On The Problem of Packing Steiner Trees of a Graph

by

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Abstract

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Let $G = (V, E)$ be an undirected graph with vertex-set V, edge-set E, and a subset of distinguished terminal vertices K of V, where $|K| \geq 2$ (the vertices in $V - K$ are called the non-terminal vertices). The degree of a vertex v of G is the number of edges incident at v in G. A K-Steiner tree T of G is a tree containing the terminal vertex-set K, and any vertex of degree one in T must belong to K. The K-edge-connectivity, denoted as $\lambda_K(G)$, of a connected graph G with terminal vertex-set K, is the minimum number of edges whose removal from G disconnects at least two vertices of K.

The Steiner Tree Packing problem (STPP for short) is the problem of finding the maximum number of edge-disjoint K-Steiner trees, $t_K(G)$, contained in G. Specifically, in this thesis, we are interested in finding a lower bound on $t_K(G)$ with respect to the K-edge-connectivity, $\lambda_K(G)$.

The Steiner Tree Packing problem has attracted considerable attention from many researchers in different areas because of its wide applicability. It has applications in routing problems arising in VLSI circuit design, where an effective way of sharing different signals among cells in a circuit can be achieved by the use of edge-disjoint Steiner trees. It also has a variety of computer network applications such as multicasting, video-conferencing, and network information flow where simultaneous communications can be facilitated by using edge-disjoint Steiner trees.

In 2003, Kriesell conjectured that any graph $G = (V, E)$ with terminal vertex-set $K \subseteq V$ has at least $\lfloor \lambda_K(G) / 2 \rfloor$ edge-disjoint K-Steiner trees. In this thesis we give a background on this problem and we present most of the major results obtained so far. Moreover, we show that Kriesell's conjecture can be answered affirmatively if the edges of G can be partitioned into K-Steiner trees. This result yields bounds for the problem of packing K-Steiner trees with certain intersection properties in a graph. In addition we show that for any graph G with terminal vertex-set K, $t_K(G) \geq \lfloor \lambda_K(G) / 2 \rfloor - |V - K| / 2 - 1$. Moreover, we study a vulnerability index called the edge-toughness, which connects the connectivity of a network with the problem of packing Steiner trees of a graph.

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Figure 5.4 A Complete bipartite graph (τ'' edge-disjoint K-Steiner trees)......................44

Chapter 1

1.1 Introduction and Preliminaries

We are concerned with undirected graphs $G = (V, E)$, with vertex-set V and edge-set E, and with a distinguished subset of vertices K, called terminal vertices. Given a graph $G = (V, E)$ with terminal vertices $K \subseteq V$, the K-edge-connectivity of G, $\lambda_K(G)$, is the minimum number of edges whose removal from G disconnect at least two vertices $s, t \in K$.

The degree of a vertex u of G, denoted by $d_G(u)$, is the number of edges incident at u. A tree is a connected graph with no cycles. A K-Steiner tree of G is a tree of G that contains all the vertices of K, and any pendant vertex (i.e., a vertex of degree 1) is also a terminal vertex. A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$; G' is a spanning subgraph of G if $V' = V$.

A path between two vertices s and t of a graph $G = (V, E)$, P, is an alternate sequence of vertices and edges $< v_1 = s, e_1, v_2, e_2, v_3 ... e_{r-1}, v_r = t >$, where all the vertices of the sequence are distinct; the length of P is $r-1$.

Let $t_K(G)$ represents the maximum number of edge-disjoint K-Steiner trees of G. For the case in which $K = \{s, t\}$, $s, t \in V$, it is a well-known fact, by Menger's theorem [1], that the maximum number of edge-disjoint $\{s, t\}$-Steiner trees is equal to $\lambda_K(G)$. At the other end, when $K = V$, a K-Steiner tree is called a spanning tree. Nash-Williams [2] and Tutte [3] independently obtained a result that implies that $t_V(G) \geq \left\lfloor \frac{\lambda_V(G)}{2} \right\rfloor$.

An important problem is to try to find the maximum number of edge-disjoint K-Steiner trees in a graph with terminal vertices K. In the current literature, the problem also is known as the Steiner trees Packing Problem. The objective of this problem is to find a set of maximum number of edge-disjoint K-Steiner trees contained in G, for a given graph $G = (V, E)$ and $K \subseteq V$.

In 2003, Kriesell [13] conjectured the following:

Conjecture 1: *For any given graph* $G = (V,E)$, *and any arbitrary* $K \subseteq V$,

$t_K(G) \geq \left\lfloor \frac{\lambda_K(G)}{2} \right\rfloor$.

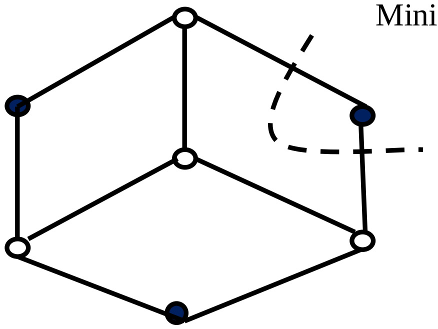
In this thesis we will discuss several important results obtained hitherto and we present new results. The Steiner trees Packing Problem has attracted considerable attention from researchers in different areas because of its wide applicability. It has applications in routing problems arising in VLSI circuit design, where an effective way of sharing different signals amongst cells in a circuit can be achieved by the use of edge-disjoint Steiner trees [19, 20]. It also has a variety of computer network applications such as multicasting, video-conferencing, and network information flow where simultaneous communications can be facilitated by using edge-disjoint Steiner trees.

For example in multicasting a sender or the source must transmit all of its data (packets) to a set of users, simultaneously. Multicasting is often used for streaming media and Internet television applications where each stream of data will be directed using different edge-disjoint Steiner trees (i.e., the number of edge-disjoint Steiner trees is equal to the maximum number of simultaneous packets that could be sent).

In our study we interested only in the case where the intermediate nodes (i.e., non-terminal nodes) in the network between the source and the users are allowed to merely store and forward information packets.

To give an intuition of the problem of how the maximum number of edge-disjoint Steiner trees is related to the *K*-edge-connectivity, consider the example depicted in Figure 1.1. Here we show a graph where the sender *s* (also called the source) could be any of the terminal vertices (black vertices) and the remaining black vertices are the receivers (the white vertices are the non-terminal vertices $V - K$). Given a graph $G = (V, E)$, $S \subseteq E$ is a *K*-edge disconnecting set of *G* if the removal of the edges of *S* from *G* disconnect at least two vertices of *K*. Let $P_G(s, K)$ be the maximum number of simultaneous packets that can be sent from the source *s* to all the designated users *K*.

Let us assume that each edge can be used only once by a packet (i.e., the capacity of each edge is one). Clearly $P_G(s, K)$ is bounded by $\min\{|S|: S \text{ is a } K\text{-edge disconnecting set of } G\}$ (i.e., $\lambda_K(G)$) (see Figure 1.1). Thus, $\lambda_K(G)$ is an upper bound on the maximum number of simultaneous packets that can be sent by the source *s*.


Minimum *K*-edge-disconnecting set = 2

$t_K(G) = 1$

Figure 1.1: Illustrates the relationship between the maximum number of edge-disjoint *K*-Steiner trees and the maximum number of simultaneous packets can be sent by the source. $\lambda_K(G) = 2$ and $t_K(G) = 1$.

Moreover $P_G(s, K) = t_K(G)$ as each packet must be sent through a distinct *K*-Steiner tree of *G*.

This thesis is organized as follows. In Chapter 2 we introduce some background and we will review major results relating $t_K(G)$ and $\lambda_K(G)$ for the case when we have two terminal vertices (i.e., $K = \{s, t\}$) and for the case where $K = V$. In particular for the case $K = V$, $t_K(G)$ and $\lambda_K(G)$ are related through a vulnerability index called the edge-toughness of G, denoted as $\eta_K(G)$. In Chapter 3, we present results relating $t_K(G)$ with $\lambda_K(G)$ for an arbitrary set of terminal vertex-set $K \subseteq V$.

In Chapter 4, we show that Kriesell's conjecture can be answered affirmatively if the edges of G can be partitioned into K-Steiner trees. This result yields bounds for the problem of packing K-Steiner trees with certain intersection properties in a graph. In addition we show that for any graph G with terminal vertex-set K, $t_K(G) \geq \lfloor \lambda_K(G) / 2 \rfloor - |V - K| / 2 - 1$.

Finally in Chapter 5, we present some new results implicating a vulnerability index of a graph called the K-edge-toughness that relates the connectivity aspects of a graph and the problem of packing K-Steiner trees. Moreover, we'll study some classes of graphs for which $t_K(G) = \lfloor \eta_K(G) \rfloor$. The importance of the last equality is that when met it implies Conjecture 1.

Chapter 2

Background and Well-known Results

2.1 Menger's Theorem: Let $K = \{s, t\}$

In 1927, Menger [1] showed that for a graph $G = (V, E)$ and vertices $s, t \in V$, the edge-connectivity $\lambda_{\{s,t\}}(G)$ is equal to the maximum number of edge-disjoint paths between s, and t (see Figure 2.1). For $K = \{s, t\}$, an $\{s, t\}$-Steiner tree is a path between s and t.

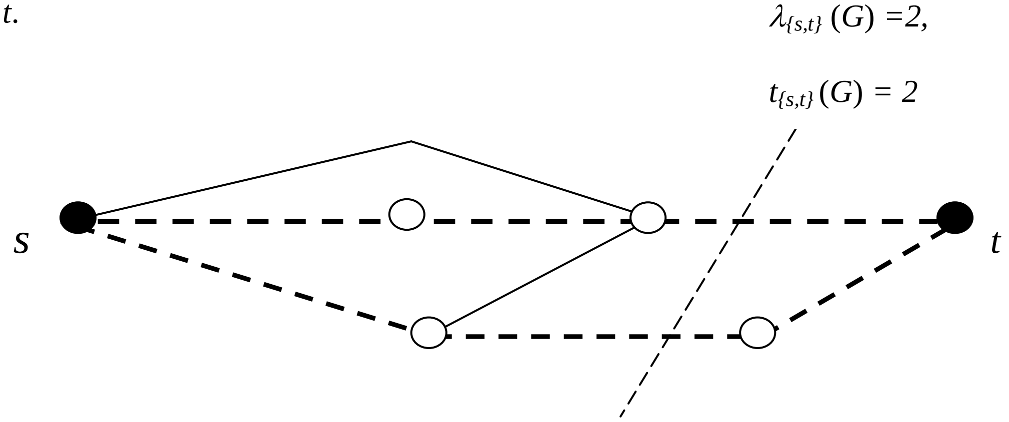


Figure 2.1: Example of the case where $K = \{s, t\}$. By Menger's theorem $t_K(G) = \lambda_{\{s, t\}}(G)$.

We can embed the result obtained by Menger [1] into the network flows problem. Let network N be modeled by a directed graph $G = (V, E)$ consisting of:

a) The arcs of G are assigned nonnegative integers, where $c(e)$ represents the capacity of the arc e.

b) Two distinguished vertices, s and t of G, called the source and sink, respectively, such that the source has only outgoing arcs and the sink has only incoming arcs.

A flow f for a network N is an assignment of an integer value $f(e)$ to each arc e that satisfies the following properties:

a) Capacity Rule: For each arc e, $0 \leq f(e) \leq c(e)$.

b) Conservation Rule: For each vertex $v \neq s, t$ $\sum_{e \in E^-(v)} f(e) = \sum_{e \in E^+(v)} f(e)$, where $E^-(v)$ and $E^+(v)$ are the incoming and outgoing arcs of v respectively.

c) The value of a flow f, denoted by $|f|$, is the total flow emanating from the source (i.e., s), which is the same as the total flow into the sink (i.e., t).

A cut $<S,T>$ of G is set of arcs emanating from a set of vertices S and converging into the set T of the remaining vertices (i.e., $T = V\text{-}S$), with the requirement that s is a vertex of S and t is a vertex of T. A minimum cut is a cut with minimum capacity among all cuts, where the capacity of a cut is the sum of the capacities of its arcs.

