TOPOLOGICAL MODELS OF BELIEF LOGICS

by

Chris Steinsvold

A dissertation submitted to the Graduate Faculty in Philosophy

in partial fulfillment of the requirements for the degree of Doctor of Philosophy,

City University Of New York

2007

UMI Number: 3283152

UMI Microform 3283152

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

This manuscript has been read and accepted for the Graduate Faculty in Philosophy in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Melvin Fitting

__________________________ ____________

Chair of Examining Committee Date

John Greenwood

__________________________ ____________

Executive Officer Date

Supervisory Committee:

Rohit Parikh, Richard Mendelsohn, Michael Levin

Main Advisor:

Rohit Parikh

Readers:

Kit Fine (NYU), Melving Fitting

City University Of New York

Abstract

TOPOLOGICAL MODELS OF BELIEF LOGICS

by

Chris Steinsvold

Adviser: Professor Rohit Parikh

We explore the technical and philosophical aspects of topological models for modal logics of belief. We focus on K4, KD4, and KD45, interpreting the diamond as the derivative operator. Completeness proofs are presented for these logics, as well as for multi-agent KD45 with common belief. Special philosophical emphasis is given to the T_1 condition, which we interpret to mean that the agent can never have a complete set of beliefs.

ACKNOWLEDGEMENTS

Gratitude is difficult to fully express. I want to thank my main advisor, Rohit Parikh; my two other advisors, Richard Mendelsohn and Michael Levin; and my two readers: Kit Fine and Mel Fitting. For different but no less important reasons I thank my mother, Virginia Steinsvold; my father, John Steinsvold; and my wife, Marie-Juliette Riot. Finally, for reasons of personal inspiration, I want to acknowledge three old friends: Frank Gerwer, Ian Bavitz, and Ben Colen.

Possibility is therefore the heaviest of all categories.

SØREN KIERKEGAARD

Contents

Part I

The Basic Approach

Chapter 1

Introduction And Basic Considerations

1.1 Introduction

I introduce my dissertation by recalling my initial thoughts.

Since learning of it, I have been impressed by the following coincidence. The modal logic S4 has a topological semantics. Also, S4 is widely used as a logic of knowledge.

It is the interior operator which obeys the S4 axioms. Otherwise put, the interior operator acts like knowledge. I had asked myself: is there a topological operator which obeys a logic of belief? The answer is yes. Given certain constraints on the topology, the dual of the derivative behaves like belief.

For several years I have been thinking about this in two ways.

The first is technical, i.e. completeness proofs. There are several logics of belief, and one naturally wants to show completeness for them. Various proofs are contained herein. A natural canonical construction is used to show completeness for K4 and KD4 (this approach carries over to the multi-agent case). It is still is not clear if this canonical approach works for KD45, so another technique is used. Also, topological completeness is shown for $\mathrm{KD45}_n^C$ (multi-agent KD45 with common belief). However, topological completeness for $\mathrm{KD4}_n^C$ and $\mathrm{K4}_n^C$ is, to me, unknown.

The second is philosophical, or interpretive. Given that there is a topological operator which acts like belief, how can we interpret the topological semantics on an intuitive level? That is, how can we interpret the topological semantics as a semantics for an agent with beliefs? This is the philosophical part of the dissertation. Initially, I was hoping to do for the topological semantics what Jaakko Hintikka has done for the relational semantics, [15]. Now, however, this hope strikes me as too bold. Speaking modestly, we do present an interpretation for the topological semantics, one which is different than the relational semantics for logics of belief. However, in our attempt to coherently interpret the semantics, we must assume a very unconventional assumption. Our interpretation of the semantics rests on the assumption that the proposition which expresses all truths at a possible world is incoherent to the agent. Put another way, our assumption is that the simple notion of 'all truths' is incoherent. One philosopher has recently argued for this assumption, namely, Patrick Grim (in [13]). In turn, we label this assumption the Grim intuition.

What makes both aspects, the technical and the philosophical, interesting is the T_1 condition. The 5 axiom implies this condition. It is this fact which makes completeness for KD45 more involved, and it is this condition which leads us to the Grim intuition.

1.2 Related Work

There are a number of pieces of related work which I am compelled to mention. My first encounter with the exploration of the relationship between topology and epistemology was the paper by Dabrowski, Moss, and Parikh [6]. It is this paper which got me thinking on the current course. Building upon the aforementioned article is the work by Georgatos [11] and Heinemann [14]. Also closely related is a chapter on topoogy and epistemology, written by Moss, Parikh, and myself, to appear in The Handbook Of Spatial Logics [2] (which should hold further interest for anyone concerned with the relationship between logic and topology).

Other, perhaps less directly related works are: Topology via Logic (Vickers) [25] , Knowledge and its Limits (Williamson) [26], and The Logic Of Reliable Inquiry (Kelly) [18] . The interplay between Topology and Epistemology is heavy in Kelly, however, it unfortunately doesn't relate to the work presented here. Williamson's epistemological text makes some use of topology, but again I see no strong connections between his work and my own. The epistemological interpretation set forth in Topology via Logic, on the other hand, may be more relevant, though I do not explore it here.

1.3 Three Logics Of Belief: K4, KD4, and KD45

Name	Axiom	Also Known As
5	$\neg B\phi \rightarrow B\neg B\phi$	Negative Introspection
4	$B\phi \rightarrow BB\phi$	Positive Introspection
D	$B\phi \rightarrow \neg B\neg\phi$	Consistency
K	$B(\phi \rightarrow \psi) \rightarrow (B\phi \rightarrow B\psi)$	Closure

The logics we'll consider are *normal* logics. That is, they are closed under Modus Ponens, Necessitation (if $\vdash \phi$ then $\vdash B\phi$), and contain all propositional tautologies.

We'll confine the discussion to three logics: K4, KD4, and KD45.[1]

One immediate philosophical question is: how should we interpret the formula: $B\phi$? For now, read $B\phi$ as:

the agent rationally believes ϕ

This interpretation is simple and idealized.[2]

[1]We use the common convention of referring to these normal logics by their axioms. Thus the logic K4 is the smallest normal modal logic whose axioms are K and 4.

[2]It is idealized in the following sense. Since the logics we consider are normal, they are closed under necessitation (if $\vdash \phi$ then $\vdash B\phi$). Since the logic contains all logical

1.3.1 Why Rational Belief?

There are three reasons why I prefer to interpret the box as rational belief, and not merely belief. Since we are already assuming that the agent is idealized (is aware of all logical truths), it is awkward to assume that such an idealized agent would hold beliefs irrationally. The second reason is due to Jonathan Alder [1], who convincingly argues that a belief without sufficient evidence simply doesn't make sense. The third reason is simple enough: it is what I am interested in.

1.4 Defining A Topology

Let W be a non-empty set. Let τ be a subset of the power set of W such that:

1. τ is closed under finite intersection
2. τ is closed under all unions
3. $W \in \tau$
4. $\emptyset \in \tau$

tautologies, our agent rationally believes all logical tautologies (as well as all theorems of the system, and all consequences of the agent's beliefs (because of the K axiom)).

τ is a *topology* on W. The members of τ are *open sets*, or *opens*. We'll use O and U as variables for open sets. A set is *closed* iff the complement is open. A set is *clopen* iff it is both closed and open. The pair $< W, \tau >$ is a *topological space*.

Further definitions will be introduced below.

1.5 Topological Models

Given a topological space $< W, \tau >$ we define a *valuation* V, as a function from atomic sentences to subsets of W. Intuitively, if $b \in V(p)$ then p is true at the world b. A topological model $M =< W, \tau, V >$ is a topological space with a valuation.

Definition of truth in a topological model is as follows:

$M, w \models p$ iff $w \in V(p)$

$M, w \models \psi \wedge \phi$ iff $M, w \models \psi$ and $M, w \models \phi$

$M, w \models \psi \vee \phi$ iff $M, w \models \psi$ or $M, w \models \phi$

$M, w \models \neg\phi$ iff $M, w \not\models \phi$

$M, w \models B\phi$ iff $(\exists O)(w \in O$ and $(\forall x \in O \setminus \{w\})(M, x \models \phi))$[3]

[3]To be sure, where X and Y are sets, $X \setminus Y$ is shorthand for $X \cap -Y$.

The definition allows for models where $B\phi \wedge \neg\phi$ is true at a world. Rational beliefs can be false.[4]

The definition captures the dual of a well known topological operator.[5] Let $< W, \tau >$ be a topological space, and let $A \subseteq W$. Consider the derived set of A, $d(A)$:

$$w \in d(A) \text{ iff } (\forall O)(\text{if } w \in O \text{ then } (\exists x \in O \setminus \{w\})(x \in A))$$

Compare this with the following consequence of our definitions:

$$M, w \models \neg B \neg \phi \text{ iff } (\forall O)(\text{ if } w \in O \text{ then } (\exists x \in O \setminus \{w\})(M, x \models \phi))$$

Let $[\phi]$ be the set of worlds where the formula ϕ is true (in some topological model). Then $[\neg B \neg \phi] = d([\phi])$. Similarly, $[B\phi] = -d(-[\phi])$.

$d(A)$ has many names: the derived set of A, the derivative of A, the Cantor-Bendixson derivative of A, the set of limit points of A, the set of accumulation points of A, and the set of cluster points of A. It is usually written as: A'

Our definition of truth in a model for $B\phi$ is meant to capture the dual of this operator. In so far as d is a modal operator, it is a diamond. To be

[4] B is essentially a box: $\Box$ (thus, $\neg B \neg$ is a diamond: $\Diamond$).

[5] First studied by Cantor, see [7].

sure, every topology determines a unique d operator. We are by no means the first to treat this operator as a modal operator, see [9] and [24].[6]

A sentence ϕ is *valid in a topological model* iff ϕ is true at every point in the model. A sentence ϕ is *valid in a (topological) space* iff ϕ is valid in every topological model based on the space.

As in the case of relational (or Kripke) semantics, the validity of our axioms corresponds to certain conditions on the topology. For instance, axiom D, $B\phi \to \neg B \neg \phi$, is valid in a topological space iff there are no open singletons.[7] The K axiom is valid for all topological spaces, as is the rule of Necessitation (if ϕ is a theorem, $B\phi$ is a theorem).

The 4 axiom, $B\phi \to BB\phi$, is valid in a space iff every derived set is closed. The 5 axiom, $\neg B\phi \to B\neg B\phi$, is valid in a space iff every derived set is open.[8]

We give some examples of topological spaces where some axioms hold and other do not.

Consider the following topology on {a,b },

[6]What we do hope to be novel is our exploration of this operator as an epistemic operator.

[7]Recall that axiom D is valid in a relational frame iff every world relates to some world.

[8]See Section B.4 for proofs, and Section D.1 for an alternative view of 4 and D.

$$\{ \{ \text{a} \} , \{ \text{a,b} \} , \{ \} \}$$

Axiom 4 is valid here, but axiom D and axiom 5 are not valid in this space.

Consider the following topology on the set of natural numbers,$\mathbb{N}$,

$$\{ [\text{n} , \infty) \mid \text{n} \in \mathbb{N} \} \cup \{ \{ \} \}$$ [9]

KD4 is valid here, but 5 is not.

Let A be an infinite subset of $\mathbb{N}$. Consider:

$$\{ O \subseteq \mathbb{N} \mid A \cap -O \text{ is finite } \} \cup \{ \{ \} \}$$

KD45 is valid in this topology. In fact, one can show completeness for KD45 with respect to it. A set is *co-finite* iff its complement is finite. Note that, If $\mathbb{N} \cap -A$ is finite, then the opens sets will be all the co-finite sets with the empty set. If $\mathbb{N} \cap -A$ is infinite, then the topology will contain all the co-finite sets, and more, for A is in the topology.

[9] $[\text{n} , \infty)$ contains n and all numbers greater than n.

1.6 The Interpretation: The Limits Of Rational Belief And The T_1 Condition

Of course, a topological space is typically given a spatial interpretation. Here, we are interested in an epistemic (and hence non-spatial) interpretation of the topological semantics.

Given a topological space $< W, \tau >$, we interpret the members of W as possible worlds. As is common, we interpret sets of possible worlds as propositions.[10] Let w be a world ($w \in W$), and let A be a proposition ($A \subseteq W$); Interpret $w \in d(A)$ to mean:

The proposition A is within the limit of rational belief at w

There are limits to what we can rationally believe. One cannot, rationally, believe a contradiction; hence a contradiction is outside the limit of rational belief (no matter what world you are at). Since we are interpreting sets of

[10] In particular, the set of all worlds where cats exist is taken to be the proposition (or intension) expressed by the sentence 'cats exist.' See [10] and [19]. Despite my adoption of this common usage, I suggest a critical alternative later on, see Appendix E.

possible worlds as proposition, the empty set represents the contradictory proposition. Thus, our interpretation coheres well with the simple fact that no world is ever a member of $d(\{\})$.

Recall that the operator d is a diamond, and hence represents a type of possibility. Roughly, $w \in d(A)$ means that the proposition A is possible for the agent at world w (where *possible* is an abbreviation for *within the limit of rational belief*).

If $w \notin d(A)$ then the proposition A is outside the limit of rational belief at world w. In this case, the agent *must* believe $-A$. Alternatively, A is beyond belief.

To make the interpretation as clear as possible, assume $w \in d(A)$. By the interpretation, this means that the proposition A is within the limit of rational belief at world w. A may represent the proposition *Mr. Brown is in Barcelona.* The negation of this sentence may also be within the limit of rational belief at world w, which we would represent as: $w \in d(-A)$. Of course, there may be another world, say v, where we are with Mr. Brown in Barcelona having drinks, and hence $v \notin d(-A)$. The proposition that Mr. Brown is not in Barcelona is outside the limit of rational belief, at v, because I am with him there and I see him in front of me. Thus, $v \notin d(-A)$ means that the agent (rationally) believes that Mr. Brown is in Barcelona. In general, for any set S, $-d(-S)$ will be the set of worlds where S is rationally believed.

At this point in the text, our interpretation of $w \in d(A)$ may seem superficial, or perhaps unnecessarily wordy. Moreover, we expect the reader to be wondering what is special about the topological semantics (in comparison to relational semantics). What is of special interest to us is the following condition:

The T_1 condition: $(\forall x)(\forall y)(x \notin d(\{y\})$ [11]

The T_1 condition is also called the T_1 separation axiom.

