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DESIGN OF NONLINEAR CONTROL SYSTEMS
VIA A TIME DOMAIN RATIO CRITERION

by

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To the four women in my life:

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ABSTRACT

This dissertation presents an investigation of the quantitative properties of Liapunov functions. A design method is developed which utilizes these quantitative properties. The design method is applicable to high order nonlinear autonomous systems containing inherent nonlinearities. The nonlinearities may be of the multiplicative type and/or expressible as general polynomials of the system state variables. The objective of the design method is to formulate a controller for a system so that the equilibrium solution of the system augmented with the controller will be asymptotically stable and the domain of attraction contains a given initial condition. It is also required that the magnitude of the deviation of the output of the augmented system from its equilibrium value, due to the given initial condition, be less than a specified value for all time greater than a specified time.

For the purpose of developing the design method new theorems are proven which present sufficient conditions

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for a surface to be closed. A method is also formulated for generating a class of Liapunov functions $V(\underline{x})$ which possess negative definite time derivatives. The vector $\underline{x}$ is a column vector of the system state variables. This method is a modification of Mekel's M-Matrix method.

In the design method a time domain ratio criterion $\eta(\underline{x})$ is formulated from the ratio of a Liapunov function $V(\underline{x})$ and its time derivative $\dot{V}(\underline{x})$ (i.e. $\eta(\underline{x}) = -\dot{V}(\underline{x})/V(\underline{x})$). The numerical values of the elements of the Liapunov function, its time derivative, and the time domain ratio criterion are constrained to satisfy inequalities which are developed in this dissertation. The inequalities are sufficient conditions for a nonlinear autonomous control system to possess the desired time response.

The design method is illustrated and analyzed via illustrative examples. It is demonstrated that although a controller is designed for a specific initial condition the augmented system's settling time is known for a set of initial conditions. Also presented is the fact that the design method yields bounds for the magnitudes of the state variables. It is further illustrated that the design method can in some cases yield sufficient conditions for the augmented system to be globally asymptotically stable.

CHAPTER 1

INTRODUCTION

1.1 Theory of Stability in the Sense of Liapunov

Liapunov⁽²⁸⁾ in 1892 introduced a criterion called the direct method for studying the stability of an equilibrium solution of nonlinear systems. This criterion gives information about stability or instability by making use of only the form of the equations describing the system but without requiring the explicit knowledge of the solution. The underlying idea of the direct method of Liapunov is based on the existence of a positive definite scalar function $V(\underline{x})$ of the state vector $\underline{x}$ with the following properties⁽¹⁴⁾.

a) $V(\underline{x})$ is continuous and has continuous first partial derivatives in a region Ω about the equilibrium point $\underline{x}=0$, where the origin is chosen as the equilibrium point.

b) $V(0) = 0$

c) Outside the origin and always in region Ω , $V(\underline{x})$ is positive and vanishes only at the origin. The origin is an isolated minimum of $V(\underline{x})$.

Since $V(\underline{x})$ has first partial derivatives it has a gradient $\underline{\nabla V}$, and one can obtain $\dot{V}(\underline{x}) = (\underline{\nabla V})' \cdot \dot{\underline{x}}$ as its time derivative, where $(\underline{\nabla V})'$ is the transpose of $\underline{\nabla V}$.

If in addition $\dot{V}(\underline{x}) \leq 0$ in Ω , $V(\underline{x})$ is called a Liapunov function. With these definitions, Liapunov's stability theory can be summarized by the following theorems.

1. Stability Theorem.

If there exists in some neighborhood Ω of the origin a Liapunov function $V(\underline{x})$, then the origin is stable.

2. Asymptotic Stability Theorem

If, in addition to the requirements of Theorem 1, $\dot{V}(\underline{x})$ is negative definite $[\dot{V}(\underline{x}) < 0]$ in Ω then the stability is asymptotic.

3. Global Asymptotic Stability Theorem

If the conditions of Theorem 2 are satisfied for all $\underline{x}$ and in addition $V(\underline{x}) \rightarrow \infty$ as $|\underline{x}| \rightarrow \infty$, where $|\underline{x}|$ is the Euclidean norm of $\underline{x}$, then the system under consideration is asymptotically stable in the large.

4. Instability Theorem

If $V(\underline{x}) > 0$, $V(0) = 0$ and $\dot{V}(\underline{x}) > 0$ in Ω , then the origin is unstable.

Liapunov functions possess both qualitative and quantitative properties. The qualitative properties have been studied and rigorously discussed in numerous publications.*

* See reference [32] for an extensive Bibliography pertaining to Liapunov's theory of stability.

The quantitative properties however have not been fully explored. In this dissertation the quantitative properties of Liapunov functions are investigated and a design method is developed which utilizes these quantitative properties. The design method is applicable to high order autonomous control systems containing inherent nonlinearities. The nonlinearities may be of the multiplicative type and/or expressible as general polynomials of the system state variables. The nonlinearities may also be odd or even functions of the state variables and their projections may lie in any quadrant of the state space. In the design method a time domain ratio criterion $\eta(\underline{x})$ is formulated from the ratio of a Liapunov function and its time derivative (i.e. $\eta(\underline{x}) = -\dot{V}(\underline{x})/V(\underline{x})$). The numerical values of the elements of the Liapunov function $V(\underline{x})$, its time derivative $\dot{V}(\underline{x})$ and the time domain ratio criterion $\eta(\underline{x})$ are constrained to satisfy inequalities which are developed in this dissertation. The inequalities are sufficient conditions for a nonlinear autonomous control system to possess a desired time response.

1.2 The Design Method

It is assumed that the block diagram of a nonlinear autonomous control system and the differential equation describing the system are given. The design method considers the following problem.

1.2.1 Statement of Design Problem

It is desired to formulate a controller so that the equilibrium solution of the original system augmented with the controller will be asymptotically stable. It is also required that the output of the augmented system, denoted by $x(t)$, satisfy the specification that

$$|x(t)| \leq x_{1s} \quad \text{for } t \geq t_s$$

for the initial condition $x(0) = x_{10}$, $\dot{x}(0) = x_{20}, \dots, x^{(n-1)}(0) = x_{n0}$, where $\dot{x} = dx/dt$, $x^{(n)} = d^n x/dt^n$ and x_{1s} and t_s are specified constants.

The equilibrium solution of the unaugmented system may be unstable, stable or asymptotically stable.

1.2.2 Design Method Approach

The design method is to achieve the objective of obtaining the desired controller by utilizing the quantitative properties of Liapunov functions. A class of $V(\underline{x})$ functions (Chapter 4) is formulated for the augmented system

whose controller output is denoted by $u(\underline{x})$. The elements of $V(\underline{x})$ are functions of the unaugmented system's parameters and the parameters of the controller output. The time domain ratio criterion $\eta(\underline{x})$ is obtained from the ratio of the $V(\underline{x})$ function and its time derivative (i.e. $\eta(\underline{x}) = -\dot{V}(\underline{x})/V(\underline{x})$). The elements of the $V(\underline{x})$ function, its time derivative $\dot{V}(\underline{x})$, and the time domain ratio criterion $\eta(\underline{x})$ are then constrained to satisfy design inequalities which are developed in this dissertation (Chapter 5). These constraints are sufficient conditions for the equilibrium solution of the augmented system to be asymptotically stable and for the output $x(t)$ to satisfy the specification that $|x(t)| \leq x_{1s}$ for $t \geq t_s$. A digital computer is employed as an engineering aid to determine whether numerical values exist for the postulated controller output parameters which enable the design constraints to be satisfied. The application of the design method to higher order nonlinear systems is made feasible only by the use of a digital computer. It is the utilization of the digital computer which enables the designer to effectively apply the direct method of Liapunov for the design of high order nonlinear systems.

In the following section the contents and contributions of each of the succeeding chapters are outlined.

1.3 Contents of Dissertation

Chapter 2 presents the historical background and the motivation for the design method. In Chapter 3 the mathematical concepts required for the development of the method are given. The sign definiteness of high order functions of several variables and the concept and properties of surfaces are discussed in that chapter. New theorems are proven which yield sufficient conditions for a surface to be closed and bounded. Test procedures are developed which enable the digital computer to be utilized for verification of these conditions. The chapter concludes with a discussion of the ratio of two quadratic forms. The extremal values of the ratio of two quadratic forms are later employed to develop a design inequality.

In chapter 4 a new method is developed for generating a class of Liapunov functions. The Liapunov functions are applicable for the design of high order nonlinear control systems. The nonlinearities may be of the multiplicative type and/or expressible as general polynomials of the system state variables. The nonlinearities may also be odd or even functions of the state variables and their projections may lie in any quadrant of the state space. The method is a modification of Mekel's⁽³²⁾ M-Matrix method. Illustrations of the method are given in the final section of the chapter.

The philosophy and objective of the design method are presented in chapter 5. The concepts and properties of the time domain ratio criterion $\eta(\underline{x}) \triangleq -\dot{V}(\underline{x})/V(\underline{x})$ are discussed in this chapter. From the quantitative properties of $\eta(\underline{x})$ new design constraints are formulated. These constraints are sufficient conditions for the equilibrium solution of the augmented system to be asymptotically stable and for the output $x(t)$ to satisfy the specification that $|x(t)| \leq x_{1s}$ for $t \geq t_s$.

Chapter 6 presents the design method in detail. Each step of the method is discussed and illustrated numerically via a specific design problem. The design method consists of the following seven steps.

- Step (1): Formulation of the Augmented System
- Step (2): Formulation of the Modified M-Matrix
Liapunov Function and its Time Derivative
- Step (3): Classification of the Controller Output
- Step (4): Reduction of the Liapunov Function
and its Time Derivative
- Step (5): Formulation of the Constraints on the
Elements of the Liapunov Function

Step (6): Formulation of the Constraints on the
Elements of the Time Derivative of the
Liapunov Function

Step (7): Application of a Digital Computer
Program for Solving the Design Constraints

The design method is further illustrated and analyzed in chapter 7. In this chapter it is demonstrated that although a controller is designed for a specific initial condition the augmented system's settling time is known for a set of initial conditions. This eliminates the previously necessary task of re-analyzing the non-linear system for each initial condition which may be of interest. Also presented is the fact that the design method yields bounds for the magnitudes of the state variables. In addition, the chapter analyzes the effect of increased controller complexity upon the response of the system as shown by the illustrative problem of chapter 6. Chapter 7 includes a second order design problem which illustrates that the design method can in some cases yield sufficient conditions for the augmented system to be globally asymptotically stable.

The historical background and the motivation for the design method are presented in the following chapter.

CHAPTER 2

HISTORICAL BACKGROUND AND MOTIVATION

2.1 Introduction

Since the late 1950's, there has been a continuing effort to develop methods for the design of nonlinear control systems and a vast outflow of research and publication has resulted. The concepts of the quantitative properties of a Liapunov function have been demonstrated for the design of linear systems⁽³⁶⁾, but the extension of such an approach to nonlinear control systems has not been fully investigated.

A survey of the literature pertaining to the application of the quantitative properties of Liapunov functions reveals that most of the work has been in the area of analysis while only a small percentage of the papers have been concerned with the area of design and a smaller percentage yet with synthesis. The following section presents a survey of the results which has been obtained to date.

2.2 Survey of the Literature

In 1960, Kalman and Bertram⁽¹⁸⁾ proposed the estimate of rapidity of the transient response of a system by means of a number η and a Liapunov function $V(\underline{x})$, such that $\dot{V}(\underline{x}) \leq -\eta V(\underline{x})$. For a quadratic Liapunov function they interpret $1/\eta$ as the system's maximum time constant. They also note that the value of η is dependent upon the Liapunov function chosen and that little is known about how to choose the form of a Liapunov function so that η may be meaningfully interpreted.

Bass⁽¹⁾ proposed a technique to optimize the transient behavior of control systems subject to saturation by the use of linear compensation which tends to maximize the value of $-\dot{V}[\underline{x}, \underline{u}(t)]$, where $\underline{u}(t)$ is the control variable subject to $|u_i| < a_i < \infty$. It is noted that Bass has approximated the condition of maximizing η , since

$$\text{Min}_{\underline{u}(t)} \left[\frac{V[\underline{x}, \underline{u}(t)]}{-\dot{V}[\underline{x}, \underline{u}(t)]} \right] < V[\underline{x}, \underline{u}(t)] \cdot \left\{ \text{Min}_{\underline{u}(t)} \left[\frac{1}{-\dot{V}[\underline{x}, \underline{u}(t)]} \right] \right\}$$

Cunningham⁽⁷⁾, in a discussion of the application of Krasovskii's method for generating Liapunov functions shows that for a stable second order linear system the minimum value of $\eta(\underline{x})$, where $\eta(\underline{x}) = -\dot{V}(\underline{x})/V(\underline{x})$, yields a

bound for the maximum time constant of the system. He states that for a nonlinear system, the choice of a Liapunov function which will allow meaningful interpretations of the properties of $\eta(\underline{x})$, is not yet known.

Gibson⁽¹¹⁾, in his book, shows that the minimum value of $\eta(\underline{x})$ can be employed as an analysis tool for linear systems. In a paper describing a procedure for the design of intentionally nonlinear controllers for the compensation of linear systems, Grayson⁽¹²⁾ utilizes an approximation to $\eta(\underline{x})$ in order to obtain an estimate of the rapidity with which the process output approaches that of a model.

In another paper, Grayson⁽¹³⁾ also specifies the form $-\dot{V}(\underline{x}) = \alpha V(\underline{x})^k$ (where α and k are constants) for a Liapunov function of the system $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{u}(\underline{x})$, where the system $\dot{\underline{x}} = \underline{f}(\underline{x})$ is stable but not asymptotically stable. For $k=1$, or $\eta(\underline{x}) = \text{constant}$, he shows that it is possible to obtain an explicit solution for time T such that $V[\underline{x}(T)] = 0$, where T is the specified terminal time.

Johnson⁽¹⁷⁾, in a paper which investigates the quasi-optimal control of nonlinear systems, proves that if a Liapunov function is chosen to be of quadratic form, then the desired equilibrium state will be exponentially

stable if $\dot{V}(\underline{x}, \underline{u}) \leq -\eta V(\underline{x})$, where η is chosen to be positive. The weakness of this method is the requirement of a quadratic Liapunov function. Grayson⁽¹³⁾ commented on this paper by noting that its practical application is limited to linear systems.

Nahi⁽³⁵⁾, for the purpose of developing a design technique for the time optimal control of systems of the form $\dot{\underline{x}} = \underline{f} [\underline{x}, \underline{u}(t)]$, developed two theorems which enable one to choose a control input $\underline{u}(t)$, which will return the system to the origin in minimum time. Nahi's technique is only applicable to systems which satisfy the condition $\text{Min}_{\underline{u}(t)} [\dot{V}(\underline{x})] = -h [V(\underline{x})]$, where $h [V(\underline{x})]$ is a function of $V(\underline{x})$ only. The types of systems which are included in the class that satisfy this condition seem to be very limited. However, Nahi states that his search has not been exhaustive.

Castrucci⁽⁵⁾ presents a method for analyzing the transient behavior of autonomous nonlinear systems and defines a concept whereby a nonlinear system can be considered to behave "better than" a linear system in a given region of the phase space. The definition used is that an autonomous nonlinear system $\dot{\underline{x}} = \underline{f}(\underline{x})$ is better than the linear system $\dot{\underline{y}} = -\lambda \underline{y}$, $\lambda > 0$, in a domain S of the phase space, relative to a Liapunov function, if when both systems depart from any point s , the system $\dot{\underline{x}} = \underline{f}(\underline{x})$

reaches any Liapunov surface before the linear system. He parallels the concept of finding the minimum value of $\eta(\underline{x})$ but does not offer any direction which would lead to a design criterion.

Vogt⁽⁴²⁾ develops a relationship between the time derivative of a Liapunov function and a specific definition of relative stability between two systems. He develops theorems which relate his concept of relative stability to the minimum and maximum values of $\eta(\underline{x})$ in a given region of the phase space. The characteristics of $\eta(\underline{x})$ are used as a qualitative tool and no attempt is made to deduce quantitative information about the system performance. In another paper, Vogt⁽⁴³⁾ also develops inequality relationships (for a given form of a Liapunov function) between the maximum and minimum values of $\eta(\underline{x})$ and the maximum and minimum real parts of the characteristic values of the linearized system's stability matrix. However, these inequalities cannot be extended to a design criterion because the ranges of the inequalities are too large.

In each of their Ph.D. dissertations, Geiss⁽¹⁰⁾ and Monopoli⁽³⁴⁾ mention that the development of methods for specifying the characteristics of nonlinear systems via constraints on a Liapunov function, is an area of future research. Each notes, however, that one will encounter

enormous algebraic difficulties and offer no solutions. Leondes⁽²⁵⁾ and Ogata⁽³⁶⁾ in their books both mention the concept of η presented by Kalman. Each concludes that the concept of η is useful only for simple forms of Liapunov functions again citing the algebraic difficulties.

In the next section the motivation for the design method is given.

2.3 Motivation for Design Method

As is noted in the previous section a desire for developing design methods which use the quantitative characteristics of a Liapunov function and its time derivative has been expressed by numerous authors. The major advantage of such methods is that the designer can learn in a systematic manner and apply the accumulated knowledge to the design of future systems. The difficulties of such a method are the high order nonlinear algebraic equations which are encountered and the formulation of explicit relationships from which the design parameters can be chosen.

In order to obtain explicit relationships some researchers (13,17,35) have identified concepts of Liapunov's direct method with the procedures used to design optimal control systems. The equations which result from the minimization or maximization of performance indices, which are usually related to a Liapunov function, yield the desired relationships which the design parameters must satisfy. Such techniques are employed in quasi-time optimal control problems and modeling techniques. Many times, however, the performance indices are chosen from experience and it is not clear how a specified index of performance relates to the behavior of the system.

The design method presented in this dissertation circumvents this problem by formulating controllers which satisfy constraints that are sufficient conditions for the system to exhibit the desired time response. The digital computer is employed in the method in order to simplify the solution of the high order algebraic equations. The design method is of the robust type. Robust in the sense that there will probably not exist a unique controller which satisfies the design constraints but instead a class of controllers. Since there is presently only meager knowledge about the conditions under which controllers can be found for nonlinear systems, it is deemed essential to develop a design method from which the designer can learn in a systematic manner and apply the accumulated knowledge to the design of future systems.

The mathematical concepts which are employed in the development of the design method are presented in the following chapter.

CHAPTER 3

MATHEMATICAL BACKGROUND

3.1 Introduction

The mathematical concepts which are required for the design method are discussed in this chapter. Section 3.2 presents the concepts necessary for the interpretation and verification of the sign definiteness of high order functions of several variables. In section 3.3 the properties of closedness and boundedness of surfaces are discussed. New theorems are proven which simplify the testing procedures required to verify these properties. It is the application of the simplified testing procedures, combined with the aid of a digital computer, which make the design method a feasible one. The final section 3.4 discusses the ratio of two quadratic forms and presents theorems which are later employed to derive design inequalities.

3.2 Definiteness of High Order Functions of Several Variables

In this section the concepts required for the interpretation and verification of the sign definiteness of high order functions of several variables are given. The concepts are presented in a form which is meaningful for the design method. The specific functions investigated in the method are the system's Liapunov function and its time derivative. The following definitions are used.⁽³⁰⁾

Definition 3.1. Positive (Negative) Definite.

The scalar function $V(\underline{x})$ is said to be positive (negative) definite if for

$$|\underline{x}| \leq h \quad \text{where} \quad |\underline{x}|^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

and h is an arbitrarily small positive number

$$V(\underline{x}) > 0 \quad \left[V(\underline{x}) < 0 \right] \quad \text{for all } \underline{x} \neq 0$$

and

$$V(0) = 0$$

Definition 3.2. Positive (Negative) Semidefinite.

The scalar function $V(\underline{x})$ is said to be positive (negative) semidefinite if for

$$|\underline{x}| \leq h \quad \text{where} \quad |\underline{x}|^2 = x_1^2 + x_2^2 \dots + x_n^2$$

and h is an arbitrarily small positive number

$$V(\underline{x}) \geq 0 \quad \left[V(\underline{x}) \leq 0 \right] \quad \text{for all } \underline{x}$$

and

$$V(\underline{0}) = 0$$

Definition 3.3. Indefinite

The scalar function $V(\underline{x})$ is said to be indefinite if it assumes both positive and negative values for

$$|\underline{x}| \leq h \quad \text{where} \quad |\underline{x}|^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

and h is an arbitrarily small positive number.

For h arbitrarily small in definitions 3.1 and 3.2, $V(\underline{x})$ will be definite in an arbitrarily small region about the origin. The larger h , the greater the region of definiteness for $V(\underline{x})$. If h is infinite, $V(\underline{x})$ is definite in the whole $\underline{x}$ space.

It is known ⁽³⁷⁾ that any polynomial $V(x)$, in the single variable x , with constant coefficients and of degree $2n$ or $2n-1$ can be written in the quadratic form

$$V(x) = \sum_{i=0}^n \sum_{j=0}^n m_{ij} x^i x^j \quad (3-1)$$

or

$$V(\underline{x}) = \underline{x}' M \underline{x} \quad (3-2)$$

where $\underline{x}$ is a column vector and $\underline{x}'$, its transpose, is given by $\underline{x}' = [1 \ x \ x^2 \ \dots \ x^n]$ and M is a constant symmetric matrix with the elements m_{ij} .

An extension of this quadratic form representation to a high order polynomial of several variables has been recently given by Bose and Li⁽²⁾. The polynomial $V(x_1, x_2, \dots, x_n)$, in the n -variables $x_1, x_2, \dots, x_n$, with constant coefficients can be written in the form

$$V(x_1, x_2, \dots, x_n) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \dots \sum_{k=0}^{n_n} m_{ij \dots k} x_1^i x_2^j \dots x_n^k \quad (3-3)$$

where $i, j, \dots, k$ are n indices, or

$$V(x_1, x_2, \dots, x_n) = \underline{y}' N \underline{y} \quad (3-4)$$

where $\underline{y}$ is a column vector and $\underline{y}'$, its transpose, is given by $\underline{y}' = [y_1, y_2, \dots, y_m]$ with elements that are products of powers of $x_1, x_2, \dots, x_n$ and N is a constant symmetric matrix. The N matrix is not unique and depends upon the choice of the vector $\underline{y}$. Equation (3-4) is illustrated by the following example.

Example 3.1

Let $V(x_1, x_2) = x_1^2 + 2x_1^2 x_2^2 + x_2^2$. The polynomial $V(x_1, x_2)$ can be written in the two forms

$$V(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}' \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix} \quad (3-6)$$

or

$$V(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \ x_2 \end{bmatrix}' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_1 \ x_2 \end{bmatrix} \quad (3-7)$$

In Eq. (3-6),

$$\underline{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (3-8)$$

and in Eq. (3-7)

$$\underline{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \ x_2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (3-9)$$

The polynomial $V(x_1, x_2, \dots, x_n)$ can also be written in the pseudo-quadratic form

$$V(\underline{x}) = \underline{x}' M(\underline{x}) \underline{x} \quad (3-10)$$

where $\underline{x}$ is a column vector and $\underline{x}'$, its transpose, is now given by $\underline{x}' = [x_1, x_2, \dots, x_n]$ and $M(\underline{x})$ is a symmetric n -th order matrix with elements that are constants and/or functions of the components of $\underline{x}$. Note that the column vector $\underline{x}$ and its transpose $\underline{x}'$ in Eq. (3-10) differ from the column vector $\underline{x}$ and its transpose $\underline{x}'$ in Eq. (3-2).

Example 3.2

As an illustration of the pseudo-quadratic form, the polynomial of example 3.1 can be written as

$$V(\underline{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1 + 2x_2^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad M(\underline{x}) = \begin{bmatrix} 1 + 2x_2^2 & 0 \\ 0 & 1 \end{bmatrix}$$

Note the nonlinear characteristic of the $M(\underline{x})$ matrix.

In the design method it is necessary to determine the properties of definiteness and semidefiniteness for

functions of several variables. In the remainder of this section the pseudo-quadratic notation and generalization of Sylvester's theorem (32) are utilized to develop simplified testing procedures for determining these properties.

The following two theorems (44) are a generalization of Sylvester's theorem, which is a necessary and sufficient condition for a scalar function $V(\underline{x})$, where $V(\underline{x}) = \underline{x}'M\underline{x}$ and M is a constant real symmetric matrix, to be positive (negative) definite or positive (negative) semidefinite in the entire $\underline{x}$ space.

Theorem 3.1

In order that the scalar function $V(\underline{x})$, where $V(\underline{x}) = \underline{x}'M\underline{x}$ and M is a constant real symmetric n -th order matrix with determinant $|M|$, be positive definite in the entire $\underline{x}$ space it is necessary and sufficient that there exists a sequence of $n-1$ distinct positive integers $u, v, \dots, w$, each not exceeding n , such that

$$\begin{aligned} |M| &> 0 \\ |M^u| &> 0 \\ |M^{uv}| &> 0 \\ &\dots \dots \dots \\ |M^{uv\dots w}| &> 0 \end{aligned}$$

For the scalar function $V(\underline{x})$ to be negative definite in the entire $\underline{x}$ space it is necessary and sufficient that

for n-odd	for n-even
$ M < 0$	$ M > 0$
$ M^u > 0$	$ M^u < 0$
$ M^{uv} < 0$	$ M^{uv} > 0$
$\dots\dots\dots$	$\dots\dots\dots$
$ M^{uv\dots w} < 0$	$ M^{uv\dots w} < 0$

Here the superscript notation is used to designate principal minors. For example $|M^{uv}|$ represents the principal minor obtained by deleting rows and columns u and v .

Theorem 3.2

In order that the scalar function $V(\underline{x})$, where $V(\underline{x}) = \underline{x}' M \underline{x}$ and M is a constant real symmetric n -th order matrix with determinant $|M|$, be positive semidefinite in the entire $\underline{x}$ space, it is necessary and sufficient, if the matrix M is of rank r , that there exists an r -th order minor $|M_r|$ and a sequence of $r-1$ distinct integers $u, v, \dots, w$, each not exceeding r , such that

$$\begin{aligned}
|M_r| &> 0 \\
|M_r^u| &> 0 \\
|M_r^{uv}| &> 0 \\
|M_r^{uv\dots w}| &> 0
\end{aligned}$$

For the scalar function $V(\underline{x})$ to be negative semidefinite in the entire $\underline{x}$ space, it is necessary and sufficient that

for r-odd	for r-even
$ M_r < 0$	$ M_r > 0$
$ M_r^u > 0$	$ M_r^u < 0$
$ M_r^{uv} < 0$	$ M_r^{uv} > 0$
.	
$ M_r^{uv\dots w} < 0$	$ M_r^{uv\dots w} < 0$

In the design method, the system's Liapunov function and its time derivative are written in the pseudo-quadratic form of Eq. (3-10). Sufficient conditions for the scalar function $V(\underline{x})$ of Eq. (3-10) to be positive (negative) definite or positive (negative) semidefinite in a region Ω of the $\underline{x}$ space are obtained by a generalization of theorems

3.1 and 3.2. The sufficient conditions are given by the following two theorems⁽³²⁾.

Theorem 3.3.

In order that the scalar function $V(\underline{x})$, where $V(\underline{x}) = \underline{x}'M(\underline{x})\underline{x}$ and $M(\underline{x})$ is a symmetric n -th order matrix with elements that are real constants and/or functions of the components of $\underline{x}$, be positive definite in a region Ω of the $\underline{x}$ space it is sufficient that there exists a sequence of $n-1$ distinct positive integers $u, v, \dots, w$, each not exceeding n , such that

$$\begin{aligned} |M(\underline{x})| &> 0 \\ |M^u(\underline{x})| &> 0 \\ |M^{uv}(\underline{x})| &> 0 \\ &\dots \dots \dots \\ |M^{uv\dots w}(\underline{x})| &> 0 \end{aligned} \quad \text{for all } \underline{x} \in \Omega$$

For the scalar function $V(\underline{x})$ to be negative definite in a region Ω of the $\underline{x}$ space it is sufficient that

for n-odd $ M(\underline{x}) < 0$ $ M^u(\underline{x}) > 0$ $ M^{uv}(\underline{x}) < 0$ $ M^{uv\dots w}(\underline{x}) < 0$	for n-even $ M(\underline{x}) > 0$ $ M^u(\underline{x}) < 0$ $ M^{uv}(\underline{x}) > 0$ $ M^{uv\dots w}(\underline{x}) < 0$	for all $\underline{x} \in \Omega$
---	--	---------------------------------------

Theorem 3.4

In order that the scalar function $V(\underline{x})$, where $V(\underline{x}) = \underline{x}'M(\underline{x})\underline{x}$ and $M(\underline{x})$ is a symmetric n-th order matrix with elements that are real constants and/or functions of the components of $\underline{x}$, be positive semi-definite in a region Ω of the $\underline{x}$ space, it is sufficient, if the matrix $M(\underline{x})$ is of rank r in Ω , that there exists an r -th order minor $|M_r(\underline{x})|$ and a sequence of $r-1$ distinct integers $u, v, \dots, w$, each not exceeding r , such that

$$\begin{aligned}
 &|M_r(\underline{x})| > 0 \\
 &|M_r^u(\underline{x})| > 0 \\
 &|M_r^{uv}(\underline{x})| > 0 \quad \text{for all } \underline{x} \in \Omega \\
 &\dots\dots\dots \\
 &|M_r^{uv\dots w}(\underline{x})| > 0
 \end{aligned}$$

For the scalar function $V(\underline{x})$ to be negative semi-

definite in the region Ω of the $\underline{x}$ space it is sufficient that

for r-odd	r-even
$ M_r(\underline{x}) < 0$	$ M_r(\underline{x}) > 0$
$ M_r^u(\underline{x}) > 0$	$ M_r^u(\underline{x}) < 0$
$ M_r^{uv}(\underline{x}) < 0$	$ M_r^{uv}(\underline{x}) > 0$ for all $\underline{x} \in \Omega$
.	
$ M_r^{uv\dots w}(\underline{x}) < 0$	$ M_r^{uv\dots w}(\underline{x}) < 0$

In the design method theorems 3.3 and 3.4 are utilized in testing the following two properties of $V(\underline{x})$.

Property 1.

The scalar function $V(\underline{x}) = \underline{x}'M(\underline{x})\underline{x}$ is greater than the positive definite quadratic function $\underline{x}'M\underline{x}$ in a region Ω of the $\underline{x}$ space. Note that the matrices $M(\underline{x})$ and M have the same definitions as given in the previous theorems.

Property 2.

The scalar function $V(\underline{x}) = \underline{x}' M(\underline{x}) \underline{x}$ is positive (negative) definite or positive (negative) semi-definite in a region Ω of the $\underline{x}$ space.

Property 1 occurs in the design method as a sufficient condition to be satisfied by the system's Liapunov function for the initial Liapunov surface to be closed (Section 3.3). Property 2 occurs as a sufficient condition on the time derivative of the Liapunov function to show that the equilibrium solution of a system is asymptotically stable (section 5.2).

In general, when testing for properties 1 and 2, a digital computer algorithm is required. The algorithm however is confined to a defined region of the $\underline{x}$ space and a large parameter search is not required. This arises from the fact that the region Ω defined for the design procedure, as developed in section 5.2, is the interior of a hyper-rectangle defined in n-dimensional space as $|x_1| = c_1, |x_2| = c_2, \dots, |x_n| = c_n$, where c_i is a positive constant.

For many problems, however, such a search is not necessary. The application of the pseudo-quadratic form notation and the sufficient conditions of theorems 3.3 and 3.4 may reduce the complexity and degree of the equations involved so that analytical methods may be employed to verify the properties. This is illustrated in the following two examples.

Example 3.3

As an illustration of the verification of property 1, let $V(\underline{x}) = 4x_1^2 + x_1^2 x_2 + 5x_2^2$ and $\underline{x}'M\underline{x} = x_1^2 + 4x_2^2$. It is desired that $V(\underline{x}) > \underline{x}'M\underline{x}$ in the region defined by $|x_1| \leq 1$ and $|x_2| \leq 2$. The function $V(\underline{x})$ is first written in the pseudo-quadratic form

$$V(\underline{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 4 + x_2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the quadratic function is written in the quadratic form

$$\underline{x}'M\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then subtracting,

$$V(\underline{x}) - \underline{x}'M\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 4 + x_2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$V(\underline{x}) - \underline{x}'M\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 3 + x_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, the minimum value of $3 + x_2$ in the region Ω is

1. Hence for $\underline{x} \in \Omega$

$$V(\underline{x}) - \underline{x}' M \underline{x} \geq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

Therefore $V(\underline{x}) > \underline{x}' M(\underline{x}) \underline{x}$ in the defined region Ω .

Example 3.4

As an illustration of the verification of property 2, let $V(\underline{x}) = 9x_1^2 + 2x_1x_2 + 6x_2^2 + 4x_1^4x_2 + 2x_2^3$. It is desired that $V(\underline{x}) \geq 0$ in the region Ω defined by $|x_1| \leq 1$ and $|x_2| \leq 2$. The function $V(\underline{x})$ is first written in the pseudo-quadratic form

$$V(\underline{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 9 + 4x_1^2x_2 & 1 \\ 1 & 6 + 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, the minimum value of $9 + 4x_1^2x_2$ in the region Ω is 1 and the minimum value of $6 + 2x_2$ in Ω is 2.

Hence, for $\underline{x} \in \Omega$

$$V(\underline{x}) \geq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

Therefore, $V(\underline{x}) > 0$ in the defined region Ω .

These simplified techniques for testing properties 1 and 2 are employed in the illustrative problems of chapters 6 and 7.

In the next section the concept of surfaces and some of their properties are discussed. New theorems are proven which simplify the testing procedures required to verify these properties. It is the application of the simplified testing procedures, combined with the aid of a digital computer, which make the design method a feasible one.

3.3 Properties of a Surface

In this section the concept of surfaces and some of their properties are presented. As developed in section 5.2, the equation

$$V[\underline{x}(t_0)] = C_0 \quad (3-11)$$

(where $V[\underline{x}(t_0)]$ is a system's Liapunov function at the initial time t_0 and C_0 its scalar value) represents an initial Liapunov surface. It is necessary that this surface be closed if the design method is to be applicable. New theorems are proven in this section which are sufficient conditions on the function $V(\underline{x})$ for the surface of Eq. (3-11) to be closed. In the design method the digital computer is utilized in a systematic manner for the verification of these sufficient conditions. The following definitions are employed in this section.

Definition 3.4. Surface⁽³⁾

A surface is defined as the set of all points in n -space which satisfy the equation $f(\underline{x}) = 0$.

Definition 3.5. Quadratic Surface⁽³⁾

If the function $f(\underline{x})$ in definition 3.4 can be written in the quadratic form $f(\underline{x}) = \underline{x}'M\underline{x} + C$ where M is a constant real symmetric matrix and C is a constant,

then the surface is called quadratic.

Definition 3.6. Closedness⁽³⁰⁾

A surface is closed, with respect to the origin, if it intersects every continuous path, at a finite distance from the origin, leading from the origin to infinity.

The concept of closedness was correlated to the concept of definiteness by Letov⁽²⁷⁾. He proved that if $V(\underline{x})$ is a positive definite function, then the equation $V(\underline{x}) = C$ (C is a positive constant) represents a set of nested closed surfaces with respect to the origin in a "sufficiently small region about the origin". He also showed that if the $V(\underline{x})$ function approaches infinity as the Euclidean norm of $\underline{x}$ approaches infinity, then the surfaces are closed for all positive finite C .

For the purpose of a design method, these results are of little practical value. For the method presented in this dissertation sufficient conditions must be developed such that the surface $V[\underline{x}(t_0)] = C_0$ (where $V[\underline{x}(t_0)]$ is a system's Liapunov function at the initial time t_0 and C_0 is its scalar value) is a closed surface for a function $V(\underline{x})$ with the following properties in an open region Ω of the $\underline{x}$ space.

- (i) $V(\underline{x})$ is positive definite
- (ii) $V(\underline{x})$ is a continuous function of $\underline{x}$
- (iii) $V(\underline{x})$ has defined and continuous first partial derivatives with respect to $\underline{x}$
- (iv) $\frac{dV(\underline{x})}{dt}$ is negative definite.