L.R. Ford and D. R. Fulkerson [21] proved the following:

Theorem 2.1 (Max-Flow Min-Cut Theorem)

(L. R. Ford and D. R. Fulkerson [21]) *In a transport network the value of a maximum flow is equal to the capacity of a minimum cut.*

The Max-Flow Min-Cut theorem is a statement in optimization theory about maximum flows in flow networks. Menger's theorem follows as a special case of Theorem 2.1. Consider an undirected graph $G = (V, E)$ and let G^d be the graph obtained from G by replacing each edge e of G by two antiparallel arcs. Moreover, let $N(G^d)$ be the s-t network obtained by assigning a unit capacity to each arc of G^d. Clearly, there is a one-to-one correspondence between the s-t paths of G and the directed s-t paths in G^d. Two s-t paths in graph G are edge-disjoint if and only if their corresponding directed s-t paths in G^d are also edge-disjoint. Moreover, the minimum number of edges in an s-t cut of graph G is equal to the minimum capacity of a cut of the digraph G^d, since each arc was assigned a capacity of 1.

In Figure 2.2, we show that when the minimum cut is 3, then the maximum number of edge-disjoint *s*-*t* paths is 3.

P. Elias and C. E. Shannon also proved the max-flow min-cut theorem independently in 1956 [18].

Theorem 2.2 (*P. Elias and C.E. Shannon* [*18*]) *The maximum amount of flow is equal to the capacity of a minimum cut.*

Edge-Disjoint Paths

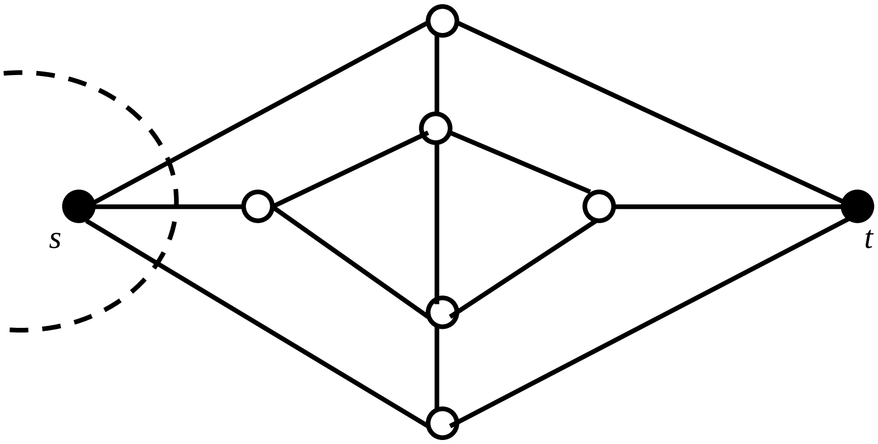
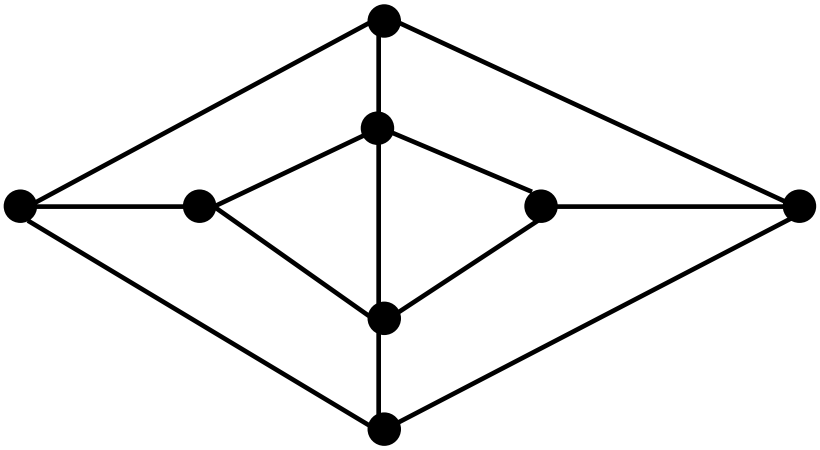


Figure 2.2: [Menger 1927] Max *s*-*t* paths = Min *s*-*t* cut, where the minimum *K*-edge disconnecting set = 3 and the maximum number of edge-disjoint *K*-Steiner trees = 3.

2.2 All Vertices are terminal vertices: Let $K = V$

The second well-known result is when $K = V$ (i.e., all the vertices are terminal vertices). For the case $K = V$, a *K*-Steiner tree of *G* is called a spanning tree. In 1961, Nash-Williams [2] and Tutte [3] independently established a formula for the size of the largest set of edge-disjoint spanning trees possible in a given graph *G*.



Spanning Tree Packing

Figure 2.3: When all the vertices are terminals, the problem is to find the maximum number of edge-disjoint spanning trees.

Nash-Williams and Tutte independently showed that a connected graph $G = (V, E)$ has m edge-disjoint connected spanning subgraphs if and only if for every subset of edges $S \subseteq E$: $m \cdot [\omega(G-S)-1] \leq |S|$, where $\omega(G-S)$ is the number of connected components obtained from G after deleting the edges of S.

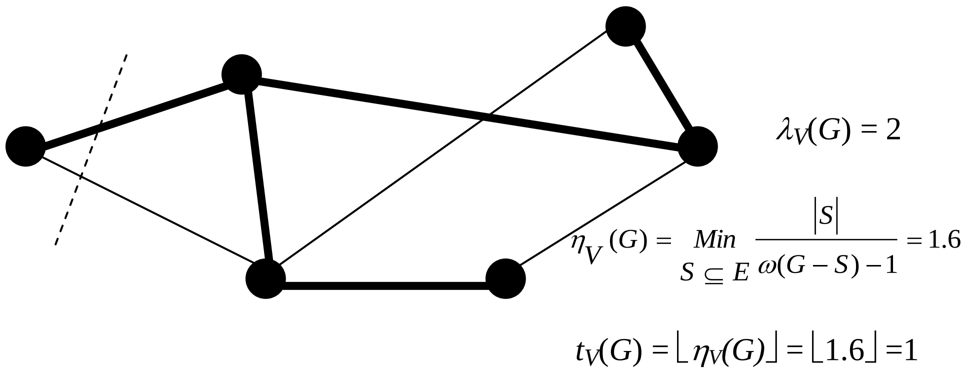
Equivalently for every disconnecting set $S \subseteq E$: $m \leq \frac{|S|}{\omega(G-S)-1}$, where $\omega(G-S) \geq 2$, thus the maximum number of edge-disjoint spanning trees is then

$$t_V(G) = \underset{S \subseteq E}{Min} \left\lfloor \frac{|S|}{\omega(G-S)-1} \right\rfloor, \qquad (2.1)$$

where S is a disconnecting set.

The vulnerability of a communication network is defined as the measurement of the global strength of its underlying graph. In 1983, Gusfield [4] introduced a vulnerability index of a graph called the edge-toughness, which follows from (2.1). The edge-toughness, $\eta_V(G)$, is defined as $\eta_V(G) = \underset{S \subseteq E}{Min} \dfrac{|S|}{\omega(G-S)-1}$. Thus, from (2.1) it follows that the maximum number of edge-disjoint Steiner trees is $\lfloor \eta_V(G) \rfloor$.

Let's say for example, we are interested in splitting a graph G into $\pi + \omega(G)$ components (where $\omega(G)$ is the number of connected components of a graph G) by the removal of edges, $\eta_V(G)$ tells us that we must remove at least $\pi \cdot \eta_V(G)$ edges from G.



$\lambda_V(G) = 2$

$\eta_V(G) = \underset{S \subseteq E}{Min} \dfrac{|S|}{\omega(G-S)-1} = 1.6$

$t_V(G) = \lfloor \eta_V(G) \rfloor = \lfloor 1.6 \rfloor = 1$

Figure 2.4: Example of the case where $K = V$ (Nash-Williams and Tutte's theorem).

Cunningham [5] generalized the concept of edge-toughness for weighted graphs, where each edge e, is assigned a positive weight $c(e)$. He called it the strength of a network and it is denoted as $\sigma(G, c)$. The weight of an edge can be viewed as the effort required to destroy the edge (i.e., to break up the connection). The strength $\sigma(G, c)$ of a network is defined as

$$\sigma(G,c) = \underset{S \subseteq E}{Min} \frac{c(S)}{\omega(G-S)-1}, \qquad (2.2)$$

where $c(S) = \sum_{e \in S} c(e)$ and $\omega(G-S)-1 > 0$.

Cunningham also found a polynomial time algorithm to compute the strength of a graph *G*, $\sigma(G, c)$, thus, in particular, he found a polynomial time algorithm to compute the toughness since $\sigma(G, 1) = \eta_V(G)$ (i.e., each edge is assigned a weight of 1).

Moreover, Cunningham generalized the concept of the edge-toughness to matroids. A matroid $M = (X, I)$ consists of a finite set *X* of elements, and a family *I* of subsets of *X*, called the independent sets, with the following properties:

1. The empty set is independent.
2. Every subset of an independent set is independent, and
3. All maximal independent sets contained in *X* have the same cardinality (called the bases of a matroid).

The rank of a subset *A* of *X*, $\rho(A)$, is the size of the largest independent set contained in *A*. Let *M* be a matroid on *X* with a rank function ρ. Define the toughness of a matroid as

$$\eta(M) = \underset{T \subset X}{Min} \; \frac{|X - T|}{\rho(X) - \rho(T)}, \tag{2.3}$$

where the minimum is taken over subsets *T* of *X* such that $\rho(T) < \rho(X)$. The following theorem is a generalization of (2.1) to matroids.

Theorem 2.3 (*Cunningham* [5]) *The maximum number of disjoint bases of a matroid is equal to* $\lfloor \eta(M) \rfloor$.

In 1989 Hobbs [22] found a polynomial time algorithm on the number of tests of independence, to determine the toughness of a matroid $\eta(M)$. Given a graph $G = (V, E)$, consider the matroid $M = (E, I)$, where the independent sets are the spanning subgraphs of *G* with no cycles (acyclic). The bases are the spanning trees of the graph *G*, and we can see that (2.3) is equivalent to the edge-toughness $\eta(G)$.

Given a graph $G = (V, E)$, one of the immediate consequences of the definition of $\eta(G)$ is that $\eta(G) \leq |E|/(n\text{-}1)$ where $n = |V|$. In [2] and [26] graphs with $\eta(G) = |E|/(n\text{-}1)$ were characterized:

Theorem 2.4 The sets of graphs G satisfying $\eta(G) = |E|/(n\text{-}1)$ includes:

a) Plane triangulations;

b) For any natural number k, the edge-disjoint union of k nontrivial spanning trees;

c) Edge-transitive graphs.

In the next chapter, we present new results that try to answer Kriesell's conjecture (see Conjecture 1 in Chapter 1) for any arbitrary set K of terminal vertices.

Chapter 3

Results for Arbitrary Terminal Set *K*

3.1 Results relating $t_K(G)$ with $\lambda_K(G)$ for arbitrary *K*

In 2000, Petingi and Rodriguez [7] presented a lower bound for $t_K(G)$, as a function of the *K*-edge-connectivity of a graph *G* and the number of vertices of *G* that do not belong to *K* (i.e., the non-terminal vertices).

In the following theorem, they established a lower bound on the maximum number of edge-disjoint Steiner trees with respect to an arbitrary subset $K \subseteq V$.

Theorem 3.1 *Given a graph $G = (V, E)$ and terminal set $K \subseteq V$, $|K| \geq 2$, then*

$$t_K(G) \geq \left\lfloor \frac{1}{2} \lambda_K(G) \cdot \left(\frac{2}{3}\right)^{|V-K|} \right\rfloor,$$

where *V-K* are the non-terminal vertices of *G*.

The proof of Theorem 3.1 was established by induction on the number of non-terminal vertices; *V-K*. When all the vertices are terminal vertices, it follows from the following theorem by Kundu [24].

Theorem 3.2 (*Kundu* [*24*]) *Let G be a graph with edge-connectivity $\lambda_V(G)$, then $t_V(G) \geq \lfloor \lambda_V(G)/2 \rfloor$.*

In 2003, Jain et al. [9] presented an algorithm for the Steiner tree Packing Problem which shows that, for a given connected graph $G = (V, E)$ with terminal set *K* ($K \subseteq V$), there are $\alpha_{|K|} . \lambda_K(G)$ edge-disjoint *K*-Steiner trees, where $\alpha_{|K|}$ is a sequence that tends to $\frac{4}{|K|}$ as $|K|$ tends to infinity. They also proved that this result is the best possible if the number of terminal vertices is 3.

