equation (2) can be rewritten in the case of incompressible transverse isotropy to show that

Table 1:

The values of various bulk moduli and constants C_{eff}^d , Λ , J , $\hat{A}_r$, and $\hat{A}_z$ defined for convenience, for cases of different material models, Cowin and Mehrabadi (2007).

	Transverse isotropy compressible	Transverse isotropy incompressible	Isotropy Compressible	Isotropy Incompressible
$\frac{1}{K_1^m}$	$\frac{3(1-\nu_{12}^m - \nu_{13}^m)}{E_1^m}$	0	$\frac{1}{K^m} = \frac{3(1-2\nu^d)}{E_1^m}$	0
$\frac{1}{K_3^m}$	$\frac{3(1-2\nu_{13}^m)}{E_3^m}$	0	$\frac{1}{K^m}$	0
$\frac{1}{K_{R_{eff}}^m}$	$\frac{2}{3K_1^m} + \frac{1}{3K_3^m}$	0	$\frac{1}{K^m}$	0
$\frac{1}{K_{R_{eff}}^d}$	$\frac{2(1-\nu_{12}^d - \nu_{13}^d)}{E_1^d} + \frac{1-2\nu_{31}^d}{E_3^d}$	$\frac{2(1-\nu_{12}^d - \nu_{13}^d)}{E_1^d} + \frac{1-2\nu_{31}^d}{E_3^d}$	$\frac{1}{K^d} = \frac{3(1-2\nu^d)}{E^d}$	$\frac{1}{K^d} = \frac{3(1-2\nu^d)}{E^d}$
C_{eff}^d	$\frac{1}{K_{R_{eff}}^d} - \frac{1}{K_{R_{eff}}^m} + \phi\left(\frac{1}{K^f} - \frac{1}{K_{R_{eff}}^m}\right)$	$\frac{1}{K_{R_{eff}}^d}$	$\frac{1}{K^d} - \frac{1}{K^m} + \phi\left(\frac{1}{K^f} - \frac{1}{K^m}\right)$	$\frac{1}{K^d}$
Λ	$C_{eff}^d - \frac{1}{K_{R_{eff}}^d} + \frac{2}{K_{R_{eff}}^m} - \frac{1}{9\Lambda_1} \left[\frac{2E_1^d}{K_1^{m^2}} + \frac{4E_1^d \nu_{31}^d}{K_1^m K_3^m} + \frac{E_3^d(1-\nu_{12}^d)}{K_3^{m^2}} \right]$	0	$C_{eff}^d(1-S^2 K^d C_{eff}^d)^*$	0
J	$\left[1 - \frac{E_1^d}{3\Lambda_1} \left(\frac{1}{K_1^m} + \frac{\nu_{31}^d}{K_3^m} \right) \right]^2 + \Lambda \frac{E_1^d \Lambda_2}{\Lambda_1}$	1	$\frac{1}{1+\nu^d} \left[3(1-\nu^d) - 2(1-2\nu^d)S^2 K^d C_{eff}^d \right] K^d C_{eff}^{d*}$	1
$\hat{A}_r$	$1 - \frac{E_1^d}{3\Lambda_1} \left(\frac{1}{K_1^m} + \frac{\nu_{31}^d}{K_3^m} \right)$	1	$1 - \frac{K^d}{K^m} = SK^d C_{eff}^{d*}$	1
$\hat{A}_z$	$1 - \frac{E_3^d}{3\Lambda_1} \left(\frac{2\nu_{13}^d}{K_1^m} + \frac{1-\nu_{12}^d}{K_3^m} \right)$	1	$1 - \frac{K^d}{K^m} = SK^d C_{eff}^{d*}$	1

* S represents Skempton parameter and is given by $S = \frac{(1/K^d) - (1/K^m)}{(1/K^d) - (1/K^m) + \phi((1/K^f) - (1/K^m))}$ for the isotropic compressible case.

$$\hat{\mathbf{A}} = \hat{\mathbf{U}} \text{ and } \frac{1}{K_1^m} = \frac{1}{K_3^m} = 0. \quad (2-6)$$

These equations will be used frequently to obtain results for the incompressible case from the similar result that is free of the assumption of incompressibility.

2-3 The fluid content

The variation in fluid content of the fluid volume ζ (dimensionless) is the variation of the fluid volume per unit volume of the porous material due to diffusive fluid mass transport. It is defined as the difference between the strain of the porous material and the strain of the fluid volume in the pore space. The first strain is due to the change in the volumetric strain in the solid matrix and the latter is due to the fluid pressure,

$$\zeta = \hat{\mathbf{A}} \cdot \hat{\mathbf{E}} + \Lambda p, \quad (2-7)$$

where ϕ is the porosity, $\hat{\mathbf{E}}$ is the strain vector, and

$$\Lambda = \hat{\mathbf{U}} \cdot \hat{\mathbf{S}}^m \cdot \hat{\mathbf{A}} + \phi \left(\frac{1}{K^f} - \frac{1}{K_{\text{Reff}}^m} \right). \quad (2-8)$$

The time rate of change of the fluid content ζ is due either to fluid flux or to volume changes.

The variation in fluid content can also be linearly related to both the stress and the pore pressure, Cowin and Mehrabadi (2007),

$$\zeta = \hat{\mathbf{A}} \cdot \hat{\mathbf{S}}^d \cdot \hat{\mathbf{T}} + C_{\text{eff}}^d p, \quad (2-9)$$

where $\hat{\mathbf{T}}$ is the stress vector and

$$C_{\text{eff}}^d = \frac{1}{K_{\text{Reff}}^d} - \frac{1}{K_{\text{Reff}}^m} + \phi \left(\frac{1}{K^f} - \frac{1}{K_{\text{Reff}}^m} \right). \quad (2-10)$$

It is important to note that changes in the fluid content due to changes in the porosity are neglected because they are higher order terms. This can easily be proved by replacing ϕ with ϕ plus a variation in ϕ or by Darcy's law (see the Appendix) which is based on the

assumption that the movement of the boundaries of the pores is a higher order term that is negligible. Thus ϕ is a reference value of porosity and not a variable. The fluid content in the case of incompressibility is obtained from equations (2-6), (2-7) and (2-8), thus

$$\zeta = \hat{\mathbf{U}} \cdot \hat{\mathbf{E}}, \text{ and } \Lambda = 0. \quad (2-11)$$

2-4 The strain-stress-pore pressure relations

In a saturated porous medium the strain $\hat{\mathbf{E}}$ is linearly related, not only to the stress $\hat{\mathbf{T}}$, but also to the fluid pressure p in the fluid-filled pores; thus one can write the strain-stress-pore pressure relation as

$$\hat{\mathbf{E}} = \hat{\mathbf{S}}^d \cdot \hat{\mathbf{T}} + \hat{\mathbf{S}}^d \cdot \hat{\mathbf{A}}p, \quad (2-12)$$

or

$$\hat{\mathbf{E}} = \hat{\mathbf{S}}^d (\hat{\mathbf{T}} + \hat{\mathbf{A}}p) = \hat{\mathbf{S}}^d \cdot \hat{\mathbf{T}}^{eff}. \quad (2-13)$$

The equations above include the effect of the pore pressure. If the pore pressure is zero the equation above will coincide with the anisotropic Hooke's law.

In the unconfined compression problem the strain can be expressed in terms of the displacement according to the relation:

$$\hat{\mathbf{E}} = [\hat{E}_{rr}, \hat{E}_{\theta\theta}, \hat{E}_{zz}, \sqrt{2}\hat{E}_{rz}, \sqrt{2}\hat{E}_{r\theta}, \sqrt{2}\hat{E}_{z\theta}]^T = \left[\frac{\partial u}{\partial r}, \frac{u}{r}, \varepsilon(t), 0, 0, 0 \right]^T, \quad (2-14)$$

and the stress according to the relation

$$\hat{\mathbf{T}} = [\hat{T}_{rr}, \hat{T}_{\theta\theta}, \hat{T}_{zz}, \sqrt{2}\hat{T}_{r\theta}, \sqrt{2}\hat{T}_{rz}, \sqrt{2}\hat{T}_{z\theta}]^T = [\hat{T}_{rr}, \hat{T}_{\theta\theta}, \hat{T}_{zz}, 0, 0, 0]^T. \quad (2-15)$$

The components of the compliance tensor $\hat{\mathbf{S}}^d$ for transverse isotropy are:

$$\hat{S}_{11} = \hat{S}_{22} = \frac{1}{E_1}, \hat{S}_{12} = \frac{-\nu_{12}}{E_1}, \hat{S}_{13} = \hat{S}_{23} = \frac{-\nu_{31}}{E_3} = \frac{-\nu_{13}}{E_1}, \hat{S}_{33} = \frac{1}{E_3}, \quad (2-16)$$

and the elasticity moduli in terms of various Young's moduli and Poisson's ratio are given by

$$\begin{aligned}\hat{C}_{rr} &= \frac{E_1(1-\nu_{13}\nu_{31})}{(1+\nu_{12})(1-\nu_{12}-2\nu_{13}\nu_{31})} = \frac{\Delta_2}{\Delta_1} E_1, \quad \hat{C}_{r\theta} = \frac{E_1(\nu_{12}+\nu_{13}\nu_{31})}{(1+\nu_{12})(1-\nu_{12}-2\nu_{13}\nu_{31})} = \frac{1-\Delta_2}{\Delta_1} E_1, \\ \hat{C}_{rz} &= \frac{E_1\nu_{31}}{(1-\nu_{12}-2\nu_{13}\nu_{31})} = \frac{\nu_{31}}{\Delta_1} E_1 = \frac{\nu_{13}}{\Delta_1} E_3, \quad \hat{C}_{zz} = \frac{E_3(1-\nu_{12})}{(1-\nu_{12}-2\nu_{13}\nu_{31})} = \frac{(1-\nu_{12})}{\Delta_1} E_3,\end{aligned}\quad (2-17)$$

where Δ_1 and Δ_2 are defined by

$$\Delta_1 = 1 - \nu_{12}^d - 2\nu_{13}^d\nu_{31}^d, \text{ and } \Delta_2 = \frac{1 - \nu_{13}^d\nu_{31}^d}{1 + \nu_{12}^d}. \quad (2-18)$$

One should not overlook the definitions of the deltas above in equation (2-18) since they will be used widely in the text. Equation (2-17) can be rewritten for the case of isotropic symmetry by making all the E 's and ν 's equal, in particular

$$\hat{C}_{rr} = \hat{C}_{\theta\theta} = \hat{C}_{zz} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad \hat{C}_{rz} = \hat{C}_{z\theta} = \hat{C}_{r\theta} = \frac{-E\nu}{(1+\nu)(1-2\nu)}. \quad (2-19)$$

The stress components are determined by substituting equations (2-3) and (2-14) through (2-16) into equation (2-12), thus

$$\begin{aligned}T_{rr} &= -\hat{A}_r p + \hat{C}_{rr}^d \frac{\partial u}{\partial r} + \hat{C}_{r\theta}^d \frac{u}{r} + \hat{C}_{rz}^d \varepsilon(t), \\ T_{\theta\theta} &= -\hat{A}_r p + \hat{C}_{r\theta}^d \frac{\partial u}{\partial r} + \hat{C}_{rr}^d \frac{u}{r} + \hat{C}_{rz}^d \varepsilon(t), \\ T_{zz} &= -\hat{A}_z p + \hat{C}_{rz}^d \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) + \hat{C}_{zz}^d \varepsilon(t).\end{aligned}\quad (2-20)$$

The stresses in the equation above must satisfy the equilibrium equation:

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \quad (2-21)$$

Substituting the non-zero stress components (2-20) into the equilibrium equation above and subsequently integrating with respect to r produces the following expression:

$$\frac{\partial u}{\partial r} + \frac{u}{r} = \frac{\partial}{r \partial r} (ru) = \frac{\hat{A}_{rr}}{\hat{C}_{rr}^d} p(r, t) + f(t), \quad (2-22)$$

where $f(t)$ is an arbitrary function of time obtained in the integration with respect to r .

Using equations (2-22), (2-14) and (2-3), the term $\hat{\mathbf{A}} \cdot \hat{\mathbf{E}}$ that appears in equation (2-7) can be evaluated:

$$\hat{\mathbf{A}} \cdot \hat{\mathbf{E}} = \hat{A}_r \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) + \hat{A}_z \varepsilon(t) = \frac{\hat{A}_r^2}{\hat{C}_{rr}^d} p(r, t) + \hat{A}_z \varepsilon(t) + \hat{A}_r f(t); \quad (2-23)$$

to find $\hat{\mathbf{A}} \cdot \hat{\mathbf{E}}$ in terms of ζ , eliminate p in equation (2-23) using equation (2-7). This elimination yields:

$$\hat{\mathbf{A}} \cdot \hat{\mathbf{E}} = \frac{1}{J} \left[\hat{A}_r^2 \zeta + \Lambda \hat{C}_{rr}^d (\hat{A}_z \varepsilon(t) + \hat{A}_r f(t)) \right], \quad (2-24)$$

where the dimensionless quantity J is introduced:

$$J = \hat{A}_r^2 + \Lambda \hat{C}_{rr}^d. \quad (2-25)$$

Using the second part of equation (2-11) and the first part of equation (2-6) it is possible to show that $J = 1$ in the case of incompressibility. Different values of J for different material models can be calculated as shown in Table (1).

2-5 The Laplace transform solution

The diffusion equation for the pressure field is the key equation to solve in many poroelastic problems. The diffusion equation for the variation in volume fraction can be obtained by substituting Darcy's law into the conservation of mass equation and subsequently eliminating the pore pressure by use of equation (2-7) as shown by Cowin and Mehrabadi (2007)

$$\frac{\partial \zeta}{\partial t} - \frac{1}{\mu \Lambda} \hat{\mathbf{K}} \cdot \hat{\mathbf{O}}[\zeta] = -\frac{1}{\mu \Lambda} \hat{\mathbf{K}} \cdot \hat{\mathbf{O}}[\hat{\mathbf{A}} \cdot \hat{\mathbf{E}}], \quad (2-26)$$

where $\hat{\mathbf{K}}$ is the permeability vector, and $\hat{\mathbf{O}}$ is a double gradient operator. Using the variation in fluid content for the solution of poroelasticity problems has been very effective, but only if the stress or strain field is known or may be calculated without reference to the variation in fluid content.

A differential equation for the pore pressure is then obtained by substituting equations (7) and (2-23) into equation (2-26), thus

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} - \frac{1}{m} \frac{\partial p}{\partial t} = \frac{\mu}{k_{rr}} \left(\hat{A}_z \frac{\partial \varepsilon}{\partial t} + \hat{A}_r \frac{\partial f(t)}{\partial t} \right). \quad (2-27)$$

A diffusive differential equation for the variation in fluid content can be obtained by substituting Equation (2-24) into equation (2-26) thus

$$\frac{\partial^2 \zeta}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial \zeta}{\partial \lambda} - \frac{\partial \zeta}{\partial \tau} = 0, \quad (2-28)$$

where the parameters

$$\tau = \frac{mt}{b^2}, \quad \lambda = \frac{r}{b} \quad \text{and} \quad m = \frac{k_{rr} \hat{C}_{rr}^d}{\mu J} \quad (2-29)$$

have been introduced. The inner radius is denoted by c and the outer radius is denoted by b in the definitions (2-29).

The Laplace transform of equation (2-28) yields a Bessel differential equation in the variation of fluid content $\tilde{\zeta}(\lambda, s)$:

$$\frac{\partial^2 \tilde{\zeta}}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial \tilde{\zeta}}{\partial \lambda} - s \tilde{\zeta} = 0. \quad (2-30)$$

where the tilde ($\sim$) over the function indicates that the independent variable of the function has been shifted from τ to s by the Laplace transformation. The equation above is the key to a convenient approach to the solution of cylinder problems in poroelasticity as pointed out by Detourney et al. (1993, Section 5.6.2.1, equation (162)). It is also interesting to see that equation (2-30) holds for generalized plane strain in the isotropic plane in a transversely isotropic material. The solution for equation (2-30) is given by

$$\tilde{\zeta}(\lambda, s) = Jg_1(s)I_0(\lambda\sqrt{s}) + Jg_2(s)K_0(\lambda\sqrt{s}), \quad (2-31)$$

where $I_o(\lambda\sqrt{s})$ and $K_o(\lambda\sqrt{s})$ are modified Bessel functions of the first and second kinds respectively. The coefficient J , defined in equation (2-25), is introduced in the solution (2-31) for convenience. The arbitrary functions of the transformed variable in the solution (2-31), $g_1(s)$ and $g_2(s)$, have to be determined. The existence of the $K_o(\lambda\sqrt{s})$ term in equation (2-28) is necessary in the case of an annular cylinder, but it is not necessary for obtaining a particular solution for a solid disk, which is the case when $c = 0$ and $a = 0$. Substitution of equations (2-31) and (2-24) into equation (2-7) and imposing of the Laplace transform yields an expression for the transformed pressure field,

$$p(\lambda, s) = \frac{\hat{C}_{rr}^d}{J} \left(Jg_1(s)I_o(\lambda\sqrt{s}) + Jg_2(s)K_o(\lambda\sqrt{s}) - \hat{A}_z p(s) - \hat{A}_r p(s) \right). \quad (2-32)$$

First Condition (BC1)

This solution requires sealing the inner boundary and allowing free fluid flow through the outer boundary. The boundary conditions reflecting this requirement are $p(1, s) = 0$ and

$$\frac{\partial p(a, s)}{\partial \lambda} = 0 \text{ at the outer and inner boundaries respectively.}$$

Second Condition (BC2)

This solution requires sealing the outer boundary and allowing free fluid flow through the

inner boundary. The boundary conditions reflecting this requirement are $\frac{\partial p(1, s)}{\partial \lambda} = 0$ and

$$p(a, s) = 0.$$

Third Condition (BC3)

To allow fluid flow from the outer and inner boundaries, the boundary conditions to be satisfied are $p(1, s) = 0$ and $p(a, s) = 0$. Please note that $p(a, s) = 0$ means that the annular region is freely drained. There are several alternatives to setting $p(a, s) = 0$; for example, the

pressure could be specified as some non-zero value controlled in an experiment, or the drainage of the annular region could be constrained to be radially through the specimen and the pressure in the annular region would then vary in the posed boundary value problem.

The function $\hat{p}(s)$ appearing in equation (2-32) depends on the draining condition and it is calculated by substituting the boundary condition for the pressure at the inner or the outer boundaries into equation (2-32). One can substitute the pressure or the pressure gradient from BC1, BC2, or BC3 in equation (2-32) to determine $\hat{p}(s)$. For instance, the flow through the outer boundary of BC1 yields

$$\hat{p}(s) = \frac{1}{\hat{A}_r} \left(Jg_1(s)I_o(\sqrt{s}) + Jg_2(s)K_o(\sqrt{s}) - \hat{A}_z \hat{p}(s) \right). \quad (2-33)$$

The three drainage conditions above lead to different solutions for $\hat{p}(s)$. Integration of equation (2-22) and applying the Laplace transform to the result, and then substituting the expression of $\hat{p}(s)$ for the appropriate boundary condition (BC1, BC2, BC3) yields a desired expression for the Laplace transform of the radial displacement; for BC1 one obtains

$$\hat{u}(\lambda, s) = \frac{b\lambda}{2} \left\{ \hat{A}_r g_1(s) I_o(\sqrt{s}) \left[2h_1(\lambda, s) + \frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_{rr}^2} \right] + \hat{A}_r g_2(s) K_o(\sqrt{s}) \left[\frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_{rr}^2} - 2h_2(\lambda, s) \right] - \frac{\hat{A}_z}{\hat{A}_r} \hat{p}(s) \right\}. \quad (2-34)$$

The solutions for $g_1(s)$ and $g_2(s)$ also depend on the imposed drainage condition. Two conditions are needed to obtain those solutions. The first condition can be taken from any drainage condition other than the one that is used for the determination of $\hat{p}(s)$. The second condition arises from the fact that the radial stresses are zero on the boundary that allows fluid flow, thus

$$\mathcal{P}_{rr}^0 = 0 = -\hat{A}_r \mathcal{P}(\lambda, s) + \hat{C}_{rr}^d \frac{\partial \mathcal{P}^0}{b \partial \lambda} + \hat{C}_{r\theta}^d \frac{\mathcal{P}^0}{b \lambda} + \hat{C}_{rz}^d \mathcal{P}(s). \quad (2-35)$$

The solutions for $g_1(s)$ and $g_2(s)$ are shown in Table (2). The functions $h_1(\lambda, s)$ and $h_2(\lambda, s)$ have been introduced and they are shown in Table (2) for convenience. The function $h_1(\lambda, s)$ monotonically decreases from a value of one half at $s = 0$ to a value of 0 as s tends to infinity, while $h_2(\lambda, s)$ decreases from a value of infinity as $s = 0$ to a value of 0 as s tends to infinity. Therefore, equation (2-35) above can be used in obtaining the solutions for $g_1(s)$ and $g_2(s)$ as shown in Table 2.

2.5 Results

The total force $P(t)$ applied to the circular flat plate can be calculated now. As a first step in the calculation process, the third relation in equation (2-20) is used to express $P(t)$ as

$$P(t) = 2\pi \int_c^b T_{zz} r dr = -\frac{2\pi}{\hat{C}_{rr}^d} (\hat{A}_z \hat{C}_{rr}^d - \hat{A}_r \hat{C}_{rz}^d) \int_c^b p r dr + \pi(b^2 - c^2) (\hat{C}_{zz}^d \varepsilon(t) + \hat{C}_{rz}^d f(t)). \quad (2-36)$$

The Laplace transform of equation (2-36) above yields

$$\frac{\mathcal{P}(s)}{\pi(b^2 - c^2)} = 2\pi \int_a^1 \mathcal{P}_{zz}^0 \lambda d\lambda = -\frac{2b^2}{\hat{C}_{rr}^d} \frac{(\hat{A}_z \hat{C}_{rr}^d - \hat{A}_r \hat{C}_{rz}^d)}{(b^2 - c^2)} \int_a^1 \mathcal{P}(\lambda, s) \lambda d\lambda + \hat{C}_{zz}^d \mathcal{P}(s) + \hat{C}_{rz}^d \mathcal{P}(s). \quad (2-37)$$

Equation (2-32) can be used to express equation (2-37) in terms of known functions and parameters. One obtains for all drainage conditions, after some rearrangement of terms, the equations for $\mathcal{P}(s)$, $\mathcal{P}(\lambda, s)$, and $\mathcal{P}^0(\lambda, s)$ as shown in Table 4. For instance, $\mathcal{P}(s)$ for BC1 is

$$\frac{\hat{P}(\lambda, s)}{\pi(b^2 - c^2)(\hat{C}_{rr}^d - \hat{C}_{r\theta}^d)} = \frac{\left\{ \begin{aligned} &\left[\frac{-N}{2} + a^2 F \right] \frac{K_1(a\sqrt{s})I_o(\sqrt{s})h_1(1, s) - Ya^2 K_1(a\sqrt{s})I_o(\sqrt{s})h_1(a, s) + \frac{M}{2} K_1(a\sqrt{s})I_o(\sqrt{s}) -}{(1-a^2)} \\ &\left[\frac{-N}{2} + a^2 F \right] \frac{K_o(\sqrt{s})I_1(a\sqrt{s})h_2(1, s) + Ya^2 K_o(\sqrt{s})I_1(a\sqrt{s})h_2(a, s) + \frac{M}{2} K_o(\sqrt{s})I_1(a\sqrt{s})}{(1-a^2)} \end{aligned} \right\} \hat{P}(\lambda, s)}{I_1(a\sqrt{s})K_o(\sqrt{s})[C + Dh_2(1, s)] + I_o(\sqrt{s})K_1(a\sqrt{s})[C - Dh_1(1, s)]} \quad (2-38)$$

The constants B , C , D , M and N are the same constants that appeared in equations (91), (99) and (100) in Cowin and Mehrabadi (2007), see Table 3 for formulas for various special cases. These constants are defined as follows:

$$\begin{aligned} B &= \frac{1}{2\hat{C}_{rr}^d} \left[\hat{A}_z (\hat{C}_{rr}^d + \hat{C}_{r\theta}^d) - 2\hat{A}_r C_{rz}^d \right], \quad C = \hat{A}_r^2 + \frac{\Lambda}{2} (\hat{C}_{rr}^d + \hat{C}_{r\theta}^d), \quad D = \frac{\hat{A}_r^2}{\hat{C}_{rr}^d} (\hat{C}_{rr}^d - \hat{C}_{r\theta}^d), \\ M &= \frac{1}{(C_{rr}^d - C_{r\theta}^d)} \left[2\hat{A}_r^2 C_{zz}^d - 4\hat{A}_r \hat{A}_z C_{rz}^d + (C_{rr} + C_{r\theta}) \hat{A}_z^2 + \Lambda (C_{rz}^d C_{zz}^d + C_{r\theta}^d C_{zz}^d - 2C_{rz}^{d2}) \right], \\ N &= \frac{2}{C_{rr}^d (C_{rr}^d - C_{r\theta}^d)} \left[\left(\hat{A}_r^2 (C_{zz}^d C_{rr}^d - C_{zz}^d C_{r\theta}^d + 2C_{rz}^{d2}) - 4\hat{A}_r \hat{A}_z C_{rr}^d C_{rz}^d + \hat{A}_z^2 (C_{rr}^d + C_{r\theta}^d) C_{rr}^d \right) \right]. \end{aligned} \quad (2-39)$$

