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DEMAND MICROECONOMICS OF URBAN RESIDENTIAL LOCATION

by

BARBARA BANDES BROWN

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in partial fulfillment of the requirements for the degree of
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Harold M Hochman
Chairman of Examining Committee

5/16/79
date

Executive Officer

Harold M. Hochman

Elliot Zupnick

Michael Grossman

Supervisory Committee

The City University of New York

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

CHAPTER ONE

INTRODUCTION

This dissertation is written in a spirit of advocacy; I am advocating a particular version of urban residential location theory. The version I advocate does not originate with me, but its implications have never been worked out to any great extent, which is why, I imagine, two competing versions have succeeded in holding the field. By here presenting for the first time a thorough and classical working out of the implications of the version I advocate I arrive at many distinctive new conclusions through which the theory can be empirically tested. This version has (in my opinion, anyway) more realistic assumptions than competing versions, and I analyze it using the most standard methods of microeconomics, in order to bring residential location theory to the point where it can withstand the type of attack[1] to which it has recently been subjected.

Urban residential location theory addresses the question of exactly where people will choose to live in the world of the so-called "monocentric city." This is a strange abstract world consisting of a featureless plain whose center is the world's only city. All the people of this world are either self-sufficient or else employed in the single city center. What follows is a thumbnail sketch of the monocentric city literature.

Of the three competing versions of the theory of the monocentric city, the most influential is Richard Muth's, as presented in his Cities and Housing; it provides the theoretical basis for much of the current

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[1] William Wheaton (1977c) tested and rejected the two leading theories, and on this basis concluded that income-location patterns are best explained as flight from social and racial externalities and from the urban tax burden of supporting the poor; see my Chapter Five.

empirical work in the area. Muth's theory is that an individual chooses where to live by maximizing a utility function whose two arguments are housing space and expenditure on all other goods including leisure, subject to an income constraint which includes the value of travel time in both total income and expenditure, and which also includes the value of leisure time in total income. Joseph de Salvo extends Muth's work by presenting its formal comparative statics. The work of Robert Solow falls into this category (of utility-maximization models omitting distance from the utility function). It differs from Muth's in having a conventional income constraint involving only money costs.

The second version -- the version whose acceptance I am advocating in this dissertation -- is that distance from the city center should be one of the arguments of the utility function (in addition to the other two), and that the income constraint should be the familiar one of price theory, i.e. should include only money income and expenditure. If people do in fact prefer one location over another, all other things equal, and if having such preferences is in any way different from there being time costs of commuting, this version will explain location decisions better than one which does not take account of such preferences. Even though many location theorists have constituted the model as I do, nevertheless this version of location theory is relatively little known and is seldom used in empirical work, I surmise because the theory has never been given full-scale exposition and analysis.

Distance was first put into the utility function by Alonso, who initially used the utility-maximization approach (see below), showed it presented difficult theoretical problems in deriving land demand curves, and then presented his bid price theory as an alternative. Muth also put distance in the utility function in a modification of his basic model, but rejected this approach for the core of his work as having inadequate empirical content. Martin Beckmann and Aldo Montesano put distance in the utility function, but only for the case in which the utility function is log-linear in form. Montesano showed that Beckmann's analysis is faulty. In his own enquiry, which was directed entirely to the question of how location varies with income, his result was that the richer will live farther out; I shall show later that this pattern is a special implication of the log-linear utility function and does not necessarily hold for a general utility function. Herbert Mohring (1976) also put distance in the utility function; his work is very suggestive, but is brief, and again mostly focuses on the log-linear utility function. To summarize, the bare bones of this version of location theory have been stated many times, but it has been so little analyzed that it has had minimal influence.

The third version of residential location theory, constituted by the bid price model of William Alonso and its elaboration by William Wheaton (1974, 1977a), represents a departure from direct utility maximization. In this model the individual has, for each utility level, a bid price function giving, for each location, the rent level which would enable him to reach the specified utility level; the individual locates where-ever his is the highest bid price; equilibrium utility levels result from achievement of supply-demand equilibrium. I have chosen in my own work to focus entirely on the direct utility-maximization approach, because of my feeling that the familiar and classical methods of microeconomics can be expected to be as illuminating in explaining urban residential location as they are in microeconomics in general.

To reiterate, the three versions of residential location theory are 1) the utility-maximization model including distance in the utility function, 2) the utility-maximization model including distance in the utility function, and 3) the bid price model. My position is that while the first and third models, as propounded by Muth and Alonso respectively, dominate the field, actually the second model is preferable. To make this point, I will carefully present and analyze the second model, showing that it yields satisfying explanations and new results. I will be at pains to show that my model does differ critically from Muth's -- that the inclusion of time costs in the income constraint *a la* Muth is not the equivalent of including distance (i.e. commuting time) in the utility function. I will defer until Chapter Five a discussion of the essence of the difference between my work and Muth's.

In all of this my intellectual debt to William Alonso is especially great. In his *Location and Land Use* he originated the model of the monocentric city, and analyzed it with thoroughness, clarity, and insight. In his initial chapters he uses a utility-maximization model and includes distance in the utility function (on the same basis as I do), but he encounters difficulties in deriving demand functions, and so switches to his bid price analysis. What I have done herein, really, is to pick up his straight utility maximization model where he left off. To the extent that he and I cover the same ground, our work is almost identical, there being but a very few points on which we disagree, and in fact I hereby refer the reader to his Chapter 2 for elucidation of anything that might be unclear in the earlier part of my Chapter Two. Moreover, his bid price analysis is by and large consistent with my work. But I have taken on the task of fully analyzing his utility-maximization model in the conviction that it would provide more results, would more lend itself to expansion, and would ultimately be more intellectually appealing to the profession by reason of the greater familiarity of the techniques.

A goal of this dissertation is to generate new explanations and new results. The explanatory core of the location theory I present in the following chapters can be summarized very briefly as follows: The normal good "closeness" has a shadow price, which I identify, which varies with both income and distance; since rent, the price of housing space, also varies with distance, the price ratio between closeness and housing also varies with income and distance; this unusual price ratio is the slope of the budget surface trace in housing-and-distance 2-space; and the effects of parameter changes on location and demand for housing can be understood by seeing how the parameter changes shift this budget trace. The case is clearly very different from the classical two-good case, in which, of course, the price ratio does not vary with the goods or with income. So far as I know, there is no presentation of what happens in location-and-housing 2-space in the literature.

The question of how location varies with income dominates residential location theory. I can perhaps demonstrate the insights afforded by my analysis by here summarizing briefly my theory on this point: 1) if an income increase (which is, of course, a parameter change) could occur unaccompanied by price ratio changes, it would result in movement closer to the center, since closeness is a normal good; 2) an income increase does always change the price ratio: it increases the relative price of closeness; 3) the increase in the price of closeness is a force making for consumption of less closeness; and 4) the net movement with income depends on the relative strength of these two forces. Again, this explanation does not appear in the literature. While Alonso, Muth, Mohring, and Wheaton all concur that it is theoretically possible for the richer to live either farther from or closer to the city center than do the poorer, they do not show that it is the net effect of income changes on shifting price ratios which determines the outcome.

I would also like to mention here some new points that are developed in my study.

In Chapter Four I present a large number of new comparative static results. These are by and large very different from those de Salvo obtained on the basis of Muth's model. As such they lend themselves to comparative testing of Muth's model and mine, though no such testing is attempted in this paper. De Salvo's are the only other formal comparative statics in the literature for utility-maximization theory.

My analysis explains why we don't find the poor living far from the city center: at these distances and for these incomes the shadow price of closeness can be negative. As I will show, the shadow price of closeness is the negative of the sum of rent savings and commuting cost increases occasioned by increased distance; this price can become negative if the commuting cost increase swamps the rent savings. The price can be simultaneously negative for the poor and positive for the rich because the poor consume less housing and therefore achieve smaller rent savings with increased distance. This explanation, taken together with the result that income increases can lead to location change in either direction, goes far toward rationalizing the pattern of location distribution by income that we actually observe -- a pattern of the well-to-do locating in both central city and suburbs and of the poor who work in the CBD locating only in the central city.

Another result is my explanation of the relation between the value of travel time (which is an important quantity used in cost-benefit analyses of proposed transportation projects) and the wage. Empirical studies have uniformly yielded values of travel time considerably below the wage rate, despite the expectations of many economists that the value of travel time would be equal to the wage. I explain herein why in fact we should usually expect the value of travel time to be less than the wage.

In summary, I have as my goals in this dissertation to 1) present and fully analyze the distance-included utility-maximization model of urban residential location, 2) derive new explanations and results, and 3) show that this model is different from and preferable to alternative models. A by-product will be critical analysis of the other monocentric city models and results. In what follows, in Chapter Two I will present and discuss the structural equations and the equilibrium conditions. In Chapter Three I will discuss the relation between location and income. Chapter Four will be devoted to the comparative statics. Chapter Five will present conclusions and policy implications.

CHAPTER TWO

STRUCTURAL EQUATIONS AND EQUILIBRIUM CONDITIONS

We assume throughout this study a featureless plain with transportation possible in all directions, availability of employment only at the center of the city, absence of all transportation-requiring activities other than work, and absence of taxes and public goods.

We start at once with a consumer who maximizes his utility subject to the constraint imposed by his income and prices; he acts as if he maximizes an ordinal utility function subject to a budget constraint function. Housing space, and a single good which represents all goods and services other than housing space and location, are taken to be two of the arguments of his utility function.

If we believe that he does care where he lives in relation to the center of our monocentric city -- that, all other things being equal, he prefers one location to another -- then distance should be an additional argument in his utility function. An explanation of where he chooses to live which does not take account of location preference by including location in the utility function might or might not turn out to be correct, but we can make sure location's influence is taken account of by explicitly including it in the utility function.

A problem is that in the real world it is not distance from the center per se that directly affects preference, but rather many different factors that vary, or seem to vary, with distance. Prominent among these are traveling time to and from work, traveling time to and from places of preferred leisure activities (e.g. theaters, cross-country skiing), pollution of various sorts, crime rates, traffic and other congestion, proximity to low-income and minority groups, accessibility to local shopping, and quality of schools and other public goods: the list is almost a catalogue of the concerns of urban economics. For many of the items on the list, the direction of their variation with distance is by no means completely obvious. Moreover, since some of these factors increase utility and others decrease it with the increase of distance, there is no clear overall pattern of the direction in which utility should vary with the increase in distance.

As a simplification I shall focus on only one of these factors that vary with distance, that factor being travelling time to and from work; all the other factors are explicitly excluded. The choice is an obvious one, since travel time to work enters one way or another into all the models surveyed in Chapter One except Mohring's. My reason here for putting travel time into the utility function is the subjective disutility of traveling back and forth -- the irksomeness of being in a crowded subway car, of waiting for a bus in the rain, of being stuck in a traffic jam in an overheating car, of losing all that time in transit even when conditions are at their best. I shall neglect the variation in irksomeness with mode of transportation, by assuming there exists only a single mode. I assume also that travel time is proportional to distance -- that there is no traffic congestion to slow down traffic in central areas so that it takes longer, there, to traverse a given distance. I define a unit of time to be the amount of time required to traverse a unit of distance. I assume too that the number of trips to and from work is given and invariant. I am assuming, then, not only that distance is an argument of the utility function but also that the marginal utility of distance is negative.

Actually, in a full analysis of urban location we would not have distance in the utility function, but rather all of the factors listed above, as well as any others of significance, which directly affect preference. (In this sense, the argument of this dissertation is not that distance cannot be excluded from the utility function, but rather that the preference factors that vary with distance cannot be excluded from the utility function.) The relation between each factor and distance would have to be analyzed, and the relationships would be expressed in constraint functions. It is permissible to put distance directly into the utility function only if the factor which by rights should be in the utility function can properly be defined to be the equivalent of distance. In this study commuting time, the factor which should be in the utility function, has been made equal to distance by the assumption that commuting time is proportional to distance and the definition that a unit of time is the amount of time required to traverse a unit of distance.