Theorem 3.3 (*Jain et al.* [9]) *Let G be a connected graph with terminal vertex-set K. Then G contains* $\lfloor \alpha_{|K|} . \lambda_K(G) \rfloor$ *edge-disjoint K-Steiner trees, where* α_i *can be defined recursively by:* $\alpha_2 = 1, \quad \alpha_i = \alpha_{i-1} - \frac{\alpha_{i-1}^2}{4}$, *for i > 2.*

To show Theorem 3.3, Jain et al. combined a collection of edge-disjoint paths with *K*-Steiner trees using an iterative algorithm.

An independent set, or stable set in a graph *G* = (*V*, *E*), is a set of vertices $V' \subseteq V$, such that for every pair of vertices $u, v \in V'$, $(u, v) \notin E$ (i.e., *u* and *v* are not adjacent in *G*).

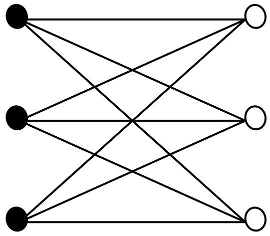


Figure 3.1: Example of an independent vertex set: the black vertices represent an independent set.

In 2003, Kriesell [13] considered the special case of a graph *G* = (*V*, *E*), and terminal vertices *K*: If *V*-*K* (i.e., the non-terminal vertices) is an independent set, and $\lambda_K(G) \geq r(r+1)$, then *G* contains *r*-edge-disjoint *K*-Steiner trees.

A stronger version of this result appeared in Frank, Király, and Kriesell [12], where they weaken the requirement for the *K*-edge connectivity to *3r*-edge-connectedness (i.e., $\lambda_K(G) \geq 3r$). That was based on a generalization of the Nash-Williams [2] and

Tutte [3] theorems applied to hypergraphs using matroid theory, where a hypergraph is a pair (*V*, *E)* where *V* is a set of elements, called nodes or vertices, and *E* is a set of subsets of *V*, called hyperedges.

Theorem 3.4 (*Frank et al.* [*12*]) *Let G* = (*V*, *E*) *be an undirected graph and* $K \subseteq V$ *be a subset of nodes so that* $V - K$ *is stable* (*independent set*) *and G is 3r-edge-connected in K* (*i.e.,* $\lambda_K(G) \geq 3r$), *then G contains r edge-disjoint K-Steiner trees.*

3.2 Most recent important result relating the maximum number of edge-disjoint *K*-Steiner trees to the *K*-edge-connectivity of *G*

The most recent important result was given by Lap Chi Lau [10] which presents the first polynomial time constant factor approximation algorithm for the Steiner tree Packing Problem. He showed that if $\lambda_K(G) \geq 26r$, then a collection of *r* edge-disjoint *K*-Steiner trees could be constructed in polynomial time.

Given an undirected multigraph *G* and a subset of vertices $K \subseteq V(G)$, the Steiner tree Packing Problem is to find a largest collection of edge-disjoint *K*-Steiner trees. Prior to Lau's work, there was not even an approximation algorithm for the Packing Steiner tree Problem. Lau closed the gap by presenting the first polynomial time constant factor approximation algorithm for this problem.

The main theorem is an approximation min-max relation between the maximum number of edge-disjoint *K*-Steiner trees and the minimum size of a disconnecting set of edges that disconnect some pair of vertices in *K*, (i.e., $\lambda_K(G)$). The techniques used to prove his theorem are purely combinatorial.

The vertices in K are the black vertices of G (which are also called the terminal vertices) while the vertices in $V - K$ are white (also known as the non-terminal vertices). An edge is white if it connects two white vertices.

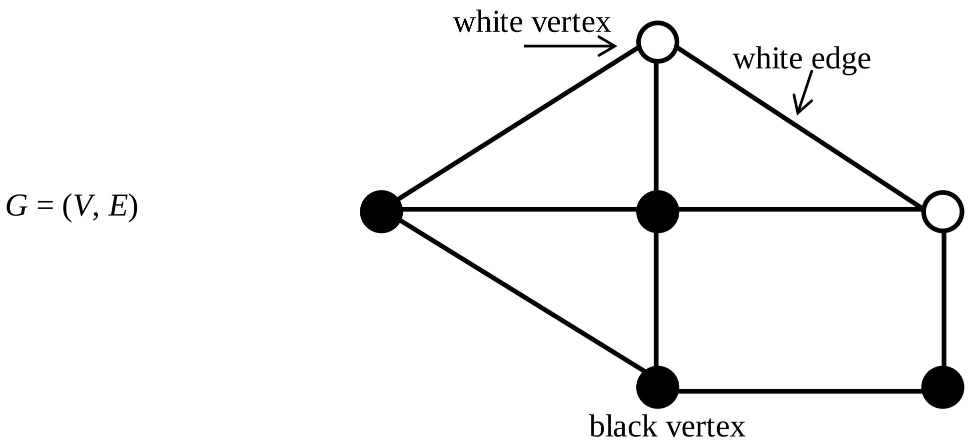


Figure 3.2: An example of Graph $G = (V, E)$, showing a white edge, black vertices K, and white vertices $V - K$.

Theorem 3.5 (*Lap Chi Lau* [*10*]) *If* $G = (V, E)$ *is* $26r$ *K-edge-connected, then G has r edge-disjoint K-Steiner trees.*

The proof of Theorem 3.5 is actually based on a new idea of graph decomposition, the edge splitting lemma by Mader [15], and the result by Frank, Király and Kriesell [12]. The proof is constructive so if G is $26r$ K-edge connected, then a collection of r edge-disjoint K-Steiner trees can be constructed in polynomial time (i.e., $t_K(G) \geq \left\lfloor \frac{\lambda_K(G)}{26} \right\rfloor$).

Theorem 3.6 (*Lap Chi Lau* [*10*]) *There is a polynomial time algorithm to construct a collection of at least* $\left\lfloor \frac{\lambda_K(G)}{26} \right\rfloor$ *edge-disjoint K-Steiner trees.*

Given an instance of the Steiner tree packing problem, Lau's method is to reduce the general case to a restrictive case where there are no white edges. The key observation is that if Theorem 3.5 holds then it holds with a rich combinatorial property, which is called the "extension property".

A small degree vertex is a black vertex whose degree is equal to $\lambda_K(G)$. The extension property says that for any edge-partition of the edges incident to a small degree vertex, the edge-partition can be extended to edge-disjoint K-Steiner trees such that each class in the edge-partition is contained in one K-Steiner tree.

If the graph G has a small degree vertex, hence the theorem holds. Otherwise we proceed to decompose the graph into two new graphs and we apply contraction to find a small degree vertex.

The proof can be divided into two steps:

1. Given a graph G with m white edges, and we search for a minimum disconnecting set in G with a white edge, and we decompose the graph G through the cut, resulting in two graphs G_1 and G_2 with a total of at most $m-1$ white edges. If there is no white edge in a minimum cut then we just remove that white edge (i.e., we can remove the white edge without decreasing the K-edge-connectivity). The cut decomposition lemma which we will present in section 3.2.2 shows that if Theorem 3.5 holds in both G_1 and G_2 with the extension property, then we can always "piece' together the solutions in G_1 and G_2 so that Theorem 3.5 holds in G with the extension property. By applying the cut decomposition lemma recursively, we remove the difficulty of having white edges, and we reduce an instance with m white edges to at most $m+1$ instances without a white edge.

2. The second step is to prove that Theorem 3.5 does hold with the extension property when there are no white edges in the graph. As explained in section 3.2.3, by using Mader's splitting lemma, we assume that every white vertex is of degree three, and this will give us a set of good paths.

3.2.1 The Setup

Given $G = (V, E)$ an undirected multigraph, with K-edge-connectivity $\lambda_K(G)$. A small vertex is a black vertex of degree $\lambda_K(G)$ in G. Let $E(u)$ be the set of edges that are incident at a vertex u, and $P_r(u) = \{E_1, E_2, \ldots, E_r\}$ is a balanced edge-subpartition of u if $E_1 \cup E_2 \cup \ldots \cup E_r \subseteq E(u)$, where $|E_i| \geq 2$ *for* $1 \leq i \leq r$ *, and* $E_i \cap E_j = 0$ *for* $i \neq j$.

We denote the set of vertices neighbors of u in E_i by $N_{Ei}(u)$. A subgraph H spans a subset of vertices U if $U \subseteq V(H)$. H is a K-subgraph of G if it is a connected subgraph of G that spans all the vertices of K, and H is a double K-subgraph of G if it is a K-subgraph and every vertex of K is of degree at least 2 in H.

The Extension Property

Given G, $K \subseteq V(G)$, and a balanced edge-subpartition $P_r(v) = \{E_1, E_2, \ldots, E_r\}$ of a small vertex v, let $\{H_1, H_2, \ldots, H_r\}$ be r edge-disjoint K-subgraphs that extend $P_r(v)$ if for $1 \leq i \leq r$:

1. $E_i \subseteq E(H_i)$.
2. $H_i - v$ is a $(K-v)$ subgraph that spans all the neighbors of v; $N_{Ei}(v)$.

Theorem 3.7 (*Lap Chi Lau* [*10*]) (*The Extension Theorem*) *If G is 26r K-edge-connected, then G has r edge-disjoint double K-subgraphs. Furthermore, for any balanced edge-subpartition* $P_r(v)$ *of any small vertex v, G has r edge-disjoint double K-subgraphs that extend* $P_r(v)$.

The proof of Theorem 3.7 is done by contradiction, where we assume that G^* is a connected graph and a counterexample to Theorem 3.7 with a "*minimum number of edges"*. Moreover, let $Q = 26$. So the plan is to show that G^* does not exist and thus Theorem 3.7 holds. The proof is divided into three parts:

1. In section 3.2.2, we show that G^* has no white edge by using the cut decomposition lemma.
2. In section 3.2.3, we show that every white vertex of G^* is of degree three by using Mader's Splitting lemma.
3. Finally, in section 3.2.4, we show that the extension property does hold in G^* and thus G^* does not exist.

3.2.2 Cut Decomposition

The idea in this section is to reduce Theorem 3.7 from the general case to a restrictive case where there is no white edge.

Lemma 3.1 (*The Cut Decomposition lemma*) *G^* has no white edge.*

Proof. Let e be a white edge. If $G^* - e$ is still Qr-K-connected, then by the choice of G^*, we get our desired edge-disjoint double K-subgraphs in $G^* - e$ and thus in G^*.

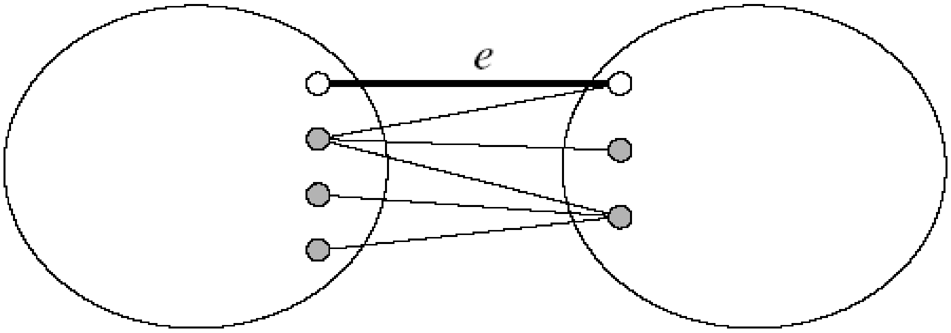


Figure 3.3: The edge e is a white edge between two white vertices ($\lambda_k(G^*) = 26r$).

Now, we consider that there is a K-cut $T = \{e_1, e_2, ..., e_{Qr}\}$ containing e. By the minimality of T, we get exactly two connected components C_1 and C_2 in $G^*\text{-}T$. Then, we construct a new multigraph G_1 by contracting C_2 to a single black vertex v_1, keeping all the edges from v_1 to C_1, (see Figure 3.4).

Similarly, we construct another new multigraph G_2 by contracting C_1 to a single black vertex v_2. Furthermore, G_1 and G_2 have fewer edges than G^*, since every component has at least a white vertex and a black vertex. Next, since G^* is still Qr-K-connected, then G_1 is Qr-K_1-connected and G_2 is Qr-K_2-connected, where K_1 = the black vertices in C_1 plus v_1, and K_2 = the black vertices in C_2 plus v_2. Therefore, by the choice of G^*, Theorem 3.7 holds in G_1 and G_2.