A new pair of constants, F and Y , has also been introduced:

$$\begin{aligned} F &= \hat{A}_r^2 (C_{zz}^d C_{rr}^d - C_{zz}^d C_{r\theta}^d) + \hat{A}_r \hat{A}_z (C_{rz}^d C_{r\theta}^d - C_{rr}^d C_{rz}^d) \\ Y &= \frac{A_r A_z C_{rz} (3C_{rr}^d + C_{r\theta}^d) - A_z^2 C_{rr}^d (C_{rr}^d + C_{r\theta}^d) - 2A_r^2 C_{rz}^{d2}}{(C_{rr}^d - C_{r\theta}^d) C_{rr}^d}. \end{aligned} \quad (2-40)$$

Using the representation for $\hat{P}(\lambda, s)$ obtained above, the total load $P(t)$ generated on a specimen by an arbitrary strain history $\varepsilon(t)$ can be determined. Alternatively, if the load history $P(t)$ is prescribed, then the equation can be used to determine $\varepsilon(t)$. The displacement field $u(\lambda, s)$

Table 2:

The solutions for $g_1(s)$ and $g_2(s)$. The definitions of $h_1(\lambda, s)$ and $h_2(\lambda, s)$ are shown for convenience. Gailani and Cowin (2008)

	$g_1(s)$	$g_2(s)$	$h_1(\lambda, s)$	$h_2(\lambda, s)$
BC1	$\frac{B\tilde{\varepsilon}(s)K_1(a\sqrt{s})}{I_1(a\sqrt{s})K_o(\sqrt{s})[C + Dh_2(1, s)] + I_o(\sqrt{s})K_1(a\sqrt{s})[C - Dh_1(1, s)]}$	$\frac{B\tilde{\varepsilon}(s)I_1(a\sqrt{s})}{I_1(a\sqrt{s})K_o(\sqrt{s})[C + Dh_2(1, s)] + I_o(\sqrt{s})K_1(a\sqrt{s})[C - Dh_1(1, s)]}$	$\frac{I_1(\lambda\sqrt{s})}{\lambda\sqrt{s}I_o(\sqrt{s})}$	$\frac{K_1(\lambda\sqrt{s})}{\lambda\sqrt{s}K_o(\sqrt{s})}$
BC2	$\frac{B\tilde{\varepsilon}(s)K_1(\sqrt{s})}{I_1(\sqrt{s})K_o(a\sqrt{s})[C + Dh_2(a, s)] + I_o(a\sqrt{s})K_1(\sqrt{s})[C - Dh_1(a, s)]}$	$\frac{B\tilde{\varepsilon}(s)I_1(\sqrt{s})}{I_1(\sqrt{s})K_o(a\sqrt{s})[C + Dh_2(a, s)] + I_o(a\sqrt{s})K_1(\sqrt{s})[C - Dh_1(a, s)]}$	$\frac{I_1(\lambda\sqrt{s})}{\lambda\sqrt{s}I_o(a\sqrt{s})}$	$\frac{K_1(\lambda\sqrt{s})}{\lambda\sqrt{s}K_o(a\sqrt{s})}$
BC3	$\frac{B}{\left(\frac{K(s)}{I(s)}(C - Dh_1(1, s)) + (C + Dh_2(1, s))\right)I_o(\sqrt{s})} \frac{K(s)}{I(s)} \tilde{\varepsilon}(s)^{**}$	$\frac{B}{\frac{K(s)}{I(s)}(C - Dh_1(1, s)) + (C + Dh_2(1, s))} \frac{\tilde{\varepsilon}(s)^{**}}{K_o(\sqrt{s})}$	$\frac{I_1(\lambda\sqrt{s})}{\lambda\sqrt{s}I_o(\sqrt{s})}$	$\frac{K_1(\lambda\sqrt{s})}{\lambda\sqrt{s}K_o(\sqrt{s})}$

$$^{**} I(s) = \left(\frac{I_o(a\sqrt{s})}{I_o(\sqrt{s})} - 1 \right) \quad \text{and} \quad K(s) = \left(1 - \frac{K_o(a\sqrt{s})}{K_o(\sqrt{s})} \right)$$

and the pore pressure field $p(\lambda, s)$ may be evaluated for an imposed load history, as opposed to imposed displacement history. Thus, the solution for the linear poroelastic, transversely isotropic, unconfined compression, creep and stress-relaxation analysis is completed. Before inverting these expressions, it is helpful to observe some mathematical properties pertaining to the Laplace transform solution equations for $u(\lambda, s)$, $p(\lambda, s)$ and $\dot{p}(s)$. Laplace transform theory includes two theorems that are used here to gain some insight into these results without accomplishing the complete inversion process. The initial and final value theorems are characterized by the two respective formulas (Thomson (1960)):

$$\lim_{s \rightarrow 0} sf(s) = \lim_{t \rightarrow \infty} f(t), \text{ and } \lim_{s \rightarrow \infty} sf(s) = \lim_{t \rightarrow 0} f(t). \quad (2-41)$$

The case considered as an example is the problem of an applied step loading $\varepsilon(t) = \varepsilon_o H(t)$ that has the Laplace transform $\dot{\varepsilon}(s) = \varepsilon_o/s$. Applying the initial and final value theorems it will become possible to estimate the displacements of the outer and inner boundaries at $t = 0$ and $t = \infty$ for all different boundary conditions. For example the displacement at the outer boundary at $t = \infty$ for BC1 will be

$$u(1, \infty) = \frac{b\varepsilon_o}{2} \left(\frac{2B\hat{A}_r}{(a^2D + 2C - D)} (1 - a^2 + \frac{\Lambda\hat{C}_{rr}^d}{\hat{A}_r^2}) - \frac{\hat{A}_z}{\hat{A}_r^2} \right). \quad (2-42)$$

The application of this method to $p(\lambda, s)$ and $\dot{p}(s)$ for $t = 0$ and $t = \infty$ are shown in Table 5 for all drainage conditions. The inverse Laplace transforms of equations representing the solutions $u(\lambda, s)$, $p(\lambda, s)$ and $\dot{p}(s)$ may be obtained by standard methods. We note that $u(\lambda, s)$, $p(\lambda, s)$ and $\dot{p}(s)$ are all of the fractional form $A(s)/sQ(s)$, the inverse of which is

$$f(t) = \mathfrak{S}^{-1} \frac{A(s)}{sQ(s)} = \lim_{s \rightarrow 0} \frac{A(s)}{Q(s)} + \sum_{n=2}^{\infty} \lim_{s \rightarrow \alpha_n} \frac{A(s)}{sQ'(s)} e^{st}, \quad (2-43)$$

Table 3:

The values of the constants B , C , D , F , M , N , and Y defined for convenience for different material models. Gailani and Cowin (2008)

	Transverse isotropy compressible	Transverse isotropy incompressible	Isotropy compressible	Isotropy incompressible
B	$\frac{1}{2\Delta_2}(\hat{A}_z - 2\nu_{31}^d \hat{A}_r)$	$\frac{1}{2\Delta_2}(1 - 2\nu_{31}^d)$	$\frac{1-2\nu^d}{2(1-\nu^d)} SK^d C_{eff}^d$	$\frac{1-2\nu^d}{2(1-\nu^d)}$
C	$\hat{A}_r^2 + \frac{\Lambda E_1^d}{2\Delta_1}$	1	$\frac{1}{2(1+\nu^d)} \left[3 - (1-2\nu^d) S^2 K^d C_{eff}^d \right] K^d C_{eff}^d$	1
D	$\frac{\hat{A}_r^2}{\Delta_2} (2\Delta_2 - 1)$	$\frac{1}{\Delta_2} (2\Delta_2 - 1)$	$\frac{1-2\nu^d}{1-\nu^d} (SK^d C_{eff}^d)^2$	$\frac{1-2\nu^d}{1-\nu^d}$
M	$C_{eff}^d \frac{E_3^d}{2\Delta_2 - 1}$	$\frac{1}{K_{R_{eff}}^d} \frac{E_3^d}{2\Delta_2 - 1}$	$3K^d C_{eff}^d$	3
N	$\frac{2E_3^d}{2\Delta_2 - 1} \left(C_{eff}^d - \frac{J\Delta_1}{E_1^d \Delta_2} \right)$	$\frac{2E_3^d}{2\Delta_2 - 1} \left(\frac{1}{K_{R_{eff}}^d} - \frac{\Delta_1}{E_1^d \Delta_2} \right)$	$\frac{4(1-2\nu^d)}{1-\nu^d} (SK^d C_{eff}^d)^2$	$\frac{4(1-2\nu^d)}{1-\nu^d}$
F	$\left[\hat{A}_r^2 (1 - \nu_{12}^d) - \hat{A}_r \hat{A}_z \nu_{13}^d \right] \frac{E_3^d}{\Delta_2 E_1^d}$	$\left[(1 - \nu_{12}^d) - \nu_{13}^d \right] \frac{E_3^d}{\Delta_2 E_1^d}$	$\frac{(SK^d C_{eff}^d)^2}{(1-\nu^d)}$	$\frac{1}{(1-\nu^d)}$
Y	$\frac{1}{E_1^2 \Delta_2 (2\Delta_2 - 1)} \left[\hat{A}_r \hat{A}_z E_1 E_3 \nu_{13}^d (2\Delta_2 + 1) - \hat{A}_z^2 \Delta_2 E_1^2 - 2\hat{A}_r^2 E_3^2 \nu_{13}^{d2} \right]$	$\frac{1}{E_1^2 \Delta_2 (2\Delta_2 - 1)} \left[\frac{E_1^d E_3^d \nu_{13}^d (2\Delta_2 + 1)}{-\Delta_2 E_1^d - E_3^2 \nu_{13}^{d2}} \right]$	$\frac{(3\nu^d - 1)}{1-2\nu^d} (SK^d C_{eff}^d)^2$	$\frac{(3\nu^d - 1)}{1-2\nu^d}$

Table 4:Formulas of $\tilde{u}(\lambda, s)$, $\tilde{p}(\lambda, s)$ and $\tilde{P}(s)$ for all three drainage conditions. Gailani and Cowin (2008)

	$\frac{\tilde{P}(s)}{\pi(b^2 - c^2)(C_{rr} - C_{r\theta})}$	$\frac{2\tilde{u}(\lambda, s)}{b\lambda}$	$\frac{\tilde{p}(\lambda, s)}{\hat{C}_{rr}^d}$
BC1	See equation (38)	$\left\{ \hat{A}_r g_1(s) I_o(\sqrt{s}) \left[2h_1(\lambda, s) + \frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} \right] + \right. \\ \left. \hat{A}_r g_2(s) K_o(\sqrt{s}) \left[\frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} - 2h_2(\lambda, s) \right] \right\} + \\ \left. - \frac{\hat{A}_z}{\hat{A}_r} \tilde{e}(s) \right\}$	$g_1(s) I_o(\lambda \sqrt{s}) + g_2(s) K_o(\lambda \sqrt{s}) \\ - g_1(s) I_o(\sqrt{s}) - g_2(s) K_o(\sqrt{s})$
BC2	$\left\{ \frac{Y}{(1-a^2)} K_1(\sqrt{s}) I_o(a\sqrt{s}) h_1(1, s) - \left[\frac{-N+a^2 F}{2} \right] \frac{K_1(\sqrt{s}) I_o(a\sqrt{s}) h_1(a, s)}{(1-a^2)} \right. \\ + \frac{M}{2} K_1(\sqrt{s}) I_o(a\sqrt{s}) + \left[\frac{-N+a^2 F}{2} \right] \frac{K_o(a\sqrt{s}) I_1(\sqrt{s}) h_2(a, s)}{(1-a^2)} \\ \left. - \frac{Y}{(1-a^2)} K_o(a\sqrt{s}) I_1(\sqrt{s}) h_2(1, s) + \frac{M}{2} K_o(a\sqrt{s}) I_1(\sqrt{s}) \right\} \tilde{e}(s)$	$\left\{ \hat{A}_r g_1(s) I_o(a\sqrt{s}) \left[2h_1(\lambda, s) + \frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} \right] + \right. \\ \left. + \hat{A}_r g_2(s) K_o(a\sqrt{s}) \left[\frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} - 2h_2(\lambda, s) \right] - \frac{\hat{A}_z}{\hat{A}_r} \tilde{e}(s) \right\}$	$g_1(s) I_o(\lambda \sqrt{s}) + g_2(s) K_o(\lambda \sqrt{s}) \\ - g_1(s) I_o(a\sqrt{s}) - g_2(s) K_o(a\sqrt{s})$
BC3	$\left\{ \left[\frac{-N+a^2 F}{2} \right] \frac{K(s)}{(1-a^2)} h_1(1, s) - Y a^2 \frac{K(s)}{I(s)} h_1(a, s) + \frac{M}{2} \frac{K(s)}{I(s)} \right. \\ \left. - \left[\frac{-N+a^2 F}{2} \right] \frac{h_2(1, s) + Y a^2 h_2(a, s) + \frac{M}{2}}{(1-a^2)} \right\} \tilde{e}(s)$	$\left\{ \hat{A}_r g_1(s) I_o(\sqrt{s}) \left[\frac{2I_1(\lambda \sqrt{s})}{\lambda \sqrt{s} I_o(\sqrt{s})} + \frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} \right] + \right. \\ \left. \hat{A}_r g_2(s) K_o(\sqrt{s}) \left[\frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} - \frac{2K_1(\lambda \sqrt{s})}{\lambda \sqrt{s} K_o(\sqrt{s})} \right] \right\} + \\ \left. - \frac{\hat{A}_z}{\hat{A}_r} \tilde{e}(s) \right\}$	$g_1(s) I_o(\lambda \sqrt{s}) + g_2(s) K_o(\lambda \sqrt{s}) \\ - g_1(s) I_o(\sqrt{s}) - g_2(s) K_o(\sqrt{s})$

where $A(s)$ and $Q(s)$ are analytic at $s = \alpha_n$, $A(\alpha_n) \neq 0$ and α_n are simple poles of $Q(s) = 0$. For example calculating $P(t)$ for a step load by using equation (41) for BC1 yields

$$\frac{\Delta_1 P(t)}{\pi(b^2 - c^2)(2\Delta_2 - 1)} = \frac{\left(\frac{-N}{2} + a^2 F + M\right)}{(a^2 D + 2C - D)} E_1 \varepsilon_o + \sum_{n=2}^{\infty} \frac{\left[\frac{C}{D} \left(\frac{-\frac{N}{2} + a^2 F}{(1 - a^2)} \right) + \frac{M}{2} \right]}{\left(C - \frac{\alpha_n^2 C^2}{2D} - \frac{D}{2} + \frac{a}{2} \varphi_n \right)} E_1 \varepsilon_o e^{-\alpha_n^2 m t / b^2}, \quad (2-44a)$$

The estimate of $P(t)$ as t tends to infinity is recorded in Table 5,

$$P(\infty) = \lim_{s \rightarrow 0} \frac{A(s)}{Q(s)} = \frac{\pi(b^2 - c^2)(2\Delta_2 - 1) \varepsilon_o E_1 \left(\frac{-N}{2} + a^2 F + M \right)}{\Delta_1 (a^2 D + 2C - D)}. \quad (2-44b)$$

Notice that $P(\infty)$ can be reduced to $-\pi b^2 E_3^d \varepsilon_o$ for an incompressible transverse isotropy solid porous disk. Equation (2-44a) can be rewritten for simplification purposes by using equation (2-44b)

$$P(t) = P(\infty) + \frac{\pi(b^2 - c^2)(2\Delta_2 - 1)}{\Delta_1} \sum_{n=2}^{\infty} A_n E_1 \varepsilon_o e^{-\alpha_n^2 m t / b^2}, \quad (2-44c)$$

where

$$A_n = \frac{\left[\frac{C}{D} \left(\frac{-\frac{N}{2} + a^2 F}{(1 - a^2)} \right) + \frac{M}{2} \right]}{\left(C - \frac{\alpha_n^2 C^2}{2D} - \frac{D}{2} + \frac{a}{2} \varphi_n \right)}. \quad (2-44d)$$

The term φ_n in equation (2-44a) above is associated with some complexity and needs to be evaluated. The evaluation process will convert the associated Bessel modified functions to Bessel functions of the first and second kind (see appendix). Thus the expression for φ_n is

$$\varphi_n = \frac{C \alpha_n (J_o(a \alpha_n) Y_o(\alpha_n) - J_o(\alpha_n) Y_o(a \alpha_n))}{(J_1(a \alpha_n) Y_o(\alpha_n) - J_o(\alpha_n) Y_1(a \alpha_n))} + \frac{D (J_1(\alpha_n) Y_o(a \alpha_n) - J_o(a \alpha_n) Y_1(\alpha_n))}{(J_1(a \alpha_n) Y_o(\alpha_n) - J_o(\alpha_n) Y_1(a \alpha_n))}. \quad (2-45)$$

In the case of incompressible transverse isotropy, the summation term in equations (44a) and (44c) may vanish for large values of the poles α_n . Applying the same method to BC2 yields

$$\frac{P(t)}{\pi(b^2 - c^2)(C_{rr} - C_{r\theta})} = \frac{\left(\frac{-N}{2} + a^2 F + Ma^2\right)}{(D + 2a^2 C - a^2 D)} \varepsilon_o + \sum_{n=2}^{\infty} \frac{\left(\frac{M}{2} - \frac{C}{D} \left[\frac{-N}{2} + a^2 F\right] \frac{1}{(1-a^2)}\right) \varepsilon_o e^{-\alpha_n^2 m t / b^2}}{\frac{C}{2} \chi_n - \frac{D}{2a} \eta_n - \left(\frac{a^2 \alpha_n^2 C^2}{2D} + \frac{D}{2} - C\right)}, \quad (2-46)$$

where

$$\eta_n = \frac{[J_o(\alpha_n)Y_1(a\alpha_n) - J_1(a\alpha_n)Y_o(\alpha_n)]}{[J_1(\alpha_n)Y_o(a\alpha_n) - J_o(a\alpha_n)Y_1(\alpha_n)]} \text{ and } \chi_n = \frac{\alpha_n [J_o(\alpha_n)Y_o(a\alpha_n) - J_o(a\alpha_n)Y_o(\alpha_n)]}{[J_1(\alpha_n)Y_o(a\alpha_n) - J_o(a\alpha_n)Y_1(\alpha_n)]}. \quad (2-47)$$

Equation (2-46) shows that BC2 in the special case where the annular central region vanishes, that is to say when $a = c = 0$, leads to the special case in which flow cannot occur and the resulting disk functions as an elastic solid with the undrained elastic constants; recall that BC2 implies flow through the inner boundaries only. To see how this result is deduced from (46) note that the last term in (2-46) contains $(1/a)$ in the denominator, thus as $a \rightarrow 0$ this term tends to infinity; then (2-46) is no longer a function of time.

Cohen et al. (1998) also obtained an analytical solution for the experimentally significant case of a ramp loading for the transverse isotropic incompressible material. The ramp loading is characterized by a constant strain rate $\dot{\varepsilon}_o$ and a period of duration t_o during which the applied strain is rising linearly from a value of 0 to $t_o \dot{\varepsilon}_o$. After time t_o the value of $\varepsilon(t)$ is constant at $t_o \dot{\varepsilon}_o$. The expression in terms of the Heaviside step function $H(t)$ is

$$\varepsilon(t) = \dot{\varepsilon}_o [tH(t) - (t - t_o)H(t - t_o)]. \quad (2-48)$$

Table 5:

Formulas to be used to calculate the pressure, displacement and total force at $t = 0$ and $t = \infty$ for a step load. Gailani and Cowin (2008)

	BC1	BC2	BC3
$p(\lambda, 0)$	$-\frac{\hat{C}_{rr}^d B \varepsilon_o}{C}$	$-\frac{\hat{C}_{rr}^d B \varepsilon_o}{C}$	$-\frac{\hat{C}_{rr}^d B \varepsilon_o}{C}$
$p(\lambda, \infty)$	0	0	0
$u(1, 0)$	$\frac{b \varepsilon_o}{2 \hat{A}_r} \left(\frac{B \Lambda \hat{C}_{rr}^d}{C} - \hat{A}_z \right)$	$\frac{b \varepsilon_o}{2 \hat{A}_r} \left(\frac{B \Lambda \hat{C}_{rr}^d}{C} - \hat{A}_z \right)$	$\frac{b \varepsilon_o}{2 \hat{A}_r} \left(\frac{B \Lambda \hat{C}_{rr}^d}{C} - \hat{A}_z \right)$
$u(1, \infty)$	$\frac{b \varepsilon_o}{2} \left(\frac{2 B \hat{A}_r}{(a^2 D + 2C - D)} (1 - a^2 + \frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_R^2}) - \frac{\hat{A}_z}{\hat{A}_R^2} \right)$	$\frac{b \varepsilon_o}{2 \hat{A}_r} \left(\frac{2 B \Lambda \hat{C}_{rr}^d}{(D + 2a^2 C - a^2 D)} - \hat{A}_z \right)$	$-\frac{b \varepsilon_o}{2} \frac{(\hat{A}_z)}{\hat{A}_r}$
$u(a, 0)$	$\frac{c \varepsilon_o}{2 \hat{A}_r} \left(\frac{B \Lambda \hat{C}_{rr}^d}{C} - \hat{A}_z \right)$	$\frac{c \varepsilon_o}{2 \hat{A}_r} \left(\frac{B \Lambda \hat{C}_{rr}^d}{C} - \hat{A}_z \right)$	$\frac{c \varepsilon_o}{2 \hat{A}_r} \left(\frac{B \Lambda \hat{C}_{rr}^d}{C} - \hat{A}_z \right)$
$u(a, \infty)$	$\frac{b \varepsilon_o}{2 \hat{A}_r} \left(\frac{2 B \Lambda \hat{C}_{rr}^d}{(a^2 D + 2C - D)} - \hat{A}_z \right)$	$\frac{b \varepsilon_o}{2} \left(\frac{2 B \hat{A}_r}{(D + 2a^2 C - a^2 D)} (1 - a^2 + \frac{\Lambda \hat{C}_{rr}^d}{\hat{A}_r^2} - \frac{\hat{A}_z}{\hat{A}_r^2}) \right)$	$-\frac{c \varepsilon_o}{2} \frac{(\hat{A}_z)}{\hat{A}_r}$
$P(0)$	$\frac{\pi(b^2 - c^2)(2\Delta_2 - 1)\varepsilon_o E_1 M}{2\Delta_1 C}$	$\frac{\pi(b^2 - c^2)(2\Delta_2 - 1)\varepsilon_o E_1 M}{2\Delta_1 C}$	$\frac{\pi(b^2 - c^2)(2\Delta_2 - 1)\varepsilon_o E_1 M}{2\Delta_1 C}$
$P(\infty)$	$\frac{\pi(b^2 - c^2)(2\Delta_2 - 1)\varepsilon_o E_1 (\frac{-N}{2} + a^2 F + M)}{\Delta_1(a^2 D + 2C - D)}$	$\frac{\pi(b^2 - c^2)(2\Delta_2 - 1)\varepsilon_o E_1 (\frac{-N}{2} + a^2 F + M a^2)}{\Delta_1(D + 2a^2 C - a^2 D)}$	$\frac{\pi(b^2 - c^2)(2\Delta_2 - 1)\varepsilon_o E_1 (\frac{-N}{2} + a^2 F - a^2 Y + M)}{\Delta_1(2C - D)}$

The equation for $P(t)$ for the ramp loading in equation (48) when $0 \leq t \leq t_o$ is

$$\frac{\Delta_1 P(t)}{\pi(b^2 - c^2)(2\Delta_2 - 1)} = A_1 E_1 \&_o^t + E_1 \frac{b^2}{m} \&_o^t \sum_{n=2}^{\infty} \frac{A_n}{\alpha_n^2} (1 - e^{-\alpha_n^2 m t / b^2}), \quad (2-49)$$

and for $t \geq t_o$

$$\frac{\Delta_1 P(t)}{\pi(b^2 - c^2)(2\Delta_2 - 1)} = A_1 E_1 \&_o^{t_o} + E_1 \frac{b^2}{m} \&_o^t \sum_{n=2}^{\infty} \frac{A_n}{\alpha_n^2} (e^{-\alpha_n^2 m (t-t_o) / b^2} - e^{-\alpha_n^2 m t / b^2}). \quad (2-50)$$

The analysis can provide the pressure distribution in the radial direction for different time intervals that can show the draining process of the disk. Using equation (2-43), the pressure formula of BC1 in Table (4) can be inverted for a ramp displacement when $t \leq t_o$

$$p(\lambda, t) = \frac{b^2}{m \alpha_n^2} \sum_{n=2}^{\infty} \frac{(T_n(\lambda) - 1) BC_{rr}^d}{\left(C - \frac{\alpha_n^2 C^2}{2D} - \frac{D}{2} + \frac{a}{2} \varphi_n \right)} \&_o^t (1 - e^{-\alpha_n^2 m t / b^2}), \quad (51)$$

where

$$T_n(\lambda) = \frac{(J_1(a\alpha_n)Y_o(\lambda\alpha_n) - J_o(\lambda\alpha_n)Y_1(a\alpha_n))}{(J_1(a\alpha_n)Y_o(\alpha_n) - J_o(\alpha_n)Y_1(a\alpha_n))}, \quad (52)$$

A compressible transverse isotropy model of a bone tissue is selected in order to examine the pressure distribution in the radial direction in the case of an imposed displacement. The bone tissue selected is that containing only the small-dimensioned lacunar-canalicular porosity and not the larger sized vascular porosity. The approximate transverse isotropic poroelastic constants for this level of bone tissue are assumed to be $E_1^m = 18.6$ GPa, $E_3^m = 22.32$ GPa, $\nu_{12}^m = \nu_{21}^m = 0.322$, $\nu_{13}^m = \nu_{23}^m = 0.255$, $\nu_{31}^m = \nu_{32}^m = 0.312$, $G_{12}^m = 7.28$ GPa and $G_{13}^m = G_{23}^m = 7.6$ GPa for the matrix material; $E_1^d = 15.17$ GPa, $E_3^d = 15.96$ GPa, $\nu_{12}^d = \nu_{21}^d = 0.27$, $\nu_{13}^d = \nu_{23}^d = 0.27$, $\nu_{31}^d = \nu_{32}^d = 0.285$, $G_{12}^d = 5.97$ GPa and $G_{13}^d = G_{23}^d = 6.74$ GPa for the drained elastic constants. In this paper, the canalicular intrinsic permeability is taken to be $k_{rr} = 2 \times 10^{-20}$ m². The lacunar canalicular porosity ϕ is 0.05, the compressibility of water is $K^f = 2.3$ GPa, the viscosity of

water $\mu = 1$ centipoise (cP) = 0.01 Poise = 0.01 dyne.s/cm² and the strain ε_o is 1.25×10^{-4} .

