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ENTRAINMENT EFFECTS ON THE DISTRIBUTION OF SALINITY IN
THE HUDSON ESTUARY

City University of New York

Ph.D.

1980

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ENTRAINMENT EFFECTS ON THE DISTRIBUTION OF SALINITY
IN THE HUDSON ESTUARY

by

CHENG-PING DAVID FU

A dissertation submitted to the Graduate Faculty in
Physics in partial fulfillment of the requirements
for the degree of Doctor of Philosophy,
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1980

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1980

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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LIST OF SYMBOLS AND ABBREVIATIONS

A_x	Dispersion coefficient
B	Amplification factor
C	Phase velocity
C_1, C_2, C_3	Constants
exp	Exponential (e)
g	Acceleration due to gravity
H_1, H_2	Thickness in the upper and lower layers, respectively
HO, HH, HP	The predicted values of H in n'th, (n+1/2)'th, (n+1)'th time step
k	Wave number ($= 2\pi/L$)
K	Vertical diffusivity
L	Wavelength (Chap. 2) Length of salt intrusion (other chapters)
M.P.	Mile point
N	vertical viscosity
N.S.	Navier-Stokes
P_o	Surface atmosphere pressure
P_1, P_2	Pressure in upper and lower layer, respectively
PP	Entrainment increase with x
Q	Horizontal transport
R	Roughness coefficient
S_1, S_2	Salinity in the upper and lower layers, respectively
SO, SH, SP	The predicted value of S in n'th, (n+1/2)'th, (n+1)'th time step.

S	The corrected values of S in $(n+1)$ 'th time step
s'	Perturbation from the mean value for S
t	Time
T	Tidal period
U_1, U_2	X-directed velocity components in the upper and lower layers, respectively
UO, UH, UP	The predicted values of U in n 'th, $(n+1/2)$ 'th, $(n+1)$ 'th time step
U	The corrected values of U in $(n+1)$ 'th time step
u'	Perturbation from the mean value for U
U_*	Frictional velocity
W	Vertical velocity
w'	Perturbation from the mean value for w
W_a	Vertical advective velocity
W_e	Entrainment
W_{2+}	$W_{2+\epsilon}$
W_{2-}	$W_{2-\epsilon}$
x	Horizontal coordinate along channel and positive seaward
x'	Dimensionless parameter ($= x/L$)
x_0	Phase
z	Vertical coordinate and positive upward

GREEK SYMBOLS

α	Parameter in the salt equation
β	Proportional constant in the eq. of state
η_1, η_2	Surface and interface elevations relative to the bottom ($z=0$), respectively

η_{2+}	$\eta_{2+\epsilon}$
η_{2-}	$\eta_{2-\epsilon}$
λ	Stability criterion (Chap. 2)
ρ_1, ρ_2	Density in the upper and lower layers, respectively
τ	Surface wind stress

CHAPTER 1

ESTUARY AND ENTRAINMENT

1.1 General Description of Estuary

1.1.1 Definition of estuaries

An estuary is defined as a semi-enclosed body of water having a free connection with the open sea and within which sea water is measurably diluted with fresh water derived from land drainage (Cameron and Pritchard 1963). Traditionally the term estuary has been applied to the lower reaches of a river into which sea water intrudes and mixes with fresh water draining seaward from the land.

1.1.2 Classification of estuaries

Various attempts have been made to classify estuaries by type. Such a classification scheme for an estuary as a whole is difficult to devise, for there are hardly any two estuaries that are exactly the same as regards to geometric and bathmetric configuration, physical oceanographic characteristics of circulation and mixing, or both.

The physical oceanographic conditions in the estuary are functions of the amount of fresh water influx, oceanic salinity, intensity of tidal action, wind stress, intensity of local turbulence, and variations of width and bathymetry.

According to the physical oceanographic conditions relating to the salinity distribution, there are four types: highly stratified salt wedge; highly stratified fjord; partially mixed; and homogeneous estuaries (Pritchard 1955, Cameron and Pritchard 1963).

The dimensionless ratio of the tidal flow to the freshwater flow is called the vertical stratification factor (VSF). The VSF, or a function of it, has been used by several investigators (Pritchard 1967, Bowden 1967, Simmons 1955) as an indicator in determining the type of circulation in estuaries. The VSF of fjord, salt-wedge of highly stratified estuaries is less than 10. The VSF of a partially mixed estuary is between 20 and 200. The VSF of well-mixed estuaries is more than 2,000.

1.2 Physical Processes in Partially Mixed Estuaries

1.2.1 Density current

The basic difference in the dynamics for a zero density difference and any other density difference, is that for the density difference regime, any mixing, whether mechanically or convectively induced, will contribute to an apparent longitudinal diffusive process due to an associated internal gravitational circulation initiated by that mixing.

In partially mixed estuaries the gravitational, or density current, velocities are commonly an order of magnitude smaller than the amplitude of the tidal currents. Because of their reversal at mid-depth, however, the vertical shear in the density current is often greater than in the tidal current and the dispersive effect associated with it is correspondingly greater.

1.2.2 Advection and mixing

Water movements alone cause an advective transport of properties and lead to dispersion when the current velocity varies with position. Density current is one kind of tideless motion. In general, tidal motion superimposes and oscillates with density current. When averaged over a tidal cycle, a net flux of salt into the estuary is produced.

Turbulent mixing has significant effects upon all mixing properties. This will be discussed in a later section of this chapter.

1.3 Vertical Mixing and Diffusion

1.3.1 Vertical mixing

The vertical mixing in estuaries is carried out by two processes --entrainment and diffusion (Carstens 1970, Dyer 1973). Entrainment is a one way process in which a less turbulent water mass becomes drawn into a more turbulent layer. Diffusion, however, is a two way process in which equal volumes of water are exchanged between the two layers. In general, both methods of mixing can work at the same time, their relative importance depends on the degree of turbulence in the two layers. Salt is transported upwards and the potential energy of the water column is increased by both methods.

1.3.2 Vertical diffusion

Vertical or horizontal shear in the current stretches or distorts an initially compact distribution of a pollutant, for example, producing sharp concentration gradients. Thus the rate of turbulent diffusion, which is usually taken as being proportional to the concentration gradient, is increased.

In view of the difficulties of measuring turbulent fluctuations and eddy fluxes of momentum or matter directly, it is not surprising that few such observations have been made. There is much more extensive literature on turbulent transport processes which select a form based on an analogy to molecular processes. That is, the terms involve a turbulent viscosity for the internal stress terms and a turbulent diffusivity for the diffusion terms. They may be expressed as

$$\langle w's' \rangle = -K \frac{\partial S}{\partial z} \quad (1.1)$$

and

$$\langle u'w' \rangle = -N \frac{\partial U}{\partial z} \quad (1.2)$$

The variables embraced by brackets $\langle \quad \rangle$ are the time averaged values, and K is the coefficient of vertical eddy diffusion and N is the coefficient of vertical eddy viscosity.

Molecular diffusion has been neglected, as in most problems involving dispersion in the sea or estuaries its effects are several orders of magnitude smaller than those of turbulent diffusion.

Even with the form of these terms (equations 1.1 & 1.2) selected, the values of the coefficients which enter these terms are not a priori determined but must be found by observation. The procedure is, however, indirect. Usually any numerical model is run with various values for the coefficients until a satisfactory match of model results with observed velocities and salinities are obtained.

It is generally accepted that the coefficient of vertical diffusion, K , decreases with increasing Richardson number.

Rossby and Montgomery (1935) proposed:

$$K = K_0 (1 + 2.5 Ri)^{-2} \quad (1.3)$$

where K_0 is the value of K under neutrally stable conditions. Ri is the Richardson number, which is given by:

$$Ri = g (\partial \rho / \partial z) / \rho (\partial U / \partial z)^2 \quad (1.4)$$

where g denotes the gravitational acceleration, ρ is the density, U is the horizontal velocity, and z is the vertical coordinate.

Using different field data, Munk and Anderson (1948) arrived at the same form as equation (1.3), but with different constants:

$$K = K_0 (1 + 3.33 Ri)^{-1.5} \quad (1.5)$$

To remove the singularity at negative Ri , Mamayev (1958) proposed an alternate to equations (1.3) and (1.5):

$$K = K_0 \exp (-0.8 Ri) \quad (1.6)$$

But the exponential form which was found to give better approximations to the data for large Ri (Delft Hydraulic Laboratory, 1974) is:

$$K = K_0 \exp (-3.0 Ri) \quad (1.7)$$

1.3.3 Ratio between salt and momentum coefficients

In homogeneous flow the coefficients N and K may be expected to be approximately equal to reach their maximum values at mid-depth. The presence of a stable density gradient tends to reduce them both, but reduces K at a faster rate than N so that the ratio (K/N) is also decreased.

Bowden (1967) and recently Bowden and Gilligan (1971), discussed the ratio (K/N) observed in the Mersey Estuary over a range of Richardson numbers. Bowden makes the interesting point that this curve is very close to that derived by Munk and Anderson (1948) in connection with a theory of the thermocline. They proposed the following empirical relations

$$N = A (1 + 10 \quad Ri)^{-0.5}$$

$$K = A (1 + 3.33 \quad Ri)^{-1.5}$$

where A is the mixing coefficient for adiabatic conditions. The ratio K/N is 1.0 under adiabatic conditions and $K < N$ under stable conditions.

Observed and theoretically assumed values for eddy coefficients for tidally averaged flow in partially mixed estuaries are presented in Table 1-1.

1.3.3.1 Ratio (K/N) under adiabatic condition

In the earlier theory concerning the transport of salt* (or heat) in turbulent flows, it is simply assumed that there is complete analogy between transport of momentum and transport of salt. It is

*The coefficient for heat diffusivity is assumed equal to the coefficient for salt diffusivity in this paragraph.

Table 1-1

Assumed and Observed Values for Vertical Eddy

Coefficients of Previous Investigators.

Values Correspond to Tidally Averaged Flow.

INVESTIGATORS	K (cm ² /sec)	N (cm ² /sec)	LOCATION
<u>Theoretically assumed constant coefficients</u>			
Rattray and Hansen (1962)	2	4	James River
Fisher et al. (1972)	.16	.22	Flume
	1.6-5	4.5-12	James River
Hansen and Rattray (1972)	0.5	5	Baltimore Harbor
Overland (1973)	10,25	10,25	Hudson Estuary
Bowden and Hamilton (1975)	20	40	Rotterdam Waterway
<u>Calculated from data</u>			
Pritchard (1960)			
Max. value in vertical profile	6.5	--	James River
Min. value in vertical profile	1.0	--	James River
Bowden (1965)			
Max. value in vertical profile	32	--	Mersey Estuary
Min. value in vertical profile	6	--	Mersey Estuary
Bowden and Sharaf El Din (1966)			
Max. value in vertical profile	17-28	29	Mersey Estuary
Min. value in vertical profile	1-3	8.5	Mersey Estuary
Gordon (1974)	--	2-5	Choptank River

frequently assumed that the ratio is 1.0, but we have found that this need not necessarily be so. Ellison and Turner (1960) made some experiments on stratified pipe flow. In their data analysis, based on the theoretical work of Ellison (1957), they found that the ratio in the adiabatic condition is 1.4. Record and Cramer (1966) found that the ratio is 1.2 in the atmosphere. Businger et al. (1970) found that the ratio is 1.35 in the atmosphere. Webster (1964) made an experimental study in which turbulence was measured in a wind tunnel and he found that the ratio is 2.2. Because of considerable difficulties in measurement of temperature fluctuations, the accuracy of Webster's work is low. The ratio is 1.15 from the experimental data in the wind tunnel (Mery et al. 1974). However, there are many other examples of ratios less than 1.0 (e.g., Hinze 1976 and Table 1-1).

There is much controversy because of the difference in values obtained for this ratio. And, that ratio is a very important factor in understanding the governing theories not only for natural estuaries but also for fluid mechanics.

1.4 Entrainment in Estuaries

1.4.1 Laboratory information

Most of the information on entrainment has been derived from laboratory experiments. From experiments with moving fresh water and stationary sugar solutions, Keulegan (1949) found an empirical relation between velocity of fresh water (V_1), the critical velocity of mixing (V_c) and velocity of entrainment (W_e) which is expressed by the following equation:

$$W_e = 3.5 \cdot 10^{-4} (V_1 - 1.15 V_c)$$

and

$$V_c = g^{1/3} (\nu_2 \Delta \rho / \rho)^{1/3} \quad (1.0/0.178)$$

where ν_2 (cm²/sec) is viscosity of the lower flow and $\Delta \rho$ is density difference between the upper and lower layers.

In Keulegan's result, the maximum Reynolds number of his experiments is 10^4 and the critical velocity would be of order of a few centimeters per second and a wavelength of a few centimeters. However, in natural estuaries, Reynolds numbers are of the order of 10^7 , velocity is of the order of $10^0 \sim 10^1$ cm/sec and wavelength of the order of 10^4 cm.

According to the results of Keulegan, values obtained in actual field experiments for the above parameters are always above the critical condition.

Lofquist (1960) extended Keulegan's work, using a dense liquid moving beneath a stationary upper layer. Macagno and Rouse (1962) have studied mixing between two layers moving in opposite directions. The turbulent entrainment characteristic of an inclined system, salt water moving beneath a stationary upper layer of fresh water, has been studied experimentally by Ellison and Turner (1959). Chu and Vanvari (1976) extended, experimentally, that part of Ellison and Turner's work which pertained to a surface jet. There is considerable difficulty in comparing these experimental results because of the various experimental conditions and differing definitions of the length involved in the

Froude numbers or Richardson numbers.

It is also difficult to draw from these experiments any direct quantitative conclusions that might be relevant to environmental conditions, both because of the scale effects and because of the lack of information in the experiments about the large-scale structure of the turbulence (Phillips 1966).

However, from all of the above references, there are some common conclusions and they are as follows:

(1) Entrainment is inversely related to the Richardson number or directly related to the internal Froude number.

(2) There is little information about negative Richardson number which is in another category (overturning will occur).

(3) Entrainment is proportional to the relative turbulence or the relative velocity between two layers.

In addition, over most of the entraining region of the jet in the work of Chu and Vanvari (1976) the entrainment velocity is nearly proportional to longitudinal turbulent intensity. This fact suggests that the local entrainment process is not strongly affected by the buoyancy.

1.4.2 Theoretical approach

Most investigators assume, based on dimensional arguments, that the entrainment velocity is proportional to one kind of velocity. Parameterizations of entrainment velocity fell into following general categories:

(1) Frictional Velocity:

Assuming a constant uniform stress (τ) applied by the wind over the sea surface, and an elapsed time interval sufficiently smaller than the inverse of the Coriolis frequency (i.e., for fully turbulent entrainment), Phillips (1966) discusses the process of thermocline erosion in the upper ocean and argues that the entrainment velocity is a constant value and proportional to the frictional velocity (U_*), where U_* is related to wind stress (τ) by $U_* = \sqrt{\tau/\rho}$.

A simple model is set up by Nihoul (1972). He uses the time-dependent results derived from his model to explain the mechanism of entrainment and thermocline erosion during the first stage of the process (i.e., before limitation occurs due to cessation of wind or inhibition by buoyancy).

(2) Flow Velocity:

Some investigators (Morton, Taylor & Turner 1956, Scorer 1957, Turner 1963) assumed that the entrainment velocity is everywhere proportional to the upward velocity of parcels, in these cases they analyzed the motion of buoyant parcels rising through the atmosphere.

1.5 General Characteristics of a Partially Mixed Estuary

1.5.1 Circulation pattern

In general, fresh water, driven by the horizontal pressure gradient generated by river runoff, enters the estuary at the head and flows toward the ocean. At the same time, the heavier sea water intrudes upstream due to tidal force.

These two advective movements will be coupled with vertical turbulent mixing due to turbulent eddies generated by tidal motion and by density induced circulation due to the density difference between fresh and sea water. The consequence of these factors will be two-layer circulation in the partially-mixed estuary case. Examples include the Mersey, the Southampton Water, and most coastal plain estuaries of the east coast of United States such as the James River. The James River is very famous because of extensive field studies and detailed data analysis by Pritchard and other related members of the Chesapeake Bay Institute.

There is seaward flow in the upper layer and landward flow in the bottom layer. At the same time, there is turbulent mixing between the two layers. Pritchard has calculated that in the James estuary the seaward flow in the upper layer at the sea-end is twenty times the river flow and the compensating inflow on the bottom is nineteen times the river flow.

1.5.2 Salinity distribution

In the James Estuary, the form of the vertical salinity profile has the general shape of an inverse tangent function and does not change much along the estuary. The salinity of the upper and lower layers are almost homogeneous along the depth.

There is normally a zone of high salinity gradient at about mid-depth, and this halocline is still sufficiently weak to allow some exchange of volume between upper and lower layers due to turbulent mixing.

The surface salinity increases steadily down the estuary and undiluted fresh water only occurs very near the head of the estuary. Also in the saline water on the bottom there is a longitudinal gradient of salinity. Consequently there is a large section in the middle reaches of this type of estuary in which the horizontal salinity gradients are almost uniform.

1.6 More Recent Considerations on the Characteristics of Physical Processes

Recently, there are reports modifying significantly the circulation pattern in estuaries as a result of wind stress on the surface of the estuary. There are three-layered or reversed two-layered circulation patterns (Elliott 1976, Murray 1972). Significant lateral variations in salinity and velocity are reported (Elliott 1976, Dyer 1976).

The resulting lateral distribution of salinity tends to balance the Coriolis force associated with the longitudinal velocities (Elliott 1976). There are some contributions on lateral distribution in velocities due to effects of channel curvature and cross-sectional shape (Dyer 1976). Some investigators (Fisher 1976, Pritchard 1979, Posmentier and Raymont 1979) suggest that three dimensional model is necessary to explain the effects due to lateral variations in velocities and salinities on circulation pattern and salinity distribution.

According to Murray's report, data taken in the vicinity of the Mississippi River fit the reversed two-layered circulation pattern. The pattern exists only temporarily for periods of wind blowing and for cases of moderate or high wind speed. His data are still regarded as classical two-layered circulation pattern either after wind blowing

or after filtering out the wind stress during wind blowing.

Wind driven current has been studied extensively in open ocean (which are rather deep and far from boundaries) for many years. Most existing formulas, representing the effects of wind stress on wind driven current, are still considered as inconclusive or even arguable (Neumann and Pierson 1966). For estuaries, the situation is more inconclusive due to additional unconfirmed effects of shallow and narrow boundaries on current. Therefore, observations in estuaries are often made during periods of relatively light winds. For the same reason, either analytic or numerical models for estuaries are often developed for cases where wind effects are negligible.

CHAPTER 2

NUMERICAL INSTABILITIES

2.1 The Rationale for a Numerical Approach.

It is difficult to obtain an analytic solution because there are non-linear terms in the coupled, simultaneous equations.

Finite-difference methods for solving partial differential equations were discussed by Courant, Friedrichs and Lewy (1928). But the limited application of numerical approaches at that time was due to practical considerations. In recent years, not only the development of the appropriate numerical techniques but also the tremendous improvement of computer speed and capacity have made numerical models possible.

2.2 Numerical Instabilities in Modeling Estuaries

It is often difficult to find the cause of the numerical instabilities that occur. Generally, a rather lengthy and time-consuming analysis must be made to locate the places and time sequences of growing instabilities. Even when these instabilities are located and documented, their analyses are often experimental (i.e., including change in formulations, coefficients or methods of solution). All the criteria and rules described in documentations just give us some basic concepts and guide us to solve these problems.

Generally simultaneous differential equations are often solved by experiences through numerical experiments; it is equivalent to the

trial and error method.

