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Studies in valuation of derivative instruments

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STUDIES IN VALUATION OF DERIVATIVE INSTRUMENTS

by

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Chapter 1

INTRODUCTION

In the first part of this paper, I discuss some aspects of the theory of the Black-Scholes equation and its derivation as explained in the Black-Scholes paper and in articles by R. Merton. Another solution to the Black-Scholes equation is obtained, which is shown to be the unique solution to the model if the Stratonovich integral is assumed in place of the Ito integral. The work is also done to show directly that if the log-normal distribution is assumed, the Black-Scholes formula is the time-discounted expected value of the European call option. In the second part of the paper the random walk model of financial asset price will be modified to reflect the fact that some traders take into account the profitability and salvage value of the underlying businesses so that the price of a going business will not become less than a certain non-zero minimum. Two models will be explored, in the first of which the price of the asset will be allowed to exceed any preset bound, and in the second of which the price will not be allowed to exceed a certain maximum. So that the models can reflect the change in the value of the security between interest or dividend payments, the upper and lower bounds are taken to be time dependent. The first is a modification to allow for the cases of tulips, Mississippi land, Tokyo real estate, and the like. I hope that the second model, in which the asset price varies between two bounds ,will aid in the further study of market behavior along the lines suggested by P. Cootner, as discussed below.The two models could be obtained as the log-normal model of P. Samuelson is obtained in the continuous model of random walk by replacing dS by dS/S, the singularity at zero being replaced by singularities at the upper and lower bounds. This method of introduction is not used, because it can lead to difficulties in interpretation, as explored in the discussion in the early parts of the paper, and because a finite model is better suited to discussion of the economic issues involved. The models obtained are applied to the valuation of options.

Chapter 2

THE EQUATIONS FOR VALUATION OF A CALL OPTION

In this chapter I discuss some equations that have been studied in the application of the random walk hypothesis to the valuation of stock options. Consider first the equation 7 of F. Black and M. Scholes, [2], pg. 643, and equation 21, pg. 646 of the same reference.

$$w_2 = rw - rxw_1 - \frac{1}{2}v^2x^2w_{11}.$$

Here $w = w(x,t)$ is the option value, x is the stock value, t is the time, r is the risk free interest rate, and v^2 is the variance of the rate of return on the stock. Equation 44 of R. Merton, [9], pg. 170 follows.

$$\frac{1}{2}\sigma^2 S^2 W_{11} + (rS - D)W_1 - W_2 - rW = 0.$$

Here $W = W(S,\tau)$. The subscripts denote partial differentiation, W is the value of the option, S is the price of the share, t* is the maturity date of the option, t is the time, τ=t*-t is the time remaining to maturity, $D = D(S,\tau)$ is the dividend per share per unit time. The approximation of continuous dividend leads to taking $D = \rho S$ for ρ equal to a constant rate . The equation number 2 of E. Schwartz, [13], is the same except that D is taken to be zero except for lump sums at discrete times and more suggestive literal subscript notation is used.

$$\frac{\sigma^2}{2} S^2 W_{SS} + rSW_S - W_\tau - Wr = 0.$$

I will discuss the equation of R. Merton's paper with D=ρS but use the literal subscripts as in E. Schwartz. Please note at this point that the cited equations are equivalent except for the choice of D.

2.1 REDUCTION TO THE HEAT EQUATION

It is important to relate the study of this equation to the existing literature on partial differential equations by reducing the equations to a well known standard form, called the Heat Equation. Of course, the initial and boundary data given in any problem must be correspondingly transformed. At this point I will go through the reduction of this equation to the heat equation. The benefits of the

reduction will be discussed afterward. The substitution λ=ln S can be used to remove the S and S^2 factors. The chain rule implies the following equations.

$$S^2W_{11} = S^2W_{SS} = W_{\lambda\lambda} - W_\lambda.\ SW_1 = SW_S = W_\lambda.$$

It follows that Merton's equation 44 is equivalent to

$$\frac{\sigma^2}{2}W_{\lambda\lambda} + (r - \rho - \frac{\sigma^2}{2})W_\lambda - W_\tau - Wr = 0.$$

The first derivative with respect to λ can be removed by replacing W by a new unknown function U related to W by the equation $W = L(\lambda)U(\lambda, \tau)$. Here L is an appropriately chosen function. The equations

$$W_\lambda = L'U + LU_\lambda. \text{ and },\ W_{\lambda\lambda} = L''U + 2L'U_\lambda + LU_{\lambda\lambda}.$$

yield the equivalent equation

$$\frac{\sigma^2}{2}LU_{\lambda\lambda} + [2L'\frac{\sigma^2}{2} + (r-\rho-\frac{\sigma^2}{2})L]U_\lambda - LU\tau + [(r-\rho-\frac{\sigma^2}{2})L' + \frac{\sigma^2}{2}L'' - rL]U = 0.$$

The function L is chosen to be a solution of the differential equation

$$\sigma^2L' + (r - \rho - \frac{\sigma^2}{2})L = 0.$$

That is, $L = \exp(-[r - \rho - \frac{\sigma^2}{2}]\lambda\sigma^{-2})$. This choice implies that the coefficient of U_λ is 0. It follows from the differential equation that $L' = -\frac{L}{\sigma^2}(r - \rho - \frac{\sigma^2}{2})$ and $L'' = \frac{L}{\sigma^4}(r - \rho - \frac{\sigma^2}{2})^2$. If these values are substituted in the equivalent equation above, every term of the new equation obtained has the factor L, which may be divided out since the exponential function is never zero. The equation for U is then as follows:

$$\frac{\sigma^2}{2}U_{\lambda\lambda} - U_\tau - [\frac{1}{2\sigma^2}(r - \rho - \frac{\sigma^2}{2})^2 + r]U = 0.$$

The term not involving a derivative is eliminated by the substitution $U = \vartheta(\tau)V(\lambda, \tau)$, where V is a new unknown function and ϑ is chosen so that the coefficient of V in the equivalent equation for V is zero. Since ϑ is a function of τ alone, $U_{\lambda\lambda} = \vartheta V_{\lambda\lambda}$ and $U_\tau = \vartheta' V + \vartheta V_\tau$ with $\vartheta' = \frac{d\vartheta}{d\tau}$. If these values are substituted in the equation for U, the equation for V is then as follows:

$$\frac{\sigma^2}{2}\vartheta V_{\lambda\lambda} - \vartheta V_\tau - [\vartheta' + (\frac{1}{2\sigma^2}(r - \rho - \frac{\sigma^2}{2}) + r)\vartheta]V = 0.$$

The function ϑ is chosen to solve the differential equation obtained by setting the coefficient of V equal to zero. The solution is $\vartheta = \exp(-[\frac{1}{2\sigma^2}(r - \rho - \frac{\sigma^2}{2}) + r]\tau)$ Since ϑ is never zero, ϑ can be divided out of the equation. The equation for the unknown function V is $\frac{\sigma^2}{2}V_{\lambda\lambda} - V_\tau = 0$. The final part of the reduction of the original equation to the Heat Equation is the replacement of the variable τ by the new variable $\theta = \frac{\sigma^2}{2}\tau$. It follows from the chain rule that $V_\tau = V_\theta \frac{\partial\theta}{\partial\tau} = \frac{\sigma^2}{2}V_\theta$. The equation for $V = V(\lambda, \theta)$ is the Heat Equation as follows:

$$V_{\lambda\lambda} = V_\theta.$$

Now that we have found the explicit relation between the given equation and the Heat Equation in standard form we can apply the theory that has been developed over the centuries to theoretical questions and planning the calculation of solutions, for part of the theory of the Heat Equation is the theory of numerical approximation of solutions. It is known that some obvious methods of numerical solution of the Heat Equation based on approximation of the equation by difference equations are divergent. This is also of interest in studying the numerical methods used in the calculation of solutions to the equation in the case of $D = 0$. except when lump sum dividends are paid as in the paper of E. Schwartz cited above.

A similar reduction is summarized with some misprints in equation 9 of the cited reference by F. Black and M. Scholes.

2.2 SOLUTIONS TO THE EQUATIONS

The solution of the equation given by F. Black and M. Scholes is obtained by use of the Fourier Transform. c. f. J. Fourier, [6], article 363 and F. John, [7] , pg. 204 . Also in the book by F. John is a proof that the problem of solving the Heat Equation with initial conditions given on an infinite interval has infinitely many solutions. F. John has a uniqueness theorem in his book, that the Fourier transform method results in a solution which is the only function in a class of functions characterized by their behavior at infinity that satisfies the equation and which has the initial conditions as limits as $t \to t*$ or $\tau \to 0$. Unfortunately it is sometimes difficult to realistically specify the behavior of a solution at infinity in the applications of the Heat Equation to either Finance or to Physics.

To find a solution alternative to the solution given by the Fourier Transform, we need only look to the solutions linear in x, where the separate imposition of the condition that the option need not be exercised results only in a discontinuity of the first derivatives of the solution. Recall the Black-Scholes equation

$$w_2 = rw - rxw_1 - \frac{1}{2}v^2x^2w_{11}.$$

It follows by substitution that $w = ax.$ is a solution for all constants a. The equation for solutions that are functions of t alone is $w_t = rw$. Here w_t replaces w_2, and w_1 and w_{11} are zero. The solution to this equation is $w = be^{rt}$. Since the partial differential equation is linear and homogeneous, the sum of two solutions is again a solution. Thus one tries to find coefficients a and b such that $w = ax + be^{rt}$ matches the initial conditions, that $w = \max(0, x - E)$. for $t = t*$. Thus we set $a = 1.$ and $b = -Ee^{-rt*}$ to obtain $w = \max(0, x - Ee^{-r(t*-t)})$. Thus $w = x - Ee^{-r(t*-t)}$ for $\ln x \geq \ln(E) - r(t* - t)$ and $w = 0.$ otherwise.

2.3 USE OF THE STRATONOVICH INTEGRAL

It will be shown below that the solution linear in x also follows from the Black-Scholes difference equation

$$\Delta x - \Delta w/w_1 = (x - w/w_1)r\Delta t,$$

derived from the sequence of formulas on pp. 642, and 643 of Black-Scholes, [2], omitting the application of the Ito lemma, and using the Stratonovich integral to obtain a first order equation.

