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A CHARACTERIZATION OF LATTICE NILPOTENT GROUPS OF CLASSES TWO AND THREE

by

CAROL HEINES JACOBOWITZ

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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SECTION I. INTRODUCTION

Let C be a compact manifold and $\mathfrak{n}$ a connected simply connected nilpotent Lie group. We say that the group $\mathfrak{n}$ acts on C if there exists a mapping

$$\varphi: C \times \mathfrak{n} \to C$$

which depends continuously on both variables and, when we denote $\varphi(c,n)$ by cn ($c \in C$, $n \in \mathfrak{n}$), satisfies

(i) $(cn_1)n_2 = c(n_1n_2)$ and

(ii) $ce = c$

where c is an arbitrary point in C, n_1 and n_2 are arbitrary elements of $\mathfrak{n}$ and e is the identity of $\mathfrak{n}$.

The set $\mathfrak{z}$ of elements of $\mathfrak{n}$ which leave all points of C fixed (i.e., $\mathfrak{z} = \{f \in \mathfrak{n}: cf = c \text{ for all } c \in C\}$) is a closed normal subgroup of $\mathfrak{n}$. If $\mathfrak{z} = \{e\}$ we say that $\mathfrak{n}$ acts effectively on C. We say that $\mathfrak{n}$ acts transitively on C if, for any two points c_1 and c_2 of C, there exists an element n of $\mathfrak{n}$ such that $c_1n = c_2$.

Let us assume that $\mathfrak{n}$ acts effectively and transitively on C. Then C is called a nilmanifold.

For a given point $c \in C$, the set Γ of elements n in $\mathfrak{n}$ for which $cn = c$ is a closed subgroup of $\mathfrak{n}$ and is called the stable subgroup of $\mathfrak{n}$ corresponding to the point c. The nilmanifold C is isomorphic to the space of cosets of $\mathfrak{n}$ by Γ.

A. I. Mal'cev, in [5], showed that with the above assumptions the stable subgroups Γ of $\mathfrak{N}$ are discrete.

The lower central series of $\mathfrak{N}$ is defined to be the system of normal subgroups

$$\mathfrak{N} = \mathfrak{N}^1 \supset \mathfrak{N}^2 \supset \mathfrak{N}^3 \supset \cdots \supset \mathfrak{N}^\ell \supset \mathfrak{N}^{\ell+1} = e$$

where $\mathfrak{N}^i$ is generated by the elements $g^{-1}h^{-1}gh$, $g \in \mathfrak{N}$, $h \in \mathfrak{N}^{i-1}$, for $i = 2,3,\ldots,\ell+1$. Let N be the Lie algebra of $\mathfrak{N}$. The lower central series of N is defined to be the system of ideals

$$N = N^1 \supset N^2 \supset \ldots \supset N^\ell \supset N^{\ell+1} = 0$$

where N^1 is generated by the elements $[a,b]$, $a \in N$, $b \in N^{i-1}$ for $i = 2,3,\ldots,\ell+1$. If the lower central series of $\mathfrak{N}$ (and hence also that of N) is of length ℓ i.e. contains ℓ non-trivial subgroups (ideals), then we say that $\mathfrak{N}$ (and hence N) is ℓ-step nilpotent.

Since $\mathfrak{N}$ is simply connected, connected and nilpotent, the exponential mapping, denoted by exp, is a homeomorphism of N onto $\mathfrak{N}$. Its inverse mapping is called the logarithm mapping and is denoted by log. The Campbell-Hausdorff formula gives the relationship between the group multiplication in $\mathfrak{N}$ and the Lie algebra operations in N: If x, y and z are elements of N such that

$$\exp(x)\exp(y) = \exp(z) ,$$

then

$$z = \sum_{m=1}^{\infty} \sum_{p_i,q_i=1}^{\infty} \frac{(-1)^{m-1}}{m \sum_{i=1}^{m} (p_i+q_i)}$$

$$\frac{\overbrace{[\ldots[x,x],\ldots,x]}^{p_1},\overbrace{y],\ldots,y]}^{q_1},\ldots,\overbrace{x],\ldots,x]}^{p_m},\overbrace{y],\ldots,y]}^{q_m}}{p_1!\,q_1!\,\ldots\,p_m!\,q_m!}$$

(cf. [4], p. 173). Computing the first few terms we obtain that

$$z = x + y + \tfrac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]]$$
$$- \frac{1}{24}[x,[y,[x,y]]] + \ldots \quad .$$

Since N is nilpotent there are finitely many remaining terms.

In this paper we shall be concerned with discrete cocompact subgroups Γ of $\mathfrak{N}$ having the special property that $\log(\Gamma)$ is an additive subgroup of N. If it has this property, Γ is called a lattice nilpotent group. In the case where $\mathfrak{N}$ is abelian, every discrete cocompact subgroup is a lattice group. If $\mathfrak{N}$ is the Lie group of three-dimensional matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,y,z \in \mathbb{R},$$

it contains the following two discrete cocompact subgroups:

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} : n_1, n_2, n_3 \in \mathbf{Z} \right\}$$

and

$$\Gamma_2 = \left\{ \begin{pmatrix} 1 & n_1 & n_3/2 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} : n_1, n_2, n_3 \in \mathbf{Z} \right\}$$

In this case N consists of matrices of the form

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} , \quad x,y,z \in R ,$$

and log: $\mathcal{N} \to N$ is given by

$$\log(M) = M - I - \frac{(M - I)^2}{2} ,$$

where M is a matrix in $\mathcal{N}$ and I is the identity matrix. Using this equation it can be shown that $\log(\Gamma_2)$ is a lattice in N, but $\log(\Gamma_1)$ is not.

We shall produce a condition which is necessary and sufficient for a discrete cocompact subgroup Γ of a connected simply connected nilpotent Lie group $\mathcal{N}$ to be a lattice group, in both the two-step and three-step cases. These conditions enable us to describe all the lattice nilpotent groups of the free two-step nilpotent Lie group and the free three-step nilpotent Lie group. We shall prove that every lattice subgroup of a connected simply connected nilpotent Lie group $\mathfrak{n}$ is the homomorphic image of a lattice subgroup of a free nilpotent Lie group; hence we indirectly describe all the lattice subgroups of $\mathfrak{n}$ when $\mathfrak{n}$ is two- or three-step nilpotent.

SECTION II. CONNECTED SIMPLY CONNECTED NILPOTENT LIE GROUPS AND THEIR DISCRETE COCOMPACT SUBGROUPS

In this section we give some definitions and facts concerning the above groups which reveal a great deal about their structure. Further details as well as proofs can be found in [5].

Canonical bases. We define a nilpotent group canonical basis of an abstract nilpotent group G to be a finite set of elements $g_1, g_2, \ldots, g_r$ of G such that

(i) every element g of G can be written in the form $g = g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$ where $n_1, n_2, \ldots, n_r$ are integers;

(ii) $G_i = \{g \in G : g = g_i^{n_i} g_{i+1}^{n_{i+1}} \ldots g_r^{n_r}\}$ is a normal subgroup of G, for $i = 1,2,\ldots,r$; and if we set $G_{r+1} = \{1\}$ then

(iii) G_i/G_{i+1} is infinite cyclic for $i = 1,2,\ldots,r$.

We define a nilpotent Lie algebra canonical basis to be a basis $e_1, e_2, \ldots, e_r$ of a nilpotent Lie algebra N which has the property that the collection of elements spanned by $e_i, e_{i+1}, \ldots, e_r$ is an ideal of N for $i = 1,2,\ldots,r$.

We will shorten these terms to "canonical basis" in each case so long as no confusion results.

Coordinate systems. Let $\mathfrak{n}$ be a connected simply connected nilpotent Lie group and let N be its Lie algebra.

For a given canonical basis $e_1, e_2, \ldots, e_r$ of N, every element a in N has a representation of the form $a = \sum_{i=1}^{r} a_i e_i$ where $a_1, a_2, \ldots, a_r$ are real numbers. We say that $e_1, e_2, \ldots, e_r$ is a coordinate system of the first type for $\mathfrak{n}$, and $\alpha = \exp(a)$ has coordinates $a_1, a_2, \ldots, a_r$ with respect to this system.

Suppose $\mathfrak{n}$ has a system of one-parameter subgroups $x_1(t), x_2(t), \ldots, x_r(t)$ such that

(i) each element α in $\mathfrak{n}$ can be represented in the form $x_1(t_1)x_2(t_2)\ldots x_r(t_r)$ where $t_1, t_2, \ldots, t_r$ are real numbers;

(ii) $\mathfrak{n}_i = \{\alpha \in \mathfrak{n} : \alpha$ can be represented as $x_i(t_i)x_{i+1}(t_{i+1})\ldots x_r(t_r)\}$ is a closed normal subgroup of $\mathfrak{n}$; and if we set $\mathfrak{n}_{r+1} = \{1\}$ then

(iii) $\mathfrak{n}_i/\mathfrak{n}_{i+1}$ is a one-parameter vector group for $i = 1, 2, \ldots, r$.

Then $x_1(t), x_2(t), \ldots, x_r(t)$ are said to be a coordinate system of the second type for $\mathfrak{n}$, and if $\alpha = x_1(t_1)x_2(t_2)\ldots x_r(t_r)$ we say α has coordinates $t_1, t_2, \ldots, t_r$ with respect to this system.

A system of coordinates of the second type $x_1(t), x_2(t), \ldots, x_r(t)$ determines a corresponding system of coordinates of the first type $e_1, e_2, \ldots, e_r$ where $x_i(1) = \exp(e_i)$ for $i = 1, 2, \ldots, r$. Likewise a system of the first type determines a system of the second type.

