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CANONICAL QUANTIZATION OF NON-ABELIAN GAUGE THEORY IN THE SCHRÖDINGER PICTURE: APPLICATIONS TO MONOPOLES AND INSTANTONS

by

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Abstract

Canonical Quantization of Non-Abelian Gauge Theory in the Schrödinger Picture: Applications to Monopoles and Instantons

by

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In this thesis we present a detailed formulation of the quantum theory of non-abelian gauge fields in the Schrödinger picture. In the first part we apply it to the semi-classical quantization of the t'Hooft-Polyakov monopole paying special attention to the treatment of boundary conditions and, local and global gauge symmetry. The perturbation expansion is then discussed using standard collective co-ordinates. In the Prasad-Sommerfield limit we present all the eigenfunctions of the fluctuation equation; construct the groundstate wave function in terms of gauge and translation invariant co-ordinates and compute its total angular momentum to be zero.

The second part of the thesis is devoted to aspects of instanton phenomena, in the Schrödinger picture, elucidating the role of euclidean time. We demonstrate the precise relationship between boundary conditions, choice of gauge and the corresponding picture of the semi-classical vacuum.

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CONTENTS

I: INTRODUCTION AND SUMMARY

In recent years there has been important progress in our understanding of quantum field theories. This progress is in large part due to the development of approximation methods which revealed that field theories possess a richer structure than what one suspects in ordinary perturbation theory, where the basic set of states around which one performs the perturbation expansion are obtained by diagonalizing the free field Hamiltonian.(1) All amplitudes computed within such an approximation are analytic in the coupling constant to any finite order in perturbation theory.

To go beyond such an approximation two different approaches have developed. One is the method of redefining the field theory on a discrete space-time lattice;(2) such a field theory is a well defined quantum system for any value of the coupling constant. Statistical mechanics methods especially the renormalization group have been widely used to study these field theories. The other is the semi-classical method which is a carry over of the WKB approximation to the continuum field theory.(3) Our discussion will be entirely within the realm of the latter development encompassing both real and imaginary time phenomena.

We first discuss real time phenomena. An essential role is played by certain classical solutions of the complete non-linear equations; their field energy is localized in space, and they appear as classical extended objects. If we perform a Lorentz transformation on the solution the energy-momentum relation of relativistic mechanics is satisfied. These field configurations may also give rise to non-zero values for other constants of motion of the field theory like angular momentum, electric charge etc. Such classical extended objects are called solitons.(4) Under certain assumptions of stability, which we shall shortly explain, the solitons lead to new states in the quantum theory.

Soliton solutions are known in many field theories; for example in 1+1 dimensions the $\lambda\phi^4$ scalar field theory possesses a static kink solution; all solutions of the sine-gordon theory are known because it is a completely integrable system. Beyond 1+1 dimensions there are no static soliton solutions to purely scalar field theories (This is usually called Derrick's theorem;(4) appendix 1). However time dependent solitons do exist in such theories. In 2+1 dimensions the abelian Higgs model has a soliton which is a concentration of discrete magnetic flux. The same model in 3+1 dimensions contains as a solution magnetic flux tubes with finite energy per unit length; these are the vortices of Neilsen and Olesen.(5) Non-Abelian gauge theories in 3+1 dimensions have no soliton solution unless coupled to Higgs fields (appendix 1); the monopole solution of t'Hooft and Polyakov occurs in the Georgi-Glashow model:(6) SU(2) gauge theory with spontaneous symmetry breaking by a triplet of Higgs fields.

Several methods have been developed to study the relevance of these classical solitons in the quantum theory;(7), (8), (9) the simplest of these closely mimics the quantization of field theories with spontaneous symmetry breaking.(9) We briefly discuss it for static solitons. To simplify the notation denote a generic field variable by the symbol $\vec{r}$ (space, Lorentz and internal symmetry indices label the components of $\vec{r}$). The Hamiltonian is H = $\frac{\vec{p}^2}{2}$ + $\frac{V(g\vec{r})}{g^2}$; 'g' is a coupling constant. In terms of the scaled co-ordinate $\vec{q}=g\vec{r}$ and momentum $\vec{p}'=\frac{\vec{p}}{g}$, we have $H = \frac{1}{g^2}\left(g^4\frac{\vec{p}'^2}{2} + V(\vec{q})\right)$; since $\vec{p}'=-i\hbar\frac{\partial}{\partial\vec{q}}$, $g^4\vec{p}'^2=-(g^2\hbar)^2\frac{\partial^2}{\partial\vec{q}^2}$; hence '$\hbar$' and '$g^2$' always appear as a product implying the semi-classical WKB expansion in powers of $\hbar$ and the weak coupling expansion in powers of 'g' are one and the same thing. Henceforth we set $\hbar$ = 1.

It is also clear that the leading term of the lowest eigenvalue of H is given by $\frac{1}{g^2}V_{min}$, which is reached by those configurations which satisfy the static classical equations $\left(\frac{\partial V}{\partial\vec{q}}\right)_{\vec{q}_{cl}}=0$. Further the quadratic form $V^{(2)}_{\alpha\beta}=\frac{\partial^2 V[\vec{q}_{cl}]}{\partial q_\alpha\partial q_\beta}$ must have no negative eigenvalue to ensure that we are atleast around a local minimum $\vec{q}_{cl}$ in configuration space. Note that the classical solution $\bar{r}_{cl}=\bar{q}_{cl}/g$ and the classical part of the energy are not analytic in 'g' and in the limit of weak coupling represent large effects.

After these observations one determines the Fock space of the $O(g^0)$ fluctuations $\vec{\eta}=\vec{r}-\vec{r}_{cl}$. Expanding H in powers of 'g' one gets

$$H = \frac{V_{min}(\vec{q}_{cl})}{g^2} + \frac{1}{2}\sum_{\alpha}\left(-\frac{\partial^2}{\partial\eta_\alpha^2} + V^{(2)}_{\alpha\beta}[\vec{q}_{cl}]\eta_\alpha\eta_\beta\right) + O(g)$$

The O(1) quadratic form defines the Fock space around the classical solution and the soliton quantum state to O(1) is the ground state of this Fock space.

However the above perturbation expansion is invalid being infested with infrared divergences arising from the zero eigenmodes $\delta\vec{q}_{cl}$ of $V^{(2)}[\vec{q}_{cl}]_{\alpha\beta}$; $\delta\vec{q}_{cl}$ is a result of infinitisimal symmetry group action on the classical solution. A correct treatment is provided by the method of collective co-ordinates[9] where one performs a seperation of the degrees of freedom into group parameters and group invariant degrees of freedom by fixing a gauge which is typically of the form $\delta\vec{q}_{cl}\cdot\vec{\eta}=0$. The perturbation expansion is done in the space of group invariant variables.

The above method has been successfully applied to the quantization of solitons in 1+1 dimensional field theories.[3] We would like to discuss its application to the quantization of the t'Hooft-Polyakov monopole as an example of the semi-classical method in gauge theories. We have devoted chapter II of this thesis to a detailed discussion of this problem.

Any attempt to quantize the t'Hooft-Polyakov monopole raises certain basic questions in the quantization of gauge fields, especially when the fields are long ranged. For this reason, we have chosen to formulate the theory in the Schrödinger picture of quantum mechanics, using the method of Dirac.[10] In this method it is well known that the wave function satisfies Gauss' law, which can be interpreted to mean that the wave function carries a trivial representation of the subgroup of gauge transformations whose parameters vanish at spatial infinity. However the Hamiltonian of the system commutes with a much larger symmetry group including the gauge transformation which does not become trivial at spatial infinity. One of the questions we address ourselves is the representation the wave function carries of these gauge transformations.

We have presented a detailed formulation of Hamilton's variational principle.[(11)] It was pointed out by Regge and Teitelboim,[(12)] in the context of the theory of gravitation, that the equations of motion are stationary points of the action provided certain surface integrals that appear in the variation of the action are absent. Using the variational principle as a guide we discuss the boundary conditions on the gauge and Higgs fields. We are also led to the conclusion that if the wave function is to carry a non-trivial representation of the asymptotic gauge group, the configuration space of the theory needs extention to include the parameters of the gauge group on the boundary, (we mean the boundary of a large three dimensional box which eventually goes to infinity) and thereby one picks up an additional subsidiary condition on the wave function. In the extended configuration space we have a generalization of Gauss' law; the wave function once more carries a trivial representation of the entire gauge group. This sort of conclusion is very reminiscent of the method of collective co-ordinates, the parameters of the gauge group on the boundary are collective coordinates of global gauge symmetry.

The asymptotic gauge group is actually an infinite parameter group: a direct product of local SU(2) groups at each point on the boundary. However due to the admissible boundary conditions in a spontaneously broken theory only a one parameter global U(1) symmetry is realized on the physical states in this model.[(13), (11)] An important conceptual point emerges: the local gauge symmetry with vanishing gauge parameters at spatial infinity is <u>never broken</u> because the physical states manifestly satisfy Gauss' law; it is only the asymptotic gauge group that is broken in the sense that it is not completely (or may be not at all) realized in the space of physical states; most of the charges of the asymptotic gauge group are screened.

As we mentioned, the wave function in the extended configuration space, which now includes the global U(1) parameter as a collective co-ordinate, is gauge invariant to the class of gauge transformations which are asymptotically U(1). Hence it is only a function of the orbits of the gauge group in configuration space. Following Faddeev and Popov[(14)] one fixes a gauge to select out one point per orbit, achieving a separation of the gauge degrees of freedom from the gauge invariant ones.[(15)] Several well known gauges are discussed, e.g. the axial gauge and its generalization achieves a complete separation of the degrees of freedom, however it is almost impossible to develop a perturbation expansion in this gauge. The Coulomb gauge and its generalization the background gauge are computationally viable, but the orbits of gauge fields may have many intersections with these gauge fixing surfaces.[16]

However if the fluctuation around a given classical solution does not become 'large' it is possible to prove that for the validity of a certain gauge, the Faddeev-Popov operator for <u>at least</u> one intersection of the orbit of the classical solution with the gauge fixing surface, must not develop a normalizable zero mode, in which case it is not invertible. The orbit may have several other intersections and at some of these the Faddeev-Popov operator may not be invertible, however we may use the gauge invariance of the wavefunction to peak it at the point where the operator is invertible. We find that both the Coulomb and background gauges satisfy the above criterion in perturbation theory around the t'Hooft-Polyakov monopole. We should mention that the Coulomb and the background gauges fix the gauge only upto the homogeneous solution of the Faddeev-Popov operator; in this model the homogeneous solution can be shown to be a one parameter function. In order to fix this parameter we need to generalize these gauges by imposing an additional condition.

It is most convenient to develop the perturbation expansion in the generalized background gauge. As far as the t'Hooft-Polyakov solution is concerned we have no handle on the solution of the linearized fluctuation equations; we have just set up a formal perturbation expansion, treating the translation zeromodes by standard collective co-ordinates. However in the Prasad-Sommerfield limit[17] where the Higgs coupling is absent the classical solution actually represents a 'static' euclidean field configuration with the Higgs field identified with the fourth component of a euclidean gauge field.[18] With this hint we carry over techniques of instanton physics to discuss the fluctuation equations; using the methods of t'Hooft[19] and Brown et. al.[20] We present the complete set of eigenfunctions of the fluctuation equations in terms of the eigenfunctions of the Faddeev-Popov operator. Since the homogeneous solution of the Faddeev-Popov operator can be exactly determined, we find in the generalized background gauge, exactly three normalizable zero modes which correspond to spatial translations of the classical solution. We find that these eigenfunctions have definite transformation properties under space rotations; hence in the space of gauge and translation invariant fluctuations we can explicitly construct the ground state wave function in the Fock space, and compute the expectation value of the total angular momentum. We find it to be zero. The monopole has spin zero at least as described as the ground state in the Fock space.

In the final section of chapter II we discuss angular momentum in a gauge theory and show that the phenomenon of isospin turning into spin, discovered by Hasenfratz and t'Hooft and Jackiw and Rebbi,[21] follows just from the necessity of having fixed boundary conditions for the Higgs field in a spontaneously broken theory. With this section we end our long and detailed exposition of the semiclassical method in a gauge theory.

One of the motivations for the development of the semi-classical method was to discuss extended objects within the frame work of local field theory with the hope that it may shed light on a Lorentz invariant theory of extended hadron structure. However, within quantum chromodynamics (QCD), which is a most promising candidate for a theory of the strong interactions, consisting of coloured quarks interacting gauge invariantly with SU(3) coloured gauge fields, there are no soliton like classical solutions of the type we discussed earlier. This is simply because there is no inherent length scale in the classical version of the theory.

The understanding and proof of extended hadron structure within QCD is perhaps the most important and challenging problem. There have been several interesting approaches to this problem, to mention, the renormalization group approach on a euclidean lattice[2] and the variational approach.[22] With analogies from statistical mechanics, Polyakov[23] suggested that in the weak coupling regime the euclidean path integral of the continuum version of QCD may be approximated by a gas of pseudo-particles or instantons which are approximate stationary points of the euclidean action; these pseudo-particles in QCD have non-trivial topological properties. Subsequently t'Hooft, and Callan, Dashen, Gross, and Jackiw and Rebbi,[24] interpreted these localized euclidean excitations to be tunnelling events in imaginary time. The perturbation theory groundstate is modified by tunnelling transitions and is parametrized by an angle θ. This statistical mechanics approach has been further extended towards a theory of hadronic structure by Callan, Dashen and Gross.[24]

The discussion of tunnelling phenomena in a non-abelian gauge theory is subtle in regard to the discussion of boundary condition and gauge fixing. Our discussion is presented in the Schrödinger picture of quantum mechanics, where the euclidean time of the instanton parametrizes a tunnelling path in configuration space (Gervais and Sakita[25]).

Alternative pictures of the gauge theory vacuum arise depending on the choice of boundary conditions and gauge conditions which may or may not be consistent with these boundary conditions. A trivial gauge group on the boundary of three dimensional space partitions the gauge group into homotopy classes and leads to a Bloch wave picture of the semi-classical ground state. However such a choice of boundary conditions is inconsistent with the choice of the axial gauge.

A correct treatment of this gauge requires an extended configuration space which includes the gauge group on the boundary of three dimensional space.(26) The Bloch wave picture is no longer valid and the picture is analogous to a pendulum performing topologically distinct traverses in the classically forbidden region. In either case the important gauge invariant point is that the wave function carries an irreducible representation of the homotopy group $\pi_3(SU(2)) \cong \mathbb{Z}$, specified by a Bloch parameter 'θ'.

II: CANONICAL QUANTIZATION OF NON-ABELIAN GAUGE FIELDS APPLICATIONS TO MONOPOLE SOLUTIONS:

In this chapter we present a detailed discussion of the canonical formulation of non-abelian gauge theory. Our discussion is within a specific model of SU(2) gauge fields coupled to a triplet of Higgs scalars. In section 1 we discuss the t'Hooft-Polyakov monopole which is a classical solution of the model, with special emphasis on topological properties. In section 2 we discuss the canonical formulation closely following the method of Dirac. Special emphasis is on surface terms, boundary conditions and global gauge transformations. In section 3 we discuss expressing the wave function in terms of gauge invariant co-ordinates by fixing a gauge, thereby ensuring the subsidiary condition of the wave function. As examples we discuss the unitary, axial, Coulomb and background gauges with special emphasis on the treatment of global symmetry. In section 4 the formal perturbation expansion around the classical solution is discussed in the background gauge, using standard collective co-ordinates to treat the translation mode. In section 5 we discuss the Prasad-Sommerfield limit and its analogue to euclidean gauge fields. We present all the eigenfunctions of the small fluctuation equations in terms of the eigenfunction of the Faddeev-Popov operator, discuss their completeness and rotational properties. The wave function of the monopole has zero spin. In section 6 we discuss angular momentum in a gauge theory; having fixed boundary conditions for the Higgs field turns isospin into spin.

Section 1: The t'Hooft-Polyakov monopole solution

The monopole solution of t'Hooft and Polyakov occurs in the Georgi-Glashow model in 3+1 dimensions.[6] The Lagrangian of the system is

$$\mathcal{L} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu} - \frac{1}{2}\nabla_\mu\vec{\Phi}\,\nabla_\mu\vec{\Phi} - \frac{\lambda}{4}\left(\vec{\Phi}^2 - \frac{\mu^2}{\lambda}\right)^2 \qquad (1.1)$$

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\,\vec{A}_\mu\times\vec{A}_\nu$$

$$\nabla_\mu\vec{\Phi} = \partial_\mu\vec{\Phi} + g\,\vec{A}_\mu\times\vec{\Phi}$$

Vector notation is used for iso-spin indices. $\vec{A}_\mu$ is an SU(2) gauge field; $\vec{\Phi}$ is a triplet of Higgs fields in the adjoint representation which is isomorphic to the group SO(3). The cross product notation is defined in terms of the structure constants of the group, e.g.

$$\left(\vec{A}_\mu\times\vec{A}_\nu\right)^a = \epsilon^{abc}A^b_\mu A^c_\nu$$

The Lagrangian (1.1) is invariant with respect to the gauge transformation

$$\frac{\tau^a}{2i}A^{a\prime}_\mu = U^+\frac{\tau^a}{2i}A^a_\mu U + \frac{1}{g}U^+\partial_\mu U$$

$$\frac{\tau^a}{2i}\Phi^{a\prime} = U^+\frac{\tau^a}{2i}\Phi^a U$$

U is an SU(2) matrix and τ^a are the Pauli matrices.

The potential energy corresponding to the Lagrangian (1.1) is

$$V = \int d\bar{x}\left[\frac{1}{4}\bar{F}_{ij}\cdot\bar{F}_{ij} + \frac{1}{2}\nabla_i\bar{\Phi}\cdot\nabla_i\bar{\Phi} + \frac{\lambda}{4}\left(\bar{\Phi}^2 - \frac{\mu^2}{\lambda}\right)^2\right] \tag{1.2}$$

Since all three terms of the potential energy density are non-negative the absolute minimum of V is zero, and it is reached for field configurations which satisfy

$$\bar{F}_{ij} = \nabla_i\vec{\Phi} = \vec{\Phi}^2 - \frac{\mu^2}{\lambda} = 0 \tag{1.3}$$

The solution to (1.3) upto a continuous gauge transformation is

$$\bar{A}_i = 0 = \partial_i\vec{\Phi} = \vec{\Phi}^2 - \frac{\mu^2}{\lambda} = 0$$

The classical ground state is a constant Higgs field of constant magnitude $\mu/\sqrt{\lambda}$. The direction of the Higgs field is arbitrary and there is a classical ground state corresponding to each such direction. These degenerate classical ground states are related to each other by a gauge transformation which is an element of the coset SO(3)/U(1). In order to do perturbation theory one fixes the boundary condition of the Higgs field to be a fixed direction in iso-space. This is spontaneous symmetry breaking: the fixed boundary condition of the Higgs field violates the symmetry of the original Lagrangian. Now one shifts the full quantum field $\vec{\Phi}$ by its large background value (λ is small) and expands the potential energy V to O(1) in λ; the spectrum of small oscillations is a pair of charged massive vector bosons of mass $M_W = g\mu/\sqrt{\lambda}$, a massless photon corresponding to the unbroken U(1) gauge group and a neutral scalar meson of mass $M_H = \sqrt{2}\mu$. Due to local gauge invariance of the physical states the goldstone modes can be eliminated by a choice of gauge. This is the Higgs mechanism. The particle states carry an irreducible representation of the global group U(1); the global SO(3) symmetry of the original Lagrangian has been broken down to U(1). We have just briefly summarized the perturbation expansion in the 'vacuum' sector of a spontaneously broken theory.

In accordance with the ideas of the semiclassical method described in the introduction, one now looks for other non-trivial minima of the potential energy V. In order to interpret these minima as particle states in a systematic quantum expansion one looks for regular classical solutions which give a finite value of V. This is the classical energy of the soliton. For this to happen each of the three positive terms in (1.2) must go to zero as $r \to \infty$ at least as fast as:

$$\vec{\Phi}^2 \longrightarrow \mu^2/\lambda \quad + \quad o\left(r^{-\frac{3}{2}-\epsilon}\right) \tag{1.4a}$$

$$\nabla_i \vec{\Phi} \longrightarrow \quad o\left(r^{-\frac{3}{2}-\epsilon}\right) \tag{1.4b}$$

$$\vec{A}_i \longrightarrow \quad o\left(r^{-1}\right) \tag{1.4c}$$

(1.4a) means that as $r \to \infty$, the equation

$$\frac{\lambda}{\mu^2}\, \vec{\Phi}^2(\infty,\omega) = 1$$

defines a mapping of S^2_∞ (the 2 dimensional sphere at spatial ∞) onto the unit Higgs sphere which in this case is also $S^2 = SO(3)/U(1)$. It is known that such mappings fall into homotopy classes specified by an integer winding number(27) :

$$d = \frac{1}{8\pi} \int d^2\sigma_i \; \epsilon_{ijk} \; \hat{\Phi} \cdot \partial_j \hat{\Phi} \times \partial_k \hat{\Phi} \tag{1.5}$$

where $\hat{\Phi}(\infty,\omega) = \frac{\lambda}{\mu^2} \vec{\Phi}(\infty,\omega)$. ω is a parametrization of S^2_∞.

In the vacuum sector $\hat{\Phi} = (0,0,1)$ or any other constant and $d = 0$. The simplest choice that gives $d = +1$ is

$$\hat{\Phi}_a = \hat{x}_a = x_a / |\vec{x}| \tag{1.6}$$

Since (1.5) is a topological invariant it is invariant under any continuous deformation of $\hat{\Phi}$, in particular by a continuous gauge transformation. So (1.6) is unique upto a continouos gauge transformation. The asymptotic form of the gauge field is almost determined by (1.4b) and (1.6):

$$\partial_i \hat{\Phi} + g \bar{A}_i \times \hat{\Phi} = 0$$

implies for $\hat{\Phi} = \hat{x}$

$$\bar{A}_i = -\frac{1}{g} \hat{x} \times \partial_i \hat{x} + \hat{x} (\bar{A}_i \cdot \hat{x})$$

if we choose $\bar{A}_i \cdot \hat{x} = 0$,

$$A_{ai} = \epsilon_{aij} \hat{x}_j / g r \tag{1.7}$$

An illuminating picture of what is going on can be given as follows: We note that any solution of (1.4b) is given by

$$\hat{\Phi}(\omega) = U(\omega)\hat{z}$$
$$U(\omega) = P\left\{\exp ig \int_N^{w} \bar{A}_i \cdot \bar{T}\, dx_i\right\} \tag{1.8}$$

$U(\omega)$ is the path dependent phase factor which is infact an element of the gauge group.(4) The path 'P' is taken over the sphere S^2_∞; N denotes the north pole (0,0,1) where the Higgs field points in the z = (0,0,1) direction in iso-space; 'w' denotes another point on the sphere where the Higgs field is $\hat{\Phi}(\omega)$. We also note that

$$A_i = U^{\dagger} \frac{\partial}{\partial x_i} U \tag{1.9}$$

Equations (1.8) and (1.9) mean that the monopole boundary conditions are gauge rotations of the boundary conditions $A_i = 0$ and $\hat{\Phi} = \hat{z}$ of the vacuum sector. This gauge transformation is discontinuous and leads to a non-vanishing $1/r^2$ component of the field strength $\vec{F}_{ij}$.

The continuation of the asymptotic forms (1.6) and (1.7), to the interior constitutes the ansatz of t'Hooft and Polyakov:

$$A_{ai} = \frac{\epsilon_{aij}\hat{x}_j}{gr}\left(1 - K(r)\right)$$
$$\Phi_a = \hat{x}_a \frac{H(r)}{gr} \tag{1.10}$$

The functions K(r) and H(r) are determined by extremizing the energy functional (1.2) which in terms of K(r) and H(r) becomes:

$$V = \frac{4\pi}{g^2}\int_0^\infty dr\left[K'^2 + \frac{(K^2-1)^2}{2r^2} + \frac{H^2K^2}{r^2} + \frac{(rH'-H)^2}{2r^2} - \frac{\mu^2}{2}H^2 + \frac{\lambda H^4}{4g^2r^2}\right] \tag{1.11}$$

$$\frac{\delta V}{\delta K(r)} = r^2K'' - K^3 - KH^2 + 1 = 0$$

$$\frac{\delta V}{\delta H(r)} = r^2H'' - 2HK^2 - \beta^2H^3 + \mu^2H = 0 \tag{1.12}$$

These coupled equations have regular solutions with boundary conditions:

$$\begin{aligned} K(r) &\to \exp(-\beta^{-1}\mu r) \quad &, \quad r\to\infty \\ \frac{H(r)}{r} &\to \beta^{-1}\mu \quad &, \quad r\to\infty \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} K(r) &\to 1 + C_1 r^2 \quad &, \quad r\to 0 \\ H(r) &\to C_2 r^2 \quad &, \quad r\to 0 \end{aligned} \tag{1.14}$$

$\beta = \lambda/g^2$ is the <u>fixed</u> ratio of the Higgs and gauge coupling; it is 0(1) for the semiclassical method to work. In fact we have just one expansion parameter g^2 which occurs multiplied with $\hbar$ in the quantum expansion. The constants C_1 and C_2 are determined to fit the behaviour as $r \to \infty$. The value of the static energy was estimated by t'Hooft to be

$$E_{cl} = \frac{4\pi}{g^2} M_W C(\beta) \tag{1.15}$$

$C(\beta)$ is almost independent of β. It varies from 1.1 for $\beta = 0.1$ to 1.44 for $\beta = 10$. $M_W = 2\mu/\sqrt{\lambda}$ is the charged vector boson mass. As expected the topological soliton has a large mass compared to the quarta of the vacuum sector. Because of its non-trivial topological properties the Hilbert space of small fluctuations around the monopole has no overlap with that around the constant Higgs background.

t'Hooft also identified the abelian electromagnetic field tensor throughout space as

$$\vec{F}_{\mu\nu} = \hat{\phi}\cdot\vec{F}_{\mu\nu} - \frac{1}{g}\hat{\phi}\cdot\nabla_\mu\hat{\phi}\times\nabla_\nu\hat{\phi} \tag{1.16}$$

The solution (1.7) is electrically neutral; however it carries magnetic charge:

$$B_k = \frac{1}{2}\epsilon_{kij}F_{ij} = -\frac{1}{g}\hat{x}_k/r^2$$

giving

$$M = \frac{1}{4\pi}\int d^2\sigma_k B_k = -\frac{1}{g} \tag{1.17}$$

for the magnetic charge. In fact rewriting

$$F_{\mu\nu} = \partial_\mu(\vec{A}_\nu\cdot\hat{\phi}) - \partial_\nu(\vec{A}_\mu\cdot\hat{\phi}) - \frac{1}{g}\hat{\phi}\cdot\partial_\mu\hat{\phi}\times\partial_\nu\hat{\phi}$$

we see that the magnetic charge is precisely the winding number of the mapp $\hat{x}: \omega \mapsto \hat{x}(\omega)$ times $-1/g$.

