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The fine structure of the Medvedev lattice and the partial degrees

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THE FINE STRUCTURE OF THE MEDVEDEV LATTICE AND THE PARTIAL DEGREES

by

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INTRODUCTION

This paper investigates the Medvedev lattice, the Mučnick lattice, the Dyment lattice, and some of their sublattices. These are distributive lattices which arise in a natural way within the study (and as generalizations) of two of the most well known recursion theoretic reducibility orderings, namely the T-degrees and the partial degrees.

Notwithstanding their richness and, at the same time, the pleasant regularity of their properties, these structures have not been extensively studied. A happy consequence of this fact is that this paper is essentially self contained.

Consequently, the purpose of these introductory pages is neither to review background material nor to analize at length (nor even to list) the main results. We want simply to sketch a historical outline of the subject and to indicate the lines along which this paper develops.

The Medvedev lattice was first introduced in MEDVEDEV (1955), in the attempt to make precise the idea, due to Kolmogorov, which consists in identifying true propositional formulas with identically "solvable" problems. Following Medvedev, one says that a mass problem (i.e. a collection of total functions from the natural numbers into the natural numbers) corresponds to a given informal problem (for example the problem of enumerating the elements of a non-empty set A of natural numbers) if any of its functions can solve the problem (in our example, can enumerate the set A) and any of them can be obtained from any "solution" to the problem. The notion of "solvable problem" is made precise by saying that a mass problem is <u>solvable</u> if it contains a recursive function. Corresponding to this definition of solvable mass problem, there is an intuitive notion of reducibility between mass problems: given two mass problems $\mathcal{A}$ and $\mathcal{B}$ one can say that $\mathcal{A}$ is "reducible" to $\mathcal{B}$ if, given any "solution" to $\mathcal{B}$ (i.e. any recursive element of $\mathcal{B}$) we can effectively get a "solution" to $\mathcal{A}$ (i.e. a recursive element of $\mathcal{A}$); Medvedev proposes to make this notion precise by saying that $\mathcal{A}$ is <u>reducible</u> to $\mathcal{B}$ (notation $\mathcal{A} \leq \mathcal{B}$) if there exists a recursive operator Ψ such that $\Psi(\mathcal{B}) \subseteq \mathcal{A}$. The relation $\leq$ is a preordering relation on the set of mass problems. The next step toward the definition of the Medvedev lattice consists in dividing this set by the equivalence relation corresponding to $\leq$ ($\mathcal{A}$ is <u>equivalent</u> to $\mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$): the partial order that we obtain in this way is in fact a distributive lattice with a least and a greatest element. Its elements are called <u>degrees of difficulty</u>.

Medvedev's paper of 1955 gained a bit of notoriety with the exposition of its contents by Rogers (see ROGERS

(1967)) who proposed also several open problems about the lattice. Remarkable is the solution to one of these problems, given by Dyment (see DYMENT (1876)) who showed that the property of being of degree of solvability (i.e. a degree of difficulty containing a mass problem of the form {f}, for some function f) is lattice-theoretic. In Chapter I we prove some results about the degrees of enumerability (i.e. degrees of difficulty containing a mass problem of the form {f: range(f)=A}, for some set A of natural numbers); these results may turn out to be useful to face the open problem of establishing whether the property of being a degree of enumerability is lattice-theoretic (see again ROGERS (1967)).

The Dyment lattice (DYMENT (1976)) is a generalization of the Medvedev lattice which consists in considering mass problems of partial functions and defining the notion of reducibility in terms of partial recursive operators instead of recursive operators. The Mučnick lattice (MUČNICK (1963)) is obtained by dropping the requirement of effectiveness when defining the notion of reducibility between mass problems: in order that $\mathcal{A}$ be reducible to $\mathcal{B}$ one simply requires that $(\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leq_T g]$.

A thorough investigation of the Mučnick lattice is given as well as a detailed exposition of the relationships between these three lattices in terms of lattice theoretic homomorphisms.

As shown in MEDVEDEV (1955), DYMENT (1976), MUČNICK (1963) respectively, the Medvedev lattice, the Dyment lattice and the Mucnick lattice are in fact Brouwer algebras. We shall see that the Mucnick lattice is also a Heyting algebra, whereas the Medvedev and the Dyment lattices are not. We shall introduce several sublattices of these lattices which turn out to be of interest in that, as intitionistic diagonalizable algebras, they enable us to regard the jump operator in the T-degrees and the partial degrees as a τ-operator in the corresponding diagonalizable algebras.

Chapter II is devoted to the problem of embedding finite Heyting and Brouwer algebras in the Mucnick lattice and finite Brouwer algebras in the Medvedev and the Dyment lattices (Chapter I contains also a characterization of the countable distributive lattices with least and greatest elements which are embeddable in these lattices). The results concerning these embeddings enable us to characterize the set of propositional formulas which are true in our lattices regarded as Brouwer algebras (or as a Heyting algebra in the case of the Mučnick lattice; as is well known every Heyting algebra and every Brouwer algebra can be viewed as a model for the intuitionistic propositional calculus), thus solving an open problem in MUČNICK (1963) and showing that the claim in ROGERS (1967), p. 289, that the propositional formulas which are true in the Medvedev lattice (therein called identities) are exactly the theorems of the intuitionistic propositional calculus is false.

DYMENT (1976) is the only paper which studies at some length filters and ideals of the Medvedev lattice. We have tryed to carry this study forward, by investigating the algebraic structure of several quotient lattices of the Medvedev and Mucnick lattices. A particular attention has been paid to embedding finite Heyting and Brouwer algebra into these quotient lattices with consequent interesting conclusions in terms of intermediate logics.

Some of these results make use of theorems about the partial degrees, which we prove in Chapter III. The paper ends with an appendix (Chapter III, Section 2) in which we show also that a conjecture due to Case (see CASE (1971)) is false.

DEFINITIONS AND NOTATION

We will use without further comment the following definitions and notation.

The set of natural numbers $\{0,1,\ldots\}$ is denoted by ω. ${}^{\omega}\omega$ denotes the set of all total functions from ω into ω. Elements of this set are denoted by the lower case Roman letters f,g,h,...

$({}^{\omega}\omega)^*$ denotes the set of all partial functions from ω into ω. We use the lower case Greek letters $\varphi, \psi, \chi, \ldots$ to denote partial functions from ω into ω, with the following exceptions:
partial functions from ω into ω whose domain is finite (called <u>finite segments</u>) are denoted by $\tilde{\varphi}, \tilde{\psi}, \tilde{\chi}, \ldots$;
partial functions whose domain is a finite initial segment of ω (called <u>finite initial segments</u>) are denoted by $\tilde{f}, \tilde{g}, \tilde{h}, \ldots$, with the exception of finite initial segments of 0-1 valued functions which are usually denoted by the lower case Greek letters $\alpha, \beta, \gamma \ldots$ (In Chapter II we use the lower case Greek letters as variables ranging also through the set of propositional formulas of a given propositional language).

If $\tilde{f}$ is a finite initial segment, then $lh(\tilde{f})$ denotes the least natural number x such that $x \notin dom(\tilde{f})$ (for any partial function φ, $dom(\varphi)$ denotes the domain of φ).

Unless otherwise specified, throughout this paper a partial function is understood to be a partial function from ω into ω; likewise, a total function is a total function from ω into ω. The word "total" will be frequently dropped: thus, a function is a total function. Occasionally, we use Curch's lambda notation for functions.

Given a partial function φ and $x\in\omega$, we write $\varphi(x)\downarrow$ ($\varphi(x)$ <u>converges</u>) if $x\in dom(\varphi)$; analogously, we write $\varphi(x)\uparrow$ ($\varphi(x)$ <u>diverges</u>) if $x\notin dom(\varphi)$. If φ and ψ are partial functions and $x\in\omega$, then $\varphi(x)\downarrow=\psi(x)\downarrow$ abbreviates $\varphi(x)\downarrow$ & $\psi(x)\downarrow$ & $\varphi(x)=\psi(x)$. On the other hand, $\varphi(x)=\psi(x)$ means $[[\varphi(x)\uparrow$ & $\psi(x)\uparrow]$ or $[\varphi(x)\downarrow=\psi(x)\downarrow]]$.

Given two finite segments $\tilde{\varphi}$, $\tilde{\psi}$ we say that $\tilde{\varphi}$ <u>lexicographically precedes</u> $\tilde{\psi}$ if, for the least x such that $\tilde{\varphi}(x)\neq\tilde{\psi}(x)$, either

1) $\tilde{\varphi}(x)\uparrow$ or

2) $\tilde{\varphi}(x)\downarrow$ & $\tilde{\psi}(x)\downarrow$ & $\tilde{\varphi}(x)\leq\tilde{\psi}(x)$.

Given a set A, its characteristic function is denoted by c_A; sometimes and when this does not raise any confusion, a set is identified with its characteristic function. Thus, for instance, sometimes we write A(x) to mean $c_A(x)$. ${}^{\omega}2$ denotes the subset of ${}^{\omega}\omega$ constituted by all 0-1 valued functions, i.e. all characteristic functions of subsets of the set of natural numbers.

${}^{\omega}\omega$ is given the <u>Baire topology</u>. If $\tilde{f}$ is a finite initial segment, let $S_{\tilde{f}}=\{g: \tilde{f}\subseteq g\}$. Then the family $\{S_{\tilde{f}}: \tilde{f}$ is a finite initial segment$\}$ is a basis for this topology. With this topology, ${}^{\omega}\omega$ is a complete metric space.

The relativization of the Baire topology to ${}^{\omega}2$ gives the <u>Cantor topology</u> for the subspace ${}^{\omega}2$. ${}^{\omega}2$ equipped with this topology is a complete and compact metric space (called the <u>Cantor space</u>).

Capital Italic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ are used, possibly together with subscripts, to denote subsets of ${}^{\omega}\omega$ or $({}^{\omega}\omega)^*$ We use the term <u>mass problem</u> to denote subsets of ${}^{\omega}\omega$ and the term <u>mass problem of partial functions</u> to denote subsets of $({}^{\omega}\omega)^*$.

If φ and ψ are partial functions, then $\varphi \vee \psi$ denotes the partial function defined by

$$\varphi \vee \psi(x) = \begin{cases} \varphi(y) & \text{if } x=2y \\ \\ \psi(y) & \text{if } x=2y+1 \end{cases}$$

Most of our terminology and notation for recursion theory is standard and can be found in ROGERS (1967), with a few changes and additions. Maybe the most significant change is the notation for the pairing function. We fix a 1-1 recursive function $\lambda xy.\langle x,y\rangle$ from ω^2 onto ω and we fix also two recursive functions $\lambda x.(x)_0, \lambda x.(x)_1$ such that for all $x,y \in \omega$, $(\langle x,y\rangle)_0 = x$, $(\langle x,y\rangle)_1 = y$ and $\langle (x)_0,(x)_1\rangle = x$.

If $R \subseteq \omega^2$, then $\tau(R)$ denotes the set $\{\langle x,y\rangle : (x,y) \in R\}$ and if $A \subseteq \omega$ then $\tau^{-1}(A)$ denotes the set $\{(x,y) : \langle x,y\rangle \in A\}$. A set A is <u>single-valued</u> if $\tau^{-1}(A)$ is a partial function. A set $A \subseteq \omega$ is said to be <u>total</u> if $\tau^{-1}(A)$ is a total function.

$\{\varphi_z\}_{z\in\omega}$ is an acceptable enumeration of all partial recursive functions. Accordingly, an enumeration $\{W_z\}_{z\in\omega}$ of all recursively enumerable (shortly: r.e.) sets is given, where $W_z=\mathrm{dom}(\varphi_z)$. Let r be a fixed recursive function such that, for every $z\in\omega$, $W_z=\mathrm{range}(\varphi_{r(z)})$ and $\varphi_{r(z)}$ is 1-1 and $(\forall x)(\forall y)[x\in\mathrm{dom}(\varphi_{r(z)})\ \&\ y<x \Longrightarrow y\in\mathrm{dom}(\varphi_{r(z)})]$ (for the existence of such a function see e.g. ROGERS (1967)). W_z denotes the finite set $\{x: (\exists y<s)[x=\varphi_{r(z)}(y)]\}$.

If $F\subseteq\omega$ is finite then the number $\sum\{2^x: x\in F\}$ is called the <u>canonical index</u> of F; we have a 1-1 enumeration $\{D_u: u\in\omega\}$ of all finite sets, where D_u is the finite set having canonical index u.

Φ_z is the enumeration operator defined by the r.e. set W_z: thus, for all $A\subseteq\omega$, $\Phi_z(A)=\{x: (\exists u)[<x,u>\in W_z\ \&\ D_u\subseteq A]\}$. If Φ is an enumeration operator and φ is a partial function, then $\Phi(\varphi)$ denotes the set $\Phi(\tau(\varphi))$.

A <u>partial recursive operator</u> Ω is a partial mapping of $({}^{\omega}\omega)^*$ into $({}^{\omega}\omega)^*$ which is defined by means of an enumeration operator, i.e. there exists an enumeration operator Φ such that $(\forall\varphi\in\mathrm{dom}(\Omega))[\Omega(\varphi)=\tau^{-1}(\Phi(\varphi))]$. Ω_z is the partial recursive operator defined by the enumeration operator Φ_z.

A partial recursive operator which is total is called <u>recursive operator</u>. Thus, a recursive operator is a total mapping Ψ from $({}^{\omega}\omega)^*$ into $({}^{\omega}\omega)^*$ for which there exists an enumeration operator Φ such that $(\forall\varphi\in({}^{\omega}\omega)^*)$ $[\Psi(\varphi)=\tau^{-1}(\Phi(\varphi))]$. We recall the following theorem (see ROGERS (1967), p. 149):

Fundamental Operator Theorem There is a recursive function σ such that for every $z\in\omega$, $\Phi_{\sigma(z)}$ defines a recursive operator and, for every $f\in{}^{\omega}\omega$, if $\Phi_z(f)$ is single-valued then $\Phi_{\sigma(z)}(f)=\Phi_z(f)$.

Henceforth Ψ_z will denote the recursive operator determined by the enumeration operator $\Phi_{\sigma(z)}$. Thus ($\forall$recursive operator Ψ)$(\exists z)(\forall f\in{}^{\omega}\omega)[\Psi(f)=\Psi_z(f)]$.

Detailed definitions of Turing reducibility (notation: $\leq_T$) and enumeration reducibility (shortly, e-reducibility; notation: $\leq_e$) are to be found elsewhere (see e.g. ROGERS (1967)). For every $A\subseteq\omega$, let $[A]_T=\{B: A\equiv_T B\}$ and $[A]_e=\{B: A\equiv_e B\}$. $a_T, b_T, c_T, \ldots$ denote Turing degrees (shortly T-degrees); o_T is the least T-degree. $a_e, b_e, c_e, \ldots$ are used to denote enumeration degrees (e-degrees or also partial degrees); o_e is the least e-degree. Whenever we need to use subscripts when writing out T-degrees or e-degrees we shall write exspressions like $(a_i)_T$ or $(a_i)_e$, where $\{(a_i)_T: i\in I\}$ and $\{(a_i)_e: i\in I\}$ are two families of T-degrees and e-dgrees respectively, indexed with the set I.

Let $\leq \in \{\leq_T, \leq_e\}$: if φ and ψ are partial functions and $A \subseteq \omega$, then $\varphi \leq \psi$, $A \leq \varphi$, $\varphi \leq A$ denote respectively $\tau(\varphi) \leq \tau(\psi)$, $A \leq \tau(\varphi)$, $\tau(\varphi) \leq A$. We recall that $(\forall f, g \in {}^{\omega}\omega)$ $[f \leq_T g \Longleftrightarrow (\exists z)\ [\Psi_z(g) = f]$ and $(\forall f, g \in {}^{\omega}\omega)[f \leq_T g \Longleftrightarrow f \leq_e g]$.

Corollary (to the Fundamental Operator Theorem). There is a recursive function δ such that, if Ψ is a partial mapping from ${}^{\omega}\omega$ into ${}^{\omega}\omega$ for which there exists $z \in \omega$ such that for every $f \in \mathrm{dom}(\Psi)$, $\Psi(f) \leq_T f$ via the Turing reduction with Godel-number z, then Ψ is the restriction, to its domain, of $\Psi_{\delta(z)}$.

Proof. Let Ψ and z be as in the corollary. Uniformly in z we can find an r.e. set W such that for every $f \in \mathrm{dom}(\Psi)$, $\Psi(f) = \{x: (\exists u)[\langle x,u\rangle \in W\ \&\ D_u \subseteq \tau(f)]\}$: this can be easily done since we restrict our attention to total functions. Then the corollary follows from the Fundamental Operator Theorem.

An e-degree a_e is <u>non-total</u> if $(\forall A \in a_e)(\forall f \in {}^{\omega}\omega)[A \not\equiv_e f]$; a_e is <u>quasiminimal</u> if $a_e >_e o_e$ & $(\forall A \in a_e)(\forall f \in {}^{\omega}\omega)[f \leq_e A \Longrightarrow f$ recursive]. The upper semilattice of T-degrees is denoted by $\mathfrak{D}_T$; the upper semilattice of e-degrees is denoted by $\mathfrak{D}_e$. The use of capital Gothic letters to denote structures will be frequent in this paper.

If A is any set, then the symbol $|A|$ denotes the cardinality of A. If $A,B \subseteq \omega$, then $A \vee B$ denotes the set $\{2x: x \in A\} \cup \{2x+1: x \in B\}$.

Let $\mathcal{L}=\langle L,\leq\rangle$ be a partial order; if $a,b \in L$ and $a \leq b$ then standard interval notation $\mathcal{L}([a,b])$, $\mathcal{L}((a,b])$, $\mathcal{L}([a,b))$, $\mathcal{L}((a,b))$ will be used: we shall omit the specification of the structure (thus writing $[a,b]$ instead of $\mathcal{L}([a,b])$, etc.) when this does not cause any confusion. $a|b$ means that neither $a \leq b$ nor $b \leq a$. If $X,Y \subseteq L$ then $X|Y$ means $(\forall x \in X)(\forall y \in Y)\ [x|y]$. A subset X of L is an <u>antichain</u> if $(\forall x,y \in X)[x \neq y \Rightarrow x|y]$.

Occasionally, language from category theory will be used. A partial order $\mathcal{L}=\langle L,\leq\rangle$ can be regarded as a category in the following sense: $ob(\mathcal{L})=L$ (i.e. the objects of $\mathcal{L}$ are the elements of L); for every $a,b \in ob(\mathcal{L})$, the morphism set $\mathcal{L}(a,b)$ is a singleton if $a \leq b$, is the empty set otherwise. Clearly there exists a bijective correspondence between homomorphisms of partial orders and functors of the corresponding categories. If $\mathcal{L}_1$, $\mathcal{L}_2$ are partial orders and $F: \mathcal{L}_1 \longrightarrow \mathcal{L}_2$, $G: \mathcal{L}_2 \longrightarrow \mathcal{L}_1$ are homomorphisms, then in view of this correspondence, we say that F is <u>left adjoint</u> to G or G is <u>right adjoint</u> to F (notation: $F \dashv G$) if $\mathcal{L}_2(F(a),b) = \mathcal{L}_1(a,G(b))$ for every $a \in \mathcal{L}_1$, $b \in \mathcal{L}_2$ (here and often in the following, we use the same symbol to denote both the structure and the universe of the structure; we adopt this use to avoid pedantic specifications). For every partial order $\mathcal{L}$,

$\mathcal{L}^{op}$ denotes the dual of $\mathcal{L}$, in the sense of category theory. If $\mathcal{L}$ is a partial order and $a, b \in \mathcal{L}$, then $a \vee b$ denotes, if it exists, the least upper bound of a and b, and $a \wedge b$ denotes, if it exists, the greatest lower bound of a and b. This notation, which is almost universally adopted, conflicts however with the one used for the lattices $\mathfrak{M}$, $\mathfrak{M}_w$, $\mathfrak{M}_e$ (which we are going to define in the next sections) in the papers where these lattices were originally defined (see MEDVEDEV (1955), MUCNICK (1963), DYMENT (1976) respectively): in these papers $a \vee b$ denotes the greatest lower bound and $a \wedge b$ denotes the least upper bound. The notation for $\mathfrak{M}$ in ROGERS (1967), POULSEN (1970), DYMENT (1976), (1980) is the same as in MEDVEDEV (1955).

CHAPTER I

ALGEBRAIC PROPERTIES OF THE MEDVEDEV AND MUCNICK LATTICES

1. The Medvedev lattice $\mathfrak{M}$.

The Medvedev lattice was first defined in MEDVEDEV (1955). Our exposition follows ROGERS (1967) except for the notation for the greatest lower bound and the least upper bound, as mentioned in the Introduction.

Let $\mathcal{A}$, $\mathcal{B}$ be mass problems. We say (MEDVEDEV (1955)) that $\mathcal{A} \leq \mathcal{B}$ if $(\exists$ recursive operator $\Psi)(\forall f \in \mathcal{B})[\Psi(f)$ is total & $\Psi(f) \in \mathcal{A}]$. Clearly $\leq$ is a preordering relation. The relation defined by $\mathcal{A} \equiv \mathcal{B} \Longleftrightarrow [\mathcal{A} \leq \mathcal{B}$ & $\mathcal{B} \leq \mathcal{A}]$ is an equivalence relation. The set M of equivalence classes of this equivalence relation is the set of <u>degrees of difficulty</u>. $[\mathcal{A}]$ denotes the equivalence class of the mass problem $\mathcal{A}$. A,B,C,... denote degrees of difficulty.

In M one can anambiguously define a partial ordering (for which we still use the symbol $\leq$) by letting $[\mathcal{A}] \leq [\mathcal{B}]$ if $\mathcal{A} \leq \mathcal{B}$.

Theorem 1.1 (MEDVEDEV (1955)) The partial order $\mathfrak{M}=\langle M, \leq \rangle$ is a distributive lattice with a least and a greatest element.

Proof. 1) $[\mathcal{A}] \wedge [\mathcal{B}] = [0*\mathcal{A} \cup 1*\mathcal{B}]$ where, for $i \in \omega$ and a mass problem $\mathcal{A}$, $i*\mathcal{A} = \{i*f : f \in \mathcal{A}\}$, $i*f$ being defined by

$$i*f(x)=\begin{cases} i & \text{if } x=0 \\ f(x-1) & \text{if } x>0 \end{cases}$$

The notation $\mathcal{A}\wedge\mathcal{B}=0*\mathcal{A}\cup 1*\mathcal{B}$ will be also used.

2) $[\mathcal{A}]\vee[\mathcal{B}]=[\{f\vee g\colon f\in\mathcal{A}\ \&\ g\in\mathcal{B}\}]$ (see Introduction for the definition of $f\vee g$). The notation $\mathcal{A}\vee\mathcal{B}=\{f\vee g\colon f\in\mathcal{A}\ \&\ g\in\mathcal{B}\}$ will be used.

3) $[\emptyset]$ is the greatest element, henceforth denoted by 1.

4) $[\{f\colon f$ recursive$\}]$ is the least element, henceforth denoted by 0.

The above definitions are unambiguous and make $\mathcal{M}$ a distributive lattice.

Valuable sources for the Medvedev lattice are DYMENT (1976), (1980), POULSEN (1970). We recall some basic notions and known facts.

Fact 1.1 (PLATEK (1970)) The cardinality of $\mathcal{M}$ is $2^{2^{\aleph_0}}$.

Definition 1.1 (MEDVEDEV (1955)) A degree of difficulty S is a <u>degree of solvability</u> if $S=[\{f\}]$ for some function f.

Fact 1.2 The degree of difficulty 0 is a degree of solvability; indeed, for every recursive function f, $0=[\{f\}]$.

Proof. Obvious.

Notice that the identity $I: {}^{\omega}\omega \longrightarrow {}^{\omega}\omega$ is a recursive operator: thus $(\forall \mathcal{A}, \mathcal{B} \subseteq {}^{\omega}\omega)[\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{B} \leq \mathcal{A}]$.

Fact 1.3 (MEDVEDEV (1955)) The mapping $I: \mathcal{D}_T \longrightarrow \mathfrak{M}$ defined by $I([A]_T) = [\{c_A\}]$ is an embedding of the upper semilattice $\mathcal{D}_T$ onto the degrees of solvability of $\mathfrak{M}$.

Theorem 1.2 (MEDVEDEV (1955), DYMENT (1976)) The property of being a degree of solvability is lattice-theoretic; in fact, S is a degree of solvability $\Longleftrightarrow$ $(\exists T)[S<T$ & $(\forall A)[S<A \Longrightarrow T \leq A]]$.

Proof. The implication $\Longrightarrow$ is proved in MEDVEDEV (1955): if $S=[\{f\}]$ is a degree of solvability, then $T=[\{z*g\colon \Psi_z(g)=f$ & $(\forall n)[\Psi_n(f) \neq g]\}]$ is the desired degree.

The other implication, which was posed in ROGERS (1967) as an open problem is proved in DYMENT (1976).

It is useful at this point to fix some additional pieces of notation. The degree of solvability $[\{f\}]$ is denoted by S_f. If $S=[\{f\}]$ is a degree of solvability, then the degree T whose existence is stated in Theorem 1.2 is denoted by T_S or T_f. We use also the notation $\mathcal{C}_f = \{z*g\colon \Psi_z(g)=f$ & $(\forall n)[\Psi_n(f) \neq g]\}$; thus $T_f = [\mathcal{C}_f]$.

Corollary 1.1 $\mathfrak{M}$ is atomic; in fact $\mathfrak{M}$ has only one atom.

Proof. T_0 is the desired atom.

Following DYMENT (1976), we denote T_0 by $0'$; we introduce also the following notation: $\mathcal{O}=\{f: f$ recursive$\}$ $\mathcal{O}'=\{f: f$ non-recursive$\}$. Thus $0=[\mathcal{O}]$ and $0'=[\mathcal{O}']$.

<u>Definition 1.2</u> (MEDVEDEV (1955)) A degree of difficulty E is a <u>degree of enumerability</u> if, for some $A\subseteq\omega$, $E=[\{f: \text{range}(f)=A\}]$. In this case E is called <u>the degree of enumerability of A</u> and denoted by E_A (We shall also use the notation $\mathcal{E}_A=\{f: \text{range}(f)=A\}$).

Fact 1.4 (MEDVEDEV (1955)) The mapping $I: \mathfrak{D}_e \longrightarrow \mathfrak{M}$ defined by $I([A]_e)=E_A$ is an embedding of the upper semilattice $\mathfrak{D}_e$ onto the degrees of enumerabilty.

2. The lattice $\mathfrak{M}_e$.

$\mathfrak{M}_e$ is defined in DYMENT (1976) and is to $\mathfrak{M}$ as $\mathfrak{D}_e$ is to $\mathfrak{D}_T$.

We consider the set $({}^{\omega}\omega)^*$ of all partial functions. Mass problems are now subsets $\mathcal{A} \subseteq ({}^{\omega}\omega)^*$, i.e. $\mathcal{A}$ is a set of partial functions. One defines $\mathcal{A} \leq_e \mathcal{B}$ if $(\exists$ partial recursive operator $\Omega)(\forall \varphi \in \mathcal{B})[\varphi \in \mathrm{dom}(\Omega)$ & $\Omega(\varphi) \in \mathcal{A}]$ (we apologize for using the same symbol as for e-reducibility). Let us consider the equivalence relation $\equiv_e$ given by $\mathcal{A} \equiv_e \mathcal{B}$ if $\mathcal{A} \leq_e \mathcal{B}$ & $\mathcal{B} \leq_e \mathcal{A}$. The set M_e of equivalence classes is partially ordered by a relation which we still denote by $\leq_e$ and given by $[\mathcal{A}]_e \leq_e [\mathcal{B}]_e$ if $\mathcal{A} \leq_e \mathcal{B}$ (here, of course, $[\mathcal{A}]_e$ denotes the equivalence class of $\mathcal{A}$ under $\equiv_e$). Equivalence classes $A_e, B_e, C_e, \ldots$ are called <u>partial degrees of difficulty</u> or <u>degrees of difficulty in $\mathfrak{M}_e$</u>.

Theorem 2.1 (DYMENT (1976)) $\mathfrak{M}_e = \langle M_e, \leq_e \rangle$ is a distributive lattice with a least and a greatest element.

Proof. Similar to tha proof of Theorem 1.1: if $i \in \omega$ and φ is a partial function then $i * \varphi$ has the obvious meaning. Notice that the least and the greatest element are given by $0_e = [\{\varphi : \varphi \text{ is partial recursive}\}]_e$ and $1_e = [\emptyset]_e$, respectively.

Definition 2.1 (DYMENT (1976)) A partial degree of difficulty E_e is a <u>partial degree of enumerability</u> (or, simply, a <u>degree of enumerability in $\mathcal{M}_e$</u>) if $E_e=[\{\varphi\}]_e$ for some partial function φ.

A theorem similar to Theorem 2.1 holds:

<u>Theorem 2.2</u> (DYMENT (1976)) The property of being a partial degree of enumerability is lattice-theoretic in $\mathcal{M}_e$. In fact, E_e is a partial degree of enumerability $\Longleftrightarrow$ $(\exists T_e)[E_e<_e T_e \ \& \ (\forall A_e)[E_e<_e A_e \Longrightarrow T_e \leq_e A_e]]$.

3. The lattice $\mathcal{M}_w$.

Between mass problems we now define another preordering relation due to Mučnick (see MUČNICK (1963)).

We let $\mathcal{A} \leq_w \mathcal{B}$ if $(\forall g\in \mathcal{B})(\exists f\in \mathcal{A})[f\leq_T g]$. Since, of course, $\mathcal{A} \leq_w \mathcal{B} \Longleftrightarrow (\forall g\in \quad)(\exists$ recursive operator $\Psi)[\Psi(g)\in \quad]$, $\leq_w$ is actually defined by changing the quantifier prefix $\exists\forall$ in the definition of $\leq$ to the prefix $\forall\exists$. As an immediate consequence of this fact, we have that $\leq \subseteq \leq_w$. The converse inclusion does not hold in general (see Example 3.1 below), i.e. $\leq_w$ is a reducibility no doubt weaker than $\leq$. This is perhaps the reason for the name <u>weak reducibility</u> given to $\leq_w$ by Mučnick. Again we define an equivalence relation $\equiv_w$ by letting $\mathcal{A} \equiv_w \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ & $\mathcal{B} \leq_w \mathcal{A}$ and, between equivalence classes, we define $[\mathcal{A}]_w \leq_w [\mathcal{B}]_w$ if $\mathcal{A} \leq_w \mathcal{B}$ (once again using the same notation for the preordering relation and the partial ordering originated by that preordering on the set of equivalence classes): this definition is independent of the choice of mass problems $\mathcal{A}, \mathcal{B}$ in the respective equivalence classes.

Definition 3.1 For every mass problem $\mathcal{A}$, let $C(\mathcal{A})=\{f: (\exists g\in \mathcal{A})[g\leq_T f]\}$.

Notice that $C(\mathcal{A})$ is upwards closed under $\leq_T$, i.e. $(\forall f)(\forall g)[f\in C(\mathcal{A})\ \&\ f\leq_T g \Longrightarrow g\in C(\mathcal{A})]$.

Fact 3.1 a) $C(\mathcal{A})=C(C(\mathcal{A}))$;

b) $\mathcal{A} \equiv_w C(\mathcal{A})$;

c) $\mathcal{A} \leq_w \mathcal{B} \iff \mathcal{B} \subseteq C(\mathcal{A})$.

Proof. Obvious.

Example 3.1 For every function f, $\{f\} \equiv_w \{g: f \leq_T g\}$. This follows immediately from Fact 3.1 c), since $C(\{f\})=\{g: f \leq_T g\}$.

On the other hand, if f is not recursive then we have $\{g: f \leq_T g\} < \{f\}$: indeed, from the fact that $\{f\} \subseteq \{g: f \leq_T g\}$ it follows that $\{g: f \leq_T g\} \leq \{f\}$. That $\{f\} \not\leq \{g: f \leq_T g\}$ holds, follows from Theorem 3.1 of DYMENT (1976). This theorem states that for every mass problem $\mathcal{A}$ not containing any recursive function and such that $\mathcal{A}$ is a discrete set in the Baire topology, then $(\forall \mathcal{B})[\mathcal{A} \leq \mathcal{B} \Longrightarrow \mathcal{B}$ is nowhere dense in the Baire topology]. In our example, $\{f\}$ is discrete, but the set $\{g: f \leq_T g\}$ is not nowhere dense (in fact, $\{g: f \leq_T g\}$ is dense!). Thus $\{f\} \not\leq \{g: f \leq_T g\}$. This example shows that $\leq_w \not\subseteq \leq$.

Definition 3.2 Let $\mathcal{A}$ be a mass problem. A suset $\mathcal{B} \subseteq \mathcal{A}$ is a <u>basis</u> of $\mathcal{A}$ if $\mathcal{A}=C(\mathcal{B})$ and $(\forall f \in \mathcal{B})(\forall g \in \mathcal{A})[g \not<_T f]$. A degree of difficulty A is said to have <u>finite</u> (<u>countable</u>, <u>at most countable</u>) <u>basis</u> if $A=[\mathcal{A}]$ where $\mathcal{A}$ has a finite (countable, at most countable) basis.

Let now M_w be the set oe equivalence classes of the relation $\equiv_w$. Elements $A_w, B_w, C_w, \ldots$ of M_w are called <u>weak degrees of difficulty</u>.

Theorem 3.1 (MUČNICK (1963)) $\mathfrak{M}_w = \langle M_w, \leq_w \rangle$ is a distributive lattice with a least and a greatest element.

Proof. 1) $[\mathcal{A}]_w \wedge [\mathcal{B}]_w = [\mathcal{A} \cup \mathcal{B}]_w$;

2) $[\mathcal{A}]_w \vee [\mathcal{B}]_w = [\mathcal{A} \vee \mathcal{B}]_w = [C(\mathcal{A}) \cap C(\mathcal{B})]_w$;

3) $1_w = [\emptyset]_w$;

4) $0_w = [\mathcal{O}]_w$.

The above definitions are unambiguous and make $\mathfrak{M}_w$ a distributive lattice.

Theorem 3.2 (MUČNICK (1963)) $\mathfrak{M}_w$ is complete.

Proof. Let $\{(A_i)_w = \{[\mathcal{A}_i]_w : i \in I\}$ be a family of weak degrees of difficulty. It is easily seen that

$\bigwedge \{(A_i)_w : i \in I\} = [\bigcup \{\mathcal{A}_i : i \in I\}]_w$;

$\bigvee \{(A_i)_w : i \in I\} = [\cap \{C(\mathcal{A}_i) : i \in I\}]_w$.

We now list some properties of $\mathfrak{M}_w$.

Theorem 3.3 The cardinality of $\mathfrak{M}_w$ is $2^{2^{\aleph_0}}$.

Proof. Let $\mathcal{A}$ be a set of functions whose T-degrees constitute an antichain in $\mathfrak{D}_T$ and let the cardinality of $\mathcal{A}$ be $2^{\aleph_0}$ (for the existence of such an antichain in $\mathfrak{D}_T$, see for example SACKS (1961)). Let us index the elements of $\mathfrak{A}$ using the set $\mathbb{R}$ of real numbers; thus $\mathfrak{A}=\{f_i: i\in \mathbb{R}\}$. Given a subset $\mathfrak{B}$ of $\mathfrak{A}$, let us say that $\mathfrak{B}$ is <u>antisymmetric</u> if $(\forall i\in \mathbb{R})[f_i\in\mathfrak{B} \Longleftrightarrow f_{-i}\notin \mathfrak{B}]$. Now, there exist $2^{2^{\aleph_0}}$ antisymmetric subsets of $\mathfrak{A}$ and if $\mathfrak{B}_1$, $\mathfrak{B}_2$ are antisymmetric and distinct, then $\mathfrak{B}_1|_w \mathfrak{B}_2$. The proof actually shows that in $\mathfrak{M}_w$ there are antichains of cardinality $2^{2^{\aleph_0}}$. On the other hand $2^{2^{\aleph_0}}$ is clearly an upper bound for the cardinality of $\mathfrak{M}_w$. Thus we have shown that tha cardinality of $\mathfrak{M}_w$ is $2^{2^{\aleph_0}}$.

Definition 3.3 (MUČNICK (1963)) A weak degree of difficulty S_W is a <u>weak degree of solvability</u> if, for some function f, $S_w=[\{f\}]_w$ (hence $S_w=[\{g: f\le_T g\}]_w$).

Definition 3.4 (DYMENT (1976)) A degree of difficulty A (a weak degree of difficulty A_w) is said to be <u>finite</u> (<u>countable</u>, <u>at most countable</u>) if, for some $\mathfrak{A}$, $A=[\mathfrak{A}]$ ($A_w=[\mathfrak{A}]_w$) and $\mathfrak{A}$ is finite (countable, at most countable).

Maybe it should be emphasized that a weak degree of difficulty A_w is finite (countable, at most countable) if and only if A_w has a finite (countable, at most countable) basis (see Fact 3.1), once we define the notion of basis for weak degrees of difficulty in the same way as for degrees of difficulty. Thus the notion of basis for weak degrees of difficulty is not an interesting one, whereas it leads to several remarkable conclusions for degrees of difficulty. For example, in DYMENT (1976) it is proved that the property of being a degree of difficulty containing a mass problem having a basis is lattice-theoretic in $\mathcal{M}$.

Theorem 3.4 The property of being a weak degree of solvability is lattice-theoretic in $\mathcal{M}_w$.

Proof. We prove that S_w is a weak degree of solvability $\Longleftrightarrow (\exists T_w)[S_w <_w T_w \ \& \ (\forall A_w)[S_w <_w A_w \Longrightarrow T_w \leq_w A_w]]$.
Proof of $\Longrightarrow$. Let $S_w = [\{f\}]_w$. It suffices to take $T_w = [\{g: f <_T g\}]_w$.
Proof of $\Longleftarrow$. Assume that S_w is not a weak degree of solvability and let $S_w = [\mathcal{B}]_w$. Also assume that $S_w <_w A_w$, where $A_w = [C(\mathcal{A})]_w$. Since $C(\mathcal{A}) \not\leq_w \mathcal{B}$, we have (see Fact 3.1) that $\mathcal{B} \not\subseteq C(\mathcal{A})$ i.e. $(\exists f \in \mathcal{B})[f \notin C(\mathcal{A})]$. For such a function f, we conclude that $\mathcal{B} <_w \{f\}$ (since $\mathcal{B} \leq_w \{f\}$, but $[\mathcal{B}]_w$ is not a weak degree of solvability) and $C(\mathcal{A}) \not\leq_w \{f\}$; hence $S_w <_w [\{f\}]_w$ but $A_w \not\leq_w [\{f\}]_w$. Therefore there is no minimum of the weak degrees greater than S_w.

Corollary 3.1 $\mathcal{M}_w$ is atomic and has only one atom (henceforth denoted by $0'_w$; $0_w = [\mathcal{O}']_w$).

Proof. Immediate, since 0_w is clearly a weak degree of solvability.

The following is actually an improvement of the analogous result in DYMENT (1976).

Theorem 3.5 $(\forall A_w \notin \{0_w, 0'_w, 1_w\})(\exists$ weak degree of solvability $S_w)[A_w |_w S_w]$.

Proof. Let A_w be such that $0'_w <_w A_w <_w 1_w$. Then there exists a non-recursive function h such that $A_w \not\leq_w [\{h\}]_w$, for otherwise it would be $A_w \leq_w 0'_w$. If also $[\{h\}]_w \not\leq_w A_w$, then we have the desired result, for in this case $A_w |_w [\{h\}]_w$. Thus suppose that $A_w \not\leq_w [\{h\}]_w$ and $[\{h\}]_w \leq_w A_w$ and let $A_w = [\mathcal{A}]_w$. In this case, we show that there exists a function g such that $\{g\} \not\leq_w \mathcal{A}$ and $h \not\leq_T g$: then $\{g\} \not\leq_w \mathcal{A}$ and also $\mathcal{A} \not\leq_w \{g\}$ (for otherwise it would be $\{h\} \leq_w \{g\}$, i.e. $h \leq_T g$, a contradiction), thus proving the theorem. Indeed, if no such function existed then it would be (see Fact 3.1) $\mathcal{A} \subseteq \bigcap\{\{f: g \leq_T f\}: h \not\leq_T g\}$. But $\bigcap\{\{f: g \leq_T f\}: h \not\leq_T g\} = \emptyset$, since one can easily show that $(\forall f)(\exists g)[h \not\leq_T g \;\&\; g \not\leq_T f]$: hence we would have $\mathcal{A} = \emptyset$, i.e. $A_w = 1_w$, a contradiction.

A remark on maximal antichains in $\mathcal{M}_w$ is appropriate at this point. We already know that $\mathcal{M}_w$ has antichains of cardinality $2^{2^{\aleph_0}}$ (see proof of Theorem 3.1). Now, for every function f, let $(B_f)_w = [\{g: g \not\leq_T f\}]_w$. If f belongs to a minimal T-degree, then it is easily seen that the set $\{(B_f)_w, [\{f\}]_w\}$ is a maximal antichain of $\mathcal{M}_w$ (maximal with respect to inclusion, of course). Then in $\mathcal{M}_w$ there exist non-trivial maximal antichains of cardinality 2 (this fact conflicts with the case of $\mathcal{D}_T$ where every non-trivial maximal antichain has cardinality $2^{\aleph_0}$). In DYMENT (1976) it is proved that also $\mathcal{M}$ has non-trivial maximal antichains of cardinality 2.

It follows from Theorem 3.4 that, given a weak degree of solvability $S_w=[\{f\}]_w$, there is a weak degree T_w such that $(\forall A_w)[S_w<_w A_w \Rightarrow T_w \leq_w A_w]$; in the proof of that theorem it is shown that $T_w=[\{g: f<_T g\}]_w$. Throughout Theorem 3.6 below, we shall use $(T_S)_w$ to denote such a degree T_w (consistently with the use of T_S for $\mathcal{M}$); also, we shall use $\mathcal{T}_f$ to denote the mass problem $\{g: f<_T g\}$.

Theorem 3.6 Let A_w, B_w be weak degrees of difficulty such that $A_w<_w B_w$. Then $(A_w,B_w)=\emptyset \iff (\exists$ weak degree of solvability $S_w)[A_w=B_w \wedge S_w$ & $B_w \not\leq_w S_w$ & $B_w \leq_w (T_S)_w]$.

Proof. Let $A_w<_w B_w$ and $A_w=[\mathcal{A}]_w$, $B_w=[\mathcal{B}]_w$ where $\mathcal{A}=c(\mathcal{A})$, $\mathcal{B}=c(\mathcal{B})$.

Proof of $\Rightarrow$. Suppose that $(A_w,B_w)=\emptyset$. Since $\mathcal{A}<_w\mathcal{B}$, $(\exists f\in\mathcal{A})[f\notin\mathcal{B}]$. Thus $\mathcal{A}\leq_w\{f\}$ and $\mathcal{B}\not\leq_w\{f\}$. It follows that $\mathcal{A}\leq_w\mathcal{B}\wedge\{f\}<_w\mathcal{B}$; thus $\mathcal{A}\equiv_w\mathcal{B}\wedge\{f\}$ and $\mathcal{A}\leq_w\mathcal{B}\wedge\mathcal{T}_f$; but $\mathcal{A}\equiv_w\mathcal{B}\wedge\mathcal{T}_f \Rightarrow \mathcal{B}\wedge\mathcal{T}_f\leq_w\{f\}$; hence $\mathcal{B}\leq_w\{f\}$, a contradiction. It follows that $\mathcal{A}<_w\mathcal{B}\wedge\mathcal{T}_f\leq_w\mathcal{B}$; hence, by hypothesis, $\mathcal{B}\wedge\mathcal{T}_f\equiv_w\mathcal{B}$ i.e. $\mathcal{B}\leq_w\mathcal{T}_f$.

Proof of $\Longleftarrow$. Suppose that $\mathcal{A} \equiv_w \mathcal{B} \wedge \{f\}$, $\mathcal{B} \not\leq_w \{f\}$ and $\mathcal{B} \leq_w \mathcal{T}_f$. Assume that $(\exists \mathcal{C})[\mathcal{A} <_w \mathcal{C} <_w \mathcal{B}]$. Let $\mathcal{C}_1 = \{g \in \mathcal{C} : f \leq_T g\}$. Since $\mathcal{C}_1 \subseteq \mathcal{C}$, we have that $\mathcal{C} \leq_w \mathcal{C}_1$; moreover, $\{f\} \leq_w \mathcal{C}_1$. On the other hand, it must be $\mathcal{C}_1 \not\leq_w \{f\}$, for otherwise we would have $\mathcal{C} \leq_w \mathcal{B} \wedge \{f\}$ and since $\mathcal{A} \equiv_w \mathcal{B} \wedge \{f\}$ this would imply $\mathcal{C} \leq_w \mathcal{A}$, a contradiction. Thus, $\{f\} <_w \mathcal{C}_1$; but then also $\mathcal{T}_f \leq_w \mathcal{C}_1$ which in turn implies $\mathcal{B} \leq_w \mathcal{C}_1$. On the other hand, since $\mathcal{A} <_w \mathcal{C}$ and $\mathcal{A} \equiv_w \mathcal{B} \wedge \{f\}$ and $(\forall g \in \mathcal{C} - \mathcal{C}_1)[f \not\leq_T g]$, we have that $(\forall g \in \mathcal{C} - \mathcal{C}_1)(\exists h \in \mathcal{B})[h \leq_T g]$. Therefore we conclude that $\mathcal{B} \leq_w \mathcal{C}$, a contradiction.

Remark 3.1 A very similar characterization of empty intervals of $\mathcal{M}$ is given in DYMENT (1976); namely, it is proved that if $A<B$ then $(A,B)=\emptyset \Longleftrightarrow (\exists$ degree of solvability $S)[A=B \wedge S$ & $B \not\leq S$ & $B \leq T_S]$.

Theorem 3.7 If $A_w <_w B_w$ and B_w is countable then $(\exists C_w)\ [A_w <_w C_w <_w B_w]$.

Proof. The proof follows from Theorem 3.6, since for no countable mass problem $\mathcal{B}$ and non-recursive function f can we have $\mathcal{B} \leq_w \mathcal{T}_f$, if $[\mathcal{B}]_w \neq 0_w$.

Remark 3.2 An embedding $I: \mathcal{D}_T \longrightarrow \mathcal{M}_w$ similar to that of Fact 1.3 exists. The proof is obvious.

Definition 3.5 (LERMAN (1983)) Let $\mathcal{L}=\langle L,\leq\rangle$ be a partial order and let $A\subseteq L$. A is an <u>automorphism base</u> for $\mathcal{L}$ if every function $F: A \longrightarrow A$ has at most one extension $F: L \longrightarrow L$ which is an automorphism of $\mathcal{L}$.

Theorem 3.8 The set of weak degrees of solvability is an automorphism base for $\mathcal{M}_w$.

Proof. If $B_w=[\mathcal{B}]_w$ then $B_w=\bigwedge\{[\{f\}]_w: f\in\mathcal{B}\}$. The theorem then follows from the observation that if $F: \mathcal{M}_w \longrightarrow \mathcal{M}_w$ is an automorphism then $F(\bigwedge\{(A_i)_w: i\in I\})=\bigwedge\{F((A_i)_w): i\in I\}$: indeed, $\bigwedge\{(A_i)_w: i\in I\}\leq_w (A_i)_w \Longrightarrow F(\bigwedge\{(A_i)_w: i\in I\})\leq_w F((A_i)_w)$; suppose that $(\exists B_w)(\forall i)\ [B_w\leq_w F((A_i)_w)\ \&\ B_w \not\leq_w F(\bigwedge\{(A_i)_w: i\in I\})]$: let $B_w=F(C_w)$, then $(\forall i)[C_w\leq_w (A_i)_w]$ but $C_w \not\leq_w \bigwedge\{(A_i)_w: i\in I\}$, a contradiction.

Remark 3.3 Let $\mathrm{Aut}(\mathcal{D}_T)$, $\mathrm{Aut}(\mathcal{M}_w)$ denote the automorphism groups of $\mathcal{D}_T$ and $\mathcal{M}_w$ respectively. From Theorem 3.4 and Theorem 3.8 it follows that there exists an isomorphism $\Gamma: \mathrm{Aut}(\mathcal{D}_T) \longrightarrow \mathrm{Aut}(\mathcal{M}_w)$. Indeed, suppose that $F\in\mathrm{Aut}(\mathcal{D}_T)$ and for every f let $g_f\in F([f]_T)$; then F is mapped by Γ to the automorphism G of $\mathcal{M}_w$ defined by $G([\mathcal{A}])=G(\bigwedge\{[\{f\}]_w: f\in\mathcal{A}\})=\bigwedge\{[\{g_f\}]_w: f\in\mathcal{A}\}$.

Definition 3.6 A weak degree of difficulty E_w is a <u>weak degree of enumerability</u> if for some set $A\subseteq\omega$,

$E_w=[\{f: \text{range}(f)=A\}]_w$. The weak degree of enumerability of A is denoted by $(E_A)_w$.

Fact 3.2 For every $A\subseteq\omega$, $(E_A)_w=[\{f: A\leq_e f\}]_w$; thus if $[A]_e$ is a non-total e-degree then $(E_A)_w=[\{f: A<_e f\}]_w$.

Proof. It is enough to show that $(\forall f)[A\leq_e f \Rightarrow (\exists g)[g\leq_T f \ \&\ \text{range}(g)=A]]$. So, suppose that $A\leq_e f$; then for some z, $A=\Phi_z(f)$. We use this fact to construct a function g such that range(g)=A and g is r.e. in f. By totality, this also implies that $g\leq_T f$ and the proof is complete.

Theorem 3.9 If $C(\mathcal{Q})\leq_w\{f: A\leq_e f\}$ and $\mathcal{Q}$ is countable then $(\exists f\in\mathcal{Q})[f\leq_e A]$.

Proof. The proof is similar to the proof of Theorem 3.4 of DYMENT (1976). Let $\mathcal{Q}=\{f_i: i\in\omega\}$ and assume that $(\forall i)[f_i \not\leq_e A]$. It is enough to construct a function g such that range(g)=A and satisfying the following requirements for every $n,i\in\omega$:

$P_{<n,i>}$: $\Psi_n(g) \neq f_i$.

Let $A=\{x_0,x_1,\dots\}$. We define by induction a sequence $\{\tilde{g}_s\}_{s\in\omega}$ of finite initial segments. Assume $\tilde{g}_{-1}=\emptyset$.

Step 0) Assume that $\tilde{g}_{2s-1}$ is already defined. Define

$$\tilde{g}_{2s}(x)=\begin{cases}\tilde{g}_{2s-1}(x) & \text{if } x<lh(\tilde{g}_{2s-1})\\ x_s & \text{if } x=lh(\tilde{g}_{2s-1})\end{cases}$$

Step 2s+1) Assume that $\tilde{g}_{2s}$ is already defined and let $s=<n,i>$.

Case a) $(\exists x)\ (\exists \tilde{g} \supseteq \tilde{g}_{2s})$ [range$(\tilde{g}) \subseteq A$ & $\Psi_n(\tilde{g})(x)\downarrow$ & $\Psi_n(\tilde{g})(x) \neq f_i(x)$].

Then let $\tilde{g}_{2s+1}$ be the least such $\tilde{g}$ in the lexicographical ordering defined in the Introduction.

Case b) Otherwise let $\tilde{g}_{2s+1}=\tilde{g}_{2s}$.

Finally, let $g= \bigcup\{\tilde{g}_s: s\in\omega\}$.

It easily follows that range$(g)=A$. Suppose now that $\Psi_n(g)=f_i$ for some $n,i\in\omega$; then $f_i= \bigcup\{\Psi_n(\tilde{g}): \tilde{g} \supseteq \tilde{g}_{2<n,i>}$ & range$(\tilde{g}) \subseteq A\}$. Thus $f_i\leq_e A$, a contradiction.

Remark 3.3 If $[A]_e$ is a non-total e-degree, then $(E_A)_w$ is not countable (this is an immediate consequence of the previous theorem). Of course, if $[A]_e$ is a total e-degree, then $(E_A)_w=(S_A)_w$ since if $g\in[A]_e$ then $(E_A)_w= [\{f: g\leq_e f\}]_w = [\{g\}]_w=(S_A)_w$.

Theorem 3.10 The mapping $I: \mathfrak{D}_e \longrightarrow \mathcal{M}_w$ given by $I([A]_e)=(E_A)_w$ is an embedding of the upper semilattice $\mathfrak{D}_e$ onto the weak degrees of enumerability.

Proof. We will show that $A\leq_e B \Longleftrightarrow \{f: A\leq_e f\}\leq_w\{f: B\leq_e f\}$.

Proof of $\Longrightarrow$. This is obvious, since $A\leq_e B$ implies $\{f: B\leq_e f\} \subseteq \{f: A\leq_e f\}$.

Proof of $\Longleftarrow$. Suppose that $A \not\leq_e B$. We show that there exists a function f such that range$(f)=B$ and satisfying the following requirements, for every $n\in\omega$:

P_n: $\Phi_n(f) \neq A$.

For such an f we have $B \leq_e f$ and $(\forall g)[A \leq_e g \Rightarrow g \not\leq_T f]$. The construction is similar to that of Theorem 3.9. Set $\tilde{f}_{-1} = \emptyset$ and let $B = \{b_0, b_1, \ldots\}$.

Step 2s) Define

$$\tilde{f}_{2s}(x) = \begin{cases} \tilde{f}_{2s-1}(x) & \text{if } x < lh(\tilde{f}_{2s-1}) \\ b_s & \text{if } x = lh(\tilde{f}_{2s-1}) \end{cases}$$

Step 2s+1)

Case a) $(\exists \tilde{f} \supseteq \tilde{f}_{2s})[\mathrm{range}(\tilde{f}) \subseteq B \ \&\ \Phi_s(\tilde{f}) \not\subseteq A]$.

Let $\tilde{f}_{2s+1}$ be the least such $\tilde{f}$.

Case b) Otherwise, let $\tilde{f}_{2s+1} = \tilde{f}_{2s}$.

Finally, let $f = \bigcup \{\tilde{f}_s : s \in \omega\}$.

Clearly $\mathrm{range}(f) = B$. Suppose now that $\Phi_s(f) = A$. Then $(\forall x \in A)(\exists \tilde{f})[\tilde{f} \supseteq \tilde{f}_{2s} \ \&\ \mathrm{range}(\tilde{f}) \subseteq B \ \&\ x \in \Phi_s(\tilde{f})]$. Furthermore, since $\Phi_s(f) = A$, $(\forall \tilde{f})[\tilde{f} \supseteq \tilde{f}_{2s} \ \&\ \mathrm{range}(\tilde{f}) \subseteq B \Rightarrow \Phi_s(\tilde{f}) \subseteq A]$. Thus $A = \bigcup \{\Phi_s(\tilde{f}) : \tilde{f} \supseteq \tilde{f}_{2s} \ \&\ \mathrm{range}(\tilde{f}) \subseteq B\}$; hence $A \leq_e B$, a contradiction.

4. Degrees of enumerability.

Fact 4.1 For every $A \subseteq \omega$, $\mathcal{E}_A \equiv \{z*f: \Phi_z(f)=A\}$.

Proof. Let $A \subseteq \omega$ be given and let $\mathcal{B}=\{z*f: \Phi_z(f) = A\}$. If A is finite, then $[\mathcal{E}_A]=[\mathcal{B}]=0$ and the claim is proved. Suppose that A is infinite. Let h be a recursive function such that, for every $u \in \omega$, $D_{h(u)}=\{\langle x+1,y\rangle: \langle x,y\rangle \in D_u\}$.

By the Recursion Theorem, there exists an $e \in \omega$ such that

$$W_e=\{\langle\langle x,y\rangle,u\rangle: \tau^{-1}(D_u) \text{ is a finite initial segment } \& \ (\exists z)[\langle 0,z\rangle \in D_u \ \& \ (\exists s)(\exists v)[\langle y,v\rangle \in W_z^s \ \& \ D_{h(v)} \subseteq D_u \& (\forall a \in W_z^s)(\forall b \in D_{h((a)_1)})[(b)_0 < lh(\tau^{-1}(D_u))] \ \& \ (\forall i<x)(\exists r)(\exists w)[r \neq y \ \& \ \langle\langle i,r\rangle,w\rangle \in W_e \ \& \ D_w \subseteq D_u] \ \& \ (\forall t<s)(\forall a \in W_z^t)[(a)_0 \neq y \ \& \ D_{h(a)_1} \subseteq D_u \Rightarrow (\exists i<x)(\exists w)[\langle\langle i,(a)_0\rangle,w\rangle \in W_e \ \& \ D_w \subseteq D_u]]]]\}.$$

Let Ω_e be the partial recursive operator defined by the enumeration operator Φ_e and, given any partial function φ, let φ^+ denote the partial function given by $\varphi^+(x)=\varphi(x+1)$. It is not difficult to see that if f is a total function (warning: totality is crucial here) then $f \in dom(\Omega_e)$ and $\Omega_e(f)$ can be informally computed as follows:

Step 0) If $f(0)=z$, say, then enumerate W_z until you find a number $\langle y,u\rangle \in W_z$ such that $D_u \subseteq \tau(f^+)$. If such a number exists, for the first such number $\langle y,u\rangle$ in the enumeration of W_z, let $\Omega_e(f)(0)=y$.

Otherwise, $\Omega_e(f)(0)\uparrow$.

Step n+1) If $(\exists k\le n)[\Omega_e(f)(k)\uparrow]$ then $\Omega_e(f)(n+1)\uparrow$. Otherwise, suppose that $f(0)=z$ and, for every $k\le n$, let $\Omega_e(f)(k)=y_k$. Enumerate W_z until you find a number $\langle y,u\rangle\in W_z$ such that $(\forall k\le n)[y\ne y_k]$ and $D_u\subseteq\tau(f^+)$. For the first such number, if it exists, in the enumeration of W_z, let $\Omega_e(f)(n+1)=y$. If no such a number exists, let $\Omega_e(f)(n+1)\uparrow$.

Using the fact that A is infinite, one easily concludes that if $z*f\in\mathcal{B}$, then $\Omega_e(z*f)$ is total and $\mathrm{range}(\Omega_e(z*f))=A$. Therefore, by the Fundamental Operator Theorem, we have that $\mathcal{E}_A\le\mathcal{B}$ via Ψ_e, since Ψ_e is a recursive operator (the one determined by the enumeration operator $\Phi_{\sigma(e)}$) which coincides with Ω_e on all total functions in the domain of Ω_e.

For the converse it is enough to show that $(\forall\mathcal{C})[\mathcal{E}_A\le\mathcal{C}\Longrightarrow\mathcal{B}\le\mathcal{C}]$. To this end, let $\mathcal{E}_A\le\mathcal{C}$ via Ψ_z; hence $(\forall f\in\mathcal{C})[\Psi_z(f)$ is total & $\mathrm{range}(\Psi_z(f))=A]$. Let $W_w=\{\langle x,2^{\langle u,x\rangle}\rangle: x,u\in\omega\}$ and let $\Phi_v=\Phi_w\circ\Phi_{\sigma(z)}$ (see Introduction for the definition of σ). Finally, if Ψ is a recursive operator such that, for every $f\in{}^{\omega}\omega$, $\Psi(f)=v*f$, then $\mathcal{B}\le\mathcal{C}$ via Ψ : indeed, since $\Phi_v(f)=\Phi_w(\Phi_{\sigma(z)}(f))$, we have that, for every $x\in\omega$, $x\in\Phi_v(f)\Longleftrightarrow(\exists u)[\langle x,2^{\langle u,x\rangle}\rangle\in W_w$ & $\langle u,x\rangle\in\Phi_{\sigma(z)}(f)]\Longleftrightarrow x\in\mathrm{range}(\Psi_z(f))\Leftrightarrow x\in A$.

Aside. In the proof of Fact 4.1, we have given a formal definition of Ψ_e, by first defining the r.e. set

W_e. Usually, formal definitions of recursive operators are burdensome and scarcely intuitive. For this reason, in the following we often give only informal descriptions of recursive operators. With the help of the Fundamental Operator Theorem and the Corollary to the Fundamental Operator Theorem exposed in the Introduction, going from an informal definition to a formal one is most often a matter of routine.

Remark 4.1 If $[A]_e$ is a non-total e-degree then $\mathcal{E}_A \equiv \{z*f: \Phi_z(f)=A \ \& \ (\forall e)[\Phi_e(A) \neq f]\}$.

Definition 4.1 (DYMENT (1976)) A degree of difficulty A is <u>dense</u> if $A=[\mathcal{A}]$, for some mass problem $\mathcal{A}$ which is dense in the Baire topology.

Definition 4.2 (DYMENT (1976)) A degree of difficulty A is <u>discrete</u> if $A=[\mathcal{A}]$, for some mass problem $\mathcal{A}$ which is discrete in the Baire topology.

Definition 4.3 (DYMENT (1976)) A degree of difficulty A is <u>effectively discrete</u> if, for some mass problem $\mathcal{A}$, $A=[\mathcal{A}]$ and $(\forall f\in\mathcal{A})(\forall g\in\mathcal{A})[f(0)=g(0) \Rightarrow f=g]$.

The name effectively discrete comes from the following observation:

$(\forall f\in\mathcal{A})(\forall g\in\mathcal{A})[f(0)=g(0)\Rightarrow f=g] \Longleftrightarrow (\exists$ r.e. set S of finite initial segments)[$(\forall g\in\mathcal{A})(\exists\tilde{f}\in S)[\tilde{f}\subseteq g]$ & $(\forall\tilde{f}\in S)$ $[|S_{\tilde{f}}\cap\mathcal{A}|\leq 1]]$.

