

PRESCRIBED MEAN CURVATURE GRAPHS IN HYPERBOLIC
SPACE

by

TIMOTHY BOCCHI

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of
the requirements for the degree of Doctor of Philosophy,
The City University of New York

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Abstract

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Timothy Bocchi

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In this work we examine mean curvature in hyperbolic space. We seek the existence and uniqueness of smooth hypersurfaces with prescribed mean curvature H and given ideal boundary Γ . We prove that in the half-space model $\mathbb{H}^{n+1} = \mathbb{R}^n \times (0, \infty)$ of hyperbolic space, unique radial graphs over the semi-sphere $S^n \cap \mathbb{H}^{n+1}$ exist provided $H \in C^1(\overline{S_+^n})$, $|H| \leq 1$, $|H| < 1$ on $\partial\overline{S_+^n}$, and Γ is the radial graph in $\mathbb{R}^n \times \{0\}$ of a continuous function on $\partial\overline{S_+^n}$. Classical methods in the theory of elliptic partial differential equations are employed. If one varies the mean curvature or the ideal boundary in an appropriate way, the resulting surfaces are shown to foliate \mathbb{H}^{n+1} .

DEDICATION

For My Mother and Father

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CHAPTER 1.

INTRODUCTION

1.1 Background

Consider $\mathbb{R}^n \times (0, \infty)$ with metric $g_{ij}(x, t) = \frac{1}{t^2} \delta_{ij}$ for $(x, t) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$, the half-space model of $(n + 1)$ -dimensional hyperbolic space. The authors in [3] demonstrate the existence and uniqueness of radial graphs with constant mean curvature $-1 < H < 1$ and prescribed asymptotic boundary Γ , where Γ is the boundary of a $C^{1,1}$ star-shaped domain lying in \mathbb{R}^n . Their graph is a C^∞ surface having regularity of order $C^{1,1}$ at Γ . What is meant by a radial graph, is the image of the map $x \mapsto \exp(f(x))x$ where $x \in S_+^n = \{x : |x| = 1, x \in \mathbb{R}_+^{n+1}\}$ and f is a real valued function on S_+^n .

The authors in [1] considered the same problem, but for non-constant H . They used a more "natural" coordinate system for \mathbb{H}^{n+1} , so that radial graphs will have rectangular coordinates. In fact, define the diffeomorphism $\Psi : \mathbb{D}^n \times \mathbb{R} \rightarrow \mathbb{R}_+^{n+1}$ by

$$\Psi(x, t) = \frac{2 \exp(t)}{1 + |x|^2} \left(x, \frac{1 - |x|^2}{2} \right), \quad (1.1)$$

where $\mathbb{D}^n = \{x : |x| < 1, x \in \mathbb{R}^n\}$. It was then shown that the pull-back metric γ on the cylinder $\mathbb{D}^n \times \mathbb{R}$ is

$$[\gamma_{ij}(x, t)] = \left\{ \begin{array}{cc} \frac{4}{(1-|x|^2)^2} I_n & 0 \\ 0 & \frac{(1+|x|^2)^2}{(1-|x|^2)^2} \end{array} \right\}. \quad (1.2)$$

Note that the image under Ψ of $\mathbb{D}^n \times \{0\}$ is S_+^n , and the image of a vertical graph is radial. Moreover, vertical translations in the cylinder correspond to dilations in the half-space model. Thus, vertical translations are isometries in the cylinder model (which can also be seen by the absence of t in (1..2)). We also note that the inherited metric on $\mathbb{D}^n \times \{0\}$ is just the metric of the ball model, that is, \mathbb{D}^n with $\gamma_{ij}(x) = \delta_{ij} \frac{4}{(1-|x|^2)^2}$.

In this setting, the graph $\{(x, u(x)) : x \in \mathbb{D}^n\}$ of a C^2 function u has mean curvature $H \iff Q(u) = nH$. Here,

$$\begin{aligned} Qu(x) &= a^{ij}(x, Du)u_{ij}(x) + b(x, Du) \\ &= \beta \left(\delta_{ij} - \frac{u_i u_j}{\alpha^2} \right) u_{ij} + Du \cdot (\beta k + l), \end{aligned} \quad (1.3)$$

where

$$\alpha = \left(\frac{4}{(1+|x|^2)^2} + |Du|^2 \right)^{1/2}, \quad \beta = \frac{1-|x|^2}{2\alpha}, \quad (1.4)$$

$$l = -\frac{x}{\alpha} + \frac{4x(1-|x|^2)}{\alpha^3(1+|x|^2)^3}, \quad \text{and} \quad k = \frac{4x+2nx(1+|x|^2)}{(1-|x|^2)(1+|x|^2)}. \quad (1.5)$$

The operator Q is quasilinear elliptic. The eigenvalues of $a^{ij}(x, Du)$ are $\lambda = \frac{2(1-|x|^2)}{(1+|x|^2)^2 \alpha^3}$ with multiplicity 1 and

$$\Lambda = \beta = \frac{1-|x|^2}{2\alpha}. \quad (1.6)$$

with multiplicity $n-1$, $\lambda \leq \Lambda$.

The authors of [1] then prove:

Theorem 1 *Let Ω be a bounded domain (with respect to the hyperbolic metric) and let $H \in C^1(\overline{\Omega})$, $-1 \leq H \leq 1$. Then, if $|H(y)| < H'(y)$ everywhere on $\partial\Omega$, where*

H' denotes the hyperbolic mean curvature of the cylinder over $\partial\Omega$, for every $\varphi \in C^{2,\alpha}(\overline{\Omega})$ there is a unique solution to the Dirichlet problem $Qu = nH$ in Ω , $u = \varphi$ on $\partial\Omega$.

One of our main results in this work is to extend Theorem 1 to include the case where $\Omega = \mathbb{D}^n$. This also serves as an extension of the aforementioned result contained in [3]. Moreover, if the boundary data or the prescribed mean curvature function is replaced by an appropriately defined one parameter family, the resulting surfaces foliate hyperbolic space.

1.2 Notes on PDE

We would like to discuss several ideas from the general theory of second order quasilinear elliptic partial differential equations.

An important feature of Q in equation (1.3), is that it obeys the comparison principle:

$$Qu \leq Qv \text{ on } \Omega \text{ and } v \leq u \text{ on } \partial\Omega \implies v \leq u \text{ on } \Omega. \quad (1.7)$$

It is important for our use of the comparison principle to note that if either inequality is strict in the "if" part of (1.7), then so is the resulting inequality on all of Ω . See Theorem 10.1 in [2] for a proof of (1.7). Also of importance are the following interior estimates. Roughly speaking, a solution u to a quasilinear elliptic PDE (or family of PDE) of order 2, with an interior gradient estimate (uniform in the case of a family), has an interior (apriori) Holder estimate for its gradient.

$$\sup_{\substack{x,y \in K \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x - y|^\alpha} \leq C, \quad (1.8)$$

where K is a compact subset of the domain of u , and C depends on (among other things) $\sup_{x \in K} |Du(x)|$. This is essentially Theorem 13.6 in [2]. The Schauder interior estimates give a $C^{2,\alpha}$ (apriori) interior estimate for a solution u (or family of solutions) having an interior $C^{1,\alpha}$ estimate (uniform in the case of a family).

$$\begin{aligned} & \sup_{x \in K} (|u(x)| + |Du(x)| + |D^2u(x)|) + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha} \\ & \leq C, \end{aligned} \tag{1.9}$$

where C depends on (among other things)

$$\sup_{x \in K} (|u(x)| + |Du(x)|) + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x - y|^\alpha}.$$

This is essentially Theorem 6.2 in [2].

1.3 Geometric Preliminaries

In a general Riemannian manifold M , the divergence of a smooth vector field X at a point $p \in M$ can be defined as the trace of the linear map $Y(p) \rightarrow \nabla_Y X(p)$. ∇ will always refer to the Levi-Civita connection on M . In local coordinates,

$$\operatorname{div} X = \frac{1}{\sqrt{g}} \sum \partial_i (\sqrt{g} X^i),$$

where $X = \sum X^i \partial_i$, $g_{ij} = \langle \partial_i, \partial_j \rangle$, and $g = \det [g_{ij}]$. For a smooth function f on M , we define $\operatorname{grad} f$ to be the vector field on M satisfying $\langle \operatorname{grad} f(p), \nu \rangle = Df(p)(\nu)$ for each $p \in M$ and every $\nu \in T_p(M)$. The laplacian of f is given by $\Delta f = \operatorname{div}(\operatorname{grad} f)$.

In local coordinates,

$$\nabla f = \operatorname{grad} f = \sum g^{ij} (\partial_i f) \partial_j \quad \text{and} \quad \Delta f = \frac{1}{\sqrt{g}} \sum \partial_i (\sqrt{g} g^{ij} (\partial_j f)).$$

We note that $\|X\|$, $\operatorname{div}X$, $\operatorname{grad}f = \nabla f$, and Δf will always refer to the hyperbolic metric. We will write $|x|$, Df , and $\Delta_E f$ for their Euclidean counterparts.

Suppose M has dimension $n + 1$. We will view an embedded n -dimensional submanifold of M as a smooth hypersurface Σ , and it will always be given the Riemannian structure inherited from M . If η is a smooth unit vector field defined in a neighborhood of $p \in \Sigma$ and normal to Σ , then $\operatorname{div}\eta(p) = -nH(p)$, where H is the mean curvature of Σ with respect to the local normal field η .

In \mathbb{H}^{n+1} , our first important surface is a geodesic sphere of (hyperbolic) radius d . One can show that

$$H = -\coth d. \tag{1.10}$$

Here, the mean curvature is taken with respect to the outer unit normal. The next surface of interest is a "spherical cap". In fact, in the upper half space model of \mathbb{H}^{n+1} , suppose $p = (x, t) \in \mathbb{R}^n \times \mathbb{R}$ and let

$$\mathcal{S}(p, R) = \partial B(p; R) \cap \mathbb{H}^{n+1}, \tag{1.11}$$

where $B(p; R)$ is the Euclidean ball of radius R centered at p . Then its mean curvature with respect to the outer unit normal is the constant

$$H = -\frac{t}{R}.$$

Finally, in the coordinates of the upper half space model, the set $\{t = 1\}$, $(x, t) \in \mathbb{R}^n \times (0, \infty)$, is a horosphere whose mean curvature with respect to the upper unit normal is 1. It is not hard to show that its preimage under Ψ (see (1.1)) is the graph of the equation $u(x) = \ln(1 + |x|^2) - \ln(1 - |x|^2)$.

CHAPTER 2.

INTERIOR GRADIENT ESTIMATE

2.1 An identity for Δw

The main result of this chapter is Theorem 7, in which we derive an interior gradient estimate for a solution of the prescribed mean curvature equation. Our derivation follows the same line of reasoning as in Chapter 16 of [2], where the authors deal only with the Euclidean case.

The present section consists of a supporting Lemma which is interesting in its own right. Though we state the result for hypersurfaces in \mathbb{H}^{n+1} , the proof works equally well in a general Riemannian manifold.

Let us begin by considering a smooth hypersurface Σ in \mathbb{H}^{n+1} with a continuous unit normal field η . We choose local coordinates $u : U \rightarrow \Sigma$ and define the coordinates (for a sufficiently small ϵ) $\hat{u} : U \times (-\epsilon, \epsilon) \rightarrow \mathbb{H}^{n+1}$ by $\hat{u}(x, t) = \gamma_p(t)$, where $u(x) = p$ and γ_p is the unique unit speed geodesic with $\gamma_p(0) = p$ and $\gamma_p'(0) = \eta(p)$. With these coordinates, the metric $[g_{ij}]$, where $g_{ij} = \langle D\hat{u}(e_i), D\hat{u}(e_j) \rangle$, takes the block form $\begin{bmatrix} a_{ij} & 0 \\ 0 & 1 \end{bmatrix}$, where the vector fields $e_1, e_2, \dots, e_n, e_{n+1}$ form the standard frame for $U \times (-\epsilon, \epsilon)$. Hence, a given vector field $Y = \sum_{i=1}^{n+1} y^i e_i$ has divergence

$$\text{Div}(Y) = \frac{1}{\sqrt{g}} \sum_{k=1}^{n+1} \partial_k(\sqrt{g}y^k) = \frac{1}{\sqrt{a}} \sum_{k=1}^n \partial_k(\sqrt{a}y^k) + \frac{\partial_{n+1}(\sqrt{a})}{\sqrt{a}}y^{n+1} + \partial_{n+1}y^{n+1},$$

where g and a are the determinants $\det [g_{ij}]$ and $\det [a_{ij}]$, respectively. Note that for each $t \in (-\epsilon, \epsilon)$,

$$D\hat{u}_{(x,t)}(e_{n+1}(x, t)) = \gamma_p'(t) := \nu(p, t), \tag{2..1}$$

is a unit vector normal to the surface $\Sigma_t := \widehat{u}(U \times \{t\})$ at $\widehat{u}(x, t) = \gamma_p(t)$. Thus for a smooth vector field $Y \in \Gamma^1(\mathbb{H}^{n+1})$, we have at points on Σ ,

$$Div(Y) = Div_\Sigma(Y^T) - nH \langle Y, \nu \rangle + \nabla_\nu \langle Y, \nu \rangle, \quad (2..2)$$

where H is the mean curvature of Σ , Div_Σ is the divergence operator with respect to Σ , and Y^T is the component of Y tangent to Σ .

For the following lemma, we suppose we have a function $\Phi \in C^3(\mathbb{H}^{n+1})$ with $\text{grad } \Phi \neq \mathbf{0}$ and we let Σ denote an arbitrary level surface of Φ . We introduce the function $w = \ln(\|\nabla\Phi\|)$ and the vector field $\eta := \frac{\nabla\Phi}{\|\nabla\Phi\|}$, which is normal to Σ .

Lemma 2 *We have*

$$\Delta_\Sigma w = C^2 + \|\nabla_\Sigma w\|^2 - n\nabla_\eta H, \quad (2..3)$$

where C^2 is the sum of the squares of the principal curvatures of Σ and $H(p)$ is the mean curvature (with respect to η) at p of the level surface passing through $p \in \mathbb{H}^{n+1}$.

Proof. Let Σ_0 be a fixed level surface of Φ . Recall the vector field ν defined in (2..1) which we take to be normal to Σ_0 . Calculating $\Delta_{\Sigma_0} w$ with the help of (2..2) gives $\Delta_{\Sigma_0} w = Div_{\Sigma_0} \nabla_{\Sigma_0} w = [Div(\nabla_\Sigma w) + nH \langle \nabla_{\Sigma_0} w, \nu \rangle - \partial_\nu \langle \nabla_\Sigma w, \nu \rangle] |_{\Sigma_0}$. Here, and in what follows, the symbol $|_{\Sigma_0}$ denotes the restriction of a function to Σ_0 . Since $\langle \nabla_{\Sigma_0} w, \nu \rangle = 0$, we have

$$\Delta_{\Sigma_0} w = Div(\nabla_\Sigma w) |_{\Sigma_0} - \partial_\nu \langle \nabla_\Sigma w, \nu \rangle |_{\Sigma_0}. \quad (2..4)$$

In order to see that the right hand sides of (2..3) and (2..4) are equivalent, we must first show that

$$\nabla_\eta \eta = \nabla_\Sigma w. \quad (2..5)$$

Now,

$$\begin{aligned}
\nabla_\eta \eta &= \nabla_{\frac{\nabla \Phi}{\|\nabla \Phi\|}} \frac{\nabla \Phi}{\|\nabla \Phi\|} = \left(\nabla_{\frac{\nabla \Phi}{\|\nabla \Phi\|}} (1/\|\nabla \Phi\|) \right) \nabla \Phi + (1/\|\nabla \Phi\|) \nabla_{\frac{\nabla \Phi}{\|\nabla \Phi\|}} \nabla \Phi \\
&= -\frac{1}{\|\nabla \Phi\|^3} (\nabla_{\nabla \Phi} \|\nabla \Phi\|) \nabla \Phi + \frac{1}{\|\nabla \Phi\|^2} \nabla_{\nabla \Phi} \nabla \Phi \\
&= -\frac{1}{\|\nabla \Phi\|^3} \langle \nabla \|\nabla \Phi\|, \nabla \Phi \rangle \nabla \Phi + \frac{1}{\|\nabla \Phi\|^2} \nabla_{\nabla \Phi} \nabla \Phi. \tag{2.6}
\end{aligned}$$

We now express $\nabla_{\nabla \Phi} \nabla \Phi$ in terms of a local geodesic frame u_1, u_2, \dots, u_{n+1} centered at a point $p_0 \in \mathbb{H}^{n+1}$ (that is, the u_i form a local orthonormal frame and $\nabla_{u_i} u_j(p_0) = \mathbf{0}$ for $i, j \in \{1, 2, \dots, n+1\}$). Now, at points near p_0 , we have $\nabla \Phi = \Phi_i u_i$ so that

$$\nabla_{\nabla \Phi} \nabla \Phi = \nabla_{\Phi_i u_i} \Phi_j u_j = \Phi_i \Phi_{ji} u_j + \Phi_i \Phi_j \nabla_{u_i} u_j, \tag{2.7}$$

where $\Phi_i = \nabla_{u_i} \Phi$, $\Phi_{ji} = \nabla_{u_i} (\nabla_{u_j} \Phi)$, and summation is understood. Since $\nabla_{u_i} u_j(p_0) = \mathbf{0}$ and $\Phi_{ij}(p_0) = \Phi_{ji}(p_0)$, we have by substituting (2.7) into (2.6),

$$\nabla_\eta \eta = -\frac{1}{\|\nabla \Phi\|^3} \langle \nabla \|\nabla \Phi\|, \nabla \Phi \rangle \nabla \Phi + \frac{\Phi_i \Phi_{ij}}{\|\nabla \Phi\|^2} u_j \tag{2.8}$$

at p_0 . On the other hand,

$$\begin{aligned}
\nabla_\Sigma w &= \nabla w - \langle \nabla w, \eta \rangle \eta = \frac{\nabla \|\nabla \Phi\|}{\|\nabla \Phi\|} - \left\langle \frac{\nabla \|\nabla \Phi\|}{\|\nabla \Phi\|}, \frac{\nabla \Phi}{\|\nabla \Phi\|} \right\rangle \frac{\nabla \Phi}{\|\nabla \Phi\|} \\
&= \frac{\nabla \|\nabla \Phi\|^2}{2\|\nabla \Phi\|^2} - \frac{1}{\|\nabla \Phi\|^3} \langle \nabla \|\nabla \Phi\|, \nabla \Phi \rangle \nabla \Phi. \tag{2.9}
\end{aligned}$$

Since $\nabla \|\nabla \Phi\|^2 = (\Phi_i^2)_j u_j = 2\Phi_i \Phi_{ij} u_j$, when we substitute this into (2.9) we conclude, by comparing it to (2.8), the identity (2.5).