The consumer's budget constraint function reflects the fact that he has a given income and faces a given set of prices. I use the word "rent" to denote the price per unit of time of the services of a unit of housing space. A rent is given for each distance from the center, so that rent is a given function of distance. Commuting costs are also a function of distance: a money commuting cost is associated with each distance, with commuting cost increasing as a function of distance. Time costs of commuting, which Muth includes in the budget constraint, I explicitly exclude from the budget constraint.

Our utility function and budget constraint function are therefore written

$$U = U(z, q, t) \qquad (1)$$

$$pz + R(t)q + K(t) = y \qquad (2)$$

where z is the single good; q is housing space, measured in units of floor space; t is distance of the consumer's residence from the city center; p is the price of the composite good; R is rent; K is total money travel cost; and y is money income. We assume that the utility function is continuous and possesses continuous partial derivatives to the desired order, and that

$$U_z > 0$$
$$U_q > 0$$
$$U_t < 0 \quad .$$

On these definitions and assumptions we can prove that the rent function is continuous; for the proof see Appendix A.

We also assume

$$U_{zz} < 0$$
$$U_{qq} < 0$$
$$U_{tt} < 0 \quad .$$

In the case of z and q we are assuming diminishing marginal utility; in the case of t we are assuming increasing marginal disutility. These assumptions, taken together, are also assumptions that these would be normal goods in the standard analysis. For example, if we had two typical consumer goods, x_1 and x_2, and if an individual had income y, we would have $\partial x_1/\partial y > 0$ if we had $U_{x_2x_2} < 0$.

We assume, moreover, that our goods are independent:

$$U_{zq} = U_{zt} = U_{qt} = U_{qz} = U_{tz} = U_{tq} = 0.$$

We can now proceed as in the general analysis of consumer choice. We maximize the Lagrangian

$$L = U(z, q, t) + \lambda[y - pz - R(t)q - K(t)] \quad ,$$

where λ is a Lagrangian multiplier, by taking its first partial derivatives with respect to z, q, t, and λ and setting them equal to zero. We get

$$L_z = U_z - \lambda p = 0 \tag{3a}$$

$$L_q = U_q - \lambda R(t) = 0 \tag{3b}$$

$$L_t = U_t - \lambda[qR'(t) + K'(t)] = 0 \tag{3c}$$

$$L_\lambda = y - pz - R(t)q - K(t) = 0 \quad . \tag{3d}$$

This set of simultaneous equations, the first-order conditions, can be solved for z^*, q^*, and t^*, the individual's utility-maximizing set. The conditions also solve for λ^*.

There is nothing original about these structural equations and first-order conditions; they have appeared in the work of every economist who has put distance into the utility function. But as we proceed now to analyze their meaning, and the entailed comparative statics and location patterns, we shall find ourselves coming up with new material different from what we have seen before. We shall note the more striking differences where appropriate.

Looking towards the meaning of demand equilibrium, we note in equation (3c) the expression $qR'(t) + K'(t)$. It is essential to recognize that this term is the shadow price of t, for if z and q are held constant and t is increased a unit, the increase in expenditures is equal to $qR'(t) + K'(t)$. We define

$$A \equiv qR'(t) + K'(t) \quad .$$

We note that A is a very peculiar price, since it varies with both q and t.

Once we recognize that A is the price of t, we see that our first-order conditions are in the form of the standard first-order condition equations in the theory of consumer behavior. Like the standard equations, equations (3a), (3b), and (3c) can be restated in the form

$$\frac{U_z}{p} = \frac{U_q}{R(t)} = \frac{U_t}{A} = \lambda \quad . \tag{4}$$

The meaning of equations (4) is, that of the (z, q, t) sets established as available by the budget constraint, the consumer will choose the set for which all three commodities have the same ratio of marginal utility to price, λ *. The ratio of a commodity's marginal utility to its price gives the rate at which utility would increase if an additional dollar were spent on that commodity; at the point at which marginal utility per dollar is equal for all commodities, the consumer cannot increase his utility by consuming a dollar's worth less of one commodity in order to consume a dollar's worth more of another commodity.

To interpret the meaning of the ratio U_t/A, we note that our variable t is a discommodity rather than a commodity, in the sense that people would prefer to have less of it, not more of it. Thus we can think of consumers as trying to decrease distance, or alternatively as trying to increase "closeness," an increase in the commodity "closeness" being equivalent to a decrease in the discommodity "distance." The marginal utility of an additional unit of closeness is therefore $-U_t$. By the same reasoning, the price of closeness is -A. Once we write

$$\frac{U_t}{A} = \frac{-U_t}{-A}$$

we see that this is the ratio of closeness's marginal utility to its price. The equilibrium location is therefore the one for which the individual could not increase his utility by choosing a dollar's worth less or a dollar's worth more of closeness.

We assume the satisfaction of the second-order condition for (z^*, q^*, t^*) being the utility-maximizing set. The condition is that for the bordered Hessian

$$|H|^* \equiv \begin{vmatrix} 0 & p & R(t) & A \\ p & U_{zz} & U_{qz} & U_{tz} \\ R(t) & U_{zq} & U_{qq} & B \\ A & U_{zt} & B & C \end{vmatrix}$$

where $B \equiv U_{tq} - \lambda R'(t)$ and $C \equiv U_{tt} - \lambda A_t$, that $|\bar{H}|^* < 0$ and that every appropriately obtained third-order principal minor of $|\bar{H}|^*$ will be positive; by appropriately obtained third-order minor we mean a minor obtained by deleting any row and the corresponding column from the last three rows and columns of $|\bar{H}|^*$.

The geometrical representation of individual demand equilibrium is in three dimensions, since we have three decision variables, z, q, and t. The point of equilibrium is the point at which the (three-dimensional) budget surface is tangent to a (three-dimensional) indifference surface. This point is the only point at which there are two simultaneous two-dimensional tangencies, one in the (q, t) plane and one in the (z, q) plane.

First, to see the geometry of what happens in the (q, t) plane, we hold z constant at z^*, the equilibrium value of z, and so confine our attention to the two-dimensional plane for which $z = z^*$. We measure t on the horizontal axis and q on the vertical axis.

The trace of an indifference surface in a (q, t) plane, i.e. z held constant, has the equation

$$dU = U_q dq + U_t dt = 0 \quad .$$

The slope, $(dq/dt)_I$, of the indifference surface trace is therefore

$$\left(\frac{dq}{dt}\right)_I = -\frac{U_t}{U_q} > 0$$

on our assumptions $U_t < 0$ and $U_q > 0$, thus differing from the familiar indifference curve of consumer demand theory in having a positive rather than a negative slope. The reason is that since distance confers disutility rather than utility, a consumer moving farther out requires a positive increment of housing space in order to remain indifferent. The slope of our indifference surface trace increases with t:

$$\left(\frac{d^2q}{dt^2}\right)_I = \frac{-1}{U_q}(U_q{}^2 U_{tt} + U_{qq} U_z{}^2) > 0 \quad .$$

The trace of the individual's budget surface in the (q, t) plane for $z = z^*$ has the equation

$$q = \frac{y - pz^* - K(t)}{R(t)} \quad .$$

Its slope, $(dq/dt)_y$, is

$$\left(\frac{dq}{dt}\right)_y = \frac{-A}{R(t)} \quad .$$

The sign of the slope depends on the sign of A, which we don't know thus far. However, equations (4) imply that for equilibrium we must have

$$\frac{-U_t}{U_q} = \frac{-A}{R(t)} \quad ,$$

which is a statement that at the equilibrium point the slope of the budget trace must be equal to the slope of the indifference trace. Since we already know that the indifference traces are positively sloped, we must conclude that at a point of equilibrium the budget curve must also be positively sloped:

$$\left(\frac{dq}{dt}\right)_y > 0 \quad .$$

This condition needs to hold only at a point of equilibrium, and I will show later that there can in fact be negatively sloped portions of budget traces.

The slope of the budget surface trace in the (q, t) plane is a most unusual one, in that it varies with t — both numerator and denominator in fact varying with t — in contrast to the always constant prices in the standard consumer choice analysis. We have

$$\left(\frac{d^2q}{dt^2}\right)_y = \frac{2AR'(t) - R(t)qR''(t)}{R(t)^2}$$

on the assumption that $K''(t) = 0$. Hence

$$\left(\frac{d^2q}{dt^2}\right)_y \gtreqless \quad \text{iff} \quad 2AR'(t) - R(t)qR''(t) \gtreqless 0,$$

$$R''(t) \lesseqgtr \frac{2AR'(t)}{R(t)q} \quad .$$

So at a point of equilibrium, where the slope of the budget trace must be positive, the way this slope changes with t depends on whether the rate of change of the slope of the rent function is greater than, equal to, or less than some positive number. Thus far we have no results regarding the value of $R''(t)$, so the provisional conclusion must be that the budget line can be concave, convex, or straight. Dragging in the empirical evidence doesn't affect the provisional conclusion, since the evidence, which is that $R''(t)$ is positive, is consistent with all the alternatives.

Figure 1 shows possible individual demand equilibrium configurations in the relevant (q, t) plane. The diagrams should be thought of as applying only in the neighborhood of equilibrium.

The other two-dimension tangency required for consumer demand equilibrium is in the (z, q) plane for which $t = t^*$. An indifference trace here has the equation

$$dU = U_z dz + U_q dq = 0 \qquad ;$$

its slope is negative,

$$\left(\frac{dz}{dq}\right)_I = \frac{-U_q}{U_z} < 0 \qquad ;$$

the slope increases with q,

$$\left(\frac{d^2z}{dq^2}\right)_I = \frac{-U_{qq}}{U_z} > 0 \qquad .$$

The budget line has the equation

$$z = \frac{y - K(t^*) - qR(t^*)}{p} \qquad ;$$

its slope is negative,

$$\left(\frac{dz}{dq}\right)_y = \frac{-R(t^*)}{p} < 0 \qquad ;$$

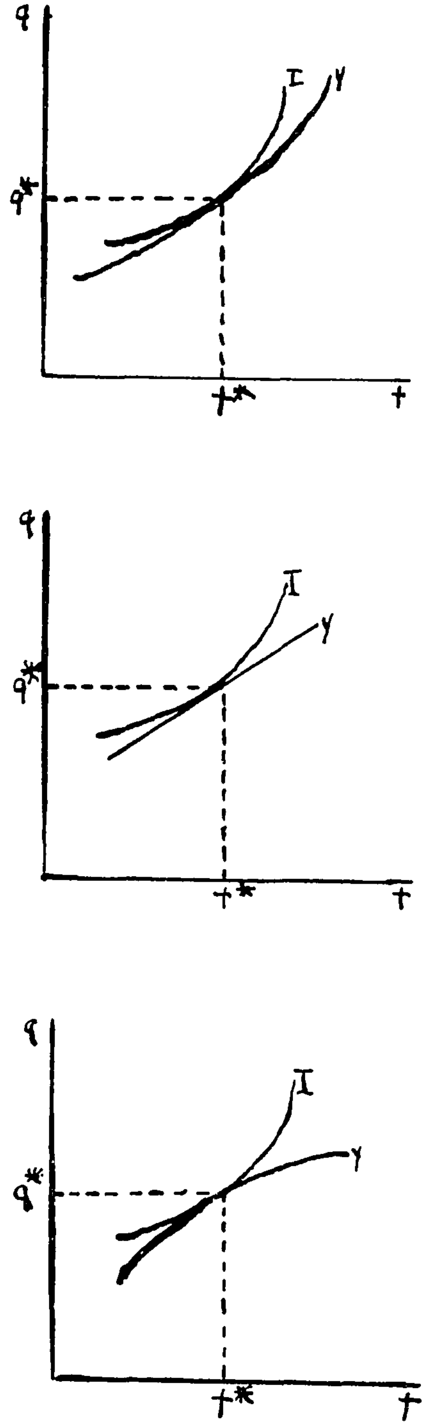

FIGURE 1. EQUILIBRIUM CONFIGURATIONS.

and the slope doesn't vary with q or z. So what we have is precisely the standard case of consumer choice among typical goods.