Definition 4.4 (MUČNICK (1963)) A mass problem $\mathcal{A}$ is <u>uniform</u> if $(\forall\hat{f})[S_{\tilde{f}}\cap\mathcal{A}\neq\emptyset\Rightarrow S_{\tilde{f}}\cap\mathcal{A}\leq\mathcal{A}]$. A degree of difficulty is <u>uniform</u> if it contains a uniform mass problem.

Remark 4.2 It follows from the definition that if $\mathcal{A}$ is uniform then $(\forall\tilde{f})[S_{\tilde{f}}\cap\mathcal{A}\neq\emptyset\Leftrightarrow S_{\tilde{f}}\cap\mathcal{A}\equiv\mathcal{A}]$.

Example 4.1 Examples of uniform mass problems are: the singleton $\{f\}$, for every function f; problems of extendability and problems of separability (see e.g. ROGERS (1967) for the definitions); every mass problem $\mathcal{A}$ satisfying: $f\in\mathcal{A}$ & $f\equiv_T g\Rightarrow g\in\mathcal{A}$ (or $f\in\mathcal{A}$ & $f\leq_T g\Rightarrow g\in\mathcal{A}$, etc.).

One should not be led to think that every dense mass problem is uniform. The following is an example of a mass problem which is dense but not uniform.

Fix an enumeration without repetitions $\{\tilde{f}_i : i\in\omega\}$ of all finite initial segments and define a sequence of functions $\{f_n : n\in\omega\}$ satisfyting:

1) $f_0\supseteq\tilde{f}_0$ & f_0 is not recursive;

2) $f_{n+1}\supseteq\tilde{f}_{n+1}$ & $(\forall k\leq n)[f_{n+1}|_T f_k]$.

This sequence clearly satisfies $(\forall i,j)[i\neq j\Rightarrow f_i|_T f_j]$.

The mass problem $\mathcal{A}=\{f_n: n\in\omega\}$ is dense, since $(\forall\tilde{f})[S_{\tilde{f}}\cap\mathcal{A}\neq\emptyset]$. But we have also that $(\forall\tilde{f}\neq\emptyset)[S_{\tilde{f}}\cap\mathcal{A}\not\leq\mathcal{A}]$, thus proving that is not uniform. To show that $(\forall\tilde{f}\neq\emptyset)[S_{\tilde{f}}\cap\mathcal{A}\not\leq\mathcal{A}]$, let $\tilde{f}\neq\emptyset$ be given and let $\tilde{g}$ be such that $\tilde{f}\not\subseteq\tilde{g}$. If, say, $\tilde{g}=\tilde{f}_j$, then $(\forall h\in S_{\tilde{f}}\cap\mathcal{A})[h|_T f_j]$; therefore for no recursive operator Ψ can we have that $\Psi(f_j)$ is total and $\Psi(f_j)\in S_{\tilde{f}}\cap\mathcal{A}$, hence $S_{\tilde{f}}\cap\mathcal{A}\not\leq\mathcal{A}$.

Fact 4.2 (MUČNICK (1963)) Every degree of enumerability is uniform.

Proof. Let $S_{\tilde{f}}\cap\mathcal{E}_A\neq\emptyset$; then range$(\tilde{f})\subseteq A$. If Ψ is a recursive operator such that $(\forall g\in{}^{\omega}\omega)[\Psi(g)=\tilde{f}*g]$, where

$$\tilde{f}*g(x)=\begin{cases}\tilde{f}(x) & \text{if } x<lh(\tilde{f})\\ g(x-lh(\tilde{f})) & \text{if } x\geq lh(\tilde{f}),\end{cases}$$

then $S_{\tilde{f}}\cap\mathcal{E}_A\leq\mathcal{E}_A$ via Ψ.

Definition 4.5 Let $\mathcal{L}=\langle L,\leq,\vee,\wedge\rangle$ be a lattice. An element $a\in L$ is <u>join-irreducible</u> if $(\forall b,c)[a=b\vee c\Longrightarrow[a\leq b$ or $a\leq c]]$; $a\in L$ is <u>meet-irreducible</u> if $(\forall b,c\in L)[a=b\wedge c\Longrightarrow[b\leq a$ or $c\leq a]]$.

Lemma 4.1 Every uniform degree of difficulty is meet-irreducible.

Proof. We shall show that if $\mathcal{A}$ is uniform and $\mathcal{B} \not\leq \mathcal{A}$ & $\mathcal{C} \not\leq \mathcal{A}$ then $\mathcal{B} \wedge \mathcal{C} \not\leq \mathcal{A}$. Suppose that $\mathcal{B} \wedge \mathcal{C} \leq \mathcal{A}$ via Ψ and $\mathcal{B} \not\leq \mathcal{A}$. Then $\Psi(\mathcal{A}) \cap 1 * \mathcal{C} \neq \emptyset$, i.e. $(\exists f \in \mathcal{A})[\Psi(f)(0)=1]$ whence $(\exists \tilde{f})(\exists f)[\tilde{f} \subseteq f$ & $f \in \mathcal{A}$ & $\Psi(\tilde{f})(0)=1]$. Now, for such an $\tilde{f}$, $S_{\tilde{f}} \cap \mathcal{A} \neq \emptyset$; so, $S_{\tilde{f}} \cap \mathcal{A} \leq \mathcal{A}$. On the other hand, since $(\forall f \in \mathcal{A})[\Psi(f)$ is total & $\Psi(f) \in 0 * \mathcal{B} \cup 1 * \mathcal{C}]$, it must be $(\forall f \in S_{\tilde{f}} \cap \mathcal{A})[\Psi(f) \in 1 * \mathcal{C}]$. Therefore $1 * \mathcal{C} \equiv \mathcal{C} \leq S_{\tilde{f}} \cap \mathcal{A} \leq \mathcal{A}$.

Corollary 4.1 Every degree of enumerability is meet-irreducible.

Proof. Obvious by Fact 4.2 and Lemma 4.1

Lemma 4.2 Let $\mathcal{A}$ be a mass problem. If $(\exists f \in \mathcal{A})(\exists g_1)$ $(\exists g_2)[f \leq_T g_1 \vee g_2$ & $g_1 |_T g_2$ & $g_1 \notin C(\mathcal{A})$ & $g_2 \notin C(\mathcal{A})]$ then $[\mathcal{A}]$ is join-reducible.

Proof. Let f, g_1, g_2 be as in the statement of the lemma. Define $\mathcal{A}_1 = 0 * \mathcal{A} \cup \{1 * g_1\}$, $\mathcal{A}_2 = 0 * \mathcal{A} \cup \{2 * g_2\}$; let z be such that $\Psi_z(g_1 \vee g_2) = f$ and, given a partial function φ, let φ^{++} denote the partial function given by $\varphi^{++}(x) = \varphi(x+2)$. Clearly, $\mathcal{A}_1 | \mathcal{A}_2$, hence $\mathcal{A}_1 < \mathcal{A}$, $\mathcal{A}_2 < \mathcal{A}$. Consider now a mapping $\Psi : ({}^{\omega}\omega)^* \longrightarrow ({}^{\omega}\omega)^*$ defined by

$$\Psi(\varphi)(x)=\begin{cases}\uparrow & \text{if } \varphi(0)\uparrow \text{ or } \varphi(1)\uparrow\\ \varphi(2(x+1)) & \text{if } \varphi(0)\downarrow \ \& \ \varphi(1)\downarrow \ \& \ \varphi(0)=0\\ \varphi(2(x+1)+1) & \text{if } \varphi(0)\downarrow \& \ \varphi(1)\downarrow \& \ \varphi(0)\neq 0 \ \& \ \varphi(1)=0\\ \Psi_z(\quad ++)(x) & \text{if } \varphi(0)\downarrow \& \ \varphi(1)\downarrow \& \ \varphi(0)\neq 0 \ \& \ \varphi(1)\neq 0\end{cases}$$

Thus, for every functions f,g and numbers $i,j\in\omega$,

$$\Psi(i*f \vee j*g)=\begin{cases} f & \text{if } i=0\\ g & \text{if } i\neq 0 \quad j=0\\ \Psi_z(f \quad g) & \text{otherwise}\end{cases}$$

Since clearly Ψ is a recursive operator, we have that $\mathcal{A} \leq \mathcal{A}_1 \vee \mathcal{A}_2$ via Ψ ; so $[\mathcal{A}]$ is join-reducible.

In JOCKUSH-POSNER (1981), it is proved that if a set $D \subseteq \mathcal{D}_T$ is comeager (a set $D \subseteq \mathcal{D}_T$ is comeager if $\{c_A: [A]_T \in D\}$ is comeager in the Cantor topology) then $(\forall a_T)(\exists (m_i)_T: i=1,\dots,4)[(m_i)_T \in D \ \& \ a_T=((m_i)_T \vee (m_2)_T) \wedge \wedge((m_3)_T \vee (m_4)_T)]$.

Given a mass problem $\mathcal{A}$, let us say that $\mathcal{A}$ is <u>T-meager</u> (<u>T-comeager</u>) if $\{[f]_T: f\in\mathcal{A}\}$ is meager (comeager) in the sense specified above.

Corollary 4.2 Let $\mathcal{A}$ be a any mass problem such that $C(\mathcal{A})$ is T-meager; then $[\mathcal{A}]$ is join reducible.

Proof. If $C(\mathcal{A})$ is T-meager, then ${}^{\omega}\omega - C(\mathcal{A})$ is T-comeager. It follows that $(\forall f)(\exists g_i: i=1,\dots,4)[g_i \notin C(\mathcal{A}) \ \& \ [f]_T = ([g_1]_T \vee [g_2]_T) \wedge ([g_3]_T \vee [g_4]_T)$. Therefore $(\forall f\in\mathcal{A})(\exists g_1)(\exists g_2) \quad [g_1 \notin C(\mathcal{A}) \ \& \ g_2 \notin C(\mathcal{A}) \ \&$

$f \leq_T g_1 \vee g_2]$. Fix an $f \in \mathcal{A}$ and let g_1, g_2 be as above; then $f \leq_T g_1 \vee g_2$ & $g_1, g_2 \notin C(\mathcal{A})$ & $g_1 |_T g_2$. It follows from Lemma 4.2 that $[\mathcal{A}]$ is join-reducible.

Corollary 4.3 Every degree of enumerability greater than 0 is join-reducible.

Proof. Notice that $C(\mathcal{E}_A) = \{f: A \leq_e f\}$ (see Fact 3.2) and this latter set is T-meager if A is not r.e.; thus $[\mathcal{E}_A]$ is join-reducible by Corollary 4.2.

Corollary 4.4 Let $A \subsetneq \omega$ be such that $[A]_e$ is a non-total e-degree. Then $(\forall \mathcal{B})[\mathcal{E}_A < \mathcal{B} \Longrightarrow (\exists \mathcal{C})[\mathcal{E}_A < \mathcal{C} < \mathcal{B}]$.

Proof. By Corollary 4.1 and the characterization of empty intervals of $\mathcal{M}$, mentioned in Remark 3.1.

Theorem 4.1 Let $A \subsetneq \omega$ be any set. Then $(\forall \mathcal{B} < \mathcal{E}_A)(\exists \mathcal{C})[\mathcal{B} < \mathcal{C} < \mathcal{E}_A]$.

Proof. In view of the characterization of empty intervals of $\mathcal{M}$ mentioned in Remark 3.1, it suffices to show that $(\forall f)[\mathcal{E}_A \not\leq \{f\} \Longrightarrow \mathcal{E}_A \not\leq \mathcal{C}_f]$ (recall that $\mathcal{C}_f = \{z*g: \Psi_z(g)=f$ & $(\forall n)[\Psi_n(f) \neq g]\}$). Our goal is to prove that $(\forall f)[A \not\leq_e f \Longrightarrow (\exists g)[f <_T g$ & $\mathcal{E}_A \not\leq \{g\}]]$.
We say that a finite initial segment $\tilde{g}$ is <u>compatible with</u> f (notation: $\tilde{g}$ cpt f) if $(\forall x)[2x < lh(\tilde{g}) \Longrightarrow \tilde{g}(2x) = f(x)]$. Let f be given. In order to construct the desired

function g we shall satisfy the following requirements for all $n \in \omega$:

P_{2n}: $g \neq \Psi_n(f)$;

P_{2n+1}: $\Psi_n(g) \notin \mathcal{E}_A$.

We start with $\tilde{g}_{-1} = \emptyset$.

Step 2n) Let $x_n = \mu x.[x$ is odd & $\tilde{g}_{2n-1}(x)\uparrow]$. Define a finite initial segment $\tilde{g}_{2n} \supseteq \tilde{g}_{2n-1}$ such that $lh(\tilde{g}_{2n}) = x_n + 1$ by letting $\tilde{g}_{2n}(x) = f(x)$ if x even & $x < x_n$ & $\tilde{g}_{2n-1}(x)\uparrow$ and

$$\tilde{g}_{2n}(x_n) = \begin{cases} 0 & \text{if } \Psi_n(f)(x_n)\uparrow \\ 1 \dot{-} \Psi_n(f)(x_n) & \text{otherwise} \end{cases}$$

Notice that $\tilde{g}_{2n}$ cpt f.

Step 2n+1)

Case a) $(\exists \tilde{g})(\tilde{g}$ cpt f & $\tilde{g} \supseteq \tilde{g}_{2n}$ & range$(\Psi_n(\tilde{g})) \nsubseteq A]$.

In this case, let $\tilde{g}_{2n+1}$ be the least such $\tilde{g}$.

Case b) otherwise, let $\tilde{g}_{2n+1} = \tilde{g}_{2n}$.

Finally, let $g = \bigcup \{\tilde{g}_n : n \in \omega\}$.

Clearly $f \leq_T g$, but $g \nleq_T f$ (because all requirements P_{2n} are satisfied by g). Suppose now that range$(\Psi_n(g)) = A$, for some $n \in \omega$. Then $A = \bigcup \{$range$(\Psi_n(\tilde{g}))$: $\tilde{g} \supseteq \tilde{g}_{2n}$ & $\tilde{g}$ cpt f$\}$, which implies A is r.e. in f and, since f is total, $A \leq_e f$, a contradiction.

Having proved the existence of such a function g, let us suppose that $\mathcal{E}_A \leq \mathcal{C}_f$ via a recursive operator Ψ and let $z \in \omega$ be such that $\Psi_z(g) = f$ (such a z exists since $f \leq_T g$). Then $z * g \in \mathcal{C}_f$ and range$(\Psi(z*g)) = A$; thus range$(\Psi'(g)) = A$ where Ψ' is a recursive operator such that, for every function h, $\Psi'(h)=\Psi(z*h)$. This fact contradicts previous conclusions.

Unfortunately, Corollaries 4.1, 4.3, 4.4 and Theorem 4.1 do not characterize degrees of enumerability in the same sense as Theorem 1.2 characterizes the degrees of solvability. The conclusions described in the above quoted corollaries and in Theorem 4.1 are shared by other classes of degrees of difficulty as well, such as for instance degrees of the form $[\{f: f_0 \equiv_T f\}]$, for every non recursive function f_o (we refer to DYMENT (1976) for a study of these degrees).

We conclude this section with the following simple remark.

Remark 4.3 $(\forall A \subseteq \omega)(\exists$ finite degree of difficulty $F > E_A)$ $(\forall B \subseteq \omega)[E_B \notin (E_A, F)]$.

Proof. In ROZINAS (1978) it is proved that $(\forall A \subseteq \omega)(\exists f)(\exists g)[f |_e g \ \& \ [A]_e = [f]_e \wedge [g]_e]$. Let $A \subseteq \omega$ be given and let $f, g \in {}^\omega\omega$ be such that $f |_e g$ & $[A]_e = [f]_e \wedge [g]_e$. Since there exist two enumeration operators Φ_u, Φ_v such that $\Phi_u(f) = A$ and $\Phi_v(g) = A$, by Fact 4.1 we have that $\mathcal{E}_A \leq \{f,g\}$ (being $\{u*f, v*g\} \equiv \{f,g\}$). On the other hand, it must be $\{f,g\} \not\leq \mathcal{E}_A$ because $\{f,g\} \leq \mathcal{E}_A$ would imply $f \leq_e A$ or $g \leq_e A$ (see e.g. Theorem 3.9 and the fact that $\leq \subseteq \leq_w$ or, else, use the fact that $[\mathcal{E}_A]$ is meet-irreducible by Corollary 4.1). Therefore $\mathcal{E}_A < \{f,g\}$.

Let us suppose now that for some $B \subseteq \omega$, $\mathcal{E}_A < \mathcal{E}_B < \{f,g\}$. But then $B \leq_e f$ or $B \leq_e g$, hence $B \leq_e A$, a contradiction.

5. Relationships between $\mathfrak{M}$, $\mathfrak{M}_e$ and $\mathfrak{M}_w$.

We first compare $\mathfrak{M}$ and $\mathfrak{M}_e$, by showing the existence of a certain pair of homomorphisms $F_e: \mathfrak{M}_e \longrightarrow \mathfrak{M}$, $I_e: \mathfrak{M} \longrightarrow \mathfrak{M}_e$.

For a given mass problem $\mathcal{A} \subsetneq ({}^{\omega}\omega)^*$ of partial functions, let $\mathcal{A}^*$ denote (see DYMENT (1976)) the mass problem $\{f: \text{range}(f) \in \tau(\mathcal{A})\}$, where, of course, $\tau(\mathcal{A}) = \{\tau(\varphi): \varphi \in \mathcal{A}\}$. Crucial to Proposition 5.1 below is the following fact

Fact 5.1 (DYMENT (1976)) Let $\mathcal{A}, \mathcal{B}$ be mass problems of partial functions. Then

a) $\mathcal{A} \leq_e \mathcal{B} \Longrightarrow \mathcal{A}^* \leq \mathcal{B}^*$;

b) $\mathcal{A} \leq_e \mathcal{A}^*$ & $(\forall \mathcal{C} \subseteq {}^{\omega}\omega)[\mathcal{A} \leq_e \mathcal{C} \Longrightarrow \mathcal{A}^* \leq_e \mathcal{C}]$ (hence $[\mathcal{A}^*]_e$ is the minimum of the degrees in $\mathfrak{M}_e$ greater than or equal to $[\mathcal{A}]_e$ and containing a mass problem which consists of total functions).

Proposition 5.1 The function $F_e: \mathfrak{M}_e \longrightarrow \mathfrak{M}$ given by $F_e([\mathcal{A}]_e) = [\mathcal{A}^*]$ is a lattice-theoretic epimorphism.

Proof. That F_e is well defined follows from Fact 5.1 a). From Fact 5.1 b), we conclude also that F_e is onto. Indeed, let $A = [\mathcal{A}]$ be a degree of difficulty in $\mathfrak{M}$; since the elements of $\mathcal{A}$ are total functions, we have that $\mathcal{A} \equiv_e \mathcal{A}^*$. But also the elements of $\mathcal{A}^*$ are total functions; hence, by definitions of $\leq_e$ and $\leq$, it must be $\mathcal{A} \equiv \mathcal{A}^*$. Therefore $F_e([\mathcal{A}]_e) = [\mathcal{A}^*] = [\mathcal{A}] = A$.

Clearly F_e preserves the least and the greatest element.

It is left to show 1) $(\forall A_e, B_e \in \mathcal{M}_e)[F_e(A_e \wedge B_e) = F_e(A_e) \wedge F_e(B_e)]$ and 2) $(\forall A_e, B_e \in \mathcal{M}_e)[F_e(A_e \vee B_e) = F_e(A_e) \vee F_e(B_e)]$.

Let us show 1). Let $\mathcal{A}, \mathcal{B}$ be mass problems of partial functions. Since $\mathcal{A} \wedge \mathcal{B} \leq_e \mathcal{A}$ and $\mathcal{A} \wedge \mathcal{B} \leq_e \mathcal{B}$, we have at once that $(\mathcal{A} \wedge \mathcal{B})^* \leq \mathcal{A}^* \wedge \mathcal{B}^*$ (see Fact 5.1 a)).

To show the converse, recall that $\mathcal{A} \wedge \mathcal{B} = \{0 * \varphi : \varphi \in \mathcal{A}\} \cup \{1 * \psi : \psi \in \mathcal{B}\}$; thus $(\mathcal{A} \wedge \mathcal{B})^* = \{f : \text{range}(f) \in \tau(\mathcal{A} \wedge \mathcal{B})\} = \{f : (\exists \varphi \in \mathcal{A})[\text{range}(f) = \tau(0 * \varphi)]\} \cup \{f : (\exists \psi \in \mathcal{B})[\text{range}(f) = \tau(1 * \psi)]\}$.

On the other hand $\mathcal{A}^* \wedge \mathcal{B}^* = \{0 * f : \text{range}(f) \in \tau(\mathcal{A})\} \cup \{1 * f : \text{range}(f) \in \tau(\mathcal{B})\}$.

Consider the mapping $\Psi : ({}^{\omega}\omega)^* \longrightarrow ({}^{\omega}\omega)^*$ defined by

$$\Psi(\varphi)(x) = \begin{cases} 0 & \text{if } x=0 \ \& \ (\exists u)[\varphi(u)\downarrow \ \& \ \varphi(u) = \langle 0,0\rangle \\ & \quad \& \ (\forall v<u)[\varphi(v)\downarrow \ \& \ \varphi(v) \neq \langle 0,1\rangle]] \\ 1 & \text{if } x=0 \ \& \ (\exists u)[\varphi(u)\downarrow \ \& \ \varphi(u) = \langle 0,1\rangle \\ & \quad \& \ (\forall v<u)[\varphi(v)\downarrow \ \& \ \varphi(v) \neq \langle 0,0\rangle]] \\ \uparrow & \text{if } x=0 \ \& \text{ otherwise} \\ \langle (r)_0 - 1, (r)_1\rangle & \text{if } x>0 \ \& \ (\exists v)[v \geq x-1 \ \& \ (\forall z \leq v) \\ & \quad [\varphi(z)\downarrow \ \& \ (\varphi(v))_0 \neq 0 \ \& \ (\forall z)[x-1 \leq z < \\ & \quad v \Rightarrow (\varphi(z))_0 = 0] \ \& \ r = \varphi(v)] \\ \uparrow & \text{if } x>0 \ \& \text{ otherwise} \end{cases}$$

To check beyond any doubt that Ψ is a recursive operator, consider the following r.e.set:

$W = \{\langle\langle x,y\rangle,u\rangle :$ $[x=0 \ \& \ y=0 \ \& \ (\exists v)[\langle v,\langle 0,0\rangle\rangle \in D_u \ \& \ (\forall w<v)$
$(\exists z)[z \neq \langle 0,1\rangle \ \& \ \langle w,z\rangle \in D_u]]]$ or $[x=0 \ \& \ y=1 \ \&$

$(\exists v)[\langle v,\langle 0,1\rangle\rangle \in D_u$ & $(\forall w<v)(\exists z)[z \neq \langle 0,0\rangle$ & $\langle w,z\rangle \in D_u]]]$ or $[x \neq 0$ & $(\exists v)[v \geq x-1$ & $(\forall z \leq v)(\exists r)[\langle z,r\rangle \in D_u]$ & $(\exists r)[\langle v,r\rangle \in D_u$ & $(r)_0 \neq 0]$ & $(\forall z)[x-1 \leq z<v \Rightarrow (\forall w \in D_u)[(w)_0=z \Rightarrow (w)_1=0]]]\}$

It is now easy to see that the enumeration operator corresponding to the r.e. set W can be taken as a defining enumeration operator for Ψ.

Let us now return to the proof of Proposition 5.1. In order to show that $\mathcal{A}^* \wedge \mathcal{B}^* \leq (\mathcal{A} \wedge \mathcal{B})^*$ via Ψ, we must show that, for every $f \in (\mathcal{A} \wedge \mathcal{B})^*$, $\Psi(f)$ is total & $\Psi(f) \in \mathcal{A}^* \wedge \mathcal{B}^*$. For instance, let range(f)=$\tau(0*\varphi)$, for some $\varphi \in \mathcal{A}$: that $\Psi(f)$ is total is obvious; also $\langle 0,0\rangle \in$range(f) and $\langle 0,1\rangle \notin$ range(f), hence $\Psi(f)(0)=0$. Furthermore, one is able to show that range$((\Psi(f))^+)=\tau(\varphi)$ (the superscript $^+$ has the same meaning as in the proof of Fact 4.1): indeed,

$u \in \text{range}((\Psi(f))^+) \Longleftrightarrow (\exists v)[(v+1,u) \in \Psi(f)] \Longleftrightarrow (\exists v)(\exists w)(\exists z)\ [w \geq v\ \&\ (w,z) \in f\ \&\ (z)_0 \neq 0\ \&\ u=\langle (z)_0-1,(z)_1\rangle] \Longleftrightarrow (\exists z)[z \in \text{range}(f)\ \&\ u=\langle (z)_0-1,(z)_1\rangle] \Longleftrightarrow u \in \tau(\varphi)$.

Therefore, $\Psi(f) \in \mathcal{A}^*$. The case range(f)=$\tau(1*\psi)$, for some $\psi \in \mathcal{B}$, is similar and we can conclude that $\mathcal{A}^* \wedge \mathcal{B}^* \leq (\mathcal{A} \wedge \mathcal{B})^*$ via Ψ.

2) Let $\mathcal{A}, \mathcal{B}$ be mass problems of partial functions. Since $\mathcal{A} \leq_e \mathcal{A} \vee \mathcal{B}$ and $\mathcal{B} \leq_e \mathcal{A} \vee \mathcal{B}$ we have (by Fact 5.1 a)) that $\mathcal{A}^* \vee \mathcal{B}^* \leq (\mathcal{A} \vee \mathcal{B})^*$.

To show that $(\mathcal{A}\vee\mathcal{B})^* \leq \mathcal{A}^* \vee \mathcal{B}^*$, remember that $(\mathcal{A}\vee\mathcal{B})^* = \{f \vee g:\ \text{range}(f)\in\tau(\mathcal{A})\ \&\ \text{range}(g)\in\tau(\mathcal{B})\}$. Let $\Psi: ({}^{\omega}\omega)^* \longrightarrow ({}^{\omega}\omega)^*$ be given by

$$\Psi(\varphi)(x) = \begin{cases} \uparrow & \text{if } \varphi(x)\uparrow \\ \langle 2u,v\rangle & \text{if } \varphi(x)\downarrow\ \&\ x \text{ even } \&\ \varphi(x)=\langle u,v\rangle \\ \langle 2u+1,v\rangle & \text{if } \varphi(x)\downarrow\ \&\ x \text{ odd } \&\ \varphi(x)=\langle u,v\rangle \end{cases}$$

Clearly Ψ is a recursive operator and $(\mathcal{A}\vee\mathcal{B})^* \leq \mathcal{A}^* \vee \mathcal{B}^*$ via Ψ.

Corollary 5.1 The cardinality of $\mathcal{M}_e$ is $2^{2^{\aleph_0}}$.

Proof. Immediate by Proposition 5.1, since the cardinality of $\mathcal{M}$ is 2^{2} (see Fact 1.1). On the other hand, by a proof similar to that of Theorem 3.2, one can show also that $\mathcal{M}_e$ has antichains of cardinality $2^{2^{\aleph_0}}$.

$\mathcal{M}$ can be viewed as a sublattice of $\mathcal{M}_e$. Indeed, one can define the embedding $I_e: \mathcal{M} \longrightarrow \mathcal{M}_e$ by $I([\mathcal{A}])=[\mathcal{A}]_e$, for every mass problem $\mathcal{A}$.

Proposition 5.2 F_e is left adjoint to I_e. Furthermore $F_e \circ I_e = I_{\mathcal{M}}$ ($I_{\mathcal{M}}$ is the identity automorphism of $\mathcal{M}$).

Proof. It must be shown that, for every $A_e \in \mathcal{M}_e$ and $B \in \mathcal{M}$, $F_e(A_e) \leq B \iff A_e \leq_e I_e(B)$. This follows from Fact 5.1 b) (in fact it is equivalent to Fact 5.1 b)). That $F_e \circ I_e = I_{\mathcal{M}}$ holds, is obvious (and follows anyway from the general theory of adjoint functors applied to this particular case).

Let us turn our attention to $\mathcal{M}$ and $\mathcal{M}_w$. We have here two obvious mappings:

$F_w: \mathcal{M} \longrightarrow \mathcal{M}_w$ givem by $F_w([\mathcal{A}])=[\mathcal{A}]_w$ and

$I_w: \mathcal{M}_w \longrightarrow \mathcal{M}$ given by $I_w([\mathcal{A}]_w)=[C(\mathcal{A})]$.

F_w, I_w are easily seen to be well defined. Furthermore F_w is an epimorphism (as shown in MUČNICK (1963)); I_w is 1-1 and is a homomorphism of partial orders which preserves $\vee$: well, as $[\mathcal{A}]_w \vee [\mathcal{B}]_w=[C(\mathcal{A})\cap C(\mathcal{B})]_w$ (notice that $C(\mathcal{A})\cap C(\mathcal{B})$ is C-closed); on the other hand $[C(\mathcal{A})\cap C(\mathcal{B})]=[C(\mathcal{A})] \vee [C(\mathcal{B})]$.

I_w does not preserve $\wedge$, however: in fact, $(\forall A_w)(\forall B_w)[A_w |_w B_w \Longrightarrow I_w(A_w \wedge B_w) \neq I_w(A_w) \wedge I_w(B_w)]$. To see this, notice that $(\forall \mathcal{A}, \mathcal{B} \subseteq {}^{\omega}\omega)[C(\mathcal{A}\cup\mathcal{B})$ is uniform] (see Example 4.1), thus the degree of difficulty $[C(\mathcal{A}\cup\mathcal{B})]$ is meet-irreducible (by Lemma 4.1). Therefore if $A_w |_w B_w$ and, say, $A_w=[\mathcal{A}]_w$, $B_w=[\mathcal{B}]_w$, then $I_w(A_w \wedge B_w)=I_w([\mathcal{A}\cup\mathcal{B}]_w) = [C(\mathcal{A}\cup\mathcal{B})]<I_w(A_w) \wedge I_w(B_w)$.

Proposition 5.3 I_w is left adjoint to F_w. Also, $F_w \circ I_w = I_{\mathcal{M}_w}$.

Proof. We must show that for every $A_w \in \mathcal{M}_w$ and $B \in \mathcal{M}$, $I_w(A_w) \leq B \Longleftrightarrow A_w \leq_w F_w(B)$. Let $A_w=[\mathcal{A}]_w$ and $B=[\mathcal{B}]$; clearly $C(\mathcal{A}) \leq \mathcal{B} \Longleftrightarrow \mathcal{A} \leq_w \mathcal{B}$, which implies $I_w([\mathcal{A}]_w) \leq [\mathcal{B}] \Longleftrightarrow [\mathcal{A}] \leq_w [\mathcal{B}]_w$ and, thus, the first part of the Proposition is proved. In particular $I_w(F_w(B))$ is the minimum degree of difficulty of $\mathfrak{M}$ among those degrees containing C-closed mass problems (i.e. problems $\mathcal{A}$ such that $C(\mathcal{A})=\mathcal{A}$) and which are less than or equal to B.

Thinking of partial orders as categories and using language borrowed from category theory, we can say, in view of Proposition 5.2 and Proposition 5.3, that $\mathfrak{M}$ is a reflective subcategory of both $\mathfrak{M}_e$ and $\mathfrak{M}_w$.

$\mathfrak{M}$, $\mathfrak{M}_w$ are not isomorphic, of course: for example, $\mathfrak{M}_w$ is complete, whereas $\mathfrak{M}$ is not. Actually, Dyment proves the following strong result (DYMENT (1980)):

Fact 5.2 1) Let $\{A_n\colon n\in\omega\}$ be a countable collection of degrees of difficulty. Then, $\bigwedge\{A_n\colon n\in\omega\}$ exists $\Longrightarrow$ $(\exists F\subsetneq\omega)[F$ is finite & $\bigwedge\{A_n\colon n\in\omega\}=\bigwedge\{A_n\colon n\in F\}$.

2) Let $\{A_n\colon n\in\omega\}$ be a countable collection of effectively discete degrees of difficulty. Then $\bigvee\{A_n\colon n\in\omega\}$ exists $\Longrightarrow(\exists F\subsetneq\omega)[F$ finite & $\bigvee\{A_n\colon n\in\omega\}$ = $\bigvee\{A_n\colon n\in F\}$.

It should be clear, however, that the sup in $\mathfrak{M}$ of an arbitrary family of degrees of difficulty containing C-closed mass problems always exists: indeed, let $\{A_i\colon i\in I\}$ be any collection of degrees of difficulty such that, for every $i\in I$, $A_i=[\mathcal{A}_i]$ and $C(\mathcal{A}_i)=\mathcal{A}_i$; then

$\bigvee\{A_i: i\in I\} = [\cap\{C(\mathcal{A}_i): i\in I\}]$, as is easily seen. We have in $\mathfrak{M}$ a lack of duality between $\wedge$ and $\vee$, which is well illustrated by the following example.

Example 5.1 a) $\bigwedge\{S: S\neq 0$ & S is a degree of solvability$\}$ does not exist. Indeed, suppose that for some mass problem $\mathcal{A}$, $[\mathcal{A}] = \bigwedge\{S: S\neq 0$ & S is a degree of solvability$\}$. Notice that if $[f]_T$ is minimal, then ($\forall$non recursive g) $[\mathcal{B}_f \wedge \{f\} \leq \{g\}]$ (where $\mathcal{B}_f = \{h: h \not\leq_T f\}$); to see this, notice that either $g \in \mathcal{B}_f$ or $f \equiv_T g$. It follows that $(\forall f)[[f]_T$ is minimal $\Rightarrow \mathcal{B}_f \wedge \{f\} \leq \mathcal{A}]$. Since there are continuously many functions whose T-degrees are minimal in $\mathfrak{D}_T$, we must have that $(\exists n)(\exists f_0)(\exists f_1)[f_0, f_1$ belong to minimal T-degrees & $f_0\neq f_1$ & $\Psi_n(\mathcal{A}) \subseteq \mathcal{B}_{f_0} \wedge \{f_0\}$ & $\Psi_n(\mathcal{A}) \subseteq \mathcal{B}_{f_1} \wedge \{f_1\}]]$.
Then, by definition of $\wedge$, either $\Psi_n(\mathcal{A}) \subseteq 0 * \mathcal{B}_{f_0}$ or $\Psi_n(\mathcal{A}) \subseteq 0 * \mathcal{B}_{f_1}$. In both cases we get a contradiction: if for example, $\Psi_n(\mathcal{A}) \subseteq 0 * \mathcal{B}_{f_0}$, then $(\forall g)[\mathcal{B}_f < \{g\}]$, a contradiction.
However, $\bigvee\{S: S$ is a degree of solvability$\}=1$.
b) In $\mathfrak{M}_w$, $\bigwedge\{S_w: S_w\neq 0_w$ & S_w is a degree of solvability$\} = 0_w'$.

$\mathfrak{M}$, $\mathfrak{M}_e$, $\mathfrak{M}_w$ are pairwisely non isomorphic. In fact we have the following theorem, where we regard $\mathfrak{M}$, $\mathfrak{M}_e$, $\mathfrak{M}_w$ as structures for the language having signature $<\leq>$.

Theorem 5.1 $\mathcal{M}$, $\mathcal{M}_e$, $\mathcal{M}_w$ are pairwisely non elemenatarily equivalent.

Proof. 1) Let $S(x)$ be the first order formula $(\exists u)(x<u\ \&\ (\forall v)(x<v \longrightarrow u\le v))$.

It follows from Theorem 1.2 and Theorem 2.2 that for every $A\in \mathcal{M}$ and $A_e\in \mathcal{M}_e$

$\mathcal{M} \models S[A]$ if and only if A is a degree of solvability

$\mathcal{M}_e \models S[A_e]$ if and only if A_e is a degree of enumerability in $\mathcal{M}_e$.

Now, $\mathcal{M} \models (\exists x)(S(x)\ \&\ (\forall y<x)(S(y) \longrightarrow (\forall z)(y\le z)))$, but $\mathcal{M}_e \not\models (\exists x)(S(x)\ \&\ (\forall y<x)(S(y) \longrightarrow (\forall z)(y\le z)))$.

This is so because there exist minimal T-degrees, but do not exist minimal e-degrees (see e.g. COOPER (1982)). Therefore, we have shown that $\mathcal{M} \not\equiv_{ee} \mathcal{M}_e$ (denoting the relation of elementary equivalence by $\equiv_{ee}$).

b) To show that $\mathcal{M} \not\equiv_{ee} \mathcal{M}_w$, notice that the three-chain

$$\begin{array}{l} \bullet\ [\mathcal{B}_f]_w \\ | \\ \bullet\ 0'_w \\ | \\ \bullet\ 0_w \end{array}$$

(where $[f]_T$ is a minimal T-degree, and $\mathcal{B}_f=\{g\colon g \not\le_T f\}$) is an initial segment of $\mathcal{M}_w$; indeed, if $\mathcal{C} <_w \mathcal{B}_f$ then $(\exists g\in \mathcal{C})[g\le_T f]$. So let $g\in \mathcal{C}$ be such that $g\le_T f$; it follows that for every non recursive function h, either $h \not\le_T f$ and in this case $h\in \mathcal{B}_f$, or $h\le_T f$ and in this case $g\le_T h$. Therefore $\mathcal{C} \le_w \mathcal{O}'$ i.e. $[\mathcal{C}]_w \le_w 0'_w$.

On the other hand, the only finite initial segment of $\mathfrak{M}$ is the two-chain

$$\begin{array}{l} \bullet\ 0' \\ | \\ \bullet\ 0 \end{array}$$

This is so because $0'=[\mathcal{O}']$ and the mass problem $\mathcal{O}'$ is uniform. Thus (by the characterization of empty intervals of $\mathfrak{M}$ mentioned in Remark 3.1) $(\forall A>0')(\exists B)[0'<B<A]$.

3) The proof of $\mathfrak{M}_e \not\equiv_{ee} \mathfrak{M}_w$ is similar to 1), since, by Theorem 3.4,

$$\mathfrak{M}_w \models (\exists x)(S(x) \;\&\; (\forall y<x)(S(y) \longrightarrow (\forall z)(y\leq z))).$$

We denote by $Th(\mathcal{L},\leq)$ the first order theory of a given partial order $\mathcal{L}$ (in the language having signature $<\leq>$. The following theorem holds.

Theorem 5.2 The first order theory of Second Order Arithmetic is 1-reducible to $Th(\mathfrak{M},\leq)$.

Proof. By Simpson's theorem (see SIMPSON (1977)), the first order theory of Second Order Arithmetic is 1-reducible to $Th(\mathcal{D}_T,\leq_T)$. Therefore it is enough to show that $Th(\mathcal{D}_T,\leq_T)$ is 1-reducible to $Th(\mathfrak{M},\leq)$. Consider the following mapping * carrying sentences into sentences and defined by induction on the complexity of the formulas: for every formulas A,B

if A is atomic then $*(A)=A$;

$*(A \vee B)=*(A) \vee *(B)$;

$*(A\&B)=*(A)\&*(B)$;

$*(A \longrightarrow B)=*(A)\longrightarrow *(B)$;

$*(\neg A)=\neg *(A)$;

$*((\exists x)A(x))=(\exists x)(S(x)\ \&\ *(A(x)))$;

$*((\forall x)A(x))=(\forall x)(S(x)\longrightarrow *(A(x))$;

i.e. a standard relativization of quantifiers to the universe of degrees of solvability in $\mathfrak{M}$. Then, for every sentence A,

$$\mathfrak{D}_T \models A \quad \text{if and only if} \quad \mathfrak{M} \models *(A)$$

Problem Is the first order theory of Second Order Arithmetic recursively isomorphic to Th($\mathfrak{M}$, $\leq$)?

In view, again, of Theorem 1.2, one can ask homogeneity questions. Let $\mathfrak{M}^f=\{A: S_f \leq A\}$.

Fact 5.3 There exist functions f such that $\mathfrak{M}$ is not isomorphic to $\mathfrak{M}^f$ (we denote the relation of being isomorphic by the symbol $\simeq$)

Proof. Suppose that f is a function such that $\mathfrak{M} \simeq \mathfrak{M}^f$; then, by Theorem 1.2, we have that $\mathfrak{D}_T \simeq \mathfrak{D}_T([[f]_T,\infty))$. It is known (see SHORE (1979)) that $(\exists a_t)(\forall b_T \geq_T a_T)[\mathfrak{D}_T \not\simeq \mathfrak{D}_T([b_T,\infty))]$ (a_T can be chosen to be $o_T^{(\omega)}$, vertex of a cone of minimal covers over o_T'' !!). Therefore, for any a_T, b_T as above, if $f \in b_T$ then $\mathfrak{M} \not\simeq \mathfrak{M}^f$.

Fact 5.3 can actually be strengthened to

Fact 5.4 $(\exists f)[\ \mathcal{M} \not\equiv_{ee} \mathcal{M}^f]$

Proof. For every sentence A, let $*(A)$ have the same meaning as in Theorem 5.2 and suppose that $\mathcal{M} \equiv_{ee} \mathcal{M}^f$. Then $\mathcal{D}_T \models A \Leftrightarrow \mathcal{M} \models *(A) \Leftrightarrow$ $\mathcal{M}^f \models *(A) \Leftrightarrow \mathcal{D}_T([[f]_T,\infty)) \models A$.

Again Shore (see SHORE (1982)) shows that $(\forall b_T \geq_T 0_T^{(\omega)})\ [\ \mathcal{D}_T \not\equiv_{ee} \mathcal{D}_T([b_T,\infty))]$. So the assertion is proved.

"Fragments" of $Th(\mathcal{M},\leq)$ can be studied also. Let us say that a formula in the language having signature $<\leq>$ is a $\exists_0$-formula if it contains no quantifiers; a formula is $\forall_n$ if its negation is logically equivalent to a $\exists_n$-formula; finally a formula is $\exists_{n+1}$ if it is of the form $(\exists \bar{x})A(\bar{x})$ where $A(\bar{x})$ is a $\forall_n$-formula.

Remark 5.1 $\exists_1 \cap Th(\mathcal{M},\leq)$ is decidable.

Proof. Immediate, since every finite partial order can be embedded in $\mathcal{D}_T$ and thus in $\mathcal{M}$ (see Fact 1.3).

Remark 5.2 $\forall_4 \cap Th(\mathcal{M},\leq)$ is undecidable.

Proof. It is known (see e.g. LERMAN (1983)) that $\forall_3 \cap Th(\mathcal{D}_T,\leq_T)$ is undecidable. The translation discuseed in Theorem 5.2 takes $\forall_3$-sentences into $\forall_4$-sentences. Hence the result follows.

Remark 5.3 Theorem 5.2, Fact 5.3, Fact 5.4, Remark 5.1 and Remark 5.2 hold also with $\mathfrak{M}_w$ in place of $\mathfrak{M}$.

Proof. Immediate.

It would be an interesting problem to determine the "best" result for Remark 5.2.

Remark 5.4 As already noticed, the three-chain is not embeddable in $\mathfrak{M}$ as an initial segment. In fact, by what we have mentioned at the end of Theorem 5.1, every degree of difficulty greater than 0' has at least $\aleph_0$ predecessors.

Degrees of difficulty different from 1 and having $2^{2^{\aleph_0}}$ predecessors of course exist. For example, let A be a degree of difficulty having an at most countable basis, i.e. $A=[C(\mathcal{A})]$, where $\mathcal{A}$ is at most countable and let $\mathbb{R}$ denote the set of real numbers. Let $\{f_i: i\in \mathbb{R}\}$ be a family of functions such that $\{[f_i]_T: i\in \mathbb{R}\}$ is an antichain in $\mathcal{D}_T$ and $(\forall i\in \mathbb{R})[f_i \notin C(\mathcal{A})]$ (see for instance SACKS (1961)). Thus, if $\mathcal{B},\mathcal{C} \subsetneq \{f_i: i\in \mathbb{R}\}$, $\mathcal{B} \neq \mathcal{C}$ and $\mathcal{B}, \mathcal{C}$ are antisymmetric (see Theorem 3.3 for the terminology), then it follows $\mathcal{B} \cup C(\mathcal{A})<C(\mathcal{A})$, $\mathcal{C} \cup C(\mathcal{A})<C(\mathcal{A})$ and $\mathcal{B} \cup C(\mathcal{A}) | \mathcal{C} \cup C(\mathcal{A})$. By the existence of $2^{2^{\aleph_0}}$ such antisymmetric sets, the claim is proved.

A similar remark holds for $\mathcal{M}_w$ (with a similar proof).

As to successors, one can easily show that every degree of difficulty different from 1 has $2^{2^{\aleph_0}}$ successors. Indeed, let f be given. Consider a family of functions $\{f_i: i \in \mathbb{R}\}$ such that $(\forall i,j \in \mathbb{R})[i \neq j \Rightarrow f <_T f_i$ & $f_i |_T f_j]$. For each $i \in \mathbb{R}$, fix z_i such that $\Psi_{z_i}(f_i)=f$. Then if $\mathcal{B}, \mathcal{C} \subseteq \{z_i * f_i: i \in \mathbb{R}\}$, $\mathcal{B} \neq \mathcal{C}$ and $\mathcal{B}, \mathcal{C}$ are antisymmetric, we have that $\{f\} < \mathcal{B}$, $\{f\} < \mathcal{C}$ and $\mathcal{B} | \mathcal{C}$. Since for every degree of difficulty $A \neq 1$ there exists a degree of solvability greater than A, the assertion follows.

A similar remark holds for $\mathcal{M}_w$.

6. Heyting algebras and Brouwer algebras.

Let $\mathcal{L}$ be a lattice. If we think of $\mathcal{L}$ as a category, then for every $a\in\mathcal{L}$ the functions $G_a:\mathcal{L}\longrightarrow\mathcal{L}$, $H_a:\mathcal{L}\longrightarrow\mathcal{L}$ defined by $G_a(b)=a\wedge b$, $H_a(b)=a\vee b$ are functors, being homomorphisms of partial orders. Let us denote G_a, H_a by $a\wedge\_$, $a\vee\_$ respectively.

Definition 6.1 A distributive lattice $\mathcal{L}=\langle L,\vee,\wedge,0,1\rangle$ is a Heyting algebra if, for every $a\in L$, the functor $a\wedge\_$ has a right adjoint (hence, $\mathcal{L}$ is a Cartesian closed category).

Definition 6.2 A distributive lattice $\mathcal{L}=\langle L,\vee,\wedge,0,1\rangle$ is a Brouwer algebra if, for every $a\in L$, the functor $a\vee\_$ has a left adjoint (hence $\mathcal{L}^{op}$ is a Cartesian closed category).

Let us denote the adjoint functor in both cases by $a\longrightarrow\_$. We remark that Definition 6.1 is equivalent to $(\forall a,b,x\in L)$ $[a\wedge x\le b \Leftrightarrow x\le a\longrightarrow b]$ and Definition 6.2 is equivalent to $(\forall a,b,x\in L)[a\longrightarrow b\le x \Leftrightarrow b\le a\vee x]$.

In a Heyting algebra one can introduce the unary operation $\neg$ by $\neg a=a\longrightarrow 0$. In a Brouwer algebra one can introduce the operation $\neg$ by $\neg a=a\longrightarrow 1$.

We are thus led to the more common definitions:

Definition 6.1 bis A <u>Heyting algebra</u> is an algebra $\mathcal{L} = \langle L, \vee, \wedge, \longrightarrow, \neg, 0, 1\rangle$ where $\langle L, \vee, \wedge, 0, 1\rangle$ is a distributive lattice and for every $a,b,x \in L$,

$a \wedge x \leq b \Longleftrightarrow x \leq a \longrightarrow b$;

$\neg a = a \longrightarrow 0$.

Definition 6.2 bis A <u>Brouwer algebra</u> is an algebra $\mathcal{L} = \langle L, \vee, \wedge, \longrightarrow, \neg, 0, 1\rangle$ where $\langle L, \vee, \wedge, 0, 1\rangle$ is a distributive lattice and for every $a,b,x \in L$

$a \longrightarrow b \leq x \Longleftrightarrow b \leq a \vee x$;

$\neg a = a \longrightarrow 1$.

Other equivalent statements are:

1) a distributive lattice $\mathcal{L} = \langle L, \vee, \wedge, 0, 1\rangle$ is a Heyting algebra if and only if in $\mathcal{L}$ one can introduce a binary operation $a \longrightarrow b$ such that, for every $a,b \in L$, the set $\{x \in L\colon a \wedge x \leq b\}$ has a maximum and this maximum equals $a \longrightarrow b$;

2) a distributive lattice $\mathcal{L} = \langle L, \vee, \wedge, 0, 1\rangle$ is a Brouwer algebra if and only if in $\mathcal{L}$ one can introduce a binary operation $a \longrightarrow b$ such that, for every $a,b \in L$, the set $\{x \in L\colon b \leq a \vee x\}$ has a minimum and this minimum equals $a \longrightarrow b$.

Notice that in a Heyting algebra $a \rightarrow b = 1 \Longleftrightarrow a \leq b$ and in a Brouwer algebra $a \longrightarrow b = 0 \Longleftrightarrow b \leq a$. It is clear that $\mathcal{L}$ is a Heyting algebra if and only if $\mathcal{L}^{op}$ is a Brouwer algebra. Also, every finite distributive lattice is both a Heyting algebra and a Brouwer algebra.

For details on Heyting algebras and Brouwer algebras see respectively RASIOWA-SIKORSKY (1963) and MC KINSEY-TARSKI (1946), where they are thoroughly investigated.

We now turn our attention to which of the preceding properties are enjoyed by our lattices.

Theorem 6.1 $\mathfrak{M}$, $\mathfrak{M}_e$ are Brouwer algebras.

Proof. (For $\mathfrak{M}$ the result was proved by Medvedev in MEDVEDEV (1955); as usual our exposition follows ROGERS (1967)). To show that $\mathfrak{M}$ is a Brouwer algebra, define the operation $\longrightarrow$ on degrees of difficulty by letting

$[\mathcal{A}] \longrightarrow [\mathcal{B}] = [\{z * g : (\forall f \in \mathcal{A})[\Psi_z(f \vee g) \in \mathcal{B}]\}]$.

$\longrightarrow$ is well defined.

The definition of $\longrightarrow$ in $\mathfrak{M}_e$ is similar using partial recursive operators instead of recursive operators.

Theorem 6.2 $\mathfrak{M}_w$ is both a Heyting algebra and a Brouwer algebra.

Proof. (that $\mathfrak{M}_w$ is a Brouwer algebra was proved by Mucnick in MUCNICK (1963)). Let $A_w = [\mathcal{A}]_w$, $B_w = [\mathcal{B}]_w$.

1) To show that $\mathfrak{M}_w$ is a Heyting algebra, we can define $A_w \longrightarrow B_w = C_w$ where $C_w = [\mathcal{C}]_w$ and $\mathcal{C} = \{f \in \mathcal{B} : (\forall g \in \mathcal{A})[g \not\leq_T f]\}$. Indeed, that $\mathcal{A} \wedge \mathcal{C} \leq_w \mathcal{B}$ holds is obvious. On the other hand, suppose that $\mathcal{A} \wedge \mathcal{X} \leq_w \mathcal{B}$ and let $f \in \mathcal{C}$; since $f \in \mathcal{B}$ & $(\forall g \in \mathcal{A})[g \not\leq_T f]$, it must follow that

$(\exists h \in \mathcal{X})[h \leq_T f]$. Hence $\mathcal{X} \leq_w \mathcal{C}$. The operation $\longrightarrow$ is clearly well defined.

2) To show that $\mathcal{M}_w$ is a Brouwer algebra, define $A_w \rightarrow B_w = C_w$ where $C_w = [\mathcal{C}]_w$ and $\mathcal{C} = \{f: (\forall g \in \mathcal{A})(\exists h \in \mathcal{B})[h \leq_T g \vee f]\}$. With this definition, one can easily see that $\mathcal{B} \leq_w \mathcal{A} \vee \mathcal{C}$. Suppose that $\mathcal{B} \leq_w \mathcal{A} \vee \mathcal{X}$: then $\mathcal{X} \subseteq \mathcal{C}$; therefore $\mathcal{C} \leq \mathcal{X}$ and a fortiori $\mathcal{C} \leq_w \mathcal{X}$. Again, $\longrightarrow$ is well defined.

When, in reference to $\mathcal{M}_w$, it becomes necessary to distinguish typographically between the right adjoint to $a \wedge \_$ from the left adjoint to $a \vee \_$, we will simply write $a \rightarrow_H \_$ for the left adjoint and $a \rightarrow_B \_$ for the right adjoint.

The above definitions can actually be regarded as definitions of binary operations on mass problems. For example, with respect to the reducibility $\leq$, we have actually defined $\mathcal{A} \longrightarrow \mathcal{B} = \{z * g: (\forall f \in \mathcal{A})[\Psi_z(f \vee g) \in \mathcal{B}]\}$, and, with respect to $\leq_w$, we have defined $\mathcal{A} \longrightarrow_H \mathcal{B} = \{f \in \mathcal{B}: (\forall g \in \mathcal{A})[g \nleq_T f]\}$ and $\mathcal{A} \longrightarrow_B \mathcal{B} = \{f: (\forall g \in \mathcal{A})(\exists h \in \mathcal{B})[h \leq_T g \vee f]\}$.

This is the meaning that we give, in the following, to expressions like $\mathcal{A} \rightarrow \mathcal{B}$, $\mathcal{A} \rightarrow_H \mathcal{B}$, etc.

Corollary 6.1 In $\mathcal{M}_w$, if $\mathcal{A} = C(\mathcal{A})$, then $[\mathcal{A}]_w \longrightarrow_H [\mathcal{B}]_w = [C(\mathcal{B} \cap \overline{\mathcal{A}})]_w$ (where $\overline{\mathcal{A}}$ denotes the complement of $\mathcal{A}$, i.e. $\overline{\mathcal{A}} = {}^{\omega}\omega - \mathcal{A}$).

Proof. We shall prove: 1) $[\mathcal{A}]_w \wedge [C(\mathcal{B}\cap\bar{\mathcal{A}})]_w \le_w [\mathcal{B}]_w$ and 2) $(\forall X_w)[[\mathcal{A}]_w \wedge X_w \le_w [\mathcal{B}]_w \Rightarrow X_w \le_w [C(\mathcal{B}\cap\bar{\mathcal{A}})]_w]$.

1): we show that $\mathcal{B}\subseteq \mathcal{A}\cup C(\mathcal{B}\cap\bar{\mathcal{A}})$: if $f\in\mathcal{B}$ and $f\notin\mathcal{A}$ then $f\in\mathcal{B}\cap\bar{\mathcal{A}}$ and, trivially, $f\in C(\mathcal{B}\cap\bar{\mathcal{A}})$.

2): we show that $(\forall\mathcal{X})[\mathcal{B}\subseteq\mathcal{A}\cup C(\mathcal{X})\Rightarrow C(\mathcal{B}\cap\bar{\mathcal{A}})\subseteq C(\mathcal{X})]$ (notice that $\mathcal{B}\subseteq\mathcal{A}\cup C(\mathcal{X})\Longrightarrow \mathcal{A}\wedge\mathcal{X}\le_w\mathcal{B}$): if $\mathcal{B}\subseteq\mathcal{A}\cup C(\mathcal{X})$ and $f\in C(\mathcal{B}\cap\bar{\mathcal{A}})$ then $(\exists g\in\mathcal{B}\cap\bar{\mathcal{A}})[g\le_T f]$, which implies that $g\in\mathcal{X}$ and hence $f\in C(\mathcal{X})$.

The following theorem shows that $\mathcal{M}$ is not a Heyting algebra. We shall make use of the following fact.

Fact 6.1 (DYMENT (1976)) Let $\mathcal{A}$ be a mass problem such that $[\mathcal{A}]>0$ and $\mathcal{A}$ is discrete in the Baire topology. Then $(\forall\mathcal{B})[\mathcal{A}\le\mathcal{B}\Rightarrow\mathcal{B}$ is nowhere dense$]$.

Theorem 6.3 For every degree of solvability $S>0$, there exists an effectively discrete degree B such that the set $\{C: S\wedge C\le B\}$ does not have a maximum.

Proof. Let $S=[\{g\}]$ be given and suppose that g is not recursive. We first define a sequence of functions $\{f_n: n\in\omega\}$ such that for every $m,n\in\omega$

i) $g<_T f_n$;

ii) $m\neq n \Longrightarrow f_m(0)\neq f_n(0)\ \&\ f_m|_T f_n$;

iii) $\Psi_n(f_n)\neq g$.

This sequence is defined by induction as follows:

<u>Step 0)</u> Let f_0 be such that $g<_T f_0$ and $\Psi_0(f_0)\neq g$.

Such a function f_0 exists because otherwise we would have $(\forall f)[g <_T f \Rightarrow \Psi_0(f)$ is total$]$ & $\Psi_0(\{f: g <_T f\}) \subseteq \{g\}$; from this it would follow that $\{g\} \leq \{f: g <_T f\}$ via Ψ_0 and therefore, by Fact 6.1, $\{f: g <_T f\}$ would be nowhere dense, but $\{f: g <_T f\}$ is manifestly not nowhere dense, a contradiction.

<u>Step n+1)</u> Consider the mass problem $\mathcal{B}_{n+1}=\{f: g <_T f$ & $(\forall i \leq n)[f |_T f_i$ & $f(0) \neq f_i]\}$. Let us first show that $\mathcal{B}_{n+1}$ is not nowhere dense in the Baire topology. To this end we must prove that $(\exists \tilde{h})(\forall \tilde{f} \supseteq \tilde{h})[S_{\tilde{f}} \cap \mathcal{B}_{n+1} \neq \emptyset]$. We shall actually prove that if $\tilde{h}$ is any finite initial segment such that $(\forall i \leq n)[\tilde{h}(0) \neq f_i(0)]$, then $(\forall \tilde{f} \supseteq \tilde{h})(\exists f \supseteq \tilde{f})[f \in \mathcal{B}_{n+1}]$. Indeed, let $\tilde{h}$ be a finite initial segment such that $(\forall i \leq n)$ $[\tilde{h}(0) \neq f_i(0)]$ and let $\tilde{f} \supseteq \tilde{h}$. By a classical relativized Kleene-Post construction, find a function k such that $g <_T k$ & $(\forall i \leq n)$ $[k |_T f_i]$, and consider the function f defined by

$$f(x)=\begin{cases} \tilde{f}(x) & \text{if } x<lh(\tilde{f}) \\ \\ k(x-lh(\tilde{f})) & \text{if } x \geq lh(\tilde{f}) \end{cases}$$

Since $f \equiv_T k$, we have $(\forall i \leq n)[f |_T f_i]$ and $g \leq_T f$. On the other hand, we can not have $f \leq_T g$, for, otherwise, we would have $(\forall i \leq n)[f \leq_T f_i]$, a contradiction. Thus $g <_T f$. Clearly, $f \supseteq \tilde{f}$ and $f \in \mathcal{B}_{n+1}$, hence $\mathcal{B}_{n+1}$ is not nowhere dense in the Baire topology and, as in Step 0), we can conclude that there exists a function f_{n+1} such that

$f_{n+1} \in \mathcal{B}_{n+1}$ and $\Psi_{n+1}(f_{n+1}) \neq g$. Therefore, the desired n+1-th element of our sequence of functions $\{f_n: n \in \omega\}$ has been defined.

Let now $\mathcal{J} = \{g\}$ and $\mathcal{B} = \{f_n: n \in \omega\}$. Since the elements of $\mathcal{B}$ satisfy ii), $[\mathcal{B}]$ is an effectively discrete degree of difficulty and, because of iii), for no Ψ_n can we have $\mathcal{J} \leq \mathcal{B}$ via Ψ_n; hence $\mathcal{J} \nleq \mathcal{B}$.

Suppose now that $\mathcal{J} \wedge \mathcal{C} \leq \mathcal{B}$ via a recursive operator Ψ. Since $\mathcal{J} \nleq \mathcal{B}$, we have that $\mathcal{B}' = \mathcal{B} \cap \{f: \Psi(f)(0)\downarrow \ \& \ \Psi(f)(0)=1\} \neq \emptyset$. Clearly, $\mathcal{B} \leq \mathcal{B}'$ and $\mathcal{C} \leq \mathcal{B}'$.
Moreover, $\mathcal{J} \wedge \mathcal{B}' \leq \mathcal{B}$ via the recursive operator Ψ' defined by

$$\Psi'(\varphi)(x) = \begin{cases} \uparrow & \text{if } \Psi(\varphi)(0)\uparrow \text{ or } [\Psi(\varphi)(0)\downarrow \ \& \\ & \quad \Psi(\varphi)(0) \neq 0 \ \& \ \Psi(\varphi)(0) \neq 1] \\ \Psi(\varphi)(x) & \text{if } \Psi(\varphi)(0)\downarrow \ \& \ \Psi(\varphi)(0)=0 \\ 1 & \text{if } x=0 \ \& \ \Psi(\varphi)(0)\downarrow \ \& \ \Psi(\varphi)(0)=1 \\ \varphi(x-1) & \text{if } x>0 \ \& \ \Psi(\varphi)(0)\downarrow \ \& \ \Psi(\varphi)(0)=1 \end{cases}$$

Indeed, for every $f \in \mathcal{B}$, $\Psi'(f) \in 0 * \mathcal{J}$ if $\Psi(f)(0)=0$, and $\Psi'(f) = 1 * f \in 1 * \mathcal{B}'$ if $\Psi(f)(0)=1$.