Next, let us evaluate the second term in (2.4). By (2.5), $\nabla_\nu \langle \nabla_\Sigma w, \nu \rangle = \nabla_\nu \langle \nabla_\eta \eta, \nu \rangle = \langle \nabla_\nu (\nabla_\eta \eta), \nu \rangle + \langle \nabla_\eta \eta, \nabla_\nu \nu \rangle = \langle \nabla_\nu (\nabla_\eta \eta), \nu \rangle$, where the last equality follows from the observation that $\nabla_\nu \nu = \mathbf{0}$. Now

$$\langle \nabla_\nu (\nabla_\eta \eta), \nu \rangle |_{\Sigma_0} = \langle \nabla_\nu (\nabla_\eta \eta), \eta \rangle |_{\Sigma_0} = \langle \nabla_\eta (\nabla_\eta \eta), \eta \rangle |_{\Sigma_0}$$

and

$$\langle \nabla_\eta(\nabla_\eta \eta), \eta \rangle = -\langle \nabla_\eta \eta, \nabla_\eta \eta \rangle = -\|\nabla_\eta \eta\|^2 = -\|\nabla_\Sigma w\|^2,$$

hence

$$-\partial_\nu \langle \nabla_\Sigma w, \nu \rangle |_{\Sigma_0} = \|\nabla_{\Sigma_0} w\|^2. \quad (2..10)$$

We now look at the first term in (2..4). Again, by (2..5), we have

$$Div(\nabla_\Sigma w) = Div(\nabla_\eta \eta) = \nabla_{u_i} \langle \nabla_\eta \eta, u_i \rangle, \quad (2..11)$$

where the u_i are as above (but $p_0 \in \Sigma_0$) and summation is understood. Now

$$\nabla_{u_i} \langle \nabla_\eta \eta, u_i \rangle = \nabla_{u_i} \langle \nabla_\eta(\eta^j u_j), u_i \rangle = \nabla_{u_i} \langle (\nabla_\eta \eta^j) u_j + \eta^j \nabla_\eta u_j, u_i \rangle \quad (2..12)$$

$$= \nabla_{u_i} (\nabla_\eta \eta^i) + \nabla_{u_i} (\eta^j \langle \nabla_\eta u_j, u_i \rangle). \quad (2..13)$$

By expanding the first term in (2..13), $\nabla_{u_i} (\nabla_\eta \eta^i) = \nabla_{u_i} (\eta^j \eta_j^i) = \eta_{j,i}^i \eta^j + \eta_i^j \eta_j^i$, one sees from equations (2..11), (2..12), and (2..13) that

$$Div(\nabla_\Sigma w) = \eta_{j,i}^i \eta^j + \eta_i^j \eta_j^i + \nabla_{u_i} (\eta^j \langle \nabla_\eta u_j, u_i \rangle) \quad (2..14)$$

We know that at any point $p \in \mathbb{H}^{n+1}$, $Div(\eta) = -nH$, where H is the mean curvature (at p) of the level surface of Φ passing through p . So

$$\begin{aligned} -n\nabla_\eta H &= \nabla_\eta Div(\eta) = \nabla_\eta \langle \nabla_{u_i} \eta, u_i \rangle \\ &= \nabla_\eta \langle \nabla_{u_i}(\eta^j u_j), u_i \rangle = \nabla_\eta \langle \eta_i^j u_j + \eta^j \nabla_{u_i} u_j, u_i \rangle \\ &= \nabla_\eta \eta_i^i + \nabla_\eta \langle \eta^j \nabla_{u_i} u_j, u_i \rangle = \eta_{i,j}^i \eta^j + \nabla_\eta \langle \eta^j \nabla_{u_i} u_j, u_i \rangle. \end{aligned}$$

Thus we conclude

$$-n\nabla_\eta H = \eta_{ij}^i \eta^j + (\nabla_\eta \eta^j) \langle \nabla_{u_i} u_j, u_i \rangle + \eta^j \nabla_\eta \langle \nabla_{u_i} u_j, u_i \rangle. \quad (2..15)$$

Recalling that $\eta_{ji}^i = \eta_{ij}^i$ and $\nabla_{u_i} u_j = 0$ at the point p_0 and substituting equation (2..15) into (2..14), we have

$$Div(\nabla_\Sigma w) = -n\nabla_\eta H + \eta_i^j \eta_j^i - \eta^j \nabla_\eta \langle \nabla_{u_i} u_j, u_i \rangle + \nabla_{u_i} (\eta^j \langle \nabla_\eta u_j, u_i \rangle) \text{ (at } p_0). \quad (2..16)$$

Claim: $\eta_i^j \eta_j^i = \mathbf{C}^2$ and $-\eta^j \nabla_\eta \langle \nabla_{u_i} u_j, u_i \rangle|_{\Sigma_0} + \nabla_{u_i} (\eta^j \langle \nabla_\eta u_j, u_i \rangle)|_{\Sigma_0} = 0$ at p_0 . The claim, along with equations (2..16), (2..10) and (2..4) completes the proof of the lemma. Now, we know the principal curvatures of Σ at p_0 are the eigenvalues of the symmetric linear operator $S : T_{p_0}(\Sigma) \rightarrow T_{p_0}(\Sigma)$ given by $v \rightarrow -\nabla_v \eta$. Therefore, $\mathbf{C}^2 = Tr(S^2)$. On the other hand, the mapping $v \rightarrow -\nabla_v \eta$ is defined on $T_{p_0}(\mathbb{H}^{n+1})$ and naturally extends S to a linear operator \widehat{S} on $T_{p_0}(\mathbb{H}^{n+1})$. We have

$$\begin{aligned} Tr(S^2) &= Tr(\widehat{S}^2) = \langle \nabla_{u_i} \eta, u_j \rangle \langle \nabla_{u_j} \eta, u_i \rangle|_{p_0} \\ &= \langle \nabla_{u_i} (\eta^l u_l), u_j \rangle \langle \nabla_{u_j} (\eta^m u_m), u_i \rangle|_{p_0} = \langle \eta_i^l u_l + \eta^l \nabla_{u_i} u_l, u_j \rangle \langle \eta_j^m u_m + \eta^m \nabla_{u_j} u_m, u_i \rangle|_{p_0} \\ &= \langle \eta_i^l u_l, u_j \rangle \langle \eta_j^m u_m, u_i \rangle|_{p_0} = \eta_i^j \eta_j^i|_{p_0}. \end{aligned}$$

This proves the first part of the claim. We now show

$$-\eta^j \nabla_\eta \langle \nabla_{u_i} u_j, u_i \rangle|_{\Sigma_0} + \nabla_{u_i} (\eta^j \langle \nabla_\eta u_j, u_i \rangle)|_{\Sigma_0} = 0. \quad (2..17)$$

Indeed,

$$\begin{aligned}
\nabla_{u_i}(\eta^j \langle \nabla_\eta u_j, u_i \rangle) &= \nabla_{u_i} \eta^j \langle \nabla_\eta u_j, u_i \rangle + \eta^j \nabla_{u_i} \langle \nabla_\eta u_j, u_i \rangle \\
&= \nabla_{u_i} \eta^j \langle \nabla_\eta u_j, u_i \rangle + \eta^j \nabla_{u_i} \langle \eta^l \nabla_{u_l} u_j, u_i \rangle \\
&= \nabla_{u_i} \eta^j \langle \nabla_\eta u_j, u_i \rangle + \eta^j \eta_i^l \langle \nabla_{u_l} u_j, u_i \rangle + \eta^j \eta^l \nabla_{u_i} \langle \nabla_{u_l} u_j, u_i \rangle \\
&= \nabla_{u_i} \eta^j \langle \nabla_\eta u_j, u_i \rangle + \eta^j \eta_i^l \langle \nabla_{u_l} u_j, u_i \rangle + \eta^j \eta^l \langle \nabla_{u_i} (\nabla_{u_l} u_j), u_i \rangle + \eta^j \eta^l \langle \nabla_{u_l} u_j, \nabla_{u_i} u_i \rangle.
\end{aligned}$$

Evaluating at p_0 , we have

$$\nabla_{u_i}(\eta^j \langle \nabla_\eta u_j, u_i \rangle) = \eta^j \eta_i^l \langle \nabla_{u_l} (\nabla_{u_i} u_j), u_i \rangle \quad (\text{at } p_0). \quad (2..18)$$

On the other hand,

$$\eta^j \nabla_\eta \langle \nabla_{u_i} u_j, u_i \rangle = \eta^j \langle \nabla_\eta (\nabla_{u_i} u_j), u_i \rangle + \eta^j \langle \nabla_{u_i} u_j, \nabla_\eta u_i \rangle = \eta^j \eta_i^l \langle \nabla_{u_l} (\nabla_{u_i} u_j), u_i \rangle + \eta^j \langle \nabla_{u_i} u_j, \nabla_\eta u_i \rangle.$$

Again, we evaluate at p_0 and conclude

$$\eta^j \nabla_\eta \langle \nabla_{u_i} u_j, u_i \rangle = \eta^j \eta_i^l \langle \nabla_{u_l} (\nabla_{u_i} u_j), u_i \rangle \quad (\text{at } p_0). \quad (2..19)$$

Equations (2..18) and (2..19) imply (2..17), verifying the claim. ■

2.2 Distance and Preliminary Interior Estimate

Our goal in this section is Corollary 5, in which we obtain a pointwise estimate for w in terms of the local geometry of Σ and an L^1 estimate.

In what follows, let Σ, H , and ν be as in the discussion preceding equation (2..2), $p_0 \in \Sigma$, and $d : \mathbb{H}^{n+1} \rightarrow [0, \infty)$ be defined by $d = d(p) = \text{dist}_{\mathbb{H}^{n+1}}(p, p_0)$, where $\text{dist}_{\mathbb{H}^{n+1}}(p, p_0)$ is the hyperbolic distance from p to p_0 . Let us also write $B(p; d) = \{q \mid \text{dist}_{\mathbb{H}^{n+1}}(q, p) < d, q \in \mathbb{H}^{n+1}\}$ and $S(p; d) = \partial B(p; d)$. We see from (2..2) that for

any point $p \in \Sigma$, $Div(\nabla d) = Div_\Sigma(\nabla_\Sigma d) - nH \langle \nabla d, \nu \rangle + \nabla_\nu \langle \nabla d, \nu \rangle$. That is, $\Delta d = \Delta_\Sigma d - nH \langle \nabla d, \nu \rangle + \nabla_\nu \langle \nabla d, \nu \rangle$ or

$$\Delta_\Sigma d = \Delta d + nH \langle \nabla d, \nu \rangle - \nabla_\nu \langle \nabla d, \nu \rangle \quad (2..20)$$

on Σ .

We would next like to find expressions for Δd and $\nabla_\nu \langle \nabla d, \nu \rangle$. We recall from (1..10), that $S(p_0; d)$ has constant mean curvature (with respect to the outward pointing normal) $-\coth d$. Therefore, at points on $S(p_0; d)$ we have $\Delta d = Div(\nabla d) = Div(\frac{\nabla d}{\|\nabla d\|})$. Hence,

$$\Delta d = n \coth d. \quad (2..21)$$

Next we calculate: $\nabla_\nu \langle \nabla d, \nu \rangle = \langle \nabla_\nu \nabla d, \nu \rangle + \langle \nabla d, \nabla_\nu \nu \rangle = \langle \nabla_\nu \nabla d, \nu \rangle$. Letting $\nu = \nu^N + \nu^T$, where $\nu^T = \langle \nabla d, \nu \rangle \nabla d$ and $\nu^N = \nu - \langle \nabla d, \nu \rangle \nabla d$, and noting that the integral curves of ∇d are geodesics emanating from p_0 , we see

$$\begin{aligned} \nabla_\nu \langle \nabla d, \nu \rangle &= \langle \nabla_\nu \nabla d, \nu \rangle = \langle \nabla_{\nu^N + \nu^T} \nabla d, \nu^N + \nu^T \rangle \\ &= \langle \nabla_{\nu^N} \nabla d, \nu^N \rangle = \|\nu^N\|^2 \left\langle \nabla_{\frac{\nu^N}{\|\nu^N\|}} \nabla d, \frac{\nu^N}{\|\nu^N\|} \right\rangle. \end{aligned} \quad (2..22)$$

Now $\|\nu^N\|^2 = \|\nu\|^2 - \|\langle \nabla d, \nu \rangle \nabla d\|^2 = 1 - \langle \nabla d, \nu \rangle^2$, and at any point on Σ , we have $1 - \langle \nabla d, \nu \rangle^2 = \|\nabla_\Sigma d\|^2$. So we have by (2..22),

$$\nabla_\nu \langle \nabla d, \nu \rangle = \|\nabla_\Sigma d\|^2 \left\langle \nabla_{\frac{\nu^N}{\|\nu^N\|}} \nabla d, \frac{\nu^N}{\|\nu^N\|} \right\rangle \quad (2..23)$$

on Σ . It is easy to see that at any point p on $S(p_0; d)$ and for any unit vector $u \in T_p(S(p_0; d))$,

$$-\langle \nabla_u \nabla d, u \rangle = -\coth d, \quad (2..24)$$

the mean curvature of $S(p_0; d)$. Indeed, the operator $u \rightarrow -\nabla_u \nabla d$ has only one eigenvalue λ (with multiplicity n). Hence $\lambda = -\coth d$. Substituting $\frac{\nu^N}{\|\nu^N\|}$ for u in (2.24), we may replace (2.23) with

$$\nabla_\nu \langle \nabla d, \nu \rangle = \|\nabla_\Sigma d\|^2 \coth d, \quad (2.25)$$

which is valid on $\Sigma \cap S(p_0; d)$. Note that in (2.22), we assumed $\nu^N \neq 0$. In case $\nu^N = 0$, (2.25) is clearly true. Thus, equations (2.25) and (2.21) together with (2.20) give us

Lemma 3 *On $\Sigma \cap S(p_0; d)$, we have*

$$\Delta_\Sigma d = n \coth d + nH \langle \nabla d, \nu \rangle - \|\nabla_\Sigma d\|^2 \coth d. \quad (2.26)$$

We note that $\Delta_\Sigma \psi \circ u = (\psi'' \circ u) \|\nabla_\Sigma u\|^2 + (\psi' \circ u) \Delta_\Sigma u = \psi'' \|\nabla_\Sigma u\|^2 + \psi' \Delta_\Sigma u$ for $u \in C^2(\Sigma)$ and $\psi \in C^2(\mathbb{R})$. Thus, substituting $\psi \circ d$ into equation (2.26), we have

$$\begin{aligned} \Delta_\Sigma (\psi \circ d) &= \psi' n \coth d + (\psi'' - \psi' \coth d) \|\nabla_\Sigma d\|^2 \coth d + \psi' nH \langle \nabla d, \nu \rangle \\ &= \psi' (n - 1) \coth d + \psi'' - (\psi'' - \psi' \coth d) (1 - \|\nabla_\Sigma d\|^2) + \psi' nH \langle \nabla d, \nu \rangle. \end{aligned} \quad (2.27)$$

Let us define $\psi_\epsilon(d) = \int_d^\infty \tanh(\tau) \chi_\epsilon(\tau) d\tau$ for $d > 0$ and let $\chi_\epsilon \in C_c^1[0, \infty)$, which is yet to be defined. We calculate

1. $\begin{aligned} \psi'_\epsilon(d) (n - 1) \coth d + \psi'' &= -\tanh(d) \chi_\epsilon(d) (n - 1) \coth d - \sec h^2(d) \chi_\epsilon(d) - \tanh(d) \chi'_\epsilon(d) \\ &= -(n - 1) \chi_\epsilon(d) - \sec h^2(d) \chi_\epsilon(d) - \tanh(d) \chi'_\epsilon(d) \\ &= \chi_\epsilon(d) (\tanh^2(d) - n) - \chi'_\epsilon(d) \tanh d. \end{aligned}$
2. $\begin{aligned} \psi''_\epsilon(d) - \psi'_\epsilon(d) \coth d &= -\sec h^2(d) \chi_\epsilon(d) - \tanh(d) \chi'_\epsilon(d) + \tanh(d) \chi_\epsilon(d) \coth d \\ &= \tanh^2(d) \chi_\epsilon(d) - \tanh(d) \chi'_\epsilon(d). \end{aligned}$

Substituting into (2.27) gives

$$\begin{aligned} \Delta_{\Sigma}(\psi_{\epsilon} \circ d) &= \chi_{\epsilon}(d) (\tanh^2(d) - n) - \chi'_{\epsilon}(d) \tanh d \\ &- (\tanh^2(d) \chi_{\epsilon}(d) - \tanh(d) \chi'_{\epsilon}(d)) (1 - \|\nabla_{\Sigma} d\|^2) - \tanh(d) \chi_{\epsilon}(d) nH \langle \nabla d, \nu \rangle. \end{aligned} \quad (2.28)$$

In what follows, let $\Sigma_t = \Sigma \cap B(p_0, t)$ for $t > 0$, μ denote n -dimensional hyperbolic surface measure, and V_t be the volume of an n -dimensional hyperbolic ball of radius t .