Important implications about the rent function and about the value of travel time can be drawn directly from the first-order conditions.

The first-order conditions imply that the rent function must be negatively sloped: equations (3a) and (3c) together imply

$$R'(t) = \frac{p\frac{U_t}{U_z} - K'(t)}{q} \quad , \tag{5}$$

so on our assumptions, R'(t) must be negative. The underlying meaning is that for marginal utility per dollar of closeness to be equal to the (positive) marginal utility per dollar of other goods, the price of closeness must be positive and hence R'(t) must be negative. In other words, the "price" of moving farther out must be negative — the consumer has to receive something, rather than paying something. No one will locate at a point where the rent isn't falling, so the rent function must be everywhere decreasing within the city.

This proof that the rent function must be everywhere negatively sloped rests on our assumption that U_t is negative. That assumption was intimately related to our initially excluding all factors that vary with distance except travel time to work. Once those factors are admitted, it's not clear what we should assume about U_t, so for a moment we drop the assumption that U_t is negative to look at the evidence. Equation (5) then implies

$$U_t \underset{<}{\overset{>}{=}} 0 \quad \text{iff} \quad R'(t) \underset{<}{\overset{>}{=}} -\frac{K'(t)}{q} \quad .$$

In the central areas of cities, where mass transit is the dominant mode, $K'(t)$ is typically equal to zero. Then for this region, the very fact that we observe $R'(t)$ to be negative implies that U_t is negative — and now we are considering U_t to refer jointly to all those factors which singly enter the utility function and are themselves functions of distance.

Muth considers and rejects inclusion of distance in the utility function partly because of his equivalent of equation (5). He says that this equation "implies that $R'(t)$ could be positive, even though $K'(t)$ is positive, if U_t is positive and the ratio (U_t/U_z) is sufficiently large. Introducing preferences for location as done above, therefore, renders the theory devoid of any empirical content" (p. 41). Muth's rejection on this basis would seem to be unfounded, since all that is needed to imply that $R'(t)$ is negative -- if such a requirement is indeed necessary — is to assume that U_t is non-positive. In effect, Muth makes the drastic assumption that U_t equals zero when the more general assumption that U_t is non-positive would suffice for his particular purposes.

According to Muth the first-order conditions also imply that $R''(t)$ must be positive. Muth shows that his first-order condition

$$qR'(t) + T'(t) = 0 \quad ,$$

where T is total transportation costs, can hold true only if the partial derivative of its left-hand side with respect to t is non-positive,

$$-qR''(t) - R'(t)\,\frac{\partial q}{\partial t} - T''(t) \leqq 0 \quad .$$

From this he concludes (by manipulation, and with some intermediate steps) that $R''(t)$ is positive. What is happening in his analysis is that his $-[qR'(t) + T'(t)]$ is actually the shadow price of closeness, and is equal to zero in his demand equilibrium because his leaving distance out of the utility function implies $U_t = 0$ — zero being the equilibrium price for something of no value. As such, $-[qR'(t) + T'(t)]/R(t)$ is the slope of the trace of the budget surface in a (q, t) plane. Utility is maximized at the point where q is maximized, which is the point where the slope of the budget trace equals zero; see Figure 2

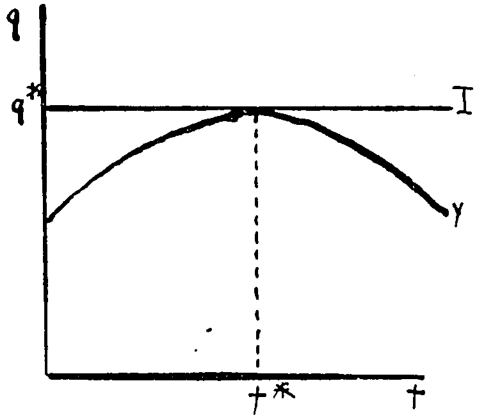

FIGURE 2. MUTH'S EQUILIBRIUM CONFIGURATION.

For this kind of equilibrium, the slope of the budget trace must be decreasing and hence R"(t) must be positive. However, when distance is included in the utility function, equilibrium can be in any of the forms shown in Figure 1, so that there are no implications for the sign of R"(t). Muth's result holds, then, only for the case of the individual who doesn't care at all where he locates, the case of $U_t = 0$ at all distances.

Mohring (1976) worked on the theory of the monocentric city in order to develop a measure of the value of travel time, the latter being a quantity which is not directly observable, yet which we need to know in order to evaluate proposed public investment in transportation projects, analytically to deal with traffic congestion, and to make decisions on highway and mass transit taxes, tolls, and subsidies. Here I will present Mohring's analysis and then show how his measure is biased. First, to define the value of travel time, Mohring shows that the slope of an indifference curve in (z, t) space is

$$\left(\frac{dz}{dt}\right)_I = -\frac{U_t}{U_z} ,$$

and that it represents the amount of z an individual would exchange for a unit of closeness. The money value of a unit of closeness is therefore $-pU_t/U_z$, so if we keep our definition of a unit of travel time (that one unit of time is the amount of time it takes to traverse one unit of distance), and if we define V to be the value of a unit of travel time, we have

$$V = -p\frac{U_t}{U_z} . \tag{6}$$

Equations (3a), (3c), and (5) yield

$$V = -[qR'(t) + K'(t)] , \tag{7}$$

which states, quite reasonably, that in equilibrium the value of travel time will be equal to its price. Since q, R'(t), and K'(t) are readily observable, equation (7) gives an empirical measure of the value of travel time. Muth's measure has the same basis as Mohring's.

An expanded version of the model shows, however, that this measure of the value of travel time is biased. The expansion involves taking into account all those factors which are given to the individual as varying with distance. These many factors, partly enumerated earlier, each enter into the model in their own specific and often complicated way. For the present purpose I will consider only one factor, which I will call trees per acre, but which can be thought of as a proxy for all factors which vary inversely with land use intensity, and whose direct money cost to the individual can be neglected. Since it is no longer only travel time which varies with distance, we can no longer simply put distance in the utility function. Instead, letting h stand for travel time and tr stand for trees per acre, we rewrite our Lagrangian

$$L = U\,[z,\ q,\ h(t),\ tr(t)] + \lambda[y - pz - R(t)q - K(t)] \quad .$$

Our first-order conditions are now (3a), (3b), and (3d) as before and

$$U_h + U_{tr}tr'(t) - \lambda[qR'(t) + K'(t)] = 0 \quad . \qquad (3c')$$

Dividing (3c') by (3a), we get

$$V = -\,[qR'(t) + K'(t)] + p\,\frac{U_{tr}tr'(t)}{U_z} \quad .$$

I suspect that on balance, the factors for which trees are a proxy mainly involve an increase of utility with distance. If so, then the equation above shows that Mohring's measure gives an underestimate of the value of travel time. Moreover, the correct measure includes terms which are not directly observable.

Empirical studies of the value of travel time by Beesley, Mohring, Pendleton, Muth, Friedlaender, and McFadden all yield the result that the value of travel time is a fairly small fraction of the wage rate, around 1/3. This result has been puzzling in the face of expectations[2] that the value of travel time would be equal to the wage. The explanation lies in looking at a more general version of our model, one in which the individual's supply of labor is included among his decision variables. The Lagrangian becomes

$$L = U(z, q, t, w) + \lambda[Ww - pz - R(t)q - K(t)] \quad ,$$

where w is units of working time, W is the wage, and once again travel time is the only utility-function factor in the model which varies with distance. The first-order conditions are as before (except that the constraint condition has Ww in place of y), the only change being that there is an additional condition,

$$L_w = U_w + \lambda W = 0 \quad . \tag{3e}$$

Dividing (3a) by (3e) and rearranging gives

$$\frac{W}{U_w} = \frac{-p}{U_z} \quad ,$$

and substituting this result into (6) gives

$$V = \frac{U_t}{U_w} W \quad . \tag{8}$$

Equation (8) says that the value of travel time is proportional to the wage, and that the proportion is the ratio of the marginal utility of travel time to the marginal utility of work time. If the marginal disutility of travel time is less than the marginal disutility of work time, V will be smaller than W.

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[2] See, for instance, Moses and Williamson.

CHAPTER THREE

INCOME AND LOCATION

The single most provocative question in urban location theory is that of the relation between location and income, probably because the fact of concentrations of the poor, and of "urban problems," in the slums of the central cities, requires explanation: we need to know why the poor are in the central cities, and why the richer are presumably elsewhere, in order to figure out what to do about the problems. Location theory has become ideologically charged, to an extent, around this question, specifically with William Wheaton's (1977c) contention that observed patterns of location result not from utility maximization as described by Alonso and Muth but from the flight of the rich away from the racial and social characteristics and tax burden of the poor. There are strong policy implications, for if the pattern results from utility maximization involving "neutral" variables, some sort of optimum is achieved through the free operation of markets, and laissez-faire would be the best policy, whereas if the pattern results from the flight of the rich for morally repugnant reasons, the remedy would involve positive policy measures to prevent flight, particularly fiscal consolidation of central city and suburbs. In this chapter I will discuss the theory which Wheaton is attacking; consideration of Wheaton's attack will be deferred to Chapter Five, where I discuss policy.

The literature on location and income is fairly extensive, I surmise because of the policy importance of the question. The studies can be divided into two groups, those whose conclusion is that location distance increases with income, and those whose conclusion is that distance can increase, decrease, or remain constant with income.

The theorists whose conclusion is that distance increases with income are Beckmann, Montesano, and Solow. Montesano showed that Beckmann's reasoning was faulty. I shall show, later in this chapter, that Montesano's own result (that distance increases with income) is entirely due to accidental, and not generally true, implications of the log-linear utility function he uses. Solow is one of those who leaves distance out of the utility function (see the preceding chapter), and I shall show why his result does not generally hold for the case where distance is included in the utility function.

Those who conclude that distance can either increase or decrease with income are Muth and Mohring (1976). (Alonso and Wheaton (1977c) also conclude that distance can increase or decrease, but their work is within the bid price framework and hence outside the purview of this dissertation.) My own finding, to be presented in this chapter, is also that distance can increase or decrease with income; my result is different from those of Muth and Mohring in its identification of the factors determining the direction of change with income. Mohring does not really identify the critical factors, while Muth's critical factors are different from those I will identify.

My initial interest in urban location theory was rooted in the sense that the existing explanations of the relation between location and income were deeply puzzling. It seemed to me that distance belonged in the utility function, and that one would expect closeness to have the characteristics of a typical normal good. If so, reasoning by analogy would yield the implication that the richer would live closer in. This seeming implication is so clearly in conflict with the empirical evidence as perhaps to convince some theorists that distance does not belong in the utility function. My own effort has been to discover how a utility function in which distance appears, implies a relation between location and income that accords with the empirical evidence. I sought a theory which would not only predict what we observe, but which would also satisfyingly explain what was happening in terms of traditional price theory. This chapter give the results of that effort.

As to the "facts" to be explained by theory: the conventional wisdom is that in general the poor live in the central cities and the rich live in the suburbs. But it is also the case, certainly in New York City anyway, that many of the very richest live very close to the center. There seems to be a mix of incomes at any given distance (often in distinct neighborhoods or communities), except that one doesn't find the poor locating far out and commuting long distances. Adequate theory should be able to predict these facts.

I will use the traditional mathematical methods of comparative statics to get my results, and after amplifying through use of geometry, will proceed with more detailed discussion of the literature I have already alluded to.