The appearance of the term $\frac{1}{2}w_{11}v^2x^2$ in the Black-Scholes equation and the term $\frac{1}{2}\sigma^2S^2W_{SS}$ in Merton's equation is due to the interpretation of the differential as the inverse of the Ito stochastic integral. If the differential is interpreted as the inverse of the Stratonovich stochastic integral, no second order term appears in the partial differential equations, but except for the deletion of the second order term the equations remain the same as above. Please assume $r \neq \rho$ and consider the equation

$$(r-\rho)SW_S - W_\tau - Wr = 0.$$

If $\lambda = \ln(S)$, then $W_S = \frac{1}{S}W_\lambda$. The above equation is equivalent, in terms of λ and τ, to the equation

$$(r-\rho)W_\lambda - W_\tau - Wr = 0.$$

If $W = \exp(-r\tau)U(\lambda,\tau)$, then $W_\tau = -rW + \exp(-r\tau)U_\tau$ follows from the chain rule. If this is substituted in the equation, the term rW is cancelled. The coefficient $\exp(-r\tau)$ can then be divided out. In terms of the unknown function U, the equation is

$$(r-\rho)U_\lambda - U_\tau = 0.$$

If $r-\rho$ is not 0, define new variables ξ and η by the equations $\lambda = (r-\rho)\xi$, and $\tau = -\xi + \eta$. The inverse of this transformation is $\xi = \frac{\lambda}{r-\rho}$, $\eta = \tau + \frac{\lambda}{r-\rho}$. By the chain rule, $U_\xi = U_\lambda \frac{\partial\lambda}{\partial\xi} + U_\tau \frac{\partial\tau}{\partial\xi}$, so that $U_\xi = (r-\rho)U_\lambda - U_\tau$. The equation for U in terms of ξ and η is

$$U_\xi = 0.$$

The solution of this equation is that U is a function of η alone. The formula for W in terms of S and τ is

$$W = \exp(-r\tau) f(\tau + \frac{\ln(S)}{r-\rho}).$$

The function f is a function of a single variable to be determined from the initial and boundary values of the problem. At $\tau = 0$, $W = f(\frac{\ln(S)}{r-\rho})$. Since $\tau = 0$ on the maturity date of the option, $W = 0$ for $S \leq E$ and $W = S - E$ for $S \geq E$. Since $S - E = \exp[(r-\rho)\frac{\ln S}{r-\rho}] - E$, the function f is given in terms of a variable α by the formula $f(\alpha) = \exp[(r-\rho)\alpha] - E$. For $\alpha = \tau + \frac{\ln(S)}{r-\rho}$, $f(\tau + \frac{\ln(S)}{r-\rho}) = \exp[(r-\rho)\tau + \ln(S)] - E$. The condition $W = 0$ for $S \leq E$ at t=0 implies $f(\alpha) = 0$ for $\alpha \leq \frac{\ln(E)}{r-\rho}$. Thus,

$$W = S\exp(-\rho\tau) - E\exp(-r\tau) \textbf{ for } \tau(r-\rho) + \ln(S) \geq \ln(E).$$

$$W = 0 \textbf{ for } \tau(r-\rho) + \ln(S) \leq \ln(E)$$

The derivation shows that this is the unique solution to the first order partial differential equation satisfying the initial conditions. Equivalently, one has $W = S\exp(-\rho\tau) - E\exp(-r\tau)$ for all values of S and τ for which the formula is not negative, and $W = 0$ otherwise. One could have found each of the terms $S\exp(-\rho\tau)$ and $E\exp(-r\tau)$ by the method of separation of variables, and combined the solutions to satisfy the initial conditions. In this case, one would have to look elsewhere for a proof that this is the unique solution of the equation with the given initial conditions. Recall that it was shown in the preceding section that this value of W is a solution of the Black-Scholes equation and therefore of the equation 44 of R. Merton cited above. If $r = \rho$, the equation is

$$-W_\tau - Wr = 0.$$

This equation has solutions $W = Ce^{-rt}$. Here C may be any function of S not involving t. For $t = 0$, $W = C$ so that $C = 0$ for $S \leq E$, and $C = S - E$ otherwise. Thus, the solution is as above. Please note at this point that the first order partial differential equation considered in this section does not reduce to the Heat Equation.

2.4 THE CHOICE OF INTEGRALS

Unfortunately, there is an infinity of integrals available to provide a differential equation approximating the Black-Scholes difference equation. There is an example of this in the book of Z. Schuss, [12], page 72. The Ito and Stratonovich definitions are most popular. The paper of S. Sethi and J. Lehoczky, [14], explores the choice between the two definitions in some financial contexts. In their paper they find valid applications of both integrals.

Chapter 3

MODIFICATIONS OF THE RANDOM WALK HYPOTHESIS

The idea of a random walk was applied by L. Bachelier, [1], to the study of the prices of securities in a thesis on the theory of speculation. The main objection to the direct application of the random walk to stock prices is that if a stock price follows a random walk, zero and negative stock prices would be likely in the model, but, of course, not in reality. The solution was that the logarithm of the stock price be the variable to which the random walk hypothesis applies, giving the stock price a log-normal distribution, c. f. P. Samuelson, [11]. A random variable S is log-normally distributed if and only if the random variable $\ln(S)$ has a normal distribution for $S > 0$. If the stock price is near zero, the logarithm of the stock price changes by a large amount for a small change in the stock price. If the stock price is not near zero, the slope of the logarithmic curve is nearly constant so that data which fits the normal distribution also fits the log-normal distribution, as long as the data is not near zero. If the stock price is near zero, and the logarithm of the stock price follows a random walk, the stock price would move lethargically. The theory explains the slow motion near zero by the notion that traders are interested in the percentage change in the stock price, rather than the change in the price itself, as is defended in detail in a paper by M. Osbourne, [10].

P. Cootner, [3], studied a model of stock market behavior in which the stock price follows a random walk with reflections at upper and lower barriers, the random walk being the result of trading by the uninformed, the barriers being the result of intervention by informed traders either hunting bargains or taking profits. The hypothesis of random walk with reflecting barriers implies that the distribution of the stock prices tends to the uniform distribution with mean the average of the two barriers, therefore independent of the value of the stock price at any given time. The uniform distribution may result from repeated interventions by the informed traders causing reflections from the barriers. The intervention should slow price changes near the ends of the trading range. It is clear that in the time interval before the first reflection, the random walk and the random walk with reflections produce identical distributions as more and more reflections become possible, the random walk with reflections produces more complicated results, as discussed by W. Feller, [5]. As an alternative, I propose to generalize the log-normal idea to fit the situation of the price having just a lower bound and, separately, the situation of the stock price having both upper and lower bounds by changing the idea of having the logarithm of the stock price follow a random walk to having $\ln(S-A)$ follow a random walk where A is the lower bound and there is no upper bound and in the case of both upper and lower bounds, $\ln[(S-A)/(B-S)] = \ln(S-A) - \ln(B-S)$ follows a random walk. This idea is like the introduction of Verhulst's logistic distribution into the stochastic differential equations of Population Biology, as discussed by M. Turelli, [15]. A big difference is the introduction of a non-zero lower bound in this discussion, while in Population Biology, the lower bound is always zero. There is some suspicion in Population Biology of the applicability of this model because zero is a repelling barrier, so that species do not become extinct. There is no such problem in using a Verhulst type model with non-zero lower bound as a model of stock market behavior because the upper and lower bounds, according to Cootner's ideas, are adjusted by the cognoscenti, so the lower is zero only after the business is defunct and the stock has no chance of paying a return. The second big difference is the dynamic character of the upper and lower bounds. There is recognition of the role of outside influences on the market in addition to the random walk. As shown below, $\ln[(S-A)/(B-S)]$ is approximately a linear function of the stock price for most of the interval between$A$ and B, the trading range, so that data which fits the hypothesis that the stock price follows a random walk will also fit the hypothesis that $\ln[(S-A)/(B-S)]$ follows a random walk except at stock values near the upper and lower ends of the trading range. Corresponding to the generalization of the variable following a ran-

dom walk, the normal distribution is generalized in two different ways as follows: A random variable X has a log-normal distribution with lower bound A if and only if $\ln(X-A)$ has a normal distribution for $X > A$. This is the usual log-normal distribution if $A = 0$. Elementary Probability Theory can be applied to generalize the known results about the log-normal distribution to this case. A random variable X has a log-normal distribution with lower bound A and upper bound B if and only if $\ln(X - A) - \ln(B - X)$ has a normal distribution for X between A and B.

This discussion of the random walk is based on the discrete model as explained in W. Feller, [5], pg. 324, market activity being finite. I paraphrase this model to present the modifications. Let the random variable X subject to the random walk start with value 0 at the starting time $t = 0$. Whenever the time is a multiple of h, X is either increased by an amount k with probability p or decreased by an amount k with probability $q = 1 - p$. For fixed h and k, the displacement X at time $t = Nh$ is the sum of N independent random variables, each having mean $(p - q)k$ and variance $4pqk$. The mean of X is therefore $N(p - q)k = t(p - q)k/h = 2ct$, and its variance is $4pqNk = 4pqk/h = 2Dt$. The number c is called the drift coefficient and the number D is called the diffusion coefficient. The constants c and D are alternatively defined by the equations $2D = k/h, p = 1/2+ck/2D$, and $q = 1/2-ck/2D$. The probability that at time t the displacement X lies between X_0 and X_1 , for $X_0 < X_1$ tends to the probability that a normally distributed random variable with mean zero and variance 1 lies between $(X_0 - 2ct)/(2Dt)^{\frac{1}{2}}$ and $(X_1 - 2ct)/(2Dt)^{\frac{1}{2}}$, that is $\frac{1}{\sqrt{2\pi}}\int_{y_0}^{y_1} e^{-\frac{1}{2}\lambda^2}d\lambda$, where $y_0 = (X_0 - 2ct)/(2Dt)^{\frac{1}{2}}$ and

$y_1 = (X_1 - 2ct)/(2DT)^{\frac{1}{2}}$.

The first application of the random walk to study variables other than the stock price is to study the logarithm of the stock price. Since the initial value of the random variable X is zero for $t = 0$, the random variable X is taken to be $\ln(S) - \ln(S_0)$, where S_0 is the stock price at $t = 0$. At this point, the mean and variance of the log-normal variable will be stated in terms of the parameters c and D of the discussion of the random walk hypothesis. The result is based on the well known theorem that the moment generating function for the normal distribution, the function $E(\exp(XT))$ for X normally distributed with mean μ and standard deviation σ, and T an independent variable is $\exp(\mu T + \sigma^2 T^2/2)$. If $X = \ln(S/S_0)$, then $S = S_0 \exp(X)$. If one sets $T = 1$, in the moment generating function, one obtains that the mean of S is $S_0 \exp(2ct + Dt)$. It seems reasonable at this point to set $2c + D$ equal to the interest rate r. The second moment of S is found by setting $T = 2$ in the moment generating function and multiplying by S_0^2. One obtains $S_0^2 \exp(4ct + 4Dt)$. The variance is found by subtracting the square of the mean. The variance is $S_0^2 \exp(4ct + 2Dt)(\exp(2Dt) - 1)$, which is approximately equal to $S_0^2 \exp(2rt)\exp(2Dt)$ for large t. The probability density function of the logarithm of the stock price is $\delta(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2Dt}}e^{-\frac{1}{2}[\frac{1}{\sqrt{2Dt}}(X-2ct)]^2}$. The profit realized on exercise of the option or warrant at time t is $\max(0, S - E)$ where E is the exercise price. In terms of $X = \ln(S/S_0)$, the profit is $P = \max(0, S_0 e^X - E)$. The expected value of the profit is

$$\epsilon(P) = \int_{\ln E - \ln S_0}^{\infty} (S_0 e^X - E)\delta(X)dX.$$

Let $\alpha = \frac{1}{\sqrt{2Dt}}[X - 2(c + D)t]$, so that $d\alpha = \frac{dX}{\sqrt{2Dt}}$.