Julia and Zee[28] found an electrically charged magnetic monopole by looking for solutions which satisfy the boundary condition:

$$Q = \int d^2\sigma_k \, \hat{x} \cdot \vec{E}_k$$

Q is the classical value of the electric charge. They supplemented the ansatz (1.10) of t'Hooft and Polyakov by fixing the classical value of $\vec{A}_0$ to be

$$A_0^a = \hat{x}^a \frac{J(r)}{gr} \tag{1.18}$$

with boundary conditions

$$\begin{aligned} K(r) &\to \exp\left(-[\beta^{-1}\mu - M^2]^{1/2} r\right), && r \to \infty \\ H(r)/r &\to \beta^{-1}\mu, && r \to \infty \\ J(r) &\to Mr + b, && r \to \infty \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} K(r) &\to 1 + C_1' r^2, && r \to 0 \\ H(r) &\to C_2' r^2, && r \to 0 \\ J(r) &\to C_3' r^2, && r \to 0 \end{aligned} \tag{1.20}$$

there does exist a finite energy solution. If we eliminate A_0 by a gauge transformation, the space components A_{ai} become time dependent. Finally we mention that Prasad and Sommerfield[17] found an exact solution, which is also an absolute minimum of $V(\vec{x})$, in the limit

$$\lambda \to 0 \qquad \text{with} \qquad H(r)/r \to \mu\beta^{-1} = C \quad \text{as} \quad r \to \infty \qquad (1.21)$$

We quote their result:

$$J(r) = \sinh\gamma\,[Cr \coth(Cr) - 1]$$

$$K(r) = Cr/\sinh(Cr)$$

$$H(r) = \cosh\gamma\,[Cr \coth(Cr) - 1] \qquad (1.22)$$

γ is an arbitrary constant. The classical mass and electric charge are given by:

$$E_{cl} = (C/\alpha)\cosh^2\gamma \quad , \quad \alpha = g^2/4\pi$$

$$Q_{cl} = 4\pi/g \sinh\gamma$$

Before we conclude this discussion of the classical solution we make note of two very important symmetry properties of the classical solution (1.10). A vector field $A_{ai}(\vec{x})$ and a scalar field $\phi_a(\vec{x})$ transform under space and isospin rotations as follows:

$$A'_{ai}(\bar{x}) = R_{ij}(\Omega)\,A_{aj}(R^{-1}(\Omega)\bar{x})$$
$$\phi'_a(\bar{x}) = \phi_a(R^{-1}(\Omega)\bar{x})$$

(space rotations)

Ω is a parameter of the rotation group. $R^T(\Omega)R(\Omega)=1$ and $\det R(\Omega) = +1$.

$$A'_{ai}(\bar{x}) = R_{ab}(\tilde{\Omega})\,A_{bi}(\bar{x})$$
$$\phi'_a(\bar{x}) = R_{ab}(\tilde{\Omega})\,\phi_b(\bar{x})$$

(iso-space rotations)

$\tilde{\Omega}$ is a parameter of the isospin group. $R(\tilde{\Omega})$ is in the adjoint representation: $R^T(\tilde{\Omega})R(\tilde{\Omega}) = 1$ and $\det R(\tilde{\Omega}) = +1$.

Now if we identify the parameters of these two distinct groups, then their simultaneous action leaves the classical solution (1.10) invariant:

$$A'^{u}_{ai}(\bar{x}) = R_{ij}(\Omega)\,R^{-1}_{kl}(\Omega)\,R_{ab}(\Omega)\,\epsilon_{bjk}\,\hat{x}_l\,(1-K)/gr$$
$$= \det R\,\epsilon_{aij}\,\hat{x}_j\,(1-K)/gr$$
$$= A^{u}_{ai}(\bar{x}) \qquad \text{(since } \det R = +1\text{)}$$
$$\phi'^{u}_a(\bar{x}) = R_{ab}(\Omega)\,R^{-1}(\Omega)_{bc}\,\hat{x}_c\,H(r)/gr$$
$$\doteq \phi^{u}_a(\bar{x})$$

Further since $\vec{\Phi}$ is a pseudo-scalar, the classical solution is also invariant under parity transformations:

$$A'_{ai}(\bar{x}) = -A_{ai}(-\bar{x})$$

and

$$\phi'_a(\bar{x}) = -\phi_a(-\bar{x})$$

Section 2: Hamiltonian formulation with surface terms

The action corresponding to the Lagrangian (1.1) can be written in Hamiltonian form:

$$S = \int_{t_1}^{t_2} dt \left[\int d\bar{x} \left(\vec{E}_i \cdot \dot{\vec{A}}_i + \vec{\pi} \cdot \dot{\vec{\Phi}} \right) - H \right] \tag{2.1}$$

$$H = \int d\bar{x} \left(\frac{\vec{E}_i^{\,2}}{2} + \frac{\vec{\pi}^{\,2}}{2} + \frac{1}{4} \vec{F}_{ij} \cdot \vec{F}_{ij} + \frac{1}{2} \nabla_i \vec{\Phi} \cdot \nabla_i \vec{\Phi} + V(\vec{\Phi}) \right)$$
$$- \int d\bar{x}\, \vec{A}_0 \cdot \left(\nabla_i \vec{E}_i + g \vec{\Phi} \times \vec{\pi} \right) + \int \vec{A}_0 \cdot \vec{E}_i \, d^2\sigma_i \tag{2.2}$$

is the Hamiltonian. We have identified $\vec{E}_i = \vec{F}_{0i}$ and $\vec{\pi} = \nabla_0 \vec{\Phi}$ as the canonical momenta conjugate to $\vec{A}_i$ and $\vec{\Phi}$ respectively; the commutation rules are:

$$\left[A^a_i(\bar{x}) , E^b_j(\bar{y}) \right] = i\, \delta_{ij}\, \delta^{ab} \delta^3(\bar{x} - \bar{y})$$

$$\left[\phi^a(\bar{x}) , \pi^b(\bar{y}) \right] = i\, \delta^{ab}\, \delta^3(\bar{x} - \bar{y}) \tag{2.3}$$

All other commutators are zero. The function $\vec{A}_0(x,t)$ that appears in the action is a Lagrange multiplier and reflects the gauge invariance of the Lagrangian. Since the time development of field configurations that enter the action is a continuous deformation, the trajectories entering (2.1) belong to the homotopy class of the initial field configuration at time t_1. By homotopy class we precisely mean the fixed time homotopy of the mapp

$$\vec{x} \mapsto f(\vec{x};t_0) \tag{2.4}$$

$f(x;t_o)$ is a typical field variable at time t_o. If $f(x;t_o)$ is a continuous function of t_o the homotopy class of (2.4) cannot change with time. If the homotopy class changes during time evolution, it must mean that $f(x;t_o)$ is discontinuous at some time and the action

$$\int_{t_1}^{t_2} dt \int d\vec{x}\ \Pi_f \frac{\partial}{\partial t} f(\vec{x},t)$$

is infinite. From now on we will stay within a fixed homotopy class.(29)

Hamilton's variational principle says that the equations of motion are stationary points of the action with fixed initial and final configurations. As usual one considers variations around the critical trajectory peaked at some time between t_1 and t_2. Then

$$\delta S = \int_{t_1}^{t_2} dt \int_V d\vec{x} \left[\frac{\delta S}{\delta \vec{A}_i(\vec{x},t)} \delta\vec{A}_i(\vec{x},t) + \frac{\delta S}{\delta \vec{A}_0(\vec{x},t)} \delta\vec{A}_0(\vec{x},t) + \ldots\ldots \right]$$

$$+ \int_{t_1}^{t_2} dt \int_S d^2\sigma_i \left[\frac{1}{2} \vec{F}_{ij} \cdot \delta\vec{A}_j + \frac{1}{2} \nabla_i\vec{\Phi} \cdot \delta\vec{\Phi} + \vec{E}_i \cdot \delta\vec{A}_0 \right] \tag{2.5}$$

the volume integral contains the usual equation of motion terms. If the variation is peaked around t_o with width Δt say,

$$\delta S = \Delta t \int_V d\bar{x} \left[\frac{\delta S}{\delta \vec{A}_i} \delta \vec{A}_i + \cdots \right]_{t_o} \tag{2.6}$$

$$+ \Delta t \int_S d^2\sigma_i \left[\vec{E}_i \cdot \delta \vec{A}_o + \frac{1}{2} \nabla_i \vec{\Phi} \cdot \delta \vec{\Phi} + \frac{1}{2} \vec{F}_{ij} \cdot \delta \vec{A}_j \right]_{t_o}$$

the surface integral by definition is evaluated on the boundary S of a very large volume V of any shape and then the volume is sent to infinity. The equations of motion are solutions to the variational problem $\delta S = 0$, provided the surface terms in (2.6) are zero. This last observation was made by Regge and Teitelboim in the context of general relativity.

We begin by discussing the first surface integral:

$$\text{(a)} \quad \int_S d^2\sigma_i \, \vec{E}_i \cdot \delta \vec{A}_o \tag{2.7}$$

If the electric field has a $o(r^{-2})$ component, the only way to drop (2.5) is by setting $\delta \vec{A}_o = 0$ on the boundary. Since $\vec{A}_o$ is actually a parameter of the gauge transformation this would mean that we do not admit gauge transformations with non-trivial asymptotic behaviour which e.g. includes global isotopic spin rotations. The only way to retain the possibility of performing asymptotically non-trivial gauge transformations and yet be consistent with the variational principle is to append the original action (2.1) with a surface term and force it to cancel (2.7). We denote this additional surface term by

$$-g\int_S d\omega\, I^a A_0^a$$

'w' is a parametrization of the boundary. We will determine the nature of the dynamical variable I^a a little later, at the moment we only assume that it is independent of $\vec{A}_i, \vec{\Phi}, \vec{E}_i$ & $\vec{\pi}$. Then as far as the variation of $\vec{A}_0$ is concerned Hamilton's variational principle leads to the constraints:

$$\nabla_i \vec{E}_i + g\vec{\Phi}\times\vec{\pi} \approx 0 \tag{2.8a}$$

$$\lim_{R\to\infty} \vec{E}_n(R;\omega)R^2 \approx g\vec{I}(\omega) \tag{2.8b}$$

$\vec{E}_n$ in (2.8b) denotes the normal component of the electric field on the boundary; R is the distance of the point 'w' on the boundary S. Equation (2.8a) is the Gauss constraint. It does not form a closed algebra of gauge rotations if we admit gauge parameters which do not vanish on the boundary. To discuss the full gauge group we must consider both constraints (2.8a) and (2.8b) atonce and form the generator

$$G_\Lambda = \int_V d\vec{x}\, \vec{\Lambda}\cdot(\nabla_i\vec{E}_i + g\vec{\Phi}\times\vec{\pi}) - \int_S d\omega\, \vec{\Lambda}\cdot(\lim \vec{E}_n R^2 - \vec{I}) \tag{2.9}$$

which is a linear combination. Its action on the field variables $\vec{A}_i$ and $\vec{\Phi}$ is given unambiguously by

$$\delta\vec{A}_i = i[\vec{A}_i, G_\Lambda] = \nabla_i\vec{\Lambda}$$

$$\delta\vec{\Phi} = i[\vec{\Phi}, G_\Lambda] = g\vec{\Phi}\times\vec{\Lambda} \tag{2.10}$$

In view of the commutation rules (2.3), the constraint (2.9) must be understood as a subsidiary condition on the state vector (Dirac)[10]

$$G_\Lambda |\Psi(t)\rangle = 0 \tag{2.11}$$

The subsidiary condition (2.11) is in the Schrödinger picture; it would be consistent only if the constraints G_Λ form a closed algebra. We must have

$$[G_\Lambda, G_{\Lambda'}]|\Psi\rangle = ig\, G_{\Lambda\times\Lambda'}|\Psi\rangle \tag{2.12}$$

$$(\vec{\Lambda}\times\vec{\Lambda}')_a = \epsilon_{abc}\Lambda_b\Lambda'_c$$

After a partial integration G_Λ can be written as

$$G_\Lambda = \tilde{G}_\Lambda + g\int \vec{I}\cdot\vec{\Lambda}\, d\omega$$

$$\tilde{G}_\Lambda = -\int d\vec{x}\left(\nabla_i\vec{\Lambda}\cdot\vec{E}_i + g\vec{\Phi}\times\vec{\Lambda}\cdot\vec{\pi}\right) \tag{2.13}$$

$\tilde{G}_\Lambda$ satisfies the gauge algebra without ambiguity via the commutation rules (2.3):

$$[\tilde{G}_\Lambda, \tilde{G}_{\Lambda'}] = ig\, \tilde{G}_{\Lambda\times\Lambda'} \tag{2.14}$$

From (2.12) and (2.14) it follows that

$$\left[\int \vec{I}\cdot\vec{\Lambda}, \int \vec{I}\cdot\vec{\Lambda}'\right]|\Psi\rangle = \int \vec{I}\cdot\vec{\Lambda}\times\vec{\Lambda}'|\Psi\rangle$$

or in other words

$$[I^a(\omega), I^b(\omega')] \approx i\epsilon^{abc} I^c(\omega)\delta^2(\omega-\omega') \tag{2.15}$$

The dynamical variable I^a satisfies a local isospin algebra on the boundary. The 'conjugate' variable to $I^a(\omega)$ is the parameter of the gauge group at the point 'w' on the boundary. Let us denote it by $\theta^a(\omega)$, a = 1,2,3. We choose to work in a representation in which $A_i^a(\vec{x})$, $\phi_a(\vec{x})$ and $\theta^a(\omega)$ are diagonal and form the wave function:

$$\Psi[\vec{A},\vec{\phi},\vec{\theta};t] = \langle \vec{A},\vec{\phi},\vec{\theta} \mid \psi(t) \rangle \tag{2.16}$$

Further, it is possible to have a canonical realization of the algebra (2.15); we can introduce a canonical momentum $P^a(\omega)$ conjugate to $\theta^a(\omega)$:

$$\left[\theta^a(\omega) , P^b(\omega') \right] = i\, \delta^{ab} \delta^2(\omega-\omega') \tag{2.17}$$

then using (2.12)

$$I^a = R^{ab}(\vec{\theta}) V_c^b(\vec{\theta}) P^c$$

satisfies the algebra (2.15). This is proved in appendix 2. The matrix $R(\vec{\theta})ab$ is an SU(2) matrix in the adjoint representation and the matrix $V_c^b(\vec{\theta})$ is defined in terms of the function $F^a(\vec{\theta},\vec{\eta})$ which specifies the group law:

$$\exp i\frac{\vec{\tau}}{2}\cdot\vec{\theta}\ \exp i\frac{\vec{\tau}}{2}\cdot\vec{\eta} = \exp i\vec{F}(\vec{\theta},\vec{\eta})\cdot\frac{\vec{\tau}}{2}$$

$$V_b^a(\vec{\theta}) = \frac{\partial F^a(\vec{\theta},0)}{\partial \eta_b}$$

From (2.10) we see that the wave function is a singlet for the subgroup of gauge transformations with vanishing parameters on the boundary. These gauge transformations are termed local. For non-vanishing parameters the wave function can, in principle, carry any representation of the local algebra (2.15). These gauge transformations will be termed global.[11] Precisely which global gauge transformations are allowed is a model dependent question and we will answer it for the t'Hooft-Polyakov model in the following discussion.

We now turn to the surface integral

(b) $\int_S \delta\vec{\Phi}\cdot\nabla_i\vec{\Phi}\, d^2\sigma_i$

In section II.1 we presented the minimal large 'r' behaviour of the fields to ensure a finite classical energy for the soliton. See (1.4a, 1.4b, 1.4c). From equations (1.4a) and (1.4b) it follows that as the volume goes to infinity the above surface integral would diverge as $O(r^{1/2-\epsilon})$, unless the variation of the direction of the Higgs field is set equal to zero on the boundary,

$$\delta\vec{\Phi} = \frac{\mu}{\sqrt{\lambda}}\,\delta\hat{\Phi} = 0 \tag{2.18}$$

i.e. the direction of the Higgs field on the boundary is fixed. As we have seen the finite energy condition (1.4a) defines a topology on the manifold of Higgs fields and a continuous time development requires the field to stay within a fixed homotopy class. We choose this homotopy class to be specified by d = +1. Then from the discussion of section II.1 and (2.18), it is seen that the direction of the Higgs field on the boundary is any particular continuous gauge transformation of $\hat{x}_a$:

$$\hat{\Phi}(w)=\Omega(w)\hat{x}(w), \quad \Omega \in SO(3)/U(1) \tag{2.19a}$$

anticipating the background gauge we fix $\Omega(w)=1$ i.e. $\hat{\Phi}=\hat{x}$ (2.19b)

(c) The surface integral

$$\frac{1}{2}\int_S d^2\sigma_i\, \bar{F}_{ij}\cdot\delta\bar{A}_j$$

can be reduced to a simple form. Using the fact that the direction of the Higgs field is fixed to be $\hat{x}_a$, the finite energy condition (1.4b)

$$\nabla_i\hat{\Phi} \sim o(r^{-\frac{3}{2}-\epsilon})$$

can be written as

$$A_{ai}=\epsilon_{aij}\hat{x}_j/gr + \hat{x}_a(\hat{x}\cdot\bar{A}_i) \tag{2.20}$$

then assuming the boundary S to be sufficiently far away

$$\frac{1}{2}\int_S d^2\sigma_i\, \bar{F}_{ij}\cdot\delta\bar{A}_j = \frac{1}{2}\int d^2\sigma_i\, \hat{x}\cdot\bar{F}_{ij}\,\delta A_j \tag{2.21}$$

$A_j = \hat{x}.\vec{A}_j$ is the 'electromagnetic' vector potential. Using formula (1.16), we can write (2.21) as

$$\int_S d\omega R^2\, \hat{n}\times\delta\vec{A}\cdot\vec{B} \tag{2.22}$$

$\vec{B}=\nabla\times\vec{A}$ is the U(1) magnetic field. $\hat{n}$ is a unit normal to the boundary. We will set this surface term equal to zero by demanding

$$\hat{n} \times \delta \vec{A} = 0 \tag{2.23}$$

on the boundary. We will see in a moment that the boundary condition (2.23) actually fixes the electromagnetic gauge on the boundary S.

Now that all surface terms in the variation of the modified action vanish the equations of motion are solutions of the variational problem $\delta S = 0$.

$$\begin{aligned}
\dot{\vec{A}}_i &= \vec{E}_i + \nabla_i \vec{A}_0 \\
\dot{\vec{E}}_i &= \nabla_j \vec{F}_{ij} + g \vec{\Phi} \times \nabla_i \vec{\Phi} + g \vec{E}_i \times \vec{A}_0 \\
\dot{\vec{\Phi}} &= \vec{\pi} + g \vec{\Phi} \times \vec{A}_0 \\
\dot{\vec{\pi}} &= \nabla_i \nabla_i \vec{\Phi} + \frac{\delta V}{\delta \vec{\Phi}} + g \vec{\pi} \times \vec{A}_0
\end{aligned} \tag{2.24}$$

We now come to the question of the allowed global gauge transformations in this model. In (2.14a) and (2.14b) we have fixed the group element belonging to the coset SO(3)/U(1) to be $\Omega = 1$. This means that the only gauge transformations allowed on the boundary belong to the little group U(1) which leaves $\hat{x}_a$ invariant. Hence the parameters of the gauge group on the boundary take the form

$$\theta^a(\omega) = \theta(\omega)\, \hat{x}_a(\omega) \tag{2.25}$$

There is a local U(1) associated with each point w on the boundary. The gauge transformation of the electromagnetic vector potential $A_i = \hat{x}.\vec{A}_i$ on the boundary is seen to be

$$\delta A_i = \partial_i \theta(w) \tag{2.26}$$

substituting in (2.18) we have

$$\hat{n} \times \vec{\nabla}\theta = 0 \tag{2.27}$$

which means that the parameter of the local U(1) group is independent of w, and (2.25) becomes

$$\theta^a(w) = \theta\, \hat{x}_a(w) \quad , \quad 0 \leq \theta \leq 2\pi$$

We can define a momentum conjugate to θ by the equation

$$p = \int_S \frac{dw}{4\pi} P_a \hat{x}_a$$

then by the commutation rules (2.12) one can verify

$$[\theta, p] = i$$

The local iso-spin generator $I^a(w)$ enters the theory only through the commuting combination

$$\int_S \hat{x}\cdot\vec{I}(w)\, dw$$

which is equal to $4\pi p$. The wave function (2.16) now depends on θ. Since the range of θ is compact, the spectrum of p is the set of integers.

Having specified the allowed global gauge rotations the subsidiary condition (2.11) becomes

$$G_\lambda \Psi(t) = 0 \tag{2.28}$$

with

$$G_\lambda = -\int d\bar{x}\, \bar{\Lambda}\cdot(\nabla_i \bar{E}_i + g\,\bar{\phi}\times\bar{\pi}) + \lambda\left(\int d^2\sigma_i\, \hat{x}\cdot\bar{E}_i - 4\pi g p\right) \tag{2.29}$$

where we have used

$$\lim_{|\bar{x}|\to\infty} \vec{\Lambda}(\vec{x}) = \lambda\hat{x}$$

Since gp is the electric charge operator, the wave function can be chosen to be an eigenstate

$$\Psi_n[\vec{A}_i,\vec{\phi},\theta] = e^{in\theta}\,{}^{\wedge}\!\Psi_n[\vec{A},\vec{\phi}] \tag{2.30}$$

From (2.28) and (2.30) it follows that under finite gauge transformation U(x), $U(|x|\to\infty) = \exp(i\lambda\hat{x}\cdot\bar{\tau}/2)$

$${}^{\wedge}\!\Psi_n[\vec{A}^U,\vec{\phi}^U] = e^{in\lambda}\,{}^{\wedge}\!\Psi_n[\vec{A},\vec{\phi}] \tag{2.31}$$

i.e. ${}^{\wedge}\!\Psi_n$ carries an irreducible representation of global U(1).

The Schrödinger equation satisfied by Ψ_n is

$$i\frac{\partial}{\partial t}\Psi[\cdot,t] = H_0\Psi[\cdot,t]$$

with

$$H_0 = \int d\bar{x}\left(\frac{1}{2}\bar{E}_i^2 + \frac{1}{2}\bar{\pi}^2 + \frac{1}{4}\vec{F}_{ij}\cdot\vec{F}_{ij} + \frac{1}{2}\nabla_i\vec{\phi}\cdot\nabla_i\vec{\phi} + V(\vec{\phi})\right) \tag{2.32}$$

Note that H_0 is gauge invariant

$$[G_\lambda, H_0] = 0 \tag{2.33}$$

The Schrödinger equation can be solved formally

$$\Psi_n(\cdot, t) = \exp(-itH_0)\Psi_n(\cdot, 0)$$

Using (2.33) the subsidiary condition becomes

$$G_\lambda \Psi_n(\cdot, 0) = 0 \tag{2.34}$$

If we choose $\Psi_n(\cdot, 0)$ to be an eigenstate of H_0, (2.34) together with the time independent Schrödinger equation

$$H_0 \Psi_{n,E} = E\, \Psi_{n,E}$$

is a complete system of equations to solve in the Schrödinger picture.

This completes the canonical formulation.

Section 3: Subsidiary condition and gauge fixing

In the previous section we concluded that the complete set of equations to solve in the Schrödinger picture are

$$H_0 \Psi_{n,E} = E\, \Psi_{n,E} \tag{3.1a}$$

$$G_\lambda \Psi_{n,E} = 0 \tag{3.1b}$$

H_0 and G_λ are given in (2.29) and (2.32) respectively. One accepted approach to solve such a complex system of equations is by eliminating the subsidiary condition (3.1b), before solving the Schrödinger equation. This is done by fixing a gauge: since equation (3.1b) is a statement of the gauge invariance of the wave function.

The differential form (3.1b)

$$\left[\int d\vec{x}\left(\nabla_i \vec{\Lambda}\,\frac{\delta}{\delta \vec{A}_i} + g\,\vec{\Phi}\times\vec{\Lambda}\cdot\frac{\delta}{\delta\vec{\Phi}}\right) + 4\pi\lambda g\frac{\partial}{\partial\theta}\right]\Psi_{n,E} = 0 \tag{3.2}$$

can be integrated for a connected gauge group to read

$$\Psi_{n,E}[A_i^U, \phi^U, \theta-\lambda] = \Psi_{n,E}[A_i, \phi, \theta] \tag{3.3}$$

where A_i^U and ϕ^U are finite gauge transforms of A_i and ϕ. The variable θ undergoes a U(1) translation by λ which is the asymptotic parameter of the gauge transformation U.

Now if we consider the orbit $[A^U, \phi^U, \theta-\lambda]$ of $\vec{A}_i, \vec{\Phi}, \theta$ under the set G of gauge transformations U(x), $U(|x|\to\infty) = \exp(i\lambda\hat{x}\cdot\vec{\tau}/2)$; we can define an equivalence relation on the set $M = \{\vec{A}_i, \vec{\Phi}, \theta\}$, by defining two points X and Y to be equivalent ($X\sim Y$) iff they are connected by a gauge transformation which belongs to G. It can be verified that the above relation on M is indeed an equivalence relation i.e.

$$X \sim X$$

$$X\sim Y \Rightarrow Y \sim X$$

$$X\sim Y + Y\sim Z \Rightarrow X\sim Z$$

This means that the manifold M is a disjoint union of equivalence classes and the wavefunction is actually a function over the quotient space M/G of equivalence classes or orbits rather than a function over M. Hence we can specify the wave function by representatives of the orbits which are usually determined by the solution of functional equations (gauge conditions which define a surface in M),

$$F[A^U, \Phi^U] = 0$$

It is usually required that the solution of the functional equation for the group element U(x) be unique i.e. one may select only one representative per orbit.

As we shall see in those cases where such F are used e.g. the axial gauges, it is indeed possible to satisfy this requirement, but it is almost impossible to develop a viable computational scheme. Gauges like the Coulomb and its generalization (the background gauge), are computationally viable but the orbit of a given field may have multiple intersections ($F[A^U, \Phi^U] = 0$ has more than one solution for U; see figure 1.)

However in doing a perturbation expansion around a given field configuration, call it A_{cl}, multiple intersections may not be a problem. If we can show that a small fluctuation around the orbit of A_{cl} can be <u>uniquely</u>

gauge transformed to a fluctuation on the gauge fixing surface near A_a , we can use the gauge invariance of the wave function (3.3) to peak it around A_a . We will qualify the use of the term small fluctuation when we discuss the Coulomb and background gauges. (see figure 2)

Before we move on to discussing specific gauges we discuss the characterization of the intersection point of a given orbit with a gauge fixing surface. To simplify the discussion we consider the gauge fields only and denote the gauge fixing surface by $F^a[A;\vec{x}] = 0$; 'a' labels the number of group generators and '$\vec{x}$' is a space point. Then the normal to this surface at the intersection point A_{ai} is given by (see figure 1.)

$$N^{a,\vec{y}}_{b,i,\vec{x}} = \frac{\delta F^a[A;\vec{y}]}{\delta A_{ai}(\vec{x})}$$

The tangent to the orbit at the point A_{ai} is,

$$T^{a,y}_{b,i,x} = \frac{\delta A^{\Lambda}_{bi}(\vec{x})}{\delta \Lambda_a(\vec{y})} = \nabla_i^{ba}(A(x))\,\delta^{(3)}(\vec{x}-\vec{y})$$

(A^{Λ} is an infinitismal gauge transformation, ∇_i is the covariant derivative at A_{ai}).

The Faddeev-Popov operator (F-P)

$$\Box^{c,a}_{z,y} = \int d\vec{x}\; N^{a,y}_{b,i,x}\; T^{c,z}_{b,i,x}$$

$$= \nabla_i(z)^{cb}\,\frac{\delta F^a[A;y]}{\delta A_{bi}(z)}$$

in a sense measures the 'angle' between the orbit and the surface at the intersection point. A zero eigenvalue signals a point of tangency, at which $\Box^{ca}$ cannot be inverted.

Now, to a discussion of some well known gauges.

(a) Unitary Gauge[30]

In this gauge one requires the Higgs field to have the same direction in isospin space at each space point. We noted in section 1 that for the magnetic monopole it is impossible to deform the Higgs field to point in a constant direction by a continuous gauge transformation. A gauge transformation discontinuous along a string emanating from the monopole can reach the unitary gauge, however the gauge field is now singular along the string. Since the string singularity is a gauge artifact (a 'bad' choice of co-ordinates) some useful observations about magnetic monopole were made in this gauge.[13] However it is not clear how to do a perturbation expansion with a string singularity. Another version of the unitary gauge requires the Higgs field to point in the direction of the classical solution at each space point. However if the classical solution has non-trivial topology as in the case of the monopole the Higgs field necessarily develops a zero at the position of the monopole where this gauge condition would be singular.

(b) Axial Gauge:[31] $A_3^a = 0$

It is always possible to choose this gauge; given an arbitrary field configuration A_i, it is easy to construct a gauge transformation U that satisfies

$$\frac{1}{g} U^\dagger \partial_3 U + U^\dagger A_3' U = A_3 = 0 \tag{3.4}$$

(We have used matrix notation for the gauge field $A_i = \vec{A}_i \cdot \vec{\tau}/2i$, τ^a are the Pauli matrices) The solution for U is written in a path dependent way like the solution of the time evolution operator in quantum mechanics.

$$U(x_1,x_2,x_3) = P\exp\left[-ig\int_{-\infty}^{x_3} dx_3' A_3'\right] \tag{3.5}$$

Now, in equation (3.3) $A_3^a = 0$ and U depends only on x_1 and x_2. Further fixation of gauge is usually done by a choice of boundary conditions. One may choose the components A_1 and A_2 to be fixed on the surface of a plane (x,y,z_o) where z_o is very large and eventually goes to infinity. This eliminates the freedom to perform all but constant gauge transformations. Instead of fixing the components of the gauge field on the plane one may choose to impose the unitary gauge on the plane.[32] Another possibility is the so called generalized axial gauge which consists of imposing[4]

$$A_2^a(x_1, x_2, 0) = 0 \tag{3.6}$$

$$A_1^a(x_1, 0, 0) = 0 \tag{3.7}$$

on lower dimensional manifolds. It is always possible to choose this gauge for an arbitrary field configuration because we can perform a gauge transformation $V(x_1 x_2)$ from the configuration $A_i'^a(x)$, $i = 1,2$ to $A_i(x)$ which satisfies (3.6)

$$\frac{1}{g} V^\dagger(x_1 x_2)\partial_2 V(x_1 x_2) + V^\dagger(x_1 x_2) A_2'(x_1 x_2 0) V(x_1 x_2) = 0$$

then

$$V(x_1 x_2) = P\exp\left[-ig\int_{-\infty}^{x_2} A_2'(x_1, x_2', 0)\,dx_2'\right]$$

Similarly one can show that (3.7) is achieved. The important point about the generalized axial gauge is that in principle it is possible to eliminate the subsidiary condition (3.2) completely. The only allowed gauge freedom is a constant gauge transformation. However it is quite clear that it is very difficult even in principle, to develop a perturbation scheme. This observation has already been made for the lattice version of the theory.