Recently, in [23] Sarenac and van Benthem wrote "... even a weak separation axiom like T_0 is not plausible epistemically." The tone of the quote impugns the plausibility of any higher separation axioms as well.[12] Thus, they imply that the T_1 condition is implausible epistemically.[13]

What does the T_1 condition mean, given our interpretation? Again, we

[11] The initial reason this condition concerned the author was due to the fact that axiom 5 implies this condition (proof below). However, the author now thinks this condition may be desirable, even if one rejects the 5 axiom as implausible.

[12] The T_0 condition can be defined as: for all a, b, either $a \notin d(\{b\})$ or $b \notin d(\{a\})$.

[13] In a sense, by saying this, Sarenac and van Benthem impugn thier own work. Why? Because if one only considers spaces below T_1, one may as well stay within the realm of

are interpreting sets of worlds as propositions. Thus, the set of worlds where cats exist is the proposition expressed by the sentence: cats exist. Analogously, a singleton set containing exactly one world, say w, is the proposition which expresses all the sentences which are true at w. Succinctly put, singletons represent maximal propositions.

Thus, given our interpretation, the T_1 condition means that the proposition $\{y\}$ is always outside the limit of rational belief. Alternatively, maximal propositions are always outside the limit of rational belief.

Why should this be the case? It is tempting to say that, being finite creatures, we would never be in a position to have a maximal (and rational) set of beliefs, and so the T_1 condition makes sense. However, such practical considerations are irrelevant here, given the level of idealization we are assuming for our agents (for instance, due to the rule of necessitation our agents are logically omniscient). Again, it is unclear why such idealized agents would be limited by such practical concerns.

Alternatively, one might suppose that since there is no finite sentence which could express a maximal proposition, perhaps this is a good reason why a singleton proposition is always outside the limit of rational belief. But

relational semantics (as such spaces are easily transformed into relations on sets).

this doesn't seem satisfactory for the following reason. For any worlds w and y, the T_1 condition tells us that $w \in -d(\{y\})$. This means that the agent rationally believes the proposition $-\{y\}$. But if a proposition cannot be expressed by a finite sentence, then the negation of that proposition cannot either. $\{y\}$ represents a maximal conjunction, and so the negation of it is a maximal disjunction, which could not be represented by a finite sentence. Thus, the assumption that our idealized agent is limited to propositions which can only be expressed by finite sentences can not help us here.

Unfortunately, there doesn't seem to be a common-sensical way to explain why singleton propositions should always be outside the limit of rational belief for idealized agents. Nonetheless, the main project here is to make sense of the topological semantics from an epistemic interpretation. As we continue we will examine the work of Patrick Grim, and see if his philosophical work may help us here.

1.7 A Way To Judge The Coherence Of The Interpretation

Initially, we had defined a topology in terms of open sets. Here we define a topology by an equivalent method, in hopes to answer a philosophical question. The following is a minor variation of a theorem in [8], p. 73. Note the four conditions used to define a topology.

Theorem 1.7.1. *Let X be a set, and let $f : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ be a function satisfying:*

1. $f(\emptyset) = \emptyset$
2. $f(f(A)) \subseteq f(A)$
3. $f(A \cup B) = f(A) \cup f(B)$
4. *For all* $z, y \in X$, $z \notin f(\{y\})$

Then $\tau = \{-(A \cup f(A)) | A \in \mathcal{P}(X)\}$ *is a topology on* X *and* $f(A) = d(A)$ *That is,* f *is the derivative function,* d*, for the topology* τ*.*

The philosophical question we hope to answer is simply: does it make sense to interpet the topological operator d as the limit of rational belief for an agent?

Since the four conditions above can define a topology, it is reasonable to say that as long as we can make epistemic sense out of these four conditions, then our interpretation does make sense. Keep in mind that the function f in the theorem is d.

The first condition was already discussed. A contradictory proposition (the empty set) is always outside the limit of rational belief, at any world, hence $d(\{\}) = \{\}$. This condition holds for all topologies.

The second condition is essentially axiom 4. That is, Axiom 4 is valid in a topology iff $d(d(A)) \subseteq d(A)$. To be sure, axiom 4 is not valid in all topologies.[14]

The third condition is surely plausible. If A is within the limit of rational belief, then so is $A \cup B$. Conversely if the proposition $A \cup B$ is within the limit of rational belief, then either A or B is within the limit. This condition holds in all topologies.

The final condition is the T_1 condition. Thus, given that the previous three conditions cohere with our interpretation, this is the last condition to make sense of. Of course, the T_1 condition does not hold for all topologies.[15]

[14] In Dugundji's original theorem, the weaker condition is: $d(d(A)) \subseteq d(A) \cup A$ (which is true for all spaces).

[15] In Dugundji's original theorem, the weaker condition is: $x \notin d(\{x\})$ for all x (which

We now look to the ideas of Patrick Grim to help us make sense of the T_1 condition.

is true for all spaces).

1.8 The Grim Intuition

We state the Grim intuition as follows: there is something incoherent about the totality of all truths. [16] This is loosely stated, and is intended as such.[17] Speaking more particularly, Grim attacks the notion of a collection of all truths. Similarly, he attacks the possibility that anyone (including God) could know all truths.

We can summarize his intuition in his own words (123-124),

> The surprise is that at the point we are tempted to speak of " all truths" or " all propositions" we already face incoherence.

It is this intuition that the very notion of " everything which is true" is in some sense an incoherent notion which is fundamental. Our claim is that the T_1 condition captures this intuition.

The connection should be clear. A singleton, considered as a proposition in the semantics, that is, a set containing a single world, represents a maximal proposition. The singleton, as a proposition, represents everything which is

[16] In the introductory chapter, Grim states that his book is about exploring " ... the logical impossibility of *totalities* of knowledge and truth"

[17] Grim speaks of " ... the suspicion that there is a single deep problem regarding truth, knowledge, and totality."

true at the world in the set. If Grim's intuition is correct, then it makes perfect sense to assume that singletons will always be outside the limit of rational belief. Moreover, this is what the T_1 condition *says*, given our interpretation. At any world w, $\{y\}$ is outside the limit of rational belief. In effect, we are substituting the word 'incoherent' for 'outside the limit of rational belief.'

To be sure, Grim himself could not accept our representation of his intuition. Grim argues against propositions expressing all truths, as well as possible worlds themselves. This step we can't take (that is, if we want to make sense of the semantics – we are of course assuming that points are possible worlds). Rather, we are left with the view that the proposition expressing all truths (at a world) exists, even though it is incoherent to the agent.

For some discussion of Grim's arguments, see Appendix E.

1.9 Theological Considerations

If the notion of " everything which is true" is incoherent, then *omniscience* is incoherent as well. Thus, the notion of God is incoherent (i.e. outside the limit of rational belief). Grim dedicates much space towards attacking the coherency of omniscience, focusing on the work of Kaplan and Montague [17], [21].

It seems that the semantics reflects this as well. Leaving aside knowledge for the moment, and just focusing on rational belief, we have the following.

Proposition 1.9.1. *$\phi \to B\phi$ is valid in a topology iff $B\bot$ is valid*

To see this, it is enough to show that both sentences are valid iff every set is open. This is true for arbitrary topologies. This simple topological fact mirrors Gődel's result: PA is complete iff it is inconsistent. Certainly, Grim is suggesting exactly this about our rational beliefs, it is impossible for them to be complete.

1.10 Final Thoughts

Our goal has been to interpret the topological semantics, and we have attempted to show that, given basic assumptions, the semantics seems to say that a proposition expressing all truths at a given world is incoherent (no matter what world the agent is at). This part Grim agrees with. However, Grim disagrees that possible worlds exist in the first place.

The natural classical assumption is that if something is incoherent, it cannot exist. Certainly, this holds for things in the world. We are suggesting that it does not hold for the world as a whole. The idea that a rule may hold inside a world, but not hold for the world as a whole is not unprecedented. For instance, though nothing in the world can accelerate up to the speed of light, inflationary cosmologists suggest that, at some point, the speed of the expansion of the universe (as a whole) had reached and exceeded the speed of light.

Again, this is merely to suggest that a law may hold for things in a world, but not apply to the world as a whole. The assumption that an incoherent object exists may be too wild for most readers. Those who are hypnotized by the constraints of classical logic won't pay much attention to an argument that says otherwise, no matter how cogent. The only real question is: is this assumption worth exploring? The only way to find out the answer here is to explore and find out later. I can only hope that this text represents a start.

To recapitulate, we've tried to find a home for the interpretation of the topological semantics, and only found ourselves partially welcomed by Grim. One philosopher I found who explicitly agrees with our interpretation (that reality is contradictory and yet at the same time, real), is the Buddhist philosopher Nāgārjuna (see [22]).[18]

However, whereas Nāgārjuna explicitly agrees, it seems Graham Priest implicitly agrees. It is the work of Graham Priest which led me to Patrick Grim and Nāgārjuna. Priest writes,

> I claim that reality is, in a certain sense, contradictory.

There are certain statements which he finds contradictory, though nonetheless true. Thus, whereas Grim uses his arguments to eliminate the idea that reality is everything that is the case, it seems Priest accepts Grim's arguments, but rejects the conclusion that the world does not exist.[19] This is entirely consonant with our interpretation, however radical.

[18] To give a quick sample from Nāgārjuna, consider:

> Everything is real and is not real,
>
> Both real and not real,
>
> Niether real nor not real.
>
> This is the lord Buddha's teaching.

[19] See Priest's section on Grim in [22]

I, among others, have always found paradoxes themselves much more interesting than various solutions. Such solutions always seem like we lose something. In any case, for those who have a taste for dialetheism, the topological semantics may suit them well.[20]

[20]As Richard Mendelsohn has discussed with me, there is a less controversial way to view all of this. Instead of turning towards Priest's philosophy, we may alternatively assert that singletons, or any finite sets, in our semantics simply are not propositions at all. Furthermore, Kit Fine strongly suggested one solution I had previously dismissed: we may suppose that our reasoners, though idealized, are nonetheless incapable of believing a proposition which expresses all truths. This may be for a number of reasons, for example: if we suppose that every belief have a finitary basis.

1.11 A Note On Knowledge And Gettier

With the previous technical details in mind, we now engage in some basic epistemology. As mentioned at the outset, I had been intrigued for some time by the somewhat well known fact that the interior operator (defined below) seems to behave like a knowledge operator. [21]

Given that there is also a topological operator which acts like justified belief, we should ask ourselves: how do these two operators interact? This is the question we explore in this section. We draw no major conclusions here. Rather, we modestly suggest that the topological semantics *may* be an interesting way to consider the Gettier problem in epistemology.[22]

The *interior of* A is the union of all open subsets of A. As mentioned, the interior operator acts like knowledge. Consider the following definition of truth for $K\phi$ in a topological model,

$$M, w \models K\phi \text{ iff } (\exists O)(w \in O \text{ and } (\forall x \in O)(M, x \models \phi))$$

Using this, one can show soundness and completeness for S4.[23] Quick

[21] they both obey the S4 axioms, see following footnote.

[22] Given the perplexity of the Gettier problem, I assume *any* new angle on it is worthwhile.

[23] See [2] for a simple proof. The result is originally due to Mckinsey and Tarski, see [20].

inspection should convince the reader that the following is always valid.

$$K\phi \leftrightarrow (B\phi \wedge \phi)$$

Thus, the topological semantics seems to support the thesis that knowledge is true justified belief (Avoiding subtleties, we'll assuming that rational belief and justified belief are the same). Up until 1963 this would not have been controversial. Since then, Gettier's celebrated article [12] has totally undermined this simple thesis.

For those unfamiliar with Gettier examples, consider the following. Bob is in a field. In the field there is a dog which looks exactly like a sheep. From this, Bob develops a justified belief that there is a sheep in the field (he has strong evidence that there is a sheep in the field). Unbeknownst to Bob, there is a real sheep in the field behind a tree. Now, Bob justifiably believes that there is a sheep in the field, and it is true, but epistemologists typically agree that Bob doesn't *know* there is a sheep in the field.

Because of examples like this, knowledge can't be true justified belief. However, we can accommodate Gettier's criticism by simply reading $K\phi$ as:

To be sure, the axioms for S4 are: $K\phi \to \phi$, $K\phi \to KK\phi$, $K(\phi \to \psi) \to (K\phi \to K\psi)$, with Necessitation and MP.

justified true belief in ϕ (instead of knowledge of ϕ). Given this, we can ask: what topological operator, if any, *truly* represents knowledge?

Consider the following suggestion.

Let $F(A)$ be the union of all clopen subsets of A (where a set is *clopen* when it is both open and closed). $F(A)$ will be a stronger operator than the interior operator. *Redefine* $K\phi$ with:

$$M, w \models K\phi \text{ iff } (\exists Z)(w \in Z \text{ and } (\forall x \in Z)(M, x \models \phi))$$

Where $\exists Z$ quantifies over clopen sets.

Thus, $K\phi$ is true at w when there is some clopen set Z, $w \in Z$ and everything in Z satisfies ϕ. This way, we get the validity of $K\phi \rightarrow (B\phi \wedge \phi)$, but not the converse. Many topologies can be used to show that the converse fails.

For instance, let τ be the co-finite topology on $\mathbb{N}$. Let $V(p) = -\{1\}$. $Bp \wedge p$ will be true at 2, and the only clopen set containing 2 is $\mathbb{N}$, which contains 1 and 1 satisfies $\neg p$. Thus Kp fails at 2.

At least formally, this operator, $F(A)$, appears to point in the right direction. That is, we have that knowledge implies true justified belief, but not the converse. The big question at this point is: can any of this give us any insight into the Gettier problem? To this we have no interesting answer. We only hope that it may. If the reader feels that our suggestion is too speculative, this criticism is well taken.

1.12 Comments

- A clopen set is closed and hence contains all worlds which are *close* to it. This seems philosophically relevant. That condition which will take us from true justified belief to knowledge may very well depend on which worlds are epistemically close. Though, again, it isn't clear to me how to explicate this further. 'Epistemically close' may mean 'epistemically relevant' or 'causally relevant' worlds.

- In reference to the last comment, we may want to specify, given some world w, which worlds are epistemically close to w. I suggest those worlds which are in the smallest clopen set containing w. In some topologies there may be no such smallest clopen set, and thus we should confine ourselves to those which have them. I have in mind those topologies where the smallest clopen set of a given point is the component[24] of the point.