Note that $-\frac{1}{2}\alpha^2 = -\frac{1}{4Dt}[(X-2ct)^2 - 4Dt(X-2ct) + 4D^2t^2]$
$= -\frac{1}{4Dt}(X-2ct)^2 + X - 2ct - Dt,$
so that $\frac{1}{\sqrt{2\pi}}e^{(2c+D)t}e^{-\frac{1}{2}\alpha^2}d\alpha = e^X\delta(X)dX$
If $X = \ln E - \ln S_0$, **then** $\alpha = \frac{1}{\sqrt{2Dt}}[\ln E - \ln S_0 - 2(c+D)t] = \alpha_0$.
Thus $\int_{\ln E - \ln S_0}^{\infty} S_0e^X\delta(X)dX = S_0e^{(2c+D)t}\frac{1}{\sqrt{2\pi}}\int_{\alpha_0}^{\infty} e^{-\frac{1}{2}\alpha^2}d\alpha$.
If we set $\beta = -\alpha$ **in this integral we obtain**
$-\int_{-\alpha_0}^{-\infty} e^{-\frac{1}{2}\beta^2}d\beta = \int_{-\infty}^{-\alpha_0} e^{-\frac{1}{2}\beta^2}d\beta = \sqrt{2\pi}N(-\alpha_0),$
so that $S_0e^{(2c+D)t}\frac{1}{\sqrt{2\pi}}\int_{\alpha_0}^{\infty} e^{-\frac{1}{2}\alpha^2}d\alpha = S_0e^{(2c+D)t}N(-\alpha_0)$.
Here $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}dz$.
Note that $2(c+D)t = (2c+D)t + Dt = rt + \frac{1}{2}\sigma^2 = rt + \frac{1}{2}v^2t$ **for** $v^2 = 2D$
Thus $-\alpha_0 = \frac{1}{v\sqrt{t}}[\ln S_0 - \ln E + (r + \frac{1}{2}v^2)t]$.
The other integral to be evaluated to find $\epsilon(P)$ **is**
$\int_{\ln E - \ln S_0}^{\infty} E\delta(X)dX = \frac{E}{\sqrt{2\pi}\sqrt{2Dt}}\int_{\ln E - \ln S_0}^{\infty} e^{-\frac{1}{2}Z^2}dX.$
Here $Z = \frac{1}{\sqrt{2Dt}}(X - 2ct)$, $dZ = \frac{1}{\sqrt{2Dt}}dX$.
It follows that the integral equals $\frac{E}{\sqrt{2\pi}}\int_{Z_0}^{\infty} e^{-\frac{1}{2}Z^2}dZ$
$= \frac{E}{\sqrt{2\pi}}\int_{-\infty}^{-Z_0} e^{-\frac{1}{2}Z^2}dZ = EN(-Z_0).$
Note that $-Z_0 = -\frac{1}{\sqrt{2Dt}}(\ln E - \ln S_0 - 2ct)$
$= \frac{1}{v\sqrt{t}}[\ln S_0 - \ln E + (2c+D)t - Dt]$
$= \frac{1}{v\sqrt{t}}[\ln S_0 - \ln E + (r - \frac{1}{2}v^2)t]$. **Thus,**

$$e^{-rt}\epsilon(P) = S_0N(-\alpha_0) - Ee^{-rt}N(-Z_0).$$

The right hand side of the equation is the Black-Scholes formula, the left hand side of the equation is the value at $t = 0$ **of the expected profit at the time of exercise. In the formulas for** $-\alpha_0$ **and** $-Z_0$, $v\sqrt{t}$ **replaces** $\sqrt{2Dt}$. **Note that** S_0 **is the price of the stock at some time before the exercise of the option** ($t = 0$), **and** t **is the length of the time interval from** $t = 0$ **to exercise of the call option(this is** $t^* - t$ **in**

Black-Scholes notation.) The second moment of P is given by

$$\mu_2(P) = \int_{\ln E - \ln S_0}^{\infty} (S_0 e^X - E)^2 \delta(X) dX.$$

The evaluation of $\mu_2(P)$ in terms of elementary functions and $N(x)$ reduces to the evaluation of three integrals,

$\int_{\ln E - \ln S_0}^{\infty} E^2 \delta(X) dX = E^2 N(-Z_0)$,

$\int_{\ln E - \ln S_0}^{\infty} -2S_0 E e^X \delta(X) dX = -2S_0 E e^{(2c+D)t} N(-\alpha_0)$,

and $\int_{\ln E - \ln S_0}^{\infty} S_0^2 e^{2X} \delta(X) dX$.

The first two integrals are constant multiples of the integrals evaluated above. The third is different, but one can use the identity derived above, that

$\frac{1}{\sqrt{2\pi}} e^{(2c+D)t} e^{-\frac{1}{2}\alpha^2} d\alpha = \delta(X) e^X dX$,

and the formula $X = \alpha\sqrt{2Dt} + 2(c+D)t$, to obtain

$S_0^2 e^{2X} \delta(X) = S_0^2 e^{\alpha\sqrt{2Dt}+2(c+D)t} \frac{1}{\sqrt{2\pi}} e^{(2c+D)t} e^{-\frac{1}{2}\alpha^2} =$

$\frac{1}{\sqrt{2\pi}} S_0^2 e^{2(c+D)t+(2c+D)t} e^{-\frac{1}{2}[\alpha^2 - 2\alpha\sqrt{2Dt}+2Dt]} e^{Dt} d\alpha =$

$\frac{1}{\sqrt{2\pi}} S_0^2 e^{4(c+D)t} e^{-\frac{1}{2}(\alpha-\sqrt{2Dt})^2} d\alpha$.

Let $\zeta = \alpha - \sqrt{2Dt}$. If $X = \ln E - \ln S_0$, then $\alpha = \alpha_0$

for $\zeta_0 = \alpha_0 - \sqrt{2Dt} = \frac{1}{\sqrt{2Dt}}[\ln E - \ln S_0 - 2(c+D)t - 2Dt]$,

so that $\zeta_0 = \frac{1}{\sqrt{2Dt}}[\ln E - \ln S_0 - (2c+4D)t]$.

Thus the third integral above equals $S_0^2 e^{4(c+D)t} N(-\zeta_0)$. As before, one can express each of these integrals in terms of r and v^2 to obtain some nice formulas, but the variance $\sigma^2(P)$ is the difference $\mu_2(P) - \epsilon(P)^2$. Cancellation will not happen in the computation of $\sigma^2(P)$in the absence of nice formulas for $N(-A)N(-B)$ and the like, so that I will not combine the formulas to obtain a single long formula for σ^2. Next, let us find the mean of the stock price S if the variable $V = \ln[(S-A)/(B-S)]$ follows a random walk with zero drift. In terms of the scaled dimensionless variable $s = [S - (A+B)/2]/[(B-A)/2]$, we obtain $V = \ln[(s+1)/(1-s)]$. Note that $s = 0$ for $S = (A+B)/2$, the middle of the trading range, $s = +1$ for $S = B$, the upper end of the trading range, and $s = -1$ for $S = A$, the lower end of the trading range. Since V is a function of s alone, the value accumulated over time is entirely accounted for by making A and B functions of the interest rate r, and the time t, rather than by postulating a nonzero drift. Let s_0 be the value of s for $S = S_0$, and let V_0 be the value of V for $s = s_0$. The variable $X = V - V_0$ has mean 0 and variance $2Dt$. Let $I = [(s_0+1)/(1-s_0)]$ so $\exp(X) = [(s+1)/(1-s)]/I$. This implies $s = (I\exp(X)-1)/(I\exp(X)+1)$. The derivation of the mean of the log-normal distribution was eased by the existence of the formula for the moment generating function of the normal distribution. This formula is not of obvious help in this case. I use numerical integration to approximate the mean and variance in a number of cases. The results are presented as graphs in the following sections.

3.1 CALCULATION OF THE AVERAGE AND VARIANCE OF STOCK PRICE

The following program computes the average and the variance of the price of a stock if the hypotheses discussed in the beginning of the paper are true. The program is inserted here because together with its comments, it summarizes and illustrates the relations between some of the variables of Part 1. The initial comment lines have been made into a separate paragraph for easy reading. The Fortran language is used in an early form. In order to follow the style conventions of the rest of the paper, the format required by the original Fortran is not strictly followed. The program can easily be restored to the original format.

3.1.1 INITIAL COMMENTS

The notation of the program follows. The variables A and B are the lower and upper limits of the trading range. These variables do not appear in the computation below, since a normalized variable is introduced. SDX is the instantaneous standard deviation of the variable doing the random walk. It is the square root of 2Dt,where t is the time from the start of the random walk and D is a constant of the random walk. The variable S is the stock price at time t, and S0 is the initial value of S at t=0. The function U(S)=(S-.5*(B+A))/.5*(B-A),is a normalized dimensionless variable related to S. In terms of U(S), S=.5*(B+A) +.5*(B-A)*U(S) Let U0=U(S0). Let F(h)=(1.+h)/(1.-h)=(2.-(1.-h))/(1-h)=2./(1-h) -1. Thus h=1.-2./(F(h)+1.) . This yields the formula S=.5*(B+A)+.5*(B-A)*(1.-2./(F(U)+1.)) , where U is written for U(S). The variable X=LOG(F(U)/F(U0)) is the variable assumed to take the random walk, so that X has a normal distribution with mean zero and variance 2Dt. Clearly, F(U)=F(U0)*exp(X) where exp is the exponential function. The program will use DEXP the double precision exponential function. The stock price S is represented by the the formula S=.5*(B+A)+.5*(B-A)*(1.-2./(F(U0)*DEXP(X)+1.). S is a linear function of U(S), so the mean and variance of S can be found from the values of the mean and variance of U(S)=1.-2./(F(U0)*DEXP(X)+1.) Note that F(U0) is a constant during the evaluation of each integral. Note also that the time enters the calculation only in the choice of SDX and the terms .5(B+A), and .5(B-A), not used below. The growth of value between dividend payments and the loss at dividend times is accounted for by the choice of the pair A,B. The changes in S corresponding to changes in X during the random walk are taken to be large at times when .5(B-A) is large, and small at times when .5(B-A) is small, so there is an implicit assumption that the traders are concerned with the change in S as a percentage of the trading range. The program calculates the mean and variance for representative values of U0 and SDX. The program uses the trapezoidal rule because of the easy error analysis available for monotonic integrands. The step size of the integration is .0005.Infinity is taken to be 3.92. The maximum error in the integral is less than or equal to .00027 .