Setting $\mathcal{B}'' = \mathcal{B} \cap \{f: \Psi'(f)(0)\downarrow \ \& \ \Psi'(f)(0)=0\}$, we have $\mathcal{B}'' \subsetneq \mathcal{B}$ and, say, $f_n \in \mathcal{B} - \mathcal{B}'' = \mathcal{B}'$. Therefore $(\forall f \in \mathcal{B}'')[f_n(0) \neq f(0)]$. Finally, let Ψ_{z_n} be such that $\Psi_{z_n}(f_n)=g$ (use here that the elements of $\mathcal{B}$ satisfy i), thus $g <_T f_n$), and define Ψ'': $({}^{\omega}\omega)^* \longrightarrow ({}^{\omega}\omega)^*$ by

$$\Psi''(\varphi)(x)=\begin{cases}\uparrow & \text{if } \Psi'(\varphi)(0)\uparrow \text{ or } [\Psi'(\varphi)(0)\downarrow \;\&\; \Psi'(\varphi)(0)\neq 0 \;\&\; \Psi'(\varphi)(0)=1] \text{ or } \varphi(0)\uparrow \\ \Psi'(\varphi)(x) & \text{if } \Psi'(\varphi)(0)\downarrow \;\&\; \varphi(0)\downarrow \;\&\; [\Psi'(\varphi)(0)=0 \text{ or } [\Psi'(\varphi)(0)=1 \;\&\; \varphi(0)\neq f_n(0)]] \\ 0 & \text{if } x=0 \;\&\; \Psi'(\varphi)(0)\downarrow \;\&\; \varphi(0)\downarrow \;\&\; \Psi'(\varphi)(0)=1 \;\&\; \varphi(0)=f_n(0) \\ \Psi_{z_n}(\varphi)(x-1) & \text{if } x>0 \;\&\; \Psi'(\varphi)(0)\downarrow \;\&\; \varphi(0)\downarrow \;\&\; \Psi'(\varphi)(0)=1 \;\&\; \varphi(0)=f_n(0)\end{cases}$$

Ψ'' is clearly a recursive operator and $\mathcal{S}\wedge(\mathcal{B}'-\{f_n\})\leq\mathcal{B}$ via Ψ'' (use that $[\mathcal{B}]$ is effectively discrete).

But, now, $\mathcal{B}'<\mathcal{B}'-\{f_n\}$, because if there exists a recursive operator Ψ such that $\Psi(\mathcal{B}')\subseteq\mathcal{B}'-\{f_n\}$ then, for such a Ψ, there is a function f_m with $m\neq n$ and $\Psi(f_n)=f_m$; since $f_n|_T f_m$, we have a contradiction. Therefore we are forced to conclude that $\mathcal{C}<\mathcal{B}'-\{f_n\}$.

We have proved that $(\forall\mathcal{C})[\mathcal{S}\wedge\mathcal{C}\leq\mathcal{B}\implies(\exists\mathcal{X})[\mathcal{C}<\mathcal{X} \;\&\; \mathcal{S}\wedge\mathcal{X}\leq\mathcal{B}]]$; hence the set $\{C: S\wedge C\leq B\}$ does not have a maximum.

Remark 6.1 In a similar way we can see that $\mathcal{M}_e$ is not a Heyting algebra.

We now examine the behavior of F_w, I_w, F_e, I_e, as defined in Section 5, with respect to the operation $\longrightarrow$.

Fact 6.2 The epimorphism $F_w: \mathcal{M} \to \mathcal{M}_w$ preserves $\neg$ but does not preserve $\longrightarrow$ (here of course we regard $\mathcal{M}_w$ as a Brouwer algebra, so $\longrightarrow$ is $\longrightarrow_B$ and $\neg$ is defined accordingly).

Proof. That F_w preserves $\neg$ is obvious, since for every degree of difficulty A

$$\neg A = \begin{cases} 0 & \text{if } A=1 \\ 1 & \text{if } A \neq 1 \end{cases}$$

On the other hand, consider $B=[\{f_0\}]$, $A=[\{f: f_0 \leq_T f\}]$, where f_0 is not recursive. Since $\{f: f_0 \leq_T f\}$ is not nowhere dense in the Baire topology, by Fact 6.1 it follows that $A<B$, hence $A \to B \neq 0$ and $F_w(A \to B) \neq 0_w$. Nevertheless $F_w(B)=[\{f_0\}]_w=[\{f: f_0 \leq_T f\}]_w=F_w(A)$, hence $F_w(A) \to F_w(B)=0_w$ and the proof is complete.

Fact 6.3. $I_w: \mathcal{M}_w \longrightarrow \mathcal{M}$ preserves $\neg$ and $\longrightarrow$. Hence I_w is a $\vee, \neg, \longrightarrow$-homomorphism.

Proof. Again, the part concerning $\neg$ is trivial since

$$\neg A_w = \begin{cases} 0_w & \text{if } A_w=1_w \\ 1_w & \text{if } A_w \neq 1_w \end{cases}$$

The other part easily follows from the definitions of I_w and $\longrightarrow_B$ (see Theorem 6.2).

Remark 6.2 An immediate consequence of Fact 6.3 is that for degrees in $\mathcal{M}$ containing C-closed mass problems, $\longrightarrow$ is computed in the same way as $\longrightarrow_B$ in w.

The following lemma shows that I_e does not preserve $\longrightarrow$.

Lemma 6.1 ($\forall$ non recursive f)($\exists g$)($\exists \varphi$)[$f \not\leq_T g$ & $f \leq_e g \vee \varphi$ & φ belongs to a quasiminimal e-degree].

Proof. Throughout this proof, if Φ is an enumeration operator and φ is a partial function then $\Phi(\varphi)$ will denote the function $\tau^{-1}(\Phi(\varphi))$.

Let f non recursive be given. We want to find a function g and a partial function φ such that $f \leq_e g \vee \varphi$ and satisfying, for all $s \in \omega$, the following requirements:

P_s: $\Psi_s(g) \neq f$;

R_s: if $\Phi_s(\varphi)$ is a total function then $\Phi_s(\varphi)$ is recursive.

Indeed, if we can satisfy all these requirements then we also guarantee that φ belongs to a quasiminimal e-degree, since automatically φ is not partial recursive: if φ were partial recursive then $g \vee \varphi \equiv_e g$ and thus $f \leq_e g$ i.e. $f \leq_T g$, a contradiction.

We will take $g = \bigcup \{\tilde{g}_s : s \in \omega\}$, $\varphi = \bigcup \{\tilde{\varphi}_s : s \in \omega\}$ where $\tilde{g}_s$, $\tilde{\varphi}_s$ are defined by induction as follows.

We set $\tilde{g}_{-1} = \emptyset$, $\tilde{\varphi}_{-1} = \emptyset$.

<u>Step 3s)</u>

Case a) $(\exists \tilde{g} \supseteq \tilde{g}_{3s-1})(\exists x)[\Psi_s(\tilde{g})(x)\downarrow$ & $\Psi_s(\tilde{g})(x) \neq f(x)]$.

In this case, let $\tilde{g}_{3s}$ be the least $\tilde{g}$ for which such an x exists and let $\tilde{\varphi}_{3s} = \tilde{\varphi}_{3s-1}$.

Case b) Otherwise, let $\tilde{g}_{3s} = \tilde{g}_{3s-1}$, $\tilde{\varphi}_{3s} = \tilde{\varphi}_{3s-1}$.

Step 3s+1) Throughout this step, $\tilde{\varphi} \trianglerighteq \tilde{\psi}$ means $\tilde{\varphi} \supseteq \tilde{\psi}$ & $\mu z.[z \in \mathrm{dom}(\tilde{\varphi}) - \mathrm{dom}(\tilde{\psi})] \geq \mathrm{lh}(\tilde{g}_{3s})$.

Case a) See whether

(*) $\quad (\exists x)(\exists y)(\exists z)(\exists \tilde{\varphi} \trianglerighteq \tilde{\varphi}_{3s})[y \neq z$ & $(x,y) \in \Phi_s(\tilde{\varphi})$ & $(x,z) \in \Phi_s(\tilde{\varphi})]$.

If so, let $\tilde{\varphi}_{3s+1}$ be the least such $\tilde{\varphi}$ and let $\tilde{g}_{3s+1}$ be the least $\tilde{g}$ such that $\tilde{g} \supseteq \tilde{g}_{3s}$ and $\mathrm{dom}(\tilde{\varphi}_{3s+1}) \subseteq \mathrm{dom}(g)$ and $(\forall x)[x \in \mathrm{dom}(\tilde{\varphi}_{3s+1}) - \mathrm{dom}(\tilde{\varphi}_{3s}) \Longrightarrow \tilde{\varphi}_{3s+1}(x) \neq \tilde{g}(x)]$.

If (*) does not hold, go to Case b).

Case b) See whether

(**) $\quad (\exists x)(\exists \tilde{\psi} \trianglerighteq \tilde{\varphi}_{3s})(\exists \tilde{\chi} \trianglerighteq \tilde{\varphi}_{3s})[\Phi_s(\tilde{\psi})(x)\downarrow$ & $\Phi_s(\tilde{\chi})(x)\downarrow$ & $\Phi_s(\tilde{\psi})(x) \neq \Phi_s(\tilde{\chi})(x)]$.

If so, choose $\tilde{\psi}, \tilde{\chi}$ having this property and let $z = \mu x.[(\forall y)[y \in \mathrm{dom}(\tilde{\psi}) \cup \mathrm{dom}(\tilde{\chi}) \Longrightarrow y < x]]$, and define $\tilde{\varphi}_{3s+1} = \tilde{\varphi}_{3s} \cup \{(z,0)\}$ and let $\tilde{g}_{3s+1}$ be the least extension $\tilde{g}$ of $\tilde{g}_{3s}$ such that $\mathrm{lh}(\tilde{g}_{3s+1}) = z+1$ and $\tilde{g}_{3s+1}(z) \neq 0$.

If (**) does not hold, define $\tilde{\varphi}_{3s+1} = \tilde{\varphi}_{3s}$ and $\tilde{g}_{3s+1} = \tilde{g}_{3s}$.

Step 3s+2) Throughout this step, we suppose to have fixed a 1-1 recursive function from the set of all finite sequences of natural numbers onto ω : the image of $(x_0, \ldots, x_k)$ under this function will be denoted by $\langle x_0, \ldots, x_k \rangle$. Together with this function, we fix also two recursive functions $\lambda x.\widetilde{lh}(x)$, $\lambda x,i.(x)_i$ having the properties: $\widetilde{lh}(<x_0,\ldots,x_k>)=k+1$ and $(<x_0,\ldots,x_k>)_i=x_i$, if $i\leq k$.

Let $x_s=lh(\tilde{g}_{3s+1})$. Define

$\tilde{\varphi}_{3s+2}=\tilde{\varphi}_{3s+1}\cup\{(x_s,<f(0),\ldots,f(s)>)\}$

$\tilde{g}_{3s+2}=\tilde{g}_{3s+1}\cup\{(x_s,<f(0),\ldots,f(s)>)\}$

This ends the construction. We have to check that the requirements are met.

Step 3s) takes care of the requirement P_s: let s be given. If Case a) of Step 3s) does not hold, then $f=\bigcup\{\Psi_s(\tilde{g}):\ \tilde{g}\supseteq\tilde{g}_{3s-1}\}$; hence f would be recursive, a contradiction.

Let us show now that Step 3s+1) makes the requirements R_s satisfied. If, in Step 3s+1), (*) holds, then $\Phi_s(\varphi)$ is not single-valued and R_s is met. If (*) does not hold and (**) holds and $z,\tilde{\psi},\tilde{\chi},\tilde{\varphi}_{3s+1}$ are as in Case b) of Step 3s+1) and x satisfies (**) then $x\notin\text{dom}(\Phi_s(\varphi))$, for otherwise the union of a suitable finite segment of φ together with $\tilde{\psi}$ or $\tilde{\chi}$ would furnish a finite segment $\tilde{\varphi}\supseteq\tilde{\varphi}_{3s}$ such that $\Phi_s(\tilde{\varphi})$ is not single-valued, contradicting the assumption that (*) does not hold. Therefore $\Phi_s(\varphi)$ is not total and, again, R_s is met.

Finally, suppose that neither (*) nor (**) holds and $\Phi_s(\varphi)$ is a total function. In this case, we show that $\Phi_s(\varphi)=\{(x,y):\ (\exists\tilde{\psi}\supseteq\tilde{\varphi}_{3s})[(x,y)\in\Phi_s(\tilde{\psi})]\}$: indeed, the inclusion $\subseteq$ is clear; to show that the latter set is included in $\Phi_s(\varphi)$, suppose that for some (x,y) and

$\tilde{\Psi} \supseteq \tilde{\varphi}_{3s}$, $(x,y) \in \Phi_s(\tilde{\Psi})$ but $(x,y) \notin \Phi_s(\varphi)$. Since $\Phi_s(\varphi)$ is total, there exists $z \neq y$ and $\tilde{\chi} \supseteq \tilde{\varphi}_{3s}$ such that $(x,z) \in \Phi_s(\tilde{\chi})$. But this would imply that (**) holds, a contradiction.

To see that $f \leq_e g \vee \varphi$, notice that for every $x, y \in \omega$, $f(x)=y$ $\Longleftrightarrow (\exists u)(\exists v)[g(u)=v \ \& \ \varphi(u)=v \ \& \ \widehat{lh}(v)=x \ \& \ y=(v)_{\widehat{lh}(v)\dot{-}1}]$.

Corollary 6.1 $I_e: \mathcal{M} \longrightarrow \mathcal{M}_e$ does not preserve $\longrightarrow$.

Proof. Let f non recursive be given and, by Lemma 6.1, let g, φ be such that $f \not\leq_T g$ & $f \leq_e g \vee \varphi$ & φ belongs to a quasiminimal e-degree. We shall show that $[\{g\}]_e \longrightarrow [\{f\}]_e \neq [\{g\}] \longrightarrow [\{f\}]$: remember that, by Theorem 6.1, $[\{g\}]_e \longrightarrow [\{f\}]_e = [\{z * \psi : g \vee \psi \in dom(\Omega_z)$ & $\Omega_z(g \vee \psi)=f\}]_e$ and $[\{g\}] \longrightarrow [\{f\}] = [\{z * h: \Psi_z(g \vee h)=f\}]$.

Let $u \in \omega$ be such that $\Omega_u(g \vee \varphi)=f$ (such a u exists because $f \leq_e g \vee$ and partial recursive operators are defined by enumeration operators). Then $\{z*h: \Psi_z(g \vee h)=f\} \not\leq_e \{z * \varphi : \Omega_z(g \vee \varphi)=f\}$ since the latter set contains $u * \varphi$, but clearly $u * \varphi$ belongs to a quasiminimal e-degree (being $u * \varphi \equiv_e \varphi$) and for every partial recursive operator Ω, for every h, if $\Omega(u * \varphi)=h$ then h is recursive. On the other hand, since $\{f\} \not\leq \{g\}$, $\{z*h: \Psi_z(g \vee h)=f\}$ does not contain any recursive function.

Fact 6.4 $F_e: \mathcal{M}_e \longrightarrow \mathcal{M}$ does not preserve $\longrightarrow$.

Proof. Let φ belong to a quasiminimal e-degree and let $\mathcal{A} = \{\varphi\}$ and $\mathcal{B} = \mathcal{A}^*$ (* is defined as in Section 5).

Then $\mathcal{B} \nleq_e \mathcal{A}$ and since $[\mathcal{A}^*] \rightarrow [\mathcal{B}^*]=0$, $[\mathcal{A}]_e \rightarrow [\mathcal{B}]_e \neq 0_e$, we have $F_e([\mathcal{A}]_e \rightarrow [\mathcal{B}]_e) \neq 0$, whereas $F_e([\mathcal{A}]) \rightarrow F_e([\mathcal{B}]_e) = [\mathcal{A}^*] \rightarrow [\mathcal{B}^*]=0$.

The positive results about F_e, I_e are contained in the following proposition.

Proposition 6.1 F_e, I_e are $\neg$-homomorphisms; moreover, for every A_e, $B_e \in \mathcal{M}_e$, $F_e(A_e) \rightarrow F_e(B_e) \leq F_e(A_e \rightarrow B_e)$ and, for every A, $B \in \mathcal{M}$, $I_e(A) \rightarrow I_e(B) \leq I_e(A \rightarrow B)$.

Proof. Obvious.

7. $\mathcal{M}_F$, $\mathcal{M}_E$.

Definition 7.1 $\mathcal{M}_F$ is the sublattice of the Medvedev lattice constituted by all finite degrees, i.e. the sublattice generated by the degrees of solvability.

Definition 7.2 $\mathcal{M}_E$ is the sublattice of the Medvedev lattice generated by the degrees of enumerability.

Remark 7.1 Clearly, $\mathcal{M}_F$ and $\mathcal{M}_E$ are embeddable (as lattices having a least and a greatest element) in both $\mathcal{M}$ and $\mathcal{M}_w$.

The property of being a degree of solvability is lattice-theoretic in $\mathcal{M}_F$: in $\mathcal{M}_F$, A is a degree of solvability $\Longleftrightarrow$ A is meet-irreducible. Analogously, in $\mathcal{M}_E$ the property of being a degree of enumerability is lattice-theoretic: in $\mathcal{M}_E$, A is a degree of enumerability $\Longleftrightarrow$ A is meet-irreducible (see Corollary 4.1).

When talking about elements of $\mathcal{M}_F$, i.e. finite degrees of difficulty A, without loss of generality we always consider a representative $\mathcal{A}$ (i.e. $A=[\mathcal{A}]$) such that, for some $k\in\omega$, $\mathcal{A}=\{f_0,\dots,f_k\}$ and $(\forall i,j\le k)[i\ne j \Rightarrow f_i(0)\ne f_j(0)\ \&\ f_i\,|_T\,f_j]$.

Lemma 7.1 In $\mathcal{M}$, $(\forall A)(\forall B)[B\in \mathcal{M}_F \Rightarrow \{C: A\wedge C\leq B\}$ has a maximum].

Proof. Each element $B\in \mathcal{M}_F$ has the form $B=S_0\wedge\ldots\wedge S_n$ where $(\forall i\leq n)[S_i$ is a degree of solvability]. Then $\max\{C: A\wedge C\leq B\}=\bigwedge\{S_i: A \not\leq S_i\}$. Indeed, $A\wedge\bigwedge\{S_i: A\not\leq S_i\}\leq B$; now, let $A\wedge C\leq B$. Then $(\forall i\leq n)[A\wedge C\leq S_i]$; since each S_i is meet-irreducible, this implies that $(\forall i\leq n)[A\leq S_i$ or $C\leq S_i]$ (see BIRKHOFF (1944)). Thus $C\leq\bigwedge\{S_i: A\not\leq S_i\}$. Of course $\bigwedge\emptyset$ is interpreted as 1.

Lemma 7.2 In $\mathcal{M}$, $(\forall A)(\forall B)[B\in \mathcal{M}_E \Rightarrow \{C: A\wedge C\leq B\}$ has a maximum].

Proof. Similar to that of Lemma 7.1, bearing in mind that each element B of $\mathcal{M}_E$ has the form $B=E_{A_0}\wedge\ldots\wedge E_{A_n}$ and each E_A is meet-irreducible (see Corollary 4.1) Thus $\max\{C: A\wedge C\leq B\}=\bigwedge\{E_A : A\not\leq E_{A_i}\}$ if $B=E_{A_0}\wedge\ldots\wedge E_{A_n}$.

Corollary 7.1 $\mathcal{M}_F$, $\mathcal{M}_E$ are Heyting algebras. Indeed, in $\mathcal{M}_F$, if $B_1=\bigwedge\{S_i: i\in I\}$, $B_2=\bigwedge\{S_j: j\in J\}$ (where I,J are finite and each S_i, S_j is a degree of solvability) then $B_1\rightarrow B_2=\bigwedge\{S_j: j\in J \ \&\ (\forall i\in I)[S_i\not\leq S_j]\}$. In $\mathcal{M}_E$, if $B_1=\bigwedge\{E_{A_i}: i\in I\}$, $B_2=\bigwedge\{E_{A_j}: j\in J\}$ (where I, J are finite and each E_{A_i}, E_{A_j} is a degree of enumerability) then $B_1\rightarrow B_2=\bigwedge\{E_A : j\in J\ \&\ (\forall i\in I)[A_i\not\leq_e A_j]\}$.

Proof. Let us prove for instance the part concerning $\mathcal{M}_F$. By Lemma 7.1, $B_1 \rightarrow B_2 = \bigwedge\{S_j: j \in J \ \& \ B_1 \not\leq S_j\}$. On the other hand, since, for every $i \in I$, each S_i is meet-irreducible, we have that $B_1 \not\leq S_j \Longleftrightarrow (\forall i \in I)[S_i \not\leq S_j]$.

Remark 7.2 $\mathcal{M}_F$ is a sub Heyting algebra of $\mathcal{M}_E$. $\mathcal{M}_F$, $\mathcal{M}_E$ are sub Heyting algebras of $\mathcal{M}_w$.

Proof. Obvious.

Remark 7.3 A lemma similar to Lemma 7.1 holds also for other sublattices of $\mathcal{M}$. For example, one can define $\mathcal{M}_U$ to be the sublattice generated by the uniform degrees (thus $\mathcal{M}_F$ and $\mathcal{M}_E$ are embeddable in $\mathcal{M}_U$). Notice that each element $A \in \mathcal{M}_U$ has the form $A = U_1 \wedge \ldots \wedge U_n$, where $(\forall i \leq n)[U_i$ is uniform$]$. This is so because if A, B are uniform, then so is $A \vee B$. Indeed, let $A = [\mathcal{A}]$, $B = [\mathcal{B}]$ where $\mathcal{A}$, $\mathcal{B}$ are uniform mass problems. Suppose that $S_{\tilde{f}} \cap (\mathcal{A} \vee \mathcal{B}) \neq \emptyset$ and let $\tilde{f}$ be of the form $\tilde{g}_0 \vee \tilde{g}_1$. Thus $S_{\tilde{f}} = S_{\tilde{g}_0 \vee \tilde{g}_1}$ and $S_{\tilde{g}_0} \cap \mathcal{A} \leq \mathcal{A}$ (via Ψ_{z_0}, say), $S_{\tilde{g}_1} \cap \mathcal{B} \leq \mathcal{B}$ (via Ψ_{z_1}, say). To show that $S_{\tilde{f}} \cap (\mathcal{A} \vee \mathcal{B}) \leq \mathcal{A} \vee \mathcal{B}$, let

$$W = \{\langle\langle x,y\rangle,u\rangle: (\exists a)(\exists v)[x=2a \ \& \ D_u = \{\langle 2b,c\rangle: \langle b,c\rangle \in D_v\} \ \& \ \langle\langle a,y\rangle,v\rangle \in W_{\sigma(z_0)}] \text{ or } (\exists a)(\exists v)[x=2a+1 \ \& \ D_u = \{\langle 2b+1,c\rangle: \langle b,c\rangle \in D_v\} \ \& \ \langle\langle a,y\rangle,v\rangle \in W_{\sigma(z_1)}]\}.$$

Clearly, W is r.e. and the enumeration operator defined by W determines a recursive operator Ψ such that for every $f, g \in {}^{\omega}\omega$, if $\Psi_{z_0}(f)$, $\Psi_{z_1}(g)$ are total then $\Psi(f \vee g)$ is total and $\Psi(f \vee g) = \Psi_{z_0}(f) \vee \Psi_{z_1}(g)$. Hence $S_F^{e} \cap (\mathcal{A} \vee \mathcal{B}) \leq \mathcal{A} \vee \mathcal{B}$ via Ψ. By an argument quite similar to the one in Corollary 7.1 and using that each uniform degree is meet-irreducible (see Lemma 4.1) it follows that if $A = \bigwedge\{U_i\colon i \in I\}$, $B = \bigwedge\{U_j\colon j \in J\}$ (where I, J are finite and each of U_i, U_j is uniform) then $A \longrightarrow B = \bigwedge\{U_j\colon j \in J \ \& \ (\forall i \in I)[U_i \not\leq U_j]\}$.

We conclude this remark by adding that, in $\mathcal{M}_U$, the property of being a uniform degree of difficulty is lattice-theoretic: In $\mathcal{M}_U$ the uniform degrees are exactly the meet-irreducible degrees.

In a similar fashion, one can define $\mathcal{M}_{e,F}$ to be the sublattice of $\mathcal{M}_e$ constituted by all degrees in $\mathcal{M}_e$ containing a finite mass problem of partial functions. Remarks similar to those for $\mathcal{M}_F$ hold, the proof being slightly more complicated since now $\mathcal{M}_{e,F}$ is not generated, as a sublattice, by the degrees of $\mathcal{M}_e$ containing a mass problem of the form $\{\varphi\}$, where is a partial function (see Example 7.1 below). Nevertheless we have the following lemma.

Lemma 7.3 In $\mathcal{M}_e$, $(\forall A_e)(\forall B_e)[B_e \in \mathcal{M}_{e,F} \Rightarrow \{C_e\colon A_e \wedge C_e \leq_e B_e\}$ has a maximum].

Proof. Let $A_e = [\mathcal{A}]_e$, $B_e = [\mathcal{B}]_e$, where $\mathcal{A}$, $\mathcal{B}$ are mass problems of partial functions. Suppose that $\mathcal{B}$ is finite and consider all non-empty subsets $\mathcal{C} \subseteq \mathcal{B}$ such that there exists a partial recursive operator Ω having the property:

$(\forall \varphi \in \mathcal{B})[\varphi \in dom(\Omega)]$ & $(\forall \varphi \in \mathcal{C})[\Omega(\varphi)(0)\downarrow$ & $\Omega(\varphi)(0)=1]$ & $(\forall \varphi \in \mathcal{B}-\mathcal{C})[\Omega(\varphi) \in 0*\mathcal{A}]$.

Let $\mathcal{C}_0, \ldots, \mathcal{C}_k$ be a list of all such subsets of $\mathcal{B}$ (notice that $\mathcal{B}$ is one of these $\mathcal{C}_i$) and for every $i \le k$ let Ω_{z_i} be a partial recursive operator having the above property with respect to $\mathcal{C}_i$. We claim that $\max\{C_e: A_e \wedge C_e \le_e B_e\} = [\cap\{\mathcal{C}_i: i \le k\}]_e$.

We first show that $\mathcal{A} \wedge \cap\{\mathcal{C}_i: i \le k\} \le_e \mathcal{B}$. Let

$$
\begin{aligned}
W=\{\langle\langle x,y\rangle,u\rangle:\ & (\forall i \le k)(\exists v)[\langle\langle 0,1\rangle,v\rangle \in W_{z_i} \ \&\ D_v \subseteq D_u \ \&\ [[x=0 \\
& \&\ y=1] \text{ or } [x>0 \ \&\ \langle x-1,y\rangle \in D_u]]] \text{ or} \\
& (\exists i \le k)(\exists v)[\langle\langle 0,0\rangle,v\rangle \in W_{z_i} \ \&\ D_v \subseteq D_u \ \& \\
& (\exists w)[\langle\langle x,y\rangle,w\rangle \in W_{z_i} \ \&\ D_w \subseteq D_u] \ \& \\
& (\forall j<i)(\exists z)[\langle\langle 0,1\rangle,z\rangle \in W_{z_j} \ \&\ D_z \subseteq D_u]]\}.
\end{aligned}
$$

Clearly W is r.e. and defines an enumeration operator which in turn determines a partial recursive operator Ω. Since, for every $\varphi \in \mathcal{B}$ and every $i \le k$, $\varphi \in dom(\Omega_{z_i})$ and $\Omega_{z_i}(\varphi)(0)$ is defined and equals either 0 or 1 we have that, for every $\varphi \in \mathcal{B}$,

$$
\Omega(\varphi)=\begin{cases} \Omega_{z_i}(\varphi) & \text{if, for some } j \le k,\ \Omega_{z_j}(\varphi) \in 0* \text{ and} \\ & \ i \text{ is the least such } j \\ 1*\varphi & \text{otherwise} \end{cases}
$$

Therefore, for every $\varphi \in \mathcal{B}$, if $\Omega(\varphi) \notin 0*\mathcal{A}$ then $\Omega(\varphi) \in 1*\cap\{\mathcal{C}_i: i \le k\}$ and thus $\mathcal{A} \wedge \cap\{\mathcal{C}_i: i \le k\} \le_e \mathcal{B}$ via Ω.

On the other hand, suppose that $\mathcal{A} \wedge \mathcal{C} \leq_e \mathcal{B}$ via a partial recursive operator Ω_z. Let $\mathcal{D} = \{\varphi \in \mathcal{B} : \Omega_z(\varphi)(0) = 1\}$. Clearly $\mathcal{D} = \mathcal{C}_i$ for some $i \leq k$, since the partial recursive operator Ω' determined by the r.e. set

$$W' = \{\langle\langle x,y\rangle,u\rangle : \quad (\exists v)[\langle\langle 0,0\rangle,v\rangle \in W_z \;\&\; D_v \subseteq D_u \;\& \\ (\exists w)[\langle\langle x,y\rangle,w\rangle \in W_z \;\&\; D_w \subseteq D_u]] \text{ or} \\ (\exists v)[\langle\langle 0,1\rangle,v\rangle \in W_z \;\&\; D_v \subseteq D_u \;\&\; [[x=0 \;\&\; y=1] \\ \text{or } [x>0 \;\&\; \langle x-1,y\rangle \in D_u]]]\}.$$

satisfies

$$\Omega'(\varphi) = \begin{cases} \Omega_z(\varphi) & \text{if } \varphi \in \mathcal{B} \;\&\; \Omega_z(\varphi)(0)=0 \\ \\ 1*\varphi & \text{if } \varphi \in \mathcal{B} \;\&\; \Omega_z(\varphi)(0)=1 \end{cases}$$

and thus satisfies also, with respect to $\mathcal{D}$, the property mentioned at the beginning of the proof. Therefore $\cap\{\mathcal{C}_i : i \leq k\} \subseteq \mathcal{D}$; thus $\mathcal{D} \leq_e \cap\{\mathcal{C}_i : i \leq k\}$ and, since clearly $\mathcal{C} \leq_e \mathcal{D}$, we have that $\mathcal{C} \leq_e \cap\{\mathcal{C}_i : i \leq k\}$. This completes the proof.

Corollary 7.2 $\mathcal{M}_{e,F}$ is a Heyting algebra.

Proof. By Lemma 7.3.

Example 7.1 Throughout this example, $\longrightarrow$ is the binary operation that makes $\mathcal{M}_{e,F}$ a Heyting algebra. Let φ_1 φ_2 be partial functions such that $\varphi_1 \subseteq \varphi_2$ & $\varphi_1 |_e \varphi_2$ (for examples of such partial functions see DYMENT (1976)). Then $[\{\varphi_1\}]_e \longrightarrow [\{\varphi_1, \varphi_2\}]_e = [\{\varphi_1, \varphi_2\}]_e$. Indeed, suppose that $\{\varphi_1\} \wedge \mathcal{A} \leq_e \{\varphi_1, \varphi_2\}$ via some partial recursive operator Ω. We can not have $\Omega(\varphi_1)=0*\varphi_1$ & $\Omega(\varphi_2)\in 1*\alpha$ since it must be $\Omega(\varphi_1) \subsetneq \Omega(\varphi_2)$. Therefore it follows that $\Omega(\{\varphi_1,\varphi_2\}) \subseteq 1*\alpha$, hence $\alpha \leq_e \{\varphi_1,\varphi_2\}$.

Another example in DYMENT (1976) shows the existence of two partial functions φ_1,φ_2 such that $\varphi_1 \leq_e \varphi_2$ & φ_1 non partial recursive & ($\forall$ partial recursive operator Ω)[$\varphi_1 \in \mathrm{dom}(\Omega)$ & $\Omega(\varphi_1)$ non partial recursive $\Longrightarrow \Omega(\varphi_1) \neq \Omega(\varphi_2)$].

In this case $[\{\varphi_1\}]_e \longrightarrow [\{\varphi_1,\varphi_2\}]_e=[\{\varphi_1,\varphi_2\}]_e$: to see this, let $\{\varphi_1\}\wedge\alpha \leq_e \{\varphi_1,\varphi_2\}$ via Ω and suppose that $\Omega(\varphi_1)=0*\varphi_1$ & $\Omega(\varphi_2)\in 1*\alpha$. Let z be such that $\Omega_z(\varphi_2)=\varphi_1$ (such a z exists since $\varphi_1 \leq_e \varphi_2$ and thus, for some z, $\varphi_1 = \tau^{-1}(\Phi_z(\varphi_2))$). By an argument similar to those in the proof of Lemma 7.3, there exists a partial recursive operator Ω' such that $(\forall\varphi \in \mathrm{dom}(\Omega))[[\Omega(\varphi)(0)\downarrow$ & $\Omega(\varphi)(0)=0 \Rightarrow \Omega'(\varphi)=\varphi]$ & $[\Omega(\varphi)(0)\downarrow$ & $\Omega(\varphi)(0)=1 \Rightarrow \Omega'(\varphi)=\Omega_z(\varphi)]]$.

This implies $\Omega'(\varphi_1)=\varphi_1=\Omega'(\varphi_2)$, a contradiction. The conclusion is the same if we suppose that $\Omega(\varphi_2)=0*\varphi_1$ and $\Omega(\varphi_1)\in 1*\alpha$. On the other hand, we can not have $\Omega(\{\varphi_1,\varphi_2\}) \subseteq \{0*\varphi_1\}$. Hence $\Omega(\{\varphi_1,\varphi_2\}) \subseteq 1*\alpha$, i.e. $\alpha \leq_e \{\varphi_1,\varphi_2\}$ as desired.

Given a function f, let f' denote $c_{(\tau(f))'}$ i.e. the characteristic function of the set obtained by applying the jump operation to $\tau(f)$.

Lemma 7.4 Let $T: \mathcal{M} \longrightarrow \mathcal{M}$ be defined by $T([\mathcal{A}])=[\{f': f\in \mathcal{A}\}]$. T is well defined; T preserves $\wedge$ and $T(1)=1$.

Proof. 1) let us show that T is well defined. For every mass problem $\mathcal{A}$, let $J(\mathcal{A})=\{f': f\in \mathcal{A}\}$. We want to show that $\mathcal{A} \le \mathcal{B} \Rightarrow J(\mathcal{A}) \le J(\mathcal{B})$. Suppose that $\mathcal{A} \le \mathcal{B}$ via Ψ_z; let g be a recursive function such that $(\forall f)[\Psi_z(f)$ total $\Longrightarrow \Psi_{g(z)}(f')=(\Psi_z(f))'$ (that such a recursive function exists can be proved using Corollary I(c), p. 255, ROGERS (1967) and Corollary to the Fundamental Operator Theorem stated in the Introduction). Then $J(\mathcal{A}) \le J(\mathcal{B})$ via $\Psi_{g(z)}$.

2) We want to show that $J(\mathcal{A} \wedge \mathcal{B}) \equiv J(\mathcal{A}) \wedge J(\mathcal{B})$. Let Ψ_{z_1}, Ψ_{z_2} be recursive operators such that $\Psi_{z_1}(n*h)=h$, for every $n\in\omega$ and $h\in{}^{\omega}\omega$, and $\Psi_{z_2}(h')=h$. Let Ψ be a recursive operator such that, for every f, if $\Psi_{z_2}(f)(0)\downarrow$ then $\Psi(f) = \Psi_{z_2}(f)(0)*\Psi_{g(z_1)}(f)$ (where g is as in 1); that such a recursive operator exists is easily seen). Suppose that $h\in J(\mathcal{A} \wedge \mathcal{B})$, i.e. $h=(i*f)'$ where $[i=0\ \&\ f\in \mathcal{A}]$ or $[i=1\ \&\ f\in \mathcal{B}]$. Then $\Psi(h)=\Psi((i*f)')=\Psi_{z_2}((i*f)')(0)*\Psi_{g(z_1)}((i*f)') = (i*f(0))*(\Psi_{z_1}(i*f))'=i*f'$. Therefore $\Psi(h)\in J(\mathcal{A}) \wedge J(\mathcal{B})$. Since $\mathcal{A} \wedge \mathcal{B} \le \mathcal{A}$ and $\mathcal{A} \wedge \mathcal{B} \le \mathcal{B}$, we already know by 1) that $J(\mathcal{A} \wedge \mathcal{B}) \le J(\mathcal{A}) \wedge J(\mathcal{B})$. Thus the claim is proved.

3) That $T(1)=1$ holds is obvious.

Fact 7.1 $(\forall A)(\forall B)[T(A) \vee T(B) \leq T(A \vee B)]$ and $(\exists A)(\exists B)[T(A) \vee T(B) < T(A \vee B)]$.

Proof. That $T(A) \vee T(B) \leq T(A \vee B)$ holds, follows again from 1) in the proof of the previous lemma. On the other hand, it is known that $(\forall b_T)(\exists a_T)[a_T' = a_T \vee o_T' = b_T \vee o_T']$ (FRIEDBERG (1957)). Take for example $b_T = o_T''$. Hence, for some a_T, $a_T' = a_T \vee o_T' = o_T''$; thus $(a_T \vee o_T')' = o_T'''$. Taking A such that $a_T = [A]_T$, we have that $J(\{c_A\})$ $J(\{c_{\emptyset'}\}) \equiv \{c_{\emptyset''}\} \vee \{c_{\emptyset''}\} \equiv \{c_{\emptyset''}\} < \{c_{\emptyset'''}\} \equiv J(\{c_A\} \vee \{c_{\emptyset'}\})$.

Corollary 7.3 The restriction of T to $\mathcal{M}_F$ gives a mapping $T: \mathcal{M}_F \longrightarrow \mathcal{M}_F$ satisfying $T(A \wedge B) = T(A) \wedge T(B)$ for every A, B$\in \mathcal{M}_F$, and $T(1)=1$.

Proof. Obvious by Lemma 7.3.

Definition 7.3 An <u>intuitionistic diagonalizable algebra</u> (shortly <u>i.d.a</u>) is a pair $<\mathcal{L}, \tau>$ where $\mathcal{L}$ is a Heyting algebra and τ is a unary operation on $\mathcal{L}$ satisfying $\tau(1)=1$, $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$, $\tau(\tau(a) \longrightarrow a) \leq \tau(a)$.

Intuitionistic diagonalizable algebras have been thoroughly studied in SAMBIN (1976), MAGARI (1978), URSINI (1979).

Theorem 7.1 $<\mathcal{M}_F, T>$ is an i.d.a.

Proof. That $T(1)=1$ and $T(A \wedge B) = T(A) \wedge T(B)$ hold, has been already proved in Corollary 7.3. Notice that

$(\forall A\in \mathfrak{M}_F)[T(A)\longrightarrow A=A]$. To see this consider any finite mass problem $\mathcal{A}$ (chosen according to the observation preceding Lemma 7.1): we must show that $[J(\mathcal{A})]\longrightarrow[\mathcal{A}]=[\mathcal{A}]$; but $[J(\mathcal{A})]\longrightarrow[\mathcal{A}]=[\{f\in\mathcal{A}: (\forall g\in J(\mathcal{A}))[g \not\leq_T f]\}]$ (see Corollary 7.1). Now, if $f\in\mathcal{A}$ and $g=h'\leq_T f$ for some $h\in\mathcal{A}$ then $h\leq_T f$, contradicting our choice of $\mathcal{A}$. Therefore $T(T(A)\longrightarrow A)=T(A)$.

Remark 7.4 1) $(\forall A\in\mathfrak{M}_F)[A\neq 1\Rightarrow A<T(A)]$;
2) both $T(A\longrightarrow B)=T(A)\longrightarrow T(B)$ and $T(A\longrightarrow B)<T(A)\longrightarrow T(B)$ can occur. Indeed, for $\mathcal{A}=\{c_{\emptyset''}\}$, $\mathcal{B}=\{c_{\emptyset'}\}$, we have $T([\mathcal{A}]\longrightarrow[\mathcal{B}])=T([\{c_{\emptyset'}\}])=[\{c_{\emptyset''}\}]=[\{c_{\emptyset'''}\}]\longrightarrow[\{c_{\emptyset''}\}] = T([\mathcal{A}])\longrightarrow T([\mathcal{B}])$, whereas for $\mathcal{A}=\{f\}$, $\mathcal{B}=\{g\}$ where $f\,|_T\, g$ and $f'\equiv_T g'$ (the existence of such functions is again a consequence of the result by Friedberg quoted in the proof of Fact 7.1), we have that $T(\{f\}]\longrightarrow[\{g\}])=T([\{g\}])=[\{g'\}]$ and $T([\{f\}])\longrightarrow T([\{g\}])=[\{f'\}]\longrightarrow[\{g'\}]$; hence $T([\{f\}])\quad T([\{g\}])=1$;
3) the set of regular elements (i.e. degrees A such that $\neg\neg A=A$) is $\{0,1\}$; hence it is not closed under T.

Proposition 7.1 $\mathfrak{M}_F$ is not a Brouwer algebra.

Proof. Let us embed the u.s.l. $\mathcal{L}=\langle\{a,b,a_n: n\in\omega\},\leq\rangle$ in $\mathcal{D}_T$ as an initial segment of $\mathcal{D}_T$, where $(\forall n\in\omega)(\forall m\in\omega)[a<a_n<b \;\&\; [m\neq n\Rightarrow a_n\,|\,a_m]]$ (we use here the result that every countable upper semilattice with a least element is embeddable in $\mathcal{D}_T$ as an initial segment:

see LACHLAN-LEBEUF (1976)). Let a_T, b_T, $(a_n)_T$ $(n\in\omega)$ be the corresponding T-degrees and choose $g\in b_T$, $f_n\in(a_n)_T$ for every $n\in\omega$. Then $(\forall n>0)[g\equiv_T f_0\vee f_n]$; for every $n\in\omega$ let z_n be such that $\Psi_{z_n}(f_0\vee f_n)=g$. In $\mathfrak{M}$, $\{f_0\}\rightarrow\{g\}=\{z*h: \Psi_z(f_0\vee h)=g\}$ (for this use of the symbol $\longrightarrow$ as an operation on mass problems, see the observation preceding Corollary 6.1). Thus, in $\mathfrak{M}$, $\{f_0\}\longrightarrow\{g\}\leq\{z_n*f_n: n>0\}$. Suppose now, for the sake of contradiction, that there exists a mass problem $\mathcal{F}>\mathcal{O}$ such that $\{g\}\leq\{f_0\}\vee\mathcal{F}$ & $(\forall\text{finite }\mathcal{G})[\{g\}\leq\{f\}\vee\mathcal{G}\Rightarrow\mathcal{F}\leq\mathcal{G}]$; then, for every finite $\mathcal{G}$ such that $\mathcal{G}\subseteq\{z_n*f_n: n>0\}$ we would have $\mathcal{F}\leq\mathcal{G}$. By cardinality arguments, and since each f_n belongs to a minimal T-degree, this would lead us to a contradiction: when the cardinality of $\mathcal{G}$ exceeds that of $\mathcal{F}$, we would have that there exist $f\in\mathcal{F}$ and n_1, n_2 (with $n_1\neq n_2$) such that $f\leq_T f_{n_1}$ and $f\leq f_{n_2}$, a contradiction, since $[f_{n_1}]_T$, $[f_{n_2}]_T$ are minimal.

Fact 7.2 F: $\mathfrak{M}\longrightarrow\mathfrak{M}$ defined by $F([\mathcal{A}])=[C(\mathcal{A})]$ preserves $\vee$ and $F(0)=0$, $F(1)=1$. Moreover $(\forall A)(\forall B)[F(A\wedge B)\leq F(A)\wedge F(B)]$ and $(\exists A)(\exists B)[F(A\wedge B)<F(A)\wedge F(B)]$.

Proof. The proof is essentially the same as for I_W (see Section 5). As to show that equality does not hold in general in the second part of the statement, notice that $C(\mathcal{A}\wedge\mathcal{B})$ is uniform for every $\mathcal{A},\mathcal{B}\subseteq{}^{\omega}\omega$ and so belongs to a meet-irreducible degree of difficulty. Thus if $F(A \wedge B) < F(A)$ and $F(A \wedge B) < F(B)$, then also $F(A \wedge B) < F(A) \wedge F(B)$.

Rremark 7.5 $(\forall A)[F(F(A) \rightarrow A) \leq F(A)]$; however, equality does not hold in general, otherwise $< \mathfrak{M}^{op}, F>$ would be an i.d.a., but F has fixed points different form 1 (in fact, $(\forall A)[F(F(A))=F(A)]$, whereas in an i.d.a. $< \mathcal{L}, \tau>$, $\tau(a)=a \Longrightarrow a=1$ (see URSINI (1979)).

Remark 7.6 The first order theory of Second Order Arithmetic is recursively isomorphic to $Th(\mathfrak{M}_F, \leq)$.

Proof. Similar to the proof of Theorem 5.2. Indeed, it is easy to see that $Th(\mathfrak{D}_T, \leq_T)$ is 1-reducible to $Th(\mathfrak{M}_F, \leq)$, since, by Remark 7.1, in $\mathfrak{M}_F$ A is a degree of solvability if and only if $(\forall B)(\forall C)[A=B \wedge C \Rightarrow [B \leq A$ or $C \leq A]]$.

We now devote our attention to a preliminary study of the operation $\longrightarrow$ in $\mathfrak{M}$.

Notice that, in $\mathfrak{M}$, $(\forall B>0')[0' \rightarrow B=B]$. Useful information is given also by the following fact.

Fact 7.3 (DYMENT (1976)) If $[\mathcal{B}]$ is effectively discrete and $\mathcal{B} \not\leq \mathcal{A}$ then $(\forall \mathcal{C})[\mathcal{B} \leq \mathcal{A} \vee \mathcal{C} \Longrightarrow \mathcal{C}$ nowhere dense]. Hence all mass problems in $[\mathcal{A}] \rightarrow [\mathcal{B}]$ are nowhere dense.

Proposition 7.2 If $\mathcal{A}$ is dense and $\mathcal{B}$ is closed in the Baire topology then $[\mathcal{A}]\longrightarrow[\mathcal{B}]=[\mathcal{B}]$.

Proof. That $[\mathcal{A}]\longrightarrow[\mathcal{B}]\leq[\mathcal{B}]$ holds, is always true. We have only to show that $\mathcal{B}\leq\{z*g: (\forall f\in\mathcal{A})[\Psi_z(f\vee g)\in\mathcal{B}]\}$.

Let h be the recursive function defined in the proof of Fact 4.1; for every $u\in\omega$, let $D_u^{(0)}$, $D_u^{(1)}$ denote respectively the sets $\{<x,y>: <2x,y>\in D_u\}$ and $\{<x,y>: <2x+1,y>\in D_u\}$; also, for every $u\in\omega$ such that D_u is single-valued, let $\delta(D_u)$ denote the finite initial segment having length equal to $n=\max\{x: (\exists y)[<x,y>\in D_u]\}+1$ and defined by

$$\delta(D_u)(x)=\begin{cases}\tau^{-1}(D_u)(x) & \text{if } x\in\mathrm{dom}(\tau^{-1}(D_u))\\ 0 & \text{if } x<n \ \&\ \text{otherwise}\end{cases}$$

Finally, given two finite initial segments $\tilde{f},\tilde{g}$ let us say that $\tilde{f}$ is <u>comparable</u> with $\tilde{g}$ if $\tilde{f}\subseteq\tilde{g}$ or $\tilde{g}\subseteq\tilde{f}$.

By the Recursion Theorem let $e\in\omega$ be such that

$W_e=\{<x,y,u,v>: \tau^{-1}(D_u)$ is a finite initial segment & D_v is single-valued & $\tau(\delta(D^{(1)}_{h(v)}))\subseteq D_u$ & $(\exists z)[<0,z>\in D_u$ & $(\exists s)[<<,x,y>,v>\in W_z^s$ & $(\forall a\in W_z^s)$ $(\forall b\in D^{(1)}_{h(a)_1})$ $[(b)_0<lh(\tau(D_u))]$ & $(\forall i<x)(\exists j)$ $(\exists w)(\exists k)[<i,j,w,k>\in W_e$ & $D_w\subseteq D_u$ & $\delta(D_k^{(0)})$ is comparable with $\delta(D_v^{(0)})]$ & $(\forall t<s)(\forall a\in W_z^t)[D_{(a)_1}$ single-valued & $\tau(\delta(D^{(1)}_{h((a)_1)}))\subseteq D_u \Longrightarrow (\exists i<x)$ $(\exists j)(\exists w)(\exists k)$ $[<i,j,w,k>\in W_e$ & $D_w\subseteq D_u$ & $\delta(D_k^{(0)})$ non comparable with $\delta(D^{(0)}_{(a)_1})]]]]]\}$.

Let $W=\{<x,y>,u>: (\exists v)[<x,y,u,v>\in W_e]\}$.

It is not difficult to see that W determines an enumeration operator which in turn defines a partial recursive operator Ω such that, for every total function f, $f\in dom(\Omega)$ and $\Omega(f)$ is informally computed as follows:

Let $f(0)=z$

Step 0) Enumerate W_z until you find a number, if it exists, $<<0,y>,v>\in W_z$ such that $D_v^{(1)}\subseteq \tau(f^+)$ (where $f^+=\lambda x.f(x+1)$) & $D_v^{(0)}$ is single valued. If no such number exists, then let $\Omega(f)(0)\uparrow$. Otherwise, let $<<0,y_0>,v_0>$ be the first such number to appear in the enumeration of W_z and let $\Omega(f)(0)=y_0$.

Step n+1) If $(\exists i\leq n)[\Omega(f)(i)\uparrow]$ then put $\Omega(f)(n+1)\uparrow$. Otherwise, enumerate W_z until you find a number, if it exists, $<<n+1,y>,v>\in W_z$ such that $D_v^{(1)}\subseteq\tau(f^+)$ & $D_v^{(0)}$ is single-valued & $(\forall i\leq n)[\delta(D_{v_i}^{(0)})$ is comparable with $\delta(D_v^{(0)})]$. If no such number exists, then let $\Omega(f)(n+1)\uparrow$; otherwise, let $<<n+1,y_{n+1}>,v_{n+1}>$ be the first such number to appear in the enumeration of W_z and put $\Omega(f)(n+1)=y_{n+1}$.

Notice that, if for every $n\in\omega$ $\Omega(f)(n)\downarrow$ then for every $n\in\omega$ a finite initial segment $\tilde{f}_n$ is defined, namely $\tilde{f}_n = \bigcup\{\delta(D_{v_i}^{(0)}): i\leq n\}$.

Let $\Omega=\Omega_{e'}$, and let $\Psi = \Psi_{\sigma(e')}$: Thus, by the Fundamental Operator Theorem, for every total f, $\Omega(f)=\Psi(f)$.

Now, if $z*g\in \mathcal{A}\longrightarrow\mathcal{B}$, then $(\forall f\in\mathcal{A})[\Psi_z(f\vee g)$ is total & $\Psi_z(f\vee g)\in\mathcal{B}$]; moreover, since $\mathcal{A}$ is dense, one can certainly find, for every $n\in\omega$, a number v_n as above, so

that the corresponding $\tilde{f}_n$ is defined and $(\forall n)[\Psi_z(\tilde{f}_n \vee g)$ is an initial segment & $(\forall i \le n)[i \in dom(\Psi_z(\tilde{f}_n \vee g)$ & $\Psi_z(\tilde{f}_n \vee g) \subseteq \Psi_z(\tilde{f}_{n+1} \vee g)]]$. Letting $S_n = \{h: (\forall i \le n)[h(i) = \Psi_z(\tilde{f}_n \vee g)(i)]\}$ we have that for every n there exists a function $h_n \in \mathcal{B}$ such that $h_n \in S_n$. Since $\mathcal{B}$ is closed, we conclude that $\lim_n h_n = \Psi(z*g)$ and thus $\Psi_z(z*g) \in \mathcal{B}$.
Then $\mathcal{B} \le \mathcal{A} \rightarrow \mathcal{B}$ via Ψ , thus proving the proposition.

Corollary 7.3 If $\mathcal{A}$ is dense and $\mathcal{B}$ is closed and $[\mathcal{B}]>0$ then $\mathcal{B} \not\le \mathcal{A}$.

Proof. Obvious by Proposition 7.2, because if $\mathcal{B} \le \mathcal{A}$ then $[\mathcal{A}] \rightarrow [\mathcal{B}]=0$.

Remark 6.2 makes it easy, in a sense, to compute $[\mathcal{A}] \rightarrow [\mathcal{B}]$ in $\mathcal{M}$, for $\mathcal{A}, \mathcal{B}$ C-closed .

Example 7.2 1) Let $\mathcal{A} = \{f: f_0 \le_T f\}$ and $\mathcal{B} = \{f: f_1 \le_T f\}$: then $[\mathcal{A}] \rightarrow [\mathcal{B}] = [\{g: (\forall f)[f_0 \le_T f \Longrightarrow f_1 \le_T f \vee g]\}]$ (notice that if $f_1 \le_T f_0$ then the latter set is ${}^{\omega}\omega$ hence, in this case, $[\mathcal{A}] \rightarrow [\mathcal{B}]=0$.

2) Let $\mathcal{A} = \{f:$ f non recursive & $f_0 \not\le_T f\}$ and $\mathcal{B} = \{f: f_0 \le_T f\}$ where f_0 is not recursive. Then $[\mathcal{A}] \rightarrow [\mathcal{B}] = [\{g: (\forall f)[f$ non recursive & $f_0 \not\le_T f \Longrightarrow f_0 \le_T f \vee g]\}] = [\{g: f_0 \le_T g\}] = [\mathcal{B}]$.

8. An embedding theorem for a certain class of countable distributive lattices.

Distributive lattices with a least and a greatest element and lattice-theoretic homomorphisms which preserve the least and the greatest element constitute a category which we will denote by $\underline{DL}$.

If $\mathcal{L}$ is an object of $\underline{DL}$ then the least and the greatest element of $\mathcal{L}$ will be denoted by $0_{\mathcal{L}}$ and $1_{\mathcal{L}}$ respectively (or simply by 0 and 1, when clearness is not affected by doing so).

We will use also the following terminology:

$\underline{T(DL)}$ is the full subcategory of $\underline{DL}$ whose objects are distributive lattices which are coatomic and have exactly one coatom. If $\mathcal{L}$ is an object of $\underline{T(DL)}$ then its coatom will be denoted by $t_{\mathcal{L}}$ or simply by t when this does not cause any confusion.

$\underline{B(DL)}$ is the full subcategory of $\underline{DL}$ whose objects are atomic distributive lattices having exactly one atom, denoted by $b_{\mathcal{L}}$ or simply by b.

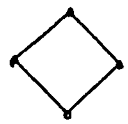
$\underline{TB(DL)}$ is the full subcategory of both $\underline{T(DL)}$ and $\underline{B(DL)}$ and thus of $\underline{DL}$ such that $ob(TB(DL))=ob(\underline{T(DL)})\cap ob(\underline{B(DL)})$.

One has two obvious functors

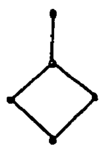
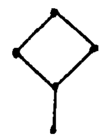
$T\colon \underline{DL} \longrightarrow \underline{T(DL)}$

$B\colon \underline{DL} \longrightarrow \underline{B(DL)}$

where, for every $\mathcal{L} \in ob(\underline{DL})$, $T(\mathcal{L})$ is the lattice obtained from $\mathcal{L}$ by adding an element greater than all elements of $\mathcal{L}$ and $B(\mathcal{L})$ is the lattice obtained from $\mathcal{L}$ by adding an element which is smaller than all elements of $\mathcal{L}$. Thus, for example, if

$\mathcal{L}$ = 

then we have

$T(\mathcal{L})$ =  and $B(\mathcal{L})$ = 

Notice that $\mathcal{L}$ can be regarded as a subset of $T(\mathcal{L})$ and, under this identification, $1_{\mathcal{L}}$ becomes $t_{T(\mathcal{L})}$; analogously, $\mathcal{L}$ can be regarded as a subset of $B(\mathcal{L})$, in such a way that $0_{\mathcal{L}}$ is identified with $b_{B(\mathcal{L})}$.

T and B are defined in the obvious way on morphisms: let $f: \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ be a morphism in <u>DL</u> and let us regard f as a function from $\mathcal{L}_1$, viewed as a subset of $T(\mathcal{L}_1)$, into $\mathcal{L}_2$ viewed as a subset of $T(\mathcal{L}_2)$. Then define $T(f): T(\mathcal{L}_1) \longrightarrow T(\mathcal{L}_2)$ by

$$T(f)(a)= \begin{cases} f(a) & \text{if } a \neq 1_{T(\mathcal{L}_1)} \\ \\ 1_{T(\mathcal{L}_2)} & \text{if } a = 1_{T(\mathcal{L}_1)} \end{cases}$$

Analogously, $B(f):B(\mathcal{L}_1) \longrightarrow B(\mathcal{L}_2)$ is defined by

$$B(f)(a)= \begin{cases} f(a) & \text{if } a \neq 0_{B(\mathcal{L}_1)} \\ \\ 0_{B(\mathcal{L}_2)} & \text{if } a = 0_{B(\mathcal{L}_1)} \end{cases}$$

Clearly $T \circ B = B \circ T$; moreover, we get a functor $T \circ B$: <u>DL</u> $\longrightarrow$ <u>TB(DL)</u>. T, B, $T \circ B$ are faithful and <u>almost</u> surjective on objects (except for the two-element Boolean algebra, every object of <u>T(DL)</u> is in the range of T; except for the two element Boolean algebra, every object of <u>B(DL)</u> is in the range of B; finally, except for the two element Boolean algebra and the lattice consisting of the three element chain, every object of <u>TB(DL)</u> is in the range of $T \circ B$): this is one of the reason for the names <u>T(DL)</u>, <u>B(DL)</u>, <u>TB(DL)</u> given to the above defined categories.

Finally, notice also that if we denote in each case by I the inclusion functors I: <u>T(DL)</u> $\longrightarrow$ <u>DL</u>, I: <u>B(DL)</u> $\longrightarrow$ <u>DL</u> and I : <u>TB(DL)</u> $\longrightarrow$ <u>DL</u> then we have $I \dashv T$, $I \dashv B$ and $I \dashv T \circ B$. Thus <u>T(DL)</u>, <u>B(DL)</u> and <u>TB(DL)</u> are coreflective subcategories of <u>DL</u> (for this terminology, see MC LANE (1971).

Clearly, $\mathfrak{M}$, $\mathfrak{M}_w$, $\mathfrak{M}_e$ are objects of <u>B(DL)</u>

Theorem 8.1 Every countable $\mathcal{L} \in$ <u>TB(DL)</u> is embeddable in $\mathfrak{M}$ via an embedding which is a morphism of <u>DL</u>

Proof. Clearly, the two element Boolean algebra and the three element chain are emebeddable in $\mathfrak{M}$. As to show that every countable object of <u>TB(DL)</u> having the form TB($\mathcal{L}$) is embeddable in $\mathfrak{M}$, we can argue as follows. Let $\mathcal{B}$ be a Boolean algebra satisfying the following conditions (the reason for excluding 0 will be clear shortly):

1) $\mathcal{B}$ is a sub Boolean algebra of the Boolean algebra $\mathcal{B}_\omega$ of all subsets of $\omega-\{0\}$;

2) $\mathcal{B}$ is countable and atomless;

3) the elements of $\mathcal{B}$, except the least element, i.e. the empty set, are infinite recursive subsets of $\omega-\{0\}$.

That such an algebra exists can be seen as follows. Let $\underset{\sim}{K}$ be the classical propositional calculus (see RASIOWA-SIKORKI (1963) or ChapterII, Section 1); and let Form($\underset{\sim}{K}$) denote the set of sentences of $\underset{\sim}{K}$. Finally, suppose that $\ulcorner\ \urcorner$:Form($\underset{\sim}{K}$) $\longrightarrow \omega-\{0\}$ is an a one-one, onto recursive coding and, for every $\alpha\in$Form($\underset{\sim}{K}$), let $[\alpha]=\{\ulcorner\beta\urcorner : \underset{\sim}{K} \vdash \alpha\leftrightarrow\beta\}$ Clearly, $\{[\alpha]: \alpha\in$Form($\underset{\sim}{K}$)$\}$ is a partition of $\omega-\{0\}$ into infinite recursive sets; thus the Boolean algebra generated, as a subalgebra of $\mathcal{B}_\omega$, by $\{[\alpha]: \alpha\in$Form($\underset{\sim}{K}$)$\}$ satisfies 1), 2), and 3).

Now, let g, $\{g_n: n\in\omega-\{0\}\}$ be functions such that $g(0)=0$ and for every $m,n\in\omega-\{0\}$

i) $g_m(0)=m$;

ii) $m\neq n \Longrightarrow g_m \mid_T g_n \;\&\; g \equiv_T g_m \vee g_n$;

we also assume :iii) there exists a two variable recursive function h such that for every $m,n\in\omega-\{0\}$, if $m\neq n$ then $\Psi_{h(m,n)}(g_m\vee g_n)=g$.

For example, it is possible to embed the poset used in the proof of Proposition 7.1 so that the uniformity condition referred to in iii) holds (see LERMAN (1981)). Thus, functions satisfying i),ii) and iii) exist.

Notice that the mass problem $\{g_n : n \in \omega - \{0\}\}$ is effectively discrete. For every $X \subseteq \omega - \{0\}$ define

$$\mathcal{A}_X = \begin{cases} \{g_x : x \in X\} & \text{if } X \neq \emptyset \\ \{g\} & \text{if } X = \emptyset \end{cases}$$

Then

a) $\mathcal{A}_X \wedge \mathcal{A}_Y \equiv \mathcal{A}_X \cup \mathcal{A}_Y = \mathcal{A}_{X \cup Y}$

To prove this, it is enough to show that $\mathcal{A}_X \wedge \mathcal{A}_Y \leq \mathcal{A}_X \cup \mathcal{A}_Y$.