The above conditions on $V(\underline{x})$ are sufficient conditions for a Liapunov function to show that the equilibrium solution of a system is asymptotically stable.

Two theorems* are now stated which are employed in the proofs of the sufficiency conditions for closedness. These theorems present known results in a form more useful for our purposes.

Theorem 3.5

The quadratic surfaces $\underline{x}'M\underline{x} = C > 0$ are closed with respect to the origin if and only if M is a positive definite matrix.

Theorem 3.6

If the quadratic function of theorem 3.5 is positive definite then the surfaces $\underline{x}'M\underline{x} = C > 0$ are nested.

* The proofs of these theorems are given in Appendix B.

The following lemma is stated and proven.

Lemma

Given the scalar function $V(\underline{x})$ which satisfies the following conditions in the open region Ω defined as the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = C > 0$ (where M_{s1} is a positive definite matrix.)

- (i) $V(\underline{x}) = 0$ only for $\underline{x} = 0$
- (ii) $V(\underline{x})$ is a continuous function of $\underline{x}$

If $V(\underline{x}) > \underline{x}' M_{s1} \underline{x}$ in the interior and on the surface of the quadratic surface, then the open region Ω contains at least one closed portion with respect to the origin of the surface defined by $V(\underline{x}) = C$.

Proof:

On the quadratic surface, $V(\underline{x}) > C$. As one progresses along any continuous path from the origin to the quadratic surface, $V(\underline{x})$ varies continuously from zero to $C' > C$. Therefore, along any continuous path from the origin to the quadratic surface, $V(\underline{x})$ must assume the value of C for at least one point. Hence there is at least one closed portion of the surface defined by $V(\underline{x}) = C$. It is also obtained from the same reasoning that there is at least one closed portion of the surface defined by

$$V(\underline{x}) = C', \text{ for } 0 < C' < C. \blacksquare$$

The proof of the lemma follows the one given by Malkin⁽³⁰⁾ to show that the surface $V(\underline{x}) = C$ will be closed for sufficiently small values of $C > 0$ if $V(\underline{x})$ is a positive definite function.

The next theorem presents sufficient conditions for all the portions of the surface defined by $V(\underline{x}) = C$, which are in the region Ω defined in the lemma, to be closed with respect to the origin. For the purpose of proving these sufficient conditions the Implicit Function Theorem⁽³⁾ is introduced.

Theorem (IFT)

Let $F(\underline{x})$ be a function of n variables which is continuous and its first partial derivatives are defined and continuous in an open region Ω and let $\underline{x}_0$ be a point in Ω for which $F(\underline{x}_0) = 0$. Suppose

that $\frac{\partial F(\underline{x}_0)}{\partial x_i} \neq 0$. Then there is a function ϕ

which is continuous and its first partial derivatives are defined and continuous in a neighborhood N of

$\underline{x}_0$ such that $x_i = \phi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ is a solution of $F(\underline{x}) = 0$ for $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ in N and such that $\phi(x_{10}, \dots, x_{(i-1)0}, x_{(i+1)0}, \dots, x_{n0}) = x_{i0}$.

The theorem can be interpreted in the following manner. If $V(\underline{x}) - C = F(\underline{x})$ is a continuous function of $\underline{x}$ and has defined and continuous first partial derivatives in Ω , then the surface $V(\underline{x}) - C = 0$ does not possess any abrupt end points or isolated points. The following theorem is now proven:

Theorem I

Given a scalar function $V(\underline{x})$ which satisfies all the conditions of the lemma and in addition:

- (iii) $V(\underline{x})$ has defined and continuous first partial derivatives in Ω

then all the portions of the surface defined by $V(\underline{x}) = C$ and in the region Ω , are closed with respect to the origin.

Proof:

The proof will be given by contradiction. Since any portion of the $V(\underline{x}) = C$ surface cannot cross the quadratic surface $\underline{x}'M_{s1}\underline{x} = C$, then the portions of the surface $V(\underline{x}) = C$, if not closed with respect to the origin, are closed with respect to another interior point or possess an abrupt end point.

As C is decreased to C' , the surface $V(\underline{x}) = C'$ must remain interior to the quadratic surface $\underline{x}'M_{s1}\underline{x} = C'$.

As C' is reduced to zero, the quadratic surface reduces to the origin and hence so does the surface $V(\underline{x}) = C'$. Therefore, the surface $V(\underline{x}) = C$ cannot be closed with respect to another interior point. The surface $V(\underline{x}) = C$ cannot possess an abrupt end point or an isolated point since the function $V(\underline{x}) - C = F(\underline{x})$ satisfies the conditions of the Implicit Function Theorem. Hence, all the portions of the surface defined by $V(\underline{x}) = C$ and in the region Ω are closed with respect to the origin. ■

The theorem can be illustrated in the following way. If the surface $V(\underline{x}) = C$ is projected onto the plane of any two variables x_i and x_j then theorem I excludes the cases shown in Fig. 3.1 and Fig. 3.2.

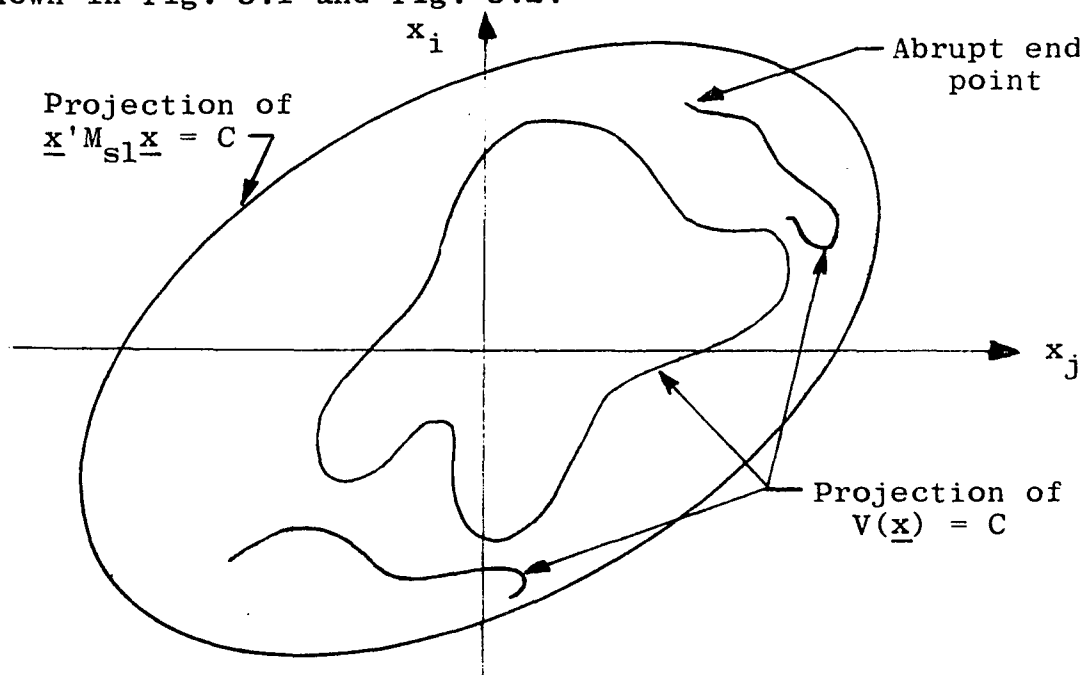


ILLUSTRATION OF THEOREM I
FIGURE 3.1

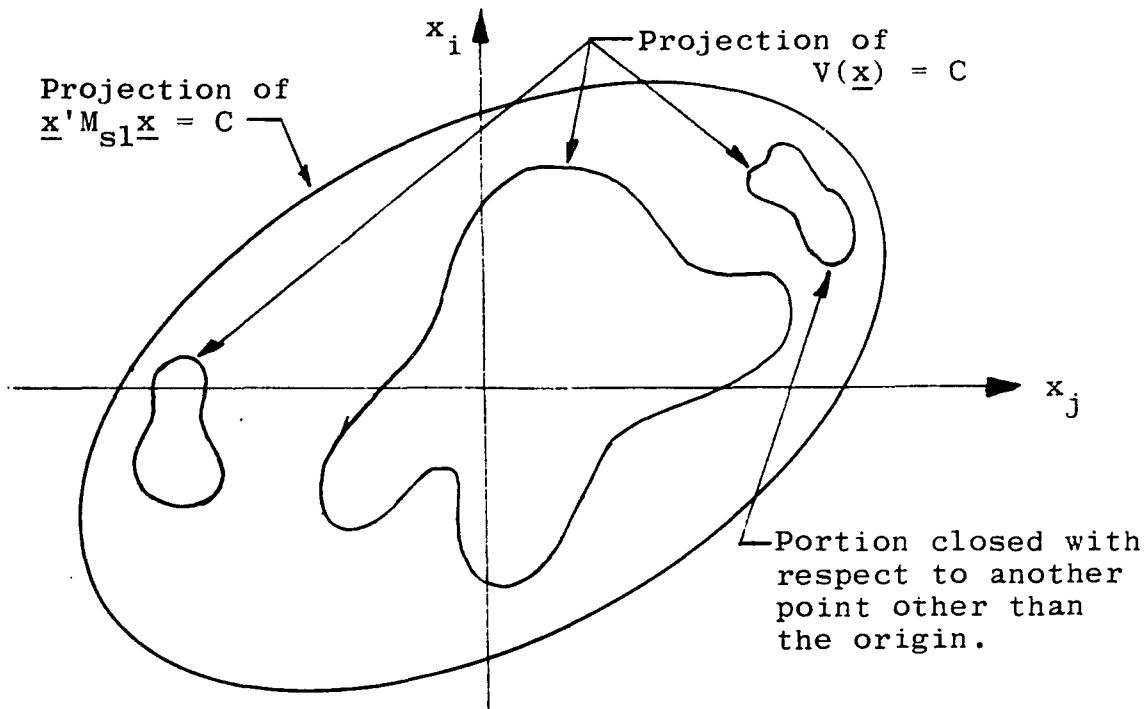


ILLUSTRATION OF THEOREM I
FIGURE 3.2

The case depicted in Fig. 3.1 is excluded since it shows the surface $V(\underline{x}) = C$ having abrupt end points. This is a contradiction of the Implicit Function Theorem.

The case depicted in Fig. 3.2 is excluded since it shows the surface $V(\underline{x}) = C$ possessing portions which are closed with respect to a point other than the origin. These portions would prevent the surface from contracting to the origin as C approaches zero.

The last theorem proven in this section gives sufficient conditions for the surface $V(\underline{x}) = C$ to contain one and only one closed portion with respect to the origin in the

interior of the quadratic surface. The case depicted in Figure 3.3 is excluded by this theorem since it shows the surface $V(\underline{x}) = C$ having more than one portion closed with respect to the origin.

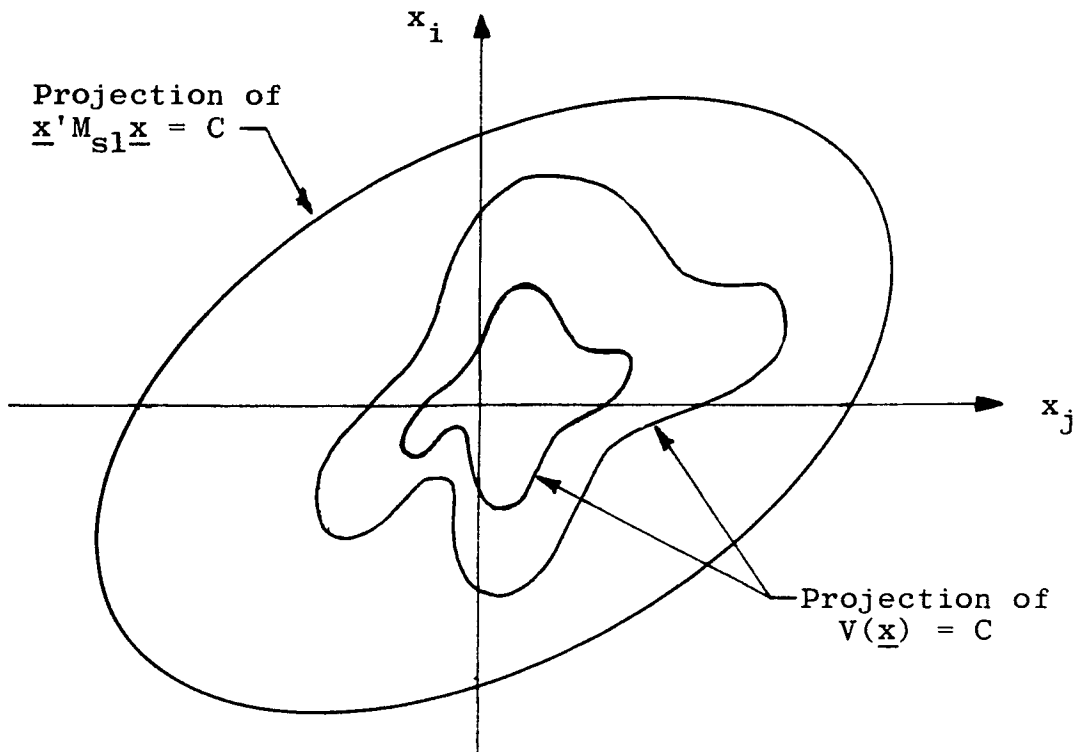


ILLUSTRATION FOR THEOREM II
FIGURE 3.3

In the following theorem it is assumed that the components of the vector $\underline{x}$ are a function of time t , or $\underline{x} = [x_1(t), x_2(t), \dots, x_n(t)]'$.

Theorem II

Given a scalar function $V(\underline{x})$ which satisfies the following conditions in the open region Ω defined as the interior of the quadratic surface $\underline{x}'M_{s1}\underline{x} = C > 0$ (where M_{s1} is a positive definite matrix)

- (i) $V(\underline{x})$ is positive definite
- (ii) $V(\underline{x})$ is a continuous function of $\underline{x}$
- (iii) $V(\underline{x})$ has defined and continuous first partial derivatives with respect to $\underline{x}$
- (iv) $\frac{dV(\underline{x})}{dt}$ is negative definite.

If $V(\underline{x}) > \underline{x}'M_{s1}\underline{x}$ in the interior and on the surface of the quadratic surface, then the open region Ω contains one and only one closed portion with respect to the origin of the surface defined by $V(\underline{x}) = C$.

Proof:

Since (i), (ii) and (iii) are the conditions for theorem I to be satisfied, there is at least one closed portion of $V(\underline{x}) = C$ in the interior of $\underline{x}'M_{s1}\underline{x} = C$, and all the portions of this surface in the interior are closed with respect to the origin. It now remains to be shown that there is only one portion.

The proof is given by contradiction. Assuming that there is more than one closed portion of the surface $V(\underline{x}) = C$, let the value of $\underline{x}(t_0)$ at time t_0 be such that it is a point on the furthest such portion. As t becomes infinite, the value of $V[\underline{x}(t)]$ approaches zero and the vector $\underline{x}$ approaches the origin along some continuous path. Along this path, $V(\underline{x})$ decreases monotonically and hence the path does not cross another $V(\underline{x}) = C$ surface. Hence the initial surface was the only surface on which $V(\underline{x}) = C$. ■

Theorem II states that all projections of the surface $V(\underline{x}) = C$ onto the plane of any two variables x_i and x_j must have the form depicted in Fig. 3.4 below.

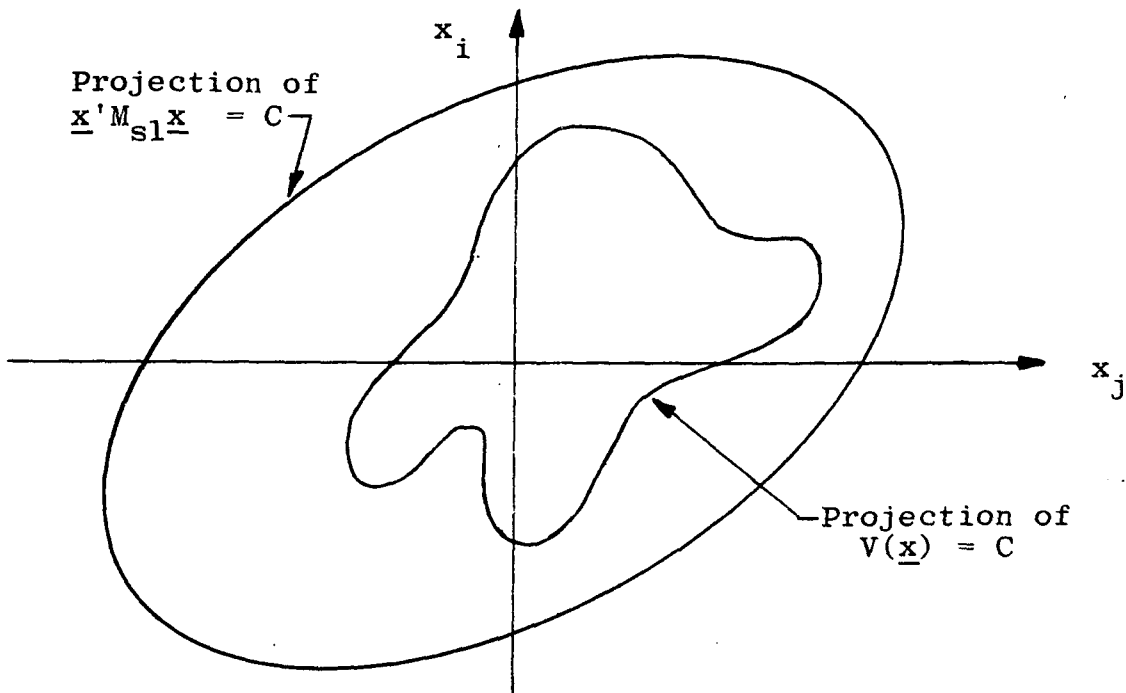


ILLUSTRATION OF THEOREM II
FIGURE 3.4

It is noted that the conditions on $V(\underline{x})$ in theorem II are sufficient conditions for a Liapunov function to show that the equilibrium solution of a system is asymptotically stable. It is also noted that the property

$$V(\underline{x}) > \underline{x}' M_{s1} \underline{x} \quad (3-12)$$

in the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = C$ in property 1 of section 3.2 and hence the techniques discussed in that section can be employed to verify this inequality.

In the following section, the theory for determining the extremal values of the ratio of two quadratic forms is presented. This theory is necessary for the derivation of the design inequalities of sections 5.3 and 5.4.

3.4 Extremal Values of the Ratio of Quadratic Forms

In this section the theory necessary for determining the extremal values of the ratio of two quadratic forms is presented. The results of this section are used in sections (5.3) and (5.4) for the derivation of design inequalities. Specifically it is desired to determine the extremal values of the ratio

$$\frac{\underline{x}' D_O \underline{x}}{\underline{x}' M_O \underline{x}} \quad (3-13)$$

in the entire $\underline{x}$ space, where $\underline{x}$ is a column vector and $\underline{x}'$ its transpose. M_O and D_O are constant symmetric matrices and M_O is positive definite.

The study of pencils of quadratic form presented by Gantmacher (9) is now discussed.

Definition 3.7. Pencil of Quadratic Forms

A pencil of quadratic forms given by

$$\underline{x}' D_O \underline{x} - \lambda \underline{x}' M_O \underline{x} \quad (3-14)$$

is determined by two hermitian matrices M_O and D_O , where λ is a parameter.

Note that a matrix M_O is defined as hermitian if $M_O^* = M_O$ where the asterisk denotes the conjugate

transpose operator and therefore a real symmetric matrix is a special case of a hermitian matrix.

Definition 3.8 Regular Pencil of Quadratic Forms

A regular pencil of quadratic forms is defined for (3-14) if the matrix M_0 is positive definite.

Using these definitions the following theorems are proven by Gantmacher*.

Theorem 3.7.

The characteristic equation

$$\det | D_0 - \lambda M_0 | = 0 \quad (3-15)$$

of a regular pencil of forms $\underline{x}' D_0 \underline{x} - \lambda \underline{x}' M_0 \underline{x}$ always has n real roots λ_k with the corresponding principal vectors $\underline{z}^k = (z_{1k}, z_{2k}, \dots, z_{nk})$ ($k=1, 2, \dots, n$) where

$$D_0 \underline{z}^k = \lambda_k M_0 \underline{z}^k \quad (3-16)$$

These principal vectors $\underline{z}^k$ can be chosen such that the relations

$$\underline{z}^{i'} M_0 \underline{z}^k = \delta_{ik} \quad (3-17)$$

are satisfied. $(i, k = 1, 2, \dots, n)$

δ_{ik} = Kronecker Delta.

*The proofs given are actually for real symmetric matrices. However, the extension to hermitian matrices is a trivial one.

It is noted that the n real roots λ_k are also the characteristic values of the matrix $D_0 M_0^{-1}$ since it follows from Eq. (3-15) that

$$\det | D_0 - \lambda M_0 | \det | M_0^{-1} | = 0 \quad (3-18)$$

or

$$\det | D_0 M_0^{-1} - \lambda | = 0 \quad (3-19)$$

which is the characteristic equation for the matrix $D_0 M_0^{-1}$.

Numbering the characteristic values in non-descending order,

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad (3-20)$$

the following theorem is presented.

Theorem 3.8

The smallest (largest) characteristic value of the regular pencil $\underline{x}' D_0 \underline{x} - \lambda \underline{x}' M_0 \underline{x}$ is the minimum (maximum) of the ratio of the forms $\underline{x}' D_0 \underline{x}$ and $\underline{x}' M_0 \underline{x}$, or

$$\min \underline{x} \left[\frac{\underline{x}' D_o \underline{x}}{\underline{x}' M_o \underline{x}} \right] = \lambda_1 = \min \lambda \left[D_o M_o^{-1} \right] \quad (3-21)$$

and

$$\max \underline{x} \left[\frac{\underline{x}' D_o \underline{x}}{\underline{x}' M_o \underline{x}} \right] = \lambda_n = \max \lambda \left[D_o M_o^{-1} \right] \quad (3-22)$$

and the minimum (maximum) is only assumed for the principal vectors of the characteristic value $\lambda_1 (\lambda_n)$.

Here the notation $\min \lambda \left[D_o M_o^{-1} \right]$ and $\max \lambda \left[D_o M_o^{-1} \right]$ is used to designate the minimum and maximum characteristic values of the matrix $D_o M_o^{-1}$, respectively. The results of theorem 3.8 are employed in sections (5.3) and (5.4) for the derivation of design inequalities.

In the next chapter, a matrix method for formulating a class of Liapunov functions for nonlinear autonomous control systems with multiple nonlinearities is presented.

CHAPTER 4

MODIFIED M-MATRIX FORMULATION OF LIAPUNOV FUNCTIONS

4.1 Introduction

This chapter presents a matrix formulation of a Liapunov function $V(\underline{x})$ and its time derivative. The formulation is applicable to the design of high order nonlinear autonomous systems which may contain multiple nonlinearities and is a modification of Mekel's⁽³²⁾ M-Matrix formulation. The major difference between the formulations is that the modified method yields a Liapunov function with a negative definite time derivative while the standard M-Matrix formulation yields a negative semidefinite time derivative. The negative definiteness is a condition necessary for the application of the design philosophy presented in chapter 5. A detailed discussion of the M-Matrix method is presented in Appendix A.

The modified method begins with an assumed form of $V(\underline{x})$ and its time derivative is then obtained. A postulated relation between the quadratic portions of $V(\underline{x})$ and $\dot{V}(\underline{x})$ and the considerations that $V(\underline{x}) > 0$ and $\dot{V}(\underline{x}) < 0$ determine the Liapunov function and its time derivative. The

quadratic portions of $V(\underline{x})$ and $\dot{V}(\underline{x})$ are formulated to be positive definite and negative definite, respectively. This assures that the Liapunov function and its time derivative are positive and negative definite, respectively in a neighborhood of the origin.

In the following section, other methods for the generation of Liapunov functions are discussed and the reasons for selecting and modifying the M-Matrix method are presented. The first part of section 4.3 presents the formulation of the quadratic portion of the modified M-Matrix Liapunov function and the second part of that section presents the formulation of the higher order terms. Section 4.4 concludes this chapter with illustrative formulations.

4.2 Background

A major weakness of Liapunov's direct method has been the construction of Liapunov functions. In an effort to solve this problem, a number of techniques for the generation of such functions have been developed.^(19, 29, 32, 40, 45) For the purpose of design, however, most of these techniques appear to be non-applicable.

Lurie⁽²⁹⁾ established a method to find sufficient conditions for global asymptotic stability of systems with one nonlinearity. He introduced a canonic form of the system equations and recommended a positive definite form for a Liapunov function in terms of the canonic variables. Lurie's method yields sufficient conditions for global asymptotic stability only for nonlinearities which lie entirely in the first and third quadrants. This method is therefore non-applicable for the investigation of systems with multiple nonlinearities or which may be asymptotically stable in a finite region of the state space.

Zubov's⁽⁴⁵⁾ method is an excellent analysis tool in the sense that it generates a Liapunov function which defines the exact boundary of the domain of attraction⁽³¹⁾. However, due to the necessity of solving linear partial differential equations, this method does not lend itself to design.

The Variable Gradient⁽⁴⁰⁾ method is based upon the assumption of a variable gradient function ∇V , from which V , the Liapunov function and $\dot{V}$, its time derivative can be determined. The determination of V and $\dot{V}$ is accomplished by imposing constraints on $\dot{V}$ and using generalized curl equations. When considering systems of high order, relations must be assumed among the elements of the variable gradient function in order to obtain solutions to the constraints. The assumed relations yield negative semi-definite time derivatives of the Liapunov function. Hence this method has restrictions for the current design method..

When considering nonlinear systems of high order that may contain multiple nonlinearities which may be of the multiplicative type the M-Matrix method⁽³²⁾ appears to be the most applicable. Unfortunately, this method also yields a negative semidefinite time derivative. Further study and investigation has found however that the method can be modified to yield a negative definite time derivative for a class of systems. The modified M-Matrix method is presented in the next section.

4.3 Formulation Of Liapunov Functions

The modified M-Matrix method permits the formulation of a Liapunov function and its time derivative using the forms

$$V(\underline{x}) = \underline{x}' M_o \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x} \quad (4-1)$$

and

$$\dot{V}(\underline{x}) = 2\underline{x}' M_o A_o \underline{x} + \underline{x}' D_{on}(\underline{x}) \underline{x} \quad (4-2)$$

where M_o and $M_o A_o$ are positive definite and negative definite real matrices, respectively. The subscripts o and n denote the order of the system and the number of nonlinearities in the system, respectively. The matrices $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ are square matrices with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$.

The modified M-Matrix method can be employed to generate a Liapunov function and its time derivative for the class of nonlinear autonomous systems which can be described by a vector differential equation of the form

$$\dot{\underline{x}} = A_o \underline{x} + A_{on}(\underline{x}) \underline{x} \quad (4-3)$$

where A_o is a constant square matrix with only negative real characteristic values and simple elementary divisors.

$A_{on}(\underline{x})$ is a square matrix with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$. The vector $A_{on}(\underline{x})\underline{x}$ contains no linear terms. The subscripts o and n denote the order of the system and the number of nonlinearities, respectively.

As proven in the next section, the requirement that the characteristic values of A_o to be negative real is a necessary condition for the M_o and $M_o A_o$ matrices to be positive and negative definite, respectively. The conditions on the elements of $A_{on}(\underline{x})$ are sufficient for the conditions on the elements of the $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ matrices to be satisfied.

The class defined by Eq. (4-3) includes systems which are of high order and contain multiple nonlinearities which may be of the multiplicative type. A multiplicative nonlinearity is defined as a nonlinear function $f(\underline{x})$ which contains a term $C_{ij}x_i^n x_j^m$, where n , m and C_{ij} are nonzero constants, for some x_i and x_j components of the vector $\underline{x}$.

The form of the $M_{on}(\underline{x})$ matrix is assumed and its elements are chosen in a logical and systematic manner so as to be related to the elements of the $A_{on}(\underline{x})$ matrix. The expressions for the M_o , $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ matrices are formulated in the following section.

4.3.1 M₀ Matrix

For the application of the design method philosophy which is presented in Chapter 5, it is desired to formulate a positive definite M_0 matrix so that

$$M_0 A_0 = (M_0 A_0)' \quad (\text{symmetric}) \quad (4-4)$$

$$M_0 A_0 < 0 \quad (\text{negative definite}) \quad (4-5)$$

where the superscript (') denotes the transpose operator.

Necessary and sufficient conditions on the matrix A_0 for the satisfaction of Eqs. (4-4) and (4-5) are given by the following three theorems.

Theorem III

The matrix $M_0 A_0$, where M_0 is a positive definite matrix, is symmetric only if A_0 has real characteristic values.

Proof

If $M_0 A_0$ is symmetric, then

$$M_0 A_0 = (M_0 A_0)' = A_0' M_0 \quad (T1)$$

Since M_0 is a positive definite matrix, there exists a non-singular matrix C such that

$$M_0 = C' C \quad (T2)$$

$$\text{Therefore, } C'CA_O = A_O' C'C \quad (T3)$$

or

$$CA_OC^{-1} = C'^{-1}A_O'C' = (CA_OC^{-1})' \quad (T4)$$

Denoting $CA_OC^{-1} = P$, it is noted that P is a symmetric matrix and has the same characteristic values as A_O . Since a symmetric matrix has real characteristic values, A_O must have real characteristic values if Eq. (T1) is to be satisfied.

Theorem IV

Given an A_O matrix, it is possible to find a positive definite matrix M_O which causes the matrix M_OA_O to be symmetric if there exists a nonsingular matrix C such that $CA_OC^{-1} = P$ where P is a real symmetric matrix.

Proof

From the hypothesis,

$$CA_OC^{-1} = P = P' = C^{-1}'A_O'C' \quad (T1)$$

$$\text{or } C'CA_O = A_O'C'C \quad (T2)$$

If the matrix M_O is formed from

$$M_O = C'C \quad (T3)$$

then M_O is positive definite and from Eq. (T2)

$$M_O A_O = A_O' M_O \quad (T4)$$

or $M_O A_O$ is a symmetric matrix

Theorem V

If $M_O A_O$ is a symmetric matrix and M_O is positive definite, then $M_O A_O$ is negative definite if and only if A_O has negative real characteristic values.

Proof:

This proof follows that of Lefschetz⁽²⁴⁾ for which $M_O A_O$ is not assumed symmetric.

To prove necessity, denote $M_O A_O = T$ and note that there exists a vector $\underline{u} \neq 0$ such that

$$A_O \underline{u} = \lambda \underline{u} \quad (T1)$$

where λ is a characteristic value of A_O

(From Theorem III all the λ 's are real). From the hypothesis,

$$A_O' M_O + M_O A_O = 2T \quad (T2)$$

$$\text{or } \underline{u}' (A_O' M_O + M_O A_O) \underline{u} = 2 \underline{u}' T \underline{u} \quad (T3)$$

$$\text{or } 2 \lambda \underline{u}' M_O \underline{u} = 2 \underline{u}' T \underline{u} \quad (T4)$$

Since M_O is positive definite, $\underline{u}' M_O \underline{u} > 0$.

Therefore, for $M_0 A_0 = T$ to be negative definite, it is necessary for $\lambda < 0$.

To prove sufficiency, it is noted from (T4) that if $\lambda < 0$, then $M_0 A_0 = T$ is negative definite.

A sufficient condition for the hypothesis of theorem IV, i.e. a real symmetric matrix P exists such that $CA_0C^{-1} = P$, to be satisfied is that the A_0 matrix possess simple elementary divisors. This is a result of the fact that the matrix A_0 is of simple structure (or similar to a diagonal matrix) if and only if it possesses simple elementary divisors. The hypothesis of theorem IV can then be satisfied by choosing the P matrix equal to $\bar{\lambda}$ where $\bar{\lambda}$ is a diagonal matrix with elements that are the characteristic values of A_0^* . Other choices for the P matrix can then be obtained from symmetric matrices which are similar to $\bar{\lambda}$.

In summary of the results of theorems III, IV and V, if a constant square matrix A_0 has negative real characteristic values and simple elementary divisors, then a positive definite M_0 matrix exists so that $M_0 A_0$ is a negative definite matrix and that M_0 matrix can be generated by the following steps:

*Vogt(43) has given sufficient conditions on A_0 for the existence of a special class of Liapunov functions for the special case $P = \bar{\lambda}$ only.

- (1) Choose a symmetric matrix P which has the same characteristic values as A_o .
- (2) Obtain C from the equation

$$CA_oC^{-1} = P \quad (4-6)$$

- (3) Formulate M_o as

$$M_o = C'C \quad (4-7)$$

The formulation of the $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ matrices is presented next.

4.3.2 $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ Matrices

In choosing the $M_{on}(\underline{x})$ matrix, Mekel's M-Matrix method given in Appendix A is followed. The form of $M_{on}(\underline{x})$ is assumed and its elements are chosen to be directly related to the elements of the system matrix $A_{on}(\underline{x})$. A compendium of $M_{on}(\underline{x})$ matrices is presented in reference [32] .

The general formulation is now illustrated and further examples are presented in section 4.4. The $M_{on}(\underline{x})$ matrix is of the form,

$\frac{F_1(\underline{x})}{x_1^2}$	$\frac{f_1(x_1)}{x_1}$	$\frac{f_2(x_1)}{x_1}$	$\dots$	$\frac{f_{n-2}(x_1)}{x_1}$	0
$\frac{f_1(x_1)}{x_1}$	$\frac{F_2(\underline{x})}{x_2^2}$	$\frac{g_1(x_2)}{x_2}$	$\dots$	$\frac{g_{n-3}(x_2)}{x_2}$	0
$\frac{f_2(x_1)}{x_1}$	$\frac{g_1(x_2)}{x_2}$	$\frac{F_3(\underline{x})}{x_3^2}$	$\dots$	$\frac{\emptyset_{n-4}(x_3)}{x_3}$	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\frac{f_{n-2}(x_1)}{x_1}$	$\frac{g_{n-3}(x_2)}{x_2}$	$\frac{\emptyset_{n-4}(x_3)}{x_3}$	$\dots$	$\frac{F_{n-1}(\underline{x})}{x_{n-1}^2}$	0
0	0	0	$\dots$	0	0

(4-8)

The forms of the terms $F_i(\underline{x})$, $f_i(x_i)$, $g_i(x_i)$, etc., are dependent upon the forms of the elements of the $A_{on}(\underline{x})$ matrix. In order to illustrate this formulation, consider the differential equation describing a fourth

$$\ddot{\ddot{x}} + a_4 \ddot{x} + a_3 \dot{x} + g(x) + a_2 \dot{x} + a_1 x + f(x) = 0$$

(4-9)

The vector form of Eq. (4-9) is

$$\dot{\underline{x}} = A_4 \underline{x} + A_{42}(\underline{x}) \underline{x} \quad (4-10)$$

where

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix} \quad (4-11)$$

and

$$A_{42}(\underline{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-f(x_1)}{x_1} & \frac{-g(x_2)}{x_2} & 0 & 0 \end{bmatrix} \quad (4-12)$$

The form of $M_{42}(\underline{x})$ is obtained as

$$M_{42}(\underline{x}) = \begin{bmatrix} \frac{F_1(x_1)}{x_1^2} & \frac{f_1(x_1)}{x_1} & \frac{f_2(x_1)}{x_1} & 0 \\ \frac{f_1(x_1)}{x_1} & \frac{F_2(x_2)}{x_2^2} - \frac{df_2(x_1)}{dx_1} & \frac{g_1(x_2)}{x_2} & 0 \\ \frac{f_2(x_2)}{x_1} & \frac{g_1(x_2)}{x_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(4-13)

Once the form of the $M_{on}(\underline{x})$ matrix has been formulated the expressions for its elements must be generated. This is accomplished by obtaining the expression for the time derivative of $V(\underline{x})$, which is given by

$$\begin{aligned} \dot{V}(\underline{x}) = & \underline{x}' A_O' M_O \underline{x} + \underline{x}' A'_{on}(\underline{x}) M_O \underline{x} + \underline{x}' M_O A_O \underline{x} + \underline{x}' M_O A_{on}(\underline{x}) \underline{x} \\ & + \underline{x}' A_O' M_{on}(\underline{x}) \underline{x} + \underline{x}' A'_{on}(\underline{x}) M_{on}(\underline{x}) \underline{x} + \underline{x}' M_{on}(\underline{x}) A_O \underline{x} \\ & + \underline{x}' M_{on}(\underline{x}) A_{on}(\underline{x}) \underline{x} + \underline{x}' \dot{M}_{on}(\underline{x}) \underline{x} \end{aligned} \quad (4-14)$$

or

$$\dot{V}(\underline{x}) = \underline{x}' D_O \underline{x} + \underline{x}' D_{on}(\underline{x}) \underline{x} \quad (4-15)$$

where

$$D_O = A_O' M_O + M_O A_O = 2M_O A_O \quad (4-16)$$

for an M_O matrix as described in section 4.3.1 and

$$\begin{aligned} D_{on}(\underline{x}) = & A'_{on}(\underline{x}) M_O + M_O A_{on}(\underline{x}) + A_O' M_{on}(\underline{x}) + M_{on}(\underline{x}) A_O \\ & + A_{on}'(\underline{x}) M_{on}(\underline{x}) + M_{on}(\underline{x}) A_{on}(\underline{x}) + \dot{M}_{on}(\underline{x}) \end{aligned} \quad (4-17)$$

It is desired to formulate a $\dot{V}(\underline{x})$ which is negative definite. Since, in view of theorem V. and the fact that the A_O matrices considered have negative real characteristic

values, D_0 is a negative definite matrix and it is therefore sufficient that the matrix $D_{on}(\underline{x})$ be negative semidefinite for $\dot{V}(\underline{x})$ to be negative definite. The elements of the $D_{on}(\underline{x})$ matrix, denoted by $d_{ii}(\underline{x})$ and $d_{ij}(\underline{x})$ may therefore be constrained to satisfy the following conditions.

$$\text{and} \quad d_{ii}(\underline{x}) \leq 0 \quad (4-18)$$

$$d_{ij}(\underline{x}) + d_{ji}(\underline{x}) = 0 \quad (4-19)$$

where the subscripts i and j denote the row and column of the matrix respectively. These conditions permit the evaluation of the unknown elements of the $D_{on}(\underline{x})$ and $M_{on}(\underline{x})$ matrices. Substitution of these elements into equations (4-1) and (4-2) yield the Liapunov function and its time derivative.