We now define $\chi_{\epsilon}(t) = \begin{cases} \frac{1}{V_{\epsilon}} - \frac{1}{V_D} & 0 \leq t \leq \epsilon \\ \frac{1}{V_t} - \frac{1}{V_D} & \epsilon \leq t \leq D \\ 0 & D \leq t \end{cases}$. Though χ_{ϵ} is not C^1 , it is clear that $\chi'_{\epsilon} \circ d \in C_b(\Sigma_D - S(p_0, \epsilon))$ for each $\epsilon > 0$.

Lemma 4 *Let D be a fixed hyperbolic radius and $0 \leq g \in C_b^2(\Sigma_D)$. Then*

$$\begin{aligned} g(p_0) &\leq \frac{1}{V_D} \int_{\Sigma_D} g \|\nabla_{\Sigma} d\|^2 d\mu + \int_{\Sigma_D} \left[- \left(\frac{1}{V_d} \right)' \frac{\tanh d}{n} - \frac{1}{V_d} \right] g d\mu + \\ &\frac{1}{n} \int_{\Sigma_D} \left[\tanh(d) \left(\frac{1}{V_d} \right)' + \frac{1}{V_d} \right] (1 - \|\nabla_{\Sigma} d\|^2) g d\mu \\ &+ \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) (H^2/4 + \|\nabla_{\Sigma} d\|^2/n) g d\mu - \frac{1}{n} \int_{\Sigma_D} \psi \circ d \Delta_{\Sigma} g d\mu, \end{aligned} \quad (2.29)$$

where $\psi(t) = \lim_{\epsilon \rightarrow 0^+} \psi_{\epsilon}(t) = \int_t^D \tanh(\tau) \left(\frac{1}{V_{\tau}} - \frac{1}{V_D} \right) d\tau$ for $t > 0$. If we only assume $g \in C_b^1(\Sigma_D)$, then the term $-\frac{1}{n} \int_{\Sigma_D} \psi \circ d \Delta_{\Sigma} g d\mu$ is replaced by $\frac{1}{n} \int_{\Sigma_D} \langle \nabla_{\Sigma}(\psi \circ d), \nabla_{\Sigma} g \rangle d\mu$.

Before we prove the lemma, we would like to show that

$$1 - \|\nabla_{\Sigma} d\|^2 \leq cd^2 \quad (2.30)$$

as $d \rightarrow 0^+$ for some constant $c \geq 0$. Indeed, let us assume that Σ lies in the ball model of $\mathbb{H}^{n+1} := \{(x, t) : x \in \mathbb{D}^n, t \in \mathbb{R}, |x|^2 + t^2 < 1\}$. Let us also assume that $p_0 = (0, 0)$

and that Σ is locally the graph of a C^2 function with $Du(0) = 0$. Then by Taylor's theorem,

$$u \leq c_1 |x|^2 \quad \text{and} \quad |Du| \leq c_2 |x| \quad (2..31)$$

as $x \rightarrow 0$ and $c_i \geq 0$. At the point $(x, u(x)) \in \Sigma$, the unit surface normal $\eta = \frac{1}{N_1} (-Du(x), 1)$, where $N_1 = \frac{2}{1-|x|^2} \sqrt{1 + |Du(x)|^2}$. Since Euclidean lines through the origin are geodesics, $\nabla d(x, u(x)) = \frac{1}{N_2} (x, u(x))$ where $N_2 = \frac{2}{1-|x|^2} \sqrt{|x|^2 + |u(x)|^2}$.

Therefore,

$$\begin{aligned} 1 - \|\nabla_{\Sigma} d\|^2 &= |\langle \nabla d, \eta \rangle|^2 = \frac{\frac{16}{(1-|x|^2)^4} (-Du \cdot x + u(x))^2}{N_1^2 N_2^2} \\ &= \frac{(-Du \cdot x + u(x))^2}{(1 + |Du(x)|^2) (|x|^2 + |u(x)|^2)}. \end{aligned} \quad (2..32)$$

and hence

$$\frac{1 - \|\nabla_{\Sigma} d\|^2}{|x|^2} \leq \frac{\left(\frac{|Du|}{|x|} + \frac{|u(x)|^2}{|x|^2} \right)^2}{\left(1 + \frac{|u(x)|^2}{|x|^2} \right)}.$$

Now using (2..31) and the fact that $|x| \sim cd(x)$ (c constant) as $x \rightarrow 0$ gives us (2..30).

Proof of Lemma 4. We multiply (2..28) through by g and integrate over Σ_D

to obtain

$$\begin{aligned} \int_{\Sigma_D} \Delta_{\Sigma} (\psi_{\epsilon} \circ d) g d\mu &= \int_{\Sigma_D} [\chi_{\epsilon}(d) (\tanh^2(d) - n) - \chi'_{\epsilon}(d) \tanh d] g d\mu \\ &- \int_{\Sigma_D} [\tanh^2(d) \chi_{\epsilon}(d) - \tanh(d) \chi'_{\epsilon}(d)] (1 - \|\nabla_{\Sigma} d\|^2) g d\mu - \int_{\Sigma_D} \tanh(d) \chi_{\epsilon}(d) nH \langle \nabla d, \nu \rangle g d\mu. \end{aligned}$$

Now,

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_D} \Delta_{\Sigma} (\psi_{\epsilon} \circ d) g d\mu = \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_D - \Sigma_{\epsilon}} \Delta_{\Sigma} (\psi_{\epsilon} \circ d) g d\mu + \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_{\epsilon}} \Delta_{\Sigma} (\psi_{\epsilon} \circ d) g d\mu. \quad (2..33)$$

But

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_\epsilon} \Delta_\Sigma (\psi_\epsilon \circ d) g d\mu = \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_\epsilon} \left[\left(\frac{1}{V_\epsilon} - \frac{1}{V_D} \right) (\tanh^2(d) - n) \right] g d\mu \\
& - \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_\epsilon} \tanh^2(d) \left(\frac{1}{V_\epsilon} - \frac{1}{V_D} \right) (1 - \|\nabla_\Sigma d\|^2) g d\mu - \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_D} \tanh(d) \left(\frac{1}{V_\epsilon} - \frac{1}{V_D} \right) nH \langle \nabla d, \nu \rangle g d\mu \\
& = -ng(p_0),
\end{aligned}$$

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_D - \Sigma_\epsilon} \Delta_\Sigma (\psi_\epsilon \circ d) g d\mu = \int_{\Sigma_D} \Delta_\Sigma (\psi \circ d) g d\mu, \text{ and } \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_D} \Delta_\Sigma (\psi_\epsilon \circ d) g d\mu = \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma_D} \psi_\epsilon \circ \\
& d\Delta_\Sigma g d\mu = \int_{\Sigma_D} \psi \circ d\Delta_\Sigma g d\mu. \text{ Hence (2.33) becomes}
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma_D} \psi \circ d\Delta_\Sigma g d\mu = \int_{\Sigma_D} \left[\left(\frac{1}{V_d} - \frac{1}{V_D} \right) (\tanh^2(d) - n) - \left(\frac{1}{V_d} \right)' \tanh d \right] g d\mu \\
& - \int_{\Sigma_D} \left[\tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) - \tanh(d) \left(\frac{1}{V_d} \right)' \right] (1 - \|\nabla_\Sigma d\|^2) g d\mu \\
& - \int_{\Sigma_D} \left[\tanh(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) nH \langle \nabla d, \nu \rangle g \right] d\mu - ng(p_0),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
g(p_0) &= \frac{1}{n} \int_{\Sigma_D} \left[\left(\frac{1}{V_d} - \frac{1}{V_D} \right) (\tanh^2(d) - n) - \left(\frac{1}{V_d} \right)' \tanh d \right] g d\mu \\
& - \frac{1}{n} \int_{\Sigma_D} \left[\tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) - \tanh(d) \left(\frac{1}{V_d} \right)' \right] (1 - \|\nabla_\Sigma d\|^2) g d\mu \\
& - \int_{\Sigma_D} \tanh(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) H \langle \nabla d, \nu \rangle g d\mu - \frac{1}{n} \int_{\Sigma_D} \psi \circ d\Delta_\Sigma g d\mu. \tag{2.34}
\end{aligned}$$

We note that by using power series, one can see that the terms $\left(\frac{1}{V_d} - \frac{1}{V_D} \right) (-n) - \left(\frac{1}{V_d} \right)' \tanh d$ and $\tanh(d) \left(\frac{1}{V_d} \right)$ are asymptotic (as $d \rightarrow 0^+$) to cd^{-n+1} , where c is some constant. We also have from (2.30) that $\tanh(d) \left(\frac{1}{V_d} \right)' (1 - \|\nabla_\Sigma d\|^2) \leq cd^{-n+2}$ for

small enough d . Thus, the above integrands belong to $L^1(\Sigma_D)$. Let us rearrange the first two integrals in (2..34).

$$\begin{aligned}
& \frac{1}{n} \int_{\Sigma_D} \left[\left(\frac{1}{V_d} - \frac{1}{V_D} \right) (\tanh^2(d) - n) - \left(\frac{1}{V_d} \right)' \tanh d \right] g d\mu \\
& - \frac{1}{n} \int_{\Sigma_D} \left[\tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) - \tanh(d) \left(\frac{1}{V_d} \right)' \right] (1 - \|\nabla_{\Sigma} d\|^2) g d\mu \\
= & \frac{1}{n} \int_{\Sigma_D} \left[\left(\frac{1}{V_d} - \frac{1}{V_D} \right) (-n) - \left(\frac{1}{V_d} \right)' \tanh d \right] g d\mu + \frac{1}{n} \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) \|\nabla_{\Sigma} d\|^2 g d\mu \\
& + \frac{1}{n} \int_{\Sigma_D} \tanh(d) \left(\frac{1}{V_d} \right)' (1 - \|\nabla_{\Sigma} d\|^2) g d\mu,
\end{aligned}$$

and we express the right side of this equality by $I_1 + I_2 + I_3$. By using the inequality $\tanh(d) |H \langle \nabla d, \nu \rangle| \leq \langle \nabla d, \nu \rangle^2 + \frac{\tanh^2(d) H^2}{4} = (1 - \|\nabla_{\Sigma} d\|^2) + \frac{\tanh^2(d) H^2}{4}$, we see that the the third term of (2..34) is bounded above by $\int_{\Sigma_D} \left(\frac{1}{V_d} - \frac{1}{V_D} \right) \frac{\tanh^2(d) H^2}{4} g d\mu - \int_{\Sigma_D} \frac{g}{V_D} (1 - \|\nabla_{\Sigma} d\|^2) d\mu + \int_{\Sigma_D} \frac{1}{V_d} (1 - \|\nabla_{\Sigma} d\|^2) g d\mu := I_4 - I_5 + I_6$. We now obtain from (2..34) and the preceding formulations, the inequality

$$g(p_0) \leq I_1 + I_2 + I_3 + I_4 - I_5 + I_6 - \frac{1}{n} \int_{\Sigma_D} \psi \circ d \Delta_{\Sigma} g d\mu. \quad (2..35)$$

Finally,

$$I_1 - I_5 = \frac{1}{V_D} \int_{\Sigma_D} g \|\nabla_{\Sigma} d\|^2 d\mu + \int_{\Sigma_D} \left[- \left(\frac{1}{V_d} \right)' \frac{\tanh d}{n} - \frac{1}{V_d} \right] g d\mu, \quad (2..36)$$

$$I_3 + I_6 = \frac{1}{n} \int_{\Sigma_D} \left[\tanh(d) \left(\frac{1}{V_d} \right)' + \frac{1}{V_d} \right] (1 - \|\nabla_{\Sigma} d\|^2) g d\mu, \quad (2..37)$$

and

$$I_2 + I_4 = \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) (H^2/4 + \|\nabla_{\Sigma} d\|^2/n) g d\mu. \quad (2..38)$$

The lemma now follows from (2.35), (2.36), (2.37), and (2.38). ■

The following corollary is a simplified version of (2.29) which we will use to derive our interior estimate. Our method of proof, however, places restrictions on the size of D . Here, w and Σ are the same as in Lemma (2). Let us define $H_{0,p_0,D} = \sup_{p \in \Sigma_D} |H(p)|$, $H_{1,p_0,D} = \sup_{p \in \Sigma_D} \|\nabla H(p)\|$, and $A(t) = \mu(\Sigma_t)$.

Corollary 5 *If $w \geq 0$ and D is small enough to ensure that $\|\nabla_\Sigma d\| > 0$ on $\bar{\Sigma}_D$, then*

$$w(p_0) \leq \frac{1}{V_D} \int_{\Sigma_D} w d\mu + \frac{(H_{0,p_0,D}^2 + 4)}{4n} \int_0^D \tanh^2(t) \frac{V_t'}{(V_t)^2} \left[\int_{\Sigma_t} w d\mu \right] dt + H_{1,p_0,D} \int_0^D \tanh(t) \frac{1}{V_t} A(t) dt. \quad (2.39)$$

Proof. Since $w \geq 0$, we may replace g in (2.29) with w to obtain

$$\begin{aligned} w(p_0) &\leq \frac{1}{V_D} \int_{\Sigma_D} w \|\nabla_\Sigma d\|^2 d\mu + \int_{\Sigma_D} \left[- \left(\frac{1}{V_d} \right)' \frac{\tanh d}{n} - \frac{1}{V_d} \right] w d\mu + \\ &\frac{1}{n} \int_{\Sigma_D} \left[\tanh(d) \left(\frac{1}{V_d} \right)' + \frac{1}{V_d} \right] (1 - \|\nabla_\Sigma d\|^2) w d\mu \\ &+ \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) (H^2/4 + \|\nabla_\Sigma d\|^2/n) w d\mu - \frac{1}{n} \int_{\Sigma_D} \psi \circ d \Delta_\Sigma w d\mu. \end{aligned} \quad (2.40)$$

Next, we show

$$- \left(\frac{1}{V_d} \right)' \frac{\tanh d}{n} - \frac{1}{V_d} \leq 0 \text{ and } \tanh(d) \left(\frac{1}{V_d} \right)' + \frac{1}{V_d} \leq 0. \quad (2.41)$$

Now,

$$\tanh(d) \left(\frac{1}{V_d} \right)' + \frac{1}{V_d} = \frac{-\tanh(d) V_d' + V_d}{V_d^2}$$

and the numerator $-\tanh(d) V_d' + V_d := f(d)$ has $f(0) = 0$. Moreover,

$$f'(d) = V_d'(1 - \sec^2 d) - \tanh(d) V_d'' = V_d' \tanh^2(d) - (n-1) V_d'$$

$$= V_d' (\tanh^2(d) - (n-1)) \leq 0.$$

A similar argument establishes

$$- \left(\frac{1}{V_d} \right)' \frac{\tanh d}{n} - \frac{1}{V_d} \leq 0.$$

Therefore, using (2.40), (2.41), the relation $\|\nabla_{\Sigma} d\|^2 \leq \|\nabla d\|^2 = 1$, and Lemma 2,

which implies that $-\frac{1}{n} \int_{\Sigma_D} \psi \circ d \Delta_{\Sigma} w d\mu \leq \int_{\Sigma_D} (\psi \circ d) \nabla_{\eta} H d\mu$, we have

$$\begin{aligned} w(p_0) &\leq \frac{1}{V_D} \int_{\Sigma_D} w d\mu \\ &+ \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) (H^2/4 + 1/n) w d\mu + \int_{\Sigma_D} (\psi \circ d) \nabla_{\eta} H d\mu. \end{aligned} \quad (2.42)$$

Next, we find upper bounds for $\int_{\Sigma_D} (\psi \circ d) \nabla_{\eta} H d\mu$ and $\int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) (H^2/4 + 1/n) w d\mu$.