We can now state a number of results which we will draw upon later.

A1. Every finitely generated torsion-free nilpotent group can be embedded in a simply connected nilpotent analytic group as a discrete cocompact subgroup.

A2. Every discrete cocompact subgroup of a connected simply connected nilpotent Lie group contains at least one canonical basis. (Hence such a subgroup is finitely generated and torsion-free.)

A3. If Γ_1 and Γ_2 are discrete cocompact subgroups of the connected simply connected nilpotent Lie groups $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively, then every isomorphism between Γ_1 and Γ_2 can be extended to a topological isomorphism between $\mathfrak{n}_1$ and $\mathfrak{n}_2$.

A4. Suppose $\mathfrak{n}$ is a connected, simply connected nilpotent Lie group with Lie algebra N and $e_1,\dots,e_r$ is a canonical basis for N. Furthermore, suppose the constants of structure of N with respect to $e_1,\dots,e_r$ are rational. Then $\exp(e_1),\dots,\exp(e_r)$ generate a discrete cocompact subgroup of $\mathfrak{n}$.

A5. If $x_1(t),x_2(t),\dots,x_r(t)$ is a coordinate system of the second type for $\mathfrak{n}$ and G is a subgroup of $\mathfrak{n}$ containing $x_1(1),x_2(1),\dots,x_r(1)$, then $\mathfrak{n}/\overline{G}$ is compact.

A6. If $\gamma_1, \gamma_2, \ldots, \gamma_r$ is a canonical basis for the discrete cocompact subgroup Γ of $\mathfrak{h}$, then the one-parameter subgroups through $\gamma_1, \gamma_2, \ldots, \gamma_r$ form a coordinate system of the second type for $\mathfrak{h}$. $\operatorname{Log}(\gamma_1), \log(\gamma_2), \ldots, \log(\gamma_r)$ is the corresponding coordinate system of the first type, and N has rational constants of structure with respect to this basis.

A7. If Γ is a discrete cocompact subgroup of $\mathfrak{h}$ and $\gamma_1, \gamma_2, \ldots, \gamma_r$ is a canonical basis for Γ, then the subalgebra of N consisting of all rational linear combinations of $\log(\gamma_1), \log(\gamma_2), \ldots, \log(\gamma_r)$ is called the rational algebra of Γ and is denoted $Q(\Gamma)$. $Q(\Gamma)$ is independent of the choice of canonical basis for Γ.

Unipotent algebraic groups. The Lie groups we are studying may be looked at from another point of view, and it will be extremely helpful to do so. We refer the reader to [7] for definitions of the terms used below and for proofs of (1), (3), and (4).

Let M(n) be the collection of all $n \times n$ real matrices. Let T(n) be the collection of all nil upper triangular matrices in M(n), and U(n) all unipotent upper triangular matrices in M(n). Let $\mathfrak{h}$ be a simply connected nilpotent analytic group.

B1. U(n) is an algebraic group with Lie algebra T(n), and exp: $T(n) \to U(n)$ is a homeomorphism under the Zariski topology.

B2. (Birkhoff embedding theorem. Cf. [2].) There exists a faithful analytic group representation $\varphi:\mathfrak{h} \rightarrow U(n)$ for some positive integer n.

B3. An algebraic subgroup of $\mathfrak{h}$ (i.e. of $\varphi(\mathfrak{h})$) is connected. Conversely a connected subgroup of $U(n)$ is algebraic. In particular, $\varphi(\mathfrak{h})$ is an algebraic group.

If S is a subset of an algebraic group, then the algebraic hull of S is defined to be the smallest algebraic subgroup containing S and is denoted $\mathcal{G}(S)$.

B4. Let C be a closed subgroup of $\mathfrak{h}$. Then $\mathfrak{h}/C$ is compact if and only if $\mathcal{G}(C) = \mathfrak{h}$.

B5. (Cf. [3], p. 105.) Let G_1 be a subgroup of $\mathfrak{h}$ and G_2 a normal subgroup of G_1. Then $\mathcal{G}(G_2)$ is a normal subgroup of $\mathcal{G}(G_1)$.

SECTION III. LATTICE NILPOTENT GROUPS

Theorem (C. C. Moore): Let Γ be a discrete cocompact subgroup of the connected simply connected nilpotent Lie group $\mathfrak{n}$. Then there exist lattice nilpotent groups Γ_1 and Γ_2 of $\mathfrak{n}$ such that $\Gamma_1 \supset \Gamma \supset \Gamma_2$.

The proof of this theorem appears in [6], pages 155-158.

Lemma 1: Let Γ be a discrete cocompact subgroup of a connected simply connected nilpotent Lie group $\mathfrak{n}$, and K a normal subgroup of Γ such that Γ/K is torsion-free. Then $\mathfrak{G}(K) \cap \Gamma = K$.

Proof: We note that since Γ is discrete, its subgroups K and $\mathfrak{G}(K) \cap \Gamma$ are discrete. By (B3), $\mathfrak{G}(K)$ is a connected subgroup of $\mathfrak{n}$; therefore $\mathfrak{G}(K)$ is simply connected. Thus $\mathfrak{G}(K)/K$ must be compact by (B4). Since $\mathfrak{G}(K) \cap \Gamma$ contains K, $\mathfrak{G}(K)/(\mathfrak{G}(K)\cap\Gamma)$ is also compact.

Suppose γ is an element of the group $\mathfrak{G}(K) \cap \Gamma$ but γ is not in K. Let

$$\pi\colon \mathfrak{G}(K) \cap \Gamma \to (\mathfrak{G}(K) \cap \Gamma)/K$$

be the projection mapping. Since Γ/K is torsion-free, its subgroup $(\mathfrak{G}(K) \cap \Gamma)/K$ is torsion-free. Hence the sequence

$$\{\pi(\gamma^n)\colon n = 1,2,\dots\}$$

is an infinite subset of $(\mathfrak{G}(K) \cap \Gamma)/K$. But since $\mathfrak{G}(K) \cap \Gamma$

and K are both discrete cocompact subgroups of $\mathfrak{G}(K)$, $(\mathfrak{G}(K) \cap \Gamma)/K$ must be discrete and compact, hence finite. This contradiction implies that γ must be an element of K, and therefore $\mathfrak{G}(K) \cap \Gamma = K$.

Theorem A: Let Γ be a lattice nilpotent group of $\mathfrak{h}$, and let G be a torsion-free homomorphic image of Γ. Then G is a lattice nilpotent group.

Proof: G is the homomorphic image of a finitely generated nilpotent group, hence is itself finitely generated and nilpotent. Since G is also torsion-free, it can be embedded in a simply connected nilpotent analytic group $\mathfrak{m}$ as a discrete cocompact subgroup.

Suppose K is the normal subgroup of Γ such that Γ/K is isomorphic to G, and let $\varphi\colon \Gamma \to G$ be the homomorphism with kernel K. Since K is normal in Γ, $\mathfrak{G}(K)$ is normal in $\mathfrak{G}(\Gamma)$. But $\mathfrak{G}(\Gamma) = \mathfrak{h}$; hence $\mathfrak{G}(K)$ is normal in $\mathfrak{h}$. Moreover, $\mathfrak{G}(K)$ is connected, so $\mathfrak{h}/\mathfrak{G}(K)$ is a simply connected analytic group. Let $\pi\colon \mathfrak{h} \to \mathfrak{h}/\mathfrak{G}(K)$ be the projection mapping. $\pi(\Gamma) = \Gamma \cdot \mathfrak{G}(K)/\mathfrak{G}(K) \cong \Gamma/[\mathfrak{G}(K) \cap \Gamma]$. Therefore, by lemma 1, $\pi(\Gamma)$ is isomorphic to Γ/K. Since $\pi(\Gamma)$ is cocompact and discrete in $\mathfrak{h}/\mathfrak{G}(K)$ and $\pi(\Gamma)$ is isomorphic to G, we must have $\mathfrak{m}$ isomorphic to $\mathfrak{h}/\mathfrak{G}(K)$.

Let $\theta\colon \mathfrak{h} \to \mathfrak{m}$ be the homomorphism with kernel $\mathfrak{G}(K)$. Note that the restriction of θ to Γ is the homomorphism

$\varphi: \Gamma \to G$. If N and M are the Lie algebras of $\mathfrak{n}$ and $\mathfrak{m}$ respectively and $d\theta$ is the differential of θ, the following diagram is commutative:

$$\begin{tikzcd}
N \arrow[r, "d\theta"] \arrow[d, "\exp"'] & M \arrow[d, "\exp"] \\
\mathfrak{n} \arrow[r, "\theta"'] & \mathfrak{m}
\end{tikzcd}$$

$d\theta \circ \log(\Gamma) = \log \circ\, \theta(\Gamma) = \log(G)$. Since $\log(\Gamma)$ is a lattice and $d\theta$ is a Lie algebra homomorphism, $\log(G)$ is also a lattice.

SECTION IV. FREE NILPOTENT LIE GROUPS

Let $x_1, x_2, \ldots, x_k$ be a finite set of symbols. We form the free Lie algebra L_k on these symbols as follows (cf. [1]):

Let W be the set of non-associative words in $x_1, x_2, \ldots, x_k$ and let M(W) be the set of linear combinations over the real numbers of elements of W. Define a bilinear mapping $M(W) \times M(W) \to M(W)$ by $(u,v) \to uv$ as follows:

If $u = \sum_{i=1}^{n} a_i w_i$ and $v = \sum_{j=1}^{n'} b_j w_j'$ where a_i and b_j are real numbers and w_i and w_j' are elements of W for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n'$, then

$$uv = \sum_{i=1}^{n} \sum_{j=1}^{n'} a_i b_j w_i w_j'$$

where $w_i w_j'$ is the word obtained by writing w_i and w_j' in juxtaposition.