Detailed formulations of the modified M-Matrix Liapunov functions are presented in the next section.

4.4 Illustrative Formulations of Liapunov Functions

In this section modified M-Matrix Liapunov functions are formulated for various systems.

4.4.1 M₀ Matrices

The steps for generating the quadratic portions of the Liapunov functions which were presented in section 4.3.1 are demonstrated for second and third order systems. One does not encounter new theoretical difficulties when generating the M₀ matrix for higher order systems.

For a second order system, we have

$$A_2 = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \quad (4-20)$$

Following the steps of section 4.3.1, one obtains

$$(1) \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \quad (4-21)$$

and

$$(2) \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad (4-22)$$

where

$$c_{11} = \pm \sqrt{\frac{1}{2}(p_{12} + p_{22})}$$

$$\begin{aligned}
c_{22} &= \pm \sqrt{\frac{1}{2}} \\
c_{12} &= \mp \sqrt{\frac{1}{2}} \\
c_{21} &= \mp \sqrt{\frac{1}{2}} (p_{12} + p_{11})
\end{aligned} \tag{4-23}$$

and (3)

$$M_2 = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & 1 \end{bmatrix} \tag{4-24}$$

$$\text{where } m_{11} = c_{11}^2 + c_{21}^2 \tag{4-25}$$

$$m_{12} = c_{11} c_{12} + c_{21} c_{22}$$

For a third order system we have

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \tag{4-26}$$

Following the steps of section 4.3.1 we have

$$(1) \quad P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \tag{4-27}$$

and

(2)

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (4-28)$$

where

$$\begin{aligned} \Delta \cdot c_{11} = & p_{22}^2 p_{33} - p_{22} p_{33}^2 + p_{11} p_{12} p_{23} - p_{11} p_{13} p_{23} \\ & + p_{22} p_{13} p_{23} - p_{22} p_{12} p_{23} + p_{33} p_{13} p_{23} - p_{33} p_{12} p_{23} \\ & - p_{11} p_{22} p_{13} + p_{11} p_{33} p_{12} + p_{22} p_{33} p_{12} - p_{22} p_{33} p_{13} \\ & + p_{22}^2 p_{13} - p_{22} p_{13}^2 - p_{22} p_{23}^2 + p_{33} p_{23}^2 + p_{33} p_{12}^2 \\ & - p_{33}^2 p_{12} + 2p_{13} p_{23}^2 - 2p_{12} p_{23}^2 \end{aligned}$$

$$\begin{aligned} \Delta \cdot c_{22} = & p_{11}^2 - p_{33}^2 + p_{11} p_{12} + p_{22} p_{12} - p_{22} p_{23} - p_{33} p_{23} \\ & - p_{12} p_{13} + p_{13} p_{23} + p_{12}^2 - p_{23}^2 \end{aligned}$$

$$\Delta \cdot c_{33} = p_{11} - p_{22} + p_{13} - p_{23} \quad (4-29)$$

$$\Delta.c_{12} = -p_{22}^2 + p_{33}^2 - p_{11} p_{12} + p_{11} p_{13} - p_{22} p_{12} \\ + p_{33} p_{13} + p_{12} p_{23} - p_{13} p_{23} - p_{12}^2 + p_{13}^2$$

$$\Delta.c_{13} = p_{22} - p_{33} + p_{12} - p_{13}$$

$$\Delta.c_{23} = -p_{11} + p_{33} - p_{12} + p_{23}$$

$$\Delta.c_{31} = p_{11}^2 p_{22} - p_{11} p_{22}^2 + p_{11} p_{12} p_{23} - p_{11} p_{12} p_{13} \\ + p_{22} p_{12} p_{23} - p_{22} p_{12} p_{13} + p_{33} p_{12} p_{13} \\ - p_{33} p_{12} p_{23} + p_{11} p_{22} p_{13} - p_{11} p_{22} p_{23} - p_{11} p_{33} p_{23} \\ + p_{22} p_{33} p_{13} + p_{11}^2 p_{23} - p_{11} p_{23}^2 - p_{11} p_{12}^2 + p_{22} p_{12}^2 \\ + p_{22} p_{13}^2 - p_{22}^2 p_{13} + 2p_{12}^2 p_{23} - 2p_{12}^2 p_{13}$$

$$\Delta.c_{21} = p_{11} p_{33}^2 - p_{11}^2 p_{33} + p_{11} p_{12} p_{13} - p_{11} p_{13} p_{23} \\ + p_{22} p_{13} p_{23} - p_{22} p_{12} p_{13} + p_{33} p_{12} p_{13} \\ - p_{33} p_{13} p_{23} + p_{11} p_{22} p_{23} + p_{11} p_{33} p_{23} \\ - p_{11} p_{33} p_{12} - p_{22} p_{33} p_{12} + p_{11} p_{23}^2 - p_{11}^2 p_{23} \\ + p_{11} p_{13}^2 + p_{33}^2 p_{12} - p_{33} p_{12}^2 - p_{33} p_{13}^2 + 2p_{12} p_{13}^2 \\ - 2p_{13}^2 p_{23}$$

$$\Delta \cdot c_{32} = -p_{11}^2 + p_{22}^2 - p_{11} p_{13} + p_{22} p_{23} + p_{33} p_{23} \\ - p_{33} p_{13} + p_{12} p_{13} - p_{12} p_{23} - p_{13}^2 + p_{23}^2$$

where

$$\Delta = p_{11} - p_{22} + p_{13} - p_{23}$$

and

(3)

$$M_3 = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (4-30)$$

4.4.2 $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ Matrices

The $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ matrices for various systems are now formulated.

Case 2.1

The system differential equation is

$$\ddot{x} + a_2 \dot{x} + a_1 x + f(x) = 0 \quad (4-31)$$

or in vector form

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ \frac{-f(x_1)}{x_1} & 0 \end{bmatrix} \underline{x} \quad (4-32)$$

The assumed form of $M_{21}(\underline{x})$ is

$$M_{21}(\underline{x}) = \begin{bmatrix} \frac{F_1(x_1)}{x_1^2} & 0 \\ 0 & 0 \end{bmatrix} \quad (4-33)$$

$D_{21}(\underline{x})$ is obtained from

$$\begin{aligned} D_{21}(\underline{x}) &= A_{21}''(\underline{x})M_2 + M_2 A_{21}(\underline{x}) + A_2' M_{21}(\underline{x}) \\ &+ M_{21}(\underline{x})A_2 + M_{21}(\underline{x}) A_{21}(\underline{x}) + A_{21}'(\underline{x}) M_{21}(\underline{x}) \\ &+ \dot{M}_{21}(\underline{x}) \end{aligned} \quad (4-34)$$

or

$$D_{21}(\underline{x}) = 2 \left[\begin{array}{c|c} -m_{12} \frac{f(x_1)}{x_1} & \frac{F_1(x_1)}{x_1^2} + \frac{1}{2} x_1 \frac{d}{dx_1} \left[\frac{F_1(x_1)}{x_1^2} \right] \\ \hline -\frac{f(x_1)}{x_1} & 0 \end{array} \right] \quad (4-35)$$

Satisfying the constraint

$$d_{12}(\underline{x}) + d_{21}(\underline{x}) = 0 \quad (4-36)$$

yields

$$\frac{f(x_1)}{x_1} = \frac{F_1(x_1)}{x_1^2} + \frac{1}{2} x_1 \frac{d}{dx_1} \left[\frac{F_1(x_1)}{x_1^2} \right] \quad (4-37)$$

or

$$\frac{F_1(x_1)}{x_1^2} = \frac{2}{x_1^2} \int_0^{x_1} f(x_1) dx_1 \quad (4-38)$$

Therefore,

$$V(\underline{x}) = \underline{x}' M_2 \underline{x} + \underline{x}' \begin{bmatrix} \frac{2}{x_1^2} \int_0^{x_1} f(x_1) dx_1 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}$$

(4-39)

and

$$\dot{V}(\underline{x}) = 2\underline{x}' M_2 A_2 \underline{x} + \underline{x}' \begin{bmatrix} \frac{-2m_{12} f(x_1)}{x_1} & 0 \\ 0 & 0 \end{bmatrix} \underline{x}$$

(4-40)

Case 2.2

The system differential equation is

$$\ddot{\underline{x}} + (a_2 + \psi(x))\dot{\underline{x}} + a_1 \underline{x} = 0 \quad (4-41)$$

or in vector form,

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 0 & -\psi(x_1) \end{bmatrix} \underline{x} \quad (4-42)$$

The assumed form of $M_{21}(\underline{x})$ is

$$M_{21}(\underline{x}) = 0 \text{ (null matrix)} \quad (4-43)$$

and $D_{21}(\underline{x})$ is obtained as

$$D_{21}(\underline{x}) = 2 \begin{bmatrix} 0 & -m_{12}\psi(x_1) \\ 0 & -\psi(x_1) \end{bmatrix} \quad (4-44)$$

Therefore,

$$V(\underline{x}) = \underline{x}' M_2 \underline{x} \quad (4-45)$$

and

$$\dot{\underline{V}}(\underline{x}) = 2\underline{x}' M_2 A_2 \underline{x} + 2\underline{x}' \begin{bmatrix} 0 & -m_{12}\psi(x_1) \\ 0 & -\psi(x_1) \end{bmatrix} \underline{x} \quad (4-46)$$

Case 2.3

The system differential equation is

$$\ddot{x} + (a_2 + \psi(x))\dot{x} + a_1 x + f(x) = 0 \quad (4-47)$$

or in vector form

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ \frac{-f(x_1)}{x_1} & -\psi(x_1) \end{bmatrix} \underline{x} \quad (4-48)$$

The assumed form of $M_{22}(\underline{x})$ is

$$M_{22}(\underline{x}) = \begin{bmatrix} \frac{F_1(x_1)}{x_1^2} & 0 \\ 0 & 0 \end{bmatrix} \quad (4-49)$$

and $D_{22}(\underline{x})$ is obtained as

$$D_{22}(\underline{x}) = 2 \left[\begin{array}{c|c} -m_{12} \frac{f(x_1)}{x_1} & \frac{F_1(x_1)}{x_1^2} + \frac{1}{2} x_1 \frac{d}{dx_1} \left[\frac{F_1(x_1)}{x_1^2} \right] - m_{12} \psi(x_1) \\ \hline \frac{-f(x_1)}{x_1} & -\psi(x_1) \end{array} \right] \quad (4-50)$$

Satisfying the constraint

$$d_{12}(\underline{x}) + d_{21}(\underline{x}) = 0 \quad (4-51)$$

yields

$$\frac{f(x_1)}{x_1} + m_{12} \psi(x_1) = \frac{F_1(x_1)}{x_1^2} + \frac{1}{2} x_1 \frac{d}{dx_1} \left[\frac{F_1(x_1)}{x_1^2} \right] \quad (4-52)$$

or

$$\frac{F_1(x_1)}{x_1^2} = \frac{2}{x_1^2} \int_0^{x_1} \left[f(x_1) + m_{12} x_1 \psi(x_1) \right] dx_1 \quad (4-53)$$

Therefore,

$$V(\underline{x}) = \underline{x}' M_2 \underline{x} + \underline{x}' \begin{bmatrix} \frac{2}{x_1^2} \int_0^{x_1} [f(x_1) + m_{12} x_1 \psi(x_1)] dx_1 & 0 \\ \hline 0 & 0 \end{bmatrix} \underline{x} \quad (4-54)$$

and

$$\dot{V}(\underline{x}) = 2\underline{x}' M_2 A_2 \underline{x} + \underline{x}' \begin{bmatrix} -2m_{12} \frac{f(x_1)}{x_1} & 0 \\ \hline 0 & -2\psi(x_1) \end{bmatrix} \underline{x} \quad (4-55)$$

Case 3.1

The system differential equation is

$$\ddot{\ddot{x}} + a_3 \ddot{x} + (a_2 + \psi(x)) \dot{x} + a_1 x = 0 \quad (4-56)$$

or in vector form

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\psi(x_1) & 0 \end{bmatrix} \underline{x} \quad (4-57)$$

The assumed form of $M_{31}(\underline{x})$ is

$$M_{31}(\underline{x}) = \begin{bmatrix} 0 & \frac{f_1(x_1)}{x_1} & 0 \\ \frac{f_1(x_1)}{x_1} & \frac{F_2(\underline{x})}{x_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4-58)$$

and $D_{31}(\underline{x})$ is obtained as

$$D_{31}(\underline{x}) = \begin{bmatrix} 0 & -m_{13}\psi(x_1) & \frac{f_1(x_1)}{x_1} \\ \frac{d}{dx_1} \left[\frac{f_1(x_1)}{x_1} \right] x_2 & \frac{1}{2} \frac{d}{dt} \left[\frac{F_2(\underline{x})}{x_2^2} \right] + \frac{f_1(x_1)}{x_1} - m_{23}\psi(x_1) & \frac{F_2(\underline{x})}{x_2^2} \\ 0 & -\psi(x_1) & 0 \end{bmatrix} \quad (4-59)$$

Satisfying the constraint

$$d_{23}(\underline{x}) + d_{32}(\underline{x}) = 0 \quad (4-60)$$

yields

$$F_2(\underline{x}) = \psi(x_1)x_2^2 \quad (4-61)$$

Satisfying the constraint

$$d_{13}(\underline{x}) + d_{31}(\underline{x}) = 0 \quad (4-62)$$

yields

$$f_1(x_1) = 0 \quad (4-63)$$

Therefore,

$$V(\underline{x}) = \underline{x}' M_3 \underline{x} + \underline{x}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & \psi(x_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} \quad (4-64)$$

and

$$\dot{V}(\underline{x}) = \underline{x}' D_3 \underline{x} + 2 \underline{x}' \begin{bmatrix} 0 & -m_{13}\psi(x_1) & 0 \\ 0 & -m_{23}\psi(x_1) + \frac{1}{2} \left[\frac{d\psi(x_1)}{dx_1} \right] x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x}$$

(4-65)

Additional expressions for $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ matrices are presented in Appendix A and reference [32] .

4.5 Summary

The modified M-Matrix method for generating Liapunov functions has been presented in detail. The Liapunov function generated possesses a positive definite quadratic portion which is independent of the high order components and a quadratic portion for its time derivative $\dot{V}(\underline{x})$ is formulated to be negative definite. This assures that the Liapunov function and its time derivative are positive and negative definite, respectively, in a neighborhood of the origin. The negative definiteness of $\dot{V}(\underline{x})$ is a necessary condition for the application of the design method, the philosophy and objective of which are presented in the next chapter.

CHAPTER 5

PHILOSOPHY AND OBJECTIVES OF THE DESIGN METHOD

5.1 Introduction

This chapter presents the philosophy and objectives of the design method. The design method is applicable to the class of high order nonlinear control systems containing inherent nonlinearities which when augmented with a controller can be described by a vector differential equation of the form

$$\dot{\underline{x}} = A_o \underline{x} + A_{on}(\underline{x}) \underline{x} \quad (5-1)$$

where A_o is a constant square matrix with only negative real characteristic values and simple elementary divisors. $A_{on}(\underline{x})$ is a square matrix with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$. The vector $A_{on}(\underline{x}) \underline{x}$ contains no linear terms and the subscripts o and n denote the order of the system and the number of nonlinearities, respectively.

The major objective of the design method is to formulate a controller so that the augmented system satisfies a generalized settling time specification. This specification is stated as follows:

Generalized Settling Time Specification

Denoting a system's output by $x_1(t)$ and the system's phase space vector by $\underline{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ it is required that the system's equilibrium solution be asymptotically stable and that

$$|x_1(t)| \leq x_{1s} \quad \text{for } t \geq t_s \quad (5-2)$$

$$\text{for } \underline{x}(t_0) = [x_{10}, x_{20}, \dots, x_{n0}]^T \quad (5-3)$$

where $\underline{x}(t_0)$ is the value of the state vector $\underline{x}(t)$ at the time t_0 and x_{1s} and t_s are specified constants.

The objective of the design method is accomplished by utilizing Liapunov's theory and the quantitative properties of a Liapunov function $V(\underline{x})$, its time derivative $\dot{V}(\underline{x})$ and a time domain ratio criterion $\eta(\underline{x})$ which is formulated from the ratio $-\dot{V}(\underline{x})/V(\underline{x})$.

Section 5.2 formulates the constraints which the Liapunov function and its time derivative must satisfy for the application of the design method. In sections 5.3 and 5.4 constraints for the time domain ratio criterion are developed which are sufficient conditions for the augmented system to satisfy the generalized settling time specification.

5.2 Constraints for the Liapunov Function and its Time Derivative

In this section the constraints which the Liapunov function and its time derivative must satisfy for the application of the design method are formulated. These constraints are sufficient conditions for the system's equilibrium solution (assumed to be the origin of the state space) to be asymptotically stable for a specified initial condition $\underline{x}(t_0) = [x_{10}, x_{20}, \dots, x_{n0}]'$

The following theorem has been proven by LaSalle and Lefschetz.(23)

Theorem 5.1

Let $V(\underline{x})$ be a scalar function with continuous first partial derivatives. Let Ω designate the region where $V(\underline{x}) < L$. Assume that Ω is bounded and that within Ω

$$\begin{aligned} V(\underline{x}) &> 0 \text{ and } \dot{V}(\underline{x}) < 0 \text{ for } \underline{x} \neq 0 \\ V(\underline{0}) &= 0 \text{ and } \dot{V}(\underline{0}) = 0 \end{aligned}$$

then the origin is asymptotically stable and every solution in Ω tends to the origin as $t \rightarrow \infty$.

The theorem can be interpreted in the following manner. The origin is asymptotically stable for a given initial condition if the initial condition is within an open region Ω , defined by $V(\underline{x}) < L$, which is bounded and within Ω $V(\underline{x})$ and $\dot{V}(\underline{x})$ are positive and negative definite, respectively. The problem arises as to the establishment of the existence of such a bounded region Ω . The concepts of surfaces which were presented in section 3.3 will now be utilized to solve this problem. The concepts are used to formulate constraints for $V(\underline{x})$ and $\dot{V}(\underline{x})$ which are sufficient conditions for the existence of the desired bounded region Ω .

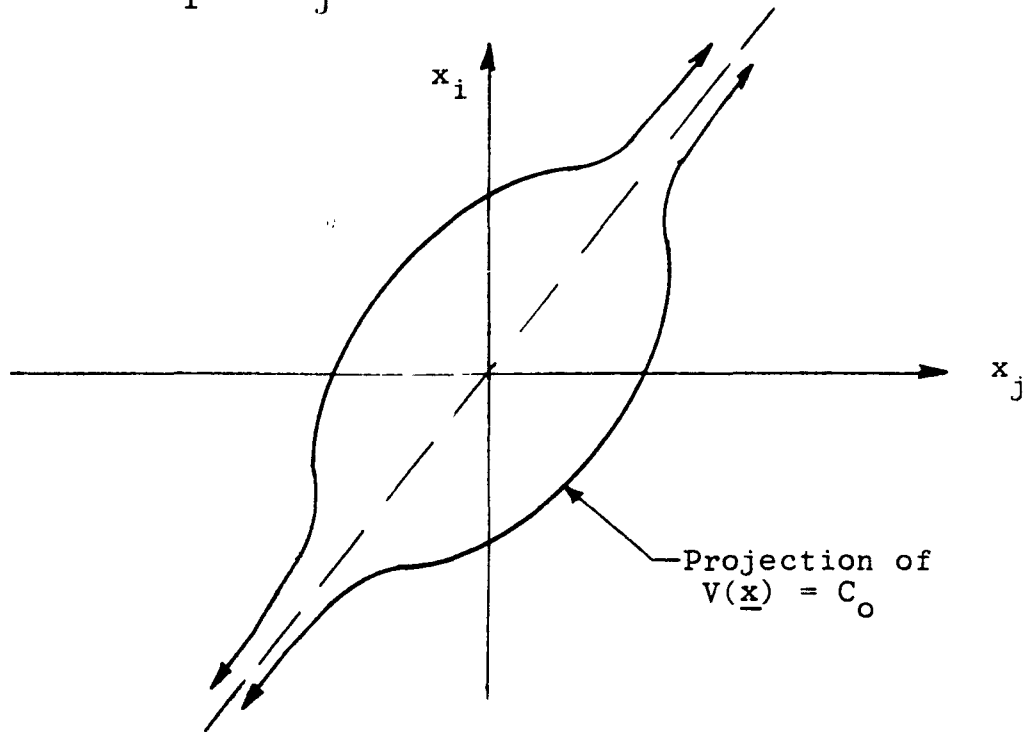
Assume that a scalar function $V[\underline{x}(t)]$ has been formulated for the augmented system, where $\underline{x}(t)$ represents the system state vector. At the initial time t_0 , the state vector will have the initial value $\underline{x}(t_0) = [x_{10}, x_{20}, \dots, x_{n0}]'$ and the scalar function will have the initial value $V[\underline{x}(t_0)] = C_0$.

As discussed in section 3.3., all the points satisfying the equation

$$V(\underline{x}) = C_0 \quad (5-4)$$

form a surface. If the surface is not closed, then the projection of the surface onto the plane of any two state

variables x_i and x_j could have the following form.



POSSIBLE PROJECTION OF OPEN SURFACE
FIGURE 5.1

The interior of an open surface is obviously not a bounded region. If, however, the initial surface given by Eq. (5-4) is closed, then the interior of this surface is bounded and the surface can be enclosed by a quadratic surface. Applying the results of theorem II of section 3.3, if a positive definite matrix M_{s1} can be found such

that*

$$V(\underline{x}) > \underline{x}' M_{s1} \underline{x} \quad (5-5)$$

in the interior and on the quadratic surface of

$$\underline{x}' M_{s1} \underline{x} = C_0$$

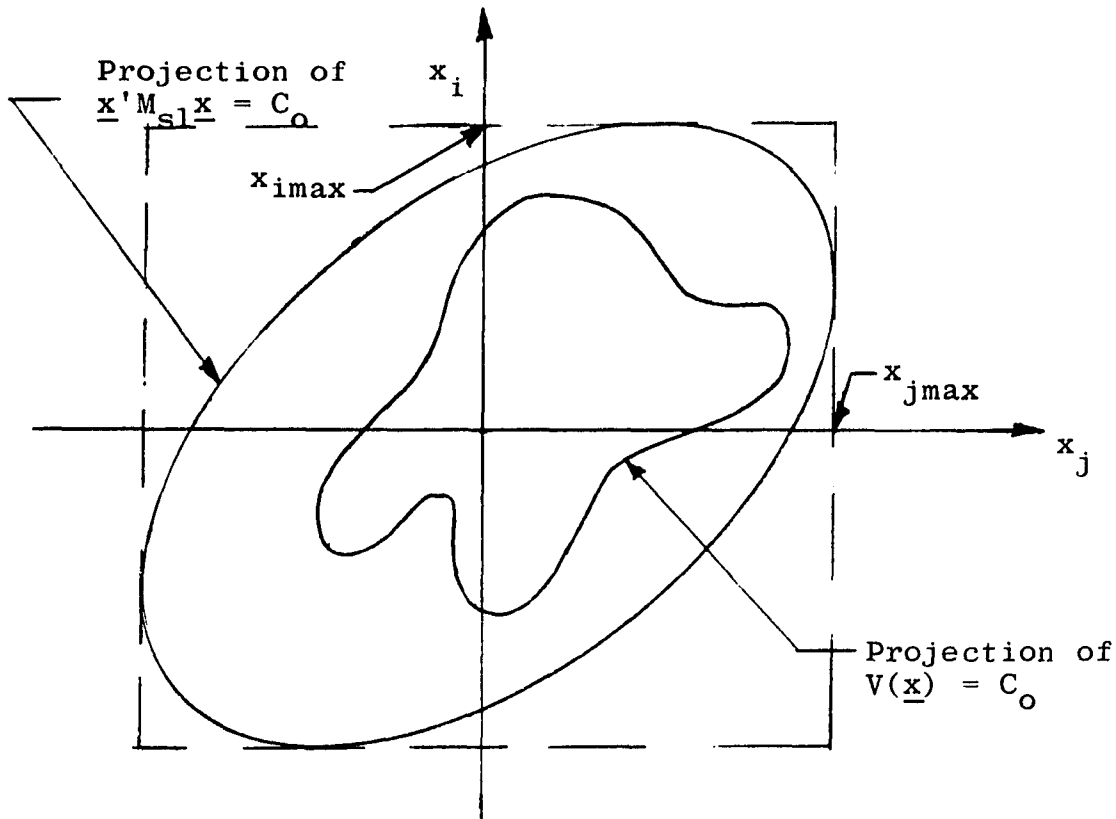
and if also in the interior of the quadratic surface

- (i) $V(\underline{x})$ is positive definite
- (ii) $V(\underline{x})$ is a continuous function of $\underline{x}$
- (iii) $V(\underline{x})$ has defined and continuous first partial derivatives with respect to $\underline{x}$
- (iv) $\dot{V}(\underline{x})$ is negative definite.

then the quadratic surface contains one and only one closed portion of the surface defined by $V(\underline{x}) = C_0$.

This is depicted in the following figure.

* When the $V(\underline{x})$ function is generated by the modified M-Matrix method of Chapter 4, Eq. (5-5) may be written as $\underline{x}' M_0 \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x} > \underline{x}' M_{s1} \underline{x}$ or $\underline{x}' M_0 \underline{x} > \underline{x}' [M_{s1} - M_{on}(\underline{x})] \underline{x}$ where both the M_0 and M_{s1} matrices are positive definite and the $M_{on}(\underline{x})$ matrix may be indefinite. As depicted in Fig. 5.2, the quadratic surface $\underline{x}' M_{s1} \underline{x} = C_0$ bounds the Liapunov surface $\underline{x}' M_0 \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x} = C_0$.



RELATION OF QUADRATIC SURFACE TO LIAPUNOV SURFACE

FIGURE 5.2

Since the initial surface $V(\underline{x}) = C_0$ is bounded by the quadratic surface, the hyper-rectangular region defined by $|x_1| \leq x_{1\max}$, $|x_2| \leq x_{2\max}, \dots, |x_n| \leq x_{n\max}$ (where $x_{1\max}, x_{2\max}, \dots, x_{n\max}$ are the maximum values of the state variables on the quadratic surface) also contains the initial surface $V(\underline{x}) = C_0$. Therefore, if it is established that $V(\underline{x})$ and $\dot{V}(\underline{x})$ are positive and negative definite respectively in this hyper-rectangle, then $V(\underline{x})$ and $\dot{V}(\underline{x})$ are positive and negative definite respectively

in the region Ω defined as the interior of the quadratic surface $\underline{x}'M_{s1}\underline{x} = C_0$.

Hence, one obtains the result that if inequality (5-5) and properties (i) through (iv) are satisfied in the interior and surface of the hyper-rectangular region defined by $|x_1| \leq x_{1max}, |x_2| \leq x_{2max}, \dots, |x_n| \leq x_{nmax}$ (where $x_{1max}, x_{2max}, \dots, x_{nmax}$ are the maximum values of the state variables on the quadratic surface $\underline{x}'M_{s1}\underline{x} = V(\underline{x}(t_0)) = C_0$) then the equilibrium solution is asymptotically stable for the initial condition $\underline{x}(t_0)$.

The testing procedures of section 3.2 will now be applied. It is stated that the function $V(\underline{x})$ will be formulated by the modified M-Matrix method and can therefore be written in the pseudo-quadratic form of Eq. (4-1) or

$$V(\underline{x}) = \underline{x}'M_0\underline{x} + \underline{x}'M_{on}(\underline{x})\underline{x} \quad (5-6)$$

where the M_0 and $M_{on}(\underline{x})$ matrices are as previously defined. The $V(\underline{x})$ function will therefore satisfy conditions (ii) and (iii). Inequality (5-5) becomes

$$\underline{x}' \left[M_0 + M_{on}(\underline{x}) \right] \underline{x} > \underline{x}'M_{s1}\underline{x} \quad (5-7)$$

in a defined region of the $\underline{x}$ -space, which is property 1 of section 3.2. Similarly, conditions (i) and (iv) become

property 2 of that section. The testing procedures discussed in that section are therefore applied for the verification of these properties.

The application of the results of this section and the testing procedures of section 3.2 are illustrated in the design problems of chapters 6 and 7.

In the following section, the properties of the time domain ratio criterion $\eta(\underline{x})$ are discussed. In the final section of this chapter these properties are interpreted to formulate constraints which, when combined with the conditions developed in the present section, are sufficient for the generalized settling time specification to be satisfied.

5.3 The Time Domain Ratio Criterion and its Properties

In this section a discussion of the time domain ratio criterion $\eta(\underline{x})$ is presented. $\eta(\underline{x})$ is defined⁽¹⁸⁾ as,

$$\eta(\underline{x}) \triangleq - \frac{\dot{V}(\underline{x})}{V(\underline{x})} \quad (5-8)$$

where $V(\underline{x})$ and $\dot{V}(\underline{x})$ are a Liapunov function and its time derivative respectively. The quantities $\eta_{\min} \Omega$ and $\eta_{\max} \Omega$ will denote the minimum and maximum values of $\eta(\underline{x})$ in a region Ω of the $\underline{x}$ space, or

$$\eta_{\min} \Omega \triangleq \min_{\underline{x} \in \Omega} \left[\frac{-\dot{V}(\underline{x})}{V(\underline{x})} \right] \quad (5-9)$$

and

$$\eta_{\max} \Omega \triangleq \max_{\underline{x} \in \Omega} \left[\frac{-\dot{V}(\underline{x})}{V(\underline{x})} \right] \quad (5-10)$$

It then follows that

$$-\dot{V}(\underline{x}) \geq \eta_{\min} \Omega V(\underline{x}) \quad \text{for } \underline{x} \in \Omega \quad (5-11)$$

and

$$-\dot{V}(\underline{x}) \leq \eta_{\max} \Omega V(\underline{x}) \quad \text{for } \underline{x} \in \Omega \quad (5-12)$$

Inequalities (5-11) and (5-12) yield the relations

$$V[\underline{x}(t)] \leq V[\underline{x}(t_0)] \exp[-\eta_{\min} \Omega (t-t_0)] \quad (5-13)$$

and

$$V[\underline{x}(t)] \geq V[\underline{x}(t_0)] \exp[-\eta_{\max} \Omega(t-t_0)] \quad (5-14)$$

where the time dependence of the state vector has been explicitly shown as $\underline{x}(t)$. Functions $V[\underline{x}(t_0)]$ and $V[\underline{x}(t)]$ are the scalar values of the Liapunov function at an initial time t_0 and the present value of time t . Inequalities (5-13) and (5-14) establish upper and lower limits for the value of $V(\underline{x})$ at a time t if its value is known at a previous time t_0 .

The extremal values of $\eta(\underline{x})$ will now be investigated for linear time invariant systems and nonlinear autonomous systems.

5.3.1 Linear Time Invariant Systems

Consider the linear time invariant system described by the vector differential equation

$$\dot{\underline{x}} = A_0 \underline{x} \quad (5-15)$$

where A_0 has only characteristic values with negative real parts. From Liapunov's direct method, it is known⁽¹⁴⁾ that a positive definite real matrix M_0 exists such that the matrix D_0 , defined as

$$D_o \triangleq A_o' M_o + M_o A_o \quad (5-16)$$

is negative definite. Therefore a quadratic Liapunov function

$$V(\underline{x}) = \underline{x}' M_o \underline{x} \quad (5-17)$$

can be chosen which possesses a negative definite time derivative given by

$$\dot{V}(\underline{x}) = \underline{x}' (A_o' M_o + M_o A_o) \underline{x} = \underline{x}' D_o \underline{x} \quad (5-18)$$

Employing this quadratic Liapunov function the expression for $\eta(\underline{x})$ becomes

$$\eta(\underline{x}) = \frac{-\underline{x}' D_o \underline{x}}{\underline{x}' M_o \underline{x}} \quad (5-19)$$

The extremal values of $\eta(\underline{x})$ are therefore

$$\eta(\underline{x})_{\min \Omega} = \min_{\underline{x} \in \Omega} \left[\frac{-\underline{x}' D_o \underline{x}}{\underline{x}' M_o \underline{x}} \right] \quad (5-20)$$

and

$$\eta(\underline{x})_{\max \Omega} = \max_{\underline{x} \in \Omega} \left[\frac{-\underline{x}' D_o \underline{x}}{\underline{x}' M_o \underline{x}} \right] \quad (5-21)$$

Since $\eta(\underline{x})$ of (5-19) is constant along a ray of the $\underline{x}$ space, the region Ω can be defined as the entire $\underline{x}$ space. In view of theorem 3.8 of section 3.4, which gives the extremal values for the ratio of quadratic forms, Eqs. (5-20) and (5-21) become

$$\eta(\underline{x})_{\min \Omega} = \eta_{\min} = \min \lambda \left[-D_o M_o^{-1} \right] \quad (5-22)$$

and

$$\eta(\underline{x})_{\max \Omega} = \eta_{\max} = \max \lambda \left[-D_o M_o^{-1} \right] \quad (5-23)$$

where $\min \lambda \left[-D_o M_o^{-1} \right]$ and $\max \lambda \left[-D_o M_o^{-1} \right]$ are respectively the minimum and maximum characteristic values of the matrix $-D_o M_o^{-1}$. Denoting the characteristic values of the A_o matrix by $\alpha_i + j\beta_i$, Vogt⁽⁴³⁾ has proven that

$$\min \lambda \left[-D_o M_o^{-1} \right] \leq -2\max(\alpha_i) \quad (5-24)$$

and

$$\max \lambda \left[-D_o M_o^{-1} \right] \geq -2\min(\alpha_i) \quad (5-25)$$

Substitution of inequalities (5-24) and (5-25) into Eqs. (5-22) and (5-23) yields

$$\eta_{\min} \leq -2\max(\alpha_i) \quad (5-26)$$

and

$$\eta_{\max} \geq -2\min(\alpha_i) \quad (5-27)$$

It is now noted that the equalities of (5-26) and (5-27) can be made strict if the results of theorems III, IV and V of sections 4.3.1 are utilized. Restating the results of these theorems, if a constant square matrix A_o has only negative real characteristic values and simple elementary divisors then a positive definite matrix M_o can be chosen so that

$$M_o A_o = A_o' M_o \quad (\text{symmetric}) \quad (5-28)$$

and

$$M_o A_o < 0 \quad (\text{negative definite}) \quad (5-29)$$

For the class of quadratic Liapunov functions given by $V(\underline{x}) = \underline{x}' M_o \underline{x}$ that possess an M_o matrix which satisfies conditions (5-28) and (5-29) the extremal values of $\eta(\underline{x})$ become

$$\eta_{\min} = \min \lambda \left[-D_o M_o^{-1} \right] = \min \lambda \left[(-A_o' M_o - M_o A_o) M_o^{-1} \right] \quad (5-30)$$

or

$$\eta_{\min} = \min \lambda \left[-2A_o' M_o M_o^{-1} \right] \quad (5-31)$$

or

$$\eta_{\min} = -2\max\lambda [A_0] \quad (5-32)$$

Similarly,

$$\eta_{\max} = -2\min\lambda [A_0] \quad (5-33)$$

Hence, for a linear time invariant system which has only negative real characteristic values and simple elementary divisors, a class of quadratic Liapunov functions exists which satisfies Eq. (5-28) and (5-29) and relates the extremal values of $\eta(\underline{x})$ to the minimum and maximum characteristic values of the system matrix A_0 .