3.1.2 THE STOCK PROGRAM

```
PROGRAM STOCK
REAL*8 APM,APV,C1,C2,E,V,G,H,P,PI,R,STEP,SDX,
c U0,VUS,Y,Z
INTEGER K
DIMENSION E(20),V(20)
OPEN(3,FILE='N344.TXT',STATUS='NEW')
OPEN(4,FILE='N444.TXT',STATUS='NEW')
OPEN(5,FILE='N544.TXT',STATUS='NEW')
OPEN(6,FILE='N644.TXT',STATUS='NEW')
OPEN(7,FILE='N744.TXT',STATUS='NEW')
OPEN(8,FILE='N844.TXT',STATUS='NEW')
```

```
OPEN(9,FILE='N944.TXT',STATUS='NEW')
OPEN(10,FILE='N1044.TXT',STATUS='NEW')
OPEN(11,FILE='N1144.TXT',STATUS='NEW')
OPEN(12,FILE='N1244.TXT',STATUS='NEW')
WRITE(*,1)
PI=3.14159265359
R=1./((2.*PI)**.5)
WRITE(*,2)R
STEP=.0005
U0=-.9 100 CONTINUE
c Before each return to this location, U0 IS increased.
P=(1.+U0)/(1.-U0)
c P is F(U0)
c The initial value of SDX is the same for each choice of U0.
SDX=.1
L=1
200 CONTINUE
c Before each return to this location, SDX is increased
Y=-3.92
c This initial value of Y=X/SDX is the lower limit of integration
c after a change of variable to Y. It is the same for each choice of
c values of SDX and U0.
APM=0.
APV=0.
K=1
300 CONTINUE
c Before each return to this location, Y is increased.
```

```
Z=Y**2
C1=DEXP(-Z/2.)
C2=1.-2./(P*DEXP(Y*SDX)+1.)
G=C1*C2
H=C1*(C2**2)
IF(K.EQ.1)G=G/2.
IF(K.EQ.1)H=H/2.
K=K+1
APM=APM+G
APV=APV+H
Y=Y+STEP
IF(Y.LT.3.92)GO TO 300
APM=(APM-G/2.)*STEP*R
APV=(APV-H/2.)*STEP*R
VUS=APV-APM**2
WRITE(*,3)U0,SDX,APM,VUS,APM/U0,VUS/SDX**2
WRITE(3,3)U0,SDX,APM,VUS,APM/U0,VUS/SDX**2
WRITE(4,4)U0,APM
E(L)=APM
V(L)=VUS
SDX=SDX+1.
L=L+1
IF(SDX.LT.10.)GO TO 200
WRITE(5,5)U0,(E(J),J=1,5)
WRITE(6,5)U0,(E(J) J=6,10)
WRITE(7,5)U0,(E(J) J=11,15)
WRITE(8,5)U0,(E(J) J=16,19)
```

```
WRITE(9,5)U0,(V(J) J=1,5)
WRITE(10,5)U0,(V(J) J=6,1)
WRITE(11,5)U0,(V(J) J=11,15)
WRITE(12,5)U0,(V(J) J=16,19)
U0=U0+.1
IF(U0.LT.1.)GO TO 100
1 FORMAT(' START OF PROGRAM STOCK.FOR')
2 FORMAT(' ',E25.15)
3 FORMAT(' U0',E8.2,' SDX',E8.2,' AVG',E10.4,' VAR',E10.4,
c 'A/U0',E10.4,' RVR',E10.4)
4 FORMAT(E8.2,E10.4)
5 FORMAT(E8.2,5E10.4)
STOP
END
```

3.2 CALCULATION OF THE VALUE OF A CALL OPTION

The following program computes the mean and the variance of the price of a call option on a stock if the hypotheses discussed in part 1 of the paper are true. The program is inserted here because together with its comments,it summarizes and illustrates the relations between some of the variables of Part 1. The initial comment lines are collected in a separate paragraph. The Fortran language is used in an early form. In order to follow the style conventions of the rest of the paper, the format required by the original Fortran is not strictly followed. The program can easily be restored to the original format.

3.2.1 INITIAL COMMENTS

The notation of the program follows. The variables A and B are the lower and upper limits of the trading range. These variables do not appear in the computation below, since a normalized variable is introduced. SDX is the instantaneous standard deviation of the variable doing the random walk. It is the square root of 2Dt, where t is the time from the start of the random walk and D is a constant of the random walk. The variable S is the stock price at time t, and S0 is the initial value of S at t=0. The function U(S)=(S-.5*(B+A))/.5*(B-A), is a normalized dimensionless variable related to S. In terms of U(S), S=.5*(B+A) +.5*(B-A)*U(S) Let U0=U(S0). Let F(h)=(1.+h)/(1.-h)=(2.-(1.-h))/(1-h)=2./(1-h) -1. Thus h=1.-2./(F(h)+1.). This yields the formula S=.5*(B+A)+.5*(B-A)*(1.-2./(F(U)+1.)), where U is written for U(S). The variable X=LOG(F(U)/F(U0)) is the variable assumed to take the random walk, so that X has a normal distribution with mean zero and variance 2Dt. Clearly, F(U)=F(U0)*exp(X) where exp is the exponential function. The program will use DEXP, the double precision exponential function. The stock price S is represented by the formula S=.5*(B+A)+.5*(B-A)*(1.-2./(F(U0)*DEXP(X)+1.). The exercise price for the option is taken to be S0. The gain upon exercise at a time t>0 is S-S0 if S-S0 is positive and 0 if S-S0 is negative or zero. Recall that the definition of X implies that X=0 if S=S0, and X>0 for S>S0 Thus the lower limit of integration in the calculation of the mean and variance of the gain on exercise is 0. From the formulas above we obtain S-S0=.5*(B-A)(-2./(F(U0)*DEXP(X)+1.)-(-2./(F(U0)+1.)) Note that the terms .5(B+A) cancel and X=0 for S=S0, so DEXP(X)=1 for S=S0. The trading range B-A can be factored out of the above expression for S and the mean and variance of the gain of the option can be found from the values of the mean and variance of 1./(F(U0)+1.)-1./(F(U0)*DEXP(X)+1.) by multiplying by (B-A) and (B-A)**2 respectively. F(U0) is a constant during the evaluation of each integral.and that the time enters the calculation only in the choice of SDX and the size of the trading range,B-A. The growth of value between dividend payments and the ex-dividend loss is accounted for by the choice of the pair A,B. The changes in S corresponding to changes in X during the random walk are taken to be large at times when .5*(B-A) is large, and small at times when .5*(B-A) is small, so there is an implicit assumption that the traders are concerned with the change in S as a percentage of the trading range. The program calculates the mean and variance for representative values of U0 and SDX. The program uses the trapezoidal rule because of the easy error analysis available for monotonic integrands. The step size of the integration is .0005. Infinity is taken to be 3.92.

3.2.2 THE OPTION PROGRAM

```
PROGRAM OPTION
REAL*8 APM,APV,C1,C2,G,H,P,PI,Q,R,STEP,SDX,
c U0,VUS,Y,Z
INTEGER K
OPEN(3,FILE='O344.TXT',STATUS='NEW')
```

```
OPEN(4,FILE='O444.TXT',STATUS='NEW')
OPEN(5,FILE='O544.TXT',STATUS='NEW')
OPEN(6,FILE='O644.TXT',STATUS='NEW')
OPEN(7,FILE='O744.TXT',STATUS='NEW')
OPEN(8,FILE='O844.TXT',STATUS='NEW')
OPEN(9,FILE='O944.TXT',STATUS='NEW')
OPEN(10,FILE='O1044.TXT',STATUS='NEW')
OPEN(11,FILE='O1144.TXT',STATUS='NEW')
OPEN(12,FILE='O1244.TXT',STATUS='NEW')
PI=3.14159265359
R=1./((2.*PI)**.5)
STEP=.0005
U0=-.9
100 CONTINUE
c Before each return to this location, U0 IS increased
P=(1.+U0)/(1.-U0)
c P is F(U0)
c Q=1./(P+1.)
Q=(1.-U0)*.5
c The initial value of SDX is the same for each choice of U0.
SDX=.1
L=1
200 CONTINUE
c Before each return to this location, SDX is increased Y=0.
c This initial value of Y=X/SDX is the lower limit of integration
c after the change of variable to Y. It is the same for each choice of
c values of SDX and U0.
```

```
APM=0.
APV=0.
K=1
300 CONTINUE
c Before each return to this location, Y is increased
Z=Y**2
C1=DEXP(-Z/2.)
C2=Q-1./(P*DEXP(Y*SDX)+1.)
G=C1*C2
H=C1*(C2**2)
IF(K.EQ.1)G=G/2.
IF(K.EQ.1)H=H/2.
K=K+1
APM=APM+G
APV=APV+H
Y=Y+STEP
IF(Y.LT.3.92)GO TO 300
APM=(APM-G/2.)*STEP*R
APV=(APV-H/2.)*STEP*R
VUS=APV-APM**2
WRITE(*,3)U0,SDX,APM,VUS,APM/U0,VUS/SDX**2
WRITE(3,3)U0,SDX,APM,VUS,APM/U0,VUS/SDX**2
WRITE(4,4)U0,APM
E(L)=APM
V(L)=VUS
SDX=SDX+1.
L=L+1
```

```
IF(SDX.LT.10.)GO TO 200
WRITE(5,5)U0,(E(J),J=1,5)
WRITE(6,5)U0,(E(J) J=6,10)
WRITE(7,5)U0,(E(J) J=11,15)
WRITE(8,5)U0,(E(J) J=16,19)
WRITE(9,5)U0,(V(J) J=1,5)
WRITE(10,5)U0,(V(J) J=6,1)
WRITE(11,5)U0,(V(J) J=11,15)
WRITE(12,5)U0,(V(J) J=16,19)
U0=U0+.1
IF(U0.LT.1.)GO TO 100
1 FORMAT(' START OF PROGRAM OPTION')
2 FORMAT(' ',E25.15)
3 FORMAT(' U0',E8.2,' SDX',E8.2,' AVG',E10.4,' VAR',E10.4,' A/U0',

cE10.4,' RVR',E10.4)
4 FORMAT(E8.2,E10.4)
5 FORMAT(E8.2,5E10.4)
STOP
END
```

3.3 THE RESULTS OF THE CALCULATIONS

The results of the calculations are presented as graphs. The horizontal axis of each graph is labeled U0, a normalized dimensionless variable associated with S_0 ,the price of the stock at the time of the purchase of the option,by the formula U0 $= U(S_0) =$

$[S_0 - \frac{1}{2}(B + A)]/[\frac{1}{2}(B - A)]$. Here A and B are the lower and upper limits of the trading range, U0 $= -1$ for $S_0 = A$, the lower end of the trading range, U0 $= 0$ for $S_0 = \frac{1}{2}(B + A)$, at the middle of the trading range, and U0 $= +1$ for $S_0 = B$, at the upper end of the trading range. The variable U0=is only weakly related to the time, in that it is expected to increase with time until a dividend is paid, but it is independent of the time interval from the purchase of the option to the time of possible exercise of the option. The volatility of the stock and the time from purchase of the option to the time of possible exercise of the option are reflected in the standard deviation of the random walk, which is used as a label on the U0 axis of each graph. The vertical axes representing variances are marked off in multiples of $(B - A)^2$. The vertical axes representing the average prices of the stock are Values of the Function $U(S)$ and thus are given relative to the center of the trading range, $\frac{1}{2}(B + A)$ in multiples of $\frac{1}{2}(B - A)$. To restore these values to ordinary units one must multiply by $\frac{1}{2}(B - A)$ and add $\frac{1}{2}(B + A)$. The vertical axes representing the average value of an option are marked off in multiples of $(B - A)$, the length of the trading range. When interpreting the graphs, one must take into account that the vertical axes have been marked off to optimize the readability of each graph, and the scale changes from graph to graph.