Now,

$$\begin{aligned} \int_{\Sigma_D} (\psi \circ d) \nabla_{\eta} H d\mu &\leq H_{1,p_0,D} \int_{\Sigma_D} (\psi \circ d) d\mu \\ &= H_{1,p_0,D} \int_0^D \psi(t) A'(t) dt \\ &= -H_{1,p_0,D} \int_0^D \psi'(t) A(t) dt \quad (\text{integration by parts}) \\ &= H_{1,p_0,D} \int_0^D \tanh(t) \left(\frac{1}{V_t} - \frac{1}{V_D} \right) A(t) dt \\ &\leq H_{1,p_0,D} \int_0^D \tanh(t) \frac{1}{V_t} A(t) dt, \end{aligned} \quad (2.43)$$

and

$$\begin{aligned}
& \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) (H^2/4 + 1/n) w d\mu \\
& \leq \frac{(H_{0,p_0,D}^2 + 4)}{4n} \int_{\Sigma_D} \tanh^2(d) \left(\frac{1}{V_d} - \frac{1}{V_D} \right) w d\mu \\
& = \frac{(H_{0,p_0,D}^2 + 4)}{4n} \int_0^D \tanh^2(t) \left(\frac{1}{V_t} - \frac{1}{V_D} \right) \left[\int_{\partial\Sigma_t} \frac{wd(\partial\Sigma_t)}{\|\nabla_\Sigma d\|} \right] dt \text{ (by coarea formula)} \\
& = \frac{(H_{0,p_0,D}^2 + 4)}{4n} \int_0^D \tanh^2(t) \left[\int_t^D \frac{V'_s}{(V_s)^2} ds \right] \left[\int_{\partial\Sigma_t} \frac{wd(\partial\Sigma_t)}{\|\nabla_\Sigma d\|} \right] dt \\
& = \frac{(H_{0,p_0,D}^2 + 4)}{4n} \int_0^D \frac{V'_t}{(V_t)^2} \int_0^t \tanh^2(s) \left[\int_{\partial\Sigma_s} \frac{wd(\partial\Sigma_s)}{\|\nabla_\Sigma d\|} \right] ds dt \text{ (by Fubini's theorem)} \\
& \leq \frac{(H_{0,p_0,D}^2 + 4)}{4n} \int_0^D \tanh^2(t) \frac{V'_t}{(V_t)^2} \left[\int_{\Sigma_t} w d\mu \right] dt. \tag{2..44}
\end{aligned}$$

The result now follows from inequalities (2..42), (2..43), and (2..44). ■

2.3 Main Interior Estimate

We are now in the position to derive an interior gradient estimate for a solution of the prescribed mean curvature equation. Our model for $(n + 1)$ -dimensional

hyperbolic space will be the cylinder $\mathbb{D}^n \times \mathbb{R}$ endowed with the metric $g_{ij}(x, t) =$

$$\left\{ \begin{array}{ll} \frac{4}{(1-|x|^2)^2} & 1 \leq i = j \leq n \\ \frac{(1+|x|^2)^2}{(1-|x|^2)^2} & i = j = n + 1 \\ 0 & \text{otherwise} \end{array} \right\} \text{ (see 1..2). We define } \Phi : \mathbb{D}^n \times \mathbb{R} \rightarrow \mathbb{R} \text{ by } \Phi(x, t) = t - u(x),$$

where $u \in C^2(\Omega)$ and $\Omega \subseteq \mathbb{D}^n$ is a domain. The vector field $\frac{\nabla\Phi}{\|\nabla\Phi\|}$ is the *upward* pointing

normal field to the level surfaces of Φ , so that

$$\operatorname{div} \left(\frac{\nabla\Phi}{\|\nabla\Phi\|} \right) |_{(x,t)} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \frac{\Phi_j}{\|\nabla\Phi\|}) |_{(x,t)} = -nH(x), \tag{2..45}$$

$H(x)$ being the mean curvature of the graph of u at $(x, u(x))$ for $x \in \Omega$. We set

$A = \frac{4}{(1-|x|^2)^2}$, $B = \frac{(1+|x|^2)^2}{(1-|x|^2)^2}$, and $v = \|\nabla\Phi\|$. Now for any C^1 function $F : \mathbb{D}^n \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\nabla F = \left(\frac{1}{A}F_1, \dots, \frac{1}{A}F_n, \frac{1}{B}F_{n+1} \right), \quad (2.46)$$

hence $\nabla\Phi = \left(\frac{-1}{A}u_1, \dots, \frac{-1}{A}u_n, \frac{1}{B} \right)$. Therefore, (2.45) now reads

$$\sum_{i=1}^n \left(\sqrt{A^n B} \left[\frac{-u_i}{Av} \right] \right)_i = -nH\sqrt{A^n B} \quad (2.47)$$

or

$$\left(\frac{-u_i}{Av} \right)_i \sqrt{A^n B} - \frac{u_i}{Av} \left(\sqrt{A^n B} \right)_i = -nH\sqrt{A^n B}.$$

We multiply this through by a function $\phi \in C_c^1(\Omega)$ and integrate by parts to obtain

$$\int_{\Omega} \frac{u_i}{Av} \sqrt{B} \phi_i \sqrt{A^n} dm = \int_{\Omega} -nH\phi \sqrt{B} \sqrt{A^n} dm$$

or

$$\int_{\Omega} \frac{u_i}{Av} \sqrt{B} \phi_i dx = - \int_{\Omega} nH\phi \sqrt{B} dx, \quad (2.48)$$

where m is Lebesgue measure, $dx = \sqrt{A^n} dm$ is the volume element for the hyperbolic disc \mathbb{D}^n , and summation is understood. Equation (2.48) is a weak form of the prescribed mean curvature equation. We mention that $v = \sqrt{\frac{1}{B} + \frac{1}{A} |Du|^2}$ and the area element for the graph of u is $d\mu = \sigma dx$, where $\sigma = \sqrt{B}v = \sqrt{1 + \frac{B}{A} |Du|^2}$.

Lemma 6 *Let Σ be the graph of u , and suppose $f \in C^1(\Sigma)$. Then $|Df| \leq 2\sigma\sqrt{B} \|\nabla_{\Sigma} f\|$.*

Proof. Let $\eta = \frac{\nabla\Phi}{\|\nabla\Phi\|}$ and define f on all of $\mathbb{D}^n \times \mathbb{R}$ by $f(x, t) = f(x)$. Then

$\nabla f = \nabla_{\Sigma} f + \langle \nabla f, \eta \rangle \eta$ and (therefore)

$$\|\nabla_{\Sigma} f\|^2 = \|\nabla f\|^2 - |\langle \nabla f, \eta \rangle|^2. \quad (2.49)$$

Since $\langle \nabla f, e^{n+1} \rangle = 0$, we have from (2.49) and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|\nabla_{\Sigma} f\|^2 &\geq \|\nabla f\|^2 - \left(\sum_{i=1}^n \frac{1}{A} f_i^2 \right) \left(\sum_{i=1}^n A \eta_i^2 \right) \\
&= \|\nabla f\|^2 - \|\nabla f\|^2 (1 - B \eta_{n+1}^2) \\
&= \|\nabla f\|^2 B \eta_{n+1}^2.
\end{aligned} \tag{2.50}$$

Now $\|\nabla f\| = \frac{1}{A} \|Df\| = \frac{1}{A} \sqrt{A} |Df|$, and $\eta_{n+1} = \frac{1}{Bv}$, hence

$$\eta_{n+1} \sqrt{B} \|\nabla f\| = \frac{|Df|}{v \sqrt{AB}}. \tag{2.51}$$

From (2.50) and (2.51), we have

$$\|\nabla_{\Sigma} f\| \geq \eta_{n+1} \sqrt{B} \|\nabla f\| = \frac{|Df|}{v \sqrt{AB}} \geq \frac{|Df|}{2vB},$$

since $\sqrt{AB} \leq 2B$. Now combining this with the equality $\sigma = \sqrt{B}v$, completes the proof. ■

For the theorem below, let $B(x_0; t) = \{p \in \mathbb{H}^{n+1} : \text{dist}_{\mathbb{H}^{n+1}}(p, (x_0, 0)) < t\}$ and define the quantities $g_{x_0, t} = \sup_{(x, y) \in B(x_0; t)} d(x)$ and $h_{x_0, t} = \sup_{(x, y) \in B(x_0; t)} |y|$, where $d(x) = \text{dist}_{\mathbb{H}^n}(x, x_0)$ for $x_0, x \in \mathbb{D}^n$. We also let $\sqrt{B_{x_0, t}} = \sup_{d(x) \leq t} \sqrt{B}$, $H_{0, x_0, t} = \sup_{d(x) \leq t} |H|$, $H_{1, x_0, t} = \sup_{d(x) \leq t} |DH|$, and for a given hyperbolic radius t , $t' = g_{x_0, t} + h_{x_0, t}$ and $t'' = g_{x_0, t} + 2h_{x_0, t}$. We wish to make the following remarks.

1. In Corollary 5, we defined $H_{1, p_0, t} = \sup_{p \in \Sigma_t} \|\nabla H(p)\|$. However, we have $\|\nabla H(x, u(x))\| \leq |DH(x, u(x))| = |DH(x)|$. Thus, we may replace $H_{1, p_0, t}$ in Corollary 5 with $H_{1, x_0, t}$, $p_0 = (x_0, 0)$.
2. A gradient estimate can be obtained from Corollary 5 provided we have estimates for $\int_{\Sigma_t} w d\mu$ and $A(t)$ for each $0 < t \leq D$. However, Corollary 5 cannot be applied

directly because the inequality $w \geq 0$ in Ω may be false, thus we define $\widehat{w} = \ln(\sqrt{B_{x_0, D''} v})$. Since \widehat{w} and w differ by a constant, clearly Lemma 2 and (hence) Corollary 5 remain valid with \widehat{w} in place of w .

3. We would also like to weaken the smoothness requirements of Σ in Lemma 2. To do this, consider the weak form of (2.3), which reads

$$\int_{\Sigma} \langle \nabla_{\Sigma} w, \nabla_{\Sigma} \phi \rangle d\mu = \int_{\Sigma} (-C^2 \phi - \|\nabla_{\Sigma} w\|^2 \phi + n \nabla_{\eta} H \phi) d\mu \quad (2.52)$$

for every function $\phi \in C_c^1(\Sigma)$. If Σ is the graph of u and $u \in C^2(\Omega)$ with mean curvature $H \in C^1(\Omega)$, then (2.52) remains valid. This can be seen by taking a sequence of C^3 functions converging to u in the space $C^2(\Omega)$. Now for positive test functions $\phi \in C_c^1(\Sigma)$, we have the weak inequality

$$\int_{\Sigma} \langle \nabla_{\Sigma} w, \nabla_{\Sigma} \phi \rangle d\mu \leq \int_{\Sigma} n \nabla_{\eta} H \phi d\mu. \quad (2.53)$$

Using this in (2.42), we see Corollary 5 is still valid.

Theorem 7 *Suppose $u \in C^2(\Omega)$ and that its mean curvature $H \in C^1(\Omega)$, where $\Omega \subset \mathbb{D}^n$ is open. Fix $x_0 \in \Omega$ and choose D small enough to use Corollary 5, and to ensure $\text{dist}_{\mathbb{H}^n}(x_0, \partial\Omega) > D''$. Then $|Du(x_0)| \leq C_2 \exp C_1 \left(\sup_{d(x) \leq D''} \frac{u(x) - u(x_0)}{h_{x_0, D}} \right)$, where $C_1 = C_1(x_0, n, D'', H_{0, x_0, D''}, H_{1, x_0, D''})$ and $C_2 = C_2(x_0, n, D'', H_{0, x_0, D''}, H_{1, x_0, D''}, \sup_{d(x) \leq D''} u(x) - u(x_0))$.*

Proof. We first assume $u(x_0) = 0$. Define the function $u_t = \left\{ \begin{array}{ll} h_{x_0, t} & u \geq h_{x_0, t} \\ -h_{x_0, t} & u \leq -h_{x_0, t} \\ u & \text{otherwise} \end{array} \right\}$,

and let $\rho \in C^1(\Omega)$ satisfy $|\rho| \leq 1$, $|D\rho| \leq \frac{1}{h_{x_0, t}}$, $\rho(x) = 1$ when $d(x) \leq g_{x_0, t}$, and

$\rho(x) = 0$ when $d(x) \geq g_{x_0,t} + h_{x_0,t} = t'$. Now using $\phi = u_t \rho$ in (2.48), one obtains

$$\int_{\Omega} \frac{\sqrt{B}}{A} \frac{u_i}{v} (u_t)_i \rho dx = - \int_{\Omega} \left(\frac{\sqrt{B}}{A} \frac{u_i}{v} u_t \rho_i + n H \sqrt{B} \rho u_t \right) dx,$$

which implies

$$\int_S \frac{\sqrt{B}}{A} \frac{|Du|^2}{v} dx \leq h_{x_0,t} \int_{d(x) < t'} \left(\frac{\sqrt{B}}{A} \frac{|Du| |D\rho|}{v} + n |H| \sqrt{B} \right) dx, \quad (2.54)$$

where $S = \{x : d(x) < g_{x_0,t}, |u(x)| < h_{x_0,t}\}$. Now

$$A(t) = \int_{\pi(\Sigma_t)} \sigma dx \leq \int_S \sigma dx, \quad (2.55)$$

where $\pi(x, t) = x$. Letting $S_1 = S \cap \{x : |Du(x)| \leq \sqrt{\frac{A}{B}}\}$ and $S_2 = S \cap \{x : |Du(x)| \geq \sqrt{\frac{A}{B}}\}$,

and using $\sqrt{B}v = \sigma$, we have from (2.55)

$$\begin{aligned} A(t) &\leq \int_{S_1} \sigma dx + \int_{S_2} \sigma dx \\ &\leq \int_{S_1} \sqrt{2} dx + \int_{S_2} \sqrt{B} v dx \\ &\leq \sqrt{2} V_{g_{x_0,t}} + \int_{S_2} \sqrt{B} v dx. \end{aligned} \quad (2.56)$$

Now $v^2 = \frac{1}{B} + \frac{1}{A} |Du|^2 \leq \frac{2|Du|^2}{A}$ on S_2 , hence on S_2 we have

$$\sqrt{B}v \leq \frac{2\sqrt{B}}{A} \frac{|Du|^2}{v}. \quad (2.57)$$

So (2.54), (2.56), and (2.57) give

$$\begin{aligned} A(t) &\leq \sqrt{2} V_{g_{x_0,t}} + 2 \int_S \frac{\sqrt{B}}{A} \frac{|Du|^2}{v} dx \\ &\leq \sqrt{2} V_{g_{x_0,t}} + 2h_{x_0,t} \int_{d(x) < t'} \left(\frac{\sqrt{B}}{A} \frac{|Du| |D\rho|}{v} + n |H| \sqrt{B} \right) dx. \end{aligned} \quad (2.58)$$

But

$$\begin{aligned}
& \frac{\sqrt{B} |Du|}{A v} \\
= & \frac{B}{A} \frac{|Du|}{\sqrt{1 + \frac{B}{A} |Du|^2}} \leq \frac{2 |Du|}{\sqrt{|Du|^2 + 4}} \quad (\text{using } \frac{1}{4} \leq \frac{B}{A} \leq 1) \\
\leq & 2.
\end{aligned} \tag{2.59}$$

Hence, combining (2.58) and (2.59), we have

$$\begin{aligned}
A(t) & \leq \sqrt{2} V_{g_{x_0,t}} + 2h_{x_0,t} \int_{d(x) < t'} \left(2 \cdot \frac{1}{h_{x_0,t}} + n |H| \sqrt{B} \right) dx \\
& \leq \sqrt{2} V_{g_{x_0,t}} + 4V_{t'} + 2nh_{x_0,t} H_{0,x_0,t'} \sqrt{B_{x_0,t'}} V_{t'} \\
& \leq CV_{t'} \left(1 + nh_{x_0,t} H_{0,x_0,t'} \sqrt{B_{x_0,t'}} \right).
\end{aligned} \tag{2.60}$$

It is also clear that this same estimate is valid if $u(x_0) \neq 0$. We now estimate $\int_{\Sigma_t} \widehat{w} d\mu$.

$$\int_{\Sigma_t} \widehat{w} d\mu = \int_{\pi(\Sigma_t)} \widehat{w} \sigma dx \leq \int_{S_1} \widehat{w} \sigma dx + \int_{S_2} \widehat{w} \sigma dx. \tag{2.61}$$

As before, $\sigma \leq \sqrt{2}$ and $v \leq \sqrt{2}$ on S_1 , and $\sigma \leq \frac{2\sqrt{B}|Du|^2}{Av}$ on S_2 . Since $\widehat{w} =$

$\ln \sqrt{B_{x_0,D''} v} \leq \sqrt{B_{x_0,D''} v}$, (2.61) now reads

$$\begin{aligned}
\int_{\Sigma_t} \widehat{w} d\mu & \leq \int_{S_1} 2\sqrt{B_{x_0,D''}} dx + \int_{S_2} \frac{2\sqrt{B} |Du|^2}{Av} \widehat{w} dx \\
& \leq 2\sqrt{B_{x_0,D''}} V_{g_{x_0,t}} + \int_S \frac{2\sqrt{B} |Du|^2}{Av} \widehat{w} dx.
\end{aligned} \tag{2.62}$$

To estimate the second integral in (2..62), we use for ϕ in (2..48) the function $(u_t + h_{x_0,t}) \rho \widehat{w}$.

