A NASH EQUILIBRIUM SOLUTION FOR STOCK MARKET CRASHES

by

LEVENT AKSOY

A dissertation submitted to the Graduate Faculty in Economics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1997

UMI Number: 9807898

UMI
300 North Zeeb Road
Ann Arbor, MI 48103

This manuscript has been read and accepted for the Graduate Faculty in Economics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

July 7, 1997
Date

Chair of Examining Committee

July 7, 1997
Date

Executive Officer

Michael Grossman

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

A Nash Equilibrium Solution for Stock Market Crashes

by

Levent Aksoy

Adviser: Professor Salih N. Neftçi

Throughout history, large, unsubstantiated movements in asset prices have long puzzeled researchers. Irrational human behaviour is blamed for the cause of such speculative bubbles and market crashes. This thesis investigates a possible underlying rationality that might trigger such changes.

There are two main sections: In the first section a game theoretical model of market is proposed. If the decision to buy or sell an asset is based on expectations of other players' decision, then, a state where all participants buy or sell at the same time can be in a Nash equilibrium. A sudden change in the players' strategies may cause a bubble as well as a market crash.

In the second part, computer simulations based on adaptive expectations are performed to support the theory. Indeed, if the right kind of conditions are met, one can observe bursts of trading activity. These periods of activity as they start, may suddenly disappear.

Acknowledgements

I am grateful to my advisor, Salih Neftçi, for his guidance in matters academic, professional and personal. If I hadn't met him, I would never have thought of joining a doctoral program.

I owe my most of my knowledge of economics and economic thinking to Michael Grossman. I've been mining him from the very first day in the program to the very last.

I also extend my thanks Ted Joyce for putting his faith in me and in this research, Linda Edwards for convincing me to finish my studies, and, to all friends and family who supported me throughout my studies in every way they could.

I dedicate this work to my father, who opened my eyes to the world, and to my daughter together we will discover it all over again.

Contents

Abstract **iv**

Acknowledgements **v**

Table of Contents **vi**

List of Figures **viii**

1 INTRODUCTION **1**

1.1 Bubbles and Crashes 2

1.2 Overview 4

2 THE GAME **7**

2.1 Introduction 7

2.2 Rationality in the capital markets 9

2.3 Bias in decision making 10

2.4 Identifying the Crash 13

2.5 Outbreak of Un-Cooperation 15

2.6 The Agents 17

2.7 Two-Player Game 20

2.7.1 Payoffs 21

2.7.2 Nash Equilibria 23

2.7.3 Adjustments 24

2.7.4 Pareto domination 26

2.7.5 Results 28

2.8 Bubble Formation 29

2.8.1 Nash Equilibria 31

2.8.2 Focal points and Pareto domination 31

2.9 The Case of the Speculator 33

2.10 Macro factors 35

2.11 Conclusion 36

3 DYNAMICS **37**

3.1 Introduction 37

3.2 Expected Returns 39

3.3 Computer Experiments 43

3.4 Results 44

4 CONCLUSION **46**

5 APPENDIX **48**

6 BIBLIOGRAPHY **63**

List of Figures

1 Number of Traders ($r_l = 0.0205$) 54

2 Stock Prices ($r_l = 0.0205$) 55

3 Expected Returns ($r_l = 0.0205$) 56

4 Number of traders ($r_l = 0.0195$) 57

5 Stock Prices ($r_l = 0.0195$) 58

6 Expected Returns ($r_l = 0.0195$) 59

7 Number of traders ($r_l = 0.0200$) 60

8 Stock Prices ($r_l = 0.0200$) 61

9 Expected Returns ($r_l = 0.0200$) 62

1 INTRODUCTION

In economics, as well as in other fields, there's a tradion to proceed from the study of simple systems to more complex ones. In economics the simplest case is usually a two-variable model. Consumers are to choose from two types of goods; production requires two factors–capital and labor. Economists have already started to use complex models. One example is the determination of capital asset prices (Sharpe, 1964). The prices of derivative securities, such as options, now require solutions of differential equations (Hull, 1993).

Following the development of portfolio theory by Markowitz, there have been two major theories put forth to derive a model for the valuation of risky assets (Reilly & Norton, 1995). The first is the Capital Asset pricing Model (hence, CAPM) developed by William Sahrpe, John Lintner and Jan Mossin which deals with the central aspects of equilibrium in capital markets (Sharpe, 1991). The model predicts the price of a capital asset in terms of its expected return and the risk it bears. The other is the Arbitrage Pricing Model (Hence, APM) developed by Stephen Ross (Reilly & Norton, 1982, p180). APM has fewer assumptions, especially, doesn't require specific assumptions on the investors' utility functions (quadratic utility function) and normally distributed returns. Furthermore, it takes into consideration of outside information such as the GNP.

The price of the asset is thus assumed to have a basic intrinsic value. This value is determined by examining variables such as future earnings, interest rates and risk variables. There may be discrepancies between the actual value and the intrinsic value of an asset, but eventually the investors will recognize the discrepancy and correct it (Reilly & Norton, 1995). This price adjustment under the "Efficient Market Hypohesis" (EMH) is rather quick in response to the arrival of new information, and therefore, the current price reflects all the information about the security. The "fundamental analysts," who believe the markets are efficient, try to perform a superior job of estimating the intrinsic value of the asset, and acquire undervalued securities, thus generate above-average returns.

The EMH is challanged by the "technical analysts." A basic premise is that the newly arriving information requires time to be fully analyzed by investors before taking an action and that the prices cannot adjust quickly. The prices will then move in "trends" to a new equilibrium, and one may take advantage of this movement by early detection (Reilly & Norton, 1995).

1.1 Bubbles and Crashes

Economists believe that under the assumption of rational behavior and rational expectations, the price of an asset must reflect the market fundementals. Financial-market participants on the other hand have quite a different view:

fundementals play only a partial role in determining the prices. Extraneous events may well influence the price if believed by other participants, or simply stated crowd psychology becomes important. If the expectations of investors change in such a way that they believe they will be able to sell an asset for a higher price in the future than they had been expecting then the current price of the asset will rise. If the reason that the price is high today is only because investors believe that the selling price will be high tomorrow then a bubble exists (Stiglitz, 1990).

It is proved that under general equilibrium, the bubbles cannot arise when theres a finite number of individuals with infinite horizons (Tirole, 1982). The argument is that, whenever the price is different than the fundemental price, the players will try either to buy or to sell the asset creating an excess demand or supply which will contradict the equilibrium assumption.

Rationality alone, however, does not imply that the price of an asset be equal to its fundemental value. That is there can be rational bubbles (Blanchard and Watson, 1982). It is possible to have bubbles even in general equilibrium with infinite horizon when there are new players coming into the game over time (Blanchard & Fischer, 1989). With a finite horizon when the economy is not efficient (interest rate is less than the growth rate) there will also be a positively valued bubble at the steady state.

Other research introduced alternatives to the efficient markets such as the

"Noise Trader" approach. In this approach, traders are not fully rational and their demand for risky assets are affected by their beliefs. Also arbitrage opportunities are risky and therefore limited (Schleifer and Summers, 1990). In a study, Youssefmir et.al. showed using computer experiments that under similar conditions, the price can inflate and deflate suggesting a bubble (Youssefmir, Huberman, Hogg, 1995).

The study of bubbles centered mostly around the existence of such events and how they can be identified. One peculiarity of the markets still remains unexplained. When the markets starts to decline the prices sometimes adjusts rather quickly. This sharp decline in the prices which is named properly as a "market crash" causes panic among investors. Why mostly "rational" agents suddenly turn out to be absolutely "irrational" has not been fully investigated. The study of group behavior where players faced with prisoners dilemma, switch strategies between cooperation or defection in a sudden and unexpected way (Glance and Huberman, 1993) may shed some light on what happens when a bubble bursts. It may also explain why bubbles start in the first place.

1.2 Overview

Mathematical modelling of processes involving humans is quite tricky. On one hand theres a need to fully understand and predict the behavior, on the other hand the model has to fit into existing theories. In order to come up with the "correct" model, you have to expand it to include many aspects of your observatios, meanwhile, simpler models are preferred for the sake of mathematical beauty. This means a trade-off between beauty and models ability to predict the real-world. The challange is to improve on both sides. This study is divided into two main parts: The first part depicts the behavior of market agents as a 2-player game. In the second part I will present a computer simulation to test the predictions of the model.

The first part includes the verbal descriptions of observed behavior during market a crash. Based on these descriptive information I will try to build a mathematical (game-theoretical) model where players can choose between two starategies: *Hold* or *Sell*. The utilities will include the return on the stocks as well as the return on a risk free asset. This model has two pure-strategy equilibria, where both players *Hold*, and where both *Sell*. The third being an unstable mixed-strategy equilibrium will simply be mentioned. In the tradition of simplistic economic modelling, when one of the players is assumed to be "all other participants," both players selling strategy suggests a market crash. Then I will question whether the predictions of this model coincides with the economic theory. A change in the returns and interest rates will effect the outcome of the game. The number of players will also effect the results. The section will continue with the analysis of another type of game with a third possible action. When both players switch to a *Buy* strategy a bubble will form. Finally, the assumption of single-unit transaction will be relaxed.

In the second part, the model will be tested using computer experiments. Agents with adaptive expectations and very short time horizon will involve in *Buy* and *Sell* decisions. The difference between the number of buyers and sellers (i.e. the excess supply or demand) will determine the direction and the rate of the price changes. The results show periods with sudden and large price movements in otherwise mostly random changes. These sudden changes in the prices can be considered as bubbles and market crashes.

2 THE GAME

2.1 Introduction

To explain consumer behavior an economist starts with a set of preferences. Based on those preferences and a corresponding utility function, he will try to predict how the consumer will react to changes in prices, income, health or the weather. Although we agree that these preferences depend on that persons psychology, we usually take them as given (exogenous) and assume that they will not be affected by the market or the prices.

How realistic can this assumption be? In the short run, our preferences will be affected by changes in our perception. Our perception is not prone to supply-side factors like advertising. No matter how educated a person is, or how much he knows the effects of advertising, one form of advertising will eventuallly effect his decision making. Similarly demand for a good may affect a consumers demand for that good. Becker studied this phenomenon on restaurants and proposed a roller-coaster shaped demand curve. We initially have a downward sloping demand; as the quantity (the number of customers) incrases the restaurant becomes 'hip' or fashionable and demand builds up creating more demand. In the assets market, with a fixed amount of equity outstanding, the current price is determined by the intersection of investor demands with the existing supply. But, equilibrium demand depends upon the current equity price and the beliefs of agents about the equity prices in the future. Therefore, the current price depends on the expectations of the future price and the expectations of the future price depends on the current price (Flood & Hodrick, 1990).

Even when using fundamental analysis similar but more indirect effects can be found. The fundamental (or intrinsic) value of a firm depends on the expected sales, profits, interest rates, macroeconomic environment, sectoral variables, etc. In general, everthing else being the same, the more profitable a firm is expected to be, the higher the value of the firm will be; thus a higher stock price. The profitability or rather the survivability of a firm depends on the strength of the company today. This strength affects a wide range areas from a new technology being accepted by others to money raising capability of the firm. This is especially true with new technology companies.