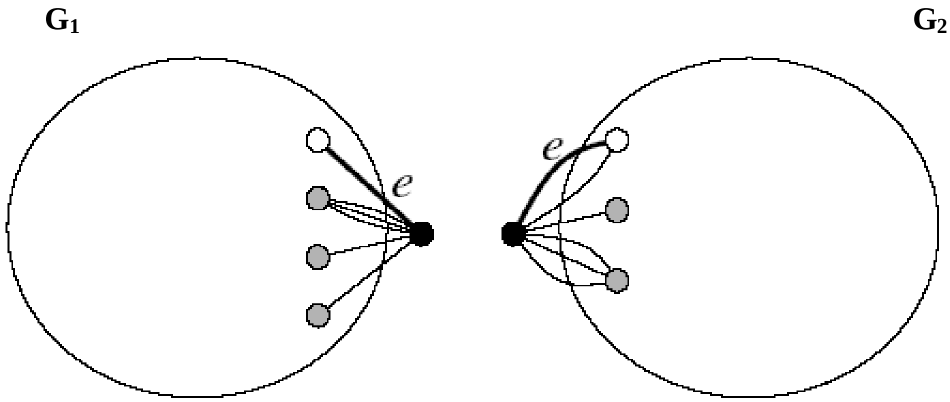


Figure 3.4: G_1 and G_2 are two connected graphs with $\lambda_K(G_1) = \lambda_K(G_2) = 26r$.

Our goal is to show that G^* has r edge-disjoint double K-subgraphs that extend $P_r(v)$. Then the plan is to combine r edge-disjoint double K_1-subgraphs in G_1 that extend $P_r(v_1)$ and r edge-disjoint double K_2-subgraphs in G_2 that extend $P_r(v_2)$, to obtain r edge-disjoint double K-subgraphs in G^* that extend $P_r(v)$.

We define a subgraph H_i of G^*, by setting $E(H_i)$ to be the union of $E(H_i^1)$ and $E(H_i^2)$. Hence, an edge of T in G_1 (or in G_2) becomes in H_i the corresponding edge in G^*.

H_i^1 and H_i^2 use exactly the same edges in T. H_i^1 and H_j^1 are edge-disjoint for $i \neq j$, H_i^2 and H_j^2 are edge-disjoint for $i \neq j$, so H_i and H_j are edge-disjoint for $i \neq j$.

Therefore, H_i is a double K-subgraph of G^*. As a result, $\{H_1, H_2, \ldots, H_r\}$ are r edge-disjoint double K-subgraphs of G^* that extend $P_r(v)$. Since v and $P_r(v)$ are picked arbitrarily this shows that Theorem 3.7 holds in G^*; a contradiction. Therefore, no counterexample and G^* has no white edge.

3.2.3 Edge Splitting

The main tool in the proof of Theorem 3.7 is Mader's [15] splitting lemma. Let G be a graph, $e_1 = (x,y)$, $e_2 = (x,z)$, be two edges where $y \neq z$. The operation of obtaining a graph $G(e_1, e_2)$ from G by deleting e_1 and e_2, and then adding exactly one new edge between y and z is called splitting at x.

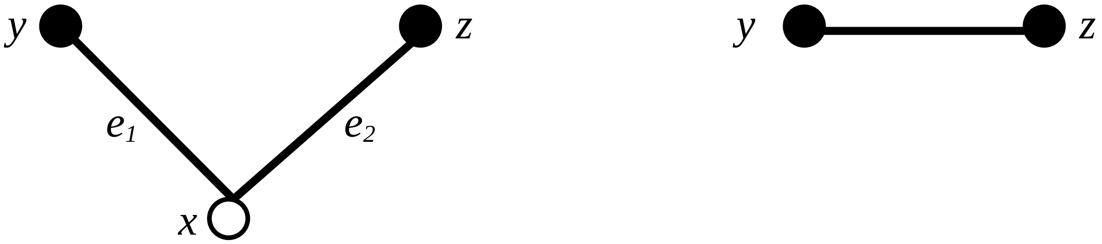


Figure 3.5: Example of Edge-Splitting

The splitting at x is called suitable if the number of edge-disjoint a, b-paths in $G(e_1, e_2)$ after deleting the two edges and replacing them by one edge, is at least the same number of edge-disjoint a, b-paths in G for every pair of vertices $a, b \in (V(G) - x)$. Moreover, if we perform a suitable splitting at a white vertex, then it will not decrease the K-edge-connectivity.

A cut-vertex is a vertex of a graph G such as its removal causes an increase in the number of the connected components of G. It can be shown that G^* does not have cut-vertex that is also a white vertex.

Lemma 3.2 (*Mader's Splitting Lemma*) [*15*] *Let x be a vertex of a connected graph G and suppose that x is not a cut-vertex and that x is incident with at least four edges and adjacent to at least two vertices. Then there is a suitable splitting of G at x.*

The following claim was shown in [10]:

Claim 3.1 *G^* does not contain any parallel edges between end-points u and v, where either u or v is a white vertex.*

Thus from Lemma 3.2 and Claim 3.1, it follows that:

Lemma 3.3 (*Lap Chi Lau* [*10*]) *Every white vertex in G^* is incident with exactly three edges and adjacent to exactly three vertices.*

3.2.4 The Extension Property

In this section we proceed to prove Theorem 3.7. When $|K| = 2$, then Theorem 3.7 follows from Menger's Theorem [1]. So, we assume that $|K| \geq 3$.

The Setup

Let v be a small vertex of G^*, and let $P_r(v) = \{E_1, E_2, \ldots, E_r\}$ be a balanced edge-subpartition of v, where $W = \{w_1, w_2, \ldots, w_\alpha\}$ be the set of white neighbors of v, and $B = \{b_1, b_2, \ldots, b_\gamma\}$ be the set of black neighbors of v. Each white vertex is incident with exactly three edges and adjacent to exactly three vertices by Lemma 3.3, and let $N_{G^*}(w_i) = \{v, x_i, y_i\}$, so the neighbors of w_i are black vertices since there is no white edge in G^*. We call $\{x_i, y_i\}$ a couple. For each black neighbor b_i of v, the weight of b_i or $c(b_i)$, is the number of multiple edges between v and b_i. Since G^* is Qr K-edge-connected, then by

Menger's Theorem, there are Qr-K-edge-disjoint paths denoted by $P(u) = \{P_1(u), P_2(u), \ldots, P_{Qr}(u)\}$, from u to v where u is a black vertex $u \neq v$. Each path in $P(u)$ uses exactly one edge in $E(v)$ since v is a small vertex.

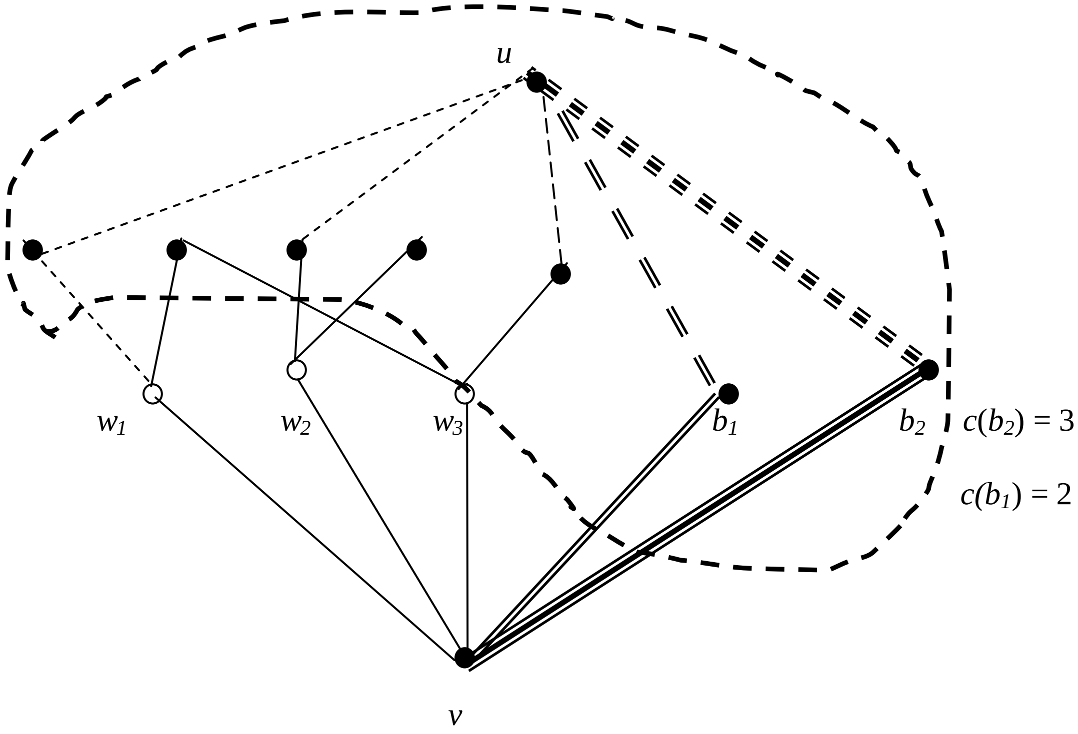


Figure 3.6: The graph of $G' = G^* - v - W$.

Let Z be a minimum (K-v)-cut of G' and $\{C_1, C_2, \ldots, C_m\}$ be the connected components of $G' - Z$. By the minimality of Z, each edge e in Z connects two vertices in different components, and we call it a crossing edge. Similarly, we call $\{x_i, y_i\}$ a crossing couple if x_i and y_i are in different components. Let X_i be the collection of couples with both vertices in the same component C_i and X_c is the collection of crossing couples.

Next we show an outline of the proof of the Extension Theorem 3.7, where G^* has no white edge and every white vertex is of degree 3 and adjacent to exactly three vertices. Moreover, Theorem 3.4 (i.e., if G^* is $3r$-K-connected then G^* have r edge-disjoint K-Steiner trees) is a critical ingredient to prove the following lemmas.

Lemma 3.4 *G' is at most $(6r-1) - (K-v)$-connected.*

If G' is $6r$ $(K-v)$-connected, then there are $2r$ edge-disjoint $(K-v)$ subgraphs of G' by Theorem 3.4. Since the union of two edge-disjoint $(K-v)$ subgraphs is a double $(K-v)$ subgraph, then there are r edge-disjoint double $(K-v)$ subgraphs in G'. In Lemma 3.4 is shown that the Extension Theorem also will be applied to G^*, then G^* will have also r edge-disjoint double K subgraphs. The $|Z|$ must be less than $6r$.

Lemma 3.5 *G'-Z has two connected components.*

Lemma 3.6 *There are at least $Qr-2|Z|$ crossing couples, that is $|X_c| \geq Qr-2|Z|$.*

By considering the edge-disjoint paths from one component to the other, the number of couples within each component and the number of crossing couples between the two components we show that $Qr = c(B_1) + |X_1| + c(B_2) + |X_2| + |X_c|$, since $c(B_1) + |X_1|$ and $c(B_2) + |X_2|$ are $\leq |Z|$. Hence, $|X_c| \geq Qr-2|Z|$.

With respect to the next lemma, we say that v_1 and v_2 have δ common paths if there are δ edge-disjoint paths starting from v_1, δ edge-disjoint paths starting from v_2, and there is a one-to-one mapping of the paths from v_1 to the paths from v_2, so each pair of the paths in the mapping ends in the same vertex.

Lemma 3.7 *If v_1 and v_2 have $2\lambda+1$ common paths in G, then there exist $\lambda+1$ edge-disjoint paths from v_1 to v_2 in G.*

Lemma 3.8 *Each component C_i $(i \in \{1,2\})$ in G'-Z is $7r$-K_i-connected.*

Consider any two black vertices $u_1, u_2 \in C_i$, by using the paths from u_1 and u_2 and using the results from Lemma 3.4 and Lemma 3.6 with the result from Lemma 3.7, we show that there are at least $7r$ edge-disjoint paths from u_1 to u_2 in C_i.

Finally, we reserve at most r edges in each component C_i and we can use those reserved edges (to be defined in Lemma 3.9) along with the (crossing) connecting edges in Z to connect the K_i-subgraphs to form r edge-disjoint double K-subgraphs of G^* that extend $P_r(v)$ " a balanced edge-subpartition"; a contradiction. Hence, we conclude that Theorem 3.7 holds in G^* and a counterexample does not exist.