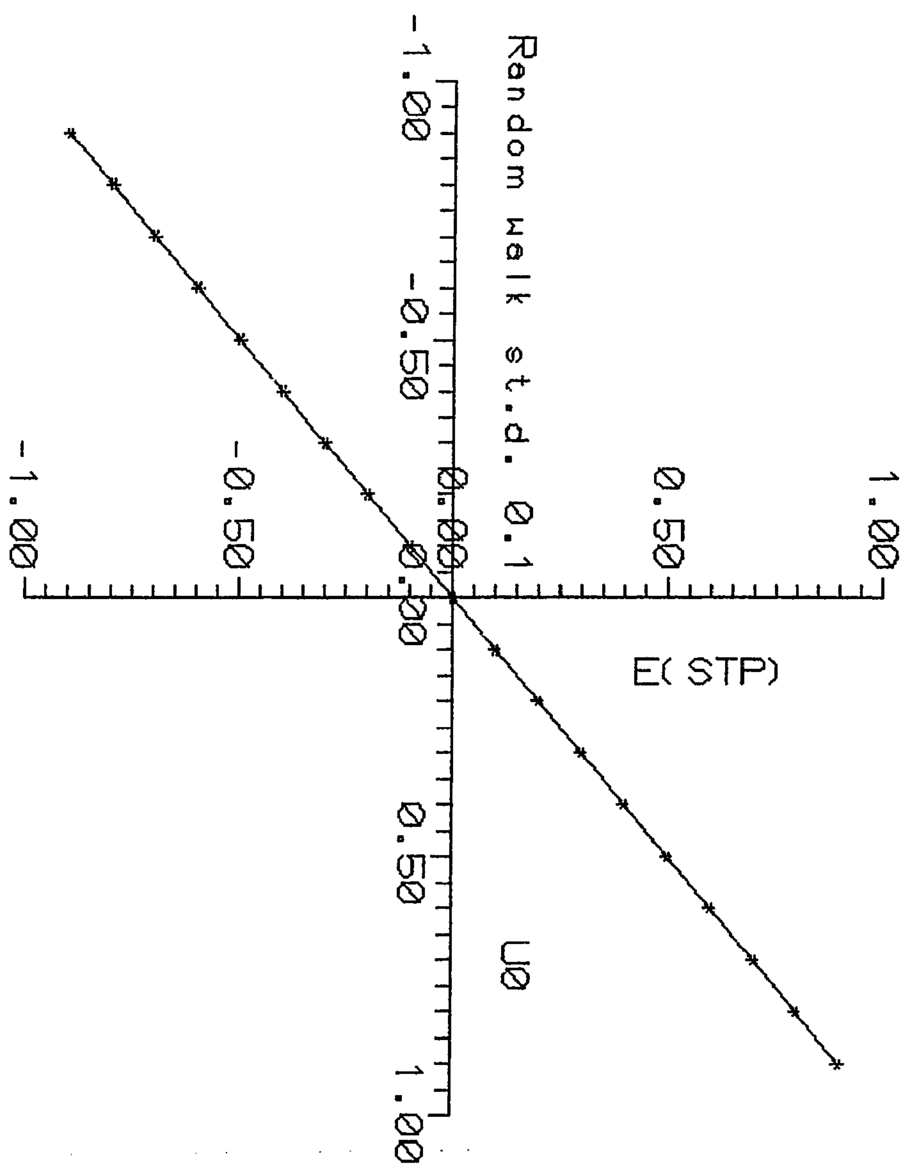
The three graphs representing the expected value of the of the stock price, suitably scaled and normalized, show the effects of reflection at the lower and upper ends of the trading range. For a small standard deviation as 0.1, the expected stock prices approximately equal $U(S_0)$,the maxima and minima on each graph being

+1 and -1 respectively. As the random walk standard deviation increases, the maxima and minima decrease in absolute value, until for random walk standard deviation 9.1, the expected value is within .3 of a unit from the center of the trading range. One would expect that for very large standard deviations , from very volatile stocks or long times, the average value would be almost zero on the graph, corresponding to the center of the trading range.

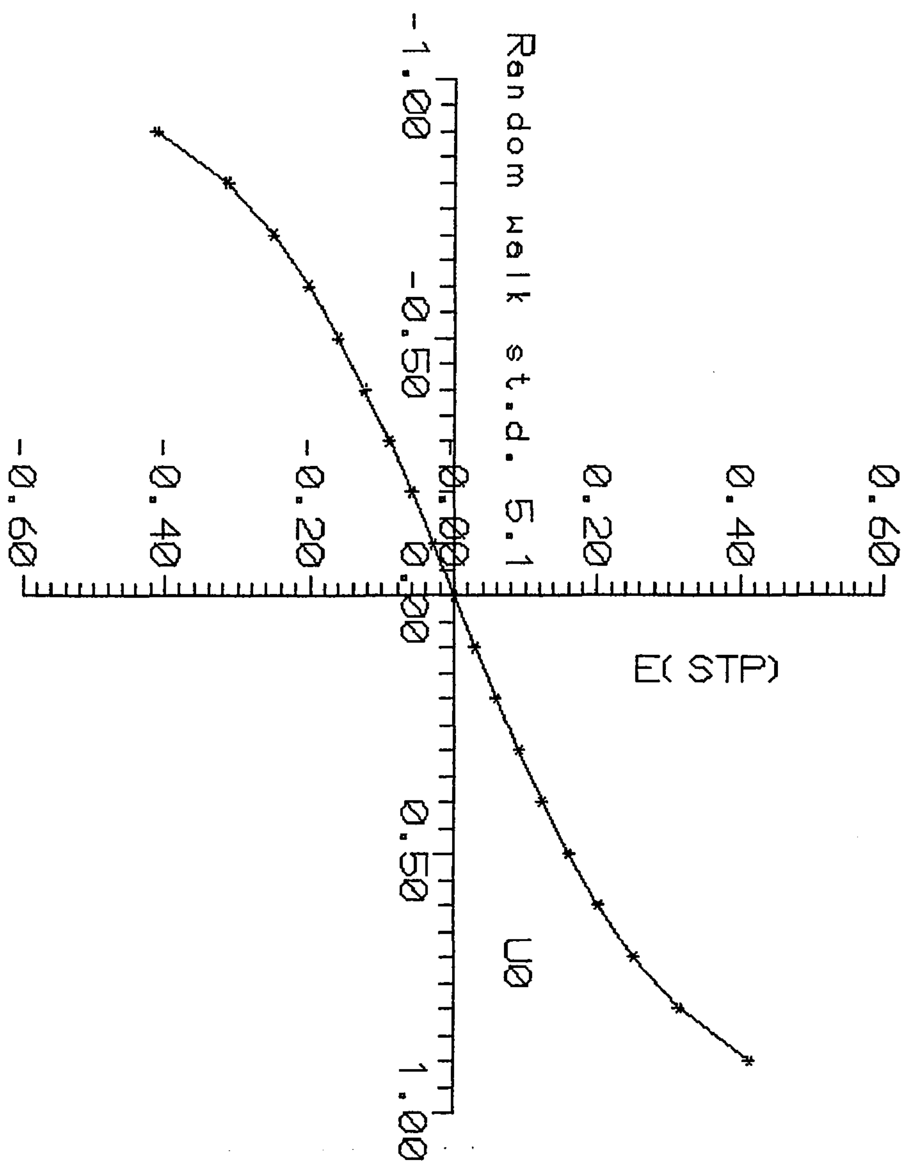
One cannot go directly from the expected value of the stock price to the expected option value because the option need not be exercised. In the graphs shown in the following, the exercise price is equal to the stock price at the time of purchase of the option. Notice that the maximum expected value of the option for each value of the standard deviation of the random walk occurs for S_0 smaller than the center of the trading range, and that option values become smaller as the stock price at time of purchase approaches the upper limit of the trading range.

The variance of the option value peaks where the expected value is highest, as shown in the last three graphs of this set.

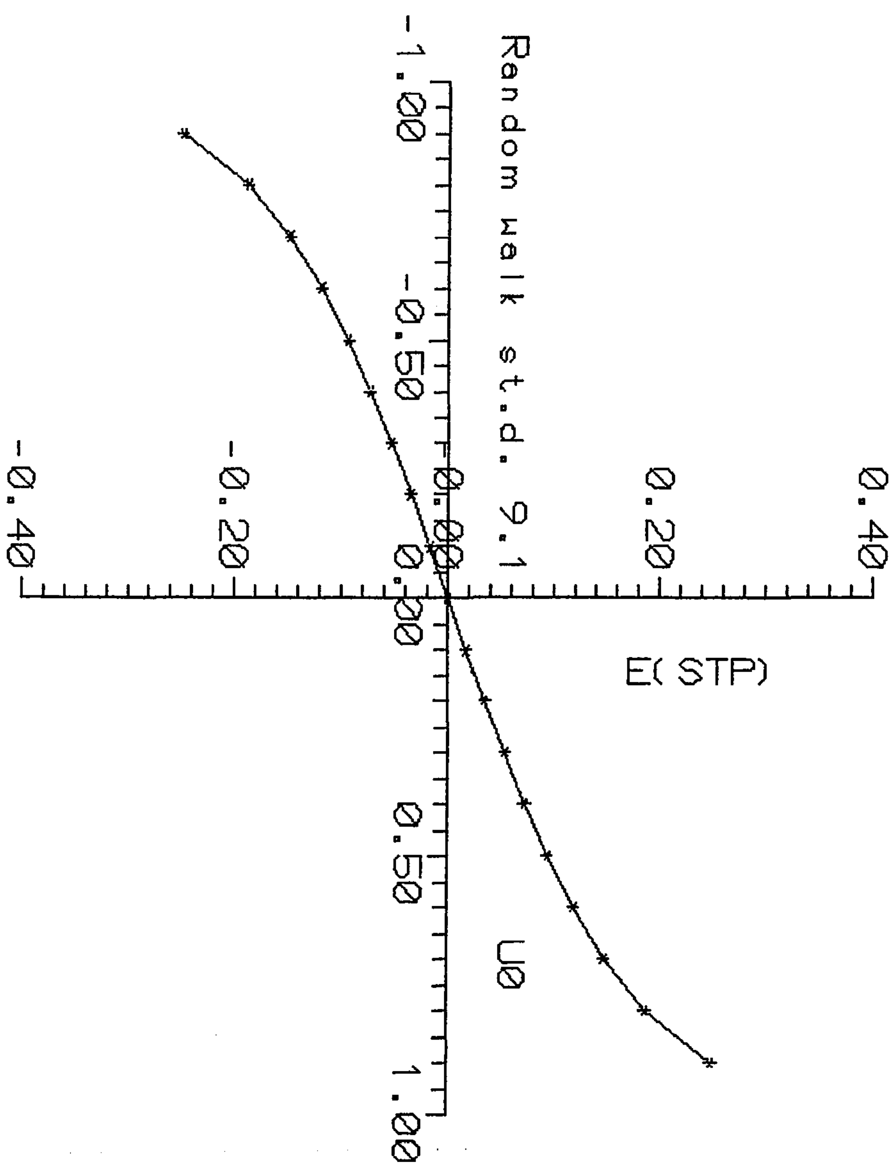
The expected value of the stock price for the standard deviation of the random walk equal to 0.1