- Perhaps the topological semantics may be given a contextualist analysis of knowledge. That is, by reading a clopen set as a context. By contextualists' standards, we can know things in one context (the everyday context), and not know them in another context (say, the epistemological context, where we can't eliminate the possibility of an evil demon). Of course, a world may be a member of more than one clopen set. Let F_1 and F_2 be two clopens such that w is a member and $F_1 \subset F_2$. Perhaps we can interpret F_2 as being the wider context (the epistemological), and F_1 as the more everyday context (where we don't need to consider the brain in a vat hypothesis). This way, we can know something in one context, and not in another, at the same world. This is only a suggestion.

- There is a (non-Gettier) problem with defining truth in a model for $K\Phi$ in terms of open sets, rather than clopens; pointed out by Kit Fine. Let $V(p) = \{w\}$, then, for all q, $w \models Bq$ iff $w \models K(p \vee q)$. This is curious. If we define $K\Phi$ in terms of clopen sets, the left to right direction of the equivalence fails, though direction from right to left holds.

[24]See Section B.3 for a definition.

- A set S is open iff $S \subseteq -d(-S)$. Thus, the open sets are those which, when true, are rationally believed (note: they may be rationally believed without being true, however). As an example: in most cases, if I have a hand, then I will rationally believe it. Of course, in extreme cases, I may have a hand and not believe it; and in those cases, one simply should not interpret this proposition to be an open set. These open sets play the role of justifications in the topology (considering the definition of truth in a model for $B\Phi$). This is at least tangentially relevant to the Gettier cases as well. One early response to Gettier's original examples was to require that the belief in question not be derived from a false belief. Recall, $w \models B\Phi$ means there is some O which w is a member of, i.e. O is a proposition which is true at w, and is used as a basis for a belief in ϕ. Thus, the topological semantics has this requirement built in.

Appendix A

A Canonical Topological Model For Extensions of K4

In this section we use a canonical approach to show completeness for K4 and KD4. Our proof relies on the following theorem from Dugundji [8], p. 73.

Theorem A.0.1 (Dugundji)**.** *Let X be a set, and let $f : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ be a function satisfying:*

1. $f(\emptyset) = \emptyset$
2. $f(f(A)) \subseteq f(A) \cup A$
3. $f(A \cup B) = f(A) \cup f(B)$

4. For all $z \in X$, $z \notin f(\{z\})$

Then $\tau = \{-(A \cup f(A)) | A \in \mathcal{P}(X)\}$ is a topology on X and $f(A) = A'$

A.1 Defining the canonical function

Let $\mathcal{L}$ be any extension of K4, and let $W^{\mathcal{L}}$ be the set of all maximally consistent sets of $\mathcal{L}$. For any formula ϕ , let $\widehat{\phi} = \{z \in W^{\mathcal{L}} | \phi \in z\}$. For any set A, let $|A|$ be the cardinality of A.

We define $f : \mathcal{P}(W^{\mathcal{L}}) \mapsto \mathcal{P}(W^{\mathcal{L}})$ as follows. For all $A \subseteq W^{\mathcal{L}}$,

$$f(A) = \{z \in W^{\mathcal{L}} | \text{ for all } B\phi, \text{ if } B\phi \in z \text{ , then } |\widehat{\phi} \cap A| \text{ is infinite } \}$$

In other words,

$$x \in f(A) \text{ iff for all } B\phi \in x, \widehat{\phi} \cap A \text{ is an infinite set.}$$

Theorem A.1.1. *f is the derivative function for a topology on $W^{\mathcal{L}}$*

Proof. Where F is any finite set, $f(F)$ is empty, hence conditions 1 and 4 of Theorem A.0.1 hold.

To show condition 2 holds, we show $f(f(A)) \subseteq f(A)$. Assume $w \in f(f(A))$ and assume $B\phi \in w$. Since our logic is some extension of K4,

$B\phi \to BB\phi \in w$, and so $BB\phi \in w$. Thus, by the definition of f, $|\widehat{B\phi} \cap f(A)|$ is infinite. But then $|\widehat{\phi} \cap A|$ is infinite. Since $B\phi$ was arbitrary, $w \in f(A)$.

Finally, condition 3 holds. Assume $w \notin f(A \cup C)$. Then some $B\phi \in w$ and $|\widehat{\phi} \cap (A \cup C)|$ is finite. So $|\widehat{\phi} \cap A|$ and $|\widehat{\phi} \cap C|$ must be finite, which implies that $w \notin f(A)$ and $w \notin f(C)$. Thus $f(A) \cup f(C) \subseteq f(A \cup C)$.

To see the converse inclusion assume $w \notin f(A)$ and $w \notin f(C)$. Thus there is some $B\phi \wedge B\psi \in w$, such that $|\widehat{\phi} \cap A|$ and $|\widehat{\psi} \cap C|$ are finite. This implies that $|\widehat{\phi \wedge \psi} \cap A|$ and $|\widehat{\phi \wedge \psi} \cap C|$ are finite as well. Thus the following is finite,

$$|(\widehat{\phi \wedge \psi} \cap A) \cup (\widehat{\phi \wedge \psi} \cap C)|$$

And since $((X \cap Y) \cup (X \cap Z)) = (X \cap (Y \cup Z))$, $|(\widehat{\phi \wedge \psi}) \cap (A \cup C)|$ is finite. Now since $B\phi \wedge B\psi \in w$, and since $(B\phi \wedge B\psi) \to B(\phi \wedge \psi)$ is a theorem of K, $B(\phi \wedge \psi) \in w$. Hence $w \notin f(A \cup C)$. □

Let $\tau^{\mathcal{L}}$ be the topology defined by our function f. This gives us our canonical topological space,

$$S^{\mathcal{L}} = < W^{\mathcal{L}}, \tau^{\mathcal{L}} >$$

Let $w \in W^{\mathcal{L}}$, we define the canonical valuation as,

$$w \in V^{\mathcal{L}}(p) \text{ iff } p \in w$$

This in turn gives us our canonical topological model,

$$M^{\mathcal{L}} =< W^{\mathcal{L}}, \tau^{\mathcal{L}}, V^{\mathcal{L}} >$$

A.2 Truth lemma

The following theorem is used to show the truth lemma.

Theorem A.2.1. *For any formula ϕ, $\widehat{\neg B \neg \phi} = f(\widehat{\phi})$*

Proof. If $w \notin \widehat{\neg B \neg \phi}$ then $w \in \widehat{B \neg \phi}$, and so $w \notin f(\widehat{\phi})$.

To see the converse inclusion assume $w \notin f(\widehat{\phi})$. Then some $B\psi \in w$ and $|\widehat{\psi \wedge \phi}|$ is finite. Since $\widehat{\psi \wedge \phi}$ is a finite set, it must be empty (because if a formula γ is consistent, then $|\widehat{\gamma}|$ is infinite). Thus $\psi \rightarrow \neg\phi$ is a theorem. So $B\psi \rightarrow B\neg\phi$ is a theorem. This tells us that $\widehat{B\psi} \subseteq \widehat{B\neg\phi}$. Thus $w \in \widehat{B\neg\phi}$. That is, $w \notin \widehat{\neg B \neg \phi}$. □

Note that the previous theorem implies, for any formula ϕ,

$$\widehat{B\phi} = -f(-\widehat{\phi})$$

Theorem A.2.2 (Truth Lemma)**.** *For all $x \in W^{\mathcal{L}}$,*

$$M^{\mathcal{L}}, x \models \phi \textit{ iff } x \in \widehat{\phi}$$

Proof. The case of the propositional variable is immediate, and the cases for the logical connectives are straightforward. If $M^{\mathcal{L}}, x \models B\phi$, then there is some $O \in \tau^{\mathcal{L}}$, $x \in O$, and all worlds in $O \cap -\{x\}$ satisfy ϕ. By Induction hypothesis, $O \cap -\{x\} \subseteq \widehat{\phi}$. So $x \in -f(-\widehat{\phi})$. And Theorem 3 implies, $-f(-\widehat{\phi}) = \widehat{B\phi}$. Conversely, Assume $x \in \widehat{B\phi}$. By Theorem 3, $x \in -f(-\widehat{\phi})$, and thus there is some O, $x \in O$, and $O \cap -\{x\} \subseteq \widehat{\phi}$. By induction hypothesis, all worlds in $O \cap -\{x\}$ satisfy ϕ. Thus $M^{\mathcal{L}}, x \models B\phi$. □

By corollary, we know that ϕ is a theorem of $\mathcal{L}$ iff ϕ is valid in the canonical topological model for $\mathcal{L}$.

Call a logic $\mathcal{L}$ *canonical* iff the theorems of the logic are valid in the canonical topological space.

Theorem A.2.3. *K4 is canonical.*

Proof. In fact, we've already shown that $f(f(A)) \subseteq f(A)$ holds, in showing Theorem A.1.1, and so axiom 4 is valid. Thus K4 is canonical. □

Theorem A.2.4. *KD4 is canonical*

Proof. The D axiom is $B\phi \to \neg B \neg \phi$, and is valid iff $\neg B \neg \top$ is valid. Furthermore, $\neg B \neg \top$ is valid in a space iff there are no open singletons. Finally, it is straightforward to show, for all $x \in W^{\mathcal{L}}$,

$$\{x\} \in \tau^{\mathcal{L}} \text{ iff } B\bot \in x$$

Clearly, for every point x in the canonical model for KD4, $B\bot \notin x$, hence axiom D is valid, and KD4 is canonical. □

A.3 Comments

- Canonicity immediately holds over to $K4_n$ and $KD4_n$.

- The canonical space is a T_1 space, $f(F)$ is always empty, where F is finite. Unfortunately, it isn't clear whether KD45 is canonical. If the reader should find a proof or disproof, I would be delighted to see it.

- Clearly, this canonical construction doesn't rely on relational semantics. Hopefully, this suggests that the topological semantics is a natural semantics for modal logic.

- As noted in Hughes and Cresswell [16], p. 152, K4 is characterized by the class of all transitive and irreflexive relational frames. Given this result, it is easy to show topological completeness for K4. One takes the relational frame and replaces aRb with $a \in d(\{b\})$, and the same valuation immediately gives topological completeness.

Appendix B

Topological Completeness For $\mathbf{KD45}_n^C$

We show topological completeness for multi-agent KD45 with common belief. We review the logical system KD45_n^C and the relational semantics for it. The topological semantics is then introduced and explored. In the relational semantics, one uses a non-empty set with multiple relations on it. Analogously, we use a non-empty set with multiple topologies on it (a topological frame).

We assume no familiarity with topology.

Mckinsey and Tarski seem to be the first to suggest the derivative as a modal operator, in [20].[1] Some, but overall few, have taken up this suggestion and explored it (e.g. Leo Esakia [9], and Valentin Shehtman [24]).[2]

Showing soundness for $KD45_n^C$ is not as straightforward as one may think. In fact, we only show soundness for a restricted class of topological frames. Within that restricted class, common belief turns out to be the intersection of the other topologies in the frame.

[1] Whereas Cantor seems to be the first to study the derivative *per se*, see [7].

[2] I thank Guram Bezhanishvili for informing me of others with this interest (including himself, see [3].) A topological completeness proof for $KD45_n$ due to the author is set to appear in [2] as part of a chapter on topology and epistemic logic, co-written with Larry Moss and Rohit Parikh.

B.1 The Logical System $\mathbf{KD45}_n^C$

5 $\neg B_i\phi \to B_i\neg B_i\phi$

4 $B_i\phi \to B_iB_i\phi$

D $B_i\phi \to \neg B_i\neg\phi$

K $B_i(\phi \to \psi) \to (B_i\phi \to B_i\psi)$

E $E\phi \leftrightarrow (B_1\phi \wedge B_2\phi \wedge ... \wedge B_n\phi)$

C $C\phi \to E(\phi \wedge C\phi)$

N from $\vdash \phi$ infer $\vdash B_i\phi$

CR from $\vdash \phi \to E(\phi \wedge \psi)$ infer $\vdash \phi \to C\psi$

Plus all tautologies and MP.

To be sure, we show the converse of axiom C is provable. That is,

$$\mathrm{KD45}_n^C \vdash E(\phi \wedge C\phi) \to C\phi$$

1. $\vdash C\phi \to E(\phi \wedge C\phi)$

(Axiom C)

2. $\vdash EC\phi \to EE(\phi \wedge C\phi)$

(from 1, $\vdash A \to B$ entails $EA \to EB$)

3. $\vdash (E\phi \wedge EC\phi) \to (EE(\phi \wedge C\phi) \wedge E\phi)$

(from 2, truth functionally)

4. $\vdash E(\phi \wedge C\phi) \to E(E(\phi \wedge C\phi) \wedge \phi)$

(from 3, using $E(A \wedge B) \leftrightarrow (EA \wedge EB)$ twice)

5. $\vdash E(\phi \wedge C\phi) \rightarrow C\phi$

(from 4 using rule CR)

B.2 Relational semantics

A *frame* $F =< W, R_1, R_2, ..., R_n >$ is a non-empty set W with n relations on W. A *valuation* V is a function from propositional variables to subsets of W. A *model* $M =< W, R_1, R_2, ..., R_n, V >$ is a frame with a valuation. Truth in a model is defined as follows,

$M, w \models p$ iff $w \in V(p)$

$M, w \models \psi \vee \phi$ iff $M, w \models \psi$ or $M, w \models \phi$

$M, w \models \neg\phi$ iff $M, w \not\models \phi$

$M, w \models B_i\phi$ iff $(\forall z)($ if wR_iz then $M, z \models \phi)$

$M, w \models E\phi$ iff $M, w \models B_i\phi$ for all $i \in \{1, 2, ..., n\}$

$M, w \models C\phi$ iff $M, w \models E^k\phi$ for all $k > 0$

Let $R^+ = \bigcup\{R_i \mid i \in \{1, 2, ..., n\}\}$. Let R^* be the transitive closure of R^+. One can show, $M, w \models C\phi$ iff $(\forall z)($ if wR^*z then $M, z \models \phi)$

ϕ is *valid in a model* iff ϕ is true at every point in the model. ϕ is *valid* *in a frame* iff ϕ is valid in every model based on the frame.

The following correspondences are well known,

$\neg B_i\phi \rightarrow B_i\neg B_i\phi$ is valid in F iff $(\forall x)(\forall y)(\forall z)($ if $xR_iy \wedge xR_iz$ then $yR_iz)$

$B_i\phi \rightarrow B_iB_i\phi$ is valid in F iff $(\forall x)(\forall y)(\forall z)($ if $xR_iy \wedge yR_iz$ then $xR_iz)$

$B_i\phi \rightarrow \neg B_i\neg\phi$ is valid in F iff $(\forall x)(\exists y)(xR_iy)$

The conditions corresponding to the validity of axioms 5, 4, and D are respectively called *Euclidean*, *transitive*, and *serial*.