Let $a, b \in \omega$ be such that $X = W_a$, $Y = W_b$ (recall that $X, Y \in \mathcal{R}$ thus X, Y are recursive sets) and let W be the set defined as follows

$W = \{\langle\langle x,y\rangle,u\rangle : (\exists v)[\langle 0,v\rangle \in D_u \;\&\; (\exists s)[[v \in W_a^s \;\&\; (\forall t<s)$
$[v \notin W_b^t] \;\&\; [[x=0 \;\&\; y=0] \text{ or } [x>0 \;\&\; \langle x-1,y\rangle \in D_u]]] \text{ or}$
$[v \in W_b^s \;\&\; (\forall t \leq s)[v \notin W_a^t] \;\&\; [[x=0 \;\&\; y=1] \text{ or } [x>0 \;\&$
$\langle x-1,y\rangle \in D_u]]]]]\}$

Clearly W is r.e. and defines an enumeration operator which in turn determines a recursive operator Ψ such that $\mathcal{A}_X \wedge \mathcal{A}_Y \leq \mathcal{A}_X \cup \mathcal{A}_Y$ via Ψ.

b) $\mathcal{A}_X \vee \mathcal{A}_Y \equiv \mathcal{A}_{X \cap Y}$.

Indeed,

$\mathcal{A}_X \vee \mathcal{A}_Y = \{g_x \vee g_y : x \in X \;\&\; y \in Y\} = \{g_z \vee g_z : z \in X \cap Y\} \cup \{g_x \vee g_y : x \in X \;\&\; y \in Y \;\&\; x \neq y\} \equiv \{g_z : z \in X \cap Y\} \cup \{g\} \equiv \{g_z : z \in X \cap Y\} = \mathcal{A}_{X \cap Y}$: let us show for instance that $\{g_z : z \in X \cap Y\} \cup \{g\} \equiv \{g_z \vee g_z : z \in X \cap Y\} \cup \{g_x \vee g_y : x \in X \;\&\; y \in Y \;\&\; x \neq y\}$: to show that $\geq$ holds, remember that for every $n\in\omega-\{0\}$, $g_n(0)\neq g(0)$ and $g_n\leq_T g$ (and thus $(\forall z\in X\cap Y)[g_z\vee g_z\leq_T g]$; to show that $\leq$ holds, one has simply to show that there exists a recursive operator Ψ such that

$$\Psi(f)=\begin{cases}\Psi_{h(f(0),f(1))}(f) & \text{if } f(0)\neq f(1)\\ \lambda x.f(2x) & \text{if } f(0)=f(1)\end{cases}$$

and, thus,

$$\Psi(g_x\vee g_y)=\begin{cases} g & \text{if } x\neq y\\ g_x & \text{if } x=y\end{cases}$$

It is easily seen that such a recursive operator exists.

For every $X\subseteq\omega-\{0\}$, let $\neg X$ denote $(\omega-\{0\})-X$; define a function $F:\mathcal{B}\longrightarrow\mathcal{M}$ by $F(X)=[\mathcal{a}_{\neg X}]$.

From a) and b) it follows that

$F(X\cap Y)=[\mathcal{a}_{\neg(X\cap Y)}]=[\mathcal{a}_{\neg X\cup\neg Y}]=[\mathcal{a}_{\neg X}\wedge\mathcal{a}_{\neg Y}]=$

$=[\mathcal{a}_{\neg X}]\wedge[\mathcal{a}_{\neg Y}]=F(x)\wedge F(Y)$;

$F(X\cup Y)=[\mathcal{a}_{\neg(X\cup Y)}]=[\mathcal{a}_{\neg X\cap\neg Y}]=[\mathcal{a}_{\neg X}\vee\mathcal{a}_{\neg Y}]=$

$=[\mathcal{a}_{\neg X}]\vee[\mathcal{a}_{\neg Y}]=F(X)\vee F(Y)$.

Moreover, by ii), since $m\neq n\implies g_m\mid_T g_n$, it follows that F is one-one.

Since $\mathcal{B}$ is a countable atomless Boolean algebra, every countable distributive lattice is embeddable in $\mathcal{B}$, via an embedding that preserves the least element (see for example BURRIS-SANKAPPANAVAR (1980)). Thus, we are in a position to conclude the proof of the theorem. Indeed, let $TB(\mathcal{L})\in$ <u>TB(DL)</u> be countable. Let

$J: \mathcal{L} \longrightarrow \mathcal{B}$ be an embedding morphism of DL. Define a funcyion $I: \mathcal{L} \longrightarrow \mathcal{M}$ by

$$I(a) = \begin{cases} 0 & \text{if } a = 0_{TB(\mathcal{L})} \\ F(J(a)) & \text{if } a \notin \{0_{TB(\mathcal{L})}, 1_{TB(\mathcal{L})}\} \\ 1 & \text{if } a = 1_{TB(\mathcal{L})} \end{cases}$$

(again, $\mathcal{L}$ is viewed here as a subset of $TB(\mathcal{L})$: notice that $I(b_{TB(\mathcal{L})}) = F(J(0_{\mathcal{L}})) = F(\emptyset) = [\mathcal{a}_{\omega - \{0\}}]$ and $I(t_{TB(\mathcal{L})}) = F(J(1_{\mathcal{L}})) = F(\omega - \{0\}) = [\{g\}]$.

I is the desired embedding.

Remark 8.1 If we define $F(X) = [\mathcal{a}_X]$ instead of $F(X) = [\mathcal{a}_{\neg X}]$ then we can use such an F to show that every countable object of TB(DL) is embeddable in $\mathcal{M}^{op}$, via a morphism of DL.

Corollary 8.1 Every countable $\mathcal{L} \in$ TB(DL) is embeddable in $\mathcal{M}_e$ and in $\mathcal{M}_w$ (via a morphism of DL).

Proof. Let $TB(\mathcal{L})$ be countable and let $I: TB(\mathcal{L}) \longrightarrow \mathcal{M}$ be the embedding of the previous theorem. Then $I_e \circ I: TB(\mathcal{L}) \longrightarrow \mathcal{M}_e$ and $F_w \circ I: TB(\mathcal{L}) \longrightarrow \mathcal{M}_w$ are embeddings of the category DL (I_e and F_w are defined in Section 5); F_w is not 1-1, but F_w I is 1-1).

Since clearly 1 is not the sup of any countable family of degrees strictly less than 1, we have

Corollary 8.2 A countable object $\mathcal{L}$ of <u>DL</u> is embeddable in $\mathfrak{M}$ if and only if $\mathcal{L}$ is an object of <u>TB(DL)</u>.

The proof of Theorem 8.1 shows that the range of the embedding function is contained into the set of effectively discrete degrees. On the other hand, we have

Lemma 8.1 The effectively discrete degrees form a sublattice of $\mathfrak{M}$.

Proof. Since 0, 1 are clearly effectively discrete,it suffices to show that if A_1, A_2 are effectively discrete, then so are $A_1 \vee A_2$ and $A_1 \wedge A_2$.

Suppose that A_1, A_2 are effectively discrete degrees and let $\mathcal{A}_1$, $\mathcal{A}_2$ be mass problems such that $A_1=[\mathcal{A}_1]$ and $A_2=[\mathcal{A}_2]$ and assume that for, every $i\in\{1,2\}$, for every $f,g\in \mathcal{A}_i$, $f(0)=g(0) \Longrightarrow f=g$ (that every effectively discrete degree contains a mass problem $\mathcal{A}$ satisfying $(\forall f,g\in \mathcal{A})[f(0)=g(0) \Longrightarrow f=g]$ is remarked in DYMENT (1976)). Thus, the set $S=\{\tilde{f}: \tilde{f}$ is an initial segment & $lh(\tilde{f})=2\}$ is clearly r.e. and witnesses the fact that $A_1 \vee A_2=[\mathcal{A}_1 \vee \mathcal{A}_2]$ is effectively discrete.
Finally, to show that $A_1\wedge A_2=[\mathcal{A}_1 \wedge \mathcal{A}_2]$ is effectively discrete, consider the r.e. set of initial segments $\{0*\tilde{f}: \tilde{f}$ is an initial segment & $lh(\tilde{f})=1\} \cup \{1*\tilde{f}: \tilde{f}$ is an initial segment & $lh(\tilde{f})=1\}$.

If $\mathcal{M}_{ED}$ denotes the sublattice of effectively discrete degrees, then we can conclude

Corollary 8.3 Every countable $\mathcal{L} \in \underline{TB(DL)}$ is embeddable in $\mathcal{M}_{ED}$ via a morphism of $\underline{DL}$.

We conclude this section by introducing the notion of Kripke frame, which will be used later in this paper (an excellent introduction to the theory of Kripke frames and related notions is FITTING (1969)).

Definition 8.1 A Kripke frame is a partial order $\mathcal{P} = \langle P, \leq \rangle$, where $P \neq \emptyset$.

There is a useful correspondence between Heyting algebras and Kripke frames; this correspondence will turn out to be quite useful for the topics treated in the next chapter.

Definition 8.2 Let $\mathcal{L}$ be a Heyting algebra and let $K(\mathcal{L}) = \{F\colon F$ is a proper prime filter of $\mathcal{L}\}$.
Then the partial order $\mathcal{K}(\mathcal{L}) = \langle K(\mathcal{L}), \subseteq \rangle$ will be referred to as the Kripke frame corresponding to $\mathcal{L}$.

On the other hand, given a Kripke frame $\mathcal{P} = \langle P, \leq \rangle$, let us say that a subset $Q \subseteq P$ is $\leq$-closed if

$(\forall x)(\forall y)[x \in Q \ \& \ x \leq y \Rightarrow y \in Q]$ and let $A(\mathcal{P}) = \{Q \subseteq P: Q$ is $\leq$-closed$\}$.

Definition 8.3 *The Heyting algebra corresponding to* $\mathcal{P}$ is the algebra $\mathfrak{A}(\mathcal{P})$ having $A(\mathcal{P})$ as universe and whose operations are defined as follows: for every $Q_1, Q_2 \in A(\mathcal{P})$

1) $Q_1 \wedge Q_2 = Q_1 \cap Q_2$;

2) $Q_1 \vee Q_2 = Q_1 \cup Q_2$;

3) $Q_1 \longrightarrow Q_2 = \{x \in P: (\forall y \in P)[x \leq y \Rightarrow [y \in Q_1 \text{ or } y \in Q_2]]\}$;

4) $Q_1 = Q_1 \longrightarrow \emptyset$.

For the sake of completeness, we will carry a little farther this sort of duality between Heyting algebras and Kripke frames by recalling the notion of open homomorphism. However, we will not make any use of this notion in this thesis.

Definition 8.3 Let $\mathcal{P}_1 = \langle P_1, \leq_1 \rangle$, $\mathcal{P}_2 = \langle P_2, \leq_2 \rangle$ be Kripke frames. A function $f: P_1 \longrightarrow P_2$ is an *open homomorphism of Kripke frames* if

1) $(\forall x, y \in P_1)[x \leq_1 y \Longrightarrow f(x) \leq_2 f(y)]$;

2) $(\forall x \in P_1)(\forall y \in P_2)[f(x) \leq_2 y \Longrightarrow (\exists z \in P_1)[f(z) = y]]$.

Theorem 8.2 (DE JONGH-TROELSTRA (1966)) Let $\mathcal{P}_1$, $\mathcal{P}_2$ be finite Kripke frames. Then $\mathfrak{A}(\mathcal{P}_1)$ is embeddable as an

Heyting algebra in $\mathfrak{A}(\mathcal{P}_2)$ if and only if there exists an open homomorphism of $\mathcal{P}_1$ onto $\mathcal{P}_2$.

CHAPTER II

THE LATICES $\mathcal{M}$, $\mathcal{M}_e$, $\mathcal{M}_w$, $\mathcal{M}_F$ AND INTERMEDIATE LOGICS.

1. Intermediate logics and embeddings.

Heyting algebras constitute a subcategory of $\underline{DL}$, its morphisms being those morphisms of $\underline{DL}$ which preserve the operations $\longrightarrow$ and $\neg$.

Analogously, Brouwer algebras are a subcategory of $\underline{DL}$: the morphisms of this category are defined in a similar way.

Throughout this chapter homomorphisms are understood to be morphisms of one of these two subcategories, the particular choice being clearly determined by the context.

Let $\underline{L}$ be a propositional language with connectives $\vee$, $\wedge$, $\longrightarrow$, $\neg$ and a countable infinite set of propositional variables. We warn the reader that we use the same symbols to denote logical connectives of $\underline{L}$ as well as operations of Heyting (or Brouwer) algebras: however this should be harmless enough and should not cause any confusion.

Form($\underline{L}$) will denote the set of formulas of $\underline{L}$. Throughout this chapter we use the lower-case Greek letters α, β, γ as variables running through Form($\underline{L}$).

Let $\underline{H}$ denote the intuitionistic propositional calculus, i.e. the deductive closure, using only Modus Ponens as rule of inference, of the following axiom schemes:

$\alpha \longrightarrow (\beta \longrightarrow \alpha)$

$(\alpha \longrightarrow (\beta \longrightarrow \gamma)) \longrightarrow ((\alpha \longrightarrow \beta) \longrightarrow (\alpha \longrightarrow \gamma))$

$\alpha \wedge \beta \longrightarrow \alpha$

$\alpha \wedge \beta \longrightarrow \beta$

$\alpha \longrightarrow (\beta \longrightarrow (\alpha \wedge \beta))$

$\alpha \longrightarrow (\alpha \vee \beta)$

$\beta \longrightarrow (\alpha \vee \beta)$

$(\alpha \longrightarrow \gamma) \longrightarrow ((\beta \longrightarrow \gamma) \longrightarrow (\alpha \vee \beta \longrightarrow \gamma))$

$(\alpha \longrightarrow \beta) \longrightarrow ((\alpha \longrightarrow \neg \beta) \longrightarrow \neg \alpha)$

$\neg \alpha \longrightarrow (\alpha \longrightarrow \beta)$

$\underline{K}$ will denote the classical propositional calculus, i.e. the deductive closure under Modus Ponens of the set of formulas $\underline{H} \cup \{\alpha \vee \neg \alpha : \alpha \in Form(\underline{L})$.

Definition 1.1 $\underline{IL}=\{\underline{L}: \underline{H} \subseteq \underline{L} \subseteq \underline{K}\}$ is the set of <u>intermediate logics</u>.

For a thorough investigation of this set see HOSOI (1967), HOSOI (1969), HOSOI-ONO (1973): HOSOI-ONO (1973) is a survey containing valuable information also as regards references, historical remarks etc. However, we will have ourselves the opportunity of making several remarks on this set in the remainder of this chapter.

Heyting algebras and Brouwer algebras are used for a semantics of intermediate logics in the following way: Let $\mathcal{L}$ be a Heyting algebra: a function $h:\mathrm{Form}(\underline{L}) \longrightarrow \mathcal{L}$ is an H-homomorphism if it satisfies, for every $\alpha,\beta \in \mathrm{Form}(\underline{L})$:

$h(\alpha \wedge \beta)=h(\alpha) \wedge h(\beta)$

$h(\alpha \vee \beta)=h(\alpha) \vee h(\beta)$

$h(\neg \alpha)=\neg h(\alpha)$

$h(\alpha \longrightarrow \beta)=h(\alpha) \longrightarrow h(\beta)$

where the symbols $\wedge$, $\vee$, $\longrightarrow$, $\neg$ denote, as previously remarked, on the left side of these equations the logical connectives of $\underline{L}$ and on the right side the operations of $\mathcal{L}$

Definition 1.2 Let $\mathcal{L}$ be a Heyting algebra and let $\alpha \in \mathrm{Form}(\underline{L})$. Then $\mathcal{L} \models_H \alpha$ if for every H-homomorphism h, $h(\alpha)=1$;

$\underline{L}_H(\mathcal{L})=\{\alpha \in \mathrm{Form}(\underline{L}) : \mathcal{L} \models_H \alpha\}$.

Dually, given a Brouwer algebra $\mathcal{L}$, a function h: $\mathrm{Form}(\underline{L}) \longrightarrow \mathcal{L}$ is a B-homomorphism if it satisfies

$h(\alpha \wedge \beta)=h(\alpha) \vee h(\beta)$

$h(\alpha \vee \beta)=h(\alpha) \wedge h(\beta)$

$h(\neg\alpha)=\neg h(\alpha)$

$h(\alpha \longrightarrow \beta)=h(\alpha) \longrightarrow h(\beta)$

Definition 1.3 Let $\mathcal{L}$ be a Brouwer algebra and let $\alpha \in$ Form($\underline{L}$). Then $\mathcal{L} \models_B \alpha$ if for every B-homomrphism h, $h(\alpha)=0$;

$\underline{L}_B(\mathcal{L})=\{\alpha\in Form(\underline{L}): \mathcal{L} \models_B \alpha\}$.

For every Heyting algebra $\mathcal{L}$, we have $\underline{H} \subseteq \underline{L}_H(\mathcal{L}) \subseteq$ K; for every Brouwer algebra $\mathcal{L}$, $\underline{H} \subseteq \underline{L}_B(\mathcal{L}) \subseteq \underline{K}$.

The first significative application of this (or at least very similar) semantics was perhaps in GODEL (1932).

There is also another semantics for intermediate logics which uses Kripke frames. This semantics was discovered by Kripke (KRIPKE (1965)). We would like to sketch here also this semantics, though extraneous to our purposes, just for the sake of completeness.

Let $\mathcal{P}=\langle P,\leq\rangle$ be a Kripke frame. A $\underline{\mathcal{P}\text{-valuation}}$ (with respect to $\underline{L}$) is a function $v:Form(\underline{L})\times P \longrightarrow \{0,1\}$ such that for every $x,y \in P$ and $\alpha,\beta \in Form(\underline{L})$,

$v(\alpha,x)=1 \;\&\; x\leq y \;\Longrightarrow\; v(\alpha,y)=1$

$v(\alpha\wedge\beta,x)=1 \iff v(\alpha,x)=1 \;\&\; v(\beta,x)=1$

$v(\alpha\vee\beta,x)=1 \iff v(\alpha,x)=1$ or $v(\beta,x)=1$

$v(\alpha\longrightarrow\beta,x)=1 \iff (\forall z\geq x)[v(\alpha,z)=1 \Longrightarrow v(\beta,z)=1]$

$v(\neg\alpha,x)=1 \iff (\forall z\geq x)[v(\alpha,z)=0]$

Then one defines $\underline{L}(\mathcal{P}) = \{\alpha \in Form(\underline{L})$: (for every $\mathcal{P}$-valuation $v)(\forall x \in P)[v(\alpha, x) = 1]\}$.

The following holds (see ONO (1971)): $\underline{L}(\mathcal{P}) = \underline{L}_H(\mathcal{Q}(\mathcal{P}))$; if $\mathcal{L}$ is a Heyting algebra then $\underline{L}_H(\mathcal{L}) \subseteq \underline{L}(\mathcal{K}(\mathcal{L}))$; moreover, if $\mathcal{L}$ is finite then $\underline{L}_H(\mathcal{L}) = \underline{L}(\mathcal{K}(\mathcal{L}))$.

We recall the following useful fact (see for example RASIOWA-SIKORSKI (1963)).

Fact 1.1 Let $\mathcal{L}_1$, $\mathcal{L}_2$ be Heyting algebras (Brouwer algebras). If F: $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is a monomorphism then $\underline{L}_H(\mathcal{L}_2) \subseteq \underline{L}_H(\mathcal{L}_1)$ ($\underline{L}_B(\mathcal{L}_2) \subseteq \underline{L}_B(\mathcal{L}_1)$). If F: $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is an epimorphism, then $\underline{L}_H(\mathcal{L}_1) \subseteq \underline{L}_H(\mathcal{L}_2)$ (and similarly, $\underline{L}_B(\mathcal{L}_1) \subseteq \underline{L}_B(\mathcal{L}_2)$).

Proof. Almost immediate. For example, suppose that $\mathcal{L}_1$, $\mathcal{L}_2$ are Heyting algebras and let F: $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ be an epimorphism. Let $\alpha \notin \underline{L}_H(\mathcal{L}_2)$; then there exists an H-homomorphism $h: Form(\underline{L}) \longrightarrow \mathcal{L}_2$ such that $h(\alpha) \neq 1_{\mathcal{L}_2}$. Let R be the set of propositional variables which appear in α; of course R is finite. For every $p \in R$, choose $ap \in \mathcal{L}_1$ such that $F(ap) = h(p)$. In order to show that $\alpha \notin \underline{L}_H(\mathcal{L}_1)$, consider for example the H-homomorphism h' obtained by extending in the obvious way to the set Form($\underline{L}$) the function h' defined on the set of propositional variables by

$$h'(p)=\begin{cases} ap & \text{if } p\in R \\ 1_{\mathcal{L}_1} & \text{if } p\notin R \end{cases}$$

$h':\mathrm{Form}(\underline{L}) \longrightarrow \mathcal{L}_1$ and $h'(\alpha)\neq 1_{\mathcal{L}_1}$,for otherwise $h(\alpha)= F(h'(\alpha))=1_{\mathcal{L}_2}$, a contradiction.

Definition 1.4 Let $\underline{J}$ be the deductive closure under Modus Ponens of $\underline{H} \cup \{\neg\alpha \vee \neg\neg\alpha:\ \alpha\in\mathrm{Form}(\underline{L})\}$.

The intermediate logic $\underline{J}$ has been investigated in several papers by Jankov (see e.g. JANKOV(1963), JANKOV (1968)); this is the reason for the symbol $\underline{J}$ (after Jankov) given by us to the previous logic.

An intermediate logic $\underline{L}$ is said to have <u>the finite model property</u> if there exists a set $\{\mathcal{L}_i:\ i\in I\}$ of finite Heyting algebras such that $\underline{L}= \cap\ \{\underline{L}_H(\mathcal{L}_i):\ i\in I\}$.

Fact 1.2 Let $\{\mathcal{L}_i : i\in\omega\}$ be the set of all finite Heyting algebras. Then

a) $\underline{H}= \cap\ \{\underline{L}_H(\mathcal{L}_i):i\in\omega\}= \cap\ \{\underline{L}_H(T(\mathcal{L}_i)):i\in\omega\}$;

b) $\underline{J}= \cap\ \{\underline{L}_H(B(\mathcal{L}_i):i\in\omega\}= \cap\ \{\underline{L}_H(TB(\mathcal{L}_i)):i\in\omega\}$.

(T,B are as defined in Chapter I, Section 8).

Proof. a) was proved in (or, at least, is an easy consequence of) JASKOWSKI (1936); b) is proved in JANKOV (1968).

Theorem 1.1 a) If $\mathcal{L} \in \{ \mathfrak{M}_F, \mathfrak{M}_E, \mathfrak{M}_{e,F}, \mathfrak{M}_w, \}$ then $\underline{J} \subseteq L_H(\mathcal{L})$;

b) if $\mathcal{L} \in \{ \mathfrak{M}, \mathfrak{M}_e, \mathfrak{M}_w \}$ then $\underline{J} \subseteq L_B(\mathcal{L})$.

(Remember that by Chapter I, Theorem 6.2, $\mathfrak{M}_w$ is both a Heyting algebra and a Brouwer algebra).

Proof. a) Let $\mathcal{L} \in \{ \mathfrak{M}_F, \mathfrak{M}_E, \mathfrak{M}_{e,F}, \mathfrak{M}_w \}$: it suffices to show that, for every $\alpha \in Form(\underline{L})$, $\neg\alpha \vee \neg\neg\alpha \in L_H(\mathcal{L})$. Using the fact that in $\mathfrak{M}_F$, $\mathfrak{M}_E$, $\mathfrak{M}_{e,F}$, and $\mathfrak{M}_w$ the least element 0 is meet irreducible, one easily sees that for every degree of difficulty $A \in \mathcal{L}$

$$\neg A = \begin{cases} 1 & \text{if } A=0 \\ 0 & \text{if } A \neq 0 \end{cases}$$

(here, of course, if $\mathcal{L} = \mathfrak{M}_w$, the operation $\longrightarrow$ is the one pertinent to $\mathfrak{M}_w$ as a Heyting algebra). Thus, for every H-homomorphism h and every $\alpha \in Form(\underline{L})$, we have $h(\neg\alpha \vee \neg\neg\alpha) = \neg h(\alpha) \vee \neg\neg h(\alpha) = 1$; hence $\neg\alpha \vee \neg\neg\alpha \in \underline{L}_H(\mathcal{L})$.

b) Let $\mathcal{L} \in \{ \mathfrak{M}, \mathfrak{M}_e, \mathfrak{M}_w \}$ and let $\mathfrak{M}_w$ be regarded as a Brouwer algebra. Since in $\mathfrak{M}$, $\mathfrak{M}_e$, $\mathfrak{M}_w$ the greatest element 1 is join-irreducible, for every degree of difficulty $A \in \mathcal{L}$ we have

$$\neg A = \begin{cases} 0 & \text{if } A=1 \\ 1 & \text{if } A \neq 1 \end{cases}$$

Hence, for every B-homomorphism h and every $\alpha \in Form(\underline{L})$, we have $h(\neg\alpha \vee \neg\neg\alpha) = \neg h(\alpha) \wedge \neg\neg h(\alpha) = 0$ and the proof is complete.

F_e, I_e, F_w, I_w as defined in Chapter I, Section 6 give preliminary information about the relationships between the logics of the lattices mentioned in Theorem 1.1. For example

Corollary 1.1 Let $Form^{\wedge,\longrightarrow,\neg}(\underline{L})=\{\alpha\in Form(\underline{L})$: α contains only $\wedge$, $\longrightarrow$, $\neg$ as logical connectives$\}$ and, for every $\underline{L}\in\underline{IL}$, let $\underline{L}^{\wedge,\longrightarrow,\neg}=\underline{L}\cap Form^{\wedge,\longrightarrow,\neg}(\underline{L})$. Then $\underline{L}_B{}^{\wedge,\longrightarrow,\neg}(\mathcal{M}) \subseteq \underline{L}_B{}^{\wedge,\longrightarrow,\neg}(\mathcal{M}_w)$.

Proof The proof follows from Fact 1.1 and Chapter I, Fact 6.3, as I_w is a $\vee,\longrightarrow,\neg$-monomorphism: notice that I_w preserves $\vee$ and not $\wedge$, but this is consistent with our claim since $\mathcal{M}$ and $\mathcal{M}_w$ are regarded as Brouwer algebras.

For the next theorem we first recall some preliminary facts about distributive lattices (the reader is referred to BIRKHOFF (1944)).

Let $\mathcal{L}$ be a distributive lattice. First, we remark that for every $p \in \mathcal{L}$,

p is meet-irreducible $\Longleftrightarrow$ $(\forall q,r\in \mathcal{L})[q\wedge r\leq p \Longrightarrow q\leq p$ or $r\leq p]$;

p is join-irreducible $\Longleftrightarrow$ $(\forall q,r\in \mathcal{L})[p\leq q\vee r \Longrightarrow p\leq q$ or $p\leq r]$.

A finite set $\{a_i\colon i\in I\}$ of elements of $\mathcal{L}$ is said to be

<u>$\wedge$-irredundant</u> if

$(\forall i\in I)[a_i$ is meet-irreducible$]$ and $(\forall j\in I)[\bigwedge\{a_i : i\in I\} < \bigwedge\{a_i : i\in I \ \&\ i\neq j\}]$.

Dually, $\{a_i : i\in I\}$ is <u>$\vee$-irredundant</u> if $(\forall i\in I)[a_i$ is join irreducible$]$ and $(\forall j\in I)$ $[\bigvee\{a_i : i\in I \ \&\ i\neq j\} < \bigvee\{a_i : i\in I\}]$.

Lemma 1.1 (BIRKHOFF (1944)) Let $\mathcal{L}$ be a finite distributive lattice. Then for every $a\in\mathcal{L}$ there exists one and only one $\wedge$-irredundant set $\{a_i : i\in I\}$ of elements of such that $a=\bigwedge\{a_i : i\in I\}$.
Dually, for every $a\in\mathcal{L}$ there exists one and only one $\vee$-irredundant set $\{a_i : i\in I\}$ such that $a=\bigvee\{a_i : i\in I\}$.

Theorem 1.2 Every finite Heyting algebra having exactly one atom and one coatom is embeddable in $\mathfrak{M}_F$.

Proof. Since every finite distributive lattice is a Heyting algebra (as remarked in Chapter I, Section 6), it is enough to show that for every finite object $\mathcal{L}$ of the category <u>DL</u>, the lattice $TB(\mathcal{L})$ is embeddable in $\mathfrak{M}_F$ as a Heyting algebra (we skip the case of the two-element Boolean algebra and the three-element chain, which are clearly embeddable in $\mathfrak{M}_F$).

Suppose that $\mathcal{L}$ is a finite distributive lattice and let $P=\{p\in\mathcal{L} : p \text{ is meet-irreducible}\}$.

Define a function lv: $\mathcal{L} \longrightarrow \omega$ as follows:

$lv(p)=0$ if $p=1_{\mathcal{L}}$;

$lv(p)=n+1$ if $(\exists q)[lv(q)=n \ \& \ p<q \ \& \ \neg(\exists r)[p<r<q]]$.

We use this function to define a finite partial order $\langle A, \circledS \rangle$ by letting $A= \bigcup\{A_p: p\in P\}$, $\circledS = \bigcup\{\circledS_n : n\in\omega\}$ where $\{A_p : p\in P\}$, $\{\circledS_n : n\in\omega\}$ are defined by induction: at step n) we define a finite set A_p for each $p\in P$ such that $lv(p)=n$, together with a partial ordering $\circledS_n$ on the set $\bigcup\{A_p : lv(p)\le n\}$.

The sequence $\{\circledS_n : n\in\omega\}$ will satisfy $(\forall m,n\in\omega)[m\le n \Longrightarrow \circledS_m \subseteq \circledS_n]$, which is enough to ensure that $\circledS = \bigcup\{\circledS_n : n\in\omega\}$ is a partial ordering.

In order to get a suitable indexing of the elements of each set A_p, we suppose also that we have fixed a linear ordering $\precsim$ on P.

In the course of the induction below, for every $p\in P$ we define a number $n(p)$ which will turn out to equal $|A_p|-1$.

Step 0) Let $p\in P$ be such that $lv(p)=0$.

Then define $A_p=\{a_p^0\}$. Notice that if $lv(p)=0$ then $p=1_{\mathcal{L}}$.

$\circledS_0$ is the identity relation on $A_{1_{\mathcal{L}}}$.

Step n+1) If there is no $p \in P$ such that $lv(p)=n+1$ then there is no A_p to define and we simply let $\circledS_{n+1}= \circledS_n$.

Otherwise, for every $p\in P$ such that $lv(p)=n+1$, we carry out the following construction, where we distinguish two cases;this distinction is not necessary in the sense that for every p we could define A_p in the same way as it is defined in Case a) below. However this would superfluously complicate the definition of A_p when $(\forall q\in P)[p\le q$ or $q\le p]$. Indeed, given p, the construction aims to find a function $F:P \longrightarrow \mathcal{M}_F$ such that, for every $q\in P$, $F(p\wedge q)=F(p)\wedge F(q)$, $F(p\vee q)=F(p)\vee F(q)$, $F(p\rightarrow q)=F(p)\rightarrow F(q)$. These conditions are automatically satisfied when p is comparable with all the elements of P, once we have only made sure that F is an order isomorphism and F(p) is meet-irreducible in $\mathcal{M}_F$.

Case a) $(\exists q)[q \in P \ \&\ p \mid q]$.

By Lemma 1.1, there exists a $\wedge$-irredundant set $\{q_0,\dots,q_v\}$ such that $\bigwedge\{q\in P \ :p<q\}= \bigwedge\{q_h:h\le v\}$: we may also suppose that $q_0 \not\le \dots \not\le q_v$.

Clearly, $(\forall h\le v)[lv(q_h)\le n]$; thus the sets $A_{q_0},\dots,A_{q_v}$ are already defined.

Let $n(p)= \left|\{A_h:\ h\le v\}\right|-1$ and define $A_p=\{a_p^0,\dots,a_p^{n(p)}\}$.

Let also $\circledless^p_{n+1}$ be the smallest preordering (i.e. antisymmetric and transitive) relation on $A_p \cup \bigcup\{A_q:\ lv(q)\le n\}$ satisfying

1) $\circledless_n \subseteq \circledless^p_{n+1}$;

2) $(\forall h\le v)(\forall i\le n(q_h))[a_p^{\Sigma\{n(q_j):j\le h-1\}+h+i} \circledless^p_{n+1} a^i_{q_h}]$

(where the otherwise meaningless symbol $\Sigma\{n(q_j):j\le -1\}$ is interpreted as 0;

Case b) Otherwise, i.e. $(\forall q\in P)[p\le q$ or $q\le p]$.

Let again $\bigwedge\{q\in P: p<q\} = \bigwedge\{q_h: h\le v\}$ where $\{q_0,\dots,q_v\}$ is $\wedge$-irredundant. Define

$A_p=\{a^o_p\}$ and let $\textcircled{<}^p_{n+1}$ be the smallest preordering relation on $A_p\cup\bigcup\{A_q: lv(q)\le n\}$ satisfying

1) $\textcircled{\le}_n \subseteq \textcircled{<}^p_{n+1}$;

2) $(\forall h\le v)(\forall i\le n(q_h))[a^o_p \textcircled{<}^p_{n+1} a^i_{q_h}]$.

Finally, to conclude Step n+1), let $\textcircled{\le}_{n+1}$ be the partial order on $\bigcup\{A_p: lv(p)\le n+1\}$ generated by $\bigcup\{\textcircled{<}^p_{n+1}: lv(p)= n+1\}$.

However, we will frequently happen to use a different notation for several of the above defined notions: suppose that A_p is defined at Step n+1) through Case a); then, for each $h\le v$ and $i\le n(q_h)$, let

$a^{q_h,i}_p = a_p{}^{\Sigma\{n(q_j):j\le h-1\}+h+i}$ and let

$A^{q_h}_p=\{a^{q_h,o}_p,\dots,a^{q_h,n(q_h)}_p\}$.

It follows that $A_p=\bigcup\{A^{q_h}_p: h\le v\}$ and, according to the definition of $\textcircled{<}^p_{n+1}$ above, we have that $(\forall h\le v)(\forall i\le n(q_h))[a^{q_h,i}_p \textcircled{<}^p_{n+1} a^i_{q_h}]$.

We list now several claims which will allow us to prove the theorem.

Claim 1 $(\forall q\in P)(\forall j\le n(q))[a^j_q \textcircled{\le} a^o_{1_{\mathcal{L}}}]$.

Proof of Claim 1. By induction on $n=lv(q)$.

Step 0) If $lv(q)=0$ then the claim is trivially true;

Step n+1) Suppose that the claim is true for every $q\in P$ such that $lv(q)\le n$ and let $q\in P$ be such that $lv(q)=n+1$. Let $\{s_0,\dots,s_w\}$ be a $\wedge$-irredundant set such that

$\bigwedge\{s\in P: q<s\}=\bigwedge\{s_h: h\leq w\}$. If A_q is defined through Case a) then $(\forall j\leq n(q))(\forall h\leq w)(\exists i\leq n(s_h))[a_q^j \leq a_{s_h}^i]$; on the other hand, if A_q is defined through Case b) then $n(q)=0$ and $(\forall h\leq w)(\forall i\leq n(s_h))[a_q^0 \leq a_{s_h}^i]$. In both cases the claim follows by induction, since $(\forall h\leq w)[lv(s_h)\leq n]$.

Given two subsets $B,C \subsetneq A$ let us say that $B \boxed{\leq} C$ if $(\forall c\in C)(\exists b\in B)[b \leq c]$.

Claim 2 $(\forall p\in P)(\forall q\in P)[p\leq q \Longrightarrow A_p \boxed{\leq} A_q]$.

Proof of Claim 2. By induction on $n=lv(p)$.

Step0) If $lv(p)=0$ then $p=1_{\mathcal{L}}$ and the claim follows trivially.

Step n+1) Let $p\in P$ and $lv(p)=n+1$ and suppose that the claim is true for every $p\in P$ such that $lv(p)\leq n$. Let $\bigwedge\{r\in P: p<r\}=\bigwedge\{r_h: h\leq v\}$ where $\{r_0,\ldots r_v\}$ is a $\wedge$-irreduntant set. Suppose that $q\in P$ and $p<q$ (the case $p=q$ is trivial): then $q\in \{r: p<r\}$ and, thus, $\bigwedge\{r_h: h\leq v\}\leq q$. Since q is meet-irreducible, we have $r_k\leq q$ for some $k\leq v$: but $lv(r_k)\leq n$, so the claim follows by induction and by the definitions of $\leq$ and $\boxed{\leq}$.

Claim 3 $(\forall p\in P)(\forall \text{finite } B\subsetneq A))(\forall r\in P)[A_p \cup B \boxed{\leq} A_r \Longrightarrow A_p \boxed{\leq} A_r$ or $B \boxed{\leq} A_r]$.

Proof of Claim 3. By induction on $n=lv(p)$.

Step 0) Let $lv(p)=0$; then $p=1_{\mathcal{L}}$. By Claim 1, we have that $(\forall B\subsetneq A)[B \boxed{\leq} A_{1_{\mathcal{L}}}]$. Thus, for every $r\in P$, we have

$A_{1_{\mathcal{L}}} \cup B \boxed{\leq} A_r \Longrightarrow B \boxed{\leq} A_r$.

Step n+1) Assume that the claim is true for every $p \in P$ such that $lv(p) \leq n$ and let $B \subseteq A$, $p, r \in P$ be such that $A_p \cup B \boxed{\leq} A_r$ and $lv(p)=n+1$. If $A_p = A_r$ then the claim is trivially true; otherwise, let $\bigwedge\{s \in P: p<s\} = \bigwedge\{s_h: h \leq w\}$ where $\{s_0, \ldots, s_w\}$ is $\wedge$-irredundant; by the construction and the definitions of $\circled{\leq}$ and $\boxed{\leq}$, $A_p \cup B \boxed{\leq} A_r$ implies $\bigcup\{A_{s_h}: h \leq w\} \cup B \boxed{\leq} A_r$ or, equivalently, $A_{s_0} \cup (\bigcup\{A_{s_h}: 1 \leq h \leq w\} \cup B) \boxed{\leq} A_r$.

Since, for every $h \leq w$, $lv(s_h) \leq n$, we conclude by induction that either $A_{s_0} \boxed{\leq} A_r$ or $\bigcup\{A_{s_h}: 1 \leq h \leq w\} \cup B \boxed{\leq} A_r$: if $A_{s_0} \boxed{\leq} A_r$ then the proof is complete since, clearly, $A_p \boxed{\leq} A_s$; if, instead, $\bigcup\{A_{s_h}: 1 \leq h \leq w\} \cup B \boxed{\leq} A_r$, then we proceed in the same way using the induction hypothesis, until we get an $i \leq w$ such that $A_{s_i} \boxed{\leq} A_r$, whence $A_p \boxed{\leq} A_r$, or otherwise $(\forall h \leq w)[A_{s_h} \not\boxed{\leq} A_r]$ and $B \boxed{\leq} A_r$.

Claim 4 $(\forall p \in P)(\forall q \in P)[A_p \boxed{\leq} A_q \Longrightarrow p \leq q]$.

Proof of Claim 4. By induction on $n = lv(p)$.

Step 0) If $lv(p)=0$ then the claim follows immediately since, by Claim 1, $A_{1_{\mathcal{L}}} \boxed{\leq} A_q \Longrightarrow A_{1_{\mathcal{L}}} = A_q$ and thus $q = 1_{\mathcal{L}}$.

Step n+1) Suppose that the claim is true for every $p \in P$ such that $lv(p) \leq n$ and let $p, q \in P$ be given such that $lv(p)=n+1$ and $A_p \boxed{\leq} A_q$. If $A_p = A_q$ then $p=q$ and the claim is trivial; otherwise, let $\bigwedge\{r \in P: p<r\} = \bigwedge\{r_h: h \leq v\}$ where $\{r_0, \ldots, r_v\}$ is $\wedge$-irredundant: clearly, $A_p \boxed{\leq} A_q$ implies $\bigcup\{A_{r_h}: h \leq v\} \boxed{\leq} A_q$. By Claim 3, there exists $i \leq v$ such that

A_{r_i} $\boxed{\leq}$ A_q: since $lv(r_i)\leq n$, it follows by induction that $r_i \leq q$. But $p\leq \bigwedge\{r_h\colon h\leq v\}$, thus the claim follows.

Claim 5 Let $p\in P$ be such that $n(p)=0$. Then
$(\forall q\in P)(\forall j\leq n(q))\ [p\leq q \Longrightarrow a_p^o \textcircled{\le} a_q^j\ \&\ q\leq p \Longrightarrow a_q^j \textcircled{\le} a_p^o]$.
Proof of Claim 5. Let $p\in P$ be such that $n(p)=0$: we argue by induction on $n=lv(q)$ to show that $(\forall q\in P)(\forall j\leq n(q))$ $[p\leq q \Longrightarrow a_p^o \textcircled{\le} a_q^j\ \&\ q\leq p \Longrightarrow a_q^j \textcircled{\le} a_p^o]$.
Step 0) If $lv(q)=0$ then the claim follows by Claim 1.
Step n+1) Suppose that the claim is true for every q such that $lv(q)<n$ and let $q\in P$ be given such that $lv(q)=n+1$. If $p\leq q$ then the claim follows by Claim 2. Thus, let $q<p$ and let $\bigwedge\{s\in P\colon q<s\}=\bigwedge\{s_h\colon h\leq w\}$ where $\{s_0,\ldots s_w\}$ is $\wedge$-irredundant: the claim follows by induction, since $(\forall h\leq w)[lv(s_h)\leq n]$ and $(\forall j\leq n(q))(\exists h\leq w)(\exists i\leq n(s_h))$ $[a_q^j \textcircled{\le} a_s^i]$.

Claim 6 $\langle A, \textcircled{\le}\rangle$ is an upper-semilattice.
Proof of Claim 6. We want to show that $(\forall p\in P)(\forall q\in P)$ $(\forall i\leq n(p))(\forall j\leq n(q))[a_p^i$ and a_q^j have a least upper bound].
We argue by induction on $n=lv(p)$.
Step 0) If $lv(p)=0$ then $p=1_{\mathcal{L}}$ and the claim follows by Claim 1.
Step n+1) Suppose that the claim is true for every $p\in P$ such that $lv(p)\leq n$. Let $p\in P$ be such that $lv(p)=n+1$ and let $\{r_0,\ldots r_v\}$ be a -irredundant set such that $\bigwedge\{r\in P\colon p<r\}=\bigwedge\{r_h\colon h\leq v\}$.

We first suppose that A_p is defined through Case a). Let $q \in P$ and let $i,j \in \omega$ be such that $i \leq n(p)$, $j \leq n(q)$. Referring to the alternative notation decribed earlier, there exist $k \leq v$ and $l \leq n(r_k)$ such that $a_p^i = a_p^{r_k, l}$. But $lv(r_k) \leq n$ and, therefore, by induction the least upper bound of $a_{r_k}^l$ and a_q^j exists: it is not difficult to see that this least upper bound is also the least upper bound of a_p^i and a_q^j.

If A_p is defined thorugh Case b), then the claim follows by Claim 5.

Let $H: \langle A, \circledS \rangle \longrightarrow \mathfrak{D}_T - \{o_T\}$ be an embedding which preserves suprema. Such an embedding exists, since every finite lattice is embeddable in $\mathfrak{D}_T$ via an embedding which preserves infima and suprema (as a reference for this fact, see for example LERMAN (1981), where it is proved that every finite lattice is embeddable in $\mathfrak{D}_T$ via an embedding which preserves infima and suprema and embeds as an initial segment of $\mathfrak{D}_T$) and since every finite upper-semilattice, if not already a lattice, becomes a lattice simply by adding a least element. We can suppose that $o_T \notin H(A)$: otherwise one can consider a new upper semilattice A^* obtained from A by adding an element smaller than all the elements of A; since also A^* is embeddable via, say, H^*, the restriction H of H^* to A is the desired embedding.

For every $p\in P$ and $i\le n(p)$, let $H(a_p^i)=(a_p^i)_T$ and let f_p^i be a function such that $f_p^i \in (a_p^i)_T$; let also $\mathcal{A}_p=\{f_p^0,\ldots,f_p^{n(p)}\}$.

By Lemma 1.1, every element of $\mathcal{L}$ is the infimum of a unique $\wedge$-irredundant set: if $a\in\mathcal{L}$ and $a=\bigwedge\{p_i:i\in I\}$ where $\{p_i:i\in I\}$ is $\wedge$-irredundant, let $\mathcal{A}_a=\bigcup\{\mathcal{A}_{p_i}: i\in I\}$.

Finally, let $F:\mathcal{L}\longrightarrow\mathcal{M}_F$ by given by $F(a)=[\mathcal{A}_a]$.

Claim 7 $(\forall a\in\mathcal{L})(\forall b\in\mathcal{L})[a\le b \Longleftrightarrow F(a)\le F(b)]$.

Proof of Claim 7. Since H is an embedding of partial orders we have that $(\forall p\in P)(\forall q\in P)[\mathcal{A}_p\le\mathcal{A}_q \Longleftrightarrow A_p \boxed{\le} A_q]$ (recall that $\mathcal{A}_p\le\mathcal{A}_q \Longleftrightarrow (\forall f\in\mathcal{A}_q)(\exists g\in\mathcal{A}_p)[g\le_T f]$: this is so by Chapter I, Remark 7.1, since $\mathcal{M}_F$ is embeddable in $\mathcal{M}_w$). Hence, by Claim 2 and Claim 4, $(\forall p\in P)(\forall q\in P)[p\le q \Longleftrightarrow \mathcal{A}_p\le\mathcal{A}_q]$.
Let now $a,b\in\mathcal{L}$ and let $a=\bigwedge\{p_i: i\in I\}$, $b=\bigwedge\{q_j: j\in J\}$ where $\{p_i: i\in I\}$, $\{q_j: j\in J\}$ are $\wedge$-irredundant.
If $a\le b$ then, since each q_j is meet-irreducible, we have $(\forall j\in J)(\exists i\in I)[p_i\le q_j]$; hence, for what has been just remarked, $(\forall j\in J)(\exists i\in I)[\mathcal{A}_{p_i}\le\mathcal{A}_{q_j}]$, which implies $\bigcup\{\mathcal{A}_{p_i}:i\in I\}\le\bigcup\{\mathcal{A}_{q_j}:j\in J\}$, i.e. $F(a)\le F(b)$.
On the other hand, suppose $F(a)\le F(b)$, i.e. $\mathcal{A}_a\le\mathcal{A}_b$: thus $\bigcup\{\mathcal{A}_{p_i}:i\in I\}\le\bigcup\{\mathcal{A}_{q_j}:j\in J\}$ and, again since H is an embedding of partial orders, this implies $\bigcup\{A_{p_i}:i\in I\}$

$\boxed{\leq} \bigcup\{A_{q_j} : j\in J\}$. By Claim 3, it follows that $(\forall j\in J)(\exists i\in I)\ [A_{p_i} \boxed{\leq} A_{q_j}]$ and by Claim 4 we have $(\forall j\in J)(\exists i\in I)[p_i \leq q_j]$: but then we conclude $\bigwedge\{p_i: i\in I\} \leq \bigwedge\{q_j : j\in J\}$ and the claim is proved.

Claim 8 $(\forall a\in \mathcal{L})(\forall b\in \mathcal{L})[F(a\wedge b)=F(a)\wedge F(b)]$.

Proof of Claim 8. Let $a,b\in \mathcal{L}$ be given. Since $a\wedge b\leq a$ and $a\wedge b \leq b$, by Claim 7 we easily conclude that $F(a\wedge b)\leq F(a)\wedge F(b)$.

On the other hand, notice that if $\{p_i: i\in I\}$ and $\{q_j: j\in J\}$ are $\wedge$-irredundant sets such that respectively $a=\bigwedge\{p_i: i\in I\}$ and $b=\bigwedge\{q_j: j\in J\}$, then there exists a $\wedge$-irredundant set $R \subseteq \{p_i: i\in I\} \cup \{q_j: j\in J\}$ such that $a\wedge b=\bigwedge R$. Thus $\bigcup\{\mathcal{Q}_r: r\in R\} \subseteq \bigcup\{\mathcal{Q}_{p_i}: i\in I\} \cup \bigcup\{\mathcal{Q}_{q_j}: j\in J\}$ and therefore $\bigcup\{\mathcal{Q}_{p_i}: i\in I\} \wedge \bigcup\{\mathcal{Q}_{q_j}: j\in J\} \leq \bigcup\{\mathcal{Q}_r: r\in R\}$, i.e. $F(a)\wedge F(b)\leq F(a\wedge b)$, as desired.

Claim 9 $(\forall a\in \mathcal{L})(\forall b\in \mathcal{L})[F(a\vee b)=F(a)\vee F(b)]$.

Proof of Claim 9. We first show by induction on $n=lv(p)$ that

(*) $(\forall p\in P)(\forall q\in P)[F(p\vee q)=F(p)\vee F(q)]$.

<u>Step 0)</u> If $lv(p)=0$ then (*) follows from Claim 7, since $p=1_{\mathcal{L}}$ and $(\forall q\in P)[F(q)\leq F(1_{\mathcal{L}})]$.

<u>Step n+1)</u> Let (*) be true for every $p\in P$ such that $lv(p)\leq n$ and let $p,q\in P$ where $lv(p)=n+1$; suppose also that $\bigwedge\{r\in P: p<r\}=\bigwedge\{r_h: h\leq v\}$ where $\{r_0,\dots,r_v\}$ is a $\wedge$-irredundant set. If $p\leq q$ or $q\leq p$ then (*) follows again

from Claim 7. If $p \mid q$ then one easily sees that $p \vee q = (\bigwedge\{r_h: h \leq v\}) \vee q$; moreover, by the construction and the definitions of $\mathfrak{S}$ and H, we have that $\mathfrak{a}_p \vee \mathfrak{a}_q \equiv (\bigcup\{\mathfrak{a}_{r_h}: h \leq v\}) \vee \mathfrak{a}_q$. Therefore, since $(\forall h \leq v)[lv(r_h) \leq n]$, we have

$F(p \vee q) = F((\bigwedge\{r_h: h \leq v\} \vee q) = F(\bigwedge\{r_h \vee q: h \leq v\}) =$ (apply Claim 8) $= \bigwedge\{F(r_h \vee q): h \leq v\} =$ (apply the induction hypothesis) $= \bigwedge\{F(r_h) \vee F(q): h \leq v\} = \bigwedge\{[\mathfrak{a}_{r_h}] \vee [\mathfrak{a}_q]: h \leq v\} = \bigwedge\{[\mathfrak{a}_{r_h} \vee \mathfrak{a}_q]: h \leq v\} = [(\bigcup\{\mathfrak{a}_{r_h}: h \leq v\}) \vee \mathfrak{a}_q] = [\mathfrak{a}_p \vee \mathfrak{a}_q] = [\mathfrak{a}_p] \vee [\mathfrak{a}_q] = F(p) \vee F(q)$.

Let us now return to the proof of Claim 8 and let $a, b \in \mathcal{L}$, with $a = \bigwedge\{p_i: i \in I\}$, $b = \bigwedge\{q_j: j \in J\}$, where $\{p_i: i \in I\}$ and $\{q_j: j \in J\}$ are $\wedge$-irredundant. We have

$F(a \vee b) = F((\bigwedge\{p_i: i \in I\}) \vee (\bigwedge\{q_j: j \in J\})) = F(\bigwedge\{p_i \vee q_j: i \in I \ \& \ j \in J\}) =$ (apply Claim 8) $= \bigwedge\{F(p_i \vee q_j): i \in I \ \& \ j \in J\} =$ (apply (*)) $= \bigwedge\{F(p_i) \vee F(q_j): i \in I \ \& \ j \in J\} =$ (apply Claim 8) $= F(\bigwedge\{p_i: i \in I\}) \vee F(\bigwedge\{q_j: j \in J\}) = F(a) \vee F(b)$.

Let us now define $I: TB(\mathcal{L}) \longrightarrow \mathfrak{M}_F$ by

$$I(a) = \begin{cases} 0 & \text{if } a = 0_{TB(\mathcal{L})} \\ 1 & \text{if } a = 1_{TB(\mathcal{L})} \\ F(a) & \text{if } a \in \mathcal{L}. \end{cases}$$

By Claims 7,8 and 9, I is a lattice-theoretic embedding. To conclude the proof of the theorem we have only to show

Claim 10 $(\forall a \in TB(\mathcal{L}))(\forall b \in TB(\mathcal{L}))[I(a \rightarrow b) = I(a) \rightarrow I(b)]$.

Proof of Claim 10. Suppose that $a,b \in \mathcal{L}$ and let $a = \bigwedge\{p_i: i \in I\}$, $b = \bigwedge\{q_j: j \in J\}$ where $\{p_i: i \in I\}$, $\{q_j: j \in J\}$ are $\wedge$-irredundant. Then

$a \longrightarrow b = \bigwedge\{q_j: j \in J \ \&\ (\forall i \in I)[p_i \not\leq q_j]\}$; also, by Chapter I Corollary 7.1, $(\bigcup\{\mathcal{A}_{p_i}: i \in I\}) \longrightarrow (\bigcup\{\mathcal{A}_{q_j}: j \in J\}) = \{f \in \bigcup\{\mathcal{A}_{q_j}: j \in J\}: (\forall g \in \bigcup\{\mathcal{A}_{p_i}: i \in I\})[g \not\leq_T f]\}$.

By Claims 2 and 4 and since H is an embedding of upper-semilattices, it follows that this set equals $\bigcup\{\mathcal{A}_{q_j}: j \in J \ \&\ (\forall i \in I)[\mathcal{A}_{p_i} \not\leq \mathcal{A}_{q_j}]\}$; hence

$$(\bigcup\{\mathcal{A}_{p_i}: i \in I\}) \rightarrow (\bigcup\{\mathcal{A}_{q_j}: j \in J\}) = \mathcal{A}_{\wedge\{p_i: i \in I\}} \xrightarrow{\quad} \mathcal{A}_{\wedge\{q_j: j \in J\}}.$$

If $\{a,b\} \cap \{0_{TB(\mathcal{L})}, 1_{TB(\mathcal{L})}\} \neq \emptyset$ then we can show that $I(a \longrightarrow b) = I(a) \longrightarrow I(b)$ case by case: for example, let $a = 1_{\mathcal{L}}$ and $b \neq a$. Then $a \longrightarrow b = b$; on the other hand $I(a) \longrightarrow I(b) = I(b)$: thus $I(a \longrightarrow b) = I(a) \longrightarrow I(b)$.

Corollary 1.2 A finite Heyting algebra $\mathcal{L}$ is embeddable in $\mathcal{M}_F$ if and only if $\mathcal{L}$ is an object of TB(DL).

Proof. The proof is obvious since in $\mathcal{M}_F$ 0 is meet-irreducible and 1 is join-irreducible.

Remark 1.1 Let $\mathcal{L}$, P be as in Theorem 1.2 and let $\mathcal{P} = \langle\{A_p: p \in P\}, \boxed{\leq}\rangle$; then $\mathcal{P}$ is isomorphic, as a partial order, to the dual of the Kripke frame corresponding to $T(\mathcal{L})$ (see Chapter I, Definition 8.2). This can be seen as follows. First recall that, by Claims 2 and 4 in the proof of Theorem 1.2, we have $(\forall p\in P)(\forall q\in P)[A_p \boxed{\leq} A_q \Longleftrightarrow p\leq q]$. For every $a\in T(\mathcal{L})$, let J_a denote the principal ideal generated by a, i.e. $J_a=\{b\in T(\mathcal{L}): b\leq a\}$: since in a finite distributive lattice every prime ideal is the principal ideal generated by some meet-irreducible element and since $(\forall p\in P)(\forall q\in P)[p\leq q \Longleftrightarrow J_p \subseteq J_q]$, it follows that $\mathcal{P}$ is isomorphic with the partial order $<\{J: J$ is a proper prime ideal of $T(\mathcal{L})\}, \subsetneq>$. But this latter partial order is isomorphic with the dual of the Kripke frame corresponding to $T(\mathcal{L})$, since, for every filter F of $T(\mathcal{L})$, F is a proper prime filter if and only if $\{a\in T(\mathcal{L}): a\notin F\}$ is a proper prime ideal.

Clearly, $\mathcal{P}$ is isomorphic (as a partial order) to $<A, \circledS>$ if and only if $(\forall p\in P)[n(p)=0]$: so, in this case, $<A, \circledS>$ is isomorphic to the dual of the Kripke frame corresponding to $T(\mathcal{L})$.

We conclude by remarking the following two easy properties:

1) $(\forall p\in P)[n(p)=0] \Longleftrightarrow (\forall p\in P)[\bigwedge\{r\in P: p<r\}\in P$ or $(\forall q\in P)[p\leq q$ or $q\leq p]]$;

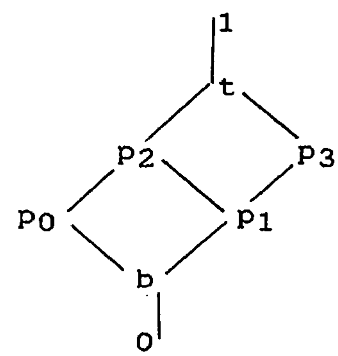
2) $(\forall p\in P)[n(p)=0] \Longrightarrow (\forall p\in P)(\forall q\in P)[p\vee q\in P]$.

In both cases the proof is by induction.

We now give several examples of the procedure exposed in the proof of Theorem 1.2. Each example consists of three pictures: the first picture represents the lattice which we want to embed in $\mathcal{M}_F$ (each lattice being a finite Heyting algebra with exactly one atom and one coatom); the second picture represents the partial order $\langle A, \circledS \rangle$; the third picture represents the image of the lattice under the embedding.

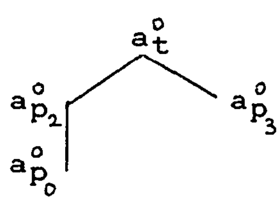
Example 1.1

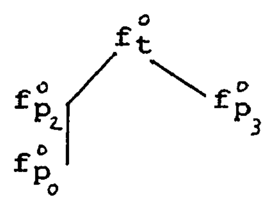
1)

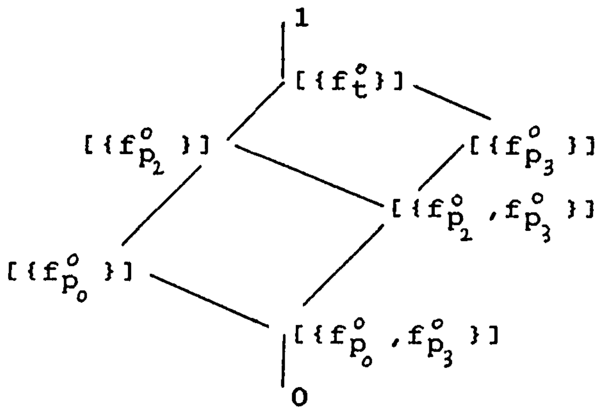


$P=\{t,p_2,p_3,p_0\}$.

A:

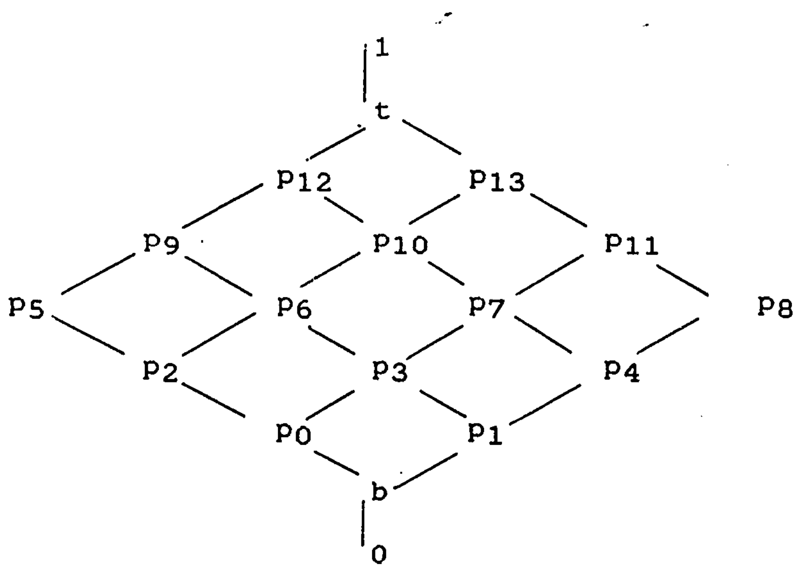






For example, $p_0 \longrightarrow p_1 = p_3$: indeed, $\{f^o_{p_0}\} \longrightarrow \{f^o_{p_2}, f^o_{p_3}\} \equiv \{f^o_{p_3}\}$, as $f^o_{p_0} <_T f^o_{p_2}$.