When a small firm invents a new technology standard which competes others in the market, in order to be successful, the firm needs to advertise to gain acceptance. However, its marketing efforts largely depends on its resources. Even if the technology is superior to other existing technologies, the firm is bound to disappear when it cannot convince investors to raise capital. If the stock market puts a high value on a firm, then the firm can more easily borrow or sell stock at a higher price. This money can be used in marketing or research. Meanwhile, other firms are more likely to follow a larger firm in terms of accepting new technology or establishing standards.

This process may eventually become a self fullfilling prophecy. If people beive that a firm will become successful, then it is more likely that the firm will be, than not. In a way, the market will make the firm.

2.2 Rationality in the capital markets

In studying capital markets one assumption is that the majority of players in capital markets have enough sophistication at least to accept that there will be well founded mathematical methods to estimate the price of a capital asset. There are well established models such as the CAPM and APM. In major markets as in New York, London, Hong Kong, traders make use of computers and such models to estimate the the "fundamental value" of a security. These models became so sophisticated that sometimes the computers (machines) spell out the decisions. In less developed markets like emerging markets of the Eastern Europe and China, we should expect to see larger deviations from the fundamental value of the stock, as the traders are less likely to be able to calculate exact value of the firm.

The use of computers and the large number of players both favor the EMH. Computers which use sophisticated pricing models can very quickly evaluate any new information, and produce results with great accuracy. The number of players, on the other hand, can "average" or "smooth" the irrational behavior by investors towards an unbiased demand. But are all of these conditions enough to prevent a "collective panic" condition? Or at the very extreme, can computers have their own "psychology?"[1]

The assumption of EMH and rational traders can be tested in two ways: By looking at the market in general and testing for a random walk in the price level under the assumption that new information arrives in a random fashion. The second is by looking at the traders individually. The irrational behavior of market participants does not necessitate the market in general is not efficient as long as this irrationality is not biased. If a significant percentage of the investors have similar biased views about the market, then we should not expect the EMH to hold.

2.3 Bias in decision making

The bias in decision making can grouped in two major categories. The first assumes that the investors are rational but they cannot eliminate systematic biases. Second assumes that the investors are simply not rational.

It is possible that the rational traders cannot eliminate systematic biases. They may not take advantage of arbitrage opportunities partly because they cannot absolutely be sure that such opportunities exist, because the fundamental value of an asset is not certain (Shleifer & Summers, 1990).

[1]Since computers are programmed by humans, the underlying algorithms can embed human biases or psychology.

There has been studies about the psychology of the traders. One survey, for example, found that professional currency forecasters during the mid 80s, expected that in the long run market fundamentals to hold, but in the short run price trends to persist (Day & Huang, 1990). The type of trders with systematic biases away from the fundamental values are called "noisy" traders. Such noise traders have trend chasing tendencies. The investors bias is exemplified by the "Theory of Reflexivity" presented by George Soros, who stresses the importance of investor bias in security analysis (George Soros, 1987).

The "Technical Analysis" of the stock prices which range from moving averages to drawing lines, also shows a bias away from the market fundemantals. The importance of such theories and methods is not in their mathematical or economic merit, but in the number traders (individuals or institutions) who believe in them. The number of "theoretical" books read by investors may exceed couplefold the number of journals read.

Portfolio "insurance" is another deviation from market fundemantals. Such traders use computerized strategies which dictate buying after market rises and selling after the market declines a certain percentage points. In other words, "they can afford to take greater risks in rising markets because the portfolio insurance offers a disciplined way of avoiding risk in declines" (Leland & Rubinstein, 1990).

The assumption of "infinitely lived" agents may not be suitable as well. In the case of mutual funds, the investors do look at the past behaviour of the fund assuming that it is an indicator of how well the funds are managed. The funds themselves advertise their past performance; third parties like Morningstar and Lipper and less formal Consumer Report compare funds using past returns as well as other figures. These, in turn, further affect the investors maybe to an extent to be conditioned only on the past performance. Since the funds are in competition theres a pressure on the fund managers to perform well in the short run to attract new investors causing a deviation from rational behaviour.

A survey done on the behavior of investors reports very interesting results (Shiller, 1990). Following the market crash on October 19, 1987, questionnaires were sent to the investors (institutional and individual), asking about their own personal experiences. Among the answers for the causes of the stock market crash, market being overpiced, existance of "insurance" (also stop-loss, program, computer) traders, and irrationality of investors were the common themes. For the reason why a "rebound" would occur, "intuition" was very frequent. The research found no recognizable exogenous trigger for the crash.

Meanwhile, a recent research found that intiution and gut feelings have biological and mental basis. Human brain has a covert system which is activated long before people are consciously aware that they had decided anything. When subjects were drawing cards from two deck - one providing higher returns than the other - they subconsciously became aware of the difference

(Sandra Blakeslee, 1997). Similarly a computationally more powerful brain may be working in the background and providing rational decisions. However, rationality should refer to decisions as a result of mathematical calculations which can be repeated by others, and not vague decisions conscious or otherwise.

A very good description of the boom and crash of 1929 can be found in Galbraiths classic book (John Kenneth Galbraith, 1988). The explanation of events emphasizes the emotional responses by and irrationality of investor decisions. Although the economy was well into a depression by the fall of 1929, depression alone cannot explain why the prices adjusted in a matter of days instead of a smoother and and flatter decline. Also in the case of 1929 crash (like in 1987 and 1946 (Miller, 1990)) there was not an obvious exogenous trigger.

All of these factors are real-life counter-examples to the assumptions uderlying the research on non-existance of speculatice bubbles. Unless those assumptions are satisfied we cannot conclude that speculative bubbles do not exist.

2.4 Identifying the Crash

In the crash of 1929 Galbraith describes the lack of equilibrium as "Often tehre were no buyers, and only after vertical declines could anyone be induced to bid." Also the ticker tape showed lagging price quotes. The panic stopped as quickly as it started when larger financial institutions decided to take action. After a couple days of steady price levels, stocks started to fall again, this time without recovery. Without an "organized support" prices kept falling; and as usual the ticker "ran late." A day later larger blocks were offered, but without buyers their prices continued to fall. There were also unfilled orders and multiple sales of the same stocks etc. as a result of clerical errors due to human fatigue.

In the crash of 1987 too, both the buyers and sellers did not have a clear picture of what the prices were. There were substantial intervals where no buyers could be found to pair with sellers (Eichengreen, 1990). Many traders found themselves with an overhang of unfilled sell orders going into the next day (Leland & Rubinstein, 1990).

Among the characteristics of the market crashes, three are quite important. First is delayed or mis-information, second is the absence of equilibrium, and third is the failure of the markets in terms of inoperability. High volume and volatility also accompanies the large price drops. These characteristics also challange the assumptions made by Tirole and others. Thus we are safe to assume that agents are not always rational in their decision making and the market does not have to be in equilibrium. Furthermore, the panic among investors as they try to sell, but not being able to fulfill their orders suggest that the market itself may cease to exist during a crash.

2.5 Outbreak of Un-Cooperation

The collective action problem among intentional agents whose choices depend not only on the past but also on their expectations as to how their actions will affect those of others have been studied extensively by Glance and Huberman (Glance & Huberman, 1993). When faced with "Prisoners' Dilemma" agents will have to choose between a strategy to free-ride or to defect and one where the benefit of cooperation offsets individual costs. This problem is presented as a repeated n-person prisoners' dilemma, wherein the benefit obtained by an individual from cooperating in producing the good is outweighed by the cost of cooperating for the one-shot game.

Along with imperfect information, it is shown that the onset of overall cooperation can take place in a sudden and unexpected way. Similarly, defection can appear out of nowhere in very large, previously cooperating groups.

The authours introduce an optimality (or energy) function as a fraction of cooperating agents.[2]

This function has two local minimums and one also bing the global. The

[2]If the mean probability that cooperation is preferred is $\rho(f_c)$ then the optimality function Ω is given by:

$$\Omega(f_c) = \int_0^{f_c} df'_c[f'_c - \rho_c(f'_c)]$$

global minimum is depicted as the optimal state of the system whereas the local (which is not the global one) minimum as the metastable equilibrium. The difference between the minimums and maximum is called the 'barier.' If somehow the sytem is trapped in the metastable equilibrium, the fluctiations caused by strategy switching agents away from the local minimum relax back to it. Over a longer period, larger fluctiations can push the system over the barrier maximum. Once a critical mass is reached, remaining agents rapidly switch into this new strategy that corresponds to the optimal Nash Equilibrium.

The optimality function varies as the size of the group increases. Of the two strategies, one may be optimal for small groups, whereas it may become the metastable for larger groups.

One interesting result is that the system moving one state to the other happens rather quickly. In the example given by Galnce and Huberman, the time for recovery from a metastable state was many orders of magnitude larger than the crssover time. This switching property of collective action provides a starting point for the study of dynamics of market crashes. The following sections will search for a similar behavioral property in groups trading stocks. The switching property will hint that the players can move from ane market state to the other quickly. We will then look for possible answers why the prices sometimes move in rather large steps within a day or even hours.

2.6 The Agents

Consider a group of agents involved in trading securities. Their actions not only depend on the expected return on these securities but also are a determinant of future prices.

In this hypothetical world, possible actions are buying, holding and selling the security. When they decide to buy the security the demand for it will increase (i.e. shift to right). Assuming a more or less constant supply, the price of the security thus increase. When they decide to sell the security, this time the supply will increase (i.e. rightward shift) causing the price to fall. When they hold the stock, since neither the demand nor the supply will be affected, the price will remain constant.[3]

These agents have utility functions which depend on the relative return they earn in the market (as opposed to absolute return on their investment). This utility is consistent with the behavior of real-world investors. The mutual funds not only advertise about their past returns of the fund but they also compare them with the market return and how they performed above the market.[4] You can also find individual investors bragging about how they beat the market or how they avoided losses during declines. On the other hand a

[3]Constant is used here to denote a steady price level where although minor price fluctiations might occur, is not considered a major trendy shift.

[4]These ads are common in the Business section of New York Times.

higher return always gives a higher satisfaction no matter what the market return is. Based on these observations a utility function should be such that: a) a player will be better off as the return on his investment increases given a market return (marginal utility of return is positive); b) given the return on investment his utility will decrease as the market return increases (marginal utility of market return is negative); c) if the return on investment is constant relative[5] to the market return, then the utility will increase with the incresing return (weighted sum of marginal utilities is constant).