An improper numerical scheme will cause problems with the stability of the computation. But there are still other factors that cause this problem. We separated them into two categories: physical reasons and numerical reasons.

2.2.1 Physical reasons

2.2.1.1 Improper coefficients

In the development of a numerical model, the finite difference approximation over the vertical direction will be one or two orders in magnitude smaller than over the horizontal direction. As a consequence, the exchange coefficients in the vertical direction will probably be more sensitive in the adjustment of the model than the coefficients in the horizontal direction (Leendertse and Liu 1975, 1977; Hamilton 1975).

Models are particularly sensitive to the magnitude of the bottom frictional coefficient (Leendertse and Liu 1975, 1977; Harleman and Lee 1969).

2.2.1.2 Improper boundary conditions

Improper descriptions of the boundary will cause instability. It occurs often, especially at the seaward end of models (Leendertse and Liu 1975; Hamilton 1975).

2.2.1.3 Unstable physical mode and improper initial conditions

The flow being simulated is generally unstable, and cannot always be resolved properly on computational grids.

The initial conditions can greatly influence the number of iterations required for convergence so the most realistic variation is desirable (Hess 1976; Chap. 6).

2.2.1.4 Jump

The hydraulic jump due to a sudden geometric change causes turbulence which induces strong physical instability. A density jump will also cause physical instability (Chu and Vanvari 1976).

2.2.1.5 Improper mathematical formulation

Insufficient and controversial formulations of wind-driven (Naumann and Pierson 1966), bottom frictional and interfacial frictional forces will cause instability (Leendertse and Liu 1975). In addition, improper treatment of entrainment will cause instability (Vreugdenhil 1969; Leendertse and Liu 1977; Chap. 6).

2.2.2 Numerical reasons

(1) Violation of the stability criteria belonging to the chosen methods of solution.

(2) Computational mode.

(3) Nonlinearity.

(4) Coupling.

(5) Aliasing and other restraints (Marchuk, Kochergin, Klimok and Sukhorukov 1977).

The discussion of instabilities due to numerical reasons will be described in the following sections of this chapter.

2.3 Finite Difference Approximations and Truncation Errors

2.3.1 Truncation errors

The truncation error is defined to be the difference between the differential and the difference equations. In general, the higher the order of the truncation error with respect to Δx , the more accurate is the finite difference approximation. From the following analysis (Chap. 2.3.2), it will be shown that the central difference approximation has a smaller truncation error than any other method.

2.3.2 Derivative approximation

Consider an arbitrary function $f(x)$ and its expansion in a convergent power series:

$$\begin{aligned} f(x \pm \Delta x) = & f(x) \pm f'(x) (\Delta x) + f''(x) (\Delta x)^2 / 2! \\ & \pm f'''(x) (\Delta x)^3 / 3! + \dots \end{aligned} \quad (2.1)$$

where x represents either the space or time coordinate, Δx represents the interval of variable x , $f'(x)$, $f''(x)$... represent the derivatives of first, second,... order respectively.

2.3.2.1 Central difference approximation

$$f'(x) = (f(x + \Delta x) - f(x - \Delta x)) / (2\Delta x), \quad O(\Delta x)^2 \quad (2.2)$$

2.3.2.2 Forward difference approximation

$$f'(x) = (f(x + \Delta x) - f(x)) / (\Delta x), \quad O(\Delta x) \quad (2.3)$$

2.3.2.3 Backward difference approximation

$$f'(x) = (f(x) - f(x-\Delta x))/(\Delta x), \quad O(\Delta x) \quad (2.4)$$

where $O(\Delta x)$ represents a larger error than the $O(\Delta x)^2$.

2.3.3 Derivative approximation for time

Consider a differential equation with the following form:

$$\partial f / \partial t = G(t, x, f'(t, x), f''(t, x), \dots) \quad (2.5)$$

where $f', f'' \dots$ represent the derivatives of x for the first, second, ...order respectively.

A central difference approximation is

$$\frac{f_m^{n+1} - f_m^{n-1}}{2\Delta t} = G_m^n$$

where superscripts of $n-1, n, n+1$ are the number of time steps

subscript m is the m 'th grid point

and $f_m^{n+1} = f(x=m(\Delta x), t=(n+1)(\Delta t))$, $G_m^n = G(x=m(\Delta x), t=n(\Delta t))$

and Δt is a time interval and Δx is a distance between two consecutive grid points.

A forward difference approximation is

$$\frac{f_m^{n+1} - f_m^n}{\Delta t} = G_m^n$$

A backward difference approximation is

$$\frac{f_m^{n+1} - f_m^n}{\Delta t} = G_m^{n+1}$$

2.3.3.1 Central time differencing

Applied to each grid point, the approximation contains only one unknown value of the dependent variable at the $n+1$ time level for each grid point. So it is solved explicitly in terms of the known values at the n and $n-1$ time levels. It is an explicit scheme.

Due to the characteristics of central difference for three time levels, $n-1, n, n+1$, the solution of the central finite difference equation has two solutions, a physical mode and a computational mode, rather than one solution of the original differential equation.

A central difference for time is stable only under certain conditions to be explained in section (2.4)

2.3.3.2 Forward time differencing

It is an explicit scheme. From the content in the next section (2.4), conclusions for a forward difference approximation for time are the following:

The computational scheme with central difference approximation for the advection term is always unstable. The computational scheme with either forward or backward difference approximation is unstable for the case of the same direction of phase velocity and advective gradient.

2.3.3.3 Backward time differencing

These finite difference equations involve several unknowns in each equation. When the difference equations are applied to all grid points, the simultaneous difference equations with unknowns at the $(n+1)$ 'th time level are to be solved. The technique of inverting a matrix is needed to solve the equations. In the next section, it will be shown that this approximation is unconditionally stable.

In addition, both forward and backward schemes are two-time levels. No two-time-level schemes have a computational mode.

2.3.3.4 Trapezoidal implicit scheme

Approximate eq. (2.5) to the following form:

$$\frac{f_m^{n+1} - f_m^n}{\Delta t} = \frac{G_m^{n+1} + G_m^n}{2}$$

Central difference is used, but only midway between two consecutive time steps, e.g., at $(n+1/2)$. Since $(n+1/2)$ of the variables are off-time points, averages of the on-time values at time levels n & $(n+1)$ must be used in the right side of eq. (2.5). So, it is still one kind of implicit scheme.

This scheme preserves all the advantages of the original central-difference scheme, including a smaller truncation error, and removes the computational mode of central-difference because of the characteristics of two-time levels. In addition, it has the advantage of an implicit scheme, unconditionally stable.

2.4 Computational Instabilities

Truncation error was introduced in section 2.3.1. It is by no means the only source of errors when numerical methods are utilized to solve differential equations. In this section, computational instabilities will be introduced including the spurious computational mode, linear instabilities and non-linear instabilities.

2.4.1 Computational mode

When a linear equation such as eq. (2.5) or eq. (2.6) is treated, there is no computational mode in the two time level schemes. But the computational mode occurs in three and higher time level schemes.

In the three time level scheme, a quadratic equation will be solved. There are two solutions in the quadratic equation--physical mode and computational mode. The latter arises just from the finite difference approximation.

2.4.2 Linear instabilities

Two simple and basic cases, advection and diffusion, will be discussed in this section. These cases are important and occur often.

2.4.2.1 Linear instabilities (Advection Case)

2.4.2.1.1 Differential equation and its general analytic solution

The differential equation of advective motion with positive speed c is

$$\frac{\partial F}{\partial t} + c \frac{\partial F}{\partial x} = 0.0 \quad (2.6)$$

The general analytic solution is

$$F(x-ct) = A \text{ EXP } (iK (x-ct)) \quad (2.7)$$

where $K = 2 \pi / L$

Eq. (2.7) is a single harmonic Of a Fourier Spectrum with wavelength L , phase speed c , and a constant amplitude A .

2.4.2.1.2 General solution of the finite difference equation and general criteria

The general solution of finite difference approximation is assumed in the following form:

$$F = B^t \text{ EXP } (iKx) \quad (2.8)$$

where B is the amplification factor and will be discussed later.

Eq. (2.8) is equivalent to eq. (2.9).

$$F_m^n = B^{n\Delta t} \text{ EXP } (iKm\Delta x) \quad (2.9)$$

where n represents the n 'th time level and m represents the m 'th grid point.

The expression (2.9) is just one component of the Fourier Series (2.10).

$$F = \sum_k A B^{n\Delta t} \text{ EXP } (iKm\Delta x) \quad (2.10)$$

where A is the amplitude. And eq. (2.10) satisfies the differential equation (2.6).

Furthermore

$$(F_m^{n+1} / F_m^n) = B^{\Delta t} \quad (2.11)$$

This implies that $B^{\Delta t}$ is the amplification factor of the wavelength related to K. It means that

$$\begin{aligned} B^{\Delta t} &> 1.0 \dots\dots\dots \text{Amplified,} \\ B^{\Delta t} &= 1.0 \dots\dots\dots \text{Neutral,} \\ B^{\Delta t} &< 1.0 \dots\dots\dots \text{Damped.} \end{aligned} \quad (2.12)$$

It is required for stability that no component of (2.10) be amplified without bounds in the solution. It is seen that the factor $B^{\Delta t}$ is raised to n'th power. It is clear that to prevent unbounded growth of any component, the criterion is

$$\max_K |B^{\Delta t}| \leq 1.0 \quad (2.13)$$

2.4.2.1.3 Summary

By using the similar analysis as in the previous section, it is easy to derive the following results:

(1) Central differencing in time and space.

The scheme has numerical instability while the value of λ , ($= c \Delta t / \Delta x$), is greater than 1.

The stability of the scheme is neutral while the value of λ is less than 1.

- (2) Central differencing in time and forward differencing in space.

The scheme has numerical instability while the value of λ ($= 2c\Delta t/\Delta x$) is either much greater or much smaller than one.

- (3) Central differencing in time and backward differencing in space.

The stability conditions of this scheme are the same as those of case (2).

- (4) Forward differencing in time and central differencing in space.

The stability of the scheme is always unstable no matter what the value of λ , ($= \frac{c\Delta t}{2\Delta x}$), is.

- (5) Forward differencing in time and space.

The stability of the scheme is always unstable no matter what the value of λ , ($= \frac{c\Delta t}{\Delta x}$), is.

- (6) Forward differencing in time and backward differencing in space.

The stability of the scheme is always unstable no matter what the value of λ , ($= \frac{c\Delta t}{\Delta x}$), is.

- (7) Backward differencing in time and central differencing in space.

The solution of scheme is damped for all values of λ where $\lambda = \frac{c\Delta t}{2\Delta x}$.

(8) Backward differencing in time and forward differencing in space.

The solution of the scheme is damped for the condition of $\lambda > 1$, where $\lambda = \frac{c\Delta t}{\Delta x}$.

The stability of the scheme is unstable for the condition of $0 < \lambda < 1$.

(9) Backward differencing in time and space.

The stability conditions of the scheme are the same as those of case (8).

(10) Trapezoidal differencing in time and central differencing in space.

The stability of the scheme is always stable no matter what the value of λ , ($= \frac{c\Delta t}{\Delta x}$), is.

2.4.2.1.4 Summary of methods used in finite differencing and their related stability

The following general equation

$$\frac{\partial f}{\partial t} = F_1 + F_2 = F \quad (2.14)$$

where F_1 and F_2 represent non-linear and linear terms, respectively.

This equation can be solved by many types of computational schemes.

A very excellent summary of the properties of these schemes is given by Haltiner (1971) for eq. (2.14).

2.4.2.2 Linear instabilities (Diffusion Case)

2.4.2.2.1 The differential equation and its general analytic solution

$$\frac{\partial F}{\partial t} = c \frac{\partial^2 F}{\partial x^2} \quad (2.15)$$

The above equation is a diffusion equation with constant c .

The analytic solution is

$$F = \sum_k A_k \text{EXP} (iKx - k^2 ct)$$

2.4.2.2.2 Forward time differencing (explicit scheme)

The corresponding finite difference equation is

$$(F_m^{n+1} - F_m^n)/\Delta t = c (F_{m+1}^n - 2F_m^n + F_{m-1}^n)/(\Delta x) \quad (2.16)$$

The solution of eq. (2.16) is assumed in the following

$$F_m^n = B^{n\Delta t} \text{EXP} (iK_m \Delta x) \quad (2.17)$$

After substituting (2.17) into (2.16), simplifying and reorganizing, it reduces to

$$B^{\Delta t} = 1 - 2\lambda (1 - \cos (K\Delta x)) \quad (2.18)$$

where $\lambda = c\Delta t/(\Delta x)$.

Eq. (2.18) implies that the factor $B^{\Delta t}$ is real and does not exceed +1 for all values of K . The condition that $B^{\Delta t} > -1$ for such values of K is that

$$2\lambda \leq 1 \quad (2.19)$$

this is the stability criterion for eq. (2.16). This criterion has the unfortunate constraint that if Δx is chosen rather small in the interest of accuracy, the allowed Δt should be so small that large number of cycles of calculation is required to complete a problem.

2.4.2.2.3 Backward time differencing (implicit scheme)

The corresponding finite difference equation is

$$\frac{F_m^{n+1} - F_m^n}{\Delta t} = c \frac{F_{m+1}^{n+1} - 2F_m^{n+1} + F_{m-1}^{n+1}}{(\Delta x)^2} \quad (2.20)$$

After applying the similar procedures as the previous section (2.4.2.2.2), eq. (2.20) leads to

$$B^{\Delta t} = \frac{1}{1 + 2\lambda (1 - \cos(K\Delta x))} \leq 1.0$$

where $\lambda = c\Delta t/(\Delta x)^2$. This scheme is always computationally stable for all wavelengths.

2.4.2.2.4 Partly implicit scheme

This scheme is treated in considerable detail by Richtmyer and Morton (1967). The corresponding finite difference equation is

$$\frac{F_m^{n+1} - F_m^n}{c\Delta t} = r \frac{F_{m+1}^{n+1} - 2F_m^{n+1} + F_{m-1}^{n+1}}{(\Delta x)^2} + (1-r) \frac{F_{m+1}^n - 2F_m^n + F_{m-1}^n}{(\Delta x)^2} \quad (2.21)$$

If $r = 0.0$, then it is purely explicit
 = 1.0, then it is purely implicit
 = 0.5, then it is the Crank-Nicholson method.

The conclusions from the stability criterion are

$$2\lambda \leq 1/(1-2r), \quad \text{if } 0.0 \leq r \leq 0.5$$

$$\text{No restriction,} \quad \text{if } 0.5 < r \leq 1.0$$

2.4.2.2.5 Summary

A very excellent summary of the properties of these schemes is given by Richtmyer and Morton (1967) for eq. (2.15).

2.4.3 Non-linear Instabilities

2.4.3.1 Aliasing

$$L \geq 2 \Delta x, \quad L \text{ is a wavelength}$$

This criterion is required for all the finite difference equations. If it is not satisfied, the solution cannot be resolved and will be misrepresented. This misrepresentation is known as aliasing.

2.4.3.2 Non-linear instabilities

Non-linear instability is removed neither by decreasing the quantity Δt nor by decreasing Δx . This fact was pointed out by Norman Phillips (1959).

Usually, the solution of a finite difference equation is the form of $\text{EXP}(ikm\Delta x)$ or a harmonic function. Now suppose there are two single waves, one with wave number m_1 and another with wave number m_2 . The product of these two waves using trigonometric identities produces two new waves with wave numbers (m_1+m_2) , (m_1-m_2) . One of them is shorter in wavelength than both original waves. As the solutions for future time are continuously generated, the new wavelengths become shorter and shorter. Eventually, wavelengths will be smaller than a two-grid-length. Aliasing will be caused by this process due to the effect of a non-linear term. This spurious computational phenomenon is known as non-linear instability.

2.4.3.3 Linearization

Consider a general nonlinear equation

$$\partial F / \partial t = f_1 (\partial F / \partial x) + f_2 (\partial^2 F / \partial x^2) + f_3 \quad (2.22)$$

where f_1, f_2, f_3 may be functions of x, t, F . If f_1, f_2, f_3 are replaced by f_1^*, f_2^*, f_3^* , then eq. (2.22) will become a linear equation.

Making good approximations will remove the nonlinearity from nonlinear terms. Only linear terms will exist in the new approximated difference equation. Usually, some unknown nonlinear terms will be replaced by the known values or predictors in order to linearize them.

2.4.3.4 Predictor-Corrector method

There are many ways to choose f^* as follows:

(1) Extrapolation predictor method.

Rubin and Steinhardt (1963) made a linear extrapolation from two previous time levels. That is

$$f(n+1, x, (F^{n+1})^*) = f(n+1, x, F^n, F^{n-1})$$

where $(F^{n+1})^*$ is a linear extrapolation from F^n and F^{n-1} .

(2) Predictor with an explicit formula

This is one kind of predictor-corrector scheme. There is one example in the following:

$$(F_m^{n+1} - F_m^n)/(\Delta t) = (f_1^*)^{n+1/2} \frac{((\partial F/\partial x)_m^{n+1} + (\partial F/\partial x)_m^n)}{2} + \quad (2.23)$$

$$(f_2^*)^{n+1/2} \frac{((\partial^2 F/\partial x^2)_m^{n+1} + (\partial^2 F/\partial x^2)_m^n)}{2} + (f_3^*)^{n+1/2}$$

where f^* are derived from an explicit formula, e.g.,

$$f^*(n+1/2, F^{n+1/2}) = f(n+1/2, F^n) \quad (2.24)$$

eq. (2.24) is equivalent to eq. (2.25).

$$(F_m^{n+1/2} - F_m^n)/\Delta t/2 = f_1^n ((F_{m+1}^n - F_{m-1}^n)/(2\Delta x)) + \quad (2.25)$$

$$f_2^n ((F_{m+1}^n - 2F_m^n + F_{m-1}^n)/(\Delta x)^2) + f_3^n$$

Eq. (2.25) is a predictor; and eq. (2.23) is a corrector.

(3) Predictor with an implicit method.

This is one kind of predictor-corrector scheme. The following example is the Douglas predictor-corrector scheme (Douglas 1962), where the corrector is the same as (2) but his predictor is implicit.

$$\begin{aligned} (F_m^{n+1/2} - F_m^n) / (\Delta t / 2) = f_1^n \frac{((\partial F / \partial x)_m^{n+1/2} + (\partial F / \partial x)_m^n)}{2} + \\ \frac{((\partial^2 F / \partial x^2)_m^{n+1/2} + (\partial^2 F / \partial x^2)_m^n)}{2} + f_3^n \end{aligned} \quad (2.26)$$

where eq. (2.26) is a predictor. There is other example in this category where both predictor and corrector are implicit (Douglas-Jones 1963).

2.4.3.5 Iteration

The iterative procedure is generally terminated when two successive values of predictor are close to each other. The degree of accuracy is chosen by user.

This method is the best from the accuracy standpoint, but it is time-consuming. All methods in the section 2.5.4 can be combined with iterative methods.

CHAPTER 3

NUMERICAL MODELS OF THE HUDSON ESTUARY

3.1 General Description of the Hudson Estuary3.1.1 Topography

The Hudson Estuary (Fig. 3-1) flows from upstate New York into the Atlantic Ocean near New York City. Its downstream boundary was chosen as the Battery by many investigators (Harleman et al. 1972, Overland 1973, Abood 1974). Seaward of the Battery, the channel broadens into New York Harbor where irregular geometry and complicated tidal and salinity conditions resulting from the influence of the East River and Long Island Sound and of the Narrows preclude any direct connection of the Hudson Estuary with the Atlantic Ocean. Its upstream boundary was chosen by those investigators at the Federal Dam in the Troy Locks (212.4 kilometers north of the Battery), making it a closed-end estuary.