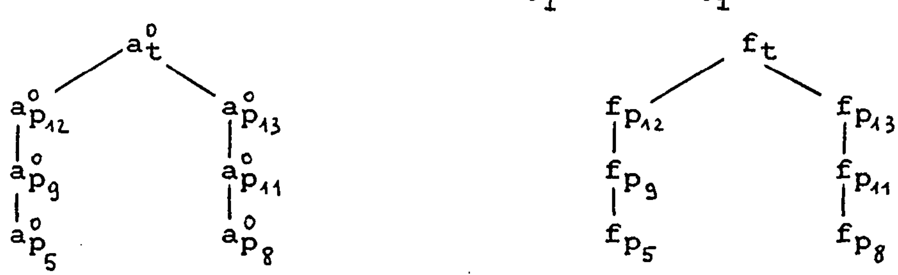
2)

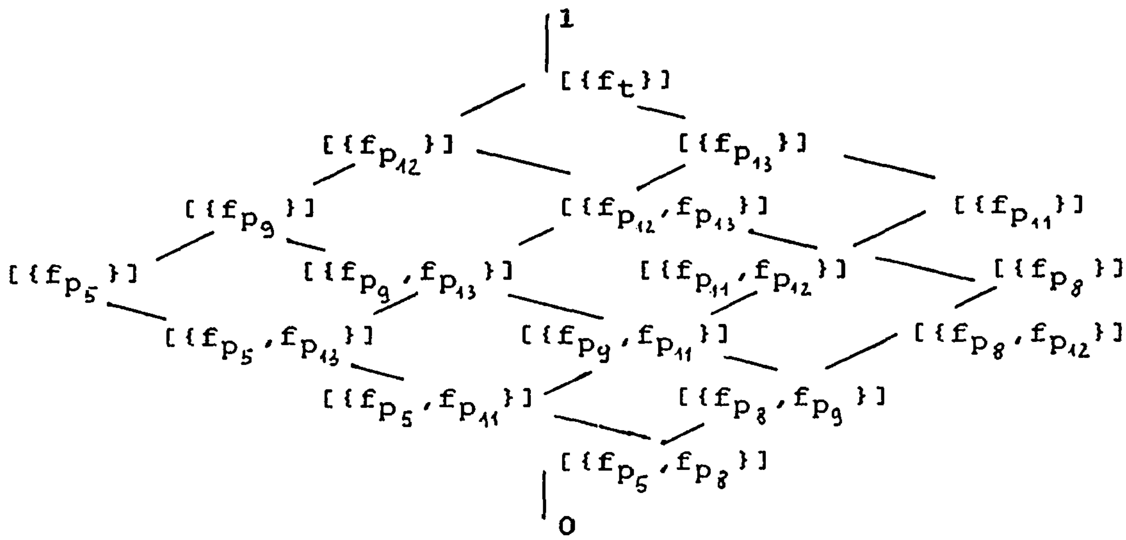


$P = \{t, p_{12}, p_{13}, p_9, p_{11}, p_5, p_8\}$.

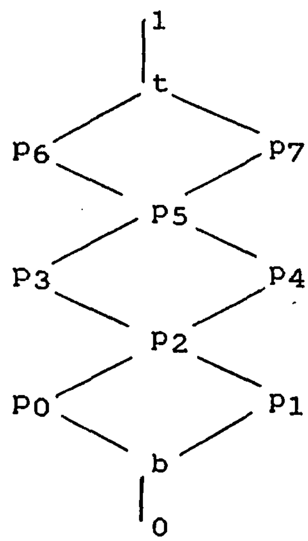
In the following picture write f_{p_i} for $f^o_{p_i}$:

A:

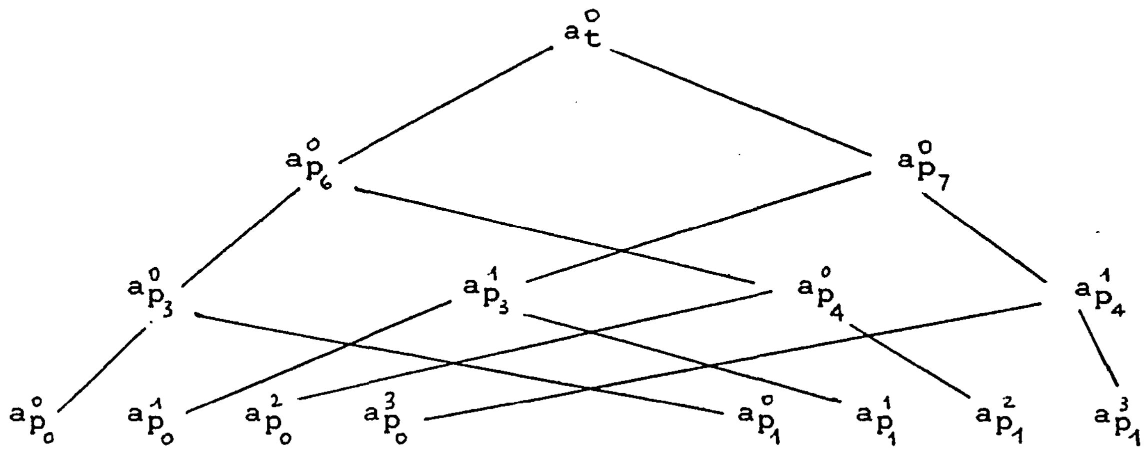



1
[{f_t}]
[{fp12}]
[{fp13}]
[{fp9}]
[{fp12, fp13}]
[{fp11}]
[{fp5}]
[{fp9, fp13}]
[{fp11, fp12}]
[{fp8}]
[{fp5, fp13}]
[{fp9, fp11}]
[{fp8, fp12}]
[{fp5, fp11}]
[{fp8, fp9}]
[{fp5, fp8}]
0

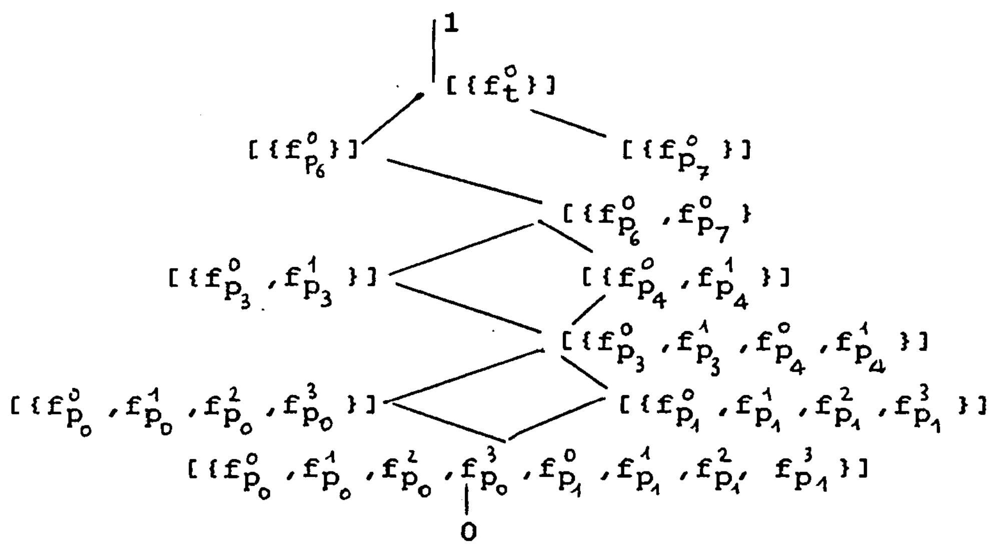
3)


1
t
p6
p7
p5
p3
p4
p2
p0
p1
b
0

$P=\{t, p_6, p_7, p_3, p_4, p_0, p_1\}$.



Using the notation employed in the proof of Theorem 1.2, the image of the preceding Heyting algebra under the embedding looks like

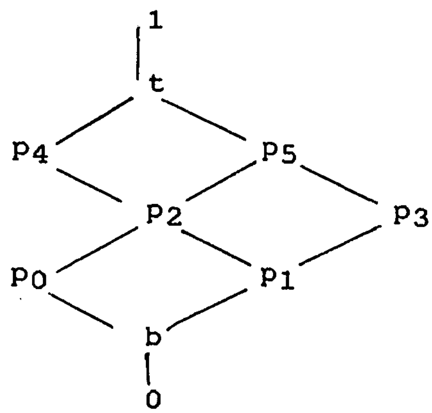


For instance,

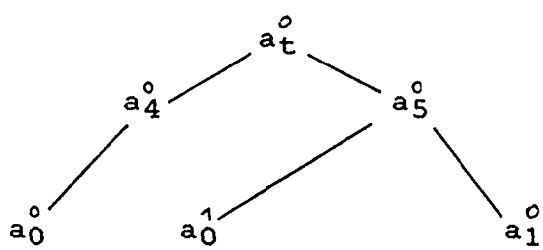
$\{f^0_{p_0}, f^1_{p_0}, f^2_{p_0}, f^3_{p_0}\} \vee \{f^0_{p_1}, f^1_{p_1}, f^2_{p_1}, f^3_{p_1}\} = \{f^0_{p_0} \vee f^0_{p_1}, f^0_{p_0} \vee f^1_{p_1}, f^0_{p_0} \vee f^2_{p_1}, f^0_{p_0} \vee f^3_{p_1}, f^1_{p_0} \vee f^0_{p_1}, f^1_{p_0} \vee f^1_{p_1}, f^1_{p_0} \vee f^2_{p_1}, f^1_{p_0} \vee f^3_{p_1}, f^2_{p_0} \vee f^0_{p_1}, f^2_{p_0} \vee f^1_{p_1}, f^2_{p_0} \vee f^2_{p_1}, f^2_{p_0} \vee f^3_{p_1}, f^3_{p_0} \vee f^0_{p_1}, f^3_{p_0} \vee f^1_{p_1}, f^3_{p_0} \vee f^2_{p_1}, f^3_{p_0} \vee f^3_{p_1}\} \equiv \{f^0_{p_3}, f^0_t, f^0_{p_6}, f^0_t, f^0_t, f^1_{p_3}, f^0_t, f^0_{p_7}, f^0_{p_6}, f^0_t, f^0_{p_4}, f^0_t, f^0_t, f^0_{p_7}, f^0_t, f^1_{p_4}\} \equiv \{f^0_{p_3}, f^1_{p_3}, f^0_{p_4}, f^1_{p_4}\}$;

$$\{f^0_{p_0}, f^1_{p_0}, f^2_{p_0}, f^3_{p_0}\} \longrightarrow \{f^0_{p_1}, f^1_{p_1}, f^2_{p_1}, f^3_{p_1}\} \equiv \{f^0_{p_1}, f^1_{p_1}, f^2_{p_1}, f^3_{p_1}\};$$
$$\{f^0_{p_4}, f^1_{p_4}\} \longrightarrow \{f^0_{p_0}, f^1_{p_0}, f^2_{p_0}, f^3_{p_0}\} \equiv \{f^0_{p_0}, f^1_{p_0}, f^2_{p_0}, f^3_{p_0}\}$$

4)

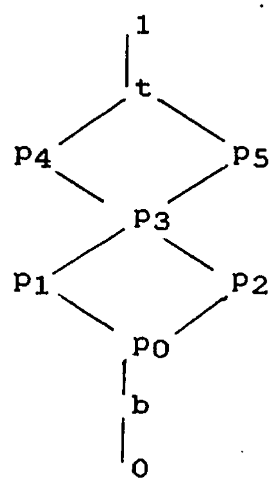


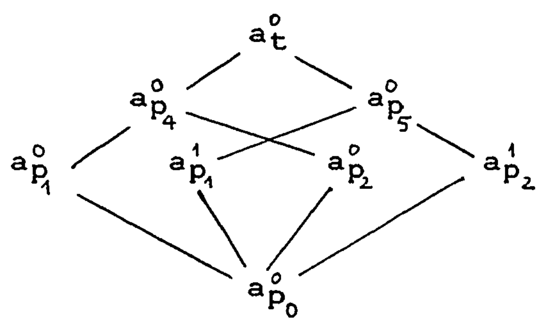
A:



It is left to the reader to complete the example. This example shows also that the converse implication in b) of Remark 1.1 does not hold: indeed $(\forall p \in P)(\forall q \in P)[p \vee q \in P]$, but $n(p_0)=1$.

5)





It is left to the reader to complete the example.

Corollary 1.3 If $\mathcal{L} \in \{\mathfrak{M}_F, \mathfrak{M}_E, \mathfrak{M}_{e,F}, \mathfrak{M}_w\}$ (where $\mathfrak{M}_w$ is regarded as a Heyting algebra), then $\underline{L}_H(\mathcal{L}) = \underline{J}$.

Proof. In view of Theorem 1.1, we need only to show $\underline{L}_H(\mathcal{L}) \subseteq \underline{J}$. Since $\mathfrak{M}_F$ is a sub-Heyting algebra of each of the lattices mentioned in the statement of the Corollary, the result follows from Theorem 1.2, Fact 1.1 and Fact 1.2 b).

We now define a class of mass problems and prove several results about the degrees of these mass problems. These degrees will play an important role in the remainder of this chapter.

Definition 1.5 For every $f\in {}^{\omega}\omega$, let $\mathcal{B}_f=\{g: g \not\leq_T f\}$ (notice that if f is recursive then $\mathcal{B}_f= \mathcal{O}'$). Let also $B_f=[\mathcal{B}_f]$ and $(B_f)_w=[\mathcal{B}_f]_w$. For the sake of semplicity, also $[\mathcal{B}_f]_w$ will be denoted by B_f when no possibility of confusion arises by doing so.

Fact 1.3 In $\mathcal{M}$, B_f is uniform and, thus, meet irreducible.

Proof. Obvious, as $\mathcal{B}_f$ is closed under $\leq_T$.

Fact 1.4 In $\mathcal{M}$ $\{A: B_f\not\leq A\}$ is an ideal and concides with the principal ideal generated by $[\{f\}]$; analogously, in $\mathcal{M}_w$ $\{A_w: B_f \not\leq_w A_w\}$ is an ideal and coincides with the principal ideal generated by $[\{f\}]_w$.

Proof If $\mathcal{B}_f \not\leq \mathcal{A}$ then $\mathcal{A} \not\subseteq \mathcal{B}_f$; hence $(\exists g\in \mathcal{A})[g\leq_T f]$, which implies $\mathcal{A} \leq\{f\}$. On the other hand, if $\mathcal{A} \leq\{f\}$ then $\mathcal{B}_f \not\leq \mathcal{A}$: otherwise it would be $\mathcal{B}_f\leq\{f\}$, a contradiction.
The proof for $\mathcal{M}_w$ is similar.

Fact 1.5 For every function f, $\{A:A<B_f\}=\{A: A\leq B_f \wedge [\{f\}]\}$ and $\{A_w: A_w <_w B_f\}=\{A_w: A_w \leq_w B_f \wedge [\{f\}]_w\}$.

(Again the result implies that a certain ideal is principal).

Proof Clearly $B_f \wedge [\{f\}] < B_f$; thus $\{A: A \le B_f \wedge [\{f\}]\} \subseteq \{A: A < B_f\}$.

On the other hand $A < B_f \Longrightarrow B_f \not\le A$; hence, by Fact 1.4, if $A < B_f$ then $A \le [\{f\}]$ which implies $A \le B_f \wedge [\{f\}]$. The proof for $_w$ is similar.

Using language borrowed from the theory of Turing degrees (see e.g. LERMAN (1984)), B_f is a strong minimal cover of $B_f \wedge [\{f\}]$ (both in $\mathfrak{M}$ and in $\mathfrak{M}_w$, upon placing the subscript $_w$ in a suitable way): this is just the content of Fact 1.5.

Corollary 1.4 In both $\mathfrak{M}$ and $\mathfrak{M}_w$, B_f is join-irreducible.

Proof. Obvious by Fact 1.5

Lemma 1.2 The function I: $\mathfrak{D}_T \longrightarrow \mathfrak{M}$ given by $I([f]_T) = B_f$ is an embedding of partial orders.

Proof. For every $f,g \in {}^{\omega}\omega$, $f \le_T g \Longleftrightarrow \mathcal{B}_g \subseteq \mathcal{B}_f$; hence $f \le_T g \Longleftrightarrow B_f \le B_g$ and the lemma is proved.

Lemma 1.3 The function I: $\mathfrak{D}_T \longrightarrow \mathfrak{M}_w$ given by $I([f]_T) = B_f$ is an embedding of partial orders which preserves infima

Proof Assume that $[f]_T \wedge [g]_T$ exists and let h be a function such that $[h]_T = [f]_T \wedge [g]_T$. We need to show that $\mathcal{B}_f \wedge \mathcal{B}_g \equiv_w \mathcal{B}_h$: indeed, one can actually prove that $\mathcal{B}_f \cup \mathcal{B}_g = \mathcal{B}_h$.

Corollary 1.5 In $\mathcal{M}_w$, each B_f is meet-reducible.

Proof. The proof follows from Lemma 1.3, noticing that each T-degree is the infimum of two greater T-degrees.

Although in $\mathcal{M}$, for every function f, the interval $(B_f \wedge [\{f\}], B_f)$ is empty, below $B_f \wedge [\{f\}]$, if f is not recursive, we have density as is shown by the following fact.

Fact 1.6 $(\forall \mathcal{A})(\forall f)[\ \mathcal{A} < \mathcal{B}_f \wedge \{f\} \Longrightarrow (\exists \mathcal{C})[\ \mathcal{A} < \mathcal{C} < \mathcal{B}_f \wedge \{f\}]]$.

Proof. We use the characterization of empty intervals of $\mathcal{M}$ stated in Chapter I, Remark 3.1: we want to show that for every mass problem $\mathcal{A}$ and every $f,g \in {}^{\omega}\omega$ such that $\mathcal{B}_f \wedge \{f\} \not\leq \{g\}$, if $\mathcal{A} \equiv (\mathcal{B}_f \wedge \{f\}) \wedge \{g\}$ then $\mathcal{B}_f \wedge \{f\} \not\leq \mathcal{T}_g$ (recall that $\mathcal{T}_g = \{z * h:\ \Psi_z(h) = g\ \&\ h \not\leq_T g\}$).

Suppose that $\mathcal{B}_f \wedge \{f\} \not\leq \{g\}$; notice that $\mathcal{B}_f \wedge \{f\} \not\leq \{g\} \Longrightarrow g <_T f$. Let Ψ_z be a recursive operator such that for every $l,m \in {}^{\omega}\omega$, $\Psi_z(l \vee m) = l$ and assume that there exists a recursive operator Ψ such that

$(\forall h\in \mathcal{C}_g)[\Psi(h)\in \mathcal{B}_f \wedge \{f\}]$. Now, $(\forall h)[h\equiv_T f \Rightarrow z*(g\vee h)\in \mathcal{C}_g]$: indeed, $\Psi_z(g\vee h)=g$ and $g\vee h \not\leq_T g$ since $g\vee h\equiv_T f$; but, also, $h\equiv_T f \Longrightarrow \Psi(z*(g\vee h)) \leq_T f$, since $\Psi(z*(g\vee h)) \leq_T g\vee h$; hence $\Psi(z*(g\vee h))=1*f$. Therefore, it follows that $\{f\} \leq z*(\{g\}\vee\{h: h\equiv_T f\})\equiv\{g\}\vee\{h: h\equiv_T f\}$. Since $\{f\} \not\leq \{g\}$, by Chapter I, Fact 7.3 this would imply that $\{h: h\equiv_T f\}$ is nowhere dense, a contradiction for $\{h: h\equiv_T f\}$ is dense in the Baire topology. The contradiction arises from assuming that $\mathcal{B}_f \wedge\{f\} \not\leq \{g\}$ and $\mathcal{B}_f \wedge\{f\} \leq \mathcal{C}_g$, thus the proof is complete.

We add also that, both in $\mathcal{M}$ and in $\mathcal{M}_w$, although B_f is join-irreducible, $B_f \wedge [\{f\}]$ (or $B_f \wedge [\{f\}]_w$ if we are in $\mathcal{M}_w$) need not be join-irreducible. For example, if $A <_e f$, $B <_e f$ and $A\vee B\equiv_e f$ then, in $\mathcal{M}$, $B_f\wedge E_A < B_f \wedge [\{f\}]$, $B_f\wedge E_B < B_f \wedge[\{f\}]$, but $(B_f \wedge E_A) \vee (B_f \wedge E_B)=B_f \wedge [\{f\}]$: (to show that $B_f \wedge E_A < B_f \wedge [\{f\}]$ and $B_f\wedge E_B < B_f \wedge[\{f\}]$ use that each degree of enumerability in $\mathcal{M}$ is meet-irreducible).

Until esplicitely stated to the contrary, from now on we regard $\mathcal{M}_w$ as a Brouwer algebra. Hence $\longrightarrow$ is understood to be $\longrightarrow_B$ and, accordingly, $\neg$ satisfies $\neg A_w=A_w \longrightarrow 1_w$. From now on we systematically drop the subscript $_w$ when writing out weak degrees of difficulty, provided that this is not a reason of confusion.

Lemma 1.4 In $\mathfrak{M}_w$, $\bigvee\{B_{f_i} : i\le m\} \longrightarrow \bigvee\{B_{g_j} : j\le n\} = \bigvee\{B_{g_j} : j\le n \ \&\ (\forall i\le m)[g_j \not\le_T f_i]\}$.

Proof. The proof is obvious, since each B_{f_i} is join-irreducible (see Corollary 1.4) and $(\forall f,g\in{}^{\omega}\omega)[B_f\le_w \le_w B_g \iff f\le_T g]$ (see Lemma 1.3).

Lemma 1.3, Lemma 1.4 and Corollary 1.5 suggest a duality of behavior between the degrees of solvability in $\mathfrak{M}_F$ and the degrees having the form B_f, for some $f\in{}^{\omega}\omega$, in $\mathfrak{M}_w$. In particular we have the following theorem.

Theorem 1.3 Each finite Brouwer algebra with exactly one atom and one coatom is embeddable in $\mathfrak{M}_w$.

Proof. The proof is similar to the proof of Theorem 1.2; only, we deal here with the dual situation. The role played in that proof by the degrees of solvability is now played by the degrees of the form B_f, for some function f.

As in Theorem 1.2, we want to show that, given any finite Brouwer algebra $\mathcal{L}$, $TB(\mathcal{L})$ is embeddable in $\mathfrak{M}_w$ as a Brouwer algebra (this is enough to prove the theorem since clearly the two-element Boolean algebra and the three-element chain are embeddable in $\mathfrak{M}_w$ as Brouwer algebras).

Let $\mathcal{L}$ be given and let $P=\{p\in \mathcal{L} : p \text{ is join-irreducible}\}$. Define a function $lv:\mathcal{L}\longrightarrow\omega$ by

$lv(p)=0$ if $p=0_{\mathcal{L}}$;

$lv(p)=n+1$ if $(\exists q)[lv(q)=n \ \& \ q<p \ \& \ \neg(\exists r)[q<r<p]]$.

Then define a partial order $\langle A, \circledS \rangle$ using P and the function lv formally in the same way as in Theorem 1.2 except for the following changes:

in Step n+1), Case a) if $p \in P$ is such that $lv(p)=n+1$, then by Lemma 1.1 let $\{q_0, \ldots q_v\}$ be a $\vee$-irredundant set such that $\bigvee\{q \in P: q<p\} = \bigvee\{q_h: h \leq v\}$. Define A_p formally in the same way as in Theorem 1.2, but let $\circledS^{p}_{n+1}$ be the smallest preordering relation on $A_p \cup \bigcup\{A_q: lv(q) \leq n\}$ satisfying:

1) $\circledS_n \subseteq \circledS^{p}_{n+1}$;

2) $(\forall h \leq v)(\forall i \leq n(q_h))[a^i_q \circledS^{p}_{n+1} a_p^{\Sigma\{n(q\ ):j \leq h-1\}+h+i}]$.

In Step n+1), Case b), one again considers a $\vee$-irredundant set $\{q_0, \ldots q_v\}$ such that $\bigvee\{q \in P: q<p\} = \bigvee\{q_h: h \leq v\}$ and lets $\circledS^{p}_{n+1}$ be the smallest preordering relation on $A_p \cup \bigcup\{A_q: lv(q) \leq n\}$ satisfying:

1) $\circledS_n \subseteq \circledS^{p}_{n+1}$;

2) $(\forall h \leq v)(\forall i \leq n(q_h))[a^i_q \circledS^{p}_{n+1} a^o_p]$.

Using methods quite similar to the ones used in the proof of Theorem 1.2 (only dualizing once in a while) one can show that $\langle A, \circledS \rangle$ is a lower semi-lattice.

Let $H: \langle A, \circledS \rangle \longrightarrow \mathfrak{D}_T$ be an embedding of partial orders which preserves infima (for the existence of such an H, the reference mentioned in tha proof of Theorem 1.2 is still valid); for every $p \in P$ and every $i \leq n(p)$, let $f^i_p \in H(a^i_p)$ and define $F: P \longrightarrow \mathcal{M}_w$ by $F(p) = \bigvee\{B_{f^i_p}: a^i_p \in A_p\}$.

Extend F to all the elements of $\mathcal{L}$ by letting $F(a)=\bigvee\{F(p_h);\ h\leq v\}$ if $a=\bigvee\{p_h:\ h\leq v\}$ and $\{p_0,\ldots,p_v\}$ is a $\vee$-irredundant set.

Finally, let $I\colon TB(\mathcal{L})\longrightarrow \mathcal{M}_w$ be defined by

$$I(a)=\begin{cases} 0 & \text{if } a=0_{TB(\mathcal{L})} \\ 1 & \text{if } a=1_{TB(\mathcal{L})} \\ F(a) & \text{if } a\in\mathcal{L}\ . \end{cases}$$

That I is the desired embedding follows formally in the same way as in Theorem 1.2, dualizing whatever we need to dualize, using Lemma 1.3, Corollary 1.5 and using Lemma 1.4 instead of Chapter I, Corollary 7.1.

Corollary 1.6 A finite Brouwer algebra $\mathcal{L}$ is embeddable in $\mathcal{M}_w$ if and only if $\mathcal{L}$ is an object of <u>TB(DL)</u>.

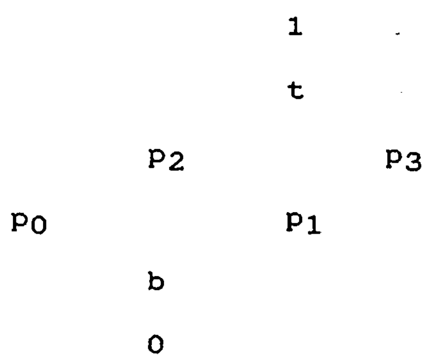
Proof. Obvious, since in $\mathcal{M}_w$ the least element is meet-irreducible and the greatest element is join-irreducible.

Remark 1.2 Let us be given a finite Brouwer algebra of the form $TB(\mathcal{L})$. Let $\langle A, \circledS\rangle$ be as in the proof of Theorem 1.3, where $A=\bigcup\{A_p:\ p\in P\}$ and $P=\{p\in\mathcal{L}:\ p$ is join-irreducible$\}$; for every $B,C\subseteq A$ define $B \boxed{\leq} C$ if $(\forall b\in B)(\exists c\in C)[b \circledS c]$. Then we can show that $(\forall p\in P)(\forall q\in P)[p\leq q \Longleftrightarrow A_p \boxed{\leq} A_q]$. The reader who has carefully read Remark 1.1 should have no difficulty to convince himself that the partial order $\mathcal{P}=\langle\{A_p:\ p\in P\}, \boxed{\leq}\rangle$

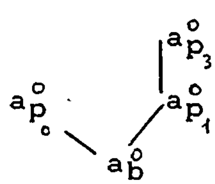
is isomorphic to the dual of the Kripke frame corresponding to $B(\mathcal{L})$ (simply use here that $(\forall p \in P)(\forall q \in P)[p \leq q \Longleftrightarrow F_q \subseteq F_p]$, where, for every $a \in B(\mathcal{L})$, F_a denotes the principal filter $\{b \in B(\mathcal{L}) : a \leq b\}$).

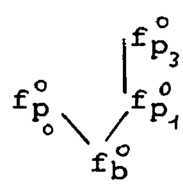
Example 1.2

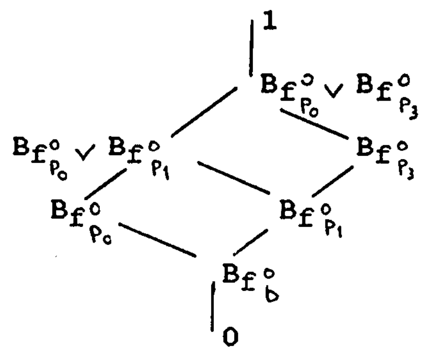
1)



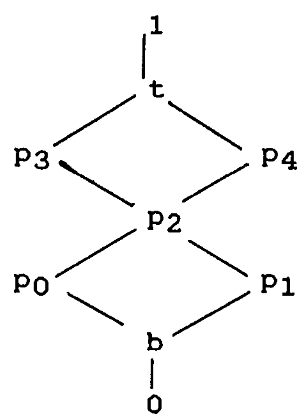
A:



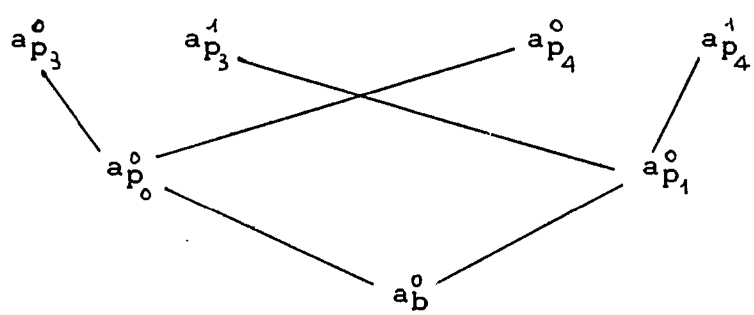


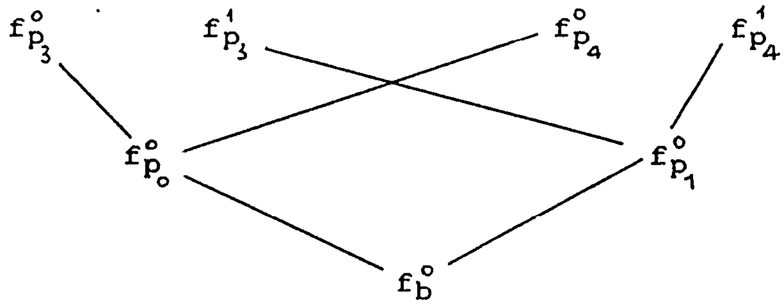

1
$Bf^{0}_{p_0} \vee Bf^{0}_{p_3}$
$Bf^{0}_{p_0} \vee Bf^{0}_{p_1}$
$Bf^{0}_{p_3}$
$Bf^{0}_{p_0}$
$Bf^{0}_{p_1}$
Bf^{0}_{b}
0

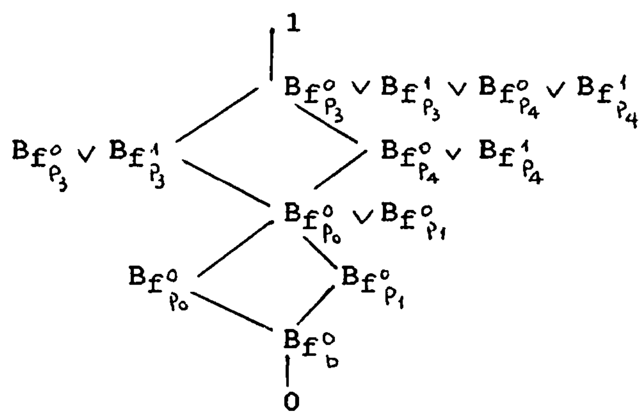
2)


1
t
p3
p4
p2
p0
p1
b
0

A:


$a^{0}_{p_3}$
$a^{1}_{p_3}$
$a^{0}_{p_4}$
$a^{1}_{p_4}$
$a^{0}_{p_0}$
$a^{0}_{p_1}$
a^{0}_{b}





For example:

$$(Bf^0_{p_3} \vee Bf^1_{p_3}) \wedge (Bf^0_{p_4} \vee Bf^1_{p_4}) = (Bf^0_{p_3} \wedge Bf^0_{p_4}) \vee (Bf^0_{p_3} \wedge Bf^1_{p_4}) \vee (Bf^1_{p_3} \wedge Bf^0_{p_4}) \vee (Bf^1_{p_3} \wedge Bf^1_{p_4}) = Bf^0_{p_0} \vee Bf^0_b \vee Bf^0_b \vee Bf^0_{p_1} = Bf^0_{p_0} \vee Bf^0_{p_1}.$$

The following corollary provides an answer to an open problem in MUČNICK (1963).

Corollary 1.7 Let $\mathcal{M}_w$ be regarded as a Brouwer algebra; then $\underline{L}_B(\mathcal{M}_w)=\underline{J}$.

Proof. In view of Theorem 1.1, we need only to show that $\underline{L}_B(\mathcal{M}_w) \subseteq \underline{J}$. Since the finite model property established for $\underline{J}$ in Fact 1.2 obviously still holds in its dual form (i.e. considering Brouwer algebras instead of Heyting algebras), Fact 1.1 and Theorem 1.3 imply that $\underline{L}_B(\mathcal{M}_w) \subseteq \underline{J}$.

We now turn our attention to the more complex problem of embedding Brouwer algebras in $\mathcal{M}$. The difficulty here arises from the fact that we do not have at hand a "nice" class of degrees of difficulty which are at the same time join-irreducible and meet-irreducible (as were in $\mathcal{M}_w$ the degrees of the form B_f, for $f \in {}^{\omega}\omega$). Indeed, by Fact 1.3 and Corollary 1.4 each B_f is both meet-irreducible and join-irreducible in $\mathcal{M}$. Nevertheless these degrees prove suitable for an ample variety of embeddings as we shall see below. We first remark the following Theorem, where the operation $\longrightarrow$ is the one that makes $\mathcal{M}$ a Brouwer algebra.

Theorem 1.4 Let $X = \bigvee_{j\in J}(\bigwedge_{i\in I_j} B_{f_i^j})$, $Y = \bigvee_{v\in V}(\bigwedge_{u\in U_v} B_{g_u^v})$, where J, V and each of I_j, U_v are finite. Then $X \longrightarrow Y = \bigvee\{(\bigwedge_{u\in U_v} B_{g_u^v}) : v\in V \ \&\ (\forall j\in J)[\bigwedge_{u\in U_v} B_{g_u^v} \nleq \bigwedge_{i\in I_j} B_{f_i^j}]\}$.

Proof. Let v be such that $(\forall j\in J)[\bigwedge_{u\in U_v}\mathcal{B}_g \nleq \bigwedge_{i\in I_j}\mathcal{B}_{f_i^j}]$ and let $\mathcal{X} = \bigvee_{j\in J}(\bigwedge_{i\in I_j}\mathcal{B}_{f_i^j})$. It is enough to show that $(\forall \mathcal{Z})[\bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \le \mathcal{X}\vee\mathcal{Z} \implies \bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \le \mathcal{Z}]$. Let $\mathcal{Z}$ be given such that $\bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \le \mathcal{X}\vee\mathcal{Z}$. If $(\forall j\in J)[\bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \nleq \bigwedge_{i\in I_j}\mathcal{B}_{f_i^j}]$ then $(\forall j\in J)(\exists i\in I_j)[\bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \nleq \mathcal{B}_{f_i^j}]$; since the degree of difficulty of each $\mathcal{B}_{f_i^j}$ is meet-irreducible (see Fact 1.3), this is actually equivalent to

(*) $(\forall j\in J)(\exists i\in I_j)(\forall u\in U_v)[\mathcal{B}_{g_u^v} \nleq \mathcal{B}_{f_i^j}]$.

Since, by distributivity, $\mathcal{X} \equiv \bigwedge_{f\in\prod_{j\in J} I_j}(\bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j})$, from (*) and the fact that the degree of each $\mathcal{B}_{g_u^v}$ is join-irreducible (see Corollary 4) we conclude that $(\exists f\in\prod_{j\in J} I_j)(\forall u\in U_v)[\mathcal{B}_{g_u^v} \nleq \bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j}]$.

Nevertheless, by our assumptions, it must be $\bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \le \bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j} \vee \mathcal{Z}$.

By the definition of the operation $\wedge$ on mass problems, we have that $\bigwedge_{u\in U_v}\mathcal{B}_{g_u^v} \equiv \bigcup_{u\in U_v} u*\mathcal{B}_{g_u^v}$; therefore, $\bigcup_{u\in U_v} u*\mathcal{B}_{g_u^v} \le (\bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j})\vee\mathcal{Z}$ via some recursive operator Ψ. Let $(\bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j})^{(u)} \subseteq \bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j}$ and $\mathcal{Z}^{(u)} \subseteq \mathcal{Z}$ be such that $(\bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j})^{(u)} \vee \mathcal{Z}^{(u)} = \{h\in(\bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j})\vee\mathcal{Z} : \Psi(h)(0)=u\}$. Clearly, $\mathcal{B}_{g_u^v} < (\bigvee_{j\in J}\mathcal{B}_{f_{f(j)}^j})^{(u)} \vee \mathcal{Z}^{(u)}$ via, say, a recursive operator Ψ' such that $\Psi'(h) = \lambda x.\Psi(h)(x+1)$ (it is easy to convince oneself that such a recursive operator exists).

Claim 1 $\mathcal{Z} \equiv \bigwedge_{u\in U_v}\mathcal{Z}^{(u)}$.

Proof of Claim 1. Certainly $\mathcal{Z} \leq \bigwedge_{u\in U_v} \mathcal{Z}^{(u)}$, since $\bigwedge_{u\in U_v} \mathcal{Z}^{(u)} \equiv \bigcup_{u\in U_v} u * \mathcal{Z}^{(u)}$ and each $\mathcal{Z}^{(u)}$ is a subset of $\mathcal{Z}$.

To show the converse, suppose that the above mentioned recursive operator Ψ comes from the enumeration operator defined via the r.e. set W_a. Let W be the r.e. set given by

$W=\{<<x,y>,z>:$ $[x=0$ & $\tau^{-1}(D_z)$ is a finite initial segment & $(\exists s)(\exists w)[<<0,y>,w>\in W_a^s$ & $\{<b,c>: <2b,c>\in D_w\}$ is single-valued & $\{<b,c>: <2b+1,c>\in D_w\} \subseteq D_z$ & $(\forall d\in W_a^s)(\forall e\in D_{(d)_1})[(e)_0<lh(\tau^{-1}(D_z))]$ & $(\forall t<s)$ $(\forall d\in W_a^t)$ $[\{<b,c>: <2b,c>\in D_{(d)_1}\}$ single-valued $\Longrightarrow \{<b,c>: <2b+1,c> \in D_{(d)_1}\} \not\subseteq D_z]]]$ or $[x\neq 0$ & $D_z=\{<x-1,y>\}]\}$

and let m be an r.e. index for W; with a little work, we can convince ourselves that for every total function h, $h\in dom(\Omega_m)$ (warning: the fact that h is total is essential); so, by the Fundamental Operator Theorem, $(\forall f)$ $[\Omega_m(f)=\Psi_m(f)]$.

We want to show that $\bigwedge_{u\in U_v} \mathcal{Z}^{(u)} \leq \mathcal{Z}$ via Ψ_m. Let $l\in \mathcal{Z}$; clearly $(\forall x)[x\neq 0 \Longrightarrow \Psi_m(l)(x)\downarrow$ & $\Psi_m(l)(x)=l(x-1)]$ (this is true of every total function l: we are not using here the assumption that $l\in \mathcal{Z}$); therefore we have only to check that $0\in dom(\Psi_m(l))$ and $\Psi_m(l)(0)\in U_v$. To this end, let $s\in \omega$ be the first stage in the enumeration $\{W_a^s\}_{s\in\omega}$ of W_a such that $(\exists y)(\exists w)[<<0,y>,w>\in W_a^s$ & $\{<b,c>: <2b,c>\in D_w\}$ is single-valued & $\{<b,c>: <2b,c>\in D_w\} \subseteq \tau(l)\}]$; let $\tilde{l}$ be a finite initial segment such that $\tilde{l} \subseteq l$ and $(\forall d\in W_a^s)$ $(\forall e\in D_{(d)_1})[(e)_0<lh(\tilde{l})]$; let y,w be as

before and let z be the canonical index of $\tau(\tilde{1})$: then, clearly, $\langle\langle 0,y\rangle,z\rangle \in W$ & $D_z \subseteq \tau(1)$. Moreover, since the mass problem $\bigvee_{j\in J} \mathcal{B}_{f^j_{\xi(j)}}$ is dense in the Baire topology, there certainly exists $f \in \bigvee_{j\in J} \mathcal{B}_{f^j_{\xi(j)}}$ such that $\{\langle b,c\rangle\ \langle 2b,c\rangle \in D_W\} \subseteq \tau(f)$. But, then, $\Psi_m(1)(0) = =\Omega_m(1)(0)=y= \Psi(f \vee 1)(0)$ and, since $\Psi(f \vee 1)(0) \in U_v$, the claim is proved.

Claim 2 $(\forall u \in U_v)[\ \mathcal{B}_{g^\vee_u} \leq \mathcal{Z}^{(u)}]$.

Proof of Claim 2. Suppose that there exists $u \in U_v$ such that $\mathcal{B}_{g^\vee_u} \not\leq \mathcal{Z}^{(u)}$; thus $\mathcal{Z}^{(u)} \not\subseteq \mathcal{B}_{g^\vee_u}$ and therefore there exists a function $l \in \mathcal{Z}^{(u)}$ such that $l \leq_T g^\vee_u$.

Since $(\forall j \in J)[g^\vee_u \not\leq_T f^j_{\xi(j)}]$, the set $\{\bigvee_{j\in J} f_j: (\forall j \in J)[f_j \leq_T f^j_{\xi(j)}]$ & $\bigvee_{j\in J} f_j \leq_T g^\vee_u\}$ is dense in the Baire topology; hence $\{\bigvee_{j\in J} f_j: (\forall j \in J)[f_j \not\leq_T f^j_{\xi(j)}]$ & $\bigvee_{j\in J} f_j \leq_T g^\vee_u\} \cap \{f: \Psi(f \vee 1)(0)\downarrow$ & $\Psi(f \vee 1)(0)=u\} \neq \emptyset$, as the set $\{f: \Psi(f \vee 1)(0)\downarrow$ & $\Psi(f \vee 1)(0)=u\}$ is manifestly open in the Baire topology.

Furthermore, $\{\bigvee_{j\in J} f_j: (\forall j \in J)[f_j \not\leq_T f^j_{\xi(j)}]$ & $\bigvee_{j\in J} f_j \leq_T g^\vee_u\} \subseteq \mathcal{B}_{f^j_{\xi(j)}}$; thus we can conclude that $(\exists f)[f \in \bigvee_{j\in J} \mathcal{B}_{f^j_{\xi(j)}}$ & $f \leq_T g^\vee_u$ & $\Psi(f \vee 1)(0)=u]$.

Therefore for such a function f, $\Psi(f \vee 1) \in u * \mathcal{B}_{g^\vee_u}$, a contradiction since $\Psi(f \vee 1) \leq_T f \vee 1$ and thus $\Psi(f \vee 1) \leq_T g^\vee_u$.

The contradiction has arisen from assuming that there exists $u \in U_v$ such that $\mathcal{B}_{g^\vee_u} \not\leq \mathcal{Z}^{(u)}$. This concludes the proof of Claim 2.

By Claim 2 we have $\bigwedge_{u\in U_v} \mathcal{B}_{g^\vee_u} \leq \bigwedge_{u\in U_v} \mathcal{Z}^{(u)}$; since, by Claim 1, $\bigwedge_{u\in U_v} \mathcal{Z}^{(u)} \equiv \mathcal{Z}$, this implies $\bigwedge_{u\in U_v} \mathcal{B}_{g^\vee_u} \leq \mathcal{Z}$, as desired.

Let $\mathcal{M}'$ be the sublattice of $\mathcal{M}$ generated by the set $\{B_f$: f is a total function$\}$.

Corollary 1.8 $\mathcal{M}'$ is a sub-Brouwer algebra of $\mathcal{M}$.

Proof. Immediate from Theorem 1.4.

Notice that each element of $\mathcal{M}'$ different from 0, 1 can be written in the form $\bigvee_{j\in J} \bigwedge_{i\in I_j} B_{f_i^j}$ for some finite sets J, I_i $(i\in J)$. Such an expression will be called <u>reduced</u> if

$(\forall j,j'\in J)(\forall i\in I_j)(\forall i'\in I_{j'})[j\neq j' \Longrightarrow B_{f_i^j} \mid B_{f_i^{j'}}]$ and

$(\forall j\in J)(\forall i,i'\in I_j)[i\neq i' \Longrightarrow B_{f_i^j} \mid B_{f_{i'}^j}]$.

Clearly, each element of $\mathcal{M}'$ different from 0, 1 can be written in a reduced expression. Henceforth, when we write an expression of the form $\bigvee_{j\in J} \bigwedge_{i\in I_j} B_{f_i^j}$ we shall suppose (unless specified to the contrary) that this expression is reduced.

If $\mathcal{L}$ is a finite Brouwer algebra and H: $TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ is an embedding, then let

$D_H(\mathcal{L})=\{B\in \mathcal{M}'$: $(\exists$ reduced expression $\bigvee_{j\in J} \bigwedge_{i\in I_j} B_{f_i^j})$

$[\bigvee_{j\in J} \bigwedge_{i\in I_j} B_{f_i^j} \in H(\mathcal{L})$ & $(\exists j\in J)(\exists i\in I_j)[B=B_{f_i^j}]]\}$

and let

$\triangle_H(\mathcal{L})=\{a_T\in \mathfrak{D}_T$: $(\exists f\in a_T)[B_f\in D_H(\mathcal{L})]\}$.

Moreover, we will denote by $\mathcal{B}_H(\mathcal{L})$ the sublattice of $\mathcal{M}'$ generated by the set $\{B_f: (\exists a_T \in \Delta_H(\mathcal{L}))[f \in a_T]\}$. We have the following

Lemma 1.5 Let $\mathcal{L}$ be a finite Brouwer algebra and let $H: TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ be an embedding. Then for every $n \in \omega$ there exist n embeddings $H_0: TB(\mathcal{L}) \longrightarrow \mathcal{M}', \ldots, H_{n-1}: TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ such that $(\forall r,s<n)(\forall B \in \mathcal{B}_H(\mathcal{L}))(\forall C \in \mathcal{B}_H(\mathcal{L}))\ [r \neq s \ \&\ B \notin \{0,1\} \ \&\ C \notin \{0,1\} \Rightarrow B \mid C]$.

Proof. Let $H: TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ be an embedding, where $\mathcal{L}$ is a finite Brouwer algebra.

Notice that $\Delta_H(\mathcal{L})$ is a finite partial order, in fact a finite sub-partial order of $\mathcal{D}_T$.

Let now $n \in \omega$ be given and let us denote by $\bigsqcup_{r<n} \Delta_H(\mathcal{L})$ the n-folded coproduct of $\Delta_H(\mathcal{L})$. Since $\bigsqcup_{r<n} \Delta_H(\mathcal{L})$ is a finite partial order, there exists an embedding of partial orders $F: \bigsqcup_{r<n} \Delta_H(\mathcal{L}) \xrightarrow{I_s} \mathcal{D}_T$. For every $s<n$, let F_s be the composite map $\Delta_H(\mathcal{L}) \longrightarrow \bigsqcup_{r<n} \Delta_H(\mathcal{L}) \xrightarrow{F} \mathcal{D}_T$ where I_s is the s-th injection of the coproduct; moreover, if $a_T \in \Delta_H(\mathcal{L})$ and $f \in a_T$, let us also agree that $F_s(f)$ denotes a representative of $F_s(a_T)$.

For every $r<n$, define $H_r: TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ by

$$H_r(a) = \begin{cases} \bigvee_{j \in J} \bigwedge_{i \in I_j} B_{F_r(f_i^j)} & \text{if } H(a) = \bigvee_{j \in J} \bigwedge_{i \in I_j} B_{f_i^j} \text{ and} \\ & \bigvee_{j \in J} \bigwedge_{i \in I_j} B_{f_i^j} \text{ is reduced} \\ 1 & \text{if } a = 1_{TB(\mathcal{L})} \\ 0 & \text{if } a = 0_{TB(\mathcal{L})} \end{cases}$$

We claim that $H_0,\dots,H_{n-1}$ are the desired embeddings.

Let us show first the incomparability property: let $r,s<n$ be such that $r\neq s$; let $B\in\mathcal{B}_{H_r}(\mathcal{L})$, $C\in\mathcal{B}_{H_s}(\mathcal{L})$ be such that neither $B\in\{0,1\}$ nor $C\in\{0,1\})$ and suppose, for the sake of contradiction, that $B\leq C$. Let $B=\bigvee_{j\in J}\bigwedge_{i\in I_j}B_{f_i^j}$ and $C=\bigvee_{v\in V}\bigwedge_{u\in U_v}B_{g_u^v}$. Since $\bigvee_{j\in J}\bigwedge_{i\in I_j}B_{f_i^j}\leq\bigvee_{v\in V}\bigwedge_{u\in U_v}B_{g_u^v}$, by Theorem 1.4 we have that $(\forall j\in J)(\exists v\in V)[\bigwedge_{i\in I_j}B_{f_i^j}\leq\bigwedge_{u\in U_v}B_{g_u^v}]$; hence (since each $B_{g_u^v}$ is meet-irreducible), $(\forall j\in J)(\exists v\in V)(\forall u\in U_v)(\exists i\in I_j)[B_{f_i^j}\leq B_{g_u^v}]$ and (by Lemma 1.2) $(\forall j\in J)(\exists v\in V)(\forall u\in U_v)(\exists i\in I_j)[f_i^j\leq_T g_u^v]$. Since $[f_i^j]_T\in F_r(\triangle_H(\mathcal{L}))$ and $[g_u]_T\in F_s(\triangle_H(\mathcal{L}))$, this implies that there are elements $a\in I_r(\triangle_H(\mathcal{L}))$, $b\in I_s(\triangle_H(\mathcal{L}))$ such that $a\leq b$, a contradiction. In a similar way we rule out the possibility $C\leq B$. Therefore $B\,|\,C$ as desired.

To show that, for every $r<n$, H_r is an order-theoretic embedding, i.e. $(\forall a,b\in TB(\mathcal{L}))[a\leq b\iff H_r(a)\leq H_r(b)]$, it suffices to show that $(\forall a,b\in TB(\mathcal{L}))[H(a)\leq H(b)\iff H_r(a)\leq H_r(b)]$. Thus, let $r<n$ be given and let $a,b\in\mathcal{L}$ (the case $\{a,b\}\cap\{0_{TB(\mathcal{L})},1_{TB(\mathcal{L})}\}\neq\emptyset$ is trivial). Suppose $H(a)\leq H(b)$, and let $H(a)=\bigvee_{j\in J}\bigwedge_{i\in I_j}B_{f_i^j}$, $H(b)=\bigvee_{v\in V}\bigwedge_{u\in U_v}B_{g_u^v}$.

As before, we get $(\forall j\in J)(\exists v\in V)(\forall u\in U_v)(\exists i\in I_j)[B_{f_i^j}\leq B_{g_u^v}]$; by our assumptions on F_r and by Lemma1.2, this implies $(\forall j\in J)(\exists v\in V)(\forall u\in U_v)(\exists i\in I_j)[B_{F_r(f_i^j)}\leq B_{F_r(g_u^v)}]$; hence $\bigvee_{j\in J}\bigwedge_{i\in I_j}B_{F_r(f_i^j)}\leq\bigvee_{v\in V}\bigwedge_{u\in U_v}B_{F_r(g_u^v)}$ i.e. $H_r(a)\leq H_r(b)$.

That $H_r(a)\leq H_r(b)\implies H(a)\leq H(b)$ holds, can be seen in a similar way.

Finally, the proofs that for every $r<n$ and for every $a,b \in TB(\mathcal{L})$, $H_r(a \vee b)=H_r(a) \vee H_r(b)$, $H_r(a \wedge b)= H_r(a) \wedge H_r(b)$, $H_r(a \longrightarrow b)=H_r(a) \longrightarrow H_r(b)$ and $H_r(\neg a) = \neg H_r(a)$ follow along the same pattern and are omitted.

Theorem 1.5 Let $\mathcal{L}$ be a finite Brouwer algebra. If $TB(\mathcal{L})$ is embeddable in $\mathcal{M}'$ then for every $n \in \omega$ $TB((B(\mathcal{L}))^n)$ is embeddable in $\mathcal{M}'$.

Proof. Suppose that $H: TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ is an embedding, where $\mathcal{L}$ is a finite Brouwer algebra. If this is the case then we can also find an embedding H': $TB(B(\mathcal{L})) \longrightarrow \mathcal{M}'$. To see this, notice first that we can suppose that $0' \notin H(TB(\mathcal{L}))$: this is certainly the case if $o_T \notin \triangle_H(\mathcal{L})$; on the other hand, if $o_T \in \triangle_H(\mathcal{L})$, then let $\triangle' \subseteq \mathcal{D}_T$ be order-isomorphic to $\triangle_H(\mathcal{L})$ and such that $o_T \notin \triangle'$; if we define an embedding $\bar{H}$: $TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ by using $\triangle'$ as in the proof of Lemma 1.5, then it easily follows that $0' \notin H(TB(\mathcal{L}))$. If $0' \notin H(TB(\mathcal{L}))$ then one can define H': $TB(B(\mathcal{L})) \longrightarrow \mathcal{M}'$ by

$$H'(a)=\begin{cases} H(a) & \text{if } a \notin \{0_{TB(B(\mathcal{L}))}, b_{TB(B(\mathcal{L}))}\} \\ 0 & \text{if } a=0_{TB(B(\mathcal{L}))} \\ 0' & \text{if } a=b_{TB(B(\mathcal{L}))} \end{cases}$$

(as usual, we view $TB(\mathcal{L})$ as a subset of $TB(B(\mathcal{L}))$).
Let now $n \in \omega$ and, by Lemma 1.5, let H_0: $TB(B(\mathcal{L})) \longrightarrow \mathcal{M}', \ldots, H_{n-1}: TB(B(\mathcal{L})) \longrightarrow \mathcal{M}'$ be embeddings such that

(*) $(\forall r,s<n)(\forall B\in \mathcal{B}_{H_r}(\mathcal{L}))(\forall C\in \mathcal{B}_{H_s}(\mathcal{L}))[r\neq s$ & $B\notin\{0,1\}$ & $C\notin\{0,1\} \Longrightarrow B \mid C]$.

Let $\{p_0,\ldots p_{k-1}\}$ be the join-irreducible elements of $B(\mathcal{L})$ (assume $p_0=0_{B(\mathcal{L})}$).

For every $i<n$ and for every $j<k$, let $H_i(p_j)=A^i_j$.

The join-irreducible elements of $(B(\mathcal{L}))^n$ are then $\{(p_{\alpha(0)},\ldots,p_{\alpha(n-1)}):\ \alpha:n \longrightarrow k$ & $(\forall i\in n$ with the exception of at most one $i)[\alpha(i)=p_0]\}$.

Define $I: TB((B(\mathcal{L}))^n) \longrightarrow \mathcal{M}'$ by

$$I(a)=\begin{cases} 0 & \text{if } a=0_{TB((B(\mathcal{L}))^n)} \\ 1 & \text{if } a=1_{TB((B(\mathcal{L}))^n)} \\ \bigvee_{\alpha\in A}(A^0_{\alpha(0)}\wedge\ldots\wedge A^{n-1}_{\alpha(n-1)}) & \text{if } a=\bigvee_{\alpha\in A}(p_{\alpha(0)},\ldots,p_{\alpha(n-1)}) \text{ and } \\ & \{(p_{\alpha(0)},\ldots p_{\alpha(n-1)}):\ \alpha\in A\} \text{ is a } \\ & \vee\text{-irredundant set.} \end{cases}$$

(For the third clause in the definition of I, we refer of course to Lemma 1.1).

We now check that I is the desired embedding.

- I is one-one: we shall actually show that $(\forall a,b\in TB((B(\mathcal{L}))^n))[a\leq b \Longleftrightarrow I(a)\leq I(b)]$.

Let $a,b\in(B(\mathcal{L}))^n$ (the case $\{a,b\}\cap\{0_{TB((B(\mathcal{L}))^n)}, 1_{TB((B(\mathcal{L}))^n)}\}\neq\emptyset$ is trivial) and let $a=\bigvee_{\alpha\in A}(p_{\alpha(0)},\ldots,p_{\alpha(n-1)})$, $b=\bigvee_{\beta\in B}(p_{\beta(0)},\ldots,p_{\beta(n-1)})$ where $\{(p_{\alpha(0)},\ldots p_{\alpha(n-1)}):\ \alpha\in A\}$ and $\{(p_{\beta(0)},\ldots p_{\beta(n-1)}):\ \beta\in B\}$ are $\vee$-irredundant. Then

$a\leq b \Longleftrightarrow \bigvee_{\alpha\in A}(p_{\alpha(0)},\ldots,p_{\alpha(n-1)}) \leq \bigvee_{\beta\in B}(p_{\beta(0)},\ldots,p_{\beta(n-1)}) \Longleftrightarrow$

$(\forall\alpha\in A)(\exists\beta\in B)[(p_{\alpha(0)},\ldots,p_{\alpha(n-1)})\le(p_{\beta(0)},\ldots,p_{\beta(n-1)})]$ $\Longleftrightarrow(\forall\alpha\in A)(\exists\beta\in B)[p_{\alpha(0)}\le p_{\beta(0)}\ \&\ldots\&\ p_{\alpha(n-1)}\le p_{\beta(n-1)}]\Longleftrightarrow$ $(\forall\alpha\in A)(\exists\beta\in B)[A^{0}_{\alpha(0)}\le A^{0}_{\beta(0)}\ \&\ldots\&\ A^{n-1}_{\alpha(n-1)}\le A^{n-1}_{\beta(n-1)}]$ (we use here that each H_i is an embedding) $\Longleftrightarrow$ $(\forall\alpha\in A)(\exists\beta\in B)$ $[\bigwedge_{r<n}A^{r}_{\alpha(r)}\le\bigwedge_{r<n}A^{r}_{\beta(r)}]$.

For this last equivalence we use our assumptions on the embeddings $H_0,\ldots H_{n-1}$ as well as Theorem 1.4: indeed, suppose that $\bigwedge_{r<n}A^{r}_{\alpha(r)}\le A^{t}_{\beta(t)}$, for some $t<n$. Let $A^{t}_{\beta(t)}=\bigvee_{j\in J^{t}_{\beta}}(\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i})$ and for every $r<n$ let $A^{r}_{\alpha(r)}=\bigvee_{j\in J^{r}_{\alpha}}(\bigwedge_{i\in I^{r}_{\alpha,j}}B(f^{r}_{\alpha})^{j}_{i})$; since $\bigwedge_{r<n}A^{r}_{\alpha(r)}\le A^{t}_{\beta(t)}$, using distributivity we have that $\bigvee_{\xi\in\prod_{s<n}J^{s}_{\alpha}}(\bigwedge_{r<n}\bigwedge_{i\in I^{r}_{\alpha,\xi(r)}}B(f^{r}_{\alpha})^{\xi(r)}_{i})\le$ $\bigvee_{j\in J^{t}_{\beta}}(\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i})$; then, by Theorem 1.4, $(\forall\xi\in\prod_{s<n}J^{s}_{\alpha})$ $(\exists j\in J^{t}_{\beta})[\bigwedge_{r<n}\bigwedge_{i\in I^{r}_{\alpha,\xi(r)}}B(f^{r}_{\alpha})^{\xi(r)}_{i}\le\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i}]$.

Let now $j'\in J^{t}_{\alpha}$ and let $\xi\in\prod_{s<n}J^{s}_{\alpha}$ be such that $\xi(t)=j'$; then for such a ξ there exists $j\in J^{t}_{\beta}$ such that $\bigwedge_{r<n}\bigwedge_{i\in I^{r}_{\alpha,j'}}B(f^{r}_{\alpha})^{j'}_{i}\le\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i}$ and, since the embeddings $H_0,\ldots,H_{n-1}$ satisfy $(*)$ and each $B(f^{t}_{\beta})^{j}_{i}$ is meet-irreducible, we must have $\bigwedge_{i\in I^{t}_{\alpha,j'}}B(f^{t}_{\alpha})^{j'}_{i}\le\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i}$. Therefore $(\forall j'\in J^{t}_{\alpha})(\exists j\in J^{t}_{\beta})[\bigwedge_{i\in I^{t}_{\alpha,j'}}B(f^{t}_{\alpha})^{j'}_{i}\le\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i}]$ and, thus, $\bigvee_{j\in J^{t}_{\alpha}}\bigwedge_{i\in I^{t}_{\alpha,j}}B(f^{t}_{\alpha})^{j}_{i}\le\bigvee_{j\in J^{t}_{\beta}}\bigwedge_{i\in I^{t}_{\beta,j}}B(f^{t}_{\beta})^{j}_{i}$ i.e $A^{t}_{\beta(t)}\le A^{t}_{\alpha(t)}$.

We conclude that $a\le b\Longleftrightarrow I(a)\le I(b)$, as desired.

- $I(a\vee b)=I(a)\vee I(b)$: this is immediate from the definition of I.

- $I(a\wedge b)=I(a)\wedge I(b)$. We may again assume that $\{a,b\}\cap\{0_{TB((B(\mathcal{L}))}),1_{TB((B(\mathcal{L}))})\}=\emptyset$; let a

$= \bigvee_{\alpha\in A}(p_{\alpha(0)},\ldots,p_{\alpha(n-1)})$, $b= \bigvee_{\beta\in B}(p_{\beta(0)},\ldots,p_{\beta(n-1)})$ where $\{(p_{\alpha(0)},\ldots,p_{\alpha(n-1)}): \alpha\in A\}$, $\{(p_{\beta(0)},\ldots,p_{\beta(n-1)}): \beta\in B\}$ are $\vee$-irredundant. Then

$$a \wedge b= \bigvee_{\alpha\in A}(p_{\alpha(0)},\ldots,p_{\alpha(n-1)}) \wedge \bigvee_{\beta\in B}(p_{\beta(0)},\ldots,p_{\beta(n-1)})=$$
$$= \bigvee_{\alpha\in A}\bigvee_{\beta\in B}(p_{\alpha(0)}\wedge p_{\beta(0)},\ldots,p_{\alpha(n-1)}\wedge p_{\beta(n-1)}).$$

Let $\alpha\in A$ and $\beta\in B$. We distinguish two cases:

Case 1) There exist $i<n$ and $s,t<k$ such that for every $j<n$, the j-th projection

$$(p_{\alpha(0)}\wedge p_{\beta(0)},\ldots,p_{\alpha(n-1)}\wedge p_{\beta(n-1)})_j= \begin{cases} p_0 & \text{if } j<n \ \&\ j\neq i \\ p_s \quad p_t & \text{if } j<n\&j=i \end{cases}$$

Case 2) $(p_{\alpha(0)} \wedge p_{\beta(0)},\ldots,p_{\alpha(n-1)}\wedge p_{\beta(n-1)})=(p_0,\ldots,p_0)$.