In other words if the utility of an investor is given by $U_i = f(r_i, r_m)$ where r_i is the return of player i and r_m is the market return, then:

$$\text{a)}\quad MU_{r_i} = \left.\frac{\partial U_i}{\partial r_i}\right|_{r_m} > 0 \qquad ;$$

$$\text{b)}\quad MU_{r_m} = \left.\frac{\partial U_i}{\partial r_m}\right|_{r_i} < 0 \qquad ;\text{ and}$$

c) Given $r_i - r_m = c$

$$\left.\frac{\partial U_i}{\partial r_m}\right|_{r_i - r_m = c} = \left.\frac{\partial U_i}{\partial r_i}\right|_{r_i - r_m = c} = \frac{\partial}{\partial r_i} f(r_i, r_i - c) > 0$$

A utility function satisfying above conditions is a linear combination of market return and return on player's investment:

[5]Relative is used for the difference between individual's return and the market return; a ratio can also be used instead. See appendix for a formal comparison

$$U_i = f(r_i, r_m) = ar_i - br_m \tag{1}$$

where a and b are positive numbers and their difference is also positive.

$$(a, b > 0) \quad \text{and} \quad (a - b > 0)$$

The marginal utilities for the given utility function are:

$$MU_{r_i} = \frac{\partial U_i}{\partial r_i} = \frac{\partial}{\partial r_i}(ar_i - br_m) = a > 0$$

$$MU_{r_m} = \frac{\partial U_i}{\partial r_m} = \frac{\partial}{\partial r_m}(ar_i - br_m) = -b < 0$$

and when $r_i - r_m = c$

$$\frac{\partial U_i}{\partial r_i} = \frac{\partial}{\partial r_i}(ar_i - br_m) = \frac{\partial}{\partial r_i}(ar_i - b(r_i - c)) = a - b > 0$$

Since the agents do not choose the magnitude of returns r_i and r_m, the second order conditions for utility maximization do not apply here. The players will prefer higher return as it increases their total wealth; they will then choose between finite number of possible actions in order to maximize their returns.

Although the players can play many times, the time horizon is not infinite. They may choose to or forced to exit the game, but other players can replace them. The total number of players can increase or decrease. Except for major shocks to the economy or society, since the population grows, we should expect this number to increase in the long run.

I would like to point out once again that the model derived here is based on the observed behavior of the investor and not on a previously developed financial model.

2.7 Two-Player Game

Let's assume that the abovementioned agents are involved in trading securities (stocks) in such a way that they only have two possible strategies: *Hold* or *Sell* the security. Let's further assume that each transaction consists of a sale of a single unit of security. In this setup, there's a risk-free rate of return at which the players can lend or borrow. The security can be an index portfolio which brings the market return.

If the players hold the security they will recieve the market return; if they sell the stock they can lend their money at the risk-free interest rate and recieve $\bar{r}$.

The stock market itself can be in one of the two states *Steady* and *Low*. In the *Steady* state the expected return of the market, r_m, will be equal to r_S, and in the *Low* state the return will be r_L.

Since a risk is involved with an investment in the stock market, we expect the *Steady* state return, r_S, to be greater than the risk free rate, $\bar{r}$, and, the *Low* state market return to be less than the risk free rate.

$$r_L < \bar{r} < r_S$$

2.7.1 Payoffs

Using the utility function given by (1), the player will get the following utilities for each of his actions and the state the market is in:

	Steady	Low
Hold	$(ar_S - br_S)$	$(ar_L - br_L)$
Sell	$(a\bar{r} - br_S)$	$(a\bar{r} - br_L)$

Since r_S is greater than $\bar{r}$, the player will be better of if he *Holds* the stock when the market is *Steady*. If $r_S > \bar{r}$, then

$$U_{i,\text{Hold, Steady}} = ar_S - br_S > a\bar{r} - br_S = U_{i,\text{Sell, Steady}}$$

Similarly he will obtain a higher utility when he *Sells* the stock when the market is *Low* since r_L is less than $\bar{r}$. If $r_L < \bar{r}$, then

$$U_{i,\text{Hold, Low}} = ar_S - br_S < a\bar{r} - br_S = U_{i,\text{Sell, Low}}$$

In order to maximize his utility the player should therefore implement the following strategy:

Hold when the market is **Steady, Sell** when the market is **Low**

When there are many agents trading stocks in a similar way, their actions will affect the demand and supply, therefore current prices as well as the market return. A two-player game may capture the essence of a multi-player game if we consider the *other* player as *all other* players. This assumption is consistent with general economic practice such as assuming one good as *all other* goods in indifference curves.

In this kind of market, if the majority of the players are *Holding*, the market will be in a *Steady* state; when the majority of the players decide to *Sell* then the market will switch to a *Low* state. Accordingly, if the *other* player chooses to *Hold* the market will be in the *Steady* state, whereas if he chooses to *Sell* then the market will switch to a *Low* state. A game between these two players will have a strategic form as given below:

Player 2

		Hold	*Sell*
Player 1	*Hold*	$((a-b)r_S, (a-b)r_S)$	$((a-b)r_L, a\bar{r} - br_S)$
	Sell	$(a\bar{r} - br_S, (a-b)r_L)$	$(a\bar{r} - br_L, a\bar{r} - br_L)$

Payoffs to: (Player 1, Player 2)

When the *Other* player switches from *Hold* to *Sell* (i.e. market declines), the utility of the first player will decrease if he *Holds*, and increase if he *Sells*. This puts an incentive on the player to switch from *Hold* to *Sell* when the market declines.

The following payoff matrix is a numerical example to the game above and will help to visualize it:

P 2

P 1		*H*	*S*
	H	(5,5)	(1,2)
	S	(2,1)	(4,4)

2.7.2 Nash Equilibria

This game is a version of 'Coordination Game', where the players are better off if their actions match In other words, players will accept prefer playing safe and recieve the risk free return, although staying in a *Steady* market is definitely preferred. Worst thing that can happen is to keep the stock in a declining market.

There are two pure strategy and one mixed strategy equilibria to the game. The two pure strategies are (*Hold*,*Hold*) and (*Sell*,*Sell*) and the mixed strategy equilibrium is

$$P_{Hold} = \frac{\bar{r} - r_L}{r_S - r_L} \tag{2}$$

for both players. The derivation of the mixed strategy equilibrium is given in the appendix.

The derivatives of the probability to hold are:

When we take the derivatives of P_{Hold}

$$\begin{aligned}
\frac{\partial P_{Hold}}{\partial r_S} &= \tfrac{\partial}{\partial r_S}\left(\tfrac{\bar{r}-r_L}{r_S-r_L}\right) &&= -1 < 0 \\
\frac{\partial P_{Hold}}{\partial r_L} &= \tfrac{\partial}{\partial r_L}\left(\tfrac{\bar{r}-r_L}{r_S-r_L}\right) &&= \frac{-(r_S-\bar{r})}{(r_S-r_L)^2} < 0 \\
\frac{\partial P_{Hold}}{\partial \bar{r}} &= \tfrac{\partial}{\partial \bar{r}}\left(\tfrac{\bar{r}-r_L}{r_S-r_L}\right) &&= 1 > 0
\end{aligned}$$

Mixed strategy equilibria are not so intuitive as pure strategy equilibria. One objection to mixed strategies is that people in the real world do not take random actions (Rasmusen p72). The other is that a player who selects a a mixed strategy is indifferent between two pure strategies. When there's a small deviation in the probability selected by either players, destroys the equilibrium. Another way to interpret the game is that among the many stock market participants a certain percentage will choose one pure strategy and the rest the other strategy. If, for example, $P_{Hold} = 0.90$ then we should expect 90% of the players will choose a *Hold* strategy, whereas the remaining 10% will choose to *Sell*.

2.7.3 Adjustments

The mixed strategy probabilities given above (2) shouldn't be used directly to show the relation between the risk free interest rate and the market return. The derivative of probability to *Hold* with respect to the risk free interest rate being positive, for example, doesn't mean that the market will rise when the interest rates drop. It simply means that the mixed strategy equilibrium will be at a higher probability. We should look at the sign of $\frac{\partial E_1}{\partial p_1}$ instead, for the effects of a change in any of the parameters ($\bar{r}$, r_S, r_L).

Since the ith player determine his mixed strategy depending on the derivative of the payoff function (response function), we have to look at its sign around zero.

$$\frac{\partial E_i}{\partial p_i} = a(r_S - r_L)p_j + a(r_L - \bar{r}) = mp_j + n$$

Since $r_S > r_L$ and $r_L < \bar{r}$, $m > 0$ and $n < 0$. This is the equation for a line with a positive slope and a negative intercept. The mixed strategy Nash equilibrium is obtained when $\frac{\partial E_i}{\partial p_i} = 0$, or when $p_j = -\frac{n}{m}$, for any value of p_j greater than $-\frac{n}{m}$, $\frac{\partial E_i}{\partial p_i}$ will be positive.

A change in either $\bar{r}$, r_S, or r_L will shift and/or rotate this line resulting in another equilibrium point p_i and p_j.

The mixed strategy equilibrium is not a stable equilibrium. When p_2 increases slightly above the equilibrium, $\frac{\partial E_1}{\partial p_1}$ becomes positive causing the first player to adjust his strategy such that p_1 also increases to improve his expected payoff (E_1). Since the players have symmetric payoffs, the second player will also adjust his strategy to further increase p_2. This will in turn collapse the mixed strategy equilibrium towards the pure strategy equilibrium of (*Hold,Hold*).

The reverse is true when either of the players slightly decrease their probability of *Holding*. $\frac{\partial E_i}{\partial p_i}$ will become negative resulting in player i to decrease $p_{i,Hold}$ (so that the expected returns will increase).

One interesting property of this mixed equilibrium is that when either of the rates change the sign of $\frac{\partial E_i}{\partial p_i}$ will change from 0 to either positive or negative, which in turn collapse the mixed equilibrium toward either of the two pure equilibrium. The direction of the change is consistent with real-world observations and other theories. For example if the risk free interest rate $\bar{r}$ increases, $\frac{\partial E_i}{\partial p_i}$ will become more negative causing both players to decrease their probabilities (to *Hold*, $p_{i,Hold}$) until it becomes 0 (i.e. pure (*Sell*,*Sell*) strategy equilibrium). Any fall in the risk free interest rate, $\bar{r}$, will have the reverse effect, cusing players to switch from a mixed strategy to a pure (*Hold*,*Hold*) strategy.

Similarly an increase in the *Steady* market return will have a positive effect on the market. The effect of a decreasing *Low* market return is also negative on the market.