Lemma 3.9 *G^* has r edge-disjoint double K-subgraphs $\{H_1, H_2, \ldots, H_r\}$ that extend $P_r(v)$.*

We pick an arbitrary min$\{r, |Z|\}$ edges in Z and call them the connecting edges. For each connecting edge e with a white endpoint w in C_i, we remove one edge e' in C_i, which is incident with w, since there are no white edges, then the other endpoint of e' must be black, and we call e' a reserve edge. By Lemma 3.7, the number of crossing edges is at least $Qr\text{-}2|Z|$. Since $P_r(v)$ is a balanced edge-subpartition $|E_i| \geq 2$ for $1 \leq i \leq r$, there are at most min$\{r, |Z|\}$ classes of $P_r(v)$ with no edges in the corresponding couples that are crossing. Hence, there is at most min$\{r, |Z|\}$ of subgraphs, let's say $\{H_1, H_2, \ldots, H_{min(r,|Z|)}\}$ that are not connected by the crossing couples. Next, by adding each connecting edge and its reserve edge (if any) to a different subgraph that has not been connected by a crossing couple, we get $\{H_1, H_2, \ldots, H_r\}$ are r-edge-disjoint K-subgraphs of G^* that extend $P_r(v)$. Since every black vertex is of degree 2, then we get our desired edge-disjoint double K-subgraphs.

Chapter 4

Our Results

We are considering undirected graphs with possible multiple edges. Let $G = (V, E)$ be an undirected graph with finite sets V and E, representing the vertices and edges of G, respectively (we also use the alternative notation $V(G)$ and $E(G)$ to refer to the vertex-set and edge-set of G, respectively). A trivial graph is a graph with just one vertex, and all other graphs are nontrivial. Let $K \subseteq V$, $|K| \geq 2$ denotes a distinguished set of vertices of G, called terminal vertices. A graph $G = (V, E)$ can be decomposed into K-Steiner trees if there exists a set $T = \{T_1, T_2, \ldots, T_r\}$ of r edge-disjoint K-Steiner trees of G such that $V = \bigcup_{i=1}^{r} V(T_i)$ and $E = \bigcup_{i=1}^{r} E(T_i)$.

In this chapter, in section 4.1, we show that Kriesell's conjecture (Conjecture 1) can be answered affirmatively if the edges of a graph G can be partitioned into K-Steiner trees. This result yields lower and upper bounds for the problem of packing K-Steiner trees with certain intersection properties in a graph G. In addition to that, in section 4.2 we show that for any graph G with terminal vertex-set K, $t_K(G) \geq \lfloor \lambda_K(G)/2 \rfloor - |V-K|/2-1$.

4.1 Decomposing a graph into *K*-Steiner trees

In this section we show that if a graph G with terminal vertex-set K can be decomposed into K-Steiner trees (i.e., G is the union of edge-disjoint K-Steiner trees), then Kriesell's conjecture can be answered affirmatively. Moreover, this result provides a lower bound and an upper bound for the problem of packing several K-Steiner trees (not necessarily edge-disjoint) of a graph G.

Consider an undirected graph G with terminal vertex-set K, and $k = |K|$. The following lemma plays an important role in our result.

Lemma 4.1 Let $T = (V(T), E(T))$ be a K-Steiner tree of a graph $G = (V, E)$, then

$$\sum_{v \in K} d_T(v) \le 2(k-1)$$

Proof. Let n be the number of vertices of T. Since the sum of the degrees of the vertices of T is $2(n\text{-}1)$, one gets

$$\sum_{v \in K} d_T(v) + \sum_{v \in V(T)-K} d_T(v) = 2(n-1) \tag{4.1}$$

But by definition of a K-Steiner tree, any non-terminal vertex in T must be of degree at least two. Thus from (4.1) we obtain

$$\sum_{v \in K} d_T(v) = 2(n-1) - \sum_{v \in V(T)-K} d_T(v) \le 2(n-1) - 2(n-k) = 2(k-1). \blacksquare$$

The above Lemma 4.1 yields the following theorem:

Theorem 4.1 *Suppose that a graph G with terminal vertex-set K can be decomposed into K-Steiner trees. Then G contains at least* $\frac{k}{k-1}\frac{\lambda_K(G)}{2}$ edge-disjoint K-Steiner trees.

Proof. Suppose that G can be decomposed into a set $\mathcal{T} = \{T_1, T_2 \ldots T_r\}$ of r edge-disjoint K-Steiner trees. As G is formed by the union of the K-Steiner trees of $\mathcal{T}$, we have that for any vertex v of G, $d_G(v) = \sum_{i=1}^{r} d_{T_i}(v)$. With regard to the sum of the degrees of the terminal vertices of G and interchanging sums, we have

$$\sum_{v \in K} d_G(v) = \sum_{v \in K}\sum_{i=1}^{r} d_{T_i}(v) = \sum_{i=1}^{r}\sum_{v \in K} d_{T_i}(v). \tag{4.2}$$

But from Lemma 4.1, the sum of the degrees of the terminal vertices of a *K*-Steiner tree is at most 2(*k*-1), thus from (4.2) we obtain

$$\sum_{v\in K} d_G(v) = \sum_{i=1}^{r}\sum_{v\in K} d_{T_i}(v) \le 2r(k-1) \tag{4.3}$$

Next we realize that $\lambda_K(G)$ is less than or equal to the degree of any terminal vertex in *G*. Thus from (4.3), it follows that $k\lambda_K(G) \le \sum_{v\in K} d_G(v) \le 2r(k-1)$. But the maximum number of edge-disjoint *K*-Steiner trees, $t_K(G)$, is at least *r*, giving

$$t_K(G) \ge \frac{k}{k-1}\frac{\lambda_K(G)}{2}\ .\blacksquare$$

Thus when *G* can be decomposed into *K*-Steiner trees, Kriesell's conjecture is true. Moreover Theorem 4.1 gives a better bound than the one stated in Conjecture 1 [13]. For example, when *K* = {*s*, *t*}, the Theorem 4.1 bound is $t_K(G) \ge \lambda_K(G)$, which is in fact tight.

For the specific case where *K* = *V*, (a *K*-Steiner tree is called a spanning tree), a vulnerability index of a graph *G*, called the edge-toughness, and denoted by $\eta(G)$, was originally introduced by Gusfield [4]. Let $\omega(G)$ represents the number of connected components of a graph *G*. For any connected graph *G* = (*V*, *E*), the edge-toughness $\eta(G)$ of *G* is defined by

$$\eta(G) = \underset{S\subseteq E}{Min} \frac{|S|}{\omega(G-S)-1}\ ,$$

over all subsets $S \subseteq E$, given that $\omega(G-S) > 1$, where $\omega(G-S)$ denotes the number of connected components when we delete the edge-set *S* from *G*.

In 1961, Nash-Williams [2] and Tutte [3] independently proved that a result which is equivalent to the following

Theorem 4.2 *For any nontrivial connected graph G, the maximum number of edge-disjoint spanning trees of G is* $\lfloor \eta(G) \rfloor$.

Furthermore, Cunningham [5] found a generalization of Theorem 4.2.

Theorem 4.3 *For any nontrivial connected graph G and any natural numbers* ***a*** *and* ***b****,* $\eta(G) \geq a/b$ *if and only if G has a family* $\mathcal{T}$ *of* ***a*** *spanning trees such that each edge of G lies in at most* ***b*** *trees of* $\mathcal{T}$.

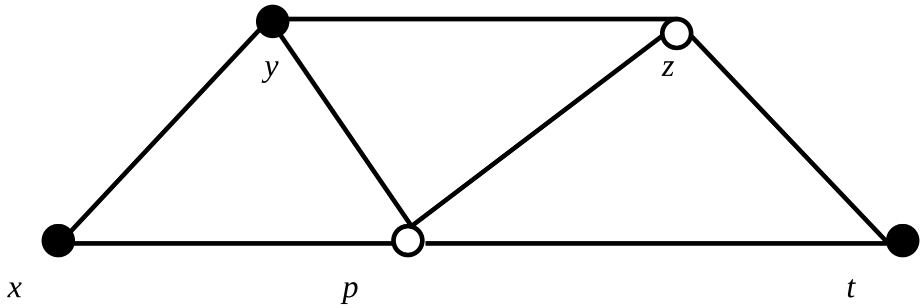
The concept of edge-toughness was generalized for any arbitrary subset $K \subseteq V$. In [6], Petingi and Rodriguez defined the *K*-edge-toughness of a connected graph *G* = (*V*, *E*) as

$$\eta_K(G) = \min_{S \subseteq E} \frac{|S|}{\omega_K(G-S)-1},$$

over all subsets $S \subseteq E$, given that $\omega_K(G-S) > 1$. In this case $\omega_K(G-S)$ represents the number of connected components that contains at least a vertex of *K* when we delete the set of edges *S* from *G*.

The following generalization of Theorem 4.3 for an arbitrary set of terminal vertices $K \subseteq V$ was proved in [7]

Theorem 4.4 *For any connected graph G* = (*V*, *E*)*, and a set* $K \subseteq V$, $|K| \geq 2$, *if G has a family* $\mathcal{T}$ *of* ***a*** *K-Steiner trees such that each edge of G lies in at most* ***b*** *K-Steiner trees of* $\mathcal{T}$, *then* $\eta_K(G) \geq a/b$.



Given an undirected graph, $G = (V, E)$.

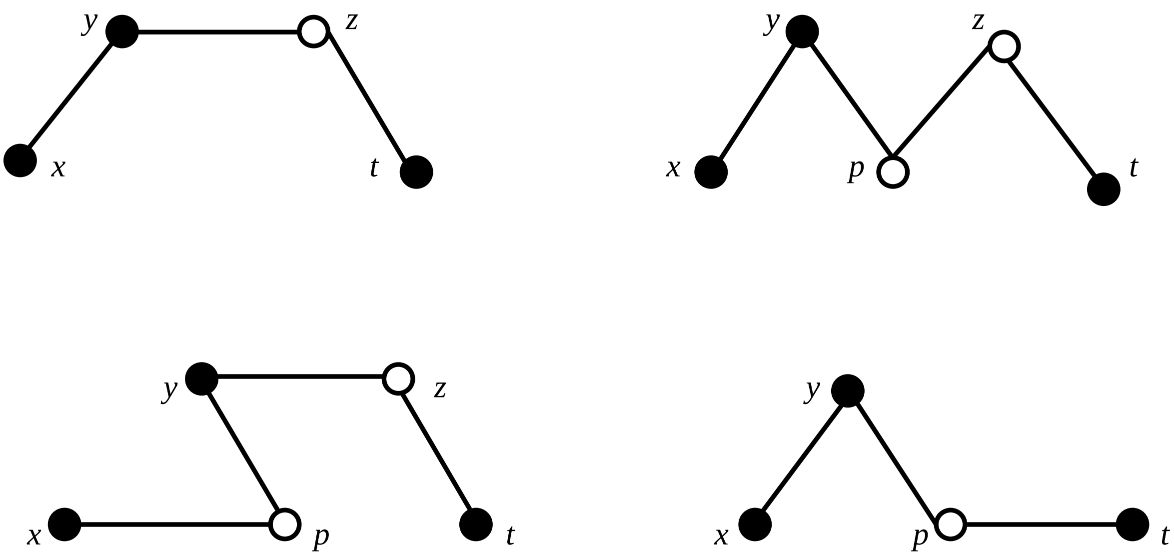


Figure 4.1: A Collection of four K-Steiner trees, not edge-disjoint, where the black vertices are the terminal vertices and the white vertices are the non-terminal vertices. In this Figure; $a = 4$ and $b = 3$.

Unlike Theorem 4.3, the converse of Theorem 4.4 is not necessarily true; that is given any natural numbers a and b such that $\eta_K(G) \geq a / b$, it is not necessary for G to have a family $\mathcal{T}$ of a Steiner trees such that each edge of G lies in at most b Steiner trees of $\mathcal{T}$. In [7] a graph G was depicted with $t_K(G) = 1$ and $\eta_K(G) = 2$.

Suppose that $\mathcal{T}$ is a family of K-Steiner trees of G (not necessarily edge-disjoint). Denote by $G_{\mathcal{T}}$ the graph obtained by replacing an edge $e = (u, v)$ of G with b parallel edges between end-points u and v, if e appears in b K-Steiner trees of $\mathcal{T}$, $b \geq 1$. We first notice that if there exists a family of a K-Steiner trees of a graph G, $\mathcal{T}$, then $a \leq \lambda_K(G_{\mathcal{T}})$. This fact together with Theorem 4.1 yields the following result

Theorem 4.5 *For any connected graph $G = (V, E)$, and a set $K \subseteq V$, $|K| \geq 2$, if G has a family $\mathcal{T}$ of* ***a*** *K-Steiner trees such that each edge of G lies in at most* ***b*** *K-Steiner trees of $\mathcal{T}$, then* $\frac{k}{k-1}\frac{\lambda_K(G_T)}{2\mathrm{b}} \leq a/b \leq \frac{\lambda_K(G_T)}{b}$.