The Hudson, just north of the Battery, has a cross-section with an average width and depth of approximately 1,220 meters and 12.2 meters respectively. At approximately 32 kilometers north of the Battery, however, the river begins to widen and becomes more shallow until it reaches its maximum width of approximately 5,800 meters and its minimum mean depth of approximately 3.7 meters at the site which is 53.9 kilometers north of the Battery. Within the next 6 kilometers the

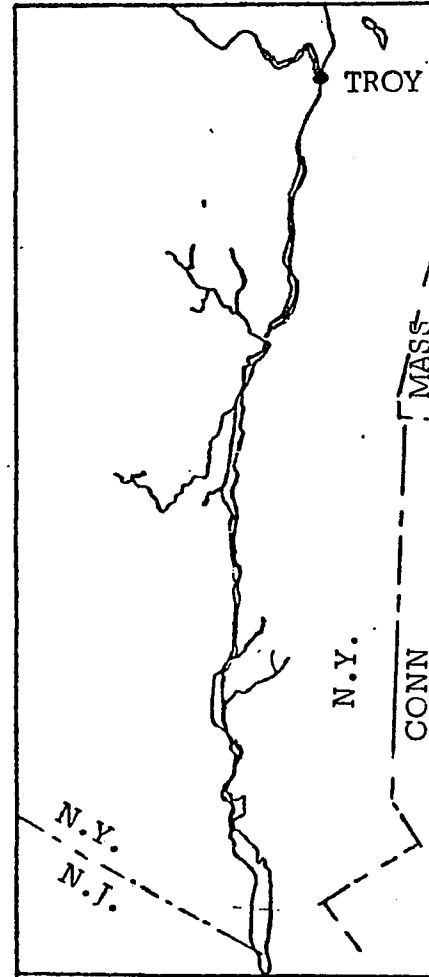
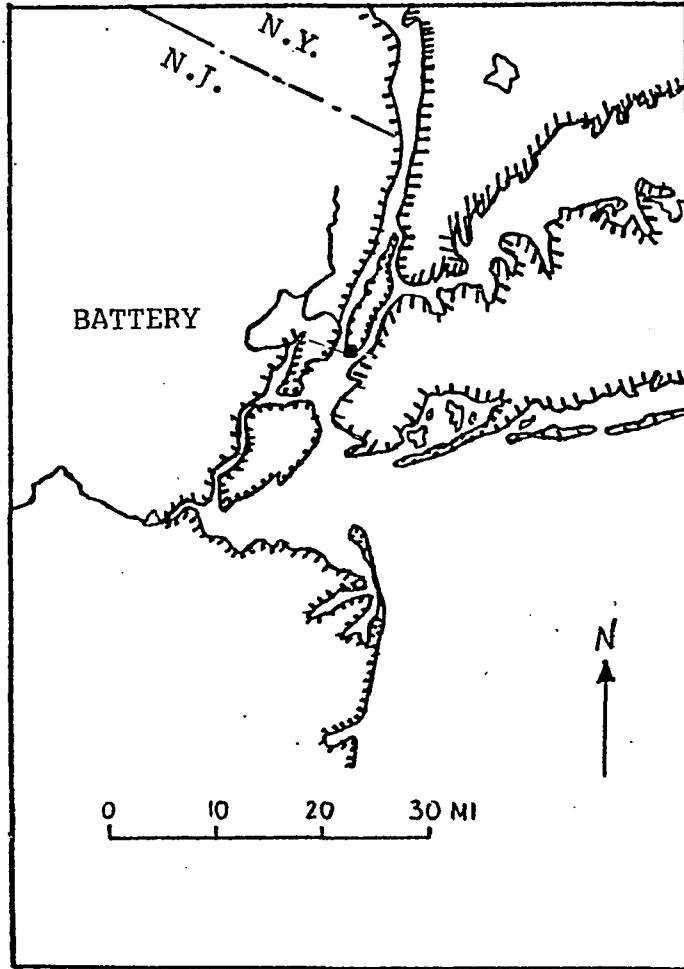


FIGURE 3-1

river's width drops to less than 800 meters and its mean depth increases to 22 meters. From there the river gradually gets narrower and more shallow until Saugerties where there is another discontinuity in the river's width.

3.1.2 Fresh water influx

As a fairly large estuary, the Hudson exhibits a large degree of variation and instability in its salinity distribution. One measure of flushing time is the ratio (R) of the total estuarine water volume (V) to mean annual fresh water flow (Q). According to Simpson et al. (1974) $V = 2.97 \times 10^9 \text{ m}^3$, $Q = 550 \text{ m}^3/\text{sec}$ which yields the value of R equal to 0.18 years. For the Chesapeake Bay, the value of R is 1.2 years and it is 0.6 years for the Delaware Bay. Therefore, the Hudson has a lower ratio of fresh water flow and a higher salinity instability.

3.1.3.1 Tidal range and current

The mean tidal ranges (NOAA: 1974 Tidal Table) are:
at the Battery--1.37 m.; at the George Washington Bridge--1.19 m.;
at Newburgh--0.88 m.; at Poughkeepsie--0.94 m.; at Albany--1.40 m.;
and at Troy--1.43 m.

Mean current velocities (NOAA: 1974 Tidal Current Table) at these stations respectively are: 0.75 m/sec (1.5 knots) flood and 1.15 m/sec (2.3 knots) ebb; 0.8 m/sec (1.6 knots) flood and 1.1 m/sec (2.2 knots) ebb; 0.45 m/sec (0.9 knots) flood and 0.55 m/sec (1.1 knots) ebb; 0.55 m/sec (1.1 knots) flood and 0.6 m/sec (1.2 knots) ebb;

0.15 m/sec (0.3 knots) flood and 0.4 m/sec (0.8 knots) ebb, and 0.35 m/sec (0.7 knots) ebb (no flood).

3.1.3.2 Stratification VSF

The values of the VSF in the Hudson, computed by Abood (1974), are shown in the following:

<u>Annual Averages (VSF)</u>			
<u>Distance</u> *	<u>Maximum</u>	<u>Mean</u>	<u>Minimum</u>
0	11	28	60
59.5	4	10	21
69.2	4	9	20
106.2	3	8	17

These numbers indicate that not only does the Hudson stratification vary radically with time and with distance but also that this combination of salinity instability and stratification inhomogeneity bears out the complicated and unique difficulties to be expected when modeling the Hudson.

3.1.4 Tidal dynamics

A tidal wave is usually a shallow water wave in estuaries, because the wavelength, L , of a tidal wave in estuaries is much longer than the depth, H , of the estuaries. The approximate formula for a shallow water wave is, $C = \sqrt{gH}$ (Neumann & Pierson 1967), to calculate the celerity of the tidal wave, C . The depth (H) of the Hudson is

* Distances are measured in kilometers north of Battery, N.Y.

approximately 10 meters, therefore the celerity- (C) in the Hudson is approximately 10 m/sec.

The period of a lunar semi-diurnal tidal wave, T, is 12.42 hours in the Hudson. There is a basic relationship, $L=TC$, between wavelength, period and celerity. The wavelength (L) is 282 miles from a calculation using the above formula. Since this approximation, $C = \sqrt{gH}$, neglects the effects of friction, it would be expected that the real value for the average C is somewhat less than 10 m/sec with a correspondingly smaller L.

3.2 Historical Description of Numerical Models of the Hudson Estuary

3.2.1 MIT model

3.2.1.1 Tidal dynamics model

A one-dimensional tidal dynamics model of an estuary without salinity effect was developed by Harleman and Lee (1969). It is an explicit finite difference scheme. Their scheme should obey the Courant criterion.

They solved the simultaneous equations which included the continuity and the N.S. equations. There are five terms in the N.S. equation including acceleration, inertia, longitudinal change of cross-section, longitudinal change of depth, and bottom frictional force terms. There is no wind stress.

They assumed a constant roughness coefficient. The coefficient was an important factor in the verification of their tidal dynamics model and was chosen numerically from a steady-state case.

3.2.1.2 Salt balance model

There are three terms in their salt balance equation including salinity which changes with time, advection and a longitudinal dispersive term.

The salt balance model was developed by Thatcher and Harleman (1972). It is an implicit finite difference scheme.

The longitudinal dispersive coefficient was determined from a correlation method for the degree of stratification which was based upon their WES (Water Experiment Station) data and several field data reports for the Delaware Estuary.

3.2.1.3 Application to the Hudson Estuary

Harleman et al. (1972) applied their tidal dynamics model and salt balance model to the Hudson Estuary. There is an additional term, longitudinal density gradient, in the N.S. equation. Those two models are coupled by the linear salinity-density relationship. It predicts the transient longitudinal distribution of salinities along the estuary.

The necessary inputs to execute their model are water surface elevations, maximum flood salinities at the downstream boundary and freshwater inflows along the estuary at all of the predictive periods.

The assumptions and empirical methods, which were employed by them, are summarized in the following:

- (1) negligible Coriolis force
- (2) constant roughness coefficient with time and space

- (3) no wind stress
- (4) a linear equation of state
- (5) time-dependent salinities at the sea-end boundary
- (6) artificial time elapse to change from the lowest to highest salinity at the sea-end boundary.
- (7) constant longitudinal dispersive coefficient of the Hudson.

The model is capable of predicting the time-history of salinities in estuaries. The model can be applied to estuaries with complicated topography, different regional hydrograph and various cross sections. The possibility of applying one dimensional salt equation to estuaries is mostly based upon the precision of longitudinal dispersion coefficient. However, for the Hudson estuary, the result of the correlation between longitudinal dispersion coefficient and the degree of stratification in their studies is not as good as others (p. 15-16, Harleman et al. 1972).

Comparisons on the Hudson estuary showed several anomalies. One of the reasons of such disagreements may be due to the lack of more complete field data regarding salinity distributions to verify the mathematical model.

More detailed field data regarding salinity distributions and more research work (e.g., Fisher 1976, Posmentier and Rayment 1979) relating to determine longitudinal dispersion coefficient may provide a better base to apply their model to predict the time-history of salinities in the Hudson estuary.

The numerical scheme of their model is a good example of decoupling the salt balance model with the tidal dynamics model. Such technique encourages mathematical modelers to modify some existed models with additional functional terms in the subprogram rather than to develop new coupled models.

3.2.2 Overland's intrusion model

A two-dimensional, fourteen layer model for a salt intrusion in partially mixed estuaries was developed by Overland (1973). It presents the steady-state solutions of vertical and horizontal distributions of salinity and velocity in response to thirteen different combinations of run-off, and ratio (K/N) in terms of stratification factor. The major intent is to calculate directly the differential advectations of salt rather than estimating an apparent horizontal dispersion coefficient.

He solved the simultaneous equations involving the continuity, N.S. and salt balance equations. There are five terms in the N.S. equation that include acceleration, longitudinal advection, surface slope, longitudinal density gradient and vertical diffusion terms. There are four terms in salt balance equation including salt change with time, salt transports of horizontal and vertical advection, vertical salt diffusion terms. He employed a linear equation of state. The model used a time invariant salinity profile at the sea-end boundary. There is no wind stress.

From the above description, there are three simultaneous, coupled, non-linear and time-dependent advection-diffusion equations. Overland employed a central difference operator for the time derivative. He employed a Jacobian type scheme suggested by Arakawa (1966) and tested successfully in weather prediction for the advection term. He also employed an implicit scheme for the diffusion term. It is really a semi-implicit scheme. In addition, he employed several empirical methods to eliminate numerical instabilities including: (1) Using forward time scheme occasionally to control weak divergence due to the effect of central time differencing, (2) Using the empirical (1,2,1) smoother upon the salt-balance equation for every iteration to save the growth of spurious computational mode, (3) Using artificial infinity eddy coefficients when an unstable situation developed.

The assumptions and empirical methods are summarized in the following:

- (1) Coriolis force is neglected.
- (2) Width and depth are constant respectively all along the estuary.
- (3) Wind stress is negligible.
- (4) A linear equation of state is assumed.
- (5) Forward time derivative scheme is employed occasionally.
- (6) Artificial smoother is employed periodically.
- (7) Artificial infinity eddy coefficients are imposed occasionally.
- (8) Empirical relationships in terms of stratification parameters for N and K are employed.

Under the circumstances and lack of data for the vertical variation of the tidal mean flow for the Hudson Estuary, he gets the similar circulation pattern and the same right order of magnitude for the velocities value as the James estuary and other partially mixed estuaries.

The model of Overland is still not practical, because the tidally average vertical salinity profile at the sea-end boundary is time-independent. There are several difficulties to model a real estuary including unconfirmed turbulent processes, ratio (K/N), bottom frictional force and wind stress on shallow water, and complicated topography, cross section and geometrical sea-end boundary. It is harder for the Hudson Estuary (Chap. 3.1).

But it is some contribution from the scientific standpoint to find out several characteristics for many partially mixed regions and to develop the numerical technique of the advanced model.

3.2.3 Vachtsevanos's model

Vachtsevanos proposed (1974) a laterally-averaged unsteady thermal and salinity model for the Hudson Estuary. Including the effects of thermal-induced density gradient and the corresponding thermal transport is one of main features of this model.

He solved the simultaneous equations of the continuity, N.S., salt transport, and thermal transport equations. There are six terms in the N.S. equations which include: acceleration, horizontal and vertical advection, surface slope, density gradient and viscosity terms.

There are four terms in salt transport equation which include: salt change with time, horizontal and vertical salt transport, and vertical diffusion terms. There are six terms in thermal transport equation which include: temperature change with time, horizontal and vertical heat transports, heat transports due to horizontal and vertical eddy conductivity, and heat source terms. He employed a linear equation of state between density, salinity and temperature.

Surface elevations and velocities were, approximately and explicitly, derived from two simultaneous equations including one-dimensional equations of continuity and momentum. It was an implicit scheme for four simultaneous, two-dimensional equations which were linearized in terms of surface slope and advection by these predictors (elevations and velocities).

He employed several empirical formulas, which were fitted by Bowden (1967), for all four turbulent transport coefficients including eddy viscosity, eddy diffusivity, vertical and longitudinal thermal diffusivities. But in these formulas there are still several parameters which will be determined, empirically and numerically, by the modeler. It is required to specify oceanic salinity at the sea-end boundary during the entire flood cycle, and to determine the salinity empirically during the period of the ebb. He also assumed that the sea-end boundary condition of temperature is similar to those of salinity.

There is no report about the result of the model and the flexibility, sensitivity, restrictions and time interval of the numerical scheme.

CHAPTER 4

MATHEMATICAL FORMULATION

4.1 Basic Governing Equations

Quantitative derivation is based upon conservation principles for water, momentum and salt.

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0. \quad (4.1)$$

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} + \frac{\partial}{\partial z} \langle u'w' \rangle = \frac{-1}{\rho} \left(\frac{\partial P}{\partial x} \right) \quad (4.2)$$

$$\frac{\partial S}{\partial t} + U \frac{\partial S}{\partial x} + W \frac{\partial S}{\partial z} + \frac{\partial}{\partial z} \langle s'w' \rangle = 0. \quad (4.3)$$

where the notation is as follows:

- x horizontal coordinate along channel and positive seaward,
- z vertical coordinate and positive upward,
- t time,
- U tidally, laterally and vertically averaged velocity component in the x direction,
- W tidally and laterally averaged velocity component in the z direction,
- S tidally, laterally and vertically averaged salinity,
- u',w',s' perturbation from the mean value for U,W,S respectively,

ρ tidally, laterally and vertically averaged density,
 P Pressure

4.2 Formulation of the Model

In this report, entrainment and diffusion are imposed on a classical two-layered tidally averaged circulation model (see Fig. 4.1) which was described in Chap. 3.2. Those variables (U, W, S, H, η) with subscript indexes 1 and 2 will denote the upper and lower layers respectively (see Fig. 4.2 and Appendix 1). And η is the height of water above the bottom.

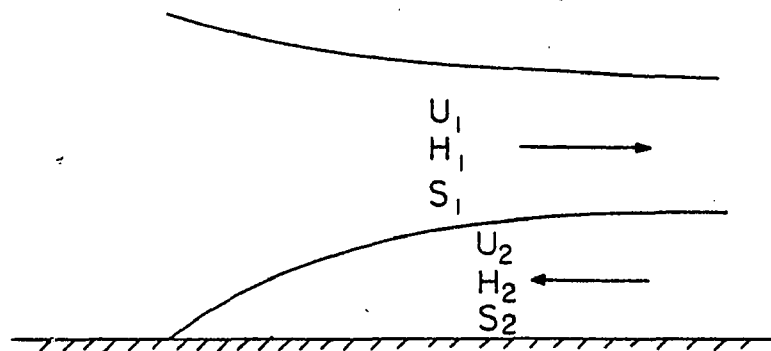


FIGURE 4-1

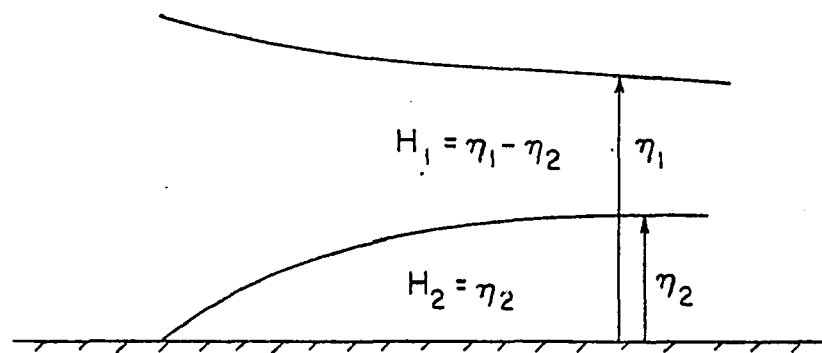


FIGURE 4-2

4.2.1 Assumptions and restrictions

The principal assumptions and restrictions used in the development of the model are:

- (1) Assume complete mixing (including momentum, density) within the individual layers except across the interface.
- (2) Fluid is incompressible.
- (3) The interface of density and the interface of velocity coincide.
- (4) Wind effects are neglected. This assumption makes the model somewhat atypical. However, this term could easily be added if the model were to be applied to an area where the effects of wind are significant.
- (5) Surface gravity waves are filtered out by employing rigid-lid approximation.
- (6) Assume flat bottom.
- (7) Tides are neglected.
- (8) The hydrostatic assumption is used.
- (9) A linear equation of state is employed.

Assumption (8) specifies that variations of density are dynamically important only in determining the vertical pressure gradient.

$$P(x, z) = P_0(x) + g \int_z^{\eta_1} \rho \, dz \quad (4.4)$$

where g is acceleration due to gravity, P_0 is the atmospheric pressure and η_1 is the coordinate of surface.

Assumption (9) is formulated by the equation

$$\rho = \rho_0 (1 + \beta s) \quad (4.5)$$

where ρ_0 is the density of fresh water and β is a constant, and numerically equal to $7.8 \cdot 10^{-4}$ (1/PPT).^{*} This is regarded as a good enough approximation to the equation of state for conditions in estuaries (Bowden 1967).

4.2.2 Vertical flux, vertical velocity and entrainment velocity

Assumption (1) implies that the entrainment velocity is zero everywhere except near the interface. Total vertical velocity (W) includes two parts: one part is the entrainment velocity (W_e) which is defined as the volume across the interface per unit normal area of interface and per unit time; the other part is a traditionally defined vertical advective velocity (W_a). Therefore the following two equations are satisfied.

$$W = W_e + W_a \quad (4.6)$$

$$W_a = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad (4.7)$$

Velocity (W) satisfies the continuity equation. The detailed explanation and consistency between W , W_e and W_a are presented in Appendices 1 and 2.