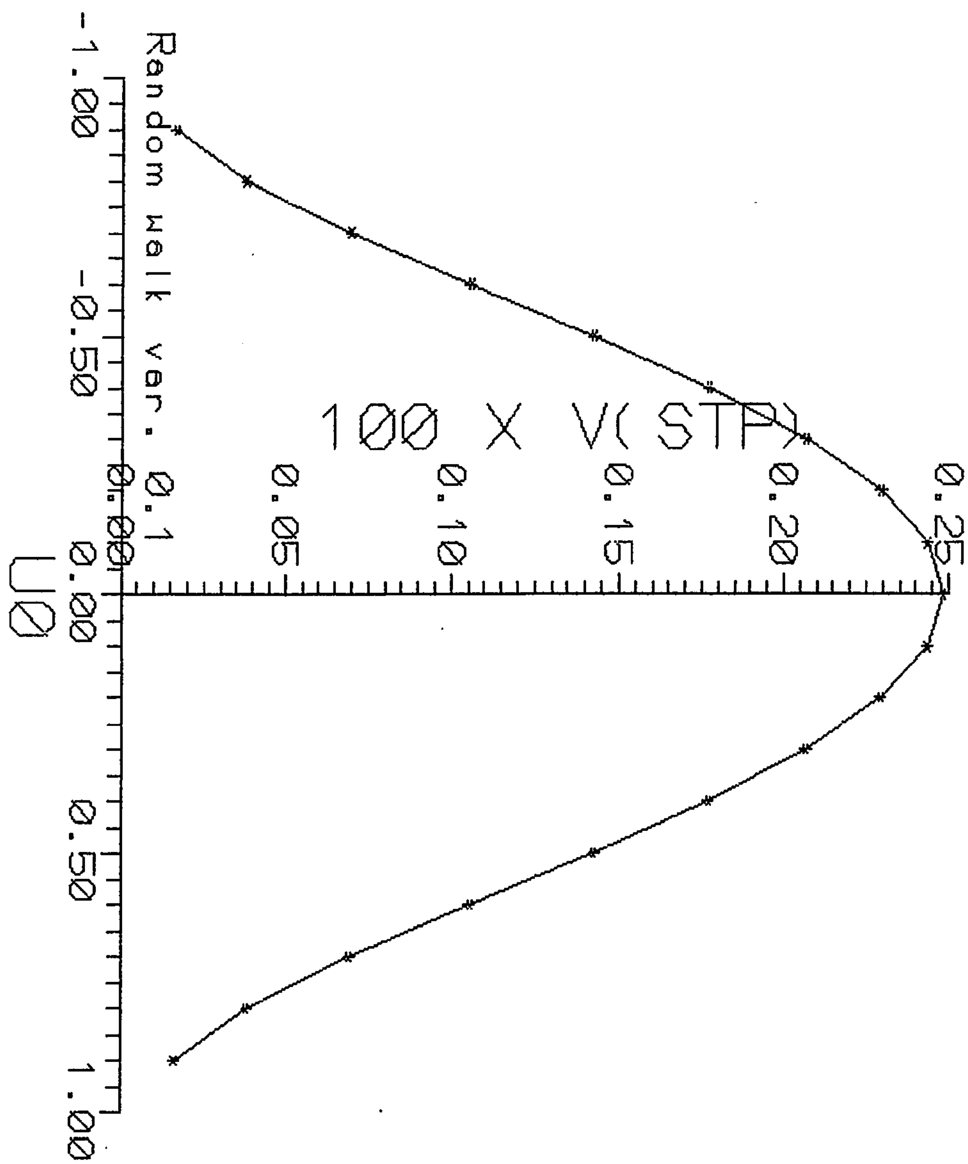
The expected value of the stock price for the standard deviation of the random walk equal to 5.1



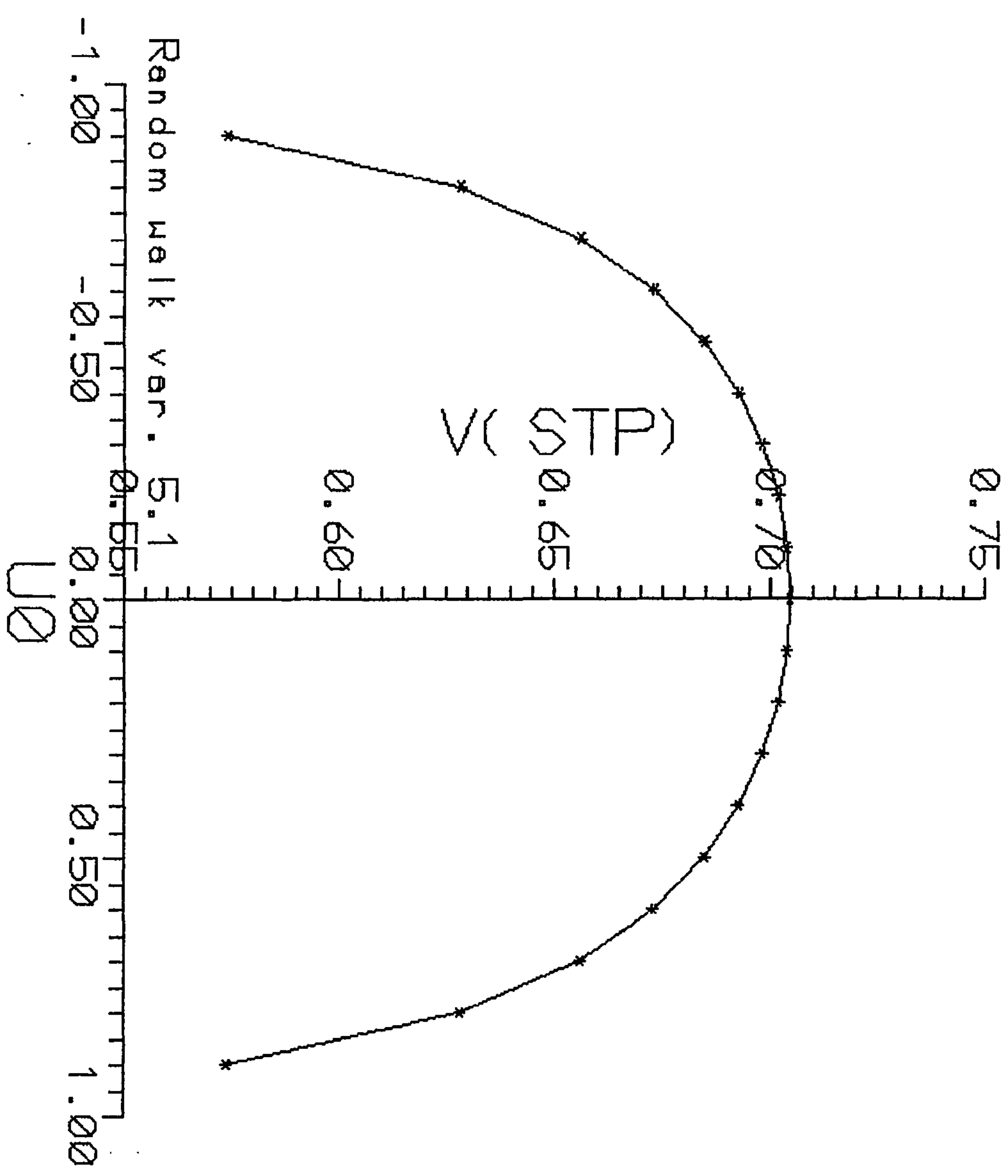
The expected value of the stock price for the standard deviation of the random walk equal to 9.1



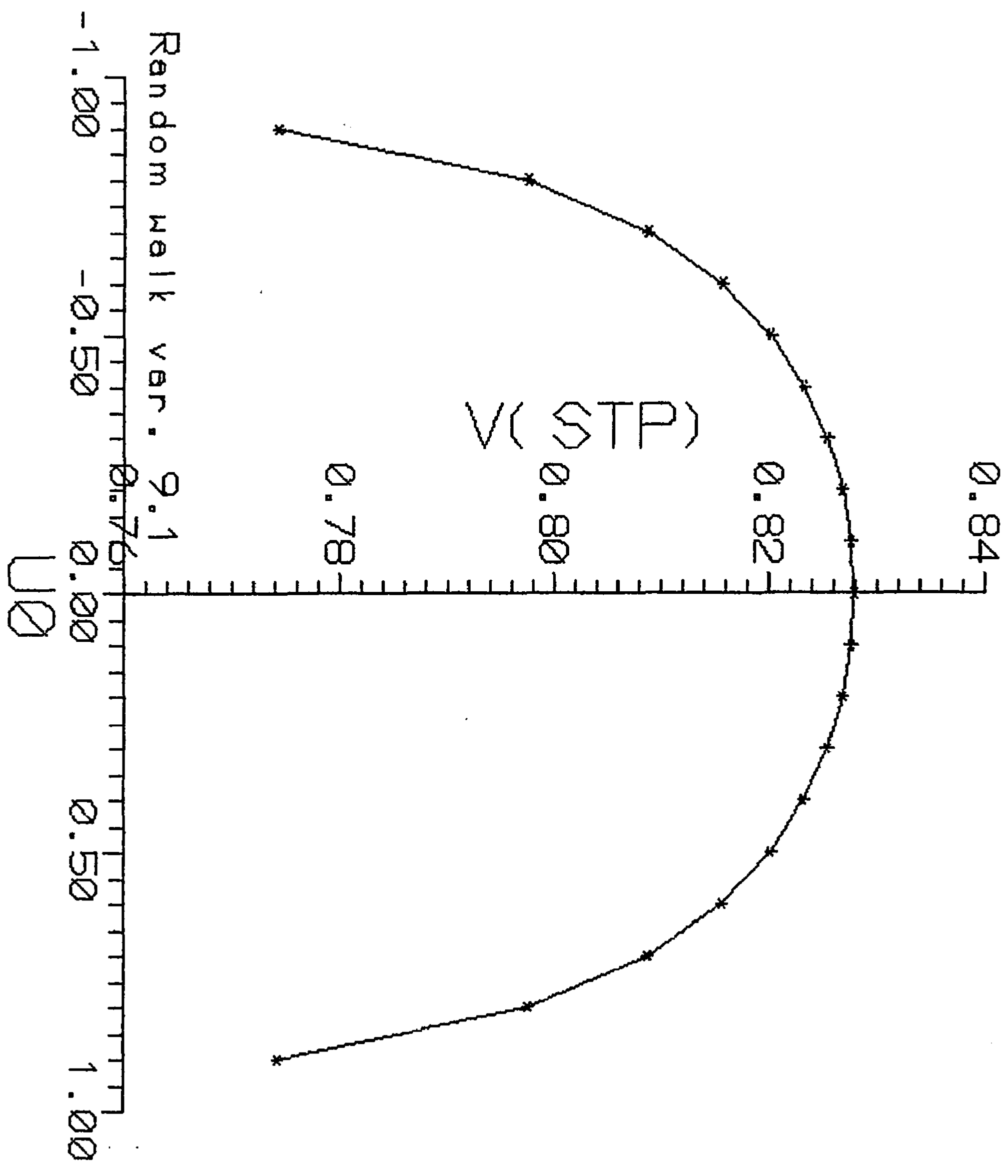
One hundred times the variance of the stock price for the standard deviation of the random walk equal to 0.1