In the algebra M(W) consider the ideal I generated by the relations

(i) (anti-symmetry) $uu = 0$;

(ii) (Jacobi identity) $u(vw) + v(wu) + w(uv) = 0$ for $u,v,w \in M(W)$.

We define L_k to be $M(W)/I$.

Now let J_ℓ be the ideal in L_k generated by all words of length greater than ℓ. Then L_k/J_ℓ is called the free

ℓ-step nilpotent Lie algebra on k symbols and is denoted $N_{k,\ell}$. Let $\mathfrak{n}_{k,\ell}$ be the simply connected analytic group whose Lie algebra is $N_{k,\ell}$. $\mathfrak{n}_{k,\ell}$ is called the free ℓ-step nilpotent Lie group on the symbols $\exp(x_1),\exp(x_2),\ldots,\exp(x_k)$. If M(W) is taken over the rational numbers instead of the real numbers, we obtain the free ℓ-step nilpotent rational Lie algebra $N^Q_{k,\ell}$ on k symbols. The rational group corresponding to $N^Q_{k,\ell}$ is the free ℓ-step nilpotent rational group $\mathfrak{n}^Q_{k,\ell}$ on the symbols $\exp(x_1),\exp(x_2),\ldots,\exp(x_k)$. (For a discussion of rational groups see [7].)

<u>Lemma 2</u>: Let N be a nilpotent Lie algebra. A subset S generates N if and only if the cosets $s + N^2$, $s \in S$, generate N/N^2.

Proof: (Necessity.) Let $\pi\colon N \to N/N^2$ be the projection mapping. If S generates N then every $n \in N$ can be written as a finite product of elements of S. If $\pi(n) \neq 0$ then $\pi(n)=\sum_{i=1}^{k} s_i + N^2$ for some $s_i \in S$. Thus $\{s + N^2,\ s \in S\}$ generates N/N^2.

(Sufficiency.) N/N^2 is isomorphic to a vector subspace V of N such that $N = V \oplus N^2$ (as vector spaces). Without loss of generality we may assume $S \subseteq V$. By hypothesis S generates V, and we will now show that V generates N^2. Every $n \in N^2$ can be written as a finite linear combination of commutators i.e. $n = \sum_{i=1}^{k} \alpha_i[n_i,m_i]$ where α_i is a real number and n_i and m_i are elements of N. Suppose some n_i or

m_i is not in V. Say $n_1 \notin V$, for argument's sake. Then

$$n_1 = v' + \sum_{i=1}^{k'} \alpha_i' [n_i', m_i'] ,$$

where $v_1 \in V$, the α_i' are real numbers and n_i' and m_i' are elements of N. Say $n_1' \notin V$. Then

$$n_1' = v'' + \sum_{i=1}^{k''} \alpha_i'' \ [n_i'', m_i'']$$

etc. Since N is nilpotent we must eventually obtain an expression for n as a linear combination of finite products of elements of V. Otherwise we would have n a product of infinite length, a contradiction.

If N is an ℓ-step nilpotent Lie algebra and N/N^2 is k-dimensional, we can define a one-to-one linear mapping from $N_{k,\ell}/(N_{k,\ell})^2$ onto N/N^2 . It follows from lemma 2 that we can find a homomorphism of $N_{k,\ell}$ onto N. Hence every nilpotent Lie group is the homomorphic image of a free nilpotent Lie group. In the same way we see that every rational nilpotent group is the homomorphic image of a free nilpotent rational group.

Lemma 3: A finitely generated subgroup F of a rational nilpotent group $\mathfrak{n}^Q$ is discrete.

Proof: We use induction on the nilpotency class of $\mathfrak{n}^Q$. In the abelian case, $\mathfrak{n}^Q$ is a vector group over the rational numbers, say of dimension k. Suppose $\{f_1, f_2, \ldots, f_r\}$ is a set of generators of F. Let $\{f_1, f_2, \ldots, f_R\}$ be a maximal linearly independent subset, re-ordering if necessary. Then we can express the remaining generators as rational linear combinations of $f_1, f_2, \ldots, f_R$ i.e.

$$f_j = \sum_{i=1}^{R} q_{ji} f_i \quad ,$$

where $j = R + 1, R + 2, \ldots, r$ and $q_{ji} \in Q$. Now F is a subgroup of the vector group V generated by

$f_1, f_2, \ldots, f_R$, $\{q_{ji} f_i : j = R + 1, R + 2, \ldots, r$ and $i = 1, 2, \ldots, R\}$.

Since $f_1, f_2, \ldots, f_R$ are linearly independent, V is isomorphic to $V_1 \oplus V_2 \oplus \ldots \oplus V_R$ where V_i is the subgroup of Qf_i generated by

(4) $f_i, q_{R+1,i} f_i , q_{R+2,i} f_i , \ldots , q_{r,i} f_i$.

Thus if V_i is discrete for $i = 1, 2, \ldots, R$ then V (and hence F) is discrete. But V_i is indeed discrete, because the elements (4) generate the discrete group $\{n/D\ f_i : n \in \mathbf{Z}\}$ where D is the least common denominator of $q_{R+1,i}, q_{R+2,i}, \ldots, q_{ri}$. This proves the theorem for the abelian case.

Suppose $\mathfrak{h}^Q$ is ℓ-step nilpotent, and let $\mathfrak{h}_*^Q = \mathfrak{h}^Q/(\mathfrak{h}^Q)^{\ell}$. If $\pi: \mathfrak{h}^Q \to \mathfrak{h}_*^Q$ is the projection mapping, then $\pi(F)$ is finitely generated, since it is the homomorphic image of a finitely generated group. By the induction hypothesis, $\pi(F)$ is discrete in $\mathfrak{n}^Q_*$. Now $\pi(F)$ is isomorphic to the factor group $F/F\cap(\mathfrak{n}^Q)^{\ell}$. Therefore F is a fibre bundle over $\pi(F)$ with fibre $F_1 = F\cap(\mathfrak{n}^Q)^{\ell}$. But F_1 is a subgroup of the finitely generated nilpotent group F and hence is finitely generated. Moreover, F_1 is abelian, so we have F_1 discrete by the proof for the abelian case. This gives us a discrete base space and a discrete fibre, which implies that the bundle F is discrete.

Lemma 4. Let $\mathfrak{n}$ be a rational nilpotent Lie group, and let S be a finite subset of $\mathfrak{n}$ such that the cosets $s\mathfrak{n}^2$, $s \in S$, linearly span $\mathfrak{n}/\mathfrak{n}^2$. Then S generates a discrete cocompact subgroup of $\mathfrak{n}$.

Proof: By lemma 3, S must generate a discrete subgroup G. To show cocompactness of G we use induction on the nilpotency class of $\mathfrak{n}$.

If $\mathfrak{n}$ is abelian then $\mathfrak{n} = \mathfrak{n}/\mathfrak{n}^2$ and S must contain a basis B of $\mathfrak{n}$. B generates a lattice L in $\mathfrak{n}$, and $\mathfrak{n}/L$ is a torus, hence compact. $\mathfrak{n}/G$ is the homomorphic image of $\mathfrak{n}/L$, and the homomorphic image of a compact space is compact. Therefore G is cocompact in $\mathfrak{n}$.

Suppose $\mathfrak{n}$ has class ℓ i.e. $\mathfrak{n}^{\ell}$ is the last nontrivial subgroup of the lower central series of $\mathfrak{n}$. Let $\pi : \mathfrak{n} \to \mathfrak{n}/\mathfrak{n}^{\ell}$ be the projection mapping. Since $\mathfrak{n}^{\ell}$ is a normal subgroup of $\mathfrak{N}^2$, π induces an isomorphism of $\mathfrak{N}/\mathfrak{N}^2$ onto $(\mathfrak{N}/\mathfrak{N}^{\ell})/(\mathfrak{N}^2/\mathfrak{N}^{\ell})$. Therefore the cosets $\pi(s)\mathfrak{N}^2/\mathfrak{N}^{\ell}$ linearly span $(\mathfrak{N}/\mathfrak{N}^{\ell})/(\mathfrak{N}^2/\mathfrak{N}^{\ell})$. By the induction hypothesis $\pi(S)$ generates a discrete cocompact subgroup G^* of $\mathfrak{N}/\mathfrak{N}^{\ell}$.