Assumption (1) implies that vertical flux is negligible everywhere except across the interface.

^{*} PPT = part per thousand.

Assume

$$u'w' \Big|_{\eta_{2-}}^{\eta_{2+}} = -W_e (U_1 - U_2) - N \frac{\partial U}{\partial z} \Big|_{\eta_{2-}}^{\eta_{2+}} \quad (4.8)$$

$$\left(N \frac{\partial U}{\partial z} \right) \Big|_{\eta_{2+}} = \left(N \frac{\partial U}{\partial z} \right) \Big|_{\eta_{2-}} = N' (U_1 - U_2) \quad (4.8')$$

and assume

$$s'w' \Big|_{\eta_{2-}}^{\eta_{2+}} = -W_e (S_1 - S_2) - K \frac{\partial S}{\partial z} \Big|_{\eta_{2-}}^{\eta_{2+}} \quad (4.9)$$

$$\left(K \frac{\partial S}{\partial z} \right) \Big|_{\eta_{2+}} = \left(K \frac{\partial S}{\partial z} \right) \Big|_{\eta_{2-}} = K' (S_1 - S_2) \quad (4.9')$$

where N is the coefficient of vertical viscosity,

K is the coefficient of vertical diffusivity,

N' is the viscosity parameter,

K' is the diffusivity parameter.

4.2.3 Boundary conditions

4.2.3.1 Surface

(A) Wind effects are neglected (assumption 4),

therefore

$$\langle u'w' \rangle = 0.$$

(B) There is no salt flux through the surface, therefore

$$\langle w's' \rangle = 0.0$$

(C) The kinematic condition of the surface is

$$w_a = \frac{\partial \eta_1}{\partial t} + U_1 \frac{\partial \eta_1}{\partial x}$$

4.2.3.2 Interface

The horizontal velocity (U), total vertical velocity (W) and vertically advective velocity (W_a) are discontinuous but pressure (P) is continuous.

That is:

$$\lim_{\epsilon \rightarrow 0} U_{\eta_{2+}} \neq \lim_{\epsilon \rightarrow 0} U_{\eta_{2-}}$$

$$\lim_{\epsilon \rightarrow 0} W_{\eta_{2+}} \neq \lim_{\epsilon \rightarrow 0} W_{\eta_{2-}}$$

$$\lim_{\epsilon \rightarrow 0} W_{a\eta_{2+}} \neq \lim_{\epsilon \rightarrow 0} W_{a\eta_{2-}}$$

$$\lim_{\epsilon \rightarrow 0} P_{\eta_{2+}} = \lim_{\epsilon \rightarrow 0} P_{\eta_{2-}}$$

And further, we can show (see Appendix 1) in the following:

$$\lim_{\epsilon \rightarrow 0} \eta_{2+} = \lim_{\epsilon \rightarrow 0} \eta_{2-}$$

$$W_{\eta_{2+}} - W_{\eta_{2-}} = (U_1 - U_2) \left(\frac{\partial \eta_2}{\partial x} \right)$$

and for a steady state

$$(W_{\eta_{2+}} - W_e) / (W_{\eta_{2-}} - W_e) = U_1 / U_2$$

4.2.3.3 Bottom

(A) There is no flow through the bottom, therefore

$$W = 0.$$

(B) There is no salt flux through the bed of the estuary, therefore

$$\langle w's' \rangle = 0.$$

(C) The stress of the fluid on the bed is assumed to obey the quadratic friction law, hence

$$\langle u'w' \rangle = -RU|U|$$

where R is roughness coefficient of bottom.

4.3 Final Formulation of the Model

4.3.1 Continuity equation

Integrating eq. (4.1), we obtain (see Appendix 2)

$$\frac{\partial H_2}{\partial t} = -W_e - \frac{\partial}{\partial x} H_2 U_2 \quad (4.10)$$

And according to assumptions (5) and (6),

$$\frac{\partial H_1}{\partial t} + \frac{\partial H_2}{\partial t} \approx 0. \quad (4.11)$$

4.3.2 Navier-stokes (N.S.) equation

Bottom Layer: After integrating eq. (4.2) from η_3 to η_{2-} ,
we obtain (see Appendix 3)

$$H_2 \frac{\partial U_2}{\partial t} + H_2 U_2 \frac{\partial U_2}{\partial X} = N (U_1 - U_2) - R U_2 |U_2| - \int_0^{\eta_2} \frac{1}{\rho} \left(\frac{\partial P}{\partial X} \right) dz$$

Upper Layer: After integrating eq. (4.2) from η_{2+} to η_1 ,
we obtain (see Appendix 4)

$$H_1 \frac{\partial U_1}{\partial X} + H_1 U_1 \frac{\partial U_1}{\partial X} = -N (U_1 - U_2) - W_e (U_1 - U_2) - \int_{\eta_{2+}}^{\eta_1} \frac{1}{\rho} \left(\frac{\partial P}{\partial X} \right) dz$$

Applies eq. (4.4) to the upper layer

$$P(x, z) = P_o(x) + g \rho_1 (\eta_1 - z) \quad (4.12)$$

and to the lower layer

$$P(x, z) = P_o(x) + g \rho_1 H_1 + g \rho_2 (\eta_2 - z) \quad (4.13)$$

Then

$$\int_{\eta_{2+}}^{\eta_1} \frac{1}{\rho_1} \left(\frac{\partial P_1(x, z)}{\partial X} \right) dz = \frac{H_1}{\rho_1} \frac{\partial P_o(x)}{\partial x} + \frac{g}{\rho_1} \frac{\partial \rho_1}{\partial x} \frac{H_1^2}{2} + g H_1 \frac{\partial \eta_1}{\partial x}$$

and

$$\int_{\eta_3}^{\eta_2} \frac{1}{\rho_2} \left(\frac{\partial P_2(xz)}{\partial x} \right) dz = \frac{H_2}{\rho_2} \frac{\partial P_o(x)}{\partial x} + \frac{g}{\rho_2} \frac{\partial \rho_1}{\partial x} H_1 H_2 +$$

$$g \frac{\rho_1}{\rho_2} H_2 \frac{\partial H_1}{\partial x} + \frac{g}{\rho_2} \frac{\partial \rho_2}{\partial x} \frac{H_2^2}{2} + g H_2 \frac{\partial H_2}{\partial x}$$

Therefore, final forms of N.S. equations become

$$H_2 \frac{\partial U_2}{\partial t} = -H_2 U_2 \frac{\partial U_2}{\partial x} - R U_2 |U_2| + N(U_1 - U_2) - \frac{H_2}{\rho_2} \frac{\partial P_o(x)}{\partial x} -$$

$$\frac{g}{\rho_2} \frac{\partial \rho_1}{\partial x} H_1 H_2 - g \frac{1}{\rho_2} H_2 \frac{\partial H_1}{\partial x} - \frac{g}{\rho_2} \frac{\partial \rho_2}{\partial x} \frac{H_2^2}{2} - g H_2 \frac{\partial H_2}{\partial x} \quad (4.14)$$

in the lower layer, and

$$H_1 \frac{\partial U_1}{\partial t} = -H_1 U_1 \frac{\partial U_1}{\partial x} - N(U_1 - U_2) - W_e(U_1 - U_2) -$$

$$\frac{H_1}{\rho_1} \frac{\partial P_o(x)}{\partial x} - \frac{g}{\rho_1} \frac{\partial \rho_1}{\partial x} \frac{H_1^2}{2} - g H_1 \frac{\partial \eta_1}{\partial x} \quad (4.15)$$

in the upper layer.

There are nine terms in the equation of the lower layer:

acceleration, horizontal advection, bottom stress, interfacial stress, atmospheric pressure gradient, density gradient of the upper layer, thickness gradient of upper layer, density gradient of the lower layer, and the thickness gradient of the lower layer.

There are seven terms in the equation of the upper layer: acceleration, horizontal advection, interfacial stress, entrainment, atmospheric pressure gradient, density gradient, and the surface slope.

4.3.3 Conservation equation of salt

Bottom Layer: After integrating eq. (4.3) from bottom ($Z = 0.0$) to η_{2-} , we obtain (see Appendix 5)

$$H_2 \frac{\partial S_2}{\partial t} + H_2 U_2 \frac{\partial S_2}{\partial x} = K (S_1 - S_2) \quad (4.16)$$

After combining above equation with eq. (4.10), we obtain

$$\frac{\partial}{\partial t} (H_2 S_2) + \frac{\partial}{\partial x} (H_2 U_2 S_2) = K (S_1 - S_2) - W_e S_2 \quad (4.17)$$

Upper Layer: After integrating from η_{2+} to η_1 , we obtain (see Appendix 6)

$$H_1 \frac{\partial S_1}{\partial t} + H_1 U_1 \frac{\partial S_1}{\partial x} = -K (S_1 - S_2) - W_e (S_1 - S_2) \quad (4.18)$$

After combining above equation with the continuity equation, we obtain

$$\frac{\partial}{\partial t} (H S_1) + \frac{\partial}{\partial x} (H_1 U_1 S_1) = -K (S_1 - S_2) + W_e S_2 \quad (4.19)$$

After combining equations (4.17) & (4.19), we obtain

$$\frac{\partial}{\partial t} (H_1 S_1 + H_2 S_2) + \frac{\partial}{\partial x} (H_1 U_1 S_1 - H_2 U_2 S_2) = 0 \quad (4.20)$$

CHAPTER 5

METHOD OF SOLUTION

5.1 Numerical Scheme

5.1.1 Predictor-corrector scheme

When simultaneous difference equations with diffusion are treated in an explicit scheme, both the diffusion term and the coupling which are introduced through the exchange coefficients often cause instabilities. There are a series of reports describing the development of a three-dimensional model for estuaries and coastal seas from the research group of Rand Corporation. They had spent many years developing an explicit scheme (Leendertse, Alexander & Liu, Vol. 1, 1973; Leendertse & Liu, Vol. 2, 1975; Leendertse, Liu & Nelson, Vol. 3, 1975). Because of instabilities which were caused by the coupled exchange coefficients, they had to give up their previous explicit scheme and change to the implicit scheme which is tedious, time-consuming to program but provided a better basis to work from (Leendertse, Liu, Vol. 4, 1977; Liu, Nelson, Vol. 5, 1977).

This research emphasizes the entrainment effects on salinity distribution which is averaged over tidal cycle. The coupled continuity, salt and momentum equations are solved numerically by the implicit scheme and linearized by the predictor-corrector method (Chap. 2.4). If there are any non-linear instabilities, an iterative method will be employed

to eliminate it (Chap. 2.4).

5.1.2 Flow diagram

A flow diagram of the computational scheme is given in Fig. 5-1. The computer program was written in FORTRAN. H_0 , U_0 and S_0 correspond to height, velocity and salinity at time $t=N$. HH' , UH' and SH' correspond to those predicted values at time $t=N+(1/2)$. HH , UH and SH correspond to those final predicted values at time $t=N+1$. H , U and S correspond to those corrected values at time $t=N+1$.

Pathway 1 in Fig. 5-1 is a forward time difference using the known values of (H_0, U_0, S_0) to predict the values of (HH', UH', SH') . Pathway 2 in the Figure is a central time difference using the known values of (H_0, U_0, S_0) and (HH', UH', SH') to predict the values of (HH, UH, SH) . Pathway 3 in the Figure is a backward time difference using the known values of (H_0, U_0, S_0) , (HH', UH', SH') and (HH, UH, SH) to derive the values of (H, U, S) . A method of iteration would be employed to eliminate those discrepancies between predicted and corrected values in case of non-linear instabilities due to disagreement. In the development of the scheme, an iterative method was employed for several cases. In the final program, there is no need for the help of iteration in all cases of two minute time intervals.

5.1.3 Initial values

A linear distribution in x for the heights, the velocities in the bottom layer and for the heights in the upper layer were used in the model as the initial values. The velocities in the upper layer were

FLOW CHART

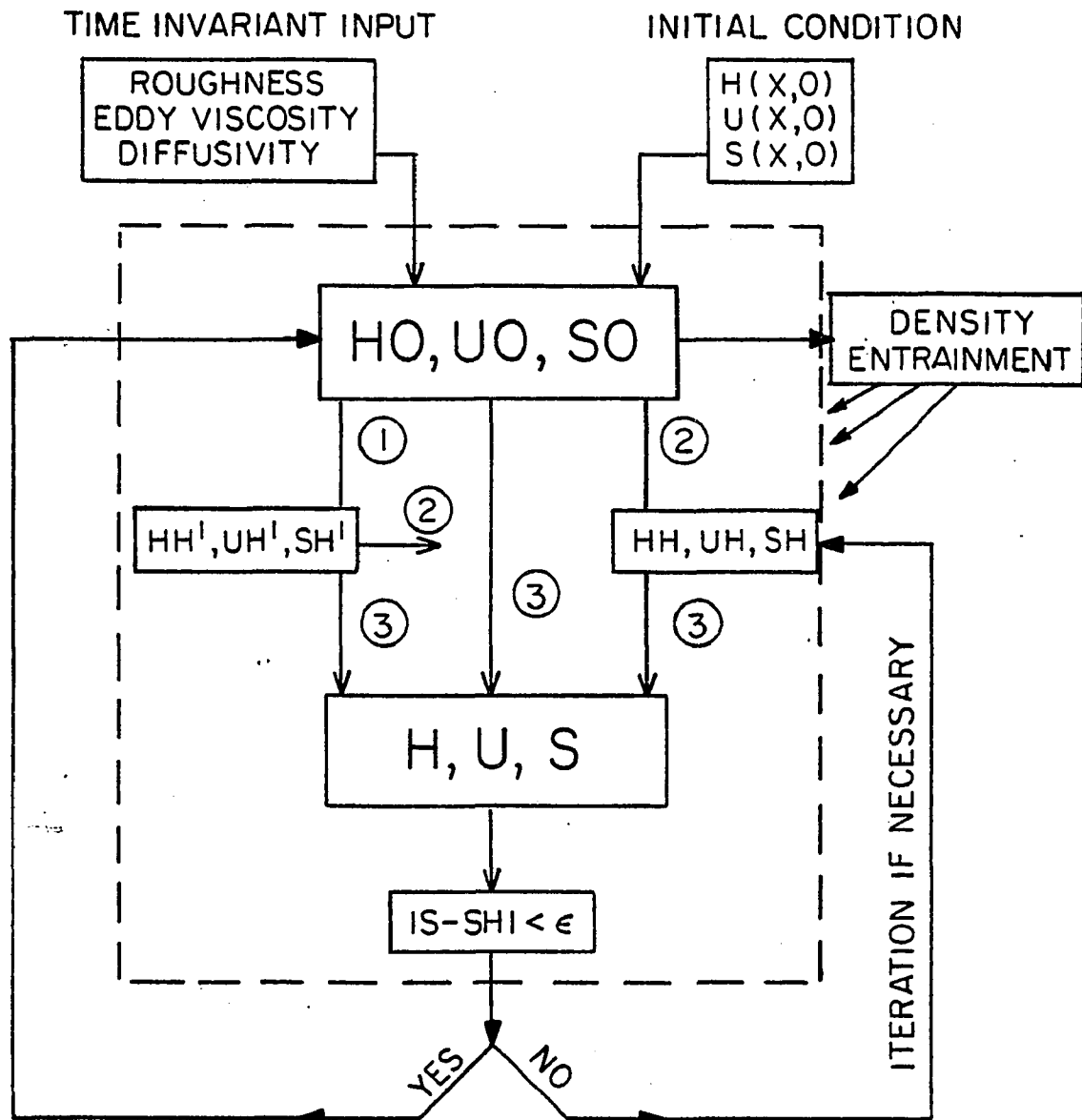


FIGURE 5-1

derived from the continuity equation and considered as initial values. A linear distribution in x (its results in part B of Chap. 6) for the salinities of both layers was considered as initial values. However, the model with the derived field data as the initial values for the salinities (its results in part A of Chap. 6) even leads to more stable results than those with linear distribution.

When initial values are closer to final results, convergence of the model calculation will be faster. If there is significant evidence that some available values are closer to final results than the original initial values, which have been described in the last paragraph, then they can be chosen to replace the initial values. For example, the case of $pp = 0.6$ (pp is a dimensionless parameter which notates an increase of entrainment in x direction; see Chap. 6.2 for details), the initial values were described in the last paragraph and results derived from the model are convergent. However, for the case of $pp = 0.8$, it may not be possible to get a convergent solution if the same initial values were chosen as that of $pp = 0.6$. So convergent results of $pp = 0.6$ are chosen as initial values for the case of $pp = 0.8$. However, nearly all initial values in this model are chosen from the linear distribution.

5.1.4 Steady state solution

From the equations of (4.1), (4.2), (4.3), in Chapter 4, there are time changes of the properties (i.e., $\partial H/\partial t$, $\partial U/\partial t$, $\partial S/\partial t$). To include these terms in the simultaneous differential equations requires including one more dimension (t), adding considerable difficulty to their solution. However, the inclusion of the time

dimension does not add to the difficulty of solving the finite difference numerical equations. Assume that there are M simultaneous differential equations and N grid points, therefore there are $N M$ difference equations to be solved. Every unknown is a function of time t . So there are actually $(N M + 1)$ unknowns to be solved, and the effort employed is not any different for $N M$ and $(N M + 1)$ unknowns in the difference equations.

As time goes on, if the terms $(\partial H / \partial t, \partial U / \partial t, \partial S / \partial t)$ become smaller and smaller, and all unknowns approach a steady state solution (Example: Fig. 5-2; Table 6-8). If the initial values are real, the results derived from the model will represent not only the steady state solution but also the time-dependent physical processes (time evolution).