Next let bG be the graph obtained from G by replacing each edge $e = (u, v)$ in G with b parallel edges between end-points u and v. Clearly $\lambda_K(bG) = b\lambda_K(G)$. Moreover $G_{\mathcal{T}}$ is a subgraph of bG, thus one gets

Theorem 4.6 *For any connected graph $G = (V, E)$, and a set $K \subseteq V$, $|K| \geq 2$, if G has a family $\mathcal{T}$ of* ***a*** *K-Steiner trees such that each edge of G lies in at most* ***b*** *K-Steiner trees of $\mathcal{T}$, then* $\frac{k}{k-1}\frac{\lambda_K(G_T)}{2\mathrm{b}} \leq a/b \leq \frac{\lambda_K(G_T)}{b} \leq \lambda_K(G)$.

Here we conjecture the following:

Conjecture 4.2 *For any connected graph $G = (V, E)$, and a set $K \subseteq V$, $|K| \geq 2$, for any family $\mathcal{T}$ of* ***a*** *K-Steiner trees of G such that each edge of G lies in at most* ***b*** *K-Steiner trees of $\mathcal{T}$, $t_K(\mathrm{G}) \geq \lfloor a/b \rfloor$.*

If the above conjecture can be answered affirmatively, Kriesell's conjecture is also true, since it is known that G contains a family of $\lambda_K(G)$ K-Steiner trees where each edge of G lies in at most two K-Steiner trees of G [13].

4.2 A lower bound on the maximum number of edge-disjoint *K*-Steiner trees of a graph

In [13] Kriesell proved the following result:

Theorem 4.7 *Let* $G = (V, E)$ *be a given graph with terminal vertices K, where the non-terminal vertices are of even degree. Then* $t_K(G) \geq \lfloor \lambda_K(G) / 2 \rfloor$.

In the same paper, Kriesell showed that by duplicating the edges of a subforest of G on at most $|V- K|$ edges, we can obtain a supergraph of G, where the non-terminal vertices are of even degree. From this fact it follows that $t_K(G) \geq \lfloor \lambda_K(G)/2 \rfloor - | V-K |$. In the next theorem we show that Kriesell's bound can be improved significantly.

Theorem 4.8 *For any connected graph* $G = (V, E)$*, with terminal vertex-set K,* $t_K(G) \geq \lfloor \lambda_K(G) / 2 \rfloor - |V - K| / 2 - 1$.

Proof. Let X be a minimum K-edge-disconnecting set of G (i.e., $|X| = \lambda_K(G)$). Let C_1 and C_2 be the two connected components of G-X (i.e., the graph obtained by deleting the edges of X from G). Since X is minimum K-edge-disconnecting set of G with respect to the terminal vertex-set K, C_1 and C_2 contain terminal vertices $K_1 \neq 0$ and $K_2 \neq 0$, respectively. Moreover $K = K_1 \bigcup K_2$.

In G, let O_1 and O_2 be the set of non-terminal vertices of odd degree corresponding to the non-terminal vertices of C_1 and C_2, respectively. Let P_1 represents an arbitrary collection of $\lfloor |O_1|/ 2 \rfloor$ disjoint pairs of vertices from the vertices of O_1. Similarly we construct a collection P_2 of disjoint pairs of vertices from the vertices of O_2.

Construct a graph $G' = (V', E')$ from $G = (V, E)$, such that $V = V'$ and $E' = E \cup E_1 \cup E_2$, where the edge-set $E_1 = \{(u, v): (u, v) \in P_1\}$ and $E_2 = \{(u,v) : (u,v) \in P_2\}$

(i.e., G' is obtained from G by adding edges between pairs of non-terminal vertices of odd degree in C_1, and adding edges between pairs of vertices of odd degree in C_2). If the cardinality of O_1 is odd, there exists a non-terminal vertex u, which is not in any of the pairs of P_1. In this case, add an edge $e = (u, v)$ to E_1, where $v \in K_1$. Similarly if there exists a vertex, which is not in any of the pairs of P_2, add an edge to E_2 whose end-points are this vertex and a terminal vertex of K_2 (see Figure 4.2). By construction $\lambda_K(G') = \lambda_K(G)$, thus from Theorem 4.7, one gets

$$t_K(G') \geq \lfloor \lambda_K(G') / 2 \rfloor = \lfloor \lambda_K(G) / 2 \rfloor. \tag{4.4}$$

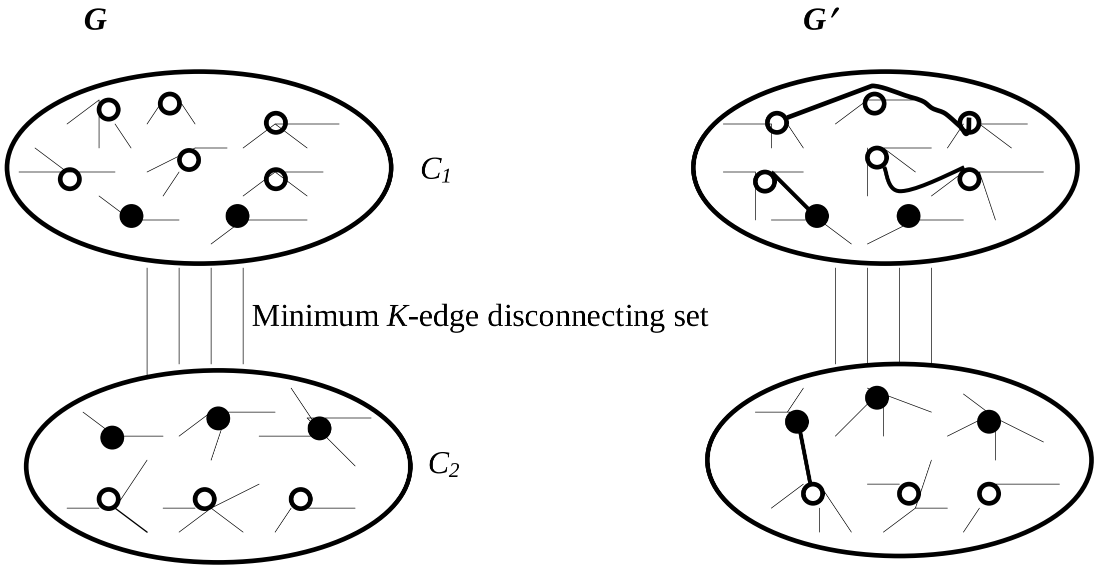


Figure 4.2: Adding edges between pairs of non-terminal vertices of odd degree.

Let $T_K(G')$ be a set of edge-disjoint K-Steiner trees of G' such that $|T_K(G')| = t_K(G')$. Moreover, let $T_K(G') = T_1 \cup T_2$, where T_1 is the set of K-Steiner trees of $T_K(G')$ that contains at least an edge of $E_1 \cup E_2$, and T_2 includes the others. But

$$|T_1| \leq |E_1| + |E_2| \leq \frac{|O_1|+1}{2} + \frac{|O_2|+1}{2} = \frac{|O_1|+|O_2|}{2} + 1. \tag{4.5}$$

As O_1 and O_2 are disjoint sets, (4.5) yields

$$|T_1| \le |E_1| + |E_2| \le \frac{|V-K|}{2} + 1. \qquad (4.6)$$

With regard to T_2, let G_{T_2} represents the graph obtained by the union of the K-Steiner trees of T_2; but the trees of T_2 are edge-disjoint, thus G_{T_2} is a subgraph of G containing the terminal vertex-set K, therefore

$$|T_2| \le t_K(G). \qquad (4.7)$$

From (4.4), (4.6), and (4.7), we can conclude that

$$t_K(G) + \frac{|V-K|}{2} + 1 \ge t_K(G') \ge \lfloor \lambda_K(G') / 2 \rfloor = \lfloor \lambda_K(G) / 2 \rfloor,$$

or equivalently, $t_K(\mathrm{G}) \ge \lfloor \lambda_K(G) / 2 \rfloor - \frac{|V-K|}{2} - 1$, thus the theorem follows. ■

As previously mentioned in Chapter 2, Nash-Williams and Tutte independently showed a result, which is equivalent to stating that $t_K(G) = \lfloor \eta_K(G) \rfloor$ for the particular case $K = V$. In the next chapter we will try to connect the maximum number of edge-disjoint Steiner trees $t_K(G)$ to the K-edge-toughness $\eta_K(G)$, for arbitrary terminal set K.

Chapter 5

Edge-Toughness and Packing *K*-Steiner Trees

5.1 Edge-Toughness

A vulnerability index of a graph G was introduced by Gusfield [4], called the edge-toughness, and denoted by $\eta(G)$. The edge-toughness tells us that in order to split a graph into $k + \omega(G)$, where $\omega(G)$ represents the number of connected components of G, we must remove at least $k\eta(G)$ edges from G; thus $\eta_V(G)$ measures how tough it is to break up G. The edge-toughness is defined as follows

$$\eta_V(G) = \underset{S \subseteq E}{Min} \frac{|S|}{\omega(G-S)-1}, \tag{5.1}$$

where the minimum is taken over every subset $S \subseteq E$ that separates G into $\omega(G-S)$ components, given that $\omega(G-S) > 1$. Moreover in 1961, Nash-Williams [2] and Tutte [3] independently proved the following result connecting the edge-toughness with the problem of packing spanning trees,

Theorem 5.1 *For any nontrivial connected graph G, the maximum number of edge-disjoint spanning trees of G is* $\lfloor \eta_V(G) \rfloor$.

In 1985, Cunningham [5] found a generalization of Theorem 5.1, implicating the spanning trees of a graph.

Theorem 5.2 *For any nontrivial graph G and any natural numbers **a** and **b**,* $\eta(G) \geq a / b$ *if and only if G has a family* τ *of **a** spanning trees, known as a t-packing, such that each edge of G lies in at most **b** trees of* τ.

In 1997 Petingi and Rodriguez [6] proposed a generalized edge-toughness index of a graph G, $\eta_K(G)$, this index tells us how tough is to break up a communication between the vertices of an arbitrary set $K \subseteq V$, $|K| \geq 2$. Moreover, they showed that some of the properties of edge-toughness for the particular case $K = V$ are extended to any arbitrary subset $K \subseteq V$.

Given a connected graph $G = (V, E)$ and a set $K \subseteq V$, $|K| \geq 2$, the edge-toughness is defined as $\eta_K(G) = \underset{S \subseteq E}{Min} \frac{|S|}{\omega_K(G-S)-1}$, where $\omega_K(G\text{-}S)$ represents the number of connected components that contains at least a vertex of K (i.e., a terminal vertex) when we delete the set $S \subseteq E$ from G, given that $\omega_K(G\text{-}S) > 1$. Petingi and Rodriguez [6] generalized Theorem 5.2 for an arbitrary subset $K \subseteq V$, which implicates this time the K-edge-toughness with the packing of a set of K-Steiner trees.

Theorem 5.3 *For any nontrivial graph* $G = (V, E)$, *a set* $K \subseteq V$, $|K| \geq 2$, *and any natural numbers* ***a*** *and* ***b***, $\eta_K(G) \geq a / b$ *if* G *has a family* τ *of* ***a*** *K-Steiner trees, such that each edge of G lies in at most* ***b*** *K- Steiner trees of* τ.

Unlike Theorem 5.2, the converse of Theorem 5.3 is not necessarily true; that is given any natural numbers a and b such that $\eta_K(G) \geq a / b$, it is not necessary for G to have a family τ of a Steiner trees, such that each edge of G lies in at most b Steiner trees of τ. In Figure 5.1, a graph G is depicted with $t_K(G) = 1$ and $\eta_K(G) = 2$ (i.e., in this case $a = 2$ and $b = 1$).

