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EXTREMA CONCERNING ASYMMETRIC GRAPHS

by

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This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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L. V. Q.

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1. Introduction. By a graph we mean a finite undirected graph (as defined in [1; p. 2]) without loops and without multiple edges. The automorphism group of a graph consists of those permutations of the vertex set of the graph which preserve adjacency relations (cf. [1; p. 239]). A graph is called asymmetric if its automorphism group consists only of the identity automorphism (cf. [2]). If a graph has a nonidentity automorphism group the graph is called symmetric. In this paper we study questions pertaining to the existence, structure, and enumeration of asymmetric graphs. Our first theorem establishes the extreme values of p and q for which there exist asymmetric graphs having p vertices and q edges.

Let a_n denote the number of asymmetric trees having n vertices. The numbers a_n have been determined by F. Harary and G. Prins [3; p. 155]. These numbers are here employed in the manner indicated below to assign to each integer p ($p \geq 8$) a pair of integers (N, w).

$$(1.1)\quad \begin{cases} \text{For each integer } p \ (p \geq 8)\text{, let } N \text{ and } w \text{ be defined by} \\ \sum_{n=1}^{N} a_n n \leq p < \sum_{n=1}^{N+1} a_n n, \quad \text{and} \\ p = \sum_{n=1}^{N} a_n n + w(N+1) + r \quad (0 \leq w < a_{N+1};\ 0 \leq r < N+1). \end{cases}$$

Theorem 1. If K is an asymmetric graph having p vertices and q edges, then

(i) $p = 1$ or $p \geq 6$, and

(ii) $m_p \leq q \leq M_p$, where

$$m_p = \begin{cases} 0 & \text{if } p = 1 \\ 6 & \text{if } p = 6, 7 \\ p - \sum_{n=1}^{N} a_n - w & \text{if } p \geq 8, \text{ and} \end{cases}$$

$$M_p = \begin{cases} 0 & \text{if } p = 1 \\ 9 & \text{if } p = 6 \\ 15 & \text{if } p = 7 \\ p(p-3)/2 + \sum_{n=1}^{N} a_n + w & \text{if } p \geq 8, \text{ where} \end{cases}$$

N and w are as defined in (1.1).

The bounds m_p and M_p are the best possible in the sense that for each p ($p = 1$, $p \geq 6$) there exist asymmetric graphs having p vertices and respectively the minimum m_p and the maximum M_p number of edges.

The preceding theorem is proven in the class of all graphs. We denote this class by G. We have also established the values p, m_p, and M_p for the following three classes of graphs: C = the class of connected graphs, G^t = the class of graphs having no vertices of degree 2, henceforth called topological graphs, and C^t = the class of connected topological graphs (cf. §2).

We next turn our attention to the structure of asymmetric graphs. We note that the order of the automorphism group of a graph can in some sense be considered a measure of the "symmetry" of the graph. This suggests the following question, due to P. Erdös and A. Rényi (cf. [2]): <u>how can one measure the "asymmetry" of an asymmetric graph</u>?

With respect to the above question we observe that the complete graph on p vertices, U_p, is symmetric, except when p = 1. Thus, it is clear that any asymmetric graph K ($K \neq U_1$) can be made symmetric, i.e., a symmetric graph can be obtained from K by adjoining a sufficient number of edges to K. It is also possible to make K symmetric by deleting edges from K. However, the number of edges which must be adjoined to an asymmetric graph in order to make it symmetric is in general not the same number of edges which must be deleted from the given graph in order to make it symmetric. Examples of this latter fact are given in §3. These observations suggest the following definitions for <u>measures of asymmetry</u> of a graph.

If K is a graph we define the <u>positive asymmetry</u> $A^+[K]$ of K as the least number of edges which when adjoined to K yields a symmetric graph, the <u>negative asymmetry</u> $A^-[K]$ of K as the least number of edges which when deleted from K yields a symmetric graph, and the <u>asymmetry</u> A[K] of K as the minimum of $\alpha + \delta$ where a symmetric graph can be obtained from K by adjoining α edges and deleting δ edges. In §3 we obtain upper bounds for the values $A^+[K]$, $A^-[K]$, and $A[K]$ where K ranges over the class of all graphs having p vertices and q edges. In particular, Theorem 5 of §3 sharpens a result, obtained by P. Erdős and A. Rényi, concerning the function A (cf. Theorem 3 [2; p. 311]).

In §4, for each class of graphs G, C, G^t, and C^t, we determine the least value $G(p,1)$ for which there exists a graph K with p vertices, $A[K] = 1$ ($A^+[K] = 1$, $A^-[K] = 1$), and $G(p,1)$ edges. Our theorem for the class G contains the solution to a problem posed by P. Erdős and A. Rényi (cf. [2; p. 312]). This result is an immediate consequence of our Theorem 1. Our theorem with respect to the class C^t is stated below.

Let $C^t(p,1)$ denote the least number of edges for which there exists an asymmetric graph K in C^t having p vertices, $A[K] = 1$, and $C^t(p,1)$ edges, where <u>we require that the symmetric graph obtained from K is a graph in</u> C^t.

<u>Theorem 9</u>. $C^t(p,1)$ <u>is undefined for</u> $p = 1, 2, \ldots, 6$, $C^t(7,1) = 11$, <u>and</u>

$$C^t(p,1) = \begin{cases} p + 2 & \text{for } p = 8 + 2n \quad (n = 0, 1, 2, \ldots) \\ p + 1 & \text{for } p = 9 + 2n \quad (n = 0, 1, 2, \ldots). \end{cases}$$

<u>The values of</u> $C^t(p,1)$ <u>with respect to the function</u>

<u>A^+ are the same as those with respect to the function A</u>.

<u>The values of</u> $C^t(p,1)$ <u>with respect to the function A^- are as follows</u>: $C^t(p,1)$ <u>is undefined for</u> $p = 1, 2, \ldots, 6$, $C^t(7,1) = 11$, $C^t(9,1) = 12$, <u>and</u> $C^t(p,1) = p + 2$ <u>for</u> $p = 8$ <u>and</u> $p \geq 10$.

In the process of proving these theorems we noted the particularly simple structure of the asymmetric topological graphs having 2 independent cycles. As a consequence of this we were able to enumerate these graphs (cf. §5). This result is a partial solution of one of the unsolved problems tabulated by F. Harary in [4; p. 188].

2. Existence theorems. Proof of Theorem 1. If $p = 1$, then $K = U_1$ and $m_1 = M_1 = 0$. In [2; p. 296] it is shown that, if a graph has 2, 3, 4, or 5 vertices, then it is a symmetric graph. It is further noted that the graph in Fig. 2.1 has 6 vertices and is asymmetric.

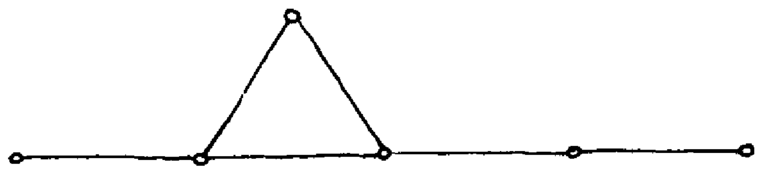

Fig. 2.1

In the proof of Theorem 4 [2; p. 311] it is shown that there exists an asymmetric tree having p vertices for each $p \geq 7$. This proves part (i) of the theorem.

We remark here that the minimum value 6 for the number of vertices of an asymmetric graph K ($K \neq U_1$) was first observed by I. N. Kagno (cf. [5; footnote p. 859]) and that the existence of asymmetric trees having p vertices for each $p \geq 7$ is due to R. Frucht (cf. [6; p. 241]).

We now seek the extrema m_p and M_p for each p ($p \geq 6$). If $p = 6$, we have, by referring to the graph in Fig. 2.1, that $m_6 \leq 6$. If a graph K has 6 vertices and 5 edges or less, then either K is a tree or K is not connected. Since a tree with 6 vertices is symmetric and a graph with 6 vertices which is not connected must contain a component which is symmetric or in the extreme case have no edges, we must have $m_6 = 6$. Since a graph and its complement have the same autômorphism group (cf. [5; p. 860]), the complement of the graph in Fig. 2.1 is asymmetric and consequently yields the maximum value $M_6 = (6 \cdot 5/2) - 6 = 9$. If $p = 7$, the values m_7 and M_7 are obtained by considering the asymmetric tree having 7 vertices (cf. Fig. 2.2) and the complement of this tree respectively. These graphs yield the values $m_7 = 6$ and $M_7 = (7 \cdot 6/2) - 6 = 15$.

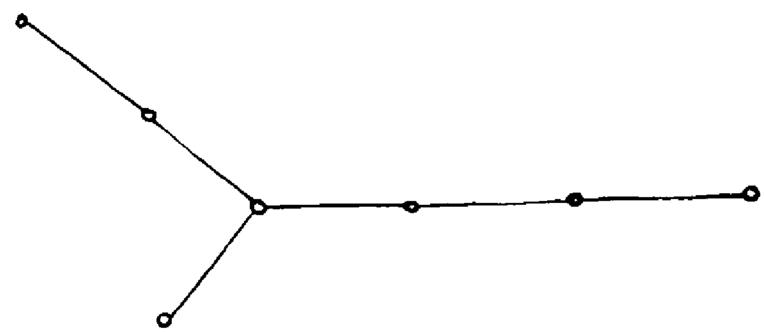

Fig. 2.2

If $p \geqq 8$ we claim that the minimum value m_p is equal to the number of edges of an asymmetric forest having p vertices and a minimal number of edges. This assertion is proven by showing that an asymmetric graph having p vertices cannot have less edges than a minimal edge asymmetric forest with p vertices.

We first show that a minimal edge asymmetric forest F_p having p vertices has a maximal number of components $c(F_p)$ and that for minimal edge asymmetric forests the number of components $c(F_p)$ is a monotone increasing function of p.