5.2 Finite Difference Equations

5.2.1 Continuity equation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} (UH) = \pm W_e \quad (5.1)$$

where the positive entrainment (W_e) in the above equation corresponds to the upper layer, and the negative sign corresponds to the lower layer (see Appendix, eq. (A2.12) and (A 2.15)). We may rewrite the above conservative form into the following advective form:

$$\frac{\partial H}{\partial t} + U \frac{\partial H}{\partial x} + H \frac{\partial U}{\partial x} = \pm W_e \quad (5.2)$$

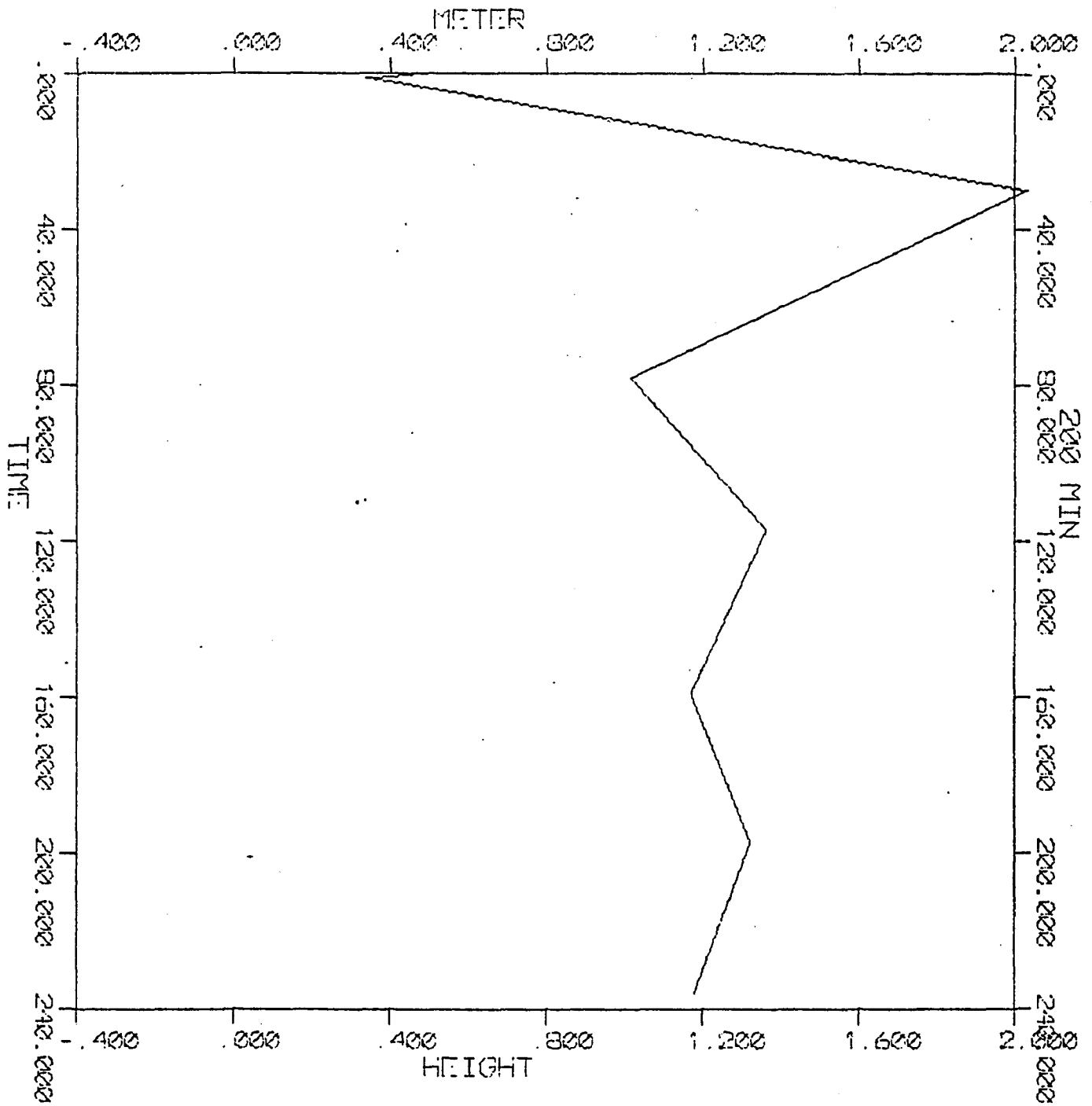


FIGURE 5-2

The above two differential equations are equivalent. But their corresponding difference equations are different in respect to the numerical instabilities they produce. In order to get numerically convergent results, all their corresponding difference equations should satisfy the condition,

$$\frac{\partial}{\partial H_{ij}} \left(\frac{\partial H_{ij}}{\partial t} \right) < 0 \quad (5.3)$$

where subscripts (i,j) correspond to ith and jth grid points in the x and z directions respectively. We tested the following cases.

ADVECTIVE FORM

$$\begin{aligned} \frac{H_{ij} - H_{0ij}}{\Delta t} + UH_{ij} \frac{H_{i+1,j} - H_{i-1,j}}{2\Delta x} + \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} HH_{i,j} &= \pm (W_e)_i \\ \frac{\partial}{\partial H_{ij}} \left(\frac{\partial H_{ij}}{\partial t} \right) &= \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} \end{aligned} \quad (5.4)$$

where Δt is the time interval for every time step, and

Δx is the distance between two consecutive grid points.

For the case of upper layer, eq. (5.4) satisfies the criterion of eq.

(5.3). It does not apply to the case of lower layer.

CONSERVATIVE FORM

$$\begin{aligned} \frac{H_{ij} - H_{0ij}}{\Delta t} + \frac{U_{i+1,j} H_{i+1,j} - U_{i-1,j} H_{i-1,j}}{2\Delta x} &= \pm W_e \\ \frac{\partial}{\partial H_{ij}} \left(\frac{\partial H_{ij}}{\partial t} \right) &= 0.0 \end{aligned} \quad (5.5)$$

Eq. (5.5) implies that the conservative continuity equation (5.1) includes the neutral condition of instability for both layers.

5.2.1.1 Finite difference equation

Equations (4.11) and (4.10) were employed using a central difference for the space gradient operator.

Upper Layer

$$(H_{i,1} + H_{i,2} - H_{i,1} - H_{i,2})/\Delta t = 0$$

Lower Layer

$$\begin{aligned} \frac{(H_{i,2} - H_{i,2})}{\Delta t} + \frac{U_{i+1,2} H_{i+1,2} - U_{i-1,2} H_{i-1,2}}{4 \Delta x} \\ + \frac{U H_{i+1,2} H_{i+1,2} - U H_{i-1,2} H_{i-1,2}}{4 \Delta x} = - (W_e)_i \end{aligned}$$

5.2.2 Navier-Stokes equation

Equations (4.14) & (4.15) are employed.

Upper Layer

$$\begin{aligned} \frac{U_{i,1} - U_{i,1}}{\Delta t} + U_{i,1} \frac{U_{i+1,1} - U_{i-1,1}}{2 \Delta x} + \frac{(W_e)_i}{H_{i,1}} (U_{i,1} - U_{i,2}) \\ = -g \frac{H_{i+1,1} + H_{i+1,2} - H_{i-1,1} - H_{i-1,2}}{2 \Delta x} + \frac{N}{H_{i,1}} (U_{i,2} - U_{i,1}) \\ - \frac{g H_{i,1}}{2 \rho_{i,1}} \frac{1000 \beta (S_{i+1,1} - S_{i-1,1})}{2 \Delta x} - \frac{1}{\rho_{i,1}} \frac{(Po)_{i+1} - (Po)_{i-1}}{2 \Delta x} \end{aligned}$$

Lower Layer

$$\frac{U_{i,2} - U_{i,2}^{0}}{\Delta t} + U_{H,i,2} \frac{U_{i+1,2} - U_{i-1,2}}{2\Delta x} = \frac{-g}{\rho_{i,2}} \frac{H_{H,i,1}}{2} \frac{1000}{\Delta x} \beta$$

$$(S_{i+1,1} - S_{i-1,1}) - \frac{g}{\rho_{i,2}} \frac{H_{H,i,2}}{4} \frac{1000}{\Delta x} \beta (S_{i+1,2} - S_{i-1,2})$$

$$- \frac{g}{\rho_{i,2}} \frac{\rho_{i,1}}{2} \frac{1}{\Delta x} (H_{i+1,1} - H_{i-1,1}) - \frac{g}{2} \frac{1}{\Delta x} (H_{i+1,2} - H_{i-1,2})$$

$$+ \frac{N}{H_{H,i,2}} (U_{i,1} - U_{i,2}) - \frac{R}{H_{H,i,2}} U_{i,2} \text{ABS}(U_{H,i,2}) - \frac{1}{\rho_{i,2}} \frac{(Po)_{i+1} - (Po)_{i-1}}{2 \Delta x}$$

5.2.3 Salt balance equation

Equations (4.16) & (4.17) are the two salt balance equations.

Some investigators preferred the advective form (Hamilton 1975);

however, while others preferred the conservative form (Overland 1973).

Both give trouble with numerical instabilities because of the lack of a horizontal smoother (i.e.

$$K_x \left(\frac{\partial^2 S}{\partial x^2} \right)$$

term) and changing slowly salinity field compared to velocity field.

The advective form may get help from an upwind differencing.* The conservative form may get help from a smoother for every cycle. Both methods of upwind differencing and the use of a smoother are empirical. The conservative form is good for a matrix method (Young & Hirt 1972).

*The idea of upwind differencing is that the change in salinity at any point due to advection is brought about only by salt advected from upstream.

According to those reports (Hamilton & Overland), there were unrealistic results near the upstream and downstream boundaries in the development of their models. Better numerical techniques are needed at these boundaries.

Equations (4.17) & (4.20) are employed as two independent differential equations. The central difference space gradient operator applied to these equations.

$$\begin{aligned}
 & \frac{HH_{i,2} S_{i,2} + H_{i,2} SH_{i,2}}{2 \Delta t} - \frac{HO_{i,2} SO_{i,2}}{\Delta t} + \\
 & \frac{UH_{i+1,2} HH_{i+1,2} S_{i+1,2} + UH_{i+1,2} H_{i+1,2} SH_{i+1,2} - UH_{i-1,2} HH_{i-1,2} S_{i-1,2}}{4 \Delta x} \\
 & - \frac{UH_{i-1,2} H_{i-1,2} SH_{i-1,2}}{4 \Delta x} \\
 & = K (S_{i,2} - S_{i,1}) - (W_e)_i S_{i,2}
 \end{aligned}$$

And

$$\begin{aligned}
& \frac{HH_{i,1}S_{i,1} + HH_{i,2}S_{i,2} + H_{i,1}SH_{i,1} + HH_{i,2}SH_{i,2}}{2 \Delta t} - \frac{HO_{i,1}SO_{i,1} + HO_{i,2}SO_{i,2}}{\Delta t} \\
& + \frac{UH_{i+1,1}HH_{i+1,1}S_{i+1,1} + UH_{i+1,1}H_{i+1,1}SH_{i+1,1} + UH_{i+1,2}HH_{i+1,2}S_{i+1,2} + UH_{i+1,2}H_{i+1,2}SH_{i+1,2}}{4 \Delta x} \\
& - \frac{UH_{i-1,1}HH_{i-1,1}S_{i-1,1} + UH_{i-1,1}H_{i-1,1}SH_{i-1,1} + UH_{i-1,2}HH_{i-1,2}S_{i-1,2} + UH_{i-1,2}H_{i-1,2}SH_{i-1,2}}{4 \Delta x}
\end{aligned}$$

$$= 0$$

CHAPTER 6

RESULTS, DISCUSSION AND CONCLUSIONS

6.1 General Description

Results in this report are separated into two parts. Part A is described in section 6.1.1, part B is described in section 6.1.2.

For mathematical convenience, distance x is defined to be zero at the upstream boundary, and is positive in the downstream direction.

A constant depth of 10 meters and a constant cross-section of 19000 square meters were assumed.

6.1.1 Part A: Salinity distribution and entrainment Assumptions

A series of cases were investigated to determine the flexibility of the model and to determine its sensitivity to various assumptions regarding entrainment mixing. Cases (1) to (3) were related to zero entrainment. Case (4) was related to the solutions without the N.S. equation. Cases (5) to (7) were related to different assumptions regarding entrainment mixing from my model.

The results of cases (5) to (7) were derived from the model with the same initial and boundary conditions (Chap. 5.1) which corresponded to the Hudson Estuary. The upstream velocity (0.02 m/sec) and downstream velocity (0.2 & -0.145 m/sec, corresponding to the upper and lower layers respectively) compare well with Hudson field

data. The eddy viscosity was kept constant for all cases (Table 6-1) in order to find the effects of entrainment.

6.1.2 Part B: Comparison with field data

Salinity data were taken from nine cruises (1974) by the Hudson Estuary Survey program of the C.U.N.Y. Institute of Marine and Atmospheric Science. Physical, biological, chemical, and geographical observations were included among routine station operations and have been reported (Posmentier and Raymont 1979, Posmentier and Rachlin 1976, Weiss, Rachlin and Coch 1975). Salinity measurements were made in situ by an Inter Ocean 513D Probe. Raymont (1976) averaged the salinity measurements vertically and determined the vertically-averaged salinity at the time of low slack, based on the assumption of tidal advection. This treatment of the data was used successfully by Harleman et al. (1972). Based on their data, the longitudinal salinity distributions on the dispersion equation of salt (which appear in Table 6-2) have been computed by Posmentier and Raymont (1979). Comparisons between the results derived from the model proposed here and their data will appear in Figures 6-4, 6-5, and 6-6.

The lengths of salt intrusions are assumed equal to $(2.0/\alpha)$, where α is a parameter determined from the one dimensional salt equation (eq. 6.2). Their values are listed in the Table 6-2. Observed salinities were maintained at the mouth end. A constant run-off was calculated by Raymont (1976) from the data provided by U.S.G.S.

Table 6-1

Tabulation of Cases Run

Case	Entrainment Assumption	Subcase		Others	Table	Figure
		K/W _e	Other Parameter			
5 (part A)	Constant	1.85	No	Dimensionless	6-3	6-1, 6-2 6-3, 6-7
6 (part A)	$W_e(x') = C_1 + pp \cdot x' \cdot 10$	1.85	pp = 0.1 pp = 0.4 pp = 0.8	Dimensionless	6-4	6-1
6A (part A)	$W_e(x') = C_1 \cdot \exp(-C_2 \cdot x' \cdot 10)$	1.85	c = 0.1 c = 0.25	Dimensionless	6-5	6-2
7 (part A)	$W_e(x') = C \cdot (V_1(x') - V_2(x'))$	1.85 7.50	No	Dimensionless Free Sea-end Velocities	6-6	6-3
8 (part B)	$W_e(x') = C \cdot (V_1(x') - V_2(x'))$	4.286	No	Comparison with field data R1: Median Run-off	6-7 6-8	6-4
9 (part B)	$W_e(x') = C \cdot (V_1(x') - V_2(x'))$	5.24	No	Comparison with field data R2: Weakest Run-off	6-9	6-5
10 (part B)	$W_e(x') = C \cdot (V_1(x') - V_2(x'))$	2.34	No	Comparison with field data R8: Strongest Run-off	6-10	6-6

Table 6-2

Tabulation of Field Cruises¹

Cruise Number	Date 1974	Q (m ³ /sec) Flow Rate	D (m ² /sec) Equation ²	X ₀ (km) Equation ³	α (m ⁻¹) Equation ³	L (km) Length of Salt Intrusion ⁴
R1	19-20 Jan.	748	704	9.64	5.6 10 ⁻⁵	35.0
R2	2-4 Mar.	1130	663	13.99	8.9 10 ⁻⁵	22.0
R3	6-7 Apr.	1023	631	2.90	8.5 10 ⁻⁵	23.4
R4	4 May	934	728	4.04	6.8 10 ⁻⁵	29.6
R5	7 June	815	434	4.28	9.8 10 ⁻⁵	20.2
R6	13 July	842	1340	15.76	5.0 10 ⁻⁵	40.5
R7	4-5 Oct.	564	507	12.46	5.9 10 ⁻⁵	34.2
R8	2 Nov.	364	438	3.98	4.4 10 ⁻⁵	45.7
R9	6-7 Dec.	709	482	0.69	7.7 10 ⁻⁵	25.8

1. All data except L come from Table 1 (Posmentier and Raymont 1979)

2. $U \frac{\partial S}{\partial X} = D \frac{\partial^2 S}{\partial X^2}$

3. $S = 35 \exp [-\alpha(x + x_0)]$

4. L is the assumed length of salt intrusion.

Results were derived from the model for all the above cases are summarized in the Table 6-1.

6.2 Results

6.2.1 Results (Part A)

One dimensional dispersive equation of salt for the steady state is given by the following form:

$$U (\partial S / \partial x) = A_x (\partial^2 S / \partial x^2) \quad (6.1)$$

where A_x is the dispersive coefficient which was assumed as constant throughout this section.

There are several cases proposed to be discussed in the following classification.

(A) Classification by Velocity

(1) Uniform Velocity

The solution of equation (6.1) is

$$S = S_0 \exp(-\alpha(x+x_0)) \quad (6.2)$$

where S_0 , X_0 , are the integration constants and will be determined by boundary conditions, and α ($= -U/A_x = \text{const.}$) is independent of x .

This simplest case has been discussed at length and adopted very often due to practical considerations. The possibility of applying this assumption to equation (6.1) is mostly based upon the precision of constant A_x . Several suggestions to modify A_x had been

discussed (Harleman et al., 1972, Posmentier & Rayment 1979, Fisher 1976, etc.).

(2) Non-uniform Velocity

The solution of eq. (6.1) under the assumption of case (2) is the same form as eq. (6.2). In the case (2), phase x_0 may be different from case (1) and $\alpha (= -U/A_x)$ is a function of x .

(B) Classification of Entrainment (two layers)

Coupling with the Continuity Equation

(3) Zero Entrainment

The continuity equation is given in the following form:

$$\partial Q_i / \partial x = W_e = 0.$$

$$Q_i = H_i U_i, \text{ index } i \text{ is either 1 or 2}$$

where $Q = H_1 U_1 + H_2 U_2 = HU$, i.e., the discharge per unit width.

The equation (6.3) implies that

$$Q_i = \text{const.}$$

and

$$U_i = \text{const.}$$

According to the continuity equation, the transports and velocities of the upper and lower layers are constants. The equivalent net horizontal velocities are all the same in this case. This is just case (1).

(4) Non-zero but Constant Entrainment

Transport solved by the continuity equation is

$$Q_1(x) = Q_0 + W_e x \quad (6.4)$$

where Q_0 is the discharge of the fresh river water at the head of the estuary. From the above equation, $Q_1(x)$ is increased linearly with distance x . The solution of simultaneous equations of (6.4) and (6.1) has the same form as equation (6.2). In this case (4), $\alpha (= -U/A_x = -/(H A_x))$ is a function of x .

(4A) Non-zero but Constant Entrainment

Coupling with two layer salt balance equations

Stommel (1952) solved simultaneously eq. (6.4) and the salt balance equations with entrainment equivalent to the following form:

$$Q_1 (\partial S / \partial x) = K (\partial^2 S / \partial z^2) + W_e (\partial^2 S / \partial z^2) \quad (6.5)$$

These results have been reported (Stommel 1952, Officer, 1976). The dimensionless ratio of eddy diffusivity to entrainment (K/W_e) was considered as a parameter. He obtained the results with different values of the ratio (K/W_e). He found that when he used the ratio (K/W_e) 5.0, the longitudinal distribution of salinity is linear. This agrees with the data of the James Estuary (Pritchard 1952) where the longitudinal distribution of salinity is also linear while the ratio in that case is 3.6.

(C) Classification by Entrainment (Two Layers)

Coupling with Continuity & N.S. equations

From case (1) to case (4A), there are no effects of frictional force, density current, interfacial stress and momentum exchanged due to entrainment. All of class (C) can be solved only by numerical methods.

(5) Entrainment Constant with Time t and Distance x

That is:

$$\partial W_e(x)/\partial x = 0. \quad (6.6)$$

Under the assumption of case (5), the solutions of the simultaneous equations in Chap. 4 are presented in Table 6-3, Fig. 6-1. From Fig. (6-1), there are linear distribution of salinities in both the upper and lower layers. This is a feature of the James Estuary (Pritchard 1952). There is no comparison between the ratio in this case and the case of the James Estuary due to different boundary conditions. The discussion will be presented in Chap. 6.3.1.

(6) Entrainment Constant with Time t and Changing

Linearly with Distance x

That is:

$$\partial W_e(x)/\partial x = \text{const.}$$

and

$$W_e(x') = C1 + pp \quad x' \quad 10.0 \quad (6.7)$$

where $C1$ and pp are constants specified by boundary conditions, and x' ($= x/L = 0.1, 0.2, \dots, 0.9$; L : total intrusion length of salinity) is a dimensionless distance parameter.