(c) <u>Coulomb gauge:</u>[33] <u>$\partial_i A_i = 0$</u>

This is a standard gauge used in perturbation theory.

However Gribov[16] has observed that it may not be possible to specify the orbit of a given gauge field by a unique transverse field, i.e. the equation

$$\frac{\partial}{\partial x_i}\left[U^\dagger A_i U + \frac{1}{g} U^\dagger \frac{\partial}{\partial x_i} U\right] = 0 \quad , \quad U^\dagger U = 1 \tag{3.8}$$

may have more than one solution (We are assuming that there is at least one solution). Further there is a distinct possibility that a certain intersection point of the orbit and the gauge fixing surface may be a point of tangency i.e. the Faddeev-Popov operator may develop a normalizable zero mode in which case it cannot be inverted, rendering the procedure to be described useless.

Specifically in doing a perturbation expansion around the monople, the orbit of the classical solution may have several intersections with the surface $\partial_i A_i = 0$; one point is the classical solution (1.10) itself. (See figure 2)

Now consider a field configuration near the orbit of the classical solution; the orbit of this field can have complicated intersections with the Coulomb gauge surface; however if it can be shown that every such orbit has a unique intersection near the given classical solution (1.10), we can use the gauge invariance of the wave function to do a perturbation expansion around $\vec{A}_{cl}$ and $\vec{\Phi}_{cl}$. (Recall that the wave function is an eigenstate of global electric charge and we have already included the conjugate angle 'θ' in its specification (2.30), so that if Ψ' as defined in (2.30), develops a phase by gauge transformation it is automatically compensated by an equal and opposite phase developed by the dynamical variable 'θ'.

Under certain conditions which we shall discuss later such as described above is indeed possible for the magnetic monopole, for consider (figure 2)

$$A'_i(x) = A_{ai}(\Omega_0) + a'_i(x) \tag{3.9}$$

$$\left(A_{ai}(\Omega_0) = \Omega_0^\dagger A_{ai} \Omega_0 + \frac{1}{g} \Omega_0^\dagger \partial_i \Omega_0 = O(1/g);\ \Omega_0(r\to\infty) = w_0 \hat{x} \text{ and } a'(x) = O(1) \right)$$

a point near $A_{ai}(\Omega_0)$. The gauge transformation which brings it near $A_{ai}(x)$ is evidently $\Omega_0^{-1}(x)$,

$$A'_i \to A_i'^{\Omega_0^{-1}} = A_{ai} + a_i(x) \quad , \quad a_i(x) = \Omega_0 a'_i \Omega_0^\dagger$$

Now perform an infinitisimal gauge transformation with parameter $\vec{\alpha}(\vec{x})$, onto the coulomb surface

$$A_i'^{\Omega_0^{-1}}(x) \to A''_i = A_i'^{\Omega_0^{-1}} + \nabla_i (A_i'^{\Omega_0^{-1}}) \vec{\alpha} \tag{3.10}$$

and determine $\vec{\alpha}$ by demanding $\partial_i A''_i = 0$

$$\partial_i \nabla_i (A_a) \vec{\alpha} + g\, \partial_i (\vec{a}_i \times \vec{\alpha}) + \partial_i a_i = 0 \tag{3.11}$$

The solution to the above equation can be developed by expanding $\vec{\alpha}$ in a power series in 'g', provided the Faddeev-Popov operator $\partial_i \nabla_i (A_a)$ can be inverted. We will prove in appendix (3), that the spectrum of $\partial_i \nabla_i (A_a)$ is positive and that there is a homogeneous solution $\vec{\Psi}_0'$, $\partial_i \nabla_i (A_a) \vec{\Psi}_0' = 0$ with boundary condition $\vec{\Psi}_0'(r \sim \infty) \sim \hat{x}(\lambda - 1/r)$

The presence of a homogeneous solution with free parameter λ means that it is not possible to uniquely gauge transform $\vec{A}_i'(x)$ to a point near $\vec{A}_{ai}(x)$. In order to determine λ we impose an additional gauge condition

$$\int d\vec{x} \; \vec{A}_i \cdot \partial_i \vec{\Psi}_0'(\lambda = 1; \vec{x}) = 0 \tag{3.12}$$

that is to say that the gauge fixing surface is an intersection of (3.12) with the coulomb surface.

By a partial integration and using $\partial_i A_i = 0$, (3.12) becomes

$$\int d^2\sigma_i \; \hat{x} \cdot \vec{A}_i = 0 \tag{3.13}$$

and substituting A_i'' from (3.10), we have

$$\int d^2\sigma_i \, \hat{x} \cdot \partial_i \vec{\alpha} = - \int d^2\sigma_i \, \hat{x} \cdot \vec{a}_i$$

The inhomogeneous part of $\vec{\alpha}$ does not contribute to the above surface integral since it is determined by $(\partial_i \nabla_i)^{-1}$ which vanishes for large r (appendix 3). Substituting the asymptotic homogeneous solution

$$\vec{\Psi}_0'(\lambda) \sim \hat{x} \; (\lambda - 1/r)$$

we have

$$4\pi\lambda = - \int d^2\sigma_i \, \hat{x} \cdot \vec{a}_i$$

which determines λ and hence homogeneous solution.

So we see that using the gauge invariance of the wavefunction it is indeed possible to do a perturbation expansion around any one good point of intersection of the orbit with the two gauge fixing surfaces, as long as the Faddeev-Popov operator is invertible at the point of intersection; it does not matter if at certain other intersections the Faddeev-Popov operator cannot be inverted.

In the above discussion we have considered field configuration near the orbit of the classical solution, by which we specifically mean that if we consider the Fourier amplitude $\vec{a}_i(\vec{k})$ corresponding to $\vec{a}_i(\vec{x})$, then $\vec{a}_i(k\to 0)$ remains bounded. If the contrary is true, the fluctuation a_i would compete in size with the classical solution which is long ranged, ($O(1/r)$ as $r\to\infty$) and the invertibility of equation (3.11), would be in question because $\partial_i \nabla_i (A_a + g a_i)$ may develop a normalizable zero mode. Then by continuity the configuration, $A_a + g a_i \pm \delta a_i$, would develop a small negative eigenvalue in the Faddeev-Popov operator. (See figure 3)

To understand the significance of the small negative eigenvalue, following Polyakov,[16] the non-linear equation (3.8) can be formulated as the stationary point of the action

$$W = \int d\vec{x}\, \mathrm{Tr}\left[\partial_i U \partial_i U^{\dagger} - 2g\, \partial_i U U^{\dagger} A_i\right], \quad U^{\dagger}U = 1 \qquad (3.14)$$

$$(\partial_i A_i = 0)$$

Then a sufficient condition for the existence of a solution near U = 1 is that the linearized action

$$W[U=1+iV] = \int d\vec{x}\; V^a(x)\, \partial_i \nabla_i(A)^{ab} V^b(x) \;<\; 0 \tag{3.15}$$

i.e. the Faddeev-Popov operator has a negative eigenvalue.

This means that the orbit of $A_u + a_i \pm \delta a_i$ crosses the Coulomb gauge surface in two nearby points and one may need to fix the gauge further to distinguish between these.

Concluding this discussion we comment that the possibility to do a perturbation expansion around the monopole in the Coulomb gauge rested among other factors on the invertibility of the Faddeev-Popov operator at the classical solution which in turn crucially depends on the fact that the function K(r) which enters the classical solution is bounded: $0 \leq K(r) \leq 1$, $\forall r$.

We now turn to a gauge which is most convenient for a perturbation expansion and which is free of any restriction on the background field.

(d) <u>The background field gauge</u>: (34) $\underline{\tilde{\vec{\nabla}}_i(\vec{A}_i - \vec{A}^a_i) + g\,\vec{\bar{\Phi}}^a \times (\vec{\Phi} - \vec{\Phi}_\omega) = 0}$

The discussion is similar in principle to the Coulomb gauge in the previous section. The orbit of the classical solution may have several intersections with the gauge fixing surface, i.e.

$$\tilde{\nabla}_i (A^U_{ai} - A_{ai}) + g\, \bar{\Phi}_a \times (\Phi^U_a - \bar{\Phi}_a) = 0, \quad \tilde{\nabla}_i = \partial_i + g \bar{A}^a_i \times \tag{3.16}$$

may have solutions other than U = 1.

Because the wave function is gauge invariant and the ground state is a gaussian around the classical solution we once more consider a point near $A_a(\Omega_0)$ and $\phi_a(\Omega_0)$

$$\begin{aligned} A'_i &= A_{ai}(\Omega_0) + a'_i(x) \\ \phi' &= \phi_a(\Omega_0) + \varphi'(x) \end{aligned} \tag{3.17}$$

($A_{ai}(\Omega_0)$ and $\phi_a(\Omega_0)$ are $0(1/g)$ finite gauge transforms of A_a and ϕ_a; a'_i and φ' are o(1) fluctuations) and perform a finite gauge transformation Ω_0^{-1} to bring it near A_{ai}, ϕ_a:

$$\begin{aligned} A'^{\Omega_0^{-1}}_i &= A_{ai}(x) + a_i(x) \quad , \quad a_i(x) = \Omega_0 a'_i(x) \Omega_0^\dagger \\ \phi'^{\Omega_0^{-1}} &= \phi_a(x) + \varphi(x) \quad , \quad \varphi(x) = \Omega_0 \varphi'(x) \Omega_0^\dagger \end{aligned}$$

and then perform a successive infinitisimal gauge transformation with parameter $\vec{\alpha}$ to bring it onto the background gauge surface,

$$\begin{aligned} \vec{A}''_i &= \vec{A}'^{\Omega_0^{-1}}_i + \nabla_i (A'^{\Omega_0^{-1}}) \vec{\alpha} \\ \vec{\phi}'' &= \vec{\phi}'^{\Omega_0^{-1}} + \vec{\phi}'^{\Omega_0^{-1}} \times \vec{\alpha} \end{aligned} \tag{3.18}$$

$\vec{\alpha}(x)$ is to be determined from

$$\tilde{\nabla}_i(A_i'' - A_{ui}) + \bar{\Phi}_u \times (\bar{\Phi}'' - \bar{\Phi}_u) = 0\ , \qquad \text{which implies}$$

$$\tilde{\nabla}_i\tilde{\nabla}_i\vec{\alpha} + g^2\,\bar{\Phi}_u \times (\bar{\Phi}_u \times \vec{\alpha}) + \tilde{\nabla}_i(\bar{a}_i \times \vec{\alpha}) + \bar{\Phi}_u \times (\bar{\varphi} \times \vec{\alpha}) + (\tilde{\nabla}_i a_i + g\bar{\Phi}_u \times \bar{\varphi}) = 0$$

This equation can be solved as a perturbation series in 'g', provided the Faddeev-Popov operator

$$\mathcal{D} = -\left(\tilde{\nabla}_i\tilde{\nabla}_i + g^2\,\bar{\Phi}_u \times (\bar{\Phi}_u \times\ )\right) \tag{3.19}$$

is invertible.

We now quote certain properties of $\mathcal{D}$ which we have proved in appendix (4).

(a) Consider the system in a large finite box of volume V. Then in the space of functions which vanish on the boundary of the box, $\mathcal{D}$ is hermetian and all such eigenfunctions have positive eigenvalues and form a complete set. Further the eigenfunctions are co-variant as one moves along the orbit of the classical solution, the eigenvalues are invariant, hence any intersection point of the orbit of the classical solution is good to do a perturbation expansion since the spectrum of the Faddeev-Popov operator is the same at all intersections (see figure 4)

(b) Under the boundary condition $\alpha^a(r\to\infty)$ = (constant)$\hat{x}^a$ the homogeneous equation $\mathcal{D}\vec{\Psi}_0 = 0$ has a unique solution, $\vec{\Psi}_0(\bar{x};\lambda)$ = $\hat{x}\,\Psi_0(\lambda;r)$, with asymptotic behavior,

$$\Psi_0(r\to\infty) \sim \lambda - 1/r \tag{3.20}$$

In the Prasad-Sommerfield limit the homogeneous equation can be exactly solved with solution

$$\vec{\Psi}_0 = \hat{x}\,[\lambda r \operatorname{Coth}(\lambda r) - 1]/r \tag{3.21}$$

which is the classical Higgs field.

Following identically our discussion of the Coulomb gauge, the existence of the one parameter homogeneous solution means that it is not possible to uniquely gauge transform $A_i'(x)$, $\phi'(x)$ (3.17), to a point near A_{ci}, Φ_c on the gauge fixing surface. In order to determine λ we impose an additional gauge condition

$$\int d\vec{x}\left\{(\vec{A}_i - \vec{A}_{ci})\cdot\vec{\nabla}_i\vec{\Psi}_0(\lambda=1) + (\vec{\Phi}-\vec{\Phi}_c)\cdot\vec{\Phi}_c \times \vec{\Psi}_0(\lambda=1)\right\} = 0 \tag{3.22}$$

By a partial integration and using the background gauge, (3.22) becomes a surface integral

$$\int d^2\sigma_i\left\{(\vec{A}_i - \vec{A}_{ci})\cdot\hat{x} + (\vec{\Phi}-\vec{\Phi}_c)\cdot\hat{x}\right\} = 0 \tag{3.23}$$

Substituting A_i'' and ϕ'' from (3.18), into the surface integral we have

$$\lambda = (4\pi)^{-1}\int d^2\sigma_i\,\hat{x}\cdot\vec{a}_i$$

which determines the homogeneous solution.

Elimination of Subsidiary Condition:

We have used the gauge invariance of the wave function to express it in terms of points on the orbits of gauge and Higgs fields which intersect a gauge fixing surface. The action of the gauge group generated by (2.29), leaves an orbit invariant, by definition, and hence the points which specify it on the gauge fixing surface. Hence the wave function is expressed in terms of gauge invariant quantities, and the subsidiary condition (2.28) can be taken as an operator equation

$$G_\lambda = \int d\vec{x}\left(\vec{E}_i \nabla_i \vec{\Lambda} + g\vec{\Lambda}\cdot\vec{\pi}\times\vec{\Phi}\right) - 4\pi\lambda g P = 0 \tag{3.24}$$

The above expression can be simplified by expanding the gauge function $\vec{\Lambda}$ (x) in terms of the eigenfunctions of the Faddeev-Popov operator of the chosen gauge. Since we will be exclusively working with the background gauge we expand $\vec{\Lambda}$ (x) in terms of the eigenfunctions of $\mathcal{D}$ discussed in appendix 4.

Since $\vec{\Lambda}\,(r\to\infty) = \lambda\hat{x}$, we form the function $\tilde{\Lambda}$ (x) = $\vec{\Lambda}$(x) $- \vec{\Psi}_0(\lambda\,;\vec{x}\,)$, which vanishes on the boundary of the large box and expand it in terms of the complete set $\{\vec{\Psi}_n\}$,

$$\vec{\Lambda}(\vec{x}) = \vec{\Psi}_0(\lambda;\vec{x}) + \sum_n a_n \vec{\Psi}_n(\vec{x}) \tag{3.25}$$

Substituting (3.25) into (3.24) we have

$$-\sum_n a_n \int d\vec{x}\,\vec{\Psi}_n(\vec{x})\cdot\left(\nabla_i\vec{E}_i + g\,\vec{\Phi}\times\vec{\pi}\right) + \int d\vec{x}\left(\vec{E}_i\nabla_i\vec{\Psi}_0 + g\vec{\Psi}_0\cdot\vec{\pi}\times\vec{\Phi}\right) - 4\pi\lambda P = 0 \tag{3.26}$$

We have used the fact that $\vec{\Psi}_n(x)$ vanishes on the boundary to drop the surface terms $\int d^2\sigma_i \vec{E}_i \cdot \vec{\Psi}_n$.
Further for the same reason we can functionally differentiate (3.26) with respect to $\vec{\Psi}_n(x)$ and use their linear independence to arrive at

$$\nabla_i \vec{E}_i + g\vec{\Phi} \times \vec{\pi} = 0 \tag{3.27}$$

and

$$\int d\vec{x}\left(\vec{E}_i \cdot \nabla_i \vec{\Psi}_0 + g\, \vec{\Psi}_0 \cdot \vec{\pi} \times \vec{\Phi}\right) = 4\pi g\, p \tag{3.28}$$

(3.28) is actually a surface term

$$\int d^2\sigma_i\, \hat{x} \cdot \vec{E}_i = 4\pi g\, p$$

However we shall work with (3.28) since we know the explicit form of the homogeneous solution $\vec{\Psi}_0$, of the Faddeev-Popov operator (3.21).

The operator constraints (3.27) and (3.28), together with the gauge conditions are certainly incompatible with the canonical commutation rules.[10] We may choose as our independent canonical variables the gauge and the Higgs fields which satisfy the gauge conditions, and the corresponding canonical momenta being components of $\vec{E}_i$ and $\vec{\pi}$ tangent to the gauge fixing surface. As for the normal components we may use (3.27) and (3.28) to express these in terms of the independent canonical variables. We shall present a detailed discussion of this reduction procedure when we discuss the perturbation expansion in the next section.

Section 4: Perturbation expansion

In the previous section we discussed how the perturbation theory around a given classical solution can be discussed by fixing a gauge to eliminate (3.1b) as a subsidiary condition on the wave function. As we mentioned the most convenient of the gauges we discussed are the background gauges and we shall discuss the perturbation expansion imposing these.

Performing the shift

$$\vec{A}_i = \vec{A}_{ci} + \vec{a}_i$$
$$\vec{\Phi} = \vec{\Phi}_c + \vec{\varphi} \tag{4.1}$$

and expanding in powers of 'g', the Hamiltonian (2.32) becomes

$$H = \int d\vec{x} \left(\frac{\vec{E}_i^2}{2} + \frac{\vec{\Pi}^2}{2} + \mathcal{V}_0(\vec{x}) + \mathcal{V}_I(\vec{x}) \right) + E_c. \tag{4.2}$$

$$\mathcal{V}_0(\vec{x}) = \frac{1}{2} \tilde{\nabla}_i \vec{a}_j \cdot \tilde{\nabla}_i \vec{a}_j + g \vec{F}^c_{ij} \cdot \vec{a}_i \times \vec{a}_j + \frac{g^2}{2} |\vec{a}_i \times \vec{\Phi}_c|^2$$
$$+ \frac{1}{2} \tilde{\nabla}_i \vec{\varphi} \cdot \tilde{\nabla}_i \vec{\varphi} + \frac{1}{2} g^2 |\vec{\varphi} \times \vec{\Phi}_c|^2 + \frac{\lambda}{2} \vec{\varphi}^2 (\vec{\Phi}_c^2 - \mu^2/\lambda) + \lambda (\vec{\varphi} \cdot \vec{\Phi}_c)^2$$
$$+ 2g \tilde{\nabla}_i \vec{\Phi}_c \cdot \vec{a}_i \times \vec{\varphi} \tag{4.3}$$

is the O(1) potential energy density. The $O(1/g)$ term is zero by virtue of the static equations of motion

$$\tilde{\nabla}_j \vec{F}^u_{ij} + g \bar{\Phi}_u \times \tilde{\nabla}_i \bar{\Phi}_u = 0 \tag{4.4}$$

($\vec{F}^u_{ij} = \partial_i \vec{A}^u_j - \partial_j \vec{A}^u_i + g \vec{A}^u_i \times \vec{A}^u_j$ is the classical field strength)

$$\begin{aligned} \mathcal{V}_I(\vec{x}) = &\frac{g}{2} (\vec{a}_i \times \vec{a}_j) \cdot (\tilde{\nabla}_i \vec{a}_j - \tilde{\nabla}_j \vec{a}_i) + g\, \vec{a}_i \times \vec{\varphi} \cdot \tilde{\nabla}_i \vec{\varphi} \\ &+ g \left(\frac{\lambda}{g^2}\right) \bar{\Phi}_u \cdot \vec{\varphi}\, \vec{\varphi}^{\,2} \\ &+ \frac{g^2}{4} (\vec{a}_i \times \vec{a}_j)^2 + \frac{g^2}{2} (\vec{a}_i \times \vec{\varphi})^2 + \frac{g^2}{4} \left(\frac{\lambda}{g^2}\right) \vec{\varphi}^{\,4} \end{aligned} \tag{4.5}$$

is the interaction part, and

$$E_u = \int d\vec{x} \left(\frac{1}{4} \vec{F}^u_{ij} \cdot \vec{F}^u_{ij} + \frac{1}{2} \tilde{\nabla}_i \vec{\Phi}^u \cdot \tilde{\nabla}_i \vec{\Phi}^u + \frac{\lambda}{4} \left[\bar{\Phi}_u^2 - \frac{\mu^2}{\lambda} \right]^2 \right) \tag{4.6}$$

is the $O(g^{-2})$ classical mass of the monopole (1.15).

In the above expansion we have kept the ratio $\beta = \lambda / g^2$ fixed. We have also imposed the background gauges we discussed in the previous section on the fluctuation,

$$\tilde{\nabla}_i \vec{a}_i + g \vec{\Phi}_u \times \vec{\varphi} = 0 \tag{4.7a}$$

$$\int d\vec{x} \left(\vec{a}_i \cdot \tilde{\nabla}_i \vec{\Psi}_0 + g\, \vec{\varphi} \cdot \bar{\Phi}_u \times \vec{\Psi}_0 \right) = 0 \tag{4.7b}$$

The problem now is the diagonalization of the O(1) quadratic potential energy in (4.2), which after partial integration can be written as

$$V_0 = \int_V d\vec{x} \left(\frac{1}{2} a_{ai} D_1^{ai,bj} a_{bj} + \frac{1}{2} \varphi_a D_2^{ab} \varphi_b + a_{ai} D_3^{ai,b} \varphi_b \right) + \frac{1}{2} \int_S d\vec{\sigma}_i \left[a_{aj} \partial_i a_{aj} + \varphi_a \partial_i \varphi_a \right] \tag{4.8}$$

The operators D_i $i = 1,2,3$ are

$$D_1^{ai,bj} = \mathcal{D}^{ab} \delta^{ij} + 2g \bar{F}_{ij}^{ac} \epsilon^{abc}$$

$$D_2^{ab} = \mathcal{D}^{ab} + \lambda (\bar{\Phi}_a^2 - \mu^2/\lambda) \delta^{ab} + 2\lambda \bar{\Phi}_a^a \bar{\Phi}_a^b$$

$$D_3^{ab,i} = 2g \epsilon^{abc} \widetilde{\nabla}_i \bar{\Phi}_a^c$$

$\mathcal{D}^{ab} = -(\widetilde{\nabla}_i \widetilde{\nabla}_i + g^2 \bar{\Phi}_a \times (\bar{\Phi}_a \times\ ))^{ab}$ is the Faddeev-Popov operator.

In (4.8) we will drop the surface terms assuming the fluctuations are zero on the boundary. In this space of functions the operators D_i are hermetian having real eigenvalues. Now the functions that diagonalize (4.8) satisfy the linear equations

$$D_1^{ai,bj} a_{bj} + D_3^{ai,b} \varphi_b = \omega^2 a_{ai} \tag{4.9a}$$

$$D_2^{ab}\varphi_b - D_3^{ai,b}a_{bi} = \omega^2\varphi_a \tag{4.9b}$$

and the linear gauge conditions (4.7a) and (4.7b). We cannot solve these equations, hence this section will be a formal discussion of the perturbation expansion; in the next section we shall discuss the Prasad-Sommerfield limit for which substantial progress can be made towards a solution.

We do not know whether $w^2 \geqslant 0$, however for the validity of the perturbation expansion we will assume that this is so.

Translation zero modes and collective co-ordinates:

The zero frequency solutions to (4.9a) and (4.9b) are usually translates of the background classical fields which are not invariant to symmetry generators of the original Hamiltonian. We now discuss such modes arising from space translations. These are given by

$$\delta_i\vec{a}_j = \frac{\partial}{\partial x_i}\vec{A}_j^{\alpha} - \tilde{\nabla}_j\vec{\alpha}_i \quad , \quad \delta_i\vec{\varphi} = \frac{\partial}{\partial x_i}\vec{\Phi}^{\alpha} - g\,\vec{\Phi}^{\alpha}\times\vec{\alpha}_i$$

We have added pure gauge terms depending on the gauge parameters $\vec{\alpha}_i(\vec{x})$, because the naive translates of the classical solution are not gauge covariant functions of the classical solution, a property all solutions to the linear fluctuation equations (4.9a) and (4.9b) possess, by virtue of the gauge covariance of the differential operators D_1, D_2, and D_3. Also, the naive translates of the classical solution do not satisfy the gauge conditions (4.7a) and (4.7b). The choice of the function $\vec{\alpha}_i(\vec{x})$ follows easily from the gauge covariance of the modes: $\vec{\alpha}_i = \vec{A}^a_i$ and

$$\delta_i \vec{a}_j = \partial_i \vec{A}^a_j - \tilde{\nabla}_j \vec{A}^a_i = \vec{F}^a_{ij}$$

$$\delta_i \vec{\varphi} = \partial_i \vec{\Phi}^a + g \vec{A}^a_i \times \vec{\Phi}^a = \tilde{\nabla}_i \vec{\Phi}_a \tag{4.10}$$

The gauge condition (4.7a) is satisfied by virtue of the classical equations (4.4). To see that (4.7b) is satisfied, upon partial integration, and using the classical equations (4.4),

$$\int d\vec{x} \left(\vec{F}^a_{ij} \cdot \tilde{\nabla}_j \vec{\Psi}_0 + g \tilde{\nabla}_i \vec{\Phi}^a \cdot \vec{\Phi}_a \times \vec{\Psi}_0 \right)$$

$$= \int d^2\sigma_i \, \vec{F}^a_{ij} \cdot \vec{\Psi}_0 = \int d^2\sigma_i \, \vec{F}^a_{ij} \cdot \hat{x}$$

Upon using the explicit form of the classical solution $A^a_{ai} = \epsilon_{aij} \hat{x}_j (1-K(r))/gr$, it can be seen that the above surface integral vanishes.

From the form of the classical solutions it is possible to show that the modes (4.10) are orthogonal amongst themselves

$$\int d\vec{x} \left(\vec{F}^a_{ij} \cdot \vec{F}^a_{kj} + \tilde{\nabla}_i \vec{\Phi}^a \cdot \tilde{\nabla}_k \vec{\Phi}^a \right) = \delta_{ik} N' \tag{4.11}$$

The norm

$$N' = \frac{1}{3}\int d\vec{x}\left(\vec{F}^a_{ij}\cdot\vec{F}^a_{ij} + \tilde{\nabla}_i\vec{\Phi}^a\cdot\tilde{\nabla}_i\vec{\Phi}^a\right)$$

is actually equal to M_0 the classical mass (4.6). This can be proved by a scaling trick (appendix 1).

There seem to be no other zero modes to (4.9a) and (4.9b), satisfying (4.7a) and (4.7b). In the next section 5 we shall prove this to be the case for the Prasad-Sommerfield monopole.