Consider the formula (2.1) for the cyclomatic number $N(K)$ of a graph K having $q(K)$ edges, p vertices, and $c(K)$ components:

$$N(K) = q(K) - p + c(K). \quad (2.1)$$

For a forest F we have $N(F) = 0$. Thus, for fixed p, $q(F) = p - c(F)$ is minimal if and only if $c(F)$ is maximal. Now, let F_p denote a minimal edge asymmetric forest having p vertices and $c(F_p)$ components and let v denote the number of vertices in a maximal vertex component of F_p. If this component is replaced by an asymmetric tree with $v + 1$ vertices the forest obtained is asymmetric, has $p + 1$ vertices, and $c(F_p)$ components. Therefore a minimal edge asymmetric forest having $p + 1$ vertices has at least $c(F_p)$ components. This shows that the number of components in a minimal edge asymmetric forest is a monotone increasing function of the number of vertices p. We are now in a position to show that an asymmetric graph having p vertices cannot have less edges than a minimal edge asymmetric forest having p vertices.

Let K denote an asymmetric graph having p vertices and let F_p denote a minimal edge asymmetric forest having p vertices. From the cyclomatic number formula (2.1) we have

$$N(K) = q(K) - p + c(K) \quad (2.2)$$

and

$$0 = q(F_p) - p + c(F_p). \quad (2.3)$$

Let $K = F \cup H$, where F consists of the tree components of K and H consists of the nontree components of K. Then,

(2.4) $\quad N(K) = N(H)$ and $c(K) = c(H) + c(F)$.

Combining (2.2), (2.3), and (2.4) we have:

(2.5) $\quad q(K) - q(F_p) = N(H) - c(H) + c(F_p) - c(F)$.

Since H has no tree components we have $N(H) - c(H) \geq 0$. Let v denote the number of vertices in the asymmetric forest F and let F_v denote a minimal edge asymmetric forest having v vertices. Then, $c(F) \leq c(F_v)$. Since $v \leq p$ and $c(F_p)$ is monotone increasing with respect to p for minimal edge asymmetric forests we have $c(F_v) \leq c(F_p)$. Thus, $c(F) \leq c(F_p)$. These observations together with (2.5) show that $q(K) - q(F_p) \geq 0$. This completes the proof of the fact that an asymmetric graph having p vertices $(p \geq 8)$ cannot have less edges than a minimal edge asymmetric forest with p vertices. We shall now construct minimal edge asymmetric forests for each p $(p \geq 8)$ and proceed to obtain the values for m_p and M_p.

Let the set of asymmetric trees be put into a 1-1 correspondence with the positive integers in such a way that, if T_r and T_s are asymmetric trees with r and s vertices respectively, then $r < s$ implies T_s follows T_r in the ordering of the asymmetric trees induced by the selected 1-1 correspondence. Note that the ordering within a set of trees having the same number of vertices is arbitrary.

Specifically, since the first few values of a_n are $a_1 = 1$, $a_i = 0$ $(i = 2, 3, \ldots, 6)$, $a_7 = 1$, $a_8 = 1$, and $a_9 = 3$, (cf. [3; p. 156]), the preceding remarks imply that, in the ordering of the asymmetric trees obtained, U_1 is first, the asymmetric trees having 7 and 8 vertices are second and third respectively, and the three asymmetric trees having 9 vertices are fourth, fifth, and sixth. Now, given p $(p \geq 8)$, let N and w be defined as in (1.1). Keeping in mind that

$$\sum_{n=1}^{N} a_n n = 1\cdot 1 + 1\cdot 7 + 1\cdot 8 + 3\cdot 9 + \cdots + a_N N$$

we shall consider the two cases, $w = 0$ and $w \neq 0$.

If $w = 0$, then

$$p = \sum_{n=1}^{N} a_n n - N + (N + r).$$

The forest consisting of all the asymmetric trees having no more than $N - 1$ vertices plus the first $a_N - 1$ asymmetric trees having N vertices plus an asymmetric tree having $N + r$ vertices is an asymmetric forest having a minimal number of edges with respect to p. In the case where $r = 0$, the asymmetric tree having $N + 0$ vertices to be chosen should be the a_N th asymmetric tree having N vertices, thereby insuring that this tree will be distinct from the already chosen $a_N - 1$ asymmetric trees having N vertices. The assertions concerning this forest are clear, since its components are distinct asymmetric trees and the construction maximizes the number of components, hence minimizes the number of edges.

If $w \neq 0$, then

$$p = \sum_{n=1}^{N} a_n n + (w - 1)(N + 1) + (N + 1 + r).$$

In this case the minimal forest we construct consists of all the asymmetric trees having no more than N vertices plus the first w - 1 asymmetric trees having N + 1 vertices plus an asymmetric tree having N + 1 + r vertices. As before we note that if r = 0, then the asymmetric tree having N + 1 + 0 vertices to be chosen should be distinct from the already chosen w - 1 asymmetric trees having N + 1 vertices. As in the case w = 0, this forest is asymmetric and has a minimal number of edges relative to p.

We now compute the number of edges in these minimal edge forests. If w = 0, then the forest constructed has

$$\sum_{n=1}^{N-1} a_n + (a_N - 1) + 1$$

components. If $w \neq 0$, then the forest constructed has

$$\sum_{n=1}^{N} a_n + (w - 1) + 1$$

components. Thus, in both cases these minimal forests have

$$\sum_{n=1}^{N} a_n + w$$

components. Applying the defining formula (2.1) for the cyclomatic number of a graph in the case where the graph is a forest having p vertices and $\sum_{n=1}^{N} a_n + w$ components we obtain the following value for m_p:

$$m_p = p - \sum_{n=1}^{N} a_n - w \quad (p \geq 8).$$

Since the complement of any such forest is an asymmetric graph with a maximal number of edges, we obtain the following value for M_p:

$$M_p = p(p - 1)/2 - m_p = p(p - 3)/2 + \sum_{n=1}^{N} a_n + w \quad (p \geqq 8).$$

This completes the proof of part (ii) of the theorem and since we have exhibited graphs having the extreme number of edges indicated in the statement of the theorem, the proof of Theorem 1 is completed.

Theorems 2, 3, and 4 below, establish, for each class of graphs C, C^t, and G^t, the extreme values of p and q for which there exist asymmetric graphs having p vertices and q edges.

Theorem 2. *If K is an asymmetric connected graph having p vertices and q edges, then*

(i) $p = 1$ *or* $p \geq 6$, *and*

(ii) $m_p \leq q \leq M_p$, *where*

$$m_p = \begin{cases} 0 & \text{if } p = 1 \\ 6 & \text{if } p = 6 \\ p - 1 & \text{if } p \geq 7, \text{ and} \end{cases}$$

$$M_p = \begin{cases} 0 & \text{if } p = 1 \\ 9 & \text{if } p = 6 \\ 15 & \text{if } p = 7 \\ p(p-3)/2 + \sum_{n=1}^{N} a_n + w & \text{if } p \geq 8, \text{ where} \end{cases}$$

N and w are as defined in (1.1).

The bounds m_p *and* M_p *are the best possible in the sense that for each p* ($p = 1$, $p \geq 6$) *there exist asymmetric connected graphs having p vertices and respectively the minimum* m_p *and the maximum* M_p *number of edges.*

Proof. If K is a connected graph having p vertices, then K has at least p - 1 edges. In the proof of Theorem 1 we noted the existence of asymmetric trees having p vertices for any $p \geq 7$. Since a tree having p vertices has p - 1 edges, we have $m_p = p - 1 \quad (p \geq 7)$.

The values m_1 and m_6 for connected graphs cannot be less than those obtained in Theorem 1 for arbitrary graphs. In particular U_1 and the connected graph in Fig. 2.1 yield the values $m_1 = M_1 = 0$ and $m_6 = 6$ for the present case.

We next note that the extreme values M_p for connected graphs cannot exceed those given for arbitrary graphs in Theorem 1. In particular we observe that the graphs given in the proof of Theorem 1 realizing the extreme values M_p $(p \geq 6)$ are all connected graphs. Specifically, if $p = 6$ or 7 we have as our maximal edge asymmetric connected graphs the complements of the graphs in Figures 2.1 and 2.2 respectively. This yields $M_6 = 9$ and $M_7 = 15$. For $p \geq 8$ each minimal edge asymmetric forest given in the proof of Theorem 1 contains the graph U_1 as a component. This implies that the maximal edge asymmetric graphs which are complements of these graphs are connected graphs. This completes the proof of Theorem 2.

Theorem 3. *If K is an asymmetric connected topological graph having p vertices and q edges, then*

(i) $p = 1$ *or* $p \geq 7$, *and*

(ii) $m_p \leq q \leq M_p$, *where*

$$m_p = \begin{cases} 0 & \text{if } p = 1 \\ 11 & \text{if } p = 7 \\ p + 2 & \text{if } p = 8 + 2k \ (k = 0, 1, 2, \ldots) \\ p + 1 & \text{if } p = 9 + 2k \ (k = 0, 1, 2, \ldots), \text{ and} \end{cases}$$

$$M_p = \begin{cases} 0 & \text{if } p = 1 \\ 15 & \text{if } p = 7 \\ p(p-3)/2 + \sum_{n=1}^{N} a_n + w & \text{if } p \geq 8, \text{ where} \end{cases}$$

N and w are as defined in (1.1).

The bounds m_p *and* M_p *are the best possible in the sense that for each p* ($p = 1$, $P \geq 7$) *there exist asymmetric connected topological graphs having p vertices and respectively the minimum* m_p *and the maximum* M_p *number of edges.*

Remark. Since the language and initial setting of [7] is different from that used in this paper and as we will be using some results contained in [7] we make the following observations. If K is a topological graph, ($K \neq U_1$), then K can be thought of as a 1-dimensional cell complex and the automorphism group of K as a graph is isomorphic to the homeotopy group of K as a complex (cf. [7; p. 353]). In particular a 1-dimensional cell complex with homeotopy group equal to zero can be thought of as an asymmetric topological graph (cf. Lemma [7; p. 354]). Finally, we note that the theorem on p. 353 of [7] which we will explicitly use is valid only for connected 1-dimensional cell complexes.