This gives us

$$\begin{aligned}
& \int_S \frac{2\sqrt{B} |Du|^2}{Av} w dx \\
& \leq 4h_{x_0,t} \int_{\substack{d(x) < t' \\ u(x) \geq -h_{x_0,t}}} \left(\frac{\sqrt{B} |u_i \rho_i| \widehat{w}}{Av} + \frac{\sqrt{B} |u_i| |Dw| \rho}{Av} + n |H| \widehat{w} \rho \sqrt{B} \right) dx \\
& \leq 4h_{x_0,t} \int_{\substack{d(x) < t' \\ u(x) \geq -h_{x_0,t}}} (2\widehat{w} |D\rho| + 2\rho |Dw| + n |H| w \rho \widehat{w}) dx. \tag{2..63}
\end{aligned}$$

Let $\phi \in C_c^1(\mathbb{D} \times \mathbb{R})$ and use ϕ^2 in (2..52) to obtain the inequality

$$\int_{\Sigma} \|\nabla_{\Sigma} w\|^2 \phi^2 d\mu \leq - \int_{\Sigma} \langle \nabla_{\Sigma} w, \nabla_{\Sigma} \phi^2 \rangle d\mu + n \int_{\Sigma} \nabla_{\eta} H \phi^2 d\mu.$$

or

$$\int_{\Sigma} \|\nabla_{\Sigma} w\|^2 \phi^2 d\mu \leq -2 \int_{\Sigma} \langle \nabla_{\Sigma} w, \nabla_{\Sigma} \phi \rangle \phi d\mu + n \int_{\Sigma} \nabla_{\eta} H \phi^2 d\mu. \tag{2..64}$$

We use the inequality $2|\langle \nabla_{\Sigma} w, \nabla_{\Sigma} \phi \rangle \phi| \leq 2 \|\nabla_{\Sigma} w\| \|\nabla_{\Sigma} \phi\| |\phi| \leq \frac{1}{2} \|\nabla_{\Sigma} w\|^2 \phi^2 + 2 \|\nabla_{\Sigma} \phi\|^2$,

and (2..64) to conclude

$$\begin{aligned}
& \int_{\Sigma} \|\nabla_{\Sigma} w\|^2 \phi^2 d\mu \leq 4 \int_{\Sigma} \|\nabla_{\Sigma} \phi\|^2 d\mu + 2n \left(\sup_{p \in \text{supp} \phi} |\nabla_{\eta} H(p)| \right) \int_{\Sigma} \phi^2 d\mu \\
& \leq \left(4 \sup_{p \in \text{supp} \phi} \|\nabla_{\Sigma} \phi(p)\|^2 + 2n \left(\sup_{p \in \text{supp} \phi} |\nabla_{\eta} H(p)| \right) \right) \mu(\Sigma \cap \text{supp} \phi), \tag{2..65}
\end{aligned}$$

provided $\phi^2 \leq 1$. In fact, let $\phi(x, s) = \rho(x) \tau(s)$, where $\tau \in C^1(\mathbb{R})$ has $0 \leq \tau(s) \leq 1$,

$\tau(s) = 1$ for $s \in \left(-h_{x_0,t}, \sup_{d(x) \leq t'} u \right)$, $\tau(s) = 0$ for $s \notin \left(-2h_{x_0,t}, h_{x_0,t} + \sup_{d(x) \leq t'} u \right)$, and

$|\tau'| \leq \frac{1}{h_{x_0,t}}$. We now use Lemma 6 and see that $2\rho\tau \circ u |Dw| \leq 4\phi \|\nabla_{\Sigma} w\| \sigma \sqrt{B}$. Thus,

$\int_{\mathbb{D}^n} 2\rho\tau \circ u |Dw| dx \leq 2\sqrt{B_{x_0,t'}} \int_{\Sigma} \phi \|\nabla_{\Sigma} w\| d\mu$, which gives

$$\int_{\substack{d(x) \leq t' \\ u(x) \geq -h_{x_0,t}}} 2\rho |Dw| dx \leq 2\sqrt{B_{x_0,t'}} \int_{\Sigma} \phi \|\nabla_{\Sigma} w\| d\mu. \tag{2..66}$$

The inequality $\int_{\Sigma} \phi \|\nabla_{\Sigma} w\| d\mu \leq \left(\int_{\Sigma} \phi^2 \|\nabla_{\Sigma} w\|^2 d\mu \right)^{1/2} \mu(\text{supp}\phi \cap \Sigma)^{1/2}$, along with (2..65)

and (2..66), gives the estimate

$$\int_{\substack{d(x) \leq t' \\ u(x) \geq -h_{x_0, t}}} 2\rho |Dw| dx \leq 2\sqrt{B_{x_0, t'}} \left(4 \sup_{p \in \Sigma} \|\nabla_{\Sigma} \phi(p)\|^2 + 2nH_{1, t', x_0} \right)^{1/2} \mu(\text{supp}\phi \cap \Sigma). \quad (2..67)$$

We use (2..46) to estimate $\sup_{p \in \Sigma} \|\nabla_{\Sigma} \phi(p)\|^2$:

$$\begin{aligned} \|\nabla_{\Sigma} \phi(p)\|^2 &\leq \|\nabla \phi(p)\|^2 \\ &\leq \frac{1}{A} \sum_{i=1}^n \phi_i^2(p) + \frac{1}{B} \phi_{n+1}^2(p) \\ &= \frac{1}{A} \sum_{i=1}^n \rho_i^2(x) \tau^2(u(x)) + \frac{1}{B} \rho(x) (\tau')^2(u(x)) \\ &\leq \left(\frac{1}{A} + \frac{1}{B} \right) \frac{1}{h_{x_0, t}^2} \\ &\leq \frac{2}{h_{x_0, t}^2}. \end{aligned}$$

Substitute this into (2..67) to obtain

$$\int_{\substack{d(x) \leq t' \\ u(x) \geq -h_{x_0, t}}} 2\rho |Dw| dx \leq 2\sqrt{B_{x_0, t'}} (8 + 2nH_{1, t', x_0} h_{x_0, D}^2)^{1/2} \frac{1}{h_{x_0, t}} \mu(\text{supp}\phi \cap \Sigma). \quad (2..68)$$

Now

$$\mu(\text{supp}\phi \cap \Sigma) \leq \int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0, t}}} \sigma dx = \int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0, t}}} \sqrt{B} \nu dx, \quad (2..69)$$

and this is estimated in the same manner as (2..60) was obtained from (2..56). Before doing so, we note that the remaining part of the integral in (2..63) can also be estimated

using $\int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0,t}}} \sqrt{B} v dx$. In fact,

$$\begin{aligned} & \int_{\substack{d(x) \leq t' \\ u(x) \geq -h_{x_0,t}}} n |H| \widehat{w} \rho \sqrt{B} dx \leq n H_{0,x_0,t'} \int_{\substack{d(x) \leq t' \\ u(x) \geq -h_{x_0,t}}} \widehat{w} \sqrt{B} dx \\ & \leq n \sqrt{B_{x_0,D''}} H_{0,x_0,t'} \int_{\substack{d(x) < t' \\ u(x) \geq -h_{x_0,t}}} v \sqrt{B} dx \leq n \sqrt{B_{x_0,D''}} H_{0,x_0,t'} \int_{\substack{d(x) < t' \\ u(x) \geq -2h_{x_0,t}}} v \sqrt{B} dx, \end{aligned} \quad (2..70)$$

and

$$\begin{aligned} & \int_{\substack{d(x) < t' \\ u(x) \geq -h_{x_0,t}}} 2\widehat{w} |D\rho| dx \leq \frac{2}{h_{x_0,t}} \int_{\substack{d(x) < t' \\ u(x) \geq -h_{x_0,t}}} \widehat{w} \sqrt{B} dx \quad (\text{using } \sqrt{B} \geq 1) \\ & \leq \frac{2\sqrt{B_{x_0,D''}}}{h_{x_0,t}} \int_{\substack{d(x) < t' \\ u(x) \geq -h_{x_0,t}}} v \sqrt{B} dx \leq \frac{2\sqrt{B_{x_0,D''}}}{h_{x_0,t}} \int_{\substack{d(x) < t' \\ u(x) \geq -2h_{x_0,t}}} v \sqrt{B} dx. \end{aligned} \quad (2..71)$$

Therefore, by (2..62), (2..63), (2..68), (2..69), (2..70), and (2..71), we have

$$\begin{aligned} & \int_{\Sigma_t} \widehat{w} d\mu \leq 2\sqrt{B_{x_0,D''}} V_{g_{x_0,t}} \\ & \quad + 16h_{x_0,t} \left[n\sqrt{B_{x_0,D''}} H_{0,x_0,t'} + \frac{2\sqrt{B_{x_0,D''}}}{h_{x_0,t}} + \frac{\sqrt{B_{x_0,t'}}}{h_{x_0,t}} (8 + 2nH_{1,t',x_0} h_{x_0,D}^2)^{1/2} \right] \\ & \quad \cdot \int_{\substack{d(x) < t' \\ u(x) \geq -2h_{x_0,t}}} v \sqrt{B} dx \\ & \leq C\sqrt{B_{x_0,D''}} V_{g_{x_0,t}} \\ & \quad + C\sqrt{B_{x_0,D''}} \left[1 + nh_{x_0,t} H_{0,x_0,D'} + (1 + nH_{1,D',x_0} h_{x_0,D}^2)^{1/2} \right] \\ & \quad \cdot \int_{\substack{d(x) < t' \\ u(x) \geq -2h_{x_0,t}}} v \sqrt{B} dx. \end{aligned} \quad (2..72)$$

We shall now calculate.

$$\int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0,t}}} \sqrt{B} v dx \leq \sqrt{2} V_{t'} + \int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0,t}}} \frac{2\sqrt{B} |Du|^2}{Av} w dx. \quad (2.73)$$

Let us use $\phi = \rho u_t$ in (2.48), where this time $u_t = \max\{u + 2h_{x_0,t}, 0\}$ and $\rho \in C^1(\Omega)$ has $0 \leq \rho \leq 1$, $\rho = 1$ for $d(x) \leq t'$, $\rho = 0$ for $d(x) \geq t''$, and $|D\rho| \leq \frac{1}{h_{x_0,t}}$. This gives us

$$\int_{\Omega} \frac{\sqrt{B} u_i}{A} \frac{u_i}{v} (u_t)_i \rho dx = - \int_{\Omega} \left(\frac{\sqrt{B} u_i}{A} \frac{u_i}{v} u_t \rho_i + n H \sqrt{B} u_t \rho \right) dx,$$

which implies

$$\begin{aligned} & \int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0,t}}} \frac{\sqrt{B} |Du|^2}{A} \frac{1}{v} dx \\ & \leq \left[\sup_{d(x) \leq t''} u(x) + 2h_{x_0,t} \right] \int_{\substack{d(x) \leq t'' \\ u(x) \geq -2h_{x_0,t}}} \left(\frac{\sqrt{B} |Du| |D\rho|}{A} \frac{1}{v} + n |H| \sqrt{B} \rho \right) dx \\ & \leq \left[\sup_{d(x) \leq t''} u(x) + 2h_{x_0,t} \right] \int_{\substack{d(x) \leq t'' \\ u(x) \geq -2h_{x_0,t}}} \left(2 |D\rho| + n |H| \sqrt{B} \rho \right) dx \text{ (by ??)} \\ & \leq \left[\sup_{d(x) \leq t''} u(x) + 2h_{x_0,t} \right] \left[\frac{2}{h_{x_0,t}} + n H_{0,x_0,t''} \sqrt{B_{x_0,t''}} \right] V_{t''} \\ & \leq \left[\frac{1}{h_{x_0,t}} \sup_{d(x) \leq t''} u(x) + 2 \right] \left[2 + n h_{x_0,t} H_{0,x_0,t''} \sqrt{B_{x_0,t''}} \right] V_{t''}. \end{aligned} \quad (2.74)$$

The inequalities in (2.73) and (2.74) give us the estimate

$$\begin{aligned} & \int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0,t}}} \sqrt{B} v dx \\ & \leq \left(\sqrt{2} + 2 \left[\frac{1}{h_{x_0,t}} \sup_{d(x) \leq t''} u(x) + 2 \right] \left[2 + n h_{x_0,D} H_{0,x_0,t''} \sqrt{B_{x_0,t''}} \right] \right) V_{t''} \end{aligned}$$

or

$$\begin{aligned}
& \int_{\substack{d(x) \leq t' \\ u(x) \geq -2h_{x_0, t}}} \sqrt{B} v dx \\
& \leq C \left[1 + \frac{1}{h_{x_0, t}} \sup_{d(x) \leq t''} u(x) \right] \left[1 + nh_{x_0, t} H_{0, x_0, t''} \sqrt{B_{x_0, t''}} \right] V_{t''} \\
& \leq C \left[1 + \frac{1}{h_{x_0, t}} \sup_{d(x) \leq D''} u(x) \right] \left[1 + nh_{x_0, D} H_{0, x_0, D''} \sqrt{B_{x_0, D''}} \right] V_{t''}. \tag{2..75}
\end{aligned}$$

Combining (2..75) with (2..72), we have for each $0 < t \leq D$

$$\int_{\Sigma_t} \widehat{w} d\mu \leq C \left[1 + \frac{1}{h_{x_0, t}} \sup_{d(x) \leq D''} u(x) \right] V_{t''}, \tag{2..76}$$

where $C = C(x_0, n, D, H_{0, x_0, D''}, H_{1, x_0, D''})$. By replacing $u(x)$ with $u(x) - u(x_0)$ in (2..76), we may remove the restriction $u(x_0) = 0$. Thus, we obtain from (2..60), (2..76), and (2..39)

$$\begin{aligned}
\widehat{w}(x_0) & \leq C \frac{V_{D''}}{V_D} \left(1 + \frac{1}{h_{x_0, D}} \left(\sup_{d(x) \leq D''} u(x) - u(x_0) \right) \right) \\
& + C \int_0^D \tanh^2(t) \frac{V'_t}{(V_t)^2} \left[1 + \frac{1}{h_{x_0, t}} \left(\sup_{d(x) \leq D''} u(x) - u(x_0) \right) V_{t''} dt \right] \\
& + C \int_0^D \tanh(t) \frac{V_{t'}}{V_t} dt. \tag{2..77} \\
& = C_1 \frac{1}{h_{x_0, D}} \left(\sup_{d(x) \leq D''} u(x) - u(x_0) \right) + C_2,
\end{aligned}$$

where $C_1 = C_1(x_0, n, D'', H_{0, x_0, D''}, H_{1, x_0, D''})$ and $C_2 = C_2(x_0, n, D'', H_{0, x_0, D''}, H_{1, x_0, D''}, \sup_{d(x) \leq D''} u(x) - u(x_0))$. Since $\widehat{w}(x_0) = \ln(\sqrt{B_{x_0, D''}} v(x_0))$ and $v = \sqrt{\frac{1}{B} + \frac{1}{A} |Du|^2}$, the estimate follows.