2.7.4 Pareto domination

This game has two pure-strategy equilibria - (*Hold*,*Hold*) being Pareto superior. Nevertheless, it is not clear which equilibrium should be expected. The outcome not only depends on the returns but the number of players as well. With only two players, playing *Hold* is better than *Sell* provided that the single oppponent will play *Hold* with probability of 0.5 or more. But with more opponents, it is better to play *Hold* when **all** other opponents will play *Hold* with probability of 0.5. With n opponents, this translates into the nth root of 0.5.[6]

Excluding the mixed strategy equilibrium, which of the remaining two pure-strategy equilibria is more likely to played? To answer this we need to look at the expected payoffs for both strategies. If the first player believes that the second player will play *Hold* with a probability of p_2 (and *Sell* with $1 - p_2$), then he will decide by comparing his expected returns from both of his possible actions (i.e. *Hold* and *Sell*).

$$\begin{aligned} E[Hold|r_S, r_L, \bar{r}, p_2] &= p_2(ar_S - br_S) + (1 - p_2)(ar_L - br_L) \\ E[Sell|r_S, r_L, \bar{r}, p_2] &= p_2(a\bar{r} - br_S) + (1 - p_2)(a\bar{r} - br_L) \end{aligned}$$

$$\begin{aligned} &E[Hold|r_S, r_L, \bar{r}, p_2] - E[Sell|r_S, r_L, \bar{r}, p_2] \\ &= p_2(ar_S - br_S) + (1 - p_2)(ar_L - br_L) - p_2(a\bar{r} - br_S) + (1 - p_2)(a\bar{r} - br_L) \\ &= p_2(ar_S - a\bar{r}) + (1 - p_2)(ar_L - \bar{r}) = p_2 a(r_S - r_L) + a(r_L - \bar{r}) \end{aligned}$$

[6]Assuming equal probabilities to *Hold*, p, for each of the opponents, as the number of opponents increase, you'll need a higher level of confidence

$$p^n > \frac{1}{2} \Rightarrow p > \sqrt[n]{\frac{1}{2}} > \frac{1}{2}$$

This is the exact same equation we found when we were trying to optimize the expected return for Player 1. The results we find is again similar to that of real-world observations: Player 1 is more likely to play *Hold*

- as p_2 increases;[7]

- as r_S increases;[8]

- as $\bar{r}$ decreases;[9]

- as r_L increases.[10]

This last result is particularly interesting. If the expected *Low* market return is much lower than the other returns (especially $\bar{r}$) the players may not want to take the chance of losing a large amount and may want to exit the market.

2.7.5 Results

The "focalness"[11]of the Pareto dominant strategies in general and in this case epends on the players' mood and their past experiences as well as the risk of a certain strategy. When for example, the risk-free interest rate is low, and expected market returns are high the focal point of the game is *(Hold, Hold).*

[7] $\frac{\partial}{\partial p_2}(p_2 a(r_S - r_L) + a(r_L - \bar{r})) = a(r_S - r_L) > 0$

[8] $\frac{\partial}{\partial r_S}(p_2 a(r_S - r_L) + a(r_L - \bar{r})) = ap_2 > 0$

[9] $\frac{\partial}{\partial \bar{r}}(p_2 a(r_S - r_L) + a(r_L - \bar{r})) = -1 < 0$

[10] $\frac{\partial}{\partial r_L}(p_2 a(r_S - r_L) + a(r_L - \bar{r})) = a(1 - p_2) > 0$

[11] Some strategy combinations are focal points. Formalizing what makes a strategy combination is hard and depends on the context. The choice of a focal point may have psychological reasons.

When the extra gain over the risk-free rate is not that large and the expected losses ($r_L - \bar{r}$) are great, it will be much safer to play *(Sell, Sell)* abandoning the Pareto efficient strategy in favor of a risk-dominant strategy.

The expectation on the low market retun may also depend on the past experiences. Although it is easier to develop beliefs on othe players' strategy, the low market return (r_L) is harder to guess. If a "market correction" should occur, how much correction should be anticipated is hard to estimate. The question becomes whether this "correction" will turn into a market crash. At that point what was in market participants beliefs can easily become a self fulfilling prophecey.

2.8 Bubble Formation

We will now introduce a second type of game which includes a third market state and another strategy for the players. The market now can be in one of the three states: *High, Steady,* and *Low.* The returns for these states are: r_H, r_S, and r_L respectively. Naturally, the rates are ordered as $r_H > r_S > r_L$.

For simplicity, the players' utility will simply consist of the return on their investments. The players can now buy a single unit of stock in addition to the one unit they already have, by borrowing at a fixed *borrowing* rate. Similarly, they can lend their money at the fixed, risk free rate if they choose to *Sell* the stock they own. The lending rate is $\bar{r}_l$, and the borrowing rate is $\bar{r}_b$, and we expect $\bar{r}_b$ to be greater than $\bar{r}_l$.

The payoff matrix for each player will be as follows:

	High	*Steady*	*Low*
Buy	$2r_H - \bar{r}_b$	$2r_S - \bar{r}_b$	$2r_L - \bar{r}_b$
Hold	r_H	r_S	r_L
Sell	$\bar{r}_l$	$\bar{r}_l$	$\bar{r}_l$

We will again assume that the *other* player represents all other players, and that if the *other* player *sells*, the market will move to a *Low* state, and reverse will be true when the *other* player buys. Based on the above payoff matrix, a game between these two players will have the following strategic form.

		Player 2		
		Buy	*Hold*	*Sell*
	Buy	$(2r_H - \bar{r}_b, 2r_H - \bar{r}_b)$	$(2r_S - \bar{r}_b, r_H)$	$(2r_L - \bar{r}_b, \bar{r}_l)$
Player 1	*Hold*	$(r_H, 2r_S - \bar{r}_b)$	(r_S, r_S)	$(r_L, \bar{r}_l)$
	Sell	$(\bar{r}_l, 2r_L - \bar{r}_b)$	$(\bar{r}_l, r_L)$	$(\bar{r}_l, \bar{r}_l)$

Payoffs to: (Player 1, Player 2)

2.8.1 Nash Equilibria

The solution of this game depends on how borrowing and lending rates ($\bar{r}_b$ and $\bar{r}_l$) compare to the market returns, and especially to the *Steady* market return (r_S). When the order of the market returns and risk free rates is $r_H > \bar{r}_b > r_S > \bar{r}_l > r_L$ then the game will have three pure strategy equilibria: {*Buy,Buy*}, {*Hold,Hold*}, and, {*Sell,Sell*}. We will skip the mixed strategy equilibria (if there's any) for reasons discussed in the previous sections.

However if the order is changed, especially when the *Steady* market rate r_S falls below the lending rate $\bar{r}_l$ or when it exceeds the borrowing rate $\bar{r}_b$ then { *Hold,Hold* } equilibrium disappears leaving only the other two pure-strategy equilibria.

At this point it needs to be pointed out that these returns do not have to be the absolute returns on the market. We didn't make any assumption on the risk aversion behaviour of the players. If the players are *risk-neutral* then these returns will be the observed market returns. However, if the players are *risk-averse* then we can adjust these returns such that they use a *risk-corrected* measure. One way is to subtract a *risk-premium* rate from these returns.

2.8.2 Focal points and Pareto domination

If one of the players (e.g. **Player 1**) believes that the other (**Player 2**)will play *Buy* with probability p_2, play *Hold* with probability q_2 and thus play *Sell* with probability $1 - p_2 - q_2$ then the expected returns for each of his actions will be:

$$\begin{aligned} E_{1,Buy} &= (2r_H - \bar{r}_b)p_2 + (2r_S - \bar{r}_b)q_2 + (2r_L - \bar{r}_b)(1 - p_2 - q_2) \\ &= 2r_H p_2 + 2r_S q_2 + 2r_L(1 - p_2 - q_2) - \bar{r}_b \\ E_{1,Hold} &= r_H p_2 + r_S q_2 + r_L(1 - p_2 - q_2) \\ E_{1,Sell} &= \bar{r}_l p_2 + \bar{r}_l q_2 + \bar{r}_l(1 - p_2 - q_2) = \bar{r}_l \end{aligned}$$

If we rewrite $E_{1,Buy}$ as $2(r_H p_2 + r_S q_2 + r_L(1 - p_2 - q_2))$ and substitute $E_{1,Hold}$ instead of $r_H p_2 + r_S q_2 + r_L(1 - p_2 - q_2)$ then we obtain $E_{1,Buy} = 2(E_{1,Hold}) - \bar{r}_b$. For visual simplicity we will replace $E_{1,Hold}$ with e, the expected value of the return from *Holding* one unit of stock.

To decide on when to *Hold*, *Buy* or *Sell* the stock we need to compare the expected returns from each of the actions and choose the highest return.

$$2E_{1,Buy} - \bar{r}_b \quad \overset{?}{\gtrless} \quad E_{1,Hold} \quad \overset{?}{\gtrless} \quad \bar{r}_l$$

$$2e - \bar{r}_b \quad \overset{?}{\gtrless} \quad e \quad \overset{?}{\gtrless} \quad \bar{r}_l$$

$$e - \bar{r}_b \quad \overset{?}{\gtrless} \quad 0 \quad \overset{?}{\gtrless} \quad \bar{r}_l - e$$

The decision rule simplifies to:

Buy when $e > \bar{r}_b$

Sell when $e < \bar{r}_l$

Hold otherwise.

2.9 The Case of the Speculator

In the previous sections, the players could only transact one unit of the security. In this sections that constraint will also be relaxed allowing players to trade more than single unit of the stock.

The player starts the game with an endowment of k stocks. He can sell any number (n) of his stocks, or buy any number (m) by borrowing at the borrowing rate $(\bar{r}_l)$. The payoff matrix will be as follows:

	High	*Steady*	*Low*
Buy	$kr_H + m(r_H - \bar{r}_b)$	$kr_S + m(r_S - \bar{r}_b)$	$kr_L + m(r_L - \bar{r}_b)$
Hold	kr_H	kr_S	kr_L
Sell	$(k-n)r_H + n\bar{r}_l$	$(k-n)r_S + n\bar{r}_l$	$(k-n)r_L + n\bar{r}_l$

By definition both m and n are non-negative numbers. If the number of stocks sold exceeds the number of available stocks, then it will be considered as short selling.

Since the player makes his decisions by comparing the payoffs in a given column, then if we set initial endowment (k) equal to zero, then the outcome of the game will not be affected. However, since the endowment is zero and all the trabsactions are done either by borrowing or short selling, this will be the case for a pure speculator. Therefore the payoff matrix with k being equal to 0 is:

	High	*Steady*	*Low*
Buy	$m(r_H - \bar{r}_b)$	$m(r_S - \bar{r}_b)$	$m(r_L - \bar{r}_b)$
Hold	0	0	0
Sell	$n(\bar{r}_l - r_H)$	$n(\bar{r}_l - r_S)$	$n(\bar{r}_l - r_L)$

In this model, the speculator not only decides whether to *Buy* or *Sell*, he's also decides on how much (i.e. the amounts m and n). If we assume previous relation between the returns and borrowing and lending rates, the payoffs will have the following signs:

	High	*Steady*	*Low*
Buy	$m(r_H - \bar{r}_b) > 0$	$m(r_S - \bar{r}_b) < 0$	$m(r_L - \bar{r}_b) < 0$
Hold	0	0	0
Sell	$n(\bar{r}_l - r_H) < 0$	$n(\bar{r}_l - r_S) < 0$	$n(\bar{r}_l - r_L) < 0$

The solutions of this game includes the usual three pure strategy equilibria (all *Buy*, all *Hold*, or all *Sell*) and the likeliness of any of the equilibrium depends on the expected market return. If the market return isexpected to be greater than the borrowing rate ($\bar{r}_b$) then the players are more likely to play *Buy*; if the market return is expected to be less than the lending rate ($\bar{r}_l$) then the players are more likely to play *Sell*; otherwise *Hold* will be the outcome.

Once the action decision is made, the speculators also need to choose the amount of stocks to *Buy* and to *Sell*. This decison is rather easy as in order to maximize their utilities, they need to trade as many stocks as they can. In this case as many stocks as they can borrow for.