The outer radius of the osteon is about $80 \mu\text{m}$ and the inner radius is $20 \mu\text{m}$. Tables 1 and 3 should be used to determine the various constants that are involved. The pore pressure distribution for BC1 can be plotted for an imposed ramp displacement when $t \leq t_o$ as shown in Fig. (4), which is only for a transverse isotropy model with compressible constituents. The peak pressure is recorded to be 6.7 kPa.

The same approach can be used to determine the pressure distribution for BC2. The pressure distribution in BC2 a ramp displacement when $t \leq t_o$ can be expressed as

$$p(\lambda, t) = \frac{b^2}{m\alpha_n^2} \sum_{n=2}^{\infty} \frac{(F_n(\lambda) - 1) B \hat{C}_{rr}^d}{\frac{C}{2} \chi_n - \frac{D}{2a} \eta_n - \left(\frac{a^2 \alpha_n^2 C^2}{2D} + \frac{D}{2} - C \right)} \mathfrak{E}_o (1 - e^{-\alpha_n^2 m t / b^2}) \quad (2-53a)$$

and for $t \geq t_o$

$$p(\lambda, t) = \frac{b^2}{m\alpha_n^2} \sum_{n=2}^{\infty} \frac{(F_n(\lambda) - 1) B \hat{C}_{rr}^d \mathfrak{E}_o \left(e^{-\alpha_n^2 m (t - t_o) / b^2} - e^{-\alpha_n^2 m t / b^2} \right)}{\frac{C}{2} \chi_n - \frac{D}{2a} \eta_n - \left(\frac{a^2 \alpha_n^2 C^2}{2D} + \frac{D}{2} - C \right)} \quad (2-53b)$$

where

$$F_n(\lambda) = \frac{J_1(\alpha_n) Y_o(\lambda \alpha_n) - J_o(\lambda \alpha_n) Y_1(\alpha_n)}{J_1(\alpha_n) Y_o(a \alpha_n) - J_o(a \alpha_n) Y_1(\alpha_n)}. \quad (2-54)$$

Fig. 5 shows the pressure distribution for BC2 at different times. Notice how the pressure is constrained to zero at the unsealed boundary and is much larger at the sealed one. The pressure will start rising over the whole radius length. This distribution is approximately opposite to the pressure distribution of BC1 and the reason is obvious. The pressure distribution at the outer boundary of BC2 is shown in Fig. 6 for the whole loading period that includes equation (2-53a) and equation (2-53b) or for $0 \leq t \leq \infty$. However, the pressure distribution for BC3 is not in the same manner as BC1 and BC2. The peak pressure in BC2 is

shown to be 9.7 kPa. In BC3 the flow is allowed from both boundaries, thus the pressure is constrained to zero at both boundaries but it will be at the peak somewhere between the two boundaries as shown in Fig. (7).

The pressure distribution for BC3 involves a more complicated process and can be shown for a ramp displacement where $t \leq t_o$ to be

$$p(\lambda, t) = \frac{b^2}{m\alpha^2} \sum_{n=2}^{\infty} \frac{(X_n(\lambda) - 1) \hat{BC}_{rr}^d \frac{1}{\alpha} (1 - e^{-\alpha_n^2 m t / b^2})}{\frac{D}{2} (aZ_n - 1) + C - \frac{C^2 \alpha^2}{2D} + \frac{C\alpha}{2} (aR_n - L_n)}, \quad (2-55)$$

where

$$X_n(\lambda) = \frac{J_o(\lambda\alpha) [Y_o(a\alpha) - Y_o(\alpha)] + Y_o(\lambda\alpha) [J_o(\alpha) - J_o(a\alpha)]}{J_o(\alpha) Y_o(a\alpha) - J_o(a\alpha) Y_o(\alpha)}, \quad (2-56)$$

$$Z_n = \frac{J_1(\alpha) Y_1(a\alpha) - J_1(a\alpha) Y_1(\alpha)}{J_o(\alpha) Y_o(a\alpha) - J_o(a\alpha) Y_o(\alpha)}, \quad (2-57)$$

$$R_n = \frac{(J_1(a\alpha) Y_o(\alpha) - J_o(\alpha) Y_1(a\alpha))}{J_o(\alpha) Y_o(a\alpha) - J_o(a\alpha) Y_o(\alpha)} \quad (2-58)$$

and

$$L_n = \frac{J_1(\alpha) Y_o(\alpha) - J_o(\alpha) Y_1(\alpha)}{J_o(\alpha) Y_o(a\alpha) - J_o(a\alpha) Y_o(\alpha)}. \quad (2-59)$$

A comparison of the three figures shows that BC3 exhibits slow pressure rise to a peak pressure of 2.43 kPa, and this might be attributed to the condition itself which allows fluid flow in two directions, thus larger flow rate will occur.

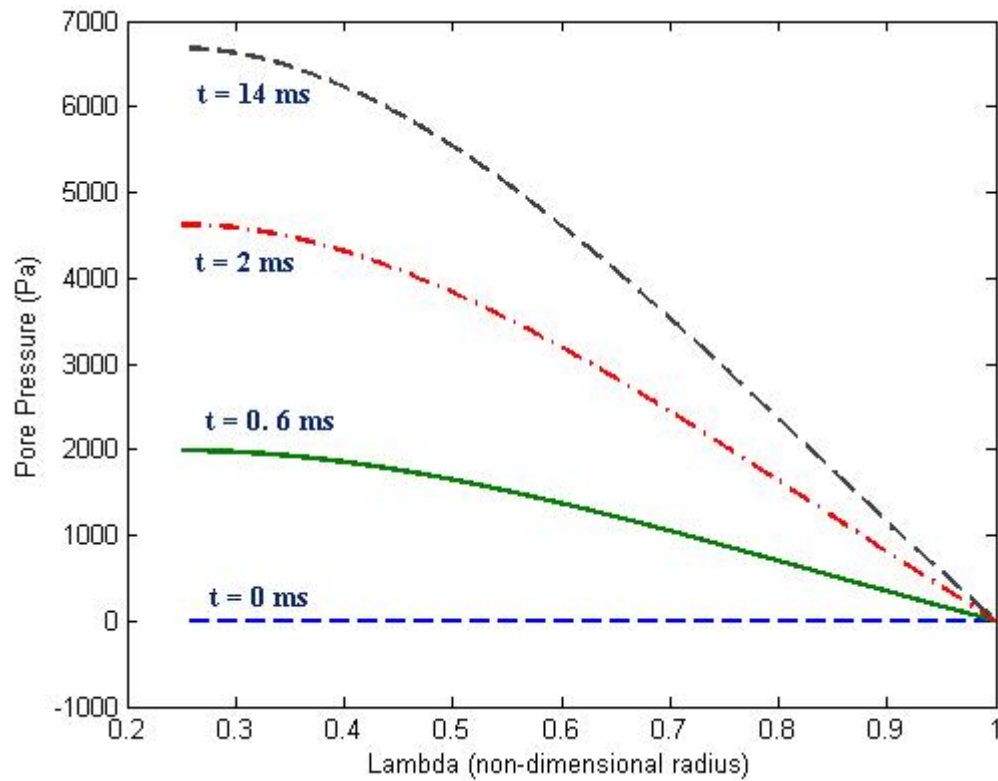


Figure 4:. The pressure distribution in BC1 for a compressible transverse isotropy model. The pressure will keep rising to the peak value of 6.7 kPa in 14 ms approximately. The figure shows only the first stage in an imposed ramp displacement, from Gailani and Cowin (2008).

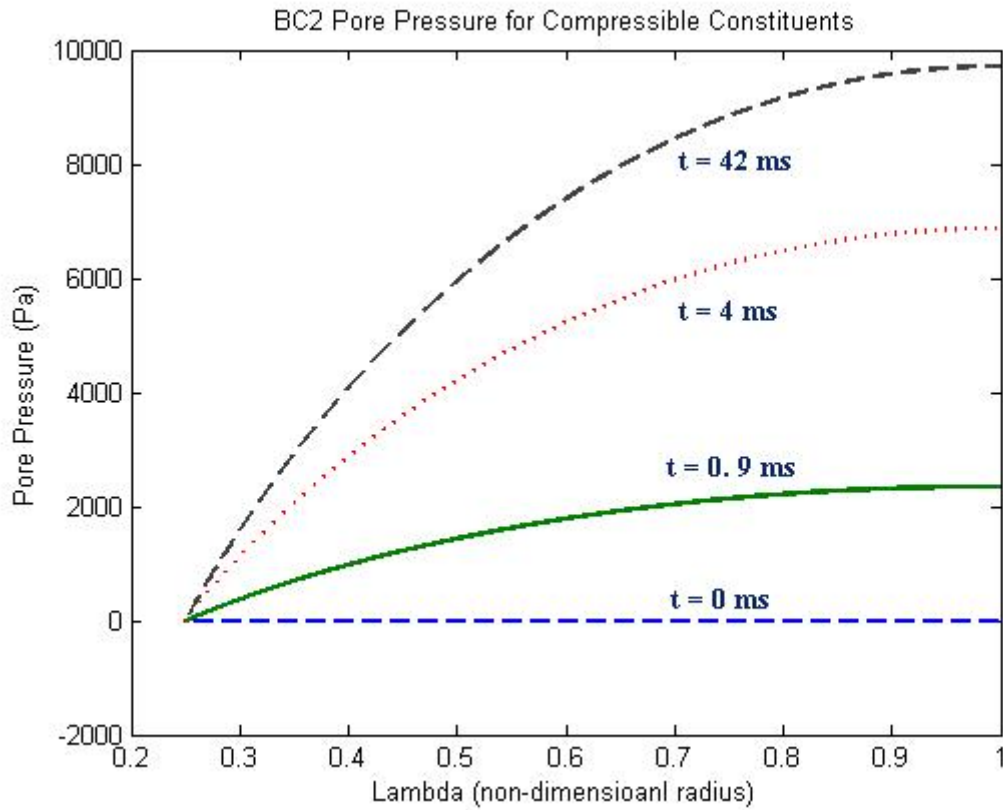


Figure 5: The pressure distribution in BC2 for a compressible transverse isotropy model. The pressure will keep rising to the peak value of 9.7 kPa in 42 ms approximately. The figure shows only the first stage in an imposed ramp displacement, from Gailani and Cowin (2008).

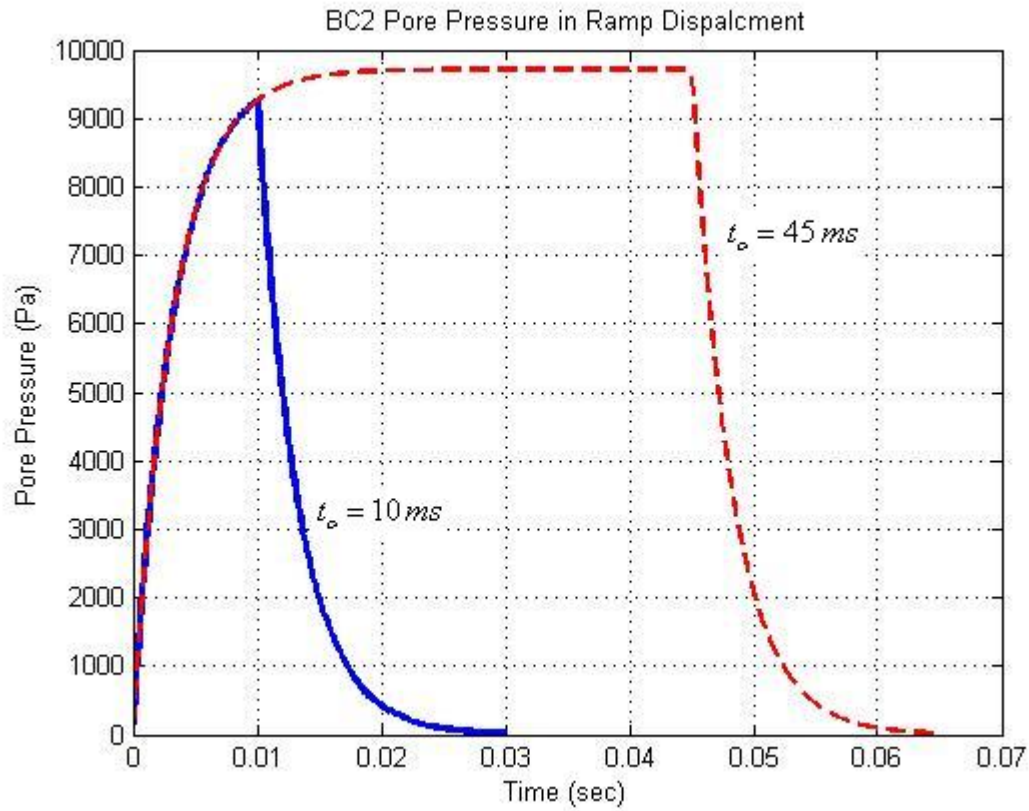


Figure 6: The pore pressure at $r = b$ for BC2. When applying a ramp displacement in the case of compressible transverse isotropy model, the time t_o is assumed to be either 10 ms or 45 ms. The pressure increases as the displacement is increasing, but when the displacement is held constant, the pressure begins to diminish rather rapidly, from Gailani and Cowin (2008).

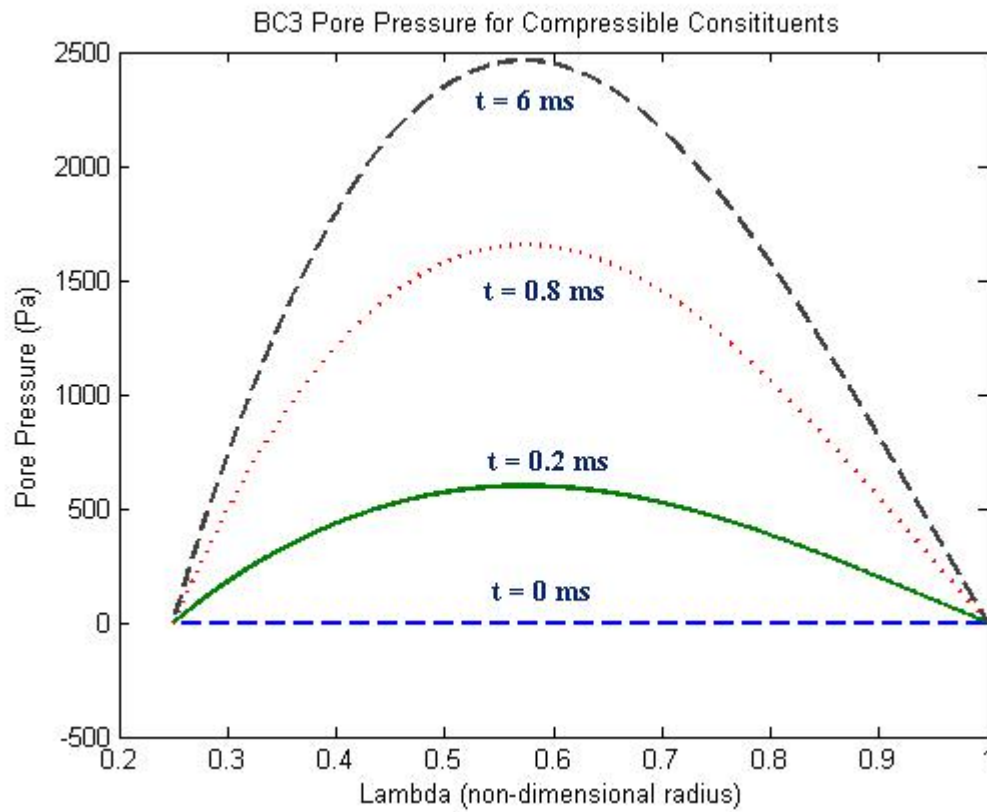


Figure 7:. The pressure distribution in BC3 for a compressible transverse isotropy model. The pressure will keep rising to the peak value of 2.43 kPa in 6 ms approximately. The figure shows only the first stage in an imposed ramp displacement, from Gailani and Cowin (2008).

2.6 The ramp loading of a compressible poroelastic material

The purpose of this section is to provide in depth analysis for the results obtained in Figure 6 earlier. Figure 6 has shown an asymptotic behavior for the pore pressure after specific time value. This behavior reflects the fact that an infinite time for the rise period doesn't necessarily produce an infinite pressure. To illustrate this, a model like the Terzaghi model that is shown in Figure 1 is considered. The model is assumed to be a simple poroelastic compressible model. In this analysis of the ramp loading the following notation is employed:

$a(t)$ the displacement of the piston downward. ramp function :

A is the area of the piston head

A_o is the area of the orifice in the piston head

b is the a constant representing the volumetric compressibility of the pore fluid

c is the constant relating $p(t)$ to $q(t)$ (a form of Darcy's law), $q(t) = cp(t)$

$F(t)$ is the force applied to the piston head to effect the ramp loading

$p(t)$ is the the pressure in the piston head chamber

p_o is the a reference value for the pressure

$q(t)$ is the the volume flow rate of fluid from the orifice in the piston head

R_F - is the the rate of volume decrease in the piston cavity due to the fluid leaving through the orifice in the piston head cavity due to the downward movement of the piston

R_C is the compression rate of the fluid volume in the piston cylinder space

R_P is the rate of volume decrease in the piston due to the downward movement of the cavity due to the downward movement of the piston,

V_o is the reference value of the volume.

In this example $a(t)$ it will be specified to be :

$$a(t) = \frac{a_o}{t_o} t \quad \text{for } t_o \geq t \geq 0$$

$$a(t) = a_o \quad \text{for } t \geq t_o,$$

and

$$b = \frac{(p(t) - p_o)}{(V(t) - V_o)}.$$

Notice that

$$\frac{dp}{dt} = b \frac{dV}{dt}$$

On the other hand R_p can be expressed as

$$R_p = A \frac{da}{dt},$$

R_F as

$$R_F = A_o q(t) = A_o c p(t),$$

and R_C as

$$R_C = \frac{dV}{dt} = \frac{dp(t)}{b dt}.$$

It is important to notice that

$$R_p = R_F + R_C$$

and

$$A \frac{da}{dt} = A_o c p(t) + \frac{dp}{b dt}$$

we can rearrange this using the fact that $a(t)$ is given by the ramp loading, it follows that

$$\frac{dp}{dt} + A_0 b c p(t) = b A \frac{a_0}{t_0} \quad \text{for } t_0 \leq t \leq 0$$

$$\frac{dp}{dt} + A_0 b c p(t) = 0 \quad \text{for } t \geq t_0$$

If the fluid is incompressible then as $b \rightarrow \infty$ we find that

$$A_0 c p(t) = A \frac{a_0}{t_0} \quad \text{for } t_0 \leq t \leq 0$$

$$A_0 c p(t) = 0 \quad \text{for } t \geq t_0$$

The solution for the ramp portion of the loading of the compressible case is

$$p(t) = A \frac{a_0}{c A_0 t_0} (1 - e^{-A_0 b c t}) \quad \text{for } t_0 \leq t \leq 0$$

The equation above shows that when t approaches infinity the pressure will tend to a constant

constant, $\frac{A a_0}{c A_0 t_0}$. The time instance can be approximated by

$$\frac{dp}{dt} = \tan(45^\circ) = \frac{A a_0 b}{t_0} e^{-A_0 b c t}.$$

The solution for the constant portion of the loading for the compressible case (for $t \geq t_0$) is

$$p(t) = p(t_0) e^{-A_0 c b t}$$

where

$$p(t_0) = A \frac{a_0}{c A_0 t_0} (1 - e^{-A_0 b c t_0})$$

This shows that $p(t) \rightarrow 0$ as $t \rightarrow \infty$.

Another approach that will show similar results to the results obtained above is by applying the following loading condition to Cowin and Mehrabadi(2007) to a *solid* poroelastic disk fully saturated under a ramp loading (rising)

$$\mathcal{P}(s) = \frac{\mathcal{R}}{s^2},$$

where s is the Laplace operator and the equation is showing the ramp loading in terms of Laplace transforms.

Using the final value theorem

$$\lim_{s \rightarrow 0} s \mathcal{P}(\lambda, s) = \lim_{t \rightarrow \infty} p(\lambda, t)$$

in Eq.(85) in Cowin and Mehrabadi(2007)

$$\mathcal{P}(\lambda, s) = \hat{C}_{rr}^d g_1(s) \left[I_o(\lambda\sqrt{s}) - I_o(a\sqrt{s}) \right].$$

Applying the theorem above to this pressure equation yields a finite value for the pressure as

$t \rightarrow \infty$

$$p(\lambda, \infty) = \frac{B \mathcal{R} \hat{C}_{rr}^d (\lambda^2 - 1)}{4 \left(C - \frac{D}{2} \right)}$$

which means that the pressure is finite when $t \rightarrow \infty$. Also the equation clearly satisfies the

boundary condition $p(1, t) = 0$

Now if we apply the same procedure but with more complexity to the pressure equation of

BC2 described in Gailani and Cowin (2008) for an *annular* disk

$$\mathcal{P}(\lambda, s) = \hat{C}_{rr}^d \left(g_1(s) \left[I_o(\lambda\sqrt{s}) - I_o(a\sqrt{s}) \right] + g_2(s) \left[K_o(\lambda\sqrt{s}) - K_o(a\sqrt{s}) \right] \right).$$

Applying the final value theorem of Laplace to get the pressure at $t \rightarrow \infty$

$$p(\lambda, \infty) = \frac{B \hat{C}_{rr}^d \mathcal{R}}{\frac{4D}{2a} + C - \frac{D}{2}} (\lambda^2 - a^2 + 2 \ln a - 2 \ln \lambda).$$

Again, the pressure is finite. Thus, it is evident that the pressure will not go to infinity as

$t \rightarrow \infty$, instead it will be uniform at each point with a unique value that depends only on λ

2.7 Discussion

The analysis of this unconfined compression test of an annular disk presents a geometric generalization of the problems solved in Mandel (1953), Armstrong et al. (1984), Cohen et al. (1998) and Cowin and Mehrabadi (2007) for the circular disk. The analysis presented the solution to three major boundary conditions. It is very interesting to see that the solutions of Cowin and Mehrabadi (2007) for a solid circular disk can be restored from this analysis, although one should not be too surprised. For example, substituting $a = 0$ in equations (2-40) and (2-42) as well as all other conditions of Table 4 for BC1, will recover equations (102), (104) and the results in Table 2 in the solution of Cowin and Mehrabadi (2007). This result is valid only for BC1, which assumes fluid flow through the outer boundary only; this assumption also characterizes the results of Armstrong et al (1984) and Cohen et al (1998).

As we noted in the introduction of BC3, there are several alternatives to the assumption that the annular region is freely drained. For example, the pressure in the annular region could be specified as some non-zero value controlled in an experiment, or the drainage of the annular region could be constrained to be radially through the specimen and the pressure in the annular region would then vary in the posed boundary value problem. It would not be difficult to use a non-zero boundary condition on the pore pressure at the inner cylindrical surface and we will do so in a subsequent study because we could drive the pore fluid in the radial direction using both the pore pressure differential between the inner and outer radii and the applied axial loading. Such an experiment would be an effective way to determine how much applied strain is required for the permeability to begin to deviate from linearity.

It is observed that the analysis has shown that the displacements, pressure, and the applied force are the same for all three boundary conditions when $t \rightarrow 0$. The physical reason for this is that there is no fluid flow at the first instance, $t = 0$, thus the disk is functioning as an object whose elasticity is determined by the undrained elastic constants. In addition to the fact that all the boundary conditions are dependent on the flow and since there is no flow in the first instance, then it is expected that the displacements, the pressure, and the applied force are the same.

The analysis has shown that BC3, the boundary condition that assumes fluid flow through both boundaries, has a different mathematical structure (Table 2 and Table 4). It can be shown that there exists a point between the inner and outer boundaries where the pressure gradient is zero, thus allowing fluid flow through the two opposite boundaries in two opposite directions. This point is not fixed as the emptying proceeds but it moves very slowly towards the outer boundary.

The advantage of this analysis is that it extends to hard tissues, through the lack of the incompressibility assumption and the annular disk shape, an analytical model for an experimental test situation. This permits a more detailed study for bone porosity and the nested system of lacunar-canalicular porosity (PLC) and vascular porosity (PV) in long bones where these different boundary conditions will give an insight into the nature and quantity of the flow between these porosities.