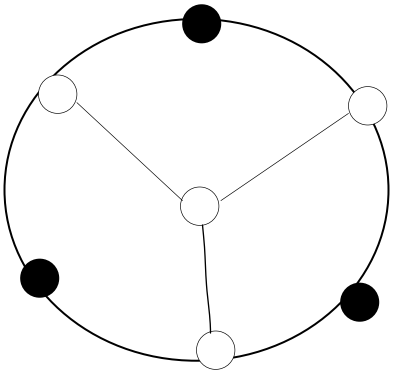


Figure 5.1: Graph G with $t_K(G) < \lfloor \eta_K(G) \rfloor$ (black vertices belong to K).

5.2 The complexity of determining the Edge-Toughness for arbitrary *K*

Let $t_V(G)$ be the maximum number of edge-disjoint spanning trees in a graph G. It follows from Theorem 5.1 that a connected graph G has $t_V(G)$ edge-disjoint spanning trees if and only if $\eta_V(G) \geq t_V(G)$.

The area of graph vulnerability is concerned with the question of how much communications are disrupted by the deletion of some edges or vertices from a graph. Indeed, the most fundamental measure of graph vulnerability of a connected graph is the edge-connectivity (i.e., the minimum number of edges whose deletion from graph G disconnects it).

One of the motivations in studying the edge-toughness of a graph is that it can be used as a more refined measure of graph vulnerability than the one based on the edge-connectivity. The edge-toughness is a more significant measurement in comparing the vulnerability of two graphs when they have the same edge-connectivity.

For an arbitrary K, from the definition of $\eta_K(G)$, it follows that $\eta_K(G) \leq \lambda_K(G)$. Moreover, we can establish a lower bound on $\eta_K(G)$ as stated in the following lemma,

Lemma 5.1 *For any given undirected graph G* = (*V*, *E*), $\frac{\lambda_K(G)}{2} < \eta_K(G)$.

Proof. Given a graph $G = (V, E)$, let S be a set of edges of G (see Figure 5.2) such that

$$\eta_K(G) = \frac{|S|}{\omega_K(G-S)-1}. \qquad (5.2)$$

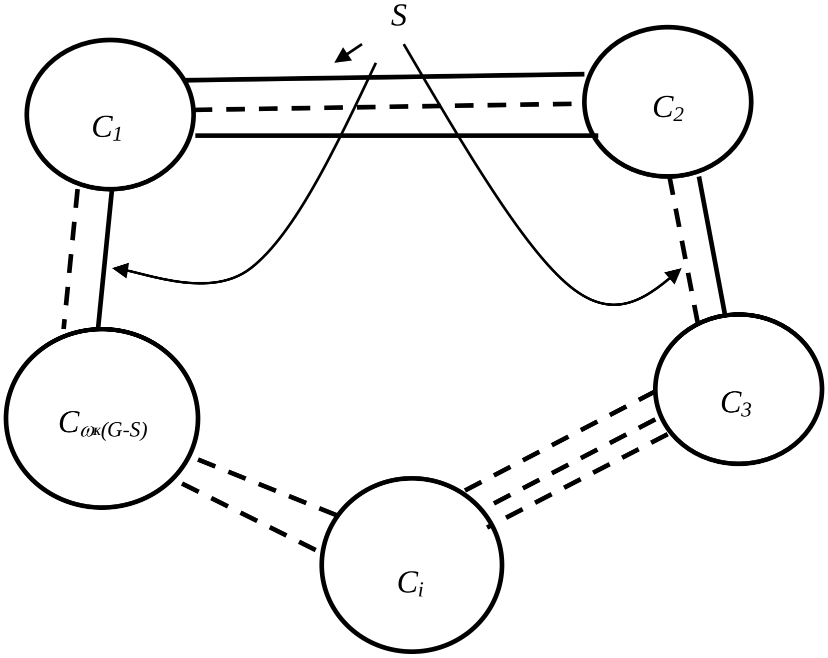


Figure 5.2: A Connected graph.

Moreover, it is easy to prove that each component C_i, $1 \leq i \leq \omega_K(G\text{-}S)$, contains at least one terminal vertex. Let $d(C_i)$ be the number of edges emanating from C_i to any other component C_j ($i \neq j$). Since each component contains at least one terminal vertex, then

$$d(C_i) \geq \lambda_K(G), \text{ for } 1 \leq i \leq \omega_K(G\text{-}S), \qquad (5.3)$$

since the edges incident at C_i disconnect at least two terminal vertices in G.

Moreover, $$|S| = \frac{\sum_{i=1}^{\omega_K(G-S)} d(C_i)}{2}. \qquad (5.4)$$

Thus from (5.3) and (5.4) one gets

$$\omega_K(G-S)\cdot\frac{\lambda_K(G)}{2}\le\frac{\sum_{i=1}^{\omega_K(G-S)} d(C_i)}{2}\ . \tag{5.5}$$

It follows then from (5.2), (5.4) and (5.5) that

$$\eta_K(G)\ge\frac{\omega_K(G-S)}{\omega_K(G-S)-1}\cdot\frac{\lambda_K(G)}{2}, \tag{5.6}$$

therefore $\eta_K(G)>\frac{\lambda_K(G)}{2}$. ■

The definition of $\eta_K(G)$ and the lower bound on $\eta_K(G)$ established in Lemma 5.1, yield the following computational result

Theorem 5.4 *There is a 2-approximation algorithm to determine* $\eta_K(G)$.

Proof. From the definition of *K*-edge-toughness and Lemma 5.1 we can establish the inequality

$$\frac{\lambda_K(G)}{2}<\eta_K(G)\le\lambda_K(G). \tag{5.7}$$

We first prove that $\lambda_K(G)=\underset{\{u,v\}\subseteq K}{Min}\,\lambda_{\{u,v\}}(G)$. For any $\{u, v\}\subseteq K$, $\lambda_{\{u,v\}}(G)\ge\lambda_K(G)$ since a min$\{u, v\}$-disconnecting set is also a *K*-disconnecting set in *G*. Conversely, let *X* be a *K*-disconnecting set of *G* such that $|X| = \lambda_K(G)$. Since the cardinality of *X* is minimum, then *G*-*X* has exactly two connected components C_1 and C_2, where each component contains at least one terminal vertex. Let *u* and *v* be (two terminals) such that *u* is a vertex *in* C_1 and *v* is a vertex in C_2. But *X* is also an $\{u,v\}$-disconnecting set, thus $\lambda_{\{u,v\}}(G)\le|X|=\lambda_K(G)$. Then, from the previous facts, we have $\lambda_K(G)=\lambda_{\{u,v\}}(G)$ for some $\{u, v\}\subseteq K$. Thus

$$\lambda_K(G) = \underset{\{u,v\}\subseteq K}{Min}\, \lambda_{\{u,v\}}(G)\,. \tag{5.8}$$

For any graph $G = (V, E)$ with n vertices and m edges, we can compute $\lambda_{\{u,v\}}(G)$ for any $\{u, v\} \subseteq K$ in $O(nm)$ by the application of Max-Flow Min-Cut Theorem (see[25], page 119) as stated in section 2.1. Since there are at most $\binom{n}{2}$ pairs of terminal vertices, we can then determine $\lambda_K(G)$ in $O(n^3m)$. But from the definition of $\eta_K(G)$ and Lemma 5.1, $\lambda_K(G)/2 < \eta_K(G) \leq \lambda_K(G)$, then there exists a 2-approximation algorithm to determine $\eta_K(G)$ in terms of $\lambda_K(G)$. ■

As discussed in section 2.2, $\eta_K(G)$ can be determined in polynomial time for the particular cases, $K = V$ and $|K| = 2$. For arbitrary terminal vertex-set K, the computational complexity for determining the K-edge-toughness remains an open problem.

5.3 Classes of graphs for which $t_K(G) = \lfloor \eta_K(G) \rfloor$

As mentioned in Section 2.1 and Section 2.2, for the particular cases $K = V$ (all vertices are terminals) and $|K| = 2$, $t_K(G) = \lfloor \eta_K(G) \rfloor$. Even though for any arbitrary terminal vertex set K, it is not true that $t_K(G) = \lfloor \eta_K(G) \rfloor$ (in Figure 5.1 a graph G is depicted in which $t_K(G) < \lfloor \eta_K(G) \rfloor$), in this section we determine some classes of graphs for which $t_K(G) = \lfloor \eta_K(G) \rfloor$.

We know that $t_K(G) \leq \eta_K(G)$ for any graph $G = (V, E)$ with terminal vertex set K. If a graph G has $t_K(G) = \lfloor \eta_K(G) \rfloor$ then from Lemma 5.1, we have $t_K(G) = \lfloor \eta_K(G) \rfloor \geq \lfloor \lambda_K(G)/2 \rfloor$, thus Conjecture 1 (Kriesell's conjecture, see Section 1.1) is true for G.

In the following theorem (see [6]) for a graph $G = (V, E)$ and a subset $W \subseteq V$, we denote $G[W]$ the subgraph of G induced by the vertex set W.

Theorem 5.5 *If $G = (V, E)$ is the complete graph on $n = |V|$ vertices (denoted as K_n), then for any set $K \subseteq V, |K| \geq 2$, the maximum number of edge-disjoint K-Steiner trees of G is $\lfloor \eta_K(G) \rfloor$.*

Proof. Let $K = \{k_1, k_2, \ldots, k_{|K|}\}$ and let $S \subseteq E$ be a disconnecting set of G such that G-S is composed of connected components $C_1 = (V_1, E_1)$, $C_2 = (V_2, E_2)$, …, $C_{|K|} = (V_{|K|}, E_{|K|})$, with $V_i = \{k_i\}$, $E_i = 0$, for each i, $2 \leq i \leq |K|$, and $C_1 = [k_1 \cup (V- K)]$.

From the definition of $\eta_K(G)$ and (5.2) we obtain

$$\eta_K(G) \leq \frac{|S|}{\omega_K(G-S)-1} = \frac{|S|}{|K|-1}. \tag{5.9}$$

The disconnecting set

$$S = E[K - k_1] \cup E^*, \tag{5.10}$$

where $E\,[K\text{-}k_1]$ is the edge set of $G[K\text{-}k_1]$ and $E^* = \{(x, y) : x \in ((V\text{-}K) \cup k_1), y \in (V\text{-}k_1)\}$.

Clearly $|E\,[K\text{-}k_1]| = \binom{|K|-1}{2}$ and $|E^*| = (n\text{-}|K|+1)(|K|\text{-} 1)$.

So from (5.9) and (5.10) we conclude

$$\eta_K(G) \leq n - (|K| / 2). \tag{5.11}$$

Furthermore, consider the subgraph of G induced by K, $G[K] = K_{|K|}$ (the complete graph on $|K|$ vertices). Any spanning tree of $G[K]$ is K-Steiner tree of G. Since $G[K]$ (complete graph) is edge-transitive, it follows from Theorem 2.4 (see page 11) that $\eta(G[K]) = (|K||K\ \text{-}1|) / (2 |K\ \text{-}1|) = |K| / 2$. Then if we denote T_1 a set with the maximum number of edge-disjoint spanning trees of $G[K]$ then

$$|T_1| = \lfloor \eta(G[K]) \rfloor = \lfloor |K| / 2 \rfloor. \tag{5.12}$$

Moreover let $V\text{-}K = \{v_1, v_2, \ldots, v_{|V-K|}\}$ and let $T_2 = \{t_1, t_2, \ldots, t_{|V\text{-}K|}\}$be a set of K-Steiner trees of G such that for each i, $1 \le i \le |V\text{–}K|$, and let $\{v_i \cup K\}$ and $\{\bigcup_{j=1}^{|K|}(v_i, k_i)\}$ represent the vertex and edge sets of t_i, respectively. Clearly T_2 is a family of edge-disjoint K-Steiner trees of G. By construction $T_1 \cap T_2 = \varnothing$, and if $T(G)$ is a family of edge-disjoint K-Steiner trees of G, such that it has the maximum number of edge-disjoint K-Steiner trees of G, then $|T(G)| \ge |T_1| + |T_2| = \lfloor |K| / 2 \rfloor + n - |K|$.