Only Case 1) deserves attention. By Lemma 1.1, let $p_s\wedge p_t== \bigvee\{p_r: r\in R_{\alpha,\beta}\}$, where $\{p_r: r\in R_{\alpha,\beta}\}$ is $\vee$-irredundant. Then

$$(p_{\alpha(0)} \wedge p_{\beta(0)},\ldots,p_{\alpha(i-1)} \wedge p_{\beta(i-1)},p_{\alpha(i)}\wedge p_{\beta(i)},p_{\alpha(i+1)} \wedge p_{\beta(i+1)},\ldots,p_{\alpha(n-1)} \wedge p_{\beta(n-1)})=(p_0,\ldots,p_0,\bigvee_{r\in R_{\alpha,\beta}}p_r, p_0,\ldots, p_0) = \bigvee_{r\in R_{\alpha,\beta}}(p_0,\ldots,p_0,p_r,p_0,\ldots,p_0).$$

Therefore $I(a \wedge b)=I(\bigvee_{\alpha\in A}\bigvee_{\beta\in B}\bigvee_{r\in R_{\alpha,\beta}}(p_0,\ldots,p_0,p_r,p_0,\ldots,p_0))=$
$$\bigvee_{\alpha\in A}\bigvee_{\beta\in B}\bigvee_{r\in R_{\alpha,\beta}}(A_0^0 \wedge\ldots\wedge A_0^{i-1}\wedge A_r^i \wedge A_0^{i+1}\wedge\ldots\wedge A_0^{n-1}).$$

On the other hand, $I(a)\wedge I(b)=I(\bigvee_{\alpha\in A}(p_{\alpha(0)},\ldots,p_{\alpha(n-1)}))$ $\wedge I(\bigvee_{\beta\in B}(p_{\beta(0)},\ldots,p_{\beta(n-1)}))= \bigvee_{\alpha\in A}(A^0_{\alpha(0)} \wedge \ldots \wedge A^{n-1}_{\alpha(n-1)})$ $\wedge\bigvee_{\beta\in B}(A^0_{\beta(0)} \wedge \ldots \wedge A^{n-1}_{\beta(n-1)})= \bigvee_{\alpha\in A}\bigvee_{\beta\in B}((A^0_{\alpha(0)}\wedge A^0_{\beta(0)})\wedge\ldots$ $\wedge(A^{n-1}_{\alpha(n-1)} \wedge A^{n-1}_{\beta(n-1)}))= \bigvee_{\alpha\in A}\bigvee_{\beta\in B}(A_0^0 \wedge\ldots\wedge A_0^{i-1} \wedge \bigvee_{r\in R_{\alpha,\beta}} A_r^i$ $\wedge A_0^{i+1}\wedge\ldots\ A_0^{n-1}).$

This last equality is justified by the fact that H_i: $TB((B(\mathcal{L}))^n) \longrightarrow \mathfrak{M}'$ is an embedding.

Finally we get

$I(a) \wedge I(b) = \bigvee_{\alpha\in A} \bigvee_{\beta\in B} \bigvee_{r\in R_{\alpha,\beta}} (A_0^0 \wedge \ldots \wedge A_0^{i-1} \wedge A_r^i \wedge A_0^{i+1} \ldots \wedge A_0^{n-1})$

as desired.

- $I(a \longrightarrow b) = I(a) \longrightarrow I(b)$. Once again the case $\{a,b\} \cap \{0_{TB((B(\mathcal{L}))^n)}, 1_{TB((B(\mathcal{L}))^n)}\} = \emptyset$ is trivial; so we may assume $a,b \in (B(\mathcal{L}))^n$.

Let $a = \bigvee_{\alpha\in A} (p_{\alpha(0)}, \ldots, p_{\alpha(n-1)})$, $b = \bigvee_{\beta\in B} (p_{\beta(0)}, \ldots, p_{\beta(n-1)})$ where $\{(p_{\alpha(0)}, \ldots, p_{\alpha(n-1)}) : \alpha\in A\}$, $\{(p_{\beta(0)}, \ldots, p_{\beta(n-1)}) : \beta\in B\}$ are $\vee$-irredundant; for every $r<n$, let also $A^r_{\alpha(r)} = \bigvee_{j\in J^r_\alpha} (\bigwedge_{i\in I^r_{\alpha,j}} B(f^r_\alpha)^j_i)$ and $A^r_{\beta(r)} = \bigvee_{j\in J^r_\beta} (\bigwedge_{i\in I^r_{\beta,r}} B(f^r_\beta)^j_i)$.

Since $a \longrightarrow b = \bigvee\{(p_{\beta(0)}, \ldots, p_{\beta(n-1)}) : \beta\in B \ \& \ (\forall\alpha\in A) [(p_{\beta(0)}, \ldots, p_{\beta(n-1)}) \nleq (p_{\alpha(0)}, \ldots, p_{\alpha(n-1)})]\}$, we have

$I(a \longrightarrow b) = \bigvee\{\bigwedge_{r<n} A^r_{\beta(r)} : \beta\in B \ \& \ (\forall\alpha\in A)[\bigwedge_{r<n} A^r_{\beta(r)} \nleq \bigwedge_{r<n} A^r_{\alpha(r)}]\}$

Then $I(a \longrightarrow b) = \bigvee\{\bigwedge_{r<n} (\bigvee_{j\in J^r_\beta} (\bigwedge_{i\in I^r_{\beta,j}} B(f^r_\beta)^j_i)) : \beta\in B \ \& \ (\forall\alpha\in A)$ $[\bigwedge_{r<n} (\bigvee_{j\in J^r_\beta} (\bigwedge_{i\in I^r_{\beta,j}} B(f^r_\beta)^j_i)) \nleq \bigwedge_{r<n} (\bigvee_{j\in J^r_\alpha} (\bigwedge_{i\in I^r_{\alpha,j}} B(f^r_\alpha)^j_i))]\} =$

$= \bigvee\{\bigvee_{\zeta\in\prod_{s<n} J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i) : \beta\in B \ \& \ (\forall\alpha\in A)$

$[\bigvee_{\zeta\in\prod_{s<n} J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i) \nleq \bigvee_{\xi\in\prod_{s<n} J^s_\alpha} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i)]\}$.

Notice that, by Theorem 1.4,

$(\forall\alpha\in A)[\bigvee_{\zeta\in\prod_{s<n} J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i) \nleq \bigvee_{\xi\in\prod_{s<n} J^s_\alpha} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i)]$

$\Longleftrightarrow (\forall\alpha\in A)(\forall\xi\in\prod_{s<n} J^s_\alpha)[\bigvee_{\zeta\in\prod_{s<n} J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i)$

$\nleq \bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i]$.

Thus

$I(a \longrightarrow b) = \bigvee\{\bigvee_{\zeta\in\prod_{s<n} J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i) : \beta\in B \ \& \ (\forall\alpha\in A)(\forall\xi\in \prod_{s<n} J^s_\alpha)[\bigvee_{\zeta\in\prod_{s<n} J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i) \nleq \bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i]\}$.

On the other hand,

$$I(a) \longrightarrow I(b)= \bigvee_{\alpha\in A} (\bigwedge_{r<n} (\bigvee_{j\in J^r_\alpha} (\bigwedge_{i\in I^r_{\alpha,j}} B(f^r_\alpha)^j_i)))$$
$$\longrightarrow \bigvee_{\beta\in B} (\bigwedge_{r<n} (\bigvee_{j\in J^r_\beta} (\bigwedge_{i\in I^r_{\beta,j}} B(f^r_\beta)^j_i))) = \bigvee_{\alpha\in A} \bigvee_{\xi\in \prod_{s<n} J^s_\alpha} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i)$$
$$\longrightarrow \bigvee_{\beta\in B} \bigvee_{\zeta\in\prod J^s_\beta} (\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i) = \text{(by Theorem 1.4)} =$$
$$\bigvee\{ \bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i : \beta\in B \ \&\ \zeta\in \prod_{s<n} J^s_\beta \ \&\ (\forall\alpha\in A)(\forall \xi\in \prod_{s<n} J^s_\alpha)$$
$$[\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i \not\leq \bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i]\}.$$

In order to show that $I(a\longrightarrow b)=I(a)\longrightarrow I(b)$ is then enough to show that, given $\beta\in B$, if there exists $\zeta\in \prod_{s<n} J^s_\beta$ such that $(\forall\alpha\in A)(\forall\xi\in \prod_{s<n} J^s_\alpha)[\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta(r)}} B(f^r_\beta)^{\zeta(r)}_i \not\leq \bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i]$, then for every $\zeta'\in \prod_{s<n} J^s_\beta$ one has

$$(\forall\alpha\in A)(\forall\xi\in \prod_{s<n} J^s_\alpha)[\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta'(r)}} B(f^r_\beta)^{\zeta'(r)}_i \not\leq \bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^r_\alpha)^{\xi(r)}_i].$$

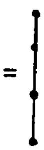
Indeed, let $\zeta'\in \prod_{s<n} J^s_\beta$, $\alpha\in A$, $\xi\in \prod_{s<n} J^s_\alpha$; our assumptions on β and ζ imply that there exists $t<n$ such that $p_{\beta(t)} \not\leq p_{\alpha(t)}$ and, therefore, since $p_{\beta(t)}$, $p_{\alpha(t)}$ are join-irreducible in $B(\mathcal{L})$, $p_{\alpha(t)} \longrightarrow p_{\beta(t)}=p_{\beta(t)}$; but H_t is an embedding of Brouwer algebras, so $A^t_{\alpha(t)} \longrightarrow A^t_{\beta(t)}=A^t_{\beta(t)}$; hence $(\forall j\in J^t_\beta)$ $(\forall j'\in J^t_\alpha)$ $[\bigwedge_{i\in I^t_{\beta,j}} B(f^t)^j_i \not\leq \bigwedge_{i\in I^t_{\alpha,\xi(t)}} B(f^t_\alpha)^{j'}_i]$: in particular, $\bigwedge_{i\in I^t_{\beta,\zeta'(t)}} B(f^t_\beta)^{\zeta'(t)}_i \not\leq \bigwedge_{i\in I^t_{\alpha,\xi(t)}} B(f^t_\alpha)^{\xi(t)}_i$.
By (*) this implies $\bigwedge_{r<n} \bigwedge_{i\in I^r_{\beta,\zeta'(r)}} B(f^t_\beta)^{\zeta'(r)}_i \not\leq \bigwedge_{r<n} \bigwedge_{i\in I^r_{\alpha,\xi(r)}} B(f^t_\alpha)^{\xi(r)}_i$ as desired.

- $I(\neg a)=\neg I(a)$: this is trivial.

Before stating some corollaries, let us see some examples of the procedure exposed in the proof of Theorem 1.5.

Example 1.3)

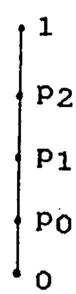
Let $\mathcal{L}$ = ; then

$B(\mathcal{L})$ =  and $TB(\mathcal{L})$ = 

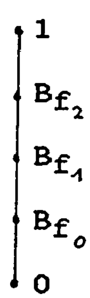
Obviously $TB(\mathcal{L})$ is embeddable in $\mathfrak{M}'$: for instance

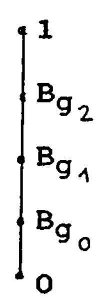
is an embedding of $TB(\mathcal{L})$ in $\mathfrak{M}'$ (g non-recursive).

Consider two different embeddings of

$TB(B(\mathcal{L}))$ = 

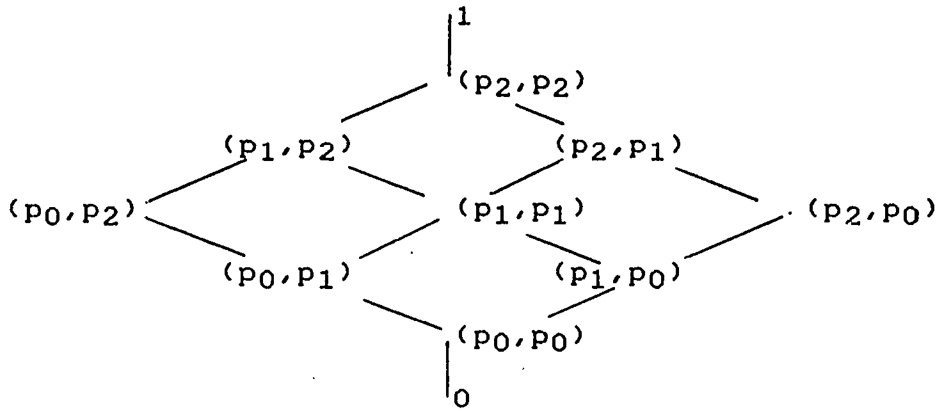
in $\mathfrak{M}'$:

$I_0 : TB(B(\mathcal{L})) \longrightarrow$ 

$I_1 : TB(B(\mathcal{L})) \longrightarrow$ 

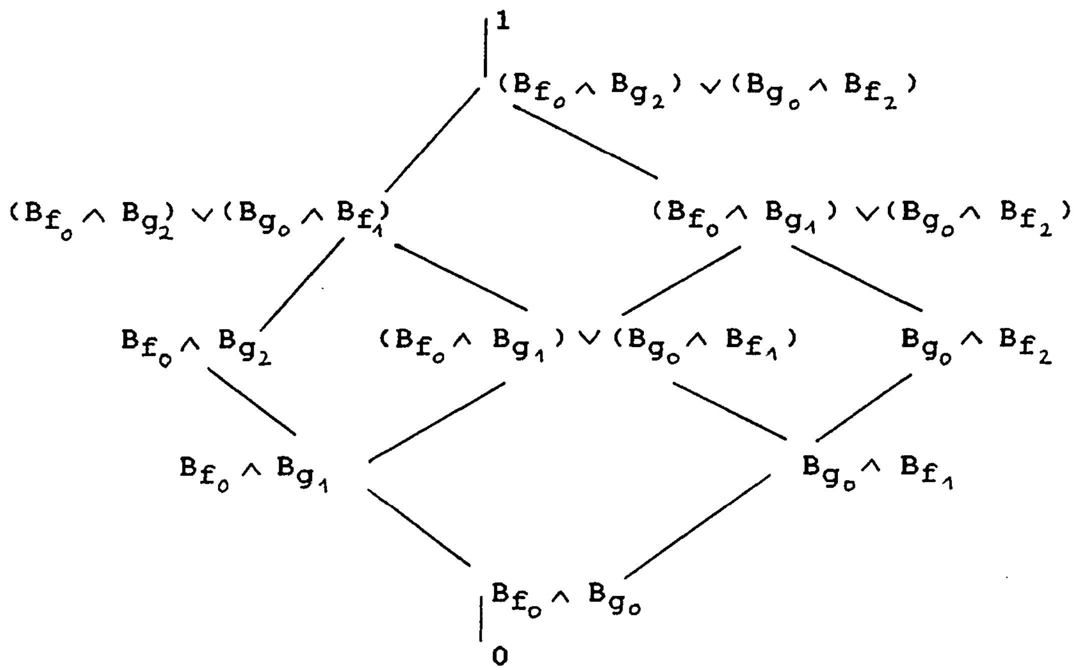
where $\{B_{f_0}, B_{f_1}, B_{f_2}\} \mid \{B_{g_0}, B_{g_1}, B_{g_2}\}$.

Hence $I_0(p_i)=B_{f_i}=A_i^0$ and $I_1(p_i)=B_{g_i}=A_i^1$. $TB((B(\mathcal{L}))^2)$ is



The join-irreducible elements of $(B(\mathcal{L}))^2$ are $\{(p_0,p_0)$. $(p_0,p_1),(p_1,p_0),(p_0,p_2)\},(p_2,p_0)$.

If we define I as in Theorem 1.5, then $I(TB(B(\mathcal{L}))^2)$ looks like:



By the methods used in the proof of Theorem 1.5, one can also prove

Theorem 1.6 Let $\mathcal{L}$ be a finite Brouwer algebra such that $TB(\mathcal{L})$ is embeddable in $\mathcal{M}'$. Then, for every $n \in \omega$, $TB(\mathcal{L}^n)$ is embeddable in $\mathcal{M}'$.

Proof. Similar to the proof of Theorem 1.5.

Corollary 1.9 If $\mathcal{L}$ is a finite Boolean algebra, then $TB(\mathcal{L})$ is embeddable in $\mathcal{M}'$.

Proof. For every finite Boolean algebra $\mathcal{L}$ there exists $n \in \omega$ such that $\mathcal{L} \simeq 2^n$, where 2 is the two-element Boolean algebra. The corollary then follows from Theorem 1.6, since TB(2) is embeddable in $\mathcal{M}'$: indeed it suffices to consider two non-recursive functions f,g such that $f <_T g$ and define $H: TB(2) \longrightarrow \mathcal{M}'$ by

$$H(a) = \begin{cases} 0 & \text{if } a = 0_{TB(2)} \\ B_f & \text{if } a = b_{TB(2)} \\ & \\ B_g & \text{if } a = t_{TB(2)} \\ 1 & \text{if } a = 1_{TB(2)} \end{cases}$$

We now define two sequences $\{\mathcal{J}^H_k\}_{k\geq 1}$, $\{\mathcal{J}^B_k\}_{k\geq 1}$ of finite distributive lattices as follows:

Definition 1.6 a) $\{\mathcal{J}^H_k\}_{k\geq 1}$ is the sequence of finite distributive lattices defined by induction as follows:

$\mathfrak{J}^H_1=2$ (i.e. the two-element Boolean algebra);

$\mathfrak{J}^H_{k+1}=T((\mathfrak{J}^H_k)^{k+1})$.

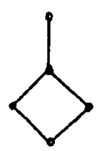
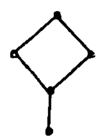
b) $\{\mathfrak{J}^B_k\}_{k\geq 1}$ is the sequence of finite distributive lattices defined by induction as follows:

$\mathfrak{J}^B_1=2$;

$\mathfrak{J}^B_{k+1}=B((\mathfrak{J}^B_k)^{k+1})$.

Thus, for instance,

$\mathfrak{J}^H_1=\mathfrak{J}^B_1=$

$\mathfrak{J}^H_2=$  $\mathfrak{J}^B_2=$ 

Fact 1.7 $\underline{H}=\bigcap\{\underline{L}_H(\mathfrak{J}^H_k)\colon k\geq 1\}=\bigcap\{\underline{L}_B(\mathfrak{J}^B_k)\colon k\geq 1\}$.

Proof. For $\underline{H}=\bigcap\{\underline{L}_H(\mathfrak{J}^H_k)\colon k\geq 1\}$ see JASKOWSKI (1936); clearly $\bigcap\{\underline{L}_H(\mathfrak{J}^H_k)\colon k\geq 1\}=\bigcap\{\underline{L}_B(\mathfrak{J}^B_k)\colon k\geq 1\}$ (actually, as is easily seen, $(\forall k\geq 1)[\underline{L}_H(\mathfrak{J}^H_k)=\underline{L}_B(\mathfrak{J}^B_k)]$.

Definition 1.7 Let $\{\mathfrak{I}^H_k\}_{k\geq 1}$, $\{\mathfrak{I}^B_k\}_{k\geq 1}$ be defined respectively as follows: for every $k\geq 1$

$\mathfrak{I}^H_k=B(\mathfrak{J}^H_k)$, $\mathfrak{I}^B_k=T(\mathfrak{J}^B_k)$.

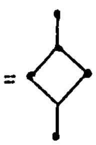
Fact 1.8 $\underline{J}=\bigcap\{\underline{L}_H(\mathfrak{I}^H_k)\colon k\geq 1\}=\bigcap\{\underline{L}_B(\mathfrak{I}^B_k)\colon k\geq 1\}$.

Proof. The equality $\underline{J}=\bigcap\{\underline{L}_H(\mathfrak{I}^H_k)\colon k\geq 1\}$ is proved in SMORYNSKI (1963) and easily follows, anyway, from a result in JANKOV (1963). $\underline{J}=\bigcap\{\underline{L}_B(\mathfrak{I}^B_k)\colon k\geq 1\}$ follows dually.

Corollary 1.10 Forevery $k\geq 1$, $\mathfrak{J}^{B}_{k}$ is embeddable in $\mathfrak{M}'$.

Proof. By induction on $k\geq 1$:

$\mathfrak{J}^{B}_{1}=T(\mathfrak{J}^{B}_{1})=$  is clearly embeddable in $\mathfrak{M}'$;

$\mathfrak{J}^{B}_{2}=T((\mathfrak{J}^{B}_{1})^{2})=$  is embeddable in $\mathfrak{M}'$: indeed,

$\mathfrak{J}^{B}_{2}\simeq TB(2^{2})$; thus we can apply Corollary 1.9.

Suppose now that $k\geq 1$ and $\mathfrak{J}^{B}_{k+1}$ is embeddable in $\mathfrak{M}'$: we want to show that $\mathfrak{J}^{B}_{k+2}$ is embeddable in $\mathfrak{M}'$.

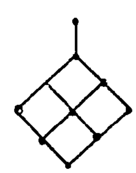
But $\mathfrak{J}^{B}_{k+1}=T(\mathfrak{J}^{B}_{k+1})=TB((\mathfrak{J}^{B}_{k})^{k+1})$: since $TB((\mathfrak{J}^{B}_{k})^{k+1})$ is embeddable in $\mathfrak{M}'$, it follows from Theorem 1.5 that $TB((B((\mathfrak{J}^{B}_{k})^{k+1}))^{k+2})$ is embeddable in $\mathfrak{M}'$; but $TB((B((\mathfrak{J}^{B}_{k})^{k+1}))^{k+2})=TB((\mathfrak{J}^{B}_{k+1})^{k+2})=T(\mathfrak{J}^{B}_{k+2})=\mathfrak{J}^{B}_{k+2}$, thus the corollary is proved.

Following JANKOV (1963) we can define also another sequence $\{(\mathfrak{J}^{H}_{k})'\}_{k\in\omega}$ by letting

$(\mathfrak{J}^{H}_{0})'=2$

$(\mathfrak{J}^{H}_{k+1})'=T(((\mathfrak{J}^{H}_{k}))'^{k+1})$.

We have $(\mathfrak{J}^{H}_{0})'=$

$(\mathfrak{J}^{H}_{1})'=$  $(\mathfrak{J}^{H}_{2})'=$ 

The dual sequence $\{(\mathfrak{J}^{B}_{k})'\}_{k\in\omega}$ is defined in the obvious way. A fact similar to Fact 1.7 also holds for the sequences $\{(\mathfrak{J}^{H}_{k})'\}_{k\in\omega}$ and $\{(\mathfrak{J}^{\beta}_{k})'\}_{k\in\omega}$. A new sequence $\{\mathfrak{H}_k\}_{k\in\omega}$ is defined in the following way: for every $k\in\omega$, $\mathfrak{H}_k=T(\{a\in(\mathfrak{J}^{\beta}_{k})':\neg a=1_{(\mathfrak{J}^{\beta}_{k})'}\})$ (see JANKOV (1963)): it is proved in the same paper that $\underline{J}=\cap\{L_B(\mathfrak{H}_k):k\in\omega\}$.

$\{a\in(\mathfrak{J}^{\beta}_{k})':\neg a=1_{(\mathfrak{J}^{\beta}_{k})'}\}$ is clearly a sub-Brouwer algebra of $(\mathfrak{J}^{\beta}_{k})'$: for instance, to check that it is closed under the operation $\longrightarrow$ let $a,b\in(\mathfrak{J}^{\beta}_{k})'$ be such that $\neg a=1_{(\mathfrak{J}^{\beta}_{k})'}$ and $\neg b=1_{(\mathfrak{J}^{\beta}_{k})'}$; then for every $x\in(\mathfrak{J}^{\beta}_{k})'$, $1_{(\mathfrak{J}^{\beta}_{k})'}\leq(a\longrightarrow b)\vee x \implies 1_{(\mathfrak{J}^{\beta}_{k})'}\leq b\vee x$ (since $a\longrightarrow b\leq b$); hence $x=1_{(\mathfrak{J}^{\beta}_{k})'}$ i.e. $\neg(a\longrightarrow b)=1_{(\mathfrak{J}^{\beta}_{k})'}$.

Therefore, for every $k\in\omega$, we have an embedding of Brouwer algebras from $\mathfrak{H}_k$ into $T((\mathfrak{J}^{\beta}_{k})')$ and the following corollary holds:

Corollary 1.11 For every $k\in\omega$, $T((\mathfrak{J}^{\beta}_{k})')$ and $\mathfrak{H}_k$ are embeddable in $\mathfrak{M}'$.

Proof. That each $T((\mathfrak{J}^{\beta}_{k})')$ is embeddable in $\mathfrak{M}'$ can be proved using methods quite similar to those in the proof of Corollary 1.10: indeed,

$T((\mathfrak{J}^{\beta}_{0})')=$  is clearly embeddable in $\mathfrak{M}'$;

$T((\mathfrak{J}^{\beta}_{1})')=$  is embeddable in $\mathfrak{M}'$.

Then proceed by induction using Theorem 1.5.

Having proved that each $T((\mathfrak{J}^{\beta}_{k})')$ is embeddable in $\mathfrak{M}'$, the claim that each $\mathfrak{H}_k$ is embeddable in $\mathfrak{M}'$ follows from the remarks immediately preceding the statement of Corollary 1.11.

Example 1.4 In Example 1.3, $TB((B(\mathcal{L}))^2) = T((\mathcal{T}_2^B)')$: using the embedding of Example 1.3, we get the following embedding of $\mathcal{H}_2$ in $\mathcal{M}'$:

$$
\mathcal{H}_2 \longrightarrow
\begin{tikzcd}[column sep=small]
 & 1 \arrow[d, dash] & \\
 & (B_{f_0} \wedge B_{g_1}) \vee (B_{g_0} \wedge B_{f_1}) \arrow[dl, dash] \arrow[dr, dash] & \\
B_{f_0} \wedge B_{g_1} \arrow[dr, dash] & & B_{g_0} \wedge B_{f_1} \arrow[dl, dash] \\
 & B_{f_0} \wedge B_{g_0} \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}
$$

Corollary 1.12 $\underline{J} = \underline{L}_B(\mathcal{M})$.

Proof. By Theorem 1.1, Fact 1.1, Corollary 1.10 and Fact 1.8 (i.e. $\underline{J} = \cap\{\underline{L}_B(\mathcal{T}_k^B) : k \geq 1\}$); instead of Corollary 1.10 and Fact 1.8, one can also use Corollary 1.11 and the above mentioned result of Jankov, namely $\underline{J} = \cap\{\underline{L}_B(\mathcal{H}_k) : k \in \omega\}$.

The results which we have proved for $\mathcal{M}$ have their counterparts for $\mathcal{M}_e$. Indeed, for every partial function φ one can define $\underline{B}_\varphi$ to be the degree in $\mathcal{M}_e$ of the mass problem of partial functions $\{\psi : \psi \not\leq_e \varphi\}$.
Theorem 1.4 and Theorem 1.5 (when suitably restated) hold also for $\mathcal{M}_e$. In particular we have

Corollary 1.13 $\underline{J} = \underline{L}_B(\mathcal{M}_e)$.

Proof. As in Corollary 1.12.

Another corollary to Theorem 1.4 is the following

Corollary 1.14 Every free distributive lattice on a set X of generators, where $|X| \leq 2^{\aleph_0}$ is embeddable in $\mathfrak{M}$ (of course as a Brouwer algebra).

Proof. Map X into a subset of $\{B_{f_i} : i \in I\}$ having the appropriate cardinality, where $\{[f_i]_T : i \in I\}$ is an antichain of $\mathfrak{D}_T$ of cardinality $2^{\aleph_0}$ and consider the sublattice of $\mathfrak{M}$ generated by the image of X under this mapping.

If $\mathcal{L}_1 = \langle L_1, \leq_1 \rangle$, $\mathcal{L}_2 = \langle L_2, \leq_2 \rangle$ are lattices, let us denote by $\mathcal{L}_1 \oplus \mathcal{L}_2$ the lattice $\langle L_1 \cup L_2, \leq \rangle$ where $\leq = \leq_1 \cup \leq_2 \cup L_1 \times L_2$ (we assume $L_1 \cap L_2 = \emptyset$). We have

Corollary 1.15 If the finite Brouwer algebras $TB(\mathcal{L}_1)$, $TB(\mathcal{L}_2)$ are embeddable in $\mathfrak{M}'$ then $B(\mathcal{L}_1) \oplus T(\mathcal{L}_2)$ and $TB(\mathcal{L}_1) \oplus TB(\mathcal{L}_2)$ are also embeddable in $\mathfrak{M}'$.

Proof. Let H_1: $TB(\mathcal{L}_1) \longrightarrow \mathfrak{M}'$, H_2: $TB(\mathcal{L}_2) \longrightarrow \mathfrak{M}'$ be embeddings and let us show for example that $B(\mathcal{L}_1) \oplus T(\mathcal{L}_2)$ is embeddable in $\mathfrak{M}'$. Let $\Delta_1 \subseteq \mathfrak{D}_T$, $\Delta_2 \subseteq \mathfrak{D}_T$ be order-isomorphic copies of $\Delta_{H_1}(\mathcal{L}_1)$ and $\Delta_{H_2}(\mathcal{L}_2)$ respectively (see the notation introduced before Lemma 1.5), such that

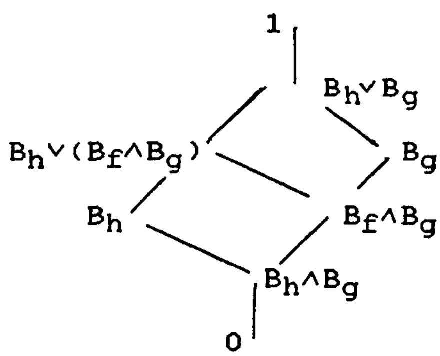
$(\forall a_T \in \triangle_1)(\forall b_T \in \triangle_2)[a_T <_T b_T]$. Then, starting from $\triangle_1$ and $\triangle_2$, by the methods used in the proof of Lemma 1.5, define two embeddings I_1: $TB(\mathcal{L}_1) \longrightarrow \mathcal{M}'$, I_2: $TB(\mathcal{L}_2) \longrightarrow \mathcal{M}'$. It follows that the function $I: B(\mathcal{L}_1) \oplus T(\mathcal{L}_2) \longrightarrow \mathcal{M}'$ defined by

$$I(a) = \begin{cases} I_1(a) & \text{if } a \in B(\quad_1) \\ \\ I_2(a) & \text{if } a \in T(\quad_2) \end{cases}$$

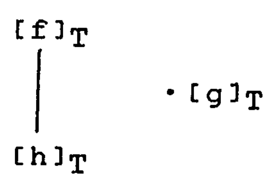
is the desired embedding.

Example 1.5 Let us see some examples of embeddings obtained by methods different from the ones described so far.

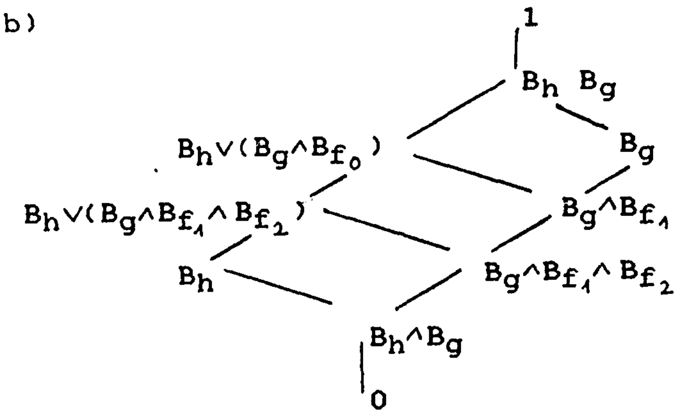
a)



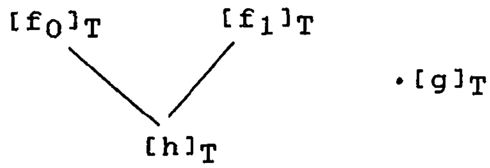
The preceding is a sub-Brouwer algebra of $\mathcal{M}$: f,g,h are chosen in such a way that the respective T-degrees are ordered as follows



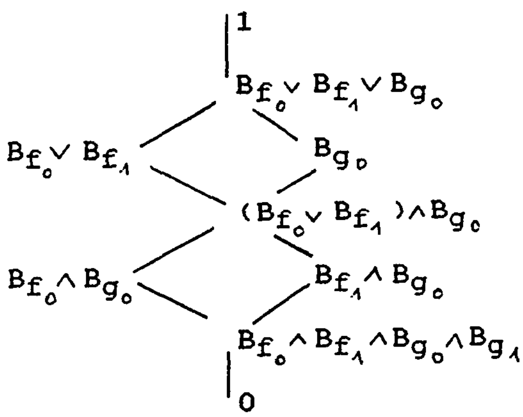
b)

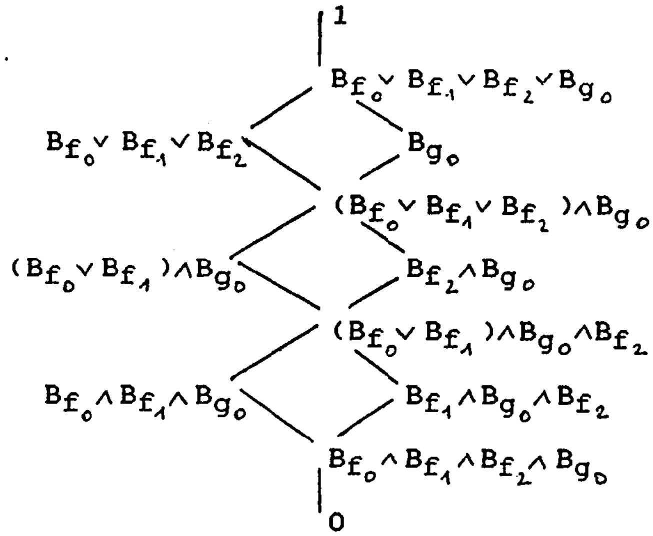

1
$B_h \; B_g$
$B_h \vee (B_g \wedge B_{f_0})$
B_g
$B_h \vee (B_g \wedge B_{f_1} \wedge B_{f_2})$
$B_g \wedge B_{f_1}$
B_h
$B_g \wedge B_{f_1} \wedge B_{f_2}$
$B_h \wedge B_g$
0

where


$[f_0]_T$
$[f_1]_T$
$\cdot [g]_T$
$[h]_T$

c)


1
$B_{f_0} \vee B_{f_1} \vee B_{g_0}$
$B_{f_0} \vee B_{f_1}$
B_{g_0}
$(B_{f_0} \vee B_{f_1}) \wedge B_{g_0}$
$B_{f_0} \wedge B_{g_0}$
$B_{f_1} \wedge B_{g_0}$
$B_{f_0} \wedge B_{f_1} \wedge B_{g_0} \wedge B_{g_1}$
0



in each of the examples in c) the functions are to be taken pairwisely T-incmparable.

2. Some quotient lattices.

In order to understand what follows, one has to recall the notation introduced in the proof of Theorems 1.2 and 1.3: in particular, in Theorem 1.3, with a given finite Brouwer algebra having the form TB($\mathcal{L}$) we have associated a finite partial order (actually a lower semi-lattice) and defined a function $lv: \mathcal{L} \longrightarrow \omega$. Moreover if $P_{\mathcal{L}} = \{p \in \mathcal{L} :$ p is join-irreducible $\}$, then with each $p \in P_{\mathcal{L}}$ we have associated a certain number n(p).

Let $W = \{TB(\mathcal{L}) : \mathcal{L}$ is a finite distributive lattice

$$\& \ (\forall p \in P_{\mathcal{L}})[n(p)=0]\}.$$

Recall that the condition $(\forall p \in P_{\mathcal{L}})[n(p)=0]$ is equivalent to say that $\langle A, \circledS \rangle$ is order-isomorphic to the dual of the Kripke frame corresponding to B($\mathcal{L}$).

Theorem 2.1 Let TB($\mathcal{L}$)$\in$W. Then there exists an embedding of Brouwer algebras $I: TB(\mathcal{L}) \longrightarrow \mathcal{M}_w$ such that I(B($\mathcal{L}$)) is an ideal of $\mathcal{M}_w$.

Proof. Let TB($\mathcal{L}$)$\in$W be given and let $H: \langle A, \circledS \rangle \longrightarrow \mathfrak{D}_T$ be an embedding which preserves infima and such that H(A) is an initial segment of $\mathfrak{D}_T$. Let $P = \{p_0, \ldots, p_k\}$ be the set of join-irreducible elements of $\mathcal{L}$ and for every $i \leq k$ choose a function f_i such that $f_i \in H(\dot{a}^{\circ}_{p_i})$.

Define $I: TB(\mathcal{L}) \longrightarrow \mathcal{M}_w$ by (for semplicity we omit the subscript $_w$ when writing out weak degrees of difficulty):

$$I(a)=\begin{cases} 0 & \text{if } a=0_{TB(\mathcal{L})} \\ 0' & \text{if } a=b_{TB(\mathcal{L})} \\ \bigvee_{j\in J} B_{f_j} & \text{if } a \neq b_{TB(\ )}\ \&\ a=\bigvee\{p_j : j\in J\} \text{ and } \{p_j : \\ & j\in J\} \text{ is } \vee\text{-irredundant} \\ 1 & \text{if } a=1_{TB(\mathcal{L})} \end{cases}$$

As in Theorem 1.3 one can prove that I is an embedding of Brouwer algebras (notice that if f,g belong to minimal T-degrees and $f|_T g$ then, in $\mathcal{M}_w$, $B_f \wedge B_g = 0'$ (we remind the reader that we are dealing here with weak degrees of difficulty and that the subscript $_w$ is omitted: in $\mathcal{M}$ 0' is meet-irreducible!). We use the assumption $(\forall p\in P)[n(p)=0]$ in that it ensures that $(\forall B)[B \in I(\mathcal{L}) \Longrightarrow (\exists J \subseteq \{0,\ldots k\})[B= \bigvee\{B_{f_j} : j\in J\}]$.

Let us now show that $I(B(\mathcal{L}))$ is an initial segment of $\mathcal{M}_w$.

First notice that $C \leq_w I(a)\ \&\ lv(a)=0 \Longrightarrow C \in I(B(\mathcal{L}))$: indeed, if $lv(a)=0$ then $a=b_{TB(\mathcal{L})}$, thus $I(a)=0'$; therefore $C\leq_w I(a) \Longrightarrow C \in I(B(\mathcal{L}))$.

Suppose now that $C \leq_w I(a)\ \&\ lv(a)\leq n \Longrightarrow C \in I(B(\mathcal{L}))$, for every weak degree of difficulty C: we intend to show also that, $C \leq_w I(a)\ \&\ lv(a)=n+1 \Longrightarrow C \in I(B(\mathcal{L}))$. Thus, suppose that $C \leq_w I(a)\ \&\ lv(a)=n+1$ and let $I(a)= \bigvee\{B_{f_j} : j\in J\}$, where $\{p_j : j\in J\}$ is $\vee$-irredundant; of course we may assume $C <_w I(a)$. Let $C =[\mathcal{C}]_w$, where $\mathcal{C}$ is C-closed; thus $\mathcal{C} <_w \cap\{\mathcal{B}_{f_j} : j\in J\}$ and so $(\exists i\in J)[\mathcal{C} \nsubseteq \mathcal{B}_{f_i}]$ Then $\{g: g\equiv_T f_i\} \subseteq \mathcal{C}$, which implies $\mathcal{C} \leq_w \cap\{\mathcal{B}_f : [f]_T \in H(A)\ \&\ f<_T f_i\} \cap \cap\{\mathcal{B}_{f_j} : j\in J-\{i\}\}$: indeed, if $g\in\bigcap\{\mathcal{B}_f: [f]_T\in H(A)\ \&\ f<_T f_i\}\cap\bigcap\{\mathcal{B}_{f_j}: j\in J-\{i\}\}$ then we have two possibilities:

1) $g\in\mathcal{B}_{f_i}$: in this case $g\in\bigcap\{\mathcal{B}_{f_j}: j\in J\}$, so $g\in\mathcal{C}$;

2) $g\notin\mathcal{B}_{f_i}$: thus, $g\le_T f_i$; we can show however that $g\not<_T f_i$, so $g\equiv_T f_i$ and $g\in\mathcal{C}$ as desired: indeed, suppose that $g<_T f_i$; since $H(A)$ is an initial segment of $\mathcal{D}_T$, this implies that there exists $f<_T f_i$ such that $[f]_T\in H(A)$ and $g\le_T f$, contradicting the assumption $g\in\bigcap\{\mathcal{B}_f: [f]_T\in H(A)\ \&\ f<_T f_i\}$.

Therefore $C\le_w \bigvee\{B_f: [f]_T\in H(A)\ \&\ f<_T f_i\}\vee\bigvee\{B_{f_j}: j\in J-\{i\}\}$

To conclude the proof is then enough to show that $lv(I^{-1}(\bigvee\{B_f: [f]_T\in H(A)\ \&\ f<_T f_i\}\vee\bigvee\{B_{f_j}: j\in J-\{i\}\}))\le n$, i.e. $\bigvee\{B_f: [f]_T\in H(A)\ \&\ f<_T f_i\}\vee\bigvee\{B_{f_j}: j\in J-\{i\}\}<_w \bigvee\{B_{f_j}: j\in J\}$: indeed, the inequality $\bigvee\{B_f: [f]_T\in H(A)\ \&\ f<_T f_i\}\vee\bigvee\{B_{f_j}: j\in J-\{i\}\}\le_w\bigvee\{B_{f_j}: j\in J\}$ is immediate; on the other hand, that $\bigvee\{B_{f_j}: j\in J\}\not\le\bigvee\{B_f: [f]_T\in H(A)\ \&\ f<_T f_i\}\vee\bigvee\{B_f : j\in J-\{i\}\}$ follows from the fact that each B_{f_j} is join-irreducible and $(\forall j\in J-\{i\})[B_{f_j}\not\le B_{f_i}]$.

Let $\mathcal{M}_w^- = \mathcal{M}_w-\{0_w\}$; clearly $\mathcal{M}_w^-$ is still both a Heyting algebra and a Brouwer algebra. The least element of $\mathcal{M}_w^-$ is $0'$.

Corollary 2.1 For every $TB(\mathcal{L})\in W$, the finite Brouwer algebra $T(\mathcal{L})$ is embeddable in $\mathcal{M}_w^-$, as an ideal of $\mathcal{M}_w^-$.

Proof. Let $T(\mathcal{L})$ be a finite Heyting algebra, such that $TB(\mathcal{L}) \in W$. Let $I: TB(\mathcal{L}) \longrightarrow \mathcal{M}_w$ be the embedding defined in the proof of Theorem 2.1. Since $I(b_{TB(\mathcal{L})})=0'$, it follows that the restriction of I to $T(\mathcal{L})$ is the desired embedding of $T(\mathcal{L})$ in $\mathcal{M}_w^-$.

For the convenience of the reader, we recall that, given any distributive lattice $\mathcal{L}$, every filter F of $\mathcal{L}$ gives rise to a congruence of $\mathcal{L}$: under this congruence, two elements x, y of the lattice are equivalent if and only if there exists an element z of F such that $x \wedge z = y \wedge z$. Let us denote by $\mathcal{L}/_F$ the distributive lattice obtained by dividing $\mathcal{L}$ by this congruence; elements of $\mathcal{L}/_F$ are thus equivalence classes $[x]_F$ of elements of $\mathcal{L}$. For the partial ordering $\leq/_F$ determined in $\mathcal{L}/_F$ by the congruence, recall that $[x]_F \leq/_F [y]_F$ if and only if there exists $z \in F$ such that $x \wedge z \leq y$. A dual situation holds, considering ideals instead of filters.

Lemma 2.1 Let $\mathcal{L}$ be a Heyting algebra and let F be a filter of $\mathcal{L}$. Then $\mathcal{L}/_F$ is a Heyting algebra and the canonical epimorphism $\mathcal{L} \longrightarrow \mathcal{L}/_F$ is also a homomorphism of Heyting algebras.

Proof. See e.g. RASIOWA-SIKORSKI (1963).

Theorem 2.1 has the following nice corollary

Corollary 2.2 Let TB($\mathcal{L}$)$\in$W. Then there exists a principal filter F of $\mathcal{M}_w$ such that $\mathcal{M}_{w/F} \simeq$ B($\mathcal{L}$) as Heyting algebras.

Proof. Embed TB($\mathcal{L}$) in $\mathcal{M}_w$ via an embedding I in such a way that I(B($\mathcal{L}$)) is an ideal of $\mathcal{M}_w$ (see Theorem 2.1). In particular, let $I(t_{TB(\mathcal{L})})=I(\bigvee\{p_i: i\in I\})= \bigvee\{B_{f_i}: i\in I\}$, where $\{p_i: i\in I\}$ is $\vee$-irredundant and let $F=\{A : I(t_{TB(\mathcal{L})})\leq_w A\}$. Then the result is immediate, since $\mathcal{M}_{w/F} \simeq [0_w, I(t_{TB(\mathcal{L})})] \simeq$ B($\mathcal{L}$) as distributive lattices but then also as Heyting algebras by Lemma 2.1

Remark 2.1 Recall from Remark 1.2 that, given any finite Brouwer algebra TB($\mathcal{L}$), if P denotes the set of join-irreducible elements of $\mathcal{L}$, then $(\forall p\in P)[n(p)=0] \Leftrightarrow (\forall p\in P)[\bigvee\{q\in P: q<p\}\in P$ or $(\forall q\in P)[p\leq q$ or $q\leq p]]$. It follows from this fact that, by looking at the Kripke frame corresponding to B($\mathcal{L}$), we have an easy way to decide whether or not TB($\mathcal{L}$)$\in$W. Indeed, let $\mathcal{K}(B(\mathcal{L}))=\langle K(B(\mathcal{L})),\leq\rangle$ be the Kripke frame corresponding to B($\mathcal{L}$); then $(\forall p\in P)$ $[n(p)=0]$ $\Leftrightarrow (\forall a\in K(B(\mathcal{L})))[a$ has exactly one immediate successor or $(\forall b\in K(B(\mathcal{L})))[a\leq b$ or $b\leq a]]$.

We have included this remark, although not used in the remainder of this paper, because the semantics of intermediate logics which one is most likely to encounter in the literature is the one that uses Kripke frames. Thus, once we have at disposal a Kripke frame (with a least element) we are able to say whether or not this Kripke frame corresponds to a Brouwer algebra of the form B($\mathcal{L}$) such that TB($\mathcal{L}$)$\in$W.

Corollary 2.3 Let $\mathcal{L}$ be a finite distributive lattice. If the dual of the Kripke frame corresponding to B($\mathcal{L}$) is a tree, then TB($\mathcal{L}$)$\in$W.

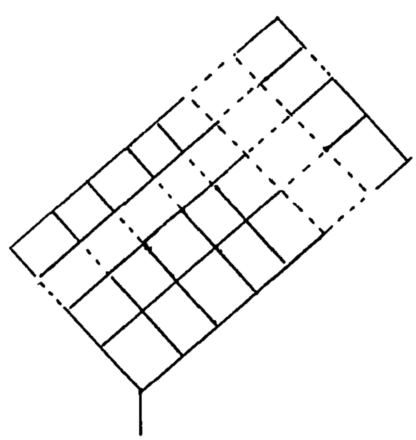
Proof. Immediate by Remark 2.1.

Corollary 2.4 If TB($\mathcal{L}_1$), TB($\mathcal{L}_2$)$\in$W then B($\mathcal{L}_1$)$\oplus$T($\mathcal{L}_2$) $\in$W.

Proof. Immediate by Remark 2.1.

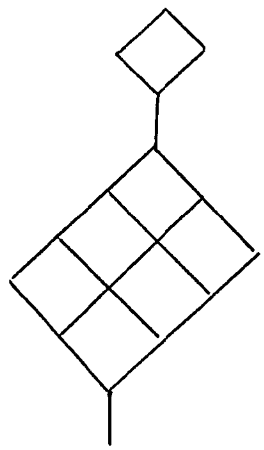
Example 2.1 Putting together Corollary 2.3 and Remark 2.1 one is able to conclude that the following finite Heyting algebras can be obtained by dividing $\mathcal{M}_w$ by suitable principal filters.

a) for every m,n$\in\omega$ the tailed mxn chessboard:



b) every B($\mathcal{L}$), where $\mathcal{L}$ is a finite Boolean algebra;

c) (by Corollary 2.3 and Corollary 2.4):



We single out the following example for its particular historical importance.

Let $\mathcal{J}_n$ denote the chain having n+1 elements. The relevance of the sequence of Heyting algebras $\{ \mathcal{J}_n : n \geq 1\}$ to the semantics of intermediate logics was pointed out by Gödel (GÖDEL (1932)): namely he showed that, letting $\underline{S}_n = \underline{L}_H(\mathcal{J}_n)$, we get a strictly increasing sequence of intermediate logics. This was the first example to show that the cardinality of the set of intermediate logics is greater than 2 (later on it has been proved that this set has cardinality $2^{\aleph_0}$: see JANKOV (1968)). The sequence $\{\underline{S}_n : n \geq 1\}$ has been thoroughly investigated in HOSOI (1966); hereby, it is shown that for every $n \geq 1$ $\underline{S}_n$ is axiomatizable.

Corollary 2.5 For every $n \geq 1$, there exists a principal filter F such that $\mathcal{M}_{W/F} \simeq \mathcal{J}_n$ as Heyting algebras.

Proof. Immediate since for every $n\geq 1$ $\mathcal{J}_n$ has the form $B(\mathcal{J}_{n-1})$ where $TB(\mathcal{J}_{n-1})\in W$ (let $\mathcal{J}_0=\cdot$).

Corollary 2.6 For every finite $TB(\ \ )\in W$ there exists a principal filter F such that $\mathcal{M}^{-}_{W/F}\simeq\mathcal{L}$.

Proof. As in Corollary 2.1 and Corollary 2.2.

Remark 2.2 Of course we can use more sophisticated results on initial segments of $\mathcal{D}_T$ to get more results here. For example, it is known (see RUBIN (1985)) that $\aleph_1$ is embeddable in $\mathcal{D}_T$ as an initial segment. For such an embedding $\{[f_\beta]_T\colon \beta<\aleph_1\}$, define the sequence $\{(A_\beta)_W\colon \beta<\aleph_1\}$ by transfinite induction as follows

$(A_0)_W=B_{f_0}$ (for semplicity,in B_{f_0} the subscript $_W$ is omitted):hence $(A_0)_W=0'_W$, since f_0 is recursive

$(A_{\beta+1})_W=B_{f_{\beta+1}}$(in $B_{f_{\beta+1}}$ the subscript $_W$ is omitted)

$(A_\lambda)_W=\bigvee_{\gamma<\lambda} B_{f_\gamma}$ if λ is a limit ordinal (in B_{f_γ} the subscript $_W$ is omitted)

Let now $\alpha<\aleph_1$ and let $F=\{A_W\colon A_W\geq_W(A_\alpha)_W\}$. Then $\mathcal{M}_{W/F}\simeq\mathcal{J}_\alpha$ where the sequence $\{\mathcal{J}_\alpha\colon \alpha$ is an ordinal$\}$ is defined by

$\mathcal{J}_0=2$

$\mathcal{J}_{\beta+1}=T(\mathcal{J}_\beta)$

$\mathcal{J}_\lambda=T(\bigcup_{\gamma<\lambda}\mathcal{J}_\gamma)$ if λ is a limit ordinal.

In particular

Corollary 2.7 There exists a principal filter F such that a) $\mathcal{M}_{w/F} \cong \mathcal{S}_\omega$;
b) $\underline{L}_H(\mathcal{M}_{w/F}) = \underline{S}_\omega = \cap\{\underline{S}_n : n \geq 1\}$.

Proof. Immediate.

$\underline{S}_\omega$ is the so-called <u>Dummet's logic</u> and equals the deductive closure of $\underline{H} \cup \{(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha): \alpha, \beta \in Form(\underline{L})\}$.

Lemma 2.1 has its dual version for Brouwer algebras and ideals. Let us now consider again $\mathcal{M}_w$ as a Brouwer algebra. Despite the previous results we have

Theorem 2.2 If I is a proper principal ideal of $\mathcal{M}_w$, then every finite Brouwer algebra of the form TB($\mathcal{L}$) is embeddable in $\mathcal{M}_{w/I}$. In particular, $\underline{L}_B(\mathcal{M}_{w/I}) = \underline{J}$.

Proof. Let $I = \{A_W : A_W \leq_w D_W\}$ be a proper principal ideal and let $D_W = [\mathcal{D}]_W$ where $\mathcal{D}$ is C-closed. Let f be a non-recursive function such that $\mathcal{B}_f \not\leq_w \mathcal{D}$. Such a function exists since otherwise we would have $(\forall f)\ [\mathcal{D} \subseteq \mathcal{B}_f]$ i.e. $\mathcal{D} \subseteq \cap\{\mathcal{B}_f : f \text{ is a function}\} = \emptyset$, a contradiction since I is assumed to be proper.
Let now TB($\mathcal{L}$) be a finite Brouwer algebra: in order to embed TB($\mathcal{L}$) into $\mathcal{M}_{w/I}$, consider the partial order $\langle A, \circledS \rangle$ as defined in the proof of Theorem 1.3. Embed $\langle A, \circledS \rangle$ into the interval $([f]_T, \infty)$ of $\mathcal{D}_T$ (i.e. relativize the embedding to the function f) and define an

embedding $H: TB(\mathcal{L}) \longrightarrow \mathfrak{M}_w$ in exactly the same way as in the proof of Theorem 1.3. We claim that the composite $\overline{H}: TB(\mathcal{L}) \xrightarrow{H} \mathfrak{M}_w \longrightarrow \mathfrak{M}_{w/I}$ is the desired embedding. $\overline{H}$ is clearly a homomorphism of Brouwer algebras (see Lemma 2.1); it is left to show that H is one-one. Suppose for the sake of contradiction that there exist $a, b \in \mathcal{L}$ such that $a \nleq b$ and $\overline{H}(a) \leq_{/I} \overline{H}(b)$: let $a = \bigvee\{p_u: u \in U\}$, $b = \bigvee\{p_j: j \in J\}$ where $\{p_u: u \in U\}$, $\{p_j: j \in J\}$ are $\vee$-irredundant. Since $\overline{H}(a) \leq_{/I} \overline{H}(b)$, it follows that $\bigvee_{u\in U} \bigvee_{i\leq n(p_u)} B_{f^i_{p_u}} \leq_w \bigvee_{j\in J} \bigvee_{r\leq n(p_j)} B_{f^r_{p_j}} \vee D_w$ (we refer here to the notation already used in the proof of Theorem 1.3); but since $H(a) \nleq_w H(b)$ (H being one-one) it must follow also that $(\exists u \in U)(\exists i \leq n(p_u))(\forall j \in J)(\forall r \leq n(p_j))[B_{f^i_{p_u}} \nleq_w B_{f^r_{p_j}}]$. Therefore, since $B_{f^i_{p_u}}$ is join-irreducible, there exist $u \in U$ and $i \leq n(p_u)$ such that $B_{f^i_{p_u}} \leq_w D_w$, which implies $B_f \leq_w D_w$, a contradiction (throughout the proof the subscript $_w$ in weak degrees of difficulty of the form B_h has been omitted).

Theorem 2.3 Let I be a proper principal ideal of $\mathfrak{M}$. For every finite Brouwer algebra $\mathcal{L}$, if $TB(\mathcal{L})$ is embeddable in $\mathfrak{M}'$ then $TB(\mathcal{L})$ is embeddable in $\mathfrak{M}/_I$.

Proof. Let $H: TB(\mathcal{L}) \longrightarrow \mathfrak{M}'$ be an embedding and let $I = \{A: A \leq D\}$ be a proper principal ideal. As in the proof of Theorem 2.2 we can choose a non-recursive function f such that $B_f \nleq D$. Let $\Delta \subseteq \mathcal{D}_T$ be order-isomorphic to $\Delta_H(\mathcal{L})$ (this notation was introduced shortly before

Lemma 1.5) and such that $(\forall a_T \in \Delta)[[f]_T \le_T a_T]$. Define an embedding $H': TB(\mathcal{L}) \longrightarrow \mathfrak{M}'$ starting from Δ (use methods employed in the proof of Lemma 1.5). Exactly as in Theorem 2.2 we can show that the composite map $\bar{H}: TB(\mathcal{L}) \longrightarrow \mathfrak{M}' \hookrightarrow \mathfrak{M} \rightarrow \mathfrak{M}/_I$ is the desired embedding.

Corollary 2.8 $\underline{L}_B(\mathfrak{M}/_I) = \underline{J}$.

Proof. Immediate, by Theorem 2.3, Corollary 1.9 and Fact 1.8.

The embeddings of Theorem 1.1 explicitly used finite weak degrees of difficulty. If F denotes the filter of $\mathfrak{M}_w$ generated by the degrees of solvability, then F contains every finite degree. Nevertheless, we have the following theorem, where we regard again $\mathfrak{M}_w$ as a Heyting algebra.

Theorem 2.4 Every finite Heyting algebra $TB(\mathcal{L})$ is embeddable in $\mathfrak{M}_{w/F}$. Hence $\underline{L}_H(\mathfrak{M}_{w/F}) = \underline{J}$.

Proof. Let $TB(\mathcal{L})$ be given and let $\langle A, \circledS \rangle$, P be as in the proof of Theorem 1.2. Let A_ω be the partial order obtained by adding a greatest element ∞ to the ω-folded coproduct $\bigsqcup_{j\in\omega} A$ of $\langle A, \circledS \rangle$ and, for every $j \in \omega$, let I_j denote the j-th canonical injection of A into $\bigsqcup_{j\in\omega} A$. Find an embedding of upper semi-lattices $K: A_\omega \longrightarrow \mathfrak{D}_T - \{o_T\}$ (A is easily seen to be an upper semilattice) and for every $j \in \omega$, $p \in P$, $i \le n(p)$ choose a function $f_p^{i,j}$ such that $K(I_j(a_p^i))=[f_p^{i,j}]_T$; moreover, let g be a function such that $K(\infty)=[g]_T$. For every function f let F_f denote the weak degree of difficulty $[\{f\}]_w (=[\{h: f\leq_T h\}]_w)$.

We claim that the function $H:TB(\mathcal{L}) \longrightarrow \mathcal{M}_w$ defined by

$$H(a)=\begin{cases} 0_w & \text{if } a=0_{TB(\mathcal{L})} \\ 1_w & \text{if } a=1_{TB(\mathcal{L})} \\ \bigwedge_{j\in\omega}\bigwedge_{u\in U}\bigwedge_{i\leq n(p_u)} F_{f^{i,j}_{p_u}} & \text{if } a=\bigwedge\{p_u:u\in U\} \text{ where } \{p_u: u\in U\} \text{ is } \wedge\text{-irredundant} \end{cases}$$

is an embedding of Heyting algebras ($H(a)$ is always defined because $\mathcal{M}_w$ is a complete lattice).

Indeed, for every $j\in\omega$, let H_j: $TB(\mathcal{L}) \longrightarrow \mathcal{M}_w$ be defined by

$$H_j(a)=\begin{cases} 0_w & \text{if } a=0_{TB(\mathcal{L})} \\ 1_w & \text{if } a=1_{TB(\mathcal{L})} \\ \bigwedge_{u\in U}\bigwedge_{i\leq n(p_u)} F_{f^{i,j}_{p_u}} & \text{see the third clause of } H(a) \end{cases}$$

As in the proof of Theorem 1.1, H_j is seen to be an embedding of Heyting algebras.

Since, for every $a\in TB(\mathcal{L})$, $H(a)=\bigwedge\{H_j(a): j\in\omega\}$, we have

- $H(a\wedge b)=\bigwedge\{H_j(a\wedge b): j\in\omega\}=\bigwedge\{H_j(a)\wedge H_j(b): j\in\omega\}=(\bigwedge\{H_j(a): j\in\omega\})\wedge(\bigwedge\{H_j(b): j\in\omega\})=H(a)\wedge H(b)$;
- $H(a\vee b)=\bigwedge\{H_j(a\vee b): j\in\omega\}=\bigwedge\{H_j(a)\vee H_j(b): j\in\omega\}$; on the other hand, $H(a)\vee H(b)=(\bigwedge\{H_j(a): j\in\omega\})\vee(\bigwedge\{H_j(b): j\in\omega\})=\bigwedge_{j\in\omega}\bigwedge_{r\in\omega}(H_j(a)\vee H_r(b))$: if $j\neq r$ and $a=\bigwedge\{p_u:u\in U\}$, $b=\bigwedge\{p_v: v\in V\}$ (with $\{p_u: u\in U\}$, $\{p_v: v\in V\}$

$\wedge$ -irredundant); then $H_j(a) \vee H_r(b) = (\bigwedge_{u\in U} \bigwedge_{i\leq n(p_u)} F_{f^{i,j}_{p_u}}) \vee (\bigwedge_{v\in V} \bigwedge_{i\leq n(p_v)} F_{f^{i,r}_{p_v}}) = \bigwedge_{u\in U} \bigwedge_{i\leq n(p_u)} \bigwedge_{v\in V} \bigwedge_{s\leq n(p_v)} (F_{f^{i,j}_{p_u}} \vee F_{f^{s,r}_{p_v}}) = F_g$.