In the previous chapter we used the first-order condition equations (3) to solve for z^*, q^*, t^*, and λ^*, the individual's utility-maximizing set, so we can write these variables as functions of the exogenous variables:

$$z^* = z^* \, [p, R(t), K(t), y]$$

$$q^* = q^* \, [p, R(t), K(t), y]$$

$$t^* = t^* \, [p, R(t), K(t), y]$$

$$\lambda^* = \lambda^* \, [p, R(t), K(t), y] \quad .$$

In the above equations, $R(t)$ and $K(t)$ are taken to be the entire rent and commuting cost functions. To solve for $\partial t^*/\partial y$, we take the total differentials of the first-order condition equations (3), we set $dp = dR(t^*) = dK(t^*) = dR'(t^*) = dK'(t^*) = 0$ because we are interested only in the effects of varying y, we divide by dy, and so get the following matrix equation:

$$\begin{bmatrix} 0 & -p & -R(t^*) & -A^* \\ -p & U_{zz} & U_{qz} & U_{tz} \\ -R(t^*) & U_{zq} & U_{qq} & B^* \\ -A^* & U_{zt} & B^* & C^* \end{bmatrix} \begin{bmatrix} \partial\lambda^*/\partial y \\ \partial z^*/\partial y \\ \partial q^*/\partial y \\ \partial t^*/\partial y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Solving by Cramer's rule, we get

$$\frac{\partial t^*}{\partial y} = \frac{1}{|J|^*} \Big\{ -p(U_{zq}B^* - U_{zt}U_{qq}) - U_{zz}\,[-R(t^*)\,B^* + A^*\,U_{qq}]$$
$$+ U_{qz}\,[-R(t^*)\,U_{zt} + A^*\,U_{zq}] \Big\}$$

where $|J|^*$ is the determinant of the Jacobian matrix above. (From here on in, I shall drop the asterisks; the reader is reminded that equilibrium values are referred to.) On our assumption that $U_{zq} = U_{zt} = U_{qt} = U_{qz} = 0$, this reduces to

$$\frac{\partial t^*}{\partial y} = \frac{-U_{zz}}{|J|}\,[AU_{qq} + R(t)\,\lambda R'(t)] \quad .$$

Therefore

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad AU_{qq} + R(t)\,\lambda R'(t) \lesseqgtr 0. \tag{9}$$

At this point we have a criterion for $\partial t^*/\partial y$, but neither its sign nor its meaning is obvious. For clarification, we look at the budget and indifference surface traces in the (q, t) plane for z held constant at z^*; we want to see what happens to the slopes of the traces when we increase q a unit while holding t constant at t^*. We get

$$\frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_Y = \frac{-R'(t)}{R(t)}$$

and

$$\frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_I = \frac{U_tU_{qq}}{U_q^2} \quad .$$

Now, going back to our criterion condition (9), we can substitute in, from equations (4),

$$A = \frac{U_tR(t)}{U_q}$$

and

$$\lambda = \frac{U_q}{R(t)} \quad .$$

and manipulate a bit, and get

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad \frac{U_tU_{qq}}{U_q^2} \lesseqgtr \frac{-R'(t)}{R(t)} \quad ,$$

and therefore

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad \frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_I \lesseqgtr \frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_y \quad . \qquad (10)$$

In words, whether an increase in income produces a move farther from or closer to the center depends on whether the increase in slope of the budget trace is larger or smaller than the increase in slope of the indifference trace, when q increases while z and t are held constant. Condition (10) is determinative because, when slopes shift such that

$$\frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_I \substack{\leq \\ >} \frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_y ,$$

the point of two tangencies [-Ut/Uq = -A/R(t) and $U_z/U_q = p/R(t)$] on the new budget surface is such that the new t* is always larger than/equal to/ smaller than the old t*.

It may help to explain the meaning of condition (10) if we contrast the relation between income and location to the relation between income and the standard normal good of the theory of consumer choice. At the crux of the matter is the unusual budget surface we are dealing with in the economics of location. We saw in Chapter Two that the slope of the trace of the budget surface in a (q, t) plane may well vary with t. It is also true that the slope of this trace increases with y, for variation in q at a given t:

$$\frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_y \frac{\partial q}{\partial y} = \frac{-\partial A/\partial q}{R(t)} \frac{\partial q}{\partial y} = \frac{-R'(t)}{R(t)^2} > 0 \quad .$$

Therefore households with different income levels are facing different price ratios at a given t, all other things equal. The price ratio, $-A/R(t)$, increases with y because -A increases with y; -A increases with y because q increases with y. We have richer persons facing a higher cost of closeness because, since they occupy more housing space, the higher rents as one moves closer to the center hit them more heavily. By contrast, in the theory of consumer choice the price ratio does not vary with income.

The relation in a (q, t) plane between indifference surface traces representing higher and lower utility levels is that at every t, the higher indifference trace has a larger slope than the lower indifference trace:

$$\frac{\partial}{\partial q}\left(\frac{dq}{dt}\right)_I = \frac{U_t U_{qq}}{U_q^2} > 0 \quad .$$

But there is the same kind of slope change in the case of consumer choice between two normal goods, so that there are no differences on the utility-function side of things to account for the unusual result. Rather, what we have in the normal-good consumer-choice case is that the increase in the indifference trace slope is <u>always</u> greater than the increase in the budget trace slope — because there is no increase at all in the budget trace slope. Hence, by our condition an income increase always causes increased consumption of the normal good in question. We can conclude that when consumption of closeness falls with income, it is because of the relatively large magnitude of the increase in the price of closeness with income.

It is clear, therefore, that even though closeness is a normal good, its consumption can decrease with an income increase: the theory does explain that a richer person can locate farther out than a poorer person with the same utility function. Note especially that we do not have to assume that closeness is an inferior good, or a non-good in the sense of not appearing in the utility function, to get this result. By the same token, it is also clear that a richer person can locate closer in than a poorer person with the same utility function.

The condition for $\partial t^*/\partial y$ derived here differs substantially from the others in the literature. Muth's condition (in its graphical version) is that $\partial t^*/\partial y$ will be positive/zero/negative according as, at the initial location, the increase in $-qR'(t)$ is greater than/equal to/less than the increase in marginal costs of transportation, the latter consisting of both money and time costs. My condition and Muth's are distinctly different from each other: manipulation does not dissolve the one into the other. Also, the two conditions do predict differently. For instance, Muth predicts that if an income increase is not the result of an increase in the wage, equilibrium distance will always increase, since in this case $-qR'(t)$ will increase at the initial location while marginal transportation costs will be unchanged (p. 30). My own prediction is that distance can increase, remain constant, or decrease, since for this case the slopes of the indifference and budget traces do both shift.

De Salvo, whose work is a mathematical expansion of Muth's, presents two versions of his condition. One is similar to Muth's condition above; the other is also different from mine, and de Salvo does not go into its meaning.

Mohring (1976) presents the only other criterion condition for $\partial t^*/\partial y$ in the literature. His condition can be stated

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff } E_{Ly} \gtreqless E_{Ty} + E_{V_y} V/(C' + V)$$

where E_{L_Y}, E_{T_Y}, and E_{V_Y} are respectively the income elasticities of demand for land, trips, and the value of travel time, V is the value of travel time, and C' is marginal money travel costs. This condition is arrived at by rearranging and totally differentiating two of the first-order conditions, without actually solving. As such, it relates change in one dependent variable to changes in others, rather than to the independent variables.

The result I have gotten on the relation between location and income contrasts sharply with the findings of Beckmann, Montesano, and Solow that an increase in income must always produce relocation farther from the city center. Montesano pointed out the error in Beckmann's work, which latter I therefore will not discuss.

Montesano includes distance in the utility function as I do, yet he gets the result that distance must always increase with income, and since his work has great mathematical sophistication, it is very provocative. Putting his utility function in the form

$$U = c_o \log q + \log \frac{1}{t} + \sum_{i=2}^{n} c_i \log z_i$$

where c_o and the c_i are constants, he uses the second-order conditions to show that dy/dt must always be positive. The result, however, depends on certain relations which hold true for this log-linear utility function but not for a general utility function, these being

$$\begin{aligned} U_q &= -qU_{qq} \\ U_{\frac{1}{t}} &= -\frac{1}{t}U_{\frac{1}{t}\frac{1}{t}} \\ U_{z_i} &= -z_iU_{z_iz_i} \qquad , \qquad i = 2, \ldots, n. \end{aligned}$$

These relations serve to cancel many terms in the second-order conditions which would not otherwise disappear, so that Montesano could extract a result from second-order conditions whereas, in general, second-order conditions cannot imply comparative-static results. Since Montesano's utility and budget equations are basically the same as mine except for his log-linear form of the utility function, his result should be the same as mine except as there are special implications of the log-linear form. If he had solved for dy/dt in the standard comparative-static manner (rather than by manipulating the second-order conditions), he would have gotten my condition for $\partial t^*/\partial y$, but stated in terms of his log-linear utility function.

Using, therefore, my condition (10) for $\partial t^*/\partial y$, and substituting into it the log-linear relation $U_{qq} = -U_q/q$, and, from (4), $U_q = \lambda R(t)$ and $U_t/\lambda = A$, we get

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad \frac{-qR'(t)}{qR(t)} \quad \frac{-K'(t)}{qR(t)} \lesseqgtr \frac{-R'(t)}{R(t)} \ ,$$

and hence

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad \frac{-K'(t)}{qR(t)} \lesseqgtr 0 \quad .$$

Since $-K'(t)/[qR(t)]$ is always negative at a point of equilibrium, the log-linear form gives the result that $\partial t^*/\partial y$ is always positive. So we can see why, when distance is included in the utility function, the log-linear form implies $\partial t^*/\partial y > 0$, whereas in the general case there is no such implication. Theory and numerical analysis based on a log-linear utility function, such as that of Mohring (1976) and Solow (1972) as well as Montesano, are therefore open to question.

Solow's argument that the rich will live farther out is based on Muth's model. For his case of a population with homogeneous tastes and two income levels, he shows that all those with a particular income level will live in a band for which the rents are such as to keep them indifferent regarding different locations within that band, according to the relation

$$R'(t) = \frac{-K'(t)}{q} \quad .$$

Arguing, next, that $R'(t)$ must be steeper in the band closer to the city center, he uses the equation above to demonstrate that a lower income level produces a steeper $R'(t)$: if $K'(t)$ is constant and q increases with y, then $R'(t)$ becomes flatter with y. (In a footnote, Solow acknowledges that if $K'(t)$ includes time costs of travel which increase enough with y, the richer will live in the more central band.) But this argument can hold only for the case where distance is left out of the utility function: with distance in the utility function the corresponding equation becomes

$$R'(t) = \frac{1}{q}\left[p\frac{U_t}{U_z} - K'(t)\right] \quad ,$$

so

$$\frac{dR'(t)}{dy} = \frac{1}{q}\left[p\frac{U_{tt}}{U_z}\frac{\partial t^*}{\partial y} - p\frac{U_t}{U_z^2}U_{zz}\frac{\partial z^*}{\partial y} - R'(t)\frac{\partial q^*}{\partial y}\right] \quad ,$$

which is of indeterminate sign.

A new result of mine on the relation of location and income is an explanation of why low-income households may not be found living at and commuting from the more distant locations. We saw in the previous chapter that the budget surface trace in a (q, t) plane may be concave, convex, or linear — that we can have $(d^2q/dt^2)_y$ greater than, equal to, or less than zero. Here we look at a special situation which may well arise when $(d^2q/dt^2)_y$ is negative. For smaller values of t (at least), the function will be increasing at a decreasing rate. One might well find, within the relevant range of distances, a distance at which the slope of the trace is equal to zero and beyond which the slope is negative. Since, as we showed earlier, at a point of equilibrium the slope of the budget trace must be positive, we would find no lower-income residents at the more distant locations.