Finally, we address the following question: Comparing the modeling of the fluid and solid constituents as incompressible versus compressible, is the extra effort justified given the increased complexity involved in the modeling of compressible constituents in a poroelastic material? We have observed that this question is one that arises from the biomechanics rather

than the geomechanics poroelastic community. The geomechanics poroelastic community is aware of the Skempton parameter, a parameter that has a range from zero to one and represents the fraction of the hydrostatic component of the stress in the matrix material that is mechanically transferred to the fluid pore pressure. (The Skempton parameter is defined in a footnote to Table 1.) In the case of soft tissues and swampy saturated soils the Skempton parameter is assumed to be about one, that is to say all of the hydrostatic component of the stress in the matrix material is mechanically transferred to the fluid pore pressure. This is because the bulk compressibility of the two materials is about the same and incompressibility of both the fluid and solid constituents is a reasonable assumption. In hard tissue and rocks the Skempton parameter is often less than one-half, thus less than one-half of the hydrostatic component of the stress in the matrix material is mechanically transferred to the fluid pore pressure. In effect the pore pressure is shielded from the matrix material hydrostatic stress by the much greater bulk compressibility of the matrix material. Neglecting the effect of relative compressibility for a Skempton parameter less than one will increase the error in a problem in proportion to the decrease in the Skempton parameter from one. For hard tissues and rocks, these are very significant errors. There is a discussion of the point in the Introduction and the compressible and incompressible assumptions are compared in Cowin and Mehrabadi (2007).

3.

Experimental Determination of the Permeability of the PLC Porosity

3.1 Introduction

Interstitial fluid flow in bone is necessary for cellular nutrition and waste removal as well as being a significant factor in the sensing of mechanical loading by bone tissue, mechanotransduction. Fluid flow in tissues occurs at different scales including the main circulation, microcirculation and interstitium. Blood flow from the circulatory system provides tissues with oxygen, nutrients and appropriate conditions (temperature, pH) for cellular metabolism, as well as a mechanism for removal of carbon dioxide and other cellular waste products via the venous and lymphatic systems. Amino acids, sugars, fatty acids, coenzymes, hormones, neurotransmitters, salts and many other substances pass from the capillary blood vessels into the tissue's interstitium. In bone, these substances travel across the capillary walls into the interstitial fluid in the vascular porosity (PV). The vascular porosity is the space inside the osteonal and Volkmann canals and exterior to the soft tissue structures, blood vessels and nerves, within these channels. The interstitial fluid in the PV also travels via canaliculi across lacunae, the matrix pores of the bone tissue where the osteocytes are encased. The fluid space composed of lacunae and canaliculi, excluding the volume occupied by osteocytes and their dendritic processes, is designated the lacunar-canalicular porosity (PLC).

Time varying mechanical loads applied to bone generate fluid pressure gradients in the PLC that contribute to the interstitial fluid flow and the transport in the PLC. The interstitial fluid flow is important for cellular nutrition and waste removal, and it is a critical factor in osteocyte mechanotransduction. Strain levels produced in the bone tissue by normal

physiological activity are generally less than 0.2%; however, it is well known that osteocytes only respond to mechanical strains higher than 0.5%. While strains in the mineralized matrix are not sufficient to stimulate osteocytes, it has been proposed that the fluid flow in the lacunar-canalicular system generated by physical activity produces a strain amplification effect in the dendritic osteocyte cytoplasmic process within the range of stimulus an osteocyte can sense. Bone fluid flow in the PLC exerts a drag force on the osteocyte process that is large enough for the cell to sense. Membrane channels, integrins, focal adhesion protein complexes, etc are among the structures located in the osteocyte process involved in the transduction of interstitial fluid flow into synthesis and release of biochemical factors implicated in the regulation of bone growth and development. Hence, the study of interstitial fluid flow is critical to better understand the mechanosensory system in bone and related bone diseases.

In this study the permeability of the PLC compartment is determined by curve fitting of a compressible transverse isotropic poroelastic model of a porous annular disk under a ramp loading history (Cowin and Mehrabadi 2007; Gailani and Cowin 2008) and experimental stress-relaxation test data for single osteons. The PLC tissue intrinsic permeability in the radial direction of the osteonal test specimen is the only free parameter in the curve fitting procedure.

3.2. Analytical solution.

A single osteon, idealized in this study as a fully saturated porous annular cylinder, is subjected to unconfined compression in stress relaxation mode. The experimental technique of subjecting circular disks of soft tissue to unconfined compression in order to determine the soft tissue's permeability, and some of its elastic constants, has become widely used in the last few decades (Armstrong et al., 1984; Cohen et al., 1998; Wu and Herzog 2006; Yin and Elliot, 2004, and many others). The original study of unconfined compression appeared in a paper by

Mandel (1953). The analysis of the experimental data from such a test is based on an analytical solution to an infinitesimally deformed saturated poroelastic disk loaded in a similar manner. For the analysis of soft tissues the assumption that the fluid and solid constituents of the poroelastic medium are both incompressible is reasonable. However in the case of hard tissues the constituent incompressibility assumption is not appropriate because the effective bulk modulus of the poroelastic solid constituent is almost an order of magnitude stiffer than that of the poroelastic fluid constituent, thus the solid constituent shields the fluid constituent from stress.

Following Mandel (1953), Armstrong et al., (1984) obtained a theoretical poroelastic solution assuming infinitesimal deformation, frictionless platen interface, isotropic material properties and incompressible constituents, and reported that a good agreement between the theoretical response and the experimental data could not be obtained. Cohen et al. (1998) proposed an incompressible transverse isotropic model and compared their theoretical solution to the unconfined tests of the growth plate and the chondroepiphysis, and showed that the unconfined compression model with incompressible constituents and transverse isotropy provided a good curve fit.

Cowin and Mehrabadi (2007) extended the previous solutions to include the compressibility assumption of the fluid and solid phases and applied this approach to bone. In Gailani and Cowin (2008) a theoretical solution for the unconfined compression of a porous annular cylindrical disk was developed. The solution assumed three draining conditions. The first condition (BC1) considered flow through the outer boundaries only, the second condition (BC2) assumed flow through the inner boundaries only, and the third condition (BC3) takes into account flow through both boundaries. A ramp loading that is characterized by a constant

strain rate $\dot{\epsilon}_o$ preceded by a period of duration t_o during which the applied strain is rising linearly from a value of 0 to $t_o \dot{\epsilon}_o$, is applied to obtain each of these solutions. After time t_o the value of $\epsilon(t)$ is constant at $t_o \dot{\epsilon}_o$. The expression of this strain loading function in terms of the Heaviside step function $H(t)$ is

$$\epsilon(t) = \dot{\epsilon}_o [tH(t) - (t - t_o)H(t - t_o)]. \quad (3.1)$$

The boundary condition of interest here is the second condition (BC2) in which it was assumed that flow occurred across the inner boundaries only. In this case Gailani and Cowin (2008) obtained the following formulas for the force $P(t)$ on the annular disk due to the applied strain loading:

$$\begin{aligned} \frac{P(t)}{\pi(b^2 - c^2)(\hat{C}_{rr}^d - \hat{C}_{r\theta}^d)} &= \frac{\left(Ma^2 + \left[\frac{-N}{2} + a^2 F \right] \right) \dot{\epsilon}_o t}{(D + a^2 [2C - D])} + \\ &\frac{b^2 \dot{\epsilon}_o}{m \alpha_n^4} \sum_{n=1}^{\infty} \frac{\left(\left[\frac{-N}{2} + a^2 F \right] + \frac{M}{2} R(\alpha_n) \right) \left(1 - e^{-\alpha_n^2 m t / b^2} \right)}{\left\{ \frac{1}{2} \left(aC - \frac{3D}{a\alpha_n^2} \right) + \frac{R(\alpha_n)}{\alpha_n^2} \left\{ \frac{C}{2} + \frac{D}{2} \right\} - \frac{C}{2} Z(\alpha_n) - V(\alpha_n) \frac{D}{2a\alpha_n^2} \right\}}, \end{aligned} \quad (3.2)$$

for $0 \leq t \leq t_o$, and

$$\frac{P(t)}{\pi(b^2 - c^2)(\hat{C}_{rr}^d - \hat{C}_{r\theta}^d)} = \frac{\left(Ma^2 + \left[\frac{-N}{2} + a^2 F \right] \right) \dot{\epsilon}_o t_o}{(D + a^2 [2C - D])} -$$

$$\frac{b^2 \mathcal{E}_0}{m \alpha_n^4} \sum_{n=1}^{\infty} \frac{\left(\left[\frac{-N}{2} + a^2 F \right] + \frac{M}{2} R(\alpha_n) \right) \left(1 - e^{\alpha_n^2 m t_0 / b^2} \right) e^{-\alpha_n^2 m t / b^2}}{\left\{ \frac{1}{2} \left(aC - \frac{3D}{a \alpha_n^2} \right) + \frac{R(\alpha_n)}{\alpha_n^2} \left\{ \frac{C}{2} + \frac{D}{2} \right\} - \frac{C}{2} Z(\alpha_n) - V(\alpha_n) \frac{D}{2 a \alpha_n^2} \right\}} \quad (3.3)$$

for $t \geq t_0$.

There is a significant amount of notation that appears in these two equations and we now identify the significance of each symbol. The inner and outer radii of the annular disk are denoted by b and c , a is the non dimensional inner radius ($a = c/b$), the constants C , D , F , M , N , and Y are defined in Table 3, $\hat{C}_{rr}^d$ and $\hat{C}_{r\theta}^d$, components of the drained transversely isotropic elasticity matrix, are defined in Eq.(17). The diffusivity of the pore fluid pressure is characterized by m ,

$$m = \frac{K_{rr} \hat{C}_{rr}^d}{\mu J}, \quad (3.4)$$

where K_{rr} is the permeability in the radial direction, $\mu = 1$ centipoise (cP) is the viscosity of the water, J is defined in Table 1. Tables 1 and 3 summarize the expressions for a compressible transverse isotropic material model. The three functions

$$R(\alpha_n) = \frac{\alpha_n \{J_1(\alpha_n)Y_0(a\alpha_n) - J_0(a\alpha_n)Y_1(\alpha_n)\}}{J_1(a\alpha_n)Y_1(\alpha_n) - J_1(\alpha_n)Y_1(a\alpha_n)}, V(\alpha_n) = \frac{\alpha_n \{J_1(a\alpha_n)Y_0(\alpha_n) - J_0(\alpha_n)Y_1(a\alpha_n)\}}{J_1(a\alpha_n)Y_1(\alpha_n) - J_1(\alpha_n)Y_1(a\alpha_n)},$$

$$Z(\alpha_n) = \frac{J_0(a\alpha_n)Y_0(\alpha_n) - J_0(\alpha_n)Y_0(a\alpha_n)}{J_1(a\alpha_n)Y_1(\alpha_n) - J_1(\alpha_n)Y_1(a\alpha_n)}, \quad (3.5)$$

are combinations of different types of Bessel functions and α_n are the simple poles obtained by setting the denominator of the Laplace transform of $P(t)$ equal to zero for BC2 as shown in Galiani & Cowin (2008).

3.3 Preparation of the specimens for experimental testing

The osteon is composed of a central Haversian canal housing a blood vessel and is surrounded by alternating mineralized collagen lamellae. The mean Haversian canal and osteon diameters in femoral bone are estimated to be 101 ± 18 and $312 \pm 48 \mu\text{m}$ respectively, Hofmann et al., (2006). Their small size and stiffness make osteons challenging to isolate, nonetheless several approaches have been described in the literature since 1959. Ascenzi and coworkers (Ascenzi and Fabri 1959; Ascenzi and Bonucci 1967 Ascenzi et al., 1985; Ascenzi et al., 1994) developed a method for extraction of osteons from a thin cross-section of bone using a sharp steel needle eccentrically inserted in a dentist's drill or using a custom made micro lathe. Osteons have also been isolated by propagating a fracture through the natural boundary of the osteon (Frasca et al., 1976) and using a precision and computer controlled osteon pushout micro-testing system (Dong et al., 2005).

Most osteons in cortical bone do not have a perfectly cylindrical shape; however, for their testing and analysis in this study, osteon samples were prepared into cylindrical shaped specimens. Fresh femoral bovine bone samples were obtained from a local slaughterhouse, kept in Phosphate-buffered Saline Solution (PBS) and processed the same day. Gross bone slides 10 mm thick were cut perpendicular to the main axis of the whole bone using a belt diamond saw DL3000 XL (Diamond Tech. Intern. USA). A second gross cut was performed perpendicular to the annular slides to obtain parallelepiped shaped pieces of about 7 to 10 mm in length. A low speed circular diamond saw was then used to correct uneven cut surfaces (Isomet, Buehler Corp.) resulting in smooth flat surfaces. Top and bottom surfaces were then polished with progressive 400-, 800- and 1200-grit SiC sand paper and alumina slurry ($0.1 \mu\text{m}$ particle size). The parallelism between both surfaces was tested at different locations using a

digital caliper, and the polishing was continued until the flatness error was smaller than 1 μm . Dehydration was avoided during the sample preparation and testing procedures by permanent irrigation with PBS.

Several cylindrical rods of approximately 1.5mm diameter were obtained from the bone cube using a variable speed Mill/Drill machine (G0463, Grizzly Industrial, Inc.) with a 1/18" diamond plated core drill bit (3.3 mm/0.13", Technology LK Inc.). These cylindrical bone samples were imaged using a high resolution Micro-CT system (SkyScan Corp., Belgium). Images from Micro-CT allowed the identification of the location and inclination angle of the Haversian canals relative to the central axis of the sample. A schematic diagram for the osteon isolation process is shown in Figure 8. Samples exhibiting a mean Haversian canal orientation parallel and concentric to the axis of the sample were retained for the osteon isolation process as shown in Figure 10. The diameter of selected bone samples was reduced up to 200-300 μm using a 4"x5" Mini Micro-lathe (Central Machinery Inc., USA), corresponding approximately to the average diameter of osteons. This cutting operation was accomplished at very low speed, with extra care as the diameter of the specimen was made thinner and thinner. Water was used as a coolant during the milling and lathe operations to prevent heating and dehydration of the sample (Currey, 1988). A 400-500 μm height segment was cut from the cylindrical specimen, resulting in an isolated osteon (n=12) suitable for the unconfined compression stress relaxation test as shown in Figure 9.

3.4 Micro-CT scanning and images post treatment.

The bone samples were scanned using a high resolution 1172 SkyScan Micro-CT system. For image acquisition, X-ray projections were generated from the sample each 0.3 degrees, obtaining 600 consecutive views with 0.9 μm pixel size. Five exposures by projection were

used to produce high contrast low noise images. The raw images were corrected for possible pixel defects in the digital detector using bright and dark fields, and a standard reconstruction algorithm (Feldkamp) included in the acquisition system was applied to generate 3-D volumes from the planar X-ray projections. Identification of the solid tissue and the Haversian pore cavity was developed based on a global thresholding method. 3-D volume reconstruction for each sample was performed using a computer workstation. A commercially available software package (Mimics, Materialise, Ann Arbor, MI) was used to generate the 3-D bone geometry from each micro-CT scan. High resolution images were calibrated for densities within the volumes using a phantom containing hydroxyapatite (the main component of bone's mineralized phase), air, and water. An image generated in the Micro-CT is shown in Figure 10.

3.5 Micromechanical testing.

A miniature uniaxial material testing stage was developed by our group for the assessment of mechanical properties in small bone samples. The computer controlled stage comprises an Intelligent Picomotor control module (8753 driver, New Focus Inc.) and an Ethernet controller (8752 Ethernet controller, New Focus Inc.). The driver was connected to a small piezoelectric motor (Picomotor, New Focus Inc.) which has a displacement resolution of 30 nm. The Z-stage held a stainless steel platen vertically aligned to a second platen in a loading chamber filled with water, and connected to a load cell (GS series gram force -10 grams- sensor, Transducer Technique Inc.). The resolution of the measurement was 0.05% (5 mg) of the load cell's full scale. Load cell output signal was amplified using an instrumentation amplifier (INA 122, Texas Instruments) and then calibrated using standard weights to display the measured force in grams. Voltages were converted into digital signals using a data acquisition

interface (USB6810, National Instruments) and visualized in real time using LabView (V8.5, National Instruments). A schematic diagram for the loading system is shown in Figure 11.

In order to perform a stress relaxation test in the sample, the loading stage was used under displacement control. The user inputs the desired strain rate in LabView, along with the type of test (compression or tension), then sends the commands to the Ethernet controller and the picomotor stage. The loading system performance was tested using a cylindrical steel sample in stress relaxation test (Goto et al. 1999). The osteon specimen was prestressed with 0.2g and then loaded using a constant strain rate $\dot{\epsilon}$ for a period of time t_0 during which the applied strain was rising linearly from a value of 0 to 2 g of force approximately. After time t_0 the value of $\epsilon(t)$ was held constant. The loading system is shown in Figure 12. The sample was left to freely undergo relaxation until the change in measured force was no longer significant. The load vs. time curve was displayed in real time on the screen and stored in the hard drive for post-processing.

3.6 Estimation of the permeability using stress relaxation test

The pore sizes in cortical bone are of approximately three discrete magnitudes; the largest pore size (approximately 50 μm diameter) is associated with the vascular porosity (PV), the second largest pore size (approximately 0.3 μm diameter) with the canaliculi in the lacunar-canalicular porosity (PLC) and the smallest pore size (approximately 10 nm diameter) is in collagen-apatite porosity (PCA) (Cowin, 1999).

In human cortical bone the cement line separates osteons from the interstitial bone tissue, which consists of remnants of primary lamellar bone or fragments of remodeled osteons.

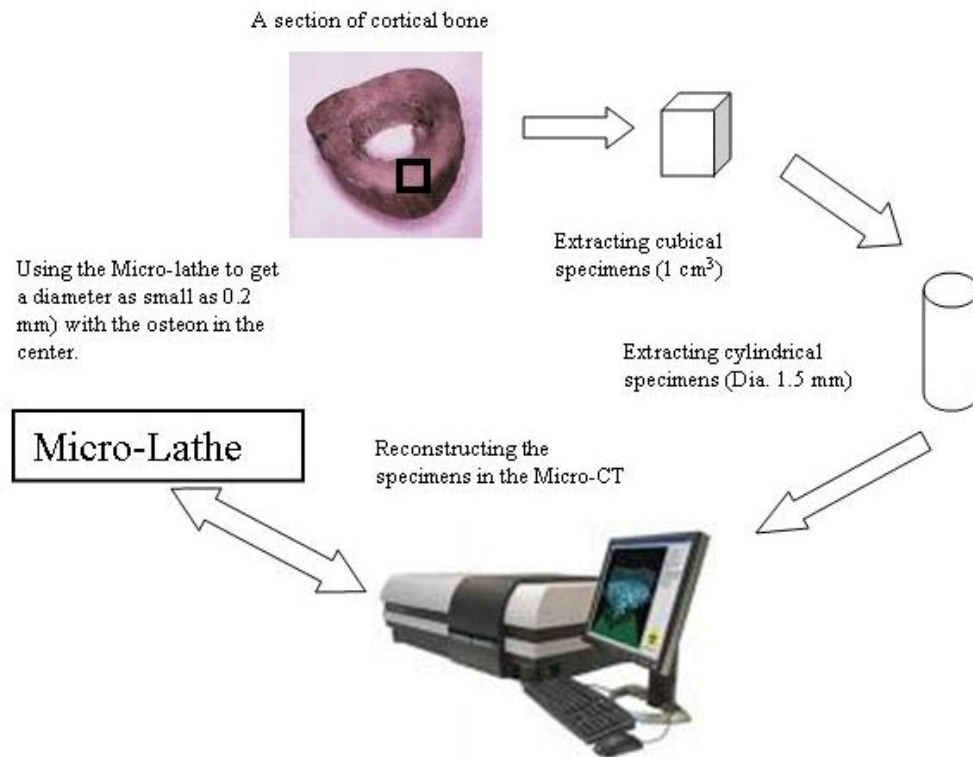


Figure 8: Osteon isolation process. A thick section was cut from the mid-diaphysis of the bone, and a cubical sample was prepared. Cylindrical rods of about 1.5 mm diameter were extracted from this bone cube and analyzed using a Micro-CT to identify the samples with Haversian canals approximately concentric and parallel to the axis of the cylindrical rod. Selected samples were processed in a Micro-Lathe to reduce the diameter of the sample to 300 mm, resulting in samples containing a single osteon. All cutting procedures are performed under continuous irrigation and low speed. Gailani et al., (2008)

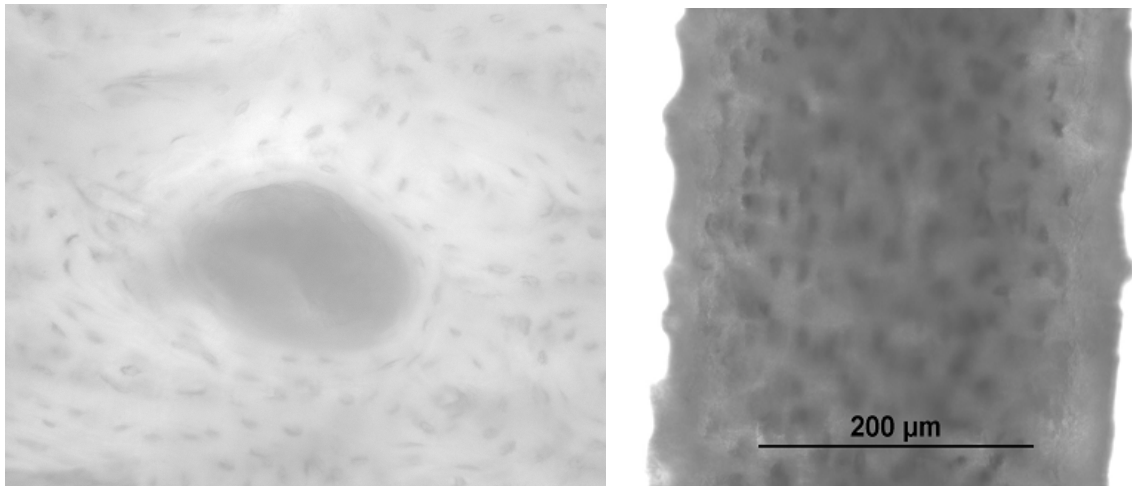


Figure 9: A top view of the Haversian canal (Left). An isolated osteon sample with an approximate diameter of 250 micrometers as seen under the microscope (Right).

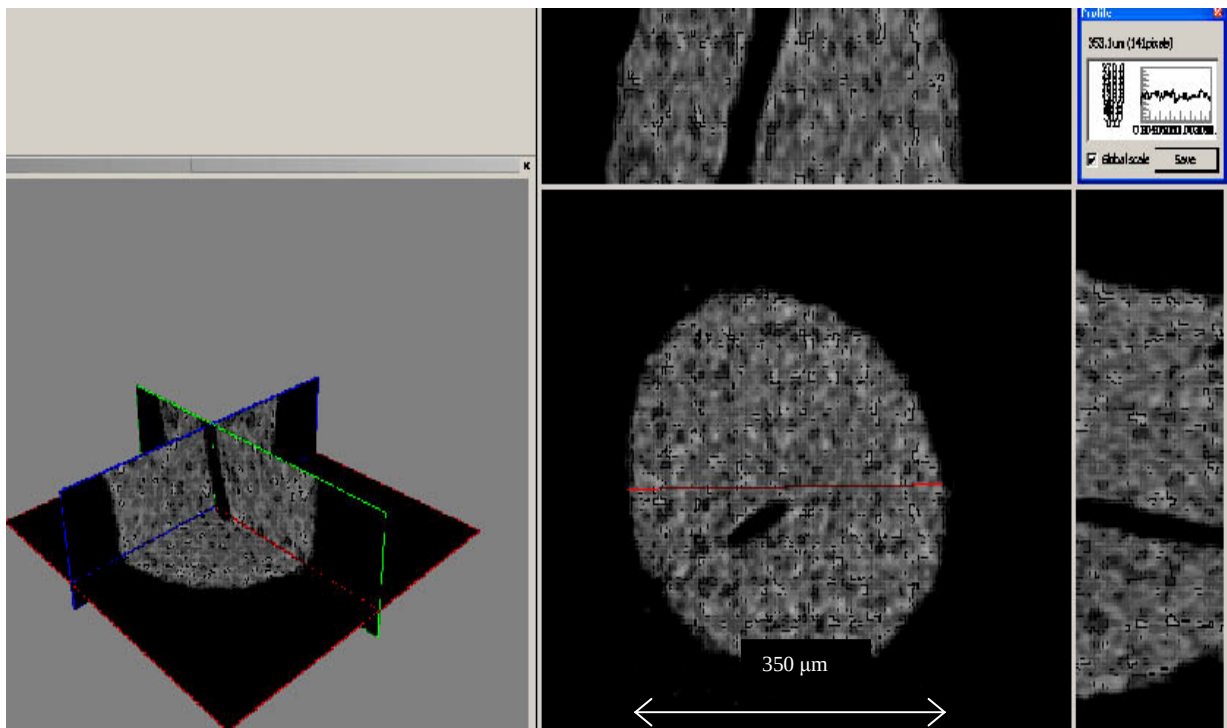


Figure 10: Micro-CT image of an isolated osteon (Diameter is 350 micrometer approximately as shown on the upper right corner). The image shows the three main views and the Haversian canal. The diameter of the Haversian canal can be measured from these images. A 3D construction of the osteon is shown (left), Gailani et al., (2008)

Estimates of the intrinsic permeability of the lacunar-canalicular porosity were provided by Zhang et al. (1998), Smit et al. (2002) and Oyen (2008) who employed a nanoindentation technique with a poroelastic analysis to provide an estimate of the permeability of the osteon. Lei and Szeri (2006) described a procedure for estimating the material parameters of biological soft tissue by fitting model prediction to experimental load–deformation data.