■

We note that C_1 and C_2 above increase if $H_{0, x_0, D''}$, $H_{1, x_0, D''}$, $\sup_{d(x) \leq D''} u(x) - u(x_0)$, $\sqrt{B_{x_0, D''}}$, $A(x_0)$, $h_{x_0, D''}$, $g_{x_0, D''}$, or $\frac{V_{D''}}{V_D}$ are allowed to increase. Now, suppose

we are given a compact subset $K \subset \Omega$. Choose D so that $D'' := \sup_{x \in K} g_{x,D} + 2h_{x,D} < \text{dist}_{\mathbb{H}^n}(K, \partial\Omega)$ and small enough for Corollary 5. Let $K_{D''} = \{x : \text{dist}_{\mathbb{H}^n}(x, K) \leq D''\}$.

We may replace the first four of these quantities; $H_{0,x_0,D''}$, $H_{1,x_0,D''}$, $\sup_{d(x) \leq D''} u(x) - u(x_0)$, $\sqrt{B_{x_0,D''}}$, with $\sup_{x \in K_{D''}} |H(x)|$, $\sup_{x \in K_{D''}} |DH(x)|$, $\sup_{x \in K_{D''}} 2|u(x)|$, and $\sup_{x \in K_{D''}} \sqrt{B(x)}$, respectively. Where as the quantities $A(x_0)$, $h_{x_0,D''}$, $g_{x_0,D''}$, and $\frac{V_{D''}}{V_D}$ are replaced by their supremum as x_0 varies over K . Therefore, we have

Corollary 8 *Let $K \subset \Omega$ be compact and D , D'' , $K_{D''}$, C_1 , and C_2 be as they are described above. Then $\sup_{x \in K} |Du| < C_2 \exp C_1(\frac{u_0}{h_D})$, where $u_0 = \sup_{x \in K_{D''}} 2|u(x)|$.*

CHAPTER 3.

EXISTENCE AND NON EXISTENCE RESULTS

3.1 The Operator Q_r and a Preliminary Existence Theorem

We recall the mean curvature operator Q from (1.3). For $0 < r \leq 1$, let us define $\mathbb{D}_r^n = \{x : |x| < r, x \in \mathbb{D}^n\}$ and $\chi_r : \mathbb{D}^n \rightarrow \mathbb{D}_r^n$ by $x \mapsto rx$. Suppose $\tilde{u} \in C^2(\mathbb{D}_r^n)$ satisfies $Q\tilde{u} = nH$. Then for $x \in \mathbb{D}^n$,

$$a^{ij}(rx, D\tilde{u}(rx))\tilde{u}_{ij}(rx) + b(rx, D\tilde{u}(rx)) = nH(rx). \quad (3.1)$$

Define u on \mathbb{D}^n by $u(x) = \tilde{u} \circ \chi_r(x) = \tilde{u}(rx)$. Calculating the partial derivatives $u_i(x) = r\tilde{u}_i(rx)$ and $u_{ij}(x) = r^2\tilde{u}_{ij}(rx)$, and substituting into (3.1), we have

$$a^{ij}(rx, \frac{1}{r}Du(x))u_{ij}(x) + r^2b(rx, \frac{1}{r}Du(x)) = r^2nH(rx), \quad (3.2)$$

for $x \in \mathbb{D}^n$. Thus we have for $0 < r \leq 1$,

$$Q_r u = r^2nH \text{ on } \mathbb{D}^n \Leftrightarrow Q[u \circ \chi_r^{-1}] = nH \circ \chi_r^{-1} \text{ on } \mathbb{D}_r^n, \quad (3.3)$$

where $Q_r u = a_r^{ij}u_{ij} + r^2b_r$, $a_r^{ij}(x, p) = a^{ij}(rx, \frac{1}{r}p)$, and $b_r(x, p) = b(rx, \frac{1}{r}p)$ for $x \in \mathbb{D}^n$ and $p \in \mathbb{R}^n$. We note that Q_r is elliptic with maximum eigenvalue $\Lambda_r(x, p) = \Lambda(rx, \frac{1}{r}p)$ and minimum eigenvalue $\lambda_r(x, p) = \lambda(rx, \frac{1}{r}p)$. It is necessary (and it is the case) for our use of the interior estimates (1.8) and (??), that for a given compact set $K \subset \mathbb{D}^n \times \mathbb{R}^n$ and $0 < r_0 < 1$, there is a $c > 0$ such that

$$c \leq \lambda_r(x, p) \leq \Lambda_r(x, p) \leq \frac{1}{c} \quad (3.4)$$

all $(x, p) \in K$ and $r_0 \leq r \leq 1$.

The following theorem is a simple consequence of Theorem 1. First, let $h(r)$ denote the mean curvature (with respect to the inward pointing unit normal) of the cylinder $\partial\mathbb{D}_r^n \times \mathbb{R}$. We claim that for $0 < r < 1$,

$$1 < h(r) \quad \text{and} \quad \lim_{r \rightarrow 1^-} h(r) = 1. \quad (3.5)$$

In fact, define $\Phi : \mathbb{D}^n \times \mathbb{R} \rightarrow \mathbb{R}$ by $\Phi(x, t) = |x|$. Then the set $\{\Phi = r\}$ is the cylinder $\partial\mathbb{D}_r^n \times \mathbb{R}$. Now $\nabla\Phi = \frac{1}{A}D\Phi = \frac{1}{A}D|x|$ and $\|D|x|\| = \sqrt{A|D|x||^2} = \sqrt{A}$. Thus,

$$\begin{aligned} nh(r) &= \operatorname{div} \left[\frac{\nabla\Phi}{\|\nabla\Phi\|} \right] \\ &= \frac{1}{\sqrt{A^n B}} \left(\sqrt{A^{n-1}B} |x|_i \right)_i \\ &= \frac{\left(\sqrt{A^{n-1}B} \right)_i |x|_i}{\sqrt{A^n B}} + \frac{1}{\sqrt{A}} |x|_{ii} \\ &= \frac{D \left(\sqrt{A^{n-1}B} \right) \cdot D|x|}{\sqrt{A^n B}} + \frac{n-1}{\sqrt{A}|x|} \end{aligned} \quad (3.6)$$

Let us simplify $\frac{D(\sqrt{A^{n-1}B})}{\sqrt{A^n B}}$. Now $D \left(\sqrt{A^{n-1}B} \right) = \frac{(n-1)A^{n-2}BDA + A^{n-1}DB}{2\sqrt{A^{n-1}B}}$, so that

$$\begin{aligned} \frac{D \left(\sqrt{A^{n-1}B} \right)}{\sqrt{A^n B}} &= \frac{(n-1)A^{n-2}BDA + A^{n-1}DB}{2A^{n-1}B\sqrt{A}} \\ &= \frac{(n-1)DA}{2A^{3/2}} + \frac{DB}{2\sqrt{AB}}. \end{aligned} \quad (3.7)$$

Since $A = \frac{4}{(1-|x|^2)^2}$ and $B = A \frac{(1+|x|^2)^2}{4}$, we have $DA = 2A^{3/2}|x|D|x| = 2A^{3/2}x$ and $DB = DA \frac{(1+|x|^2)^2}{4} + A(1+|x|^2)|x|D|x| = 2\sqrt{AB}(1 + \frac{1}{\sqrt{B}})x$. Substitute this into (3.7)

to obtain

$$\frac{D \left(\sqrt{A^{n-1}B} \right)}{\sqrt{A^n B}} = \left(n + \frac{1}{\sqrt{B}} \right) x. \quad (3.8)$$

Plugging into (3.6), we have

$$nh(r) = \left(n + \frac{1}{\sqrt{B}} \right) |x| + \frac{n-1}{\sqrt{A}|x|} = (n-1) \left(\frac{1}{\sqrt{A}|x|} + |x| \right) + |x| + \frac{|x|}{\sqrt{B}}$$

$$= (n-1) \frac{1+|x|^2}{2|x|} + \frac{2|x|}{1+|x|^2}$$

or, letting $r = |x|$,

$$h(r) = \frac{(n-1)(1+r^2)}{n \cdot 2r} + \frac{1}{n} \frac{2r}{1+r^2}. \quad (3..9)$$

We express the RHS of (3..9) as $f(t) = (1-\alpha)\frac{1}{t} + \alpha t$ where $\alpha = \frac{1}{n}$ and $t = \frac{2r}{1+r^2}$. Then $f'(t) = 0 \Leftrightarrow t = \sqrt{\frac{1-\alpha}{\alpha}}$ and $f''(\sqrt{\frac{1-\alpha}{\alpha}}) > 0$. Thus, f has an absolute minimum when $t = \sqrt{\frac{1-\alpha}{\alpha}} = \sqrt{n-1} \geq 1$. Since $f(1) = 1$, (3..5) follows.

Theorem 9 *Let $H \in C^1(\overline{\mathbb{D}^n})$ satisfy $|H| < 1$ and $\phi \in C^{2,\alpha}(\overline{\mathbb{D}^n})$ for some $\alpha \in (0, 1)$.*

Then for $r \in (0, 1)$, the Dirichlet problem

$$Q_r u = r^2 n H \quad \text{and} \quad u = \phi \text{ on } \partial \mathbb{D}^n \quad (3..10)$$

has a unique solution $u \in C^{2,\alpha}(\overline{\mathbb{D}^n})$.

Proof. According to (3..5), we have for each $r \in (0, 1)$, $|H \circ \chi_r^{-1}| < h(r)$ on $\partial \mathbb{D}_r^n$.

Theorem 1 then provides us a unique function $\tilde{u} \in C^{2,\alpha}(\overline{\mathbb{D}_r^n})$ satisfying $Q \tilde{u} = n H \circ \chi_r^{-1}$ and $\tilde{u} = \phi \circ \chi_r^{-1}$ on $\partial \overline{\mathbb{D}_r^n}$. Letting $u = \tilde{u} \circ \chi_r$ and using (3..3) completes the proof. ■

3.2 Interior C^1 and Global Height Estimates

Let us define u_r to be the solution to (3..10). In order to show that equation (3..10) has a solution when $r = 1$, (which we denote by u_1), we derive bounds for $|u_r|$ and $|Du_r|$.

Lemma 10 *Let $0 < r_0, \rho < 1$. Then $\sup_{|x| \leq \rho} (|u_r| + |Du_r|) \leq C = C(\rho, r_0, \phi, H)$ for all $r \in [r_0, 1]$. Moreover, we have a global height estimate $\sup_{|x| \leq 1} |u_r| \leq C = C(\phi, H)$ for all $r \in (0, 1]$.*

Proof. Let us choose a constant H_0 so that $\sup_{x \in \mathbb{D}^n} |H| \leq H_0 < 1$ and consider the spherical cap in (1.11) with Euclidean radius $R = \frac{1}{1-H_0}$ centered at $(0, -\frac{H_0}{1-H_0})$. The pre-image of this surface under the map Ψ in (1.1) is the graph of a smooth function $f : \mathbb{D}^n \rightarrow \mathbb{R}$ satisfying $f(0) = 0$ and having constant mean curvature H_0 . Let us define $\phi_0 = \sup_{x \in \partial \mathbb{D}^n} |\phi|$ and $f_0 = \sup_{x \in \mathbb{D}^n} |f|$. Since $u_r \circ \chi_r^{-1} = \phi \circ \chi_r^{-1}$ on $\partial \mathbb{D}_r^n$, and $\sup_{x \in \partial \mathbb{D}_r^n} |\phi \circ \chi_r^{-1}| = \sup_{x \in \partial \mathbb{D}^n} |\phi|$, we have

$$-\phi_0 - f_0 + f \leq u_r \circ \chi_r^{-1} \leq \phi_0 + f_0 - f \quad (3.11)$$

on $\partial \mathbb{D}_r^n$. But $-nH_0 \leq Q(u_r \circ \chi_r^{-1}) = nH \circ \chi_r^{-1} \leq nH_0$ in \mathbb{D}_r^n , hence the inequality in (3.11) persists throughout \mathbb{D}_r^n by the comparison principle. Thus,

$$\sup_{x \in \mathbb{D}_\rho^n} |u_r| \leq \sup_{x \in \mathbb{D}^n} |u_r| = \sup_{x \in \mathbb{D}_r^n} |u_r \circ \chi_r^{-1}| \leq \phi_0 + 2f_0 \quad \forall r \in (0, 1]. \quad (3.12)$$

We now estimate $\sup_{x \in \mathbb{D}_\rho^n} |Du_r|$. Corollary 8 gives us constants $C_1^{(r)}$ and $C_2^{(r)}$ such that

$$\sup_{x \in \mathbb{D}_{r\rho}^n} |D(u_r \circ \chi_r^{-1})(x)| \leq C_2^{(r)} \exp(C_1^{(r)} \frac{u_0}{h_{D(r)}}) \quad (3.13)$$

for $r \in [r_0, 1]$. Here, $u_0 = 2(\phi_0 + 2f_0)$ and $D(r)$ is chosen (as in Corollary 8) so that $D''(r) := \sup_{x \in \mathbb{D}_{r\rho}^n} g_{x, D(r)} + 2h_{x, D(r)} < \text{dist}_{\mathbb{H}^n}(\partial \mathbb{D}_{r\rho}^n, \partial \mathbb{D}_\rho^n)$. We also insist that the numbers $D(r)$ are bounded away from 0, so that $0 < D_0 \leq D(r)$ for some D_0 and all $r \in [r_0, 1]$. The quantities that determine $C_1^{(r)}$ and $C_2^{(r)}$ increase, hence so do $C_1^{(r)}$ and $C_2^{(r)}$, as r increases, with the exception of $\sup_{x \in B(r)} |D(H \circ \chi_r^{-1})(x)|$. Here, $B(r) = \{x : \text{dist}_{\mathbb{H}^n}(x, 0) < f(r\rho) + D''(r)\}$ and $f(r\rho)$ is the hyperbolic radius corresponding to the Euclidean radius $r\rho$. In fact,

$$\sup_{x \in B(r)} |D(H \circ \chi_r^{-1})(x)| = \sup_{x \in B(r)} \frac{1}{r} |DH \circ \chi_r^{-1}(x)|$$

We do have, however,

$$\sup_{x \in B(r)} |D(H \circ \chi_r^{-1})(x)| \leq \frac{1}{r_0} \sup_{x \in B(r)} |DH \circ \chi_r^{-1}| \quad (3.14)$$

So we may replace (3.13) with

$$\sup_{x \in \mathbb{D}_{r\rho}^n} |D(u_r \circ \chi_r^{-1})(x)| \leq C_2^{(1)} \exp(C_1^{(1)} \frac{u_0}{h_{D_0}}) \quad (3.15)$$

provided we replace $\sup_{x \in \mathbb{D}_{\rho+D''(1)}^n} |D(H \circ \chi_r^{-1})(x)|$ with $\frac{1}{r_0} \sup_{r \in [r_0, 1]} \left[\sup_{x \in B(r)} |DH \circ \chi_r^{-1}| \right]$ in the constants $C_1^{(1)}$ and $C_2^{(1)}$. Thus, by (3.15), we have

$$\sup_{x \in \mathbb{D}_\rho^n} |Du_r(x)| = \sup_{x \in \mathbb{D}_{r\rho}^n} |D(u_r \circ \chi_r^{-1})(x)| \leq C_2^{(1)} \exp(C_1^{(1)} \frac{u_0}{h_{D_0}})$$

$\forall r \in [r_0, 1]$. ■

The proof of the next lemma is the same as in Lemma 10 when $r = 1$.

Lemma 11 *Given any $H \in C(\mathbb{D}^n)$ and $\phi \in C(\partial\mathbb{D}^n)$, suppose $f \in C^2(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$ satisfies $Qf \geq n|H|$ on \mathbb{D}^n and $f(0) = 0$. Then any solution $u \in C^2(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$ to the Dirichlet problem*

$$Qu = nH \quad \text{and} \quad u = \phi \quad \text{on} \quad \partial\mathbb{D}^n \quad (3.16)$$

enjoys the height estimate

$$\sup_{x \in \mathbb{D}^n} |u| \leq \phi_0 + 2f_0 \quad (3.17)$$

where $\phi_0 = \sup_{x \in \partial\mathbb{D}^n} |\phi|$ and $f_0 = \sup_{x \in \mathbb{D}^n} |f|$. Moreover, if $H \in C^1(\mathbb{D}^n)$ we have the interior estimate $\sup_{|x| \leq \rho} (|u| + |Du|) \leq C = C(\rho, \phi_0, f_0, H)$.