<u>Proof of Theorem 3</u>. It was proven in [7; Theorem p. 353] that, if a connected topological graph is asymmetric and not equal to U_1, then it must have at least 7 vertices and that for each p ($p \geqq 7$) there exist asymmetric connected topological graphs having p vertices. This proves part (i) of the theorem.

It was also shown in [7; Theorem p. 353] that an asymmetric connected topological graph which is not equal to U_1 must have cyclomatic number (called the nullity in [7]) greater than or equal to 2. This combined with part (i) yields $p + 1 \leqq m_p$ ($p \geqq 7$).

The graphs depicted in Fig. 2.3 are asymmetric connected topological graphs having $p = 9 + 2k$ vertices and $q = 10 + 2k$ edges ($k = 0, 1, 2, \ldots$). Thus, we have

$$m_p = p + 1 \quad \text{if } p = 9 + 2k \ (k = 0, 1, 2, \ldots).$$

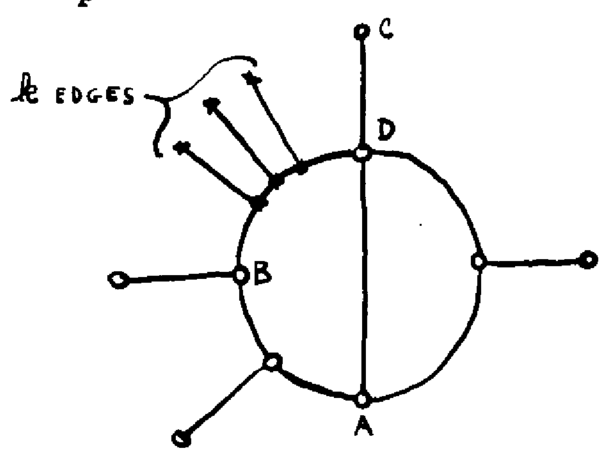

Fig. 2.3

In [7; p. 356] it is shown that every asymmetric connected topological graph having cyclomatic number equal to 2 is of the form:

$$[(u,v)_1(u,v)_2(u,v)_3(v,w)]$$

with W ($W \geq 3$) free edges (by a <u>free edge</u> (s,t) at the vertex s we mean an edge (s,t) such that the vertex t has degree 1) adjoined at isolated interior points of the parallel edges $(u,v)_i$ ($i = 1, 2, 3$) such that each of these three edges has a distinct number of free edges adjoined to it. Note that in this adjunction process we are introducing new vertices in contrast with the adjunction of edges discussed in the introduction. The number of vertices that each of these graphs has is of the form $p = 9 + 2k$ ($k = 0, 1, 2, \ldots$). Thus, if $p \neq 9 + 2k$ ($k = 0, 1, 2, \ldots$), then $p + 2 \leq m_p$ ($p \geq 7$).

The graphs depicted in Fig. 2.4 are asymmetric connected topological graphs having $p = 8 + 2k$ vertices and $q = 10 + 2k$ edges ($k = 0, 1, 2, \ldots$). Thus, we have

$$m_p = p + 2 \quad \text{if } p = 8 + 2k \ (k = 0, 1, 2, \ldots).$$

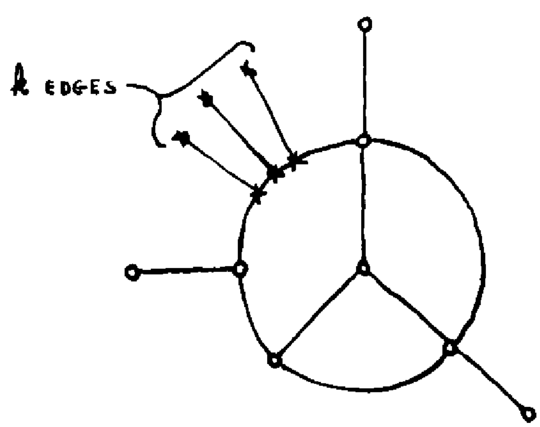

Fig. 2.4

To complete the proof of our assertion concerning m_p it remains to show that $m_7 = 11$. In [7; Theorem p. 353] it is shown that an asymmetric connected topological graph must have at least 10 edges. The graph in Fig. 2.5, which is an asymmetric connected topological graph having 7 vertices and 11 edges, together with the following lemma yields

$$m_7 = 11.$$

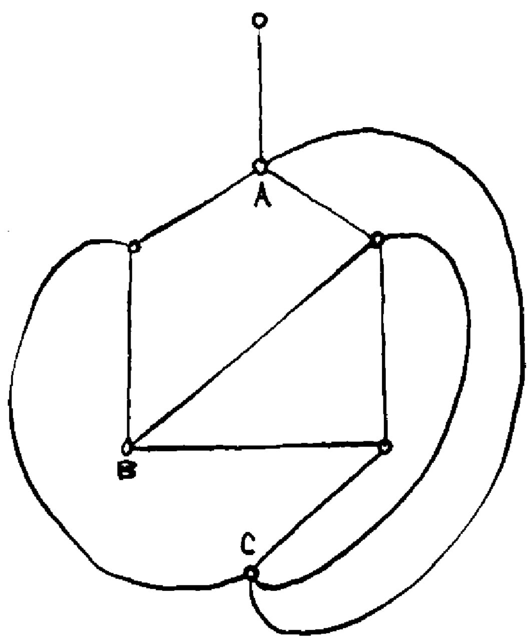

Fig. 2.5

Lemma 2.1. Every connected topological graph having 7 vertices and 10 edges is symmetric

Proof. Let K denote a connected topological graph having 7 vertices and 10 edges.

(a) K contains at least one free edge. This is seen by considering the following relation:

$$2q = \sum_{i=1}^{p} d(A_i) \qquad (2.6)$$

where q is the number of edges of a graph having p vertices and $d(A_i)$ is the degree of the vertex A_i (cf. [1; p. 7]). Applying (2.6) to K together with the assumption that K does not have any free edges yields the following contradiction:

$$2 \cdot 10 = \sum_{i=1}^{7} d(A_i) \geq 3 \cdot 7 .$$

Thus, K contains at least one free edge.

(b) If K contains exactly one free edge, then K is symmetric. Let K contain exactly one free edge (A, B), where d(A) = 1, and consider the graph K' defined as follows:

$$K' \equiv (K - \{(A, B)\}) \cup \{B\}.$$

Since (A, B) is the only free edge in K, every automorphism of K leaves (A, B) fixed. Thus, the automorphism group of K is isomorphic to the group of automorphisms of K' which leave B fixed or equivalently to the group of automorphisms of the complement $\overline{K'}$ (in U_6) of K' which leave B fixed. We shall consider $\overline{K'}$ rather than K' because $\overline{K'}$ has 6 vertices and 6 edges whereas K' has 6 vertices and 9 edges. Applying (2.6) we find that K does not have any vertex of degree greater than 4 and in fact contains at most one vertex of degree 4. This implies that $\overline{K'}$ is connected and contains at most one vertex of degree 1. We next note that K' contains at most one vertex of degree 2, the only possibility being the vertex B. This implies that $\overline{K'}$ contains at most one vertex of degree 3 (the vertex B which is to remain fixed when we consider automorphisms of $\overline{K'}$). Now, there are exactly thirteen connected graphs having 6 vertices and 6 edges (cf. Table 9

[8; p. 150]). Among these there are exactly four graphs which have at most one vertex of degree 3. We depict these candidates for the graph $\overline{K'}$ in Fig. 2.6. Note that these graphs need not be, and in fact are not, topological graphs.

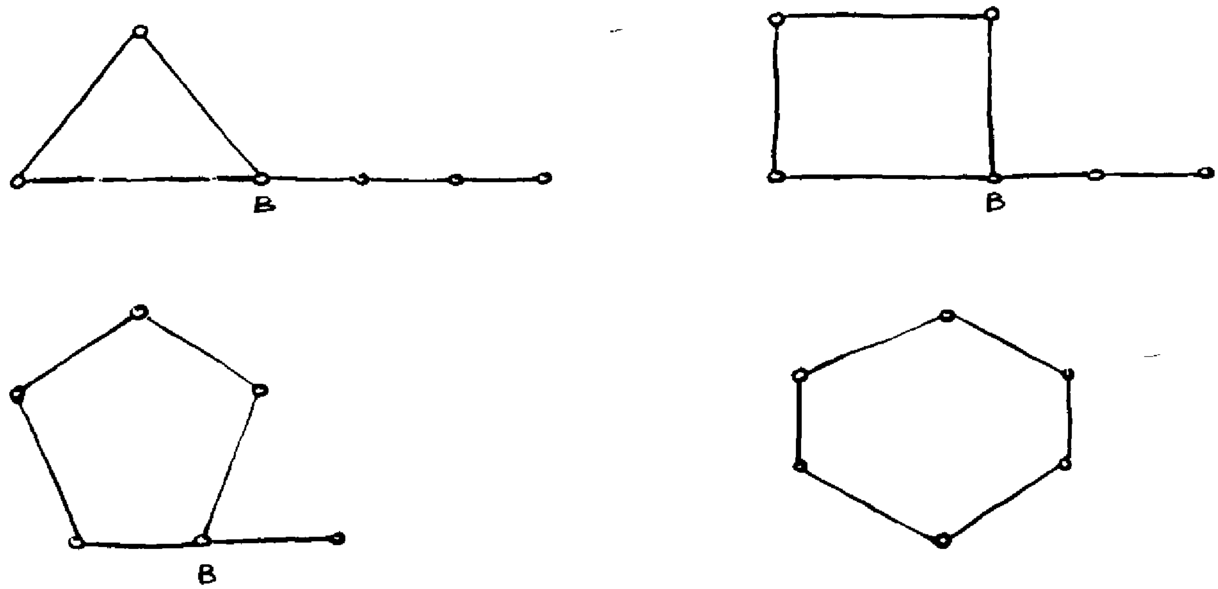

Fig. 2.6

The three possibilities for $\overline{K'}$ which have a vertex B of degree 3 each have nonidentity automorphisms which leave the vertex B fixed. The fourth graph in Fig. 2.6 has, corresponding to each vertex, a nonidentity automorphism which leaves that vertex fixed. Thus, we have shown that, if K has exactly one free edge, then K is symmetric.