The translation zero modes are treated by standard collective co-ordinate method.[9] We choose the wave-function, expressed in terms of the gauge invariant variables satisfying the gauges (4.7a) and (4.7b), to be an eigenstate of linear momentum

$$P_i\,\Psi_{n,\vec{p},E}[\vec{A},\vec{\Phi},\theta] = p_i\,\Psi_{n,\vec{p},E}[\vec{A},\vec{\Phi},\theta] \tag{4.12}$$

p_i is the eigenvalue of the gauge invariant linear momentum operator

$$P_i = \int d\vec{x}\left(\vec{E}_k\cdot\vec{F}_{ik} + \vec{\pi}\cdot\nabla_i\vec{\Phi}\right) \tag{4.13}$$

derived from the gauge invariant symmetric energy momentum tensor

$$T_{\mu\nu} = \vec{F}_{\mu\lambda}\cdot\vec{F}_{\nu\lambda} + \nabla_\mu\vec{\Phi}\cdot\nabla_\nu\vec{\Phi} - \delta_{\mu\nu}\mathcal{L} \tag{4.14}$$

P_i in (4.13), with the operator constraint (3.27), which is valid in the space of gauge invariant variables, becomes

$$P_i = \int d\vec{x}\left(\vec{E}_K \frac{\partial}{\partial x_i}\vec{A}_K + \vec{\Pi}\frac{\partial}{\partial x_i}\vec{\Phi}\right) \tag{4.15}$$

after neglecting the surface term $\int d^2\sigma_i \, \vec{E}_K \cdot \vec{A}_K$, since $\vec{A}_i \sim o(1/r)$ and $\vec{E}_i \sim o(1/r^2)$.

We now introduce centre of mass collective coordinates $\vec{X}$ by the following point transformation:

$$\vec{A}_i(\vec{x}) = \tilde{A}_i(\vec{x}+\vec{X})$$

$$\vec{\Phi}(\vec{x}) = \tilde{\Phi}(\vec{x}+\vec{X}) \tag{4.16}$$

We note that the fields $\tilde{A}_i$ and $\tilde{\Phi}$ are translation invariant since the action of the translation group on $\vec{A}_i$ and $\vec{\Phi}$ is equivalent to translating $\vec{X}$.

Using the fact that the wavefunction is a momentum eigenstate it follows that

$$\Psi_{n,\vec{P},E}[\vec{A}_i,\vec{\Phi},\theta] = \exp(i\vec{p}\cdot\vec{X})\,\Psi_{n,\vec{P},E}[\tilde{A}_i,\tilde{\Phi},\theta] \tag{4.17}$$

and the eigenvalue equation (4.12) now becomes a subsidiary condition on the wave function

$$P_i \Psi_{n,\vec{P},E} = -i\frac{\partial}{\partial X_i}\Psi_{n,\vec{P},E}$$

The wave function

$$\hat{\Psi}_{n,\vec{P},E}[\vec{A}_i,\vec{\Phi},\theta,\vec{X}] = \exp(-i\vec{p}\cdot\vec{X})\,\Psi_{n,\vec{P},E}[\vec{A}_i,\vec{\Phi},\theta]$$

is annihilated by the translation generator in the extended phase space

$$\left[P_i - \left(-i\frac{\partial}{\partial X_i}\right)\right]\hat{\Psi}_{n,\vec{P},E}[\vec{A}_i,\vec{\Phi},\theta,\vec{X}] = 0 \tag{4.18a}$$

The integrated version of (4.18) is

$$\hat{\Psi}_{n,\vec{P},E}[\vec{A}_i(\cdot+\vec{a}),\vec{\Phi}(\cdot+\vec{a}),\theta,\vec{X}-\vec{a}] = \hat{\Psi}_{n,\vec{P},E}[\vec{A}_i(\cdot),\vec{\Phi}(\cdot),\theta,\vec{X}] \tag{4.18b}$$

and we have a situation which is entirely analogous to equations (3.2) and (3.3) which express the gauge invariance of the wavefunction in the extended phase-space $\vec{A}_i$, $\vec{\Phi}$, θ .

In order to eliminate (4.18a) as a subsidiary condition, we fix a gauge:

$$\int d\vec{x}\left\{\vec{F}^a_{ij}\cdot[\vec{A}_j-\vec{A}^a_j] + \tilde{\nabla}_i\vec{\Phi}^a\cdot[\vec{\Phi}-\vec{\Phi}^a]\right\} = 0 \tag{4.19}$$

$\vec{F}^a_{ij}$ and $\tilde{\nabla}_i\vec{\Phi}^a$ are the zero modes of translation which we discussed before. In view of the fact that the fluctuations in (4.19) also satisfy the background gauges (4.7a) and (4.7b), (4.19) can be written as

$$\int d\vec{x}\left\{\partial_i\vec{A}^a_j\cdot[\vec{A}_j-\vec{A}^a_j] + \partial_i\vec{\Phi}^a\cdot[\vec{\Phi}-\vec{\Phi}^a]\right\} = 0 \tag{4.20}$$

To confirm that (4.20) is a satisfactory gauge condition consider a field configuration $\vec{A}',\vec{\Phi}'$ in the neighbourhood of a point $\vec{A}^a_i(\vec{x}+\vec{X}_0)$, $\vec{\Phi}^a(\vec{x}+\vec{X}_0)$ on the orbit of the classical solution by the translation group. (Note that this point need not, and is not on the gauge fixing surface defined by the background gauges, except when

$\vec{X}_o = 0$):

$$\vec{A}_i' = \vec{A}_i^a(\vec{x} + \vec{X}_o) + \vec{a}_i'(\vec{x})$$

$$\vec{\Phi}' = \vec{\Phi}^a(\vec{x} + \vec{X}_o) + \vec{\varphi}'(\vec{x})$$

(4.21)

($\vec{a}_i'$ and $\vec{\varphi}'$ are fluctuations of O(1)).

To see whether it is possible to uniquely translate (4.21) onto the surface (4.20), near $\vec{A}_i^a(x)$ and $\vec{\Phi}^a(x)$, perform a finite translation on $\vec{A}_i'$ and $\vec{\Phi}'$ with parameter $-\vec{X}_o$ and a successive infinitisimal translation with parameter $\vec{\Delta}$:

$$\vec{A}_i' \rightarrow \vec{A}_i^a(\vec{x}) + \vec{a}_i(\vec{x}) + \Delta_j\left(\partial_j \vec{A}_i^a + \partial_j \vec{a}_i\right)$$

$$\vec{\Phi}' \rightarrow \vec{\Phi}^a(\vec{x}) + \vec{\varphi}(\vec{x}) + \Delta_j\left(\partial_j \vec{\Phi}^a + \partial_j \vec{\varphi}\right)$$

(4.22)

where $\vec{a}_i(\vec{x}) = \vec{a}_i'(\vec{x} - \vec{X}_o)$ + $\vec{\varphi}(\vec{x}) = \vec{\varphi}'(\vec{x} - \vec{X}_o)$.

Substituting (4.22) into (4.20), $\vec{\Delta}$ is determined by

$$\int d\vec{x}\left[\partial_j \vec{A}_i^a \cdot \partial_k \vec{A}_i^a + \partial_j \vec{A}_i^a \cdot \partial_k \vec{a}_i + \partial_j \vec{\Phi}^a \cdot \partial_k \vec{\Phi}^a + \partial_j \vec{\Phi}^a \cdot \partial_k \vec{\varphi}\right]\Delta_k$$

$$= \int d\vec{x}\left[\partial_j \vec{A}_i^a \cdot \vec{a}_i + \partial_j \vec{\Phi}^a \cdot \vec{\varphi}\right]$$

(4.23)

Since $\vec{a}_i = O(1) = \vec{\varphi}$ and $\vec{A}_i^a = O(1/g) = \vec{\Phi}^a$, the matrix equation (4.23) can be solved uniquely as a perturbation series in 'g', provided the fluctuation remains small and provided the matrix

$$M_{jk} = \int d\vec{x}\left[\frac{\partial}{\partial x_j}\vec{A}_i^a \cdot \frac{\partial}{\partial x_k}\vec{A}_i^a + \frac{\partial}{\partial x_j}\vec{\Phi}^a \cdot \frac{\partial}{\partial x_k}\vec{\Phi}^a\right]$$

is invertible. Using the explicit form (1.10) of the classical solution we can evaluate

$$M_{jK} = \delta_{jK} M$$

$$M = \frac{8\pi}{3}\int_0^\infty dr \left[\frac{H(r)^2}{r^2} + \frac{(1-K(r))^2}{r^2} + 2r^2\left(\frac{H}{r}\right)'^2 + 2r^2\left(\frac{1-K}{r}\right)'^2 \right] > 0$$

which is indeed invertible. This proves that (4.20) is a viable gauge condition.

The non-zero modes and expansion of fields:

We denote the non-zero eigenmodes of the equations (4.9a) and (4.9b) satisfying the gauge conditions (4.7a) and (4.7b) by $\vec{\chi}_i^{(n)}$ and $\vec{\chi}^{(n)}$; we denote the corresponding eigenvalues by ω_n^2. We have used discrete labels for the modes because we are working in a large but finite box with vanishing boundary conditions.

From the hermetian nature of the operators D_i, it follows that the non-zero modes are orthogonal to the translation zeromodes (in the L^2 norm) and hence $\vec{\chi}_i^{(n)}$ and $\vec{\chi}^{(n)}$ satisfy the gauge condition (4.19). For the same reason modes belonging to distinct eigenvalues are orthogonal and can be chosen box normalized to unity. For notational convenience we have omitted labeling degenerate eigenmodes.

We now decompose the gauge and Higgs field fluctuations and the corresponding canonical momenta into directions normal and parallel to the gauge fixing surfaces defined by the background gauges and (4.19):

$$\vec{A}_i = \vec{A}^a_i + \sum_n q_n \vec{\chi}^{(n)}_i + \sum_j \frac{\vec{F}^a_{ij}}{\sqrt{M_0}} \tilde{q}_j + \sum_n \frac{\tilde{\nabla}_i \vec{\Psi}_n}{\sqrt{\epsilon_n}} \xi_n + \frac{\tilde{\nabla}_i \vec{\Psi}_0}{\sqrt{4\pi}} \xi_0$$

$$\vec{\Phi} = \vec{\Phi}^a + \sum_n q_n \vec{\chi}^{(n)} + \sum_j \frac{\tilde{\nabla}_j \vec{\Phi}^a}{\sqrt{M_0}} \tilde{q}_j + \sum_n g \frac{\vec{\Phi}_a \times \vec{\Psi}_n}{\sqrt{\epsilon_n}} \xi_n$$

$$\vec{E}_i = \sum_n p_n \vec{\chi}^{(n)}_i + \sum_j \frac{\vec{F}^a_{ij}}{\sqrt{M_0}} \tilde{p}_j + \sum_n \frac{\tilde{\nabla}_i \vec{\Psi}_n}{\sqrt{\epsilon_n}} \eta_n + \frac{\tilde{\nabla}_i \vec{\Psi}_0}{\sqrt{4\pi}} \eta_0$$

$$\vec{\pi} = \sum_n p_n \vec{\chi}^{(n)} + \sum_j \frac{\tilde{\nabla}_j \vec{\Phi}^a}{\sqrt{M_0}} \tilde{p}_j + \sum_n \frac{g \vec{\Phi}_a \times \vec{\Psi}_n}{\sqrt{\epsilon_n}} \eta_n \qquad (4.24)$$

$\vec{F}^a_{ij}/\sqrt{M_0}$, $\tilde{\nabla}_i \vec{\Phi}^a/\sqrt{M_0}$ are evidently normal to the surface (4.19). $\tilde{\nabla}_i \vec{\Psi}_n/\sqrt{\epsilon_n}$ and $g\vec{\Phi}_a \times \vec{\Psi}_n/\sqrt{\epsilon_n}$ are normal to the surface (4.7a); $\vec{\Psi}_n$ are the eigenvectors of the Faddeev-Popov operator $\mathcal{D}$, ϵ_n are the corresponding eigenvalues. It is not difficult to verify that

$$\int d\vec{x} \left[\tilde{\nabla}_i \vec{\Psi}_n \cdot \tilde{\nabla}_i \vec{\Psi}_m + g^2 \vec{\Phi}_a \times \vec{\Psi}_n \cdot \vec{\Phi}_a \times \vec{\Psi}_m \right] = \epsilon_n \delta_{mn} > 0 \qquad (4.25)$$

And lastly $\tilde{\nabla}_i \vec{\Psi}_0/\sqrt{4\pi}$ is normal to (4.7b), and we can verify that

$$\int d\vec{x} \left[\tilde{\nabla}_i \vec{\Psi}_0(\lambda=1;\vec{x}) \cdot \tilde{\nabla}_i \vec{\Psi}_0(\lambda=1;\vec{x}) \right] = 4\pi \qquad (4.26)$$

$\vec{\Psi}_0(\lambda=1)$, being the homogeneous solution of $\mathcal{D}$.

However since the fields $\vec{A}_i$ and $\vec{\Phi}$ have been chosen to lie on the surfaces (4.7a), (4.7b) and (4.19), the normal components are absent i.e.

$$\tilde{q}_j = \xi_n = \xi_0 = 0 \tag{4.27}$$

The co-ordinates q_n and the corresponding momenta P_n satisfy the usual commutation relations

$$[q_n, P_m] = i\,\delta_{mn} \tag{4.28}$$

because the normal modes $\vec{\chi}_i^{(n)}$ and $\vec{\chi}^{(n)}$, satisfy the gauge conditions (4.7a), (4.7b) and (4.19).

Substituting the orthonormal expansions (4.24) into the Hamiltonian (4.2), and using (4.27), we have

$$\begin{aligned} H_0 = {} & \frac{1}{2}\sum_n \eta_n^2 + \frac{1}{2}\eta_0^2 + \frac{1}{2}\sum_j \tilde{P}_j^2 \\ & + \frac{1}{2}\sum_n \left(P_n^2 + \omega_n^2 q_n^2\right) + V_I(q_n) + E_{cl} \end{aligned} \tag{4.29}$$

The first line corresponds to canonical momenta in directions normal to the gauge fixing surfaces; these are to be determined in terms of the independent co-ordinates q_n and P_n , by solving the operator constraints (3.27), (3.28)

$$\nabla_i \vec{E}_i + g\,\vec{\Phi}\times\vec{\pi} = 0$$

$$\int d\vec{x}\left[\nabla_i \vec{\Psi}_0\cdot\vec{E}_i + g\,\vec{\Phi}\times\vec{\Psi}_0\cdot\vec{\pi}\right] = -i4\pi g\frac{\partial}{\partial\theta} = 4\pi g p \tag{4.30}$$

and

$$\int d\vec{x}\left[\vec{E}_j \frac{\partial}{\partial x_i}\vec{A}_j + \vec{\pi}\frac{\partial}{\partial x_i}\vec{\Phi}\right] = -i\frac{\partial}{\partial x_i} = P_i$$

Substituting the expansions (4.24), together with (4.27), into (4.30) we obtain to O(g),

$$\eta_\ell = g \sum_{m,n} A^\ell_{mn} q_m P_n + o(g^2)$$

$$\eta_0 = g\left(P - \sum_{m,n} B_{mn} q_m P_n\right) + o(g^2)$$

$$\tilde{P}_j = M_0^{-1/2}\left(P_j - \sum_{m,n} C^j_{mn} q_m P_n\right) + o(g^2)$$

$$A^\ell_{mn} = \int d\vec{x}\ \epsilon_\ell^{-1/2}\,\bar{\Psi}_\ell\cdot\left(\vec{\chi}_i^{(m)}\times\vec{\chi}_i^{(n)} + \vec{\chi}^{(m)}\times\vec{\chi}^{(n)}\right)$$

$$B_{mn} = \int d\vec{x}\ (4\pi)^{-1/2}\left(\vec{\chi}_i^{(m)}\times\vec{\chi}_i^{(n)} + \vec{\chi}^{(m)}\times\vec{\chi}^{(n)}\right)$$

$$C^i_{mn} = \int d\vec{x}\left[\vec{\chi}_j^{(m)}\tilde{\nabla}_i\vec{\chi}_j^{(n)} + \vec{\chi}^{(m)}\tilde{\nabla}_i\vec{\chi}^{(n)}\right]$$

(4.31)

Substituting the expressions (4.31) into the Hamiltonian (4.29) we obtain to $O(g^2)$:

$$H_0 = E_{cl} + \sum_n\left(-\frac{\partial^2}{\partial q_n^2} + \omega_n^2 q_n^2\right) + V_I(q_n)$$

$$+ \frac{\vec{P}^2}{2M_0} + g^2\frac{n^2}{2} + \cdots$$

$$\cdots - \frac{g^2}{2}\left(A^{\ell}_{mn}A^{\ell}_{rs} + B_{mn}B_{rs} + C^{j}_{mn}C^{j}_{rs}\right)q_m q_r \frac{\partial^2}{\partial q_n \partial q_s}$$

$$-\frac{g^2}{2}\left((A^{\ell}A^{\ell})_{mn} + B^2_{mn} - 2ni\,B_{mn} + (C^jC^j)_{mn} - 2ip_j\,C^j_{mn}\right)q_m\frac{\partial}{\partial q_n}$$

$$+ o(g^3) + \cdots \tag{4.32}$$

Recalling that the wavefunction which is an eigen-state of electric charge and linear momentum has the form

$$\hat{\Psi}_{n,\vec{p},E}[\vec{A}_i,\vec{\Phi},\theta,\vec{x}] = \exp i(\vec{p}\cdot\vec{x}+n\theta)\Psi_{n,\vec{p},E}[\vec{A}_i,\vec{\Phi}]$$

$$= \exp i(\vec{p}\cdot\vec{x}+n\theta)\Psi_{n,\vec{p},E}(q_1,q_2\ldots) \tag{4.33}$$

The O(1) solution to the Schrödinger equation

$$H_0\hat{\Psi} = E\hat{\Psi} \tag{4.34}$$

is given by

$$\hat{\Psi}^{0}_{n,\vec{p},E}[\vec{A}_i,\vec{\Phi},\theta,\vec{x}] = \exp(i\vec{p}\cdot\vec{x}+in\theta)\prod_n\left(\frac{\omega_n}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}\sum\omega_n q_n^2\right) \tag{4.35}$$

$$E^0 = E_{cl} + \frac{1}{2}\sum_n \omega_n \tag{4.36}$$

From (4.35) we have

$$\int_{-\infty}^{+\infty} d\vec{x}\int_0^{2\pi} d\theta \int_{-\infty}^{+\infty}\prod_n dq_n\,\hat{\Psi}^{0*}_{n,\vec{p},E}\,\hat{\Psi}^{0}_{n',\vec{p}',E} = \delta(n-n')\,\delta^{(3)}(\vec{p}-\vec{p}')$$

The Fock space in the monopole sector can be constructed in the usual manner by the action of the creation operator

$$a_n^{\dagger} = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{1}{\sqrt{2\omega_n}} \frac{\partial}{\partial q_n}$$

$$\hat{\Psi}^{N_1, N_2, \ldots\ldots}_{n,\vec{p}}[\vec{x},\theta,q_n] = \prod_n \frac{(a_n^{\dagger})^{N_n}}{\sqrt{N_n!}}\, \hat{\Psi}^{o}_{n,\vec{p}}[\vec{x},\theta,q_n]$$

and $E_{N_1,N_2\ldots} = \sum_n (N_n + \frac{1}{2})\omega_n$

From the Hamiltonian (4.32), it is clear that the dependence of the energy on electric charge and linear momentum starts at $O(g^2)$.

Renormalization:

The zero point energy in (4.36) must be evaluated by comparing it with the corresponding quantity when the background fields are

$$A^{a}_{ai} = 0 \quad , \quad \phi^{a}_{a} = \hat{z}_a\, \mu^2/\lambda$$

The ground state energy is in fact

$$E^{o} = \langle \text{Monopole} | H_o | \text{Monopole} \rangle - \langle \text{vac.} | H_o | \text{vac.} \rangle$$

$$= E_a + \frac{1}{2}\sum_n \omega_n - \frac{1}{2}\sum_n \omega_{no}$$

$$= E_{cl} + \sum_n \frac{1}{2}\left(\omega_n - \omega_{no}\right) \tag{4.37}$$

As the number of oscillators goes to infinity, the sum (4.37) would be expected to have a logarithmic divergence. By t'Hooft's background field method[35] the renormalization counterterm which removes divergences to O(1), has the same form for any smooth background field. Since our classical background field is smooth and singularity free we expect this procedure to eliminate the divergence in (4.37). We remark that even though we know in principle how to renormalize the expectation value of the Hamiltonian and other operators, in the state (4.35), we do not know in principle how to renormalize the wave function itself, because the product $\prod_n \left(\frac{\omega_n}{\pi}\right)^{1/4}$ in (4.35) is evidently divergent.

Section 5: The Prasad-Sommerfield limit

We recall that the exact solution of Prasad and Sommerfield (1.22) (consider case with $\gamma = 0$), extremizes the potential energy (1.2) with $\lambda \to 0$, $\mu \to 0$ holding $\mu/\sqrt{\lambda}$ fixed:

$$V = \int d\vec{x} \left[\frac{1}{4} \vec{F}_{ij} \cdot \vec{F}_{ij} + \frac{1}{2} \nabla_i \vec{\Phi} \cdot \nabla_i \vec{\Phi} \right] \tag{5.1}$$

The boundary condition on the Higgs field is the same as that of the t'Hooft-Polyakov monopole.

Essential to our discussion of the perturbation expansion, will be certain observations of Bogolonomy and Coleman et. al.[36] which we present in the language of Euclidean field theory. Identifying the Higgs field with the time component of a Euclidean vector potential, $\vec{\Phi} \equiv \vec{A}_0$ (this $\vec{A}_0$ is distinct from the Lagrange multiplier we discussed in section 2), (5.1) can be written as

$$V = \int d^3\vec{x}\ \frac{1}{4}\ \vec{F}_{\mu\nu}(\vec{x}) \cdot \vec{F}_{\mu\nu}(\vec{x})$$

$$\vec{F}_{\mu\nu}(\vec{x}) = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu\,, \quad \vec{\Phi} = \vec{A}_0\,, \quad \partial_0 \vec{A}_\mu = 0 \tag{5.2}$$

Since the fields in (5.2) depend only on space co-ordinates (5.2) can be understood as the action per unit time of a 4-dimensional Euclidean Yang-Mills theory.

Defining the dual of $\vec{F}_{\mu\nu}$ by

$$^*\vec{F}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,\vec{F}_{\rho\sigma} \tag{5.3}$$

($\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor in 4-dimensions with $\epsilon_{0123} = +1$; note $^{**}\vec{F}_{\mu\nu} = \vec{F}_{\mu\nu}$)
We have the following inequality of Belavin, Schwartz, Polyakov and Tyupkin (BPST),

$$\int d\vec{x}\ \left(\vec{F}_{\mu\nu} \pm {}^*\vec{F}_{\mu\nu}\right)^2 \geqslant 0$$

$$\Rightarrow \quad \int d\vec{x}\ \vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu} \geqslant \left| \int d\vec{x}\ {}^*\vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu} \right|$$

$$\Rightarrow \quad V \geqslant \left| \int d\vec{x}\ \tfrac{1}{2} \epsilon_{ijk} \vec{F}_{jk} \cdot \vec{F}_{0i} \right|$$

$$= \left| \int d\vec{x}\ \tfrac{1}{2} \epsilon_{ijk} \vec{F}_{jk} \cdot \nabla_i \vec{\Phi} \right|$$

$$= \frac{\mu}{\sqrt{\lambda}} \left| \int_S d^2\sigma_i\ \tfrac{1}{2} \epsilon_{ijk} \vec{F}_{jk} \cdot \hat{\Phi} \right| \quad , \quad \epsilon_{ijk} \nabla_i \vec{F}_{jk} = 0 \tag{5.4}$$

From (1.16), the surface integral under the modulus sign is the magnetic flux, which is a constant of motion. Further, for those configurations which satisfy $\hat{\Phi} \cdot \vec{A}_\mu = 0$ on the boundary the magnetic flux can be written in terms of the topological charge 'd' in (1.5) and the inequality (5.4), becomes

$$V = g\frac{\mu}{\sqrt{\lambda}} \frac{4\pi}{g^2} |d| = C \cdot \frac{4\pi}{g^2} |d|$$

($C = g\mu/\sqrt{\lambda}$ is a parameter in the model).

It is obvious that the BPST inequality is saturated for self dual and anti-selfdual fields:

$$^*\vec{F}_{\mu\nu} = \pm \vec{F}_{\mu\nu}$$

which is equivalent to

$$\tfrac{1}{2} \epsilon_{ijk} \vec{F}_{jk} = \mp \nabla_i \vec{\Phi} \tag{5.5}$$

which are the equations of Bogolonomy and Coleman et. al. for the absolute minimum of V. The Prasad-Sommerfield solution satisfies (5.5) with negative sign and hence is self-dual. It carries magnetic charge -1. The Prasad-

Sommerfield solution with magnetic charge +1 is obtained by simply reversing the sign of the Higgs field in (5.5) i.e. it is anti- self dual.

The Hamiltonian for small fluctuations (4.1) around this solution can be obtained from (4.3) setting $\lambda = \mu = 0$, $\mu / \sqrt{\lambda} =$ constant

$$H = \int d\vec{x} [\vec{E}_i^2 + \vec{\pi}^2 + \mathcal{V}_0(\vec{x}) + \mathcal{V}_I(\vec{x})] + E_{cl}$$

$$E_{cl} = c \cdot 4\pi / g^2$$

and

$$\mathcal{V}_0(\vec{x}) = \frac{1}{2} \tilde{\nabla}_i \vec{a}_j \cdot \tilde{\nabla}_i \vec{a}_j + g \vec{F}^{cl}_{ij} \cdot \vec{a}_i \times \vec{a}_j + \frac{g^2}{2} |\vec{a}_i \times \vec{\Phi}_{cl}|^2$$

$$+ \frac{1}{2} \tilde{\nabla}_i \vec{\varphi} \cdot \tilde{\nabla}_i \vec{\varphi} + \frac{g^2}{2} |\vec{\Phi}_{cl} \times \vec{\varphi}|^2$$

$$+ 2g \tilde{\nabla}_i \vec{\Phi}_{cl} \cdot \vec{a}_i \times \vec{\varphi} \tag{5.6}$$

If we denote $\vec{\varphi} \equiv \vec{a}_0$, we can write the O(1) potential energy as

$$V_0 = \int \mathcal{V}_0 \, d\vec{x} = \int d\vec{x} \left(\frac{1}{2} \tilde{\nabla}_\mu \vec{a}_\nu \cdot \tilde{\nabla}_\mu \vec{a}_\nu + g \vec{F}^{cl}_{\mu\nu} \cdot \vec{a}_\mu \times \vec{a}_\nu \right) \tag{5.7}$$

where $\tilde{\nabla}_\mu = (\tilde{\nabla}_i \, , g \vec{\Phi}_{cl} \times)$

Recall that the fluctuations satisfy the background gauges

$$\tilde{\nabla}_i \vec{a}_i + g \vec{\Phi}_{cl} \times \vec{\varphi} = \tilde{\nabla}_\mu \vec{a}_\mu = 0 \tag{5.8}$$

$$\int d\vec{x}\, \widetilde{\nabla}_i \vec{\Psi}_0(\vec{x};\lambda=1)\cdot\vec{a}_i = \int d\vec{x}\, \widetilde{\nabla}_\mu \vec{\Psi}_0 \cdot \vec{a}_\mu = 0$$

The normal modes which diagonalize (5.7) (i.e. equations (4.9a) and (4.9b) in the Prasad-Sommerfield limit) satisfy the following linear equations

$$\left(\widetilde{\nabla}_\mu \widetilde{\nabla}_\mu \delta_{\nu\lambda} + 2g\, \vec{F}^{a}_{\nu\lambda} \times \right) \vec{a}_\lambda = -\omega^2 \vec{a}_\nu \tag{5.9}$$

The operator $\widetilde{\nabla}_\mu \widetilde{\nabla}_\mu = \widetilde{\nabla}_i \widetilde{\nabla}_i + g^2 \vec{\Phi}_{cl} \times (\vec{\Phi}_{cl} \times$ is the Faddeev-Popov operator in the background gauge. The most important property of the linear equation (5.9) is that the potential $\vec{F}^{a}_{\mu\nu}$ is self-dual or anti-self-dual. (We choose to work with anti-self-dual or magnetic charge +1 fields). This observation, which is implicit in the work of t'Hooft[19], has been made explicit and extensively used by Brown et. al.[20]; it enables one to relate the solutions of (5.9) with those of the eigenvalue equation of the Faddeev-Popov operator (Appendix 4):

$$-\mathcal{D}\vec{\Psi} = \widetilde{\nabla}_\mu \widetilde{\nabla}_\mu \vec{\Psi} = -\epsilon \vec{\Psi} \tag{5.10}$$

Before we proceed, we introduce the 'η' symbols of t'Hooft (see appendix of ref.19)

$$\eta_{a\mu\nu} = \epsilon_{a\mu\nu} \quad , \mu,\nu = 1,2,3 \ , \ a = 1,2,3$$

$$\eta_{a0\nu} = -\delta_{a\nu} \quad , \quad \eta_{a\nu 0} = \delta_{a\nu}$$

$$\eta_{a00} = 0$$

$$\bar{\eta}_{a\mu\nu} = (-1)^{\delta_{\mu 0}+\delta_{\nu 0}} \eta_{a\mu\nu}$$

As explained by t'Hooft $\eta_{a\mu\nu}(\bar{\eta}_{a\mu\nu})$ enables us to associate with a self-dual (anti-self dual) tensor a 3 - vector by means of the equations

$$A_{\mu\nu} = \eta_{a\mu\nu} A_a \quad , \quad {}^*A_{\mu\nu} = A_{\mu\nu}$$

$$\bar{A}_{\mu\nu} = \bar{\eta}_{a\mu\nu} \bar{A}_a \quad , \quad {}^*\bar{A}_{\mu\nu} = -\bar{A}_{\mu\nu}$$

Now following ref. (20), one can verify using the static equations of motion $\tilde{\nabla}_\mu F_{\mu\nu} = 0$, that

$$\vec{\chi}_\nu = \tilde{\nabla}_\nu \vec{\Psi} \tag{5.11}$$

solves (5.9), provided $\vec{\Psi}$ solves (5.10), with $\omega^2 = \epsilon$.
But since

$$\tilde{\nabla}_\nu \vec{\chi}_\nu = \tilde{\nabla}_\nu \tilde{\nabla}_\nu \vec{\Psi} = -\mathcal{D}\vec{\Psi} = -\epsilon \vec{\Psi} \neq 0$$

unless $\epsilon = 0$; and for $\epsilon = 0$,

$$\int d\vec{x}\, \tilde{\nabla}_\nu \vec{\Psi}_0 \cdot \tilde{\nabla}_\nu \vec{\Psi} = \int d\vec{x}\, \tilde{\nabla}_\nu \vec{\Psi}_0 \cdot \tilde{\nabla}_\nu \vec{\Psi}_0 \neq 0$$

(5.11) does not satisfy the gauge conditions (5.8), and hence is not an acceptable solution to the fluctuation equations (5.9).