The reason is that at these distances commuting costs get too high for those with lower incomes. If the slope of the budget trace is negative, we have

$$\frac{-A}{R(t)} < 0 \qquad ,$$

which implies

$$K'(t) > -qR'(t) \qquad .$$

That is, for these people, at these distances, marginal commuting costs are greater than marginal rent savings as t increases: commuting costs are so high, relative to rent savings, for individuals who can afford only small residences, that the cost of closeness becomes negative. We have commuting costs so high that lower-income households have nothing to gain by living farther out, and so do not.

There is no inconsistency in having the low-income budget trace negatively sloped at greater distances -- so that those with low incomes locate farther in -- while at the same time higher-income budget traces are positively sloped so that those with higher incomes do locate at these distances. The two situations are mutually consistent because, at a given distance, so long as R"(t) isn't both positive and relatively large, the change with distance of the slope of a budget trace increases with income. For all but the largest values of R"(t), a mapping of budget traces onto a (q , t) plane will show, at a given distance, either a progression from $(d^2q/dt^2)_Y < 0$ to $(d^2q/dt^2)_Y = 0$ to $(d^2q/dt^2)_Y > 0$ with income, or else all traces will have $(d^2q/dt^2)_Y > 0$; for a proof of this proposition see Appendix B. So in the case where we have the progression with income from $(d^2q/dt^2)_Y < 0$ to $(d^2q/dt^2)_Y > 0$, at farther distances there will be no low-income residents and yet there will be higher-income residents, because the former will have negative budget trace slopes while at the same time the latter have positive budget-trace slopes.

CHAPTER FOUR

COMPARATIVE STATICS

The purpose of this chapter is to examine how changes in the model's exogenous variables (household income, y; the price, p, of the composite consumer good; the rent function, R(t); and the travel cost function K(t)) produce changes in the endogenous variables (household demand for location, t; housing space, q; and the composite good, z). The material in the preceding chapter, on income and location, is of course also formally a part of the comparative statics, but it was dealt with in a separate chapter because urban location is the main focus of this study and as such received the fullest treatment. The analysis in this chapter is more terse and takes only passing notice of the very large general literature on housing demand and demand in general. Rather, typically I shall comment on housing and general demand theory only as appearing in the location theory literature. In what follows in this chapter I will write first at some length about the relation between housing demand and income, and then more briefly about the relations among the other variables.

All the comparative static results are found by initially taking the total differentials of the first-order condition equations, then setting equal to zero all changes in exogenous variables that are not of immediate interest, then solving the resulting matrix equation by Cramer's rule. Proceeding to examine the effect of varying income on household demand for housing space, we get the same matrix equation as in Chapter Three, and the solution is

$$\frac{\partial q^*}{\partial y} = \frac{-J_{13}}{|J|} = \frac{-1}{|J|} \left[-pCU_{zq} + pBU_{zt} + R(t)CU_{zz} - ABU_{zz} - R(t)U_{zt}^2 + AU_{tz}U_{zq} \right]$$

where $|J|$ is the determinant of the Jacobian. On our assumption that $U_{zq} = U_{zt} = U_{tz} = U_{tq} = 0$, we have

$$\frac{\partial q^*}{\partial y} = \frac{-U_{zz}}{|J|} \left[R(t)\left(U_{tt} - \lambda \frac{\partial A}{\partial t}\right) + A\lambda R'(t)\right] ,$$

so that

$$\frac{\partial q^*}{\partial y} \begin{smallmatrix}\geq \\ <\end{smallmatrix} 0 \quad \text{iff} \quad R(t)\left(U_{tt} - \lambda \frac{\partial A}{\partial t}\right) + A\lambda R'(t) \begin{smallmatrix}\leq \\ >\end{smallmatrix} 0 .$$

Substituting, from (4), $\lambda = U_q/R(t)$, and manipulating, we get

$$\frac{\partial q^*}{\partial y} \begin{smallmatrix}\geq \\ <\end{smallmatrix} 0 \quad \text{iff} \quad \frac{U_{tt}}{U_q} \begin{smallmatrix}\leq \\ >\end{smallmatrix} \frac{R(t)\frac{\partial A}{\partial t} - AR'(t)}{R(t)^2} .$$

Since, holding z constant,

$$-\frac{\partial}{\partial t}\left(\frac{dq}{dt}\right)_I = \frac{-\partial}{\partial t}\left(\frac{-U_t}{U_q}\right) = \frac{U_{tt}}{U_q}$$

and

$$-\frac{\partial}{\partial t}\left(\frac{dq}{dt}\right)_Y = -\frac{\partial}{\partial t}\left[-\frac{A}{R(t)}\right] = \frac{R(t)\frac{\partial A}{\partial t} - AR'(t)}{R(t)^2} ,$$

our critical inequality translates into

$$\frac{\partial q^*}{\partial y} \begin{smallmatrix}\geq \\ <\end{smallmatrix} 0 \quad \text{iff} \quad -\frac{\partial}{\partial t}\left(\frac{dq}{dt}\right)_I \begin{smallmatrix}\leq \\ >\end{smallmatrix} -\frac{\partial}{\partial t}\left(\frac{dq}{dt}\right)_Y .$$

The import of the criterion is that if we focus on the plane for which $z=z^*$ and compare conditions at the initial point with conditions at the point on the new budget trace for which q is unchanged and t is reduced a unit,

q^* will increase (remain unchanged/ decrease) if the change in slope of the intercepting indifference trace is less than (the same as/ more than) the slope change of the budget trace.

Seemingly, then, demand for housing space can decrease with income under certain conditions. This implication flies in the face of what we know from theory of consumer choice, so it is worthwhile to investigate in some detail how this can happen. As t decreases, the change in slope of the indifference surface trace is negative: $-\partial(dq/dt)_I/\partial t = U_{tt}/U_q < 0$. So for the change in slope of the indifference trace to be greater than the change in slope of the budget trace, the budget trace slope change must be negative and sufficiently great in absolute value.

The change in slope of the budget trace can be negative or positive, depending on the slope of the budget trace. Figure 3 shows the relation between the shape of the budget trace and its slope change as t decreases. The slope change from A to C equals the slope change from A to B plus the slope change from B to C, so the change from A to B equals the change from A to C minus the change from B to C. The change from B to C is positive, as we saw in the previous chapter. If the change from A to C is positive and relatively large in absolute value, the change from A to B will be positive and relatitively large in absolute value, and therefore the change from B to A will be negative and relatively large in absolute value. So housing demand can fall with income only if $(d^2q/dt^2)_Y$ is positive and sufficiently large.

Intuitively, what is happening when housing demand decreases with income is that the relative price of closeness falls when one moves from a position like B to a position like A, and falls so much that the relative marginal value of closeness at B is greater than its relative

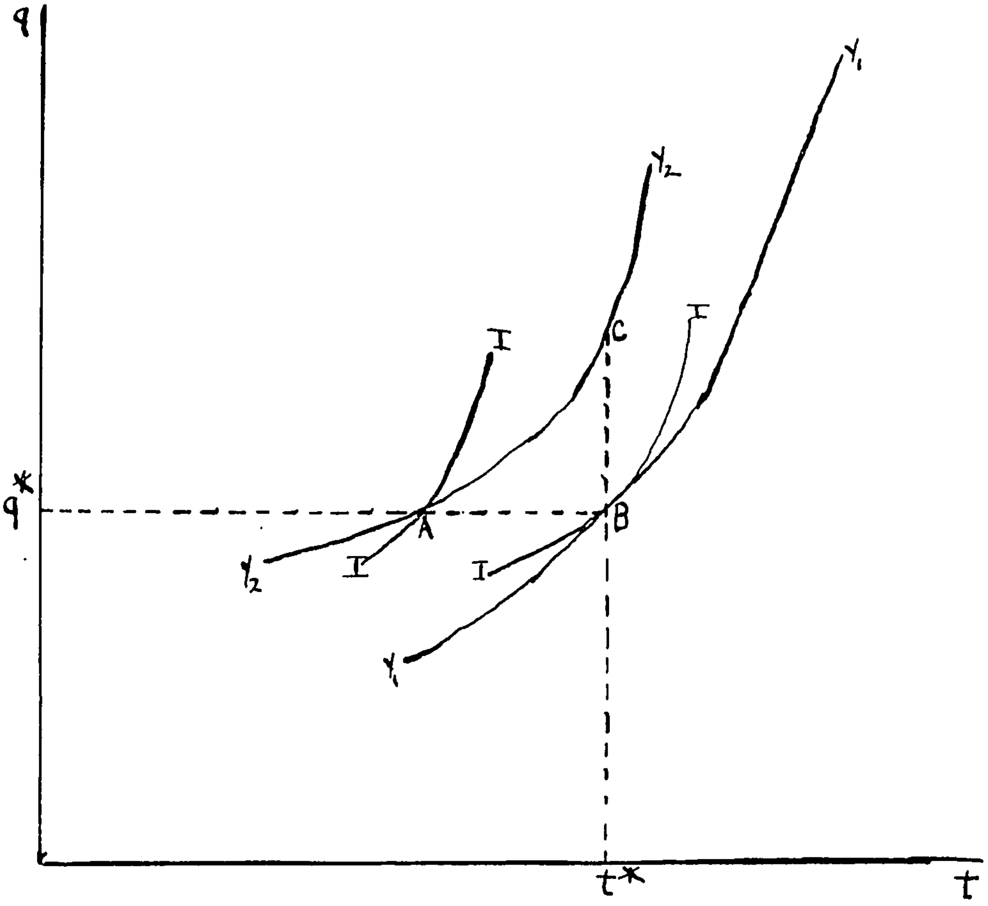

FIGURE 3. CONFIGURATION FOR $\partial q^*/\partial y < 0$

price there; the individual will therefore reach equality of relative marginal value and price only by increasing closeness to such an extent that it is at the expense of housing space. And this mechanism operates only with a budget trace with $(d^2q/dt^2)_y > 0$ because only for such a trace does the price of closeness fall as closeness increases.

Our criterion for $\partial q^*/\partial y$ also gives us additional information. We have

$$\frac{\partial q^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad R(t)U_{tt} - R(t)\lambda\frac{\partial A}{\partial t} + A\lambda R'(t) \lesseqgtr 0 \quad .$$

The second-order condition tells us

$$|\bar{H}| = -R(t)^2U_{tt} + R(t)^2\lambda\frac{\partial A}{\partial t} - 2R(t)A\lambda R'(t) - A^2U_{qq} > 0 \quad ;$$

hence

$$[R(t)U_{tt} - R(t)\lambda\frac{\partial A}{\partial t} + A\lambda R'(t)] + \frac{A}{R(t)}[R(t)\lambda R'(t) + AU_{qq}] < 0 \quad .$$

If we have $\partial q^*/\partial y \underset{<}{=} 0$, we have to have $R(t)\lambda R'(t) + AU_{qq} > 0$. But

$$\frac{\partial t^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad AU_{qq} + R(t)\lambda R'(t) \lesseqgtr 0 \quad .$$

We conclude that we can have $\partial q^*/\partial y = 0$ or $\partial q^*/\partial y < 0$ only if we have $\partial t^*/\partial y < 0$. Also, if we have $\partial t^*/\partial y > 0$ or $\partial t^*/\partial y = 0$, we have to have $\partial q^*/\partial y > 0$.