(c) <u>If K contains exactly two free edges, then K is symmetric</u>. Let K contain exactly two free edges (A, B) and (C, D), where d(A) = d(C) = 1, and consider the graph K' defined as follows:

$$K' \equiv (K - \{(A, B),\ (C, D)\}) \bigcup \{B,\ D\} .$$

Since (A,B) and (C,D) are the only free edges of K, every automorphism of K either interchanges the edges (A,B) and (C,D) or leaves both of them fixed. Thus, the automorphism group of K is isomorphic to the group of automorphisms of K' which either interchange the vertices B and D or leave both B and D fixed. As before it will be more convenient to consider the complement $\overline{K'}$ of K'. In this case $\overline{K'}$ is one of two possibilities: either $\overline{K'}$ consists of two disjoint edges plus U_1 or a chain of length two plus two copies of U_1 (cf. Fig. 2.7). This follows from the fact that K' is a graph with 5 vertices and 8 edges, i.e., K' is U_5 with two edges deleted.

Fig. 2.7

We further note that in the case where $\overline{K'}$ is the graph with the chain of length two, one of the free edges of K, say (A,B), must have been adjoined at the vertex of degree 2. For if the were not the case, K would have that vertex as a vertex of degree 2. By referring to Fig. 2,7 we see that for any choice of the vertices B and D in the first graph there is a nonidentity automorphism of the graph which either interchanges B and D or leaves B and D fixed. For any choice of D in the second graph we see that there is a nonidentity automorphism of the graph which leaves both B and D fixed. Thus, we have shown that if K has exactly two free edges, then K is symmetric.

(d) K contains at most two free edges. If K contains n ($3 \leq n < 7$) free edges, then the graph K' obtained from K by removing the free edges, in the manner indicated in the preceding paragraphs, would have $7 - n$ vertices and $10 - n$ edges. But for $n \geq 3$ this would imply that K' had multiple edges, which it does not.

Combining (a), (b), (c), and (d) completes the proof of Lemma 2.1.

In order to prove the assertions concerning M_p we show that the asymmetric connected graphs ($p = 1$, $p \geq 7$) exhibited in proving the assertions concerning M_p in Theorem 2 are topological graphs. If $p = 1$ or $p = 7$, it is clear that the connected maximal edge asymmetric graphs in question are topological. Thus, let K denote a maximal edge asymmetric graph which is the complement of a minimal edge asymmetric forest F of the type we have defined in the proof of Theorem 1 and let K have p vertices ($p \geq 8$). Then, K has a vertex x of degree 2 if and only if F has a vertex of degree $p - 3$. This follows from the equation:

$$\deg_F(x) + 2 = p - 1.$$

We now recall that F has U_1 as a component and we note that the component C(x) of F which contains x, since it contains the closed star of x, $\overline{st}(x)$, must contain at least p - 2 vertices. This leaves exactly one vertex y of F unaccounted for. Since F is asymmetric, y cannot be an isolated vertex. Thus, y is in the component C(x). Since C(x) is a tree and

$$C(x) - st(y) = \overline{st}(x),$$

y cannot be adjacent to more than one vertex of C(x). It is clear that there is, up to isomorphism, only one graph that can be obtained in this way and that this graph is symmetric. On the other hand this graph is supposed to be the graph F which is asymmetric. Thus, we have arrived at a contradiction. Therefore, K cannot contain a vertex of degree 2 and this completes the proof of Theorem 3.

Theorem 4. *If K is an asymmetric topological graph having p vertices and q edges, then*

(i) $p = 1$ *or* $p \geq 7$, *and*

(ii) $m_p \leq q \leq M_p$, *where*

$$m_p = \begin{cases} 0 & \text{if } p = 1 \\ 11 & \text{if } p = 7 \\ 10 & \text{if } p = 8 \\ p + 1 & \text{if } p = 9 + 2k \ (k = 0, 1, 2, \ldots) \\ p & \text{if } p = 10 + 2k \ (k = 0, 1, 2, \ldots), \text{ and} \end{cases}$$

$$M_p = \begin{cases} 0 & \text{if } p = 1 \\ 15 & \text{if } p = 7 \\ p(p-3)/2 + \sum_{n=1}^{N} a_n + w & \text{if } p \geq 8, \text{ where} \end{cases}$$

N and w are as defined in (1.1).

The bounds m_p *and* M_p *are the best possible in the sense that for each p* ($p = 1$, $p \geq 7$) *there exist asymmetric topological graphs having p vertices and respectively the minimum* m_p *and the maximum* M_p *number of edges.*

Proof. Part (i) of Theorem 1 implies that an asymmetric topological graph K ($K \neq U_1$) must have at least 6 vertices. However, each asymmetric graph having 6 vertices has at least one vertex of degree 2. Thus, for asymmetric topological graphs (not necessarily connected) the possible number p of vertices is $p = 1$ or $p \geq 7$. That these values are realized follows from Theorem 3. Thus, we have proven part (i) of the theorem.

Let K denote an asymmetric topological graph having p vertices, q edges, and c components. Then, by (2.1) we have, $q = p + N(K) - c$. Every asymmetric connected topological graph, other than U_1, satisfies $N(K) \geq 2$ (cf. [7; Theorem p. 353] and recall the Remark that follows the statement of Theorem 3 of this paper), and

$$N(K) = \sum_{i=1}^{c} N(K_i),$$

where K_i is a component of K. Thus, $q \geq p + 2c - c$, if U_1 is not a component of K, and $q \geq p + 2(c - 1) - c$, if U_1 is a component of K. This in turn yields

$$(2.7)\quad \begin{cases} q \geq p + c \geq p + 1, \text{ if } U_1 \text{ is not a component of K, and} \\ q \geq p + c - 2 \geq p. \text{ if } U_1 \text{ is a component of K.} \end{cases}$$

Therefore, $p \leq m_p$ $(p \geq 7)$.

We have noted that there are no asymmetric topological graphs having 6 vertices. This implies that a minimal edge asymmetric topological graph having 7 vertices must be connected. Thus, by Theorem 3, we have

$$m_7 = 11.$$

This in turn implies that a disconnected asymmetric topolog-graph having 8 vertices must have at least 11 edges. However, applying Theorem 3, we see that the minimum $m_8 = 10$ is realized by a connected asymmetric topological graph.

We next note that the graphs depicted in Fig. 2.3 each combined with the graph U_1 are asymmetric topological graphs having $p = 10 + 2k$ vertices and $q = 10 + 2k$ edges

($k = 0, 1, 2, \ldots$). Thus,

$$m_p = p \quad \text{if } p = 10 + 2k \ (k = 0, 1, 2, \ldots).$$

In view of (2.7) and Theorem 3 the above values of p are the only values for which $m_p = p$. For the remaining values of p, we must have $p + 1 \leq m_p$, and by Theorem 3 this minimum is realized by the graphs depicted in Fig. 2.3, i.e.,

$$m_p = p + 1 \quad \text{if } p = 9 + 2k \quad (k = 0, 1, 2, \ldots).$$

With respect to the values of M_p, we note that the values given in Theorem 1 are absolute maximums for arbitrary asymmetric graphs. Furthermore, as was shown in proving Theorem 3, the graphs constructed in establishing these values are topological graphs. This completes the proof of Theorem 4.

3. Asymmetry bounds. Let $A^+[K]$, $A^-[K]$, and $A[K]$ denote the positive asymmetry, negative asymmetry, and asymmetry of a graph K as defined in the introduction.

Lemma 3.1. If K denotes a graph and $\bar{K}$ its complement, then (i) $A[K] = A[\bar{K}]$ (Erdös-Rényi), (ii) $A^+[K] = A^-[\bar{K}]$, (iii) $A^-[K] = A^+[\bar{K}]$, and (iv) $A[K] \leq \min\{A^+[K], A^-[K]\}$. These relations are valid in the class G of all graphs.

Proof. Statement (i) is Lemma 1 [2; p. 295]. For the proof of (ii), let $A^+[K] = \alpha$ and let K' denote a symmetric graph obtained from K by adjoining α edges to K. Since K' is symmetric, it follows that $\overline{K'}$ is symmetric. We now note that $\overline{K'}$ can be obtained from $\bar{K}$ by deleting the edges that were adjoined to K in order to obtain K'. Thus, $A^-[K] \leq \alpha$. Since α is the least number of edges which when adjoined to K will yield a symmetric graph, α must be the least number of edges which when deleted from $\bar{K}$ will yield a symmetric graph. Therefore, $A^-[\bar{K}] = \alpha$. This proves (ii). Using the fact that $\bar{\bar{K}} = K$, and part (ii), we obtain $A^-[K] = A^+[\bar{K}]$, thereby proving (iii). The proof of (iv) is clear, for if K can be made symmetric by either adjoining α edges or deleting δ edges, then $A[K] \leq \min\{\alpha, \delta\}$.