In this study, the PLC permeability is determined from the curve fitting of experimental vs. theoretical stress relaxation curves in which the permeability is the only unknown parameter. The code to compute the stress relaxation response of the annular bone sample was implemented in MatLab (MathWorks, MA). Equations 2 and 3 describe the ramp loading and relaxation period of the sample, which are functions of the relationships in tables 1 and 2. Two sets of bone tissue elastic properties are required in the model for the estimation of the permeability. The first set involves the osteonal matrix material properties, characterized by the superscript m; they include E_1^m, E_3^m, ν_{12}^m , and ν_{13}^m . The second set involves the measured elastic constants in a poroelastic material in the drained condition, with superscript d, $E_1^d, E_3^d, \nu_{12}^d, \nu_{13}^d$, and ν_{31}^d . The transverse isotropic poroelastic constants considered in the model are summarized in Table 3. The curve fitting toolbox in MatLab was used to import the raw data and create a custom function based on the theoretical model that exclusively depends on the permeability. This custom equation is then fitted to the experimental data using a least-squares algorithm built in the toolbox. The only parameter that has not been assigned a value in the curve fitting process is the permeability in the radial direction of the sample K_r .

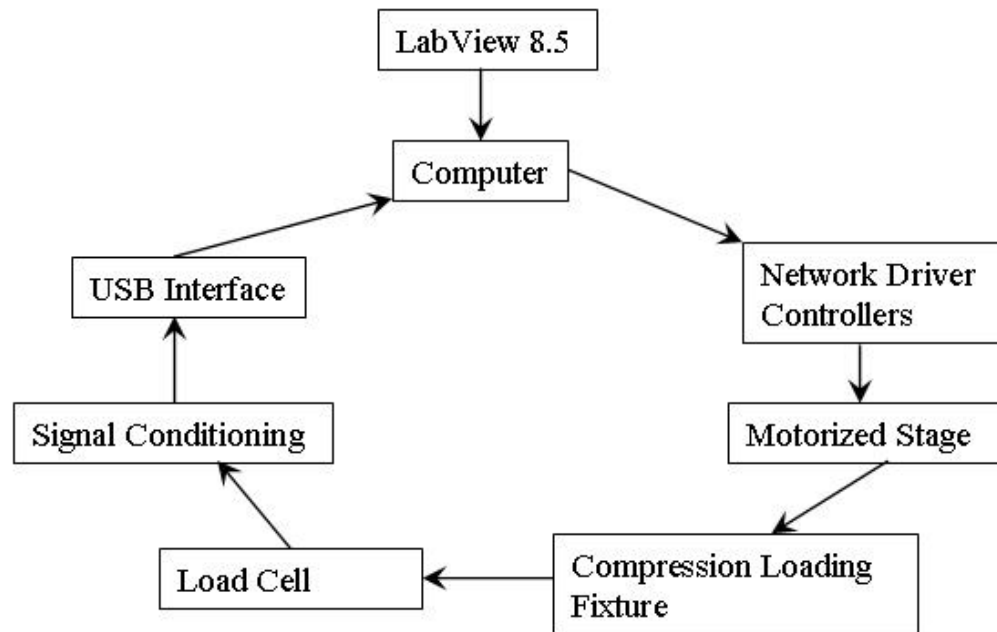


Figure 11: A uniaxial micro-mechanical testing device was constructed based on a small piezoelectric motor. The motor was integrated to a custom compression loading stage, and axially aligned to a load cell. The position, speed and acceleration of the motor was controlled from a computer via a network driver controller using a custom LabView application. Load cell measurements were amplified and digitized using a USB interface, and displayed in real time during the test and recorded for offline data analysis, from Gailani et al., (2008).

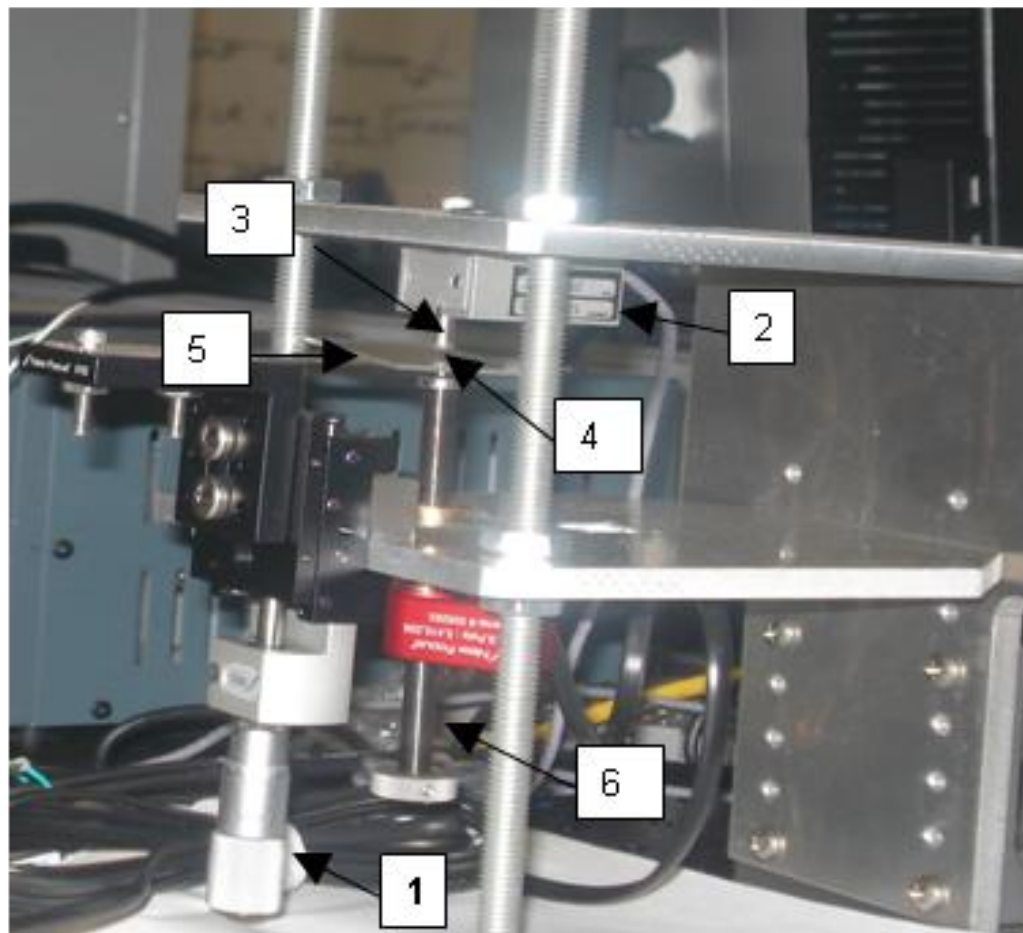


Figure 12: Photograph of the material testing stage which consists of: (1) micrometer, (2) load cell (10 grams, 0.05% FOM), (3) steel platens, (4) osteon sample, (5) sample stage, (6) picomotor linear actuator (30 nm displacement/step). Gailani et al.(2008)

3.7 Results

Figure 13 shows the stress relaxation curve for a sample made of steel and a typical stress relaxation time history of the load intensity recorded for one of the isolated osteons in response to a ramp displacement with constant strain rate ($\dot{\epsilon} \approx 10 \mu\epsilon s^{-1}$). The load intensity in both curves is normalized by their respective peak load intensity. The stress relaxation of steel was found to be practically zero. However, when the bone sample underwent the same test, the response for bone clearly exhibits an exponential decrease in the recorded load magnitude. This implies that the stress relaxation response observed in the bone sample is not a consequence of an artifact in the developed loading system.

The stress relaxation test was performed in each sample at three different strain rates, $\dot{\epsilon} \approx 10 \mu\epsilon s^{-1}$, $100 \mu\epsilon s^{-1}$, and $1000 \mu\epsilon s^{-1}$. Figure 14 shows typical stress relaxation curves for a representative osteon, where the axes display the non-dimensional load intensity versus the non-dimensional time t/t_0 . This figure shows that the relaxation of the sample is sensitive to changes within the range of strain rate tested. The most intense relaxation process occurs at the highest strain rates. However, the load intensity at equilibrium is very similar for the same sample tested at the three strain rates employed.

The intrinsic permeability was obtained by curve fitting the theoretical model characterized by equation (3.3) to the experimental stress relaxation time history (Figure 15) and using a least-squares algorithm. The permeability was the only free parameter in the fitting algorithm. The procedure was performed using 30 poles in the summation term in equations (3.2) and (3.3) with the material properties reported in Table 6. The permeability was found to be $3 \times 10^{-24} - 6 \times 10^{-25} m^2$ for the group loaded at $10 \mu\epsilon s^{-1}$, $5 \times 10^{-24} - 7 \times 10^{-25} m^2$ for the group loaded at $100 \mu\epsilon s^{-1}$, and $8 \times 10^{-24} - 5 \times 10^{-25} m^2$ for the group loaded at $1000 \mu\epsilon s^{-1}$.

Another approach that we examined was to determine all the drained elastic properties and the permeability by curve fitting the analytical results to the experimental data. However, this approach requires assigning some initial values to the drained elastic properties in order to determine the first pole and then using the least squares method to estimate the permeability. This approach shows good agreement between the experimental and analytical results in the loading period and less agreement in the relaxation period. The estimated permeability was in the range of $O(10^{-24}) - O(10^{-25})$. During loading time ($\approx 1000 \mu\text{s}$) the best curve fit was obtained with a permeability value in the range of $2 \times 10^{-24} - 3 \times 10^{-24}$. The drained elastic properties ($n=5$) were estimated to be: $E_1^d = 15.94 - 16.38$ GPa, $E_3^d = 13.1 - 13.4$ GPa, $\nu_{12}^d = \nu_{21}^d = 0.34 - 0.35$, $\nu_{13}^d = \nu_{23}^d = 0.3$, and $\nu_{31}^d = \nu_{32}^d = 0.24 - 0.25$. These values can be compared with the values previously reported in Table 6.

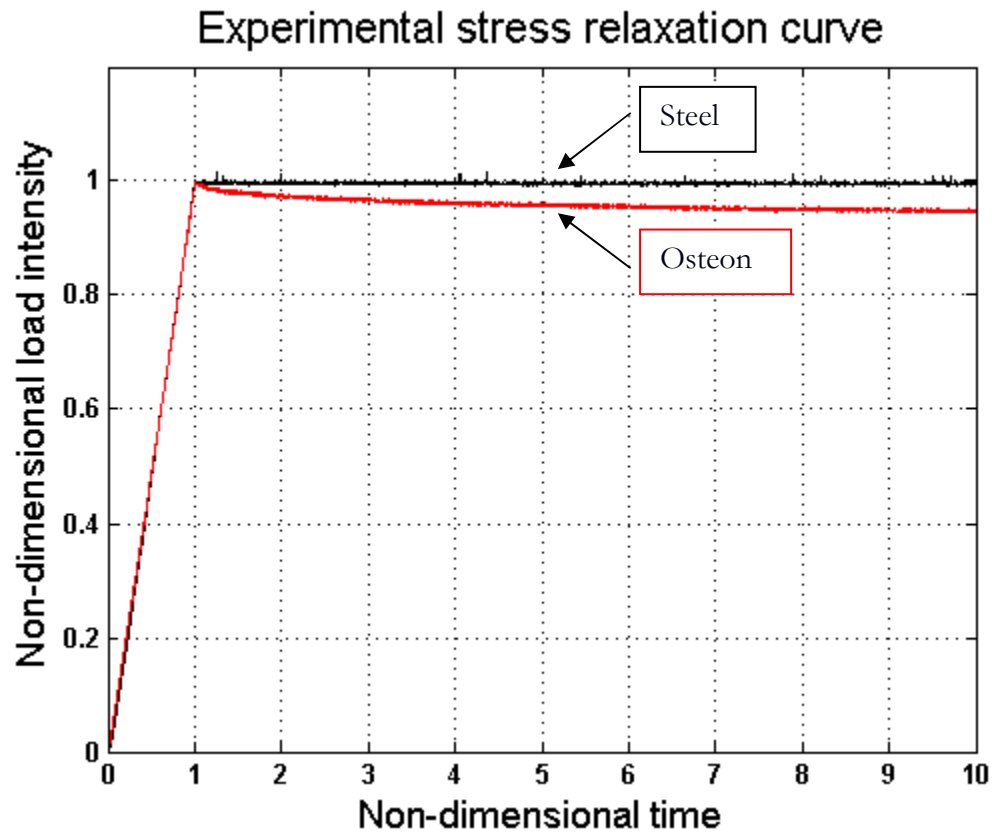


Figure 13: Stress relaxation curves for a sample made of steel and for one of the isolated osteons in response to a ramp displacement with constant strain rate ($100 \mu\text{m s}^{-1}$). The load intensity and loading time in both curves were normalized by the peak load intensity and t_0 respectively. Gailani et al., (2008)

Table 6: Elastic material properties of an osteon as previously reported. The properties include the drained (superscript d) and the matrix material properties (superscript m)

Material Parameters	PLC
$E_1^d = E_1^d$	15.17 GPa
E_3^d	15.96 GPa
$\nu_{12}^d = \nu_{21}^d$	0.27
$\nu_{31}^d = \nu_{32}^d$	0.285
$\nu_{13}^d = \nu_{23}^d$	0.27
$E_1^m = E_1^m$	18.6 GPa
E_3^m	22.32 GPa
$\nu_{12}^m = \nu_{21}^m$	0.322
$\nu_{31}^m = \nu_{32}^m$	0.312
$\nu_{13}^m = \nu_{23}^m$	0.255
K_f (Compressibility of water)	2.3 GPa
μ (Viscosity of water)	1 centi Poise
Porosity ϕ	0.05

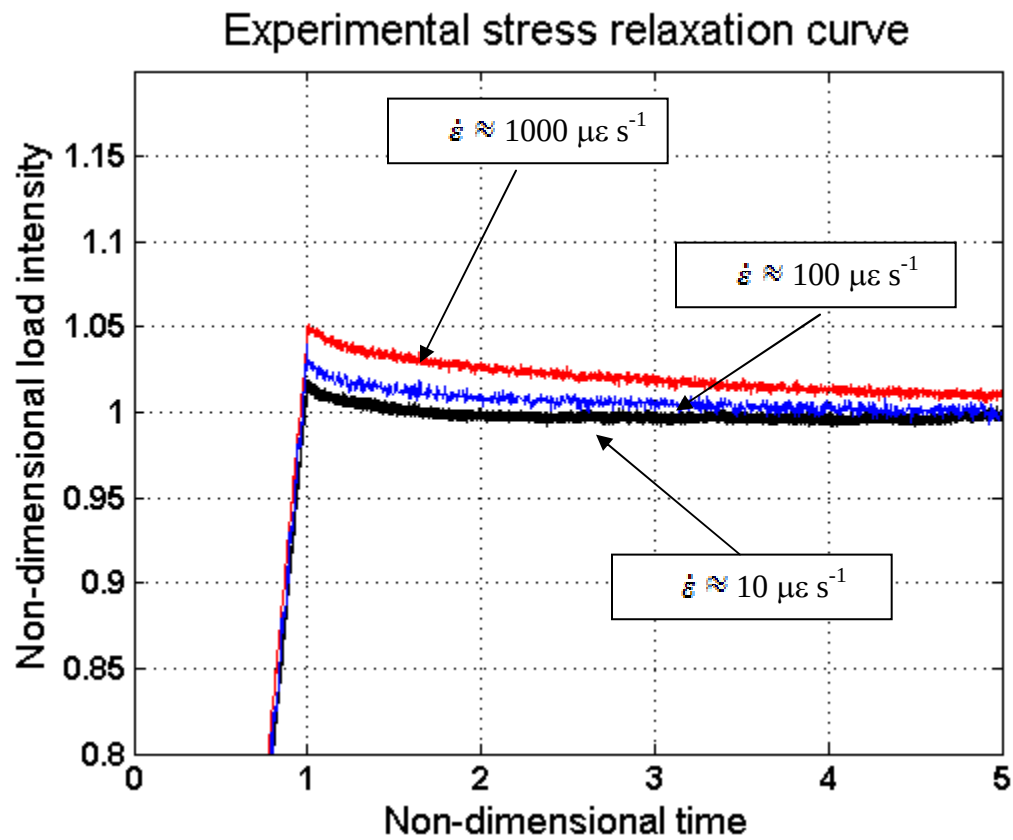


Figure 14: Stress relaxation response of a representative osteon loaded at three different strain rates, $\dot{\epsilon} \approx 10 \mu\epsilon s^{-1}$, $100 \mu\epsilon s^{-1}$, and $1000 \mu\epsilon s^{-1}$. The axes display the non-dimensional load intensity versus the non-dimensional time t/t_0 . Gailani et al., (2008)

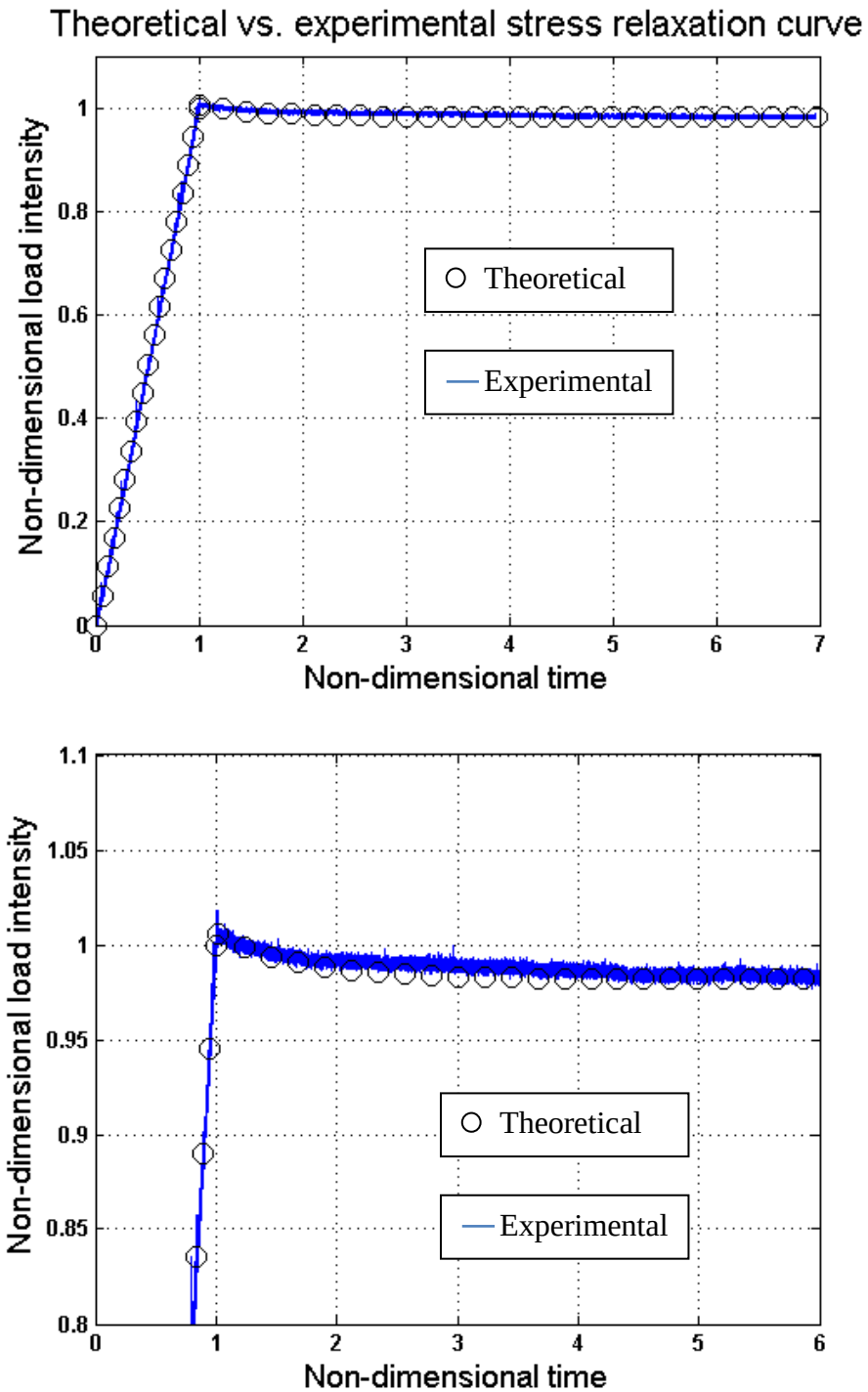


Figure 15: Curve fitting of the theoretical and experimental results of stress relaxation tests to estimate the intrinsic permeability K_{rr} . The curve fitting results have shown a good agreement in the loading period and some deviations in the relaxation period. Gailani et al., (2008)

3.8 Discussion

The analysis presented in this chapter showed clearly the complexity associated with the determination of the permeability of the PLC porosity. Part of the complexity arises from the size of the osteon. To address that problem, a new methodology to isolate the osteon is presented in that chapter. The method devised seems to be less complicated and more precise compared to the previous attempts by Ascenzi et al., (1959), (1967), (1985), (1994), Frasca et al., (1976), and Dong et al., (2005). Another component of the problem stems from the complexity of the analytical solution which requires the determination of both the matrix material properties and the drained elastic constants. While the matrix material properties can be determined by acoustic microscopy or nano-indentation (Ashman et al., 1984; Lang, 1970; Huja et al., 2006), the drained elastic properties require recording the deformation in the radial and longitudinal directions on a very small scale. However, in this analysis we used the values for both sets of elastic properties that were reported in the literature. Another approach that we examined was to determine the drained properties by curve fitting the analytical results to the experimental data. Since determination of the poles in Eq. (2) and (3) requires the knowledge *a priori* of the drained elastic properties, the simultaneous curve fitting of elastic constants and poles was not possible; therefore some of the drained elastic properties are determined by inspection, as shown in Smith et al., (2002). The estimated permeability value using the same values that are reported in the literature, Table 3, is in the range of $O(10^{-24} \text{ m}^2)$ when using a large number of poles, and $O(10^{-25} \text{ m}^2)$ when using only one pole and determining the drained elastic properties by inspection. The second term in Eq. (2) and (3) which contain the poles, has little effect on the curve fitting results and as the poles increase in ascending order the terms keeps decreasing, thus its influence on the results keeps decreasing as well. The reported

value for the intrinsic permeability is in a reasonable agreement with some of the previously reported values. Zhang et al., reported a value of $1.47 \times 10^{-20} \text{ m}^2$ based on their theoretical work. Smit et al. (2002) have determined the permeability of the osteons by inspection using a comparison of their numerical solution (FEA) to the experimental data reported by Otter et al. (1992); they reported a permeability value of $2.2 \times 10^{-22} \text{ m}^2$. More recently Oyen (2008) employed a nanoindentation technique combined with a poroelastic analysis to provide an estimate of the PLC permeability that ranged from $4.1 \times 10^{-24} \text{ m}^2$ in water to $O(10^{-26} \text{ m}^2)$ in polar solvents (methanol, ethanol, acetone). However Oyen employed an incompressible rather than a compressible poroelastic model to obtain the permeability, raising the possibility that her estimate may be smaller than the real value.

The deviation of the experimental approach from the analytical approach in this study arises from the fact that the experimental model cannot satisfy all the requirements of the analytical model. The Haversian canal is not centered in the middle of the osteon as assumed by the analytical solution and is not straight throughout the osteon either. In addition to that, it is extremely difficult to ensure that the top and bottom surfaces of the loaded specimens are flat; therefore some shear and bending may occur. The assumption of no friction between the plates and the osteon is too ideal in this scale and was not easy to satisfy. Also, there exist a natural variability in the mechanical properties of the several samples tested. In the unconfined compression test of articular cartilage, Armstrong et al. (1984) reported that a good agreement between the theoretical response and the experimental data could not be obtained. They attributed this to confinement effects due to surface adhesion at the tissue-platen interface. The porosity could not be determined in the present analysis by curve fitting, so we adopted the value reported in the literature. The adopted values have an effect on the results. A limitation

of the current approach is that the curve fitting was performed for only one parameter (the intrinsic permeability). A more sophisticated approach would consider the drained elastic parameters in the curve fitting algorithm.

4.

Russian Doll Poroelasticity

4.1 Introduction

This chapter presents a theory for the poroelasticity for porous materials with a special type of pore space structure. This particular pore space structure model is developed as a model for interstitial pore fluid flow in tissues like bone, tendon, meniscus and possibly other tissue types. The special type of pore space structure is described as “Russian Doll” because a particular pore size porosity can only drain its fluid into, and receive its fluid supply from, a nearest neighbor porosity, a “nearest neighbor” in terms of the pore size of its porosity. These different pore porosities are nested within the other pore size porosities like a nested set of Russian Dolls; the nesting or ordering criterion is the porosity pore size. A Russian nested doll or matryoshka doll is a set of dolls of decreasing sizes placed one inside another. The plural form matryoshki is used when referring to more than one doll. A set of matryoshki consists of a wooden figure that can be pulled apart to reveal another figure of the same sort inside. It has in turn another figure inside, and so on. The nested porosities are connected so the pore fluid may flow easily through each and across the boundaries between the two nearest neighbor porosities, but any particular pore size porosity may only interchange its pore fluid with the next larger pore size porosity and the next smaller pore size porosity.

The animal vascular tree is an example of a pore structure as two connected nested systems. In a microcirculatory bed blood flows from arteries to arterioles, then to capillaries and on to venules and into the veins; in each of these pore structures the pore size is

continuously variable, probably monotonically. The arterial system consists of the capillaries nested with the arterioles that are nested within the arteries. The venous system consists of the capillaries nested with the venules that are nested within the veins. The capillary plexuses of the two nested systems are connected. A special set of macroscopic balance laws were developed to model this situation by Huyghe and Van Campen (1995a,b). The pressure and flow were selectively averaged according to the prevailing hierarchical pore size structure using a formal averaging procedure developed by Whitaker (1967). This process yielded averaged equations of balance of momentum and balance of mass for the continuously variable pore size structure. The equations obtained were valid for an incompressible porous solid saturated with a continuous spectrum of intercommunicating incompressible fluids. The computational application of this model is described in Vankan et al. (1996; 1997a,b).