We now consider $\partial z^*/\partial y$. We have

$$\frac{\partial z^*}{\partial y} = -\frac{J_{12}}{|J|} = \frac{1}{|J|}[-pU_{qq}C + pB^2 + U_{qz}R(t)C - U_{qz}AB - U_{tz}R(t)B + U_{tz}U_{qq}A] \quad .$$

On our assumption of independence of marginal utilities, we have

$$\frac{\partial z^*}{\partial y} = \frac{-p}{|J|}[U_{qq}(U_{tt} - \lambda\frac{\partial A}{\partial t}) - \lambda^2R'(t)^2] \quad .$$

So

$$\frac{\partial z^*}{\partial y} \gtreqless 0 \quad \text{iff} \quad U_{qq}(U_{tt}-\lambda \frac{\partial A}{\partial t}) - \lambda^2 R'(t)^2 \gtreqless 0 \quad .$$

To know more about $\partial z^*/\partial y$ we look at $\partial \lambda^*/\partial y$:

$$\frac{\partial \lambda^*}{\partial y} = \frac{-J_{11}}{|J|} = \frac{-1}{|J|} \quad (U_{zz}U_{qq}C - U_{zz}B^2 - U_{zq}{}^2C + 2U_{qz}U_{zz}B - U_{zt}{}^2U_{qq}) \quad ,$$

which gives us, assuming independence,

$$\frac{\partial \lambda^*}{\partial y} = \frac{-U_{zz}}{|J|} [U_{qq}(U_{tt}-\lambda \frac{\partial A}{\partial t}) - \lambda^2 R'(t)^2] \quad .$$

If we now introduce the standard microeconomic assumption of diminishing marginal utility of money, i.e. that $\partial\lambda^*/\partial y < 0$, we see that

$$U_{qq}(U_{tt} - \lambda \frac{\partial A}{\partial t}) - \lambda^2 R'(t)^2 > 0 \quad .$$

We therefore conclude that $\partial z^*/\partial y$ is always positive.

In the rest of this chapter we examine the effects of varying our other exogenous variables.

To find the effect of varying p while holding the other exogenous variables constant, we take our totally differentiated first-order condition equations, set $dR(t) = dK(t) = dR'(t) = dK'(t) = dy = 0$ and get

$$\begin{bmatrix} 0 & -p & -R(t) & -A \\ -p & U_{zz} & U_{qz} & U_{tz} \\ -R(t) & U_{zq} & U_{qq} & B \\ -A & U_{zt} & B & C \end{bmatrix} \begin{bmatrix} \partial\lambda^*/\partial p \\ \partial z^*/\partial p \\ \partial q^*/\partial p \\ \partial t^*/\partial p \end{bmatrix} = \begin{bmatrix} z \\ \lambda \\ 0 \\ 0 \end{bmatrix} \quad .$$

Solving by Cramer's rule gives

$$\frac{\partial z^*}{\partial p} = z\frac{J_{12}}{|J|} + \lambda\frac{J_{22}}{|J|} = T_1 + T_2$$

$$\frac{\partial q^*}{\partial p} = z\frac{J_{13}}{|J|} + \lambda\frac{J_{23}}{|J|} = T_3 + T_4$$

$$\frac{\partial t^*}{\partial p} = z\frac{J_{14}}{|J|} + \frac{\lambda J_{24}}{|J|} = T_5 + T_6 \quad .$$

Terms T_1, T_3, and T_5 are all income effects: $T_1 = -z\, \partial z^*/\partial y$, $T_3 = -z\, \partial q^*/\partial y$, and $T_5 = -z\, \partial t^*/\partial y$. We found that, assuming all our goods are normal, $\partial z^*/\partial y$ is positive, while $\partial q^*/\partial y$ and $\partial t^*/\partial y$ are indeterminate, so T_1 is negative while T_3 and T_5 are indeterminate. Evaluating T_2, the substitution effect in $\partial z^*/\partial p$, we observe that λ is positive; $|J|$ has the same value as $|H|$, which by the second-order condition is negative; and J_{22} is a principal minor of $|\overline{H}|$ and hence by the second-order condition is positive. T_2 therefore being negative, $\partial z^*/\partial p$ is negative.

Expanding T_4, the substitution effect in $\partial q^*/\partial p$, we have, on our assumptions,

$$T_4 = \frac{\lambda J_{23}}{|J|} = \frac{\lambda p}{|J|} \left[R(t)\left(U_{tt} - \lambda\frac{\partial A}{\partial t}\right) + A\lambda R'(t)\right] = \frac{-\lambda p}{U_{zz}} \frac{\partial q^*}{\partial y} .$$

Thus T_3 is opposite in sign to $\partial q^*/\partial y$ and T_4 has the same sign as $\partial q^*/\partial y$. Since $\partial q^*/\partial y$ can have any sign, $\partial q^*/\partial p$ can have any sign.

The analysis for $\partial t^*/\partial p$ is the same as for $\partial q^*/\partial p$. We have

$$\frac{\partial t^*}{\partial p} = T_5 + T_6 = -z\frac{\partial t^*}{\partial y} \frac{-\lambda\, p \partial t^*}{U_{zz} \partial y} ,$$

so for the same reasons, $\partial t^*/\partial p$ can have any sign.

Now we look at the effects of variation in the rent function. The literature has it that the rent function is probably best approximated by a negative exponential function, but I found that form too difficult to work with for the present purpose. Instead I will use the linear form $R(t) = c - gt$, where the parameter c gives the level of the rent function at the inner edge of the residential area (where $t = 0$), and the (positive) parameter g gives the rate of rent decrease with t.

Looking first at the effects of variation in c, we get the matrix equation

$$\begin{bmatrix} 0 & -p & -R(t) & -A \\ -p & U_{zz} & U_{qz} & U_{tz} \\ -R(t) & U_{zq} & U_{qq} & B \\ -A & U_{zt} & B & C \end{bmatrix} \begin{bmatrix} \partial\lambda^*/\partial c \\ \partial z^*/\partial c \\ \partial q^*/\partial c \\ \partial t^*/\partial c \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ \lambda \\ 0 \end{bmatrix} .$$

The solution is

$$\frac{\partial z^*}{\partial c} = \frac{qJ_{12}}{|J|} + \frac{\lambda J_{32}}{|J|} = T_1 + T_2$$

$$\frac{\partial q^*}{\partial c} = \frac{qJ_{13}}{|J|} + \frac{\lambda J_{33}}{|J|} = T_3 + T_4$$

$$\frac{\partial t^*}{\partial c} = \frac{qJ_{14}}{|J|} + \frac{\lambda J_{34}}{|J|} = T_5 + T_6 .$$

T_1, T_3, and T_5 are all income effects: $T_1 = -q\partial z^*/\partial y$, $T_3 = -q\partial q^*/\partial y$, and $T_5 = -q\partial t^*/\partial y$. We know, then, that T_1 is negative while T_3 and T_5 can have any sign. As to the substitution effects:

$$T_2 = \frac{\lambda J_{32}}{|J|} = \frac{\lambda p}{|J|} \left[R(t)\left(U_{tt}-\lambda\frac{\partial A}{\partial t}\right) + A\lambda R'(t)\right]$$

$$= \frac{-\lambda p}{U_{zz}} \frac{\partial q^*}{\partial y} .$$

Since the sign of $\partial q^*/\partial y$ is indeterminate, so is the sign of T_2. T_4 is negative, since J_{33} is a principal minor of $|J|$ and is therefore positive. And we have

$$T_6 = \frac{\lambda J_{34}}{|J|} = \frac{\lambda}{|J|} \left[R(t)AU_{zz}-p^2\lambda R'(t)\right] ,$$

which evaluation reveals to be negative. Since $\partial z^*/\partial c$, $\partial q^*/\partial c$, and

$\partial t^*/\partial c$ all contain indeterminate terms, we see that all of these comparative static derivatives are indeterminate.

Now we look at the effects of varying g. The matrix equation is

$$\begin{bmatrix} 0 & -p & -R(t) & -A \\ -p & U_{zz} & U_{qz} & U_{tz} \\ -R(t) & U_{zq} & U_{qq} & B \\ -A & U_{zt} & B & C \end{bmatrix} \begin{bmatrix} \partial\lambda^*/\partial g \\ \partial z^*/\partial g \\ \partial q^*/\partial g \\ \partial t^*/\partial g \end{bmatrix} = \begin{bmatrix} -qt \\ 0 \\ -\lambda t \\ -\lambda q \end{bmatrix}$$

and the solution is

$$\frac{\partial z^*}{\partial g} = \frac{-tq^{J}12}{|J|} - \frac{\lambda t^{J}32}{|J|} - \frac{\lambda q^{J}42}{|J|} = T_1 + T_2 + T_3$$

$$\frac{\partial q^*}{\partial g} = \frac{-qt^{J}13}{|J|} - \frac{\lambda t^{J}33}{|J|} - \frac{\lambda q^{J}43}{|J|} = T_4 + T_5 + T_6$$

$$\frac{\partial t^*}{\partial g} = \frac{-qt^{J}14}{|J|} - \frac{\lambda t^{J}34}{|J|} - \frac{\lambda q^{J}44}{|J|} = T_7 + T_8 + T_9 \quad .$$

T_1, T_4, and T_7 are income effects: the increase in g reduces rent levels and therefore affects expenditures. $T_1 = qt\partial z^*/\partial y$ and is therefore positive. $T_4 = qt\partial q^*/\partial y$ and $T_7 = qt\partial t^*/\partial y$; hence both are indeterminate. T_2, T_5, and T_8 are substitution effects of the rent level changes; we have $T_2 = -t\partial z^*/\partial c_s$, $T_5 = -t\partial q^*/\partial c_s$, and $T_8 = -t\partial t^*/\partial c_s$ where the subscript s denotes the substitution effect. These effects are of opposite sign to those caused by changes in c because an increase in c raises rent levels while an increase in g reduces them. So T_2 here is indeterminate, while $T_5$and T_8 are positive. T_3, T_6, and T_9 are the direct substitution effects of decreasing the slope of the rent function and thereby decreasing A. We have

$$T_3 = \frac{-\lambda q J_{42}}{|J|} = \frac{-\lambda q}{|J|}\left\{p\,[AU_{qq} + R(t)\ \lambda\ R'(t)]\right\}$$

$$= \frac{\lambda q}{U_{zz}}\,p\,\frac{\partial t^*}{\partial y}$$

$$T_6 = \frac{-\lambda q J_{43}}{|J|} = \frac{-\lambda q}{|J|}\,[p^2\lambda R'(t) - AR(t)U_{zz}]$$

$$T_9 = \frac{-\lambda q J_{44}}{|J|}\ .$$

Since $\partial t^*/\partial y$ is indeterminate, so is T_3. By evaluation, T_6 is negative. And since J_{44} is a principal minor of $|J|$ and is therefore positive, T_9 is positive. Thus, when the rent function gets steeper, the price of closeness increases, and so less closeness is demanded. The ultimate result of all these effects is that $\partial z^*/\partial g$, $\partial q^*/\partial g$, and $\partial t^*/\partial g$ are all indeterminate.

We come finally to the effects of variation in transportation costs. We would expect the transportation cost function to be more or less linear, so we write $K(t) = b + kt$, and for this linear case we therefore consider our endogenous variables to be functions of b and k rather than of $K(t)$ (as well as of y, p, and $R(t)$). We must rewrite the first and last of our equations giving the total differentials of equations (3) to reflect this change. Then, to look at variation of b we set $dp = dR(t) = dR'(t) = dk = dy = 0$ and get

$$\begin{bmatrix} 0 & -p & -R(t) & -A \\ -p & U_{zz} & U_{qz} & U_{tz} \\ -R(t) & U_{zq} & U_{qq} & B \\ -A & U_{zt} & B & C \end{bmatrix} \begin{bmatrix} \partial\lambda^*/\partial b \\ \partial z^*/\partial b \\ \partial q^*/\partial b \\ \partial t^*/\partial b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\ .$$

Our solutions are

$$\frac{\partial z^*}{\partial b} = \frac{J_{12}}{|J|}$$

$$\frac{\partial q^*}{\partial b} = \frac{J_{13}}{|J|}$$

$$\frac{\partial t^*}{\partial b} = \frac{J_{14}}{|J|} \quad .$$

The reader can see that these partials are simply the negatives of the partials with respect to income. Thus, for example, imposition of a toll at the edge of the CBD, in such a way that it would affect all commuters, would be identical in effect to lowering all incomes: effects on q and t would be indeterminate, while consumption of z would decrease.