However from (5.11) we can construct further solutions to (5.9) using the anti-self duality of $\vec{F}^a_{\mu\nu}$; which enables us to write,

$$\vec{F}^a_{\mu\nu} = \bar{\eta}_{a\mu\nu} \vec{F}_a^{\,a}$$

Since $\eta_{a\kappa\mu}\bar{\eta}_{b\kappa\lambda} = \eta_{a\kappa\lambda}\bar{\eta}_{b\kappa\mu}$ (t'Hooft, Ref. (19); equation A-15), we see that

$$\vec{\chi}_{\mu}^{(a)} = \eta_{a\mu\nu}\tilde{\nabla}_{\nu}\vec{\Psi} \quad , \; a = 1, 2, 3 \tag{5.12}$$

are also eigenfunctions of (5.9), provided $\vec{\Psi}$ satisfies (5.10) with $\epsilon = \omega^2$.

We now prove that the eigenfunctions (5.12) do satisfy the gauge conditions (5.8):

$$\begin{aligned}
\tilde{\nabla}_{\mu}\vec{\chi}_{\mu}^{(a)} &= \eta_{a\mu\nu}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\vec{\Psi} \\
&= \eta_{a\mu\nu}\tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu}\vec{\Psi} + \eta_{a\mu\nu}\bar{\eta}_{b\mu\nu}\vec{F}_{b}^{a} \\
&= 0
\end{aligned}$$

In the above we have used the identity $\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu} - \tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu} = \vec{F}_{\mu\nu}^{a}\times$, $\eta_{a\mu\nu} = -\eta_{a\nu\mu}$ and $\eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} = 0$. Further for $\epsilon = \omega^2 \neq 0$, we have after a partial integration (we drop the surface term because $\vec{\Psi}_{\epsilon} = 0$ on the boundary for $\epsilon \neq 0$)

$$\begin{aligned}
\int d\vec{x}\,\tilde{\nabla}_{\mu}\vec{\Psi}_{0}\cdot\vec{\chi}_{\mu}^{(a)} &= \int d\vec{x}\,\tilde{\nabla}_{\mu}\vec{\Psi}_{0}\,\eta_{a\mu\nu}\tilde{\nabla}_{\nu}\vec{\Psi}_{\epsilon} \\
&= \int d\vec{x}\left(-\eta_{a\mu\nu}\tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu}\vec{\Psi}_{0}\cdot\vec{\Psi}_{\epsilon}\right) \\
&\int d\vec{x}\left(-\eta_{a\mu\nu}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\vec{\Psi}_{0}\cdot\vec{\Psi}_{\epsilon} - \eta_{a\mu\nu}\bar{\eta}_{b\mu\nu}\vec{F}_{b}^{a}\times\vec{\Psi}_{0}\cdot\vec{\Psi}_{\epsilon}\right)
\end{aligned}$$

$= 0$ (since $\vec{F}_{\mu\nu}^{a} = \bar{\eta}_{b\mu\nu}\vec{F}_{b}^{a}$ is antiself dual).

If $\epsilon = \omega^2 = 0$, we cannot drop the surface term, we find

$$\int d\vec{x}\, \tilde{\nabla}_\mu \vec{\Psi}_0 \cdot \vec{\chi}_\mu^{(a)} = \int d\vec{x}\, \tilde{\nabla}_\mu \vec{\Psi}_0 \cdot \eta_{a\mu\nu} \tilde{\nabla}_\nu \vec{\Psi}_0$$

$$= \int d\vec{x}\, \tilde{\nabla}_i \vec{\Psi}_0\, \epsilon_{aij} \tilde{\nabla}_j \vec{\Psi}_0 \tag{5.13}$$

Now recall from section 3 that the unique solution to the homogeneous equation $\mathcal{D}\vec{\Psi}_0 = 0$ under the boundary condition $\vec{\Psi}_0(|\vec{x}|\to\infty) = \hat{x}$, is the classical higgs field (3.21),

$$\vec{\Psi}_0 = \vec{\Phi}_{cl} = \hat{x}\,[r \coth r - 1]/r$$

Using the classical equations (5.5), we may write (5.13) as

$$\int d\vec{x}\, \tilde{\nabla}_\mu \vec{\Psi}_0 \cdot \vec{\chi}_\mu^{(a)} = \int d\vec{x}\, \tilde{\nabla}_i \vec{\Phi}^{cl} \cdot \vec{F}_{ai}^{cl}$$

$$= \int d\sigma_i^2\, \vec{\Phi}^{cl} \cdot \vec{F}_{ai}^{cl} = 0$$

Zero frequency modes:[37]

The eigenfunction corresponding to $\omega^2 = \epsilon = 0$ is

$$\vec{\chi}_\mu^{(a)} = \eta_{a\mu\nu} \tilde{\nabla}_\nu \vec{\Psi}_0 \quad , \quad a = 1, 2, 3 \tag{5.14}$$

and since under the boundary condition $\vec{\Psi}_0(r\to\infty) = \hat{x}$, $\vec{\Psi}_0 = \vec{\Phi}_{cl}$, we have

$$\vec{\chi}^{(a)}_{\mu} = \eta_{a\mu\nu}\widetilde{\nabla}_{\nu}\vec{\Phi}_{a} \tag{5.15}$$

which gives

$$\vec{\chi}^{(a)}_{i} = \epsilon_{aij}\widetilde{\nabla}_{j}\vec{\Phi}_{a} = \vec{F}^{a}_{ai}$$ for the vector field

and $\vec{\chi}^{(a)}_{o} = -\widetilde{\nabla}_{a}\vec{\Phi}_{a}$ for the scalar field.

These are precisely the translation zero modes we discussed in the previous section. To show that there are no other zero modes, we have to show that the solution to the homogeneous equation $\mathcal{D}\vec{\Psi}_o = 0$ is unique. Under the assumption that the zero modes (5.14) are square integrable it is possible to prove this.

Proof:

$$\int d\vec{x}\ \vec{\chi}^{(a)}_{\mu}\cdot\vec{\chi}^{(a)}_{\mu} < \infty$$

means

$$\int d\vec{x}\left(\widetilde{\nabla}_{i}\vec{\Psi}_{o}\cdot\widetilde{\nabla}_{i}\vec{\Psi}_{o} + g^{2}\,|\vec{\Phi}_{a}\times\vec{\Psi}_{o}|^{2}\right) < \infty$$

The convergence of the above integral imposes the following constraints on the asymptotic behavior of $\vec{\Psi}_o$.

$$\lim_{r\to\infty} r^{3/2}\,\widetilde{\nabla}_{i}\vec{\Psi}_{o} = 0$$

$$\lim_{r\to\infty} r^{3/2}\,\vec{\Phi}_{a}\times\vec{\Psi}_{o} = 0 \tag{5.16}$$

Now consider a $1/r$ expansion of $\vec{\Psi}_o$:

$$\vec{\Psi}_{o} = \vec{\Psi}^{(0)}_{o}(w) + \vec{\Psi}^{(1)}_{o}(w)/r + \cdots$$

W is the polar angle.

Substituting into (5.16) it follows that

$\vec{\Psi}_o^{(0)}(w) = \hat{\Phi}^a \cdot$ (constant) and $\vec{\Psi}_o^{(1)}(w) = \hat{\Phi}^a \cdot$ (constant).

Since $\hat{\Phi}_a = \hat{x}$, the only solution to $\mathcal{D}\vec{\Psi}_o = 0$ is $\vec{\Psi}_o = \vec{\Phi}_a$.

This proves that the only zero modes to (5.9) satisfying the gauge conditions (5.8) are (5.15), the translation zero modes.

The non-zero modes: (a) Orthogonality:

The non-zero modes ($\epsilon = \omega^2 \neq 0$) satisfy the following orthogonality condition

$$\int d\vec{x}\, \vec{\chi}_\mu^{(a)\omega} \cdot \vec{\chi}_\mu^{(b)\omega'} = \delta_{ab}\, \omega^2\, \delta_{\omega\omega'} \tag{5.17}$$

Proof: The presence of $\delta_{\omega\omega'}$ is easily explained by the Hermetian nature of the differential operator in the space of functions which vanish on the boundary. To prove the rest

$$\begin{aligned}
\int d\vec{x}\, \vec{\chi}_\mu^{(a)\omega} \cdot \vec{\chi}_\mu^{(b)\omega} &= \int d\vec{x}\, \eta_{a\mu\nu}\eta_{b\mu\lambda} \tilde{\nabla}_\nu \vec{\Psi}_\epsilon\, \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon \\
&= \int d\vec{x}\, (\delta_{ab}\delta_{\nu\lambda} + \epsilon_{abc}\eta_{c\nu\lambda}) \tilde{\nabla}_\nu \vec{\Psi}_\epsilon \cdot \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon \\
&= \delta_{ab} \int d\vec{x}\, \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon \cdot \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon \\
&= \delta_{ab} \int d\vec{x}\, \vec{\Psi}_\epsilon \left(-\tilde{\nabla}_i\tilde{\nabla}_i - g^2 \vec{\Phi}_a \times (\vec{\Phi}_a \times\right)\vec{\Psi}_\epsilon
\end{aligned}$$

W is the polar angle.

Substituting into (5.16) it follows that
$\vec{\Psi}_o^{(0)}(w) = \hat{\Phi}^a \cdot$ (constant) and $\vec{\Psi}_o^{(1)}(w) = \hat{\Phi}^a \cdot$ (constant).
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$$\int d\vec{x}\, \vec{\chi}_\mu^{(a)\omega} \cdot \vec{\chi}_\mu^{(b)\omega} = \int d\vec{x}\, \eta_{a\mu\nu}\eta_{b\mu\lambda} \tilde{\nabla}_\nu \vec{\Psi}_\epsilon\, \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon$$

$$= \int d\vec{x}\, (\delta_{ab}\delta_{\nu\lambda} + \epsilon_{abc}\eta_{c\nu\lambda}) \tilde{\nabla}_\nu \vec{\Psi}_\epsilon \cdot \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon$$

$$= \delta_{ab} \int d\vec{x}\, \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon \cdot \tilde{\nabla}_\lambda \vec{\Psi}_\epsilon$$

$$= \delta_{ab} \int d\vec{x}\, \vec{\Psi}_\epsilon \left(-\tilde{\nabla}_i \tilde{\nabla}_i - g^2 \vec{\Phi}_a \times (\vec{\Phi}_a \times\right) \vec{\Psi}_\epsilon$$

$$= \delta_{ab} \int d\vec{x}\, \vec{\Psi}_\epsilon \mathcal{D} \vec{\Psi}_\epsilon = \delta_{ab}\,\epsilon = \delta_{ab}\,\omega^2$$

(b) Completeness:

We now prove that the orthogonal functions

$$\vec{\chi}^{(a)}_\mu(\vec{x})_n = \eta_{a\mu\nu} \tilde{\nabla}_\nu \vec{\Psi}_n(\vec{x})$$

are complete. The discrete label is there because we have put our system in a large box with zero boundary conditions. The functions $\vec{\Psi}_n$ are discussed in appendix 4; for the moment we consider only those $\vec{\Psi}_n$ which vanish on the boundary of the box i.e. those which correspond to $\epsilon_n = \omega_n^2 \neq 0$. Recall that these functions are orthonormal and complete i.e. any function $f^a(\vec{x})$ which vanishes on the boundary of the box has an expansion

$$f^a(\vec{x}) = \sum_{n=1}^{\infty} a_n \Psi^a_n(\vec{x})$$

with $a_n = \int d\vec{x}\,(\vec{\Psi}_n \cdot \vec{f}) < \infty$. This is equivalent to the equation

$$\sum_{n=1}^{\infty} \Psi^a_n(\vec{x}) \Psi^b_n(\vec{y}) = \delta^{ab}\, \delta^3(\vec{x}-\vec{y}) \tag{5.18}$$

Now form the kernal in terms of the non-zero modes

$$K^{ab}_{\mu\nu}(\vec{x},\vec{y}) = \sum_{n,c} \chi^{(c)}_{\mu,a}(\vec{x})_n \chi^{(c)}_{\nu,b}(\vec{y})_n$$

and consider

$$I = \int d\vec{x}\, d\vec{y}\; f^a_\mu(\vec{x})\, K^{ab}_{\mu\nu}(\vec{x},\vec{y})\, g^b_\nu(\vec{y})$$

where f^a_μ and g^b_ν are functions that vanish on the boundary of the box. Using the completeness relation (5.18) it is straightforward to prove that

$$I = -\int d\vec{x} \left[f^a_\mu(\vec{x}) \left[\tilde{\nabla}_\rho \tilde{\nabla}_\rho \delta_{\mu\nu} + 2g \vec{F}^a_{\mu\nu} \times \right]_{ab} g^b_\nu(\vec{x}) \right] \tag{5.19}$$

The differential operator in the above expression is precisely the one that occurs in (5.9). To understand the meaning of (5.19) define the ket vectors $|n,(b)\rangle$ by

$$\langle \vec{x}, a, \mu | n,(b) \rangle = \chi^{(b)}_{\mu,a}(\vec{x})_n$$

and the operator K by

$$\langle \vec{x}, a, \mu | K | \vec{y}, b, \nu \rangle = -\left(\tilde{\nabla}_\rho \tilde{\nabla}_\rho \delta_{\mu\nu} + 2g \vec{F}^a_{\mu\nu} \times \right)_{ab} \delta(\vec{x}-\vec{y})$$

then (5.19) becomes

$$\sum_{n,b}^{\infty} |n,(b)\rangle \langle n,(b)| = K \tag{5.20}$$

Note that $|n\rangle$ are not normalized and from (5.17) we have

$$\langle n,(a)|m,(b)\rangle = \epsilon_n \delta_{nm} \delta_{ab}, \quad K|n,(b)\rangle = \epsilon_n |n,(b)\rangle$$

Defining the normalized eigenvector

$$|\hat{n},(b)\rangle = |n,b\rangle / \sqrt{\epsilon_n} \quad , \quad \epsilon_n \neq 0$$

(5.20) reads

$$\sum_{n=1,a}^{\infty} \epsilon_n |\hat{n},(a)\rangle \langle \hat{n},(a)| = K \tag{5.21}$$

which says that in terms of the non-zero modes

$$\langle \vec{x}, a, \mu \mid \hat{n}, (b) \rangle = \chi^{(b)}_{\mu,a}(\vec{x})_n / \sqrt{\epsilon_n}$$

we have a spectral representation of the operator K; this is equivalent to saying that the normalized non-zero modes are complete in the subspace orthogonal to the only three zero modes of translation.

(c) Rotational properties of the eigenfunctions:

We first consider the eigenvalue problem of the Faddeev-Popov operator (5.10); substituting the explicit forms of the classical solution (5.10) becomes

$$\Big[\Big(p_r^2 + \frac{\vec{L}^2}{r^2} + \frac{1}{r^2}\left[(1-K)^2 + H^2\right] \Big) \delta^{ab} + \frac{2}{r^2}(1-K)\vec{L}\cdot\vec{T}^{ab}$$

$$+ \frac{1}{r^2}\left[(1-K)^2 - H^2\right] \hat{x}^a \hat{x}^b \Big] \psi^b = \epsilon\, \psi^a \tag{5.22}$$

where $p_r^2 = -\frac{1}{r}\frac{d^2}{dr^2}(r^2\ \ )$, $\vec{L} = -i\vec{r}\times\vec{\nabla}$ and $(T^a)_{bc} = -i\,\epsilon_{abc}$.

It is natural to define the operator $\vec{J} = \vec{L} + \vec{T}$, which commutes with the differential operator in (5.22), and consider an expansion in vector spherical harmonics with 'angular momentum' J . These harmonics and their properties are also used in appendix 3. Substituting the expansion

$$\vec{\Psi} = \sum_{j}\sum_{m}\left(\frac{1}{r}A_{jm}(r)\vec{Y}_{jjm} + \frac{1}{r}B_{jm}(r)\vec{Y}_{j,j+1,m} + \frac{1}{r}C_{jm}(r)\vec{Y}_{j,j-1,m}\right)$$

into (5.22), we can separate it into three coupled linear equations

$$-A''_{jm} + \frac{\alpha_0}{r^2}A_{jm} = \epsilon A_{jm}$$

$$-B''_{jm} + \frac{\alpha_+}{r^2}B_{jm} + \frac{(K^2-H^2-2K+1)}{r^2}a_+(B_{jm}a_+ + C_{jm}a_-) = \epsilon B_{jm}$$

$$-C''_{jm} + \frac{\alpha_-}{r^2}C_{jm} + \frac{(K^2-H^2-2K+1)}{r^2}a_-(B_{jm}a_+ + C_{jm}a_-) = \epsilon C_{jm}$$

(5.23)

with

$$\alpha_0 = (1-K)^2 + H^2 - 2(1-K) + j(j+1)$$

$$\alpha_+ = (1-K)^2 + H^2 + (j+2)(j-1+2K)$$

$$\alpha_- = (1-K)^2 + H^2 + (j-1)(j+2-2K)$$

$$a_+ = -(j+1/2j+1)^{1/2}$$

$$a_- = (j/2j+1)^{1/2}$$

$$H(r) = r\coth r - 1$$
$$K(r) = 2 - r/\sinh r$$

We have not been able to solve the coupled equations (5.23), however certain important properties like the total angular momentum of the ground state can be deduced without their explicit forms.

In order to discuss the rotational properties of the functions $\chi^{(b)jm}_{ia} = \eta_{bi\nu}\tilde{\nabla}_\nu \Psi^{jm}_a$, $\chi^{(b)}_a = \eta_{b0\nu}\tilde{\nabla}_\nu \Psi^{jm}_a$ for a given eigenvalue $\epsilon = \omega^2$ we anticipate a result from the next section, that the total angular momentum operator which generates space rotations in the space of gauge invariant variables, where the Gauss constraint (3.27) is valid, is

$$\hat{J}_k = -i\int d\vec{x}\; E^a_i \left(L^k \delta_{ab}\delta_{ij} + T^k_{ab}\delta_{ij} + S^k_{ij}\delta_{ab}\right) A^b_j - i\int d\vec{x}\; \pi^a \left(L^k \delta_{ab} + T^k_{ab}\right)\phi^b \tag{5.24}$$

$$L^k = -i\epsilon_{k\ell m}x_\ell \partial_m\,,\quad T^k_{ab} = -i\,\epsilon_{kab}\,,\quad S^k_{ij} = -i\epsilon_{kij}$$

We further assume that the zero modes of translation are fixed and that we are actually examining the rotational properties of the fluctuations in the centre of mass co-ordinate system without contradicting the fact that the space translation and rotation generators do not commute.

From (5.24) it follows that the infinitisimal action of the rotation group on the eigenfunctions $\chi_{ia}^{(n)}$ and $\chi_{a}^{(n)}$ is given by

$$\delta_K \chi_{ia}^{(n)\,jm} = (J^K)_{ij,ab}\, \chi_{jb}^{(n)\,jm}$$

$$\delta_K \chi_{a}^{(n)\,jm} = (j^k)_{ab}\, \chi_{b}^{(n)\,jm} \tag{5.25}$$

where $\vec{J} = \vec{L} + \vec{T} + \vec{S}$ and $\vec{j} = \vec{L} + \vec{T}$. It is convenient for the calculation to write the eigenfunctions in terms of the operators, $\vec{S}$, $\vec{T}$ and $\vec{P} = -i\vec{\nabla}$ as

$$\chi_{ia}^{(n)\,jm} = \left[(\vec{S}\cdot\vec{P})_{ni}\,\delta_{ac} + (\vec{S}\times\hat{x})_{ni}\,\vec{T}_{ac}\left(\frac{1-K}{r}\right) + i\,\delta_{ni}\,\vec{T}_{ac}\cdot\hat{x}\,\frac{H}{r}\right]\psi_c^{jm}$$

$$\chi_{a}^{(n)\,jm} = -i\left[P_n\,\delta_{ac} + (\vec{T}\times\hat{x})^n_{ac}\left(\frac{1-K}{r}\right)\right]\psi_c^{jm}$$

After some calculations we can prove that

$$\begin{aligned}
\delta_K \chi_{ia}^{(n)\,jm} &= \left[(\vec{S}\cdot\vec{P})_{ni}\,\delta_{ab} + (\vec{S}\times\hat{x})_{ni}\,\vec{T}_{ab}\left(\frac{1-K}{r}\right) + \delta_{ni}(\vec{T}\cdot\hat{x})_{ab}\,\frac{H(r)}{r}\right] \\
&\quad \times (j^K)_{bc}\,\psi_c^{jm} \\
&\quad + (S^k)_{ln}\,\chi_{ia}^{(l)\,jm} \\
&= \sum_{m'} (j^k)_{mm'}\,\chi_{ai}^{(n),\,jm'} + \sum_{l} (S^k)_{ln}\,\chi_{ia}^{(l)\,jm}
\end{aligned}$$

$$\begin{aligned}
\delta_K \chi_{a}^{(n)\,jm} &= -i\left(P_n\,\delta_{ab} + (\vec{T}\times\hat{x})^n_{ab}\right)(j^k)_{bc}\,\psi_c^{jm} \\
&\quad + (S^k)_{ln}\,\chi_{a}^{(n)\,jm}
\end{aligned}$$

$$= \sum_{m'} (\jmath^{k})_{mm'} \chi_a^{(n),\, \jmath m'} + \sum_{\ell} (S^{k})_{\ell n} \chi_a^{(\ell)\, \jmath m}$$

$(\jmath^{k})_{mm'}$ are the matrix elements of $\vec{\jmath}$ in the basis $|\jmath, m\rangle$ $(\vec{\jmath}^{\,2}|\jmath,m\rangle = \jmath(\jmath+1)|\jmath,m\rangle$, $\jmath^3|\jmath,m\rangle = m|\jmath,m\rangle)$.

This means that the indices 'm' and '(n)' transform independently under the action of the generators $\vec{\jmath}$ and $\vec{S}$. It should be possible to form linear combinations with Clebsch-Gordon coefficients, which transform irreducibly under the action of $\vec{J} = \vec{\jmath} + \vec{S}$; we denote these by χ_{ia}^{JM} and χ_a^{JM} , where $J = \jmath, \jmath \pm 1$, is the eigenvalue of $\vec{J}^2$ and M of J^3.

Angular momentum of ground state:

We have already proved that the only zero frequency modes of the fluctuation equation (5.9) are those corresponding to space translations; in particular there are no zero frequency modes corresponding to space rotations. This means that the angular momentum (5.24) commutes with the O(1) potential energy (5.7), unlike the linear momentum (4.15), and we may proceed to simultaneously diagonalize these without necessarily introducing collective co-ordinates.

Recalling that the gauge and translation invariant fluctuations satisfy the gauge conditions (4.7a), (4.7b) and (4.19), we may expand the fields in (5.24) in term of the non-zero modes

$$\vec{A}_i = \vec{A}_i^{u} + \sum_{JM} \left(a_{JM} \vec{\chi}_i^{JM} + a^{+}_{JM} \vec{\chi}_i^{*JM} \right)/\sqrt{2V\omega_J}$$

$$\vec{\Phi} = \vec{\Phi}^{u} + \sum_{JM} \left(a_{JM} \vec{\chi}^{JM} + a^{+}_{JM} \vec{\chi}^{*JM} \right)/\sqrt{2V\omega_J}$$

$$\vec{E}_i = -i \sum_{JM} \sqrt{\frac{\omega_J}{2V}} \left(a_{JM} \vec{\chi}_i^{JM} - a^{+}_{JM} \vec{\chi}_i^{*JM} \right)$$

$$\vec{\pi} = -i \sum_{JM} \sqrt{\frac{\omega_J}{2V}} \left(a_{JM} \vec{\chi}^{JM} - a^{+}_{JM} \vec{\chi}^{*JM} \right) \tag{5.26}$$

ω_J is the non-zero frequency; a_{JM} and a^{+}_{JM} are the annihilation and creation operators related to the co-ordinates and momenta in the expansions (4.24) in the usual manner; they satisfy the commutation rules

$$\left[a_{JM} , a^{+}_{J'M'} \right] = \delta_{JJ'} \delta_{MM'} \tag{5.27}$$

all other commutators are zero. In (5.26) we have not indicated the components of the momenta $\vec{E}_i$ and $\vec{\pi}$ normal to the gauge fixing surfaces (4.7a), (4.7b) and (4.19) since they do not contribute to $\hat{J}_K$.

Before we substitute the expansions (5.26) into we remark that $\hat{J}_K$ in (5.24) as it stands is not a hermetian operator in the quantum theory so we symmetrize it by adding the hermetian conjugate and adopt

$$J_S^K = \left(\hat{J}_K + \hat{J}_K^{+} \right)/2 \tag{5.28}$$

as the angular momentum in the quantum theory. (Similar remarks apply to the linear momentum in (4.15)).

Substituting (5.26) into (5.28) and noting that

$$(J^K)_{ab,ij} A^a_{bj} = 0 = (\mathcal{J}^K)_{ab} \phi^a_b$$

we have after some calculation

$$J^K_S = \sum_J \sum_{MM'} d^{KJ}_{MM'} \left(a^+_{JM} a_{JM'} + \tfrac{1}{2} \delta_{MM'} \right) \tag{5.29}$$

$$d^{KJ}_{MM'} = \langle JM | J^K | JM' \rangle$$

is the representation matrix of $J^K = L^K + T^K + S^K$ in the basis $|JM\rangle$; $\vec{J}^2 |JM\rangle = J(J+1)|JM\rangle$ and $J^3|JM\rangle = M|JM\rangle$.

The constant term in (5.29) is in fact zero if we note that

$$\sum_J \sum_{MM'} d^{KJ}_{MM'} \delta_{MM'} = \sum_J \left(\mathrm{Tr}\, d^{KJ} \right) = 0$$

and (5.29) is

$$J^K_S = \sum_J \left(\sum_{MM'} d^{KJ}_{MM'} a^+_{JM} a_{JM'} \right)$$

and

$$\vec{J}^2_S = \sum_K J^K_S J^K_S = \sum_K \sum_{JJ'} \sum_{MM',NN'} d^{KJ}_{MM'} d^{KJ'}_{NN'} a^+_M a^+_N a_{M'} a_{N'}$$

$$+ \sum_{J,M} J(J+1) a^+_{JM} a_{JM}$$

Now since the gaussian ground state is defined by the equation

$$a_{JM}|\hat{\Psi}_0\rangle = 0$$

it follows that

$$\langle\hat{\Psi}_0|\vec{J}_s^{\,2}|\hat{\Psi}_0\rangle = 0$$

i.e. the spin of the monopole state defined by $|\hat{\Psi}_0\rangle$ is zero. This conclusion seems to be contrary to a conjucture of Montonen and Olive[38] that the spin is one.