In modeling a nested poroelastic material the pore fluid pressure is distinct in each one of the nested porosities. The boundaries of a representative volume element (RVE) for a continuum model may be established for each of the nested pore size porosities. If the interstitial bone fluid flow is unidirectional in a nested system, as it is in the vascular system, the flow from pore size porosity into another is also unidirectional.

4.2 Description of the Russian doll model

Existing theories of the poroelasticity of materials with multiple connected porosities of different characteristic pore sizes and different permeabilities do not appear to cover the case of nested porosities of interest. Porosity is considered to be homogeneous in traditional treatments of poroelasticity. There are severe mathematical difficulties associated with the solution of problems involving inhomogeneous porosity. The developmental objective for

multiple pore size porosities models is to simplify the problem of modeling materials with inhomogeneous porosity.

Fractured rock appears to be the first material with multiple pore sizes to be modeled. Barenblatt et al. (1960) introduced the dual-porosity model, in which a fractured medium with two different pore size structures is represented by two completely overlapping continua, the pores of the matrix and the fractures. The distributions of the two pore sizes were considered to be well mixed in the continuum model so the pore fluid moved back and forth easily between the two pore size porosities. The double-porosity model was further developed by Warren and Root (1963) to represent the naturally fractured reservoir as an idealized system formed by identical rock-matrix domains, separated by fractures. A multiporosity/multipermeability model was proposed by Aifantis (1980) and Abdassah and Ershaghi (1986) to represent a severely fractured reservoir with either moderate permeability (by a triple-porosity/dual-permeability model) or high permeability (by a triple-porosity/triple-permeability model).

The fractured rock models employ multiple pore size porosities that are mixed together and interconnected in the same domain, which is, indeed, a property of the fractured rock being modeled. The multiple pore size porosities are unsegregated in that all pore size porosities are mixed and intertwined and not nested in any sense. In particular it is not possible to define the boundaries of the different pore size porosities in order to establish representative volume elements (RVE) for the pore size porosities in continuum models of fractured rocks. Only one large RVE containing a proportional volume of each of the pore size porosities is possible. Due to the mixture and multiple connectivity of pore size porosities in these fractured rock models, unidirectional pore fluid flow may occur along paths from one pore

size porosity into another many times and it is not “unidirectional” in the sense of all flow being one pore size porosity into another as it is in the nested vascular system; it may be “in and out” and “in and out” of one pore size porosity into another.

The flow of interstitial fluid in tissues like bone, tendon, meniscus and possibly other tissue types is similar to blood flow in the vascular system in the sense that the different pore size porosities are nested, but unsimilar in three important aspects; (1) there are only two levels of pore size porosity, (2) there is no flow out of, or in from, the smaller pore size porosity into any even smaller pore size porosity, only into, or in from, a larger pore size porosity and (3) the fluid flow direction reverses in the normal physiological function of these tissues.

The pore sizes in cortical bone are approximately three discrete sizes; the largest pore size (approximately 50 μm diameter) is associated with the vascular porosity (PV), the second largest pore size (approximately 0.3 μm diameter) with the canaliculi in the lacunar-canalicular porosity (PLC) and the smallest pore size (approximately 10 nm diameter) is in collagen-apatite porosity (PCA). The typical pore size of the vascular porosity channels is not the blood vessel pore size; rather it is the size of the tubular tunnels (Haversian systems or osteons and Volkmann canals) containing the blood vessels, the arterioles and the venules, and considerable extra pore fluid space. The interstitial pore fluid in the collagen-apatite porosity (PCA) is considered to be bound to the solid structure and the PCA is not of interest in the present considerations. Thus, the two pore size structures of interest in the flow of bone pore fluid are the vascular porosity (PV), and the lacunar-canalicular porosity (PLC); the ratio of the pore sizes is approximately 167.

The pore fluid pressure in these two pore size porosities is also quite different. Under physiologically possible rapid rise-time loadings of bone, the pore fluid pressure may rise considerably in the PLC (Zhang et al., 1998). The decay time for this pore pressure rise is much larger in the PLC than it is in the PV. The PV is a low-pore-fluid-pressure domain because the PV permeability is sufficiently large to remit a rapid decay of a pressure pulse. This must be the case because the PV contains thin walled blood vessels carrying blood with a pressure of 40 to 60 mm of Hg.

4.3 Bone tissue porosity and permeability

The measurement of the permeability of the PV has not been accomplished with sufficient accuracy to date, primarily because of the topological intertwining of the PV with the PLC. This difficulty and others were discussed in some detail by Johnson (1984) who mentioned experiments by Rouhana et al. (1981) that reported bovine cortical bone permeabilities on the order of 10^{-14} m^2 . Johnson (1984) presented factors to suggest that the actual permeability could be much larger than that value. The PV permeabilities reported in the past are thought to be significant underestimates because these previous values represent a PLC and PV lumped measurement rather than a PV measurement alone. These lumped measurements also compromise the measurement of the permeability of the PLC as pointed out by Beno et al. 2006 in the discussion of the reported PLC permeabilities (Dillaman et al., 1991; Steck, et al., 2003). There are four reported estimates of the permeability of the PLC that were discussed in chapters 1 and 3.

The pore fluid pressures in these two pore size bone porosities are distinct and vary very differently with time under mechanical loading of the whole bone. Under physiologically

possible rapid rise-time loadings of bone, the pore fluid pressure may rise considerably in the PLC (Zhang et al., 1998). The decay time for this pore pressure rise is much larger in the PLC than it is in the PV (Johnson et al., 1982; Johnson, 1984). The PV is a low-pore-fluid-pressure domain because the PV permeability is sufficiently large to permit a rapid decay of a pressure pulse. This must be the case because the PV contains thin-walled blood vessels carrying blood with a pressure of 40 to 60 mm of Hg (5.3 kPa to 8 kPa); a pore-fluid-pressure significantly greater than 40 to 60 mm of Hg will collapse these blood vessels and a prolonged increase in the pore-fluid-pressure significantly above 40 to 60 mm of Hg would deprive the tissue of oxygen and nutrients. The thin capillary walls of the endothelium, the absence of a muscle layer, and the sparse basement membrane lower the collapse pressure of blood vessel below that of a vessel in soft tissue.

These considerations suggest that that the PV and PLC function almost independently, the prime mechanical influences for the two porosities being very different as are the time scales of their response. The mechanical loading of the whole bone moves the bone fluid in the PLC. When the bone is compressed the bone fluid is driven from the PLC into the low-pressure PV and when the bone is in tension, or the compression is reduced, the bone fluid is sucked from the PV into the PLC. These drainage and imbibing processes occur on a pressure time scale that is much larger than the short pressure adjustment relaxation time for the PV and therefore have minimum influence on the pressure in the PV. The change in interstitial pore fluid pressure in the PV due to inflow or outflow of bone fluid from the PLC is insignificant because of the time period of pressure adjustment that is much shorter (estimates of these time periods are in Zhang et al., 1998) than the pressure adjustment time period for the PLC. While the bone fluid in the PLC is significantly affected for a longer time by the

mechanical loading of the whole bone, the bone fluid pressure in the PV is hardly affected because the PV relaxes the pressure pulse by diffusion so rapidly.

4.4 One poroelastic porosity level in a hierarchy of poroelastic porosity levels

There are two objectives of this section. One is to present the model of a hierarchy of poroelastic porosity levels and the other is to summarize the governing equations of saturated, compressible, poroelasticity in the form that they will be employed in this paper. A porous material containing a hierarchical system of uniform porosity levels is considered. One typical porosity level is then focused upon. This porosity level is denoted by P , and $P + 1$ represents the next porosity level with the larger uniform pore size and the porosity level with the next uniform smaller pore size is denoted by $P - 1$. Since the pore size and, we assume, the total pore volume is different at the different porosity levels it follows that the permeability will be different at the different porosity levels as well as the elastic constants associated with the different porosity levels. The lowest uniform porosity level occurs for the matrix material level in which the porosity is zero; this is denoted by $P_0 = 0$ or P_m where m indicates the matrix material. The elastic constants at the matrix material level, P_m , will be greater than those at the $P_1 = P + 1$ level because the elastic constants at the P_1 level will be compromised by the porosity not existent at the lower level. It follows then that the elastic constants at each higher level will be further compromised because at each higher level the material contains an additional level of porosity. Thus, while the elastic constants at each higher level are reduced, the total permeability is increased at each level. Traditionally the material at this “no porosity” level has been called the matrix material; however in the hierarchical scheme of porosity levels the porosity level below the porosity level under consideration plays the role of the

“material” for the particular porosity level under consideration. Thus, only in the case of the first level of porosity above the “no porosity” level do the terminologies of matrix material level and “no porosity” level coincide.

In this section the basic equations of poroelasticity are recorded for a continuum with a P level porosity admitting the possibility of fluid exchange (sources and /or sinks) from the larger uniform pore size porosity continuum at level $P + 1$ and the next uniform smaller pore size porosity level continuum at $P - 1$. Thus a separate continuum model represents each level of porosity P . The fluid transport connection between the different continuum model porosity levels occurs through sources (sinks) connecting two different porosity levels. In this section the sources (sinks) are treated in a general way because their exact nature will vary with the particular porous material being modeled. In subsequent sections addressing bone tissue specifically, examples of the porosity level connections will be made specific.

The traditional theory of poroelasticity (Biot 1941, 1955; Rice and Cleary 1976; Thompson and Willis 1991; Cowin and Mehrabadi, 2007) is recovered from this hierarchical system of uniform porosity levels if the possibility of fluid exchange between porosity levels is excluded and the generic P level is taken to be $P_1 = P + 1$ level porosity with the $P - 1$ level taken to be the matrix material level $P_0 = 0$ or P_m ; that is to say the structure proposed is a generalization of traditional poroelasticity to a hierarchical system of uniform porosity levels that allows fluid exchange between the different levels. Each porosity level is a separate continuum level model that interacts with the other continuum porosity level models by sources and sinks that transfer fluid between the different levels.

In the remainder of this section the equations governing the generic P level porosity are recorded; they are equivalent to the traditional equations of poroelasticity except for the fluid

supply terms from the two next nearest porosity levels. There are three sets of poroelastic constitutive equations, the stress-strain-pore pressure, the fluid content-stress-pore pressure and Darcy's law. The purpose of this section is now to record the form of these constitutive equations free from the incompressibility constraint on any constituent. In the following four subsections, these equations are described in the order indicated above for the constitutive equations and followed by a subsection on the governing equations.

4.4.1 Stress-strain-pore pressure relations

The displacement vector is denoted by $\mathbf{u}$ and second order tensor representation in 3D of the strain tensor is $\mathbf{E}$. The strain-displacement relations in three dimensions are written in the form

$$\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^T + \nabla \otimes \mathbf{u}). \quad (4-1)$$

The symbol $\hat{\mathbf{E}}$ is used to represent the second order strain tensor $\mathbf{E}$ as a vector in 6D,

$$\hat{\mathbf{E}} = [\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4, \hat{E}_5, \hat{E}_6]^T = [E_{11}, E_{22}, E_{33}, \sqrt{2}E_{23}, \sqrt{2}E_{13}, \sqrt{2}E_{12}]^T.$$

The stress-strain-pore pressure constitutive relation for a saturated porous medium is that the strain $\hat{\mathbf{E}}$ in the saturated porous medium is linearly related, not only to the stress $\hat{\mathbf{T}}$, but also to the fluid pressure p in the fluid-filled pores, thus one can write the strain-stress-pore pressure relation

$$\hat{\mathbf{E}} = \hat{\mathbf{S}}^{d,P} \cdot \hat{\mathbf{T}} + \hat{\mathbf{S}}^{d,P} \cdot \hat{\mathbf{A}}^P p, \quad (4-2)$$

or the stress-pore pressure- strain relation

$$\hat{\mathbf{T}} + \hat{\mathbf{A}}^P p = \hat{\mathbf{C}}^{d,P} \cdot \hat{\mathbf{E}}, \quad (4-3)$$

where $\hat{\mathbf{S}}^{d,P}$ represents the drained anisotropic compliance elastic constants of the saturated porous medium at the porosity level P and $\hat{\mathbf{C}}^{d,P}$ is its reciprocal, the drained anisotropic elasticity tensor. They are called the drained elastic constants because they are measured in

situations in which the fluid pressure p^P in the fluid-filled pores is negligible or zero. Draining all the pores before the test or doing the test so slowly causes the pores to drain from a negligible fluid pressure. In a porous medium the pores are assumed to be connected; there are no unconnected pores that prevent the flow of fluid through them. The six-dimensional vector $\hat{\mathbf{A}}^P$ (three-dimensional symmetric second rank tensor $\mathbf{A}^P$), introduced by (2), is called the *Biot effective stress coefficient vector* (6D) or *tensor* (3D) at the porosity level P . The Biot effective stress coefficient vector $\hat{\mathbf{A}}^P$ is related to the difference between effective drained elastic constants $\hat{\mathbf{S}}^{d,P}$ and the elastic compliance tensor at the $P - 1$ porosity level, $\hat{\mathbf{S}}^{d,P-1}$, by the formula (Nur and Byerlee, 1971; Carroll, 1979; Cowin and Mehrabadi, 2007; Cowin and Doty, 2007)

$$\hat{\mathbf{A}}^P = (\hat{\mathbf{I}} - \hat{\mathbf{C}}^{d,P} \cdot \hat{\mathbf{S}}^{d,P-1}) \cdot \hat{\mathbf{U}}. \quad (4-4)$$

The Biot effective stress coefficient vector $\hat{\mathbf{A}}^P$ is so named because it is employed in the definition of the *effective stress* $\hat{\mathbf{T}}^{eff}$:

$$\hat{\mathbf{T}}^{eff} = \hat{\mathbf{T}} + \hat{\mathbf{A}}^P p^P. \quad (4-5)$$

This definition of effective stress reduces the stress-strain-pressure relation (4-2) to the same form as Hooke's law, thus

$$\hat{\mathbf{E}} = \hat{\mathbf{S}}^{d,P} \cdot \hat{\mathbf{T}}^{eff}. \quad (4-6)$$

This constitutive equation is a modification of Hooke's law to include the effect of the pore pressure. When $p^P = 0$ the stress-strain-pore pressure relations (4-2) and (4-3) or (4-5) and (4-6) coincide with anisotropic Hooke's law. The advantage of the representation (4-6) is that the fluid-saturated porous material may be thought of as an ordinary elastic material with a compliance matrix $\hat{\mathbf{S}}^{d,P}$, but one subjected to the "effective stress" $\hat{\mathbf{T}}^{eff}$ rather than (ordinary) stress $\hat{\mathbf{T}}$. The drained elastic compliance tensor $\hat{\mathbf{S}}^{d,P}$ may be evaluated from knowledge of the

pore structure of the medium and the elastic compliance tensor $\hat{\mathbf{S}}^{d,P-1}$, or vice versa, using composite or effective medium theory. At the lowest level of porosity the elastic compliance tensor $\hat{\mathbf{S}}^{d,P-1}$ will coincide with the elastic compliance tensor of the (pore free) matrix material $\hat{\mathbf{S}}^m$.

4.4.2 Fluid content-stress-pore pressure relations

The fluid content-stress-pore pressure constitutive relation involves all the basic field variables for poroelasticity, the total stress $\hat{\mathbf{T}}$, the pore pressure p^P , the strain in the solid matrix $\hat{\mathbf{E}}$, and the variation in (dimensionless) fluid content ζ^P . The variation in fluid content ζ^P is the variation of the fluid volume per unit volume of the porous material due to diffusive fluid mass transport; it is defined as the difference between the strain of the pore space and the strain of the fluid volume in the pore space. The variation in fluid content ζ^P is equal to the product of the porosity ϕ^P at the level P and the fluid density ρ_f , divided by a reference value of the fluid density ρ_{f0} , thus $\zeta^P = \phi^P \rho_f / \rho_{f0}$. The variation in fluid content ζ^P depends not only upon the strain $\hat{\mathbf{E}}$ in the solid matrix, but also upon the strain induced by the pore pressure p^P , thus

$$\zeta^P = \hat{\mathbf{A}}^P \cdot \hat{\mathbf{E}} + \Lambda^P p^P, \quad (4-7)$$

where

$$\Lambda^P = \hat{\mathbf{U}} \cdot \hat{\mathbf{S}}^{P-1} \cdot \hat{\mathbf{A}}^P + \phi^P \left(\frac{1}{K^f} - \frac{1}{K_{Reff}^{P-1}} \right), \quad (4-8)$$

and where ϕ is the porosity at the level P, K^f is the bulk modulus of the fluid and K_{Reff}^{P-1} is the Reuss effective bulk modulus of an anisotropic elastic material (Cowin and Mehrabadi, 2007),

in this case the porous material at the $P-1$ level of porosity. In the case of isotropy this expression for Λ^P reduces to

$$\Lambda^P = \frac{1}{K_{Reff}^{P-1}} \left(1 - \frac{K_{Reff}^P}{K_{Reff}^{P-1}} \right) + \phi \left(\frac{1}{K^f} - \frac{1}{K_{Reff}^{P-1}} \right). \quad (4-9)$$

The variation in fluid content ζ^P is also linearly related to both the stress $\hat{\mathbf{T}}$ and the pore pressure p^P ,

$$\zeta^P = \hat{\mathbf{A}} \cdot \hat{\mathbf{S}}^{d,P} \cdot \hat{\mathbf{T}} + C_{eff}^{d,P} p^P \quad (4-10)$$

where

$$C_{eff}^{d,P} = \frac{1}{K_{Reff}^P} - \frac{1}{K_{Reff}^{P-1}} + \phi^P \left(\frac{1}{K^f} - \frac{1}{K_{Reff}^{P-1}} \right). \quad (4-11)$$

The relation between $C_{eff}^{d,P}$ and Λ^P is

$$C_{eff}^{d,P} = \Lambda^P - \hat{\mathbf{U}} \cdot \hat{\mathbf{S}}^{d,P-1} \cdot \hat{\mathbf{A}}^P + \left(\frac{1}{K_{Reff}^P} - \frac{1}{K_{Reff}^{P-1}} \right). \quad (4-12)$$

4.4.3 Mass and momentum conservation

The conservation of mass in the form expressed by the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (4-13)$$

The form of the mass conservation equation (15) is altered to apply to the P level pore fluid volume by first replacing ρ by $\phi^P \rho_f$ in (13) and then dividing the equation through by ρ_{fo} , thus

$$\frac{1}{\rho_{fo}} \frac{\partial \phi^P \rho_f}{\partial t} + \frac{1}{\rho_{fo}} \nabla \cdot (\phi^P \rho_f \mathbf{v}) = 0. \quad (4-14)$$

This relationship is easily modified to account for fluid influxes or out-fluxes from the larger uniform pore size porosity level $P + 1$ and the next uniform smaller pore size porosity level $P - 1$, thus

$$\frac{1}{\rho_{fo}} \frac{\partial \phi^P \rho_f}{\partial t} + \frac{1}{\rho_{fo}} \nabla \cdot (\phi^P \rho_f \mathbf{v}) = \frac{\phi^P \gamma^{P-1}}{\rho_{fo}} - \frac{\phi^P \gamma^{P+1}}{\rho_{fo}}, \quad (4-15)$$

where the supply terms from the next nearest neighbor porosities are denoted by γ^{P-1} and γ^{P+1} . These supply terms have the dimensions of density over time and represent the fluxes of pore fluids between porosity levels. The form of (4-15) is changed by noting that the variation in fluid content ζ^P is given by $\zeta^P = \frac{\phi^P \rho_f}{\rho_{fo}}$ and the use of (4-14), thus

$$\frac{\partial \zeta^P}{\partial t} + \nabla \cdot \mathbf{q} = \frac{\phi^P \gamma^{P-1}}{\rho_{fo}} - \frac{\phi^P \gamma^{P+1}}{\rho_{fo}}. \quad (4-16)$$

The stress equations of motion in three dimensions at porosity level P ,

$$\rho^P \mathbf{a}^P = \nabla \cdot \mathbf{T} + \rho^P \mathbf{b}, \quad \mathbf{T} = \mathbf{T}^T, \quad (4-17)$$

have no simple representation in the 6D vector notation, and the conventional notation is employed; $\mathbf{a}^P$ represents the acceleration and $\mathbf{b}$ the body force.

4.4.4 Governing differential equations

There are many methods of approach to the solution of poroelastic problems for compressible media. The method selected depends upon the information that is provided and the fields that are to be calculated. One approach that has been effective is to solve for the variation in fluid content ζ^P if the stress or the strain field is known or may be calculated without reference to the variation in fluid content ζ^P . The diffusion equation for the variation

in volume fraction is obtained by first substituting Darcy's law into the expression (4-16) for the conservation of mass and subsequently eliminating the pore pressure by use of (4-7), thus

$$\frac{\partial \zeta^P}{\partial t} - \frac{1}{\mu \Lambda^P} \hat{\mathbf{K}}^P \cdot \hat{\mathbf{O}}[\zeta^P] = -\frac{1}{\mu \Lambda^P} \hat{\mathbf{K}}^P \cdot \hat{\mathbf{O}}[\hat{\mathbf{A}}^P \cdot \hat{\mathbf{E}}] + \frac{\phi^P \gamma^{P-1}}{\rho_{fo}} - \frac{\phi^P \gamma^{P+1}}{\rho_{fo}} \quad (4-18)$$

This shows that the time rate of change of the fluid content ζ^P is due either to fluid flux or to volume changes that are caused by the strain field. It is possible to replace the strain on the right hand side of (4-18) by stress in which case (18) becomes

$$\frac{\partial \zeta^P}{\partial t} - \frac{1}{\mu C_{eff}^{dP}} \hat{\mathbf{K}}^P \cdot \hat{\mathbf{O}}[\zeta^P] = -\frac{1}{\mu C_{eff}^{dP}} \hat{\mathbf{K}}^P \cdot \hat{\mathbf{O}}[\hat{\mathbf{A}}^P \cdot \hat{\mathbf{S}}^{dP} \cdot \hat{\mathbf{T}}] + \frac{\phi^P \gamma^{P-1}}{\rho_{fo}} - \frac{\phi^P \gamma^{P+1}}{\rho_{fo}} \quad (4-19)$$

where C_{eff}^{dP} and Λ^P are related by (4-10).

Diffusion equations for the pressure field are also employed in the solution of poroelastic problems. The first diffusion equation for the pore pressure field is obtained by first substituting Darcy's law into the expression (16) for the conservation of mass and subsequently eliminating the variation in fluid content ζ^P by use of (4-10), thus

$$\frac{\partial p^P}{\partial t} - \frac{1}{\mu \Lambda^P} \hat{\mathbf{K}}^P \cdot \hat{\mathbf{O}}[p^P] = -\frac{1}{\Lambda^P} \hat{\mathbf{A}}^P \cdot \frac{\partial \hat{\mathbf{E}}}{\partial t} + \frac{\phi^P \gamma^{P-1}}{\Lambda^P \rho_{fo}} - \frac{\phi^P \gamma^{P+1}}{\Lambda^P \rho_{fo}} \quad (4-20)$$

The alternative diffusion equation for the pore pressure field is obtained by replacing the strain on the right hand side of (20) by stress, thus

$$\frac{\partial p^P}{\partial t} - \frac{1}{\mu C_{eff}^{dP}} \hat{\mathbf{K}}^P \cdot \hat{\mathbf{O}}[p^P] = \frac{1}{C_{eff}^{dP}} (\hat{\mathbf{U}} \cdot (\hat{\mathbf{S}}^{P-1} - \hat{\mathbf{S}}^P) \cdot \frac{\partial \hat{\mathbf{T}}}{\partial t}) + \frac{\phi^P \gamma^{P-1}}{C_{eff}^{dP} \rho_{fo}} - \frac{\phi^P \gamma^{P+1}}{C_{eff}^{dP} \rho_{fo}} \quad (4-21)$$

The boundary conditions on the pore pressure field customarily employed in the solution of this differential equation are (i) that the external pore pressure p^P is specified at the boundary (a lower pressure permits flow across the boundary), (ii) that the pressure gradient ∇p^P at the

boundary is specified (a zero pressure gradient permits no flow across the boundary), (iii) that some linear combination of (i) and (ii) is specified.

4.5 The poroelastic model of the osteon or PLC

Using the hierarchical scheme described in the previous section a model is formulated in this section for the transport of bone interstitial fluid between the PV and PLC porosity levels in osteonal cortical bone. A section of this bone is illustrated in Figure 16. In this paper the PV and the PLC will both be modeled as poroelastic hollow circular cylinders. The poroelastic hollow circular cylinder model of the PLC connects through its inner cylindrical wall to the PV; the hollow part of this cylinder is actually part of the PV. The inner surface of the cylinder representing the PLC is the surface across which the two porosities exchange pore fluids. The PLC is assumed to permit flow across its inner radial boundary, but not its outer radial boundary. While other assumptions are possible, an earlier study (Wang, 1999) showed that this is a reasonable assumption. The PV is assumed to permit flow across both of its radial boundaries. The PLC hollow cylinder is the osteon of Figure 1. The PV hollow cylinder is the entire bone of Figure 1 with central lumen of the whole bone, the medullary canal, constituting the hollow part of the PV model. Both of these models are continuum models and the transport connection between the two continua is the outflow-influx that occurs across the osteonal or Volkmann inner wall between the PLC and the PV (Fritton et al., 2001). In the domain between the inner surface and outer surface of the PV cylinder there are areal or area sources-sinks that permit interchange of fluid between the two continuum models representing the PV and PLC.