Therefore $\bigwedge_{j\in\omega} \bigwedge_{r\in\omega} (H_j(a) \vee H_r(b)) = \bigwedge\{H_j(a) \vee H_j(b):\ j\in\omega\} \wedge F_g = \bigwedge\{H_j(a) \vee H_j(b):\ j\in\omega\}$ as desired;

- $H(a \longrightarrow b) = \bigwedge\{H_j(a \longrightarrow b):\ j\in\omega\} = \bigwedge\{H_j(a) \longrightarrow H_j(b):\ j\in\omega\}$. Suppose again that $a = \bigwedge\{p_u\ u\in U\}$, $b = \bigwedge\{p_v : v\in V\}$, where $\{p_u:\ u\in u\}$, $\{p_v:\ v\in V\}$ are $\wedge$-irredundant: thus $H_j(a) \longrightarrow H_j(b) = \bigwedge\{H_j(p_u):\ (\forall v\in V)[p_u \not\leq p_v]\}$.

Therefore $\bigwedge\{H_j(a) \longrightarrow H_j(b):\ j\in\omega\} = \bigwedge\{\bigwedge\{H_j(p_u):\ (\forall v\in V)[p_u \not\leq p_v]\}:\ j\in\omega\} = \bigwedge_{j\in\omega}\{\bigwedge H_j(p_u):\ (\forall v\in V)[p_u \not\leq p_v]\} = \bigwedge\{H_j(a):\ j\in\omega\} \longrightarrow \bigwedge\{H_j(b):\ j\in\omega\} = H(a) \longrightarrow H(b)$;

- $H(\neg a) = \neg H(a)$: obvious.

Now, the composite $\bar{H}$: $TB(\mathcal{L}) \xrightarrow{H} \mathcal{M}_w \longrightarrow \mathcal{M}_{w/F}$ is the embedding for which we are looking: again, $\bar{H}$ is trivially a homomorphism of Heyting algebras (by the dual version of Lemma 2.1). It is left to show that $\bar{H}$ is one-one. Suppose that $\bar{H}(a) \leq_{w/F} \bar{H}(b)$, where $a = \bigwedge\{p_u:\ u\in U\}$ and $b = \bigwedge\{p_v:\ v\in V\}$ (with $\{p_u:\ u\in U\}$, $\{p_v:\ v\in V\}$ $\wedge$ -irredundant); then for some finite set of functions $\{g_r:\ r\in R\}$ we must have $H(a) \wedge \bigwedge_{r\in R} F_{g_r} \leq_w H(b)$ i.e. $\bigwedge_{j\in\omega} \bigwedge_{u\in U} \bigwedge_{i\leq n(p_u)} F_{f^{i,j}_{p_u}} \wedge \bigwedge_{r\in R} F_{g_r} \leq_w \bigwedge_{j\in\omega} \bigwedge_{v\in V} \bigwedge_{i\leq n(p_v)} F_{f^{i,j}_{p_v}}$. From the way A_ω is defined it follows that for every $v\in V$ there exists a cofinite set $S \subseteq \omega$ and $u\in U$ such that $\bigwedge_{j\in\omega} \bigwedge_{i\leq n(p_u)} F_{f^{i,j}_{p_u}} \leq_w \bigwedge_{j\in S} \bigwedge_{i\leq n(p_v)} F_{f^{i,j}_{p_v}}$: again from the definition of A_ω this implies that $p_u \leq p_v$. Thus $(\forall v\in V)(\exists u\in U)[p_u \leq p_v]$ and therefore $a \leq b$, as desired.

There are interesting similarities between principal filters of $\mathcal{M}_w$ having the form $\{A_w: A_w \geq_w B_f\}$ and the filters of $\mathcal{M}_F$ having the form $\{F: F \not\leq [\{f\}]\}$.

Fact 2.1 Let f be a non-recursive function; then the filter $F_f=\{F: F \not\leq [\{f\}]\}$ of $\mathcal{M}_F$ is prime and non-principal.

Proof. That F_f is a prime filter is obvious. As to non-principality, notice that for every finite mass problem $\mathcal{F}$ such that $[\mathcal{F}]\neq 0$, we can find a function g such that, say, $\mathcal{F} \cup \{f\} \mid_T g$. Then $[\{g\}]\in F_f$ but $\mathcal{F} \not\leq \{g\}$.

Following Corollary 2.2, we have

Theorem 2.5 For every finite Heyting algebra $TB(\mathcal{L})\in W$, there exists a filter F such that $\mathcal{M}_{F/F} \simeq TB(\mathcal{L})$.

Proof. Let $TB(\mathcal{L})$ be given and, once again, refer to the notation introduced in the proof of Theorem 1.2. Embed $\langle A, \circledS \rangle$ in $\mathcal{D}_T$ as an initial segment, via an embedding which preserves suprema. Let f denote $f^{\circ}_{t_{TB(\mathcal{L})}}$ i.e. let f correspond to $a^{\circ}_{t_{TB(\mathcal{L})}}$ under this embedding. Then $F=F_f$ is the desired filter. The proof is essentially the same as in Theorem 2.1 and Corollary 2.2 (using Theorem 1.2 instead of Theorem 1.3).

Notice that, for every finite mass problem $\mathcal{J}$, $[\mathcal{J}] \notin F_f$ $\Rightarrow \mathcal{J} \leq \{f\}$, hence $[[\{f\}]]_F$ is the coatom of $\mathfrak{M}_{F/F}$.

For principal filters of $\mathfrak{M}_F$ one has the following

Theorem 2.6 If F is a proper principal filter of $\mathfrak{M}_F$, then every finite Heyting algebra of the form TB($\mathcal{L}$) is embeddable in $\mathfrak{M}_{F/F}$.

Proof. Suppose $F=\{F: F \geq D\}$ is our filter, where $D=[\mathcal{G}]$ and let TB($\mathcal{L}$) be a finite Heyting algebra. Again, let $\langle A, \circledS \rangle$ be as in Theorem 1.2. Use an embedding $H: \langle A, \circledS \rangle \longrightarrow \mathfrak{D}_T - \{o_T\}$ which preserves suprema and such that $(\forall a_T \in H(A))(\forall f \in \mathcal{G})[[f]_T \not\leq_T a_T]$ (this is possible since F is assumed to be proper and thus $\mathcal{G}$ does not contain any recursive function). Then define as usual I: $TB(\mathcal{L}) \longrightarrow \mathfrak{M}_F$ by

$$I(a)=\begin{cases} 0 & \text{if } a=0_{TB(\mathcal{L})} \\ 1 & \text{if } a=1_{TB(\mathcal{L})} \\ \bigwedge_{u\in U} \bigwedge_{i\leq m(p_u)} [\{f^i_{p_u}\}] & \text{if } a=\bigwedge\{p_u : u\in U\} \text{ and } \{p_u : u \in U\} \text{ is} \\ & \wedge\text{-irredundant} \end{cases}$$

It is easily seen that the composite I: $TB(\mathcal{L}) \longrightarrow \mathfrak{M}_F \rightarrow \mathfrak{M}_{F/F}$ is the desired embedding.

Let now F be a principal filter of $\mathfrak{M}$; then $\mathfrak{M}/_F$ is still a Brouwer algebra in an obvious way: indeed, if $F=\{A: D \leq A\}$ then $\mathfrak{M}/_F \simeq [0,D]$ (where the isomorphism is a morphism of the category DL); but since $[0,D]$ is a Brouwer algebra so is $\mathcal{M}_{/F}$.

A similar situation holds for $\mathcal{M}_w$ regarded as a Brouwer algebra, i.e. for every principal filter F, $\mathcal{M}_{w/F}$ is a Brouwer algebra.

The difference with the situation in which we considered principal ideals is that now the congruence determined by F is no longer a congruence of Brouwer algebras and thus there may be no epimorphism (of Brouwer algebras) $\mathcal{M} \to \mathcal{M}_{/F}$, $\mathcal{M}_w \longrightarrow \mathcal{M}_{w/F}$. As a consequence of this, it may be not the case that $J \subsetneq L_B(\mathcal{M}_{/F})$ and $J \subsetneq L_B(\mathcal{M}_{w/F})$. In particular, if F is generated by a join-reducible element then $\neg\alpha \vee \neg\neg\alpha \notin L_B(\mathcal{M}_{/F})$ $(L_B(\mathcal{M}_{w/F})$.

A similar remark applies to $\mathcal{M}_{w/I}$, $\mathcal{M}_{F/I}$ etc, where $\mathcal{M}_w$, $\mathcal{M}_F$ are Heyting algebras and I is a principal ideal.

Theorem 2.7 Let $F=\{A: D\leq A\}$ be a proper principal filter of $\mathcal{M}$, where $D>0'$ and either D is join-irreducible or contains a mass problem which is closed in the Baire topology. Then every finite Brouwer algebra of the form $TB(\mathcal{L})$ which is embeddable in $\mathcal{M}'$ is also embeddable in $\mathcal{M}_{/F}$.

Proof. Suppose that $H: TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ is an embedding of the finite Brouwer algebra $TB(\mathcal{L})$ into $\mathcal{M}'$. Let $D=[\Omega]$; we may also assume that $D\neq 1$:

otherwise the theorem follows trivially. By Theorem 1.1 of DYMENT (1976) there exists a countable mass problem $\mathcal{A}$ such that $D \not\leq [\mathcal{A}]$.

Let $\Delta' \subseteq \mathcal{D}_T$ be order-isomorphic to $\Delta_H(\mathcal{L})$ and such that $(\forall a_T \in \Delta')(\forall f \in \mathcal{A})[a_T \mid_T [f]_T]$ (this can be done since $\Delta_H(\mathcal{L})$ is finite and $\mathcal{A}$ is countable). Notice that if $[f]_T \in \Delta'$ then $\mathcal{A} \subseteq \mathcal{B}_f$ and thus $\mathcal{B}_f \leq \mathcal{A}$.

Using the tecniques of Lemma 1.5, let H': $TB(\mathcal{L}) \longrightarrow \mathcal{M}'$ be an embedding of Brouwer algebras such that $\Delta_{H'}(\mathcal{L}) = \Delta'$.

We claim that the composite function I: $TB(\mathcal{L}) \longrightarrow \mathcal{M}' \hookrightarrow \mathcal{M} \longrightarrow \mathcal{M}/_F$ is the desired embedding of Brouwer algebras.

- Clearly I is a lattice-theoretic homomorphism preserving the least and the greatest element;
- I is one-one: indeed, suppose that $I(a) \leq_{/F} I(b)$ and let $H'(a) = \bigvee_{v \in V}(\bigwedge_{u \in U_v} B_{g^v_u})$, $H'(b) = \bigvee_{j \in J}(\bigwedge_{i \in I_j} B_{f^j_i})$. Thus

$(\bigvee_{v \in V}(\bigwedge_{u \in U_v} \mathcal{B}_{g^v_u})) \wedge \mathcal{D} \leq \bigvee_{j \in J}(\bigwedge_{i \in I_j} B_{f^j_i})$ i.e.

$(\bigvee_{v \in V}(\bigwedge_{u \in U_v} \mathcal{B}_{g^v_u})) \wedge \mathcal{D} \leq \bigwedge_{\xi \in \prod_{j \in J} I_j}(\bigvee_{j \in J} \mathcal{B}_{f^j_{\xi(j)}})$.

Notice now that for no $\xi \in \prod_{j \in J} I_j$ can we have $\mathcal{D} \leq \bigvee_{j \in J} \mathcal{B}_{f^j_{\xi(j)}}$: indeed we have to consider two cases:

Case 1) if D is join-irreducible then $\mathcal{D} \leq \bigvee_{j \in J} \mathcal{B}_{f^j_{\xi(j)}}$ would imply that for some $j \in J$ $\mathcal{D} \leq \mathcal{B}_{f^j_{\xi(j)}}$; since $\mathcal{B}_{f^j_{\xi(j)}} \leq \mathcal{A}$, this would imply also that $\mathcal{D} \leq \mathcal{A}$, a contradiction;

Case 2) if $\mathcal{D}$ is closed in the Baire topology then the claim follows by Chapter I, Corollary 7.3, since each $\bigvee_{j \in J} \mathcal{B}_{f^j_{\xi(j)}}$ is dense in the Baire topology.

Therefore we conclude that $(\forall \xi \in \prod_{j \in J} I_j)$ $[\bigvee_{v \in V}(\bigwedge_{u \in U_v} \mathcal{B}_{g^v_u}) \leq$

$\bigvee_{j\in J} \mathcal{B}_{f^j_{\xi(j)}}]$; it follows that $H'(a)\le H'(b)$ and thus $a\le b$.

- $I(a \longrightarrow b)=I(a) \longrightarrow I(b)$: without loss of generality we may suppose that $a,b\in \mathcal{L}$; let $H'(a)=\bigvee_{j\in J}(\bigwedge_{i\in I_j} B_{f^j_i})$, $H'(b)=\bigvee_{v\in V}(\bigwedge_{u\in U_v} B_{g^v_u})$. It is enough to show that $(\forall Z)[I(b)\le_{/F} I(a)\vee [Z]_F) \Longrightarrow I(a\longrightarrow b)\le_{/F} [Z]_F]$ or, in other words, $(\forall Z)[H'(b)\wedge D\le H'(a)\vee Z \Rightarrow H'(a\longrightarrow b)\wedge D\le Z]$. Thus, let $Z=[\,\mathcal{Z}\,]$ be given and suppose

$(\bigvee_{v\in V}(\bigwedge_{u\in U_v}\mathcal{B}_{g^v_u}))\wedge \mathcal{D} \le \bigvee_{j\in J}(\bigwedge_{i\in I_j}\mathcal{B}_{f^j_i})\vee \mathcal{Z}$;

since $H'(a\rightarrow b)=\bigvee\{\bigwedge_{u\in U_v} B_{g^v_u}$: $v\in V$ & $(\forall j\in J)[\bigwedge_{u\in U_v} B_{g^v_u} \not\le \bigwedge_{i\in I_j} B_f\,]\}$ it is enough to show that, if $v\in V$ is such that $(\forall j\in J)[\bigwedge_{u\in U_v}\mathcal{B}_{g^v_u}\not\le \bigwedge_{i\in I_j}\mathcal{B}_{f^j_i}]$, then $\bigwedge_{u\in U_v}\mathcal{B}_{g^v_u}\wedge\mathcal{D}\le\mathcal{Z}$.

Starting from this point and trying to mimic the proof of Theorem 1.4, one gets to the point where we have, for some $\xi\in\prod_{j\in J} I_j$, $\bigwedge_{u\in U_v}\mathcal{B}_{g^v_u}\wedge\mathcal{D}\le\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}}\vee\mathcal{Z}$ and $(\forall u\in U_v)$ $[\mathcal{B}_{g^v_u} \not\le \bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}}]$. This implies (letting $n\notin U_v$), $(\bigcup_{u\in U_v} u*\mathcal{B}_{g^v_u})\cup n*\mathcal{D}\le\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}}\vee\mathcal{Z}$ thus, $(\forall u\in U_v)[\mathcal{B}_{g^v_u}\le(\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}})^{(u)}\vee\mathcal{Z}^{(u)}]$ and $\mathcal{D}\le(\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}})^{(n)}\vee\mathcal{Z}^{(n)}$ (for the notation employed here, see proof of Theorem 1.4). At this point one shows that $\bigwedge_{u\in U_v}\mathcal{Z}^{(u)}\wedge\mathcal{Z}^{(n)}\equiv\mathcal{Z}$ and $(\forall u\in U_v)[\mathcal{B}_{g^v_u}\le\mathcal{Z}^{(u)}]$. Furthermore $\mathcal{D}\le(\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}})^{(n)}\vee\mathcal{Z}^{(n)}\Longrightarrow\mathcal{D}\le\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}}\vee\mathcal{Z}^{(n)}$. To see this, suppose that $(\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}})^{(n)}\vee\mathcal{Z}^{(n)}$ is reducible to $\mathcal{D}$, via say Ψ_{z_n}: if $f\vee g\in\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}}\vee\mathcal{Z}^{(n)}$ then one can effectively find an initial segment $\tilde{h}$ such that $\Psi_{z_n}(\tilde{h}\vee g)(0)=n$; but then $\tilde{h}*f\in(\bigvee_{j\in J}\mathcal{B}_{f^j_{\xi(j)}})^{(n)}\vee\mathcal{Z}^{(n)}$ and thus $\Psi_{z_n}((\tilde{h}*f)\vee g)\in$.

Now, if D is join-irreducible, then (as before: Case 1)) $\mathcal{D}\le\mathcal{Z}^{(n)}$; if $\mathcal{D}$ is closed in the Baire topology then again $\mathcal{D} \le \mathcal{Z}^{(n)}$ (see Chapter I, Proposition 7.2). Therefore $\bigwedge_{u \in U_v} \mathcal{B}_{g^v_u} \wedge \mathcal{D} \le \mathcal{Z}$ as desired.

Corollary 2.9 Let F be as in Theorem 2.8. Then $\underline{L}_B(\mathcal{M}/_F) \subseteq \underline{J}$.

Proof. Immediate by Theorem 2.7, Corollary 1.9, Fact 1.8.

Remark 2.3 i) Of course, if D is join-irreducible and D generates the filter F, then $\underline{L}_B(\mathcal{M}/_F)=\underline{J}$, since $\neg\alpha \vee \neg\neg\alpha \in \underline{L}_B(\mathcal{M}/_F)$;
ii) If F={A: 0'≤A} then $\mathcal{M}/_F \simeq 2$ and, thus, $\underline{L}_B(\mathcal{M}/_F)=\underline{K}$;
iii) if F={A: 0'<A} then $\mathcal{M}/_F$ is still a Brouwer algebra (namely $\mathcal{J}_2$) and $\underline{L}_B(\mathcal{M}/_F)=S_2$.

Remark 2.4 In Theorem 2.7, the case D=[$\mathcal{D}$], where $\mathcal{D}$ is closed, includes also the case D is a degree of solvability.

Remark 2.5 Theorem 2.7 once more emphasizes the differences between $\mathcal{M}$ and $\mathcal{M}_w$ even at the level of intermediate logics. By Corollary 2.7, for instance, we have principal filters F of $\mathcal{M}_w$ generated by join-irreducible elements such that $\underline{L}_B(\mathcal{M}_w/_F) \nsubseteq \underline{J}$.

Finally we turn our attention to $\mathcal{M}_F$ (similar remarks will aplly of course to $\mathcal{M}_E$). Let J be the principal ideal generated by a finite degree of difficulty $F>0$. Since $\mathfrak{M}_{F/J} \simeq [F,1]$ (in the category $\underline{DL}$), $\mathfrak{M}_{F/J}$ is a Heyting algebra (the interval $[F,1]$ being so).

Theorem 2.8 Let J be a proper principal ideal of $\mathfrak{M}_F$ generated by $F=[\mathfrak{F}]$; then every finite Heyting algebra of the form $TB(\mathcal{L})$ is embeddable into $\mathfrak{M}_{F/J}$.

Proof. Let $TB(\mathcal{L})$ be a finite Heyting algebra. In order to prove that $TB(\mathcal{L})$ is embeddable into $\mathfrak{M}_{F/J}$ it suffices to show that $TB(\mathcal{L})$ is embeddable into the interval $[F,1]$ of $\mathfrak{M}_F$. The proof of this is essentially the same as in Theorem 1.2 but using a suprema-preserving embedding $H: \langle A, \circledS \rangle \longrightarrow \mathfrak{D}_T((a_T,\infty])$, where $a_T = \bigvee\{[f]_T : f \in \mathfrak{F}\}$.

Corollary 2.10 If J is as in Theorem 2.8, then $\underline{L}_H(\mathfrak{M}_{F/J}) \subseteq \underline{J}$.

Proof. Immediate by Fact 1.2 (b) and Theorem 2.8.

Remark 2.6 If F is meet-irreducible then of course $\underline{L}_H(\mathfrak{M}_{F/J}) = \underline{J}$. The situation is quite different if F is not meet-irreducible. Indeed, let F, F_1, F_2 be elements of $\mathfrak{M}_F$ such that $F_1 | F_2$ and $F=F_1 \wedge F_2$; let J be the principal ideal generated by F and let us identify $\mathfrak{M}_{F/J}$ with the interval $[F,1]$ of $\mathfrak{M}_F$. Then, for instance,

$\neg F_1 = F_2$, $\neg F_2 = F_1$; thus $\neg F_1 \vee \neg\neg F_1 = F_2 \vee F_1 \neq 1$ and, therefore, $\underline{J} \not\subset \underline{L}_H(\mathcal{M}_{F/J})$.

In fact, simple arguments show that $\cap\{\underline{L}_H(\mathcal{M}_{F/J})$: J is a proper principal ideal of $_F\} = \underline{H}$. To see this, it is enough to observe that for every finite Heyting algebra $T(\mathcal{L})$, there exists a principal ideal J such that $T(\mathcal{L})$ is embeddable in $\mathcal{M}_{F/J}$: indeed, let $H: TB(\mathcal{L}) \longrightarrow \mathcal{M}_F$ be an embedding, then $T(\mathcal{L})$ is embeddable in $\mathcal{M}_{F/J}$, where $J = \{F \in \mathcal{M}_F\colon F \leq H(b_{TB(\mathcal{L})})\}$

Out of curiosity, let us see some countermodels to some of the most well-known intermediate logics:

The <u>Scott system</u> $\underline{S}$ is the deductive closure of $\underline{H} \cup \{((\neg\neg\alpha \longrightarrow \alpha) \longrightarrow (\neg\alpha \vee \alpha)) \longrightarrow (\neg\alpha \vee \neg\neg\alpha)\colon \alpha \in Form(\underline{L})\}$.

Let f,g,h be non-recursive functions such that $f|_T g$, $g <_T h$, $f|_T h$, $h \leq_T f \vee g$; let J be the principal ideal of $\mathcal{M}_F$ generated by $[\{f,g\}]$. Then $\underline{S}$. $\underline{L}_H(\mathcal{M}_{F/J})$: indeed, if $G = [\{f,h\}]$, then $\neg G = [\{g\}]$, $\neg\neg G = [\{f\}]$, thus $\neg G \vee \neg\neg G = [\{f \vee g\}]$; moreover $\neg G \vee G = [\{f \vee g, h\}] = [\{f \vee g\}]$ and $\neg\neg G \longrightarrow G = [\{f\}] \longrightarrow [\{f,h\}] = [\{h\}]$. In conclusion $(\neg\neg G \longrightarrow G) \longrightarrow (\neg G \vee G) = [\{h\}] \longrightarrow [\{f \vee g\}] = 1$, whereas $\neg G \vee \neg\neg G = [\{f \vee g\}]$; this implies $((\neg\neg G \longrightarrow G)) \longrightarrow (\neg G \vee G)) \longrightarrow (\neg\neg G \vee \neg G) \neq 1$ and therefore $\mathcal{M}_{F/J} \not\models_H ((\neg\neg\alpha \longrightarrow \alpha) \longrightarrow (\neg\alpha \vee \alpha)) \longrightarrow (\neg\neg\alpha \vee \neg\alpha)$.

Simple arguments show that neither the <u>Kreisel-Putnam scheme</u> $((\neg\alpha \longrightarrow \beta \vee \gamma) \longrightarrow ((\neg\alpha \longrightarrow \beta) \vee (\neg\alpha \longrightarrow \gamma)))$ nor <u>Rose's scheme</u> $(((\neg\neg\alpha \rightarrow \alpha) \longrightarrow (\neg\neg\alpha \vee \neg\alpha)) \longrightarrow (\neg\neg\alpha \vee \neg\alpha))$ are satisfied in $\mathcal{M}_{F/J}$ where J is any proper principal ideal generated by a meet-reducible degree.

For the Kreisel-Putnam scheme, if J is generated by $F=[\{f,g\}]$ where $f|_T g$, choose two functions h_0, h_1 such that $f \not\leq_T h_0$, $f \not\leq_T h_1$, $g<_T h_0$, $g<_T h_1$ and $f \leq_T h_0 \vee h_1$ (this can always be done). Then $(\neg[\{g\}] \longrightarrow [\{h_0\}] \vee [\{h_1\}]) \longrightarrow ((\neg[\{g\}] \longrightarrow [\{h_0\}]) \vee (\neg[\{g\}] \longrightarrow [\{h_1\}])) \neq 1$.

The situation is similar when $|\mathfrak{F}|>2$.

Remark 2.7 In Theorem 2.7 and in Theorem 2.8 we always get intermediate logics $\underline{L}$ such that $\underline{L} \subseteq \underline{J}$. Since $\underline{H}^+=\underline{J}^+$ ($^+$ denotes the formulas without the connective $\neg$) (see JANKOV (1968)) we always have $\underline{L}^+=\underline{J}^+$.

CHAPTER III

IDEALS AND FILTERS IN SUBLATTICES OF THE MEDVEDEV LATTICE AND THEIR RELATIONS WITH THE e-DEGREES.

1. Some remarks on filters and ideals and their relationship with the e-degrees.

Let $\mathfrak{U}$, $\{\mathfrak{U}_j : j \in J\}$ be objects of a given category of algebras and homomorphisms. Then $\mathfrak{U}$ is said to be a subdirect product of the family $\{\mathfrak{U}_j : j \in J\}$ (see e.g. BURRIS-SANKAPPANAVAR (1980)) if there exists an embedding I such that

1) $I : \mathfrak{U} \longrightarrow \prod_{j \in J} \mathfrak{U}_j$;

2) $(\forall j \in J)[\pi_j \circ I(\mathfrak{U}) = \mathfrak{U}_j]$ (here π_j denotes the j-th projection).

Such an I is called a subdirect embedding. An algebra $\mathfrak{U}$ is sudirectly irreducible if for every subdirect embedding $I : \mathfrak{U} \longrightarrow \prod_{j \in J} \mathfrak{U}_j$, there exists $j \in J$ such that $\pi_j \circ I : \mathfrak{U} \longrightarrow \mathfrak{U}_j$ is an isomorphism.

For Heyting algebras and Brouwer algebras we have the following fact

Fact 1.1 A Heyting algebra $\mathcal{L}$ is subdirectly irreducible if and only if $\mathcal{L} \in T(\underline{DL})$; a Brouwer algebra $\mathcal{L}$ is subdirectly irreducible if and only if $\mathcal{L} \in B(\underline{DL})$.

Proof. See for example BURRIS-SANKAPPANAVAR (1980).

Corollary 1.1 As Brouwer algebras, $\mathfrak{M}$, $\mathfrak{M}_e$, $\mathfrak{M}_w$ are subdirectly irreducible.

Proof. Immediate.

As Heyting algebras, all the lattices with which we have been dealing are subdirectly irreducible. A theorem due to Birkhoff (see BIRKHOFF (1944)) reads

Theorem 1.1 Every algebra is isomprphic to a subdirect product of subdirectly irreducible algebras.

Using some of the filters introduced in Section 2 of the previous chapter, we are in a position to exhibit such isomorphisms for $\mathfrak{M}_w$ (regarded of course as a Heyting algebra), $\mathfrak{M}_F$ etc.

Let F_f be the filter of $\mathfrak{M}_w$ defined by $F_f=\{A_w: B_f \leq_w A_w\}$ (the subscript $_w$ is omitted in B_f). Then $\mathfrak{M}_{w/F_f}$ is subdirectly irreducible because $\mathfrak{M}_{w/F_f} \simeq [O_w, B_f]$ and clearly $[O_w, B_f] \in T(\underline{DL})$: indeed $B_f \wedge [\{f\}]_w$ is the coatom of $[O_w, B_f]$, so the claim follows by Fact 1.1.

Proposition 1.1 $\mathfrak{M}_w \simeq \prod\{\mathfrak{M}_{w/F_f} : f \in {}^{\omega}\omega\}$.

Proof. Let $I: \mathfrak{M}_w \longrightarrow \prod\{\mathfrak{M}_{w/F_f} : f \in {}^{\omega}\omega\}$ be given by $I(A_w) = \langle [A_w]_{F_f} \rangle_{f \in {}^{\omega}\omega}$: it suffices to show that I is injective. Suppose that $A_w \neq B_w$ and let $A_w = [\mathcal{A}]_w$, $B_w = [\mathcal{B}]_w$ where $\mathcal{A}, \mathcal{B}$ are both C-closed. Then

$\mathcal{A} \neq \mathcal{B}$ and, say, $(\exists f \in \mathcal{A})[f \notin \mathcal{B}]$. But then, for such an f, $[A_w]_{F_f} \neq [B_w]_{F_f}$ since otherwise $A_w \wedge B_f = B_w \wedge B_f$ i.e. $\mathcal{A} \cup \mathcal{B}_f = \mathcal{B} \cup \mathcal{B}_f$, a contradiction since $f \notin \mathcal{B} \cup \mathcal{B}_f$.

Let us now use the same symbol F_f to denote a different filter in a different lattice, namely the filter of $\mathfrak{M}_F$ $\{F: F \not\leq [\{f\}]\}$, for a given function f. Notice that $\mathfrak{M}_{F/F_f}$ is subdirectly irreducible: indeed, $\mathfrak{M}_{F/F_f} \in T(\underline{DL})$, $[\{f\}]_{F_f}$ being its coatom.

Proposition 1.2 $\mathfrak{M}_F \simeq \prod\{\mathfrak{M}_{F/F_f} : f \in {}^{\omega}\omega\}$.

Proof. Let $I: \mathfrak{M}_F \longrightarrow \prod\{\mathfrak{M}_{F/F_f} : f \in {}^{\omega}\omega\}$ be defined by $I(F) = \langle [F]_{F_f} \rangle_{f \in {}^{\omega}\omega}$. It is enough to show that I is injective. Suppose that $F, G \in \mathfrak{M}_F$, $F \neq G$ and $F = [\mathcal{F}]$, $G = [\mathcal{G}]$. Then $\mathcal{F} \neq \mathcal{G}$ and, say, $(\exists g \in \mathcal{G})(\forall f \in \mathcal{F})[f \not\leq_T g]$. We claim that for every such g, $[F]_{F_g} \neq [G]_{F_g}$: indeed, for any $\mathcal{H}$ such that $\mathcal{H} \not\leq \{g\}$, we have $\mathcal{F} \cup \mathcal{H} \neq \mathcal{G} \cup \mathcal{H}$, so (letting $H = [\mathcal{H}]$) $F \wedge H \neq G \wedge H$.

We can also show that $\mathfrak{M}_E \simeq \prod\{\mathfrak{M}_{E/F_f} : f \in {}^{\omega}\omega\}$, where now $F_f = \{E \in \mathfrak{M}_E : E \not\leq [\{f\}]\}$ is a filter of $\mathfrak{M}_E$. Indeed, notice that each $\mathfrak{M}_{E/F_f}$ is subdirectly irreducible; moreover, if $\bigwedge\{E_{A_i} : i \leq k\} \not\leq \bigwedge\{E_{B_j} : j \leq n\}$ then $(\exists j \leq n)(\forall i \leq k)[A_i \not\leq_e B_j]$. Consider such a j: if f is any total function such that $f \geq_e B_j$, then $[\bigwedge\{E_{A_i} : i \leq k\}]_{F_f} \not\leq [\bigwedge\{E_{B_j} : j \leq n\}]_{F_f}$, as desired.

We now fix the notation for some of the filters and ideals of $\mathfrak{M}$ introduced in DYMENT (1976).

1) Henceforth F will denote the filter generated by the non-zero finite degrees of difficulty; since the finite degrees of difficulty constitute a sublattice of $\mathfrak{M}$, we have $F=\{A: (\exists \text{ finite } F)[F>0 \ \& \ F\leq A\}$.

2) Di will denote the filter generated by the non-zero discrete degrees (recall that a discrete degree is one that contains a mass problem which is discrete in the Baire topology); since these degrees form a sublattice, we have $Di=\{A: (\exists \text{ discrete } D)[D>0 \ \& \ D\leq A]\}$

3) C will denote the filter generated by the at most countable non-zero degrees; as before we have $C=\{A: (\exists \text{ at most countable } C)[C>0 \ \& \ C\leq A]\}$.

We say that a degree of difficulty A has <u>at most countable generalized basis</u> if, for some mass problem $\mathcal{a}$, $A=[\mathcal{a}]$ and the set $\Gamma_{\mathcal{a}}=\{[f]_T: f\in \mathcal{a} \ \& \ (\forall g\in \mathcal{a})[g \not<_T f]\}$ is at most countable. Notice that we may have $\Gamma_{\mathcal{a}}=\emptyset$ even when $|\mathcal{a}|=2^{\aleph_0}$! For instance, let $\mathcal{a}=\mathcal{O}'-C(\{f: [f]_T \text{ is minimal}\}$; then $|\mathcal{a}|=2^{\aleph_0}$ and $\Gamma_{\mathcal{a}}=\emptyset$. Also, $\Gamma_{\mathcal{a}}=\emptyset$ for $\mathcal{a}=\{f: [f]_T \text{ r.e. } \& \ [f]_T>o_T\}$.

4) B is the filter generated by the non-zero degrees with at most countable generalized basis; as before, one concludes that $B=\{A: (\exists B)[B>0 \ \& \ B \text{ has at most countable generalized basis } \& \ B\leq A\}$.

5) De will denote the ideal generated by the dense degrees different from 1; these degrees constitute a sublattice (if $\mathcal{Q}_1$, $\mathcal{Q}_2$ are two dense mass problems, then $\mathcal{Q}_1 \vee \mathcal{Q}_2$ is clearly dense; moreover $\mathcal{Q}_1 \wedge \mathcal{Q}_2 \equiv \bigcup\{(2n) * \mathcal{Q}_1: n \in \omega\} \cup \bigcup\{(2n+1) * \mathcal{Q}_2: n \in \omega\}$, and the latter mass problem is clearly dense); thus De=$\{A: (\exists$ dense degree $D)[D<1 \ \& \ A \leq D]\}$.

None of the above filters and ideals is principal as is shown in DYMENT (1976).

In DYMENT (1976) it is also remarked that if B is infinite and meet-irreducible then $\{A: B<A\}$ is a non principal filter. To this we add that if B is in addition join-irreducible (for instance if $B=B_f$ for some non recursive function f) then $\{A: B<A\}$ is prime. Also, if B is join-irreducible then $\{A: B \leq A\}$ is a prime principal filter; if B is meet-irreducible then $\{A: A \leq B\}$ is a prime principal ideal.

To the above list we add the filter $E=\{A: (\exists$ non r.e. $A_0)\dots(\exists$ non r.e. $A_n)$ $[E_{A_0} \wedge \dots \wedge E_{A_n} \leq A]\}$, i.e. E is the filter generated by the degrees of enumerability of sets whose e-degree is greater than o_e.

Remark 1.1 E is not principal.

Proof. Given $E_{A_0}, \dots, E_{A_n}$ where $(\forall i,j \leq n)$ $[i \neq j \Longrightarrow A_i |_e A_j]$, let A be a subset of ω such that $\emptyset <_e A <_e A_0$ (see e.g COOPER (1982)); then (using the fact that every degree of enumerability is meet-irreducible: see Chapter I, Corollary 4.1), we have that $E_A \wedge E_{A_1} \wedge \ldots \wedge E_{A_n} < E_{A_0} \wedge \ldots \wedge E_{A_n}$; on the other hand, if E were principal then it would be generated by a degree of difficulty having the form $E_{A_0} \wedge \ldots \wedge E_{A_n}$ for some $A_0, \ldots, A_n \subseteq \omega$, thus the remark is proved. Another way to prove this remark is to choose a set A such that $(\forall i \leq n)[A_i |_e A]$; in this case $E_A \wedge E_{A_0} \wedge \ldots \wedge E_{A_n} < E_{A_0} \wedge \ldots \wedge E_{A_n}$].

We now calculate the cardinality of the quotient lattice $\mathcal{M}/_G$ for $G \in \{F, Di, C, B, E\}$.
Clearly, if G_1, G_2 are filters and $G_1 \subseteq G_2$ then there exists a lattice-theoretic epimorphism $\mathcal{M}/_{G_1} \longrightarrow \mathcal{M}/_{G_2}$, hence $|\mathcal{M}/_{G_1}| \geq |\mathcal{M}/_{G_2}|$.

Lemma 1.1 $|\mathcal{M}/_B| = 2^{2^{\aleph_0}}$.

Proof. Let $\{f_{(x,y)}: (x,y) \in \mathbb{R}^2\}$ be a set of functions, indexed with the pairs of real numbers, such that $(\forall (x,y),(u,v) \in \mathbb{R}^2)[[f_{(x,y)}]_T$ is minimal in $\mathcal{D}_T$ & $(x,y) \neq (u,v) \Longrightarrow f_{(x,y)} |_T f_{(u,v)}\}$. For every $A \subseteq \mathbb{R}$, let $\mathcal{A}_A = \{g: (\exists x \in A)(\exists y \in \mathbb{R})[f_{(x,y)} \leq_T g]\}$. $\mathcal{A}_A$ contains $2^{\aleph_0}$ T-incomparable T-minimal functions and if A,B are two antisymmetric subsets of $\mathbb{R}$ such that $A \neq B$ then $\mathcal{A}_A | \mathcal{A}_B$. Notice also that, for every mass problem $\mathcal{C}$, if $\mathcal{A}_A \wedge \mathcal{C} \leq \mathcal{A}_B$ then $\mathcal{C} \leq \mathcal{A}_B$ (this is so since $\mathcal{A}_A | \mathcal{A}_B$ and $\mathcal{A}_B$ is meet-irreducible, being uniform). Suppose now that $\mathcal{C}$ is a mass problem such that $\mathcal{C} \leq \mathfrak{a}_B$ via some recursive operator . Then $(\forall x \in B)(\forall y \in \mathbb{R})[\Psi(f_{(x,y)}) \in \mathcal{C}]$; but $\Psi(f_{(x,y)}) \equiv_T f_{(x,y)}$ since $f_{(x,y)}$ belongs to a minimal T-degree and for no non-recursive $g \in {}^{\omega}\omega$ can we have $g <_T f_{(x,y)}$; hence $\mathcal{C}$ has non countable basis. Therefore, if $A, B \subseteq \mathbb{R}$, $A \neq B$ are antisymmetric then $[[\mathfrak{a}_A]]_B \neq [[\mathfrak{a}_B]]_B$. Since there are $2^{2^{\aleph_0}}$ antisymmetric subsets of $\mathbb{R}$, we are able to conclude that $\mathfrak{M}/_B$ has an antichain of cardinality 2^2 .

Corollary 1.2 The cardinality of the sublattice of dense degrees has cardinality $2^{2^{\aleph_0}}$.

Proof. Immediate, since each $\mathfrak{a}_A$ in the proof of the previous lemma is dense in the Baire topology.

Corollary 1.3 For every $G \in \{F, Di, C, B\}$, the cardinality of $\mathfrak{M}/_G$ is $2^{2^{\aleph_0}}$.

Proof. It is known that $F \subseteq Di \subseteq C \subseteq B$, where all inclusions are proper (see DYMENT (1976)). Then the corollary follows from the observation preceding Lemma 1.1.

The cardinality of $\mathfrak{M}/_E$ will be calculated in Proposition 1.4.

We have already encountered several examples of proper filters and ideals, both principal and non

principal. It should be remarked that all the lattices considered so far (except for $\mathcal{M}_{w}^{-}$) have exactly one maximal ideal and one maximal filter, since in each case the least element is meet-irreducible and the greatest element is join-irreducible.
Notice that in $\mathcal{M}_{w}^{-}$ the filter generated by B_f (we omit the subscript $_w$) is maximal if and only if $[f]_T$ is minimal!

Proposition 1.3 The ideal De is prime.

Proof. Suppose that $\mathfrak{A}_0 \wedge \mathfrak{A}_1 \leq \mathfrak{A}$ (via the recursive operator Ψ) and $\mathfrak{A}$ is dense and $\mathfrak{A}_1 \nleq \mathfrak{A}$; hence $(\exists f_0 \in \mathfrak{A})[\Psi(f_0) \in 0 * \mathfrak{A}_0]$: fix such a function f_0. We shall show that $(\exists \mathcal{E})[\mathcal{E}$ is dense & $\mathfrak{A}_0 \leq \mathcal{E}]$; thus either $\mathfrak{A}_1 \leq \mathfrak{A}$, and in this case $\mathfrak{A}_1 \in$De since $\mathfrak{A}$ is dense, or $(\exists$ dense $\mathcal{E})[\mathfrak{A}_0 \leq \mathcal{E}]$ and in this case $\mathfrak{A}_0 \in$De and the proposition is proved. Let $A=\{\tilde{f}: \Psi(\tilde{f})(0)=0\}$; clearly A is an r.e. set of initial segments and $(\exists \tilde{f}_0 \in A)[\tilde{f}_0 \subseteq f_0]$. Let $\mathcal{E}=\{\tilde{g}*f: \tilde{g}$ is a finite initial segment & $f \in \mathfrak{A}$ & $(\exists \tilde{f} \in A)[\tilde{f} \subseteq f]\}$. $\mathcal{E}$ is dense, since $(\forall \tilde{g})[\tilde{g} \subseteq \tilde{g}*f_0]$ and $\tilde{g}*f_0 \in \mathcal{E}$. Let $e \in \omega$ be such that $(\forall f \in {}^{\omega}\omega)[\Psi(f)=\Psi_e(f)]$ (we are applying here the Fundamental Operator Theorem); denote by S the set of all finite initial segments and let $\{A^s: s \in \omega\}$, $\{S^s: s \in \omega\}$ be finite recursive approximation to A and S respectively. Define

$W=\{<<x,y>,u>: \tau^{-1}(D_u)$ is a finite initial segment &

$(\exists s)(\exists \tilde{g}\in S^s)(\exists \tilde{f}\in A^s)[\tilde{g}*\tilde{f} \subseteq \tau^{-1}(D_u)$ & $(\forall \tilde{l}\in S^s)(\forall \tilde{m}\in S^s)$ $[lh(\tilde{l}*\tilde{m})<lh(\tau^{-1}(D_u))]$ & $(\forall t<s)(\forall \tilde{l}\in S^t)(\forall \tilde{m}\in A^t)[\tilde{l}*\tilde{m} \not\subseteq$ $\tau^{-1}(D_u)]$ & $(\forall \tilde{l}\in S^s)[lh(\tilde{l})<lh(\tilde{g}) \Longrightarrow (\forall \tilde{m}\in A^s)$ $[\tilde{l}*\tilde{m} \not\subseteq$ $\tau^{-1}(D_u)]$ & $(\exists z)[\langle\langle x,y\rangle,z\rangle\in W_e$ & $\{(a,b):$ $a<lh(\tilde{g})$ & $\langle a,b\rangle\in D_z\} \subseteq \tilde{g}$ & $\{\langle a,b\rangle: \langle a+lh(\tilde{g}),b\rangle\in D_z\} \subseteq D_u]]\}$.

Let n be an r.e. index for W and let Ψ_n be the recursive operator defined by the enumeration operator $\Phi_{\sigma(n)}$ (we are applying again tha Fundamental Operator Theorem). It is easy to see that, for every <u>total</u> function h, $\Psi_n(h)= \Psi(\lambda x.h(lh(\tilde{g})+x))$, where $\tilde{g}$ is the finite initial segment obtained in the following way: search for the first step s such that there exist $\tilde{l}\in S^s$ and $\tilde{m}\in A^s$ such that $\tilde{l}*\tilde{m} \subseteq h$; let $\tilde{g}$ be the finite initial segment such that $(\exists \tilde{m}\in A^s)[\tilde{g}*\tilde{m} \subseteq h]$ and $(\forall \tilde{l}\in S^s)$ $[lh(\tilde{l})$ $<lh(\tilde{g}) \Longrightarrow (\forall \tilde{m}\in A^s)[\tilde{l}*\tilde{m} \not\subseteq h]]$.

Therefore, $\Omega_0 \leq \mathcal{E}$ via Ψ_n: this ends the proof.

Corollary 1.4 $(\forall \text{dense } \Omega)(\forall \Omega_0)(\forall \Omega_1)[\Omega_0 \wedge \Omega_1 \leq \Omega$ & $[\Omega_1$ discrete or Ω_1 closed$] \Longrightarrow \Omega_0 \leq \Omega\ ]$.

Proof If Ω_1 is discrete or closed, then for no dense $\mathcal{E}$ can we have $\Omega_1 \leq \mathcal{E}$ (see Chapter I, Fact 7.3 and Corollary 7.3).

We are now going to prove some results which have as corollary that none of the filters F,Di,C,B,E is prime.

Theorem 1.2 Let $A,D \subseteq \omega$, $A \leq_e D$. Then there exist B_0, B_1 such that $[B_0]_e$, $[B_1]_e$ are quasiminimal and $[A]_e = [D]_e \wedge ([B_0]_e \vee [B_1]_e)$.

Proof. Let A,D be such that $A \leq_e D$. We want to find B_0, B_1 satisfying the following requirements for all $e,s \in \omega$ and $i \in \{0,1\}$, where $(\Phi_s, \theta_s)_{s \in \omega}$ is an enumeration of all pairs of enumeration operators:

R_s: $\Phi_s(D) = \theta_s(B_0 \vee B_1) \implies \Phi_s(D) \leq_e A$

$P_{<e,i>}$: $B_i \neq W_e$

$Q_{<e,i>}$: $\Phi_e(B_i)$ total $\implies \Phi_e(B_i)$ r.e.

S: $A \leq_e B_0 \vee B_1$

Throughout this proof, β, α will be variables over the set of finite initial segments of 0-1 valued functions. We shall define two increasing sequences $\{\beta_{0,s}\}_{s \in \omega}$, $\{\beta_{1,s}\}_{s \in \omega}$ of finite initial segments of 0-1 valued functions and show that the sets $B_0 = \bigcup \{\beta_{0,s}^+ : s \in \omega\}$, $B_1 = \bigcup \{\beta_{1,s}^+ : s \in \omega\}$ satisfy the requirements (for every β, β^+ denotes the set $\{x: x < lh(\beta) \ \& \ \beta(x)=1\}$).

We say that a pair (β,α) is <u>compatible</u> if $lh(\beta)=lh(\alpha)$ and $(\forall <u,x> < lh(\beta))[\beta(<u,x>)=1 \ \& \ \alpha(<u,x>)=1 \implies x \in A]$.

Since the set $\{\beta$: β is a finite initial segment of a 0-1 valued function$\}$ is linearly ordered by the ordering defined in the Introduction, we refer in this proof to the lexicagraphical ordering induced on the set of pairs $\{(\beta,\alpha)$: β, α are finite initial segments of 0-1 valued functions$\}$.

Finally, for the sake of semplicity, in this proof we occasionally identify a set with its characteristic function (as we do, for instance, in Step 4s+2), Case a) where $W_e(x)$ denotes the value of the characteristic function of W_e on the input x).

In the construction below, at Step s) we define $\beta_{0,s}$ and $\beta_{1,s}$.

Step 0) $\beta_{0,0}=\beta_{1,0}=\emptyset$.

Step 4s+1)

Case a) $(\exists x)(\exists \beta)(\exists \alpha)[\beta_{0,4s} \subseteq \beta$ & $\beta_{1,4s} \subseteq \alpha$ & (β,α) is compatible & $x \notin \Phi_s(D)$ & $x\in\Theta_s(\beta^+ \vee \alpha^+)]$

Then let (β,α) be the least pair with this property and define $\beta_{0,4s+1}=\beta$, $\beta_{1,4s+1}=\alpha$.

Case b) Otherwise, put $\beta_{i,4s+1}=\beta_{i,4s}$ for each $i\in\{0,1\}$.

Step 4s+2)

Case a) $s=<e,i>$ & $i\in\{0,1\}$.

Let $x=lh(\beta_{i,4s+1})$ and define $\beta_{0,4s+2}$, $\beta_{1,4s+2}$ in such a way that they both have length x+1 and $\beta_{i,4s+2}(x) \neq W_e(x)=\beta_{1-i,4s+2}(x)$.

Case b) $s=<e,i>$ & $i \notin \{0,1\}$.

In this case let $\beta_{0,4s+2}=\beta_{0,4s+1}$ and $\beta_{1,4s+2}=\beta_{1,4s+1}$.

Step 4s+3)

Case a) $s=<e,i>$ & $i\in\{0,1\}$. We distinguish two subcases;

Subcase a) $(\exists \beta \supseteq \beta_{i,4s+2})(\exists x)(\exists y)(\exists z)[<x,y>\in\Phi_e(\beta^+)$ & $<x,z>\in\Phi_e(\beta^+)$ & $y \neq z]$.

Let β be the least finite initial segment of 0-1 valued functions having this property; define $\beta_{i,4s+3}=\beta$ and

$$\beta_{1-i,4s+3}(x)=\begin{cases}\beta_{1-i,4s+2}(x) & \text{if } x<lh(\beta_{1-i,4s+2})\\ 1-\beta(x) & \text{if } lh(\beta_{1-i,4s+2})\leq x<lh(\beta)\\ \uparrow & \text{if } x\geq lh(\beta)\end{cases}$$

Subcase b) Otherwise, let $\beta_{0,4s+3}=\beta_{0,4s+2}$ and $\beta_{1,4s+3}$ $=\beta_{1,4s+2}$.

Case b) If $s=<e,i>$ & $i\notin\{0,1\}$, let $\beta_{0,4s+3}=\beta_{0,4s+2}$ and $\beta_{1,4s+3}=\beta_{1,4s+2}$.

Step 4s+4)

Case a) $s\in A$.

In this case let $u=\mu u.[<u,s>\geq lh(\beta_{i,4s+3})]$. Define $\beta_{o,4s+4}$, $\beta_{1,4s+4}$ in such a way that, for every $i\in\{0,1\}$, $lh(\beta_{i,4s+4})=<u,s>+1$, $\beta_{i,4s+4}(<u,s>)=1$ and $(\forall x)$ $[lh(\beta_{i,4s+3})\leq x<<u,s>\Longrightarrow\beta_{i,4s+4}(x)\neq\beta_{1-i,4s+4}]$.

Case b) if $s\notin A$, then let $\beta_{0,4s+4}=\beta_{o,4s+3}$ and $\beta_{1,4s+4}=\beta_{4s+3}$.

We have only to check that the construction works. It easily follows by induction that for every s, the pair $(\beta_{0,s}, \beta_{1,s})$ is compatible.

-Suppose that for some $s\in\omega$, $\Phi_s(D)=\theta_s(B_0\vee B_1)$; then at Step 4s+1), Case a) does not hold. Hence $\Phi_s(D)=\{x:$ $(\exists\beta\supseteq\beta_{0,4s})$ $(\exists\alpha\supseteq\beta_{1,4s})[(\beta,\alpha)$ is compatible & $x\in\theta_s(\beta^+\vee\alpha^+)]\}$: indeed, the inclusion follows from the fact that for every $t \in \omega$, the pair $(\beta_{0,t}, \beta_{1,t})$ is compatible; the other inclusion is trivial. But then $\Phi_s(D) \leq_e A$, as desired.

-Step 4s+2) ensures that the requirement $P_{<(s)_0, (s)_1>}$ is met, if $(s)_1 \in \{0,1\}$.

-Step 4s+3) ensures thst $Q_{<(s)_0, (s)_1>}$ is met, if $(s)_1 \in \{0,1\}$. Indeed, let $s=<e,i>$, where $i \in \{0,1\}$ and suppose that $\Phi_e(B_i)$ is total. Then $\Phi_e(B_i)=\{x: (\exists \beta \supseteq \beta_{i,4s+2})[x \in \Phi_e(\beta^+)]\}$: the inclusion $\subseteq$ is immediate; on the other hand, let $<x,z> \in \Phi_e(\beta^+)$, where $\beta \supseteq \beta_{i,4s+2}$, and suppose that $<x,z> \notin \Phi_e(B_i)$; since $\Phi_e(B_i)$ is total, there exists $y \neq z$ such that $<x,y> \in \Phi_e(B_i)$: in this case, for some $\beta' \supseteq \beta_{i,4s+2}$ we must have that $<x,y> \in \Phi_e((\beta')^+)$. Define

$$\alpha(x) = \begin{cases} 1 & \text{if } [\beta(x)\downarrow \ \& \ \beta(x)=1] \text{ or } [\beta'(x)\downarrow \ \& \ \beta'(x)=1] \\ 0 & \text{if } \quad x<\max\{lh(\beta), lh(\beta')\} \ \& \ [\beta(x)\downarrow \Longrightarrow \beta(x)=0 \ \& \\ & \qquad \beta'(x)\downarrow \Longrightarrow \beta'(x)=0]] \\ \uparrow & \qquad \text{otherwise} \end{cases}$$

Clearly, we have $\alpha \supseteq \beta_{i,4s+2}$ and $<x,y>$, $<x,z> \in \Phi_e(\alpha^+)$: since $y \neq z$, Case a) of Step 4s+3) applies and the construction ensures that $\Phi_e(B_i)$ is not single-valued, contradicting the assumption that $\Phi_e(B_i)$ is total. Therefore if $\Phi_e(B_i)$ is total then it equals the above set, thus $\Phi_e(B_i)$ is r.e., as desired.

-Finally, $s \in A \Longleftrightarrow (\exists u)[\langle u,s\rangle \in B_0$ & $\langle u,s\rangle \in B_1]$; hence $A \leq_e B_0 \vee B_1$. This ends the proof.

Corollary 1.5 $(\forall A \subseteq \omega)(\exists B_0, B_1 \subseteq \omega)[B_0$, B_1 belong to quasiminimal e-degrees & $A \leq_e B_0 \vee B_1]$.

Proof. Let A=D in Theorem 1.1.

Following JOCKUSH-POSNER (1981), we may notice the following corollary:

Corollary 1.6 $(\forall a_e \in \mathcal{D}_e)(\exists$ quasiminimal $(b_0)_e$, $(b_1)_e$, $(b_2)_e$, $(b_3)_e)[a_e = ((b_0)_e \vee (b_1)_e) \wedge ((b_2)_e \vee (b_3)_e)]$.

Proof. Let $a_e = [A]_e$ be given. First find B_0, B_1 such that $[B_0]_e$, $[B_1]_e$ are quasiminimal and $A \leq_e B_0 \vee B_1$ (see Corollary 1.5) and then apply Theorem1.1 taking $D = B_0 \vee B_1$.

Corollary 1.7 The filters F, Di, C are not prime.

Proof. Fix a non-recursive function f. Then, by Corollary 1.5, there exist B_0, B_1, such that $[B_0]_e$, $[B_1]_e$ are quasiminimal and $f \leq_e B_0 \vee B_1$. Thus $[\{f\}] \leq \mathcal{E}_{B_0} \vee \mathcal{E}_{B_1}$: this means that $\mathcal{E}_{B_0} \vee \mathcal{E}_{B_1} \in F$. On the other hand suppose that, for some countable mass problem $\mathcal{C}$ we have $\mathcal{C} \leq \mathcal{E}_{B_i}$ (for some $i \in \{0,1\}$). Then by Theorem 3.4 of Dyment's paper (DYMENT (1976)) there must exist $g \in \mathcal{C}$, such that $g \leq_e B_i$, a contradiction since B_i belongs to a quasiminimal e-degree. Therefore $\mathcal{E}_{B_i} \notin C$. The proof is now complete since $F \subseteq Di \subseteq C$.

To show that B is not prime, we introduce some notation and terminology (we follow LERMAN (1983)).

Let Σ be the set of all finite initial segments of 0-1 valued functions; let ρ, σ, τ be variables over this set.

A tree is a function $T: \Sigma \longrightarrow \Sigma$ satisfying:

$\sigma \subseteq \tau \implies T(\sigma) \subseteq T(\tau)$;

$\sigma | \tau \implies T(\sigma) | T(\tau)$.

Let us denote by Tr the set of all trees. A tree T is recursive if T is a recursive function. We work here only with recursive trees: for the sake of semplicity, we shall drop the word "recursive". Id denotes the identity tree $Id: \Sigma \longrightarrow \Sigma$ (i.e. $Id(\sigma)=\sigma$ for all $\sigma \in \Sigma$). Given $\sigma \in \Sigma$ and a tree T, we say that σ is on T (notation: $\sigma \subset T$) if $\sigma \in \mathrm{range}(T)$. A set A is on T (notation: $A \subset T$) if, for infinitely many σ on T, $\sigma \subseteq c_A$. We say that T_1 is a subtree of T_2 (notation: $T_1 \subseteq T_2$) if $\mathrm{range}(T_1) \subseteq \mathrm{range}(T_2)$. If $\sigma \in \Sigma$ and T is a tree, then T^σ denotes the tree defined by $T^\sigma(\tau)=T(\sigma * \tau)$.

Given $\sigma, \tau \in \Sigma$, we say that (σ, τ) is strongly uniform if $lh(\sigma)=lh(\tau)$ and

$|\{x<lh(\sigma): \sigma(x) \neq \tau(x)\}|=1$.

A tree T is strongly uniform (notation: s.u.) if $(\forall \sigma, \tau \in \Sigma)[lh(\sigma)=lh(\tau) \implies lh(T(\sigma))=lh(T(\tau))]$ and $(\forall \rho, \sigma, \tau \in \Sigma)[T(\rho * 0)=T(\rho) * \sigma \quad \& \quad T(\rho * 1)=T(\rho) * \tau \implies (\sigma, \tau)$

strongly uniform].

Clearly, Id is s.u.; if T is s.u. then $(\forall\sigma\in\Sigma)[T^{\sigma}$ is s.u.].

If $\sigma,\tau\in\Sigma$ and $lh(\sigma)=lh(\tau)$ then let $Z(\sigma,\tau)=\{x<lh(\sigma): \sigma(x)\neq\tau(x)\}$ and let $\{z_i: i\leq k\}$ be an enumeration of $Z(\sigma,\tau)$ in order of magnitude. For a given set $A\subseteq\omega$, we say that the pair (σ,τ) is <u>A-consistent</u> if $(\forall i\leq k)[i\in A \Longleftrightarrow \sigma(z_i)=1]$. If $M_1,M_2\subseteq\omega$, then (M_1,M_2) is <u>A-consisten</u> if $(\forall\sigma,\tau\in\Sigma)[\sigma\subsetneq M_0$ & $\tau\subsetneq M_1$ & $lh(\sigma)= lh\tau)$ $\Longrightarrow (\sigma,\tau)$ A-consistent].

Finally, let $T_0,T_1\in Tr$. Then we say that (T_0,T_1) is <u>A-acceptable</u> if

1) $(T(\emptyset),T_1(\emptyset))$ is A-consistent;

2) T_0 and T_1 are strongly uniform;

3) for every $\sigma\in\Sigma$,

$$T_1(\sigma)(x)=\begin{cases} T_1(\emptyset)(x) & \text{if } x<lh(T(\emptyset)) \\ \\ T_0(\sigma)(x) & \text{if } lh(T_1(\emptyset))\leq x<lh(T_1(\sigma)) \end{cases}$$

In JOCKUSH-POSNER (1981) we find the following result: for every set $A\subseteq\omega$ there exist two sets $M_0,M_1\subseteq\omega$ such that, for all $e\in\omega$ and $i\in\{0,1\}$, the following requirements are met:

$Q_{<e,i>}$: $\varphi_e^{M_i}$ total $\Longrightarrow$ [$\varphi_e^{M_i}$ recursive or $M_i\leq_T\varphi_e^{M_i}$];

$P_{<e,i>}$: $M_i\neq\varphi_e^{\emptyset}$;

S_e: $e \in A \iff z_e \in M_0$ (where $\{z_n: n \in \omega\}$ is an enumeration in order of magnitude of $\{z: M_0(z) \neq M_1(z)\}$.

These requirements ensure that $[M_0]_T$, $[M_1]_T$ are T-minimal and $A \leq_T M_0 \vee M_1$.

Let $\mathcal{R} = \{Q_{<e,i>}, P_{<e,i>}, S_e: e \in \omega \ \& \ i \in \{0,1\}\}$ and let $\{R_m: m \in \omega\}$ be an enumeration of $\mathcal{R}$. In JOCKUSH-POSNER (1981) the following lemma is proved:

Lemma 1.2 For every A-acceptable pair of trees (T_0, T_1) and for every $m \in \omega$, there exists an A-acceptable pair of trees (T_0', T_1') such that $T_0' \subseteq T_0$, $T_1' \subseteq T_1$ and (T_0', T_1') satisfies R_m (i.e. $(\forall M_0$ on $T_0')(\forall M_1$ on $T_1')$[the pair (M_0, M_1) satisfies R_m]).

Theorem 1.3 For every $a_T \in \mathcal{D}_T$ there are two disjoint families of cardinality $2^{\aleph_0}$ of minimal T-degrees $\{(m_r^0)_T: r \in R\}$, $\{(m_s^1)_T: s \in S\}$ such that $(\forall r \in R)(\forall s \in S)[a_T \leq_T (m_r^0)_T \vee (m_s^1)_T]$.

Proof. Let a_T be given and let $A \in a_T$. The argument is the usual tree of trees argument. Define a function π: $\Sigma \times \Sigma \longrightarrow Tr \times Tr$ by induction on the number $n = lh(\sigma) + lh(\tau)$ as follows:

<u>Step 0)</u> $\pi(\emptyset, \emptyset) = (Id, Id)$;

notice that (Id, Id) is A-acceptable.

<u>Step n+1)</u> Suppose that $\pi(\sigma,\tau)$ is defined on every pair (σ,τ) such that $lh(\sigma)+lh(\tau)\le n$; for each such pair (σ,τ) let $\pi(\sigma,\tau)=(T^0_{\sigma,\tau},T^1_{\sigma,\tau})$ (thus, for instance, $(T^0_{\emptyset,\emptyset},T^1_{\emptyset,\emptyset})=(Id,Id)$) and suppose that $(T^0_{\sigma,\tau},T^1_{\sigma,\tau})$ is A-acceptable; in particular $T^0_{\sigma,\tau}$, $T^1_{\sigma,\tau}$ are strongly uniform.

Suppose now that (σ,τ) is a pair such that $lh(\sigma)+lh(\tau)=n+1$; we want first to define $\pi(\sigma*i,\tau)$, for each $i\in\{0,1\}$.