Remark. In order to illustrate the inequality of the functions A^+, A^-, and A we refer to the graph given in [2; Fig. 5 p. 297] depicted here in Fig. 3.1. This graph is asymmetric, can be made symmetric by deleting the edge (A, B), but cannot be made symmetric by adjoining one edge. Thus, in this case we have

$$A[K] = A^-[K] = 1 < A^+[K].$$

If we apply Lemma 3.1, we obtain

$$A[\bar{K}] = A^+[\bar{K}] = 1 < A^-[\bar{K}].$$

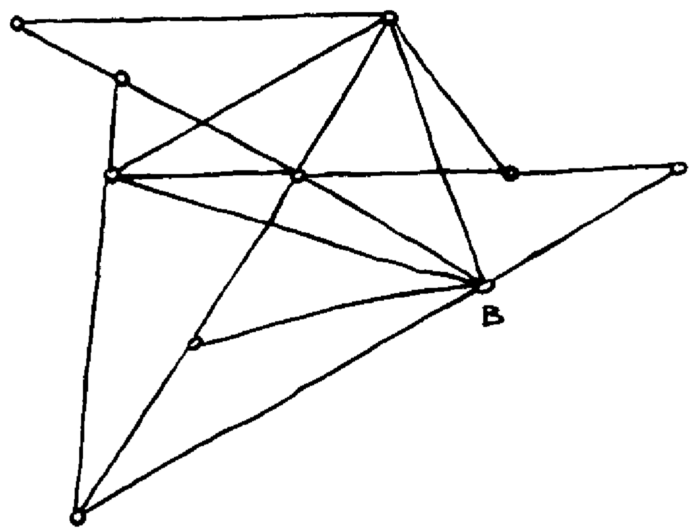

Fig. 3.1

Lemma 3.2. Let Δ_{jk} $(j \neq k)$ denote the number of vertices of a graph K which are either (1) adjacent to P_j and not adjacent to P_k or (2) adjacent to P_k and not adjacent to P_j, and let $\Delta_{jj} = 0$. Then,

(i) $A^-[K] \leq \min \Delta_{jk}$ $(j \neq k)$,

(ii) $A^+[K] \leq \min \bar{\Delta}_{jk}$ $(j \neq k)$, *where $\bar{\Delta}_{jk}$ is Δ_{jk} in $\bar{K}$, and*

(iii) $A[K] \leq \min \left\{ \min \Delta_{jk}\ (j \neq k),\ \min \bar{\Delta}_{jk}\ (j \neq k) \right\}$.

Proof. The proof of Theorem 1 [2; p. 299] explicitly yields $A^-[K] \leq \min \Delta_{jk}$ $(j \neq k)$, i.e., $A[K] \leq \min \Delta_{jk}$

$(j \neq k)$ is proven by deleting Δ_{jk} edges from K and thereby obtaining a symmetric graph. For the proof of (ii) we delete $\bar{\Delta}_{jk}$ edges from the complement of K. This yields, by (i), $A^-[\bar{K}] \leq \min \bar{\Delta}_{jk}$ $(j \neq k)$ and applying Lemma 3.1 (iii) we obtain $A^+[K] \leq \min \bar{\Delta}_{jk}$ $(j \neq k)$. Combining these facts with Lemma 3.1 (iv) we obtain (iii).

In the next two theorems we give upper bounds for the functions A^+, A^-, and A in the class G of all graphs.

Theorem 5. Let K denote an asymmetric graph $(\neq U_1)$ having p vertices and q edges, then

$$A^+[K] \leq \min\left\{M_p - q + 1,\ \bar{q} - m_p + 1\right\},$$

$$A^-[K] \leq \min\left\{M_p - \bar{q} + 1,\ q - m_p + 1\right\}, \text{ and}$$

$$A[K] \leq \min\left\{M_p - \max\{q, \bar{q}\} + 1,\ \min\{q, \bar{q}\} - m_p + 1,\ \frac{4q}{p}\left(1 - 2q/p(p - 1)\right)\right\},$$

where $\bar{q} = (p(p - 1)/2) - q$ and M_p and m_p are as defined in Theorem 1.

Proof. Clearly, $A^+[K] \leq M_p - q + 1$, $A^+[\bar{K}] \leq M_p - \bar{q} + 1$, $A^-[K] \leq q - m_p + 1$, and $A^-[\bar{K}] \leq \bar{q} - m_p + 1$. Combining this with (ii) and (iii) of Lemma 3.1 we obtain:

$$A^+[K] = A^-[\bar{K}] \leq \min\left\{M_p - q + 1,\ \bar{q} - m_p + 1\right\} \text{ and}$$

$$A^-[K] = A^+[\bar{K}] \leq \min\left\{q - m_p + 1,\ M_p - \bar{q} + 1\right\}.$$

This proves the first two assertions of the theorem.

Applying Lemma 3.1 (iv) we obtain:

$$A[K] \leq \min\left\{A^+[K],\ A^-[K]\right\}$$
$$\leq \min\left\{M_p - \max\{q,\ \bar{q}\} + 1,\ \min\{q,\ \bar{q}\} - m_p + 1\right\}.$$

Combining this with Theorem 3 [2; p. 311], which states that $A[K] \leq \frac{4q}{p}\left(1 - 2q/p(p - 1)\right)$ we obtain the last assertion of the theorem.

Remark. There are values of p and q for which the above theorem is an improvement of Theorem 3 [2; p. 311] for determining a bound for A[K]. For example let p = 15 and $q = m_{15} = 15 - 2 = 13$. The above theorem yields $A[K] \leq 1$, whereas Theorem 3 [2] yields

$$\frac{4\cdot 13}{15}\left(1 - 2\cdot 13/15\cdot 14\right) = \frac{52}{15}\left(\frac{92}{105}\right) = \frac{4784}{1575} > 3 \geq A[K].$$

In the following theorem we obtain upper bounds for the values $A^+[K]$ and $A^-[K]$ where K ranges over the class of all graphs having p vertices. In the case where a graph K is asymmetric and a symmetric graph cannot be obtained from K by the edge operations we are considering, then the corresponding functions A^+, A^-, or A shall assign to K the symbol ∞. For example, $A^+[U_1] = A^-[U_1] = A[U_1] = \infty$.

<u>Theorem 6</u>. <u>Let</u> $A^+(p)$ $(A^-(p), A(p))$ <u>denote the maximum of</u> $A^+[K]$ $(A^-[K], A[K])$ <u>where K ranges over all graphs having p vertices</u> $(p = 1, 2, 3, \ldots)$. <u>Then</u>,

(i) $A^+(1) = A^-(1) = A(1) = \infty$,

(ii) $A^+(p) = A^-(p) = A(p) = 0$ $(p = 2, 3, 4, 5)$,

(iii) $A^+(6) = A^-(6) = A(6) = 1$,

(iv) $A^+(p) \leqq [(p - 1)/2]$ $(p \geqq 7)$,

(v) $A^-(p) \leqq [(p - 1)/2]$ $(p \geqq 7)$, <u>and</u>

(vi) $A(p) \leqq [(p - 1)/2]$ $(p \geqq 7)$ (Erdös-Rényi),

<u>where [x] denotes the integral part of the real number x</u>.

<u>Proof</u>. We have already noted that, $A^+(1) = A^-(1) = A(1) = \infty$, $A^+(p) = A^-(p) = A(p) = 0$ when $p = 2, 3, 4, 5$, and $A^+(6) = A^-(6) = A(6) = 1$. Now, assume $p \geqq 7$. By Lemma 3.2 (i) we have $A^-[K] \leqq \min \Delta_{jk}$ $(j \neq k)$. Since $\min \Delta_{jk}$ $(j \neq k)$ cannot exceed the average taken over all Δ_{jk} we obtain:

$$\min \Delta_{jk} \leqq \frac{\sum_{j=1}^{p} \sum_{k=1}^{p} \Delta_{jk}}{p(p-1)} \quad (j \neq k).$$

Then as was shown in the proof of Theorem 1 [2] it follows:

$$A^-[K] \leqq [(p - 1)/2].$$

By Lemma 3.2 (ii), $A^+[K] \leqq \min \bar{\Delta}_{jk}$ $(j \neq k)$, where $\bar{\Delta}_{jk}$ is Δ_{jk} in $\bar{K}$. Thus, as above, we have

$$A^{+}[K] \leq \min \bar{\Delta}_{jk} \leq \frac{\sum_{j=1}^{p} \sum_{k=1}^{p} \bar{\Delta}_{jk}}{p(p-1)} \quad (j \neq k).$$

By (1.4) [2; p. 299], we have

$$\sum_{j=1}^{p} \sum_{k=1}^{p} \bar{\Delta}_{jk} = 2\sum_{i=1}^{p} \bar{v}_i(p - 1 - \bar{v}_i),$$

where $\bar{v}_i$ is the degree of the vertex P_i in $\bar{K}$. Using the fact that $\bar{v}_i = p - 1 - v_i$, where v_i is the degree of the vertex P_i in K, we have

$$\sum_{j=1}^{p} \sum_{k=1}^{p} \bar{\Delta}_{jk} = \sum_{j=1}^{p} \sum_{k=1}^{p} \Delta_{jk} .$$

Therefore,

$$A^{+}[K] \leq \frac{\sum_{j=1}^{p} \sum_{k=1}^{p} \Delta_{jk}}{p(p-1)} \leq [(p-1)/2].$$

The proof of the theorem is completed by noting that assertion (vi) is Theorem 1 [2; p. 298].

4. Graphs having asymmetry equal to 1. Let $G(p,1)$ $(C(p,1),\ C^t(p,1),\ G^t(p,1))$ denote the least integer for which there exists a graph K in G (C, C^t, G^t) having p vertices, asymmetry equal to 1, and $G(p,1)$ $(C(p,1),\ C^t(p,1),\ G^t(p,1))$ edges. If there are no graphs, in the particular class of graphs under consideration, having p vertices and asymmetry equal to 1, we express this fact by saying that $G(p,1)$ $(C(p,1),\ C^t(p,1),\ G^t(p,1))$ is undefined.

Theorem 7. $G(p,1)$ is undefined for $p = 1, 2, \ldots, 5$, $G(6,1) = G(7,1) = 6$, and

$$G(p,1) = p - \sum_{n=1}^{N} a_n - w \quad \text{for } p \geq 8, \text{ where}$$

N and w are as defined in (1.1).

The values of $G(p,1)$ with respect to the functions A^+ and A^- are the same as those with respect to the function A.