KD45_n^C has the finite model property.[3]

B.3 Topological definitions

Let τ be a subset of the power set of W such that: (1) $W \in \tau$, (2) $\emptyset \in \tau$, (3) τ is closed under finite intersection, (4) τ is closed under arbitrary union; τ *is a topology on* W. The members of τ are called *open sets* or *opens*. We use O, U as variables for open sets. The complement of an open set is *closed*. A set which is both open and closed is *clopen*. We refer to the members of W as points or worlds interchangeably. Where τ is a topology on W, $< W, \tau >$ is a *topological space*. Where $A \subseteq W$,

$$d(A) = \{z \in W \mid (\forall O \in \tau)(\text{ if } z \in O \text{ then } (\exists y \in O \setminus \{z\})(y \in A)\}$$

[3] A proof of this given in [10].

$d(A)$ is *the derived set of* A, and d is the *derivative operator*. $d(A)$ is usually written A' and we abandon this notation for convenience. The derived set of A goes by many names: the set of limit points of A, the set of accumulation points of A, the Cantor-Bendixson derivative of A, and the set of cluster points of A. Modally speaking d is a diamond. The multiple diamonds in the logic $\mathrm{KD45}_n^C$ will represent derivatives of topologies on the same set.

The *interior of* A, $I(A)$, is the union of all open subsets of A. The *closure of* A, $Cl(A)$, is the intersection of all closed supersets of A. The interior and closure operators are dual operators, i.e. $I(A) = -Cl(-A)$. Alternatively one can define $Cl(A)$ as $d(A) \cup A$. Thus $I(A) = -d(-A) \cap A$. A set A is open iff $A \subseteq I(A)$. It follows that, A is open iff $A \subseteq -d(-A)$.

Let $< W, \tau >$ be a topological space and let $S \subseteq W$. S is *disconnected* iff there are two open sets, O and U, such that, $O \cap S$ and $U \cap S$ are two non-empty, disjoint sets whose union is S. S is *connected* iff S is not disconnected. S is a *component* iff S is connected and all proper supersets of S are disconnected. Note that any given point is in one, and only one, component. S is *filtral* iff for all non-empty $O \subseteq S$, if $O \subseteq A$ and $A \subseteq S$, then A is open.

Let $\mathbf{B} \subseteq \tau$; $\mathbf{B}$ is a *base* for τ iff every member of τ is a union of members of **B**. The members of a base are called *basic opens*. The following is well-known.

Fact B.3.1. B is a base for some topology on W iff

(i) $W = \bigcup \mathbf{B}$, and (ii) For any $V_1, V_2 \in \mathbf{B}$, if $x \in W$ and $x \in V_1 \cap V_2$, then there is some $V_3 \in \mathbf{B}$ such that $x \in V_3 \subseteq V_1 \cap V_2$.

A topology is T_1 iff for all y, $d(\{y\})$ is empty.

This last definition is usually defined by saying: for every two points x, y there is an open O such that $x \in O$, $y \notin O$. This condition is equivalent to the assumption that every co-finite set is open.

B.4 Topological semantics

A *topological frame* $F^\tau =< W, \tau_1, \tau_2, ..., \tau_n >$ is a non-empty set W with n topologies on W. Thus a topological space is a topological frame with one topology. A *topological model* $M^\tau =< W, \tau_1, \tau_2, ..., \tau_n, V >$ is a topological frame with a valuation V.

Truth in a topological model is defined as follows,

$M^\tau, w \models p$ iff $w \in V(p)$

$M^\tau, w \models \psi \vee \phi$ iff $M^\tau, w \models \psi$ or $M^\tau, w \models \phi$

$M^\tau, w \models \neg\phi$ iff $M^\tau, w \not\models \phi$

$M^\tau, w \models B_i\phi$ iff $(\exists O \in \tau_i)($ $w \in O$ and $(\forall z \in O \setminus \{w\})(M^\tau, z \models \phi))$

$M^\tau, w \models E\phi$ iff $M^\tau, w \models B_i\phi$ for all $i \in \{1, 2, ..., n\}$

$M^\tau, w \models C\phi$ iff $M^\tau, w \models E^k\phi$ for all $k > 0$

Note the following consequence of our definitions.

$M^\tau, w \models \neg B_i\neg\phi$ iff $(\forall O \in \tau_i)($ if $w \in O$ then $(\exists z \in O \setminus \{w\})(M^\tau, z \models \phi)$

Thus the box of the language represents the dual of the derivative operator.

ϕ is *valid in a topological model* iff ϕ is true at every point in the model. ϕ is *valid in a topological frame* iff ϕ is valid in every topological model based on the frame. Generally, *validity* will mean validity in a frame (relational or topological). Often we say, for some sentence ϕ, ϕ *is valid in a topology* or ϕ *is valid in* τ, to mean ϕ is valid in some topological frame $< W, \tau >$. Given a class of frames C we write: $C \models \phi$ to mean ϕ is valid in every member of C. We now prove a number of propositions, some will be used later, the others are to increase understanding. For the rest of this section we drop the subscripts on our box B.

Proposition B.4.1. *$B\phi \rightarrow \neg B\neg\phi$ is valid in τ iff There are no open singletons*

Proof. Assume $\{w\}$ is open. Then $w \models B\neg\phi \wedge B\phi$ for any ϕ in a topological model. Conversely, assume D fails at w in some topological model. For some $O, U \in \tau$, $w \in O \cap U$, $(\forall z \in O \setminus \{w\})(z \models \phi))$, and $(\forall z \in U \setminus \{w\})(z \models \neg\phi))$. Thus $O \cap U = \{w\}$ and by finite intersection $\{w\}$ is open. □

Proposition B.4.2. *$B\phi \rightarrow BB\phi$ is valid in τ iff Every derived set is closed.*

Proof. A is closed iff $Cl(A) \subseteq A$. And $Cl(A) = d(A) \cup A$. So every derived set is closed iff $d(d(A)) \subseteq d(A)$. And $d(d(A)) \subseteq d(A)$ iff 4 is valid. □

To be sure, derived sets aren't always closed. Axiom 4 can be falsified in the following two point space: $< \{a, b\}, \{\{a, b\}, \emptyset\} >$. It is worth mentioning that K + $(Bp \wedge p) \rightarrow BBp$ is the minimal logic of the derived set.[4]

Proposition B.4.3. *$\neg B\phi \rightarrow B\neg B\phi$ is valid in τ iff Every derived set is open*

Proof. A is open iff $A \subseteq I(A)$. And $I(A) = -d(-A) \cap A$. So every derived set is open iff $d(A) \subseteq -d(-d(A))$. And $d(A) \subseteq -d(-d(A))$ iff 5 is valid. □

Proposition B.4.4. *Axiom 5 implies the T_1 condition.*

[4]Guram Bezhanishvili informed me that Leo Esakia was the first to show this (unpublished).

Proof. The T_1 condition is: $d(\{y\})$ is empty, for all y. Assume axiom 5 is valid in the topology but for two points $a \in d(\{b\})$. Let $V(p) = -\{b\}$. But then, $a \models \neg Bp \wedge \neg B \neg Bp$. Contradiction. □

Also, the validity of $B(B\phi \rightarrow \phi)$ implies the T_1 condition.

Proposition B.4.5. *There are no finite topological models for KD5.*

Proof. Assume otherwise. By Proposition B.4.4, the space is T_1, and this is equivalent to having every co-finite set open. But then all sets are open, in particular every singleton is open, contradicting the validity of axiom D. □

Proposition B.4.6. *If Axiom 5 is valid in a space, then Axiom 4 is valid.*

Proof. Assume 5 is valid. By Proposition B.4.4, our space is T_1. Assume $x \models B\phi$. For some O, $x \in O$, and all points in $O \setminus \{x\}$ satisfy ϕ. Let $z \in O \setminus \{x\}$. By T_1, $O \setminus \{x\}$ is open, and every point in $(O \setminus \{x\}) \setminus \{z\}$ satisfies ϕ. That is, $z \models B\phi$, and z was arbitrary, so $x \models BB\phi$. □

A logic which contains 5 but not 4 is topologically incomplete. Alternatively, if every derived set is open, then every derived set is closed.

Proposition B.4.7. *$B\phi \rightarrow \phi$ is never valid in a topological space.*

Proof. Let z be a point and let $V(p) = \{z\}$, then $z \models p \wedge B\neg p$. □

$B\phi \to \phi$ can be valid in a topological model, but not in a topological space.

Proposition B.4.8. *$\phi \to B\phi$ is valid iff $B\bot$ is valid*

Proof. Both of these sentences are valid iff every set is open. □

A logic which contains $\phi \to B\phi$ but not $B\bot$ is topologically incomplete. Also, this proposition curiously resembles Gődel's theorem (if you can prove everything which is true, then you can prove a contradiction).

Proposition B.4.9. *If 5 is valid and S is connected, then S is filtral.*

Proof. Assume otherwise. Thus for some non-empty O, and some non-open X, $O \subseteq X \subseteq S$. Since X is not open, there is some $z \in X \cap d(-X)$. If there is some v in $X \cap -d(-X)$ then, since v, z are in S, and $d(-X)$ is clopen (by axiom 5), this disconnects S. Thus $X \subseteq d(-X)$. But there is some $y \in O$, and $y \in X$, and so $y \in d(-X)$. Thus there is some point in $O \cap -X$, but $O \subseteq X$. Contradiction. □

Proposition B.4.10. *If 5 is valid, S is connected and open, and $x, y \in S$, then for any model and any formula ϕ, $x \models B_i\phi$ iff $y \models B_i\phi$.*

Proof. Assume $x, y \in S$, S is connected and open, $x \models B_i\phi$ and 5 is valid. For some O, all z in $O \setminus \{x\}$ satisfy ϕ. Since our space is T_1, $O \setminus \{x\}$ is open, and so $O \setminus \{x\} \cap S$ is open and is a subset of S. Since S is filtral (by proposition B.4.9), $(O \setminus \{x\} \cap S) \cup \{y\}$ is open, so $y \models B_i\phi$. □

B.5 Main theorem

Let $\mathcal{C}_n$ be the class of all F^τ such that each $\tau^i \in F_\tau$ satisfies the following three conditions: (1) every derived set is clopen, (2) there are no open singletons, and (3) the number of components is finite.

Main Theorem. $KD45_n^C \vdash \phi$ *iff* $\mathcal{C}_n \models \phi$

Thus we are only dealing with a restricted class of topological frames. We don't know whether KD45$_n^C$ is sound in the more general class where we remove condition (3). We conject it is not. The following two facts are useful.

Fact B.5.1. every point is in one, and only one, component.

Fact B.5.2. If τ has a finite number of components, the components are open.

B.5.1 $KD45_n^C$ is sound with respect to $\mathcal{C}_n$

The rule of necessitation is easily shown to preserve validity in all topological frames. The fact that topologies are closed under finite intersection entails the validity of the K axiom. Axiom E is valid. Propositions B.4.3, B.4.2, and B.4.1 immediately imply that axioms 5, 4, and D are valid in every $\tau_i \in F^\tau \in \mathcal{C}_n$. The next proposition will be used to show $\mathcal{C}_n \models C\phi \to EC\phi$.

Proposition B.5.3. *$\mathcal{C}_n \models B_iE^k\phi \to B_iE^m\phi$, where $k \geq m \geq 0$*

Proof. The validity of axioms 5 and K in $\mathcal{C}_n$ entail $\mathcal{C}_n \models B_i(B_i\phi \to \phi)$. Necessitation and the validity of $E\phi \to B_i\phi$ entail $\mathcal{C}_n \models B_i(E\phi \to B_i\phi)$. From this we derive $\mathcal{C}_n \models B_i(E\phi \to \phi)$ for each i, which entails $\mathcal{C}_n \models E(E\phi \to \phi)$.

A quick argument shows $\mathcal{C}_n \models E(\psi \to \phi) \to (E\psi \to E\phi)$. Applying this to the validity of $E(E\phi \to \phi)$ in $\mathcal{C}_n$ we get $\mathcal{C}_n \models EE\phi \to E\phi$. From this last fact and repeated application of the fact that if $\mathcal{C}_n \models A \to B$ then $\mathcal{C}_n \models EA \to EB$, we can derive $\mathcal{C}_n \models E^k\phi \to E^m\phi$ where $k \geq m > 0$. So $\mathcal{C}_n \models B_iE^k\phi \to B_iE^m\phi$ for all $k \geq m > 0$. And since $\mathcal{C}_n \models B_iE\phi \to B_i\phi$, we finally have which was to be shown: $\mathcal{C}_n \models B_iE^k\phi \to B_iE^m\phi$, where $k \geq m \geq 0$. □

The proof of the following uses the requirement that every topology in every member of $\mathcal{C}_n$ has a finite number of components.

Proposition B.5.4. $\mathcal{C}_n \models C\phi \to EC\phi$.

Proof. Assume $w \models C\phi$. This entails, for each agent i, $(\forall k \geq 0)(w \models B_i E^k \phi)$. For each agent i,

$$\text{Let } L_i = \{z \mid z \models B_i E^k \phi \text{ for all } k \geq 0\}$$

$$\text{Let } C_{L_i} = \{C_i \mid C_i \text{ is a component of } \tau_i \text{ and } C_i \cap L_i \neq \emptyset\}$$

Claim: $\bigcup C_{L_i} = L_i$. **Proof of claim**: By Fact B.5.1 every point is in some component, thus $L_i \subseteq \bigcup C_{L_i}$. Conversely, if $x \in \bigcup C_{L_i}$ then for some $C_i \in C_{L_i}$, $x \in C_i$ and $C_i \cap L_i \neq \emptyset$. Thus some $z \in C_i \cap L_i$, and $z \models B_i E^k \phi$ for all $k \geq 0$. By Fact B.5.2, C_i is open. By Proposition B.4.10, since x and z are in C_i, $x \models B_i E^k \phi$ for all $k \geq 0$. So $x \in L_i$.

$$\text{Let } G_i = \{C_i \mid C_i \text{ is a component of } \tau_i \text{ and } C_i \notin C_{L_i}\}.$$

Claim: For all $C_i \in G_i$, there exists an $m \geq 0$ such that for all v in C_i, $v \models \neg B_i E^m \phi$, and for all $k \geq m$, $v \models \neg B_i E^k \phi$.