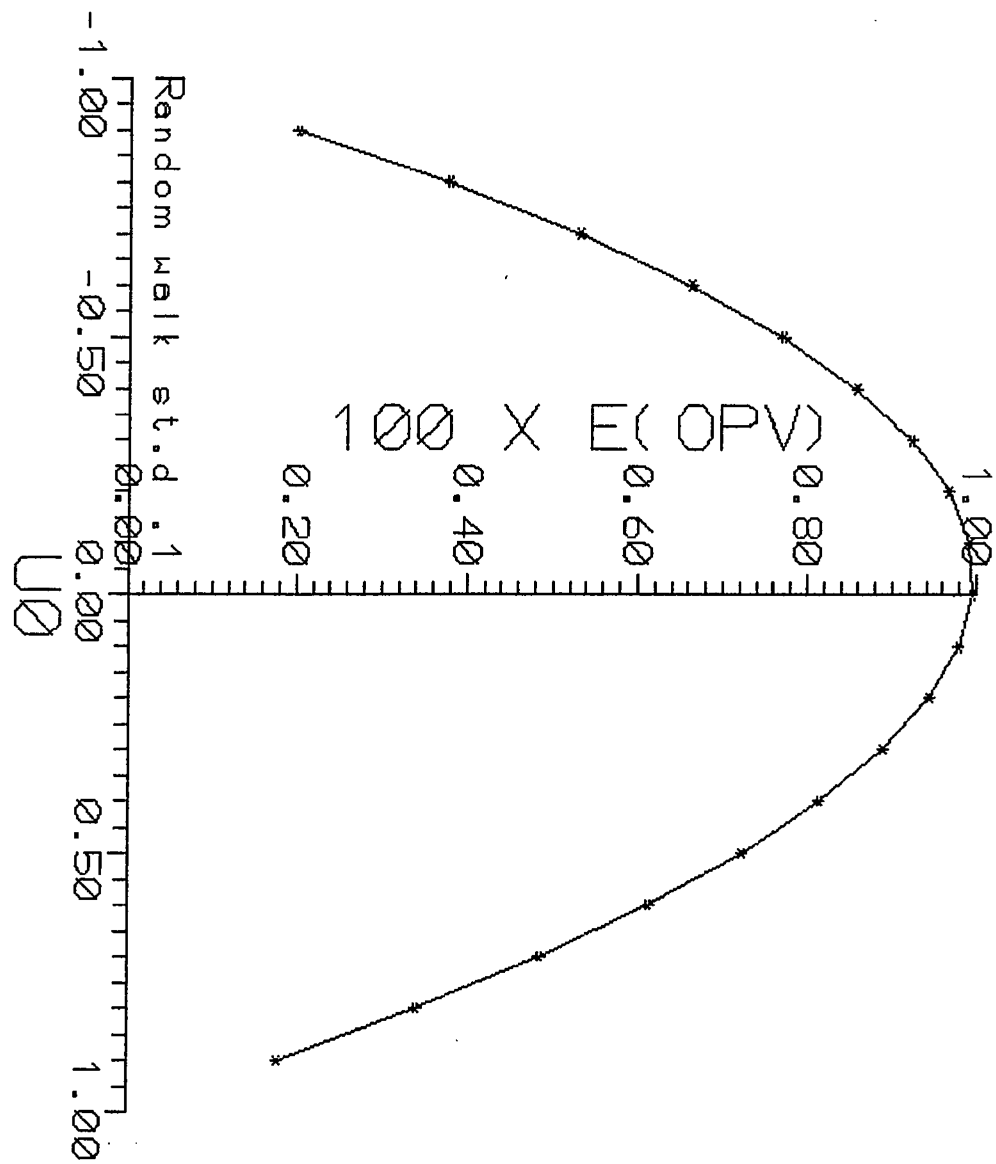
The variance of the stock price for the standard deviation of the random walk equal to 5.1



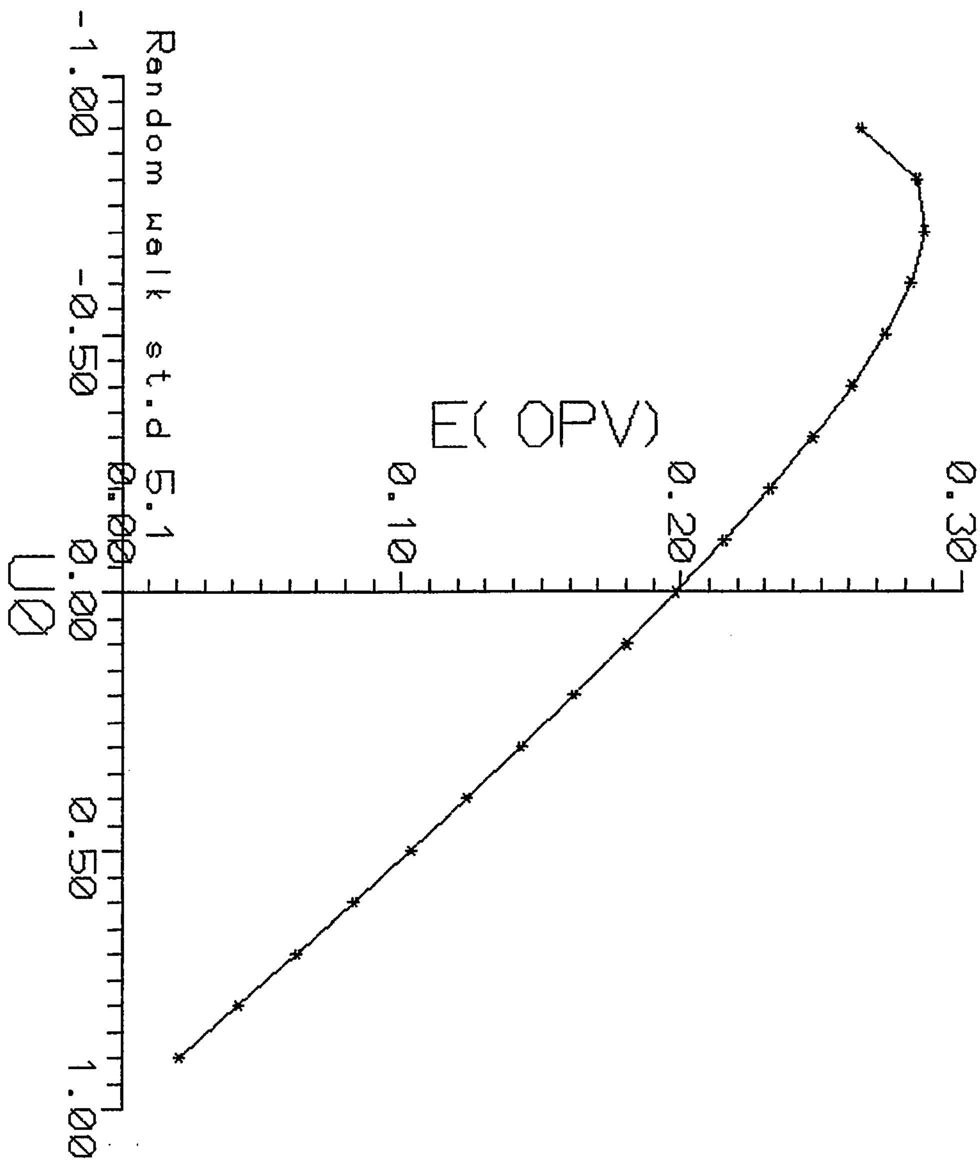
The variance of the stock price for the standard deviation of the random walk equal to 9.1



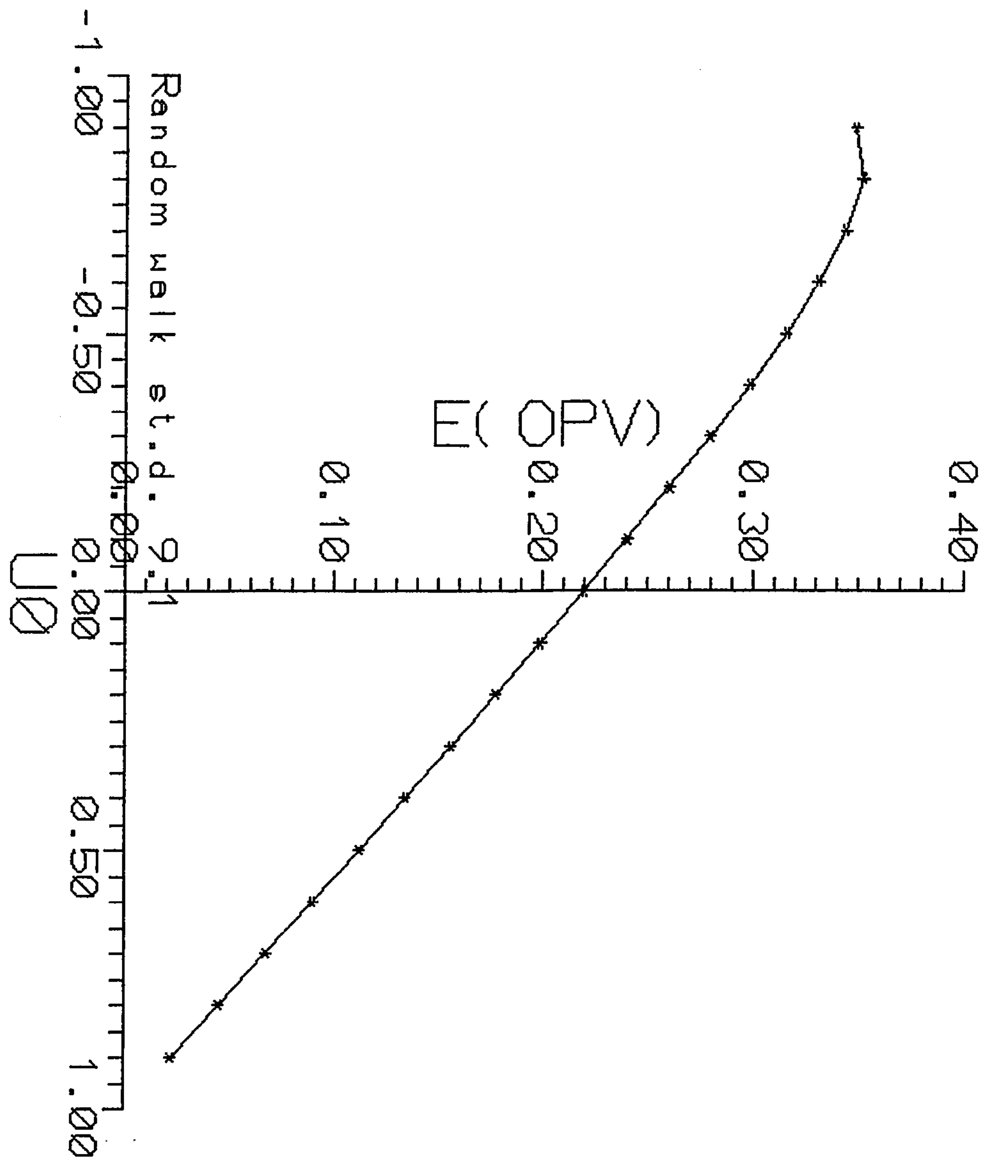
One hundred times the expected value of the option value for the standard deviation of the random walk equal to 0.1