3.3 Barrier Construction

We now construct barriers for solutions to equation (3.10). Suppose $\sup_{x \in \mathbb{D}^n} |H| < H_0 < 1$ and let $A_{\delta,1} = \{x : \delta < |x| < 1\}$ for some arbitrary $0 < \delta < 1$. We define

$$\psi(t) = \frac{1}{k} \ln(1 + Nt), \quad (3.18)$$

where $k > 0$ and $N \geq 1$ are to be chosen, and $w(x) = \psi(1 - |x|)$. Since $|Dw| = \psi'(t) = \frac{N}{k(1+Nt)} \geq \frac{1}{2k}$, we see $|Dw| \rightarrow \infty$ (independently of N) as $k \rightarrow 0$. Allowing for $|Dw|$ to be large (k small) will be necessary. Now

$$Q_r(w + \phi) = a_r^{ij}(x, Dw + D\phi)w_{ij} + a_r^{ij}(x, Dw + D\phi)\phi_{ij} + r^2 b_r(x, Dw + D\phi)$$

and $a_r^{ij} \leq n\Lambda_r$, so that

$$Q_r(w + \phi) \leq a_r^{ij}(x, Dw + D\phi)w_{ij} + n\Lambda_r |D^2\phi| + r^2 b_r(x, Dw + D\phi).$$

From our definitions in (1.4) and (1.6), we see that

$$\Lambda_r = \beta_r = \frac{1 - r^2 |x|^2}{2 \sqrt{\frac{4}{(1+r^2|x|^2)^2} + \frac{1}{r^2} |Dw + D\phi|^2}} \leq \frac{r}{2 |Dw + D\phi|}.$$

So that if $\frac{1}{2} |Dw| \geq |D\phi|$, which we assume from this point on, then

$$\beta_r \leq \frac{r}{|Dw|}. \quad (3.19)$$

Hence

$$Q_r(w + \phi) \leq a_r^{ij}(x, Dw + D\phi)w_{ij} + r^2 b_r(x, Dw + D\phi) + \epsilon \quad (3.20)$$

$\forall r > 0$ and k sufficiently small. We note that the inequality

$$\frac{\beta_r}{r^2 \alpha_r^2} \leq \frac{4r}{|Dw|^3} = \frac{4r}{(\psi')^3} \quad (3.21)$$

follows easily from (3.19) and the definition of α_r .

Let us now consider $a_r^{ij}(x, Dw + D\phi)w_{ij} =$

$$\beta_r \left(\delta_{ij} - \frac{(w_i + \phi_i)(w_j + \phi_j)}{r^2 \alpha_r^2} \right) w_{ij} = \beta_r \Delta_E w - \frac{\beta_r}{r^2 \alpha_r^2} (w_i + \phi_i)(w_j + \phi_j) w_{ij}.$$

First, $\Delta_E w = \psi'' - \psi' \frac{(n-1)}{|x|}$, so that

$$a_r^{ij}(x, Dw + D\phi)w_{ij} = -\psi' \frac{(n-1)}{|x|} \beta_r + \psi'' \beta_r - \frac{\beta_r}{r^2 \alpha_r^2} (w_i + \phi_i)(w_j + \phi_j) w_{ij},$$

hence

$$\begin{aligned} a_r^{ij}(x, Dw + D\phi)w_{ij} &\leq -\psi' \frac{(n-1)}{|x|} \beta_r + \beta_r \left(\psi'' - \frac{1}{r^2 \alpha_r^2} w_i w_j w_{ij} \right) \\ &+ \frac{\beta_r}{r^2 \alpha_r^2} |[w_i \phi_j + \phi_i (w_j + \phi_j)] w_{ij}|. \end{aligned} \quad (3..22)$$

We will show that the last two terms tend to zero as $k \rightarrow 0$. Indeed, using (3.21), we have

$$\frac{\beta_r}{r^2 \alpha_r^2} |w_i \phi_j w_{ij}| = \frac{\beta_r}{r^2 \alpha_r^2} |\psi' \psi'' D|x| \cdot D\phi| \leq \frac{\beta_r}{r^2 \alpha_r^2} \psi' |\psi''| |D\phi| \leq \frac{4|\psi''|}{(\psi')^2} r |D\phi|, \quad (3..23)$$

$$\begin{aligned} \frac{\beta_r}{r^2 \alpha_r^2} |\phi_i \phi_j w_{ij}| &= \frac{\beta_r}{r^2 \alpha_r^2} \left| \psi'' (D|x| \cdot D\phi)^2 - \frac{\psi'}{|x|} (|D\phi|^2 - (D|x| \cdot D\phi)^2) \right| \\ &\leq \frac{\beta_r |D\phi|^2}{r^2 \alpha_r^2} \left(|\psi''| + \frac{2\psi'}{\delta} \right) \\ &\leq \frac{4r |\psi''|}{(\psi')^2} |D\phi|^2 \left(\frac{1}{\psi'} + \frac{2}{|\psi''| \delta} \right), \end{aligned} \quad (3..24)$$

and

$$\beta_r \left(\psi'' - \frac{1}{r^2 \alpha_r^2} w_i w_j w_{ij} \right) = \beta_r \left(\psi'' - \frac{1}{r^2 \alpha_r^2} (\psi')^2 \psi'' \right),$$

since $w_i w_j w_{ij} = (\psi')^2 \psi''$. Moreover,

$$\beta_r \left(\psi'' - \frac{1}{r^2 \alpha_r^2} (\psi')^2 \psi'' \right) = \frac{\beta_r \psi''}{r^2 \alpha_r^2} \left(\frac{4r^2}{(1 + r^2 |x^2|)^2} + |D\phi + Dw|^2 - |Dw|^2 \right)$$

$$\begin{aligned}
&= \frac{\beta_r \psi''}{r^2 \alpha_r^2} \left(\frac{4r^2}{(1+r^2|x^2|)^2} + |D\phi|^2 + 2D\phi \cdot Dw \right) \\
&\leq \frac{4r |\psi''|}{(\psi')^3} \left(\frac{4r^2}{(1+r^2|x^2|)^2} + |D\phi|^2 + 2|D\phi| \psi' \right) \\
&\leq \frac{4r |\psi''|}{(\psi')^2} \left(\frac{4}{\psi'} + |D\phi|^2 + 2|D\phi| \right). \tag{3..25}
\end{aligned}$$

Since $\frac{|\psi''|}{(\psi')^2} = k$, the right hand side of the inequalities (3..23), (3..24), and (3..25) vanish

as $k \rightarrow 0$. This, along with (3..20) and (3..22), gives

$$Q_r(w + \phi) \leq -\psi' \frac{(n-1)}{|x|} \beta_r + r^2 b_r(x, Dw + D\phi) + \epsilon \tag{3..26}$$

for $r_0 \leq r \leq 1$ and k sufficiently small.

Next, we will consider $r^2 b_r(x, Dw + D\phi)$. Now

$$b_r(x, Dw + D\phi) = \frac{1}{r} (Dw + D\phi) \cdot (\beta_r k_r + l_r), \tag{3..27}$$

where

$$k_r = \frac{4rx + 2nrx(1+|rx|^2)}{(1-|rx|^2)(1+|rx|^2)} \text{ and } l_r = -\frac{rx}{\alpha_r} + \frac{4rx(1-|rx|^2)}{\alpha_r^3(1+|rx|^2)^3}.$$

So

$$\begin{aligned}
&r\beta_r (Dw + D\phi) \cdot k_r = -r\beta_r \psi' D|x| \cdot k_r + r\beta_r D\phi \cdot k_r \\
&= -\frac{r^2 \psi' |x|}{\alpha_r} \left[\frac{2}{(1+|rx|^2)} + n \right] + r^2 D\phi \cdot x \frac{2+n(1+|rx|^2)}{\alpha_r(1+|rx|^2)}.
\end{aligned}$$

Thus

$$r\beta_r (Dw + D\phi) \cdot k_r \leq -\frac{r^2 \psi' |x|}{\alpha_r} \left[\frac{2}{(1+|rx|^2)} + n \right] + \epsilon, \tag{3..28}$$

for small enough k . Also,

$$r(Dw + D\phi) \cdot l_r = -r^2 \psi' D|x| \cdot x \left(-\frac{1}{\alpha_r} + \frac{4(1-|rx|^2)}{\alpha_r^3(1+|rx|^2)^3} \right) + rD\phi \cdot l_r$$

$$= \frac{r^2 \psi' |x|}{\alpha_r} - r^2 \psi' |x| \frac{4(1 - |rx|^2)}{\alpha_r^3 (1 + |rx|^2)^3} + r D\phi \cdot l_r,$$

giving us

$$r(Dw + D\phi) \cdot l_r \leq \frac{r^2 \psi' |x|}{\alpha_r} + \epsilon, \quad (3..29)$$

for small enough k . Through inequalities (3..26), (3..27), (3..28), and (3..29), we obtain

$$\begin{aligned} Q_r(w + \phi) &\leq -\psi' \frac{(n-1)}{|x|} \beta_r + \frac{r^2 \psi' |x|}{\alpha_r} - \frac{r^2 \psi' |x|}{\alpha_r} \left[\frac{2}{(1 + |rx|^2)} + n \right] + \epsilon \\ &= -\frac{\psi' (n-1)(1 - |rx|^2)}{\alpha_r 2|x|} + \frac{r^2 \psi' |x|}{\alpha_r} \left(1 - n - \frac{2}{(1 + |rx|^2)} \right) + \epsilon. \end{aligned}$$

It is the case that $\psi' \sim r\alpha_r$ as $k \rightarrow 0$, hence we have

$$\begin{aligned} Q_r(w + \phi) &\leq -\frac{r(n-1)(1 - |rx|^2)}{2|x|} + r^3 |x| \left(1 - n - \frac{2}{(1 + |rx|^2)} \right) + \epsilon \\ &= -r^2 \left[(n-1) \frac{1 + |rx|^2}{2|rx|} + \frac{2|rx|}{1 + |rx|^2} \right] + \epsilon = -r^2 n h(|rx|) + \epsilon \leq -r^2 n + \epsilon. \quad (3..30) \end{aligned}$$

Since $Q_r(u_r) > -r^2 n H_0$, we choose ϵ to satisfy $0 < \epsilon < r^2 n - r^2 n H_0$ for all $r \in [r_0, 1]$ and choose k sufficiently small to accommodate this ϵ . Use the height estimate in Lemma 10 and choose N in (3..18) sufficiently large so that

$$\psi(\delta) > \sup_{x \in \mathbb{D}^n} |u_r(x)| + \sup_{x \in \mathbb{D}^n} |\phi(x)| \quad (\forall r \in [r_0, 1]). \quad (3..31)$$

We then have $Q_r(w + \phi) \leq Q_r(u_r)$ in $A_{\delta,1}$ and $w + \phi \geq u_r$ on $\partial A_{\delta,1}$. Since it is also the case that $Q_r(w - \phi) \leq -r^2 n + \epsilon$, we have $Q_r(-w + \phi) \geq r^2 n - \epsilon$ which gives us $Q_r(w + \phi) \geq Q_r(u_r)$ in $A_{\delta,1}$ and $-w + \phi \leq u_r$ on $\partial A_{\delta,1}$. Thus by the comparison principle,

$$-w + \phi \leq u_r \leq w + \phi \text{ in } A_{\delta,1}. \quad (3..32)$$

Conclusion 12 *The solutions to (3.10) satisfy (3.32) $\forall r \in [r_0, 1]$.*

Let us now consider the case $r = 1$ above.

Lemma 13 *Suppose u solves (3.16) with $\phi \in C^2(\overline{\mathbb{D}^n})$ and $H \in C(\overline{\mathbb{D}^n})$, and assume we have a height estimate $\sup_{x \in \mathbb{D}^n} |u(x)| \leq C$. Then if $|H(x)| < 1$ for all $x \in \partial \mathbb{D}^n$,*

$$-w + \phi \leq u \leq w + \phi \text{ in } A_{\delta,1}. \quad (3.33)$$

In this case, δ depends on H .

Proof. When $r = 1$, inequality (3.30) reads

$$Q(w + \phi) \leq -n + \epsilon. \quad (3.34)$$

We choose ϵ and δ so that $n |H| < n - \epsilon$ in $\overline{A}_{\delta,1}$ and choose k sufficiently small so that (3.34) is true. Next, we choose N large enough so that instead of (3.31), we have

$$\psi(\delta) > C + \sup_{x \in \mathbb{D}^n} |\phi(x)|. \quad (3.35)$$

Then

$$-w + \phi \leq u \leq w + \phi \quad (3.36)$$

holds on $\partial A_{\delta,1}$ and therefore relation (3.33) follows from the comparison principle. ■

3.4 Existence Theorems

Theorem 14 *Let $H \in C^1(\overline{\mathbb{D}^n})$ satisfy $|H| < 1$ and $\phi \in C^{2,\alpha}(\overline{\mathbb{D}^n})$ for some $0 < \alpha < 1$.*

Then the Dirichlet problem (3.16) has a unique solution $u \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$.

Proof. We choose sequences $\{r_i\}$ and $\{\rho_i\}$ with $r_0 \leq r_i < 1$ and $0 < \rho_i < 1$ converging to 1. According to Lemma 10, the sequence $\{u_{r_i}\}$ has a uniform C^1 bound in $\overline{\mathbb{D}_{\rho_1}^n}$. The bound (3.4), the interior estimates (1.8) and (1.9), along with the Arzela-Ascoli Theorem, guarantee us a subsequence $\{u_n^{(1)}\}$ of $\{u_{r_i}\}$ that converges uniformly, along with its first and second derivatives, on $\overline{\mathbb{D}_{\rho_1}^n}$. Proceeding inductively, let $\{u_n^{(m+1)}\}$ be a subsequence of $\{u_n^{(m)}\}$ converging uniformly, along with its first and second derivatives, on $\overline{\mathbb{D}_{\rho_{m+1}}^n}$. Therefore, the "diagonal" sequence $u_n^{(n)}$ converges in the space $C^2(\mathbb{D}^n)$ to a function u . That is, $u_n^{(n)}$ along with its first and second derivatives converge on compacta. We now show u solves (3.16). In fact, define $u(x) = \phi(x)$ on $\partial\mathbb{D}^n$. Now for a given $x \in \mathbb{D}^n$, choose $\rho_j > |x|$. The continuity of Q on $C^2(\mathbb{D}^n)$ therefore gives us

$$Qu(x) = \lim_{n \rightarrow \infty} Qu_n^{(n)}(x) = \lim_{n \rightarrow \infty} Qu_n^{(j)}(x) = \lim_{n \rightarrow \infty} r_{i_n}^2 H(x) = nH(x),$$

where $\{r_{i_n}\}$ is a subsequence of $\{r_i\}$. To establish continuity at the boundary, we apply the barriers defined above. Indeed, for each n , $-w + \phi \leq u_n^{(n)} \leq w + \phi$ in $A_{\delta,1}$, hence $-w + \phi \leq u \leq w + \phi$ in $A_{\delta,1}$. Thus,

$$\lim_{x \rightarrow x_0} u(x) = \phi(x_0) = u(x_0)$$

for each $x_0 \in \partial\mathbb{D}^n$. Finally, the coefficients of Q are in $C^1(\mathbb{D}^n)$, so that $u \in C^{2,\alpha}(\mathbb{D}^n)$. ■

We wish to extend Theorem 14 to allow for $H(x) = 1$ for some points $x \in \mathbb{D}^n$.