In real life the amount of borrowing and short selling are limited by variables like the amount of leverage allowed by the lending financial institutions as well as personal measures like the individuals' other wealth, prestige. These limits can be incorporated into the model as upper limits for m and n as $m \leq m_{max}$ and $n \leq n_{max}$. Since the speculator will always try to maximize his returns, he will choose the maximum amounts as optimal m or n.

2.10 Macro factors

In this model we assumed that the risk free lending and borrowing rates will not change. The interest rates may be affected directly or indirectly by these changes in the market. The direct effect can be through simple supply and demand for cash. As the investors (or speculators) puts more money in the stocks, the supply of cash to the banks will decrease while demand for it will increase. On the other hand the firms will borrow less from the banks or issue fewer bonds. Instead they can raise money by selling new toks in the market. So the net effect on the interest rates will be ambiguous.

One major cause of the interest rates comes not from market forces but from policy makers. As the stocks gain value, the Federal Reserve Banks tend to raise the interest rates. The main reson for such interventions is to prevent the formation of possible bubbles.

2.11 Conclusion

This section tried to model human behaviour using some oversimplified mathematical formulas. A complete modeling of human behavior is beyond our technical capacity today. Being approximations themselves, a better model is the one which can explain an event as close to the real world as possible. Of course, explanation does not necessarily lead to accurate prediction[12] of the future.

[12]Chaotic dynamic systems are a good example. Although one may model a system, prediction is almost impossible. No matter how precise your measurements are, due to errors, the estimated parameters will be slightly different than their actual values. The system will eventually diverge from the predictions; initially in smaller amounts, then in larger amounts until the predictions become useless.

3 DYNAMICS

3.1 Introduction

In this section a model consisting of agents participating in trade of a market asset is presented. The assumptions made for this model are as follows: a) The agents are heterogeneous in their assesment of past price changes and buiding expectations about the future returns. b) The market participation behavior of the agents is similar to as described in Youssefmir et. al. In this setup only a fraction of the agents will enter the market at any given time. The rate that they enter the market, λ, will be given by a Poisson process. c) Each agent can only trade one unit of security. This is consistent with the assumption made in sections 3.7 and 3.8 however excludes the case for the "*Pure Speculator.*"

Once the agent participates in the market, he has to make a decision whether to *Buy*, *Sell* or *Hold* the stock, based on his beliefs and expectations on the returns on the asset. When a decision to *Buy* or *Sell* is made, the market will respond to it by increasing or decreasing the price of the asset. This assumption ignores the market clearing mechanisms and the assumption that the markets are always in equilibrium. However, it does capture the simple fact that prices respond to small disequilibriums in the market.

When a player decides to sell, his decision will immediately increase the supply. The market will be in a disequilibrium state until a buyer is found. But a given demand this is only possible by letting the price of the asset to drop a bit. Once the stock is purchased again the market returns to the euilibrium state.

Assuming constant supply of the stocks the prices will increase when the demand increases. This increase in demand can be a small increment or a big jump. What we observe as price changes is the market adjustment mechanism trying to match the prices to a new equilibrium after either demand or supply moved. Actual market (auction) mechanisms are ignored for simplicity and due to the relatively large number of market participants.[13] The disequilibrium during the crash of 1929 has some support from Galbraith's observations. He provides descriptions of market trying to adjust prices for a big decline in the demand along with the increase in supply:

> "Often there were no buyers, and only after wide vertical declines could anyone be induced to bid."

This suggests that the market was in a disequilibrium state for a longer duration than usual. It should be expected that for small changes in the demand, the new equilibrium can be reached in a relatively short time. For larger shifts

[13]The choice of the model used to describe the equilibrium and the price changes depend on the number of parties (idividual or otherwise). When there's a large number of players who demand and supply commodities, we tend to use usual supply and demand analysis. With only two sides, bargaining theory applies. Auctioning falls somewhat in between these two models and is used to explain the behavior of "few," but more than two players.

in the demand, it may take longer for prices to adjust.

The market will respond to a purchase by increasing the price α and to a purchase by decreasing the price by β. This is a simple way incorporating the observation that the prices tend to rise when there are more buyers than sellers, and fall when there are more sellers.

The players look into the past beahviour of the stock and try to estimate the future price of and thus the return on the stock. This is almost a "technical analysis" type of behaviour, except that each person will have a random term to account for other news. The players are heterogeneous both in the way they calculate the return on the stock and in the news they receive.

3.2 Expected Returns

Once an agents decides to enter the market (based on a Poisson process)[14] he will make a decision to *Buy*, *Sell*, or *Hold* the stock. If λ is set to be equal to 3, for example, the player will, on the average, "wake up" every three periods. Then he will look at the price changes and will decide whether to make a

[14]To simulate a Poisson random variable with mean λ, independent uniform random (0,1) variables $U_1, U_2, \cdots$ stopping at

$$N = min\left\{n : \prod_{i=1}^{n} U_i < s^{-\lambda}\right\}$$

need to be generated. The random variable $X \equiv N - 1$ will have a Poisson distribution. That is if we continue generating random numbers until their product falls below $e^{-\lambda}$, then the number required, minus 1, is a Poisson with mean λ. (Sheldon Ross, 1994) transaction. This decision is based on the player's expectation of the future returns on the stock. The time horizon that players look into the future is quite short (only one period).

Let p_t be the price of the stock at time t, and r_t be the price change

$$r_t = p_t - p_{t-1}$$

Also let R_t be the average price changes in the past.

$$R_t = (1-a)\left\{\sum_{i=1}^{\infty} a^{i-1} r_{t-i}\right\}, \qquad 0 < a < 1 \tag{3}$$

where a being the discount factor. This average is a discrete time exponentially averaged rate of change in the price level. The farther a change into the past, the smaller its effect be in the average. The $(1-a)$ factor is to normalize the sum.

When we rewrite the terms, we find:

$$\begin{aligned}
R_t &= (1-a)\left(r_{t-1} + a r_{t-2} + a^2 r_{t-3} + \cdots\right) \\
&= (1-a)\left(r_{t-1} + a\sum_{i=2}^{\infty} a^{i-2} r_{t-i}\right) \\
&= (1-a) r_{t-1} + a(1-a)\left(\sum_{i=2}^{\infty} a^{i-2} r_{t-i}\right) \\
&= (1-a) r_{t-1} + a\left((1-a)\sum_{j=1}^{\infty} a^{j-1} r_{t-j}\right), \quad j = i-1 \\
&= (1-a) r_{t-1} + a R_{t-1}
\end{aligned}$$

which makes the average price change to be the weighted average of latest price change and the previous term's average:

$$R_t = (1-a)r_{t-1} + aR_{t-1} \tag{4}$$

The discount factor a determines how quickly a player *forgets* the past. If a is close to 1, then a^i will still be large enough to affect the average even when i is large. For smaller a, however, a^i will quickly approach to zero. We can therefore identify players with long or shorter *memory* simply by looking at this discount factor. In the simulations the players' discount factor will have a mean $\bar{a}$ and variance σ_a. Although they will behave in a similar manner, their decisions will be based on their *character* described by a.

The expectations about the future price changes are formed by observing the past price changes and reflecting that into the future as their averages. Thus the expected price change for the next period ($\hat{r}_t$) will be equal to the average past price changes R_t. There's also a small random factor (ϵ_r) added so that the expected future return will not exactly be equal to the average price. This randomness will account for all other factors influencing the decision. These factoors range from the "gut feeling" to more real effects like latest news. At any given time the random term will be different for all of the players.

$$\hat{r}_t = R_t = (1-a)r_{t-1} + aR_{t-1} + \epsilon_{t,r}$$

Remember that the players can borrow at a constant borrowing rate $\bar{r}_b$ and lend a the lending rate $\bar{r}_l$. If the expected return is less than the lending rate then player will sell his assets. Similarly, he's going to buy stocks if he expects to earn more on the stock than he's going to pay for it (i.e. the borrowing rate). We now can write the decision rules for the players:

$$\text{if} \begin{cases} \hat{r}_t > \bar{r}_b & \text{player } \textit{Buys} \\ \bar{r}_b < \hat{r}_t < \bar{r}_l & \text{player } \textit{Holds} \\ \hat{r}_t < \bar{r}_l & \text{player } \textit{Sells} \end{cases}$$

Since the players will have different discount factors (a), the observed average price (R_t) will be different for all of the players. Along with the random effect each player will have a different expected return; therefore, they will not try to trade at the same time. It will be possible while some *Holding* for example, some others will be in a *Sell* state.

The price of the stock will be determined by two factors: The number of buyers or sellers and a random factor.

$$p_t = p_{t-1} + \alpha n_{t,B} - \beta n_{t,S} + \epsilon_{t,p}$$

where α is the effect of excess demand on the prices, and β the effect of excess supply, n_B, n_S are the number of buyers and sellers at a given time t, and ϵ is the random change in the prices.

When the market in equilibrium, where the number of sellers equals the number of buyers (or when there's no excess demand or supply), the prices will follow a random walk. This property is consistent with many existing economic theories.

In an initial state where all the players are *Holding*, the random factor may create a large enough price change in one direction to draw attention of some players. If enough number of players see this price change as a trend, and believe the future prices change in large steps, then they will involve in trading.

The perception of other players' expectations is embedded in adaptive expectations. If the process of how players calculate a *trend*, then by looking at the average price a player is also forming expectations on whether they intend to *Buy* or *Sell*.

3.3 Computer Experiments

The computational experiments are rather simple simulations of price movements based on some random terms and the number of *Buyers* and *Sellers*. The program starts with random prices up to the 100th period. Since it is impossible to define infinity to a computer, it would also be impossible to calculate the average past prices. Instead a practical infinity is set to be equal to

100 periods. The error term (which is also called the remainder) involved with this exclusion is relatively small (Thomas and Finney, 1979). For a detailed analysis, please see Appendix.

The program takes about 10 hours to run 3000 periods on a PPC 7600 Macintosh computer with a 210 MHz processor. To keep each player's expectation in the memory is a serious computational penalty. The program will run significantly faster if the expectations are removed from the memory after they are used for transaction decisions and in calculating the next period's expectations. However, in order to keep track of how the players "justify" their decisions, it was necessary to keep them.

The number of players also affects the speed. Running the time up to 1,000 for a group of ten players usually returns results almost real time (minutes). As the number of players increase, the computational time seemed to increase faster than linear; we haven't looked for the compuational complexity of the program, however.

3.4 Results

There were three simulation runs. The first two lasted 1,000 rounds, and the third 3,000 rounds. The number of players were equal to 20 in all three runs, and the mean discount factor, a, was 0.95. Other parameters were: market response rate $\alpha = \beta = 0.005$, wakeup rate $\lambda = 3$, with returns averaged up to

100 periods.

In these three runs the lending and borrowing rate differ slightly. The first had the rates to be equal to $r_l = r_b = 0.0195$, the second $r_l = r_b = 0.0205$ and finally the third $r_l = r_b = 0.0200$.