5.3.2 Nonlinear Autonomous Systems

Consider the nonlinear autonomous system described by the vector differential equation

$$\dot{\underline{x}} = A_0 \underline{x} + A_{\text{on}}(\underline{x}) \underline{x} \quad (5-34)$$

where A_0 is a constant real square matrix with only negative real characteristic values and simple elementary divisors and $A_{\text{on}}(\underline{x})$ is a square matrix with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$. Note that the class of systems defined by Eq. (5-34) is the same class defined by Eq. (4-3) of section 4.3 for which the modified

M-Matrix method can be applied to formulate a Liapunov function. From Eqs. (4-1), (4-2) and (4-17) the modified M-Matrix Liapunov function and its time derivative are of the forms,

$$V(\underline{x}) = \underline{x}' M_O \underline{x} + \underline{x}' M_{On}(\underline{x}) \underline{x} \quad (5-35)$$

and

$$\dot{V}(\underline{x}) = \underline{x}' D_O \underline{x} + \underline{x}' D_{On}(\underline{x}) \underline{x} \quad (5-36)$$

where

$$M_O > 0 \quad (5-37)$$

$$D_O = 2A_O' M_O < 0 \quad (5-38)$$

and

$$\begin{aligned} D_{On}(\underline{x}) = & A_{On}'(\underline{x}) M_O + M_O A_{On}(\underline{x}) + A_O' M_{On}(\underline{x}) + M_{On}(\underline{x}) A_O \\ & + A_{On}'(\underline{x}) M_{On}(\underline{x}) + M_{On}(\underline{x}) A_{On}(\underline{x}) + \dot{M}_{On}(\underline{x}) \end{aligned} \quad (5-39)$$

The expression for $\eta(\underline{x})$ becomes,

$$\eta(\underline{x}) = \frac{-\underline{x}' D_O \underline{x} - \underline{x}' D_{On}(\underline{x}) \underline{x}}{\underline{x}' M_O \underline{x} + \underline{x}' M_{On}(\underline{x}) \underline{x}} \quad (5-40)$$

A matrix M_{s2} which is at least positive semidefinite can be chosen so that*

$$\underline{x}' M_{s2} \underline{x} \geq \underline{x}' M_{on}(\underline{x}) \underline{x} \quad (5-41)$$

in a bounded region Ω in the $\underline{x}$ space which encloses the origin. The expression for $\eta(\underline{x})$ in the region Ω therefore satisfies the inequality.

$$\eta(\underline{x}) \underset{\underline{x} \in \Omega}{\geq} \frac{-\underline{x}' D_o \underline{x} - \underline{x}' D_{on}(\underline{x}) \underline{x}}{\underline{x}' M_o \underline{x} + \underline{x}' M_{s2} \underline{x}} \quad (5-42)$$

Assume now that $\dot{V}(\underline{x})$ is negative definite in Ω . Then there exists a constant $0 \leq C_d < 1$ which must satisfy the inequality,

$$-C_d \underline{x}' D_o \underline{x} - \underline{x}' D_{on}(\underline{x}) \underline{x} \geq 0 \quad \text{for } \underline{x} \in \Omega \quad (5-43)$$

Inequality (5-43) serves as a design constraint and the constant C_d is evaluated in Step (7) of the design method as illustrated in section 6.3.7.

*The M_{s2} matrix is related to the M_o , $M_{on}(\underline{x})$ and M_{s1} matrices of section 5.2 since $\underline{x}' M_o \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x} > \underline{x}' M_{s1} \underline{x}$. Relation (5-41) then yields the result that $\underline{x}' [M_o + M_{s2}] \underline{x} > \underline{x}' M_{s1} \underline{x}$, where M_o and M_{s1} are positive definite and M_{s2} is at least positive semidefinite.

Inequality (5-42) then becomes

$$\eta(\underline{x})_{\underline{x} \in \Omega} \geq \frac{-(1-C_d)\underline{x}'D_o\underline{x}}{\underline{x}'M_o\underline{x} + \underline{x}'M_{s2}\underline{x}} \quad (5-44)$$

or

$$\eta(\underline{x})_{\underline{x} \in \Omega} \geq \frac{-\underline{x}'D_o\underline{x}}{\underline{x}'M_o\underline{x}} \cdot \frac{1 - C_d}{1 + \frac{\underline{x}'M_{s2}\underline{x}}{\underline{x}'M_o\underline{x}}} \quad (5-45)$$

or

$$\eta_{\min \Omega} \geq \min_{\underline{x} \in \Omega} \left[\frac{-\underline{x}'D_o\underline{x}}{\underline{x}'M_o\underline{x}} \right] \cdot \min_{\underline{x} \in \Omega} \left[\frac{1 - C_d}{1 + \frac{\underline{x}'M_{s2}\underline{x}}{\underline{x}'M_o\underline{x}}} \right] \quad (5-46)$$

The application of theorem 3.8 of section 3.4 reduces inequality (5-46) to

$$\eta_{\min \Omega} \geq \min \lambda \left[-D_o M_o^{-1} \right] \cdot \frac{1 - C_d}{1 + \max \lambda \left[M_{s2} M_o^{-1} \right]} \quad (5-47)$$

where $\min \lambda \left[-D_o M_o^{-1} \right]$ and $\max \lambda \left[M_{s2} M_o^{-1} \right]$ represent the minimum and maximum characteristic values of the matrices $-D_o M_o^{-1}$ and $M_{s2} M_o^{-1}$, respectively.

Since the modified M-Matrix formulation yields
 $D_o = 2A_o' M_o$, inequality (5-47) becomes

$$\eta_{\min} \Omega \geq -2\max\lambda(A_o) \frac{1 - C_d}{1 + \max\lambda [M_{s2} M_o^{-1}]} \quad (5-48)$$

An analogous expression for $\eta_{\max} \Omega$ can also be formulated. However, it will not be presented here since it is not employed in the design method.

Examining inequality (5-48) and equation (5-32), it is noted that the right hand side of inequality (5-48) reduces to that of equation (5-32) since for the linear time invariant case $M_{s2} = 0$ and $C_d = 0$.

Hence, for a nonlinear autonomous system of the class given by Eq. (5-34), if a modified M-Matrix Liapunov function can be formulated in a region Ω of the $\underline{x}$ space, then the minimum value of $\eta(\underline{x})$ in Ω can be related to the maximum characteristic value of the A_o matrix. The relationship includes a matrix M_{s2} which is at least positive semidefinite and a positive constant C_d which is less than one where

$$\underline{x}' M_{s2} \underline{x} \geq \underline{x}' M_{on}(\underline{x}) \underline{x} \quad (5-41)$$

and

$$-C_d \underline{x}' D_o \underline{x} - \underline{x}' D_{on}(\underline{x}) \underline{x} \geq 0 \quad (5-43)$$

for

$$\underline{x} \in \Omega$$

In the next section, the minimum value of $\eta(\underline{x})$ is interpreted in order to formulate a design constraint. This constraint, when combined with properties (i) through (iv) and inequality (5-5) of section 5.2, forms a complete set of sufficient conditions on the $V(\underline{x})$ and $\dot{V}(\underline{x})$ functions so that the augmented system will satisfy the generalized settling time specification.

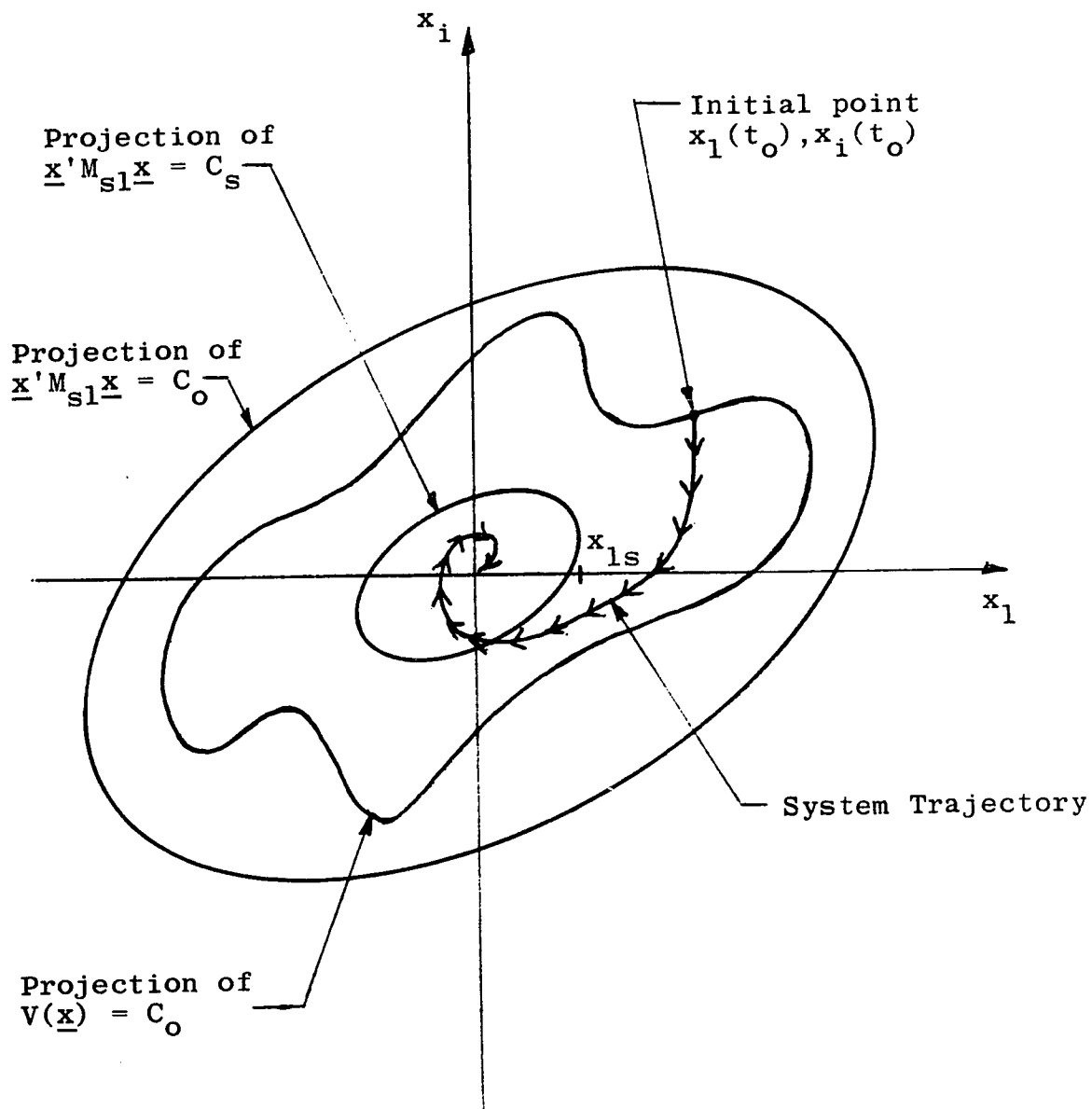
5.4 Time Domain Ratio Design Inequality

The minimum value of $\eta(\underline{x})$ is interpreted in this section and a design inequality is formulated.

The inequality, combined with properties (i) through (iv) and inequality (5-5) of section 5.2, forms a set of constraints which are sufficient conditions for the augmented system to satisfy the generalized settling time specification.

For this specification to be satisfied, the magnitude of the state variable $x_1(t)$, which is the system output and where the time dependence has been explicitly shown, must not be greater than a specified value x_{1s} for all time greater than some specified time t_s .

It is now assumed that the conditions of section 5.2 have been satisfied, i.e. inequality (5-5) and properties (i) through (iv). The system's response due to the initial condition $\underline{x}(t_0)$ will therefore asymptotically approach the origin and its trajectory in the state space projected onto the plane of the state variables x_i and x_1 is of the form depicted in Fig. 5.3.



PROJECTION OF AUGMENTED SYSTEM TRAJECTORY
FIGURE 5.3

From the above figure, it is noted that a sufficient condition for $|x_1| \leq x_{1s}$ for $t \geq t_s$ is that for $t \geq t_s$ the system trajectory be inside or on the quadratic surface given by $\underline{x}' M_{s1} \underline{x} = C_s$ on which the maximum value of x_1 is x_{1s} . Applying the results of theorem II of section 3.3, this quadratic surface encloses one and only one surface defined by $V(\underline{x}) = C_s$. Therefore, a sufficient condition for $x_1 \leq x_{1s}$ for $t \geq t_s$ is that

$$V(\underline{x}) \leq C_s \text{ for } t \geq t_s \quad (5-49)$$

From the previous section, inequality (5-13) states that

$$V[\underline{x}(t_s)] \leq V[\underline{x}(t_o)] \cdot \exp[-\eta_{\min \Omega} (t_s - t_o)] \quad (5-50)$$

Therefore $V(\underline{x}) \leq C_s$ for $t \geq t_s$ if

$$C_s \geq C_o \exp[-\eta_{\min \Omega} (t_s - t_o)] \quad (5-51)$$

Inequality (5-51) yields the relation

$$\eta_{\min \Omega} \geq \frac{1}{t_s - t_o} \ln \left[\frac{C_o}{C_s} \right] \quad (5-52)$$

Inequality (5-48), which was derived in section 5.3, states that

$$\gamma_{\min} \Omega \geq -2\max\lambda(A_o) \frac{1 - C_d}{1 + \max\lambda \left[M_{s2} M_o^{-1} \right]} \quad (5-53)$$

Combining the above inequality with inequality (5-52) yields the design constraint

$$-2\max\lambda(A_o) \frac{1 - C_d}{1 + \max\lambda \left[M_{s2} M_o^{-1} \right]} \geq \frac{1}{t_s - t_o} \ln \left[\frac{C_o}{C_s} \right] \quad (5-54)$$

Hence the design constraint given by (5-54), combined with properties (i) through (iv) of section (5.2) and inequality (5-5) of the same section, forms a complete set of sufficient conditions for the augmented system to satisfy the generalized settling time specification.

5.5 Summary

This chapter has presented the philosophy and objectives of the design method which is applicable to a class of systems described by the differential equation

$\dot{\underline{x}} = \underline{A}_0 \underline{x} + \underline{A}_{on}(\underline{x}) \underline{x}$. A generalized settling time specification was defined as follows:

Denoting a system output by $x_1(t)$ and the system phase space vector by $\underline{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]'$, it is required that the augmented system's equilibrium solution be asymptotically stable and that

$$|x_1(t)| \leq x_{1s} \text{ for } t \geq t_s$$

for $\underline{x}(t_0) = [x_{10}, x_{20}, \dots, x_{n0}]'$

where $\underline{x}(t_0)$ is the value of the state vector $\underline{x}(t)$ at the time t_0 and x_{1s} and t_s are specified constants.

A set of conditions on a scalar function $V(\underline{x})$ and its time derivative $\dot{V}(\underline{x})$ were formulated which are sufficient for the a system to satisfy the generalized settling time specification. For a modified M-Matrix Liapunov function and its time derivative, which are of the forms

$V(\underline{x}) = \underline{x}' M_O \underline{x} + \underline{x}' M_{On}(\underline{x}) \underline{x}$ and $\dot{V}(\underline{x}) = \underline{x}' D_O \underline{x} + \underline{x}' D_{On}(\underline{x}) \underline{x}$,
the sufficient conditions are stated in the following manner.

If in the open region Ω , defined as the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = V[\underline{x}(t_o)]$ (where M_{s1} is a positive definite matrix and $V[\underline{x}(t_o)]$ is the scalar value of the $V(\underline{x})$ function at time t_o) the following conditions are satisfied

- (i) $V(\underline{x})$ is positive definite
- (ii) $V(\underline{x})$ is a continuous function of $\underline{x}$
- (iii) $V(\underline{x})$ has defined and continuous first partial derivatives with respect to $\underline{x}$
- (iv) $\dot{V}(\underline{x})$ is negative definite
- (v) $V(\underline{x}) > \underline{x}' M_{s1} \underline{x}$

$$(vi) -2\max\lambda(A_o) \frac{1 - C_d}{1 + \max\lambda[M_{s2} M_o^{-1}]} \geq \frac{1}{t_s - t_o} \ln \left[\frac{C_o}{C_s} \right]$$

where

$$C_o = V[\underline{x}(t_o)]$$

M_{s2} is at least a positive semidefinite matrix which satisfies the inequality

$$\underline{x}' M_{s2} \underline{x} \geq \underline{x}' M_{on}(\underline{x}) \underline{x}$$

and C_d is a positive constant less than one,

then the system satisfies the generalized settling time specification.

In the following chapter, the design method is discussed in detail and an illustrative design problem is presented. Further illustration and analysis of the design method is given in chapter 7.

CHAPTER 6

THE DESIGN METHOD

6.1 Introduction

This chapter presents the design method in detail. The steps involved in the design method are presented for a general design problem and are then illustrated numerically for a third order nonlinear autonomous system containing an inherent multiplicative nonlinearity.

The design method is applicable to nonlinear systems which when augmented by a controller can be described by a vector differential equation of the form

$$\dot{\underline{x}} = A_o \underline{x} + A_{on}(\underline{x}) \underline{x}$$

where $\underline{x}$ is a column vector of the state variables $x_1, x_2, \dots$, etc. and A_o is a constant square matrix with only negative real characteristic values and simple elementary divisors. The matrix $A_{on}(\underline{x})$ is a square matrix with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$. The vector $A_{on}(\underline{x}) \underline{x}$ contains no linear terms and the subscripts o and n denote the order of the system and the number of nonlinearities present in the system respectively.

This class includes high order systems that contain multiple nonlinearities which may be of the multiplicative type. It is noted that the equilibrium solution of the original system may be unstable, stable or asymptotically stable.

The objective of the design method is to formulate a controller so that the original system augmented with the controller will satisfy the following generalized settling time specification. Denoting the augmented system's output by $x_1(t)$ and the system phase space vector by $\underline{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]'$, where the prime denotes the transpose, it is required that the equilibrium solution of the augmented system be asymptotically stable and that

$$|x_1(t)| \leq x_{1s} \text{ for } t \geq t_s$$

for $\underline{x}(t_0) = [x_{10}, x_{20}, \dots, x_{n0}]'$

where $\underline{x}(t_0)$ is the value of the state vector $\underline{x}(t)$ at the initial time t_0 and x_{1s} and t_s are specified constants.

The design method consists of the following seven steps.

- Step (1): Formulation of the Augmented System
- Step (2): Formulation of the Modified M-Matrix
Liapunov Function and its Time Derivative
- Step (3): Classification of the Controller Output
- Step (4): Reduction of the Liapunov Function and
its Time Derivative
- Step (5): Formulation of the Constraints on the
Elements of the Liapunov Function
- Step (6): Formulation of the Constraints on the
Elements of the Time Derivative of the
Liapunov Function
- Step (7): Application of a Digital Computer Program
for Solving the Design Constraints

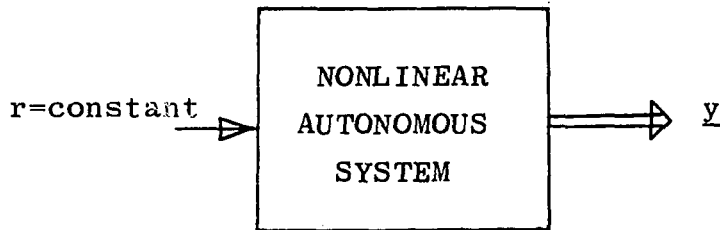
In general, a controller with an output $u(y)$ is introduced into the original system and the form of a modified M-Matrix Liapunov function is formulated for the augmented system. The elements of the Liapunov function and its time derivative are then constrained by the design inequalities given in sections 5.2 and 5.4. The design inequalities are sufficient conditions for the satisfaction

of the generalized settling time specification by the augmented system. In the final step of the design method a digital computer is employed as an engineering aid for the determination of numerical values of the postulated controller parameters which satisfy the design inequalities.

6.2 Statement of Design Problem

The design method is utilized to solve the following problem.

A system block diagram and the vector differential equation of a nonlinear autonomous system are given as presented below.



UNAugmented SYSTEM BLOCK DIAGRAM
FIGURE 6.1

The system vector differential equation is

$$\dot{\underline{y}} = B(\underline{y})\underline{y} \quad (6-1)$$

where the vector $\underline{y} = [y_1, y_2, \dots, y_n]'$ is the system's phase space vector and the matrix $B(\underline{y})$ is a square matrix with elements that are constants and/or functions of the vector $\underline{y}$.

It is desired to design a controller so that the equilibrium solution of the original system augmented with the controller will be asymptotically stable and the output

of the augmented system, denoted by $x_1(t)$, satisfies the specification that

$$|x_1(t)| \leq x_{1s} \text{ for } t \geq t_s \quad (6-2)$$

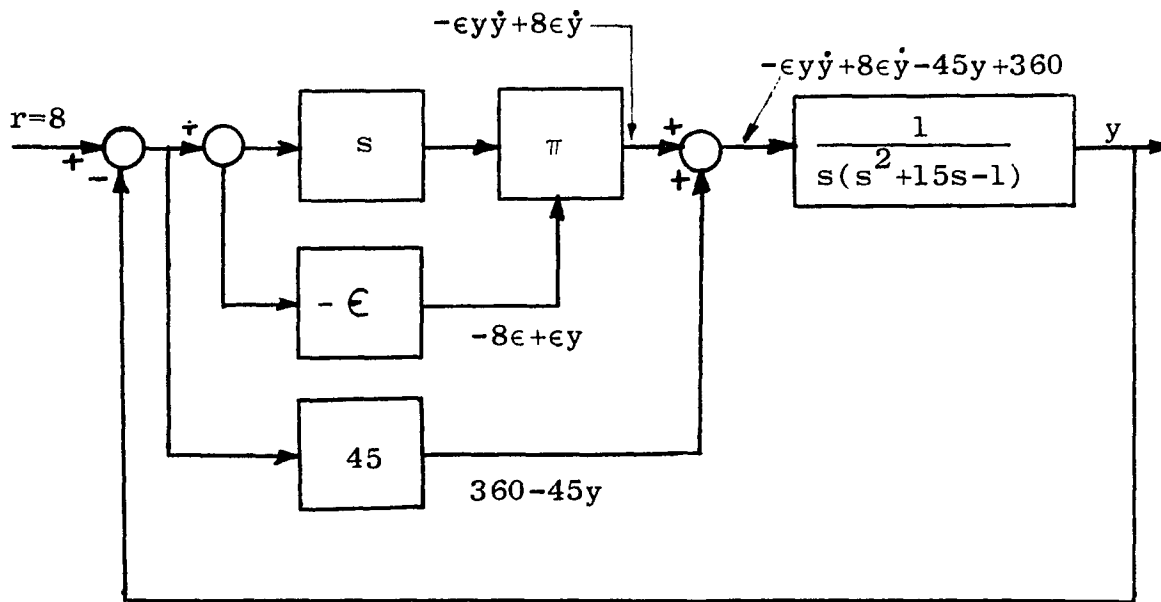
for the initial condition

$$\underline{x}(t_0) = [x_{10}, x_{20}, \dots, x_{no}]'$$

where $\underline{x}(t_0)$ is the value of the system state vector $\underline{x}(t)$ at the initial time t_0 .

It is noted that the equilibrium solution of the original system may be unstable, stable or asymptotically stable.

The specific design problem which will be employed to illustrate the design method is a third order system containing an inherent multiplicative nonlinearity and which is described by a third order Van der Pol type equation. The block diagram of the system is given in Fig. 6.2 and its differential equation is given by Eq. (6-3).



UNAugmented SYSTEM FOR ILLUSTRATIVE PROBLEM

FIGURE 6.2

$$\ddot{y} + 15\dot{y} + [\epsilon(y-8)-1] \dot{y} + 45y = 360 \quad (6-3)$$

It is desired that the equilibrium solution of the augmented system be asymptotically stable and that the output satisfy the condition

$$|y_1(t) - 8| \leq 0.2 \text{ for } t \geq 10 \text{ seconds}$$

for the initial condition $\underline{y}(0) = [10, 0, 0]^T$.

The value of the parameter ϵ in the control system will be arbitrarily chosen as $\epsilon = 2$. The effect of different numerical values of the parameter ϵ upon the results of the design method is analyzed in Chapter 7. For $\epsilon=2$ Eq. (6-3) becomes

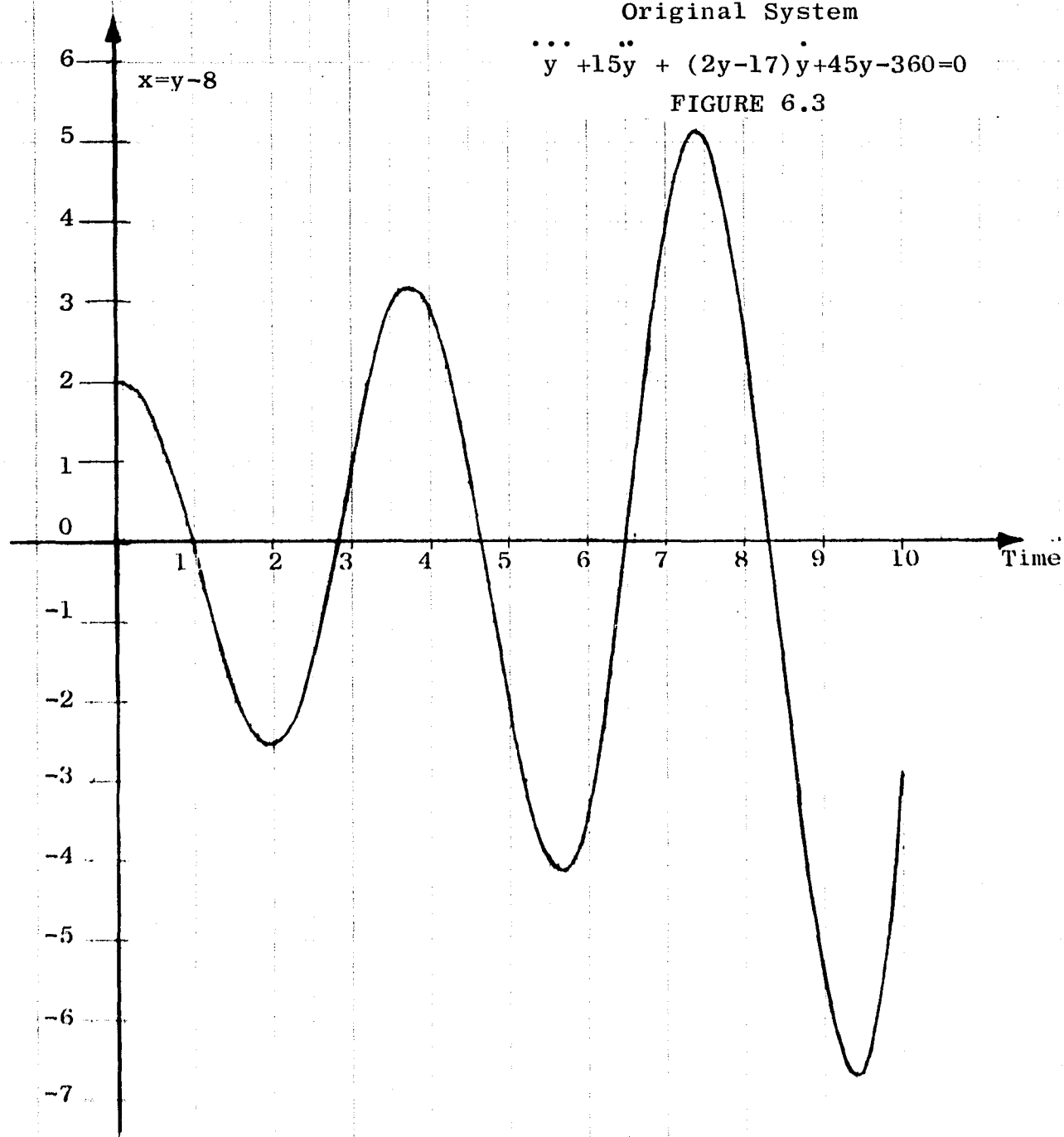
$$\ddot{\underline{y}} + 15\dot{\underline{y}} + (2\underline{y}-17)\dot{\underline{y}} + 45\underline{y} = 360 \quad (6-4)$$

Figure 6.3 depicts the response of the original system due to the initial condition $\underline{y}(0) = [10, 0, 0]^T$. It is observed that the equilibrium solution $\underline{y} = [8, 0, 0]^T$ is unstable for this initial condition.

Original System

$$\ddot{\ddot{y}} + 15\ddot{y} + (2y-17)\dot{y} + 45y - 360 = 0$$

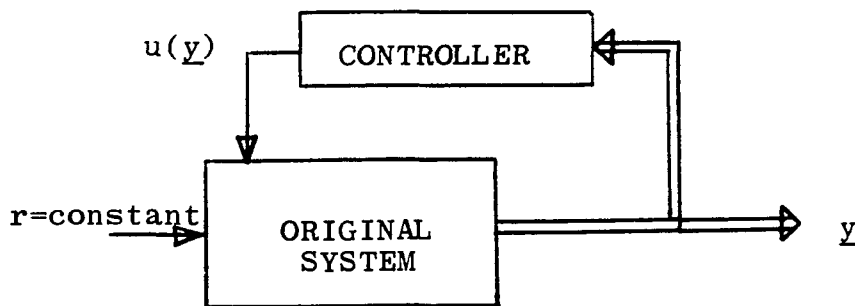
FIGURE 6.3



6.3 The Application of the Design Method

6.3.1 Step (1): Formulation of the Augmented System

The first step in the design method is to determine the position of the controller in the system. After the various possible positions are considered and a position has been determined (as shown in Fig. 6.4)



AUGMENTED SYSTEM BLOCK DIAGRAM

FIGURE 6.4

the differential equation describing the augmented system is formulated. In Fig. 6.4 the system phase space vector is denoted by the vector $\underline{y} = [y_1, y_2, \dots, y_n]'$ and the controller output is denoted by the function $u(\underline{y})$.

It is next necessary to determine whether the augmented system can be described by a differential equation of the class

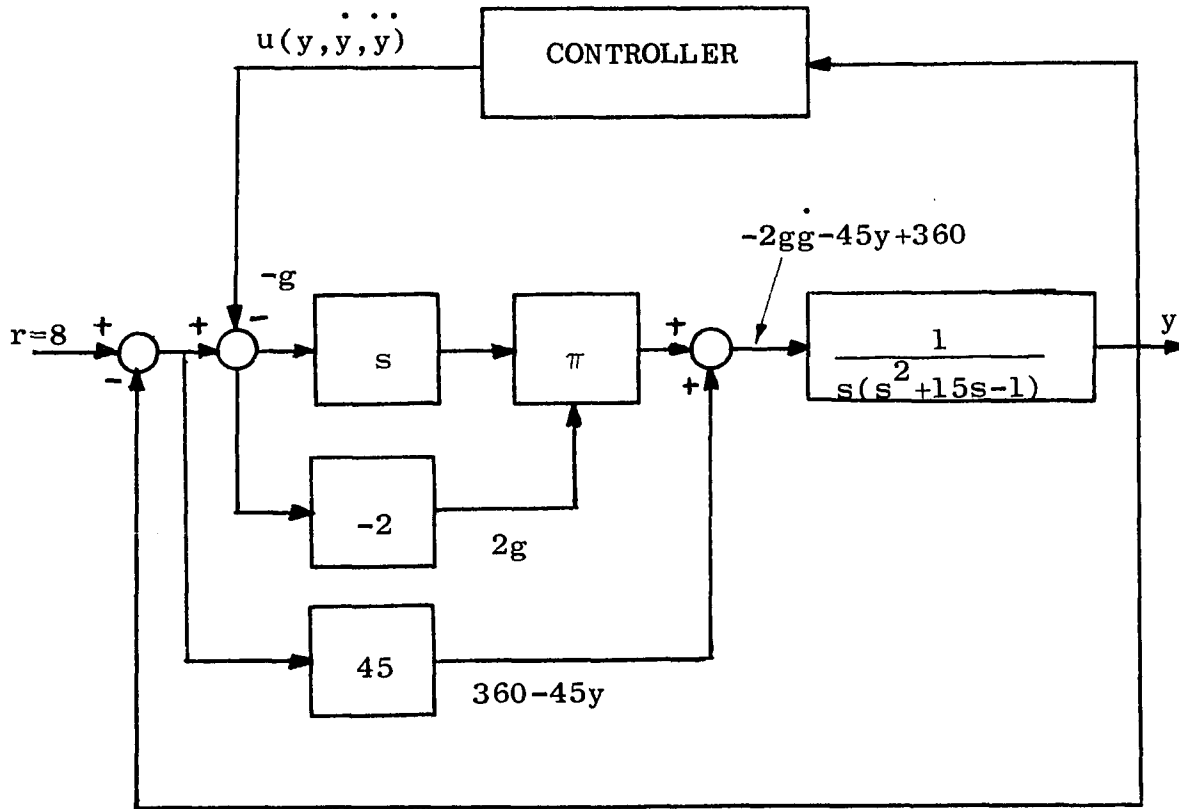
$$\dot{\underline{x}} = \underline{A}_0 \underline{x} + \underline{A}_{on}(\underline{x}) \underline{x} \quad (6-5)$$

where the vector $\underline{x} = [x_1, x_2, \dots, x_n]'$, $x_1 = y_1 - r$, $x_2 = y_2$, $\dots$, $x_n = y_n$ and A_o is a constant square matrix with only negative real characteristic values and simple elementary divisors. The matrix $A_{on}(\underline{x})$ is also a square matrix with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$. The vector $A_{on}(\underline{x})\underline{x}$ contains no linear terms. The subscripts o and n denote the order of the system and the number of nonlinearities present in the system respectively. The augmented system's differential equation must be of this class if the design method is to be applicable.

The class of systems which have the form of Eq. (6-5) is of a broad nature in the sense that it includes high order systems which contain multiple nonlinearities. These nonlinearities may be of the multiplicative type.

In the second step of the design method, a $V(\underline{x})$ function and its time derivative are formulated by the modified M-Matrix method which was presented in Chapter 4.

Demonstrating the first step of the design method by the illustrative design problem, the position of the controller is shown in Fig. 6.5.



AUGMENTED SYSTEM FOR ILLUSTRATIVE PROBLEM
FIGURE 6.5

where

$$g = u(y, \dot{y}, \ddot{y}) + y - 8 \quad (6-6)$$

and $u(y, \dot{y}, \ddot{y})$ is the controller output.