We first need a height estimate for such a solution. Let us recall the horosphere $v(x) = \tilde{v}(|x|) = \ln(1 + |x|^2) - \ln(1 - |x|^2)$ having constant mean curvature 1, and define $u_t(x) = \tilde{u}_t(|x|) = \begin{cases} \tilde{v}(|x|) & |x| \leq t \\ p_t(|x|) & t \leq |x| \leq 1 \end{cases}$, where $p_t(|x|) = \frac{1}{2}\tilde{v}''(t)(|x| - t)^2 + \tilde{v}'(t)(|x| - t) + \tilde{v}(t)$

is the second degree Taylor polynomial centered at t for \tilde{v} . One calculates to obtain

$$\tilde{v}''(t) = 4(3t^4 + 1) / (1 - t^4)^2 \quad (3.37)$$

and

$$\tilde{v}'(t) = 4t / (1 - t^4). \quad (3.38)$$

Lemma 15 *For any $\epsilon > 0$, there is a $\delta > 0$ such that $t \in [\delta, 1]$ and $|x| \geq t \Rightarrow Qu_t(x) \geq 1 - \epsilon$.*

Proof. We begin with equation (2.47) and recall $v = \sqrt{\frac{1}{B} + \frac{1}{A} |Du|^2}$

$$= \frac{1}{\sqrt{A}} \sqrt{\frac{4}{(1 + |x|^2)^2} + |Du|^2} = \frac{1}{\sqrt{A}} \alpha,$$

where $u \in C^2(\mathbb{D}^n)$ is arbitrary. So, by using (3.8), we have

$$\begin{aligned} nH &= \frac{1}{\sqrt{A^n B}} (\sqrt{A^{n-1} B} \frac{u_i}{\alpha})_i \\ &= \frac{D\sqrt{A^{n-1} B} \cdot Du}{\sqrt{A^n B} \alpha} + \frac{1}{\sqrt{A}} \left(\frac{u_i}{\alpha} \right)_i \\ &= (n + \frac{1}{\sqrt{B}}) \frac{Du \cdot x}{\alpha} + \frac{1}{\sqrt{A}} \left(\frac{u_i}{\alpha} \right)_i. \end{aligned} \quad (3.39)$$

In the case where $u(x) = f(|x|)$, use $u_i(x) = f'(|x|) |x|_i$ and $Du(x) \cdot x = |x| f'(|x|)$ in (3.39) to obtain

$$nH = (n + \frac{1}{\sqrt{B}}) \frac{|x| f'(|x|)}{\alpha} + \frac{1}{\sqrt{A}} \left(\frac{f'(|x|) |x|_i}{\alpha} \right)_i. \quad (3.40)$$

We substitute our p_t into (3.40) and differentiate to obtain

$$nH = (n + \frac{1}{\sqrt{B}}) \frac{|x| p_t'(|x|)}{\alpha} + \frac{1}{\sqrt{A}} \frac{p_t''(|x|) |x|_i^2 + (n-1) \frac{p_t'(|x|)}{|x|}}{\alpha} - \frac{1}{\sqrt{A}} \frac{p_t'(|x|) |x|_i \alpha_i}{\alpha^2}$$

$$\begin{aligned}
&\geq \left(n + \frac{1}{\sqrt{B}}\right) \frac{|x| p'_t(|x|)}{\alpha} + \frac{1}{\sqrt{A}} \frac{|x| p''_t(|x|) + (n-1)p'_t(|x|)}{|x| \alpha} - \frac{1}{\sqrt{A}} \frac{(p'_t(|x|))^2 p''_t(|x|)}{\alpha^3} \\
&= \left(n + \frac{1}{\sqrt{B}}\right) \frac{|x| p'_t(|x|)}{\alpha} + \frac{1}{\sqrt{A}} \frac{(n-1)p'_t(|x|)}{|x| \alpha} + \frac{1}{\sqrt{A}} \left[\frac{p''_t(|x|)}{\alpha} - \frac{(p'_t(|x|))^2 p''_t(|x|)}{\alpha^3} \right], \quad (3.41)
\end{aligned}$$

where we have used

$$p'_t(|x|) |x|_i \alpha_i \leq \frac{(p'_t(|x|))^2 p''_t(|x|)}{\alpha}.$$

We now use $t \leq |x| \leq 1$ in (3.41), and note that the last term in (3.41) is positive, giving us

$$nH \geq \frac{ntp'_t(|x|)}{\sqrt{4 + (p'_t(|x|))^2}} + \frac{(1-t^2)(n-1)}{2}.$$

Now using the fact that $p'_t \rightarrow \infty$ (independently of $|x|$) as $t \rightarrow 1^-$, we have for t sufficiently close to 1,

$$nH \geq n(1 - \epsilon_1) - \epsilon_2,$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are given. ■

Lemma 16 *Suppose $H \in C(\overline{\mathbb{D}^n})$ satisfies $|H| \leq 1$ and $|H| < 1$ on $\partial\mathbb{D}^n$. Then $\exists t \in (0, 1)$ such that,*

$$Qu_t(x) \geq n |H(x)|$$

for all $x \in \overline{\mathbb{D}^n}$.

Proof. First notice that $Qu_t \equiv n$ on \mathbb{D}_t^n , since the horosphere v has mean curvature 1. Choosing $\epsilon > 0$ and $\delta > 0$ so that $|H(x)| \leq 1 - \epsilon$ when $|x| \in [\delta, 1]$, we may use Lemma 15. ■

Theorem 17 *Suppose $H \in C^1(\overline{\mathbb{D}^n})$ satisfies $|H| \leq 1$ and $|H| < 1$ on $\partial\mathbb{D}^n$, and $\phi \in C^{2,\alpha}(\overline{\mathbb{D}^n})$ for some $0 < \alpha < 1$. Then the Dirichlet problem (3..16) has a unique solution $u \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$.*

Proof. Suppose $\lambda_i \in (0, 1)$ has $\lim_{i \rightarrow \infty} \lambda_i = 1$. According to Theorem 14, there exists solutions $u_i \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$ to the equation $Qu = \lambda_i nH$. Use u_t determined (by H) in Lemma 16 in place of f in Lemma 11 to obtain an interior C^1 estimate and global height estimate for the family of solutions $\{u_i\}$. As in the proof of Theorem 14, we may pass to a subsequence and assume the u_i converge in the space $C^2(\mathbb{D}^n)$ to a function u satisfying $Qu = nH$. The global height estimate serves us in Lemma 13, giving us barriers for the u_i . Again, we proceed like we have done in the proof of Theorem 14 to establish $u \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$. ■

An interior gradient estimate allows one to prove existence given only continuous boundary data. Since we haven't established any boundary regularity (except continuity), this serves to strengthen Theorem 17. This "procedure" is standard and the proof below is outlined in [2].

Theorem 18 *Suppose $H \in C^1(\overline{\mathbb{D}^n})$ satisfies $|H| \leq 1$ and $|H| < 1$ on $\partial\mathbb{D}^n$, and $\phi \in C(\partial\mathbb{D}^n)$. Then the Dirichlet problem (3..16) has a unique solution $u \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$.*

Proof. We first choose a sequence of functions $\phi_k \in C^{2,\alpha}(\overline{\mathbb{D}^n})$ having $\sup_{x \in \overline{\mathbb{D}^n, k}} |\phi_k(x)| := \phi_0 < \infty$, that converge uniformly on $\partial\mathbb{D}^n$ to ϕ . Apply Theorem 17 to obtain functions $u_k \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$ solving (3..16) with boundary values ϕ_k . By using Lemma 11, we may pass to a subsequence and assume the u_i converge in the space $C^2(\mathbb{D}^n)$ to a

function u satisfying solving (3.16). Lemma 11 also gives us a global height estimate

$$\sup_{x \in \overline{\mathbb{D}^n, k}} |u_k(x)| < C = C(\phi_0, H). \quad (3.42)$$

To show continuity up to the boundary $\partial\mathbb{D}^n$, we must modify the proof of Lemma 13 slightly. Fix $y \in \partial\mathbb{D}^n$. For any given $\epsilon > 0$, choose $\delta, K > 0$ so that $|x - y| < \delta \Rightarrow |\phi(x) - \phi(y)| < \epsilon/2$, and $k > K \Rightarrow |\phi^k(x) - \phi(x)| < \epsilon/2$ for all $x \in \partial\mathbb{D}^n$. Define $\phi^\pm \in C^2(\overline{\mathbb{D}^n})$ by

$$\phi^\pm(x) = \phi(y) \pm \left(\epsilon + \frac{2\phi_0}{\delta^2} |x - y|^2 \right). \quad (3.43)$$

One may check that the relation $\phi^-(x) \leq \phi^k(x) \leq \phi^+(x)$ holds for each $x \in \partial\mathbb{D}^n$ when $k > K$. Now, in the proof of Lemma 13, use ϕ^+ for ϕ in (3.34). In (3.35), use $\sup_{x \in \mathbb{D}^n} |\phi^+(x)|$ in place of $\sup_{x \in \mathbb{D}^n} |\phi(x)|$ and (3.42) in place of C . Therefore, instead of (3.36), we have $-w + \phi^- \leq u_k \leq w + \phi^+$ holding on $\partial A_{\delta,1}$ (for $k > K$), and hence on all of $A_{\delta,1}$ by the comparison principle. Thus,

$$-w + \phi^- \leq u \leq w + \phi^+ \quad (3.44)$$

in $A_{\delta,1}$. Now given a sequence $x_n \in A_{\delta,1}$, (3.44) implies

$$|u(x_n) - \phi(y)| \leq \epsilon + w(x_n) + \frac{2\phi_0}{\delta^2} |x_n - y|^2,$$

which in turn implies

$$\limsup_{x_n \rightarrow y} |u(x_n) - \phi(y)| \leq \epsilon.$$

Since ϵ was arbitrary, we have established continuity at y . ■

Remark 19 *Theorem 1 remains true assuming only continuous boundary values ϕ .*

Of course, the resulting solution u will only have continuity up to the boundary.

Remark 20 From (3..39), we see that if $u \in C^2(\overline{\mathbb{D}^n})$, then $|Qu| = \left| \frac{nDu \cdot x}{\alpha} \right| \leq \frac{n|Du|}{\sqrt{1+|Du|^2}} < n$ for $x \in \partial\mathbb{D}^n$. Thus, if $|H(x)| \geq 1$ at any point $x \in \partial\mathbb{D}^n$, then (3..16) has no solution $u \in C^2(\overline{\mathbb{D}^n})$.

CHAPTER 4.
FOLIATIONS

Below, we will show that when the boundary function or the mean curvature function in Theorem 18 is replaced by a family of functions satisfying certain conditions, the resulting surfaces foliate hyperbolic space. In fact, let us consider

Case 1: Suppose that in Theorem 18, $\phi \in C(\partial\mathbb{D}^n \times (-1, 1))$, and that it satisfies $\phi(x, t_1) < \phi(x, t_2)$ when $t_1 < t_2$, $\lim_{t \rightarrow 1^-} \phi(x, t) = \infty$, and $\lim_{t \rightarrow -1^+} \phi(x, t) = -\infty$, for each $x \in \partial\mathbb{D}^n$.

Case 2: Suppose that in Theorem 18, $H \in C^1(\overline{\mathbb{D}^n} \times (-1, 1))$, and that it satisfies $|H| < 1$, $H(x, t_1) < H(x, t_2)$ when $t_1 < t_2$, $\lim_{t \rightarrow 1^-} H(x, t) = 1$, and $\lim_{t \rightarrow -1^+} H(x, t) = -1$, for each $x \in \overline{\mathbb{D}^n}$.

Theorem 21 *Suppose we have either Case 1 or Case 2 above. Then Theorem 18 provides a family of solutions $u_t \in C^{2,\alpha}(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$ with the property that for each $(x, s) \in \mathbb{D}^n \times \mathbb{R}$, $\exists! t \in (-1, 1)$ such that $u_t(x) = s$.*

Proof. Clearly the u_t exist. First, we will show that in Case 1

$$\lim_{t \rightarrow -1^+} u_t(x) = -\infty \quad \text{and} \quad \lim_{t \rightarrow 1^-} u_t(x) = \infty, \tag{4.1}$$

in Case 2

$$\lim_{t \rightarrow -1^+} u_t(x) = \infty \quad \text{and} \quad \lim_{t \rightarrow 1^-} u_t(x) = -\infty, \tag{4.2}$$

and in both cases

$$\lim_{t \rightarrow t_0} u_t(x) = u_{t_0}(x). \tag{4.3}$$

Indeed, to establish (4.1), we use Lemma 16 which guarantees the existence of a function $v \in C^2(\overline{\mathbb{D}^n})$ with $Qv \geq n|H|$. We may also assume $v \equiv 0$ on $\partial\mathbb{D}^n$. Letting $M_t = \sup_{x \in \partial\mathbb{D}^n} \phi(x, t)$, we see that $-v + M_t \geq u_t$ on $\partial\mathbb{D}^n$ and $Q(-v + M_t) \leq Q(u_t)$. Thus, by the comparison principle,

$$-v + M_t \geq u_t \tag{4.4}$$

on $\overline{\mathbb{D}^n}$. Since $M_t \rightarrow -\infty$ as $t \rightarrow -1^+$, the first part of (4.1) follows from (4.4). Likewise, letting $m_t = \inf_{x \in \partial\mathbb{D}^n} \phi(x, t)$, one sees that $v + m_t \leq u_t$ on $\overline{\mathbb{D}^n}$, from which (4.1) now follows.

Let us show (4.3) for Case 1. Given $\epsilon > 0$, the continuity of ϕ allows for a $\delta > 0$ such that $|t - t_0| < \delta \Rightarrow |\phi(x, t) - \phi(x, t_0)| < \epsilon \forall x \in \partial\mathbb{D}^n$. Therefore, whenever $|t - t_0| < \delta$, we have $u_{t_0} - \epsilon < u_t < u_{t_0} + \epsilon$ on $\partial\mathbb{D}^n$, and hence on $\overline{\mathbb{D}^n}$ by the comparison principle.

We now show (4.3) for Case 2. Choose a closed interval $I \subset (-1, 1)$ whose interior contains t_0 , and a decreasing sequence $t_n \in I$ converging to t_0 . Then there is a constant C such that $\sup_{\substack{x \in \overline{\mathbb{D}^n} \\ t \in I}} |H(x, t)| < C$ and $\sup_{\substack{x' \in \overline{\mathbb{D}^n} \\ t \in I}} |D_x H(x', t)| < C$, so that we may use Lemma 11 and Lemma 13 to obtain a subsequence of the $u_{t_n}, u_{t_{n_i}}$, converging in the space $C^2(\mathbb{D}^n)$ to a function $u \in C^2(\mathbb{D}^n) \cap C(\overline{\mathbb{D}^n})$. Therefore, $\lim_{i \rightarrow \infty} Qu_{t_{n_i}}(x) = \lim_{i \rightarrow \infty} nH(x, t_{n_i}) = nH(x, t_0)$, hence $Qu = nH(x, t_0)$. Since $u = \phi$ on $\partial\mathbb{D}^n$, the uniqueness of solutions gives us $u = u_{t_0}$. Since the sequence t_n itself is decreasing, the corresponding sequence $u_{t_n}(x)$ increases (for each $x \in \mathbb{D}^n$) by the comparison principle, showing that $\lim_{n \rightarrow \infty} u_{t_n}(x) = u_{t_0}(x)$. This proves $\lim_{t \rightarrow t_0^+} u_t(x) = u_{t_0}(x)$. Similarly, $\lim_{t \rightarrow t_0^-} u_t(x) = u_{t_0}(x)$, thus establishing (4.3) for Case 2.

Lastly, we prove (4.2). Let $h_t = \inf_{x \in \mathbb{D}^n} H(x, t)$ and $\phi_0 = \sup_{x \in \partial \mathbb{D}^n} \phi(x)$. Consider the spherical cap in (1.11) with Euclidean radius $R = \frac{1}{\sqrt{1-h_t^2}}$ centered at $\left(0, \frac{-h_t}{\sqrt{1-h_t^2}}\right)$. The pre-image of this surface under the map Ψ in (1.1) is the graph of a smooth function $f_t : \mathbb{D}^n \rightarrow \mathbb{R}$ satisfying $f_t \equiv 0$ on $\partial \mathbb{D}^n$ and having constant mean curvature h_t . Since $\lim_{t \rightarrow 1^-} \frac{1}{n} Q f_t(x) = 1$, it is not hard to see that

$$\lim_{t \rightarrow 1^-} f_t(x) = -\infty \tag{4.5}$$

for $x \in \mathbb{D}^n$. Now

$$f_t + \phi_0 \geq u_t \tag{4.6}$$

on $\partial \mathbb{D}^n$ and $Q(f_t + \phi_0) \leq Q(u_t)$ in \mathbb{D}^n , so the comparison principle gives (4.6) on all of \mathbb{D}^n . Hence, by (4.5), the second part of (4.2) now follows. Similarly, if one lets $h_t = \sup_{x \in \mathbb{D}^n} H(x, t)$ and $\phi_0 = \inf_{x \in \partial \mathbb{D}^n} \phi(x)$, then (4.5) may be replaced by $\lim_{t \rightarrow -1^+} f_t(x) = \infty$ and (4.6) may be replaced by $f_t + \phi_0 \leq u_t$, leading to the first part of (4.2).

To finish the proof, we now consider a point $(x, s) \in \mathbb{D}^n \times \mathbb{R}$. Let $g(t) = u_t(x) - s$. Then $g : (-1, 1) \rightarrow \mathbb{R}$ is continuous by (4.3). By using (4.1) or (4.2), we may employ the intermediate value theorem and find a $t_0 \in (-1, 1)$ such that $g(t_0) = 0$, i.e., $u_{t_0}(x) = s$. Finally, the uniqueness of t_0 is due to the comparison principle. In fact, the function $t \rightarrow u_t(x)$ is a strictly monotone continuous function on $(-1, 1)$ whose image is $\{x\} \times \mathbb{R}$. ■

Remark 22 *A special case of Theorem 21 is when in Case 2 we let $H(x, t) = t$. This shows that the constant mean curvature surfaces found in [β] (and discussed in [♣]), sharing a common ideal boundary Γ , where Γ is the boundary of a C^0 domain in \mathbb{R}^n , foliate hyperbolic space.*

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