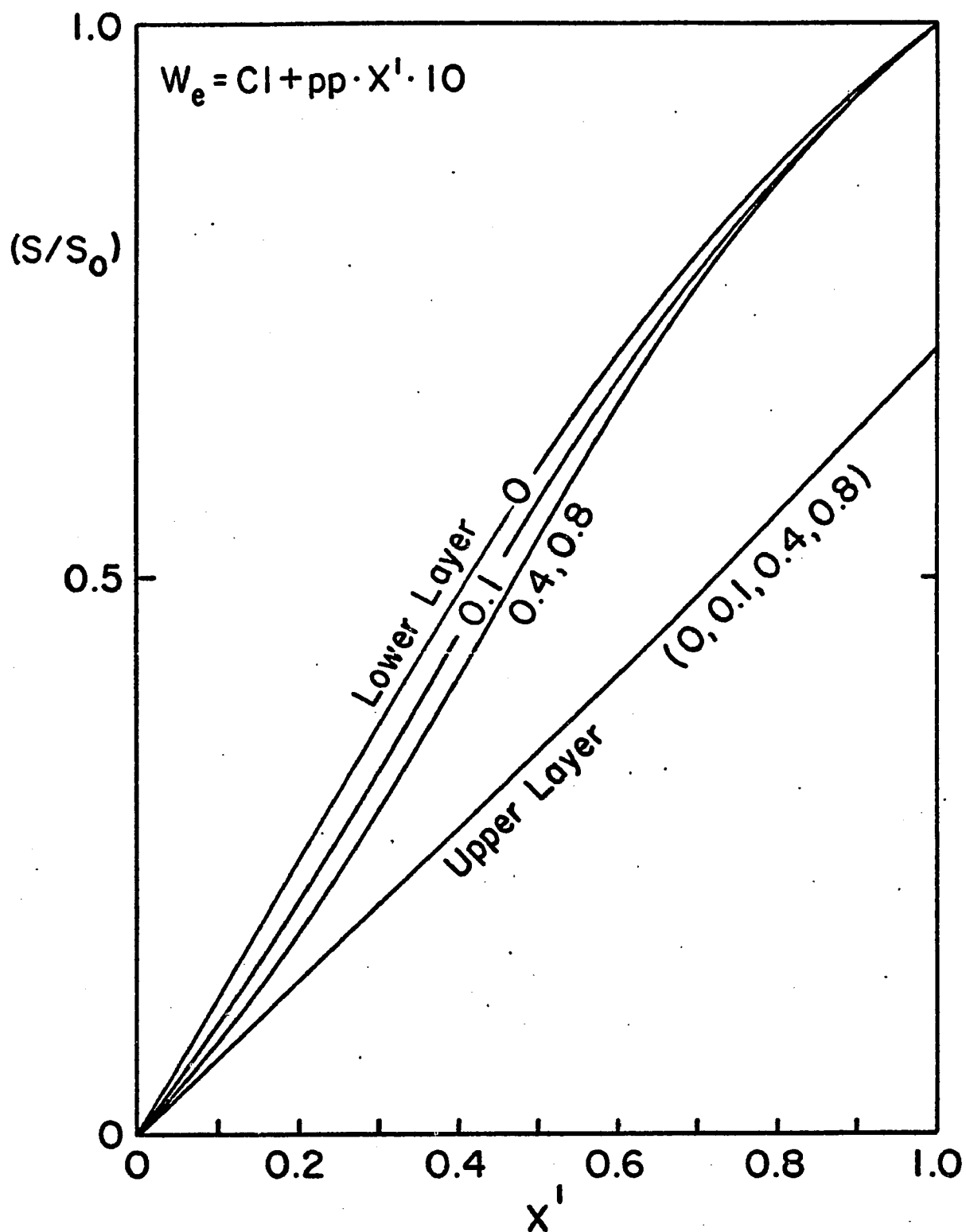


Figure 6-1

Under the assumption of case (6), the solutions of the simultaneous equations in Chap. 4 corresponding to different values of pp are presented in Table 6-4 and Fig. 6-1. The discussion will be presented in Chap. 6.3.1.

(6A) Entrainment Constant with Time t and Changing Exponentially with x

That is:

$$W_e(x') = C_1 \text{ EXP } (-C_2 x') \quad (6.8)$$

where C_1, C_2 are integration constants specified by boundary conditions.

The solutions corresponding to different values of C_2 are presented in Table 6-5, Fig. 6-2. The discussion will be presented in Chap. 6.3.1.

(7) Variable Entrainment

Assumed entrainment is a function of the frictional velocity only (Chap. 1.4). That is:

$$W_e(x') = \text{const.} \cdot (U_1(x') - U_2(x')) \quad (6.9)$$

The solutions corresponding to different values of the ratio (K/W_e) are presented in Table 6-6, Fig. 6-3. The discussion will be presented in Chap. 6.3.1.

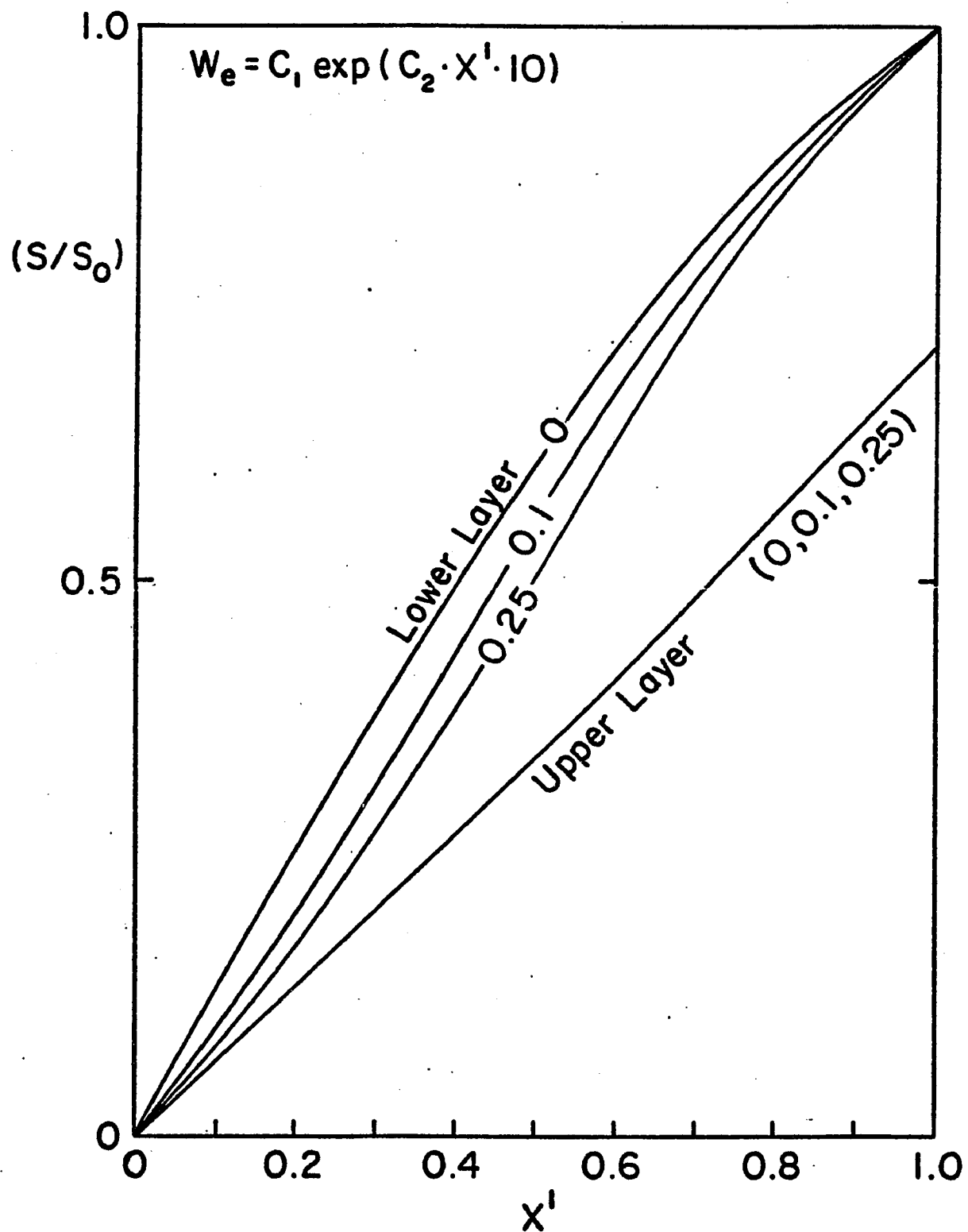


FIGURE 6-2

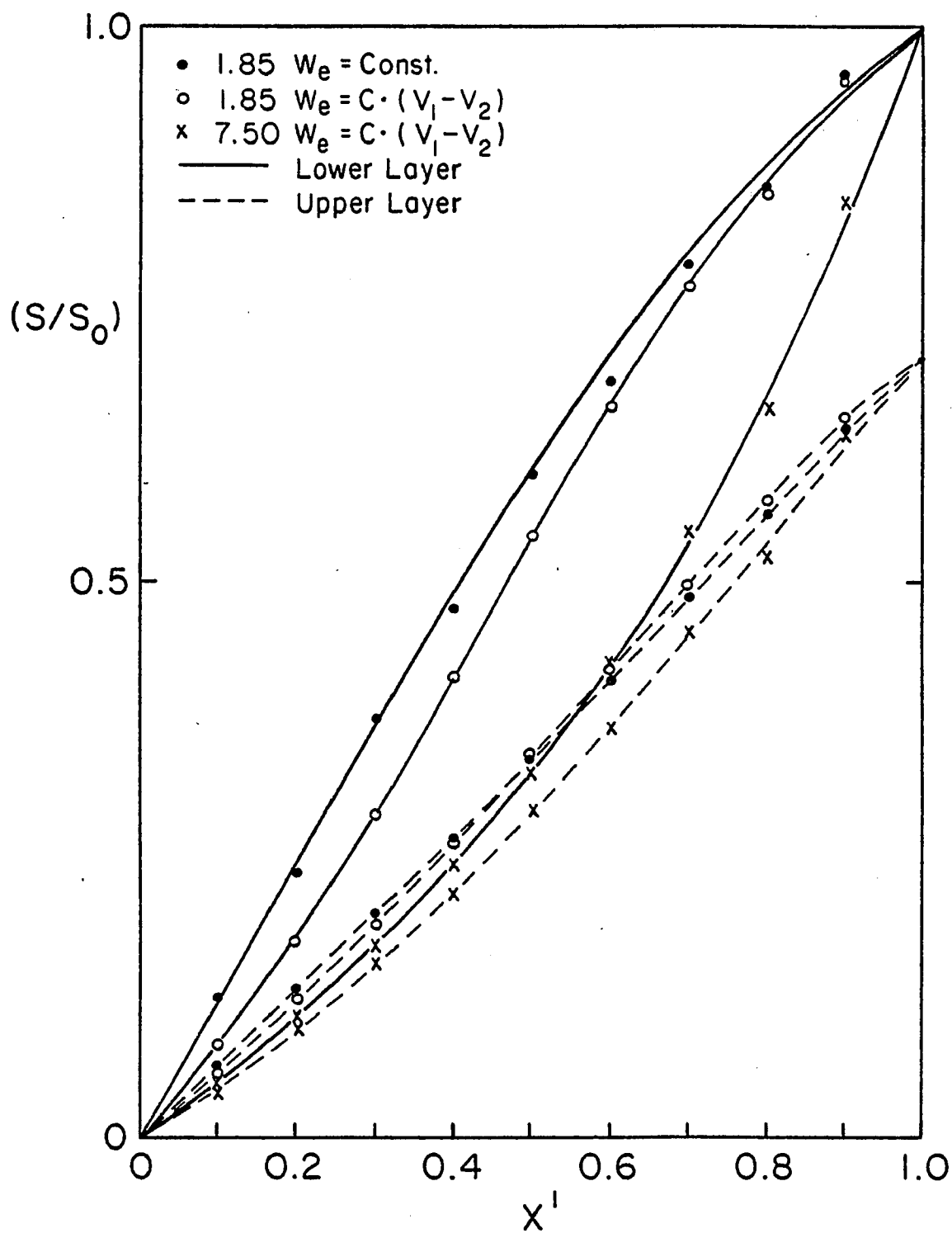


FIGURE 6-3

6.2.2 Results (Part B)

(8) Among all the run off rates corresponding to nine cruises in 1974, the lowest flow rate occurred during Cruise Two while the strongest runoff was measured on Cruise Eight with Cruise One being the median value of all cruises.

We adjusted different coefficients including coefficients of bottom frictional force, eddy viscosity and eddy diffusivity to fit the field data of Cruise One. The solutions, derived from this model with fitted coefficients, are presented for corresponding entrainment coefficients in Table 6-7 and Fig. 6-4. The field data of Cruise One are also presented in Fig. 6-4 for comparison. The separated numerical values of every term in the differential equations are listed in Table 6-8. The discussion will be presented in Chap. 6.3.2.

(9) The solutions of Cruise Two, derived from this model with proper corresponding boundary values (Chap. 5.1) and the same coefficients as Cruise One (case 8), are presented in Table 6-9 and Fig. 6-5. The field data of Cruise Two are also presented in Fig. 6-5 for comparison. The discussion will be presented in Chap. 6.3.2

(10) The solutions of Cruise Eight, derived from this model with proper corresponding boundary values (Chap. 5.1) and the same coefficients as Cruise One (case 8), are presented in Table 6-10 and Fig. 6-6. The field data of Cruise Eight are also presented in Fig. 6-6 for comparison. The discussion will be presented in Chap. 6.3.2.