Let G be the subgroup of $\mathfrak{N}$ generated by S. Then $\pi(G) = G^*$. Let $\exp(u_1),\ldots,\exp(u_t)$ be a canonical basis for G^*, and choose $x_1,\ldots,x_t \in \log(G)$ so that $\pi(\exp(x_i)) = \exp(u_i)$ for $i = 1,2,\ldots,t$. Now $u_1,u_2,\ldots,u_t$ is a canonical basis for the Lie algebra N/N^{ℓ} of $\mathfrak{N}/\mathfrak{N}^{\ell}$, where N is the Lie algebra of $\mathfrak{N}$. We construct a set $T \subseteq \log(G)$ which contains a canonical basis of N as follows: Let T contain $x_1,x_2,\ldots,x_t$ and all commutators $[x_i,x_j]$ where i and j are in $\{1,2,\ldots,t\}$ and the weights of x_i and x_j have sum ℓ. (The weight of an element x, denoted wt(x), is defined to be m if $x \in N^m$ but $x \notin N^{m+1}$.) Note that T contains a basis of N^{ℓ} since by definition N^{ℓ} is the linear span of commutators of the form $[m,n]$ where $m \in N$ and $n \in N^{\ell-1}$. Moreover, if $wt(x_i) + wt(x_j) = \ell$, then

$$\exp([x_i,x_j]) = \exp(x_i)^{-1}\exp(x_j)^{-1}\exp(x_i)\exp(x_j) \; .$$

This equation can be verified by applying the Campbell-Hausdorff formula to the right hand side. But then $\exp([x_i,x_j]) \in G$. Since T contains a basis for N^{ℓ} and also contains $x_1,\ldots,x_t$, it contains a canonical basis C for N. The one-parameter subgroups through the elements of exp(C) form a coordinate system of the second type for $\mathfrak{N}$. By (A5) the subgroup G generated by exp(T) must be cocompact.

Theorem B: Any two discrete cocompact subgroups Γ_1 and Γ_2 of the free nilpotent group $n_{k,\ell}$ are commensurable.

Proof: Suppose $\gamma_1, \gamma_2, \ldots, \gamma_r$ is a canonical basis of Γ_1 and $\delta_1, \delta_2, \ldots, \delta_r$ is a canonical basis of Γ_2. Then the lattices $\Gamma_1/(\Gamma_1)^2$ and $\Gamma_2/(\Gamma_2)^2$ are generated by the cosets $\gamma_1(\Gamma_1)^2, \gamma_2(\Gamma_1)^2, \ldots, \gamma_k(\Gamma_1)^2$ and $\delta_1(\Gamma_2)^2, \delta_2(\Gamma_2)^2, \ldots, \delta_k(\Gamma_2)^2$, respectively. Lemma 3 leads us to conclude that $\gamma_1, \gamma_2, \ldots, \gamma_k$ generate a discrete cocompact subgroup $\hat{\Gamma}_1$ of $n_{k,\ell}$. Likewise $\delta_1, \delta_2, \ldots, \delta_k$ generate a discrete cocompact subgroup $\hat{\Gamma}_2$ of $n_{k,\ell}$.

If $\log(\gamma_i) = g_i$ and $\log(\delta_i) = d_i$ for $i = 1,2,\ldots,k$, then $\{g_1 N_{k,\ell}^2, g_2 N_{k,\ell}^2, \ldots, g_k N_{k,\ell}^2\}$ and $\{d_1 N_{k,\ell}^2, d_2 N_{k,\ell}^2, \ldots, d_k N_{k,\ell}^2\}$ are bases for the vector group $N_{k,\ell}/N_{k,\ell}^2$. Let L be the linear mapping from the vector subspace V_1 of $N_{k,\ell}$ spanned by $g_1, g_2, \ldots, g_k$ to the vector subspace V_2 of $N_{k,\ell}$ spanned by $d_1, d_2, \ldots, d_k$, defined by $L(g_i) = d_i$ for $i = 1,2,\ldots,k$. By lemma 2, V_1 and V_2 each generate $N_{k,\ell}$; hence L extends to an automorphism of $N_{k,\ell}$. If λ is the corresponding automorphism of $n_{k,\ell}$ then $\lambda(\hat{\Gamma}_1) = \hat{\Gamma}_2$ i.e. $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are isomorphic. Since a discrete cocompact subgroup which is contained in another discrete cocompact subgroup must be of finite index in the latter, $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are of finite index in Γ_1 and Γ_2 respectively. Therefore Γ_1 and Γ_2 are commensurable.

Theorem C (A. I. Mal'cev): Two discrete cocompact subgroups Γ_1 and Γ_2 of a simply connected nilpotent analytic group $\mathfrak{n}$ determine isomorphic rational algebras if and only if they are commensurable.

Proof: (Sufficiency.) By hypothesis Γ_1 contains a subgroup $\hat{\Gamma}_1$ of finite index which is isomorphic to a subgroup $\hat{\Gamma}_2$ of finite index in Γ_2. $\hat{\Gamma}_1$ is cocompact, for $\mathfrak{n}/\Gamma_1$ is isomorphic to $(\mathfrak{n}/\hat{\Gamma}_1)/(\Gamma_1/\hat{\Gamma}_1)$ i.e. $\mathfrak{n}/\hat{\Gamma}_1$ is a fibre bundle over a compact space with a finite fibre. Similarly $\hat{\Gamma}_2$ is cocompact.

Let $\gamma_1,\gamma_2,\ldots,\gamma_r$ be a canonical basis of Γ_1. Since $\hat{\Gamma}_1$ is of finite index in Γ_1, for each γ_i there exists a positive integer n_i such that $\gamma_i^{n_i} \in \hat{\Gamma}_1$. Hence $n_i[\log(\gamma_i)] \in \log(\hat{\Gamma}_1)$, which implies that $\log(\gamma_i) \in Q(\hat{\Gamma}_1)$. But then $Q(\hat{\Gamma}_1) = Q(\Gamma_1)$, for both are the rational span of $\log(\gamma_1),\log(\gamma_2),\ldots,\log(\gamma_r)$. Similarly $Q(\hat{\Gamma}_2) = Q(\Gamma_2)$. Therefore, since $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are isomorphic, $Q(\Gamma_1)$ is isomorphic to $Q(\Gamma_2)$.

(Necessity.) Suppose $a: Q(\Gamma_1) \to Q(\Gamma_2)$ is an isomorphism. The Lie algebra N of $\mathfrak{n}$ is the real extension of both these algebras, so we can extend a to an automorphism $A : N \to N$. Let α be the corresponding automorphism of $\mathfrak{n}$.

Now α maps Γ_1 into a discrete cocompact subgroup $\alpha(\Gamma_1)$ of $\mathfrak{n}$. $\operatorname{Log} \circ \alpha(\Gamma_1) = A \circ \log(\Gamma_1)$ is contained in $Q(\Gamma_2)$. If $\gamma_1,\gamma_2,\ldots,\gamma_r$ is a canonical basis of Γ_1 and $\alpha(\gamma_i) = \gamma_i'$ for $i = 1,2,\ldots,r$, then $\gamma_1',\gamma_2',\ldots,\gamma_r'$ is a canonical basis of

$\alpha(\Gamma_1)$. If $\delta_1,\delta_2,\ldots,\delta_r$ is a canonical basis of Γ_2, then $\log(\delta_1),\log(\delta_2),\ldots,\log(\delta_r)$ and $\log(\gamma_1'),\log(\gamma_2'),\ldots,\log(\gamma_r')$ are both bases of $Q(\Gamma_2)$.

Let $\Gamma = \alpha(\Gamma_1) \cap \Gamma_2$. We claim that Γ is of finite index in both $\alpha(\Gamma_1)$ and Γ_2. (This is proved exactly as done by Mal'cev in his version of the theorem. Cf. [5], p. 304.) But then Γ_1 and Γ_2 are commensurable, since $\alpha^{-1}(\Gamma) \subset \Gamma_1$ and is isomorphic to Γ.

Corollary 1: Any two discrete cocompact subgroups of a free nilpotent group have isomorphic rational algebras.

Proof: This follows immediately from theorems B and C.

Corollary 2: A free nilpotent Lie algebra $N_{k,\ell}$ contains only one rational subalgebra whose dimension equals that of $N_{k,\ell}$, up to isomorphism.

Proof: Every rational subalgebra R of a real nilpotent Lie algebra N is the rational algebra $Q(\Gamma)$ of some discrete cocompact subgroup Γ of the corresponding Lie group $\mathfrak{n}$. (Choose a canonical basis for R and apply fact (A4).) If $\mathfrak{n}$ is free, $Q(\Gamma_1)$ is isomorphic to $Q(\Gamma_2)$ for any two discrete cocompact subgroups Γ_1 and Γ_2 of $\mathfrak{n}$, by corollary 1. Therefore all $Q(\Gamma)$ in N are isomorphic.

Lemma 5: Let G be a nilpotent Lie group with a simply-connected, rational identity component G_o. Suppose $\Gamma = G/G_o$

is a lattice nilpotent group. Then there exists a lattice nilpotent group $\Delta \subset G$ such that $\Delta/\Delta \cap G_o$ is isomorphic to Γ.

Proof: Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be generators of Γ and let $g_1, g_2, \ldots, g_r$ be elements in the pre-images of $\gamma_1, \gamma_2, \ldots, \gamma_r$, respectively, in G. $g_1, \ldots, g_r$ generate a discrete subgroup D of the rational nilpotent group G, by lemma 3. Let C_o be a discrete cocompact subgroup of G_o. (Any canonical basis of the Lie algebra of G_o determines a discrete cocompact subgroup of G_o, by (A4).)

Let $\overline{D} = D \cdot C_o$. Now, C_o is torsion-free, by (A2). D must be torsion-free also, for if it contained an element of finite order so would Γ. Hence $\overline{D}$ is a finitely generated, torsion-free nilpotent group, and by (A1) can be embedded in a connected simply connected rational nilpotent Lie group $\mathfrak{h}$ as a discrete cocompact subgroup.

By (B2), $\mathfrak{h}$ and G_o are unipotent algebraic groups; and it follows from (B4) that

(i) $\mathfrak{G}(\overline{D}) = \mathfrak{h}$, and

(ii) $\mathfrak{G}(C_o) = G_o$.