By looking at the trading activity (Figures 1, 4, and 7) you can see that the game is very sensitive to the interest rates. By decreasing the interest rates (in absolute terms) the activity goes from an inactive run (Fig.1) to over-active state (Fig.4). In between the two there's a set of parameters which produces results with swithing states.

4 CONCLUSION

This manuscript is far from being a complete analysis of individual behavior. Furthermore, we failed to show a very important property of bubbles. For example a crash does not necessarily follow a bubble. This may be a result of our model. It may also be possible that our view of bubbles as constant growth and burst is flawed. As of summer of 1997, the New York Stock Exchange has been gaining value for months. The announcements of the market being overvalued (up to 20 %) has had little effect this increase. The prices has gone up steadily but not smoothly. There were periods of steady prices and even declines. If we indeed are experiencing a bubble, then when the price increases slow down, a crash may not necessarily follow it.

Further analysis can be performed to investigate other properties of the aggregate behaviour of cooperating agents. These include but not limited to the following:

1. The wakeup rate can depend on the price changes rather than being constant.

2. There can be multiple character sets for players. There may be trend followers with a very ahort *memory* (small a), as well as fundemenatlists who believe in a fundamental price.

3. The financial resources can be limited. There may be an upper limit

on how much an investor bring. The auction mechanism can also be incorporated to the game.

5 APPENDIX

Mixed Strategy Equilibrium

If the player 1 plays *Hold* with probability of p_1 (and *Sell* with $1-p_1$), and the player 2 plays *Hold* with probability p_2 (and *Sell* with $1-p_2$), the first players expected payoff is

$$\begin{aligned}
E_{player1} &= p_1[(a-b)r_S p_2 + (a-b)r_L(1-p_2)] + (1-p_1)[(a\bar{r}-br_S)p_2 \\
&\quad +(a\bar{r}-br_L)(1-p_2] \\
&= (a-b)r_S p_2 p_1 - (a-b)r_L p_2 p_1 + (a-b)r_S p_1 \\
&\quad +(a\bar{r}-br_S)p_2(1-p_1) + (a\bar{r}-br_L)(1-p_2)(1-p_1) \\
&= (a-b)(r_S-r_L)p_2p_1 + (a-b)r_Lp_1 + (a\bar{r}-br_S)p_2 - (a\bar{r}-br_S)p_2p_1 \\
&\quad +(a\bar{r}-br_L)p_2p_1 + (a\bar{r}-br_L) - (a\bar{r}-br_L)p_2 - (a\bar{r}-br_L)p_1 \\
&= [(a-b)(r_S-r_L) + b(r_S-r_L)]p_2p_1 + [(a-b)r_L - (a\bar{r}-br_L)]p_1 \\
&\quad +[(a\bar{r}-br_S) - (a\bar{r}-br_L)]p_2 + [(a\bar{r}-br_L)] \\
&= [a(r_S-r_L)]p_2p_1 + [a(r_L-\bar{r})]p_1 + [b(r_L-r_S)]p_2 + (a\bar{r}-br_L)
\end{aligned}$$

If only pure strategies are allowed, p_1 equals zero or one, but in the mixed extention of the game, the first player's selection of p_1 lies on the continuum from zero to one, the pure strategies being the extreme values. Following the usual procedure for solving a maximization problem, we differentiate the payoff function with respect to the choice variable (i.e. p_1) to obtain the first order condition.

$$\frac{\partial E_1}{\partial p_1} = a(r_S - r_L)p_2 + a(r_L - \bar{r}) = 0$$

Solving for p_2 we obtain:

$$p_2 = \frac{\bar{r} - r_L}{r_S - r_L}$$

Since the game is symmetric both players will select the same probability in the mixed strategy equilibrium.

Error Term Regarding Finite Averages

If we calculate the average price changes, not until infinity but up to a finite number k, then the the two estimates will differ from each other slightly, depending on a, k, and r_t. If we call the finite average $R_t^k t$ and the remainder E_t^k, then the average returns will be equal to:

$$
\begin{aligned}
R_t &= (1-a)\left\{\sum_{i=1}^{\infty} a^{i-1} r_{t-i}\right\}, \qquad 0 < a < 1 \\
R_t &= (1-a)\left(r_{t-1} + a r_{t-2} + \ldots + a^{k-1} r_{t-k} + \left\{\sum_{i=k+1}^{\infty} a^{i-1} r_{t-i}\right\}\right) \\
R_t &= \underbrace{(1-a)\left(r_{t-1} + \ldots + a^{k-1} r_{t-k}\right)}_{R_t^k} + \underbrace{(1-a)\left\{\sum_{i=k+1}^{\infty} a^{i-1} r_{t-i}\right\}}_{E_t^k}
\end{aligned}
$$

Thus the remainder will be:

$$
\begin{aligned}
E_t^k &= (1-a)\left\{\sum_{i=k+1}^{\infty} a^{i-1} r_{t-i}\right\} \\
E_t^k &= a^k (1-a)\left\{\sum_{j=1}^{\infty} a^{i-1} r_{t-i}\right\}, \qquad j = k+1 \\
E_t^k &= a^k R_{t-k-1}
\end{aligned}
$$

Assuming R_{t-k-1} is on the order of r_t, then the error will depend on a^k. For $a = 0.95$ and $k = 100$, the error E_t^k will be on the order of 0.003.

Computer Code

```
clear;

%initial parameters
e=exp(-3); %wakeup rate
players=20;
j=1;
rb=0.0205; rl=-0.0205; %interest rates
alpha=0.005; beta=0.005; %price movements
%random number scales
s=0.05; %initial price
ss=0.001; %random price movements
sdisc=0.01; %discount factor scale

%time until 100
r(1)=randn*s; price(1)=randn*s;
for t=2:100,
r(t)=randn*s;
price(t)=price(t-1)+r(t);
end

for i=1:players,
Pi(i,1)=rand*s;
k(i)=1;
disc(i)=0.95+randn*sdisc;
discnn(i)=disc(i)^99;
R(i,1)=0;
for t=2:100,
R(i,t)=R(i,t-1)*disc(i);
end

end

sell(100)=0; buy(100)=0;

%game
for t=101:3000,
t
sell(t)=0; buy(t)=0;
%play(t)=0;
```

```
for i=1:players,
Pi(i,k(i)+1)=Pi(i,k(i))*rand;
R(i,t) = (1-disc(i))*r(t-1) + disc(i)*R(i,t-1) -
(1-disc(i))*discnn(i)*r(t-99);

if Pi(i,k(i)+1) < e, %trade

if R(i,t)+randn*ss>rb,
buy(t)=buy(t)+1;
elseif R(i,t)+randn*ss<rl
sell(t)=sell(t)+1;
end

N(j,i)=k(i); j=j+1; k(i)=1; Pi(i,1)=rand;
else
k(i)=k(i)+1; %pass
end
end

price(t) = price(t-1) + alpha*buy(t) -beta*sell(t) + randn*s;
r(t)=price(t)-price(t-1);

end

t=1:1:1000;
r=mean(R);
plot(disc), pause
plot(t,R), pause
plot(t,buy,'r',t,sell), pause
plot(price), pause
plot(t,r,t,buy/500,t,-sell/500)
```

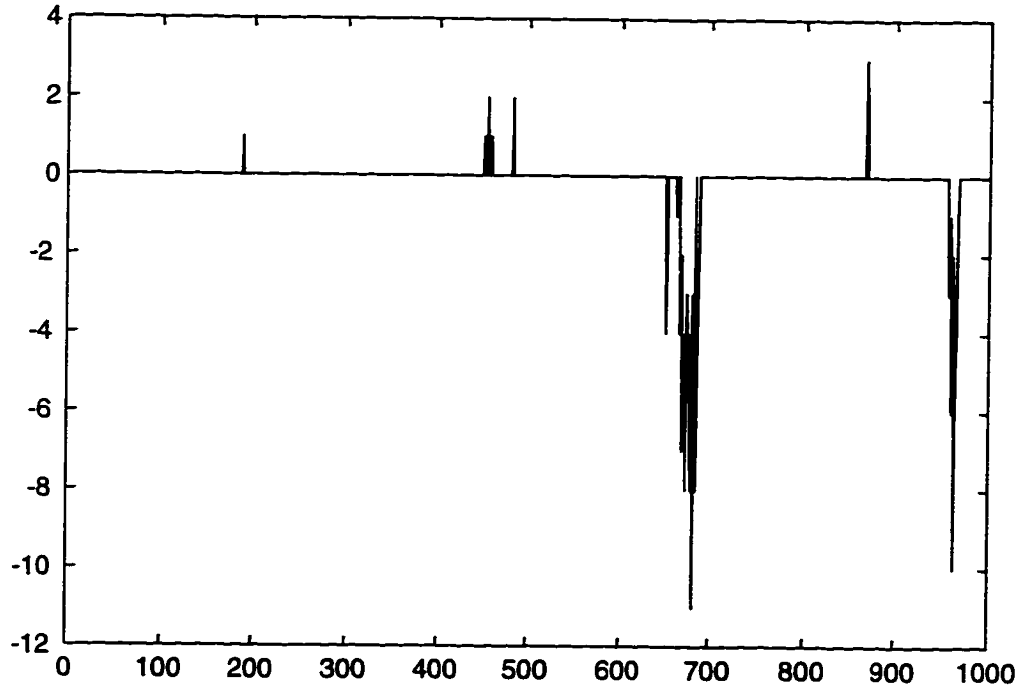

Figure 1: Number of Traders ($r_l = 0.0205$)

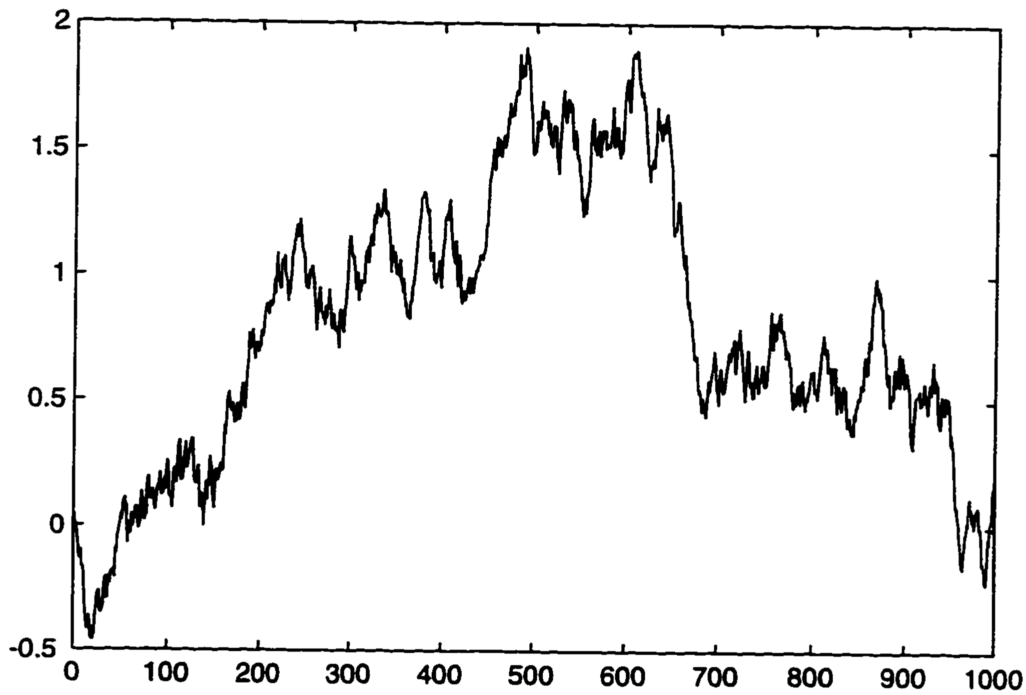

Figure 2: Stock Prices ($r_l = 0.0205$)