Proof of claim: Let a be some point in C_i such that a satisfies $\neg B_i E^m \phi$ for some $m \geq 0$ (there must be some such point otherwise $C_i \notin G_i$). By Proposition B.4.10, every $x \in C_i$ satisfies $\neg B_i E^m \phi$. But, $\mathcal{C}_n \models B_i E^k \phi \to$

$B_i E^m \phi$ for all $k \geq m$, by Proposition B.5.3. Thus, every point in C_i satisfies $\neg B_i E^k \phi$ for all $k \geq m$.

Given this last claim, *and* the requirement that each topology has only a finite number of components, we know (given some G_i): there must be some $m \geq 0$ such that for all $v \in \bigcup G_i$, $v \models \neg B_i E^m \phi$. Also, given the first claim and the definition of G_i, we know $\bigcup G_i = -L_i$. Thus, for the following claim (for each agent i) let m_i be some number such that for all $z \in -L_i$, $z \models \neg B_i E^{m_i} \phi$. Also,

$$\text{Let } L = L_1 \cap L_2 \cap ... \cap L_n$$

Claim: $L \in \tau_1 \cap \tau_2 \cap ... \cap \tau_n$. **Proof of claim**: Assume not. That is, assume L is not open for some agent h. So, for some point x, $x \in L$ and $x \notin I_h(L)$. Since $I_h(L) = L \cap -d_h(-L)$, $x \in d_h(-L)$.

$d_h(-L) = d_h(-L_1 \cup -L_2 \cup ... \cup -L_n) = d_h(-L_1) \cup d_h(-L_2) \cup ... \cup d_h(-L_n)$, and so $x \in d_h(-L_j)$ for some agent j.

$x \in L_h$ (because $x \in L$), so x satisfies $B_h E^k \phi$ for all $k \geq 0$. So x satisfies $B_h E^{m_j+1} \phi$, which entails x satisfies $B_h B_j E^{m_j} \phi$. Thus there is some $O \in \tau_h$, $x \in O$ and for all $z \in O \setminus \{x\}$, z satisfies $B_j E^{m_j} \phi$. But $x \in d_h(-L_j)$, and so there is some $v \in O \setminus \{x\}$ and $v \in -L_j$. But then $v \models \neg B_j E^{m_j} \phi$. Contradiction.

w (the initial world we started off with when we assumed $w \models C\phi$) is in L. Every world in L satisfies $E^m\phi$ for all $m > 0$, and so every world in L satisfies $C\phi$. Since L is open for each agent, $w \models B_iC\phi$ for all agents i. That is, $w \models EC\phi$.

□

Proposition B.5.5. $\mathcal{C}_n \models C\phi \rightarrow E(\phi \wedge C\phi)$

Proof. Since $\mathcal{C}_n \models C\phi \rightarrow (E\phi \wedge EC\phi)$ and $\mathcal{C}_n \models E(A \wedge B) \leftrightarrow (EA \wedge EB)$, it follows that $\mathcal{C}_n \models C\phi \rightarrow E(\phi \wedge C\phi)$ □

Thus axiom C is valid in $\mathcal{C}_n$. Finally, we show CR preserves validity in $\mathcal{C}_n$.

Proposition B.5.6. *If* $\mathcal{C}_n \models \phi \rightarrow E(\phi \wedge \psi)$*, then* $\mathcal{C}_n \models \phi \rightarrow C\psi$*.*

Proof. Assume $\mathcal{C}_n \models \phi \rightarrow E(\phi \wedge \psi)$ and $M^\tau, w \models \phi$.

So $w \models E(\phi \wedge \psi)$ which implies $w \models E^1\phi \wedge E^1\psi$.

Assume $w \models E^k\phi \wedge E^k\psi$. By repeated application of the rule that if $\models A \rightarrow B$ then $\models EA \rightarrow EB$, we get $\mathcal{C}_n \models E^k\phi \rightarrow E^{k+1}(\phi \wedge \psi)$. Thus $w \models E^{k+1}(\phi \wedge \psi)$. Since $E^{k+1}(\phi \wedge \psi) \leftrightarrow (E^{k+1}\phi \wedge E^{k+1}\psi)$, $w \models E^{k+1}\phi \wedge E^{k+1}\psi$.

Thus $w \models E^m\psi$ for all $m > 0$, i.e. $w \models C\psi$. □

This completes soundness: If $KD45_n^C \vdash \phi$, then $\mathcal{C}_n \models \phi$

We end this section with the following useful result.

Proposition B.5.7. *Let M^τ be a model based on some $< W, \tau_1, \tau_2, ..., \tau_n > \in \mathcal{C}_n$, let $w \in W$, and let $\tau^* = \tau_1 \cap \tau_2 \cap ... \cap \tau_n$, then,*

$M^\tau, w \models C\phi$ *iff* $(\exists O \in \tau^*)($ $w \in O$ *and* $(\forall z \in O \setminus \{w\})(M^\tau, z \models \phi))$

Proof. Assume for some $U \in \tau^*$, $w \in U$ and $\forall z \in U \setminus \{w\}$, $M^\tau, z \models \phi$. We use induction to show that everything in $U \setminus \{w\}$ satisfies $E^m\phi$ for all $m \geq 0$.

$E^0\phi = \phi$, thus every world in $U \setminus \{w\}$ satisfies $E^0\phi$.

Assume all points in $U \setminus \{w\}$ satisfy $E^k\phi$. Axiom 5 is valid in each τ_i, and so each is T_1, i.e. all co-finite sets are open for each agent. So $U \setminus \{w\} \in \tau^*$. Let $y \in U \setminus \{w\}$. Everything in $(U \setminus \{w\}) \setminus \{y\}$ satisfies $E^k\phi$, so y satisfies $B_iE^k\phi$ for all i, that is $EE^k\phi$. And y was arbitrary, so everything in $U \setminus \{w\}$ satisfies $EE^k\phi$, i.e. $E^{k+1}\phi$.

Thus every world in $U \setminus \{w\}$ satisfies $E^m\phi$ for all $m \geq 0$. Since $U \in \tau^*$, $w \models EE^m$ for all $m \geq 0$. That is $w \models E^m$ for all $m > 0$, so $w \models C\phi$.

Conversely, assume $M^\tau, w \models C\phi$. For any formula ψ,

$$\text{Let } [\psi] = \{z \in W \mid M^\tau, z \models \psi\}.$$

Claim: $[\phi \wedge C\phi] \in \tau^*$. **Proof of claim**: Recall that $S \in \tau_i$ iff $S \subseteq -d_i(-S)$.

So $[\phi \wedge C\phi] \in \tau_i$, for all i iff $[\phi \wedge C\phi] \subseteq -d_i(-[(\phi \wedge C\phi)])$ for all i.

Let $z \in [\phi \wedge C\phi]$. Since $\mathcal{C}_n \models C\phi \rightarrow E(\phi \wedge C\phi)$, $\mathcal{C}_n \models (\phi \wedge C\phi) \rightarrow E(\phi \wedge C\phi)$. Thus $[\phi \wedge C\phi] \subseteq [E(\phi \wedge C\phi)]$ and so $z \in [B_i(\phi \wedge C\phi)]$ for all i.

For any i, and all v, $v \in -d_i(-[(\phi \wedge C\phi)])$ iff $v \in [B_i(\phi \wedge C\phi)]$. Thus $z \in -d_i(-[(\phi \wedge C\phi)])$ for all i, and z was arbitrary.

Thus $[\phi \wedge C\phi]$ is open for each agent.

Since $w \models C\phi$ and $\mathcal{C}_n \models C\phi \rightarrow E(\phi \wedge C\phi)$, $w \models E(\phi \wedge C\phi)$.

Thus for all i, there is an $O \in \tau_i$, $w \in O$ and $O \setminus \{w\} \subseteq [\phi \wedge C\phi]$.

$O \setminus \{w\} \subseteq [\phi \wedge C\phi]$, so $O \cup [\phi \wedge C\phi] = [\phi \wedge C\phi] \cup \{w\}$ (regardless of i).

Thus, $[\phi \wedge C\phi] \cup \{w\}$ is open for all i. And everything in $[\phi \wedge C\phi] \setminus \{w\}$, satisfies ϕ in M^τ. That is, $(\exists O \in \tau^*)($ $w \in O$ and $(\forall z \in O \setminus \{w\})(M^\tau, z \models \phi))$ □

B.5.2 $KD45_n^C$ is complete with respect to $\mathcal{C}_n$

A finite relational model, M^F

With respect to the relational semantics, KD45_n^C has the finite model property. Thus every non-theorem of KD45_n^C fails in some finite relational model,

$$M^F = < W^F, F_1, ..., F_n, V^F >$$

where each F_j is a serial, transitive, and Euclidean relation on the finite set

W^F. Starting off with a finite relational model ultimately ensures that our topological model will have a finite number of components.

Another relational model, M

From M^F we create a new relational model $M =< W, R_1, ..., R_n, V >$. For each $z \in W^F$,

$$\text{Let } C(z) = \mathbf{N} \times \{z\} = \{z_1, z_2, z_3, ...\}$$

Definition B.5.8. The members of $C(x)$ are the **copies of** x. If $x \in C(y)$ and $v \in C(y)$ then x and v are **fellow copies (of** y**)**.

$$\text{Let } W = \bigcup \{C(x) | x \in W^F\}$$

For every $z \in W$ there is exactly one $y \in W^F$ such that $z \in C(y)$.

Definition B.5.9. Where x is a copy of y, call y **the original of** x.

For each propositional variable p,

$$\text{Let } V(p) = \{z \in C(y) | y \in V^F(p)\}$$

We define each R_j using F_j as follows.

$$R_j = \{(u, v) | (\exists y \in W^F)(\exists z \in W^F)(u \in C(y) \wedge\ v \in C(z) \wedge\ yF_jz\}$$

Thus we have defined $M =< W, R_1, ..., R_n, V >$.

Each R_j mimics F_j. If our finite frame is $< \{a, b\}, \{< a, b >, < b, b >\},$

then in the new frame each copy of a will relate to each copy of b and each copy of b will relate to each copy of b.

Proposition B.5.10. *for all b, c in W^F, and all j*

$$bF_jc \textit{ iff } (\forall x \in C(b))(\forall y \in C(c))(xR_jy) \textit{ iff } (\exists x \in C(b))(\exists y \in C(c))(xR_jy)$$

Proof. If bF_jc, then, by construction of R_j, $(\forall x \in C(b))(\forall y \in C(c))(xR_jy)$.

If $(\forall x \in C(b))(\forall y \in C(c))(xR_jy)$, then, since no set of copies is empty, $(\exists x \in C(b))(\exists y \in C(c))(xR_jy)$. If $(\exists x \in C(b))(\exists y \in C(c))(xR_jy)$, then, since every member of W is a member of exactly one set of copies, bF_jc. □

Proposition B.5.11. *Each R_j is serial, Euclidean, and transitive.*

Proof. If $b \in W$ then b is a copy of z, for some $z \in W^F$. Since F_j is serial, z bears F_j to some v, and b will bear R_j to the copies of v. Hence R_j is serial.

And each R_j is Euclidean, for assume aR_jb and aR_jc. Let x, y, and z be the originals of a, b, and c, respectively. By prop. B.5.10, xF_jy, and xF_jz. Since F_j is Euclidean, y bears F_j to z. Using prop. B.5.10 again, bR_jc.

That each Rj is transitive is shown similarly. □

Furthermore, copies satisfy the same formulas as the originals.

Proposition B.5.12. *For all $x \in W^F$,*

$$M^F, x \models \phi \textit{ iff } (\forall x_m \in C(x))(M, x_m \models \phi) \textit{ iff } (\exists x_m \in C(x)(M, x_m \models \phi)$$

Proof. Using prop. B.5.10, the proof is a straightforward induction on the complexity of formulas. □

It follows that if a copy satisfies a formula, then all the fellow copies do as well. This will be used to show the truth lemma.

A topological model, M^τ

For any $x \in W$,

$$\text{Let } F(x) = \{z \in W \mid z \text{ is a fellow copy of } x\}$$

To be sure, $\{F(x)|x \in W\}$ and $\{C(x)|x \in W^F\}$ are the same set, and both are finite (because W^F is finite). For any R_i in M, and any $w \in W$,

$$\text{Let } R_i(w) = \{z \in W \mid wR_iz\}$$

Now, for each $z \in W$, and each $R_i \in M$,

$$\text{Let } \tau_{i(z)} = \{V \subseteq R_i(z) \cup F(z) \mid R_i(z) \cap -V \text{ is finite}\} \cup \{\emptyset\}$$

Proposition B.5.13. *$\tau_{i(z)}$ is a topology on $R_i(z) \cup F(z)$.*

Proof. By construction, if V contains all but finitely many points of $R_i(z)$, then V is in $\tau_{i(z)}$. Thus, $R_i(z) \cup F(z) \in \tau_{i(z)}$. Any union of members of $\tau_{i(z)}$ will contain all but finitely many members of $R_i(z)$, as will any finite intersection of members of $\tau_{i(z)}$. The empty set is in $\tau_{i(z)}$ by construction. □

There are two types of $\tau_{i(z)}$. The first one is simply the co-finite topology (this is the case where $F(z) \subseteq R_i(z)$). The second is strictly finer than the co-finite topology (where $F(z)$ is disjoint from $R_i(z)$). Thus, in the second type, $R_i(z)$ will be open, but is not co-finite.

We will often appeal to the fact that $R_i(z) \in \tau_{i(z)}$. Moreover, we often appeal to the fact that where F is any finite set and $A \subseteq R_i(z) \cup F(z)$,

$$(R_i(z) \cap -F) \cup A \in \tau_{i(z)}.$$

For each R_i,

$$\text{Let } \mathbf{B}_i = \bigcup\{\tau_{i(z)} \mid z \in W\}$$

We need to show that $\mathbf{B}_i$ is a base for a topology on W. To do this we prove the following useful lemma.

Proposition B.5.14. *If $V \in \boldsymbol{B}_i$ and $x \in V$ then $R_i(x) \cap -V$ is finite.*

Proof. Assume $x \in V \in \mathbf{B}_i$. We know that $V \in \tau_{i(b)}$ for some $b \in W$. Thus $V \subseteq R_i(b) \cup F(b)$, and $R_i(b) \cap -V$ is finite.

If $x \in F(b)$, then since fellow copies relate to the same worlds, $R_i(x) = R_i(b)$. And so $R_i(x) \cap -V$ is finite.