Since C_o is contained in $\overline{D}$, we must have $G_o \subset \mathfrak{h}$. Furthermore, since G is generated by G_o and $\overline{D}$, we have $\overline{D} \subset G \subset \mathfrak{h}$. Hence $\mathfrak{G}(G) = \mathfrak{h}$. Since G_o is a normal subgroup of G, $\mathfrak{G}(G_o) = G_o$ is a normal subgroup of $\mathfrak{G}(G) = \mathfrak{h}$, by (B5).

By Moore's theorem, $\overline{D}$ is contained in a lattice subgroup $\overline{H}$ of $\mathfrak{N}$. Let $H = \overline{H} \cap G_o$. Let Δ be the subgroup of $\overline{H}$ generated by D and H. Note that $\Delta \subset G$. $\Delta \supset \overline{D}$; therefore Δ is cocompact in $\mathfrak{N}$. Since it is a subgroup of the discrete group $\overline{H}$, Δ is discrete.

Let φ be the projection homomorphism $\mathfrak{N} \rightarrow \mathfrak{N}/G_o$, and let $d\varphi$ be the differential of φ. Note that $\mathfrak{N}/G_o$ contains $\Gamma = G/G_o$ as a lattice subgroup.

Let $g \in G$. Then $g = dg_o$ for some $d \in D$ and $g_o \in G_o$. Now,

$$\log(g) = \log(d) + \log(g_o) + \ldots$$

where the remaining terms are finitely many rational multiples of brackets of $\log(d)$ and $\log(g_o)$, by the Campbell-Hausdorff formula. But since $\log(g_o)$ is in the ideal $\log(G_o)$,

$$\log(g) = \log(d) + \log(\overline{g}_o)$$

for some $\overline{g}_o \in G_o$. Hence

$$\log(G) \subseteq \log(D) + \log(G_o) \;. \tag{1}$$

Now, consider the following commutative diagram:

$$\begin{tikzcd}
\log(\mathfrak{N}) \arrow[r, "d\varphi"] & \log(\mathfrak{N}/G_o) \cong \log(\mathfrak{N})/\log(G_o) \\
\mathfrak{N} \arrow[u, "\log"] \arrow[r, "\varphi"'] & \mathfrak{N}/G_o \arrow[u, "\log"']
\end{tikzcd} \tag{2}$$

Since

$$\log(G) = \log(\varphi^{-1}(\Gamma))$$

$$\log(G) + \log(G_o) = d\varphi^{-1}(\log(\Gamma))$$

and the right hand sides are equal by the commutativity of (2), we must have

$$\log(G) = \log(G) + \log(G_o). \tag{3}$$

But $\log(D) + \log(G_o) \subseteq \log(G) + \log(G_o) = \log(G)$, and this together with (1) implies that

$$\log(G) = \log(DG_o) = \log(D) + \log(G_o) \ . \tag{4}$$

Let $\delta \in \Delta$. By (4),

$$\log(\delta) = \log(d) + \log(g_o)$$

for some $d \in D$ and $g_o \in G_o$. But $\log(\delta) - \log(d) = \log(g_o)$ is in the lattice $\log(\overline{H})$. Therefore $g_o \in G_o \cap \overline{H} = H$ and

$$\log(\Delta) \subseteq \log(D) + \log(H). \tag{5}$$

Now if $d \in D$ and $h \in H$ then

$$\log(d) + \log(h) \in \log(D) + \log(G_o) = \log(DG_o)$$

by (4). But also

$$\log(d) + \log(h) \in \log(\overline{H}).$$

Thus

$$(6) \quad \log(D) + \log(H) \subseteq \log(DG_o) \cap \log(\overline{H}) = \log(\Delta).$$

Equations (5) and (6) imply that $\log(\Delta) = \log(D) + \log(H)$.

Finally we show that $\log(D) + \log(H)$ is a lattice: Let $\log(\delta_1) = \log(d_1) + \log(h_1)$, and $\log(\delta_2) = \log(d_2) + \log(h_2)$. Then

$$(7) \quad \log(\delta_1) + \log(\delta_2) = \log(d_1) + \log(d_2) + \log(h_3)$$

for some $h_3 \in H$, since $\log(H)$ is the intersection of a lattice and an ideal, thus a lattice itself. But

$$(8) \qquad \log(d_1) + \log(d_2) = \log(d_3) + \log(k)$$

for some $d_3 \in D$ and $k \in G_o$, since $\log(G)$ is a lattice modulo $\log(G_o)$ and, by (4), $\log(G) = \log(D) + \log(G_o)$. Furthermore,

$$(9) \quad \log(d_1) + \log(d_2) - \log(d_3) = \log(k) \in \log(\overline{H})$$

since $D \subset \overline{H}$. Hence in equation (8), $k \in H$. Equation (7) may now be written

$$(10) \quad \log(\delta_1) + \log(\delta_2) = \log(d_3) + \log(k) + \log(h_3)$$
$$= \log(d_3) + \log(h_4)$$

for some $h_4 \in H$. Hence $\log(\Delta) = \log(D) + \log(H)$ is a lattice.

Since Δ contains D, $\Delta \cdot G_o = G$. Thus $\Delta / \Delta \cap G_o$ is isomorphic to G/G_o, and the lemma is proved.

Theorem D: Let Γ be a lattice nilpotent group of the connected simply connected nilpotent Lie group $\mathfrak{h}$. Then Γ is the homomorphic image of a lattice subgroup of a free nilpotent Lie group.

Proof: Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be a canonical basis of Γ. Suppose the dimension of $\mathfrak{h}/\mathfrak{h}^2$ is k, and $\mathfrak{h}$ is ℓ-step nilpotent. Let $\mathfrak{h}^Q_{k,\ell}$ be the free nilpotent rational Lie group freely generated by $x_1, x_2, \ldots, x_k$. Let θ: $\mathfrak{h}^Q_{k,\ell} \to \exp(Q(\Gamma))$ be the homomorphism determined by the mapping $x_i \to \gamma_i$ $(i = 1,2,\ldots,k)$. Finally, let K be the kernel of θ.

$\theta^{-1}(\Gamma)$ is a nilpotent Lie group with simply connected, rational identity component K. By lemma 5, there exists a lattice nilpotent group $\Delta \subset \theta^{-1}(\Gamma)$ such that $\Delta/\Delta \cap K$ is isomorphic to $\theta^{-1}(\Gamma)/K$. Hence $\theta(\Delta) = \Gamma$.

SECTION V. LATTICE NILPOTENT GROUPS OF CONNECTED SIMPLY CONNECTED TWO-STEP AND THREE-STEP NILPOTENT LIE GROUPS

Let $\mathfrak{h}$ be a connected simply connected two-step nilpotent Lie group, and let Γ be a lattice nilpotent group of $\mathfrak{h}$. Suppose $\gamma_1,\gamma_2,\ldots,\gamma_r$ is a canonical basis for Γ.

Denote $\log(\gamma_i)$ by c_i, for $i = 1,2,\ldots,r$. If i and j are in $\{1,2,\ldots,r\}$, then by the Campbell-Hausdorff formula

$$\gamma_i\gamma_j = \exp(c_i + c_j + \tfrac{1}{2}[c_i,c_j]) = \exp(c_i + c_j)\exp(\tfrac{1}{2}[c_i,c_j]).$$

Since $\log(\Gamma)$ is a lattice, $\exp(c_i + c_j) \in \Gamma$. This implies that Γ must contain $\exp(\tfrac{1}{2}[c_i,c_j])$. We shall prove that this is also a sufficient condition i.e. we shall prove

<u>Theorem E</u>: Let Γ be a discrete cocompact subgroup of a connected simply connected two-step nilpotent Lie group $\mathfrak{h}$. Let $\gamma_1,\gamma_2,\ldots,\gamma_r$ be a canonical basis for Γ. Then Γ is a lattice nilpotent group if and only if it contains

$$\exp(\tfrac{1}{2}[\log(\gamma_i),\log(\gamma_j)])$$

for $i,j = 1,2,\ldots,r$.

Proof: Necessity is proved above. Suppose α and β are elements of Γ. Then for some integers $n_1,m_1,\ldots,n_r,m_r$ we have $\alpha = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$ and $\beta = \gamma_1^{m_1} \cdots \gamma_r^{m_r}$. Now let $\log(\gamma_i) = c_i$ for $i = 1,2,\ldots,r$. Then

$$\alpha\beta = \exp(\log(\alpha) + \log(\beta) + \tfrac{1}{2}[\log(\alpha),\log(\beta)]) =$$
$$\exp(\log(\alpha) + \log(\beta)$$
$$+ \tfrac{1}{2}[n_1c_1 + \dots + n_kc_k + \dots, m_1c_1 + \dots + m_kc_k + \dots])$$

where the terms following n_kc_k and m_kc_k are central. Hence

$$\alpha\beta = \exp(\log(\alpha) + \log(\beta) + \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} n_im_j[c_i,c_j]).$$

We assume that $\exp(\frac{1}{2}[c_i,c_j]) \in \Gamma$ for $i,j = 1,2,\dots,r$. But then

$$\tfrac{1}{2}[c_i,c_j] = \sum_{\ell=k+1}^{r} p_{ij}^{\ell}c_{\ell} \qquad (p_{ij}^{\ell} \in Z).$$

Therefore $\alpha\beta = \exp(\log(\alpha) + \log(\beta) + X_{\alpha\beta})$ where $X_{\alpha\beta}$ is an integral linear combination of $c_{k+1},\dots,c_r$. Clearly $X_{\alpha\beta} \in \log(\Gamma^2)$. We then have $\alpha\beta = \exp(\log(\alpha) + \log(\beta))\exp(X_{\alpha\beta})$. Thus $\log(\alpha) + \log(\beta) = \log(\alpha\beta \exp(X_{\alpha\beta})^{-1}) \in \log(\Gamma)$, and $\log(\Gamma)$ is a lattice.