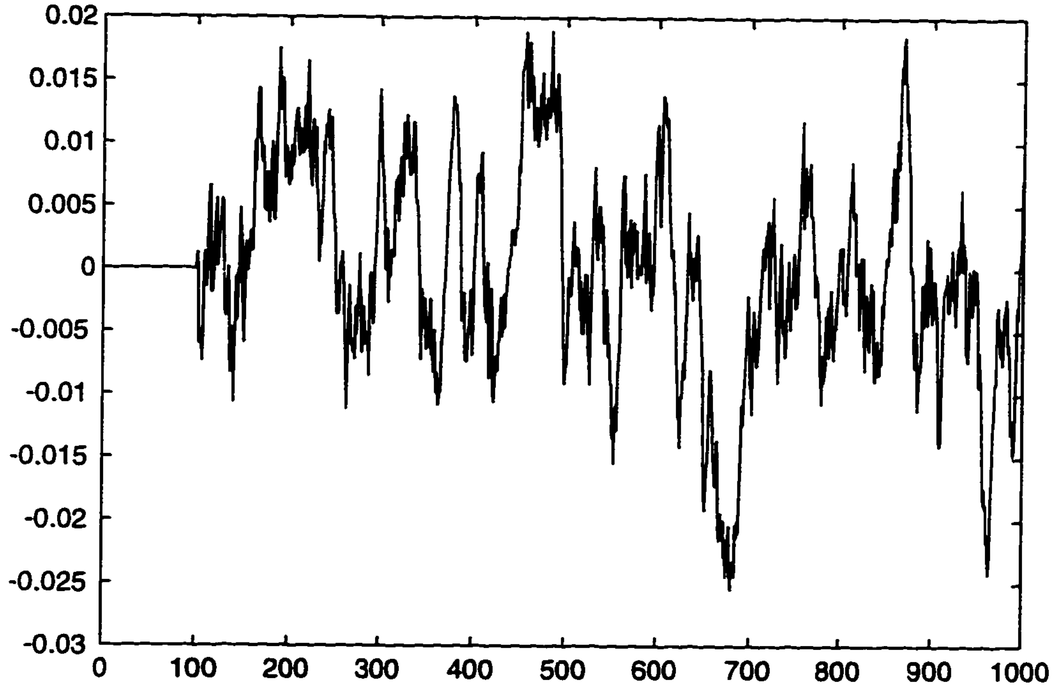

Figure 3: Expected Returns ($r_l = 0.0205$)

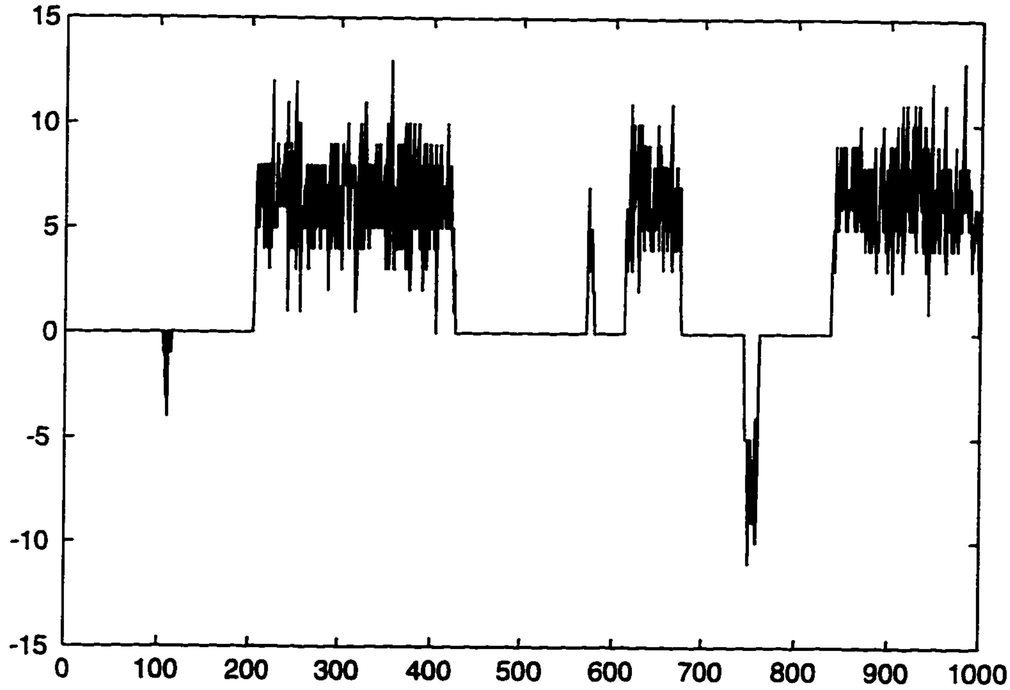

Figure 4: Number of traders ($r_l = 0.0195$)

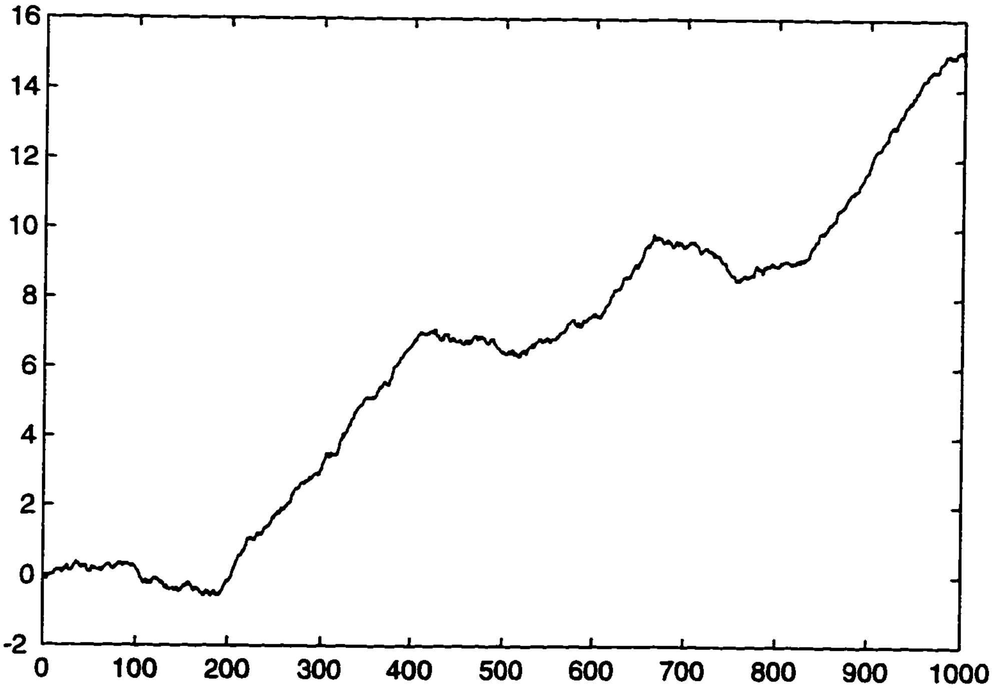

Figure 5: Stock Prices ($r_l = 0.0195$)

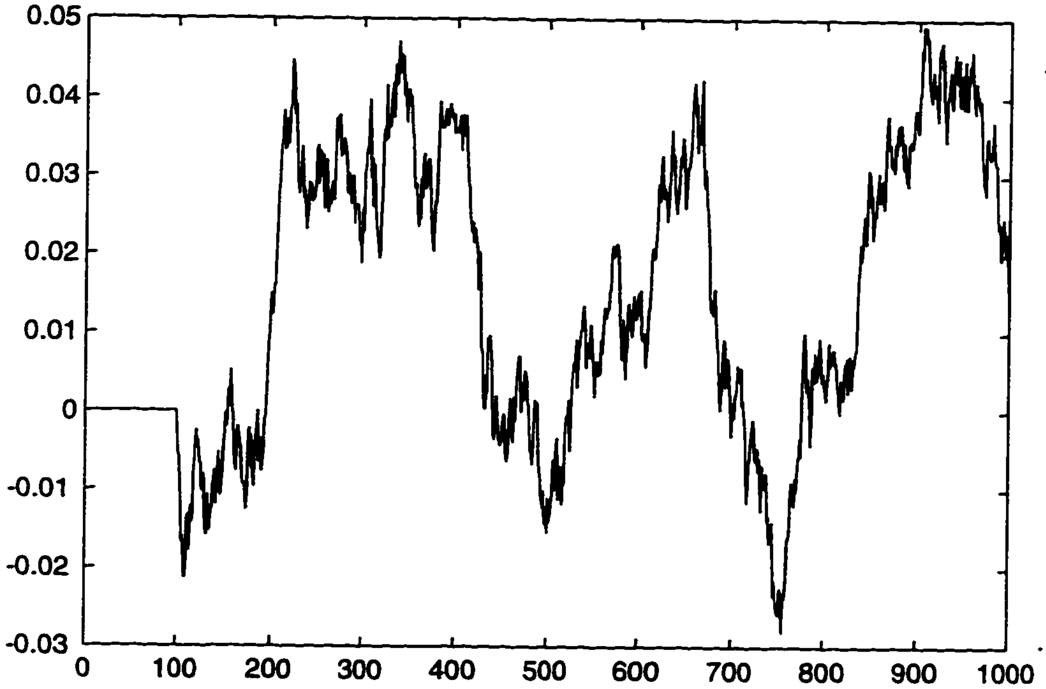

Figure 6: Expected Returns ($r_l = 0.0195$)