If $x \in R_i(b)$, then b bears R_i to x. If xR_iz, then, by transitivity, bR_iz. Thus $R_i(x) \subseteq R_i(b)$. And if bR_iz, then by the Euclidean property, xR_iz. So

$R_i(x) = R_i(b)$, and $R_i(x) \cap -V$ is finite. $\square$

Proposition B.5.15. *Each* $\boldsymbol{B}_i$ *is a basis for a topology on* W*.*

Proof. Recall Fact B.3.1. First, it is clear that $\bigcup \mathbf{B}_i = W$

Second, Let $V_1, V_2 \in \mathbf{B}_i$, and $x \in V_1 \cap V_2$.

$$\text{Let } F_1 = R_i(x) \cap -V_1, \text{ and } F_2 = R_i(x) \cap -V_2$$

By Lemma B.5.14, both F_1 and F_2 are finite.

$$\text{Let } V_3 = (R_i(x) \cap -(F_1 \cup F_2)) \cup \{x\}$$

Since $F_1 \cup F_2$ is finite, $V_3 \in \tau_{i(x)}$, and so $V_3 \in \mathbf{B}_i$. And $x \in V_3 \subseteq V_1 \cap V_2$. $\square$

Let $\tau_1, \tau_2, ..., \tau_n$ be the topologies respectively generated by $\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n$. This gives us the following topological model,

$$M^\tau = < W, \tau_1, \tau_2, ..., \tau_n, V >$$

Where the valuation V is the same one in the relational model M. We'll write τ_i to refer to an arbitrary topology in M^τ (and warn the reader not to confuse τ_i with $\tau_{i(x)}$).

Some more theorems before the truth lemma

The topological frame (which M^τ is based on) is F^τ. We show we've constructed the right frame. That is, we show that the theorems of KD45_n^C are valid in F^τ.

Proposition B.5.16. *Each τ_i is a T_1 space*

Proof. Let x and y be any two points. If $x \notin R_i(y)$, then since $R_i(y) \cup \{y\}$ is open, there is an O which contains y and not x. If $x \in R_i(y)$, then since $(R_i(y) \cap -\{x\}) \cup \{y\}$ is open and y is a member, there is an open set which contains y and not x. □

Proposition B.5.17. *The axioms of KD45$_n$ are valid in F^τ*

Proof. Axiom K and the rule of Necessitation are always valid.

Assume, for some τ_i, and some x, $\{x\} \in \tau_i$. Then $\{x\} \in \tau_{i(z)}$, for some z. But this is impossible, $R_i(z)$ is infinite, and every member of $\tau_{i(z)}$ contains all but finitely many points of $R_i(z)$. Hence Axiom D is valid.

Assume axiom 5 fails, i.e. assume some derived set is not open (for some τ_i). Thus for some subset Z and some x, $x \in d_i(Z)$ and $x \in d_i(-d_i(Z))$.

By construction, $R_i(x) \cup \{x\}$ is open. Thus there is some $v \in R_i(x)$ and $v \in -d_i(Z)$. So there is some $O \in \tau_i$, $v \in O$, and $O \setminus \{v\} \subseteq -Z$.

By Proposition B.5.14, $R_i(v) \cap -O$ is finite. Since $R_i(x)$ is a transitive and Euclidean relation, and $v \in R_i(x)$, $R_i(v) = R_i(x)$.

Thus $R_i(x) \cap -O$ is finite. And so all but finitely many points of $R_i(x)$

are in $-Z$. But by Proposition B.5.16, our space is T_1, and since $x \in d_i(Z)$ and $R_i(x) \cup \{x\}$ is open, $R_i(x) \cap Z$ is infinite. Contradiction.

By Proposition B.4.6, the validity of 5 implies the validity of axiom 4. □

We next show that each τ_i has a finite number of components.

For any set of fellow copies $F(x)$,

$$\text{Let } \beta_i(x) = \{z \in W | (\exists y \in F(x))(zR_i y \vee yR_i z)\}$$

Proposition B.5.18. *If $y \in \beta_i(x)$, then $R_i(x) = R_i(y)$.*

Proof. This follows from the fact that R_i is Euclidean and transitive. □

Proposition B.5.19. *Each $\beta_i(x)$ is connected.*

Proof. Assume otherwise. That is, assume there are some $O_1, O_2 \in \tau_i$, and $O_1 \cap \beta_i(x)$, $O_2 \cap \beta_i(x)$ are two disjoint non-empty sets whose union is $\beta_i(x)$. Let $y_1 \in O_1 \cap \beta_i(x)$ and $y_2 \in O_2 \cap \beta_i(x)$. There must be two basic open sets V_1, V_2, $y_1 \in V_1$ and $y_2 \in V_2$, where $V_1 \cap \beta_i(x)$, $V_2 \cap \beta_i(x)$ are disjoint. Of course, $R_i(x) \subseteq \beta_i(x)$. But by Proposition B.5.18, $R_i(x) = R_i(y_1) = R_i(y_2)$. Thus by Proposition B.5.14, V_1 and V_2 must both contain all but finitely many points of $R_i(x)$, and this is impossible. □

Proposition B.5.20. *Each $\beta_i(x)$ is open*

Proof. Let $z \in \beta_i(x)$. $F(z) \cup R_i(z)$ is a member of τ_i, and so is open, and $z \in F(z) \cup R_i(z)$. If $y \in F(z)$, then by Proposition B.5.10, $y \in \beta_i(x)$, and if $y \in R_i(z)$ then by Proposition B.5.18, $y \in \beta_i(x)$. Thus $F(z) \cup R_i(z) \subseteq \beta_i(x)$, and $\beta_i(x)$ is open. $\square$

Proposition B.5.21. *If xR_ix, any proper superset of $\beta_i(x)$ is disconnected.*

Proof. Let A be some non-empty set disjoint from $\beta_i(x)$, where xR_ix. Let $Z = \{R_i(y) \cup F(y) | y \in A\}$. $A \subseteq \bigcup Z$, and since every member of Z is open $\bigcup Z$ is open. By Proposition B.5.20, $\beta_i(x)$ is open. Thus, if we show that $\bigcup Z$ and $\beta_i(x)$ are disjoint, we know that $\beta_i(x) \cup A$ is disconnected.

Assume $w \in \beta_i(x) \cap \bigcup Z$. No fellow copy of a member of A can be in $\beta_i(x)$, otherwise A and $\beta_i(x)$ intersect. Thus, w can't be a fellow copy of some member of A. So for some $v \in A$, vR_iw. Now, since $w \in \beta_i(x)$, either wR_ix or xR_iw. If wR_ix, then, by transitivity, vR_ix, and so $v \in \beta_i(x)$, contradiction. If xR_iw, then, since xR_ix, wR_ix (by the Euclidean property), and so (by transitivity), vR_ix. So $v \in \beta_i(x)$, contradiction. $\square$

Proposition B.5.22. *If xR_ix, then $\beta_i(x)$ is a component.*

Proof. By Propositions B.5.19 and B.5.21. $\square$

Proposition B.5.23. *Each $\tau_i \in F^\tau$ has only a finite number of components*

Proof. The set of components partitions the space. That is, every point is in one, and only one, component. Let $Z = \{\beta_i(x) | xR_ix\}$.

By Proposition B.5.22, Z is a set of components. But every point in W is in some member of Z (because, by axiom D, every world relates to some world, and by axiom 5, that world must relate to itself). And Z is finite (because $\{F(x) | x \in W\}$ is finite). □

Proposition B.5.24. $F^\tau \in \mathcal{C}_n$

Proof. By Propositions B.5.17 and B.5.23 and the definition of $\mathcal{C}_n$. □

By Proposition B.5.24, and our previous soundness theorem, the axioms of $\mathrm{KD45}_n^C$ are valid in F^τ. We next prove two theorems used to show the common belief case of the truth lemma. To do so, we introduce a simple piece of notation.

Definition B.5.25. For any $A, B \subseteq W$,

$$AR_iB \stackrel{def}{=} (\forall v \in A)(\forall w \in B)(vR_iw)$$

Proposition B.5.26. *Where d_i is the derivative of τ_i,*

$$F(x)R_iF(y) \textit{ iff } F(x) \subseteq d_i(F(y))$$

Proof. Assume $F(x)R_iF(y)$ and let $z \in F(x)$ and $z \in O \in \tau_i$. Thus there is some basic open $V \in \mathbf{B}_i$ and $z \in V \subseteq O$. By Proposition B.5.14, V contains all but finitely many points of $R_i(z)$. Since $F(y) \subseteq R_i(z)$, V contains all but finitely many points of $F(y)$. Thus $z \in d_i(F(y))$.

Conversely, if $F(x)R_iF(y)$ fails, then no member of $F(x)$ bears R_i to any member of $F(y)$ (by Proposition B.5.10). Thus $R_i(x)$ is disjoint from $F(y)$. But $R_i(x) \cup \{x\}$ is open, and so $F(x) \nsubseteq d_i(F(y))$. □

Proposition B.5.27. *If $z \in d_i(X)$, then $\exists v$, $F(z)R_iF(v)$ and $X \cap F(v)$ is infinite.*

Proof. Assume otherwise. Let $b \in d_i(X)$, and for all v, if $F(b)R_iF(v)$ then $X \cap F(v)$ is finite. Let $Y = \bigcup\{F(x) \cap X | F(b)R_iF(x)\}$. Since $\{F(x)|x \in W\}$ is finite, Y is finite, hence closed. $(R_i(b) \cap -Y) \cup \{b\}$ is open. So there is some point $e \in (R_i(b) \cap -Y)$ and $e \in X$. But since $F(b)R_iF(e)$, $F(e) \cap X$ is finite. And $e \in F(e) \cap X$. So $e \in Y$. Contradiction. □

Truth lemma

For all $x \in W$,

$$M, x \models \phi \text{ iff } M^\tau, x \models \phi$$

Proof. The case of the propositional variables is immediate (both models share the same V). The case of the logical connectives is straightforward.

Assume $M, x \models B_i\psi$. Then everything in $R_i(x)$ satisfies ψ in M. By induction hypothesis, everything in $R_i(x)$ satisfies ψ in M^τ. By the construction of $\tau_{i(x)}$, $R_i(x) \cup \{x\} \in \tau_{i(x)}$, and so it is open for τ_i. Hence, $M^\tau, x \models B_i\psi$.

Conversely, assume $M^\tau, x \models B_i\psi$. Thus there is some $O \in \tau_i$, $x \in O$ and everything in $O \setminus \{x\}$ satisfies ψ in M^τ. By induction hypothesis, everything in $O \setminus \{x\}$ satisfies ψ in M. By Proposition B.5.14, $R_i(x) \cap -O$ is finite. And so all but finitely many members of $R_i(x)$ satisfy ψ in M.

Now assume some b in $R_i(x)$ satisfies $\neg\psi$ in M. Then all the fellow copies of b satisfy $\neg\psi$ in M. But b has infinitely many fellow copies, and since xR_ib, $F(b) \subseteq R_i(x)$. But then infinitely many points in $R_i(x)$ satisfy $\neg\psi$ in M. Contradiction. Hence $M, x \models B_i\psi$.

The case of $E\phi$ is straightforward.

Assume $M, x \not\models C\phi$. Since R^* is the transitive closure of the union of the relations in M, there is some finite chain of agents, $h, j, ..., k$, and some finite chain of worlds, $u, v, ...w, y$, such that,

$$xR_hu \ , \ uR_jv \ ... \ wR_ky$$

where $M, y \models \neg\phi$. By Proposition B.5.10, we have,

$$F(x)R_hF(u) \ , \ F(u)R_jF(v) \ ... \ F(w)R_kF(y)$$

By Proposition B.5.26, we have,

$$F(x) \subseteq d_h F(u)\ ,\ F(u) \subseteq d_j F(v)\ ...\ F(w) \subseteq d_k F(y)$$

Furthermore, for all $z \in F(y), M, z \models \neg\phi$.

By induction hypothesis, for all $z \in F(y), M^\tau, z \models \neg\phi$.

To get a contradiction, assume $M^\tau, x \models C\phi$. By Propositions B.5.24 and B.5.7, we know, there is some $O \in \tau_1 \cap \tau_2 \cap ... \cap \tau_n$, $x \in O$, and for all $w \in O \setminus \{x\}$, $M^\tau, w \models \phi$. Also, by Proposition B.5.16 every co-finite set is open in each τ_i. Thus, if we remove a finite number of points from O, we still have a member of $\tau_1 \cap \tau_2 \cap ... \cap \tau_n$.

Since $x \in O$, $O \in \tau_h$, and $x \in d_h F(u)$, $O \setminus \{x\} \cap F(u)$ is non-empty (in fact it is infinite). So some $b \in O \setminus \{x\} \cap F(u)$. Since $O \setminus \{x\} \in \tau_j$ and $b \in d_j F(v)$, $O \setminus \{x, b\} \cap F(v)$ is non-empty ...

Repeating this chain of reasoning (a finite number of times), we end up with $O \setminus F$, where F is a finite set, and $(O \setminus F) \cap F(y)$ is non-empty. Contradiction.

Conversely, assume $M^\tau, x \models \neg C\phi$.

We know, for some finite string of agents $h, j, i, ..., k$,

$$M^\tau, x \models \neg B_h B_j B_i ... B_k \phi$$

For any formula ψ,

$$\text{Let } [\psi]_{M^\tau} = \{z \in W | M^\tau, z \models \psi\}$$

We know, $x \in d_h(d_j(d_i...(d_k([\neg\phi]_{M^\tau})...)))$

By Proposition B.5.27, we know there is some world b, $F(x)R_hF(b)$, and $F(b) \cap (d_j(d_i...(d_k([\neg\phi]_{M^\tau})...))$ is infinite.

By Proposition B.5.27, $\exists c$, $F(b)R_jF(c)$ and $F(c) \cap (d_i...(d_k([\neg\phi]_{M^\tau})...)$ is infinite ...

Repeating this chain of reasoning (a finite number of times) gives us:

$$F(x)R_hF(b), F(b)R_jF(c) \ ... \ F(m)R_kF(n)$$

where $F(n) \cap [\neg\phi]_{M^\tau}$ is infinite. By induction hypothesis, $F(n) \cap [\neg\phi]_M$ is infinite. And one world satisfies a formula iff all the fellow copies of that world satisfy a formula (in M), so $F(n) \subseteq [\neg\phi]_M$.

Furthermore, we know, by Proposition B.5.10,

$$xR_hb, bR_jc \ ... \ mR_kn$$

And so $M, x \models \neg C\phi$. □

This completes the proof.

Appendix C

Completeness For KD45 and KD45F

To be sure, much that is said here assumes familiarity with the earlier parts of Appendix B, as well as a familiarity with relational semantics.