The walls of the osteonal canals are covered with cells and behind the cells are the entrances to the canaliculi. The canaliculi are passageways that run between lacunae or from the lacunae to the osteonal canal (Figure 16). Weinbaum *et al.* (1994) and Cowin *et al.* (1995) presented a detailed theoretical model of the contents of the PLC. The PLC is the porosity associated with the osteocytes; the osteocyte is the prime candidate for the mechanosensory cell in bone. Weinbaum *et al.* (1994) and Cowin *et al.* (1995) suggest that the region between the cell membrane of the osteocyte (and the osteocytic process) on one hand and the walls of the canaliculi and lacunae on the other, is filled with the cell glycocalyx (also called the surface matrix or capsule). The content of this PLC, the combination of the glycocalyx and the bone fluid, is thought to function as a gel. This model has been further developed in a number of papers, e.g., Han *et al.*, (2004).

The cross-section of an osteon containing the PLC is an annular region bounded by two roughly circular structures, the cement line on the outside and the wall of the osteonal canal on the inside. These boundaries are the two small concentric circles in Figure 16. The volume of the osteon containing the PLC is an approximately circular cylinder with an annular cross-section. The model of the osteon employed here is this volumetric PLC domain. The boundary of this domain formed by the wall of the osteonal canal is assumed to be porous and the boundary formed by the cement line is assumed to be impermeable. This assumption is in conflict with the fact that canaliculi crossing the cement line have been observed. There are three rationales for making this assumption; (1) it is not a difficult assumption to remove if necessary, (2) the number of canaliculi crossing is unknown and (3) an earlier study (Wang, *et al.*, 1999) concluded that it made little difference, if the boundary was assume to be impermeable or fully permeable, in the global predictions of physiological significance.

The pressure distribution in the PLC can be rewritten from Cowin and Mehrabadi (2007), Gailani and Cowin (2008), Eq.(4-21) or Eq.(2-27) as

$$\frac{\partial^2 p^L}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial p^L}{\partial \lambda} - \frac{b^2}{c^L} \frac{\partial p^L}{\partial t} = \frac{\mu b^2}{k_{rr}} \left(\hat{A}_z \frac{\partial \varepsilon}{\partial t} + \hat{A}_r \frac{\partial f(t)}{\partial t} \right), \quad (4-22)$$

where p^L is the pressure distribution in the PLC, b is the outer radius of the PLC, and λ is the non dimensional radius of the osteon. The rest of the terms are defined in chapter 2.(Eq. (2-29)).

Three boundary conditions are considered ,

$$(i) p^L(\lambda, 0) = 0, (ii) p^L(a, t) = p^V(\beta, t), (iii) \frac{\partial p^L(1, t)}{\partial \lambda} = 0, \quad (4-23)$$

Where $p^V(\beta, t)$ is the pressure distribution in the PV , and β is the non dimensional radius of the PV. The equation above is a nonhomogenous partial differential equation with non homogenous boundary conditions, thus a solution in the form

$$p^L(\lambda, t) = u(\lambda, t) + T(t) \quad (4-24)$$

can be assumed. Splitting the boundary and initial conditions, a solution for $T(t)$ can be obtained, thus

$$T(t) = -\frac{\mu c^L}{k_{rr}} \left(\hat{A}_z \varepsilon(t) + \hat{A}_r f(t) - \left(\hat{A}_z \varepsilon(0) + \hat{A}_r f(0) \right) \right) \quad (4-25).$$

Notice that the boundary and initial conditions for $u(\lambda, t)$ will become

$$u(\lambda, 0) = 0, u(a, t) = p^V(\beta, t) - T(t) = S(t), \frac{\partial u(1, t)}{\partial \lambda} = 0. \quad (4-26)$$

The function $u(\lambda, t)$ satisfies the homogenous equation:

$$\frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial u}{\partial \lambda} - \frac{b^2}{c^L} \frac{\partial u}{\partial t} = 0 \quad (4-27)$$

This problem contains a nonhomogenous boundary condition, therefore Duhamel's integral

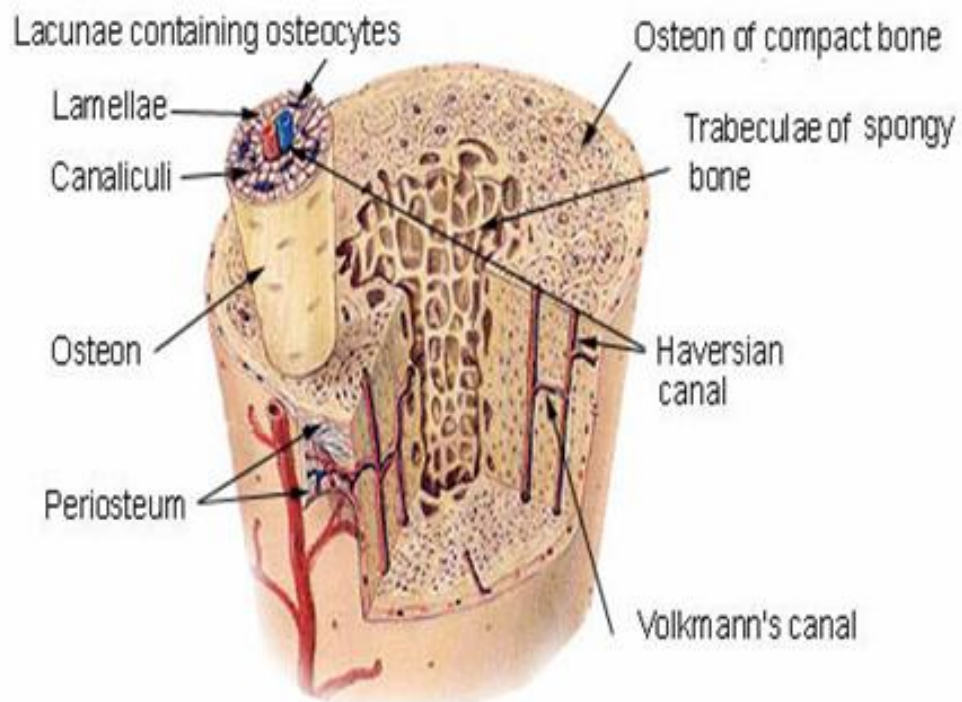


Figure (16): the PV and PLC models for fluid flow. The medullary canal can be idealized to run in the center of the bone.

can be utilized, Özisik (1980), where an auxiliary function $\Phi(\lambda, t)$ will satisfy the homogenous equation above such that

$$\frac{\partial^2 \Phi}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial \Phi}{\partial \lambda} - \frac{b^2}{c^L} \frac{\partial \Phi}{\partial t} = 0 \quad (4-28)$$

but with a new set of boundary conditions

$$\Phi(\lambda, 0) = 0, \Phi(a, t) = 1, \frac{\partial \Phi(1, t)}{\partial \lambda} = 0. \quad (4-29)$$

Then the solution of the problem (4-27) can be written in terms of the solution of the auxiliary problem (4-28) by Duhamel's

$$u(\lambda, t) = \int_{\tau=0}^t S(\tau) \frac{\partial \Phi(\lambda, t-\tau)}{\partial t} d\tau \quad (4-30)$$

Eq.(4-28) can easily be solved by the separation of variables (Sturm-Liouville problem) but after assuming another function $X(\lambda, t)$ where initial pressure is set to unity, and for times $t > 0$ the inner surface is kept at zero pressure; the solution becomes

$$\Phi(\lambda, t) = 1 - X(\lambda, t) = 1 - \sum_{n=0}^{\infty} C_n R_0(k_n \lambda) e^{-k_n^2 c^L t / b^2}, \quad (4-31)$$

Where R_0 is the space variable function that satisfy the Eigen value problem in Eq.(4-28) and can be expressed as

$$R_0(k_n \lambda) = J_0(k_n \lambda) Y_1(k_n) - J_1(k_n) Y_0(k_n \lambda), \quad (4-32)$$

k_n are the eigenvalues and can be obtained from

$$J_0(k_n a) Y_1(k_n) - J_1(k_n) Y_0(k_n a) = 0, \quad (4-33)$$

and C_n is resulting from multiplication of two constants and can be expressed in the form:

$$C_n = \frac{1}{N(k_n)} \int_a^1 \lambda' R_0(k_n \lambda') d\lambda' = \frac{a}{k_n N(k_n)} \{ J_1(k_n) Y_1(k_n a) - J_1(k_n a) Y_1(k_n) \}, \quad (4-34)$$

where the orthogonality can be determined using case 8 in Table 3-2 in Özisik (1980)

$$N(k_n) = \int_a^1 \lambda R_0^2(k_n \lambda) d\lambda = \frac{2}{\pi^2} \frac{J_0^2(k_n a) - J_1^2(k_n)}{k_n^2 J_0^2(k_n a)}. \quad (4-35)$$

Now Eq.(4-31) can be substituted into Eq.(4-30), thus

$$u(\lambda, t) = \frac{k_n^2 c^L}{b^2} \sum_{n=0}^{\infty} C_n R_0(k_n \lambda) e^{-k_n^2 c^L t / b^2} \int_{\tau=0}^t S(\tau) e^{k_n^2 c^L \tau / b^2} d\tau. \quad (4-36)$$

The integral part in Eq.(4-36) can be determined by integration by parts such that

$$I_n = \frac{b^2}{k_n^2 c^L} \left\{ S(t) e^{k_n^2 c^L t / b^2} - S(0) - \int_{\tau=0}^t e^{k_n^2 c^L \tau / b^2} d(S(\tau)) \right\}, \quad (4-37)$$

which can be substituted into Eqs.(4-36) and (4-24), thus

$$p(\lambda, t) = \frac{b^2}{k_n^2 c^L} \sum_{n=0}^{\infty} C_n R_0(k_n \lambda) e^{-k_n^2 c^L t / b^2} I_n + T(t) \quad (4-38)$$

To obtain an expression for $f(t)$ in Eq.(4-25) a condition arises from the fact that the radial stresses are zero on the boundary that allows fluid flow, Eq.(2-35), is considered, thus

$$\mathcal{Y}_{rr}^0 = 0 = -\hat{A}_r^L p^L(a, t) + \hat{C}_{rr}^{dL} \frac{\partial u}{b \partial \lambda} + \hat{C}_{r\theta}^{dL} \frac{u}{b \lambda} + \hat{C}_{rz}^{dL} \varepsilon(t). \quad (4-39)$$

The displacement u above can be obtained from Eq.(2-22) in terms of the pressure, so one can obtain after some substitutions a reasonable expression for the derivative of $f(t)$

$$\frac{df(t)}{d(t)} = B^L \left(\hat{A}_r^L F^L \frac{dp^V(\beta, t)}{d(t)} + 2 \hat{C}_{rz}^{dL} \frac{d\varepsilon(t)}{dt} \right), \quad (4-40)$$

where

$$B^L = -\frac{1}{\left\{ \hat{C}_{r\theta}^{dL} + \hat{C}_{rr}^{dL} \right\}}, \text{ and } F^L = \hat{A}_r^L \left\{ \frac{\hat{C}_{r\theta}^{dL}}{\hat{C}_{rr}^{dL}} - 1 \right\}. \quad (4-41)$$

Thus Eq.(4-38) can be rewritten by substituting Eqs.(4-40) and (4-24)

$$p^L(\lambda, t) = p^V(\beta, t) - N^L \varepsilon(t) + \sum_{n=0}^{\infty} C_n R_0(\lambda k_n) \frac{k_n^2 c^L}{b^2} e^{-k_n^2 c^L t / b^2} \left\{ M^L \int_0^t e^{k_n^2 c^L \tau / b^2} \frac{\partial p^V(\tau)}{\partial \tau} d\tau + N^L \int_0^t e^{k_n^2 c^L \tau / b^2} \varepsilon(\tau) d\tau \right\}, \quad (4-42)$$

where

$$M^L = \left[1 + \frac{\mu c^L}{k_{rr}} \hat{A}_r^L B^L F^L \right], \text{ and } N^L = \frac{\mu c^L}{k_{rr}} \left[\hat{A}_z^L + 2 \hat{A}_r^L B^L \hat{C}_{rz}^{dL} \right] \quad (4-43)$$

Eq.(4-42) can be specialized to the rising period in a ramp loading, where $\varepsilon(t) = t$ and this will give a new expression for the PLC pressure which can be simplified to

$$p^L(\lambda, t) = p^V(\beta, t) + \sum_{n=0}^{\infty} C_n R_0(\lambda k_n) \frac{b^2 (e^{-k_n^2 c^L t / b^2} - 1) N^L}{k_n^2 c^L} - \left\{ M^L \sum_{n=0}^{\infty} C_n R_0(\lambda k_n) e^{-k_n^2 c^L t / b^2} \int_0^t e^{k_n^2 c^L \tau / b^2} \frac{\partial p^V(\beta, \tau)}{\partial \tau} d\tau \right\}, \quad (4-44)$$

which gives an expression for the pore pressure distribution in the PLC when subjected to a rising load in a stress relaxation test.

4.6 The fluid exchange between the PLC and the PV

The outflow /influx between the PLC level and the PV level will be calculated directly from the pore fluid pressure gradient at the osteonal wall of the PLC level. The pressure field is then given by (4-44) above. The flux Γ from the osteon is given by

$$\Gamma = \int_{\text{luminal surface}} \mathbf{q} \cdot d\mathbf{a} = \int_{\text{luminal surface}} \left(-\frac{1}{\mu} k_{rr,L} \left(\frac{\partial p}{\partial r} \right)_{\text{osteonal lumin wall}} \right) \cdot d\mathbf{a}, \quad (4-45)$$

and have dimensions of (length)² over time because the integration is performed per unit length of the osteon, and the mass flux $\mathbf{q}$ is obtained from Darcy's law as

$$\mathbf{q} = -\frac{k_{rr,L}}{\mu} \nabla p^L(r, t). \quad (4-46)$$

For a transverse section of an osteon, let a_{cl} denote the area of an osteon inside the cement line, including the luminal domain. Thus

$$\gamma = \frac{\rho_{f0} \Gamma}{a_{cl}}. \quad (4-47)$$

Notice that γ has the dimensions of density over time and represents the flux of fluid in or out of the PV from or to the PLC per unit area of the whole bone cross section. Also γ could be expressed by using Eq.(4-45) as

$$\gamma^L = -\frac{\rho_{fo}}{a_{cl}} \frac{1}{\mu} k_{rr} \int \left(\frac{\partial p}{\partial r} \right) d\mathbf{a} = \frac{\delta \rho_{fo}}{b a_{cl}} \frac{1}{\mu} k_{rr,L} \frac{\partial p^L}{\partial \lambda}, \quad (4-48)$$

where δ is a unit length of the osteon (could be taken equal to 1) and is introduced for convenience. Thus Eq.(4-44) can be introduced into Eq.(4-48) to yield an expression for γ in terms of t and λ . The flow across the inner boundaries for the rising load is shown in Figure 17 where it is noticed that the flow pattern will follow the pressure pattern described in Figure 6. Figure 17 shows also the flow across the radial distance and the impermeability of the cement line ($\lambda = 1$). The values for the material properties that will be used for the PLC and the PV is shown in Table 7.

4.7. The nested two-pore-size-porosity model for interstitial flow in cortical bone tissue

The nested two-pore-size-porosity model for interstitial flow in cortical bone tissue is described here using Figure 16 and the properties of the anatomical structures described in the text above. To fix notation we introduce two cylindrical coordinate systems, one centered in the typical osteon in Figure 1 and one centered on the idealized whole bone cross-section in the same figure. For the larger idealized bone cross-section we assign the notation (R, ϑ, Z) to the cylindrical coordinate system and for the smaller osteonal coordinate system we assign the notation (r, θ, z) to the cylindrical coordinate system. Further, we consider the annular cross-section of a long bone in Figure 3 to be subjected to an axial stress $\sigma(r, \theta, t)$, which is a function of location on the cross-section (r, θ, z) and time t . This means that for an osteon

whose center is located at (R, ϑ, Z) in the cross-section is subjected to $\sigma^*(r^*, \theta^*, t)$ which is assumed to be uniform over the cross-section of that particular osteon, the values of r^* and θ^* serving to identify that osteon. Consistent with this use of the capital and small letters we let p denote the pore fluid pressure of the PLC of the osteon and P denotes the pore fluid pressure of the PV. Thus, p^L satisfies the condition of impermeability at the cement line and equality with p_v at the wall of the osteonal canal. The model is shown in Figure 18.

Based on all the previous results two conditions can be imposed on the pressure in the PV. The first one implies no flow through the periosteum, the outer boundary, and the second one stems from the fact that the PV is a low-pore-fluid-pressure domain because it contains thin-walled

blood vessels carrying blood with pressure of 40 to 60 mm of Hg; a pore fluid pressure significantly greater than 40 to 60 mm of Hg will collapse these blood vessels. Therefore,

it will be important to assume that the pressure on the endosteum walls (the inner boundary) is a low pressure, but not zero. This low pressure allows flow through the inner boundaries of the PV. The main concept in the nested porosity is that the fluid is flowing from the PLC into the PV and vice versa, thus $\gamma_v = h\gamma_L$,

(4-49)

where h is a constant non dimensional value of the ratio between the PV to the PLC area. An expression for the conservation of mass can be obtained by replacing ρ by $\phi\rho_f$, where ϕ is the porosity of the PV, and dividing through by ρ_{fo} as shown in Eq.(4-15), thus

$$\frac{1}{\rho_{fo}} \frac{\partial \phi \rho_f}{\partial t} + \frac{1}{\rho_{fo}} \nabla \cdot (\phi \rho_f \mathbf{v}) = h \frac{\phi_v \gamma_L}{\rho_{fo}}. \quad (4-50)$$

defined as the difference between the strain of the porous material and the strain of the fluid volume in the pore space. The first strain is due to the change in the volumetric strain in the

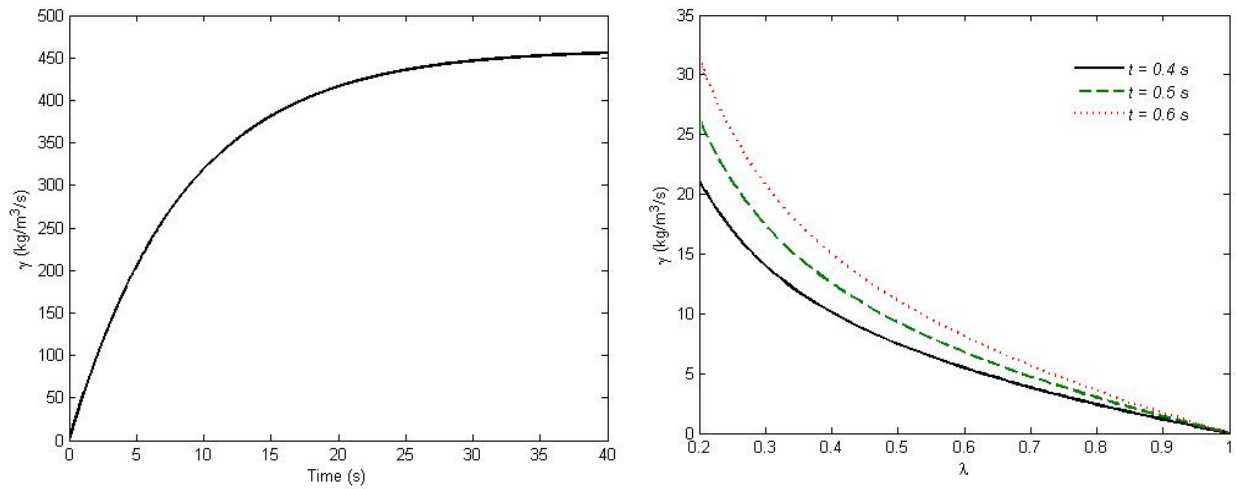


Figure 17: The flow across the osteonal walls in a ramp loading. The PLC permeability is taken to be in the order of $O(10^{-24}) \text{ m}^2$ and the strain rate is $100 \mu\epsilon/s$ and the PV effect is neglected (left). The flow across the non dimensional radial distance in a ramp loading at different times. Notice that the flow will keep increasing towards the osteonal canal at all times and is zero at the cement line to satisfy the assumed impermeability condition of the cement line. The PV effect is neglected (Right).

Table 7: The material properties reported from different sources for the elastic moduli, Poisson's ratios...etc. The superscript d reflects the drained properties and the superscript m reflects the matrix material properties. In the text the PLC is referred to by the superscript L and the PV by the superscript or subscript V. The water compressibility (K_f) is 2.3 GPa, and its viscosity (μ) is 0.001 Pa.s.

Material Parameters	PLC (L)	PV (V)
$E_1^d = E_1^d$	15.17 GPa	12.7 GPa
E_3^d	15.96 GPa	14.9 GPa
$\nu_{12}^d = \nu_{21}^d$	0.316	0.27
$\nu_{31}^d = \nu_{32}^d$	0.308	0.285
$\nu_{13}^d = \nu_{23}^d$	0.282	0.27
$E_1^m = E_1^m$	18.6 GPa	18.6 GPa
E_3^m	22.32 GPa	22.32 GPa
$\nu_{12}^m = \nu_{21}^m$	0.322	0.322
$\nu_{31}^m = \nu_{32}^m$	0.312	0.312
$\nu_{13}^m = \nu_{23}^m$	0.255	0.255
The outer radius	$r_o = 160 \mu m$	$R_o = 0.03 m$
The inner radius	$r_i = 32 \mu m$	$r_i = 0.015 m$
The Porosity ϕ	0.05	0.05
The permeability K_{rr}	$2.2 \times 10^{-24} m^2$	$6.35 \times 10^{-10} m^2$

solid matrix and the latter is due to the fluid pressure,

$$\zeta^V(R_V, t) = \frac{\rho_f \phi^V}{\rho_{fo}} = \hat{\mathbf{A}} \cdot \hat{\mathbf{E}} + \Lambda^V p^V(R_V, t), \quad (4-51)$$

where R_V is the any radius of the PV, $\hat{\mathbf{E}}$ is the strain vector, and

$$\hat{\mathbf{A}} \cdot \hat{\mathbf{E}} = \hat{A}_{r,V} \left(\frac{\partial u}{\partial R_V} + \frac{u}{R_V} \right) + \hat{A}_z^V \varepsilon(t) = \frac{\hat{A}_{r,V}^2}{\hat{C}_{rr,V}^d} p^V(R_V, t) + \hat{A}_z^V \varepsilon(t) + \hat{A}_r^V f_V(t), \quad (4-52)$$

Thus

$$\zeta^V(R_V, t) = \frac{\rho_f \phi^V}{\rho_{fo}} = \frac{J}{\hat{C}_{rr}^{dV}} p^V(R_V, t) + \hat{A}_z^V \varepsilon(t) + \hat{A}_r^V f_V(t) \quad (4-53)$$

Substitute (4-53) into (4-50) and use Darcy's law to obtain

$$\frac{R^2 k_{rrV}}{\mu} \left(\frac{\partial^2 p^V(\beta, t)}{\partial \beta^2} + \frac{1}{\beta} \frac{\partial p^V(\beta, t)}{\partial \beta} \right) - \frac{\partial p^V(\beta, t)}{\partial t} = \hat{A}_z^V \frac{\partial \varepsilon(t)}{\partial t} + \hat{A}_r^V \frac{\partial f_V(t)}{\partial t} - h \frac{\phi^V \gamma^L}{\rho_{fo}}, \quad (4-54)$$

where R is the outer radius of the osteon and β is the non dimensional radius of the PV.

Eq.(4-54) looks similar to Eq.(27) in Gailani and Cowin (2008) but with non-homogeneous

term $h \frac{\phi^V \gamma^L}{\rho_{fo}}$. This term shows clearly the coupling between the PV and the PLC porosities and

their relative dependence on each other. In other words the term $h \frac{\phi^V \gamma^L}{\rho_{fo}}$ contains the

derivative of the PLC pressure, thus

$$\gamma^L(\lambda, t) = \frac{\rho_{fo}}{a_{cl}} \frac{1}{\mu b} k_{rr,L} \left\{ \sum_{n=0}^{\infty} C_n \frac{\partial R_0(\lambda k_n)}{\partial \lambda} \frac{b^2 (e^{-k_n^2 c^L t / b^2} - 1) N \&_0}{k_n^2 c^L} - \left\{ M \sum_{n=0}^{\infty} C_n \frac{\partial R_0(\lambda k_n)}{\partial \lambda} e^{-k_n^2 c^L t / b^2} \int_0^t e^{k_n^2 c^L \tau / b^2} \frac{\partial p^V(\beta, \tau)}{\partial \tau} d\tau \right\} \right\}, \quad (4-55)$$

Therefore Eq.(4-54) involves a great complexity by having the integral of the PV pressure as

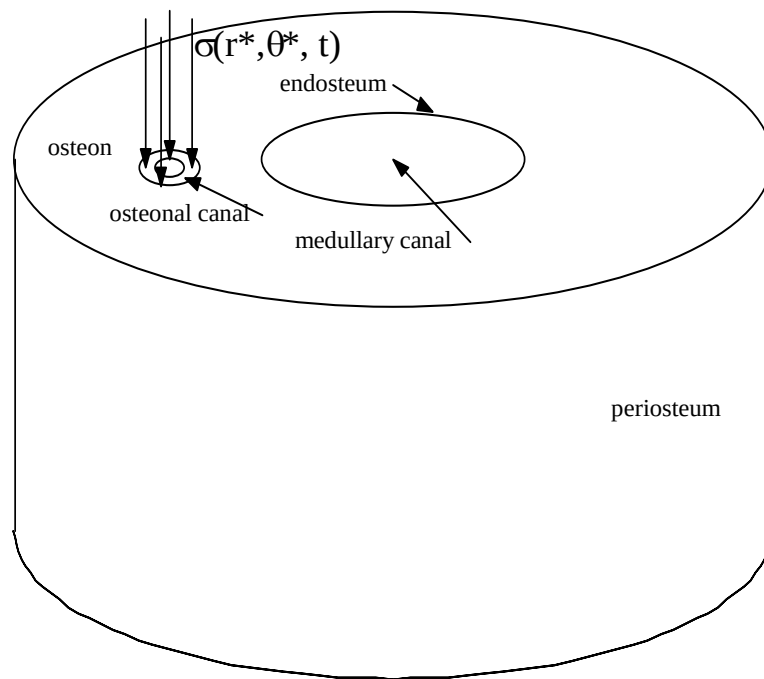


Figure 18: Constructing the model of the Russian doll. A transverse section of a long bone of an animal or human idealized as a portion of circular hollow cylinder. One of the many possible osteons in such a typical cross-section is illustrated. The osteons are subjected to stress in the z -direction and they are nested within the PV porosity. Thus the osteons act as a point source of fluids in the PV domain. The PV and PLC are idealized as circular hollow cylinders.

part of it. The solution to this equation yields some assumptions for the PV pressure either as a simple harmonic function (to include the effect of the blood pressure) or by using Green's function which will imply the assumption of more conditions (see the Appendix). However, for this time we will ignore that integral term and continue the solution with the same procedure described for the PLC and the main difference between the two solutions will be the non-homogenous term and the non-homogenous boundary conditions.