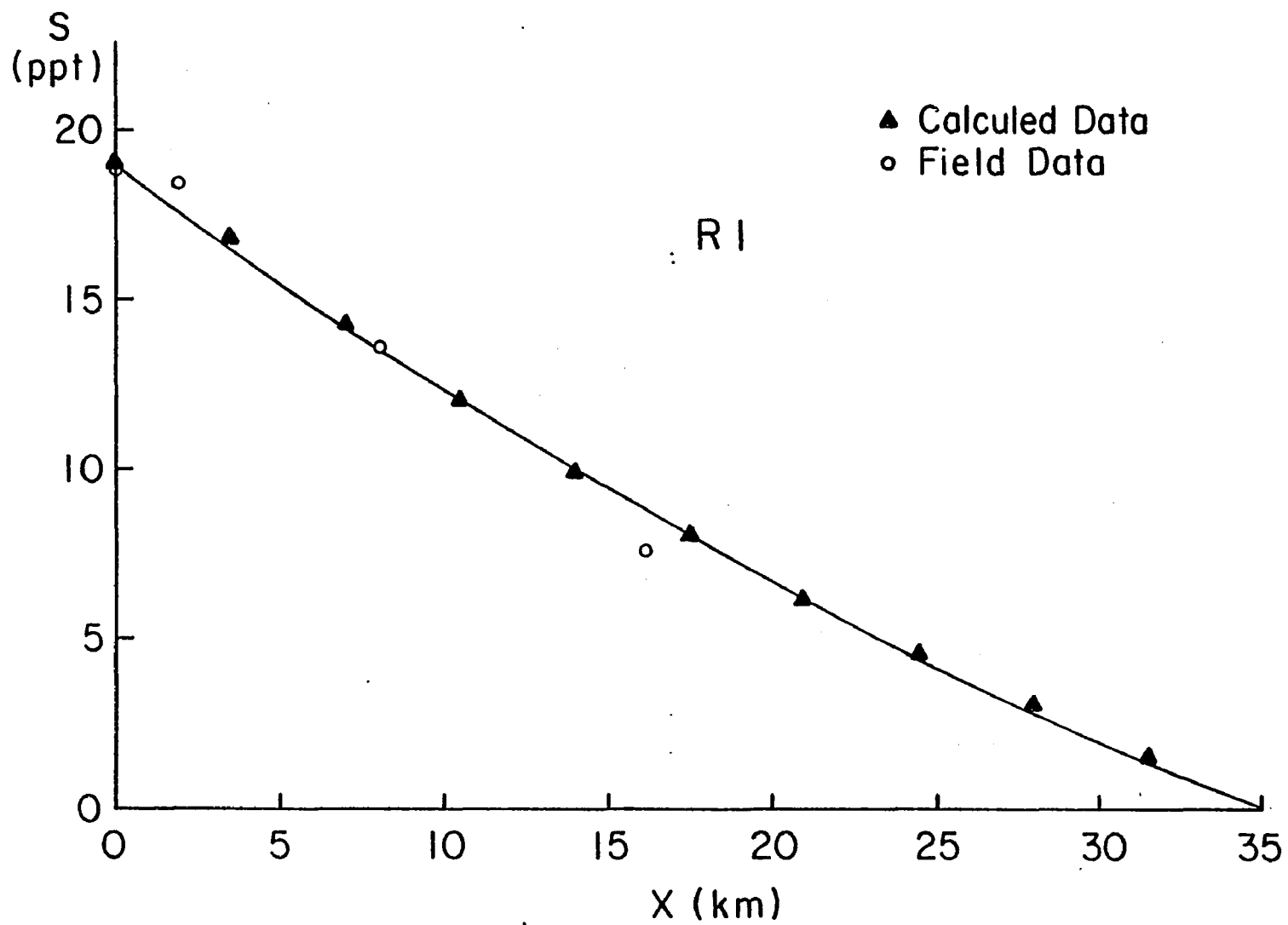


FIGURE 6-4

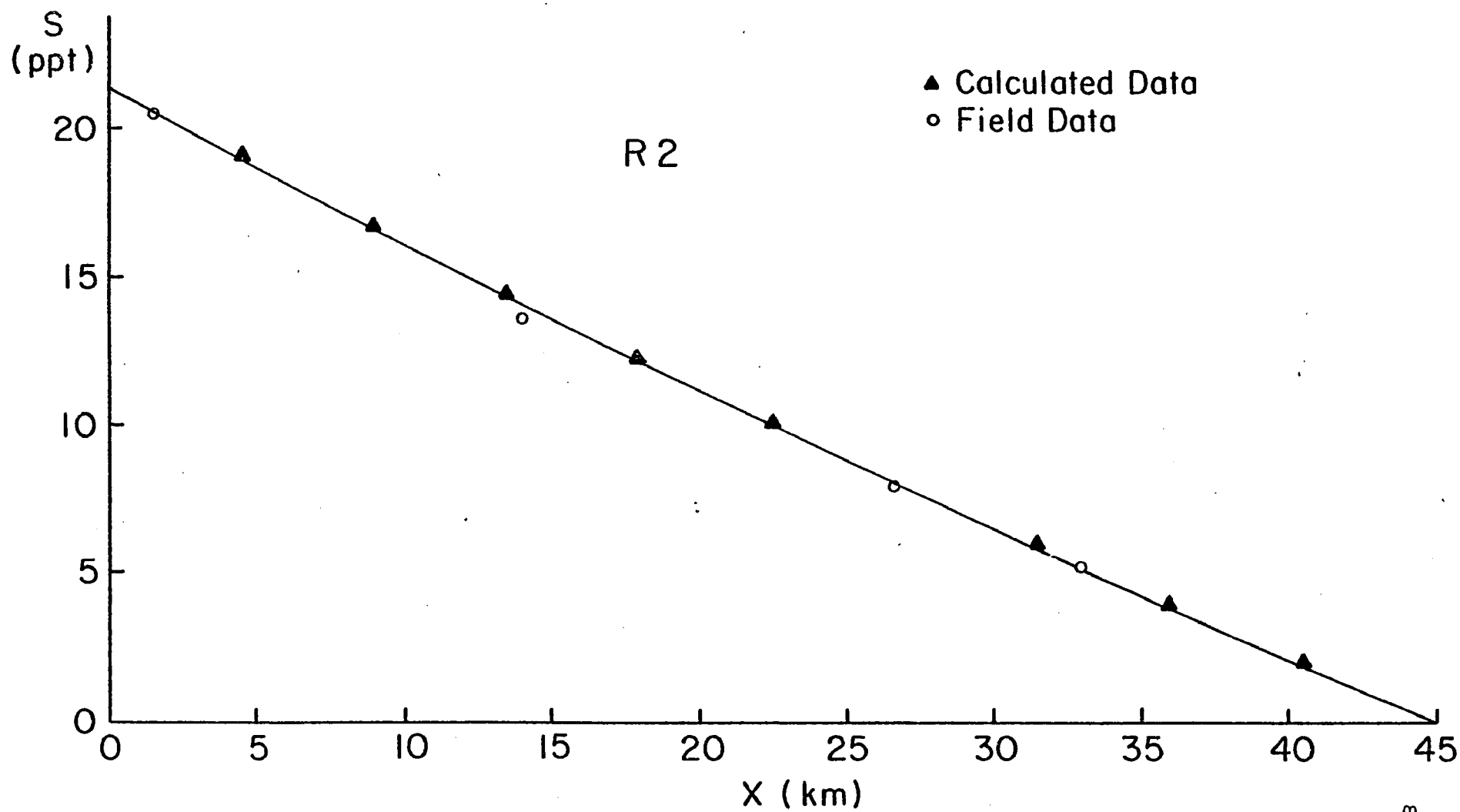


FIGURE 6-5

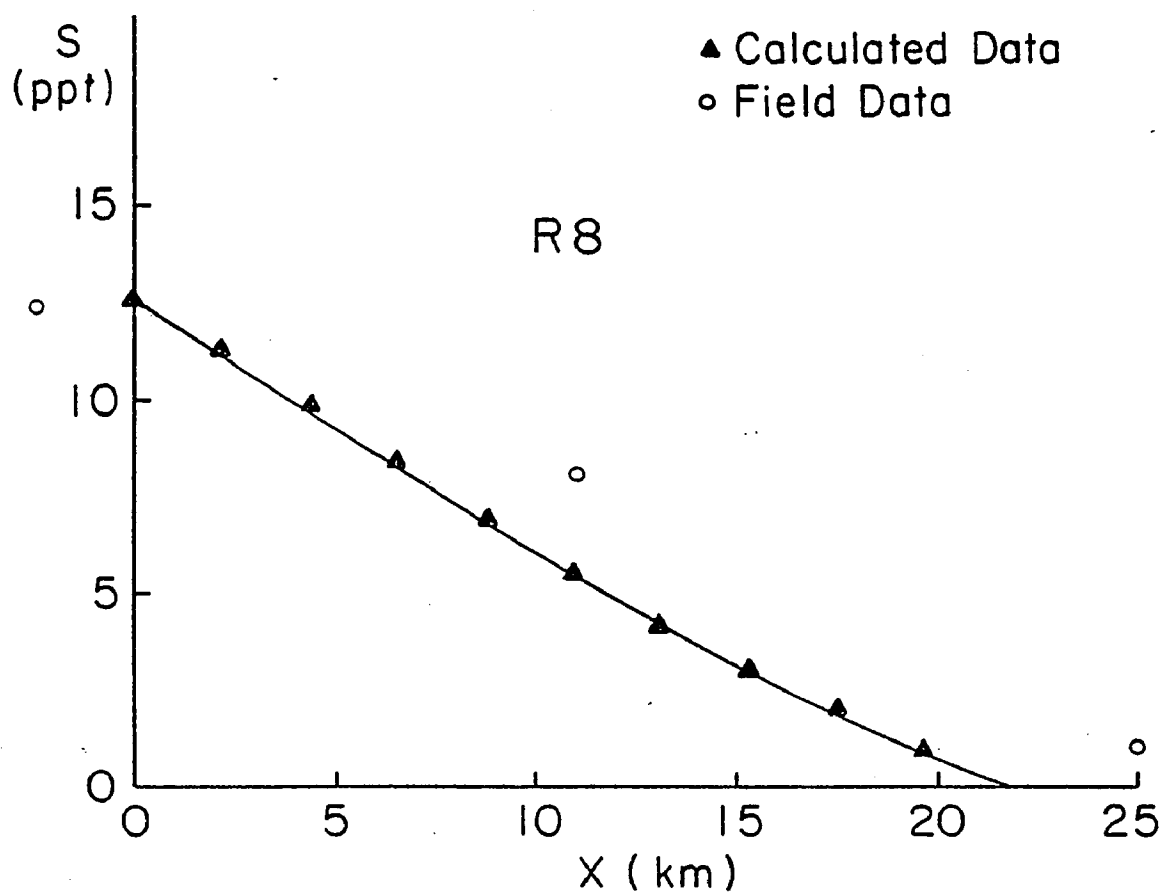


FIGURE 6-6

6.3 Discussion

6.3.1 Discussion (Part A)

6.3.1.1 Cases (5) to (7)

6.3.1.1.1 Direct and indirect entrainment effects on salinities

There are several feature of entrainment effects which are revealed in the order analysis (Table 6-8) in this section.

There is no direct effect from entrainment upon the N.S. and salt equations for the bottom layer. While in the upper layer, there is a small effect upon both equations due to entrainment. An order analysis shows that there is a large effect upon the continuity equation in both layers due to entrainment (Table 6-8). Therefore, differences of velocities and salinities are smaller than the difference of water flux during the same period in each layer. The total entrainment is constant but the frictional force does increase with higher values of pp (cases (5) to (6)) after redistribution of flux, and this is because the frictional force was assumed as having a non-linear effect upon the velocity. Therefore, the velocities and water flux decrease with higher values of pp , as does the salinity.

6.3.1.1.2 Cases (5) to (6)

Although there is little direct effect of entrainment on salinity in the bottom layer, there are still some indirect entrainment effects due to water flux and the frictional force (Chap. 6.3.1.1.1) in the cases (5).

According to Table 6-8, the effect of inter-facial stress between the two layers is negligible in comparison with the effects of the bottom frictional force. It corresponds to a small indirect effect of entrainment on salinity in both layers due to this inter-facial stress. The bottom frictional force has a dominant effect in the bottom layer and no direct effect on the upper layer (Table 6-2, eq. 4-14, and eq. 4-15). This implies that over-all entrainment effects on salinity in the upper layer are negligible in comparison with those of the lower layer. Fig. 6-1 summarizes these features.

In case (6A), the results are similar to case (6). These results are to be expected because the functional forms and the assumptions used were similar for each case.

6.3.1.1.3 Cases (7)

In the cases (7), entrainment was assumed as a function of inter-facial stress. In order to test this assumption, the velocities at the sea-end boundary were set to be free and to be a function of total entrainment.

6.3.1.1.4 Diffusion effects on salinity and velocity

From an order of magnitude analysis (Table 6-8), the diffusion term has a greater effect upon velocity than entrainment. This can be seen in case (7) where the velocity is smaller than that of case (5) because of larger bottom frictional forces which is similar to

case (6) and to the larger exchange of momentum between the two layers (Tables 6-3, 6-4, 6-6). In addition, diffusion also has a greater effect on salinity than entrainment (Table 6-8). For example, salinity in both layers for cases (7) are not very different from those of previous cases (5) and (6) for the same ratio (K/W_e). The salinities do differ greatly in both layers for different ratio of (K/W_e). This holds true for different assumptions of entrainment (cases (5) and (6)) or when the same assumption of entrainment is applied (case (7)). Fig. 6-3 summarizes these features. Diffusion therefore is the controlling factor responsible for the changes in salinity in both layers, it is a two way process (Chap. 1. 3.1). It is a reasonable result that ΔS (difference in salinity between two layers) will be smaller, the greater the diffusion.

6.3.1.2 Comparison with Stommel's Case

The results of Stommel's case are presented for one specific case of freshwater run-off. It is very difficult to compare Fig. (6-1) with those of Stommel's case directly. A comparison between cases (5) to (4A) for the same boundary condition is presented in diagram (6-7). One set of data on diagram (6-7) is calculated directly from the formula derived by Stommel (Officer 1977: pp. 168-171, Formula (5-74) & (5-75)). Another set of data is the result of case (5). These two sets of data are very close except near the upstream end of the salt wedge. Salinity results in case (5)

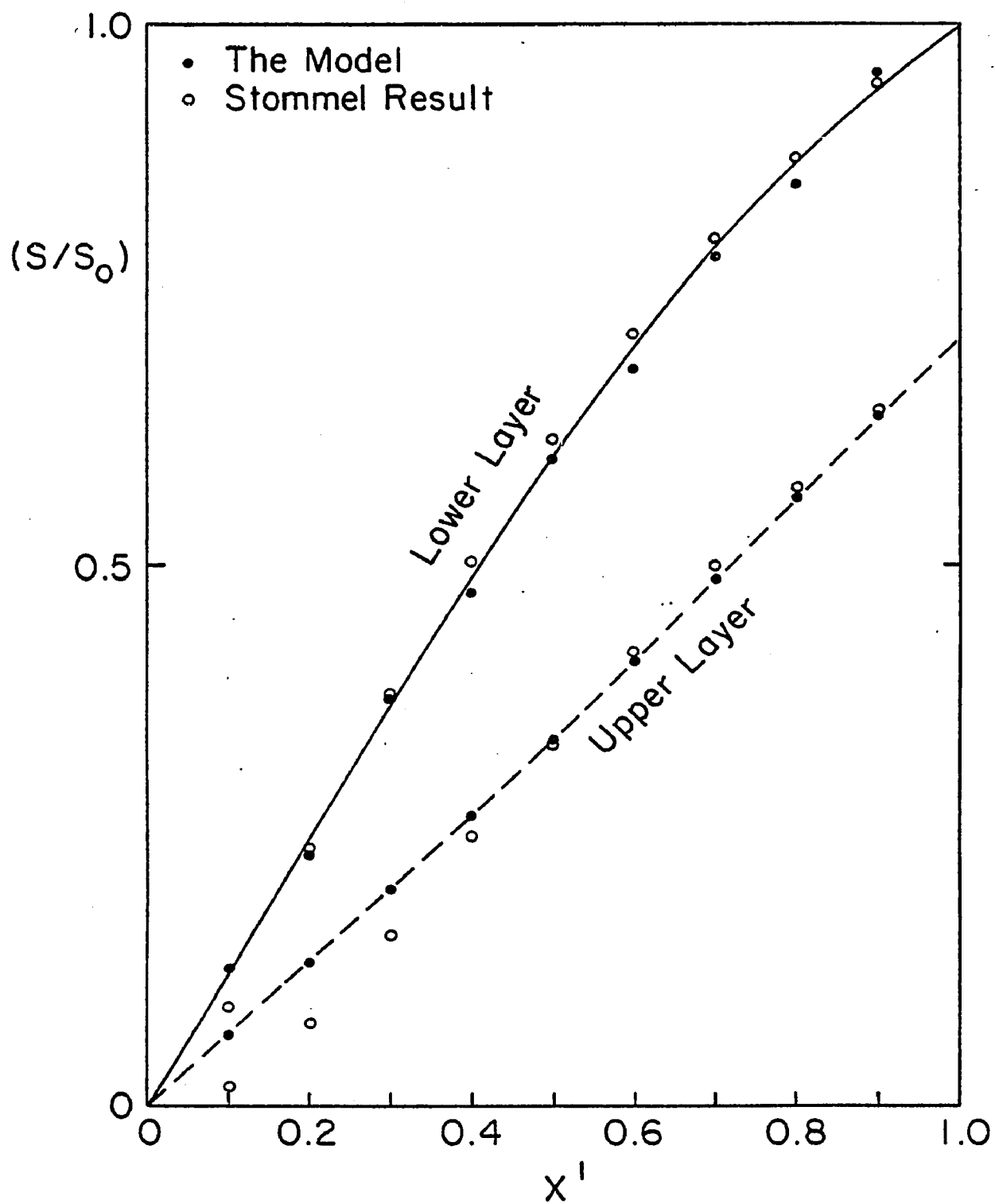


FIGURE 6-7

are smaller than the results using Stommel equation. This may be due to the effects of the bottom frictional force. The reasons of disagreement around upstream ($x' = 0.1$) may come from: (1) Some effect of longitudinal diffusion (or dispersion) beyond that region (Chap. 6.1.2; 6.3.2). (2) Unrealistic approximation in the numerical scheme at that region. This region is very sensitive to the decreasing values of thickness and velocity. It may be necessary to employ other artificial approximations in that region (Overland 1973, Hamilton 1975).

Several features that are confirmed from Figures (6-1, 2, 3, 7) are the following:

- (1) Distributions of salinities depend on the ratio (K/W_e)
- (2) Salinities are the function of run-off freshwater.
- (3) At least for certain special cases, there are linear relationships between salinity and distance x' in these diagrams. These linear relationships were described as characteristic of partially mixed estuary by Pritchard (Chap. 1.5).

Several features which were not derived by Stommel are the following:

- (1) The effects of diffusion on salinities are more dominant than the effects of entrainment.
- (2) Entrainment has a greater effect directly on thickness than on salinity and velocity.

(3) Entrainment has little direct effect on salinities but produces some indirect effect through waterflux and frictional force.

(4) Distribution of thickness is sensitive to the functional form of entrainment.

(5) Case (7) is a better assumption than other for the Hudson Estuary.

(6) Ratio (K/W_e) for the Hudson Estuary seems to be between 1.85 and 7.5.

(7) It is very difficult to obtain values for the ratio (K/W_e) greater than 7.5 using a constant eddy viscosity (case 7). But it is not for case (4A). This is because there is a maximum ratio between eddy viscosity and diffusivity (Chap. 1.3.3 & 5.1); beyond which, the values are not realistic.

(8) The bottom frictional coefficient is a very important factor in the verification of the numerical model.

(9) Improper initial condition will bring about numerical instability.

(10) Density current is an important factor in the equation of N.S. (Table 6-8).

Items (7) to (9) will be discussed further in section 6.3.3.

6.3.2 Discussion (Part B)

A comparison between calculated and measured salinities for the cases (8) to (10) shown in Figures 6-4, 6-5 and 6-6, respectively, should be made in the light of various characteristics of the field data and the mathematical model.

(1) Case (8) there is good agreement between observed and calculated values of salinity.

(2) For cases (9) and (10), a comparison between field data and the results from the mathematical model in regard to the longitudinal distribution of salinities (Figures 6-5 & 6-6) shows that the model with the same coefficients as in case (8) generally predicts accurately.

(3) The lack of detailed data of salinity around the upstream (including the length of salt intrusion) and discontinuity of salinities distribution about 64 Km north of the Battery (Raymont 1976) present a limitation on the verification of the mathematical model.

(4) Gravitational force (the combination of pressure head, surface slope, atmospheric pressure gradient and interface slope) is predominantly balanced by the density gradient and the bottom frictional forces. Terms involving advection, entrainment and inter-facial stress are negligible.

(5) The results from cases (8) to (10) confirm several previous discussions (Chap. 6.3.1.1).

(6) From Table 6-8, the sums of the items in the continuity equation, in the salt balance equation and in the Navier-Stokes equation all vanish. This agrees with the conservation of mass, salt and momentum, respectively.

6.3.3 Discussion (Numerical Scheme)

This scheme is good for a 5 minute time step, however, occasionally the technique of iteration is necessary. All results in this paper are run using a 2 minute time step without iteration.

Near the upstream end of the intrusion, the thickness of lower layer may be as small as 0.2 meter (Table 6-6). For the explicit scheme or other grid-point structure, it is necessary to satisfy the stability criteria in sections 2.4.2.1.5 & 2.4.2.2.3. They correspond to 100 and 4 second time steps for the case of strong run-off ($\Delta x = 2$ km, $(\Delta z)_{\min} = 0.2$ meter), these stability criteria are often not satisfied near the upstream end of the intrusion.

Improper bottom frictional coefficient will bring about numerical instabilities. This coefficient is different from one model to another and depends on the formula and definition of the coefficient and velocity related to the formula. To determine this coefficient one must apply a trial and error method of analysis.

For a fixed eddy viscosity (N), there is a limit for the value of diffusivity (K); otherwise, improper results will be generated. The functional form of the ratio (K/N) is still unconfirmed even under adiabatic conditions (Chap. 1.3). This point is very important to many models.

Entrainment decreases with an increased stratification factor, and falls into a separate category of possible effects (overturning will occur), when entrainment is greater than the critical entrainment

(neutral case). Of course, that will bring about physical instabilities, which is out of the scope of this model.

Initial conditions which are close to the steady-state solution will converge faster and save computer time. The original initial condition of $pp = 0.8$ of case (4) is neither convergent nor divergent; however, if the results of $pp = 0.6$ of case (4) are applied as the initial condition for $pp = 0.8$ of case (4), convergence is obtained.

The case of an open sea-end boundary condition for velocities is not only more flexible but also more stable than others. We only need to run the first 100 cycles to check the sea-end velocities for the cases (8) to (10). One must run 20,000 cycles at the minimum to be certain of convergence in case (9) and the greater the length of the salt intrusion the more cycles that must be run to produce convergence.

6.4 Conclusions

If coefficients are well chosen and velocities at the sea-end boundary are checked for their realistic results then this model will agree with the field data for the Hudson Estuary.

The assumption about entrainment as a function of velocity is the best among all other assumptions for entrainment not only for physical background but also for numerical stabilities and physical results.

The assumption, that the entrainment depends on velocities and is independent of a stratification factor, is a good approximation.

The ratio of entrainment to diffusion is a very important factor for the distribution of salinities; but the assumption of diffusion is more important than the assumption of entrainment in its effect on salinities.

This model has limited practical application, but it is useful for scientific research to explore the effects on salinities for different functional assumptions for entrainment and diffusion.

This model is under restraint due to the unconfirmed relationship between eddy viscosity and diffusivity and the exact form of the bottom frictional force. Other estuarine numerical models are under similar restraints.

The indirect effect of entrainment is much greater than its direct effect on salinity.

More data near the upstream boundary is needed to determine the length of salt-intrusion, the form of thickness of bottom layer and the functional form of entrainment.