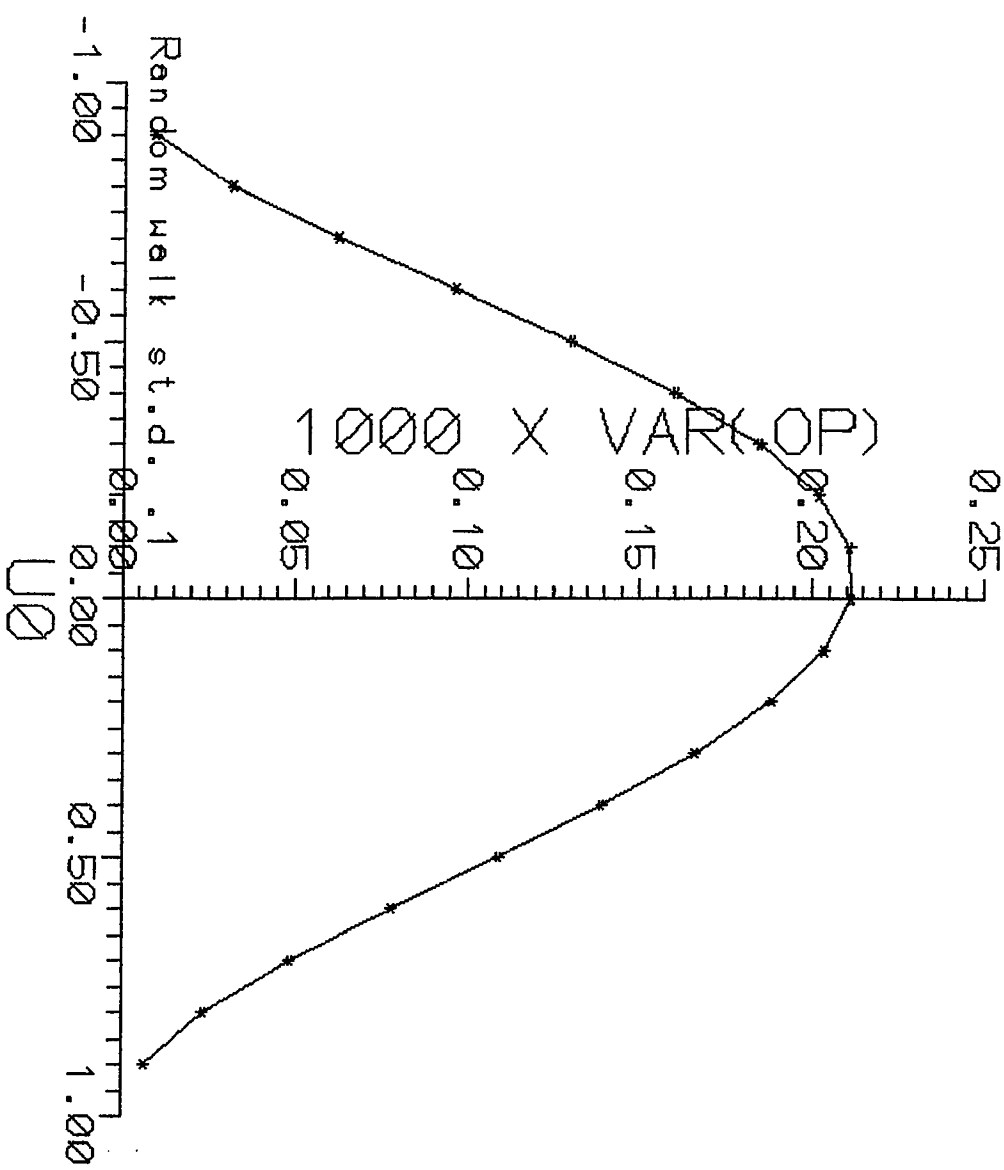
The expected value of the option value for the standard deviation of the random walk equal to 5.1



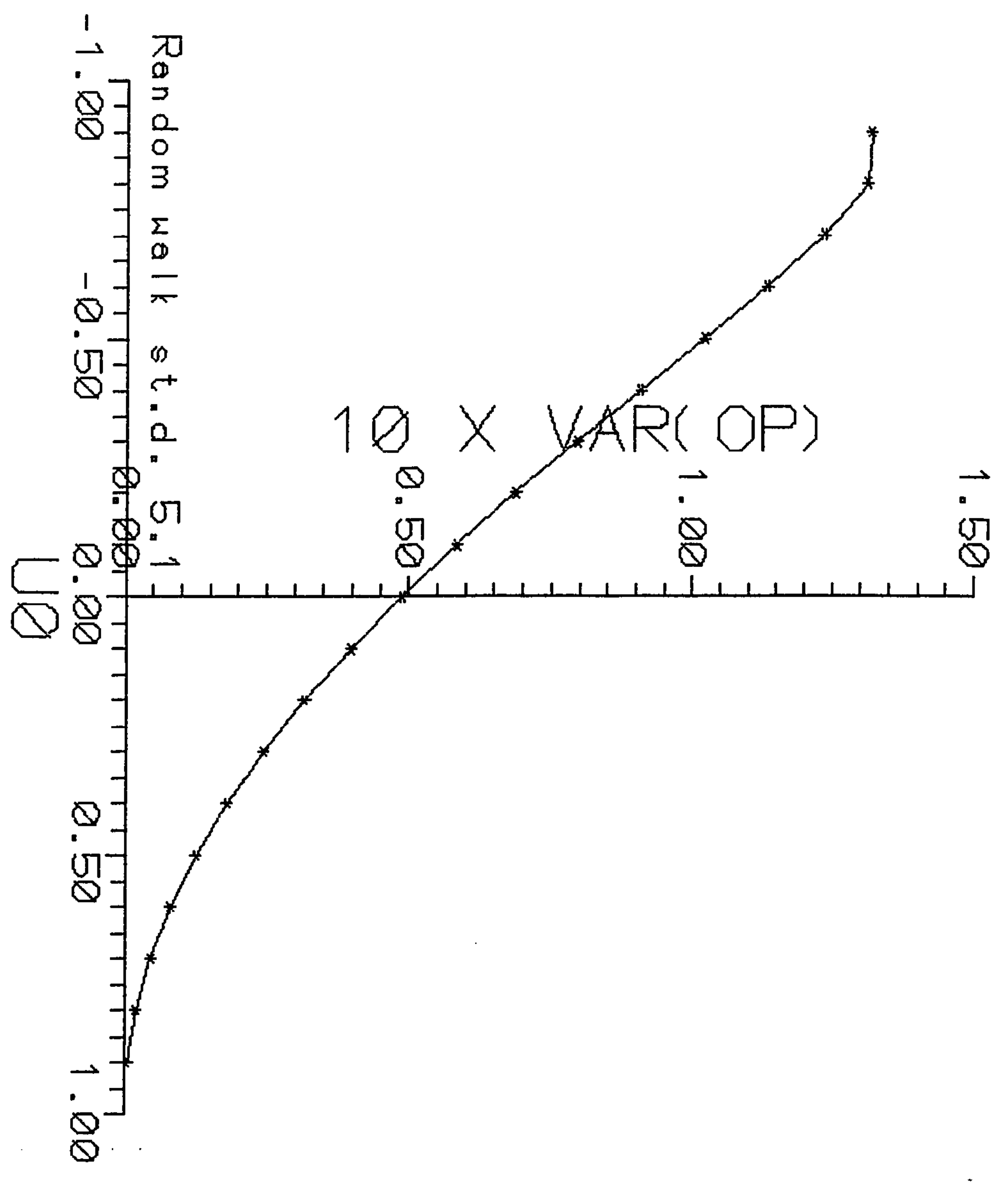
The expected value of the option value for the standard deviation of the random walk equal to 9.1



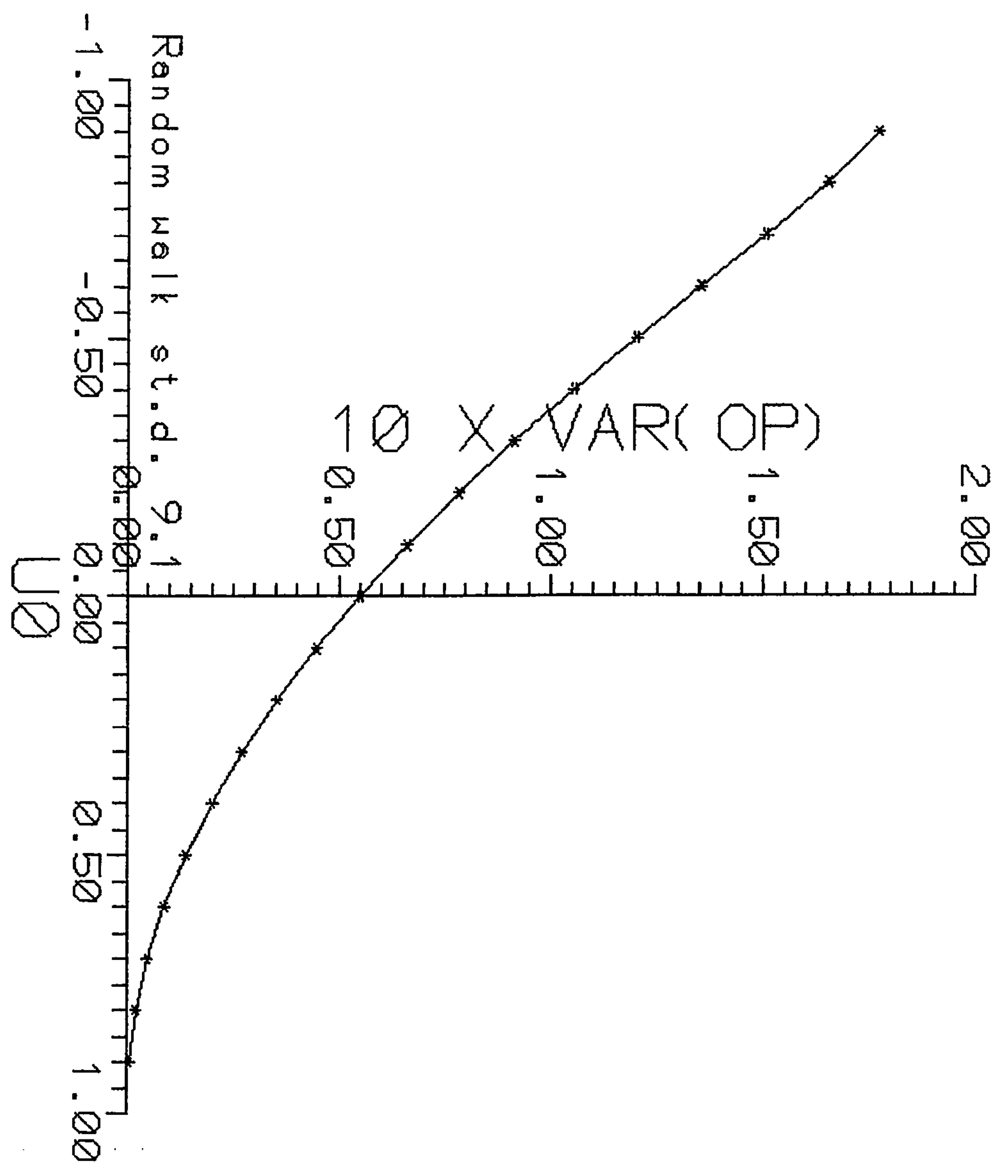
One thousand times the variance of the option value for the standard deviation of the random walk equal to 0.1



Ten times the variance of the option value for the standard deviation of the random walk equal to 5.1



Ten times the variance of the option value for the standard deviation of the random walk equal to 9.1



Chapter 4

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The collection of papers edited by P. Cootner, [4], is not recent but is still an important source. It contains a translation of the original thesis of L. Bachelier, [1], the paper of P. Cootner, [3], the paper of M. Osbourne, [10], and the paper of P. Samuelson, [11], all cited in this thesis.

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