The differential equation describing the augmented system is

$$\ddot{y} + 15\dot{y} - y + 45(y-8) + 2[u(y, \dot{y}, \ddot{y}) + y-8] \cdot [\dot{u}(y, \dot{y}, \ddot{y}) + \dot{y}] = 0 \quad (6-7)$$

The equilibrium solution

$$y=8, \ddot{y}=\dot{y}=0 \quad (6-8)$$

is then transformed to the origin by the substitution,

$$x=y-8 \quad (6-9)$$

Employing this transformation, Eq. (6-7) becomes

$$\ddot{\ddot{x}} + 15\ddot{x} - \dot{x} + 45x + 2[u(x, \dot{x}, \ddot{x}) + x] \cdot [\dot{u}(x, \dot{x}, \ddot{x}) + \dot{x}] = 0 \quad (6-10)$$

and

$$\underline{x}(0) = [2, 0, 0]^T$$

Rewriting Eq. (6-10) in matrix form,

$$\dot{\underline{x}} = B_3 \underline{x} + \underline{b}s(\underline{x}) \quad (6-11)$$

where

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -45 & 1 & -15 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \quad (6-12)$$

and

$$s(\underline{x}) = [u(\underline{x}) + x_1] \cdot [\dot{u}(\underline{x}) + x_2] \quad (6-13)$$

It is noted from Eq. (6-4) that the original system's differential equation is of the form

$$\ddot{y} + 15\dot{y} + N(y)\dot{y} + 45y - 360 = 0 \quad (6-14)$$

where

$$N(y) = 2y - 17 \quad (6-15)$$

It will now be specified that the augmented system's differential equation be of the same form as Eq. (6-4) or

$$\ddot{x} + a_3\dot{x} + [a_2 + \psi(x)]\dot{x} + a_1x = 0 \quad (6-16)$$

Rewriting Eq. (6-16) in matrix form

$$\dot{\underline{x}} = A_3\underline{x} + A_{31}(\underline{x})\underline{x} \quad (6-17)$$

where

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \quad A_{31}(\underline{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\psi(x) & 0 \end{bmatrix} \quad (6-18)$$

In the next step of the design method a modified M-Matrix Liapunov function and its time derivative are formulated for the augmented system given by Eq. (6-17).

6.3.2 Step (2): Formulation of the Modified M-Matrix

Liapunov Function and its Time Derivative

The forms of a modified M-Matrix Liapunov function $V(\underline{x})$ and its time derivative $\dot{V}(\underline{x})$ are formulated in this step. The modified M-Matrix method is presented in chapter 4 and illustrative formulations are given in section 4.4 of that chapter. The functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ are of the forms

$$V(\underline{x}) = \underline{x}' M_o \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x} \quad (6-19)$$

and

$$\dot{V}(\underline{x}) = \underline{x}' D_o \underline{x} + \underline{x}' D_{on}(\underline{x}) \underline{x} \quad (6-20)$$

where the matrices M_o and D_o are positive and negative definite matrices respectively. The subscripts o and n denote the order of the system and the number of nonlinearities present in the system respectively. The matrices $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ are square matrices with elements that are related to those of the $A_{on}(\underline{x})$ matrix.

In the next step of the design method the required forms of the controller output are formulated.

The current step of the design method will now be demonstrated by means of the illustrative problem.

The differential equation (6-16) of the augmented system is of the form considered in case 3.1 of section 4.4. The Liapunov function and its time derivative which were formulated for this case are

$$V(\underline{x}) = \underline{x}' M_3 \underline{x} + \underline{x}' M_{31}(\underline{x}) \underline{x} \quad (6-21)$$

and

$$\dot{V}(\underline{x}) = \underline{x}' D_3 \underline{x} + \underline{x}' D_{31}(\underline{x}) \underline{x} \quad (6-22)$$

where the matrices M_3 and D_3 are real matrices given by*

$$M_3 = C' C \quad (6-23)$$

and

$$D_3 = 2A_3' M_3 \quad (6-24)$$

and the matrices $M_{31}(\underline{x})$ and $D_{31}(\underline{x})$ are equal to

$$M_{31}(\underline{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \psi(x_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6-25)$$

* As developed in Section 4.3.1, the matrix C is determined from the relation $CA_3C^{-1} = P$ where P is any symmetric real matrix which has the same characteristic values of the matrix A_3 .

and

$$D_{31}(\underline{x}) = 2 \begin{bmatrix} 0 & -m_{13}\psi(x_1) & 0 \\ 0 & -m_{23}\psi(x_1) + \frac{1}{2} \left[\frac{d\psi(x_1)}{dx_1} \right] x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6-26)$$

In the next step of the design method, the forms of the controller output are formulated.

6.3.3 Step (3): Classification of the Controller Output

It is now desired to specify the forms of the controller output $u(\underline{x})$ which will enable the augmented system's differential equation to be of the form specified in step (1). At this point in the design method, the robust nature of the method is observed. The non-uniqueness of the controller is apparent since there is no unique controller output $u(\underline{x})$ which will cause the differential equation of the augmented system to have the desired form. A set U of controller outputs is therefore defined as the set of all controllers which yield the desired form of the differential equation.

After the set U has been defined, the design method proceeds in a logical and systematic manner by first investigating the simplest controller of the set U . If at the conclusion of the design method the results are found to be unsatisfactory, the designer returns to this point in the method and chooses the form of a controller output which is on the next level of complexity. Proceeding in this manner, the effect of increasing the complexity of the controller upon the system's performance can be ascertained.

In the next step of the design method, the exact forms of the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ are obtained for the controller output $u_i(\underline{x})$ which is the i -th member of the set U .

Continuing the demonstration of the design method by the illustrative problem, Eq. (6-10) is set equal to Eq. (6-16) or

$$\ddot{\ddot{x}} + 15\ddot{\dot{x}} - \dot{x} + 45x + 2 \left[u(x, \dot{x}, \ddot{x}) + x \right] \cdot \left[\dot{u}(x, \dot{x}, \ddot{x}) + \dot{x} \right] = \ddot{\ddot{x}} + a_3 \ddot{\dot{x}} + \left[a_2 + \psi(x) \right] \dot{x} + a_1 x \quad (6-27)$$

For Eq. (6-27) to be satisfied, it is noted that the controller output $u(x, \dot{x}, \ddot{x})$ can only be a function of x .

Hence the set U of allowable controller outputs is defined as

$$U = \left\{ u(x, \dot{x}, \ddot{x}) \mid u(x, \dot{x}, \ddot{x}) = u(x) \right\} \quad (6-28)$$

The general form of $u(x)$ is therefore given by the polynomial

$$u(x) = u_0 + u_1 x + u_2 x^2 + \dots + u_n x^n \quad (6-29)$$

where $u_0, u_1, \dots, u_n$ are real constants.

Substituting Eq. (6-29) into Eq. (6-27) yields

$$\begin{aligned} \ddot{x} + 15\dot{x} - \dot{x} + 45x + 2 \left[u_0 + (1+u_1)x + \dots + u_n x^n \right] \cdot \\ \left[(1+u_1)\dot{x} + \dots + nu_n x^{n-1}\dot{x} \right] = \ddot{x} + a_3 \ddot{x} + \left[a_2 + \psi(x) \right] \dot{x} + a_1 x \end{aligned} \quad (6-30)$$

Equating terms, one obtains

$$a_3 = 15 \quad (6-31)$$

$$a_2 = 2u_0(1+u_1) - 1 \quad (6-32)$$

$$a_1 = 45 \quad (6-33)$$

$$\psi(x) = 2 \left[u_0 + (1+u_1)x + u_2x^2 + \dots + u_nx^n \right] \cdot \left[(1+u_1) + 2u_2x + \dots + nu_nx^{n-1} \right] - 2u_0(1+u_1) \quad (6-34)$$

At this point it is restated that the A_3 matrix must have negative real characteristic values for the formulation of a modified M-Matrix Liapunov function.

Therefore the linear differential equation

$$\ddot{x} + 15\dot{x} + \left[2u_0(1+u_1) - 1 \right] \dot{x} + 45x = 0 \quad (6-35)$$

must have only negative real characteristic values.

The simplest form of the controller output in the set U is first postulated. If this form does not yield the desired time response, the next level of controller output complexity is chosen.

Hence, the first controller output to be investigated is of the form

$$u(x) = u_0 + u_1x \quad (6-36)$$

It is noted that the design method is repeated in chapter 7 for the controller output of the form

$$u(x) = u_0 + u_1x + u_2x^2 \quad (6-37)$$

and the results obtained for the two forms of controller outputs are compared.

In the next step of the design method the expressions for the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ are formulated for the controller output given by Eq. (6-36).

6.3.4 Step 4: Reduction of the Liapunov Function and its Time Derivative

In this step of the design method, the expression for the i -th controller output $u_i(\underline{x})$ is substituted in the expressions for the Liapunov function and its time derivative. The elements of the Liapunov function and its time derivative then become functions of the original system parameters and the unknown constant parameters of the controller output. The constraints which these elements must satisfy are formulated in the next two steps of the method.

Performing the current step of the design method for the illustrative problem, the controller output chosen is given by Eq. (6-36) or

$$u(\underline{x}) = u_0 + u_1 \underline{x} \quad (6-36)$$

From Eqs. (6-31) through (6-34), the augmented system's parameters become,

$$\begin{aligned}
 a_3 &= 15 \\
 a_2 &= 2u_0(1+u_1) - 1 \\
 a_1 &= 45 \\
 \psi(\underline{x}) &= 2(1+u_1)^2 \underline{x}
 \end{aligned} \tag{6-38}$$

Substituting Eqs. (6-38) into Eqs. (6-25) and (6-26), the expressions for the non-quadratic portions of the $V(\underline{x})$ and $\dot{V}(\underline{x})$ functions become

$$\underline{x}' M_{31}(\underline{x}) \underline{x} = \underline{x}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(1+u_1)^2 x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} \tag{6-39}$$

and

$$\underline{x}' D_{31}(\underline{x}) \underline{x} = 2\underline{x}' \begin{bmatrix} 0 & -2m_{13}(1+u_1)^2 x_1 & 0 \\ 0 & -2m_{23}(1+u_1)^2 x_1 + (1+u_1)^2 x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} \tag{6-40}$$

The expressions for the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ are therefore

$$V(\underline{x}) = m_{11}x_1^2 + m_{22}x_2^2 + x_3^2 + 2m_{12}x_1x_2 + 2m_{13}x_1x_3 + \\ 2m_{23}x_2x_3 + 2(1+u_1)^2x_1x_2^2 \quad (6-41)$$

and

$$\dot{V}(\underline{x}) = d_{11}x_1^2 + d_{22}x_2^2 + d_{33}x_3^2 + 2d_{12}x_1x_2 + 2d_{13}x_1x_3 + \\ 2d_{23}x_2x_3 - 4m_{13}(1+u_1)^2x_1^2x_2 - 4m_{23}(1+u_1)^2x_1x_2^2 + \\ 2(1+u_1)^2x_2^3 \quad (6-42)$$

In the following step of the design method the constraints which the elements of the function $V(\underline{x})$ must satisfy are formulated.

6.3.5 Step (5): Formulation of the Constraints on the Elements of the Liapunov Function

In this step the elements of the function $V(\underline{x})$ are constrained to satisfy the sufficient conditions which were formulated in Chapter 5. The conditions developed in that chapter are sufficient conditions for the augmented system to satisfy the generalized settling time specification. Since the $V(\underline{x})$ function is formulated by the modified M-Matrix method, this function is continuous with respect to $\underline{x}$ and has defined and continuous first partial

derivatives with respect to $\underline{x}$. The sufficient conditions on the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ therefore reduce to the following set of conditions for $\underline{x} \in \Omega$, where the region Ω is defined as the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = V[\underline{x}(t_0)]$

$$V(\underline{x}) = \underline{x}' M_o \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x} > \underline{x}' M_{s1} \underline{x} > 0 \quad (6-43)$$

for $\underline{x} \neq 0$ and $V(0) = 0$.

$$\dot{V}(\underline{x}) = \underline{x}' D_o \underline{x} + \underline{x}' D_{on}(\underline{x}) \underline{x} < 0 \quad (6-44)$$

for $\underline{x} \neq 0$ and $\dot{V}(0) = 0$.

$$-C_d \underline{x}' D_o \underline{x} - \underline{x}' D_{on}(\underline{x}) \underline{x} \geq 0 \quad (6-45)$$

where C_d is a positive constant less than 1 and

$$-2 \max \lambda(A_o) \frac{1 - C_d}{1 + \max \lambda[M_{s2} M_o^{-1}]} \geq \frac{1}{t_s - t_o} \ln \left[\frac{C_o}{C_s} \right] \quad (6-46)$$

where the notation $\max \lambda(A_o)$ and $\max \lambda[M_{s2} M_o^{-1}]$ is used to designate the maximum characteristic values of the matrices A_o and $M_{s2} M_o^{-1}$ respectively. The constant C_s is a positive constant such that the maximum value of the

variable x_1 on the quadratic surface $\underline{x}'M_{s1}\underline{x} = C_s$ is x_{1s} and $C_o = V[\underline{x}(t_o)]$. The matrix M_{s2} is a real matrix which is at least positive semidefinite and satisfies the inequality

$$\underline{x}'M_{s2}\underline{x} \geq \underline{x}'M_{on}(\underline{x})\underline{x} \quad (6-47)$$

Inequalities (6-43) and (6-44) are sufficient conditions for the equilibrium solution of the augmented system to be asymptotically stable. They are also sufficient conditions (as given by theorem II of section 3.3) for the quadratic surface $\underline{x}'M_{s1}\underline{x} = V[\underline{x}(t_o)]$ to contain one and only one closed portion with respect to the origin of the surface $V(\underline{x}) = V[\underline{x}(t_o)]$. These conditions therefore guarantee that the response of the augmented system, due to the initial condition $\underline{x}(t_o)$, asymptotically approaches the origin of the phase space.

Inequality (6-46) was derived in section 5.4. This design inequality, when combined with inequalities (6-43), (6-44), (6-45) and (6-47), forms a set of sufficient conditions for the augmented system output $x_1(t)$ to be less than or equal to a specified constant x_{1s} for all time $t \geq t_s$.

In the current step of the design method the constraints which are imposed on the elements of the $V(\underline{x})$ function by the inequalities (6-43), (6-46) and (6-47) are formulated. In the next step the constraints which are imposed on the elements of the $\dot{V}(\underline{x})$ function by the inequalities (6-44) and (6-45) are formulated.

Demonstrating the current step with the illustrative problem, the initial value of the $V(\underline{x})$ function for $\underline{x}(0) = [2, 0, 0]'$ is obtained from Eq. (6-41) or

$$v[\underline{x}(0)] = v[2, 0, 0] = 4m_{11} \quad (6-48)$$

Inequality (6-43) requires that

$$\underline{x}'M_3\underline{x} + \underline{x}'M_{31}(\underline{x})\underline{x} > \underline{x}'M_{s1}\underline{x} > 0 \quad (6-49)$$

in the region Ω defined as the interior of the quadratic surface

$$\underline{x}'M_{s1}\underline{x} = 4m_{11} \quad (6-50)$$

Substitution of Eq. (6-39) into inequality (6-49) yields

$$\begin{aligned}
 \underline{x}' \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & 1 \end{bmatrix} \underline{x} + \underline{x}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(1+u_1)^2 x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} \\
 > \underline{x}' M_{s1} \underline{x} \quad (6-51)
 \end{aligned}$$

From the above condition it is observed that the matrix M_{s1} can have the form

$$M_{s1} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22s1} & m_{23} \\ m_{13} & m_{23} & 1 \end{bmatrix} \quad (6-52)$$

where m_{22s1} is a positive constant which satisfies the conditions

$$m_{22} + 2(1+u_1)^2 x_1 > m_{22s1} \quad \text{for } \underline{x} \in \Omega \quad (6-53)$$

and the matrix M_{s1} is positive definite or

$$M_{s1} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22s1} & m_{23} \\ m_{13} & m_{23} & 1 \end{bmatrix} > 0 \quad (6-54)$$

The maximum value of the variable x_1 on the quadratic surface is easily found to be

$$x_{1\max} = 2 \sqrt{C_m} \quad (6-55)$$

where

$$C_m = \frac{m_{11}(m_{22s1} - m_{23}^2)}{\det |M_{s1}|} \quad (6-56)$$

and where $\det |M_{s1}|$ denotes the determinant of the matrix M_{s1} .

Substitution of the expression for $x_{1\max}$ into inequality (6-53) yields the following constraints on the elements of the $V(\underline{x})$ function and the parameters of the controller output which are required for the satisfaction of inequality (6-43).

$$\frac{m_{22} - m_{22s1}}{2\sqrt{C_m}(1+u_1)^2} > 2 \quad (6-57)$$

where the matrix M_{s1} is positive definite or

$$M_{s1} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22s1} & m_{23} \\ m_{13} & m_{23} & 1 \end{bmatrix} > 0 \quad (6-54)$$

The constraint for the satisfaction of inequality (6-47) is now formulated. Substituting Eq. (6-39) in inequality (6-47) yields

$$\underline{x}' M_{s2} \underline{x} \geq \underline{x}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(1+u_1)^2 x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} \quad (6-58)$$

for $\underline{x} \in \Omega$

The matrix M_{s2} can therefore have the form

$$M_{s2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_{22s2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6-59)$$

where m_{22s2} is a positive constant which satisfies the inequality

$$m_{22s2} \geq 2(1+u_1)^2 x_1 \quad \text{for } \underline{x} \in \Omega \quad (6-60)$$

Substitution of Eq. (6-55) into condition (6-60) yields the following constraint on the parameters of the controller output which is required for the satisfaction of inequality (6-47)

$$\frac{m_{22s2}}{2\sqrt{C_m}(1+u_1)^2} \geq 2 \quad (6-61)$$

where C_m is given by Eq. (6-56).

The constraint for the satisfaction of inequality (6-46) is now formulated. It is easily verified that

$$\frac{C_o}{C_s} = \frac{x_{1\max}^2}{x_{1s}^2} \quad (6-62)$$

Substitution of Eq. (6-62) in inequality (6-46) yields

$$-\max\lambda(A_3) \frac{1 - C_d}{1 + \max\lambda[M_{s2}M_3^{-1}]} \geq \frac{1}{10} \ln [5x_{1\max}] \quad (6-63)$$

Substituting Eq. (6-59) into the expression for the characteristic values of the matrix $M_{s2}M_3^{-1}$, it is obtained that

$$\max \lambda \left[M_{s2} M_3^{-1} \right] = \frac{m_{22}s_2(m_{11} - m_{13}^2)}{\det | M_3 |} \quad (6-64)$$

Substitution of inequality (6-61) and Eq. (6-64) into inequality (6-63) yields the following constraint on the elements of the $V(\underline{x})$ function and the parameters of the controller output which is imposed by inequality (6-46).

$$\frac{(C_d - 1) \lambda_{\max}(A_3)}{1 + \frac{2x_{1\max}(1+u_1)^2(m_{11} - m_{13}^2)}{\det | M_3 |}} \geq \frac{1}{10} \ln [5x_{1\max}] \quad (6-65)$$

where $x_{1\max}$ is given by Eq. (6-55).

The next step in the design method is to formulate the constraints on the elements of the $\dot{V}(\underline{x})$ function which are required for the satisfaction of inequalities (6-44) and (6-45).

6.3.6 Step (6): Formulation of the Constraints on the Elements
of the Time Derivative of the Liapunov
Function

In the derivation of the design inequality given by (6-46) (which was presented in sections 5.3 and 5.4) it was necessary to assume that a positive constant C_d less than one existed such that

$$-C_d \underline{x}' D_o \underline{x} - \underline{x}' D_{on}(\underline{x}) \underline{x} \geq 0 \quad (6-45)$$

for all $\underline{x}$ in a region Ω of the $\underline{x}$ space. This is inequality (5-43) of section 5.3 and its satisfaction is guaranteed in an arbitrarily small region of the origin since the matrix D_o is formulated to be negative definite.

Since $\underline{x}' D_{on}(\underline{x}) \underline{x}$ is a high order function of several variables, the establishment of condition (6-45) (as discussed in section 3.2) must in general be accomplished by numerical techniques which require the use of a digital computer. However, as illustrated in section 3.2, a technique is available which may greatly simplify the work necessary. Specifically, the function of inequality (6-45) is written in pseudo-quadratic form and a generalization of Sylvester's theorem is then applied to the elements of the pseudo-quadratic matrix.

This technique will be demonstrated by the illustrative design problem of this chapter as well as the design problems of chapter 7. In the next and final step of the design method, the digital computer is employed as a design aid in order to determine whether the constraints formulated in steps (5) and (6) can be satisfied.

Returning to the illustrative design problem, inequality (6-45) becomes

$$\begin{aligned}
 -C_d \underline{x} D_3 \underline{x} - \underline{x}' D_{31}(\underline{x}) \underline{x} = & -C_d d_{11} x_1^2 - C_d d_{22} x_2^2 - C_d d_{33} x_3^2 \\
 & -2C_d d_{12} x_1 x_2 - 2C_d d_{13} x_1 x_3 - 2C_d d_{23} x_2 x_3 - d_{n1} x_1^2 x_2 \\
 & -d_{n2} x_1 x_2^2 - d_{n4} x_2^3 > 0
 \end{aligned} \tag{6-66}$$

for $\underline{x} \in \Omega$, where

$$\begin{aligned}
 d_{n1} &= -4m_{13}(1+u_1)^2 \\
 d_{n2} &= -4m_{23}(1+u_1)^2 \\
 d_{n4} &= 2(1+u_1)^2
 \end{aligned} \tag{6-67}$$

For testing purposes, it is noted that the region for which inequality (6-66) is to be satisfied is defined as the interior of the hyper-rectangle given by

$$|x_1| \leq x_{1\max}, \quad |x_2| \leq x_{2\max} \text{ and } |x_3| \leq x_{3\max} \text{ where}$$

$x_{1\max}$, $x_{2\max}$ and $x_{3\max}$ are the maximum values of the state variables on the quadratic surface $\underline{x}' M_{s1} \underline{x} = 4m_{11}$.

It is noted that the inequality (6-66) contains the third power of x_2 . The establishment of (6-66) can be simplified by applying the technique of section 3.2. The term $C_d \underline{x}' D_3 \underline{x} + \underline{x}' D_{31}(\underline{x}) \underline{x}$ is rewritten in the following pseudo-quadratic form.

$$C_d \underline{x}' D_3 \underline{x} + \underline{x}' D_{31}(\underline{x}) \underline{x} = \underline{x}' D_p(\underline{x}) \underline{x} \quad (6-68)$$

where the matrix $D_p(\underline{x})$ is equal to

$$D_p(\underline{x}) = \begin{bmatrix} C_d d_{11} + (1-p_1) d_{n1} x_2 & C_d d_{12} + \frac{p_1}{2} d_{n1} x_1 + \frac{p_2}{2} d_{n2} x_2 & C_d d_{13} \\ C_d d_{12} + \frac{p_1}{2} d_{n1} x_1 + \frac{p_2}{2} d_{n2} x_2 & C_d d_{22} + (1-p_2) d_{n2} x_1 + d_{n4} x_2 & C_d d_{23} \\ C_d d_{13} & C_d d_{23} & C_d d_{33} \end{bmatrix} \quad (6-69)$$

Note that the two parameters p_1 and p_2 have been introduced. These parameters result from the fact that the matrix $D_p(\underline{x})$ does not have a unique representation.

In order to establish that inequality (6-45) is satisfied it need only be verified that the matrix $D_p(\underline{x})$ is negative semidefinite in the region Ω . The elements of the $D_p(\underline{x})$ matrix are denoted by

$$d_{11p}(\underline{x}) = C_d d_{11} + (1-p_1) d_{n1} x_2 \quad (6-70)$$

$$d_{22p}(\underline{x}) = C_d d_{22} + (1-p_2) d_{n2} x_1 + d_{n4} x_2 \quad (6-71)$$

$$d_{12p}(\underline{x}) = C_d d_{12} + \frac{p_1}{2} d_{n1} x_1 + \frac{p_2}{2} d_{n2} x_2 \quad (6-72)$$

The maximum values of the elements $d_{11p}(\underline{x})$ and $d_{22p}(\underline{x})$ and the minimum value of the element $d_{12p}(\underline{x})$ in the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = 4m_{11}$ are equal to

$$d_{11p\max} = C_d d_{11} + (1-p_1) d_{n1} x_{2\max} \quad (6-73)$$

$$d_{22p\max} = C_d d_{22} + (1-p_2) d_{n2} x_{1\max} + d_{n4} x_{2\max} \quad (6-74)$$

$$d_{12p\min} = C_d d_{12} + \frac{p_1}{2} d_{n1} x_{1\max} + \frac{p_2}{2} d_{n2} x_{2\max} \quad (6-75)$$

where

$$x_{1\max} = 2 \sqrt{C_m} \quad (6-76)$$

$$x_{2\max} = 2 \sqrt{C_m} \left[\frac{m_{11}^2 - m_{13}^2}{m_{22s1} - m_{23}^2} \right]^{1/2} \quad (6-77)$$

and

$$C_m = \frac{m_{11}(m_{22s1} - m_{23}^2)}{\det |M_{s1}|} \quad (6-78)$$

Therefore, a sufficient condition for inequality (6-45) to be satisfied is that the matrix $D_{p\max}$, where

$$D_{p\max} = \begin{bmatrix} d_{11p\max} & d_{12p\min} & C_d d_{13} \\ d_{12p\min} & d_{22p\max} & C_d d_{23} \\ C_d d_{13} & C_d d_{23} & C_d d_{33} \end{bmatrix} \quad (6-79)$$

is a negative semidefinite matrix.

Applying Sylvester's theorem to the $D_{p\max}$ matrix, the following sufficient conditions are obtained.

$$d_{33} < 0 \quad (6-80)$$

$$d_{22p\max} d_{33} - C_d^2 d_{23}^2 > 0 \quad (6-81)$$

$$d_{11p\max} d_{22p\max} d_{33} + 2C_d d_{12p\min} d_{23} d_{13} - C_d^2 d_{11p\max} d_{23}^2 - d_{33}^2 d_{12p\min}^2 - C_d^2 d_{22p\max} d_{13}^2 < 0 \quad (6-82)$$

Inequalities (6-80), (6-81) and (6-82) are the constraints on the elements of the $\dot{V}(\underline{x})$ function and the parameters of the controller output which are required for the satisfaction of inequality (6-45).

In the next and final step of the design method, the digital computer is employed as a design aid to determine numerical values for the parameters of the controller output which satisfy inequalities (6-54), (6-57), (6-61), (6-65), (6-80), (6-81) and (6-82).

6.3.7 Step (7): Application of a Digital Computer

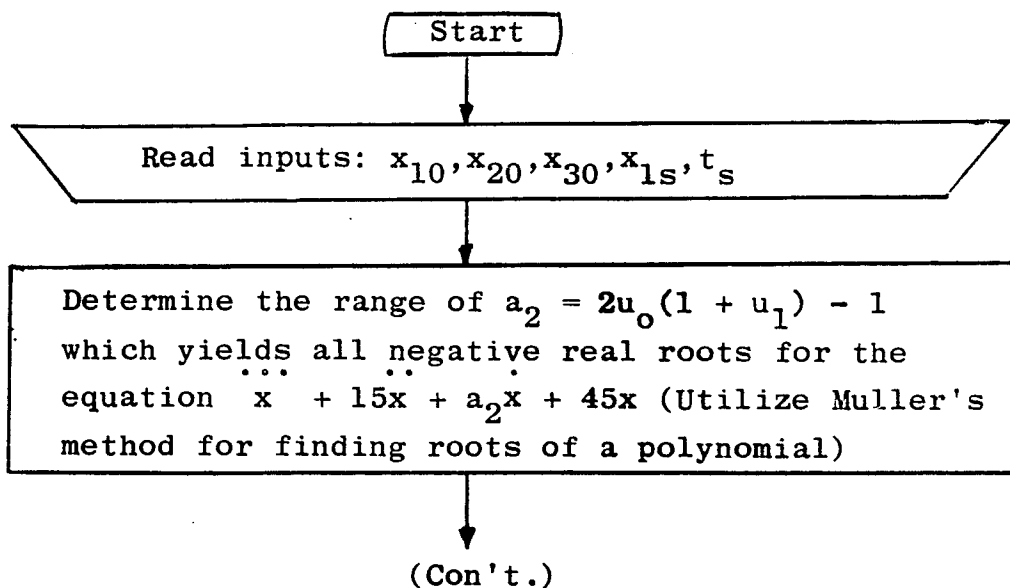
Program for Solving the Design Constraints

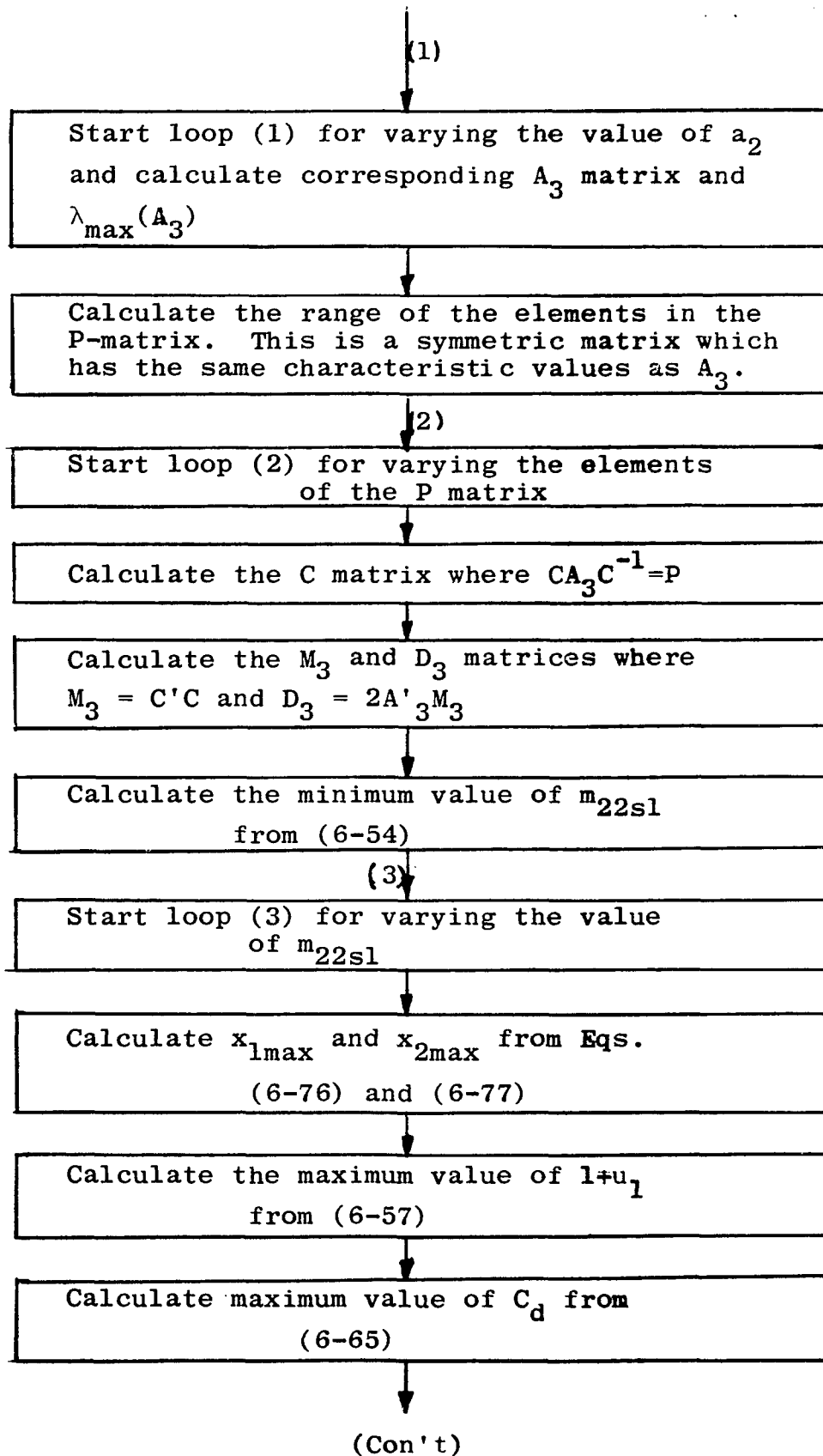
The digital computer is now utilized to solve the constraints formulated in steps (5) and (6) of the design method. When employing the digital computer as an aid intuition does not have to be heavily relied on nor is a large random parameter search required. It is the systematic and logical use of the digital computer which enables the effective application of Liapunov's direct method to the design of high order nonlinear systems.

In general, in order to obtain the allowable numerical ranges of the elements of the Liapunov function and its time derivative and the numerical ranges of the parameters of the controller output, numerical techniques such as Muller's method for finding the roots of a polynomial equation and a fourth order Runge-Kutta method to approximate the solution of a differential equation are utilized.

The programming logic necessary for the design method is best explained via the illustrative problem. The logic flow diagram of the computer program employed to solve the current design problem is now presented.

LOGIC FLOW DIAGRAM OF COMPUTER PROGRAM





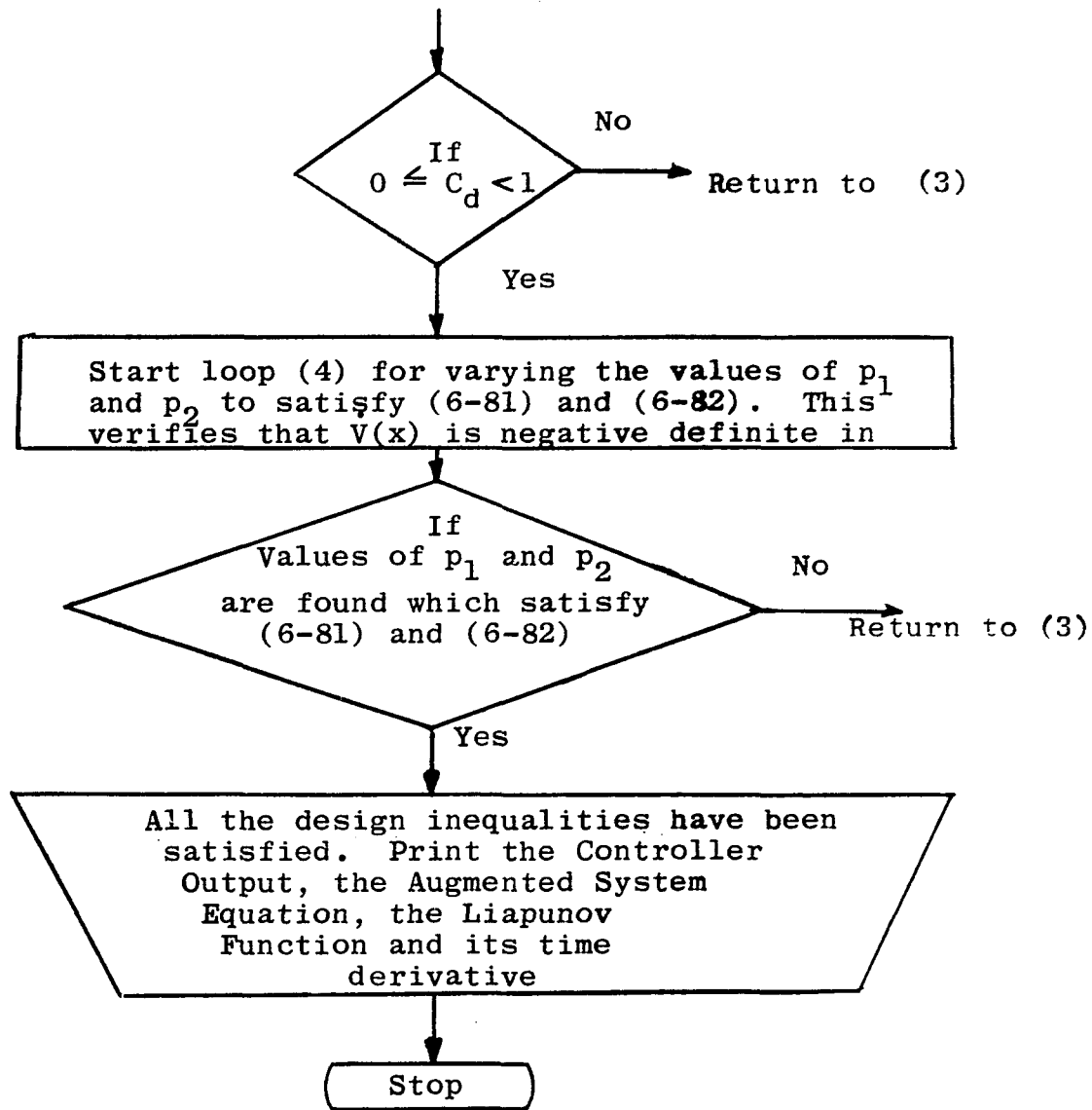


FIGURE 6.6

A set of numerical results obtained from the program is given below.