It follows from this theorem that every discrete cocompact subgroup Γ of $\mathfrak{n}_{k,2}$ with canonical basis

$$\exp(x_1),\exp(x_2),\dots,\exp(x_s)$$

where $x_1,\dots,x_k$ are free generators of $N_{k,2}$ and $s = k(k+1)/2$, is a lattice group if and only if $x_{k+1},\dots,x_s$ additively generate $\frac{1}{2}[x_i,x_j]$, $i,j = 1,2,\dots,k$. In particular, we can describe an infinite number of lattice subgroups of $\mathfrak{n}_{k,2}$ as follows: Their canonical bases are of the form

$$(*)\quad \exp(x_1),\ldots,\exp(x_k)\exp(\tfrac{1}{2L_1}[x_1,x_2]),\exp(\tfrac{1}{2L_2}[x_1,x_3]),$$
$$\ldots,\exp(\tfrac{1}{2L_{s-k}}[x_{k-1},x_k]),$$

where $x_1,\ldots,x_k$ are free generators of $N_{k,2}$ and $L_1,\ldots,L_{s-k}$ are nonzero integers. We see that (*) is indeed the canonical basis of a group by noting that

(i) the last s-k elements are central, and

(ii) the relation $\exp(x_j)^{n_j}\exp(x_i)^{n_i}$ $= \exp(x_i)^{n_i}\exp(x_j)^{n_j}\exp([x_i,x_j])^{-n_i n_j}$ shows that we can interchange factors $\exp(x_j)^{n_j}$ and $\exp(x_i)^{n_i}$ modulo a central element;

hence every product

$$\exp(x_1)^{n_1}\ \ldots\ \exp(\tfrac{1}{2L_{s-k}}[x_{k-1},x_k])^{n_s}\ \exp(x_1)^{m_1}$$
$$\ldots\ \exp(\tfrac{1}{2L_{s-k}}[x_{k-1},x_k])^{n_s}$$

where $n_1,m_1,\ldots,n_s,m_s \in \mathbf{Z}$, can be written in the form

$$\exp(x_1)^{p_1}\ \ldots\ \exp\ \frac{1}{2L_{s-k}}[x_{k-1},x_k])^{p_s}$$

for some $p_1,\ldots,p_s \in \mathbf{Z}$.

Let us denote by $\mathcal{L}_{k,2}$ the collection of subgroups with canonical bases described by (*). A discrete cocompact subgroup of $\mathfrak{n}_{k,2}$ is lattice if and only if it contains a member of $\mathcal{L}_{k,2}$.

The question arises whether every lattice subgroup of $\mathfrak{n}_{k,2}$ is in $\mathfrak{L}_{k,2}$. A counterexample follows:

Let Γ be the lattice subgroup of $\mathfrak{n}_{4,2}$ with canonical basis:

$$(1)\quad \begin{aligned}&\exp(x_1),\exp(x_2),\exp(x_3),\exp(x_4),\exp(\tfrac{1}{4}[x_1,x_2]+\tfrac{1}{4}[x_3,x_4]),\\ &\exp(\tfrac{1}{2}[x_1,x_3]),\exp(\tfrac{1}{2}[x_1,x_4]),\exp(\tfrac{1}{2}[x_2,x_3]),\\ &\exp(\tfrac{1}{2}[x_2,x_4]),\exp(\tfrac{1}{2}[x_3,x_4])\end{aligned}$$

where $\exp(x_1),\ldots,\exp(x_4)$ freely generate $\mathfrak{n}_{4,2}$. Γ clearly satisfies the conditions of theorem E. Suppose that Γ has as another canonical basis:

$$(2)\quad \begin{aligned}&\exp(y_1),\ldots,\exp(y_4),\exp(\frac{1}{2L_{12}}[y_1,y_2]),\exp(\frac{1}{2L_{13}}[y_1,y_3]),\\ &\ldots,\exp(\frac{1}{2L_{34}}[y_3,y_4])\end{aligned}$$

where $\exp(y_1),\ldots,\exp(y_4)$ freely generate $\mathfrak{n}_{4,2}$ and $L_{12},L_{13},\ldots,L_{34}$ are integers. The last six elements of (2) generate the lattice Γ^2, and so do the last six elements of (1). Hence $\frac{1}{2L_{ij}}[y_i,y_j]$ must be an integral linear combination of the last six elements of (1) for $i,j = 1,2,3,4$ and $i<j$. Furthermore each $\frac{1}{2}[x_i,x_j]$ must appear with nonzero coefficient in the expression of at least one of $\{\frac{1}{2L_{ij}}[y_i,y_j] : i<j,\ i,j = 1,2,3,4\}$. In particular there are integers i_o and j_o in $\{1,2,3,4\}$ such that $i_o<j_o$ and

$$(3)\quad \frac{1}{2L_{i_oj_o}}[y_{i_o},y_{j_o}] = \frac{k_1}{4}([x_1,x_2]+[x_3,x_4]) + \frac{k_2}{2}[x_1,x_3] + \ldots + \frac{k_6}{2}[x_3,x_4]\ ,$$

where $k_1,k_2,\ldots,k_6 \in \mathbf{Z}$ and $k_1 \neq 0$. But $\{y_1+\log(\Gamma^2),\ldots,y_4+\log(\Gamma^2)\}$ and $\{x_1+\log(\Gamma^2),\ldots,x_4+\log(\Gamma^2)\}$ are both generating sets of the lattice $\log(\Gamma)/\log(\Gamma^2)$. Hence for some integers $n_1,\ldots,n_4$ and $m_1,\ldots,m_4$ we have

$$y_{i_o} = \sum_{j=1}^{4} n_jx_j + z_{i_o} \text{ and } y_{j_o} = \sum_{k=1}^{4} m_kx_k + z_{j_o}\ ,$$

where z_{i_o} and z_{j_o} are in $\log(\Gamma^2)$. Equation (3) becomes

$$\frac{1}{2L_{i_oj_o}}\left[\sum_{j=1}^{4} n_jx_j + z_{i_o}\ ,\ \sum_{k=1}^{4} m_kx_k + z_{j_o}\right]$$

$$= \frac{1}{2L_{i_oj_o}}\left[\sum_{j=1}^{4} n_jx_j\ ,\ \sum_{k=1}^{4} m_kx_k\right]$$

$$= \frac{k_1}{4}[x_1,x_2] + \frac{k_2}{2}[x_1,x_3] + \ldots + \frac{k_5}{2}[x_2,x_4] + \left(\frac{2k_6+k_1}{4}\right)[x_3,x_4]$$

$$= \tfrac{1}{4}(k_1[x_1,x_2] + 2k_2[x_1,x_2] + \ldots + 2k_5[x_2,x_4] + (2k_6+k_1)[x_3,x_4])\ .$$

Expanding $[\sum_{j=1}^{4} n_j x_j, \sum_{k=1}^{4} m_k x_k]$ we obtain the equations:

a) $n_1 m_2 - m_1 n_2 = k_1 L_{i_o j_o}/2$

b) $n_1 m_3 - m_1 n_3 = k_2 L_{i_o j_o}$

c) $n_1 m_4 - m_1 n_4 = k_3 L_{i_o j_o}$

d) $n_2 m_3 - m_2 n_3 = k_4 L_{i_o j_o}$

e) $n_2 m_4 - m_2 n_4 = k_5 L_{i_o j_o}$

f) $n_3 m_4 - m_3 n_4 = (2k_6 + k_1) L_{i_o j_o}/2$

Now equation (a) implies that either k_1 or $L_{i_o j_o}$ is divisible by 2. Moreover, m_3(equation (a)) - m_2(equation (b)) implies that

$$L_{i_o j_o}\left(\frac{m_3 k_1}{2} - m_2 k_2\right) = - m_1(n_2 m_3 - m_2 n_3) = - m_1 k_4 L_{i_o j_o} .$$

Hence either k_1 or m_3 is divisible by 2. Similarly, m_4(equation (a)) - m_2(equation (c)) implies that either k_1 or m_4 is divisible by 2.

But then we see from equation (f) that k_1 must be divisible by 2, for if $L_{i_o j_o}$, m_3 and m_4 have this property then $(2k_6 + k_1)/2$ is an integer.

Now if no expression (3) has k_1 odd, the elements $\frac{1}{L_{ij}}[y_i, y_j]$ cannot generate the vector $\frac{1}{4}([x_1,x_2] + [x_3,x_4])$. This contradicts the existence of the canonical basis (2).

<u>Theorem F</u>: Let $\mathfrak{n}$ be a connected simply connected three-step nilpotent Lie group and Γ a discrete cocompact subgroup of $\mathfrak{n}$. Suppose $\gamma_1,\gamma_2,\ldots,\gamma_r$ is a canonical basis of Γ. Let $\log(\gamma_i) = c_i$ $(i = 1,2,\ldots,r)$. Then Γ is a lattice subgroup if and only if $\log(\Gamma)$ contains

(i) $\frac{1}{2}[c_i,c_j]$,

(ii) $\frac{1}{4}[[c_i,c_j],c_k]$ and

(iii) $\frac{1}{12}[c_i,[c_i,c_j]]$,

for $i,j,k = 1,2,\ldots,r$.