Section 6: Angular momentum in a gauge theory

We now discuss the derivation of (5.24) as the angular momentum which generates rotations in the space of gauge invariant variables, where the Gauss constraint (3.27) is valid. We give two derivations of the result.

First we consider the canonical generator:

$$J_K^{can} = \int d\vec{x}\left(E_i^a \epsilon_{K\ell m} x_\ell \partial_m A_i^a + E_i^a \epsilon_{Kij} A_j^a + \pi^a \epsilon_{K\ell m} x_\ell \partial_m \phi^a\right) \tag{6.1}$$

which generates via the commutation rules expected action of the rotation group in the space vector and scalar fields (see section 1)

$$A'_{ai}(\vec{x}) = R_{ij} A_{aj}(R^{-1}\vec{x}) \tag{6.2a}$$

$$\phi'_a(\vec{x}) = \phi_a(R^{-1}\vec{x}) \tag{6.2b}$$

where R is a rotation matrix.

From section 2 we recall that the boundary value of the Higgs field at each point 'w' of S^2_∞ is some fixed continuous gauge transformation of $\hat{x}_a$ (2.19a):

$$\hat{\phi}(w) = \Omega(w)\,\hat{x}(w) \quad , \Omega(w) \in SO(3)/U(1) \tag{6.3}$$

Further since we have fixed the fluctuation by the background gauge it is possible to choose $\Omega(w)=1$ at all points on S^2_∞ (2.19b)

$$\hat{\phi}(w) = \hat{x}(w) \tag{6.4}$$

Under the action of the rotation (6.2b) this boundary value undergoes a global isotopic spin rotation:

$$\hat{\phi}'_a = R^{-1}_{ab}\,\hat{x}_b$$

and in order to maintain the boundary condition (6.4) it is necessary to perform a compensating iso-spin rotation. Hence the true generator of rotations which leaves the Hamiltonian and boundary value of the Higgs field invariant is

$$\hat{J}_k = \overline{J}_k^{\,can.} + \overline{I}_k \tag{6.5}$$

where $I_k = \int d\vec{x}\,(E^a_i\, \epsilon_{kab} A^b_i + \pi^a \epsilon_{kab} \phi^b)$

is the generator of global isospin rotations.

Before we move on we remark that if the fluctuation is fixed by a gauge other than the background gauge it may not be possible to set $\Omega = \mathbb{1}$ at all points of S^2_∞ in (6.3). Then the action of a rotation on the boundary value of the Higgs field is a local gauge transformation

$$\hat{\Phi}'(w) = (\Omega(w)R^{-1})\hat{x}(w)$$

and in order to restore the boundary condition one performs a compensating local gauge transformation $\Omega(w)R\Omega^{\dagger}(w)$.

We now present a derivation of (6.5) starting with the gauge invariant angular momentum, defined in terms of the symmetric energy-momentum tensor $T_{\mu\nu} = \vec{F}_{\mu\lambda}\cdot\vec{F}_{\nu\lambda} + \nabla_\mu\vec{\Phi}\cdot\nabla_\nu\vec{\Phi} - \delta_{\mu\nu}\mathcal{L}$

$$J_K^{G.I.} = \epsilon_{K\ell m}\int d\vec{x}\; x_\ell\left[\vec{E}_i\cdot\vec{F}_{mi} + \vec{\pi}\cdot\nabla_m\vec{\Phi}\right] \tag{6.6}$$

which can be written as

$$J_K^{G.I} = J_K^{can.} + \int d\vec{x}\; \nabla_i\vec{\alpha}_K\cdot\vec{E}_i \tag{6.7}$$

The second term is the generator of a gauge transformation with

$$\vec{\alpha}_K = -\epsilon_{K\ell m}\, x_\ell\, \vec{A}_m$$

Upon partial integration (6.7) becomes

$$J_K^{G.I.} = J_K^{can.} + \int d^2\sigma_i\, \vec{E}_i\cdot\vec{\alpha}_K - \int d\vec{x}\; \vec{\alpha}_K\cdot\nabla_i\vec{E}_i \tag{6.8}$$

In order to evaluate the surface term which generates the global gauge rotation (if $\vec{\alpha}(\infty)_K \neq 0$), we recall that the most general form of the gauge field allowed by the finite energy condition (1.4b) for a <u>fixed</u> value $\hat{x}_a$ of the Higgs field is

$$A_{ai} = \epsilon_{aij}\hat{x}_j / gr + \hat{x}_a(\hat{x}\cdot\vec{A}_i)$$

Since the classical solution does not contribute to the second term it is part of the fluctuation and does not contribute to the surface integral in (6.8). Hence for in (6.8) we have

$$\lim_{r\to\infty} \alpha_{ak} = \frac{1}{g}\left(\delta_{ak} - \hat{x}_a\hat{x}_k\right)$$

Substituting into (6.8) we have

$$J_k^{G.I.} = J_k^{Can.} + \frac{1}{g}\int d^2\sigma_i E_i^k - \frac{1}{g}\int d^2\sigma_i \hat{x}\cdot\vec{E}_i \hat{x}_k - \int d\vec{x}\, \vec{\alpha}_k\cdot\nabla_i\vec{E}_i \qquad (6.9)$$

which gives on the physical states, using Gauss' law,

$$J_k^{G.I.}\Psi = \left(J_k^{Can.} + \int d\vec{x}\, E_i^a \epsilon_{kab} A_i^b\right)\Psi$$

which agrees with (6.5) provided

$$\int d^2\sigma_i \hat{x}_k\hat{x}_a E_i^a \Psi = 0 \qquad (6.10$$

Now the subsidiary condition (6.10) is necessary because the surface integral in (6.10) generates a global gauge rotation with asymptotic parameters $\hat{x}_k\hat{x}_a$ and such a pure gauge mode is not normalizable. This can be seen by considering the norm

$$\int d\vec{x}\left(\tilde{\nabla}_i\vec{\gamma}_k\cdot\nabla_i\vec{\gamma}_k + g^2|\vec{\Phi}_a\times\vec{\gamma}_k|^2\right)$$

with $\gamma_{ak}(r\to\infty) = \hat{x}_a\hat{x}_k$

the integrand as $r\to\infty$ can be computed to give $\frac{2}{r^2}$, which gives a divergent answer. Hence we obtain the same result for the action of the gauge invariant generator on the physical states as in (6.5).

Spin from iso-spin:

The phenomenon of having spin $\frac{1}{2}$ excitations in a field theory of bosonic fields was found by t'Hooft and Hasenfratz and by Jackiw and Rebbi.[21] We present a derivation of their results within our framework.

If besides the vector and Higgs fields, we had isospinor, Lorentz scalar fields 'ζ', the Hamiltonian (2.32) would have an additional term

$$\int d\vec{x} \left(\pi^{+}\pi + (D_i\zeta)^{+}(D_i\zeta) \right) \quad , D_i = \partial_i - i\frac{\vec{\tau}}{2}\cdot\vec{A}_i \tag{6.11}$$

π is the canonical momentum and τ_a are the Pauli matrices. We assume $\zeta = O(1)$, so that the monopole is still a soliton solution.

Repeating the previous discussion the angular momentum generator (6.5) would have an additional contribution from the iso-spinor.

$$\begin{aligned} \tilde{J}_K = \hat{J}_K &+ \int d\vec{x} \left(\pi \epsilon_{K\ell m} x_\ell \partial_m \zeta \;+\; c.c. \right) \\ &+ \int d\vec{x} \left(i\pi \frac{\tau^K}{2} \zeta \;+\; c.c. \right) \end{aligned} \tag{6.12}$$

The first of the additional terms is the usual orbital angular momentum, the second is iso-spin. This second part for the action of the gauge invariant generator on the physical states as in (6.5).

Spin from iso-spin:

The phenomenon of having spin ½ excitations in a field theory of bosonic fields was found by t'Hooft and Hasenfratz and by Jackiw and Rebbi.[21] We present a derivation of their results within our framework.

If besides the vector and Higgs fields, we had isospinor, Lorentz scalar fields 'ζ', the Hamiltonian (2.32) would have an additional term

$$\int d\vec{x}\left(\pi^{+}\pi + (D_i\zeta)^{+}(D_i\zeta)\right) \quad , D_i = \partial_i - i\frac{\vec{\tau}}{2}\cdot\vec{A}_i \tag{6.11}$$

π is the canonical momentum and τ_a are the Pauli matrices. We assume $\zeta = O(1)$, so that the monopole is still a soliton solution.

Repeating the previous discussion the angular momentum generator (6.5) would have an additional contribution from the iso-spinor.

$$\begin{aligned}\tilde{J}_K = \hat{J}_K &+ \int d\vec{x}\left(\pi\,\epsilon_{K\ell m}\,x_\ell\,\partial_m\zeta \;+\; c.c.\right)\\ &+ \int d\vec{x}\left(i\pi\,\frac{\tau^K}{2}\,\zeta \;+ c.c.\right)\end{aligned} \tag{6.12}$$

The first of the additional terms is the usual orbital angular momentum, the second is iso-spin. This second part gives rise to the possibility of realizing states with half-integer angular momentum in a field theory of Lorentz scalar and vector fields. We emphasize that this statement follows just from the necessity of having fixed boundary conditions for the Higgs field in a spontaneously broken theory.

We can diagonalize the Hamiltonian (6.11) in the external potential of the monopole solution by considering the linear equation

$$D_i^{\dagger}(\vec{A}_a)\, D_i(\vec{A}_a)\, \zeta = \nu^2 \zeta \tag{6.13}$$

since $D_i^{\dagger} D_i$ is a positive definite operator in the space of functions which vanish at infinity, (6.13) has no normalizable zero mode. The non-zero modes explicitly satisfy

$$\left(-\partial^2 + \frac{1}{2}\,\frac{(1-K)^2}{r^2} + 2\frac{(1-K)}{r}\,\frac{\vec{\tau}}{2}\cdot\vec{L}\right)\zeta = \nu^2 \zeta \tag{6.14}$$

The eigenfunctions in (6.14) may be chosen to be eigenstates of $\vec{J} = \vec{L} + \frac{\vec{\tau}}{2}$, since this operator commutes with $D_i^{\dagger} D_i$. We denote these eigenstates by $\zeta_{JM}(\vec{x})$, $J = \frac{1}{2}, 1, \frac{3}{2} \dots$ and the eigenvalues by ν_J, and consider an expansion of the fluctuations ζ and π in terms of creation and annihilation operators

$$\zeta = \sum_{JM} \frac{1}{\sqrt{2V\nu_J}} \left[b_{JM}\, \zeta_{JM} + b^{\dagger}_{JM}\, \zeta^{*}_{JM} \right]$$

$$\tilde{\Pi} = \sum_{JM} \frac{1}{i} \sqrt{\frac{\nu_J}{2V}} \left[b_{JM} \zeta_{JM} - b^{+}_{JM} \zeta^{*}_{JM} \right]$$

$[b_{JM}, b^{+}_{J'M'}] = \delta_{JJ'} \delta_{MM'}$ and all other commutators vanish.

We may once more evaluate the symmetrized angular momentum (see 5.29) and find

$$J^{k}_{S} = \sum_{J=1,2,\cdot} \sum_{MM'} d^{kJ}_{MM'} a^{+}_{JM} a_{JM'} + \sum_{J=\frac{1}{2},\frac{3}{2},\cdot} \sum_{M,M'} d^{kJ}_{MM'} b^{+}_{JM} b_{JM'}$$

The second sum is the additional contribution due to the isospinor and carries half integral angular momenta. The ground state defined by

$$a_{JM} |\Psi_0\rangle = b_{JM} |\Psi_0\rangle = 0$$

still carries zero spin since $\langle \hat{\Psi}_0 | \vec{J}^{2}_{S} | \hat{\Psi}_0 \rangle = 0$; however there do exist excited states (monopole, isospinor bound states)

$$b^{+}_{\frac{1}{2}} |\Psi_0\rangle , \; b^{+}_{\frac{3}{2}} |\Psi_0\rangle \; \cdots$$

which carry half integer angular momentum.

III: TOPOLOGICAL EXCITATIONS AND TUNNELLING IN NON-ABELIAN GAUGE THEORY IN THE SCHRÖDINGER PICTURE:

In this chapter we discuss topological excitations like instantons in euclidean space, and their relevance to tunnelling phenomena. In section 1 we review the work on the topology of gauge fields and instantons. In section 2 we present a canonical formulation of tunnelling phenomena with a detailed discussion of the role of euclidean time and the relevance of euclidean configurations in the Schrödinger picture. Section 3 is devoted to the topology of the gauge group and section 4 to a discussion of these phenomena in the axial gauge.

Section 1: Topology of Euclidean Yang-Mills fields in four dimensions

We present as background material to our discussion, a brief review of some developments in Euclidean field theory.

Consider the large time euclidean transition amplitude $\langle A|e^{-HT}|A'\rangle$, which is useful to study the ground state properties of the system since as $T\to\infty$

$$\langle A|e^{-HT}|A'\rangle \xrightarrow[T\to\infty]{} \exp(-E_0T)\langle A|0\rangle\langle 0|A'\rangle$$

Eo is the eigenvalue of H in the ground state $|0\rangle$. Polyakov[23] suggested that

$$\lim_{T\to\infty} \langle A| e^{-HT} |A'\rangle = \int d\mu(A)\, e^{-S[A]/g^2} \tag{1.1}$$

may be approximated for small coupling 'g^2', in the stationary phase approximation, by the minima of the euclidean action

$$S = -\frac{1}{2} \int d^4x \, Tr\left(F_{\mu\nu}(x) F_{\mu\nu}(x)\right) \tag{1.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad , \quad A_\mu = A^a_\mu \tau^a / 2i$$

τ^a are the Pauli matrices. The functional integral in (1.1) is over all gauge fields in a large euclidean box with $F_{\mu\nu} = 0$ on the boundary; $d\mu(A)$ is the path integral measure.

Belavin, Polyakov, Schwartz and Tyupkin[39] (BPST) discovered that euclidean gauge fields which satisfy the boundary condition $F_{\mu\nu} = 0$ or equivalently $A_\mu = \Omega^\dagger \partial_\mu \Omega$ on the boundary, fall into homotopy classes of the mapping

$$x_\mu \longmapsto \Omega(x_\mu) \tag{1.3}$$

x_μ is a point on the boundary of a large euclidean box which is topologically the 3-dimensional sphere S^3 and $\Omega(x)$ is a point in the SU(2) group manifold which is also S^3. These homotopy classes are specified by the Pontrjgin index which is an integer

$$\nu = \frac{1}{16\pi^2} \int d^4x \, Tr\left(F_{\mu\nu} {}^*F_{\mu\nu}\right)$$

$$\left({}^*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \text{ is the dual of } F_{\mu\nu}\right) \tag{1.4}$$

The topological meaning of (1.4) becomes more clear if one uses the identity

$$\frac{1}{16\pi^2}\mathrm{Tr}\left(F_{\mu\nu}{}^{*}F_{\mu\nu}\right) = \partial_\mu J_\mu^A$$

$$J_\mu^A = \frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left[A_\nu\partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta\right] \tag{1.5}$$

to write

$$\nu = \int d^4x\, \partial_\mu J_\mu^A = \int d^3\sigma_\mu\, J_\mu^A \tag{1.6}$$

which only depends on the boundary condition $F_{\mu\nu} = 0$, and can be expressed as

$$\nu = \frac{1}{24\pi^2}\int_{S_3} d^3\sigma_\mu\, \epsilon_{\mu\nu\alpha\beta}\, \mathrm{Tr}\left[\Omega^{+}\partial_\nu\Omega\, \Omega^{+}\partial_\alpha\Omega\, \Omega^{+}\partial_\beta\Omega\right] \tag{1.7}$$

$\nu = \frac{1}{24\pi^2}$

We have set $A_\mu = \Omega^{+}\partial_\mu\Omega$ on the boundary in (1.7). The integrand is precisely the jacobian of the mapping (1.3). Therefore the surface integral (1.7) is an integration over the manifold of SU(2), and since $24\pi^2$ is the group volume, ν is an integer, being the number of times the group manifold is covered by the mapping (1.3). Clearly (1.4) is invariant to continuous deformations $A_\mu \to A_\mu + \delta A_\mu$ which leave the boundary condition $F_{\mu\nu} = 0$ invariant.

The above topological considerations imply that the path integral in (1.1) is to be considered within each homotopy class and the full transition amplitude is given by (CDG ref. 24)

$$\sum_{\nu=-\infty}^{+\infty} e^{i\nu\theta}\int d\mu(A)_\nu\, e^{-S[A]/g^2} \tag{1.8}$$

$d\mu(A)_\nu$ is the measure in the ν^{th} homotopy class; $\exp(i\nu\theta)$ is a quantum mechanical phase factor and it is a representation of the homotopy group $\Pi_3(SU(2)) \cong \mathbb{Z}$ (the integers under addition). Using (1.4), (1.8) can be expressed as

$$\int d\mu(A)\, e^{-S_\theta/g^2}$$

$$S_\theta = -\frac{1}{2}\int d^4x \, \mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu} + i\theta g^2 F_{\mu\nu}{}^{*}F_{\mu\nu}\right) \tag{1.9}$$

From (1.9) it is suggested that the physical states and the energy eigenvalues are parametrized by the Bloch parameter $\theta \in [0, 2\pi)$. We shall elaborate on this point later on.

The stationary points of the action (1.2) with boundary condition $F_{\mu\nu}=0$, satisfy the euclidean equations

$$\nabla_\mu F_{\mu\nu} = 0 \quad , \quad \nabla_\mu = \partial_\mu + [A_\mu, \,] \tag{1.10}$$

The absolute minima of (1.2) which satisfy (1.10) are either self-dual or anti self-dual,

$$\pm F_{\mu\nu} = {}^{*}F_{\mu\nu} \tag{1.11}$$

This remarkable statement due to BPST[39] can be proved by saturating the inequality

$$-\int d^4x \, \mathrm{Tr}\left(F_{\mu\nu} \pm {}^{*}F_{\mu\nu}\right)^2 \geqslant 0 \iff S \geqslant \frac{1}{2}\left|\int d^4x \, \mathrm{Tr} F_{\mu\nu}{}^{*}F_{\mu\nu}\right| \tag{1.12}$$

Using (1.4) we have

$$8\pi^2 |\nu| \leq S$$

The absolute minimum of S within the topological sector $\pm\nu$ is given by $8\pi^2|\nu|$ implying (1.11).

In the $\nu=0$ sector of usual perturbation theory, the absolute minimum is $A_\mu=0$; in the $\nu=\pm 1$ sector, the absolute minimum is the BPST instanton or pseudo-particle with $S = 8\pi^2$. The explicit form of their solution is[39]

$$A^P_\mu(x) = x^2/(x^2+\rho^2) \cdot \omega^\dagger_{(1)}(x)\,\partial_\mu\,\omega_{(1)}(x) \tag{1.13}$$

$\omega_{(1)}(x) = (x_4 + i\vec{x}\cdot\vec{\tau})/\sqrt{|\vec{x}|^2+x_4^2}$ is an SU(2) group element with $\nu=1$; ρ is a scale parameter. It has been proved by Atiyah et. al.[40] that the solutions to the self duality equations exist for all values of the index ν ; hence the functional integral (1.8) really exists atleast in perturbation theory around these fields which are absolute minima of S.

<u>The Pontrjgin index in 'canonical form'</u>:

In the next section we will discuss the relevance of the euclidean field configurations in a canonical setting. Hence we wish to express the important attribute of the euclidean gauge field, its topological index (1.4), in 'canonical form'.

Defining $\vec{F}_{ij} = i\,\mathrm{Tr}(F_{ij}\vec{\tau})$ we can write (1.4) as

$$\nu = \frac{1}{8\pi^2}\int dx_4\, d\vec{x}\,\left(\partial_4\vec{A}_i - \nabla_i\vec{A}_4\right)\cdot\vec{B}_i \quad , \vec{B}_i = \tfrac{1}{2}\epsilon_{ijk}\vec{F}_{jk}$$

After a partial integration using $\nabla_i\vec{B}_i=0$ and the boundary condition $\vec{B}_i=0$

$$\nu = \frac{1}{8\pi^2}\int dx_4\, d\vec{x}\, \frac{\partial}{\partial x_4}\vec{A}_i \cdot \vec{B}_i \tag{1.14}$$

Note that the above expression is independent of A_4. If we define the functional

$$\chi[A] = -\frac{1}{8\pi^2}\int d\vec{x}\, \epsilon_{ijk}\, Tr\left[A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k\right]$$

we can prove that

$$\vec{B}_i(\vec{x}) = 8\pi^2 \frac{\delta \chi[A]}{\delta \vec{A}_i(\vec{x})} + \frac{1}{2\pi^2}\epsilon_{ijk}\int d\vec{y}\, \frac{\partial}{\partial y_j}\left[\vec{A}_k(\vec{y})\,\delta(\vec{x}-\vec{y})\right]$$

Therefore

$$\nu = \int dx_4 \int d\vec{x}\, \frac{\partial \vec{A}_i}{\partial x_4}\cdot\frac{\delta \chi}{\delta \vec{A}_i} + \frac{1}{16\pi^2}\int dx_4 \int d\vec{x}\, \frac{\partial}{\partial x_j}\left[\epsilon_{ijk}\, \vec{A}_k \cdot \frac{\partial}{\partial x_4}\vec{A}_i\right]$$

$$= \int dx_4 \frac{\partial}{\partial x_4}\chi + \frac{1}{16\pi^2}\int dx_4 \int d^2\sigma_j\, \epsilon_{ijk} \frac{\partial}{\partial x_4}\vec{A}_i \cdot \vec{A}_k$$

$$= \chi[x_4 = +\infty] - \chi[x_4 = -\infty] + \frac{1}{8\pi^2}\int dx_4 \int d^2\sigma_i\, \epsilon_{ijk}\, Tr[\partial_4 A_j A_k] \tag{1.15}$$

(1.15) is the expression for ν in canonical form.

Section 2: Canonical formulation and the meaning of Euclidean time in the Schrödinger picture

Polyakov's original suggestion[23] was to regard the euclidean functional integral (1.1) as the equilibrium partition function of a 4-dimensional system with energy $S[A]$ and temperature $T = g^2$. Subsequently t'Hooft, Callan, Dashen, Gross and Jackiw and Rebbi,[24] interpreted the instanton or for that matter any euclidean field configuration as a tunneling event in imaginary time.

However in the real time Schrödinger picture one is essentially solving the time independent Schrödinger equation for the stationary states and the significance of the euclidean time 'X_4' lies in parametrizing the tunnelling path in configuration space; the most probable of these in the WKB approximation is determined by the euclidean classical solution of the equations of motion. This important clarification is due to Gervais and Sakita.[25] We shall shortly explain their view point in the present context.

Canonical formulation:

Our discussion from this point will be in the Schrödinger picture we discussed at length in (II.2). We briefly sketch the arguments.

One begins with the real time action rewritten in Hamiltonian form

$$S = \int_{t_1}^{t_2} dt \int_V d\vec{x} \left(-\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu}\right)$$

$$= \int_{t_1}^{t_2} dt \left[\int_V d\vec{x}\left(\vec{E}_i\cdot\dot{\vec{A}}_i\right) - H\right]$$

$$H = \int_V d\vec{x}\left(\frac{\vec{E}_i^{\,2}}{2} + \frac{\vec{F}_{ij}^{\,2}}{4}\right) - \int_V d\vec{x}\,\vec{A}_0\cdot\nabla_i\vec{E}_i + \int_S d^2\sigma_i\,\vec{A}_0\cdot\vec{E}_i \tag{2.1}$$

$\vec{A}_i$ and $\vec{E}_i = \vec{F}_{0i}$ are the canonical variables and $\vec{A}_0$

appears as a Lagrange multiplier and is an arbitrary function.

The next step is the variational principle; consider

$$\delta S = \int_{t_1}^{t_2} dt \int_V d\vec{x} \left[\frac{\delta S}{\delta \vec{A}_i(\vec{x},t)} \cdot \delta \vec{A}_i(\vec{x},t) + \frac{\delta S}{\delta \vec{A}_0(\vec{x},t)} \cdot \delta \vec{A}_0(\vec{x},t) + \dots \right]$$

$$+ \int_{t_1}^{t_2} dt \int_S d^2\sigma_i \left[\frac{1}{2} \vec{F}_{ij} \cdot \delta \vec{A}_j + \vec{E}_i \cdot \delta \vec{A}_0 \right]$$

The equations of motion are true solutions to the variational problem provided the surface terms are zero. The discussion is similar to the one following (II - 2.5), except in the present case there are no higgs fields. The surface integral $\int_S \delta \vec{A}_i \cdot \vec{F}_{ij} d^2\sigma_j = 0$ because we impose the gauge invariant boundary condition $\vec{F}_{ij} = 0$ on the boundary. Since $\vec{B}_i = \frac{1}{2} \epsilon_{ijk} \vec{F}_{jk}$ and $\vec{E}_j$ do not commute as quantum mechanical operators, we may not set $\vec{E}_j = 0$ on the boundary; hence the surface term $\int_S \delta \vec{A}_0 \cdot \vec{E}_i d^2\sigma_i = 0$ only if $\delta \vec{A}_0 = 0$ on the boundary i.e. if the gauge group element $\Omega(|\vec{x}| \to \infty) = I$. If $\delta \vec{A}_0 \neq 0$ ($\Omega(|\vec{x}| \to \infty) \not\to I$) the surface term cannot be dropped and a consistent formulation necessitates (as explained in II-2), the introduction of the dynamical variable $\vec{I}$ which satisfies the constraint (II - 2.8b) and a local SU(2) algebra (II -2.15).

Summarizing this discussion we conclude that if the gauge group is trivial as $|\vec{x}| \to \infty$ the physical states satisfy the Schrödinger equation

$$\int d\vec{x} \left(\frac{\vec{E}_i^2}{2} + \frac{\vec{F}_{ij}^2}{4} \right) |\Psi\rangle = E \, |\Psi\rangle \tag{2.2}$$

and the subsidiary condition

$$\nabla_i \vec{E}_i \, |\Psi\rangle = 0 \tag{2.3}$$

which is equivalent to

$$\langle \vec{A}^{\Omega} | \Psi \rangle = \langle \vec{A} | \Psi \rangle \tag{2.4}$$

$\vec{A}^{\Omega}$ is a gauge transform of $\vec{A}$ with vanishing parameters as $|\vec{x}| \to \infty$.

If the gauge group is non-trivial as $|\vec{x}| \to \infty$ the physical states satisfy an additional subsidiary condition

$$\lim_{r \to \infty} r^2 \vec{E}_n |\Psi\rangle = g \vec{I} \, |\Psi\rangle \tag{2.5}$$

The configuration space is extended to include the parameters $\vec{\theta}$ of the SU(2) gauge group at spatial infinity; these are conjugate to $\vec{I}$ and the wave function is

$$\Psi[\vec{A};\vec{\theta}] = \{\langle \vec{A}_i | \otimes \langle \vec{\theta}_\infty | \} \, |\Psi\rangle \tag{2.6}$$

The role of euclidean time in the Schrödinger picture:

The following discussion is patterned after Gervais and Sakita.(25) Consider solving the equations (2.2) and (2.3) of pure Yang-Mills theory. In order to develop a WKB approximation in powers of $\hbar$, we scale the canonical variables by the coupling constant

$$A_{ai} \to \frac{1}{g} A_{ai}$$

$$E_{ai} \longrightarrow g\, E_{ai}$$

leaving the commutation relations $[E_{ai}, A_{bj}] = -i\hbar\delta_{ab}\delta_{ij}$ invariant. Then in the $|\vec{A}_i\rangle$ representation (2.2) and (2.3) become

$$\int d\vec{x} \left[-\frac{1}{2}(\hbar g^2)^2 \frac{\delta^2}{\delta A_{ai}(\vec{x})^2} + \frac{\vec{F}_{ij}^{\,2}}{4} \right] \Psi[\vec{A}] = g^2 E\, \Psi[\vec{A}] \tag{2.7}$$

$$(\partial_i + \vec{A}_i \times) \frac{\delta}{\delta \vec{A}_i} \Psi[\vec{A}] = 0 \tag{2.8}$$

Henceforth we set $\hbar = 1$ and note that the coupling constant has disappeared from the subsidiary condition (2.8).