The Controller Output

$$u(\underline{x}) = 95.55 - 0.71x \quad (6-83)$$

The Augmented System Equation

$$\ddot{\ddot{x}} + 15\ddot{x} + (54.57 + 0.17x)\dot{x} + 45x = 0 \quad (6-84)$$

The Liapunov function and its time derivative are

$$\begin{aligned} V(\underline{x}) = & 184.46x^2 + 86.84\dot{x}^2 + \ddot{x}^2 + 227.17x\dot{x} + 21.14x\ddot{x} \\ & + 17.45x\ddot{\ddot{x}} + 0.17x\dot{x}^2 \end{aligned} \quad (6-85)$$

and

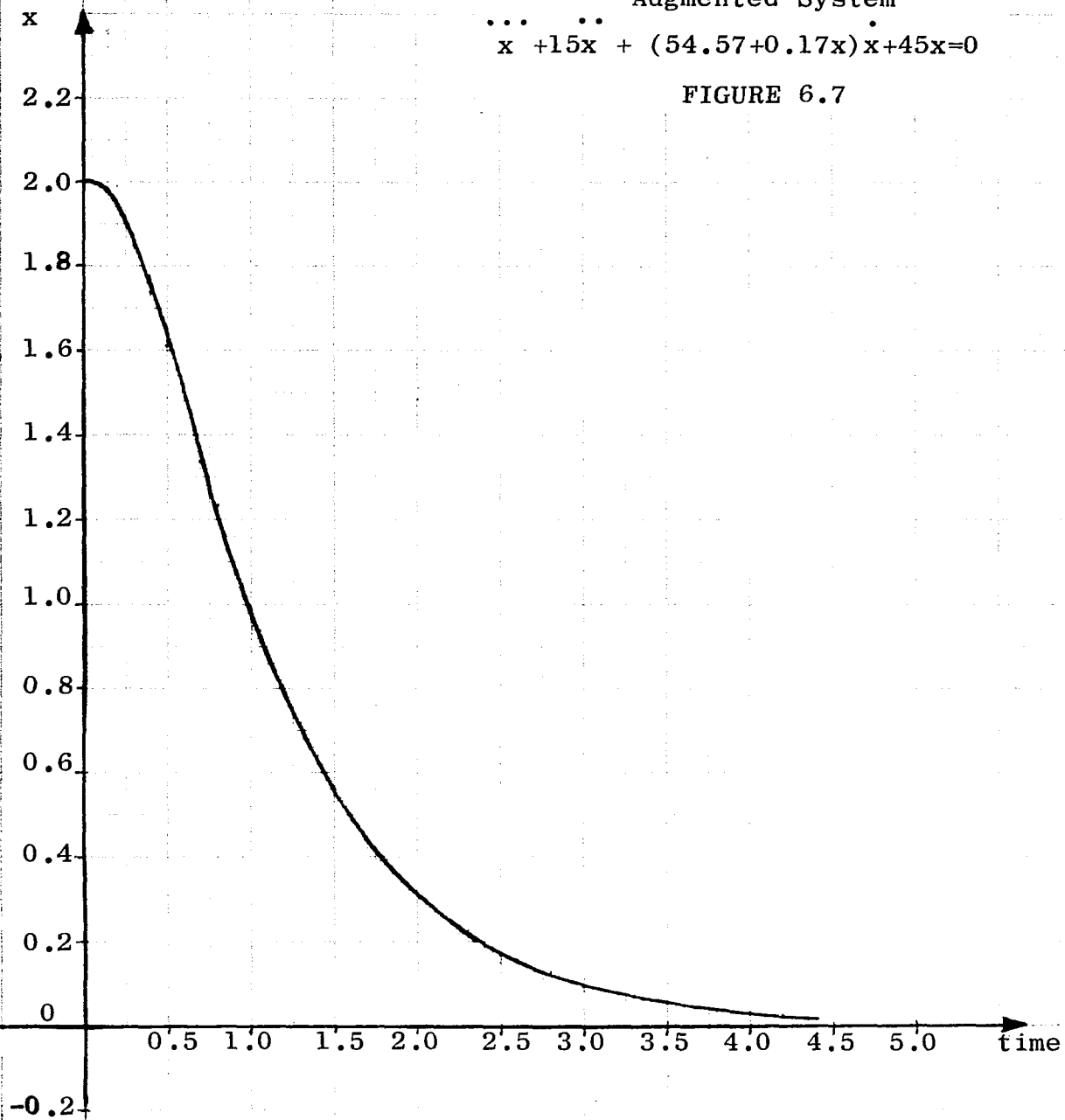
$$\begin{aligned} \dot{V}(\underline{x}) = & -951.52x^2 - 724.91\dot{x}^2 - 12.55\ddot{x}^2 - 1570.1x\dot{x} \\ & - 180x\ddot{x} - 176.01x\dot{x}\ddot{x} - 3.56x^2\dot{x} - 2.93x\dot{x}^2 \\ & + 0.17x\dot{x}^3 \end{aligned}$$

The response of the augmented system due to the initial condition $x(0) = 2$, $\dot{x}(0) = \ddot{x}(0) = 0$ is given in Fig. 6.7. It is noted that the augmented system satisfies the settling time specification that $|x_1| \leq 0.2$ for $t \geq 10$ seconds.

Augmented System

$$\ddot{x} + 15\dot{x} + (54.57 + 0.17x)\dot{x} + 45x = 0$$

FIGURE 6.7



6.4 Summary

A design method has been presented in detail via an illustrative problem. The method combines the theory of Liapunov, Matrix Algebra, the theory of surfaces and the systematic and logical use of the digital computer. The design method is of the robust type due to the fact that there does not exist a unique controller which satisfies the design inequalities. Since there is presently only meager knowledge about the conditions under which controllers for nonlinear systems can be found in general, it is deemed essential to develop heuristic design methods such as the one presented in this dissertation. With such a method one can learn in a systematic manner and apply the accumulated knowledge to the design of future systems.

In order to further understand the design method, the next chapter analyzes the results of the illustrative problem. The effect of increasing the complexity of the controller is also discussed. The chapter concludes with a second order nonlinear system which is shown to be globally asymptotically stable by application of the design method.

CHAPTER 7

FURTHER ILLUSTRATION AND ANALYSIS
OF THE DESIGN METHOD7.1 Introduction

The purpose of this chapter is to yield further understanding of the design method. In order to accomplish this objective, section 7.2 analyzes the results for the illustrative problem of chapter 6. Specifically, it is shown that although the nonlinear system was designed for a given initial condition, the settling time of the augmented system is known for a set of initial conditions. This eliminates the previously necessary task of re-analyzing a nonlinear system for each initial condition which may be of interest. Also discussed in that section are the bounds on all the system state variables which are obtained from the design method and the fact that a re-estimation of the system's settling time can be obtained after a specific controller is chosen. The dependence of the augmented system's response upon the parameter ϵ is also presented.

In section 7.3 the effect of increasing the controller complexity upon the response of the system is ascertained by means of the design method. It is found that increasing the complexity may increase the system's settling time for the initial conditions of interest. This conclusion is

verified by a digital computer simulation.

The chapter is concluded with a second order design problem for a system which is described by a Lienard type equation of the form

$$\ddot{x} + g(x)\dot{x} + f(x) = 0$$

The purpose of this problem is to present an example for which the design method yields an augmented system which is shown to be globally asymptotically stable.

7.2 Analysis of the Results of Chapter 6

In this section the results of chapter 6 are analyzed and further implications of the design method are presented.

7.2.1 Initial Conditions Defined by the Initial Liapunov Surface

The augmented system of chapter 6 was designed for the initial condition $\underline{x}(0) = [2, 0, 0]^T$. Since $V[\underline{x}(0)]$ is the initial value of the Liapunov function $V(\underline{x})$, the point $\underline{x}(0)$ is on the initial Liapunov surface defined by the equation

$$V(\underline{x}) = V[\underline{x}(0)] \quad (7-1)$$

Substitution of Eq. (6-85) into Eq. (7-1) and the fact that $V[\underline{x}(0)] = 737.84$ yields the following equation for the initial Liapunov surface.

$$184.46\dot{x}^2 + 86.84\ddot{x}^2 + \ddot{x}^2 + 227.17\dot{x}\ddot{x} + 21.14\ddot{x}\ddot{x} + 17.45\ddot{x}\ddot{x} + 0.17\ddot{x}\ddot{x}^2 = 737.84 \quad (7-2)$$

The controller output parameters were constrained to cause the Liapunov function $V(\underline{x})$ to be less than or equal to a scalar C_s for $t \geq 10$ seconds. The value of C_s is such that the maximum value of $\underline{x}(t)$ is 0.2 on the quadratic surface

$$\underline{x}' M_{s1} \underline{x} = C_s \quad (7-3)$$

where M_{s1} is a positive definite matrix. From the numerical results of the digital computer program, the M_{s1} matrix employed was

$$M_{s1} = \begin{bmatrix} 184.46 & 113.58 & 10.58 \\ 113.58 & 85.94 & 8.72 \\ 10.58 & 8.72 & 1 \end{bmatrix} \quad (7-4)$$

and the value of C_s was 1.05.

It is now noted that the design philosophy presented is concerned with the initial value of the Liapunov function and not the initial values of the state variables. The results of chapter 6 are therefore valid for any initial condition which would cause the function $V(\underline{x})$ to have the initial value 737.84. Hence the results of chapter 6 are valid for any point on the initial Liapunov surface of Eq. (7-2).

As an illustration of this fact, the point $\underline{x} = [5, -8.28, 15.635]$ is on the initial Liapunov surface. The response of the augmented system due to the initial condition $\underline{x}(0)=5$, $\dot{\underline{x}}(0) = -8.28$ and $\ddot{\underline{x}}(0) = 15.635$ is depicted

in Fig. 7.1. It is observed that the settling time specification $|x(t)| \leq 0.2$ for $t \geq 10$ seconds is satisfied.

This result eliminates the previously necessary task of re-analyzing the nonlinear control system for different initial conditions on the initial Liapunov surface.

7.2.2 Bounds on the State Variables

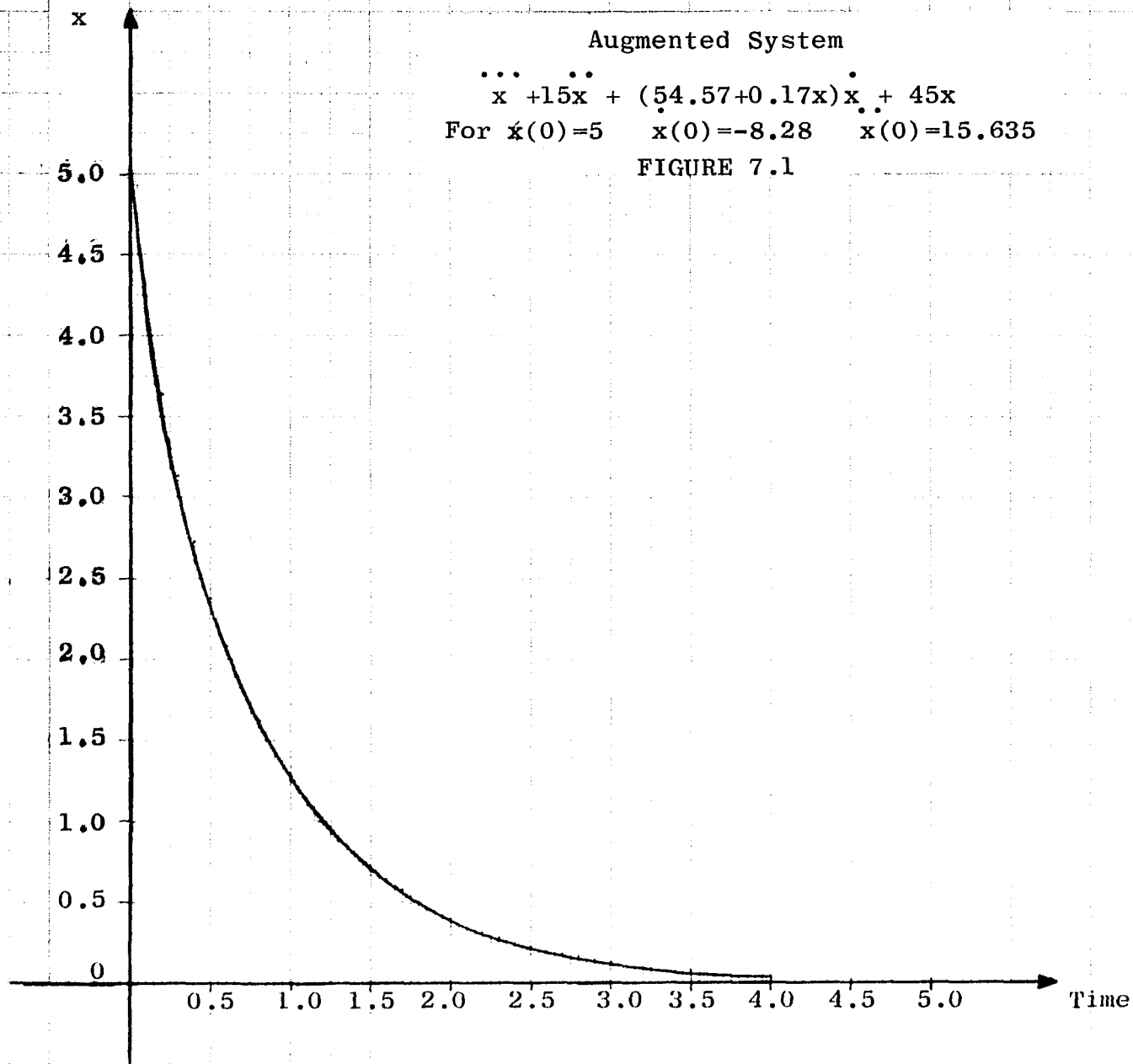
In the design method a quadratic surface is found which bounds the initial Liapunov surface. This surface is given by Eq. (6-50) of section 6.3.5 and is repeated below.

$$\underline{x}' M_{s1} \underline{x} = V [\underline{x}(0)] \quad (7-5)$$

Substitution of Eq. (7-4) into Eq. (7-5) and the fact that $V [\underline{x}(0)] = 737.84$ yields the following equation for the bounding quadratic surface.

$$\begin{aligned} 184.46x^2 + 85.94\dot{x}^2 + \ddot{x}^2 + 227.17x\dot{x} + 21.14x\ddot{x} \\ + 17.45x\ddot{x} = 737.84 \end{aligned} \quad (7-6)$$

The maximum values of the state variables on the bounding quadratic surface of Eq. (7-6) are $x_{1\max} = 5.3$, $x_{2\max} = 14.4$ and $x_{3\max} = 1243$. It is now noted that the



time derivative of $V(\underline{x})$ has been formulated to be negative definite or the $V(\underline{x})$ function is a monotonically decreasing function of time. Hence all of the Liapunov surfaces defined by $V(\underline{x}) = C(t)$, where $C(t)$ is the scalar value of the Liapunov function at time $t > 0$, are bounded by the initial Liapunov surface given by Eq. (7-4) and are therefore also bounded by the quadratic surface given by Eq. (7-6). The values of the state variables of the augmented system therefore cannot exceed the maximum values on this bounding quadratic surface. This leads to the conclusion that the design method yields bounds for the values of the augmented system's state variables and these bounds are given by $|x_1| \leq 5.3$, $|x_2| \leq 14.4$ and $|x_3| \leq 1243$.

7.2.3 Re-Estimate of the Settling Time

After a controller has been obtained, a re-estimate of the settling time for all initial conditions on the initial Liapunov surface can be calculated. The minimum value of the constant C_d is calculated which enables inequality (6-45) to be satisfied. This minimum value is then substituted into inequality (6-46) and the new value of the settling time t_s is obtained.

For the illustrative problem of chapter 6 the minimum value of C_d is found to be $C_{dmin} = 0.13$.

The new value of the settling time for the initial conditions on the initial Liapunov surface is found to be 3.9 seconds. This value is in agreement with the system responses of Figs. 6.7 and 7.1.

7.2.4 Effect of the Parameter ϵ

In section 6.2 it was stated that the effect of the numerical value of the parameter ϵ upon the results of the design method would be ascertained. This is now done.

If the parameter ϵ were included in the differential equation of the augmented system the result would be

$$\ddot{x} + 15\dot{x} \left[\epsilon u_0(1+u_1) - 1 + \epsilon(1+u_1)^2 x \right] \dot{x} + 45x = 0 \quad (7-7)$$

Assume now that the design method has been performed for one value of ϵ and it is desired to determine the controller output satisfying the same settling time specification for a different value of $\epsilon = \epsilon'$. If the following substitutions are made in Eq. (7-7),

$$u_0' = \sqrt{\frac{\epsilon}{\epsilon'}} u_0 \quad (7-8)$$

$$u_1' = \sqrt{\frac{\epsilon}{\epsilon'}} (1+u_1) - 1 \quad (7-9)$$

the resulting differential equation is

$$\ddot{x} + 15\dot{x} + \left[\epsilon' u'_0 (1+u'_1) - 1 + \epsilon' (1+u'_1)^2 x \right] \dot{x} + 45x = 0 \quad (7-10)$$

Hence the new controller output $u'(x)$ for the new value of $\epsilon=\epsilon'$ is given by $u'(x) = u'_0 + u'_1 x$ where u'_0 and u'_1 are given by Eqs. (7-8) and (7-9). As an illustration of this procedure assume that it is desired to find the new controller output for the problem of chapter 6 for an $\epsilon=200$. Substitution of the numerical quantities in Eqs. (7-8) and (7-9) yields the result $u'(x) = 9.56 - 0.97x$ for the new controller output.

In the next section the illustrative design problem of chapter 6 is repeated for a more complex controller output and the results obtained are compared with the results of chapter 6.

7.3 Increased Controller Output Complexity

It was stated in section 6.3.3 that it is desirable for the design method to enable the designer to ascertain the effect of employing a more complex controller output upon the augmented system's response. In this section, the design problem of chapter 6 is repeated for a controller output $u(x)$ which contains a component that is proportional to the square of the system's output, or

$$u(x) = u_0 + u_1x + u_2x^2 \quad (7-11)$$

where u_0 , u_1 and u_2 are unspecified constants. The results obtained for the controller given by Eq. (7-11) are compared to the results obtained in chapter 6.

7.3.1 Application of the Design Method

The results of chapter 6 were obtained for a controller output of the form

$$u(x) = u_0 + u_1x \quad (7-12)$$

It is now desired to repeat the design method for the controller output given by Eq. (7-11). We therefore return to step (4) of the design method which was presented in section 6.3.4. Employing the controller output of Eq. (7-11) the expressions for the augmented system's

parameters (which are given by Eqs. (6-38) of section 6.3.4) become

$$\begin{aligned}
 a_3 &= 15 \\
 a_2 &= 2u_0(1 + u_1) - 1 \\
 a_1 &= 45 \\
 \psi(x) &= 2 \left[(1+u_1)^2 + 2u_0u_2 \right] x + 6u_2(1+u_1)x^2 \\
 &\quad + 4u_2^2x^3
 \end{aligned} \tag{7-13}$$

The expressions for the non-quadratic portions of the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ are

$$M_{31}(\underline{x}) = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 2 \left[(1+u_1)^2 + 2u_0u_2 \right] x_1 + 6u_2(1+u_1)x_1^2 + 4u_2^2x_1^3 \\ 0 & 0 \end{array} \right] \tag{7-14}$$

and

$$D_{31}(\underline{x}) = \left[\begin{array}{c|c} 0 & -4m_{13} \left\{ \left[(1+u_1)^2 + 2u_0u_2 \right] x_1 + 3u_2(1+u_1)x_1^2 + 2u_2^2x_1^3 \right\} \\ 0 & -4m_{23} \left\{ \left[(1+u_1)^2 + 2u_0u_2 \right] x_1 + 3u_2(1+u_1)x_1^2 + 2u_2^2x_1^3 \right\} + 2 \left\{ \left[(1+u_1)^2 + 2u_0u_2 \right] x_2 + 6u_2(1+u_1)x_1x_2 + 6u_2^2x_1^2x_2 \right\} \\ 0 & 0 \end{array} \right]$$

Proceeding to step (5) of the design method, the constraints imposed by inequalities (6-43), (6-46) and (6-47) on the elements of the $V(\underline{x})$ function are next obtained. Inequality (6-43) requires that

$$m_{22} - m_{22s1} \geq -4u_2^2 x_1^3 - 6u_2(1+u_1)x_1^2 - 2 \left[(1+u_1)^2 + 2u_0 u_2 \right] x_1 \quad (7-16)$$

for $\underline{x} \in \Omega$ where the region Ω is defined as the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = 4m_{11}$.

The following constraint is required for the satisfaction of inequalities (6-46) and (6-47).

$$-\max \lambda(A_0) \frac{1 - C_d}{1 + \max \lambda \left[M_{s2} M_3^{-1} \right]} \geq \frac{1}{10} \ln \left[5x_{1\max} \right]$$

$$\text{for } \underline{x} \in \Omega \text{ where } \max \lambda \left[M_{s2} M_3^{-1} \right] =$$

$$\frac{\left\{ 2 \left[(1+u_1)^2 + 2u_0 u_2 \right] x_{1\max} + 6u_2(1+u_1)x_{1\max}^2 + 4u_2^2 x_{1\max}^3 \right\} (m_{11} - m_{13}^2)}{\det | M_3 |} \quad (7-18)$$

The quantity $x_{1\max}$ is the maximum value of the variable x_1 on the bounding quadratic surface $\underline{x}' M_{s1} \underline{x} = 4m_{11}$ and is given by

$$x_{1\max} = 2 \left[\frac{m_{11}(m_{22s1} - m_{23}^2)}{\det |M_{s1}|} \right]^{1/2} \quad (7-19)$$

Proceeding to step (6) of the design method the constraints required for the satisfaction of inequalities (6-44) and (6-45) are formulated. Inequality (6-45) is rewritten in the pseudo-quadratic form,

$$C_d \underline{x}' D_3 \underline{x} + \underline{x}' D_{31}(\underline{x}) \underline{x} = \underline{x}' D_p(\underline{x}) \underline{x} \leq 0 \quad (7-20)$$

where the matrix $D_p(\underline{x})$ is found to be

$$D_p(\underline{x}) = \begin{bmatrix} d_{11p}(\underline{x}) & d_{12p}(\underline{x}) & C_d d_{13} \\ d_{12p}(\underline{x}) & d_{22p}(\underline{x}) & C_d d_{23} \\ C_d d_{13} & C_d d_{23} & C_d d_{33} \end{bmatrix} \quad (7-21)$$

where

$$\begin{aligned} d_{11p}(\underline{x}) = & C_d d_{11} + (1-p_1) d_{n1} x_2 + p_3 d_{n3} x_2^2 + (1-p_5) d_{n5} x_1 x_2 \\ & + p_{71} d_{n7} x_1 x_2^2 + p_{81} d_{n8} x_2^3 + (1-p_9) d_{n9} x_1^2 x_2 \end{aligned} \quad (7-22)$$

$$\begin{aligned} d_{22p}(\underline{x}) = & C_d d_{22} + d_{n2} (1-p_2) x_1 + (1-p_3) d_{n3} x_1^2 + d_{n4} x_2 \\ & + (1-p_6) d_{n6} x_1^2 x_2 + p_{72} d_{n7} x_1^3 + p_{82} d_{n8} x_1^2 x_2 \end{aligned} \quad (7-23)$$

$$\begin{aligned}
d_{12p}(\underline{x}) = & C_d d_{12} + \frac{p_1}{2} d_{n1} x_1 + \frac{p_2}{2} d_{n2} x_2 + \frac{p_5}{2} d_{n5} x_1^2 + \frac{p_6}{2} d_{n6} x_2^2 \\
& + \frac{(1-p_{71}-p_{72})}{2} d_{n7} x_1^2 x_2 + \frac{(1-p_{81}-p_{82})}{2} d_{n8} x_1 x_2^2 \\
& + \frac{1}{2} p_9 d_{n9} x_1^3
\end{aligned} \tag{7-24}$$

and

$$\begin{aligned}
d_{n1} &= -4m_{13} \left[(1+u_1)^2 + 2u_1 u_2 \right] \\
d_{n2} &= -4m_{23} \left[(1+u_1)^2 + 2u_1 u_2 \right] \\
d_{n3} &= -12m_{23} u_2 (1+u_1) \\
d_{n4} &= 2 \left[(1+u_1)^2 + 2u_1 u_2 \right] \\
d_{n5} &= -12m_{13} u_2 (1+u_1) \\
d_{n6} &= 12u_2 (1+u_1) \\
d_{n7} &= -8m_{23} u_2^2 \\
d_{n8} &= 12u_2^2 \\
d_{n9} &= -8m_{13} u_2^2
\end{aligned} \tag{7-25}$$

The p parameters result from the fact that the matrix $D_p(\underline{x})$ does not have a unique representation. In order to establish that inequality (7-20) is satisfied it need only be verified that the matrix $D_p(\underline{x})$ is negative semi-definite in the region Ω . A sufficient condition for

the matrix $D_p(\underline{x})$ to be negative semidefinite is that the matrix D_{pmax} given by

$$D_{pmax} = \begin{bmatrix} d_{11pmax} & d_{12pmin} & C_d d_{13} \\ d_{12pmin} & d_{22pmax} & C_d d_{23} \\ C_d d_{13} & C_d d_{23} & C_d d_{33} \end{bmatrix} \quad (7-26)$$

is a negative semidefinite matrix. The quantities d_{11pmax} and d_{22pmax} denote the maximum values of the elements $d_{11p}(\underline{x})$ and $d_{22p}(\underline{x})$ in the region defined by $|x_1| \leq x_{1max}$, $|x_2| \leq x_{2max}$ and $|x_3| \leq x_{3max}$ where x_{1max} , x_{2max} and x_{3max} are the maximum values of the state variables on the quadratic surface $\underline{x}'M_{s1}\underline{x} = 4m_{11}$. The quantity d_{12pmin} denotes the minimum value of the element $d_{12p}(\underline{x})$ in this region. The calculus of variations is applied to find the extremal values of the elements of the matrix $D_p(\underline{x})$ and Sylvester's theorem is applied to the D_{pmax} matrix to determine if it is negative semidefinite.

The final step in the design method is the application of a digital computer program to determine whether numerical values exist for the parameters of the controller output which satisfy the design inequalities. The logic flow diagram of the computer program which was employed to

solve the current design constraints is essentially the same as that presented in Fig. 6.6 for the illustrative problem of chapter 6. The only significant differences are that there are more p 's to vary in the expression for the $D_p(\underline{x})$ matrix and the extremal values of the quantities $d_{11p}(\underline{x})$, $d_{22p}(\underline{x})$ and $d_{12p}(\underline{x})$ must be calculated.

7.3.2 Numerical Results

From the controllers which were obtained using the digital computer program, it became apparent that the added $u_2 x^2$ term in the controller output did not lead to a decreased settling time for all the initial conditions on the initial Liapunov surface. This is illustrated by the results obtained for the controller output given by

$$u(x) = 95.55 - 0.71x - 0.01x^2 \quad (7-27)$$

The results obtained by the design method for the controller output of Eq. (7-27) will now be compared with the results obtained in Chapter 6 for the controller output of Eq. (6-83) or $u(x) = 95.55 - 0.71x$.

For the controller output of Eq. (7-27) the minimum value of the constant C_d was found to be 0.4 and the corresponding predicted settling time was calculated to be

5.4 seconds. For the controller output of Eq. (6-83) the predicted settling time was calculated in section 7.2.3 to be 3.9 seconds. These results lead to the conclusion that the addition of the x^2 term may lead to a greater settling time for an initial condition on the initial Liapunov surface.

Tables 7.1 and 7.2 list the responses of the augmented system with a controller output $u(x)$ of the form

$$u(x) = 95.55 - 0.71x + u_2x^2 \quad (7-28)$$

for the two cases $u_2=0$ and $u_2=-0.01$. The tables present the responses for two initial conditions on the initial Liapunov surface. Table 7.1 lists the responses due to the initial condition $\underline{x}(0) = [-5, 8.28, -15.64]$ and Table 7.2 lists the responses due to the initial condition $\underline{x}(0) = [5, -8.28, 15.64]$

Time (seconds)	System Output For $u(x)=95.55-0.71x$	System Output For $u(x)=95.55-0.71x-0.01x^2$
0	-5.0	-5.0
1	-1.22	-1.51
2	-0.366	-0.519
3	-0.113	-0.168
4	-0.035	-0.053

COMPARISON OF AUGMENTED SYSTEMS' OUTPUT
 FOR $\underline{x}(0) = [-5, 8.28, -15.64]$
 TABLE 7.1

Time (seconds)	System Output For $u(x)=95.55-0.71x$	System Output For $u(x)=95.55-0.71x-0.01x^2$
0	5.0	5.0
1	1.27	0.954
2	0.392	0.243
3	0.121	0.072
4	0.038	0.022

COMPARISON OF AUGMENTED SYSTEMS' OUTPUT
 FOR $\underline{x}(0) = [5, -8.28, 15.64]$
 TABLE 7.2

It is observed from Table 7.1 that for the initial condition $\underline{x}(0) = [-5, 8.28, -15.64]$ the augmented system with the controller output $u(x) = 95.55 - 0.71x$ decays at a faster rate than the system with $u(x) = 95.55 - 0.71x - 0.01x^2$ as was indicated by the design method. It is noted that this conclusion is not valid for all initial conditions. This is seen from Table 7.2 which shows that for the initial condition $x(0) = [5, -8.28, 15.64]$ the response of the augmented system with the controller output $u(x) = 95.55 - 0.71x - 0.01x^2$ decays faster than the system with $u(x) = 95.55 - 0.71x$.

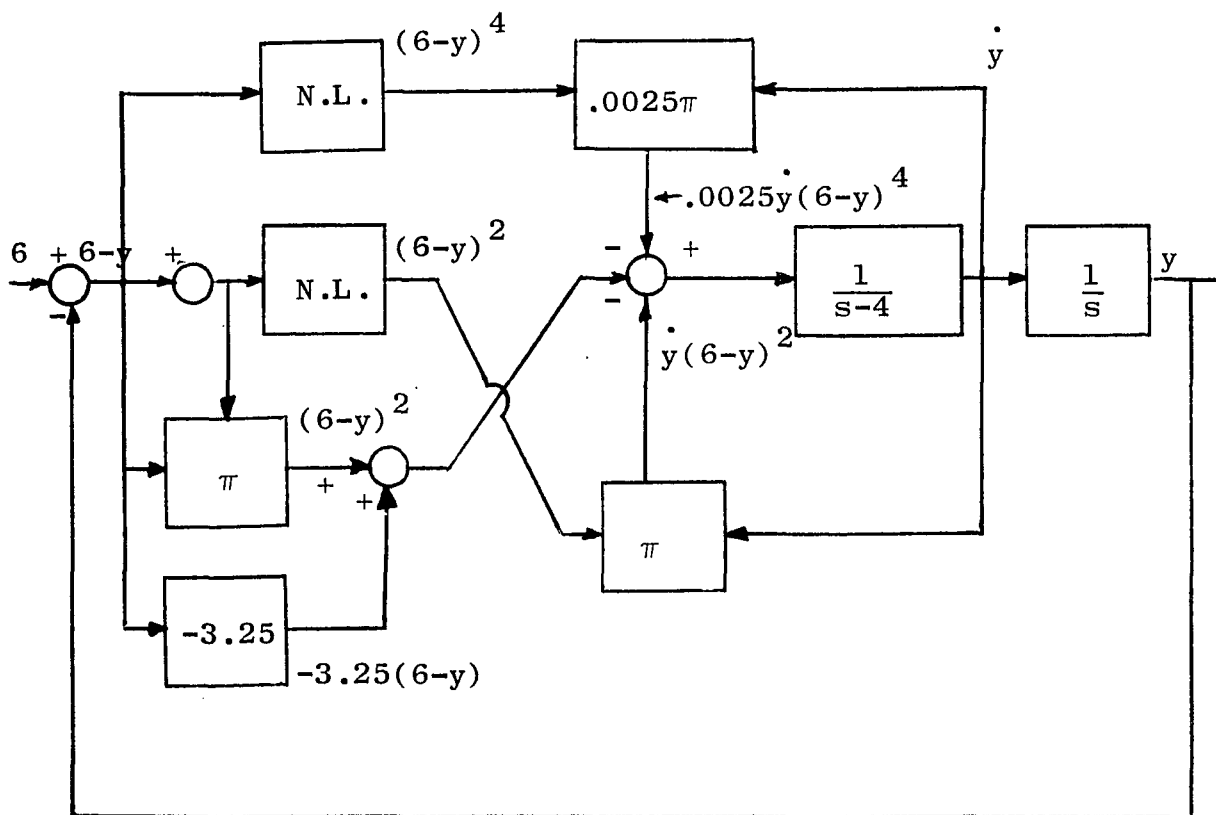
In the next section the design method is applied to a second order system which is described by a Lienard type equation.

7.4 Second Order System

In this section, the design method is applied to a second order nonlinear control system. The purpose of this section is to illustrate a problem for which the design method yields an augmented system that is globally asymptotically stable.

7.4.1 Statement of Design Problem

Consider the unaugmented system block diagram depicted in Fig. 7.2.



UNAUUGMENTED SYSTEM BLOCK DIAGRAM
FIGURE 7.2

The differential equation describing the system is

$$\ddot{y} + \left[(y-6)^2 - .0025(y-6)^4 - 4 \right] \dot{y} + 3.25(y-6) + (y-6)^2 = 0 \quad (7-29)$$

It is desired to design a controller so that the equilibrium solution of the system when augmented with the controller will be asymptotically stable and the output, denoted by $y(t)$, satisfies the condition

$$|y(t) - 6| \leq 0.2 \text{ for } t \geq 3 \text{ seconds} \quad (7-30)$$

for the initial condition $y(0) = 7$ and $\dot{y}(0) = 0$

Figure 7.3 depicts the response of the system due to the initial condition $y(0) = 7$ and $\dot{y}(0) = 0$. It is observed that the equilibrium solution $y=6$ and $\dot{y}=0$ is unstable for this initial condition.