Proof. Theorem 1 (cf. §1) states that for a given p the least number of edges that an asymmetric graph can have is m_p. Thus, if K is a minimal edge asymmetric graph, then deleting one edge necessarily yields a symmetric graph. Specifically, for a minimal edge asymmetric graph K, $A^-[K] = 1$. Furthermore, if one considers the minimal edge asymmetric graphs constructed in the proof of Theorem 1 (cf. §2), it is seen that each of these graphs can be made symmetric by adjoining one edge. This completes the proof of the theorem.

Theorem 8. $C(p,1)$ is undefined for $p = 1, 2, \ldots, 5$, $C(6,1) = 6$, and $C(p,1) = p - 1$ for $p \geq 7$ (Erdős-Rényi).

The values of $C(p,1)$ with respect to the function A^{+} are the same as those with respect to the function A.

The values of $C(p,1)$ with respect to the function A^{-} are as follows: $C(p,1)$ is undefined for $p = 1, 2, \ldots, 5$ and $C(p,1) = p$ for $p \geq 6$.

Proof. The asymmetric trees depicted in Fig. 4.1 can each be made symmetric by adjoining the edge (A, B). Thus, with respect to the functions A and A^{+} we have

$$C(p,1) = p - 1 \quad (p \geq 7).$$

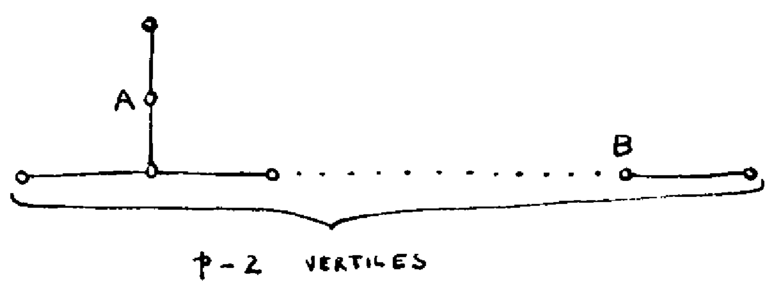

Fig. 4.1

Since a tree is disconnected if any one edge is deleted, we must have that, with respect to the function A^{-}, $C(p,1) \geq p$ $(p \geq 6)$. Consider now the graphs depicted Fig. 4.2. These graphs are asymmetric, have p vertices, p edges, and yield connected symmetric graphs when the edge (A, B) is deleted. Thus, with respect to the function A^{-}, we have

$$C(p,1) = p \quad (p \geq 6).$$

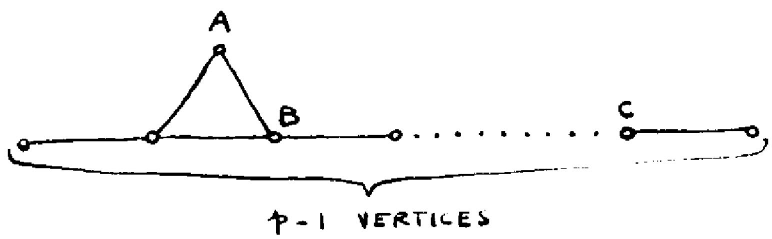

Fig. 4.2

For the case p = 6 and the function A^+, we note that a symmetric graph is obtained from the graph in Fig. 4.2 by adjoining the edge (A, C). This completes the proof of Theorem 8.

Proof of Theorem 9 (for statement of theorem see introduction). We first show that there exist minimal edge asymmetric connected topological graphs K such that $A^+[K] = 1$ and which have symmetrizations which are connected topological graphs.

The graph in Fig. 2.5 is asymmetric and has 7 vertices and 11 edges. If the edge (A, B) is adjoined to this graph we obtain a graph which has a nonidentity automorphism, namely the automorphism which interchanges the vertices B and C. Thus, $C^t(7,1) = 11$ with respect to the function A^+.

If $p = 8 + 2n$ $(n = 0, 1, 2, \ldots)$, we consider the graphs depicted in Fig. 4.3. These graphs are asymmetric and have $10 + 2n$ $(n = 0, 1, 2, \ldots)$ edges. If the edge (A, B) is adjoined to these graphs we clearly obtain a class of symmetric connected topological graphs. Thus, we have

$$C^t(p,1) = p + 2 \quad \text{if } p = 8 + 2n \ (n = 0, 1, 2, \ldots)$$

with respect to the function A^+.

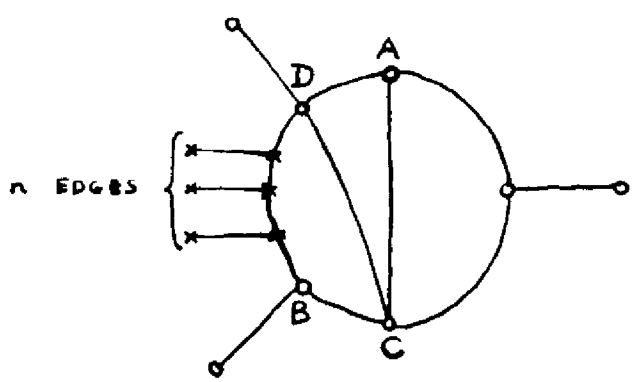

Fig. 4.3

If $p = 9 + 2n$ $(n = 0, 1, 2, \ldots)$, we consider the graphs depicted in Fig. 2.3 (with $k = n$). These graphs are asymmetric and have $10 + 2n$ $(n = 0, 1, 2, \ldots)$ edges. If the edge (A, B) is adjoined to these graphs we obtain a class of symmetric connected topological graphs. Thus, we have shown

$$C^t(p,1) = p + 1 \quad \text{if } p = 9 + 2n \ (n = 0, 1, 2, \ldots)$$

with respect to the function A^+.

For the graphs we have considered, $A[K] = A^+[K]$. Thereby proving our assertions concerning the functions A and A^+.

The deletion of edge (A, C) in the graph in Fig. 2.5 yields a connected topological graph having 7 vertices and 10 edges, which by Theorem 3 (cf. §2) must be a symmetric graph. Thus, $C^t(7,1) = 11$ with respect to the function A^-.

If $p = 8 + 2n$ $(n = 0, 1, 2, \ldots)$ we note that the deletion of edge (C, D) in the graphs depicted in Fig. 4.3 yields a class of symmetric connected topological graphs. Thus,

$$C^t(p,1) = p + 2 \quad \text{if } p = 8 + 2n \ (n = 0, 1, 2, \ldots)$$

with respect to the function A^-.

If $p = 9 + 2n$ $(n = 0, 1, 2, \ldots)$, we recall from the proof of Theorem 3 (cf. §2) the structure of asymmetric connected topological graphs and then observe that the deletion of any one edge from these graphs yields a graph which is either not connected or has a vertex of degree 2. Thus, with respect to the function A^-, we have

(4.1) $C^t(p,1) \geq p + 2$ if $p = 9 + 2n$ $(n = 0, 1, 2, \ldots)$.

We now consider the class of asymmetric graphs depicted in Fig. 4.4. These graphs have 11 + 2n vertices and 13 + 2n edges (n = 0, 1, 3, 4, ...) and can be made symmetric by deleting the edge (A,B).

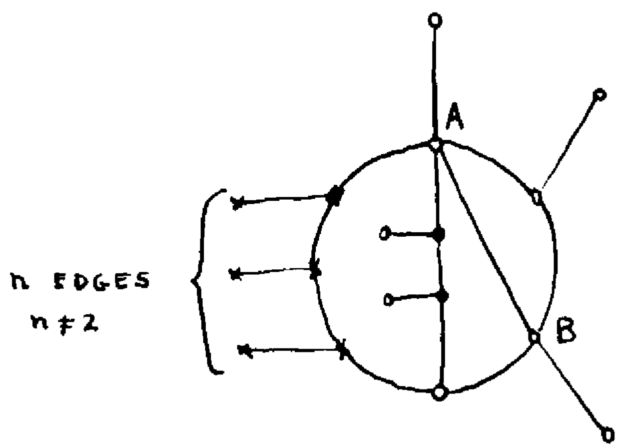

Fig. 4.4

For the case p = 15, q = 17 (the missing case n = 2, above), we consider the asymmetric graph in Fig. 4.5. Here we obtain a symmetric graph of the appropriate type when we delete the edge (A,B).

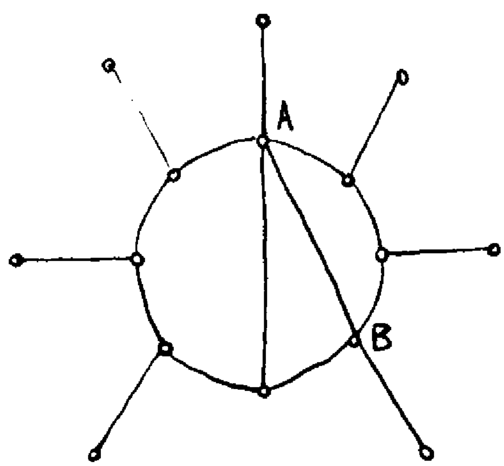

Fig. 4.5

Thus, we have shown that, with respect to the function A^-,

$$C^t(p,1) = p + 2 \quad \text{if } p = 11 + 2n \ (n = 0, 1, 2, \ldots).$$

We now consider the remaining case, $p = 9$. The graph in Fig. 4.6 is asymmetric, has 9 vertices, 12 edges, and can be made symmetric by deleting the edge (A, B).

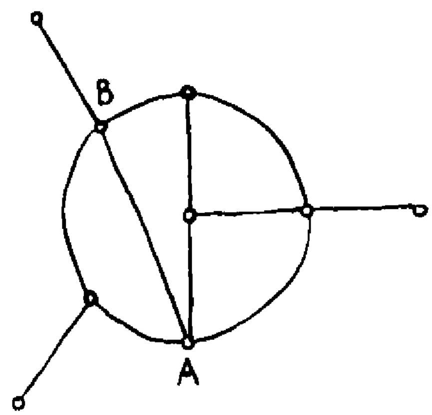

Fig. 4.6

From (4.1), we have $C^t(9,1) \geq 11$. Thus, the proof of the theorem will be complete when it is shown that there does not exist an asymmetric topological graph having 9 vertices and 11 edges from which a symmetric connected topological graph can be obtained by deleting one edge.