The matrix equation for variation of k is

$$\begin{bmatrix} 0 & -p & -R(t) & -A \\ -p & U_{zz} & U_{qz} & U_{tz} \\ -R(t) & U_{zq} & U_{qq} & B \\ -A & U_{zt} & B & C \end{bmatrix} \begin{bmatrix} \partial\lambda^*/\partial k \\ \partial z^*/\partial k \\ \partial q^*/\partial k \\ \partial t^*/\partial k \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ \lambda \end{bmatrix} \quad .$$

Solving, we get

$$\frac{\partial z^*}{\partial k} = \frac{t\ J_{12}}{|J|} + \frac{\lambda\ J_{42}}{|J|} = T_1 + T_2$$

$$\frac{\partial q^*}{\partial k} = \frac{t\ J_{13}}{|J|} + \frac{\lambda\ J_{43}}{|J|} = T_3 + T_4$$

$$\frac{\partial t^*}{\partial k} = \frac{t\ J_{14}}{|J|} + \frac{\lambda\ J_{44}}{|J|} = T_5 + T_6 \quad .$$

The terms T_1, T_3, and T_5 are the income effects. T_2, T_4, and T_6 are the same as terms produced by variation of g, but divided by -q. We can conclude that $\partial z^*/\partial k$, $\partial q^*/\partial k$, and $\partial t^*/\partial k$ are all indeterminate.

My comparative static results are summarized in Table 1. The table also shows the results derived by Muth and de Salvo, who are the only

TABLE 1. COMPARATIVE STATIC RESULTS

	My Results	Muth's Results	De Salvo's Results
$\partial z^*/\partial y$	> 0	----	Indeterminate
$\partial q^*/\partial y$	Indeterminate	> 0	> 0
$\partial t^*/\partial y$	"	Indeterminate	Indeterminate
$\partial z^*/\partial p$	< 0	----	----
$\partial q^*/\partial p$	Indeterminate	----	----
$\partial t^*/\partial p$	"	----	----
$\partial z^*/\partial c$	"	----	Indeterminate
$\partial q^*/\partial c$	"	----	< 0
$\partial t^*/\partial c$	"	< 0	"
$\partial z^*/\partial g$	"	----	Indeterminate
$\partial q^*/\partial g$	"	----	> 0
$\partial t^*/\partial g$	"	----	"
$\partial z^*/\partial b$	< 0	----	Indeterminate
$\partial q^*/\partial b$	Indeterminate	----	< 0
$\partial t^*/\partial b$	"	< 0	"
$\partial z^*/\partial k$	"	----	Indeterminate
$\partial q^*/\partial k$	"	----	< 0
$\partial t^*/\partial k$	"	< 0	"

ones to have done systematic monocentric city comparative statics within a utility-maximization framework.[3] The reader will observe that the great majority of my results are different from theirs. It should be clear, then, that my model and Muth's (on which de Salvo's work is based) are in fact vitally different, and that Muth's leaving distance out of the utility function is not offset by the other original features of his model.

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[3] Wheaton (1974) does a comparative statics study from a bid price perspective; his variables are different from those considered here.

CHAPTER FIVE

CONCLUSION

I said in Chapter One that my goals in this dissertation were to present and analyze the distance-included utility-maximization model of urban residential location, derive new explanations and results, and show that this model is different from and preferable to alternative models. Now is the time to assess the dissertation in terms of these goals. Chapters Two, Three, and Four are devoted to presentation and analysis of the model, which I shall not reiterate here. The explanations are new and center on the role of shifts in the relative shadow price of closeness (relative, that is, to the price of housing space) in producing unusual results. I have explained why richer people might consume less closeness than poorer people (i.e. might locate farther from the city center) even though closeness is a normal good; why increases in income can produce moves in either direction; what factors determine the direction of movement; why poor people do not locate beyond a certain distance; why the value of travel time is generally less than the wage. As to new results not already mentioned here, the reader is referred to Table 1 in Chapter Four. It remains for me to show in this chapter that my model is different from and preferable to alternative models.

Since what I am arguing is the superiority of including distance in the utility function, let me consider various arguments for omitting it.

1. One could argue that distance does not in fact affect people's preferences. To make this statement is to assert that, all other things being equal, an individual would be indifferent as to whether he spent a short time or a long time commuting to work.

2. One could argue, as does Solow, that since we do not know the sign of the marginal utility of distance, we should omit distance from the utility function "for simplicity" (1972, p. 162). This argument, that things we don't know enough about can be treated as nonexistent, must give way, I think, in the face of an investigation into the thing that little is known about.

3. Finally, and most importantly, there is the argument that Muth's treatment -- omission of distance from the utility function but inclusion of travel time costs in expenditures and income -- is the equivalent of my own, so that there is no need to include distance in the utility function. This argument, which is especially plausible because Muth's first-order conditions are in a sense the equivalent of mine, warrants detailed discussion.

Muth's model may be written

$$U = U(z, q)$$

$$Y = pz + R(t)q + T(t, Y)$$

$$T(t, Y) = K(t) + V(Y)t$$

where Y is income including the value of leisure and travel time (and is to be distinguished from y, my variable, money income), T is total commuting costs, V is the value of a unit of travel time, and all other variables are as previously defined. The Lagrangian is

$$L = U(z, q) + \lambda[Y - pz - R(t)q - K(t) - V(Y)t],$$

and the first-order condition resulting from partially differentiating with respect to distance is

$$-\lambda[qR'(t) + K'(t) + V(Y)] = 0 \quad ,$$

which implies

$$V(Y) = -[\ qR'(t) + K'(t)] \qquad . \qquad (3c')$$

Now, in Chapter Two I have followed Mohring in asserting my equation (6),

$$V = -p\frac{U_t}{U_z} \qquad ,$$

and this in turn implies, from equation (4),

$$V = \frac{-U_t}{\lambda} \qquad .$$

Incorporating this expression for V into Muth's model yields

$$U_t = \lambda[\ qR'(t) + K'\ (t)] \qquad .$$

But since this last is my own first-order condition (3c), one might be misled into thinking that my model and Muth's are identical.

The critical distinction lies in the different things that Muth and I have to say about V, the value of travel time. For Muth V is an exogenously given function of the wage; it varies always and only with (exogenous) wage variation. For me V is a subsidiary construct which varies in response to shifts in any and all exogenous variables, as they affect t and λ. So when Muth's (3c') is totally differentiated (to get the comparative statics matrix equation; see de Salvo), the output is something quite different from what is gotten when one totally differentiates my (3c). The differences in the forces he and I assert to be at work account for the very different comparative static results implied by Muth's model and mine. Our models, then, are definitely not equivalent.

A subsidiary goal of mine in this dissertation was to critically analyze other findings in the monocentric city literature. Here I showed, among other results, that Mohring's measure of the value of travel time is biased, that Montesano's conclusion that the rich live farther out holds only for a log-linear utility function, and that Solow's conclusion to the same effect holds only if distance is omitted from the utility function.

We come now to the important question of the policy implications of this study. The question is to some extent premature, since the work I have done is at the basic research level. But everything I have so far done does suggest that the location patterns we observe are the reflection of people's tastes for location and living space. It would seem that if cities are deserted, it is because people prefer to live at lower population densities, with more living space. If so, then in the absence of externalities there is no very good justification for urban re-development, or for subsidizing cities and urbanites at the expense of non-urban areas and people. This is not to say that the urban poor should not receive transfers, but rather that they should receive transfers because they are poor, not because they are urban. Moreover, transfers to the poor which tie them to an urban location benefit them less than do equal transfers which allow them to maximize utility with respect to location and all other goods.

It also seems fair to say at this point that policy conclusions based on models I have called into question should be subjected to re-examination, as should conclusions based on theory that ignores locational considerations where such are in fact present. Studies in transportation economics and on housing demand particularly come to mind here.

In a recent paper William Wheaton (1977c) tests models similar to mine and rejects them, his basic result being that observed location patterns are explained not by preferences for closeness and housing space but by the desires of the well-to-do to escape from the racial and social externalities and fiscal burden imposed by the urban poor. I feel constrained to comment at some length on his work to make clear why his rejection of the location-theory type of explanation is unwarranted.

Wheaton takes Alonso's and Muth's versions as the primary expositions of accepted residential location theory, to which he opposes an alternative view which he expresses (p. 620) as follows:

> It is now well documented that in American urban areas the vast majority of middle and upper income house-holds live farther from the city center in separate suburban communities... The consequences of this spatial pattern have been quite serious. The outward mobility of those with means has left American cities as segregated domains for the poor. Within city boundaries the poor can tax only themselves for necessary but deteriorating services... Having escaped this tax burden, middle income Americans enjoy a substantial "fiscal surplus" in the suburbs -- providing an implicit and regressive redistribution of income... In searching for an explanation to these problems, many economists suggest that... fiscal disparities and city decay are not only the product of middle class flight, but the cause as well... The fragmented structure of American local government and the presence of racial or social externalities are the primary forces which have generated the spatial pattern to begin with.

Wheaton tests Alonso's and Muth's theories by using them in conjunction with his own fitted utility functions to predict location-income patterns, and then by comparing the predicted patterns with the actual patterns finds that their theories are rejected by the tests; he therefore concludes by accepting the alternative running-away-from-the-underclass theory.

Testing Alonso's bid price theory, Wheaton uses utility functions he himself has fitted for different groups in an urban sample. He uses the utility functions to compute bid price functions for the different strata, and he shifts the utility levels of the bid price functions until supply-demand equilibrium is reached. The resulting pattern does not have the richer living farther out with any regularity. So if Wheaton's test is a proper one, Alonso's theory is rejected. (In the process Wheaton is also testing the location theory he himself espoused in his own prior work (1974, 1977a, 1977b): Wheaton's theory is identical to Alonso's except for Wheaton's contribution that bid price functions are arrived at through maximization of a rent function subject to a utility level constraint.) Unfortunately, Wheaton makes use of an additional hypothesis in devising his test, this hypothesis being that the parameters of the utility function themselves vary with income. In line with this, he fits and uses different utility functions for groups supposed to be similar except as to income. But for Alonso, individuals with identical tastes and different incomes have identical utility functions. This means that Alonso's hypothesis is inconsistent with Wheaton's incorporated hypothesis, that Alonso's theory does not imply the location patterns Wheaton derives. Wheaton, therefore, has not succeeded in devising a proper test of Alonso's theory.

To test Muth's theory,[4] Wheaton first shows that Muth's criterion is equivalent to saying that the rich will live farther out if, at a particular location and rent, the slopes of their bid price functions are flatter. He then uses his fitted utility functions to compute the bid price slopes for the different strata. Again, the resulting pattern does not have the richer living farther out with any regularity. This

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[4] Actually, Wheaton tests the original version of Muth's theory and also an amended version of it. The two versions are identical except that in the original version the marginal value of travel time is proportional to the wage, while in Wheaton's amended version the marginal value of travel time is equal to $-U_t/U_z$. My comments apply to Wheaton's tests of both versions.

time Wheaton's test is not a proper one for a different reason. To show the equivalence of Muth's and the bid price slope location criteria, he writes the slope of a bid price function,

$$R'(t) = \frac{1}{q}\left[\frac{U_t}{U_z} - K'(t)\right] \quad ,$$

takes its derivative with respect to income, holding location constant, and gets

$$\frac{dR'(t)}{dy} = \frac{1}{q}\left[-R'(t)\,\frac{dq}{dy} - \frac{U_t}{U_z^{2}}\,U_{zz}\,\frac{dz}{dy}\right] \quad ,$$

the expression in brackets being equivalent, according to Wheaton, to Muth's criterion expression,

$$\frac{d}{dy}\,[-qR'(t)] + \frac{dT_t}{dy} \quad ,$$

where T_t is the marginal cost of travel.