Proof: a) Necessity. Let N be the Lie algebra of $\mathfrak{n}$. Let $\pi:\mathfrak{n} \to \mathfrak{n}/\mathfrak{n}^3$ be the projection homomorphism, where $\mathfrak{n}^3$ is the last term of the lower central series of $\mathfrak{n}$, and $d\pi:N \to N/N^3$ the differential of π. Then $\pi(\Gamma)$ is the homomorphic image of the lattice nilpotent group Γ and so, by theorem A, is itself a lattice nilpotent group. Let k_1 be the dimension of $\mathfrak{n}/\mathfrak{n}^2$ and k_2 the dimension of $\mathfrak{n}/\mathfrak{n}^3$. Note that $\pi(\gamma_1),\pi(\gamma_2),\ldots,\pi(\gamma_{k_2})$ is a canonical basis for $\pi(\Gamma)$. By theorem E, $\frac{1}{2}[d\pi(c_i),d\pi(c_j)]$ is in $d\pi\circ\log(\Gamma)$ for $i,j = 1,2,\ldots,k_2$. It follows that

$$\tfrac{1}{2}[c_i,c_j] + z_{ij} = c$$

for some $c \in \log(\Gamma)$, $z_{ij} \in N^3$ and $i,j = 1,2,\ldots,k_2$. If i or j is greater than k_2 the term $\frac{1}{2}[c_i,c_j]$ is zero, hence trivially an element of $\log(\Gamma)$, and z_{ij} may be taken to be zero.

Now, for $i,j,k = 1,2,\ldots,r$,

$$\begin{aligned}&\exp(\tfrac{1}{2}[c_i,c_j] + z_{ij})\exp(c_k)\\ &\quad= \exp(\tfrac{1}{2}[c_i,c_j] + z_{ij} + c_k + \tfrac{1}{4}[[c_i,c_j],c_k])\\ &\quad= \exp(\tfrac{1}{2}[c_i,c_j] + z_{ij} + c_k)\exp(\tfrac{1}{4}[[c_i,c_j],c_k]).\end{aligned}$$

Since the left hand side and the first term on the right hand side are elements of Γ, we must have $\exp(\tfrac{1}{4}[[c_i,c_j].c_k]) \in \Gamma$.

Since $\log(\Gamma)$ contains $\tfrac{1}{2}[c_i,c_j] + z_{ij}$ and is a lattice, it must contain $-\tfrac{1}{2}[c_i,c_j] - z_{ij}$, for $i,j = 1,\ldots,r$. The equations

$$\begin{aligned}\gamma_i\gamma_j &= (\exp(c_i)\exp(c_j) =)\exp(c_i + c_j + \tfrac{1}{2}[c_i,c_j] + z_{ij})\\ &\qquad\exp(\tfrac{1}{12}\,[c_i,[c_i,c_j]] - \tfrac{1}{12}\,[c_j,[c_i,c_j] - z_{ij})\end{aligned}$$

and

$$\begin{aligned}\gamma_i^{-1}\gamma_j &= (\exp(-c_i)\exp(c_j) =)\exp(-c_i + c_j - \tfrac{1}{2}[c_i,c_j] - z_{ij})\\ &\qquad\exp(\tfrac{1}{12}\,[c_i,[c_i,c_j]] + \tfrac{1}{12}\,[c_j,[c_i,c_j] + z_{ij})\end{aligned}$$

imply that

$$\frac{1}{12}\,[c_i,[c_i,c_j]] - \frac{1}{12}\,[c_j,[c_i,c_j]] - z_{ij}$$

and

$$\frac{1}{12}\,[c_i,[c_i,c_j]) + \frac{1}{12}\,[c_j,[c_i,c_j]] + z_{ij}$$

are elements of $\log(\Gamma)$. Therefore their sum, $\frac{1}{6}\,[c_i,[c_i,c_j]]$ is in $\log(\Gamma)$. Since we have shown that $\log(\Gamma)$ also contains

$\frac{1}{4}[c_i,[c_i,c_j]]$ we have

$$(\tfrac{1}{4} - \tfrac{1}{6})[c_i,[c_i,c_j]] = \tfrac{1}{12}[c_i,[c_i,c_j]] \in \log(\Gamma) \ .$$

We have so far shown that if $\log(\Gamma)$ is a lattice it must contain $\frac{1}{2}[c_i,c_j] + z_{ij}$, $\frac{1}{4}[[c_i,c_j],c_k]$ and $\frac{1}{12}[c_i,[c_i,c_j]]$ for $i,j,k \in \{1,2,\ldots,r\}$ and $z_{ij} \in N^3$. Since c_i and c_j are in $\log(\Gamma)$,

$$d = c_i + c_j + \tfrac{1}{2}[c_i,c_j] + z_{ij} + \tfrac{1}{12}[c_i,[c_i,c_j]] - \tfrac{1}{12}[c_j,[c_i,c_j]]$$

is in $\log(\Gamma)$. But $d = \log(\gamma_i\gamma_j) + z_{ij}$. Hence $z_{ij} = d - \log(\gamma_i\gamma_j) \in \log(\Gamma)$. We may therefore conclude that $\frac{1}{2}[c_i,c_j] \in \log(\Gamma)$ for $i,j = 1,2,\ldots,r$.

b) Sufficiency. We introduce the following notation:

$$E_{ij} = \tfrac{1}{2}[c_i,c_j]$$

$$F_{ijk} = \tfrac{1}{4}[[c_i,c_j],c_k]$$

$$G_{ij} = \tfrac{1}{12}[c_i,[c_i,c_j]]$$

Let $\exp(a) = \gamma_1^{n_1}\gamma_2^{n_2} \ldots \gamma_r^{n_r}$ and $\exp(b) = \gamma_1^{m_1}\gamma_2^{m_2}\ldots\gamma_r^{m_r}$ be elements of Γ, and let k_1 and k_2 be as in (a). Then we can apply the Campbell-Hausdorff formula r times to exp(a) and exp(b) to find their expressions in terms of $c_1,c_2,\ldots,c_r$ (and bracket products of these). Adding, we obtain

$$\begin{aligned}
a + b = {} & (n_1 + m_1)c_1 + (n_2 + m_2)c_2 + (n_1 n_2 + m_1 m_2)E_{12} \\
& + (n_1^2 n_2 + m_1^2 m_2)G_{12} + (n_1 n_2^2 + m_1 m_2^2)G_{21} \\
& + (n_3 + m_3)c_3 + (n_1 n_3 + m_1 m_3)E_{13} + (n_2 n_3 + m_2 m_3)E_{23} \\
& + (n_1 n_2 n_3 + m_1 m_2 m_3)F_{123} + (n_1^2 n_3 + m_1^2 m_3)G_{13} \\
& + (n_2^2 n_3 + m_2^2 m_3)G_{23} + (n_1 n_3^2 + m_1 m_3^2)G_{31} \\
& + (n_2 n_3^2 + m_2 m_3^2)G_{32} + \ldots + (n_{k_1} + m_{k_1})c_{k_1} \\
& + (n_1 n_{k_1} + m_1 m_{k_1})E_{1 k_1} + \ldots \\
& + (n_{k_1-1} n_{k_1} + m_{k_1-1} m_{k_1})E_{k_1-1,k_1} \\
& + (n_1 n_2 n_{k_1} + m_1 m_2 m_{k_1})F_{12k_1} \\
& + (n_1 n_3 n_{k_1} + m_1 m_3 m_{k_1})F_{13\,k_1} + \ldots \\
& + (n_{k_1-2} n_{k_1-1} n_{k_1} + m_{k_1-2} m_{k_1-1} m_{k_1})F_{k_1-2,k_1-1,k_1} \\
& + (n_1^2 n_{k_1} + m_1^2 m_{k_1})G_{1\,k_1} + \ldots \\
& + (n_{k_1-1}^2 n_{k_1} + m_{k_1-1}^2 m_{k_1})G_{k_1-1,k_1} \\
& + (n_1 n_{k_1}^2 + m_1 m_{k_1}^2)G_{k_1,1} + \ldots \\
& + (n_{k_1-1} n_{k_1}^2 + m_{k_1-1} m_{k_1}^2)G_{k_1,k_1-1} \\
& + (n_{k_1+1} + m_{k_1+1})c_{k_1+1} + (n_1 n_{k_1+1} + m_1 m_{k_1+1})E_{1,k_1+1} \\
& + \ldots + (n_{k_1} n_{k_1+1} + m_{k_1} m_{k_1+1})E_{k_1,k_1+1} + \ldots \\
& + (n_{k_2} + m_{k_2})c_{k_2} + (n_1 n_{k_2} + m_1 m_{k_2})E_{1\,k_2} + \ldots \\
& + (n_{k_1} n_{k_2} + m_{k_1} m_{k_2})E_{k_1 k_2} + (n_{k_2+1} + m_{k_2+1})c_{k_2+1} \\
& + \ldots + (n_r + m_r)c_r \,.
\end{aligned}$$

Since G_{ij}, F_{ijk} and $E_{i\ell}$ are elements of $\log(\Gamma)$ by hypothesis and are central, for $i,j,k = 1,2,\ldots,r$ and $\ell = k_1+1,\ldots,r$, it is sufficient to show that the following sum is in $\log(\Gamma)$:

$$\begin{aligned} S = {} & (n_1+m_1)c_1 + (n_2+m_2)c_2 + \ldots + (n_r+m_r)c_r \\ & + (n_1n_2+m_1m_2)E_{12} + (n_1n_3+m_1m_3)E_{13} + (n_2n_3+m_2m_3)E_{23} \\ & + \ldots + (n_{k_1-1}n_{k_1}+m_{k_1-1}m_{k_1})E_{k_1-1,k_1} . \end{aligned}$$