In this section we show what the logic of the derivative for the co-finite topology is KD45. And that the logic for the topology which results from adding the empty set to a principal ultrafilter is KD45F, where the F axiom is $\neg B \neg \phi \rightarrow B\phi$ (the converse of the D axiom).

We make use of generated submodels, relational canonical models, and the following definition.

Definition C.0.28. $R(x) = \{z \mid xRz\}$

The following general lemma is useful for KD45 and KD45F.

C.1 The Splitting Lemma

We start with an arbitrary relational model, $M^A = < W^A, R^A, V^A >$

Let C be any function on W^A, such that, where $w \in W^A$,

$$C(w) \in P(\{w\} \times \mathbb{N}) \setminus \{\{\}\}.$$

Given some $w \in W^A$, $C(w)$ is the set of *copies of* w, and where $x, y \in C(w)$, x and y are fellow copies. The number of copies of w may be finite or infinite, but $C(w)$ is never empty.

$$\text{Let } W = \bigcup\{C(x) \mid x \in W^A\}$$

Each member of W is a member of one and only one $C(x)$.

$$\text{Let } R = \{< a, b > \mid (\exists x)(\exists y)(xR^Ay \wedge a \in C(x) \wedge b \in C(y))\}$$

For each p and each $w \in W^A$,

$$\text{Let } C(w) \subseteq V(p) \text{ if } w \in V^A(p)$$

This gives us a model, $M = < W, R, V >$.

Proposition C.1.1. *For all b, c in W^A,*

$$bR^Ac \text{ iff } (\forall x \in C(b))(\forall y \in C(c))(xRy) \text{ iff } (\exists x \in C(b))(\exists y \in C(c))(xRy)$$

Proof. if bR^Ac then $(\forall x \in C(b))(\forall y \in C(c))(xRy)$, by definition of R. If $(\forall x \in C(b))(\forall y \in C(c))(xRy)$ then $(\exists x \in C(b))(\exists y \in C(c))(xRy)$, since $C(w)$ is never empty, all w. If $(\exists x \in C(b))(\exists y \in C(c))(xRy)$, then, since every world in W is a member of one, and only one set of copies, bR^Ac. □

Proposition C.1.2 (Splitting Lemma)**.** *For all $w \in W^A$,*

$$M^A, w \models \phi \text{ iff } (\forall z \in C(w))(M, z \models \phi) \text{ iff } (\exists z \in C(w))(M, z \models \phi)$$

Proof. Using Proposition C.1.1, the proof goes by induction on the complexity of formulas. □

C.1.1 comments

- A version of the splitting lemma was used to show topological completeness for KD45_n^C.
- The lemma in no way depends on what type of relation R is.
- If $\neg B \neg p \to Bp$, the F axiom, is valid in $< W^A, R^A >$, it may not be valid in $< W, R >$. Thus, frame validity may be lost, but not model validity.

C.2 Soundness and Completeness for KD45

Let $\mathcal{H} = \{< W, \tau >|\ W$ is infinite and τ is the co-finite topology on $W\ \}$. We show,

$$\mathcal{H} \models \phi \text{ iff KD45} \vdash \phi$$

C.2.1 If KD45 $\vdash \phi$ then $\mathcal{H} \models \phi$

Let $< W, \tau > \in \mathcal{H}$. Since W is infinite, there are no open singletons, so D is valid. If $A \subseteq W$ is finite, $d(A)$ is empty, hence clopen. If A is infinite, then $d(A) = W$, and is hence clopen, so 5 and 4 are valid.

C.2.2 If $\mathcal{H} \models \phi$ then KD45 $\vdash \phi$

Every non-theorem of KD45 fails in $M^c =< W^c, R^c, V^c >$, where M^c is the relational canonical model. Let w be any point in the relational canonical model, and let $M^w =< W^w, R^w, V^w >$ be the submodel generated by w. Since R^c is tranitive, $W^w = R^c(w) \cup \{w\}$.

The proof now breaks into two cases. Either $w \in R^w(w)$, or not.

Case One: $w \in R^w(w)$

Since R^w is Euclidean, for all $x, y \in W^w$, xR^wy.

For each $z \in W^w$, let $C(z) = \{z\} \times \mathbb{N} = \{z_1, z_2, z_3, \ldots\}$. Thus C is a function of the type described in Section C.1, that is C is a function on W^w, and $C(x) \in P(\{x\} \times \mathbb{N}) \setminus \{\{\}\}$, for all $x \in W^w$. Define $M =< W, R, V >$ as in Section C.1. Every world in W^w has an infinite number of copies in W.

We know, for all $x \in W^w$,

$$M^w, x \models \phi \text{ iff } (\forall z \in C(x))(M, z \models \phi) \text{ iff } (\exists z \in C(x))(M, z \models \phi)$$

Furthermore,

Proposition C.2.1. *For all x, z in W, xRz.*

Proof. For all x, y in W^w, xR^wy. Assume there are two worlds in W, a and b, and a does not bear R to b. For some $x, y \in W^w$, $a \in C(x)$ and $b \in C(y)$. But then x doesn't bear R^w to y, contradiction. □

Let τ be the co-finite topology on W. This give us a topological model $M^\tau =< W, \tau, V >$, where W and V are the same as in M. We show,

Proposition C.2.2. *For all $x \in W$,*

$$M, x \models \phi \text{ iff } M^\tau, x \models \phi$$

Proof. The atomic sentences are immediate and the logical connectives are straightforward.

Assume $M, x \models B\phi$. By Proposition C.2.1 every world in W satisfies ϕ in M. By induction hypothesis (hereafter IH), every world in W satisfies ϕ in M^τ. Since W is open and x is a member, $M^\tau, x \models B\phi$.

Conversely, assume $M^\tau, x \models B\phi$. There is some O and every point in O, besides x, satifies ϕ, and by IH, every point in $O \setminus \{x\}$ satisfies ϕ in M. And O is co-finite. But every world has infinitely many fellow copies, and if one copy satisfies ϕ in M, then they all must, by Proposition C.1.2. Thus, $M, x \models B\phi$ □

Case Two: $w \notin R^w(w)$

In this case, let $C(w) = \{w_1\}$, and for all other worlds $x \in W^w$, let $C(x) = \{x\} \times \mathbb{N} = \{x_1, x_2, \dots\}$. Thus, w has only one copy, and every other world has infinitely many copies. Let $M =< W, R, V >$ be defined as in Section C.1. And let $M^\tau =< W, \tau, V >$, where τ is the co-finite topology on W.

Proposition C.2.3. *R is serial, transitive, and Euclidean.*

Proof. Since R^w Euclidean, for all $v, x \in R^w(w)$, vR^wx. And similar reasoning to that involved in Proposition C.2.1 shows that for all u, z in $R(w)$,

uRz. Furthermore, since w bears R^w to every world in $R^w(w)$, w_1 will bear R to every world in $R(w)$. Thus R is serial, transitive, and Euclidean. □

Proposition C.2.4. *For all $z \in W$,*

$$M, z \models \phi \text{ iff } M^\tau, z \models \phi$$

Proof. Assume $M, z \models B\phi$. Thus every world z bears R to satisfies ϕ. Since R is transitive and Euclidean, this means that every world in W, with the possible exception of w, satisfies ϕ, and by IH, every world (besides possibly w) satisfies ϕ in M^τ. But $W \setminus \{w\}$ is open, and so $M^\tau, z \models B\phi$ (whether $w = z$ or not).

Conversely, assume $M^\tau, z \models B\phi$. There is some co-finite O and everything in it, besides z, satisfies ϕ. By IH, all but finitely many worlds satisfy ϕ in M. But since every world, besides w, has infinitely many fellow copies, every world, with the possible exception of w, satisfies ϕ. But then, either $z = w$ or not, and either way, z satisfies $B\phi$ in M. □

Recapitulating, every non-theorem of KD45 fails at some world in the canonical model. We then took the generated submodel of a given point w. From there, there proof split into two cases. Either way, we created a new relational model where our non-theorem will fail. From there, we created a topological model for a member of $\mathcal{H}$ where the non-theorem will fail.

C.3 Soundness and Completeness for KD45F

F is the following axiom,

$$\neg B \neg p \rightarrow Bp$$

F is valid in a relational frame iff if xRy, then y is the only world x relates to. Thus, in the relational frames for KD45F, every world relates to one, and only one, world.

Let $\mathcal{G} = \{< W, F \cup \{\{\}\} >| \ F$ is a non-principal ultrafilter on $W\}$

We show,

$$\text{KD45F} \vdash \phi \text{ iff } \mathcal{G} \models \phi$$

C.3.1 If KD45F $\vdash \phi$ then $\mathcal{G} \models \phi$

Let $< W, \tau > \in \mathcal{G}$.

Since non-principal ultrafilters have no finite sets, D is valid. We next show 5 and 4 are valid.

Let $A \subseteq W$. If $A \in \tau$, then, since τ is filtral, if $x \in O$ then $O \setminus \{x\} \cap A$ is non-empty, for all $x \in W$. Thus $d(A) = W$, and is clopen. If $A \notin \tau$, then $-A \in \tau$. For any x, $-A \cup \{x\} \in \tau$. Thus, $x \notin d(A)$, for all x, and $d(A)$ is clopen. Either way, $d(A)$ is clopen, thus 5 and 4 are valid.

Clearly, if axiom F fails in τ, then for some set A and some x, $x \in d(A)$ and $x \in d(-A)$. But either $A \in \tau$ or $-A \in \tau$. But if $A \in \tau$ then $A \cup \{x\} \in \tau$, and so $x \notin d(-A)$, contradiction. And if $-A \in \tau$ then $-A \cup \{x\} \in \tau$, and $x \notin d(A)$, contradiction. Thus axiom F is valid in τ.

C.3.2 If $\mathcal{G} \models \phi$ then KD45F $\vdash \phi$

Since KD45F is relationally canonical, every non-theorem fails at some point in M^c. Let w be any such point, and let $M^w =< W^w, R^w, V^w >$ be the submodel generated by w. Since R^c is transitive, $W^w = R^c(w) \cup \{w\}$.

Now, the underlying frame of M can only be of two types. Either w relates to itself (and no other world), or w relates only to some world z, and z relates to itself (and no other world).

Case One: $w \in R^w(w)$

The underlying frame is,

$$< \{w\}, \{< w, w >\} >$$

Let $C(w) = \mathbb{N} \times \{w\} = \{w_1, w_2, \dots\}$.

Define $M =< W, R, V >$ as in Section C.1. Axiom F is not valid in the underlying frame of the model, there are infinitely many worlds and they all bear R to each other. However, ultimately this doesn't matter. Significantly, $C(w) = W$.

Let F be any non-principal ultrafilter on W, and let $\tau = F \cup \{\{\}\}$. $\tau \in \mathcal{G}$.

This gives us a topological model $M^\tau =< W, \tau, V >$.

Proposition C.3.1. *For all $x \in W$,*

$$M, x \models \phi \text{ iff } M^\tau, x \models \phi$$

Proof. Assume $M, x \models B\phi$. Then $M, z \models \phi$, for all $z \in W$. By IH, every z satisfies ϕ in M^τ. Since W is open, all z satisfy $B\phi$ in M^τ. Conversely, if some x satisfies $B\phi$ in M^τ, then in some O, $x \in O$, and everything in $O \setminus \{x\}$ satisfies ϕ. By IH, everything in $O \setminus \{x\}$ satisfies ϕ in M. Now, since every member of W is a copy of our initial world w, and some worlds in M satisfy ϕ, they all must, by Proposition C.1.2. Thus, $M, x \models B\phi$. □

Case Two: $w \notin R^w(w)$

In this case w relates to only one world, and that world relates only to itself. Call this world z. The frame of the generated submodel is,

$$< \{w, z\}, \{< w, z >, < z, z >\} >$$

Let $C(w) = \{w_1\}$ and let $C(z) = \mathbb{N} \times \{z\} = \{z_1, z_2, \dots\}$. World w has only one copy, w_1, and z has infinitely many copies. Define $M =< W, R, V >$

as in Section C.1. R will be Euclidean. Let F be any non-principal ultrafilter on W, and let $\tau = F \cup \{\{\}\}$. This gives us a topological model $M^\tau =< W, \tau, V >$.

Proposition C.3.2. *For all $x \in W$,*

$$M, x \models \phi \text{ iff } M^\tau, x \models \phi$$

Proof. Assume $M, x \models B\phi$. Either x is w_1 or x is some copy of z. Either way, every z_n satisfies ϕ in M, and by IH, every z_n satisfies ϕ in M^τ. That is, every world in $W \setminus \{w\}$ satisfies ϕ in M^τ, and $W \setminus \{w\}$ is open. Thus, $M^\tau, x \models B\phi$.

Conversely, assume $M^\tau, x \models B\phi$. Then there is some infinite O, and everything in $O \setminus \{x\}$ satisfies ϕ. By IH, everything in $O \setminus \{x\}$ satisfies ϕ in M. Since O is infinite, some copy of z satisfies ϕ in M, and so by Proposition C.1.2, every copy of z satisfies ϕ in M. Since w_1 doesn't bear R to itself, no world does (by the Euclidean property), and so $M, x \models B\phi$. □

C.3.3 comments

- What is perhaps suprising about the proof is how simple it is. Also, it is perhaps suprising that the relational models for KD45F are very simple, and yet the topological models require the axiom of choice to construct.

- Philosophically speaking, KD45F is the logic of an idealized agent who has made up thier mind about everything (Axiom F is $Bp \vee B\neg p$).

Appendix D

Various Theorems

D.1 Topological Correspondence

A sentence ϕ is said to *correspond* to some condition C iff

$$\phi \text{ is valid iff } C \text{ holds}$$

Thus, 4 is valid (in a relational frame) iff R is transitive. 4 corresponds to transitivity. Topologically, it corresponds to every derived set being closed, $d(d(A)) \subseteq d(A)$. At first glance, there doesn't seem to be a relationship between transitivity and the condition that all derived sets are closed. However, it is clear from the next theorem that there is a relationship.