Using this fact and given that $T(G)$ is a family of edge-disjoint K-Steiner trees of G, thus any edge of G lies in at most one K-Steiner trees of $T(G)$, from Theorem 5.3 and (5.11) we obtain

$$\lfloor |K| / 2 \rfloor + n - |K| \le |T(G)| \le \eta_K(G) \le n - |K| / 2, \tag{5.13}$$

but $|T(G)|$ is an integer, thus

$$\lfloor |K| / 2 \rfloor + n - |K| \le |T(G)| \le \lfloor \eta_K(G) \rfloor \le \lfloor n - |K| / 2 \rfloor. \tag{5.14}$$

Using the fact that $\lfloor |K| / 2 \rfloor + n - |K| = \lfloor n - |K| / 2 \rfloor$ and (5.13) we conclude that

$$|T(G)| = \lfloor \eta_K(G) \rfloor. \blacksquare \tag{5.15}$$

A graph $G = (V, E)$ is called bipartite if the vertex-set V can be partitioned into two sets of vertices V_1 and V_2 (i.e., $V = V_1 \cup V_2$) and every edge of G has an endpoint in V_1 and the other endpoint in V_2. If $|V_1| = n$ and $|V_2| = m$ and every vertex of V_1 is adjacent to every vertex of V_2 then we say that G is a complete bipartite graph and it is denoted as $K_{n,m}$.

Suppose that K can be partitioned into two sets of terminal vertices K_1 and K_2 with corresponding cardinalities k_1 and k_2 respectively, and where each of these sets belongs to one of two classes of the bipartition of $K_{n,m}$. Without loss of generality we'll assume that $k_1 \geq k_2$.

In the next theorem we show that if $G = (V, E)$ is a complete bipartite graph where the classes of bipartition of V have the same number of nodes, then $t_K(G) = \lfloor \eta_K(G) \rfloor$, whenever $k_2 (k_2 - 1) < (k_1 - 1)$.

Theorem 5.6 Let $G = (V, E)$ be a complete bipartite $K_{n,n}$ with terminal vertex-set $K \subseteq V$, then $t_K (G) = \lfloor \eta_K(G) \rfloor$, whenever $k_2 (k_2 - 1) < (k_1 - 1)$.

Proof. Let V_1 and V_2 be the classes of the bipartition of V with $|V_1| = n$ and $|V_2| = n$.

Let $K_1 = \{v_1, v_2, \ldots, v_r\}$ and $K_2 = \{u_1, u_2, \ldots, u_p\}$ the terminal vertex-sets of V_1 and V_2 respectively, and where $r = k_1 = |K_1|$, and $p = k_2 = |K_2|$.

Suppose without loss of generality that $k_2 \leq k_1$. We will consider the following two K-disconnecting sets:

Case 1: Let $S_1 \subseteq E$ be a K-disconnecting set of G in which G-S_1 is composed of $k_1 + k_2$ components. Moreover let one connected component be $G[v_r \cup (V_1 - K_1) \cup (V_2 - K_2)]$ (i.e., the graph induced by one terminal of K_1 and the non-terminal vertices) and the remaining $k_1 + k_2$ -1 components are the terminal vertices themselves. It is easy to see that $G[v_r \cup (V_1 - K_1) \cup (V_2 - K_2)]$ is complete bipartite graph. Consequently S_1 is composed of $n^2 - \{(n - k_1 + 1)(n - k_2)\}$ edges.

Case 2: Let $S_2 \subseteq E$ be a K-disconnecting set of G in which G-S_2 is composed of $k_1 + k_2$ components, and let one of the connected component be $G[u_p \cup (V_1 - K_1) \cup (V_2 - K_2)]$ (i.e., the graph induced by one terminal of K_2 and the non-terminal vertices) and the $k_1 + k_2 - 1$ components are the remaining terminal vertices themselves. As in Case 1, we see that $G\ [u_p \cup (V_1 - K_1) \cup (V_2 - K_2)]$ is a complete bipartite graph. Consequently S_2 is composed of $n^2 - \{(n - k_1)\ (n - k_2 + 1)\}$ edges.

By the definition of $\eta_K(G)$ and since S_1 and S_2 are K-disconnecting set of G whose deletion leave exactly $k_1 + k_2$ components (each one containing a terminal node), then

$$\eta_K(G) \le \frac{\min(|S_1|, |S_2|)}{k_1 + k_2 - 1} = \frac{|S_1|}{k_1 + k_2 - 1} \quad \text{, whenever } k_2 \le k_1. \tag{5.16}$$

Thus our problem is well-posed since we originally assumed that $k_2 \le k_1$.

Expanding (5.16) we obtain

$$\eta_K(G) \le \frac{|S_1|}{k_1 + k_2 - 1} = \frac{k_1.n + k_2.n - n - k_2(k_1 - 1)}{k_1 + k_2 - 1} = n - \frac{k_2(k_1 - 1)}{k_1 + k_2 - 1}. \tag{5.17}$$

Since n, k_1 and k_2 are positive integers, it follows from (5.17) that

$$\lfloor \eta_K(G) \rfloor \le n - \left\lceil \frac{k_2(k_1 - 1)}{k_1 + k_2 - 1} \right\rceil. \tag{5.18}$$

Next we show that $t_K(G) = \lfloor \eta_K(G) \rfloor$ whenever the inequality $k_2(k_2 - 1) < (k_1 - 1)$ is true. If $k_2(k_2 - 1) < (k_1 - 1)$, then it is easy to see that

$$\left\lceil \frac{k_2(k_1 - 1)}{k_1 + k_2 - 1} \right\rceil = k_2 \quad \text{, since } \frac{k_2(k_1 - 1)}{k_1 + k_2 - 1} \text{ is non-negative, and an increasing function on } k_1$$

which approaches k_2 for $k_1 \to \infty$ (assuming k_2 is fixed).

Thus $\lfloor \eta_K(G) \rfloor \le n - k_2$, whenever $k_2(k_2 - 1) < (k_1 - 1)$. (5.19)

We can construct $n - k_2$ edge-disjoint K-Steiner trees of G using the following approach. Let V' be k_1-k_2 non-terminal vertices of V_2, and let V'' be the remaining non-terminal vertices of V_2.

For each vertex *u* of *V'*, make *u* adjacent to all the terminal vertices of K_1. Then choose a corresponding vertex *v* of K_1 and make *v* adjacent to all the vertices of K_2. This procedure will create a *K*-Steiner tree. Repeat the procedure by choosing a different pair (*u, v*) of vertices of *V'* and K_1. This construction yields a set τ' of k_1-k_2 edge-disjoint *K*-Steiner trees (see Figure 5.3).

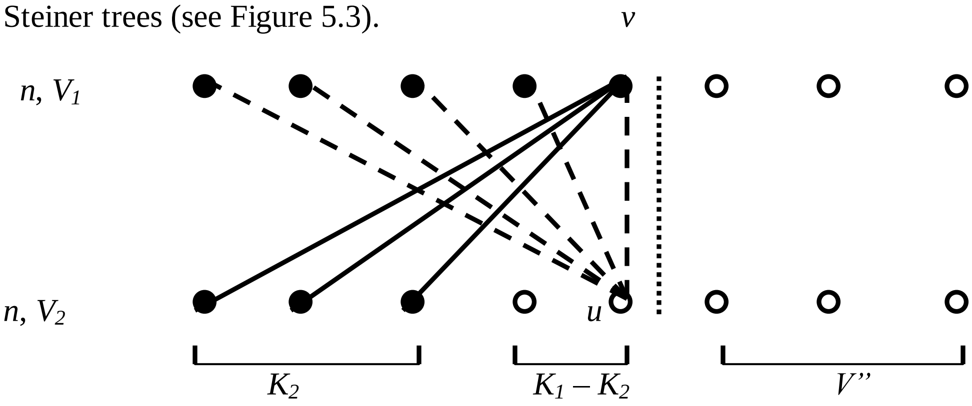


Figure 5.3: A Complete bipartite graph (τ' edge-disjoint *K*-Steiner trees).

Next for each non-terminal vertex *u* of *V”*, make this vertex adjacent to an unique vertex *v* of $V_1 – K_1$ (i.e., non-terminal vertices of V_1). Then make *u* and *v* adjacent to the terminal vertices of K_1 and K_2, respectively. This operation creates a *K*-Steiner tree. Repeat this operation for every pair (*u, v*) of vertices *V”* and ($V_1 – K_1$) yielding a set τ'' of n-k_1 edge-disjoint *K*-Steiner trees (see Figure 5.4). Moreover, the *K*-Steiner trees of τ' and τ'' are mutually edge-disjoint.

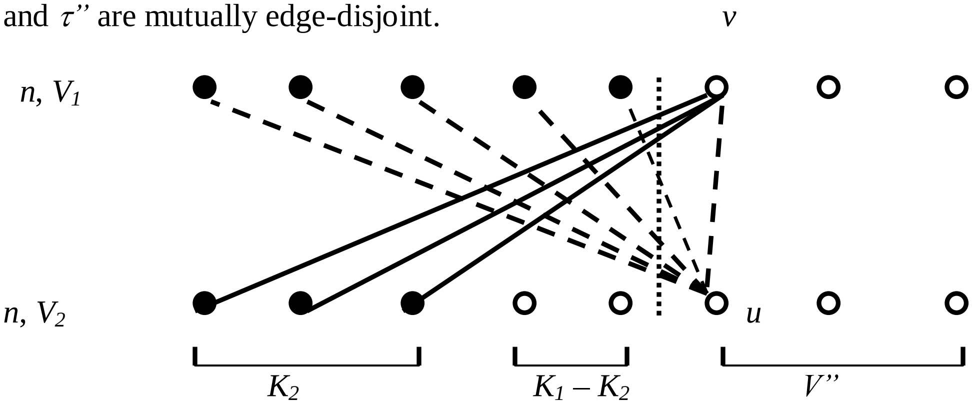


Figure 5.4: A Complete bipartite graph (τ'' edge-disjoint *K*-Steiner trees).

From Theorem 5.3 it follows that $t_K(G) \le \lfloor \eta_K(G) \rfloor$, thus from (5.19) and the construction of K-Steiner trees previously mentioned one gets,

$n - k_2 = |\tau'| + |\tau''| \le t_K(G) \le \lfloor \eta_K(G) \rfloor \le n - k_2$, whenever $k_2\,(k_2 - 1) < (k_1 - 1)$, thus the theorem follows. ■

The cases where the bipartitions are of different sizes and when they have the same size but $k_2\,(k_2 - 1) \ge (k_1 - 1)$ remain open.

5.4 Conclusions and Future Work

In this thesis, we presented the Problem of Packing Steiner trees of a graph G, with terminal vertex-set K, where the objective is to find the maximum number of edge-disjoint Steiner trees contained in G. Specifically, in 2003, Kriesell conjectured that the maximum number of edge-disjoint K-Steiner trees is bounded below by half of the K-edge-connectivity of G. Moreover this conjecture is consistent and represents a generalization of Nash-Williams and Tutte result connecting the maximum number of edge-disjoint spanning trees of G to the edge-connectivity, for the case $K = V$, and Menger's theorem, for the case $|K| = 2$.

We showed that if a graph G with terminal vertex-set K could be decomposed into K-Steiner trees (i.e., G is the union of edge-disjoint K-Steiner trees), then Kriesell's conjecture can be answered affirmatively. Moreover, the later result provides a lower and an upper bound for the problem of packing several K-Steiner trees (not necessarily edge-disjoint) of a graph G. In addition, let $\mathcal{T}$ be a set of edge-disjoint K-Steiner trees, with $|\mathcal{T}| = t_K(G)$. One interesting question is to determine how much the K-edge-connectivity of a graph G may differ with respect to the K-edge-connectivity of the subgraph of G

obtained by the union of the K-Steiner trees of $\mathcal{T}$, without incrementing $t_K(G)$. If it is determined that the two connectivities may differ by at most a constant, then from Theorem 4.1, it follows that Kriesell's conjecture is true within an additive constant.

We also showed that for any arbitrary graph G with terminal vertex-set K, G has at least $\lfloor \lambda_K(G)/2 \rfloor - |V-K|/2-1$ edge-disjoint K-Steiner trees, improving Kriesell's bound established in [13].

In chapter 5, we connected the problem of packing edge-disjoint K-Steiner trees with the K-edge-toughness $\eta_K(G)$. We also studied some properties of this vulnerability index and we showed some classes of graphs for which the maximum number of edge-disjoint K-Steiner trees is equal to $\lfloor \eta_K(G) \rfloor$. The relevance of this last equality is that Kriesell's conjecture is true for any graph that achieved the equality $t_K(G) = \lfloor \eta_K(G) \rfloor$. Another important consequence of studying the connection between $t_K(G)$ and $\eta_K(G)$ is that this analysis could help us to find a counterexample to Kriesell's conjecture. So far we found just one example of a graph for which $t_K(G) < \lfloor \eta_K(G) \rfloor$.

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