7.4.2 Step (1): Formulation of the Augmented System

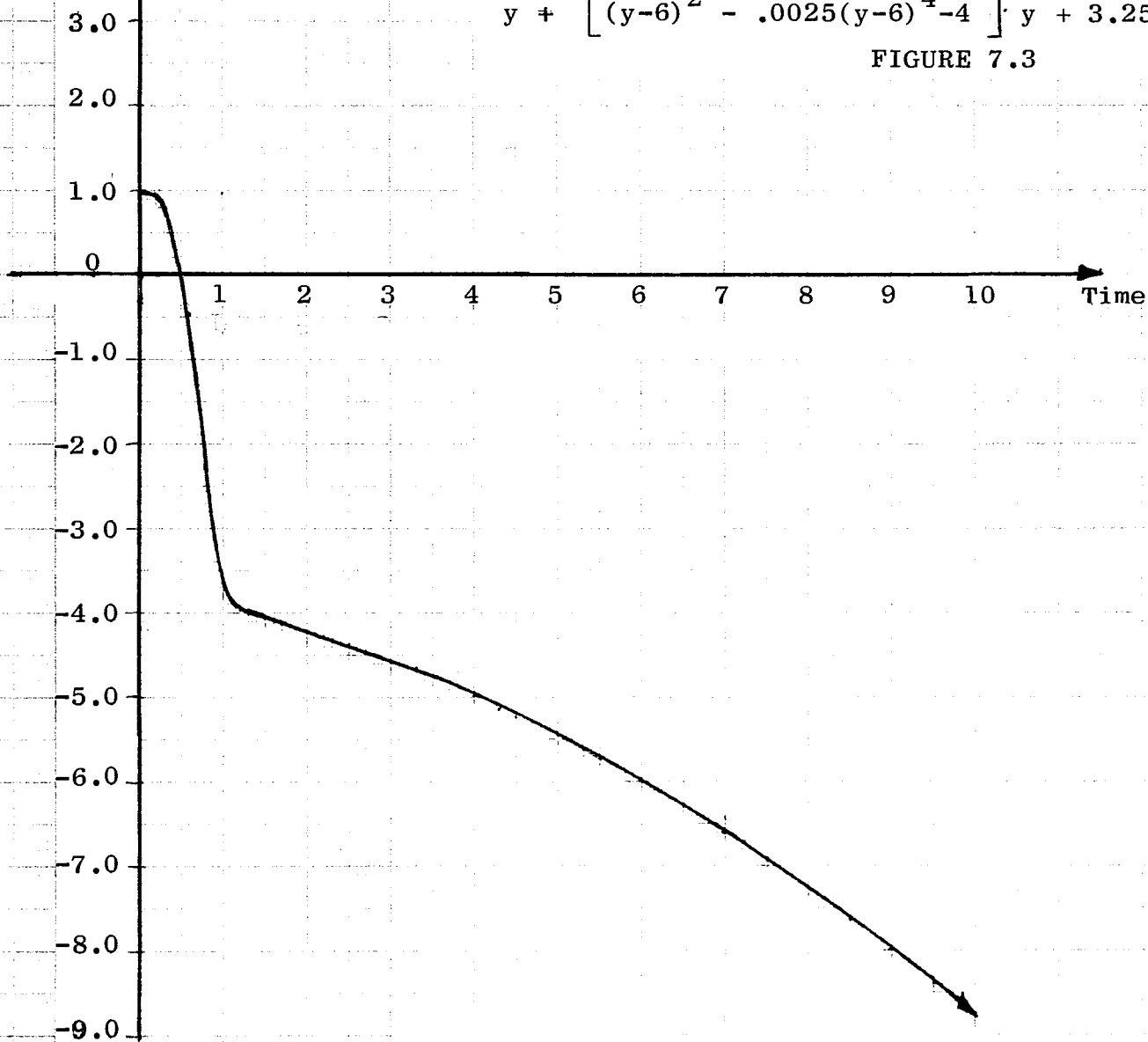
The position of the controller in the system is chosen as shown in Fig. 7.4.

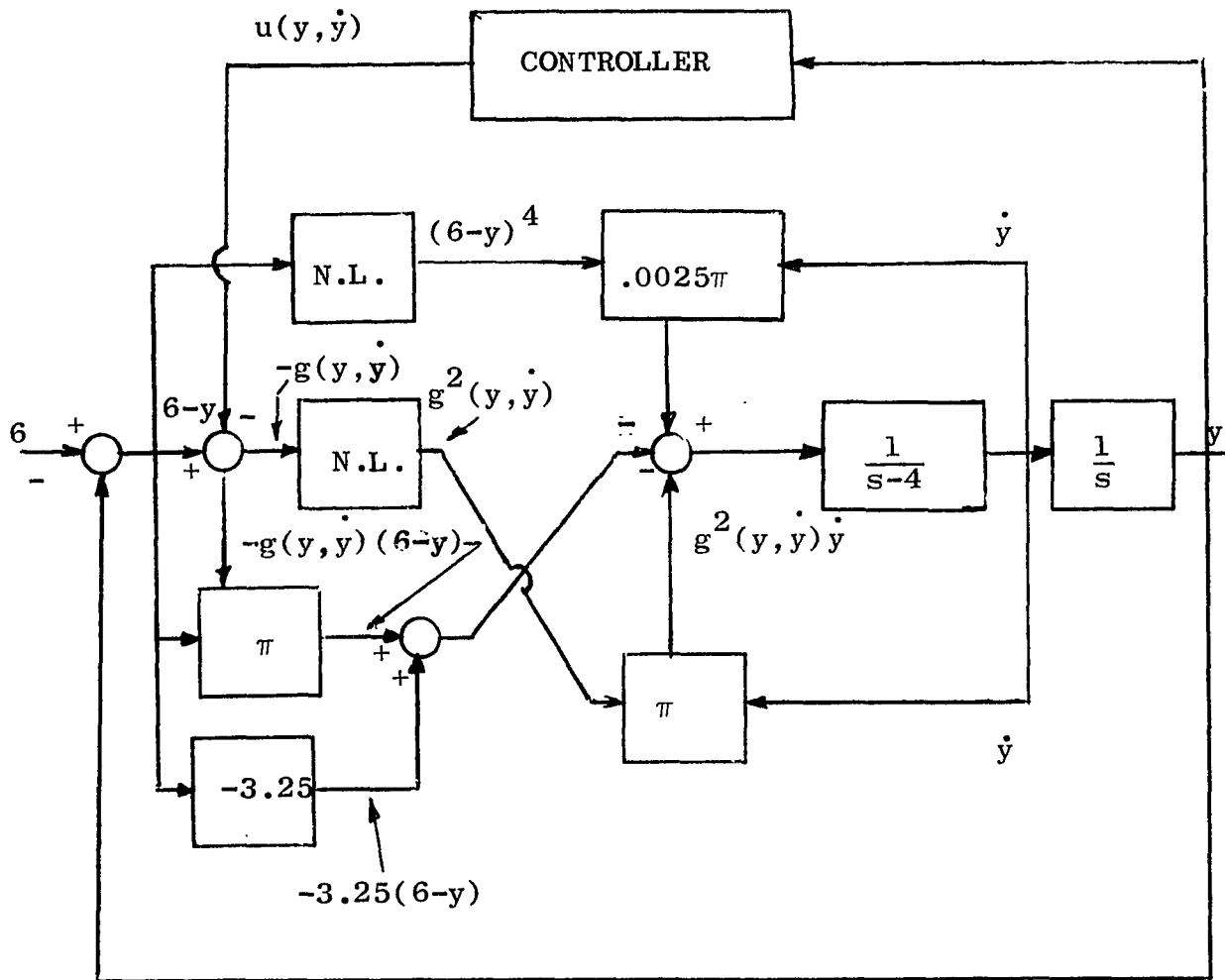
$x=y-6$

Original System

$$\ddot{y} + \left[(y-6)^2 - .0025(y-6)^4 - 4 \right] \dot{y} + 3.25(y-6) + (y-6)^2 = 0$$

FIGURE 7.3





AUGMENTED SYSTEM BLOCK DIAGRAM
FIGURE 7.4

where

$$g(y, \dot{y}) = u(y, \dot{y}) + y - 6 \quad (7-31)$$

and $u(y, \dot{y})$ is the controller output.

The differential equation describing the augmented system is

$$\ddot{y} + \left[u(y, \dot{y}) + y - 6 \right]^2 \dot{y} - .0025(y-6)^4 \dot{y} - 4\dot{y} + \left[u(y, \dot{y}) + y - 6 \right] (y-6) + 3.25(y-6) = 0 \quad (7-32)$$

Equation (7-32) has an equilibrium point at $y=6$ and $\dot{y}=0$. Shifting this equilibrium point to the origin by the substitution

$$y=x+6 \quad (7-33)$$

Eq. (7-32) becomes

$$\ddot{x} + \left[(u(x, \dot{x}) + x)^2 - .0025x^4 - 4 \right] \dot{x} + \left[u(x, \dot{x}) + x + 3.25 \right] x = 0 \quad (7-34)$$

Equation (7-34) in matrix form becomes

$$\dot{\underline{x}} = B_2 \underline{x} + b s(\underline{x}) \quad (7-35)$$

where

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 \\ -3.25 & 4 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (7-36)$$

and

$$s(\underline{x}) = \left[u(\underline{x}) + x_1 \right]^2 x_2 - .0025x_1^4 x_2 + \left[u(\underline{x}) + x_1 \right] x_1 \quad (7-37)$$

It is noted from Eq. (7-29) that the original system's differential equation is of the form

$$\ddot{x} + \psi(x)\dot{x} + f(x) = 0 \quad (7-38)$$

where

$$\psi(x) = x^2 - .0025x^4 - 4 \quad (7-39)$$

$$f(x) = 3.25x + x^2 \quad (7-40)$$

A differential equation which can be written in the form of Eq. (7-8) is known as a Lienard type equation. It is now specified that the augmented system's differential equation also be of the Lienard type or

$$\ddot{x} + [a_2 + \psi(x)]\dot{x} + a_1x + f(x) = 0 \quad (7-41)$$

In the next step of the design method a modified M-Matrix function $V(\underline{x})$ and its time derivative are formulated for Eq. (7-41).

7.4.3 Step (2): Formulation of the Modified M-Matrix

Liapunov Function and its Time Derivative

Equation (7-41) can be written in the matrix form

$$\dot{\underline{x}} = A_2\underline{x} + A_{22}(\underline{x})\underline{x} \quad (7-42)$$

where

$$A_2 = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \quad A_{22}(\underline{x}) = \begin{bmatrix} 0 & 0 \\ \frac{-f(x_1)}{x_1} & -\psi(x_1) \end{bmatrix} \quad (7-43)$$

The differential equation (7-41) is of the form considered in case 2.3 of section 4.4. The $V(\underline{x})$ function and its time derivative which were formulated for this case are

$$V(\underline{x}) = \underline{x}' M_2 \underline{x} + \underline{x}' M_{22}(\underline{x}) \underline{x} \quad (7-44)$$

and

$$\dot{V}(\underline{x}) = \underline{x}' D_2 \underline{x} + \underline{x}' D_{22}(\underline{x}) \underline{x} \quad (7-45)$$

where the matrices M_2 and D_2 are real constant matrices given by*

$$M_2 = C'C \quad (7-46)$$

* As developed in section 4.3.1, the matrix C is determined from the relation $CA_2C^{-1} = P$ where P is any symmetric real matrix which has the same characteristic values as the matrix A_2 .

and

$$D_2 = 2A_2' M_2 \quad (7-47)$$

and the matrices $M_{22}(\underline{x})$ and $D_{22}(\underline{x})$ are equal to

$$M_{22}(\underline{x}) = \begin{bmatrix} \frac{2}{x_1} \int_0^x [f(x_1) + m_{12}x_1\psi(x_1)] dx_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (7-48)$$

and

$$D_{22}(\underline{x}) = \begin{bmatrix} -2m_{12} & \frac{f(x_1)}{x_1} & 0 \\ 0 & -2\psi(x_1) \end{bmatrix} \quad (7-49)$$

In the next step the forms of the controller output are formulated.

7.4.4 Step (3): Classification of the Controller Output

The terms which must be present in the controller output are now formulated in a systematic manner so that the augmented system's differential equation will be of the form specified in step (1). Setting Eq. (7-34) equal to Eq. (7-41) yields

$$\ddot{x} + \left[(u(x, \dot{x}) + x)^2 - .0025x^4 - 4 \right] \dot{x} + \left[u(x, \dot{x}) + x + 3.25 \right] x = \ddot{x} + [a_2 + \psi(x)] \dot{x} + a_1 x + f(x) \quad (7-50)$$

In order that Eq. (7-50) be satisfied it is noted that the controller output $u(x, \dot{x})$ can only be a function of x . Hence the set U of allowable controller outputs is defined as

$$U = \{ u(x, \dot{x}) \mid u(x, \dot{x}) = u(x) \} \quad (7-51)$$

The general form of $u(x)$ is therefore given by the polynomial

$$u(x) = u_0 + u_1 x + u_2 x^2 + \dots + u_n x^n \quad (7-52)$$

where $u_0, u_1, \dots, u_n$ are real constants.

Substituting Eq. (7-52) into Eq. (7-50) yields

$$\begin{aligned}
\ddot{x} + \left[(u_0 + (1+u_1)x + u_2x^2 + \dots + u_nx^n)^2 - .0025x^4 - 4 \right] \dot{x} \\
+ \left[u_0 + 3.25 + (1+u_1)x + u_2x^2 + \dots + u_nx^n \right] x = \\
\ddot{x} + \left[a_2 + \psi(x) \right] \dot{x} + a_1x + f(x) \quad (7-53)
\end{aligned}$$

Equating terms one obtains,

$$a_1 = 3.25 + u_0 \quad (7-54)$$

$$a_2 = u_0^2 - 4 \quad (7-55)$$

$$\begin{aligned}
\psi(x) = \left[u_0 + (1+u_1)x + u_2x^2 + \dots + u_nx^n \right]^2 - .0025x_1^4 - u_0^2 \\
\quad (7-56)
\end{aligned}$$

$$f(x) = (1+u_1)x^2 + u_2x^3 + \dots + u_nx^{n+1} \quad (7-57)$$

In the next step the expressions for the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ are formulated for the controller output given by Eq. (7-52).

7.4.5 Step (4): Reduction of the Liapunov Function and its Time Derivative

Substitution of Eqs. (7-56) and (7-57) into Eqs. (7-48) and (7-49) yields the following expressions for the non-quadratic portions of the $V(\underline{x})$ and $\dot{V}(\underline{x})$ functions.

$$\underline{x}' M_{22}(\underline{x}) \underline{x} =$$

$$2\underline{x}' \left[\begin{array}{c|c} \frac{(1+u_1)}{3} x_1 + \frac{u_2}{4} x_1^2 + \dots + \frac{u_n}{n+2} x_1^n & 0 \\ \hline + m_{12} \left[\frac{2u_0(1+u_1)}{3} x_1 + \frac{u_0 u_2}{2} x_1^2 + \frac{(1+u_1)^2}{4} x_1^2 \right. & \\ \left. + \frac{2u_2(1+u_1)}{5} x_1^3 + \frac{u_2^2}{6} x_1^4 + \dots + \frac{u_n^2}{2n+2} x_1^{2n} \right] & \\ \hline - \frac{.0025}{6} x_1^4 & \\ \hline 0 & 0 \end{array} \right] \underline{x}$$

(7-58)

and

$$\underline{x}' D_{22}(\underline{x}) \underline{x} =$$

$$-2\underline{x}' \left[\begin{array}{c|c} m_{12} \left[(1+u_1) x_1 + u_2 x_1^2 + \dots \right. & 0 \\ \left. + u_n x_1^n \right] & \\ \hline 0 & \frac{2u_0(1+u_1) x_1 + 2u_0 u_2 x_1^2 + (1+u_1)^2 x_1^2 + 2u_2(1+u_1) x_1^3}{.0025 x_1^4} + u_2^2 x_1^4 + \dots + u_n^2 x_1^{2n} \end{array} \right] \underline{x}$$

(7-59)

Examination of Eqs. (7-58) and (7-59) reveals that the non-quadratic portions of the functions $V(\underline{x})$ and $\dot{V}(\underline{x})$ can be made positive semidefinite and negative semidefinite respectively in the entire $\underline{x}$ space. This can be accomplished by a controller with the output

$$u(\underline{x}) = u_0 + u_1 x + u_2 x^2 \quad (7-60)$$

where

$$u_0 > 0 \quad (7-61)$$

$$u_1 = -1 \quad (7-62)$$

$$u_2 \geq 0.05 \quad (7-63)$$

Employing the controller output given by Eq. (7-60) the expressions for the non-quadratic portions of the $V(\underline{x})$ and $\dot{V}(\underline{x})$ functions become.

$$\underline{x}' M_{22}(\underline{x}) \underline{x} = \underline{x}' \begin{bmatrix} \left(\frac{u_2}{2} + m_{12} u_0 u_2 \right) x_1^2 + \frac{m_{12}(u_2^2 - .0025)}{3} x_1^4 & 0 \\ 0 & 0 \end{bmatrix} \underline{x} \quad (7-64)$$

and $\underline{x}' D_{22}(\underline{x}) \underline{x} =$

$$-2\underline{x}' \begin{bmatrix} m_{12} u_2 x_1^2 & 0 \\ 0 & 2u_0 u_2 x_1^2 + (u_2^2 - .0025) x_1^4 \end{bmatrix} \underline{x}$$

(7-65)

It is easily verified from Eqs. (7-64) and (7-65) that the functions $\underline{x}' M_{22}(\underline{x}) \underline{x}$ and $\underline{x}' D_{22}(\underline{x}) \underline{x}$ are indeed positive and negative semidefinite respectively in the entire $\underline{x}$ space for $u_2 \geq 0.05$. Therefore if a positive definite matrix M_2 and a negative definite matrix D_2 can be found, then the augmented system will have been shown to be globally asymptotically stable.

Employing the controller output given by Eq. (7-60), the expressions for the functions $\psi(x)$ and $f(x)$ become

$$\psi(x) = 2u_0 u_2 x^2 + (u_2^2 - .0025) x^4 \quad (7-66)$$

$$f(x) = u_2 x^3 \quad (7-67)$$

In the next step the constraints which the elements of the function $V(\underline{x})$ must satisfy are formulated.

7.4.6 Step (5): Formulation of the Constraints on the Elements of the Liapunov Function

As discussed in section 6.3.5 the elements of the function $V(\underline{x})$ are constrained to satisfy the following set of inequalities for $\underline{x} \in \Omega$ (The region Ω is defined as the interior of the quadratic surface $\underline{x}' M_{s1} \underline{x} = V[\underline{x}(0)]$).

$$V(\underline{x}) = \underline{x}' M_2 \underline{x} + \underline{x}' M_{22}(\underline{x}) \underline{x} > \underline{x}' M_{s1} \underline{x} > 0 \text{ for } \underline{x} \neq 0 \quad (7-68)$$

$$V(\underline{0}) = 0 \quad (7-69)$$

and

$$-2\max\lambda(A_2) \frac{1 - C_d}{1 + \max\lambda[M_{s2} M_2^{-1}]} \geq \frac{1}{3} \ln \left[\frac{C_o}{C_s} \right] \quad (7-70)$$

where the notation $\max\lambda(A_2)$ and $\max\lambda[M_{s2} M_2^{-1}]$ is used to designate the maximum characteristic values of the matrices A_2 and $M_{s2} M_2^{-1}$ respectively. The constant C_s is a positive constant such that the maximum value of the variable x_1 on the quadratic surface $\underline{x}' M_{s1} \underline{x} = C_s$ is 0.2. Also $C_o = V[\underline{x}(0)]$ and C_d is a positive constant less than one. The matrix M_{s2} is a real matrix which is at least positive semidefinite and satisfies the inequality

$$\underline{x}' M_{s2} \underline{x} \geq \underline{x}' M_{22}(\underline{x}) \underline{x} \quad (7-71)$$

Since the matrix $M_{22}(\underline{x})$ is a positive semidefinite matrix, then the positive definite matrix M_{s1} can be chosen as

$$M_{s1} = \begin{bmatrix} m_{11}^{-\delta} & m_{12} \\ m_{12} & 1 \end{bmatrix} \quad (7-72)$$

where δ is an arbitrarily small positive constant. Inequality (7-68) is therefore satisfied.

The initial value of the $V(\underline{x})$ function for $\underline{x}(0) = (1, 0)'$ is

$$V[\underline{x}(0)] = m_{11} \quad (7-73)$$

The region Ω is therefore defined as the interior of the bounding quadratic surface

$$\underline{x}' M_{s1} \underline{x} = m_{11} \quad (7-74)$$

The constraint on the elements of the function $V(\underline{x})$ which is required for the satisfaction of inequality (7-71) is obtained by substituting Eq. (7-64) into inequality (7-71). The constraint is

$$\frac{m_{11}(1+2m_{12}u_o)u_2}{2(m_{11}-m_{12}^2)} + \frac{m_{11}^2 m_{12}(u_2^2 - .0025)}{3(m_{11}-m_{12}^2)^2} \geq 0 \quad (7-75)$$

After performing the necessary algebra, the constraint required for the satisfaction of inequality (7-70) is found to be

$$-\max\lambda(A_2) \frac{1 - C_d}{1 + \max\lambda [M_{s2} \ M_2^{-1}]} \geq \frac{1}{3} \ln 5 \left[\frac{m_{11}}{m_{11} - m_{12}^2} \right]^{1/2} \quad (7-76)$$

where $\max\lambda [M_{s2} \ M_2^{-1}] =$

$$\frac{m_{11}(1+2m_{12}u_o)u_2}{2(m_{11}-m_{12}^2)^2} + \frac{m_{11}^2 m_{12}(u_2^2 - .0025)}{3(m_{11} - m_{12}^2)^3} \quad (7-77)$$

In the next step of the design method the constraints which the elements of the function $\dot{V}(\underline{x})$ must satisfy are formulated.

7.4.7 Step (6): Formulation of the Constraints on the
Elements of the Time Derivative of the
Liapunov Function

As discussed in section 6.3.5 the elements of the function $\dot{V}(\underline{x})$ are constrained to satisfy the following inequality in the region

$$-C_d \underline{x}' D_{22} \underline{x} - \underline{x}' D_{22}(\underline{x}) \underline{x} \geq 0 \quad (7-78)$$

where C_d is a positive constant less than 1.

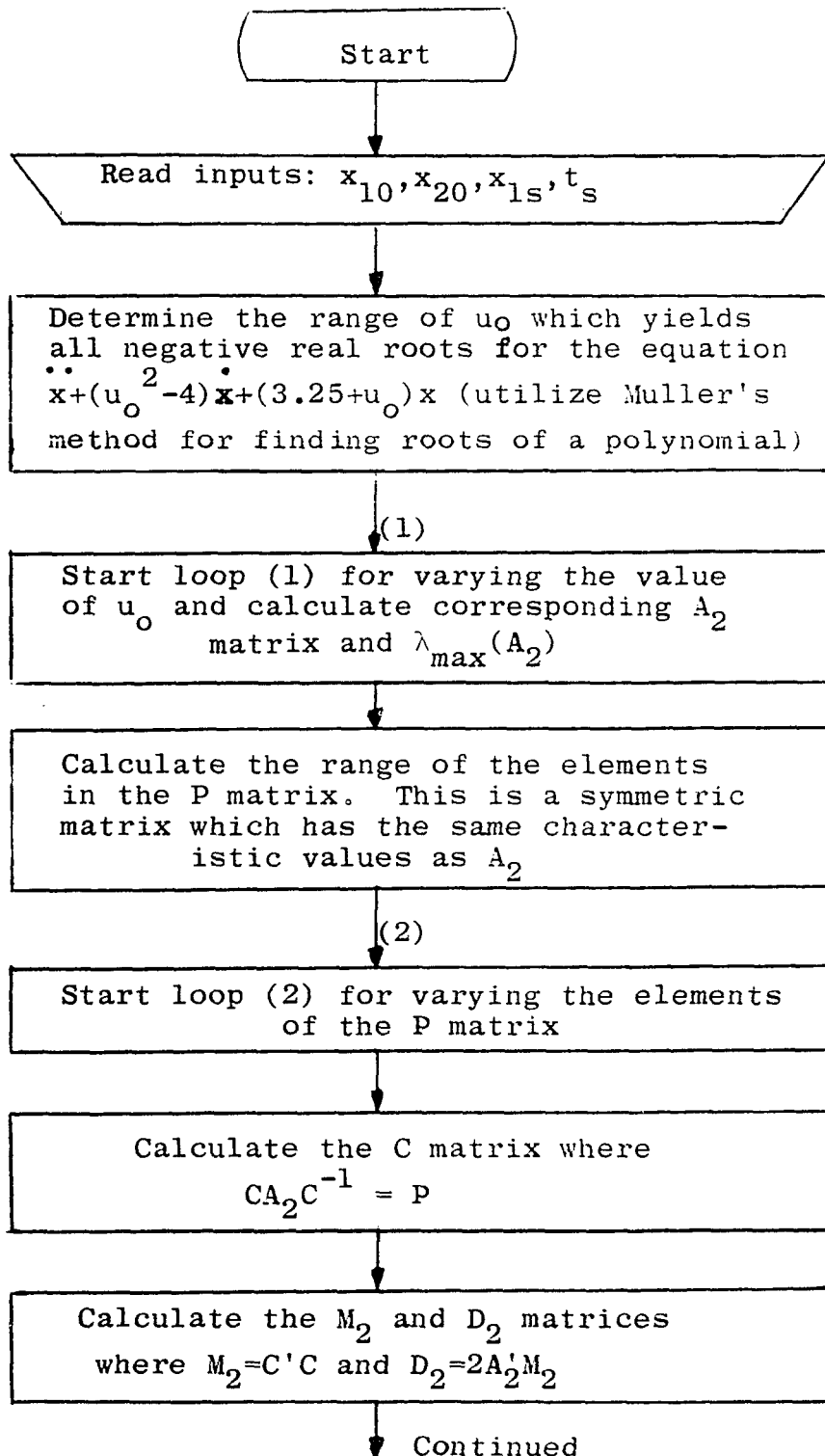
Since $D_{22}(\underline{x})$ has been formulated to be a negative semidefinite matrix, the constant C_d can be chosen equal to zero. Hence inequality (7-78) does not introduce any new constraints for the design method.

In the next section the logic flow diagram for the digital computer program employed to solve the constraints (7-72), (7-75) and (7-76) is presented and numerical result are discussed.

7.4.8 Step (7): Application of a Digital Computer Program
for Solving the Design Constraints

A logic flow diagram for the digital computer program utilized for solving the design constraints is now presented.

LOGIC FLOW DIAGRAM OF COMPUTER PROGRAM



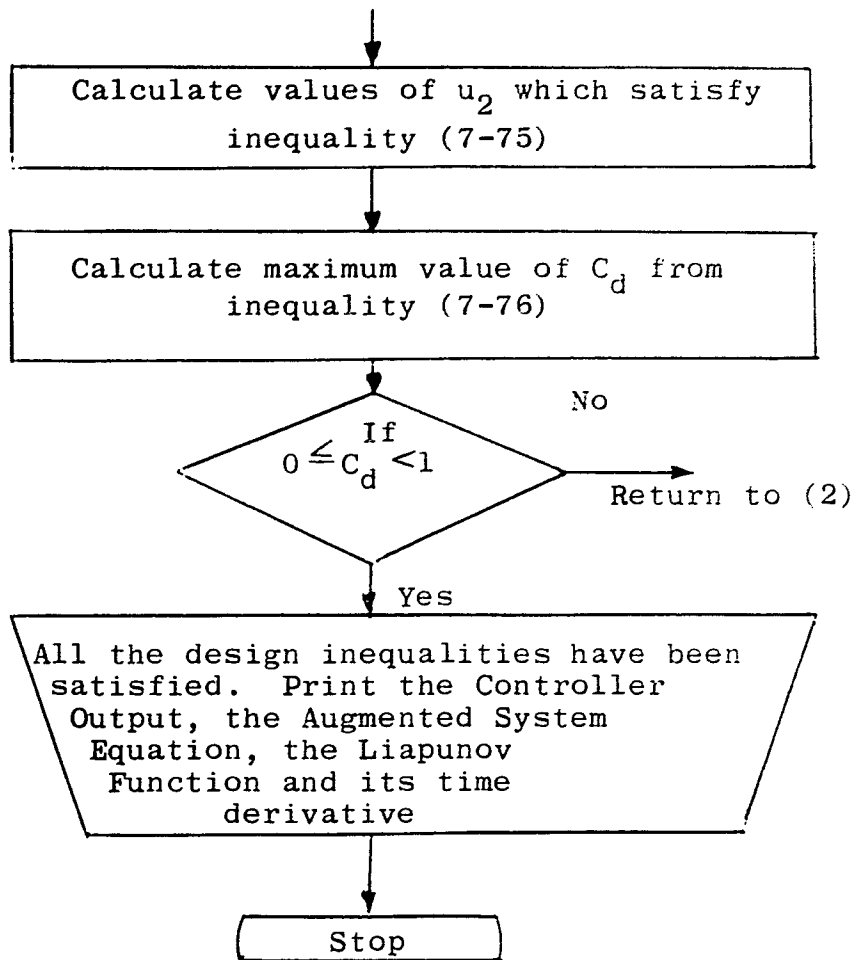


FIGURE 7.5

A set of numerical results obtained from the program is given below

The Controller Output

$$u(\underline{x}) = 3.15 - x + 0.05x^2 \quad (7-79)$$

The Augmented System Equation

$$\ddot{x} + (5.92 + 0.32x^2)\dot{x} + 6.4x + 0.05x^3 \quad (7-80)$$

The Liapunov Function and its time derivative are

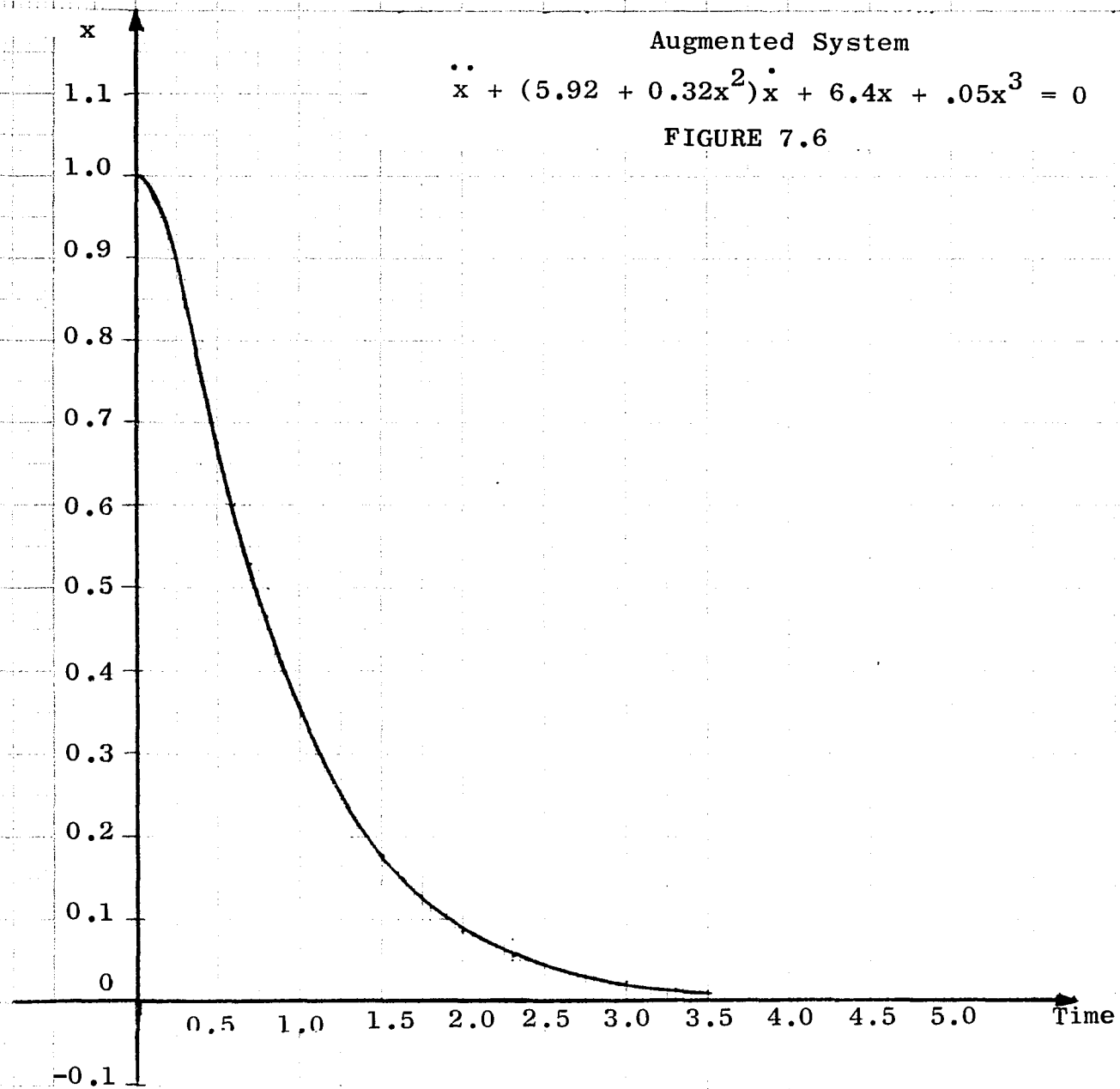
$$V(\underline{x}) = 8.86x^2 + \dot{x}^2 + 5.15x\dot{x} + 0.43x^4 \quad (7-81)$$

and

$$\begin{aligned} \dot{V}(\underline{x}) = & -32.98x^2 - 6.69\dot{x}^2 - 25.6x\dot{x} - 0.26x^4 \\ & -0.63x^2\dot{x}^2 \end{aligned} \quad (7-82)$$

The response of the augmented system due to the initial condition $x(0) = 1$ and $\dot{x}(0) = 0$ is given in Fig. 7.6. It is noted that the augmented system satisfies the settling time specification that $|x(t)| \leq 0.2$ for $t \geq 3$ seconds.

In conclusion, a controller output has been obtained so that the augmented system is globally asymptotically stable and satisfies the settling time specified. Specifically, if the u_2x^2 term were not included in the controller output, then Eqs. (7-64) and (7-65) would not yield non-quadratic portions of the Liapunov function and its time derivative which are positive and negative semidefinite respectively in the entire $\underline{x}$ space.



7.5 Summary

This chapter has presented and analyzed various aspects of the design method with the objective of gaining further understanding of the method. It is shown that the design method yields a controller output so that the augmented system will satisfy the settling time specification for a set of initial conditions. This eliminates the task of re-analyzing a nonlinear control system for each initial condition which may be of interest. It was also illustrated that the design method yields bounds for the magnitudes of the state variables. The fact was demonstrated that the effect of increasing the controller complexity upon the augmented system's response can in some case be ascertained. The chapter concluded with a design problem which illustrated that in some cases the design method can yield sufficient conditions for the augmented system to be globally asymptotically stable.

CHAPTER 8

SUMMARY AND CONCLUSIONS

8.1 Summary

This dissertation presents an investigation of the quantitative properties of Liapunov functions. A design method is developed which utilizes these quantitative properties. The design method is applicable to high order nonlinear autonomous control systems containing inherent nonlinearities. The nonlinearities may be of the multiplicative type and/or expressible as general polynomials of the system state variables. The nonlinearities may also be odd or even functions of the state variables and their projections may lie in any quadrant of the state space. The objective of the design method is to formulate a controller for a system so that the equilibrium solution of the system augmented with the controller will be asymptotically stable. It is also required that the output of the augmented system, denoted by $x(t)$, satisfy the specification that $|x(t)| \leq x_{1s}$ for $t \geq t_s$ for the initial condition $x(0) = x_{10}$, $\dot{x}(0) = x_{20}$, ..., $x^{(n-1)}(0) = x_{n0}$, where $\dot{x} = dx/dt$ and $x^{(n)} = d^n x/dt^n$.

For the purpose of developing the design method new theorems are proven in section 3.3 which present sufficient conditions for a surface to be closed. These

theorems consider a scalar function $V(\underline{x})$ which is a continuous function of $\underline{x}$ and has defined and continuous first partial derivatives with respect to $\underline{x}$ in an open region Ω . The region Ω is defined as the interior of the quadratic surface $\underline{x}'M_{s1}\underline{x} = C > 0$ where M_{s1} is a positive definite matrix. The function $V(\underline{x})$ is also assumed to be positive definite in Ω and its time derivative $\dot{V}(\underline{x})$ is assumed to be negative definite in this region. It is then proven that if $V(\underline{x}) > \underline{x}'M_{s1}\underline{x}$ in the interior of the quadratic surface, the open region Ω contains one and only one closed portion with respect to the origin of the surface defined by $V(\underline{x}) = C$.