There are exactly three asymmetric connected topological graphs having 9 vertices and 11 edges. These are shown in Fig. 4.7. The deletion of any one edge from these graphs yields a graph which is either not connected or has a vertex of degree 2 or when the edge (A, B) is deleted in the middle graph we obtain an asymmetric graph. Thus, we have shown that, with respect to the function A^- we have

$C^t(9,1) = 12$. This completes the proof of Theorem 9.

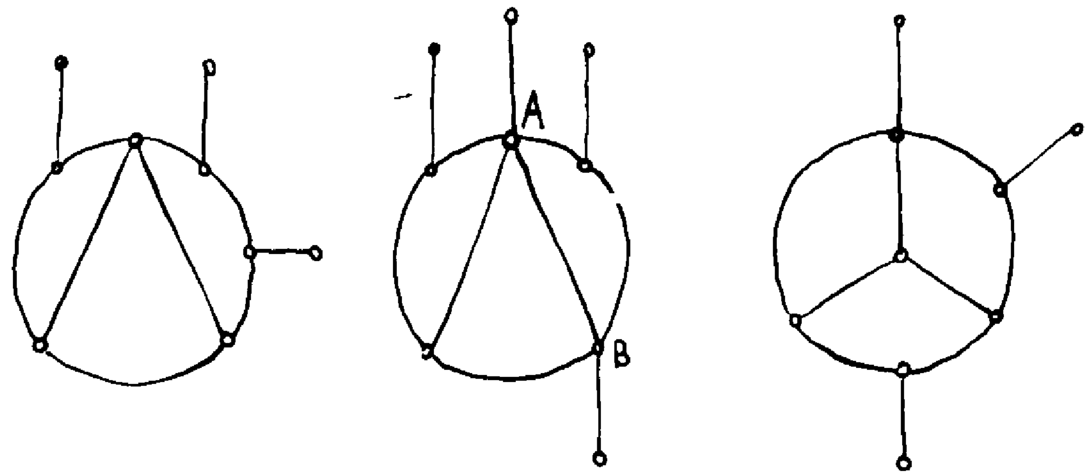

Fig. 4.7

<u>Theorem 10</u>. <u>$G^t(p,1)$ is undefined for</u> $p = 1, 2, \ldots, 6$,

$G^t(7,1) = 11$,

$G^t(8,1) = 10$, <u>and</u>

$$G^t(p,1) = \begin{cases} p + 1 & \text{for } p = 9 + 2n \quad (n = 0, 1, 2, \ldots) \\ p & \text{for } p = 10 + 2n \ (n = 0, 1, 2, \ldots). \end{cases}$$

<u>The values of $G^t(p,1)$ with respect to the functions A^+ and A^- are the same as those with respect to the function A.</u>

<u>Proof</u>. We shall show that there exist minimal edge asymmetric topological graphs K such that $A^+[K] = A^-[K] = 1$ and which have symmetrizations by one edge which are topological graphs.

From Theorem 4 (cf. §2), we have $G^t(7,1) \geq 11$, $G^t(8,1) \geq 10$, and $G^t(p,1) \geq p + 1$ if $p = 9 + 2n$ $(n = 0, 1, 2, \ldots)$. From Theorem 9, we have $C^t(7,1) = 11$ and $C^t(8,1) = 10$ with respect to the functions A^+ and A^-, and $C^t(p,1) = p + 1$ if $p = 9 + 2n$ $(n = 0, 1, 2, \ldots)$ with respect to the function A^+. Consider now the graphs depicted in Fig. 2.3. These graphs have $p = 9 + 2n$ $(n = 0, 1, 2, \ldots)$ vertices and $q = 10 + 2n$ $(n = 0, 1, 2, \ldots)$ edges. By deleting the edge (C,D) from these graphs we obtain a class of graphs which are symmetric and topological. Combining the observations of this paragraph proves the assertions of the theorem for the cases $p = 7, 8, 9 + 2n$ $(n = 0, 1, 2, \ldots)$.

If $p = 10 + 2n$ $(n = 0, 1, 2, \ldots)$, we consider the class of asymmetric graphs depicted in Fig. 4.8.

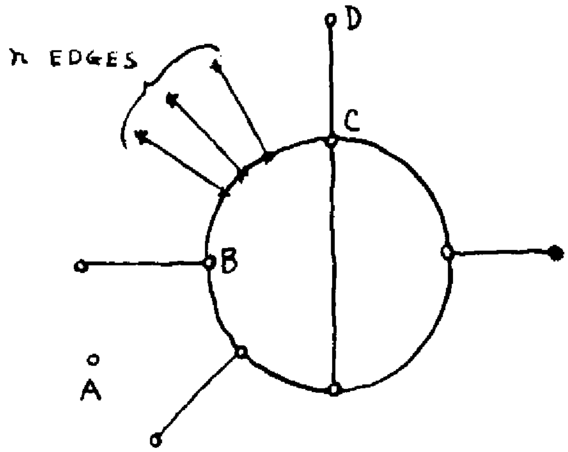

Fig. 4.8

These graphs have $10 + 2n$ $(n = 0, 1, 2, \ldots)$ edges and can be symmetrized by adjoining the edge (A, B) or by deleting the edge (C, D). In both cases the graph obtained is a topological graph. Thus,

$$G^t(p, 1) = p \quad \text{if } p = 10 + 2n \ (n = 0, 1, 2, \ldots).$$

This completes the proof of Theorem 10.

5. <u>Enumeration theorems</u>. For a recent survey of unsolved problems in the enumeration of graphs the reader is referred to F. Harary's article in [4; p. 185]. We note that in [4] topological graphs and asymmetric graphs, as defined in this paper, are called homeomorphically irreducible graphs and identity graphs respectively. In [3] F. Harary and G. Prins have enumerated topological trees and asymmetric trees. The enumeration of topological graphs having a positive number of independent cycles is an open problem as is the enumeration of asymmetric graphs (cf. [4; p. 188]). In Theorems 11 and 12 of this section we give an enumeration of asymmetric topological graphs having 2 independent cycles. This is the first case of interest in this direction, since there are no asymmetric topological trees nor asymmetric topological graphs having 1 independent cycle.

Theorem 11. *The number of asymmetric connected topological graphs having 2 independent cycles and p vertices is*

$$P_3(k) \quad \textit{if } p = 9 + 2k \quad (k = 0, 1, 2, \ldots), \textit{ and}$$

$$0 \qquad \textit{if } p \textit{ is otherwise,}$$

where $P_3(k)$ *denotes the number of partitions of k having no summand greater than 3.*

The number of asymmetric connected topological graphs having 2 independent cycles and no more than p vertices is

$$0 \quad \textit{if } p < 9,$$

$$1 \quad \textit{if } p = 9, 10, \textit{ and}$$

$$\frac{1}{72}\left[72 + N(2N^2 + 21N + 66)\right] + \begin{bmatrix} -1/8 \textit{ if } N \equiv 1 \pmod 2 \\ 0 \textit{ if } N \equiv 0 \pmod 2 \end{bmatrix} + \begin{bmatrix} -1/9 \textit{ if } N \equiv 1 \pmod 3 \\ -2/9 \textit{ if } N \equiv 2 \pmod 3 \\ 0 \textit{ if } N \equiv 0 \pmod 3 \end{bmatrix}$$

if $p = 9 + 2N$ *or* $p = 10 + 2N$ $(N = 1, 2, 3, \ldots)$.

Proof. In the proof of Theorem 3 §2 (where we refer to [7; p. 356]) we explicitly described the structure of asymmetric connected topological graphs having 2 independent cycles (cyclomatic number equal to 2). Let W be as defined in that description of these graphs. Then, it is clear that the total number of asymmetric connected topological graphs having 2 independent cycles corresponding to a given W is equal to the number of partitions of W into three unequal summands added to the number of partitions of W into two unequal summands. These graphs have $p = 3 + 2W$ ($W \geq 3$) vertices, i.e., $p \geq 9$ vertices.

The number of partitions of $n + \binom{k+1}{2}$ into k unequal summands is the same as the number of partitions of n having no summand greater than k (cf. [8; Theorem 3(b) p. 113]). Applying this to the case we are considering we have:

$$k = 3 \quad W = n + \binom{4}{2} = n + 6, \text{ and}$$

$$k = 2 \quad W = n + \binom{3}{2} = n + 3.$$

Thus, if $P_\beta(\alpha)$ denotes the number of partitions of α having no summand greater than β, we have that the total number $T(3 + 2W)$ of asymmetric connected topological graphs having 2 independent cycles and $p = 3 + 2W$ vertices is equal to:

$$P_3(W - 6) + P_2(W - 3) \quad (W \geq 3;\ P_\beta(\alpha) = 0 \text{ if } \alpha < 0).$$

If $W = k + 3$ ($k = 0, 1, 2, \ldots$), then

$$T(3 + 2W) = T(9 + 2k), \text{ and}$$

$$T(9 + 2k) = P_3(k - 3) + P_2(k) \quad (k = 0, 1, 2, \ldots).$$

Using the identity (cf. Theorem 10.1c [9; p. 201])

$$P_\beta(\alpha) = P_{\beta-1}(\alpha) + P_\beta(\alpha - \beta) \quad (1 < \beta \leq \alpha)$$

with $\alpha = k$ and $\beta = 3$ we obtain

$$P_3(k) = P_2(k) + P_3(k - 3) \quad (k \geq 3)$$

which yields

$$\text{(5.1)} \qquad T(9 + 2k) = P_3(k) \quad (k = 3, 4, 5, \ldots).$$

We further note that

$$\text{(5.2)} \qquad \begin{cases} P_3(0) = P_3(-3) + P_2(0) = 1, \\ P_3(1) = P_3(-2) + P_2(1) = 1, \text{ and} \\ P_3(2) = P_3(-1) + P_2(2) = 2. \end{cases}$$

Combining (5.1) and (5.2) we obtain

$$T(9 + 2k) = P_3(k) \quad (k = 0, 1, 2, \ldots).$$

This completes the proof of the first assertion of the theorem.