Wheaton would be quite correct as to equivalence if both he and Muth were discussing movement with income from an equilibrium starting point to the same finishing point. But, first of all, in fact Muth is talking about movement on an unchanging set of utility surfaces, while Wheaton is talking about change to a new utility function with income . So the two are talking about different dq/dy and dz/dy. Second, Muth's condition is in terms of movement from an initial equilibrium; since Muth doesn't claim that it will predict at any arbitrary location and rent, Wheaton's is not a proper test once again. Third, Wheaton's R(t) is the bid price function whereas Muth's R(t) is the actual rent function, and since bid price and rent are equal to each other only at an individual's equilibrium location, for this reason also the conditions are not equivalent for a non-equilibrium rent and location. So the two criteria are not equivalent and Wheaton's test of Muth's hypothesis is not a proper one.

We can conclude that location theory survives Wheaton's challenge, even if we unquestioningly accept Wheaton's fitted utility functions, which consistently have (from my own point of view, anyway) the wrong sign for the marginal utility of distance, and for which, also, the coefficients and exponents of distance are not very significantly different from zero. But the very fact that all of location theory could have been called into question with such vehemence, points to the need for a new version of location theory which can meet with widespread acceptance, a version which, hopefully, I have now provided.

I hope I have shown the approach that must be taken for the further development of residential location theory. Now it is necessary to proceed beyond the basics, to expansion of the model until all the significant variables are included in the utility and constraint functions, in a multi-city model with numerous workplaces within each city. On the demand side there needs to be incorporation of business location with residential location theory; the two need to be fitted together with the supply side. The full model, with fitted functions, could then be used for prediction, so that we could make best use of our urban public spending resources.

APPENDIX A

PROOF OF CONTINUITY OF THE RENT FUNCTION

Since I am here making no assumptions as to the continuity of the rent function, constrained utility is not a continuous function of distance, and therefore I am not permitted to use calculus to find the individual's optimal location. So in this Appendix I must first show how the individual, in these circumstances, finds z^*, q^*, t^*, his optimal set. That done, I also need for my rent continuity proof, a proof that the rent function must be decreasing (even if the rent function is not continuous). The third section of this Appendix will be the proof that the rent function must be continuous.

Individual Equilibrium

We have no restrictions whatever on the rent function, so that the rent at any location need not bear any specific relation to the rents at other locations. At a given location t^o, rent will be $R(t^o)$ and commuting costs will be $K(t^o)$. For each t^o the individual will find his optimal z^o and q^o, which we shall call his "semi-optimal" set. The semi-optimal set is found by setting equal to zero the partial derivatives with respect to z, q, and λ of the Lagrangian

$$L = U(z, q, t^o) + \lambda[y - pz - R(t^o)q - K(t^o)] \quad ,$$

and solving the resulting first-order condition equations. This gives

$$z^o = z^o[p, R(t^o), K(t^o), y]$$
$$q^o = q^o[p, R(t^o), K(t^o), y]$$
$$\lambda^o = \lambda^o[p, R(t^o), K(t^o), y] \quad .$$

All we can do now to solve for the overall optimum is to take the semi-optimal (z, q) pairs associated with each t and use them to compute for each t its associated utility index number according to the utility function. We find the overall optimum, which we denote (z^*, q^*, t^*), by choosing the (z^o, q^o, t^o) set with the largest index:

$$U(z^*, q^*, t^*) = \max_{\{\text{all } (z^o, q^o, t^o)\}} U(z^o, q^o, t^o) \quad .$$

We can now write

$$z^* = z^*[p, R(t), K(t), y]$$

$$q^* = q^*[p, R(t), K(t), y]$$

$$t^* = t^*[p, R(t), K(t), y] \quad .$$

The Rent Function Must Be Decreasing

We will prove, first, that at every location where rent is not lower than at all more central locations, market demand for housing space will be zero.

Consider t^*, an individual's optimal location, and any more central location $\bar{t}$. The individual's optimal set is (z^*, q^*, t^*). At $\bar{t}$ the individual has a locus of opportunities consisting of (z, q) pairs defined by his budget constraint. We focus on that pair at $\bar{t}$ whose z component is z^* and whose q component is the one paired with it by the $\bar{t}$ budget constraint; we denote this more central set $(\bar{z}, \bar{q}, \bar{t})$. We know that the individual prefers (z^*, q^*, t^*) to $(\bar{z}, \bar{q}, \bar{t})$, since the latter is not even semi-optimal whereas the former is optimal. Then we must have $q^* > \bar{q}$, because otherwise, with $\bar{z} = z^*$ and $\bar{t} < t^*$ he would surely prefer $(\bar{z}, \bar{q}, \bar{t})$ to (z^*, q^*, t^*).

From the requirement that q^* be larger than $\bar{q}$, we can deduce our condition on the rent function. We write the budget constraints for the two locations

$$y = pz^* + R(t^*)\, q^* + K(t^*)$$
$$y = p\bar{z} + R(\bar{t})\, \bar{q} + K(\bar{t}) \quad .$$

Since $z^* = \bar{z}$, we get

$$R(t^*)\, q^* + K(t^*) = R(\bar{t})\bar{q} + K(\bar{t}) \quad .$$

Manipulation gives

$$\frac{K(\bar{t}) - K(t^*) + \bar{q}[R(\bar{t}) - R(t^*)]}{R(t^*)} = q^* - \bar{q} \quad .$$

But since we have established that q^* must be larger than $\bar{q}$, we have

$$\frac{K(\bar{t}) - K(t^*) + \bar{q}[R(\bar{t}) - R(t^*)]}{R(t^*)} > 0$$

and hence

$$R(\bar{t}) - R(t^*) > \frac{K(t^*) - K(\bar{t})}{q} \quad .$$

Since the right-hand side of this inequality is positive, we see that for t^* to be optimal, we must have $R(\bar{t}) - R(t^*) > 0$ and hence the rent function must be decreasing. If t^* is not optimal for anyone, no one will locate there, and demand for housing will be zero. On the assumption that the housing supply curve has the usual form, then, at all non-decreasing rents supply-demand equilibrium will not be possible. Hence the rent function must be decreasing.

The Rent Function Must be Continuous

Consider t^{**}, a point of discontinuity. We cannot find a location in the neighborhood of t^{**} such that the rent there is $R(t^{**}) - \varepsilon$, ε being a challenge number as small as we wish. In the neighborhood of t^{**}, at every point less central than t^{**}, with a decreasing rent function the rent is therefore less than $R(t^{**}) - \varepsilon$. Let $R^{o} = R(t^{**}) - \varepsilon$. Consider $\bar{t}$, defined as any point in the neighborhood of t^{**} that is less central than t^{**}. At $\bar{t}$ the semi-optimal pair is $(\bar{z}, \bar{q})$. Another pair available to the individual at $\bar{t}$ is the pair consisting of z^{**}, the semi-optimal z at t^{**}, and the q coupled with it by the budget constraint at $\bar{t}$; this q we will call $\hat{q}$. Clearly $U(\bar{z}, \bar{q}, \bar{t}) > U(z^{**}, \hat{q}, \bar{t})$. Consider yet another pair at $\bar{t}$, this pair consisting of z^{**} and the q that would be coupled with it if the rent at $\bar{t}$ were R^{o}; this q we will call q^{o}. Since $R(\bar{t}) < R^{o}$, we have $\hat{q} > q^{o}$, and hence $U(z^{**}, \hat{q}, \bar{t}) > U(z^{**}, q^{o}, \bar{t})$.

What happens to q^{o} as $\bar{t}$ gets smaller and approaches t^{**}? We know that

$$q^{o} = \frac{y - pz^{**} - K(\bar{t})}{R^{o}} .$$

The only element in this equation that changes as $\bar{t}$ approaches t^{**} is $K(\bar{t})$; what happens to $K(\bar{t})$ is that it approaches $K(t^{**})$. If we define q^{***} as the value that q^{o} approaches as $\bar{t}$ approaches t^{**}, we have

$$q^{***} = \frac{y - pz^{**} - K(t^{**})}{R^{o}} .$$

At t^{**} the individual's semi-optimal q is q^{**}. Since

$$q^{**} = \frac{y - pz^{**} - K(t^{**})}{R(t^{**})}$$

and since $R(t^{**}) > R^{o}$, we see that $q^{***} > q^{**}$. So $U(z^{**}, q^{***}, t^{**}) > U(z^{**}, q^{**}, t^{**})$.

We have now established that as $\bar{t}$ approaches t^{**}, $(z^{**}, q^{o}, \bar{t})$ approaches $(z^{**}, q^{***}, t^{**})$, that $U(\bar{z},\bar{q},\bar{t})>U(z^{**}, q^{o}, \bar{t})$, and also that $U(z^{**}, q^{***}, t^{**}) > U(z^{**}, q^{**}, t^{**})$. We conclude, therefore, that as $\bar{t}$ approaches t^{**}, $U(\bar{z}, \bar{q}, \bar{t}) > U(z^{**}, q^{**}, t^{**})$; the individual will always prefer $\bar{t}$ to t^{**}, and hence will never locate at t^{**}, the point of discontinuity. With no one locating at t^{**}, demand for housing will be zero, and hence supply-demand equilibrium will not be possible. So the rent function must be continuous.

APPENDIX B

INCOME AND THE SHAPE OF THE BUDGET TRACE

I will show that at a given distance, the change with distance of a (q, t) plane budget trace slope increases with income, i.e.

$$\frac{d}{dy}\left(\frac{d^2q}{dt^2}\right)_y > 0$$

except for the case of $(d^2q/dt^2)_y < 0$ and relatively large in absolute value.

We have

$$\left(\frac{d^2q}{dt^2}\right)_y = \frac{2AR'(t) - R(t)qR''(t)}{R(t)^2}$$

$$= q\left[\frac{2R'(t)^2 - R(t)\ R''(t)}{R(t)^2}\right] + \frac{2K'\ (t)R'(t)}{R(t)^2}$$

and

$$\left(\frac{d^2q}{dt^2}\right)_y \gtreqless 0 \quad \text{iff} \quad 2R'(t)^2 - R(t)R''(t) \gtreqless \frac{-2K'(t)R'(t)}{q} \quad .$$

To see how this rate of change of slope varies with income, z and t held constant, we find

$$\frac{d}{dy}\left(\frac{d^2q}{dt^2}\right)_y = \frac{\partial}{\partial q}\left(\frac{d^2q}{dt^2}\right)_y \frac{\partial q}{\partial y} = \left[\frac{2R'(t)^2 - R(t)R''(t)}{R(t)^2}\right]\frac{1}{R(t)} ,$$

and

$$\frac{d}{dy}\left(\frac{d^2q}{dt^2}\right)_y \gtreqless 0 \quad \text{iff} \quad 2R'(t)^2 - R(t)R''(t) \gtreqless 0 \quad .$$

Clearly, whenever we have

$$2R'(t)^2 - R(t)R''(t) \geqq -\frac{2K'(t)R'(t)}{q} \quad ,$$

we also have $2R'(t)^2 - R(t)R''(t) > 0$, since, by evaluation, $-2K'(t)R'(t)/q$ is positive. Therefore, whenever we have $(d^2q/dt^2)_y \geqq 0$ we also have $d/dy(d^2q/dt^2)_y > 0$. For the case where $2R'(t)^2 - R(t)R''(t)$ is less than $-2K'(t)R'(t)/q$ and greater than zero, we have $(d^2q/dt^2)_y < 0$, with $d/dy(d^2q/dt^2)y$ nevertheless positive. Only when $2R'(t)^2 - R(t)R''(t) \leqq 0$ is $d/dy(d^2q/dt^2)_y \leqq 0$, and in this case $(d^2q/dt^2)_y$ is negative and relatively large in absolute value. Q. E. D.

When $(d^2q/dt^2)_y$ is negative and relatively large in absolute value, it is also true that $R''(t)$ is positive and relatively large. For then we have $2R'(t)^2 - R(t)R''(t) \leqq 0$, and hence

$$R''(t) \geqq \frac{2R'(t)^2}{R(t)} \quad .$$

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