Now let $\gamma = \gamma_1^{n_1+m_1} \gamma_2^{n_2+m_2} \ldots \gamma_r^{n_r+m_r}$. Then

$$\begin{aligned} \log(\gamma) = e = {} & (n_1+m_1)c_1 + (n_2+m_2)c_2 + \ldots + (n_r+m_r)c_r \\ & + (n_1+m_1)(n_2+m_2)E_{12} + (n_1+m_1)(n_3+m_3)E_{13} \\ & + (n_2+m_2)(n_3+m_3)E_{23} + \ldots \\ & + (n_{k_1-1}+m_{k_1-1})(n_{k_1}+m_{k_1})E_{k_1-1,k_1} + f \end{aligned}$$

where f is a sum of integral multiples of G_{ij}, F_{ijk} and $E_{i\ell}$ ($i,j,k = 1,2,\ldots,r$ and $\ell = k_1+1,\ldots,r$), hence $f \in \log(\Gamma)$. Therefore $e - f$ is in $\log(\Gamma)$, since f is central. Now,

$$\begin{aligned} S = {} & e - f - (n_1m_2+m_1n_2)E_{12} - (n_1m_3+m_1n_3)E_{13} \\ & - (n_2m_3+m_2n_3)E_{23} - \ldots - (n_{k_1-1}m_{k_1}+m_{k_1-1}n_{k_1})E_{k_1-1,k_1} . \end{aligned}$$

Denote $(n_i m_j + m_i j_j)E_{ij}$ by $\overline{E}_{ij}$. Since $e - f$ and $\overline{E}_{ij}$ are in $\log(\Gamma)$,

$$\delta = \exp(e - f)\exp(-\overline{E}_{12})\exp(-\overline{E}_{13}) \ \dots \ \exp(-\overline{E_{k_1-1,k_1}})$$

is in Γ. But $\log(\delta) = S + g$, where g is a sum of integral multiples of F_{ijk}'s. Since the F_{ijk}'s are in $\log(\Gamma)$ and are central, S itself must be in $\log(\Gamma)$, and we are done.

Let $\mathcal{L}_{k,3}$ be the collection of lattice subgroups of $\mathfrak{n}_{k,3}$ described as follows:

If $x_1,\dots,x_k$ are free generators of $N_{k,3}$ then a canonical basis for $\mathfrak{N}_{k,3}$ is

$$x_1,\dots,x_k,[x_i,x_j],[x_i,[x_i,x_j]],[x_j,[x_i,x_j]],[x_\ell,[x_i,x_j]]$$

where $i<j$, $\ell<i$ or j and $i \neq \ell \neq j$ and $i,j,\ell \in \{1,2,\dots,k\}$. There are $k(k-1)(k-2)/3$ terms of the form $[x_\ell,[x_i,x_j]]$ and $k(k-1)$ terms of the forms $[x_i,[x_i,x_j]]$ and $[x_j,[x_i,x_j]]$. Note that the Jacobi identity forces us to exclude from the basis one third of the terms $[x_N,[x_P,x_Q]]$ where $N,P,Q = 1,\dots,k$ and $N \neq P$, $N \neq Q$, $P \neq Q$, since

$$[x_N,[x_P,x_Q]] = [x_P,[x_N,x_Q]] - [x_Q,[x_N,x_P]] \ .$$

If $N<P<Q$, then $[x_N,[x_P,x_Q]]$ and $[x_P,[x_N,x_Q]]$ satisfy the above conditions but $[x_Q,[x_N,x_P]]$ does not. Thus the above conditions assure us of a linearly independent set.

Now, a canonical basis B of $N_{k,3}$ determines a discrete cocompact subgroup Γ of $\mathfrak{n}_{k,3}$ by (A4). Γ is generated by exp(B). Suppose B is of the following form, where $x_1,\ldots,x_k$ are free generators of $N_{k,3}$:

$$(**)\quad x_1,\ldots,x_k,\ \frac{1}{2L_{ij}}[x_i,x_j],\ \frac{1}{12M_{ij}}[x_i,[x_i,x_j]],$$
$$\frac{1}{12P_{ij}}[x_j,[x_i,x_j]],\ \frac{1}{4Q_{ij\ell}}[x_\ell,[x_i,x_j]]$$

where $i<j$, $\ell<i$ or j, $i\neq\ell\neq j$, $i,j,\ell\in\{1,2,\ldots,k\}$ and L_{ij}, M_{ij}, P_{ij} and $Q_{ij\ell}$ are integers. Consider the equations

$$\text{(i)}\quad \exp(\tfrac{1}{2}[x_{i_2},x_{j_2}])^{n_2}\exp(\tfrac{1}{2}[x_{i_1}x_{j_1}])^{n_1} =$$
$$\exp(\tfrac{1}{2}[x_i x_{j_1}])^{n_1}\exp(\tfrac{1}{2}[x_{i_2},x_{j_2}])^{n_2}\ ;$$

$$\text{(ii)}\quad \exp(\tfrac{1}{2}[x_i,x_j])^{n_{ij}}\exp(x_\ell)^{n_\ell} =$$
$$\exp(\frac{n_{ij}}{2}[x_i,x_j] + n_\ell x_\ell + \frac{n_{ij}n_\ell}{4}[[x_i,x_j],x_\ell]) =$$
$$\exp(x_\ell)^{n_\ell}\exp(\tfrac{1}{2}[x_i,x_j])^{n_{ij}}\exp(\tfrac{1}{2}[x_\ell,[x_i,x_j]])^{-n_\ell n_{ij}};$$

$$\text{(iii)}\quad \exp(x_j)^{n_j}\exp(x_i)^{n_i} =$$
$$\exp(n_jx_j + n_ix_i + \frac{n_in_j}{2}[x_j,x_i] + \frac{n_j^2n_i}{12}[x_j,[x_j,x_i]]$$
$$- \frac{n_i^2n_j}{12}[x_i,[x_j,x_i]]) =$$

$$\exp(x_i)^{n_i}\exp(x_j)^{n_j}\exp(\tfrac{1}{2}[x_i,x_j])^{-2n_in_j}$$

$$\exp(\frac{n_i^2n_j}{2}[x_i,[x_i,x_j]])\exp(\frac{n_j^2n_i}{2}[x_j,[x_i,x_j]])\ .$$

According to these equations we may change the order of the factors in the product $\exp(x_i)^{n_i}\exp(x_j)^{n_j}$ modulo $\exp(\frac{1}{2}[x_i,x_j])^{-2n_in_j}$ and central terms in Γ and we may change the order of the factors in the product $\exp(\frac{1}{2}[x_i,x_j])^{n_{ij}}\exp(x_\ell)^{n_\ell}$ modulo central terms in Γ, for i,

$\exp(\frac{1}{2}[x_i,x_j])^{n_{ij}}\exp(x_\ell)^{n_\ell}$ modulo central terms in Γ, for

written in the form

$$\exp(x_1)^{n_1} \ldots \exp(x_k)^{n_k}\exp(\frac{1}{L_{12}}[x_1,x_2])^{n_{k+1}} \ldots$$

$$\exp(\frac{1}{12P_{k-1,k}}[x_k,[x_{k-1},x_k]])^{n_s} .$$

This tells us that exp(B) is a canonical basis of the discrete cocompact subgroup Γ which it generates. We see from theorem F that Γ is a lattice group for any choice of integers L_{ij}, M_{ij}, P_{ij} and $Q_{ij\ell}$. We define $\mathcal{L}_{k,3}$ to be the collection of lattice nilpotent groups with canonical bases of the form exp(B). A discrete cocompact subgroup of $\mathfrak{n}_{k,3}$ is a lattice if and only if it contains a member of $\mathcal{L}_{k,3}$. As in the two-step case, it can be shown that $\mathfrak{n}_{k,3}$ contains lattice subgroups which are not in $\mathcal{L}_{k,3}$.

From the results stated in theorems E and F one would guess that in the general case of the connected simply connected ℓ-step nilpotent Lie group, there exist necessary and sufficient conditions for a discrete cocompact subgroup to be lattice, and that these arise from the Campbell-Hausdorff formula. However, attempts to prove this encounter the difficulty that the central terms analogous to z_{ij} in the proof of theorem F do not drop out so readily when ℓ is greater than 3. Moreover, the expansion of $a+b$ becomes considerably more involved, thereby complicating the sufficiency argument.

Let $q_{j,n}$ $(n = 1,2,\dots,n_j)$ be the rational coefficients of the n_j terms of length j in the Campbell-Hausdorff formula. Let $q_j = \frac{1}{N}$ where N is the greatest common denominator of $q_{j,1},\dots,q_{j,n_j}$. Note that $q_1 = 1$. Let $Q_j = q_1 q_2 \dots q_j$. We conjecture the following:

Suppose Γ is a discrete cocompact subgroup of a connected simply connected ℓ-step nilpotent Lie group $\mathfrak{n}$. Let $\exp(c_1),\dots,\exp(c_r)$ be a canonical basis for Γ. Then Γ is a lattice nilpotent group if $\log(\Gamma)$ contains

$$(Q_j)^{\ell}[c_{i_1},[c_{i_2},\dots,[c_{i_{j-1}},c_{i_j}]\dots]$$

for $j = 1,2,\dots,\ell$.

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