Proposition D.1.1. *4 is valid in a topological frame* $< W, \tau >$ *iff for all* *subsets A, B, C of W, if $A \subseteq d(B)$ and $B \subseteq d(C)$ then $A \subseteq d(C)$.*

Proof. 4 is valid iff $d(d(A)) \subseteq d(A)$. Thus assume $d(d(A)) \subseteq d(A)$ is valid and $A \subseteq d(B)$ and $B \subseteq d(C)$, but $A \not\subseteq d(C)$. So some $x \in A$ and $x \notin d(C)$. So $x \notin d(d(C))$. Since d is monotonic, $d(B) \subseteq d(d(C))$, and so $x \notin d(B)$. But then $x \notin A$, contradiction. Conversely, assume some $x \in d(d(A))$ and $x \notin d(A)$. But then $\{x\} \subseteq d(d(A))$ and $d(A) \subseteq d(A)$ but $\{x\} \not\subseteq d(A)$. Done. □

What this suggests is that the topological semantics is a higher order semantics. Relationally, the 4 axiom corresponds to transitivity on points, and topologically it corresponds to transitivity on sets of points (we've replaced aRb with $A \subseteq d(B)$).

D is relationally valid iff $(\forall x)(\exists y)(xRy)$. Compare,

Proposition D.1.2. *D is valid in $< W, \tau >$ iff $\forall A \subseteq W$, $\exists C \subseteq W$, $A \subseteq d(C)$.*

Proof. We know that D is valid iff there are no open singletons. If $A \subseteq W$, $\exists C \subseteq W$, $A \subseteq d(C)$, then, instantiating $\{w\}$ for A, we know $\{w\}$ can't be open. Thus, D is valid. Conversely, if there is a set A such that for all C, $A \not\subseteq d(C)$, then $A \not\subseteq d(W)$, and so A contains some open singleton. □

The analogy holds for some other axioms, but not all (e.g. axiom 5). To suggest a theorem: perhaps the analogy holds for an axiom Z if, and only if, $K4 + Z$ is complete with respect to the topological canonical model.

D.2 S4∩GL≠K4

Professor Parikh had been wondering whether S4∩GL=K4. The answer is no. GL is K plus the following axiom,

$$B(Bp \rightarrow p) \rightarrow Bp$$

Since the 4 axiom follows from this axiom, both S4 and GL are extensions of K4.[1] Clearly, the following is a theorem of GL and S4,

$$(B(Bp \rightarrow p) \rightarrow Bp) \vee (Bq \rightarrow q)$$

Yet it is not a theorem of K4. Consider the following frame: $< \mathbb{N}, < >$. Consider the model where $V(q) = -\{1\}$ and $V(p) = \{\}$. Then 1 satisfies $Bq \wedge \neg q$ and $B(Bp \rightarrow p) \wedge \neg Bp$. This falsifies the aforementioned theorem of GL∩S4. Since $<$ is transitive on $\mathbb{N}$, and K4 is valid in all transitive frames, this theorem of GL∩S4 is not a theorem of K4. That is, S4∩GL≠K4.

[1] See [4] for a proof of 4 using GL, and for more on GL.

D.3 Two Classes Of Topologies

In this section we simply mention, without proof, axiomatizations for two different classes of topologies.

D.3.1 K + $BB\bot$

Let W be a non-empty set and let $w \in W$. Consider,

$$\tau = \{O \subseteq W \mid w \notin O\} \cup \{W\}$$

τ is the excluded point topology on W, where w is the excluded point. Let $\mathcal{E}$ be the class of all excluded point topological spaces. One can show,

$$\text{K} + BB\bot \vdash \phi \text{ iff } \mathcal{E} \models \phi$$

We show soundness, and leave completeness as an exercise.

If K + $BB\bot \vdash \phi$ then $\mathcal{E} \models \phi$

Let $< W, \tau > \in \mathcal{E}$, and let $w \in W$. Either w is the excluded point, or not. If not, then $\{w\}$ is open, and so w satisfies $B\bot$ in any model whatsoever. And since $B\bot \rightarrow BB\bot$ is a theorem of K, w satisfies $BB\bot$ in any model. If w is the excluded point, then W is the only open set of which w is a member. As just shown, every member of W, besides w, will satisfy $B\bot$, in any model.

Thus w will satisfy $BB\bot$ in any model.

Exercise

Show,

If $\mathcal{E} \models \phi$ then K + $BB\bot \vdash \phi$.

D.3.2 GLF

GL is K plus the following axiom,

$$B(B\phi \rightarrow \phi) \rightarrow B\phi$$

GLF is GL plus the following axiom,

$$\neg B\neg p \rightarrow Bp$$

Leo Esakia [9] showed GL is the logic of scattered spaces. Where a space is *scattered* iff every non-empty subspace has an open singleton. GL is valid in all finite irreflexive and transitive frames. Here we add axiom F to GL and show soundness and completeness. Though GL is not relationally canonical [4], GLF is relationally canonical. In the frames for GLF, if xRy then y is a dead-end world, and x bears R to no other world.

$$\text{Let } \mathcal{Q} = \{< W, F \cup \{\{\}\} \mid F \text{ is a principal ultrafilter on } W \ \}$$

One can show,

$$\text{GLF} \vdash \phi \text{ iff } \mathcal{Q} \models \phi$$

We show soundness, and leave completeness as an exercise.

If GLF $\vdash \phi$ then $\mathcal{Q} \models \phi$

The members of $\mathcal{Q}$ are otherwise known as particular point topologies, formed by taking the set of all sets containing a given point, and the empty set. Assume, x satisfies $\neg B\phi$ in some model based on a member of $\mathcal{Q}$. Let z be the particular point. Since $\{x, z\}$ is open, z satisfies $\neg\phi$. Since $\{z\}$ is open, z satisfies $B\phi$. Thus $x \models \neg B\neg(B\phi \wedge \neg\phi)$. By contraposition, $B(B\phi \to \phi) \to B\phi$ is valid.

Assume axiom F fails. That is, $x \models \neg B\neg\phi \wedge \neg B\neg\neg\phi$. Let z be the particular point. x can't be z, because $\{z\}$ is open. And $\{x, z\}$ is open. Thus z satisfies ϕ and $\neg\phi$. Contradiction.

Exercise

Show,

If $\mathcal{Q} \models \phi$ then GLF $\vdash \phi$.

D.4 Closure cannot define the derivative

The claim in the title of this section is intuitively obvious. All the same, it is nice to have a proof. Since $d(A) \cup A = Cl(A)$, the derivative operator can define the closure operator. But the closure operator cannot define the derivative. To see this, recall that the D axiom, when the diamond is interpreted as the derivative, is valid iff there are no open singletons.

Now, assume the claim is false. Then there is some sentence S, such that, when the diamond is interpreted as closure, S is valid iff there are no open singletons. Since S4 is valid in all topologies, and some topologies do have open singletons, S is not a theorem of S4. However, every non-theorem of S4 can be falsified in a topology with no open singletons, [2]. Thus, S can be falsified in a topology with no open singletons. Contradiction.

Appendix E

A Discussion Of Grim's Arguments

Using a Cantorian style argument, Grim argues that there can't be a set of all true propositions. Let $T = \{t_1, t_2, ...\}$ be the set of all true propositions, where each t_j is some truth. Now consider the power set of T, $P(T)$. Now, for each non-empty $s \in P(T)$, and each $t_j \in s$, ' $t_j \in s$ ' is true (clearly). Thus, there are at least as many truths as there are members of $P(T)$. But $P(T)$ has a greater cardinality than T, and so there are more truths than there are truths. Contradiction.

Grim goes on to give a number of arguments that there can be no set of truths for various types of truths. Thus, he argues that there can be no set of contingent truths, and going further argues that there can be no set of contingent true atomic propositions, and Grim uses this last argument to conclude there are no possible worlds. However, Grim is assuming that a possible world is a maximal set of propositions (following Plantiga, *inter alia*). Here, however, we have been assuming all along that propositions are sets of possible worlds (e.g., the set of all worlds where cats exist *is* the proposition: cats exist). Thus, only at risk of circularity could we assume that worlds are sets of propositions. Furthermore, Grim states that his arguments go through regardless of whether the arguments are put in terms of sentences or propositions. This is not true. Consider the argument in the previous paragraph. In possible world semantics, every necessary truth is in effect the same proposition; namely, the set of all possible worlds. Furthermore, set theoretic claims are typically assumed to be necessary truths, so if some truth is a member of a set, it is necessarily a member of that set. Considering this, each such truth will in effect be the same proposition, and Grim's previous argument will fail.

Grim, like many, should object that having only one necessary proposition is a rather strange aspect of possible world semantics. Frankly, I agree, despite the fact that I've been using this terminology all along. However, a minor change in our how we speak about possible worlds semantics should help us. Lewis famously claimed " In my usage, any set of worlds is by definition a proposition.[1] "

The minor change I suggest is this: instead of saying that a set of worlds is a proposition, we'll say that a set of possible worlds where some sentence is true is the *extension* of the sentence. The analogy of course, is with the extension of a predicate (the set of all things the predicate applies to). Just as there are dangers of conflating a property with the extension of that property, there are dangers involved in conflating a proposition with the extension of that proposition. In the first case, the danger is losing the very useful and interesting notion that two different predicates expressing different properties may have the same extension (e.g. renates and cordates). In the second case, the danger is losing the very useful and interesting notion that two different sentences expressing different propositions may be true at the same set of worlds (in particular, necessarily true propositions).

Having made this minor change in our terminology, this certainly doesn't mean we need throw away possible worlds semantics altogether. We simply

[1] [19], p. 105

admit that the semantics can only represent the extension of a sentence, and not the intension. Despite how natural this suggestion is, it seems to be in direct contradiction with common usage amongst modal logicians.[2]

In Carnap's *Meaning and Necessity*, he writes "...we must take as extensions of sentences something that equivalent sentences have in common. The most natural choice seems the truth values." [3] On the contrary, if the extension of a predicate is defined as the set of all things the predicate is *true of*, the most natural choice for the extension of a sentence is the set of all worlds the sentence is *true at*. With all this in mind, we can now slightly update our previous interpretation as follows; if $< W, \tau >, A \subseteq W$ and $w \in W$, read $w \in d(A)$ to mean:

There is *some proposition* whose extension is A,

and that proposition is within the limit of rational belief at world w

Having made this change, we can now lay out Grim's argument, whose conclusion is that there can be no set of truths at a given world. Let $S = \{p_1, p_2 ...\}$ be the set of all true propositions at world s. Let $P(S)$ be the

[2]In [10], the intension of a sentence is defined as the set of all worlds where that sentence is true.

[3]p.26 [5]

power set of S. Consider all propositions true at s which are expressed by sentences of the form: $n \wedge (m \in D)$ (where 'n' and 'm' are names for members of S, and 'D' is a name for a member of $P(S)$). For every member of $P(S)$ there will be some particular proposition expressed by such a sentence, and so there are more true sentences at world s than there are at s. Contradiction. Thus, at least in terms of sets, the collection of all truths at a given world makes no sense.

One simple response to this is to say that such a collection is not a set, but a proper class. Technically speaking, this saves us from strict contradiction, but philosophically speaking, this is far from satisfactory. Proper classes may as well be called 'paradoxical sets'. Indeed, when Cantor anticipated this distinction (before von Neumann), he made it in terms of 'consistent multiplicities' and 'inconsistent multiplicities'.[4]

Now, in earlier sections, singletons containing a world were taken to be the proposition expressing all truths at that world. Keeping in mind our revised terminology, *propositions* are no longer in the formal semantics, all we have are extensions of sentences (sets of worlds). Considering the revised interpretation above, the T_1 condition (for all x, y, $x \notin d(\{y\})$) means:

[4]See [22] p. 122. This distinction is made in a personal letter to Dedekind.

Any proposition whose extension is $\{y\}$ is always

outside the limit of rational belief

Bibliography

[1] J. Adler. *Belief's Own Ethics.* MIT Press, Cambridge, Mass., 2002.

[2] M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors. *Handbook on logics of space.* Kluwer Academic Publishers, In preparation, 2006.

[3] G. Bezhanishvili, R. Mines, and P. Morandi. Scattered, hausdorff-reducible, and hereditary irresolvable spaces. *Topology and its Applications*, 132:291–306, 2003.

[4] G. Boolos. *The Logic Of Provability.* Cambrdige University Press, Cambridge, Mass., 1993.

[5] R. Carnap. *Meaning And Necessity.* University Of Chicago Press, Chicago, Ill., 1988.

[6] A. Dabrowski, L. Moss, and R. Parikh. Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, 78:73-110, 1996.

[7] J. W. Dauben. *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Harvard University Press, Cambridge, Massachusetts, 1979 (reprinted 1990).

[8] J. Dugudnji. *Topology*. Prentice-Hall, Englewood Cliffs, NJ, 1965.

[9] L. Esakia. Diagonal construction, Löb's formula and Cantor's scattered spaces. *Logical And Semantical Investigations*, pages 128–143, 1981. (In Russian. Tbilisi, Academy Press).

[10] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning About Knowledge*. MIT Press, Cambridge, Massachusetts, 2003.

[11] K. Georgatos. *Modal Logics For Topological Space (Ph.D. Dissertation)*. Graduate School And University Center, City University of New York, 1993.

[12] E. Gettier. Is justified true belief knowledge? *Analysis*, 23:121–123, 1963.

[13] P. Grim. *The Incomplete Universe*. MIT Press, Cambridge, Mass., 1991.

[14] Bernhard Heinemann. The hybrid logic of linear set spaces. *Logic Journal of the IGPL*, 12(3):181–198, 2004.

[15] J. Hintikka. *Knowledge And Belief.* Cornell University Press, Ithaca, New York, 1962.

[16] G.E. Hughes and M.J. Cresswell. *A New Introduction To Modal Logic.* Routledge, New York, NY, 1996.

[17] D. Kaplan and R. Montague. A paradox regained. *Notre Dame Journal of Formal Logic*, 1:79–90, 1960.

[18] K.T. Kelly. *The Logic of Reliable Inquiry.* Oxford University Press, New York, 1996.

[19] D. Lewis. *On The Plurality Of Worlds.* Blackwell, Malden, Mass., 1986.

[20] J.C.C. Mckinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[21] R. Montague. Syntactic treatments of modality with corollaries on reflexion principles and finite axiomatizability. *Acta Philosophica Fennica*, 16:153–167, 1963.

[22] G. Priest. *Beyond the Limits of Thought.* Oxford, New York, New York, 2nd edition, 2002.

[23] D. Sarenac and J. van Benthem. The geometry of knowledge. Technical Report PP-2004-20, ILLC, 2004. Also available at http://www.illc.uva.nl/Publications/reportlist.php?Year=2004.

[24] V. Shehtman. Derived sets in euclidean spaces and modal logic. Technical Report Preprint X-90-05, University of Amsterdam, 1990.

[25] S. Vickers. *Topology Via Logic.* Cambridge University Press, Cambridge, 1989.

[26] T. Williamson. *Knowledge And Its Limits.* Oxford University Press, New York, 2000.