Let T_0 be the tree defined by $T_0(\rho)=T^0_{\sigma,\tau}(i*\rho)$; thus $T_0\subsetneq T^0_{\sigma,\tau}$ and therefore T_0 is s.u. Define also T_1 by

$$T_1(\rho)(x)=\begin{cases} T^1_{\sigma,\tau}(\emptyset)(x) & \text{if } x<lh(T^1_{\sigma,\tau}(\emptyset)) \\ T_0(\rho)(x) & \text{if } lh(T^1_{\sigma,\tau}(\emptyset))\le x<lh(T_0(\rho)) \end{cases}$$

We have therefore that also $T_1\subsetneq T^1_{\sigma,\tau}$ and it is easily seen that (T_0,T_1) is A-acceptable.

By Lemma 1.2, let now $(T^0_{\sigma*i,\tau},T^1_{\sigma*i,\tau})$ be an A-acceptable pair satisfying R_n and $T^0_{\sigma*i,\tau}\subsetneq T_0$, $T^1_{\sigma*i,\tau}\subsetneq T_1$. Then define

$\pi(\sigma*i,\tau)=(T^0_{\sigma*i,\tau},T^1_{\sigma*i,\tau})$.

$\pi(\sigma,\tau*i)$ is defined in a similar fashion.

Now, for every $C\subseteq\omega$ and $i\in\{0,1\}$ let $\mathcal{G}^i_C=\{T\in Tr\colon (\exists\sigma\subseteq C)(\exists\tau\subseteq C)[T^i_{\sigma,\tau}\subseteq T]$ and let $M^C_i\in\bigcap\mathcal{G}^i_C$ (this notation is consistent with the one chosen when we have defined "A on T"). Thus, for every $C,D\subseteq\omega$, $A\le_T M^C_0\vee M^D_1$ and $C\ne C'\Rightarrow M^C_i = M^{C'}_i$. By usual cardinality arguments (since every T-degree contains only denumerably many sets) there exist two disjoint families $\{(m_r^0)_T: r\in R\}$, $\{(m_s^1)_T: s\in S\}$, each of them having cardinality $2^{\aleph_0}$ and such that $(\forall r\in R)(\forall s\in S)[a_T \le_T (m_r^0)_T \vee (m_s^1)_T]$.

Corollary 1.8 The filter B is not prime.

Proof. Fix a non-recursive function h. By Theorem 1.3 there exist two antichians of $\mathcal{D}_T$ $\{[f_r]_T: r\in R\}$, $\{[g_s]_T: s\in S\}$ having cardinality $2^{\aleph_0}$ and such that $(\forall r\in R)(\forall s\in S)$ $[[f_r]_T$ is T-minimal & $[g_s]_T$ is T-minimal & $h\le_T f_r \vee g_s]$.

Let $\mathcal{A}=\{f: h\le_T f\}$, $\mathcal{A}_1=\{f_r: r\in R\}$ and $\mathcal{A}_2=\{g_s: s\in S\}$. Then $\mathcal{A}_1 \vee \mathcal{A}_2 \subseteq \mathcal{A}$ and thus $\mathcal{A} \le \mathcal{A}_1 \vee \mathcal{A}_2$. Of course $[\mathcal{A}]\in B$; nevertheless neither $[\mathcal{A}_1]$ nor $[\mathcal{A}_2]$ belongs to B. If, for instance, $\mathcal{C}$ has at most countable generalized basis and we suppose that $\mathcal{C} \le \mathcal{A}_1$ via some recursive operator Ψ then, since $(\forall r\in R)[\Psi(f_r)\equiv_T f_r$ & $(\forall g\in \mathcal{C})[g \not\le_T f_r]]$, we get a contradiction in view of the cardinality of the set $\{[f_r]_T: r\in R\}$. Analogously, $\mathcal{C} \not\le \mathcal{A}_2$.

A consequence of Corollary 1.7 and Corollary 1.8 is that the greatest element of the lattices $\mathfrak{M}/_F$, $\mathfrak{M}/_C$, $\mathfrak{M}/_B$ is join-reducible. The proof of Corollary 1.7 shows also that the filter of $\mathfrak{M}_w$ generated by the non-zero weak degrees of solvability is not prime: let F_w denote this filter; then the greatest element of $\mathfrak{M}_w/_{F_w}$ is join-reducible, in particular the Heyting algebra $\mathcal{M}_{w/F_w}$ is subdirectly reducible (see Fact 1.1).

Proposition 1.4 $\left|\mathcal{M}/_E\right| = 2^{2^{\aleph_0}}$.

Proof. Fix a non-recursive function f. First, we prove that there exists a family (having cardinality $2^{\aleph_0}$) of functions $\{g_i: i\in \mathbb{R}\}$ (we index this family using the real numbers) such that $(\forall i,j\in \mathbb{R})[f\,|_T\, g_i$ & (f,g_i) is a minimal pair in the e-degrees & $i \neq j \implies g_i\,|_T\, g_j]$.

Let $\{\Phi_e,\theta_e\}_{e\in\omega}$ be an enumeration of all pairs of enumeration operators.

Lemma For every recursive tree T and every $e\in\omega$ there exists a recursive tree $T'\subseteq T$ such that, for every $A \subset T'$, if $\Phi_e(f)=\theta_e(c_a)$ then $\Phi_e(A)$ is r.e.

Proof. Let T be a recursive tree:

Case a) $(\exists x)(\exists \sigma\in\Sigma)[x \notin \Phi(f)$ & $x\in\theta_e(T(\sigma))]$.

In this case, let $T'=T^\sigma$ for the least σ for which Case a) holds.

Case b) Otherwise: let $T'=T$.

Clearly in both cases T' is recursive. Suppose now that $\Phi_e(f)=\theta_e(c_A)$ for some $A \subset T'$; then $\Phi_e(f)=\{x: (\exists\sigma\in\Sigma)[x\in\theta_e(T(\sigma))\}$. Indeed, the inclusion $\subseteq$ is obvious; on the other hand, let $x\in\theta_e(T(\sigma))$: then, since Case a) does not hold, $x\in\Phi_e(f)$, as desired. This ends the proof of the lemma.

Let us now return to the proof of Proposition 1.4. Define a function $\pi: \Sigma \longrightarrow Tr$ by induction as follows:

Step 0) $\pi(\emptyset)=Id$;

Step e+1) Suppose that $(\forall\sigma\in\Sigma)[lh(\sigma)\leq e \Rightarrow \pi(\sigma)$ has already been defined & $\pi(\sigma)$ is a recursive tree]; in order to define $\pi(\sigma*i)$, for $i\in\{0,1\}$, let $\pi(\sigma*i)$ be a recursive subtree $T' \subseteq (\pi(\sigma))^{\sigma*i}$ such that $(\forall A \subset T')[\Phi_e(f)=\theta_e(c_A) \Rightarrow \Phi_e(f)$ r.e.] (use the lemma). For every set $B \subseteq \omega$, let now $\mathcal{G}_B=\{T\in Tr: (\exists\sigma \subseteq c_B)[\pi(\sigma) \subset T]\}$, take $A_B \in \cap \mathcal{G}_B$ and let $g_B=c_{A_B}$. We have thus obtained a family $\{g_B: B \subseteq \omega\}$ of cardinality $2^{\aleph_0}$ such that $(\forall B,C \subseteq \omega)[B \neq C \Rightarrow g_B \neq g_C$ & $([f]_e,[g_B]_e)$ is a minimal pair in the e-degrees]. By cardinality, since each T-degree contains only denumerably many sets, we get a family $\{g_i: i\in \mathbb{R}\}$ such that $(\forall i,j\in \mathbb{R})[f|_T\, g_i$ & (f,g_i) is a minimal pair in the e-degrees & $i \neq j \Rightarrow g_i\,|_T\, g_j]$, as desired.

Now, for every antisymmetric $A \subseteq \mathbb{R}$ (for terminology, see Chapter I, Theorem 3.2) define $\mathcal{A}_A=\{g: f\leq_T g$ or $(\exists i\in A)[g_i\leq_T g]\}$. Let $A,B \subseteq \mathbb{R}$ be antisymmetric, $A \neq B$, and suppose that, for some $C_0,\ldots,C_n \subseteq \omega$, $\mathcal{A}_A \wedge (\mathcal{E}_{C_0}\wedge\ldots\wedge\mathcal{E}_{C_n}) \leq \mathcal{A}_B$ (we may assume that for every $j\leq n$ $\emptyset<_e C_j$). Since $\mathcal{A}_B$ belongs to a meet-irreducible degree of difficulty ($\mathcal{A}_B$ being uniform, see Chapter I, Lemma 4.1) and $\mathcal{A}_A \not\leq \mathcal{A}_B$ we must have, for some $j\leq n$, $\mathcal{E}_{C_j} \leq \mathcal{A}_B$; if $i\in B$ then this implies $C_j\leq_e f$ and $C_j\leq_e g_i$ (since for every set A and total function h,

$A \le_T h \Longrightarrow A \le_e h$); hence $\emptyset \equiv_e C_j$, a contradiction. Thus $[[\mathcal{a}_A]]_E \mid [[\mathcal{a}_B]]_E$ and, as in Lemma 1.1, $\mathcal{M}/_E$ has an antichain of cardinality $2^{2^{\aleph_0}}$.

Theorem 1.4 For every e-degree $a_e >_e o_e$ there exists a countable set $\{(a_i)_e: i \in \omega\}$ of e-degrees such that $(\forall i)(\forall j)[a_e \not\le_e (a_i)_e$ & $i \ne j \Longrightarrow [a_e \le_e (a_i)_e \vee (a_j)_e$ & $((a_i)_e, (a_j)_e)$ is a minimal pair in the e-degrees].

Proof. Let $a_e >_e o_e$ be given and let $A \in a_e$. We want to define a family $\{A_i: i \in \omega\}$ of subsets of ω such that the family $\{[A_i]_e: i \in \omega\}$ satisfies the theorem.

Throughout this proof α, β, τ denote variables over the set of finite initial segments of 0-1 valued functions. For every $x,y,z \in \omega$, $<x,y,z>$ denotes $<<x,y>,z>$. As usual, $(\Phi_e, \theta_e)_{e \in \omega}$ is an enumeration of all pairs of enumeration operators.

For every $i \in \omega$, we will define a sequence $\{\alpha_{i,s}\}_{s \in \omega}$ of finite initial segments of 0-1 valued functions. At the end of the construction we will let $A_i = \bigcup\{\alpha^+_{i,s}: s \in \omega\}$. The construction aims to satisfy the following requirements, for every $i,j,e \in \omega$:

$P_{<i,j>}$: $A \ne \Phi_j(A_i)$;

$Q_{<i,j>}$: $A \le_e A_i \vee A_j$ (for $i \ne j$);

$R<i,j,e>$: $\Phi_e(A_i) = \theta_e(A_j) \Longrightarrow \Phi_e(A_i)$ r.e. (for $i \ne j$).

$Q_{<i,j>}$ will be satisfied by guaranteeing that $s \in A \Longleftrightarrow (\exists m)\ [<j,<m,s>> \in A_i$ & $<i,<m,s>> \in A_j]$.

Step 0) For every $i \in \omega$, let $\alpha_{i,0} = \emptyset$.

Step 4s+1) Let $s=\langle i,j\rangle$. We distinguish two cases.

Case a) $(\exists x)(\exists \alpha \supseteq \alpha_{i,4s})[x \notin A \ \& \ x \in \Phi_j(\alpha^+)]$.

In this case, let x,α be minimal with respect to this property and let $l=\max[\{lh(\alpha)\} \cup \{\langle i,y\rangle+1: (\exists h)[\alpha(\langle h,y\rangle)=1]\}]$.

Define

$$\alpha_{i,4s+1}(y)= \begin{cases} \alpha(y) & \text{if } y<lh(\alpha) \\ 0 & \text{if } lh(\alpha)\le x<l \end{cases}$$

For every $h \neq i$ define

$$\alpha_{h,4s+1}(y)= \begin{cases} \alpha_{h,4s}(y) & \text{if } y<lh(\alpha_{h,4s}) \\ 0 & \text{if } lh(\alpha_{h,4s})\le y<l \end{cases}$$

Case b) Otherwise: for every $h \in \omega$, let $\alpha_{h,4s+1}=\alpha_{h,4s}$.

Step 4s+2)

Case a) $s=\langle i,j,e\rangle \ \& \ i \neq j$.

In this case, let m be the least number such that $\langle j,\langle m,e\rangle\rangle \notin dom(\alpha_{i,4s+1}) \ \& \ \langle i,\langle m,e\rangle\rangle \notin dom(\alpha_{j,4s+1})$.

Let $l=\max\{\langle j,\langle m,e\rangle\rangle,\langle i,\langle m,e\rangle\rangle\}$.

Define

$$\alpha_{i,4s+2}(x)=\begin{cases}\alpha_{i,4s+1}(x) & \text{if } x<lh(\alpha_{i,4s+1})\\ 1 & \text{if } x=<j,<m,e>>\\ 0 & \text{if } lh(\alpha_{i,4s+1})\le x\le l \ \& \\ & \quad x \neq <j,<m,e>>\end{cases}$$

and

$$\alpha_{j,4s+2}(x)=\begin{cases}\alpha_{j,4s+1}(x) & \text{if } x<lh(\alpha_{j,4s+1})\\ 1 & \text{if } x=<j,<m,e>>\\ 0 & \text{if } lh(\alpha_{j,4s+1})\le x\le l \ \& \\ & \quad x \neq <i,<m,e>>\end{cases}$$

For every $h \notin \{i,j\}$, define

$$\alpha_{h,4s+2}(x)=\begin{cases}\alpha_{h,4s+2}(x) & \text{if } x<lh(\alpha_{h,4s+1})\\ 0 & \text{if } lh(\alpha_{h,4s+1})\le x\le l\end{cases}$$

Case b) if $s=<i,j,e>$ & $i=j$ then, for every $h\in\omega$ define $\alpha_{h,4s+2}=\alpha_{h,4s+1}$.

<u>Step 4s+3)</u>

Case a) $s=<i,j,e>$ & $i \neq j$. Let us premise the following

<u>Definition</u> If $\alpha \supseteq \alpha_{i,4s+2}$, $\beta \supseteq \alpha_{j,4s+2}$ we say that α, β are <u>compatible</u> (notation: α cpt β) if, letting $l=\max[\{lh(\alpha_{j,4s+2})\} \cup \{<i,y>+1: \alpha(<j,y>)=1\}]$, we have $lh(\beta) \geq l$ & $(\forall x)[\beta(<i,(x)_0,(x)_1>)=1$ & $\alpha(<j,(x)_0,(x)_1>) \Rightarrow (x)_1\in A$.

In Case a) we distinguish two subcases:

Subcase a_1) $(\exists \alpha \supseteq \alpha_{i,4s+2})(\exists \beta \supseteq \alpha_{j,4s+2})(\exists x)[\ \alpha$ cpt β & $x \in \Phi_e(\alpha^+)$ & $(\forall \tau \supseteq \beta)[x \notin \theta_e(\tau^+)]]$.

Let α, β be minimal with respect to this property and let $l = \max[\{lh(\beta)\} \cup \{\langle i,y\rangle+1: (\exists h)[\alpha(\langle h,y\rangle)=1]\} \cup \{\langle j,y\rangle+1: (\exists h)[\beta(\langle h,y\rangle)=1]\}]$.

Define

$$\alpha_{i,4s+3}(y) = \begin{cases} \alpha(y) & \text{if } y<lh(\alpha) \\ 0 & \text{if } lh(\alpha) \leq y < l \end{cases}$$

and

$$\alpha_{j,4s+3}(y) = \begin{cases} \beta(y) & \text{if } y<lh(\beta) \\ 0 & \text{if } lh(\beta) \leq y < l \end{cases}$$

For every $h \notin \{i,j\}$ define

$$\alpha_{h,4s+3}(y) = \begin{cases} \alpha_{h,4s+2}(y) & \text{if } x<lh(\alpha_{h,4s+2}) \\ 0 & \text{if } lh(\alpha_{h,4s+2}) \leq y < l. \end{cases}$$

- Subcase a_2) Otherwise, for every $h \in \omega$ let $\alpha_{h,4s+3} = \alpha_{h,4s+2}$.

Case b) $s=\langle i,j,e\rangle$ & $i=j$: for every $h \in \omega$ let $\alpha_{h,4s+3}=\alpha_{h,4s+2}$.

<u>Step 4s+4</u>)

Case a) $s=\langle\langle i,j,e\rangle,x\rangle$ & $i \neq j$.

Subcase a_1) $(\exists \beta \supseteq \alpha_{j,4s+3})[x \in \theta_e(\beta^+)]$.

Let β be the least such initial segment and let $l = \max[\{lh(\beta)\} \cup \{<j,y>+1 : (\exists h)[\beta(<h,y>)=1]\}$.

$$\beta_{j,4s+4}(y) = \begin{cases} \beta(y) & \text{if } y < lh(\beta) \\ 0 & \text{if } lh(\beta) \le y < l \end{cases}$$

For $h \neq j$ define

$$\alpha_{h,4s+4}(y) = \begin{cases} \alpha_{h,4s+3}(y) & \text{if } y < lh(\alpha_{h,4s+3}) \\ 0 & \text{if } lh(\alpha_{h,4s+3}) \le y < l \end{cases}$$

Subcase a_2) Otherwise, for every $h \in \omega$ define $\alpha_{h,4s+4} = \alpha_{h,4s+3}$.

Cse b). Otherwise (i.e. $i=j$) let $\alpha_{h,4s+4} = \alpha_{h,4s+3}$ for every $h \in \omega$.

We now check that the construction works.

- At step $4s+1$) we ensure that $A \neq \Phi_j(A_i)$ (if $s=<i,j>$): indeed, if we suppose that $A=\Phi_j(A_i)$ then we are led to the conclusion $A = \{x : (\exists \alpha \supseteq \alpha_{i,4s})[x \in \Phi_j(\alpha^+)]\}$; thus A would be r.e., a contradiction.

- Step $4s+2$), if $s=<i,j,e>$ and $i \neq j$, ensures that $e \in A \Leftrightarrow (\exists m)[<i,<m,e,>> \in A_i \ \& \ <j,<m,e>> \in A_j]$.

- Finally, we have to show that, when $i \neq j$, $([A_i]_e, [A_j]_e)$ is a minimal pair in the e-degrees. Suppose $\Phi_e(A_i)=\theta_e(A_j)$. Then we claim that $\Phi_e(A_i)=\{x: (\exists \alpha \supseteq \alpha_{i,4<i,j,e>+2})[x\in\Phi_e(\alpha^+)]\}$: Indeed, the inclusion $\subseteq$ is obvious; to show that also the other inclusion holds, let $x \in \Phi_e(\alpha^+)$ where $\alpha \supseteq \alpha_{i,4<i,j,e>+2}$. We may assume that $(\forall \beta \supseteq \alpha_{j,4<i,j,e>+2})[\beta \text{ cpt } \alpha \Longrightarrow (\exists \tau \supseteq \beta)[x \in \theta_e(\tau^+)]]$, otherwise the construction would ensure $x \notin \theta_e(A_j)$, a contradiction. Now, let β be the least initial segment such that α cpt β and $lh(\beta)\geq lh(\alpha_{j,4<<i,j,e>,x>})$ and β is "filled in" by 0's. So $(\exists \tau \supseteq \beta)[x\in\theta_e(\tau^+)]$. But then $(\exists \tau \supseteq \alpha_{j,4<<i,j,e>,x>})[x\in\theta_e(\tau^+)]$, since $\beta^+ \subseteq \alpha_{j,4<<i,j,e>,x>}$ and thus Step $4<<i,j,e>,x>$ ensures $x\in\theta_e(A_j)$, so $x\in\Phi_e(A_i)$. Then also the inclusion $\supseteq$ is shown to hold and we comnclude that $\Phi_e(A_i)$ is r.e..

Remark 1.2 As in Theorem 1.1 it is possible to construct the sets $\{A_i: i\in \omega\}$ in such a way that, for every $i,e\in \omega$, the equirements

$S_{<e,i>}$: $\Phi_e(A_i)$ total $\Longrightarrow$ $\Phi_e(A_i)$ r.e.

are met; thus each $[A_i]_e$ is a quasiminimal e-degree.

Corollary 1.9 The filter E is not prime.

Proof. Let A be a non-r.e. set and let $\{A_i: i\in \omega\}$ be as in Theorem 1.3, with respect to A. Define $\mathcal{A} = \bigcup \{2i * \mathcal{E}_{A_{2i}}: i\in \omega\}$, $\mathcal{B} = \bigcup \{(2i+1) * \mathcal{E}_{A_{2i+1}}: i\in \omega\}$. Then $\mathcal{E}_A \leq \mathcal{A} \vee \mathcal{B}$ (the reduction $A\leq_e A_i \vee A_j$ is uniform in i,j). Nevertheless, neither $\mathcal{A}$ nor $\mathcal{B}$ belong to the filter E. Indeed,, suppose for example that $\mathcal{E}_{B_0}\wedge\ldots\wedge\mathcal{E}_{B_n}\leq\mathcal{A}$, for some $B_0,\ldots,B_n\subseteq\omega$ such that $(\forall i\leq n)[\emptyset<_e B_i]$. Then $(\forall i\in\omega)$ $[\mathcal{E}_{B_0}\wedge\ldots\wedge\mathcal{E}_{B_n}\leq\mathcal{E}_{A_{2i}}]$. Since $\mathcal{E}_{A_{2i}}$ is meet-irreducible, $(\exists j\leq n)(\exists i_0)(\exists i_1)[i_0\neq i_1$ & $\mathcal{E}_{B_j}\leq\mathcal{E}_{A_{2i_0}}$ & $\mathcal{E}_{B_j}\leq\mathcal{E}_{A_{2i_1}}]$. Hence $[B_j]_e\leq_e[A_{2i_0}]_e\wedge[A_{2i_1}]_e$, which implies $\emptyset\equiv_e B_j$, a contradiction.

Corollary 1.10 $\mathcal{M}_{e,F}$ is not a Brouwer algebra.

Proof. Let A, $\{A_i: i\in\omega\}$ be as in Corollary 1.9. Let

$$\varphi_A(x)=\begin{cases} x & \text{if } x\in A \\ \uparrow & \text{otherwise}\end{cases}\qquad \varphi_i(x)=\begin{cases} i & \text{if } x=0 \\ x & \text{if } x>0 \;\&\; x-1\in A \\ \uparrow & \text{otherwise}\end{cases}$$

Then, in $\mathcal{M}_e$, $\{\varphi_A\}\leq\{\varphi_0\}\vee\{\varphi_i: i>0\}$. By an argument similar to Corollary 1.9, one can easily see that the set $\{A_e\in\mathcal{M}_{e,F}: [\{\varphi_A\}]\leq_e[\{\varphi_0\}]_e\vee A_e\}$ does not have a least element.

One could consider the filter $E_{q.m.}$ generated by the degrees of enumerability of sets belonging to quasiminimal e-degrees. Obviously $E_{q.m.}$ E. The following theorem shows that in fact $E_{q.m.}=E$.

Theorem 1.5 $(\forall B)[\emptyset<_e B \Rightarrow (\exists A)[[A]_e$ quasiminimal & $\emptyset<_e A<_e B]$.

Proof. Our strategy will consist in showing that if $\emptyset <_e B$ then there exists a set A belonging to a quasiminimal e-degree such that $\emptyset <_e A \leq_e B$. The result then follows by the non-existence of minimal degrees in the upper semilattice of e-degrees. In what follows D is a varaiable over finite sets. Expressions like $\langle\langle i,j\rangle,D\rangle$ will have the meaning $\langle\langle i,j\rangle,u\rangle$ where u is the canonical index of D. We need two lemmas.

Lemma 1 $B \in \Delta_2$ & $\emptyset <_e B \Longrightarrow (\exists A)[A$ quasiminimal & $\emptyset <_e A \leq_e B]$.

Proof of Lemma 1 Given B as in the statement of Lemma 1,we want to construct an enumeration operator θ such that, letting $A=\theta(B)$, A satisfies the lemma. In order to construct such an enumeration operator θ, we define an r.e. set W by means of which θ will be defined. W will satisfy the following property: $(\forall i)(\forall j)(\forall D)[\langle\langle i,j\rangle,D\rangle \in W \Rightarrow D=\{j\}$ or $D=\emptyset]$. Thus, if $\{\theta^s\}_{s\in\omega}$ is a finite recursive approximation to θ and $\{B^s\}_{s\in\omega}$ is a Δ_2-approximation to B, then $(\forall x)[\lim_s \theta^s(B^s)(x)$ exists].

In defining W, we want to meet the following requirements for every $e \in \omega$:

P_e: $\theta(B) \neq W_e$;

R_e: $\Phi_e(\theta(B))$ total $\Longrightarrow$ $\Phi_e(\theta(B))$ r.e.

In the construction below, θ^s is the enumeration operator defined by W^s; also, given a set C and a number x, $C\restriction x$ denotes $\{y \in C: y \leq x\}$.

Step 0) $W^0=\emptyset$.

Step s+1) W^{s+1} consists of the elements of W^s plus those numbers which are put into W^{s+1} through a) and b) below, for every $e\leq s$.

a) $\langle\langle e,j\rangle,\{j\}\rangle\in W^{s+1}$ for every $\langle e,i\rangle < L(e,s)$ where $L(e,s)=\max\{x\leq s\colon (\theta^s(B^s))\restriction x=W_e^s\restriction x\}$.

b) Let $Z_e=\{\langle i,x\rangle\colon x\in\omega \ \&\ i>e\}$ and

$$\eta(e,u,s)=\begin{cases}\mu t\leq s.[(\forall n)[t\leq n\leq s \Rightarrow \Phi_e^n(D_u) \text{ non single-valued} \\ \qquad \& \ D_u\subsetneq \theta^n(B^n)\cup Z_e] \text{ if such a } t \text{ exists} \\ s+1 \quad \text{otherwise.}\end{cases}$$

$$u(e,s)=\begin{cases}\mu u\leq s.[\Phi_e^s(D_u) \text{ non single-valued } \& \ (\forall v\leq s) \\ \qquad [\Phi_e^s(D_v) \text{ non single-valued} \Rightarrow \eta(e,u,s)\leq \\ \qquad \eta(e,v,s)] \text{ if such a } u \text{ exists} \\ 0 \quad \text{otherwise}\end{cases}$$

Now, if $(\exists D)[\Phi_e^s(D)$ non single-valued $\&$ $D\subsetneq\theta^s(B^s)\cup Z_e]$, then let $D=D_{u(e,s)}$ and $(\forall d\in D\cap Z_e)[\langle d,\emptyset\rangle\in W^{s+1}]$. We say in this case that $\langle d,\emptyset\rangle$ is enumerated in W at step s+1) because of R_e.

Finally, let $W=\bigcup\{W^s\colon s\in\omega\}$.

Let now

$M_e=\{<e,j>: \ <<e,j>,\{j\}>$ is enumerated in W via a) at some step $s+1$) of the construction$\}$

$N_e=\{d: \ <d,\emptyset>$ is enumerated in W via b) because of R_e at some step $s+1$) of the construction$\}$

Sublemma 1. $(\forall k<e)[M_k$ finite & N_k finite$] \Longrightarrow M_e$ finite.

Proof of Sublemma 1. Suppose that M_e is infinite; then the function $\lambda s.L(e,s)$ is unbounded. Therefore $(\forall j)[<<e,j>,\{j\}>\in W]$. Now, $<<e,j>,\emptyset>\in W \Longleftrightarrow <e,j>\in \bigcup\{N_k: k<e\}$. Thus, for all but a finite number of j's, $<e,j>\in\theta(B) \Longleftrightarrow j\in B$.

But $\lim_s \theta^s(B^s)(x)$ exists for every x, and since $\lambda s.L(e,s)$ is unbounded, we have that $\theta(B)=W_e$, which implies B r.e., a contradiction.

Sublemma 2. $(\forall k\leq e)[M_k$ finite$] \Longrightarrow N_e$ finite.

Proof of Sublemma 2. Suppose $(\forall k\leq e)[M_k$ finite$]$ and let s be a stage such that

$(\forall i\leq e)(\forall j)[<i,j>\in\theta(B) \Longleftrightarrow (\forall t\geq s)[<i,j>\in\theta^t(B^t)]]$ (such a stage exists since $\{<<i,j>,\{j\}>\in W: i\leq e\}$ is finite & $(\forall x)[\lim_s \theta^s(B^s)(x)$ exists$]$.

Let $t\geq s$ and $D\subseteq \theta^t(B^t)\cup Z_e$ be such that $\Phi_e^t(D)$ is not single-valued. Then $\lim_s u(e,s)$ exists and, say, $\lim_s u(e,s)=u$ and $(\forall t'\geq t)[u(e,t')=u]$; in particular, after such a stage t, $<d,\emptyset>$ is enumerated in W because of R_e only if $d\in D_u$: this shows that N_e is finite.

It follows $(\forall e)[\ M_e$ finite & N_e finite].

Let us now go back to the proof of Lemma 1. First, we show that each P_e is satisfied. Suppose, for the sake of contradiction, that $\theta(B)=W_e$. Then, since $(\forall x)[\lim_s \theta^s(B^s)(x)$ exists & $\lim_s W_e^s(x)$ exists], we have that $\lambda s.L(e,s)$ is unbounded. Thus M_e is infinite, a contradiction.

To show that each R_e is satisfied, suppose that $\Phi_e(\theta(B))$ is total and let $X=\{\langle i,j\rangle: i\leq e\ \&\ \langle i,j\rangle\in\theta(B)\}$. By Sublemma 1 and Sublemma 2, X is finite. We claim that $\Phi_e(\theta(B))=\Phi_e(X\cup Z_e)$.
The inclusion $\subseteq$ is immediate. As to show the other inclusion, suppose that $\langle i,j\rangle\in\Phi_e(X\cup Z_e)-\Phi_e(\theta(B))$. Since $\Phi_e(\theta(B))$ is total, there exists $k\neq j$ such that $\langle i,k\rangle\in\Phi_e(\theta(B))$. But then $\langle i,j\rangle, \langle i,k\rangle \in \Phi_e(X\cup Z_e)$ and after a stage s such that $(\forall t\geq s)[X\subseteq\theta^t(B^t)]$, the construction will force $\Phi_e(\theta(B))$ to be non single-valued, a contradiction. Since $\Phi_e(X\cup Z_e)$ is r.e., R_e is met, as desired.

<u>Lemma 2</u>. There exists an enumeration operator θ such that, for every $B\subseteq\omega$ and $e\in\omega$

$\theta(B)$ r.e. $\Longrightarrow B\in\Delta_2$;

$\Phi_e(\theta(B))$ total $\Longrightarrow \Phi_e(\theta(B))$ r.e.

<u>Proof of Lemma 2</u>. Again we shall define an r.e. set W by

stages. Then, the enumeration operator defined by W will satisfy the lemma. At step s) we define a finite set W^s: the desired r.e. set W will be given by $W=\bigcup\{W^s: s\in\omega\}$.

Step 0) $W^0=\emptyset$.

Step s+1) W^{s+1} consists of the elements of W^s plus those numbers which are enumerated by the following procedure. Consider all triples (m,n,e) such that $<m,n>\leq s$, $e\leq s$, $m\neq n$, $(m)_0=(n)_0$: for each of these m,n consider also all finite sets F such that $\max(F) < \max\{m,n\}$.

If m,n,e and F (where F is finite) are such that $<m,n>\leq s$, $e\leq s$, $m\neq n$, $(m)_0=(n)_0$, $\max(F) < \max\{m,n\}$ and if $m,n\in\Phi_e^s(\theta^s(F)\cup S)$ (θ is as in Lemma 1) where $S\subseteq\{<i,j>: j\geq e \ \&\ j\geq\max\{m,n\}\}$, then, in correspondence with such m,n,e,F choose S having this property and minimal with respect to inclusion. Then, for every $<i,j>\in S$, let $<<i,j>,\emptyset>\in W^{s+1}$.

Finally, for every $j\leq s$, let $<<i,j>,\{j\}>\in W^{s+1}$, where $<i,j>$ is the least element of the j-th column such that $<<i,j>,\emptyset>\ \notin W^{s+1}$.

Sublemma 1 For every j, the set $\{<i,j>: <<i,j>,\emptyset>$ or $<<i,j>,\{j\}>$ is enumerated in W at some step s+1) of the construction$\}$ is finite.

Proof of Sublemma 1. Let j be fixed. There are only a finite number of triples $(e,<m,n>,D)$ such that $m\neq n$,

$(m)_0=(n)_0$, $j\geq e, \max\{m,n\}$ and D is finite and $\max(D) < \max\{m,n\}$.

If m,n are as before and $m,n\in\Phi_e^s(\theta^s(D)\cup S)$, where S is minimal such that $S \subsetneq \{<i,j>: j\geq e \ \&\ j\geq\max\{m,n\}\}$ then $(\forall <i,j>\in S)[<<i,j>,\emptyset>\in W^s]$. Thus at every stage $t\geq s+1$, the relevant set S for $(e,<m,n>,D)$ is the empty set. Therefore, only for finitely many i, do we have $<<i,j>,\emptyset>\in W$; but then after this finite set has been enumerated in W, the number $<<i,j>,\{j\}>$ which is enumerated in W never changes.

<u>Sublemma 2</u> For every $B\subseteq\omega$, $\theta(B)$ r.e. $\Longrightarrow B\in\Delta_2$.

<u>Proof of Sublemma 2</u>. In every column j, there exists $<i,j>$ such that $<<,i,j>,\{j\}>\in W \ \&\ <<i,j>,\emptyset> \notin W$.

Suppose that $\theta(B)$ is r.e., let z be an r.e. index of $\theta(B)$ and define a sequence $\{B^s\}_{s\in\omega}$ of subsets of B by $B^s=\{j: (\exists i)[<i,j>\in W_z^s \ \&\ <<i,j>,\{j\}>\in W^{s+1} \ \&\ <<i,j>,\emptyset> \notin W^{s+1}]\}$.

Then each B^s is finite and, by Sublemma 1, $(\forall j)[\lim_s B^s(j)$ exists $\&\ \lim_s B^s(j)=B(j)]$; hence $B\in \ {}_2$.

<u>Sublemma 3</u> For every $B\subseteq\omega$ and every $e\in\omega$, if $\Phi_e(\theta(B))$ is total then $\Phi_e(\theta(B))$ is r.e.

<u>Proof of Sublemma 3</u>. Suppose that $\Phi_e(\theta(B))$ is total. We shall show that $(\Phi_e(\theta(B)))\restriction_{\geq e}=(\Phi_e(\omega))\restriction_{\geq e}$ (where, for a given set C, $C\restriction_{\geq e}=\{x\in C: x\geq e\}$). From this, it follows that $\Phi_e(\theta(B))$ is r.e. as desired.

The inclusion $\subseteq$ is obvious. As to the other inclusion, let $<i,j>\in(\Phi_e(\omega))\restriction_{\geq e}$ and suppose that $<i,j> \notin (\Phi_e(\theta(B)))\restriction_{\geq e}$. But, then, $(\exists k)[j \neq k \ \& <i,k>\in\Phi_e(\theta(B))]$: let $<i,k> \in \Phi_e(\theta(D))$, where D is finite and let $F=D_{\max\{<i,j>,<i,k>\}}$; let also $r=\max\{<i,j>,<i,k>\}$: notice that $r\geq e$. Then $<i,j>,<i,k> \in \Phi_e(\theta(F)\cup S)$, where $S \subseteq \{<u,v>: v\geq r\}=\{<u,v>: v\geq r,e\}$. Therefore the construction ensures $<i,j>\in(\Phi_e(\theta(B)))\restriction_{\geq e}$.

Let us now return to the proof of Theorem 1.4 and let B be given such that $\emptyset <_e B$. Let θ be as in Lemma 2. If $\theta(B)$ is not r.e. then the theorem is proved. If $\theta(B)$ is r.e. then $B\in\Delta_2$ and the theorem is proved because of Lemma 1.

Corollary 1.11 $E=E_{q.m.}$

Proof. Immediate, since $E \subseteq E_{q.m.}$, by Theorem 1.5.

Corollary 1.5 and Theorem 1.5 suggest the following question: when do we have $A\equiv_e B_0 \vee B_1$ with B_0,B_1 quasiminimal and $B_0<_e A$, $B_1<_e A$? For example the proof of Corollary 1.5 shows that this is always the case for any A such that $[A]_e \geq_e o_e^{(2)}$ (see COOPER (1984) or MC EVOY (1984) for the definition of $o_e^{(n)}$). Does this hold for $A\in\Sigma_2$ (or even $A\in\Delta_2$) or, less ambitiously, is it true that $(\forall \text{non r.e. } A\in\Delta_2)[\exists B_0)(\exists B_1)[B_0<_e A \ \& \ B_1<_e A \ \& \ A\equiv_e B_0 \vee B_1]$? If this splitting property held, then we would have a variety of corollaries: for instance, we would be able to conclude that $(\forall \text{non r.e. } A\in\Delta_2)(\exists B\in\Delta_2)[A \mid_e B]$.

Regarding this last situation, we remark the following particular case. Recall (see COOPER-MC EVOY (1985)) that a set $B \in {}_2$ is said to be low if B has a Δ_2 approximation $\{B^s\}_{s\in\omega}$ such that $(\forall e)(\forall x)[\lim_s \Phi_e^s(B^s)(x)$ exists] (or, equivalently, if $(\forall e)[\lim_s \Phi_e^s(B^s)(e)$ exists]).

Theorem 1.6 If B is low, non r.e., then there exists A such that A is low and $A|_e B$.

Proof. Let $B \in \Delta_2$ be low, non r.e.; then $o_e <_e [B]_e <_e o_e'$ (since no set in o_e' is low). Recall that $o_e' = [\bar{K}]_e$, where $K = \{x: x \in W_x\}$. Fix a finite recursive enumeration $\{K^s\}_{s\in\omega}$ of K and, for every $s \in \omega$, let $\bar{K}^s = \{x: x \notin K^s \ \&\ x \le s\}$.

We construct a set A trying to satisfy the following requirements, for every $e \in \omega$:

P: A is low;

P_e: $A \neq \Phi_e(B)$;

N_e: $B \neq \Phi_e(A)$.

We define a sequence $\{A^s\}_{s\in\omega}$ of finite sets; at step s+1) we let

$l_P(e,s) = \max\{x \le s: A^s \restriction x = (\Phi_e^s(B^s)) \restriction x\}$

$l_N(e,s) = \max\{x \le s: B^s \restriction x = (\Phi_e^s(A^s)) \restriction x\}$

$$u(e,x,s) = \begin{cases} \mu z.[x \in \Phi_e^s((A^s)\restriction z)] & \text{if such a z exists} \\ 0 & \text{otherwise} \end{cases}$$

$L(e,s)=\max\{l_N(e,t): t\leq s\}$

$r(e,s)=\max\{e,u(e,x,s): x\leq L(e,s)\}$.

Step 0) $A^0=\emptyset$.

Step s+1) For every $e\leq s$, for every x, $<e,x>\in A^{s+1}$ if and only if

a) $<e,x>\leq lp(e,s)$ & $x\in \bar{K}^s$;

b) $<e,x>\in A^s$ & $[x\in \bar{K}^s$ or $(\exists i<e)[<e,x>\leq r(i,s)]]$.

Finally we define $A=\{x: (\exists s)(\forall t\leq s)[x\in A^t]$.

Notice that $\{A^s\}_{s\in\omega}$ is a Δ_2 approximation of A: indeed, if, for some e,x,s, $<e,x>\in A^s$ and $<e,x> \notin A^{s+1}$ then, in particular, $x\in K^s$: thus $(\forall t\leq s)[x \in A^t]$.

Lemma 1 $(\forall i<e)[\lambda s.l_N(i,s)$ bounded] & $(\forall i<e)$ $[\lambda s.lp(e,s)$ bounded] $\Longrightarrow \lambda s.lp(e,s)$ bounded.

Proof of Lemma 1. Suppose $\lambda s.lp(e,s)$ unbounded: since $\{A^s\}_{s\in\omega}$ $\{\Phi_e^s(B^s)\}_{s\in\omega}$ are Δ_2 approximations to A and $\Phi_e(B)$ respectively (we use here that $\{B^s\}_{s\in\omega}$ is a low approximation),it follows that $A=\Phi_e(B)$.

On the other hand, the assumption $(\forall i<e)(\lambda s.l_N(i,s)$ bounded] & $(\forall i<e)[\lambda s lp(i,s)$ bounded] allows us to conclude that $\{r(i,s): i<e$ & $s\in\omega\}$ is bounded (use the fact that $\{<i,x>\in A: i<e$ & $x\in\omega\}$ is finite, $\{A^s\}_{s\in\omega}$ is a Δ_2 approximation of A and clause b) of the definition of A).

Therefore, letting $R=\max\{r(i,s): i<e$ & $s\in\omega\}$, we have

that $(\forall x)[\langle e,x\rangle > R \Rightarrow [x\in\bar{K} \Longleftrightarrow \langle e,x\rangle\in A]]$. From this it follows that $\bar{K}\le_e A=\Phi_e(B)$, hence $\bar{K}\le_e B$, a contradiction.

<u>Lemma 2</u> $(\forall i\le e)[\lambda s.l_P(i,s)$ bounded] & $(\forall i<e)$ $[\lambda s.l_N(i,s)$ bounded$] \Longrightarrow \lambda s.l_N(e,s)$ bounded.

<u>Proof of Lemma 2</u>. Suppose that $\lambda s.l_N(e,s)$ is unbounded. By the assumption $(\forall i\le e)[\lambda s.l_P(i,s)$ bounded], we have that $\{\langle i,x\rangle: i\le e \ \& \ \langle i,x\rangle\in A\}$ is finite; thus let t be a stage such that $(\forall s\ge t)(\forall i\le e)(\forall x)[\langle i,x\rangle\in A \Longleftrightarrow \langle i,x\rangle\in A^s]$. It follows that, for every x, $x\in\Phi_e(A) \Longleftrightarrow (\exists s>t)[x\le L(e,s)$ & $x\in\Phi_e^s(A^s)]$: indeed, the implication $\Longrightarrow$ follows from the fact that $L(e,s)$ is unbounded; on the other hand let $x\le L(e,s)$ & $x\in\Phi_e^s(A^s)$ where $s>t$ and suppose, for the sake of contradiction, that there exists $s'>s$ such that $x\notin \Phi_e^{s'}(A^{s'})$; let s' be the least stage such that this holds; then $x\in\Phi_e^{s'-1}(A^{s'-1})$ and, for some finite set D such that $\max(D)<u(e,x,s'-1)$ and $D\subsetneq A^{s'-1}$, $x\in\Phi_e^{s'-1}(D)$. But $x\le L(e,s'-1)$, thus $\max(D)<r(e,s'-1)$:moreover, $(\forall\langle i,x\rangle\in D)$ $[i\le e \Longrightarrow \langle i,x\rangle\in A^{s'}]$ and if $\langle i,x\rangle\in D$ and $i>e$ then $\langle i,x\rangle\in A^{s'}$, because $(\forall j)(\forall x)[j>e$ & $\langle j,x\rangle \in A^{s'}-A^{s'-1} \Longrightarrow \langle j,x\rangle > r(e,s'-1)]$. We conclude that $x\in\Phi_e^{s'}(A^{s'})$, a contradiction. Since $(\forall x)[x\in\Phi_e(A) \Longleftrightarrow (\exists s>t)[x\le L(e,s)$ & $x\in\Phi_e^s(A^s)]$, $\Phi_e(A)$ is r.e.

Using that $\lambda s.l_N(e,s)$ is unbounded and arguing in a similar way, one is led to conclude also that $B=\Phi_e(A)$, which would imply that B is r.e., a contradiction.

From Lemma 1 and Lemma 2, we have $(\forall e)[\lambda s.lp(e,s)$ bounded & $\lambda s.l_N(e,s)$ bouned].

Lemma 3 A is low and $\{A^s\}_{s\in\omega}$ is a low approximation to A.

Proof of Lemma 3. Let $e,x\in\omega$ be given. Since $(\forall i\leq e)$ $[\lambda s.lp(i,s)$ bounded], we have that $\{<i,x>: i\leq e \ \& <i,x>\in A\}$ is finite. Let t be a stage such that $(\forall s\geq t)(\forall i\leq e)(\forall x)$ $[<i,x>\in A \iff <i,x>\in A^s]$. Exactly as in Lemma 2, one can show that if, for some $s>t$, $x\in\Phi_e^s(A^s)$ then $(\forall s'\geq s)[x\in\Phi_e^{s'} A^{s'})]$: hence $\lim_s\Phi_e^s(A^s)(x)$ exists.

We are now in a position to conclude the proof of the theorem. Let $e\in\omega$ be given: since $\lambda s.lp(e,s)$ is bounded, it follows that P_e is satisfied; by Lemma 3 and since $\lambda s.l_N(e,s)$ is bounded it follows that N_e is satisfied.

Let us now consider $\mathcal{M}_E$ and let F denote the filter of $\mathcal{M}_E$ generated by the non-zero degrees of solvability.

Proposition 1.5 $\underline{L}_H(\mathcal{M}_{E/F}) = \underline{J}$.

Proof. For $\mathcal{M}_{E/F}$ there hold results similar to Chapter II, Theorem 1.3, Theorem 1.4, Corollary 1.10 (considering $\{\mathcal{J}_k^H: k\geq 1\}$ instead of $\{\mathcal{J}_k^B: k\geq 1\}$): recall that every countable partial order is embeddable in the quasiminimal e-degrees: see for instance COOPESTAKE [?], where it is shown that each 1-generic set belongs to a

quasiminimal e-degree and that the e-degree of a 1-generic set is preceded by a countable set of independent e-degrees of 1-generic sets (a set A is 1-generic if for every recursively enumerable set W of finite initial segments of 0-1 valued functions either $(\exists \sigma \in W)[\sigma \subseteq A]$ or $(\exists \sigma \subseteq A)(\forall \tau \supseteq \sigma)[\tau \notin W])$.
Thus the proposition can be proved as in Chapter II, Corollary 1.12.

Remark 1.3 In $\mathcal{M}_{E/F}$, the greatest element is join-reducible, since the filter F of $\mathcal{M}_E$ is not prime: it suffices to find two incomparable quasiminimal e-degrees that join to a total e-degree (this can be done as a consequence of Corollary 1.5). Then the equivalence classes of the corresponding degrees of enumerability join to $[1]_F$ in $\mathcal{M}_{E/F}$. Nevertheless, $\mathcal{M}_{E/F}$ does not have coatoms. Indeed, we have that $(\forall [E_A]_F)(\exists [E_B]_F)[[E_A]_F < [E_B]_F < [1]_F]$, as is shown by the following

Theorem 1.7 $(\forall \text{quasiminimal } a_e)(\exists \text{quasiminimal } c_e)[a_e <_e c_e]$.

Proof. Let A be given such that $[A]_e$ is quasiminimal. We build a set C in the form $B \vee A$, such that $[C]_e$ will satisfy the theorem. It suffices to find a set B such that the following requirements are met, for every $e \in \omega$:

P_e: $B \vee A \neq \Phi_e(A)$;

R_e: $\Phi_e(B \vee A)$ total $\Longrightarrow \Phi_e(B \vee A) \leq_e A$.

At each stage s of the construction, we define a 0-1 valued initial segment β_s: at the end of the construction we shall let $B = \bigcup\{\beta_s^+ : s \in \omega\}$.

Let $\beta_{-1} = \emptyset$.

Step 2e) Let β_{2e} be such that $lh(\beta_{2e}) = lh(\beta_{2e-1})+1$ and β_{2e} is defined by

$$\beta_{2e}(x) = \begin{cases} \beta_{2e-1}(x) & \text{if } x < lh(\beta_{2e-1}) \\ 1-\Phi_e(A)(2x) & \text{if } x = lh(\beta_{2e-1}) \end{cases}$$

Step 2e+1)

Case a) $(\exists \beta \supseteq \beta_{2e})(\exists x)(\exists y)(\exists z)[y \neq z$ & $\langle x,y\rangle \in \Phi_e(\beta^+ \vee A)$ & $\langle x,z\rangle \in \Phi_e(\beta^+ \vee A)]$.

Let β be the least with this property and define $\beta_{2e+1} = \beta$.

Case b) Otherwise, let $\beta_{2e+1} = \beta_{2e}$.

P_e is satisfied at Step 2e); hence $A <_e B \vee A$.

Suppose now that $\Phi_e(B \vee A)$ is total; then $\Phi_e(B \vee A) = \{x: (\exists \beta \supseteq \beta_{2e})[x \in \Phi_e(\beta^+ \vee A)]\}$. This is seen in the usual way (see for instance Theorem 1.1). But then $\Phi_e(B \vee A) \leq_e A$, as desired.

We have left in this section several unsettled questions. For example:

- Is $\underline{L}_B(\mathcal{M}/_{D_e}) = \underline{J}$? (Certainly, $\underline{J} \subseteq \underline{L}_B(\mathcal{M}/_{D_e})$).

- What is the cardinality of $\mathcal{M}/_{De}$?

Here, certainly we have $2^{\aleph_0} \leq |\mathcal{M}/_{De}|$. Indeed, let us recall the following result from DYMENT (1976): if $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are mass problems such that $\mathcal{A}$ is effectively discrete, $\mathcal{A} \nleq \mathcal{B}$, $\mathcal{A} \leq \mathcal{B} \vee \mathcal{C}$, then $\mathcal{C}$ is nowhere dense in the Baire topology. From this result it follows that if $\mathcal{F}$, $\mathcal{G}$ are finite mass problems such that $\mathcal{F} \nleq \mathcal{G}$, then for no dense mass problem $\mathcal{D}$ can we have $\mathcal{F} \leq \mathcal{G} \vee \mathcal{D}$ (hence $[[\mathcal{F}]]_{De} \nleq [[\mathcal{G}]]_{De}$). This implies

Fact 1.2 $\mathcal{M}_F$ is embeddable in $\mathcal{M}/_{De}$.

Thus, trivially, $2^{\aleph_0} \leq |\mathcal{M}/_{De}|$.

As we have remarked, we have $F \subsetneq Di \subsetneq C \subsetneq B$ (as shown in DYMENT (1976), Theorem 3.3: as to $B \nsubseteq C$, Dyment proves for example that if f is not recursive then for no countable non-recursive mass problem $\mathcal{A}$ can we have $\mathcal{A} \leq \{g: f \leq_T g\}$).

Fact 1.3 $F \subsetneq E$; $Di \nsubseteq E$; $E \nsubseteq C$.

Proof. That $E \nsubseteq C$ holds, follows from the existence of quasiminimal e-degrees and Theorem 3.4 of DYMENT (1976): namely, if $\mathcal{A}$ is countable and $\mathcal{A} \leq \mathcal{E}_A$ then $(\exists f \in \mathcal{A})[f \leq_e A]$.

$F \subsetneq E$: this is obvious.

To show that $Di \nsubseteq E$, let $\mathcal{A} = \{f_i : i \in \omega\}$ be a discrete mass problem such that $(\forall i,j)[i \neq j \Rightarrow ([f_i]_e,[f_j]_e)$ is a minimal pair in the e-degrees] and suppose that $[\mathcal{a}]\in E$. Then there exist $A_0,\dots,A_n$ such that $\mathcal{E}_{A_0}\wedge\dots\wedge\mathcal{E}_{A_n}\leq\mathcal{a}$ (assume $(\forall i)[\emptyset <_e A_i]$): let Ψ be a recursive operator such that $\Psi(\mathcal{a})\subseteq\mathcal{E}_{A_0}\wedge\dots\wedge\mathcal{E}_{A_n}$. Then, for some i,j,k with $i \neq j$ we are forced to have $\Psi(f_i)\in\mathcal{E}_{A_k}$ and $\Psi(f_j)\in\mathcal{E}_{A_k}$, which implies $A_k\equiv_e\emptyset$, a contradiction (recall that for any set B and function f, $B\leq_T f \Rightarrow B\leq_e f$).

- Is $E \not\subseteq B$?

A positive answer to this question would be given by finding a set A and an uncountable family of sets $\{B_i: i\in I\}$ such that $[A]_e$ is quasiminimal, $(\forall i\in I)[[B_i]_T$ is T-minimal] and $(\forall i\in I)[A\leq_e B_i]$.

2. A note on quasiminimal e-degrees and total e-degrees.

In CASE (1971) it is conjectured that there are no sets A such that A lies in a total e-degree and $\bar{A}$ lies in a non-total e-degree. We shall show that this conjecture is false: in fact there exist sets A such that $[A]_e$ is quasiminimal and $[\bar{A}]_e$ is total.

Let $K=\{x: x\in W_x\}$. We recall the following

Lemma 2.1 a) $[\bar{K}]_e$ is total;

b) for every $A\in \Sigma_2$, $A\leq_e\bar{K}$.

Proof. a) That $[\bar{K}]_e$ is total follows from the fact that $\bar{K}\in \Pi_1$ and for every $A\in \Pi_1$, $[A]_e$ is total (see Lemma 6 (2) of COOPER (1984);

b) by Corollary 3.1 and Proposition 2.1 of MC EVOY (1985), for every $A\in \Sigma_2$, $A\leq_e\overline{\{x: x\in\Phi_x(\emptyset)\}}$; the claim follows since clearly $\{x: x\in\Phi_e(\emptyset)\leq_1 K$, thus $\overline{\{x: x\in\Phi_x(\emptyset)\}}\leq_1\bar{K}$ and a fortiori $\overline{\{x: x\in\Phi_x(\emptyset)\}}\leq_e\bar{K}$.

Theorem 2.1 There exists a set A such taht $[A]_e$ is quasiminimal and $[\bar{A}]_e$ is total.

Proof. Let $\{K^s\}_{s\in}$ be a finite recursive approximation to K; assume also that $<0,0>=0$.

We aim to satisfy the following requirements, for every $e\in\omega$:

P_e: $\Phi_e(A)$ total $\Longrightarrow$ $\Phi_e(A)$ r.e.

N_e: $A \neq W_e$.

We define a sequence $\{A^s\}_{s\in\omega}$ of finite sets by induction as follows.

Step 0) Let $A^0 = \emptyset$.

Step s+1) By induction on e, define the following set and functions:

$H_{e,s} = \{x: (\exists i<e)[x=r(i,s)]\}$;

$$u(e,s) = \begin{cases} \mu u \le s.[\{x \in D_u: (x)_0 > 0\} \cap H_{e,s} = \emptyset \ \& \ \{x: \langle 0,x\rangle \in D_u\} \subseteq K^s \ \& \ \Phi_e^s(D_u) \text{ non single-valued}], & \text{if such a u exists} \\ 0 & \text{otherwise} \end{cases}$$

$$m(e,s) = \mu x.[(x)_0 > 0 \ \& \ (\forall i \le e)[x > \max(D_{u(i,s)}) \ \& \ (\forall i<e)[x = m(i,s)]]];$$

$$r(e,s) = \begin{cases} \mu x.[x \in W_e^s \ \& \ (x)_0 > 0 \ \& \ x \ge m(e,s)], & \text{if such an x exists} \\ 0 & \text{otherwise} \end{cases}$$

Then define A^{s+1} by letting $x \in A^{s+1}$ if for some $e \le s$ one of the following holds:

(a) $(\exists y)[x = \langle 0,y\rangle \ \& \ y \in K^s]$;

(b) $x \in D_{u(e,s)}$;

(c) $r(e,s)=0 \ \& \ x=m(e,s)$.

This ends Step s+1).

Now define A by $x \in A$ if $(\exists t)(\forall s \ge t)[x \in A^s]$: since the relation $x \in A^s$ is recursive, A is clearly a Σ_2 set.

Sublemma 1. $(\forall e)[\lim_s u(e,s),\ \lim_s m(e,s),\ \lim_s r(e,s)$ exist].

Proof of Sublemma 1. By induction. Let $e\in\omega$ be given and assume that $(\forall i<e)[\lim_s u(i,s),\ \lim_s m(i,s),\ \lim_s r(i,s)$ exist]: for every $i<e$ let $r_i=\lim_s r(i,s)$ and let $H_e=\{x: (\exists i<e)[x=r_i]\}$. It easily follows that

$$\lim_s u(e,s)=\begin{cases}\mu u.[\{x\in D_u: (x)_0>0\}\cap H_e=\emptyset\ \&\ \{x: <0,x>\in D_u\}\subseteq K \\ \quad \&\ \Phi_e(D_u)\text{ non single-valued}],\text{ if such a u} \\ \quad \text{exists} \\ 0 \quad \text{otherwise}\end{cases}$$

The proof that $\lim_s m(e,s)$ and $\lim_s r(e,s)$ exist are similar.

For every $e\in\omega$, let $m_e=\lim_s m(e,s)$ and $r_e=\lim_s r(e,s)$.

Sublemma 2 $(\forall e)[\Phi_e(A)$ total $\Longrightarrow \Phi_e(A)$ r.e.].

Proof of Sublemma 2. Suppose that $\Phi_e(A)$ is total. Let $K_0=\{x: (\exists y\in K)[x=<0,y>]\}$ and let $H_e=\{x: (x)_0>0\ \&\ (\exists i<e)[x=r_i]\}$. We claim that $\Phi_e(A)=\Phi_e(K_0\cup\overline{H}_e)$.

Indeed, that $\Phi_e(A)$ is included in $\Phi_e(K_0\cup\overline{H}_e)$ is a consequence of the fact that $A\subseteq K_0\cup\overline{H}_e$. Suppose now that for some $<y,v>\in\omega$, $<y,v>\in\Phi_e(K_0\cup\overline{H}_e)$ and $<y,v>\notin\Phi_e(A)$; since $\Phi_e(A)$ is total, there exists $w\neq v$ such that $<y,w>\in\Phi_e(A)$. But, then, for some finite set $D\subseteq A$, we have $<y,w>\in\Phi_e(D)$ and therefore there exists a finite set E such that $E\cap H_e=\emptyset$ and $<y,v>$, $<y,w>\in\Phi_e(E)$ i.e. $\Phi_e(E)$ is not single-valued. The construction ensures that in this case $\Phi_e(A)$ is not single-valued, contradicting that assumption that $\Phi_e(A)$ is total. We have shown that if $\Phi_e(A)$ is total then $\Phi_e(A)=\Phi_e(K_0 \cup \overline{H}_e)$ but the latter set is manifestly recursively enumerable and thus the sublemma is proved.

<u>Sublemma 3</u> $(\forall e)[A \neq W_e]$.

<u>Proof of Sublemma 3</u>. Let $e \in \omega$ be given. We distinguish two cases.

Case a) $(\exists x)[x \in W_e \ \& \ (x)_0>0 \ \& \ x \geq m_e]$.

In this case r_e equals the least such x: thus $r_e \in W_e$ but, on the other hand, $r_e \notin A$ because of the priority assignment to the requirements and therefore $A \neq W_e$ as desired.

Case 2) Otherwise.

In this case $(\forall x)[x \in W_e \ \& \ (x)_0>0 \Rightarrow x<m_e]$ and, for cofinitely many s, $m_e \in A^s$; thus $m_e \in A-W_e$ and the proof is complete.

We are now in a position to conclude the proof of the theorem. Indeed, Sublemma 2 and Sublemma 3 ensure that $[A]_e$ is quasiminimal.

By Sublemma 1 we have that A is a Δ_2 set; moreover, $(\forall y)[y \in K \Leftrightarrow \langle o,y \rangle \in A]$. Thus $K \leq_1 A$ (which implies $\overline{K} \leq_e \overline{A}$) and, by Lemma 2.1 b), also $\overline{A} \leq_e \overline{K}$, since $\overline{A} \in \Delta_2$. Therefore $\overline{A} \equiv_e \overline{K}$ and, by Lemma 2.1 a), $[\overline{A}]_e$ is total.

Remark 2.1 There holds $\bar{A}\equiv_e c_A$. In general if $[B]_e$ is total then $B\equiv_e c_B$ (see COOPER (1984)) Hence in our case $\bar{A}\equiv_e c_{\bar{A}}\equiv_e c_A$. This can be seen also directly, since $c_A\leq_e\bar{K}\leq_e A$ ($c_A\leq_e\bar{K}$, since $c_A\in\Delta_2$).

The tecnique of Theorem 2.1 gives the following corollary to Theorem 1.6.

Corollary 2.1 $(\forall$non r.e. low $B)(\exists A)[A$ low & $A|_e B$ & $[A]_e$ quasiminimal].

Proof To the requirements of Theorem 1.6 add the following one, for every $e\in\omega$:

R_e: $\Phi_e(A)$ total $\Longrightarrow \Phi_e(A)$ r.e.;

also, modify Step s+1) as follows: let $l_P(e,s)$, $l_N(e,s)$, $u(e,x,s)$, $L(e,s)$ be defined as in Theorem 1.6; let also $H_e=\{x: (x)_0>e\}$,

$$v(e,s)=\begin{cases}\mu u\leq s.[D_u\subseteq A^s\cup H_e \ \&\ \Phi_e^s(D_u) \text{ non single-valued}], & \text{if such a } u \text{ exists}\\ 0 & \text{otherwise}\end{cases}$$

$r(e,s)=\max\{e,u(e,x,s),\max D_{v(e,s)}: x\leq L(e,s)\}$.

For every $e\leq s$, for every x, $\langle e,x\rangle\in A^{s+1}$ if and only if

a) $\langle e,x\rangle\leq l_P(e,s)$ & $x\in\bar{K}^s$;

b) if $(\exists D)[D\subseteq A^s\cup H_e$ & $\Phi_e^s(D)$ non single-valued], then $D_{v(e,s)}\subseteq A^{s+1}$;

c) $\langle e,x\rangle\in A^s$ & $[x\in\bar{K}^s$ or $(\exists i<e)[\langle e,x<\leq r(i,s)]]$

The proof that all requirements are met is similar to the proofs of Theorem 1.6 and Theorem 2.1.

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