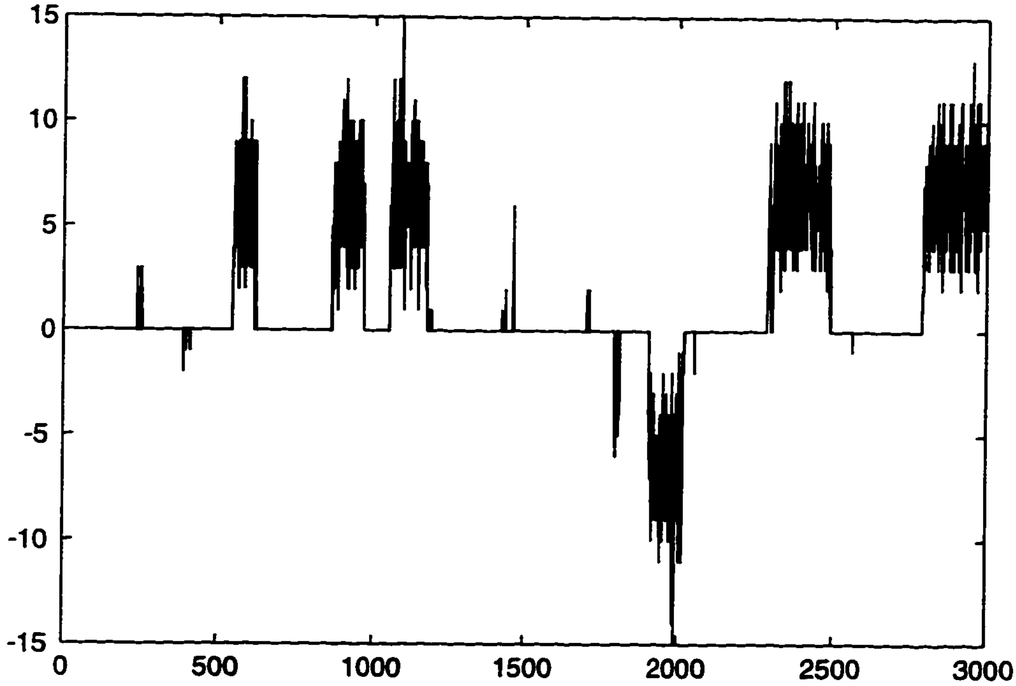

Figure 7: Number of traders ($r_l = 0.0200$)

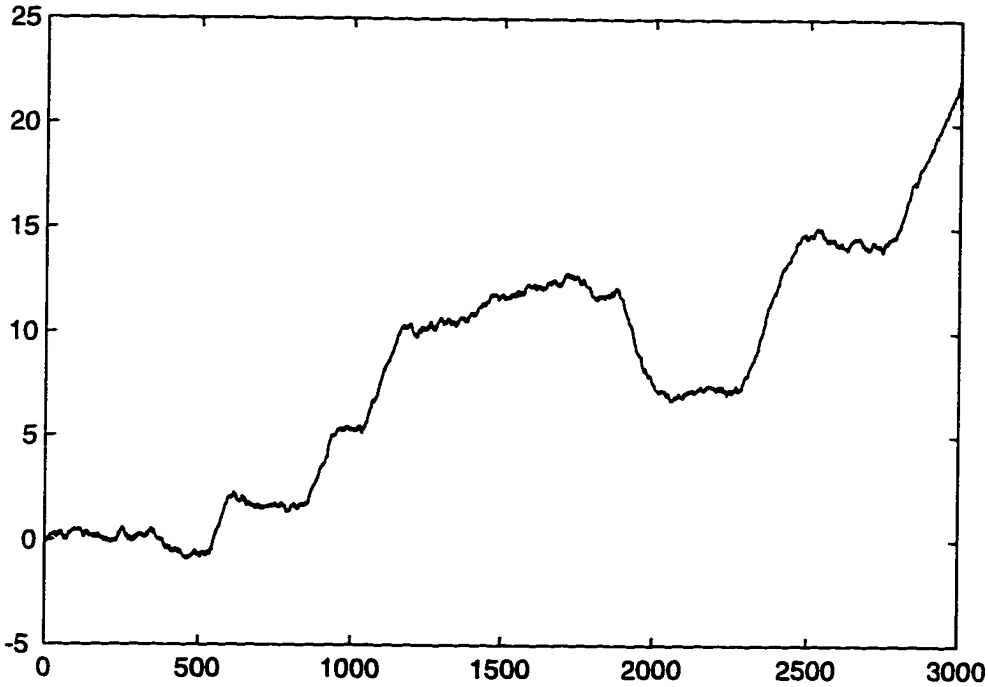

Figure 8: Stock Prices ($r_l = 0.0200$)

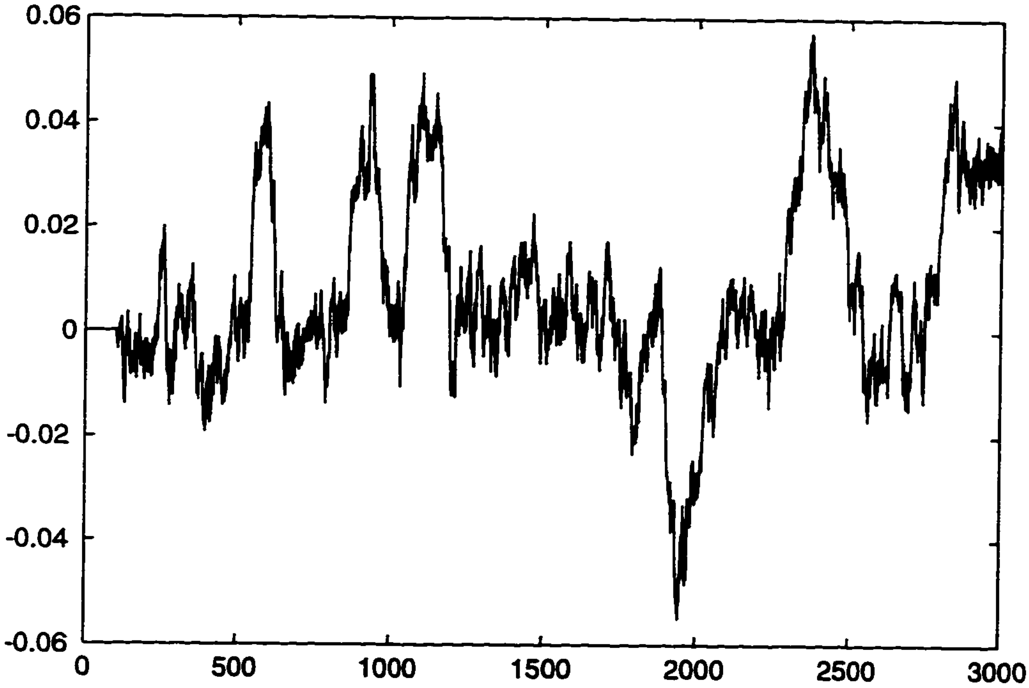

Figure 9: Expected Returns ($r_l = 0.0200$)

6 BIBLIOGRAPHY

References

[1] William F. Sharpe. Capital Asset Prices: A Theory of Market Under Conditions of Risk. *Journal of Finance,* September 1964, Vol. XIX, No. 3, pp. 489-509.

[2] John C. Hull. *Options, Futures, and Other Derivative Securities.* Prentice Hall, Englewood Cliffs, NJ, 1989. 2nd ed. 1993.

[3] Frank K. Reilly, and Edgar E. Norton. *Investments.* Dryden Press, New York, 1982. 4th ed. 1995, pp. 180, 233, 470.

[4] William F. Sharpe. Capital Asset Prices with and without Negative Holdings. *Journal of Finance,* June 1991, Vol. XLVI, No.2, pp. 489-509.

[5] Joseph E. Stiglitz. Syposium on Bubbles. *Journal of Economic Perspectives,* Vol.4 No.2, Spring 1990, pp.13-18.

[6] Jean Tirole. On the Possibility of Speculation Under Rational Expectations. *Econometrica,* September 1982, Vol. 53(6), pp. 1500-1529.

[7] Olivier J. Blanchard, and Mark W. Watson. Bubbles, Rational Expectations, and Financial Markets. In *Crises in the Economic and Financial Structure,* edited by P. Wachtel. Lexington Books, 1982.

[8] Olivier J. Blanchard, and Stanley Fischer. *Lectures on Macroeconomics.* The MIT Press, Cambridge, MA, 1989.

[9] Andrei Scleifer, and Lawrence H. Summers. The Noise Trader Approach to Finance. *Journal of Economic Perspectives,* Vol.4 No.2, Spring 1990. pp.19-33.

[10] Eugene N. White. When the Ticker Ran Late: The Stock Market Boom and Crash of 1929. In *Crashes and Panics - Lessons from History,* edited by Eugene E. White, Irwin, Homewood, IL, 1990.

[11] Jack W. Wilson, Richard E. Sylla, and Charles P. Jones. Financial Market Panics and Volatility in the Long Run. In *Crashes and Panics - Lessons from History,* edited by Eugene E. White, Irwin, Homewood, IL, 1990.

[12] Harry M. Markowitz. Foundations of Portfolio Theory. *Journal of Finance,* June 1991, Vol. XLVI, No.2, pp. 469-477.

[13] Robert P. Flood, and Peter M. Garber. Bubbles, Runs, and Gold Monetization. In *Crises in the Economic and Financial Structure*, edited by P. Wachtel, Lexington Books, 1982.

[14] Michael Youssefmir, Bernardo A. Huberman, and Tadd Hogg. Bubbles and Market Crashes. *Dynamics of Computing Group, Xerox PARC Working Paper*, 1995.

[15] Natalie S. Glance and Bernardo A. Huberman. The Outbreak of Cooperation. *Journal of Mathematical Sociology*, 1993, Vol. 17(4), pp. 281-302.

[16] Robert P. Flood and Robert J. Hodrick. On testing for Speculative Bubbles. *Journal of Economic Perspectives*, Vol.4 No.2, Spring 1990, pp.85-101.

[17] Richard H. Day and Weihong Huang. Bulls, Bears, and Market Sheep. *Journal of Economic Behavior and Organization*, 1990 Vol.14, pp 299-329.

[18] George Soros. *The Alchemy of Finance*. Wiley, New York, 1987.

[19] Hayne Leland and Mark Rubinstein. Comments on the Market Crash: Six Months After. In *Crashes and Panics - Lessons from History*, edited by Eugene E. White, Irwin, Homewood, IL, 1990.

[20] Robert J. Shiller. Speculative Prices and Popular Models. *Journal of Economic Perspectives*, Vol. 4 No.2, Spring 1990, pp. 55-65.

[21] Sandra Blakeslee. In Work on Intuition, Gut Feelings Are Tracked to Source: The Barain. *The New York Times*, March 4, 1997, pC 5.

[22] John Kenneth Galbraith. *The Great Crash 1926*. Hughton Mifflin, Boston. 1988.

[23] John Kenneth Galbraith. *A Short History of Financial Euphoria*. Whittle Books, New York, 1990.

[24] Merton Miller. The Crash of 1987 and the Crash of 1946. In *Crashes and Panics - Lessons from History*, edited by Eugene E. White, Irwin, Homewood, IL, 1990.

[25] Barry J. Eichengreen. 1929 And 1987 Parallels and Contrasts. In *Crashes and Panics - Lessons from History*, edited by Eugene E. White, Irwin, Homewood, IL, 1990.

[26] Sheldon Ross. *A First Course in Probability*. Macmillan, New York, 1976. 4th ed. 1994. p. 455.

[27] George B. Thomas and Ross L. Finney. *Calculus and Analytic Geometry.* Addison-Wesley, Reading, MA 1951. 5th ed. 1979. p. 791.

[28] Eric Rasmusen. *Games and Information.* Blackwell, Oxford, 1989.

[29] Drew Fundenberg and Jean Tirole. *Game Theory.* MIT Press, Cambridge, MA, 1991.