Based on all the previous results two conditions can be imposed on the pressure in the PV. The first one implies no flow through the periosteum, the outer boundary, and the second one stems from the fact that the PV is a low-pore-fluid-pressure domain because it contains thin-walled blood vessels carrying blood with pressure of 40 to 60 mm of Hg; a pore fluid pressure significantly greater than 40 to 60 mm of Hg will collapse these blood vessels. Therefore, it will be important to assume that the pressure on the endosteum walls (the inner boundary) is a low pressure, but not zero. This low pressure allows flow through the inner boundaries of the PV. So one can assume the following conditions

$$p^V(A, t) = p^{BO} \sin wt, \quad \frac{\partial p^V(1, t)}{\partial \beta} = 0, \quad p^V(\beta, 0) = 0 \quad (4-56),$$

where A is the non dimensional inner radius of the PV, p^{BO} is the blood pressure, should not exceed 60 mm Hg, and w is derived from the frequency of the pulses (72/s) in rad/s. Under all these conditions a general expression for the PV pressure is obtained

$$\begin{aligned} p^V(\beta, t) = & p^{BO} \sin wt - N^V \varepsilon(t) + \\ & \sum_{n=0}^{\infty} C_{nV} R_{0V}(\beta \alpha_n) \frac{\alpha_n^2 c^V}{R^2} e^{-\alpha_n^2 c^V t / R^2} \left\{ M^V \int_0^t e^{\alpha_n^2 c^V \tau / R^2} \frac{\partial p^{BV}(A, \tau)}{\partial \tau} d\tau + N^V \int_0^t e^{\alpha_n^2 c^V \tau / R^2} \varepsilon(\tau) d\tau \right\} \\ & + \sum_{n=0}^{\infty} C_{nV} R_{0V}(\beta \alpha_n) \frac{\alpha_n^2 c^V}{R^2} e^{-\alpha_n^2 c^V t / R^2} \left(\frac{-h\phi_V \mu c^V}{(\alpha_n^2 c^V / R^2 - k_n^2 c^L / b^2) k_{rV} \rho_{fo}} \frac{N k_{rr,L}}{a_{cl} \mu b} \left[e^{(\alpha_n^2 c^V / R^2 - k_n^2 c^L / b^2) t} - 1 \right] \left\{ \sum_{n=0}^{\infty} C_n \frac{\partial R_0(a k_n)}{\partial \lambda} \right\} \right), \end{aligned} \quad (4-56)$$

where the V subscripts and superscripts indicates the PV porosity, for example the constants M^V , and N^V refer to the PV and their definition is similar to the one in Eq.(4-43) but with PV constants rather than PLC constants.

Now we can specialize Eq.(4-56) to a case of a rising load in a stress relaxation test to write a new expression for the PV pressure

$$\begin{aligned}
 p^V(\beta, t) = & p^{BO} \sin wt + \\
 & \sum_{n=0}^{\infty} C_{nV} R_{0V}(\beta \alpha_n) \frac{R^2 (e^{-\alpha_n^2 c^V t / R^2} - 1) N^V}{\alpha_n^2 c^V} - \sum_{n=0}^{\infty} C_{nV} R_{0V}(\beta \alpha_n) \frac{\alpha_n^2 c^V}{R^2} e^{-\alpha_n^2 c^V t / R^2} \left\{ M^V \int_0^t e^{\alpha_n^2 c^V \tau / R^2} \frac{\partial p^{BV}(A, \tau)}{\partial \tau} d\tau \right\} \\
 & + \sum_{n=0}^{\infty} C_{nV} R_{0V}(\beta \alpha_n) \frac{\alpha_n^2 c^V}{R^2} e^{-\alpha_n^2 c^V t / R^2} \left(\frac{-h\phi_V \mu c^V}{(\alpha_n^2 c^V / R^2 - k_n^2 c^L / b^2) k_{rrV}} \frac{N k_{rr,L}}{a_{cl} \mu b} \left[e^{(\alpha_n^2 c^V / R^2 - k_n^2 c^L / b^2) t} - 1 \right] \left\{ \sum_{n=0}^{\infty} C_n \frac{\partial R_0(a k_n)}{\partial \lambda} \right\} \right)
 \end{aligned}
 \tag{4-57}$$

This expression is plotted in Figure 19 against the radial distance. The plot shows the variation of the pressure along the non dimensional radius with non zero pressure at the medullary canal (sinusoidal pressure). Notice the coupling between the PLC and the PV porosities in Eq.(4-57).

Eq.(4-57) can be substituted into Eq.(4-44) to record the effect of the PV pressure distribution on the PLC . The resulting equation is algebraically complicated due to the presence of the PV pressure as a derivative multiplied by an exponential term and under the integral. The results of this relationship are plotted in Figure 20, where we observed that the PV pressure has little influence on the PLC pressure .

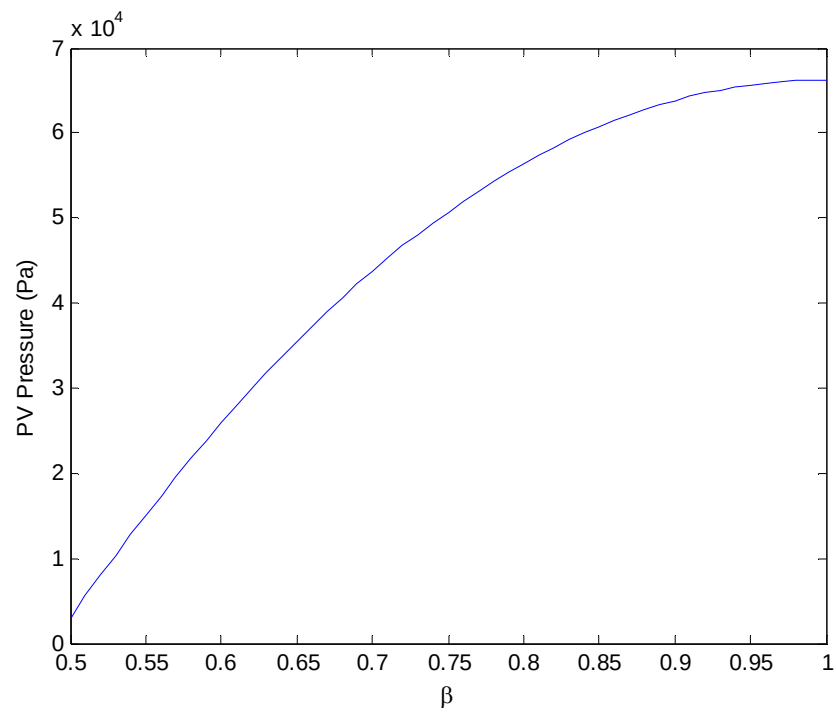


Figure 19: PV pressure along the non dimensional radius. The pressure is highest at the periosteum and non zero at the endosteum.

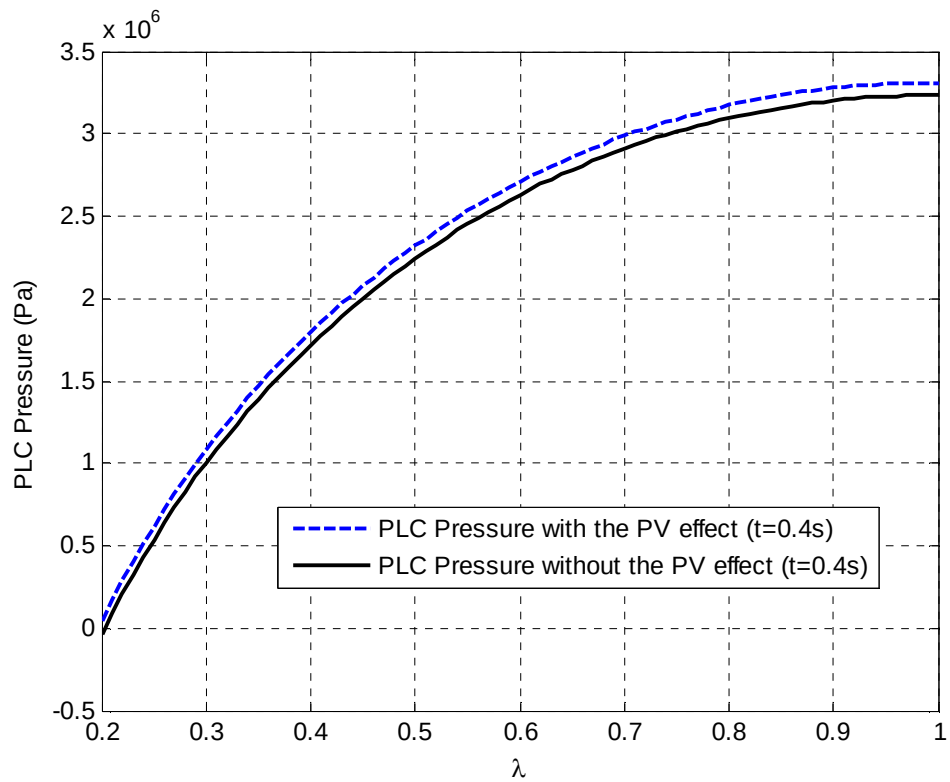


Figure 20: The effect of the PV pressure on the PLC pressure. The figure shows clearly that the PV pressure has small effect on the PLC pressure.

4-8 Discussion

The long term objective of the work, of which this paper is a part, is to develop a model of all fluid transport in bone, particularly that due to mechanical loading and blood pressure. The result presented here is the basic piece of this model. It addresses the question of pore fluid movement from the perivascular region in bone to the bone cells housed in the lacunae of the mineralized matrix. Knowledge of this fluid movement is necessary for the determination of the nutrient transport and the effect of the fluid drag on the cell processes of osteocytes in the canaliculi, which is thought to be a principal mechanotransduction mechanism. Quantification of the fluid drag on the cell processes is necessary for the evaluation of osteocytic mechanosensitivity (Weinbaum et al., 1994; Cowin et al. 1995).

A number of papers have considered the pore pressure oscillations in the PLC of an osteon without the consideration of activity in the PV. These contributions include those of Weinbaum et al. (1994), Cowin et al. (1995), Rémond and Naili (2005) and Rémond et al. (2007). Two recent papers have computationally addressed the question of the two interacting porosities. Fornells et al. (2007) considered a mixture theory based dual porosity model of the PV and the PLC. In this model the two porosities are considered to be interacting on the same level as opposed to the model in this paper where the two porosities are arranged hierarchically. As the authors point out their averaging process is associated with a typical RVE length that averages over the details of local pore pressure fields. In the paper of Goulet et al.,(2008) a finite element model whose geometry was generated from a quantitative computed tomography scan of a section of human tibia was employed. The fluid velocities and the pore fluid pressure in the PLC and in the PLC + PV were calculated for a specified applied loading consisting of combined axial loading and bending.

In this contribution the governing equations for the theory of poroelastic materials with hierarchical pore space architecture and compressible constituents undergoing small deformations are developed. These equations are applied to the problem of determining the exchange of pore fluid between the PV and the PLC in bone tissue due to cyclic mechanical loading and blood pressure oscillations. A formula for the volume of fluid that moves between the lacunar-canalicular and vascular porosity in a stress relaxation is obtained as a function of the cyclic mechanical loading and blood pressure oscillations. Formulas for the fluid pore pressure in both the lacunar-canalicular and the vascular porosity are obtained as a function of the two driving forces, the ramp loading and the blood pressure, both with specified amplitude and frequency. A general observation is that the vascular porosity and the lacunar-canalicular porosity are greatly different. The lacunar-canalicular porosity is a relatively high-pressure domain characterized by a relatively small pore size and no vascular vessels. On the other hand the vascular porosity is a relatively low-pressure domain characterized by a relatively large pore size that contains vascular vessels. The model shows that, although the two porosities exchange fluids, their influence on one another in terms of pressure change is small. Cowin et al., (2009) made the same conclusions about the pressure influence between the two porosities for a cyclic loading as shown in Figure 21.

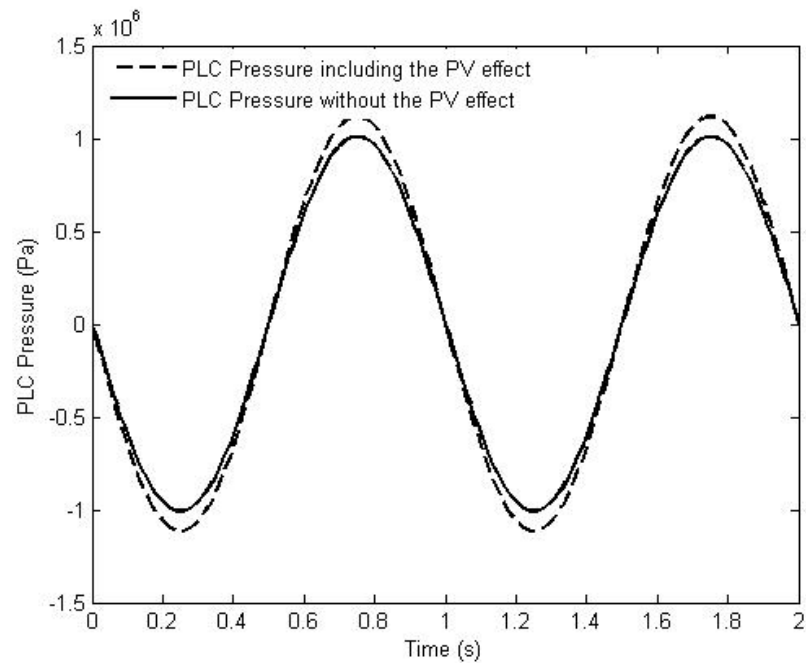


Figure 21: The influence of the PV pressure on the PLC pressure in a cyclic loading. The figure shows that the PV has little effect on the PLC, Cowin et al., (2009)

The limitations of the present work include the assumption of right circular cylindrical geometry for both the whole long bone cross-section and the osteon and the inadequacy of the lymphatics modeling. The problem of the idealized modeling may be addressed with computational models and the improvement of the lymphatics modeling depends upon our achieving a greater understanding of bone lymphatics. It may be reasonable to try to model the reabsorption of interstitial fluid into the venous system as those vessels leave the bone. It is possible that this work will lead to improvements in both of these topics. For the bone lymphatics problem the model provided could be used as the analytical basis for the determination of the location of all the interstitial bone fluid in a whole bone experiment at different times with unexpected imbalances leading to the identification of the bone lymphatics. Some computational approaches have modeled both the PV and PLC with some degree of anatomical accuracy, but computational complexity. The hierarchical approach employed here can easily be incorporated in a computational approach; one need model only the vascular porosity and develop a separate subprogram to determine the interstitial fluid supply interchange between the PLC to the PV depending upon the applied loading and the pore pressure differences between the two porosities.

Planned extensions of this work include the determination of the solute transport between the PV and the PLC and extension of the mechanical loading considered hereto the consideration of the cyclic combined axial and bending loading of a long bone. The extension is not simple because differential pressure between PV locations in a bone cross-section will cause blood to move within the vessels in the cross-section, as well as the interstitial fluid. In the case of cyclic purely axial loading of a long bone considered here, the pore pressure in the PV was homogenous at any time and did not induce blood movement within the vessels.

Appendix

Some properties of Bessel functions:

Important characteristics of Bessel functions should be observed in sketching a solution for equations such as equations (44a) and (46). The poles in equation (44a) should be calculated by setting $Q(\alpha_n) = 0$,

$$xC[J_1(ax)Y_o(x) - J_o(x)Y_1(ax)] + D[J_1(x)Y_1(ax) - J_1(ax)Y_1(x)] = 0. \quad (A1)$$

The following relationships from Hildebrand (1976) and Abramowitz (1970) should be observed:

$$J_n(ix) = i^n I_n(x), \quad (A2)$$

$$K_o(x) = \frac{i\pi}{2} H_o^{(1)}(ix), K_1(x) = -\frac{\pi}{2} H_1^{(1)}(ix), \quad (A3)$$

$$H_o^{(1)}(x) = J_o(x) + iY_o(x), H_1^{(1)}(x) = J_1(x) + iY_1(x), \quad (A4)$$

$$K_o(x) = \frac{i\pi}{2} [J_o(ix) + iY_o(ix)], \text{ and } K_1(x) = -\frac{\pi}{2} [J_1(ix) + iY_1(ix)]. \quad (A5)$$

This yields the general case:

$$K_n(z) = \frac{i^{n+1}\pi}{2} [J_n(ix) + iY_n(ix)]. \quad (A6)$$

For small values of x , Hildebrand (1976) has made the following approximations:

$$I_p(x) \approx \frac{1}{2^p p!} x^p, K_p(x) \approx 2^{p-1} (p-1)! x^{-p}, \text{ and } K_o(x) \approx -\ln x, \quad (A7)$$

with the implication that the ratio of two quantities connected by the symbol $\sim$ approaches unity as x tends to the relevant limit (here zero).

For large values of x ($x \rightarrow \infty$) we recapitulate the results listed in the preceding equation:

$$I_p(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \text{ and } K_p(x) \approx \sqrt{\frac{\pi}{2x}} \cdot e^{-x}. \quad (A8)$$

Darcy's Law:

This law relates the fluid mass flow rate to the gradient of the pore pressure p .

$$\rho_f \mathbf{v} = -\mathbf{H} \nabla p(\mathbf{x}, t), \mathbf{H} = \mathbf{H}^T \quad (\text{A9})$$

where the coefficient tensor $\mathbf{H}$ can be represented by

$$\mathbf{H} = \rho_{fo} \mathbf{K} / \mu. \quad (\text{A10})$$

Where $\mathbf{K}$ is the intrinsic Darcy's law permeability tensor, ρ_{fo} is a reference value of the fluid density and μ is the fluid viscosity. The tensor $\mathbf{K}$ has units of length squared and is a function of the porous structure only, not the fluid in the pores; thus Darcy's law (A9) takes the form

$$\mathbf{q} = (\phi \rho_f / \rho_{fo}) \mathbf{v} = -(1/\mu) \mathbf{K} \cdot \nabla p(\mathbf{x}, t), \mathbf{K} = \mathbf{K}^T \quad (\text{A11})$$

where the mass flux $\mathbf{q}$ has the dimensions of velocity. For the incompressibility condition

Darcy's law can be rewritten after setting $\rho_f = \rho_{fo}$ in equation (A11).

Green's function.

Background

Green's functions can be used in the solution of non homogenous boundary value problems .

the case considered here is the three dimensional problem

$$\nabla^2 p(r, t) + \frac{1}{\eta} g(r, t) = \frac{1}{\phi} \frac{\partial p(r, t)}{\partial t} \quad (\text{1a})$$

The solution is

$$p(r, t) = \int_A^1 r' G(r, t | r', \tau) \Big|_{\tau=0} F(r') dr' + \frac{\phi}{\eta} \int_{\tau=0}^t d\tau \int_{r'=A}^1 r' G(r, t | r', \tau) g(r', \tau) dr' - \phi \int_{\tau=0}^t d\tau [G]_{r'=1} \cdot f(\tau) d\tau \quad (\text{1b})$$

where $p(A, t) = f(t)$ and A is the non dimensional inner radius.

The solution for PV

$$\frac{k_{rrV}}{R^2 \mu} \left(\frac{\partial^2 p^V(\beta, t)}{\partial \beta^2} + \frac{1}{\beta} \frac{\partial p^V(\beta, t)}{\partial \beta} \right) - \frac{\partial p^V(\beta, t)}{\partial t} = \hat{A}_{z,V} \frac{\partial \varepsilon(t)}{\partial t} + \hat{A}_{r,V} \frac{\partial f_V(t)}{\partial t} - h \frac{\phi_V \gamma_L(\beta, t)}{\rho_{fo}}$$

(2)

Recall

$$\gamma = \frac{\rho_{fo}}{a_{cl}} \frac{1}{\mu b} k_{rr,L} \left\{ \sum_{n=0}^{\infty} C_n \frac{\partial R_0(ak_n)}{\partial \lambda} \frac{b^2 (e^{-k_n^2 c^L t / b^2} - 1) N_0}{k_n^2 c^L} - \left\{ M \sum_{n=0}^{\infty} C_n \frac{\partial R_0(ak_n)}{\partial \lambda} e^{-k_n^2 c^L t / b^2} \int_0^t e^{k_n^2 c^L \tau / b^2} \frac{\partial p^V(\beta, \tau)}{\partial \tau} d\tau \right\} \right\}, \quad (3a)$$

Eq.(2) can be rewritten

$$\frac{\partial^2 p^V(\beta, t)}{\partial \beta^2} + \frac{1}{\beta} \frac{\partial p^V(\beta, t)}{\partial \beta} - \frac{1}{\eta} \frac{\partial p^V(\beta, t)}{\partial t} = \frac{1}{\varphi} g(\beta, t) \quad (3b)$$

Where

$$\eta = \frac{k_{rrV}}{R^2 \mu} = \varphi, \text{ and} \quad (3c)$$

$$g(\beta, t) = \hat{A}_{z,V} \frac{\partial \varepsilon(t)}{\partial t} + \hat{A}_{r,V} \frac{\partial f_V(t)}{\partial t} - h \frac{\phi_V \gamma_L(\beta, t)}{\rho_{fo}}. \quad (3d)$$

The boundary conditions are

$$(i) p^V(\beta, 0) = 0, (ii) p^V(A, t) = p_o \sin wt, (iii) \frac{\partial p^V(1, t)}{\partial \beta} = 0 \quad (4)$$

To determine the appropriate Green's function, we consider the homogenous version of this problem as

$$\frac{\partial^2 \Phi}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial \Phi}{\partial \lambda} - \frac{1}{\eta} \frac{\partial \Phi}{\partial t} = 0 \quad (5a)$$

With boundary conditions:

$$(i) \Phi(\beta, 0) = F(\beta), (ii) \Phi(A, t) = 0, (iii) \frac{\partial \Phi(1, t)}{\partial \beta} = 0 \quad (5b)$$

Later on we will specialize the first boundary condition above to $\Phi(\beta, 0) = F(\beta) = 0$. The solution to the problem above is very similar to the one that used for the PLC, thus

For B2

$$f(\tau) = p_o \sin w\tau \quad (14)$$

$$[G]_{\beta'=1} = \sum_{n=0}^{\infty} R_{0V}(\alpha_n \beta) e^{-\alpha_n^2 c^V (t-\tau)/R^2} \frac{R_{0V}(\alpha_n)}{N(\alpha_n)} \quad (15)$$

Substitute Eqs. (14) and (15) into B2, thus

$$\varphi \int_{\tau=0}^t [G]_{\beta'=1} \cdot f(\tau) d\tau = \varphi \int_{\tau=0}^t \sum_{n=0}^{\infty} R_{0V}(\alpha_n \beta) e^{-\alpha_n^2 c^V (t-\tau)/R^2} p_o \sin w\tau \frac{R_{0V}(\alpha_n)}{N(\alpha_n)} d\tau \quad (16)$$

B2

Which can easily be integrated.

Now term B1

From (3d)

$$g(\beta', \tau) = \hat{A}_{z,V} \frac{\partial \varepsilon(\tau)}{\partial \tau} + \hat{A}_{r,V} \frac{\partial f_V(\tau)}{\partial \tau} - h \frac{\phi_V \gamma_L(\beta', \tau)}{\rho_{f0}} \quad (17)$$

$$\gamma_L(\beta', \tau) =$$

$$\frac{\rho_{f0}}{a_{cl}} \frac{1}{\mu b} k_{rr,L} \left\{ \left[\sum_{n=0}^{\infty} C_n \frac{\partial R_0(ak_n)}{\partial \lambda} \frac{b^2 (e^{-k_n^2 c^L t/b^2} - 1) N_{\&}}{k_n^2 c^L} - \right. \right. \\ \left. \left. M \sum_{n=0}^{\infty} C_n \frac{\partial R_0(ak_n)}{\partial \lambda} e^{-k_n^2 c^L \tau/b^2} \int_0^t e^{k_n^2 c^L \tau/b^2} \frac{\partial p^V(\beta', \tau)}{\partial \tau} d\tau \right] \right\}, \quad (18)$$

B3

Notice that term B3 needs to be integrated for $\tau \rightarrow 0 \dots t$ and then t should be replaced by τ .

However, the function $\frac{\partial p^V(\beta', \tau)}{\partial \tau}$ and further conditions should be assumed.

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