Table 6-3

Case (5) Constant Entrainment

X'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.331	0.669	0.029	-0.040	2.241	4.372
0.2	8.171	1.829	0.051	-0.072	4.585	8.165
0.3	8.000	2.000	0.056	-0.077	6.983	13.150
0.4	7.197	2.803	0.083	-0.092	9.431	16.558
0.5	7.046	2.954	0.089	-0.095	11.919	20.863
0.6	6.340	3.660	0.120	-0.105	14.453	23.781
0.7	6.159	3.841	0.128	-0.106	17.057	27.484
0.8	5.557	4.443	0.159	-0.116	19.656	29.815
0.9	5.122	4.878	0.173	-0.112	22.312	33.421

Table 6-4

Linear Change of Entrainment with Distance

pp = 0.1

X'	H ₁	H ₂	U ₁	U ₂	S ₁	S ₂
0.1	9.517	0.483	0.033	-0.016	2.235	3.653
0.2	8.505	1.495	0.042	-0.062	4.591	7.225
0.3	8.261	1.739	0.048	-0.070	7.017	11.902
0.4	7.601	2.399	0.066	-0.084	9.499	15.689
0.5	7.346	2.654	0.075	-0.091	12.024	20.053
0.6	6.731	3.269	0.099	-0.101	14.582	23.416
0.7	6.412	3.858	0.114	-0.105	17.186	27.127
0.8	5.831	4.169	0.142	-0.115	19.780	29.800
0.9	5.253	4.747	0.166	-0.114	22.383	33.226

pp = 0.4

0.1	9.344	0.656	0.029	-0.037	2.222	2.910
0.2	9.002	0.998	0.033	-0.048	4.576	6.367
0.3	8.515	1.485	0.041	-0.062	7.021	10.247
0.4	8.142	1.858	0.049	-0.073	9.537	14.663
0.5	7.647	2.353	0.063	-0.085	12.101	18.888
0.6	7.215	2.785	0.079	-0.095	14.691	23.021
0.7	6.651	2.349	0.102	-0.104	17.306	26.645
0.8	6.159	3.841	0.128	-0.113	19.902	29.825
0.9	5.344	4.656	0.162	-0.117	22.459	32.995

Table 6-4 (continued)

 $pp = 0.8$

x'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.151	0.849	0.030	-0.043	2.217	2.632
0.2	9.274	0.726	0.030	-0.038	4.573	6.206
0.3	8.557	1.443	0.039	-0.061	7.019	9.550
0.4	8.397	1.603	0.044	-0.067	9.548	14.321
0.5	7.731	2.269	0.060	-0.084	12.124	18.346
0.6	7.426	2.574	0.071	-0.092	14.727	22.896
0.7	6.710	3.290	0.099	-0.104	17.351	26.417
0.8	6.288	3.712	0.118	-0.112	19.943	29.802
0.9	5.352	4.648	0.162	-0.118	22.487	32.893

Table 6-5

Case 6(a) Exponential Change of Entrainment with Distance

$$C_2 = 0.1$$

X'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.527	0.473	0.033	-0.016	2.227	3.456
0.2	8.593	1.407	0.040	-0.059	4.580	6.907
0.3	8.350	1.650	0.045	-0.067	7.010	11.418
0.4	7.736	3.264	0.061	-0.081	9.502	15.249
0.5	7.480	2.520	0.070	-0.088	12.041	19.656
0.6	6.891	3.019	0.092	-0.099	14.614	23.156
0.7	6.560	3.440	0.107	-0.104	17.226	26.954
0.8	5.966	4.034	0.134	-0.114	19.826	29.570
0.9	5.372	4.628	0.159	-0.115	22.414	33.180

$$C_2 = 0.25$$

0.1	8.801	1.199	0.038	-0.049	2.154	2.707
0.2	9.148	0.852	0.032	-0.042	4.452	6.434
0.3	8.345	1.655	0.044	-0.063	6.845	9.156
0.4	8.470	1.530	0.043	-0.064	9.331	13.859
0.5	7.745	2.255	0.059	-0.081	11.881	17.322
0.6	7.683	2.317	0.063	-0.086	14.468	22.273
0.7	6.895	3.105	0.092	-0.099	17.123	25.687
0.8	6.610	3.390	0.101	-0.111	19.537	29.514
0.9	5.715	4.285	0.193	-0.090	22.363	32.767

Table 6-6

Entrainment with Functional Form of Frictional Stress

$$(K/W_e) = 1.85$$

X'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.352	0.648	0.031	-0.036	2.068	2.843
0.2	8.900	1.100	0.035	-0.050	4.350	6.200
0.3	8.504	1.496	0.041	-0.063	6.786	10.230
0.4	8.092	1.908	0.050	-0.075	9.336	14.561
0.5	7.675	2.325	0.062	-0.086	11.959	18.941
0.6	7.216	2.784	0.077	-0.096	14.617	23.031
0.7	6.696	3.304	0.099	-0.104	17.323	26.794
0.8	6.169	3.831	0.123	-0.116	19.971	29.875
0.9	5.273	4.727	0.166	-0.117	22.646	33.192

Effects of Entrainment and Diffusion

$$(K/W_e) = 7.50$$

X'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.349	0.651	0.045	-0.029	1.588	1.706
0.2	8.678	1.322	0.050	-0.047	3.440	3.691
0.3	8.279	1.721	0.052	-0.058	5.514	6.037
0.4	7.778	2.222	0.060	-0.070	7.798	8.648
0.5	7.382	2.618	0.069	-0.080	10.295	11.639
0.6	6.931	3.069	0.082	-0.092	12.944	14.924
0.7	6.508	3.492	0.015	-0.099	15.890	18.829
0.8	6.128	3.872	0.115	-0.121	18.311	22.916
0.9	5.983	4.017	0.246	-0.074	22.212	29.458

Table 6-7

Case (8) Comparison with Observed Results

(R1: Median Runoff)

X'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.546	0.454	0.032	-0.034	1.489	1.635
0.2	8.820	1.180	0.042	-0.063	2.988	3.304
0.3	8.481	1.159	0.050	-0.085	4.495	5.181
0.4	7.909	2.091	0.064	-0.106	6.009	7.176
0.5	7.477	2.523	0.080	-0.127	7.526	9.459
0.6	6.912	3.088	0.105	-0.147	9.038	11.939
0.7	6.413	3.587	0.134	-0.168	10.528	14.703
0.8	5.838	4.162	0.176	-0.190	11.972	17.589
0.9	5.338	4.662	0.229	-0.217	13.332	20.651

Table 6-8

Case (8) Order of Magnitude Analysis

N.S. Equation (Upper Layer)

Gravitational term	=	1.47	10^{-5}
Advection term	=	-5.0	10^{-7}
Density Gradient	=	-1.23	10^{-5}
Interfacial stress	=	-1.09	10^{-6}
Entrainment	=	9.0	10^{-7}

N.S. Equation (Lower Layer)

Gravitational term	=	1.27	10^{-5}
Advection term	=	-7.0	10^{-7}
Density Gradient term	=	-3.11	10^{-5}
Interfacial stress	=	3.30	10^{-6}
Bottom Stress	=	1.58	10^{-5}

Salt Balance Equation (Upper Layer)

Advection	=	-2.6	$\times 10^{-4}$
Entrainment	=	3.6	$\times 10^{-5}$
Diffusion	=	2.2	$\times 10^{-4}$

Salt Balance Equation (Lower Layer)

Advection	=	2.1	$\times 10^{-4}$
Diffusion	=	-2.2	$\times 10^{-4}$

Continuity Equation (Lower Layer)

Advection	=	6.38	$\times 10^{-5}$
-Advection	=	3.10	$\times 10^{-5}$
Entrainment	=	-3.28	$\times 10^{-5}$

Table 6-9

Case (9) Comparison with Observed Results

(R2: Weakest Runoff)

X'	H ₁	H ₂	U ₁	U ₂	S ₁	S ₂
0.1	8.730	1.270	0.018	-0.038	1.959	2.087
0.2	8.981	1.019	0.018	-0.061	3.922	4.280
0.3	8.368	1.632	0.027	-0.082	5.887	6.394
0.4	8.173	1.827	0.033	-0.098	7.853	8.717
0.5	7.582	2.418	0.047	-0.113	9.819	11.001
0.6	7.247	2.753	0.059	-0.127	11.782	13.490
0.7	6.630	3.370	0.084	-0.141	13.741	15.948
0.8	6.172	3.828	0.110	-0.155	15.691	18.545
0.9	5.506	4.494	0.155	-0.170	17.629	21.035

Table 6-10

Case (10) Comparison with Observed Results

(RS: Strongest Runoff)

X'	H_1	H_2	U_1	U_2	S_1	S_2
0.1	9.789	0.211	0.061	-0.025	0.963	1.136
0.2	8.701	1.299	0.079	-0.059	1.948	2.323
0.3	8.490	1.510	0.086	-0.084	2.952	3.837
0.4	7.883	2.117	0.105	-0.107	3.972	5.403
0.5	7.521	2.479	0.120	-0.126	5.002	7.200
0.6	6.912	3.088	0.146	-0.143	6.033	8.922
0.7	6.426	3.574	0.173	-0.158	7.052	10.689
0.8	5.777	4.223	0.215	-0.173	8.039	12.282
0.9	5.270	4.730	0.262	-0.192	8.972	13.862

Appendix 1

Relationship between Total Vertical,
Vertically Advective and Entrainment Velocities

Define W as a total vertical velocity

W_a as a vertically advective velocity

W_e as an entrainment velocity.

(A1) Basic relationship between W , W_a and W_e are as follows:

(A1.1) $W = W_e + W_a$

where $W_e = 0$.

$$0 \leq Z < \eta_{2-}$$

$$\eta_{2+} < Z \leq \eta_1$$

$$W_e \neq 0$$

$$\eta_{2-} \leq Z \leq \eta_{2+}$$

(A1.2) The value of W satisfies the continuity equation and is a function of Z .

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0.$$

(A1.3) $W_a(x, z) = \frac{\partial \eta(x, z)}{\partial t} + U \frac{\partial \eta(x, z)}{\partial x}$

It implies

$$(W_a)_{\eta_{2-}} = \frac{\partial \eta_2}{\partial t} + U_2 \frac{\partial \eta_2}{\partial x}$$

$$(W_a)_{\eta_{2+}} = \frac{\partial \eta_2}{\partial t} + U_1 \frac{\partial \eta_2}{\partial x}$$

$$(W_a)_{\eta_1} = \frac{\partial \eta_1}{\partial t} + U_1 \frac{\partial \eta_1}{\partial x}$$

(A2) Around interface

$$(A2.1) \quad w_{\eta_{2-}} = w_e + \frac{\partial \eta_2}{\partial t} + \frac{\partial \eta_2}{\partial x} u_2$$

$$(A2.2) \quad w_{\eta_{2+}} = w_e + \frac{\partial \eta_2}{\partial t} + u_1 \frac{\partial \eta_2}{\partial x}$$

(A2.3) For a case of steady state, equations (A2.1) and (A2.2), reduce to

$$\frac{w_{\eta_{2+}} - w_e}{w_{\eta_{2-}} - w_e} = \frac{u_1}{u_2}$$

Appendix 2

Integration of the Continuity Equation over Each Layer

After integrating eq. (4.1), the W in the lower layer ($Z < \eta_{2-}$) is derived as follows:

$$W = - \int_0^Z \frac{\partial U}{\partial X} dz = -Z \left(\frac{\partial U_2}{\partial X} \right) \quad (A2.1)$$

Then W at the position η_{2-} is

$$W = -H_2 \left(\frac{\partial U_2}{\partial X} \right) \quad (A2.2)$$

and W_a at the position η_{2-} is

$$W_a = -W_e - H_2 \frac{\partial U_2}{\partial X} \quad (A2.3)$$

From the definition of W in (1.3) of Appendix 1, the following expression is derived:

$$\begin{aligned} W_{\eta_{2+}} - W_{\eta_{2-}} &= (W_a)_{\eta_{2+}} - (W_a)_{\eta_{2-}} \\ &= (U_1 - U_2) \frac{\partial \eta_2}{\partial X} \end{aligned} \quad (A2.4)$$

Eq. (A4.4) implies

$$- \int_{\eta_{2-}}^{\eta_{2+}} \frac{\partial U}{\partial X} dz = (U_1 - U_2) \frac{\partial \eta_2}{\partial X} \quad (A2.5)$$

After integrating (4.1), the W in the upper layer ($\eta_{2+} < z \leq \eta_1$) is derived as follows:

$$W = \eta_{2+} - \int_{\eta_{2+}}^z \frac{\partial U_1}{\partial X} dz \quad (A2.6)$$

Then W at the position η_1 is

$$W = W_{\eta_{2+}} - H_1 \left(\frac{\partial U_1}{\partial X} \right) \quad (A2.7)$$

After substituting equations (A1.1) and (A1.3) into equation (A2.2), and reorganizing, we obtain

$$\frac{\partial H_2}{\partial t} + \frac{\partial}{\partial X} (H_2 U_2) + W_e = 0 \quad (A2.8)$$

After substituting equations (A1.1) and (A1.3) into equation (A2.7), it leads to

$$\frac{\partial \eta_1}{\partial t} + U_1 \left(\frac{\partial \eta_1}{\partial X} \right) = W_{\eta_{2+}} - H_1 \left(\frac{\partial U_1}{\partial X} \right) \quad (A2.9)$$

After substituting equations (A2.4) and (A2.2) into equation (A2.9) and reorganizing, it leads to

$$\frac{\partial \eta_1}{\partial t} + \frac{\partial}{\partial X} (U_1 H_1) + \frac{\partial}{\partial X} (U_2 H_2) = 0 \quad (A2.10)$$

After subtracting eq. (A2.8) from eq. (A2.10), they lead to

$$\frac{\partial H_1}{\partial t} + \frac{\partial}{\partial X} (U_1 H_1) - W_e = 0 \quad (A2.11)$$

Appendix 3

Integration of N.S. Equation over the Bottom Layer

After integrating eq. (4.2) from $z = 0$ to $z = \eta_{2-}$, each term of eq. (4.2) are given in the following expressions respectively:

$$\begin{aligned}
 \int_0^{\eta_{2-}} \frac{\partial U}{\partial t} dz &= \frac{\partial}{\partial t} \int_0^{\eta_{2-}} U dz - U_2 \frac{\partial \eta_2}{\partial t} \\
 &= \frac{\partial}{\partial t} U_2 H_2 - U_2 \frac{\partial H_2}{\partial t} \\
 &= H_2 \frac{\partial}{\partial t} U_2
 \end{aligned} \tag{A3.1}$$

where η_2 is a function of X . And

$$\begin{aligned}
 \int_0^{\eta_{2-}} \frac{\partial U^2}{\partial X} dz &= \frac{\partial}{\partial X} \int_0^{\eta_{2-}} U^2 dz - U_2^2 \frac{\partial \eta_2}{\partial X} \\
 &= H_2 \frac{\partial}{\partial X} U_2^2
 \end{aligned} \tag{A3.2}$$

$$\int_0^{\eta_{2-}} \frac{\partial}{\partial Z} (UW) dz = (UW)_{\eta_{2-}} \tag{A3.3}$$

After substituting eq. (A3.1), (A3.2) and (A3.3) into eq. (4.2), it leads to

$$H_2 \frac{\partial U_2}{\partial t} + H_2 \frac{\partial}{\partial X} U_2^2 + (UW)_{\eta_{2-}} = -(U'W')_{\eta_{2-}} - \int_0^{\eta_{2-}} \left(\frac{1}{\rho} \right) \left(\frac{\partial P}{\partial X} \right) dz \tag{A3.4}$$

Use of equations (A2.2) and (4.8) gives

$$H_2 \frac{\partial U_2}{\partial t} + H_2 U_2 \frac{\partial U_2}{\partial X} = N(U_1 - U_2) - \int_0^{\eta_{2-}} \frac{1}{\rho} \frac{\partial P}{\partial X} dz \tag{A3.5}$$

Appendix 4

Integration of N.S. Equation over the Upper Layer

After integrating eq. (4.2) from $Z = \eta_{2+}$ to $Z = \eta_1$, each term of eq. (4.2)

are given in the following expressions respectively:

$$\begin{aligned} \int_{\eta_{2+}}^{\eta_1} \frac{\partial U}{\partial t} dz &= \frac{\partial}{\partial t} \int U dz - U_1 \frac{\partial \eta_1}{\partial t} + U_1 \frac{\partial \eta_2}{\partial t} \\ &= H_1 \frac{\partial U_1}{\partial t} \end{aligned} \quad (A4.1)$$

$$\begin{aligned} \int_{\eta_{2+}}^{\eta_1} \frac{\partial}{\partial X} U^2 dz &= \frac{\partial}{\partial X} \int U^2 dz - U_1^2 \frac{\partial \eta_1}{\partial X} + U_1^2 \frac{\partial \eta_2}{\partial X} \\ &= H_1 \frac{\partial}{\partial X} U_1^2 \end{aligned} \quad (A4.2)$$

$$\begin{aligned} \int_{\eta_{2+}}^{\eta_1} \frac{\partial}{\partial Z} (UW) dz &= U_1 (W_{\eta_1} - W_{\eta_{2+}}) \\ &= -H_1 U_1 \frac{\partial U_1}{\partial X} \end{aligned} \quad (A4.3)$$

After substituting eq. (A4.1), (A4.2) and (A4.3) into eq. (4.2), it leads to

$$H_1 \frac{\partial U_1}{\partial t} + H_1 \frac{\partial U_1}{\partial t} - H_1 U_1 \frac{\partial U_1}{\partial X} = -(u'w')_{\eta_1} + (u'w')_{\eta_{2+}} - \int \frac{1}{\rho} \frac{\partial p}{\partial X} dz \quad (A4.4)$$

Use of equation (4.8) and boundary condition described in section

4.2.3.1 gives

$$H_1 \frac{\partial U_1}{\partial t} + H_1 U_1 \frac{\partial U_1}{\partial X} = -N(U_1 - U_2) - We(U_1 - U_2) - \int \frac{1}{\rho} \frac{\partial P}{\partial X} dz \quad (A4.5)$$

Appendix 5

Integration of Salt Balance Equation over the Bottom Layer

After integrating eq. (4.3) from $z = 0$ to $z = \eta_{2-}$, each term of eq. (4.3) are given in the following expressions respectively:

$$\begin{aligned} \int_0^{\eta_{2-}} \frac{\partial S}{\partial t} dz &= \frac{\partial}{\partial t} \int S dz - S_2 \frac{\partial \eta_2}{\partial t} \\ &= H_2 \frac{\partial S_2}{\partial t} \end{aligned} \quad (A5.1)$$

$$\begin{aligned} \int_0^{\eta_{2-}} \frac{\partial}{\partial X} (US) dz &= \frac{\partial}{\partial X} \int (US) dz - U_2 S_2 \frac{\partial \eta_2}{\partial X} \\ &= H_2 \frac{\partial}{\partial X} (U_2 S_2) \end{aligned} \quad (A5.2)$$

$$\begin{aligned} \int_0^{\eta_{2-}} \frac{\partial}{\partial Z} (WS) dz &= (WS)_{\eta_{2-}} \\ &= -S_2 H_2 \frac{\partial U_2}{\partial X} \end{aligned} \quad (A5.3)$$

After substituting equations (A5.1), (A5.2) and (A5.3) into eq. (4.3), it leads to

$$H_2 \frac{\partial S_2}{\partial t} + H_2 \frac{\partial}{\partial X} (U_2 S_2) - S_2 H_2 \frac{\partial U_2}{\partial X} = -(S'W')_{\eta_{2-}} \quad (A5.4)$$

Use of equation (4.9) gives

$$H_2 \frac{\partial S_2}{\partial t} + H_2 U_2 \frac{\partial S_2}{\partial x} = K(S_1 - S_2) \quad (A5.5)$$

Appendix 6

Integration of Salt Balance Equation over the Upper Layer

After integrating eq. (4.3) from $z = \eta_{2+}$ to $z = \eta_1$, each term of eq. (4.3) are given in the following equations respectively:

$$\int_{\eta_{2+}}^{\eta_1} \frac{\partial S}{\partial t} dz = \frac{\partial}{\partial t} \int S dz - S_1 \frac{\partial \eta_1}{\partial t} + S_1 \frac{\partial \eta_2}{\partial t}$$

$$= H_1 \frac{\partial S_1}{\partial t} \quad (A6.1)$$

$$\int_{\eta_{2+}}^{\eta_1} \frac{\partial}{\partial x} (US) dz = \frac{\partial}{\partial x} \int (US) dz - U_1 S_1 \frac{\partial \eta_1}{\partial x} + U_1 S_1 \frac{\partial \eta_2}{\partial x}$$

$$= H_1 \frac{\partial}{\partial x} (U_1 S_1) \quad (A6.2)$$

$$\int_{\eta_{2+}}^{\eta_1} \frac{\partial}{\partial z} (SW) dz = S_1 W_{\eta_1} - S_1 W_{\eta_{2+}}$$

$$= -S_1 H_1 \frac{\partial U_1}{\partial x} \quad (A6.3)$$

After substituting equations (A6.1), (A6.2) and (A6.3) into eq. (4.3), it leads to

$$H_1 \frac{\partial S_1}{\partial t} + H_1 \frac{\partial}{\partial x} (U_1 S_1) - H_1 S_1 \frac{\partial U_1}{\partial x} = - (S'W')_{\eta_1} + (S'W')_{\eta_2} \quad (A6.4)$$

Use of equation (4.9) and boundary condition described in section 4.3.2.1 gives

$$H_1 \frac{\partial S_1}{\partial t} + H_1 U_1 \frac{\partial S_1}{\partial X} = -K(S_1 - S_2) - W_e(S_1 - S_2) \quad (A6.5)$$

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