A method for generating a class of Liapunov functions $V(\underline{x})$ which possess negative definite time derivatives is developed in chapter 4. The negative definiteness is a condition necessary for the application of the Liapunov functions to the design method presented in this dissertation. The method is a modification of Mekel's M-Matrix method⁽³²⁾ and is applicable to the class of nonlinear autonomous systems described by a vector differential equation of the form

$$\dot{\underline{x}} = A_0 \underline{x} + A_{on}(\underline{x}) \underline{x}$$

where $\underline{x}$ is a column vector of the system state variables and A_0 is a constant square matrix with only negative real

characteristic values and simple elementary divisors. The matrix $A_{on}(\underline{x})$ is a square matrix with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$. The vector $A_{on}(\underline{x})\underline{x}$ contains no linear terms. The subscripts o and n denote the order of the system and the number of nonlinearities present respectively. The Liapunov function and its time derivative are of the forms

$$V(\underline{x}) = \underline{x}' M_o \underline{x} + \underline{x}' M_{on}(\underline{x}) \underline{x}$$

and

$$\dot{V}(\underline{x}) = 2\underline{x}' M_o A_o \underline{x} + \underline{x}' D_{on}(\underline{x}) \underline{x}$$

where the matrices M_o and $M_o A_o$ are positive definite and negative definite matrices respectively. The matrices $M_{on}(\underline{x})$ and $D_{on}(\underline{x})$ are square with elements that are continuous functions of $\underline{x}$ and have continuous first partial derivatives with respect to $\underline{x}$.

Chapter 5 investigates the quantitative properties of a time domain ratio criterion $\eta(\underline{x})$ which is formulated from the ratio of a Liapunov function and its time derivative. (i.e. $\eta(\underline{x}) = -\dot{V}(\underline{x})/V(\underline{x})$). The result of this investigation is a set of design constraints. These constraints are sufficient conditions for the equilibrium

solution of a system to be asymptotically stable and for the output $x(t)$ to satisfy the specification that $|x(t)| \leq x_{1s}$ for $t \geq t_s$. The design constraints, for a modified M-Matrix Liapunov function, are summarized in the following manner.

In the open region Ω , defined as the interior of the quadratic surface $\underline{x}'M_{s1}\underline{x} = V[\underline{x}(t_0)]$ (where M_{s1} is a positive definite matrix and $V[\underline{x}(t_0)]$ is the scalar value of the $V(\underline{x})$ function at time t_0) the following conditions are satisfied:

- (i) $V(\underline{x})$ is positive definite
- (ii) $V(\underline{x})$ is a continuous function of $\underline{x}$
- (iii) $V(\underline{x})$ has defined and continuous first partial derivatives with respect to $\underline{x}$
- (iv) $\dot{V}(\underline{x})$ is negative definite
- (v) $V(\underline{x}) > \underline{x}'M_{s1}\underline{x}$

$$(vi) -2\max\lambda(A_0) \frac{1 - C_d}{1 - \max\lambda \begin{bmatrix} M_{s2} & M_o^{-1} \end{bmatrix}} \geq \frac{1}{t_s - t_0} \ln \left[\frac{C_o}{C_s} \right]$$

where $C_o = V[\underline{x}(t_0)]$ and C_s is a positive constant such that the maximum value of the state variable x_1 on the quadratic surface $\underline{x}'M_{s1}\underline{x} = C_s$ is x_{1s} . The constant C_d is positive and less than one. The matrix M_{s2} is at least positive semi-

definite and satisfies the inequality

$$\underline{x}' M_{s2} \underline{x} \geq \underline{x}' M_{on}(\underline{x}) \underline{x}$$

then the system's equilibrium solution is asymptotically stable and the output $x(t)$ satisfies the specification that $|x(t)| \leq x_{1s}$ for $t \geq t_s$ for the initial condition $x(0) = x_{10}$, $\dot{x}(0) = x_{20}, \dots, x^{(n-1)}(0) = x_{n0}$.

The design method is discussed in detail in chapter 6 and is illustrated numerically via an illustrative example. The digital computer is employed in the design method in order to simplify the solution of high order algebraic equations. It is the utilization of the digital computer which enables the designer to effectively apply the direct method of Liapunov for the design of higher order nonlinear systems.

The design method is further illustrated and analyzed in chapter 7. In this chapter it is demonstrated that although a controller is designed for a specific initial condition the augmented system's settling time is known for a set of initial conditions. It is also shown that the design method yields bounds for the magnitudes of the state variables. The chapter also analyzes the effect of increasing the controller complexity upon the response

of the system as considered by the illustrative problem of chapter 6. Chapter 7 includes a second order design problem for which the design method yielded an augmented system which is shown to be globally asymptotically stable. The results of chapters 6 and 7 are discussed in detail in the next section.

8.2 Discussion of Results

In chapter 6 the design method is illustrated in detail for a third order nonlinear system containing an inherent multiplicative nonlinearity. The unaugmented system is described by the differential equation

$$\dddot{y} + 15\ddot{y} + (2y-17)\dot{y} + 45y - 360 = 0.$$

A controller with the output $u(x) = 95.55 - 0.71x$ (where $y=x+8$) is formulated which results in an augmented system with an asymptotically stable equilibrium solution. The output of the augmented system, denoted by $x(t)$, also satisfies the specification that $|x(t)| \leq 0.2$ for $t \geq 10$ seconds. This result is obtained for the initial condition $x(0) = 2, \dot{x}(0) = \ddot{x}(0) = 0$. The Liapunov function and its time derivative for the augmented system are formulated as

$$V(\underline{x}) = 184.46x^2 + 86.84\dot{x}^2 + \ddot{x}^2 + 227.17x\dot{x} + 21.14x\ddot{x} + 17.45\ddot{x}\dot{x} + 0.17x\dot{x}^2$$

and

$$\begin{aligned}\dot{V}(\underline{x}) = & -951.52\dot{x}^2 - 724.91\ddot{x}^2 - 12.55\ddot{x}^2 - 1570\dot{x}\ddot{x} \\ & - 180\ddot{x}\ddot{x} - 176.01\dot{x}\ddot{x} - 3.56\ddot{x}^2\dot{x} - 2.93\ddot{x}\dot{x}^2 + 0.17\dot{x}^3\end{aligned}$$

These results are analyzed in chapter 7. It is shown that the output $x(t)$ satisfies the specification $|x(t)| \leq 0.2$ for $t \geq 10$ second for all initial conditions on the initial Liapunov surface

$$\begin{aligned}184.46\dot{x}^2 + 86.84\ddot{x}^2 + \ddot{x}^2 + 227.17\ddot{x}\dot{x} + 21.14\ddot{x}\ddot{x} \\ + 17.45\ddot{x}\ddot{x} + 0.17\ddot{x}\dot{x}^2 = 737.84\end{aligned}$$

It is also demonstrated that the design method yields bounds for the magnitudes of the state variables for initial conditions on the initial Liapunov surface. The bounds calculated were $|x(t)| \leq 5.3$, $|\dot{x}(t)| \leq 14.4$ and $|\ddot{x}(t)| \leq 1243$.

The design problem of chapter 6 is repeated for a controller output which contains a component that is proportional to the square of the system's output, or $u(x) = u_0 + u_1x + u_2x^2$. It is shown that the addition of the x^2 term in the controller output does not lead to a decreased settling time for all initial conditions on the initial Liapunov surface. This is illustrated by the result that for the initial condition $x(0) = -5$, $\dot{x}(0) = 8.28$, $\ddot{x}(0) = -15.64$ the augmented system with the

controller output $u(x) = 95.55 - 0.71x$ decays at a faster rate than the system with $u(x) = 95.55 - 0.71x - 0.01x^2$. This conclusion is not valid for all initial conditions. This is illustrated by the initial condition $x(0) = 5$, $\dot{x}(0) = -8.28$, $\ddot{x}(0) = 15.64$ for which the augmented system with the controller output $u(x) = 95.55 - 0.71x - 0.01x^2$ decays at a faster rate than the system with $u(x) = 95.55 - 0.71x$.

Chapter 7 concludes with a second order design problem for an unaugmented system which is described by the Lienard type equation

$$\ddot{y} + \left[(y-6)^2 - .0025(y-6)^4 - 4 \right] \dot{y} + 3.25(y-6) + (y-6)^2 = 0$$

A controller output $u(x) = 3.15 - x + 0.05x^2$ (where $y=x+6$) is formulated which results in an augmented system with an asymptotically stable equilibrium solution and an output $x(t)$ which satisfies the specification $|x(t)| \leq 0.2$ for $t \geq 3$ seconds. The Liapunov function and its time derivative for the augmented system are obtained as

$$V(\underline{x}) = 8.86x^2 + \dot{x}^2 + 5.15x\dot{x} + 0.43x^4$$

and

$$\begin{aligned} \dot{V}(\underline{x}) = & -32.98x^2 - 6.69\dot{x}^2 - 25.6x\dot{x} - 0.26x^4 \\ & - 0.63x^2\dot{x}^2 \end{aligned}$$

It is noted that $V(\underline{x})$ and $\dot{V}(\underline{x})$ are positive definite and negative definite respectively in the entire $\underline{x}$ space. The design method has therefore yielded sufficient conditions for the augmented system to be globally asymptotically stable.

8.3 Recommendations

This work presents a design method which utilizes the quantitative properties of Liapunov functions. It has been demonstrated that quantitative information about a system's time response can be obtained from the quantitative properties of a Liapunov function for the system. The following areas of research are logical extensions of this dissertation.

1. Extension of the concepts which correlate the time response of a system to the quantitative properties of a Liapunov function for the system.

2. Extension of the present design method to systems containing nonlinearities which do not possess continuous first partial derivatives with respect to the state variables.

3. Extension of the present design method to nonautonomous systems.

4. Development and/or modification of existing methods for generating Liapunov functions to aid in the solutions of recommendations 1,2, and 3.

APPENDIX A

M-MATRIX FORMULATION

This appendix contains a paper which was presented at the Asilomar Conference on Circuits and Systems at Monterey, California on November 2, 1967 by Dr. Ralph Mekel (33). The paper presents and illustrates a formulation for generating a class of Liapunov functions.

INTRODUCTION

Since the late 1950's there has been a continuing effort to develop techniques for generating Liapunov functions and a vast outflow of research and publications has resulted. Persual of the literature indicates that there still exists a large amount of experimentation necessary for the generation of Liapunov functions for high order nonlinear control systems (39).

The purpose of this paper is to develop a logical and systematic technique for generating a class of Liapunov functions for high order nonlinear control systems. The crux of this technique is the formulation of a variable M-matrix which has constant elements in the last column and last row and all other elements are subject to change depending upon the nonlinear system.

In this communication it is shown that a large proportion of the experimentation can be eliminated by establishing a direct and logical relation between the system matrix and the form of the variable M-matrix required for the generation of this class of Liapunov functions. The form of the M-matrix depends upon the order of the nonlinear system and the number of nonlinearities in the system. The generated Liapunov functions with their time derivatives are useful for synthesis of nonlinear controllers to stabilize high order unstable plants.

CONSTRUCTION OF LIAPUNOV FUNCTIONS

In order to introduce the technique for generating the class of Liapunov functions and their time derivatives, let the differential equation describing the high order autonomous nonlinear system be given by a set of n first order differential equations expressed in matrix form

$$\dot{\underline{x}} = \underline{A}_{on}(\underline{x})\underline{x} \quad (1)$$

where $\underline{x}$ denotes a column matrix of the state variables, $\dot{\underline{x}} = dx/dt$ and $\underline{A}_{on}(\underline{x})$ is a square matrix that characterizes the system. Subscripts

o and n denote the order of the system and the number of nonlinearities in the system respectively. Matrix $\underline{A}_{on}(x)$ consists of 0's, 1's, constants and functions of the state variables.

The Liapunov function is constructed as

$$V = \underline{x}' \cdot \underline{M}_{on}(x) \cdot \underline{x} \quad (2)$$

where $\underline{x}'$ is the transpose of $\underline{x}$ and $\underline{M}_{on}(x)$ is a symmetric matrix directly related to the system matrix $\underline{A}_{on}(x)$. The time derivative of the Liapunov function is formulated as

$$\dot{V} = \underline{x}'(\underline{A}'_{on}\underline{M}_{on} + \underline{M}_{on}\underline{A}_{on} + \dot{\underline{M}}_{on})\underline{x} = \underline{x}' \cdot \underline{D}_{on}(x) \cdot \underline{x} \quad (3)$$

where the prime denotes the transpose and the dot the time derivative. For the equilibrium solution to be stable, it is required according to Liapunov's criterion that V be positive definite ($V > 0$) and $\dot{V}$ be at least negative semidefinite ($\dot{V} \leq 0$). To comply with Liapunov's criterion for $\dot{V}$, the elements of the $\underline{D}_{on}(x)$ matrix, denoted by D_{ii} and D_{ij} , are constrained to satisfy the conditions

$$D_{ii} \leq 0 \quad (4)$$

and

$$D_{ij} + D_{ji} = 0 \quad (5)$$

where i and j denote row and column respectively. These conditions permit the evaluation of the unknown elements in the $\underline{D}_{on}(x)$ matrix. Note that by determining the elements of the $\underline{D}_{on}(x)$ matrix, we determine the elements of the $\underline{M}_{on}(x)$ matrix at the same time. Substitution of the elements of $\underline{M}_{on}(x)$ and $\underline{D}_{on}(x)$ into Eqs. (2) and (3) yield the Liapunov function V and its time derivative $\dot{V}$. The crux of this development is the formulation of the $\underline{M}_{on}(x)$ matrix required for the generation of this class of V -functions. Liapunov functions and their time derivatives for seventeen types of nonlinear systems with multiple nonlinearities are given in Appendices I and II.

An example that follows illustrates the formulation of the $\underline{M}_{on}(x)$ matrix required for the generation of this class of Liapunov functions.

EXAMPLE

Consider a fourth order nonlinear control system with one nonlinearity described by the differential equation

$$\ddot{x}^{(4)} + a_4 \dot{x}^{(3)} + a_3 \ddot{x}^{(2)} + a_2 \dot{x} + f(x) = 0 \quad (6)$$

where $x^{(n)} = d^n x / dt^n$ and $\dot{x} = dx / dt$. In terms of state variables, this fourth order equation is equivalent to a set of four first order differential equations. Let $x = x_1$, then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -f(x_1) - a_2 x_2 - a_3 x_3 - a_4 x_4 \end{aligned} \quad (7)$$

Expressed in matrix form, Eq. (7) becomes

$$\dot{\underline{x}} = \underline{A}_{41}(\underline{x}) \cdot \underline{x} \quad (8)$$

where $\underline{x}$ and $\dot{\underline{x}}$ are column matrices and $\underline{A}_{41}(\underline{x})$ is a square matrix that characterizes this fourth order system. Matrix $\underline{A}_{41}(\underline{x})$ is given by

$$\underline{A}_{41}(\underline{x}) = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -\frac{f(x_1)}{x_1} & -a_2 & -a_3 & -a_4 \\ \hline \end{array} \quad (9)$$

where a_2 , a_3 and a_4 are constants. Since the system matrix given by Eq. (9) contains one nonlinearity, a Liapunov function for this system is generated by using an $\underline{M}_{41}(\underline{x})$ matrix of the form

$$\underline{M}_{41}(\underline{x}) = \begin{array}{|c|c|c|c|} \hline \frac{F(x_1)}{x_1^2} & \frac{f_1(x_1)}{x_1} & \frac{f_2(x_1)}{x_1} & m_{14} \\ \hline \frac{f_1(x_1)}{x_1} & m_{22} - f'_2(x_1) & m_{23} & m_{24} \\ \hline \frac{f_2(x_1)}{x_1} & m_{23} & m_{33} & m_{34} \\ \hline m_{14} & m_{24} & m_{34} & m_{44} \\ \hline \end{array} \quad (10)$$

where the m_{ij} are constants, $f'_2(x_1) = df_2(x_1)/dx_1$ and the remaining elements are unknown functions of the state variable x_1 .

Next consider the modified fourth order nonlinear system described by the differential equation

$$x^{(4)} + a_4 x^{(3)} + a_3 x^{(2)} + g(\dot{x}) + f(x) = 0 \quad (11)$$

Note that this system now contains two nonlinearities. The nonlinear system given by Eq. (11) is characterized by the system matrix

$$A_{42}(x) = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -\frac{f(x_1)}{x_1} & -\frac{g(x_2)}{x_2} & -a_3 & -a_4 \\ \hline \end{array} \quad (12)$$

The $M_{42}(x)$ matrix for generating a Liapunov function for this modified system is

$$M_{42}(x) = \begin{array}{|c|c|c|c|} \hline \frac{F(x_1)}{x_1^2} & \frac{f_1(x_1)}{x_1} & \frac{f_2(x_1)}{x_1} & m_{14} \\ \hline \frac{f_1(x_1)}{x_1} & \frac{G(x_2)}{x_2^2} - f'_2(x_1) + m_{22}^* & \frac{g_1(x_2)}{x_2} + m_{23}^* & m_{24} \\ \hline \frac{f_2(x_1)}{x_1} & \frac{g_1(x_2)}{x_2} + m_{23}^* & m_{33} & m_{34} \\ \hline m_{14} & m_{24} & m_{34} & m_{44} \\ \hline \end{array} \quad (13)$$

where the star denotes a modified m_{ij} . The m_{ij} 's in Eq. (13) are constants and the remaining elements are unknown functions either of the state variable x_1 or x_2 . Note the change in the $M_{42}(x)$ matrix. As the constant term a_2 in matrix $A_{41}(x)$ is replaced by a nonlinear term $\frac{g(x_2)}{x_2}$

in matrix $A_{42}(x)$, then the m_{22} and m_{23} elements of matrix $M_{42}(x)$ are changed as shown in Eq. (13). Observe that the change occurs in the second row and second column since the additional nonlinearity is a function of x_2 . Note also that the last column and last row of matrices $M_{41}(x)$ and $M_{42}(x)$ must consist of constant elements to insure definiteness and closedness of the Liapunov functions.

Substitution of $A_{on}(x)$ and $M_{on}(x)$ matrices into Eq. (3) and applying conditions given by Eqs. (4) and (5) yield Liapunov functions and their time derivatives for the above fourth order systems. The resulting V and $\dot{V}$ functions are given in Appendix II.

SUMMARY AND CONCLUSIONS

A technique is presented for generating a class of Liapunov functions for high order autonomous nonlinear control systems. It is shown that there exists a direct and logical relation between the system matrix $A_{on}(x)$ and the $M_{on}(x)$ matrix required to generate this class of Liapunov functions.

Note the horizontal and vertical addition of nonlinear functions in matrix $\underline{M}_{42}(x)$, illustrated by the example, when the system given by Eq. (6) is modified to the system described by Eq. (11). Note also the inclusion of the partial derivative $f'_2(x_1)$ in matrices $\underline{M}_{41}(x)$ and $\underline{M}_{42}(x)$.

The class of Liapunov functions generated in this communication is characterized by the $\underline{M}_{on}(x)$ matrix which has constant elements in the last row and last column and all other elements are subject to change depending upon the number of nonlinearities present in the system. As the order of the system and the number of nonlinearities in the system change, the required modification in the form of the $\underline{M}_{on}(x)$ matrix is systematic, i.e. it follows a well defined pattern.

Liapunov functions and their time derivatives for third and fourth order systems with multiple nonlinearities are given in Appendices I and II. Note the strong and clearly marked similarity of these V and $\dot{V}$ functions.

In order to conserve space all details are omitted in Appendices I and II.

APPENDIX I

1a. System Representation

$$x^{(3)} + a_3 x^{(2)} + a_2 \dot{x} + f(x) = 0 \quad (x^{(n)} = d^{(n)}x/dt^{(n)})$$

1b. Liapunov Functions

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2f(x_1)x_2 + a_2 x_2^2 + (a_3 x_2 + x_3)^2$$

$$\dot{V} = -2 \left[a_3 a_2 - f'(x_1) \right] x_2^2$$

2a. System Representation

$$x^{(3)} + a_3 x^{(2)} + h(x) \dot{x} + a_1 x = 0$$

2b. Liapunov Functions

$$V = a_1 (\sqrt{a_3} x_1 + x_2 / \sqrt{a_3})^2 + \left[h(x_1) - a_1 / a_3 \right] x_2^2 + (a_3 x_2 + x_3)^2$$

$$\dot{V} = -2 \left[a_3 h(x_1) - a_1 - \frac{1}{2} h'(x_1) x_2 \right] x_2^2$$

3a. System Representation

$$x^{(3)} + a_3 x^{(2)} + g(\dot{x}) + a_1 x = 0$$

3b. Liapunov Functions

$$V = a_1 (\sqrt{a_3} x_1 + x_2 / \sqrt{a_3})^2 + 2 \int_0^{x_2} \left[g(x_2) / x_2 - a_1 / a_3 \right] x_2 dx_2 + (a_3 x_2 + x_3)^2$$

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - a_1 \right] x_2^2$$

4a. System Representation (41)

$$x^{(3)} + \gamma(x, \dot{x})x^{(2)} + a_2\dot{x} + a_1x = 0$$

4b. Liapunov Functions

$$V = a_1(\sqrt{a_3}x_1 + x_2/\sqrt{a_3})^2 + (a_2 - a_1/a_3)x_2^2 + (a_3x_2 + x_3)^2 + 2a_3 \int_0^{x_2} [\gamma(x_1, x_2) - a_3] x_2 dx_2$$

$$\dot{V} = -2(a_3a_2 - a_1)x_2^2 - 2 \left[\gamma(x_1, x_2) - a_3 \right] x_3^2 + 2a_3x_2 \int_0^{x_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} x_2 dx_2$$

5a. System Representation

$$x^{(3)} + a_3x^{(2)} + g(\dot{x}) + f(x) = 0$$

5b. Liapunov Functions

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2f(x_1)x_2 + 2 \int_0^{x_2} g(x_2) dx_2 + (a_3x_2 + x_3)^2$$

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - f'(x_1) \right] x_2^2$$

6a. System Representation

$$x^{(3)} + a_3x^{(2)} + g(x, \dot{x}) + f(x) = 0$$

6b. Liapunov Functions

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2f(x_1)x_2 + 2 \int_0^{x_2} g(x_1, x_2) dx_2 + (a_3x_2 + x_3)^2$$

$$\dot{V} = -2 \left[a_3 \frac{g(x_1, x_2)}{x_2} - f'(x_1) + \frac{1}{x_2} \int_0^{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} dx_2 \right] x_2^2$$

7a. System Representation

$$x^{(3)} + \gamma(x, \dot{x})x^{(2)} + a_2\dot{x} + f(x) = 0$$

7b. Liapunov Functions

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2f(x_1)x_2 + a_2x_2^2 + 2a_3 \int_0^{x_2} [\gamma(x_1, x_2) - a_3] x_2 dx_2 + (a_3x_2 + x_3)^2$$

$$\dot{V} = -2[a_3 a_2 - f'(x_1)]x_2^2 - 2[\gamma(x_1, x_2) - a_3]x_3^2 + 2a_3 x_2 \int_0^{x_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} x_2 dx_2$$

8a. System Representation

$$x^{(3)} + \gamma(x, \dot{x})x^{(2)} + g(\dot{x}) + a_1 x = 0$$

8b. Liapunov Functions

$$V = a_1(\sqrt{a_3}x_1 + x_3/\sqrt{a_3})^2 + (a_3x_2 + x_3)^2 + 2 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - \frac{a_1}{a_3} \right] x_2 dx_2 +$$

$$+ 2a_3 \int_0^{x_2} [\gamma(x_1, x_2) - a_3] x_2 dx_2$$

$$\dot{V} = -2 \left[a_3 \frac{g(x_2)}{x_2} - a_1 \right] x_2^2 - 2[\gamma(x_1, x_2) - a_3] x_3^2 + 2a_3 x_2 \int_0^{x_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} x_2 dx_2$$

9a. System Representation

$$x^{(3)} + \gamma(x, \dot{x})x^{(2)} + g(\dot{x}) + f(x) = 0$$

9b. Liapunov Functions

$$V = (a_3x_2 + x_3)^2 + \frac{1}{a_2} [a_2x_2 + f(x_1)]^2 + 2 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2 dx_2 +$$

$$+ 2a_3 \int_0^{x_2} [\gamma(x_1, x_2) - a_3] x_2 dx_2 + \frac{1}{a_2} [2a_3a_2F(x_1) - f^2(x_1)]$$

$$\dot{V} = -2 \left[a_3 \frac{g(x_2)}{x_2} - f'(x_1) \right] x_2^2 - 2[\gamma(x_1, x_2) - a_3] x_3^2 + 2a_3 x_2 \int_0^{x_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} x_2 dx_2$$

APPENDIX II

1a. System Representation (4)

$$x^{(4)} + a_4x^{(3)} + a_3x^{(2)} + a_2\dot{x} + f(x) = 0 \quad (x^{(n)} = d^{(n)}x/dt^{(n)})$$

1b. Liapunov Functions

$$V = (x_4 + a_4x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} \left[x_3 + a_4x_2 + \frac{a_4}{a_2}f(x_1) \right]^2 +$$

$$+ \frac{1}{a_4} [a_2 - a_4f'(x_1)] x_2^2 + \frac{1}{a_2} [2a_2\Delta F(x_1) - a_4f^2(x_1)]$$

$$\dot{V} = -2 \left[a_2\Delta - a_4f'(x_1) + \frac{1}{2}f''(x_1)x_2 \right] x_2^2$$

2a. System Representation

$$x^{(4)} + a_4 x^{(3)} + a_3 x^{(2)} + g(\dot{x}) + a_1 x = 0$$

2b. Liapunov Functions

$$\begin{aligned} V &= (x_4 + a_4 x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} \left(x_3 + a_4 x_2 + \frac{a_4 a_1}{a_2} x_1 \right)^2 + 2 \left[g(x_2) x_3 - a_2 x_2 x_3 \right] + \\ &\quad + \frac{1}{a_4} \left[a_2 \Delta - a_4 a_1 \right] \left[x_2^2 + \frac{a_4 a_1}{a_2} x_1^2 \right] + 2 a_4 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2 dx_2 \\ \dot{V} &= -2 \left[\frac{g(x_2)}{x_2} \Delta - a_4 a_1 \right] x_2^2 - 2 \left[a_2 - g'(x_2) \right] x_3^2 \end{aligned}$$

3a. System Representation

$$x^{(4)} + a_4 x^{(3)} + \gamma(x, \dot{x}) x^{(2)} + a_2 \dot{x} + a_1 x = 0$$

3b. Liapunov Functions

$$\begin{aligned} V &= \left[x_4 + a_4 x_3 + \frac{a_4 a_1}{a_2} x_2 \right]^2 + \frac{a_2}{a_4} \left[x_3 + a_4 x_2 + \frac{a_4 a_1}{a_2} \right]^2 + \\ &\quad + \frac{1}{a_4 a_2} \left[a_4 a_2 \gamma(x_1, x_2) - a_2^2 - a_4^2 a_1 \right] x_3^2 + \\ &\quad + \frac{2 a_1}{a_2^2} \left\{ \int_0^{x_2} \left[a_4 a_2 \gamma(x_1, x_2) - a_2^2 - a_4^2 a_1 \right] x_2 dx_2 \right\} \\ \dot{V} &= - \frac{2}{a_2} \left[a_4 a_2 \gamma(x_1, x_2) - a_2^2 - a_4^2 a_1 - \frac{1}{2} a_2 \dot{\gamma}(x_1, x_2) \right] x_3^2 + \frac{a_4 a_1 x_2}{a_2} \int_0^{x_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} x_2 dx_2 \end{aligned}$$

4a. System Representation

$$x^{(4)} + a_4 x^{(3)} + a_3 x^{(2)} + g(\dot{x}) + f(x) = 0$$

4b. Liapunov Functions

$$\begin{aligned} V &= (x_4 + a_4 x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} \left[x_3 + a_4 x_2 + \frac{a_4 f(x_1)}{a_2} \right]^2 + \frac{1}{a_4} \left[a_2 \Delta - a_4 f'(x_1) \right] x_2^2 + \\ &\quad + \frac{1}{a_2} \left[2 a_2 F(x_1) - a_4 f^2(x_1) \right] + 2 a_4 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2 dx_2 + 2 \left[g(x_2) x_3 - a_2 x_2 x_3 \right] \\ \dot{V} &= -2 \left[\frac{g(x_2)}{x_2} \Delta - a_4 f'(x_1) + \frac{1}{2} f''(x_1) x_2 \right] x_2^2 - 2 \left[a_2 - g'(x_2) \right] x_3^2 \end{aligned}$$

5a. System Representation

$$x(4) + a_4 x(3) + a_3 x(2) + g(x, \dot{x}) + f(x) = 0$$

5b. Liapunov Functions

$$\begin{aligned} V &= (x_4 + a_4 x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} \left[x_3 + a_4 x_2 + \frac{a_4 f(x_1)}{a_2} \right]^2 + \\ &+ \frac{1}{a_4} \left[a_2 \Delta - a_4 f'(x_1) \right] x_2^2 + \frac{1}{a_2} \left[2a_2 \Delta F(x_1) - a_4 f^2(x_2) \right] + \\ &+ 2a_4 \int_0^{x_2} \left[\frac{g(x_1, x_2)}{x_2} - a_2 \right] x_2 dx_2 + 2 \left[g(x_1, x_2) x_3 - a_2 x_2 x_3 \right] \\ \dot{V} &= -2 \left[\frac{g(x_1, x_2)}{x_2} \Delta - a_4 f'(x_1) + \frac{1}{2} f''(x_1) x_2 - \frac{1}{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} - \right. \\ &\quad \left. - \frac{a_4}{x_2} \int_0^{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} dx_2 \right] x_2^2 - \left[a_2 - \frac{g(x_1, x_2)}{x_2} \right] x_3^2 \end{aligned}$$

6a. System Representation

$$x(4) + a_4 x(3) + \gamma(x, \dot{x}) x(2) + a_2 \dot{x} + f(x) = 0$$

6b. Liapunov Functions

$$\begin{aligned} V &= (x_4 + a_4 x_3 + \frac{a_4 a_1}{a_2} x_2)^2 + \frac{a_2}{a_4} \left[x_3 + a_4 x_2 + \frac{a_4 f(x_1)}{a_2} \right]^2 + \\ &+ \frac{1}{a_4 a_2} \left[a_4 a_2 \gamma(x_1, x_2) - a_2^2 - a_4^2 a_1 \right] x_3^2 + \frac{1}{a_4 a_2} \left[2a_1 a_4^2 F(x_1) - a_4^2 f^2(x_1) \right] + \\ &+ \frac{2a_1}{a_2^2} \left\{ \int_0^{x_2} \left[a_4 a_2 \gamma(x_1, x_2) - \frac{a_2^2}{a_1} f'(x_1) - a_4^2 a_1 \right] x_2 dx_2 \right\} \\ \dot{V} &= -2a_4 \left[a_1 - f'(x_1) \right] x_2^2 - \frac{2}{a_2} \left[a_4 a_2 \gamma(x_1, x_2) - a_2^2 - a_4^2 a_1 - \frac{1}{2} a_2 \dot{\gamma}(x_1, x_2) \right] x_3^2 + \\ &+ 2x_2 \int_0^{x_2} \left[\frac{a_4 a_1}{a_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} - f''(x_1) \right] x_2 dx_2 \end{aligned}$$

7a. System Representation

$$x(4) + a_4 x(3) + \gamma(x, \dot{x}) x(2) + g(\dot{x}) + a_1 x = 0$$

7b. Liapunov Functions

$$V = (x_4 + a_4 x_3 + \frac{a_4 a_1}{a_2} x_2)^2 + \frac{a_2}{a_4} (x_3 + a_4 x_2 + \frac{a_4 a_1}{a_2} x_1)^2 +$$

$$\begin{aligned}
& + \frac{1}{a_4 a_2} \left[a_4 a_2 \psi(x_1, x_2) - a_2^2 - a_4^2 a_1 \right] x_3^2 + 2a_4 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2 dx_2 + \\
& + \frac{2a_1}{a_2^2} \int_0^{x_2} \left[a_4 a_2 \psi(x_1, x_2) - a_2^2 - a_4^2 a_1 \right] x_2 dx_2 + 2 \left[g(x_2) x_3 - a_2 x_2 x_3 \right] \\
V = & -2 \frac{a_4 a_1}{a_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2^2 + 2 \frac{a_4 a_1}{a_2} x_2 \int_0^{x_2} \frac{\partial \psi(x_1, x_2)}{\partial x_1} x_2 dx_2 - \\
& - \frac{2}{a_2} \left[a_4 a_2 \psi(x_1, x_2) - a_2 g'(x_2) - a_4^2 a_1 - \frac{1}{2} a_2 \dot{\psi}(x_1, x_2) \right] x_3^2
\end{aligned}$$

The meaning of some notations in Appendices I and II are given below.

$$F(x_1) = \int_0^{x_1} f(x_1) dx_1, \quad G(x_2) = \int_0^{x_2} g(x_2) dx_2, \quad G(x_1, x_2) = \int_0^{x_2} g(x_1, x_2) dx_2$$

$$f'(x_1) = df(x_1)/dx_1, \quad h'(x_1) = dh(x_1)/dx_1, \quad f''(x_1) = d^2f(x_1)/dx_1^2$$

$$\dot{\psi}(x_1, x_2) = \left[\partial \psi(x_1, x_2) / \partial x_1 \right] x_2 + \left[\partial \psi(x_1, x_2) / \partial x_2 \right] x_3, \quad g'(x_2) = dg(x_2)/dx_2$$

$$\Delta = a_3 - a_2/a_4$$

A fourth order system with three nonlinearities is given in ref. 29.

APPENDIX B

QUADRATIC SURFACE THEOREMS

This appendix presents two theorems relating to the properties of quadratic surfaces.

Theorem 3.5

The quadratic surfaces $\underline{x}'M\underline{x}=C>0$ are closed with respect to the origin if and only if M is a positive definite matrix.

Proof:

Sufficient Condition: If the function $\underline{x}'M\underline{x}$ is positive definite then $\underline{x}'M\underline{x} = 0$ only at the origin $\underline{x} = 0$ and $\underline{x}'M\underline{x} \Rightarrow \infty$ as $|\underline{x}| \Rightarrow \infty$. Along any path from the origin to infinity, the function $\underline{x}'M\underline{x}$ varies continuously from the value zero to infinity. Hence at some point along the path, $\underline{x}'M\underline{x}=C$ (where C is an arbitrary positive constant). Therefore, $\underline{x}'M\underline{x}=C$ represents a closed surface.

Necessary Condition: If $\underline{x}'M\underline{x}$ is not a positive definite function, then there is some point in the $\underline{x}$ space for which $\underline{x}'M\underline{x} \leq 0$. Designating this point as $\underline{x}_N = [x_{10}, \lambda_1 x_{10}, \dots, \lambda_n x_{10}]'$ the result is obtained that

$$x_{10}^2 \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}' \begin{bmatrix} & \\ & \\ & \\ & \\ M & \\ & \\ & \\ & \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \leq 0 \quad (T-1)$$

It is noted that inequality (T-1) is now satisfied for any value of $0 \leq x_{10} < \infty$ and is therefore satisfied along a ray in the $\underline{x}$ space. However, since the surface $\underline{x}'M\underline{x}=C$ is closed, it does not intersect any ray along which $\underline{x}'M\underline{x} \leq 0$ and hence there is no such ray. The M matrix must therefore be positive definite.■

Theorem 3.6

If the quadratic function of Theorem 3.5 is positive definite then the closed surfaces $\underline{x}'M\underline{x}=C>0$ are nested.

Proof:

Along any ray in the $\underline{x}$ space,

$$\underline{x} = x_1 \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \quad \text{where } 0 < x_1 < \infty \quad (\text{T-1})$$

The value of $\underline{x}'M\underline{x}$ along a ray is therefore

$$\underline{x}'M\underline{x} = x_1^2 \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \begin{bmatrix} & \\ & M \\ & \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \quad (\text{T-2})$$

As $|x_1|$ increases monotonically along any ray from the origin to infinity, the function $\underline{x}'M\underline{x}$ increases monotonically along the ray.■

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AUTOBIOGRAPHICAL STATEMENT

Patrick Peruo, Jr. was born in Brooklyn, New York on July 8, 1939. He received a B.E.E. (1961) from The Cooper Union, New York City and an M.E.E. (1964) from The City University of New York.

During the period 1961 to 1967 Mr. Peruo was employed by The Bendix Corporation, Teterboro, New Jersey. He has held the position of an Engineering Representative to the Army Missile Command (1962) and Senior Project Engineer at The Massachusetts Institute of Technology as a member of a development group (1964 - 1966). From 1967 to 1969 he was a Lecturer at The City College. He has since rejoined The Bendix Corporation.

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