As a first step to solve these equations in the WKB approximation set

$$\Psi = \exp\left[-W[\vec{A}]/g^2 \right]$$

(we are restricting our discussion to the tunnelling region $E < V = \frac{1}{4}\int d\vec{x}\, \vec{F}_{ij}^{\,2}$)

and consider the expansion in powers of 'g',

$$W = W_0 + g^2 W_1 + \cdots$$

$$g^2 E = E_0 + g^2 E_1 + \cdots$$

to obtain for (2.7)

$$\int d\vec{x} \left[-\frac{1}{2} \frac{\delta W_0}{\delta A_{ai}(\vec{x})} \frac{\delta W_0}{\delta A_{ai}(\vec{x})} \right] + V = E_0$$

$$\int d\vec{x} \left[-\frac{1}{2} \frac{\delta W_0}{\delta A_{ai}(\vec{x})} \frac{\delta W_1}{\delta A_{ai}(\vec{x})} + \frac{1}{2} \frac{\delta^2 W_0}{\delta A_{ai}(\vec{x})^2} \right] = E_1 \tag{2.9}$$

etc.

and for (2.8)

$$\nabla_i \frac{\delta}{\delta \vec{A}_i} W_0 = 0$$

$$\nabla_i \frac{\delta}{\delta \vec{A}_i} W_1 = 0 \tag{2.10}$$

etc.

The equation (2.9) for Wo, which represents the leading approximation, is the Hamilton-Jacobi equation in the infinite dimensional configuration space $\{ A_{ai}(\vec{x}) \}$ and (2.8) is the subsidiary condition on the solution. In general it is very hard to solve the Hamilton-Jacobi equation however it is possible to construct an explicit solution along tunnelling paths in configurations space.

Consider the infinitisimal path element $dA_{ai}(\vec{x})$ in configuration space and the path element $d\ell_a(\vec{x})$ in the gauge group defined by $d\ell_a(\vec{x}) = V^a_b(\vec{\theta}(\vec{x}))\, d\theta^b(\vec{x})$; $\theta_b(\vec{x})$ are parameters of the gauge group and $V^a_b(\vec{\theta}(\vec{x}))$ is defined in terms of the function which specifies the group law (II-2). Now define the gauge invariant path length in the product space of $\vec{A}_i$ and $\vec{\theta}$

$$(ds)^2 = \int d\vec{x} \left(d\vec{A}_i - \nabla_i d\vec{\ell} \right)^2 \tag{2.11}$$

Hence $\vec{A}_i(\vec{x})$ and $\vec{\theta}(\vec{x})$ can be parametrized by 's' such that

$$\int d\vec{x} \left(\frac{d\vec{A}_i}{ds} - \nabla_i \frac{d\vec{\ell}}{ds} \right)^2 = 1 \tag{2.12}$$

Since the function $W_0[A]$ by virtue of (2.14) carries a trivial representation of the gauge group we can as well write

$$-i\frac{\delta}{\delta A_{ai}(\vec{x})}W_0 = -i\left(\frac{\delta}{\delta A_{ai}(\vec{x})} - \nabla_i^{ab}V_c^b(\theta)\frac{\delta}{\delta\theta_c(\vec{x})}\right)$$

where $I^b(\vec{x}) = -iV_b^a(\theta)\frac{\delta}{\delta\theta_a(\vec{x})}$ is the generator of the gauge transformation in group space (II-2 and appendix 2). Hence along the path $A_{ai}(\vec{x};s)$, $\theta_a(\vec{x};s)$ we can write the above expression along the 'tangent vector'

$$\left(\frac{\partial\vec{A}_i}{\partial s} - \nabla_i\frac{d\vec{\ell}}{ds}\right) \quad \text{as}$$

$$-i\left(\frac{\delta}{\delta A_{ai}(\vec{x};s)} - \nabla_i^{ab}V_b^c\frac{\delta}{\delta\theta_c(\vec{x};s)}\right)W_0 = -i\left(\frac{\partial A_{ai}}{\partial s} - \nabla_i V_b^c\frac{d\theta_c}{ds}\right)\frac{\partial}{\partial s}$$

and along the path (2.9) and (2.10) become

$$-\frac{1}{2}\left(\frac{\partial W_0}{\partial s}\right)^2 + V = E_0 \tag{2.13}$$

$$\nabla_i\left(\frac{\partial}{\partial s}\vec{A}_i - \nabla_i\frac{d\vec{\ell}}{ds}\right)\frac{\partial W_0}{\partial s} = 0 \tag{2.14}$$

The subsidiary condition (2.14) will be identically satisfied for a non-trivial Wo iff the tangent vector along the path is covariantly constant i.e.

$$\nabla_i\left(\frac{\partial}{\partial s}\vec{A}_i - \nabla_i\frac{\partial\vec{\ell}}{\partial s}\right) = 0 \tag{2.15}$$

The solution of (2.13) is

$$W_0[\vec{A}] = \int^{s_1} ds\sqrt{2(V-E_0)}\ , \quad \vec{A}_i(\vec{x};s_1) = \vec{A}_i(\vec{x}) \tag{2.16}$$

The most probable tunnelling path, minimizes the action W_0(2.16) under the constraint (2.15); to find this stationary path we introduce a Lagrange multiplier $\vec{\alpha}(\vec{x};s)$ and vary

$$\hat{W}_0 = \int_{s_0}^{s_1} ds \sqrt{2(V-E_0)} + \int_{s_0}^{s_1} ds \int d\vec{x}\ \vec{\alpha}\cdot\nabla_i\left(\frac{\partial\vec{A}_i}{\partial s} - \nabla_i\frac{\partial\vec{\ell}}{\partial s}\right) \tag{2.17}$$

freely with respect to $\vec{A}_i$, $\vec{\alpha}$ and $\vec{\ell}$ with the variations $\delta\vec{A}_i$ and $\delta\vec{\ell}$ vanishing at the end points s_0 and s_1. The variation with respect to $\vec{\alpha}$ reproduces the constraint (2.15) as a stationary point. This accomplished, it is more convenient to write

$$\hat{W}_0 = \int_{s_0}^{s_1} ds \sqrt{2(V-E_0)} - \int_{s_0}^{s_1} ds \int d\vec{x}\ \nabla_i\vec{\alpha}\cdot\left(\frac{d\vec{A}_i}{ds} - \nabla_i\frac{d\vec{\ell}}{ds}\right) \tag{2.18}$$

The surface term $\int d^2\sigma_i \vec{\alpha}\cdot\frac{\partial}{\partial s}\vec{A}_i = 0$, because $\vec{A}_i$ is pure gauge on the boundary and the normal derivative of the group element is assumed to vanish there.

We define $C(s) = -\int d\vec{x}\ \nabla_i\vec{\alpha}\cdot\left(\frac{\partial\vec{A}_i}{\partial s} - \nabla_i\frac{\partial\vec{\ell}}{\partial s}\right)$

then

$$\delta\hat{W}_0 = \int_{s_0}^{s_1} d(\delta s)\sqrt{2(V-E_0)} + \int_{s_0}^{s_1} ds\left[\delta\sqrt{2(V-E_0)} + \delta C\right],$$

in order to compute this we need $d(\delta s)$ which is found from

$$(ds)^2 = \int d\vec{x}\ \left(d\vec{A}_i - \nabla_i d\vec{\ell}\right)^2$$

$$= \int d\vec{x}\ \left[(d\vec{A}_i)^2 - (\nabla_i d\vec{\ell})^2\right]$$

$$d(\delta s) = ds \int d\vec{x}\left[\frac{d\vec{A}_i}{ds}\cdot\frac{d}{ds}\delta\vec{A}_i + \nabla_i\frac{d\vec{\ell}}{ds}\times\frac{d\vec{\ell}}{ds}\cdot\delta\vec{A}_i - \nabla_i\nabla_i\frac{d\vec{\ell}}{ds}\cdot\frac{d}{ds}\delta\vec{\ell}\right]$$

$\frac{\delta\hat{W}_0}{\delta\vec{A}_i}$ and $\frac{\delta\hat{W}_0}{\delta\vec{\ell}}$ are readily computed to give

$$\nabla_j \vec{F}_{ij} + \sqrt{2(V-E_0)}\,\frac{d}{ds}\left(\nabla_i \vec{\alpha}\right) - \sqrt{2(V-E_0)}\,\vec{\alpha}\times\frac{d}{ds}\vec{A}_i$$
$$- \sqrt{2(V-E_0)}\,\frac{d}{ds}\left(\sqrt{2(V-E_0)}\,\frac{d}{ds}\vec{A}_i\right) + 2(V-E_0)\nabla_i\frac{d\vec{\ell}}{ds}\times\frac{d\vec{\ell}}{ds} = 0 \tag{2.19}$$

and

$$\sqrt{2(V-E_0)}\,\frac{d}{ds}\left[\nabla_i\nabla_i\sqrt{2(V-E_0)}\,\frac{d\vec{\ell}}{ds}\right] + \sqrt{2(V-E_0)}\,\frac{d}{ds}\left[\nabla_i\nabla_i\vec{\alpha}\right] = 0 \tag{2.20}$$

Introducing a new parametrization defined by

$$\frac{d\tau(s)}{ds} = \sqrt{2(V-E_0)}$$

(2.19) and (2.20) become

$$\nabla_j \vec{F}_{ij} + \frac{d}{d\tau}\left(\nabla_i\vec{\alpha}\right) - \vec{\alpha}\times\frac{d}{d\tau}\vec{A}_i - \frac{d^2}{d\tau^2}\vec{A}_i + \nabla_i\frac{d\vec{\ell}}{d\tau}\times\frac{d\vec{\ell}}{d\tau} = 0 \tag{2.21}$$

$$\frac{d}{d\tau}\left[\nabla_i\nabla_i\left(\frac{d\vec{\ell}}{d\tau} + \vec{\alpha}\right)\right] = 0 \tag{2.22}$$

The only solution to (2.22) consistent with the invertibility of $\nabla_i\nabla_i$ is $\vec{\alpha} = -\frac{d\vec{\ell}}{d\tau}$ and we choose it. Substituting this into (2.21) we get

$$\frac{d^2\vec{A}_i}{d\tau^2} - \frac{d}{d\tau}\left(\nabla_i\vec{\alpha}\right) + \vec{\alpha}\times\frac{d\vec{A}_i}{d\tau} + \nabla_i\vec{\alpha}\times\vec{\alpha} + \nabla_j\vec{F}_{ji} = 0$$

which is

$$\nabla_\tau F_{\tau i} + \nabla_j F_{ji} = 0 \tag{2.23}$$

We have defined $\nabla_\tau = \partial_\tau + \vec{A}_\tau\times$, $\vec{A}_\tau \equiv \vec{\alpha}$ and $\vec{F}_{\tau i} = \partial_\tau\vec{A}_i - \nabla_i\vec{A}_\tau$

Using the same notation for (2.15) we arrive at the equations

$$\nabla_\tau F_{\tau i} + \nabla_j F_{ji} = 0$$

$$\nabla_i F_{i\tau} = 0 \tag{2.24}$$

for the most probable tunnelling path in the extended configuration space of $\vec{A}_i(\vec{x})$ and $\vec{\theta}(\vec{x})$. These are identical to the euclidean equations (1.10) provided we formally identify the path parameter τ with the euclidean time X_4. This then is the meaning of euclidean time in the Schrödinger picture.

Now that we have solved the Hamilton-Jacobi equation (2.9) and subsidiary condition (2.10) along the most probable tunnelling path parametrized by τ , we can set the element of the gauge group $U(\bar{x};\tau) = \exp\left(i\vec{\tau}\cdot\vec{\theta}(\bar{x};\tau)/2\right)$ along the tunnelling path to be the identity. This is always possible and convenient after solving the variational problem, since the gauge transformation

$$A_i(\bar{x};\tau) \rightarrow A_i(\bar{x};\tau)^{\tilde{U}(\bar{x};\tau)}$$
$$A_\tau(\bar{x};\tau) \rightarrow A_\tau(\bar{x};\tau)^{\tilde{U}(\bar{x};\tau)}$$

is a symmetry of equations (2.24) which now become

$$\frac{d^2}{d\tau^2}\vec{A}_i + \nabla_j \vec{F}_{ji} = 0$$
$$\nabla_i \frac{d}{d\tau}\vec{A}_i = 0 \tag{2.25}$$

The classical action (2.16) now reads

$$W_0[\vec{A}] = \int^{\tau} d\tau' \int d\vec{x} \left(\frac{d\vec{A}_i}{d\tau}\right)^2 \tag{2.26}$$

It can be rewritten as

$$W_0[\vec{A}] = \int^{\tau} d\tau \left[\int d\vec{x} \left\{ \frac{1}{2}\left(\frac{d\vec{A}_i}{d\tau}\right)^2 + \frac{1}{4}\vec{F}_{ij}\cdot\vec{F}_{ij} \right\} \right] - E_0\tau \tag{2.27}$$

Since $\int^{\tau} d\tau \int d\vec{x} \left(\frac{d\vec{A}_i}{d\tau} \pm \vec{B}_i\right)^2 \geqslant 0$

we have the inequality

$$W_0[\vec{A}] \geqslant \left| \int^{\tau} d\tau' \int d\vec{x} \, \frac{d}{d\tau}\vec{A}_i\cdot\vec{B}_i \right| - E_0\tau$$

and the minimum value of W_0 is reached along paths which satisfy

$$\frac{d\vec{A}_i}{d\tau} = \pm \vec{B}_i \tag{2.28}$$

which is equivalent to $E_0=0$. Equations (2.28) are the self-dual equations obtained by setting $A_4=0$ in (1.11). Note that (2.28) identically satisfies (2.25). The BPST instanton solution of (2.28) can be obtained from (1.13) by setting $A_4=0$ by a τ dependent gauge transformation. We simply present the final answer[41]

$$A_i^P(\vec{x};\tau) = \frac{1}{|\vec{x}|^2+\rho^2+\tau^2}\left[\rho^2 V^{\dagger}\partial_i V + (|\vec{x}|^2+\rho^2+\tau^2)(\omega_{(1)}V)^{\dagger}\partial_i(\omega_{(1)}V)\right]$$

$$\omega_{(1)} = \frac{\tau + \vec{x}\cdot\vec{\tau}}{\sqrt{|\vec{x}|^2+\tau^2}} \quad , V = \exp\left(i\,\hat{x}\cdot\frac{\vec{\tau}}{2}\,f(|\vec{x}|;\tau)\right)$$

$$f(|\vec{x}|;\tau) = \frac{-2|\vec{x}|}{\sqrt{|\vec{x}|^2+\rho^2}}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{\tau}{\sqrt{|\vec{x}|^2+\rho^2}}\right)\right) \tag{2.29}$$

Section 3: Topology of the gauge group in the semi-classical approximation

In the last section we saw that the leading contribution to the wave function in the WKB or semi-classical approximation is given by

$$\Psi \sim \exp\left(-\frac{1}{g^2} W_0[A]\right) \tag{3.1}$$

where $$W_0[A] = \int^{\tau} d\tau' \int d\vec{x} \left(\frac{d\vec{A}_i}{d\tau}\right)^2 \tag{3.2}$$

and the path $A_{ai}(\bar{x};\tau)$ satisfies

$$-\frac{1}{2}\int d\vec{x}\left(\frac{d}{d\tau}\vec{A}_i\right)^2 + V = E_0$$

and $\nabla_i \frac{d}{d\tau}\vec{A}_i = 0$

In order that (3.1) be meaningful we must have $W_0[A]<\infty$, which means that

$$\frac{d\vec{A}_i}{d\tau}(\bar{x};\tau) \longrightarrow 0 \qquad \text{as} \qquad |\bar{x}| \to \infty \tag{3.3}$$

at least as fast as $|\vec{x}|^{-\frac{3}{2}+\epsilon}$, where $\epsilon>0$. The boundary condition $\vec{F}_{ij}=0$ is equivalent to $A_i=\Omega^{\dagger}\partial_i\Omega$; therefore (3.3) implies $\frac{d}{d\tau}\Omega(|\bar{x}|\to\infty;\tau)=0$.
If we choose (and we can) $\Omega(|\bar{x}|\to\infty)=I$ for the initial value of the parameter τ in (3.1), then such a boundary condition is true for all τ. This has the following consequences:

(a) $\Omega(|\vec{x}|\to\infty)=I$ implies $\delta\vec{A}_0=0$ on the boundary and the surface term in the variation of the action (2.1),

$\int d^2\sigma_i\, \delta\vec{A}_0\cdot\vec{E}_i = 0$; the asymptotic gauge group continuously connected to the identity being trivial does not enter the specification of the wave function.

(b) The gauge group which satisfies the boundary condition $\Omega(|\vec{x}|\to\infty)=I$ is partitioned into homotopy classes of the mapping $\vec{x}\mapsto\Omega(\vec{x})\in SU(2)$ specified by the integer[24]

$$n[\Omega] = \frac{1}{24\pi^2}\int d\vec{x}\, \epsilon_{ijk}\, Tr\left[\Omega^{\dagger}\partial_i\Omega\,\Omega^{\dagger}\partial_j\Omega\,\Omega^{\dagger}\partial_k\Omega\right] \qquad (3.4)$$

Since explicit representatives of each class are given by

$$\Omega_n = exp\left[-in\pi\,\vec{x}\cdot\vec{\tau}\Big/\sqrt{|\vec{x}|^2+\rho^2}\right] \qquad (3.5)$$

a general element of the n^{th} homotopy class is given by

$$U_n(\vec{x}) = \Omega_n(\vec{x})\tilde{\Omega}(\vec{x}) \quad , \quad n[\tilde{\Omega}]=0 \qquad (3.6)$$

$\tilde{\Omega}(\vec{x})\in SU(2)$ has vanishing parameters as $|\vec{x}|\to\infty$.

From (3.6) we can construct the classical vacua (zero potential energy pure gauge configurations)

$$A_i^{(n)}(\vec{x}) = U_n^{\dagger}(\vec{x})\,\partial_i\, U_n(\vec{x}) \qquad (3.7)$$

A tunnelling path $A_i(\vec{x};\tau)$ interpolates between the vacua (3.7) from $\tau=-\infty$ to $\tau=+\infty$. The Pontrjgin index associated with the tunnelling path $A_i(\vec{x};\tau)$ is given by

$$\nu = \chi[\tau=+\infty] - \chi[\tau=-\infty] + \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} d\tau \int d^2\sigma_i\left[Tr\,\partial_\tau A_j A_k \epsilon_{ijk}\right]$$

In the present case the surface term vanishes since $A_i(|\vec{x}|\to\infty)\sim$ $o(|\vec{x}|^{-1-\epsilon})$, $\frac{\partial}{\partial\tau}A_i(|\vec{x}|\to\infty)\sim o(|\vec{x}|^{-3/2-\epsilon})$ and we have

$$\nu = \chi(\tau=+\infty) - \chi(\tau=-\infty)$$

Since $\lim_{\tau\to\pm\infty} A_i(\vec{x};\tau) = A_i(\vec{x})^{n\pm}$, where $A_i(\vec{x})^{n\pm}$ are given by (3.7)

$$\nu = n[U_{n+}] - n[U_{n-}]$$

$$= n[\Omega_{n+}] - n[\Omega_{n-}] \quad \text{since} \quad n[\tilde{\Omega}]=0$$

The BPST instanton (2.29) represents the most probable tunnelling path with $\nu=\pm 1$ between the asymptotic configurations

$$\lim_{\tau\to-\infty} A_i^p(\vec{x};\tau) = 0 \quad \text{and} \quad \lim_{\tau\to+\infty} A_i^p(\vec{x};\tau) = \Omega_1^\dagger \partial_i \Omega_1$$

Now we would like to determine the representation the wave function carries with respect to the gauge transformation U_n in (3.6). Let us define the operator which implements such a gauge transformation in the space of physical states by

$$T_n \Psi[\vec{A}_i] = \langle \vec{A}_i | T_n | \Psi \rangle = \langle \vec{A}_i^{U_n} | \Psi \rangle = \Psi[\vec{A}_i^{U_n}] \tag{3.8}$$

$\vec{A}_i^{U_n}(\vec{x})$ is a gauge transformation of $\vec{A}_i(\vec{x})$. By Gauss' law (2.3), (2.4), the wave function is invariant to gauge transformations whose parameters vanish as $|\vec{x}|\to\infty$ or equivalently have zero topological index (3.4). Since

$$U_n = \Omega_n \tilde{\Omega} \quad , \quad n[\tilde{\Omega}] = 0$$

we have

$$T_n \Psi[\vec{A}] = \Psi[\vec{A}^{\Omega_n \tilde{\Omega}}]$$

$$= \Psi[\vec{A}^{\Omega_n}] \neq \Psi[\vec{A}]$$

From $\Omega_n \Omega_m = \Omega_{n+m}$ it follows that $T_n \cdot T_m \approx T_{n+m}$ in the space of functions which satisfy Gauss' law. It can be shown that the adjoint of T_n, $T_n^{\dagger}$ satisfies

$$T_n^{\dagger} \Psi[\vec{A}] = \Psi[\vec{A}^{\Omega_n^{\dagger}}]$$

hence $T_n^{\dagger} T_n \approx 1$ i.e. T_n is an unitary operator. Since T_n commutes with the Hamiltonian we may choose Ψ to be an eigenstate of T_1 with eigenvalue $e^{i\theta}$

$$T_1 \Psi_\theta = e^{i\theta} \Psi_\theta \tag{3.9}$$

it follows that $T_n \Psi_\theta = (T_1)^n \Psi_\theta = e^{in\theta} \Psi_\theta$. In other words the wave function Ψ_θ, $\nabla_i E_i \Psi_\theta = 0$, is a Bloch wave in that it carries a unitary representation of the discreet group $T_n \cdot T_m \approx T_{n+m}$. (24) (15)

As we already noted the wave function is still invariant to the gauge transformation $A_i \to A_i^{\tilde{\Omega}}$, $n[\tilde{\Omega}] = 0$. At this stage one would like to realize this constraint by transforming to a set of variables which are invariant to these gauge transformations, by fixing a gauge $F[A] = 0$,

$$A_i = \tilde{A}_i^{\tilde{\Omega}} \quad , \quad F[\tilde{A}] = 0 \tag{3.10}$$

The gauge transformation $\tilde{\Omega}$ is to be determined as a solution of $F[A^{\tilde{\Omega}^{-1}}]=0$. We have already discussed these procedures in detail in chapter II. The main point here is that the gauge condition $F[A]=0$ must be such that the solution of the functional equation $F[A^{\tilde{\Omega}^{-1}}]=0$ written as $\tilde{\Omega}[A;\vec{x}]$ must be consistent with the boundary condition

$$\lim_{|\vec{x}|\to\infty} \tilde{\Omega}[A(\vec{x};\tau);\vec{x}] = I \quad \underline{\text{and}} \quad \lim_{\tau\to\pm\infty} n[\tilde{\Omega}] = 0 \tag{3.11}$$

An example of such a gauge condition in perturbation theory around the BPST tunnelling path is the background gauge,(15)

$$\partial_i[\vec{A}_i - \vec{A}_i^P] + \vec{A}_i^P \times [\vec{A}_i - \vec{A}_i^P] = 0$$

In fact it is impossible to have a single gauge condition in all of configuration space which satisfies the requirement (3.11).

Section 4: Axial gauge

We now confront ourselves with the possibility of choosing $F[A]=A_3=0$, the axial gauge. Perform the gauge transformation

$$A_i = U\tilde{A}_i U^\dagger + U\partial_i U^\dagger \quad , \quad \tilde{A}_3 = 0 \tag{4.1}$$

In order to eliminate the gauge freedom due to X_1 and X_2 dependent gauge transformations we choose to impose the boundary conditions $A_i(X_1 X_2 X_3 = -L) = 0$, on the boundary of a large box of size 2L, (See II-3). U in (4.1) satisfies the equation

$$\frac{\partial}{\partial X_3} U + A_3 U = 0$$

with solution

$$U(\bar{x};\tau) = P\exp\left[-\int_{-L}^{x_3} A_3(x_1 x_2 x_3';\tau)\right] V(x_1 x_2)$$

The arbitrary matrix $V(X_1X_2)$ can be determined from the boundary gauge condition $\tilde{A}_{1,2}(X_3=-L)=0$:

$$\tilde{A}_i(x_1 x_2 -L) = V^{+}A_i(x_1 x_2 x_3=-L)V + V^{+}\partial_i V = 0$$

But since $A_i(x_1x_2x_3) \longrightarrow I\ \partial_i I$ as $|\vec{x}| \to \infty$ we can set $A_i(x_1 x_2 x_3=-L) = 0$ giving $\partial_{1,2}V=0$ i.e. the gauge transformation in (4.1) is given by

$$U(\bar{x};\tau) = P\exp\left[-\int_{-L}^{X_3} A_3(x_1 x_2 x_3';\tau)\,dx_3'\right]$$

upto a constant which we can set to I.

It is clear that $U(\bar{x};\tau)$ has the following properties

$$\lim_{X_3\to L} U(\bar{x};\tau) = P\exp\left[-\int_{-L}^{+L} A_3\,dX_3\right] \not\to I \quad \text{as } L\to\infty \tag{4.2}$$

$$\lim_{X_3\to L} \tilde{A}_{1,2}(\bar{x};\tau) \neq 0 \quad , \quad L\to+\infty \tag{4.3}$$

Without loss of generality we now assume that $\tilde{A}_{1,2}(\bar{x};\tau\to-\infty)=0$; then using (4.1) we have $U(\bar{x};\tau\to+\infty)=\Omega_n(\bar{x})$. From these considerations it follows that

$$\tilde{A}_{1,2}(\bar{x};\tau=\pm\infty)=0 \quad \text{and} \quad n[U(\tau\to+\infty)] = n \neq 0$$

violating the conditions (3.11) which were imposed to realize Gauss' law. We emphasize that transforming to the axial gauge, though a legitimate symmetry operation of the Hamiltonian does not leave the wave function invariant

$$\Psi_\theta[\vec{A}] = \Psi_\theta[\tilde{A}^U] \neq \Psi_\theta[\tilde{A}] \quad , \tilde{A}_3 = 0$$

and it is not legitimate to implement it within the discussion of section 3.

It is quite clear that in order to admit the axial gauge one will have to work within an extended configuration space which includes the parameters of the gauge group on the boundary of the large box as dynamical variables. In this configuration space as we have seen one has the additional subsidiary condition (2.5) which together with (2.3) gives rise to a generalized Gauss' law

$$\left(\int_V d\vec{x}\, \vec{E}_i \cdot \nabla_i \vec{\Lambda} - \int_S d\omega\, \vec{I} \cdot \vec{\Lambda} \right) |\Psi\rangle = 0$$

$\vec{\Lambda}(\vec{x})$ is an arbitrary function. This is saying that $|\Psi\rangle$ is invariant to any gauge transformation.

In order to obtain a description of the tunnelling in this case where the classical vacua are necessarily $\tilde{A}_i(\vec{x}, \tau = \pm\infty) = 0$, we consider the gauge invariant Pontrjgin index (1.15) which is associated with the tunnelling path,

$$\nu = \frac{1}{8\pi^2} \int d\tau \int d^2\sigma_i\, \epsilon_{ijk} \operatorname{Tr}[\partial_\tau \tilde{A}_j \tilde{A}_k]$$

Since $\tilde{A}_3 = 0$ we have $\chi[\tau = \pm\infty] = 0$; for the same reason and $\tilde{A}_{1,2}(x_3 = -L) = 0$,

$$\nu = \frac{1}{8\pi^2}\int d\tau \int dx_1\, dx_2\, \epsilon_{3ij}\, \mathrm{Tr}\left[\partial_\tau \tilde{A}_j\, \tilde{A}_k\right]_{x_3=L} \tag{4.4}$$

where the fields in the above surface integral are evaluated on $x_3 = +L$. Since

$$\tilde{A}_{1,2} \doteq U^{+}\partial_i U \quad \text{as} \quad x_3 \to L \tag{4.5}$$

$$U(x_1 x_2\, x_3 = L; \tau) = P\exp\left[-\int_{-L}^{L} A_3(x_1 x_2 x_3'; \tau)\, dx_3'\right]$$

$$\nu = \frac{1}{8\pi^2}\int d\tau\, dx_1\, dx_2\, \mathrm{Tr}\left[U^{+}\partial_\tau U\, U^{+}\partial_i U\, U^{+}\partial_j U\right]\epsilon_{3ij}$$
$$= \frac{1}{24\pi^2}\int d\tau\, dx_1 dx_2\, \mathrm{Tr}\left[U^{+}\partial_i U\, U^{+}\partial_j U\, U^{+}\partial_k U\right]\epsilon_{ijk}$$

i, j, k run over $1, 2, \tau$. We can also verify that $U \to I$ as $\tau, x_1, x_2 \to \pm\infty$ which by itself implies that ν is an integer and is the winding number of the mapping

$$(\tau, x_1, x_2\,;\, x_3 = +L) \to U(\tau; x_1 x_2) \in SU(2) \tag{4.6}$$

ν can be written as

$$\nu = \int_{-\infty}^{+\infty} d\tau\, \frac{d}{d\tau}\Sigma(\tau) \tag{4.7}$$

$$\Sigma(\tau) = \frac{1}{24\pi^2}\int_{-\infty}^{\tau} d\tau' dx_1 dx_2\, \epsilon_{ijk}\, \mathrm{Tr}\left[U^{+}\partial_i U U^{+}\partial_j U U^{+}\partial_k U\right] \tag{4.8}$$

is the amount of the SU(2) group volume covered at τ by the mapping (4.6).