The total number $S(p)$ of asymmetric connected topological graphs having 2 independent cycles and no more than p vertices is given by

$$\text{(5.3)} \quad \begin{cases} S(p) = 0 & \text{if } p < 9, \text{ and} \\ S(p) = \sum_{k=0}^{N} P_3(k) & \text{if } p = 9 + 2N \text{ or } p = 10 + 2N. \end{cases}$$

In [10; p. xvi] the following formula for $P_3(k)$ is given

$$\text{(5.4)} \quad P_3(k) = \tfrac{1}{12}(k^2 + 6k + \tfrac{47}{6}) + (-1)^k/8 + \tfrac{1}{9}(\omega_3^k + \omega_3^{2k})$$

where $k = 1, 2, 3, \ldots$, and $\omega_3 = \exp(2\pi i/3)$. Thus, if $p = 9 + 2N$ or $p = 10 + 2N$ $(N = 1, 2, 3, \ldots)$, then

$$S(p) = 1 + \sum_{k=1}^{N} \left[\tfrac{1}{12}(k^2 + 6k + \tfrac{47}{6}) + (-1)^k/8 + \tfrac{1}{9}(\omega_3^k + \omega_3^{2k})\right].$$

Using the summation formulas:

$$\sum_{k=1}^{N} k = N(N+1)/2,$$

$$\sum_{k=1}^{N} k^2 = N(N+1)(2N+1)/6,$$

$$\sum_{k=1}^{N} (-1)^k/8 = \begin{cases} -1/8 & \text{if } N = 1(\text{mod } 2) \\ 0 & \text{if } N = 0(\text{mod } 2), \text{ and} \end{cases}$$

$$\sum_{k=1}^{N} \frac{1}{9}(\omega_3^k + \omega_3^{2k}) = \begin{cases} -1/9 & \text{if } N = 1(\text{mod } 3) \\ -2/9 & \text{if } N = 2(\text{mod } 3) \\ 0 & \text{if } N = 0(\text{mod } 3) \end{cases}$$

we obtain:

$$\begin{aligned} S(p) &= 1 + \frac{1}{12} N(N+1)(2N+1)/6 + \frac{6}{12} N(N+1)/2 + \frac{47}{72}N \\ &\quad + \begin{bmatrix} -1/8 \text{ if } N = 1(\text{mod } 2) \\ 0 \text{ if } N = 0(\text{mod } 2) \end{bmatrix} + \begin{bmatrix} -1/9 \text{ if } N = 1(\text{mod } 3) \\ -2/9 \text{ if } N = 2(\text{mod } 3) \\ 0 \text{ if } N = 0(\text{mod } 3) \end{bmatrix} \\ &= 1 + \frac{47}{72}N + \frac{1}{12} N(N+1)(2N+19)/6 \\ &\quad + \begin{bmatrix} -1/8 \text{ if } N = 1(\text{mod } 2) \\ 0 \text{ if } N = 0(\text{mod } 2) \end{bmatrix} + \begin{bmatrix} -1/9 \text{ if } N = 1(\text{mod } 3) \\ -2/9 \text{ if } N = 2(\text{mod } 3) \\ 0 \text{ if } N = 0(\text{mod } 3) \end{bmatrix} \\ &= \frac{1}{72}\left[72 + 47N + N(N+1)(2N+19)\right] \\ &\quad + \begin{bmatrix} -1/8 \text{ if } N = 1(\text{mod } 2) \\ 0 \text{ if } N = 0(\text{mod } 2) \end{bmatrix} + \begin{bmatrix} -1/9 \text{ if } N = 1(\text{mod } 3) \\ -2/9 \text{ if } N = 2(\text{mod } 3) \\ 0 \text{ if } N = 0(\text{mod } 3) \end{bmatrix}. \end{aligned}$$

Therefore,

$$S(p) = \frac{1}{72}\Big[72 + N(2N^2 + 21N + 66)\Big]$$

$$(5.5) \qquad + \begin{bmatrix} -1/8 \text{ if } N = 1 (\text{mod } 2) \\ 0 \text{ if } N = 0 (\text{mod } 2) \end{bmatrix} + \begin{bmatrix} -1/9 \text{ if } N = 1 (\text{mod } 3) \\ -2/9 \text{ if } N = 2 (\text{mod } 3) \\ 0 \text{ if } N = 0 (\text{mod } 3) \end{bmatrix},$$

if $p = 9 + 2N$ or $p = 10 + 2N$ ($N = 1, 2, 3, \ldots$).

This completes the proof of Theorem 11.

Theorem 12. The number of asymmetric topological graphs having 2 independent cycles and p vertices is

0 if $p < 9$, and

$P_3(k)$ if $p = 9 + 2k$ or $p = 10 + 2k$ $(k = 0, 1, 2, \ldots)$,

where $P_3(k)$ denotes the number of partitions of k having no summand greater than 3.

The number of asymmetric topological graphs having 2 independent cycles and no more than p vertices is

0 if $p < 9$,

1 if $p = 9$,

2 if $p = 10$,

$$\frac{1}{18}\left[22 + N(N^2 + 9N + 24)\right] + \begin{bmatrix} -1/9 \text{ if } N \equiv 1 \pmod 3 \\ -3/9 \text{ if } N \equiv 2 \pmod 3 \\ -2/9 \text{ if } N \equiv 0 \pmod 3 \end{bmatrix}$$

if $p = 9 + 2N$ $(N = 1, 2, 3, \ldots)$, and

$$\frac{1}{36}\left[72 + N(2N^2 + 21N + 66)\right] + \begin{bmatrix} -1/4 \text{ if } N \equiv 1 \pmod 2 \\ 0 \text{ if } N \equiv 0 \pmod 2 \end{bmatrix} + \begin{bmatrix} -2/9 \text{ if } N \equiv 1 \pmod 3 \\ -4/9 \text{ if } N \equiv 2 \pmod 3 \\ 0 \text{ if } N \equiv 0 \pmod 3 \end{bmatrix}$$

if $p = 10 + 2N$ $(N = 1, 2, 3, \ldots)$.

Proof. In the proof of Theorem 11 we observed that the only asymmetric connected topological graphs having 2 independent cycles are those explicitly described in the proof of Theorem 3 and that these graphs exist only when the number p of vertices is of the form $p = 9 + 2k$ $(k = 0, 1, 2, \ldots)$. In the proof of Theorem 4 we observed that asymmetric (not connected) topological graphs with 2 independent cycles exist only when the number p of vertices is of the form $p = 10 + 2k$ $(k = 0, 1, 2, \ldots)$. Each of the latter type graphs is of the form: an asymmetric connected topological graph having 2 independent cycles plus the graph U_1. Combining these facts we have that the total number $T(p)$ of asymmetric topological graphs having 2 independent cycles and p vertices is given by:

$T(p) = 0$ if $p < 9$, and

$T(p) = P_3(k)$ if $p = 9 + 2k$ or $p = 10 + 2k$ $(k = 0, 1, 2, \ldots)$,

where $P_3(k)$ denotes the number of partitions of k having no summand greater than 3. This completes the proof of the first assertion of the theorem.

It is now clear that the total number $\mathcal{S}(p)$ of asymmetric topological graphs having 2 independent cycles and no more than p vertices is given by:

$$\mathcal{S}(p) = \begin{cases} 0 & \text{if } p < 9, \\ 2\sum_{k=0}^{N} P_3(k) - P_3(N) & \text{if } p = 9 + 2N \ (N = 0, 1, 2, \ldots) \\ 2\sum_{k=0}^{N} P_3(k) & \text{if } p = 10 + 2N \ (N = 0, 1, 2, \ldots). \end{cases}$$

Making use of the result that

$$S(p) = \sum_{k=0}^{N} P_3(k) \quad \text{if } p = 9 + 2N \text{ or } p = 10 + 2N$$

(cf. (5.3)) we obtain

$$\mathcal{S}(p) = \begin{cases} 0 & \text{if } p < 9 \\ 1 & \text{if } p = 9 \\ 2 & \text{if } p = 10 \\ 2S(p) - P_3(N) & \text{if } p = 9 + 2N \ (N = 1, 2, 3, \ldots) \\ 2S(p) & \text{if } p = 10 + 2N \ (N = 1, 2, 3, \ldots), \end{cases}$$

which when combined with (5.4) and (5.5) yields the formulas for $\mathcal{S}(p)$ given in the statement of the theorem. This completes the proof of Theorem 12.

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AUTOBIOGRAPHICAL STATEMENT

Louis V. Quintas was born in New York City on January 20, 1929. He received all of his early education in the New York City public school system, graduating from the Manhattan High School of Aviation Trades in 1947. While in high school he became interested in music and, upon graduation, chose to pursue a career in music rather than one in aviation. He was a professional dance band and jazz drummer until 1955. In 1953 he had enrolled at Columbia University where he majored in mathematics receiving a B. S. in 1957 from the School of General Studies and an M. A. in 1959 from the Graduate Faculties. Between 1955 and 1957 he worked part-time as a research assistant in physics at the Thomas J. Watson Laboratory at Columbia. He taught mathematics at Columbia from 1957 to 1959 and at The City College of New York from 1959 to 1963. He has been an assistant professor of mathematics at St. John's University since 1963. In 1961 he presented his first research paper (a joint work) to the American Mathematical Society. He has subsequently published papers in functional equations, algebraic topology, and graph theory. In 1965 he enrolled in the mathematics doctoral program of The City University of New York. He is currently (1966-1967) a National Science Foundation Science Faculty Fellow.