The above picture is similar to the example of a simple pendulum in a gravitational field. The configuration space of this system is the circle $x^2 + (y-\ell)^2 = \ell^2$; ℓ is some constant. We assume that gravity points in the negative direction. Then the unique classical ground state is $X = 0 = Y$. However quantum mechanically there is a possibility of the tunnelling around the circle: if we denote $\vec{X}_a(\tau)$ the tunnelling configuration then

$$\vec{X}_a(\tau = -\infty) = 0 = \vec{X}_a(\tau = +\infty) \tag{4.9}$$

However the fact that the pendulum has gone around a trajectory which is topologically distinct from the trivial trajectory $\vec{X}=0$, causes a phase change in the wave function where the phase parameter is a free parameter of the theory. A natural way to describe this system would be to use the angle $\alpha = \tan^{-1}\left(\frac{Y-\ell}{X}\right)$, as a dynamical variable since the topologically distinct paths are naturally described by such a variable.

Now if we choose $\alpha(\tau = -\infty) = 0$, then

$$\alpha(\tau) = \frac{1}{2\pi}\int_0^{\alpha(\tau)} d\alpha' \tag{4.10}$$

is the amount of the U(1) 'group volume' covered at 'time' τ. Clearly the topological index for ν traverses is

$$\nu = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \dot{\alpha}(\tau) = \alpha(\tau = +\infty) \tag{4.11}$$

The wave function of the system carries a representation specified by a parameter 'β' of the homotopy group $\Pi_1(U(1)) \cong \mathbb{Z}$

$$\psi_\beta(\alpha + 2\pi\nu) = e^{i\nu\beta}\,\psi_\beta(\alpha) \tag{4.12}$$

Even though the classical ground states $\vec{X} = 0$, $\alpha = 2n\pi$, $n = 0, \pm 1, \cdots$

are degenerate, these are quantum mechanically distinct, and the system is equivalent to a particle moving in a periodic potential with period 2π. In this sense Ψ_β is a Bloch wave function.

The analogue of (4.9) in the non-abelian gauge theory in the axial gauge is $\tilde{A}_i(\vec{x}; \tau = \pm\infty) = 0$. The generalization of formula (4.10) is (4.8); $\Sigma(\tau)$ is the analogue of $\alpha(\tau)$ and one has the corresponding formulae (4.11) and (4.7) for the topological indices of the homotopy groups $\Pi_1(U(1)) \cong \mathbb{Z}$ and $\Pi_3(SU(2)) \cong \mathbb{Z}$ respectively. It naturally follows that the wave function of the non-abelian gauge theory must depend on the surface gauge parameters through the combination $\Sigma(\tau)$; it carries a representation of the homotopy group $\Pi_3(SU(2)) \cong \mathbb{Z}$ specified by an angle 'θ'

$$\Psi_\theta[\vec{A}; \Sigma + \nu] = e^{i\nu\theta}\,\Psi[\vec{A}; \Sigma]$$

ν is an integer.

Appendix 1: Derrick's theorem[4]

Consider a scalar field $\vec{\phi}(\vec{x})$ in n space dimensions and suppose $\vec{\phi}_0(\vec{x})$ is the static solution that is a stationary point of the energy function

$$E[\vec{\phi}] = \int d^n\vec{x}\,(\partial_i\vec{\phi})^2 + \int d^n\vec{x}\, U(\vec{\phi})$$

we assume $V = \int d^n\vec{x}\, U(\vec{\phi}) > 0$ and obviously $T = \int d^n\vec{x}\,(\partial_i\vec{\phi})^2 > 0$.

Now consider the scaled field

$$\vec{\phi}_\lambda(\vec{x}) = \vec{\phi}_0(\lambda\vec{x})$$

then

$$E(\lambda) = \lambda^{2-n}\,T + \lambda^{-n}\,V$$

and since $\vec{\phi}_0$ is a stationary point of $E[\vec{\phi}]$ we must have

$$\left(\frac{\partial E}{\partial \lambda}\right)_{\lambda=1} = 0 = V - \frac{2-n}{n}\,T$$

Since $V>0$ and $T>0$ the only solution is $n=1$. For field theories involving only scalar fields static finite energy solutions exist only in one space dimension.

The above scaling trick can also be used to prove that the monopole mass is given by (II-4.11). For consider the static energy (4.6)

$$\begin{aligned} M_0 &= \int d\vec{x}\left(\tfrac{1}{4}\vec{F}^a_{ij}\cdot\vec{F}^a_{ij} + \tfrac{1}{2}\tilde{\nabla}_i\vec{\phi}_a\cdot\tilde{\nabla}_i\vec{\phi}_a + \tfrac{\lambda}{4}\left(\vec{\phi}_a^2 - \tfrac{\mu^2}{\lambda}\right)\right) \\ &\equiv T_1 + T_2 + T_3 \end{aligned} \tag{A-I.1}$$

and define the scaled fields

$$\vec{A}_i^{\rho}(\vec{x}) = \rho\, \vec{A}_i^{a}(\rho\vec{x}) \qquad\qquad \rho > 0$$
$$\vec{\Phi}^{\rho}(\vec{x}) = \vec{\Phi}^{a}(\rho\vec{x})$$

then

$$M(\rho) = \rho T_1 + \rho^{-1} T_2 + \rho^{-3} T_3$$

and $\left(\frac{\partial M(\rho)}{\partial \rho}\right)_{\rho=1} = 0$ implies

$$T_3 = \frac{1}{3}(T_1 - T_2)$$

which when substituted into (A-I.1) gives

$$M_0 = \frac{1}{3}\int d\vec{x}\left(\vec{F}_{ij}^{a}\cdot\vec{F}_{ij}^{a} + \tilde{\nabla}_i\vec{\Phi}_a\cdot\tilde{\nabla}_i\vec{\Phi}_a\right)$$

From the above discussion it is clear that there are no soliton solutions for the gauge fields in the absence of the Higgs fields.

Appendix 2: Canonical realization of SU(2) algebra

We first prove that if

$$\tilde{I}^b = V_c^b(\theta)P^c$$

then

$$[\tilde{I}^a(\omega), \tilde{I}^b(\omega')] = -i\epsilon^{abc}\tilde{I}^c(\omega)\delta^2(\omega-\omega') \tag{A2.1}$$

Proof:

$$\begin{aligned}
[\tilde{I}^a(\omega), \tilde{I}^b(\omega')] &= [P_c V_a^c \, , P_d V_b^d] \\
&= P_c(\omega)[V_a^c(\theta(\omega)) \, , P_d(\omega)]\, V_b^d(\omega') \\
&+ V_a^c(\omega)[P_c(\omega) \, , V_b^d(\theta(\omega'))]\, P_d(\omega') \\
&= \left[i P_c V_b^d \frac{\partial V_a^c}{\partial \theta_d} - i P_d V_a^c \frac{\partial V_b^d}{\partial \theta_c} \right] \delta^2(\omega-\omega') \\
&= -i\epsilon_{abd}\tilde{I}_d(\omega)\delta^2(\omega-\omega')
\end{aligned}$$

ϵ_{abc} are the structure constants of the group and we have used the integrability condition

$$V_b^d \frac{\partial V_a^c}{\partial \theta_d} - V_a^d \frac{\partial V_b^c}{\partial \theta_d} = \epsilon_{dba} V_d^c$$

Now using the fact that the rotation matrix $R_{ab}(\theta)$ transforms in the adjoint representation under rotations we have

$$[\tilde{I}^a, R(\theta)] = R(\theta)T^a \, , \quad (T^a)_{bc} = -i\,\epsilon_{abc} \tag{A2.2}$$

From (A2.1) and (A2.2) follows the desired result

$$[I^a(\omega), I^b(\omega')] = i\,\epsilon_{abc} I^c(\omega)\,\delta^2(\omega-\omega')$$

$$I^a(\omega) = R^{ab}(\theta(\omega))\tilde{I}^b(\omega) \, .$$

Appendix 3: Faddeev-Popov operator in Coulomb gauge:

We discuss the spectrum and homogeneous solution of the differential operator

$$-\partial_i \nabla_i (A_a) = \left[\delta^{ac} \frac{\partial^2}{\partial x^2} + g \epsilon^{abc} A^a_{bi}(\vec{x}) \frac{\partial}{\partial x_i} \right] \tag{A3.1}$$

$$A^a_{ai}(\vec{x}) = \epsilon_{aij} x_j \frac{(1 - K(r))}{g r^2} \tag{A3.2}$$

The function K(r) is plotted in the paper of Julia and Zee;[28] its qualitative features are

$$K(r) \sim \exp(-ar) \quad , \quad a = \mu/\sqrt{\lambda}\, g \quad , \quad r \to \infty$$

$$K(r) \sim 1 - (\text{constant})\, r^2, \quad \text{const.} > 0 \quad , \quad r \to 0$$

and $0 \lesssim K(r) \lesssim 1$ for $0 \leq r \lesssim \infty$. In the Prasad-Sommerfield limit ($\mu \to 0$, $\lambda \to 0$, $\mu/\sqrt{\lambda} \to$ constant) $K(r) = Cr/\mathrm{Sinh}(Cr)$.

Substituting (A3.2) into (A3.1) we have

$$-\partial_i \nabla_i (A_a)^{ac} = -\delta^{ac} \partial^2 + \frac{f(r)}{r^2} (\vec{T} \cdot \vec{L})^{ac} ,$$ where

$f(r) = 1 - K(r)$, $(T^a)_{bc} = -i \epsilon_{abc}$ and $\vec{L} = -i \vec{x} \times \frac{\partial}{\partial \vec{x}}$.

The mathematical problem is of a Schrödinger problem with Hamiltonian

$$H_0 = \vec{p}^{\,2} + \frac{f(r)}{r^2} \vec{L} \cdot \vec{T} \tag{A3.3}$$

To diagonalize H_o we proceed with the following observation:

$$[T^2, H_o] = [L^2, H_o] = [\vec{J}, H_o] = 0 \qquad \text{where } \vec{J} = \vec{T} + \vec{L}$$

then

$$H_o = P_r^2 + L^2/r^2 + (J^2 - L^2 - T^2) f(r)/2r^2$$

$$P_r = -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

H_o is diagonal in the basis

$$\Psi^a_{EJML} = \Psi_{EJL}(r)\, \chi^a_{JML}(\theta, \varphi)$$

provided

$$-R''_{EJL}(r) + V(r)_{JL} R_{EJL}(r) = E\, R_{EJL}(r)$$

where $R_{EJL}(r) = r\, \Psi_{EJL}(r)$ and

$$\chi^a_{JML} = \sum_{m,a} C_{JMma} Y^L_m(\theta, \varphi)\, \xi^a$$

C_{JMma} are the Clebsch-Gordan coefficients, Y^ℓ_m the spherical harmonics and ξ^a are orthonormal vectors in isotopic spin space. The values of J and M are as usual $-|L-1| \leq J \leq |L+1|$, $-J \leq M \leq J$.

The potential which occurs in the above radial equation is

$$V_{JL}(r) = \frac{1}{r^2}\left[L(L+1) + \frac{f(r)}{2}\left(J(J+1) - L(L+1) - 2\right)\right]$$

Since $0 \leq f(r) \leq 1$ for $0 \leq r \leq \infty$, it is obvious that $V_{JL}(r) \geqslant 0$ for $0 \leqslant r \leqslant \infty$; in fact for $J=1$, $L=0$, $V_{1,0}=0$ and for $L \geqslant 1$, $V_{JL} > 0$ for $0 \leq r \leqslant \infty$, a purely repulsive potential. It is obvious that H_o has no negative eigenvalues.

The inverse of H_0 in the space of functions that vanish at infinity may now be constructed in terms of the scattering states of H_0:

$$H_0^{-1\,ab} = \sum_{JML} \int_0^\infty \left\{ \frac{dE}{E}\, n(E)\, \Psi^a_{EJML}(\vec{x})\, \Psi^b_{EJML}(\vec{x}') \right\}$$

$n(E)$ is the density of states.

Solution of the homogeneous equation:

In terms of R_{JL}, the homogeneous equation is

$$-R''_{JL} + V_{JL}(r)\, R_{JL} = 0 \tag{A3.4}$$

The case L = 0, J = 1 is trivial since $V_{1,0} = 0$. The regular solution is $a_{1,0}\,\xi^a$ which corresponds to global isotopic spin rotations. Such a solution is ruled out because it will change the boundary condition of the Higgs field which is held fixed.

We now prove that only for J = 0, L = 1 do we have a regular non-trivial solution; all other J,L have the trivial solution zero. By regular we mean the solution is finite at r = 0 and r = ∞. Consider the case when $J \neq 0$ and $L \neq 1$,

$$V_{JL} \underset{r\to\infty}{\sim} \frac{1}{2r^2}\left[L(L+1) + J(J+1) - 2\right] \equiv \frac{1}{r^2} V^\infty_{JL} > 0$$

In this limit (A3.4) becomes

$$-R''_{JL} + (V^{\infty}_{JL}/r^2)\, R_{JL} = 0$$

then $\displaystyle R_{JL} \underset{r\to\infty}{\sim} A_{JL}\, r^{s_1} + B_{JL}\, r^{s_2}$

$$2s_1 = 1 + (1 + 4V^{\infty}_{JL})^{1/2}, \quad 2s_2 = +1 - (1 + 4V^{\infty}_{JL})^{1/2}$$

Since we require $\Psi_{JL} = r R_{JL}$ is regular at $r = \infty$, we have

$$R_{JL} \sim B_{JL}\, r^{s_2}, \quad r \to \infty, \quad s_2 < 0$$

Now since $R_{JL}(r)$ has no nodes, assuming it is always non-negative, the behavior at r = 0 and r = ∞ implies the existence of a point r_0, $0 < r_0 < \infty$ such that $R'_{JL}(r_0) = 0$ and $R''_{JL}(r_0) < 0$, which contradicts equation (A3.4), unless R_{JL} = 0 for all r.

We now take up the case when J = 0 and L = 1. Equation (A3.4) becomes

$$R''_{0,1}(r) = \frac{2}{r^2} K(r)\, R_{0,1}(r)$$

or in terms of

$$\Psi''_{0,1} + \frac{2}{r}\Psi'_{0,1} = \frac{2K(r)}{r^2}\Psi_{0,1}$$

As $r \to 0$, by regularity we have $\Psi_{0,1} \sim a' r$, i.e. $\Psi_{0,1}(0) = 0$; as $r \to \infty$,

$$\Psi_{0,1} \sim \lambda - \frac{1}{r} \qquad \text{i.e.} \quad \lim_{r\to\infty} \Psi_{0,1}(r) = \lambda$$

Hence we have a one parameter homogeneous solution

$$\vec{\Psi}'_0 = \hat{x}\, \Psi_{0,1}(r;\lambda)$$

Appendix 4: Faddeev-Popov operator in background gauge:

Consider the differential operator $\mathcal{D} = -(\tilde{\nabla}_i \tilde{\nabla}_i + g^2 \bar{\Phi}_a \times (\bar{\Phi}_a \times \ ))$, $\tilde{\nabla}_i = \partial_i + g\bar{A}_{ai}\times$, in a region of space of large finite volume; then in the space of functions $\mathcal{F}$ which vanish on the boundary, we have neglecting the surface term in this case, a version of Green's identity,

$$\int d\vec{x}\, f^a(\vec{x})[\mathcal{D}^{ab} g^b(\vec{x})] = \int d\vec{x}\, g^a(\vec{x})\,[\mathcal{D}^{ab} f^b(\vec{x})] \tag{A4.1}$$

which means that in $\mathcal{F}$, $\mathcal{D}$ is hermetian and further if we put $g^a = f^a$ in (A4.1), we see that

$$\int d\vec{x}\, f^a(\vec{x}) \mathcal{D}^{ab} f^b(\vec{x}) = \int d\vec{x}\, (\tilde{\nabla}_i \vec{f} \cdot \tilde{\nabla}_i \vec{f} + g^2 |\bar{\Phi}_a \times \vec{f}\,|^2) \geqslant 0 \tag{A4.2}$$

which means $\mathcal{D}$ is non-negative in $\mathcal{F}$. In fact we can prove that $\mathcal{D}$ is strictly positive <u>in $\mathcal{F}$</u>. Suppose it were zero, then (A4.2) implies the equations

$$\tilde{\nabla}_i \vec{f} = 0 \quad , \quad \bar{\Phi}_a \times \vec{f} = 0 \quad , \quad \forall\, \vec{x}. \tag{A4.3}$$

Now $\bar{\Phi}_a \times \vec{f} = 0$ enables us to write $\vec{f} = f(\vec{x})\, \vec{\Phi}_a / |\vec{\Phi}_a|$ substituting this into $\tilde{\nabla}_i \vec{f} = 0 = \partial_i \vec{f} + g\vec{A}_{ai} \times \vec{f}$ we get $\partial_i f = 0$, which means that the only solution to (A4.3) is

$$\vec{f}(\vec{x}) = \frac{\vec{\Phi}_a}{|\bar{\Phi}_a|}(\text{constant}) = \hat{x}\,(\text{constant}) \tag{A4.4}$$

which vanishes on the boundary only if the constant is zero i.e. if $\vec{f}(x) = 0$. Note that this is also the only regular solution possible. This completes the proof of the positivity of $\mathcal{D}$ in $\mathcal{F}$.

From the above discussion we conclude that for the eigenvalue problem in $\mathcal{F}$,

$$\mathcal{D}^{ab}\Psi_n^b(\vec{x}) = \epsilon_n \Psi_n^a(\vec{x}) \tag{A4.5}$$

the eigenfunctions Ψ_n^a are complete, the eigenvalues are real and positive and if $\epsilon_n \neq \epsilon_m$ we can choose Ψ_n^a and Ψ_m^b to be orthogonal

$$\int d\vec{x}\ \vec{\Psi}_n \cdot \vec{\Psi}_m = 0 \quad , m \neq n$$

It is also evident that if we perform a gauge transformation $A_u \to A_u^{\Omega}$, $\phi_u \to \phi_u^{\Omega}$ on the classical solution entering $\mathcal{D}$, the eigenfunctions of the gauge transformed differential operator are related covariantly to $\vec{\Psi}_n$, i.e.

$$\mathcal{D}^{ab}(A_u^{\Omega}, \phi_u^{\Omega})\, \phi_n^b(x) = \epsilon_n'\, \phi_n^a(x)$$

is solved by $\phi_n(x) = \Omega_{ab}\Psi_n^b(x)$ and $\epsilon_n' = \epsilon_n$, Ψ_b satisfies (A4.5). This proves that any intersection point of the classical solution and the gauge fixing surface, is good to do a perturbation expansion around.

<u>Solution of the homogeneous equation</u>:

We first prove that under the boundary condition,

$$\Psi_0^a(r \to \infty) = \lambda \hat{x}^a \tag{A4.6}$$

the solution of the homogeneous equation

$$\mathcal{D}^{ab}\Psi_0^b = \left(\tilde{\nabla}_i\tilde{\nabla}_i\vec{\Psi}_0 + g^2\,\vec{\Phi}_{cl}\times(\vec{\Phi}_{cl}\times\vec{\Psi}_0)\right)^a = 0 \tag{A4.7}$$

is unique.

Suppose there exists another solution $\Psi_0'^b$ with the <u>same</u> boundary condition; then the function $f^a = \Psi_0^a - \Psi_0'^a$ satisfies vanishing boundary conditions and hence equations (A4.3). The only solution is $f^a = \Psi_0^a - \Psi'^a = 0$.

We now exhibit a solution with boundary condition (A4.6) which is exact in the Prasad-Sommerfield limit. We make the ansatz which is physically very reasonable

$$\vec{\Psi}_0 = (\vec{\Phi}_{cl}/|\vec{\Phi}_{cl}|)\Psi_0(\vec{x}) = \hat{x}\,\Psi_0(\vec{x})$$

The differential equation (A4.7) now becomes

$$-\left[\delta^{ab}\partial^2 - \frac{2}{r^2}(1-K(r))\vec{T}\cdot\vec{L}^{ab}\right]\hat{x}^b\Psi_0 + g^2\vec{A}_i^a\cdot\vec{A}_i^a\,\hat{x}^a\Psi_0 = 0 \tag{A4.8}$$

where we have used the form

$$A_{ai}^a = \epsilon_{aij}\hat{x}_j(1-K(r))/gr$$

The term in the Higgs field drops out. In (A4.8) $\vec{L} = -i\,\vec{x}\times\frac{\partial}{\partial\vec{x}}$ and $(T^s)_{bc} = -i\,\epsilon_{abc}$. This simplifies to

$$P_r^2\Psi_0 + \frac{1}{r^2}\left[2K^2 + \vec{L}^2\right]\Psi_0 = 0$$

or equivalently

$$-\tilde{\Psi}_{\ell m}'' + V_\ell(r)\,\tilde{\Psi}_{\ell m} = 0 \tag{A4.9}$$

$\Psi_0 = \sum Y_{\ell m}(\theta,\varphi)\,\tilde{\Psi}_{\ell m}(r)/r$, $Y_{\ell m}(\theta,\varphi)$ is a spherical harmonic and

$$V_\ell(r) = \left[2K^2 + \ell(\ell+1)\right]/r$$

(A4.9) can be easily solved in the regions $r \sim 0$ and $r \sim \infty$:

$$\tilde{\Psi}_\ell(r\sim 0) \approx \bar{a}\, r^{S_1} \quad , \; S_1 = (1 + [9 + 4\ell(\ell+1)]^{1/2})/2, \; \bar{a} > 0$$

$$\tilde{\Psi}_\ell(r\sim\infty) \approx \bar{b}\, r^{S_2} \quad , \; S_2 = (-1 + [1 + 4\ell(\ell+1)]^{1/2})/2, \; \bar{b} > 0$$

We have chosen $\bar{a}$ and $\bar{b}$ to be positive since $\vec{\Psi}_0(x)$ has no nodes. For $\ell \neq 0$, the above means that, $\tilde{\Psi}_\ell \to 0^+$ as $r \to 0$ and $r \to \infty$; since there is no node this means that there is a finite point $r_0 \neq 0$, for which the function passes through a local maximum, i.e. $\tilde{\Psi}_\ell''(r_0) < 0$. This contradicts equation (A4.9) unless $\tilde{\Psi}_\ell = 0$ everywhere, because has been chosen to be non-negative. For $\ell = 0$, the above argument does not apply since $\tilde{\Psi}_0(r)$ no longer vanishes as $r \to \infty$. In fact we have

$$\tilde{\Psi}_0(r\sim 0) \sim r^2 \;, \quad \tilde{\Psi}_0(r\sim\infty) \sim -1 + \lambda r, \; \lambda > 0$$

or equivalently

$$\vec{\Psi}_0(r\sim 0) \sim \hat{x}\, r$$
$$\vec{\Psi}_0(r\sim\infty) \sim \hat{x}\,(\lambda - 1/r) \qquad \text{(A4.10)}$$

(N.B. we have chosen $\lambda > 0$ since the nodeless function $\tilde{\Psi}_0(r)$ is chosen to be positive; if $\tilde{\Psi}_0(r)$ were chosen to be negative the asymptotic behaviour would be

$$\tilde{\Psi}_0(r\sim\infty) \sim 1 + \lambda r \quad , \quad \lambda < 0.)$$

It is reasonable to assume that there is a solution that interpolates between the two limits in (A4.10). Further by the uniqueness theorem it is the only solution with boundary condition (A4.6). In the Prasad-Sommerfield limit we have an exact solution: Since the static equations of motion become

$$\tilde{\nabla}_i \vec{\Phi}_a = \frac{1}{2} \epsilon_{ijk} \vec{F}^a_{jk}$$

we have

$$\tilde{\nabla}_i \tilde{\nabla}_i \vec{\Phi}_a = \frac{1}{2} \epsilon_{ijk} \tilde{\nabla}_i \vec{F}^a_{jk} = 0$$

(Bianchi identity)

which is equivalent to

$$\mathcal{D}^{ab} \phi^b_a = 0$$

The exact expression for the classical Higgs field is (1.22)

$$\vec{\Phi}_a = (\hat{x}/r)[\lambda r \operatorname{Coth}(\lambda r) - 1]$$

which is a one parameter solution with boundary value

$$\vec{\Phi}_a (r \to \infty) \sim \hat{x}\left(\lambda - \frac{1}{r}\right) .$$

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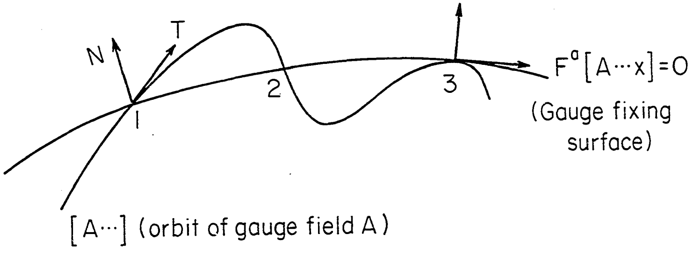
Figure 1



$N^{a,y}_{b,i,x} = \delta F^a[A;\vec{y}]/\delta A_{bi}(\vec{x})$ is the normal to

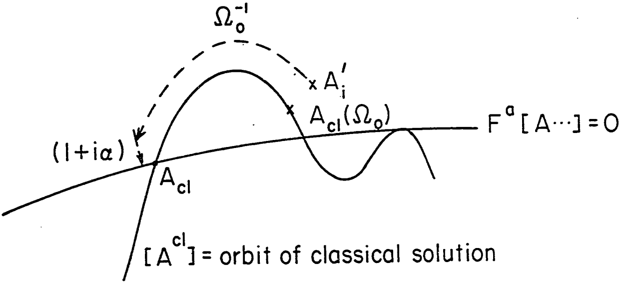
$T^{a,y}_{b,i,x} = \nabla_i^{ba}(\vec{A}(\vec{x}))\,\delta^3(\vec{x}-\vec{y})$ is the tangent to the orbit $[A]$.

The Faddeev-Popov operator $\Box^{ca} = (N,T)^{ca} = \nabla_i^{cb}\,\delta F^a/\delta A_{bi}$.

vanishes at intersection 3, a point of tangency.

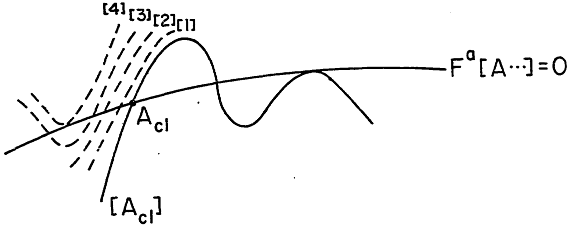
(For simplicity we assumed $F^a = 0$ independent of the Higgs field)

Figure 2



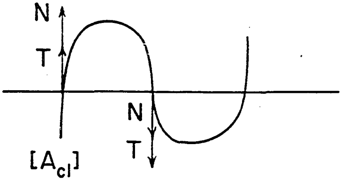
$A_i'^a$ near $A^a_{\alpha}(\Omega_0)_i$; gauge transformation $\Omega_0^{-1}(1+i\alpha)$ brings $A_i'^a$ onto gauge fixing surface near $A^a_{\alpha i}$.

Figure 3



Orbits [1] and [2] of fluctuation intersect F = 0, once near A_{cl} ; orbit [3] has two nearby intersections which must be distinguished; orbit [4] is a point of tangency, the Faddeev-Popov operator develops zero eigenvalue.

Figure 4 (Background gauge):



$$F = \tilde{\nabla}_i (A_i - A_{cli}) + g\bar{\phi}_{cl} \times (\bar{\phi} - \bar{\phi}_{cl}) = 0$$

N and T are co-incident at all intersections of the classical solution.