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ON THE UNIQUENESS OF THE

TETRAHEDRAL ASSOCIATION SCHEME

by

PETER ROLLAND

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Alan J Hoffman
Chairman of Examining Committee

1/6/76
date

R. Sacksteder
Executive Officer

H. Peyton Young

Richard Sacksteder

Supervisory Committee

The City University of New York

Abstract

ON THE UNIQUENESS OF THE TETRAHEDRAL ASSOCIATION SCHEME

by

Peter Rolland

Adviser: Professor Alan Hoffman

We define an association scheme $\mathcal{T}_n$, herein named the tetrahedral association scheme of order n, on the set of unordered triples of an n-set by the following rule: two distinct triples are i^{th} associates (i = 0, 1, or 2) iff their intersection has cardinality i. We calculate the parameters of this scheme and investigate whether or not these parameters determine the scheme up to isomorphism. The same question has been studied by others (Bose and Laskar, Aigner) regarding the tetrahedral graph, whose points are unordered triples of an n-set, two triples being adjacent iff their intersection has cardinality 2. They succeeded in resolving the question affirmatively for all meaningful values of n except n = 9, ..., 16. In this dissertation we prove that the parameters of the tetrahedral association scheme determine the scheme (up to isomorphism) for all meaningful values of n (including n = 9, ..., 16). This is somewhat surprising since Connor, Shirkhande, Chang, and Hoffman have proved that the triangular association scheme (analogous to the tetrahedral scheme except that it is defined on the set of unordered pairs of an n-set) is determined by its parameters <u>except</u> for n = 8, when there are three counterexamples.

We use eigenvalue arguments extensively, elaborating on methods published by A. J. Hoffman in his paper, "On the Uniqueness of the Triangular Association Scheme", Ann. Math. Statist. 31 (1960) 492-497.

After calculating the parameters of the tetrahedral association scheme we define association matrices A_0, A_1, and A_2 for an association scheme $\mathcal{A}_n$, whose parameters coincide with those of $\mathcal{T}_n$; the rows and columns of A_i correspond to the elements of $\mathcal{A}_n$; $(A_i)_{rs} = 1$ iff r and s are i^{th} associates, and $(A_i)_{rs} = 0$ otherwise. We show that there are certain linear relationships between the A_i, and we use these to calculate the minimal polynomials of A_1 and A_2. Having these, we determine that the minimum eigenvalue of $A_1 + 2A_2$ is -3. The Cauchy interlacing theorem gives us a necessary criterion which must be met by principal submatrices of $A_1 + 2A_2$: the minimal eigenvalue of the submatrix must not be less than -3. We also define, for each $a \in \mathcal{A}_n$, the graph G_a whose points are the second associates of a. We show that $A(G_a) + J - I$ is a principal submatrix of $A_1 + 2A_2$ and use the Cauchy interlacing theorem to establish that every principal submatrix of $A(G_a)$ has a smallest eigenvalue which is no less than -2; furthermore if the smallest eigenvalue equals -2, then the sum of the coordinates of a corresponding eigenvector must be 0. Using these criteria we establish the following:

(1) G_a does not contain a 3-claw;

(2) G_a does not contain four independent points;

(3) Given any non-adjacent pair of points x, y in G_a, x and y

have exactly two mutual neighbors in G_a, and these are not adjacent to each other.

These facts are sufficient to determine all adjacencies in G_a: G_a is the disjoint union of three cliques of order $n - 3$; furthermore, each $x \in G_a$ lies in exactly one clique of order 3, the elements of which lie in distinct $(n - 3)$-cliques. Using this information about G_a, we are able to construct an isomorphic mapping from $\mathcal{A}_n$ onto $\mathcal{T}_n$.

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CHAPTER 1

STATEMENT OF THE PROBLEM

Let $I_n = \{1,2, \ldots, n\}$. Consider $T_n = \{\{i,j,k\} | i,j,k \in I_n, i \neq j \neq k \neq i\}$, the set of unordered triples on an n-set. Let t s $\in T_n (t \neq s)$. We will say that t and s are 0^{th}, first, or second associates, and write $\alpha(t,s) = 0$, 1, or 2, respectively, depending on whether $|t \cap s| = 0$, 1, or 2. The following set of parameters define the relationship of elements of T_n to each other:

(1) There are $\binom{n}{3}$ triples;

(2) for all $t \in T_n$, the number of 0^{th}, first, and second associates of t is n_0, n_1, and n_2, respectively;

(3) for all t, $s \in T_n$, $t \neq s$, $\alpha(t,s) = i \in \{0,1,2\}$, the number of triples which are j^{th} associates of t <u>and</u> k^{th} associates of s is $p^i_{jk} (j,k \in \{0,1,2\})$.

The n_i and the p^i_{jk} are parameters of the situation; they depend on n but not on s or t. The complete set of parameter values is given in Table 1 at the end of this section.

<u>Example (1)</u>. The triple $\{1,2,3\}$ has $\binom{n-3}{3}$ 0^{th} associates, namely any $\{i,j,k\}$ where i, j, and k are distinct elements of $\{4, \ldots, n\}$. Thus $n_0 = \binom{n-3}{3}$.

Example 2. Let $t = \{1,2,3\}$ and $s = \{3,4,5\}$. t and s are first associates. There are four triples which are second associates of t and s, namely $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,4\}$, and $\{2,3,5\}$. Thus $p^1_{22} = 4$.

These parameters define an association scheme on T_n which we call the tetrahedral association scheme for historical reasons explained in the next section. Henceforth, the symbol T_n will stand for this association scheme. It is natural to inquire whether or not T_n is the unique (up to isomorphism) scheme with these parameters. In other words, if A_n is an association scheme with the same parameters as T_n, is there necessarily an isomorphism $\phi: A_n \to T_n$? In this paper we show that the answer is affirmative for $n \geq 9$. Combined with earlier results, which we shall discuss in the next section, this shows that for all $n \geq 6$ (the smallest value of n for which the problem can be posed), the parameters characterize the scheme.

Table 1

Parameter Values for $\mathcal{A}_n$ and $\mathcal{T}_n$

Note that $p^i_{jk} = p^i_{kj}$, so nearly half the p^i_{jk} have been omitted from the table.

(a) The association scheme has $\binom{n}{3}$ elements.

(b) $n_0 = \binom{n-3}{3}$

$n_1 = \frac{3}{2}(n - 3)(n - 4)$

$n_2 = 3(n - 3)$

(c)

$p^0_{00} = \binom{n-6}{3}$	$p^1_{00} = \binom{n-5}{3}$	$p^2_{00} = \binom{n-4}{3}$
$p^0_{01} = 3\binom{n-6}{3}$	$p^1_{01} = 2\binom{n-5}{2}$	$p^2_{01} = \binom{n-4}{2}$
$p^0_{11} = 9(n - 6)$	$p^1_{11} = \frac{1}{2}(n - 5)(n + 2)$	$p^2_{11} = (n - 4)^2$
$p^0_{20} = 3(n - 6)$	$p^1_{20} = n - 5$	$p^2_{20} = 0$
$p^0_{21} = 9$	$p^1_{21} = 2(n - 4)$	$p^2_{21} = 2(n - 4)$
$p^0_{22} = 0$	$p^1_{22} = 4$	$p^2_{22} = n - 2$

CHAPTER 2

HISTORICAL NOTES

The analogous question for pairs of elements of I_n was first raised by Conner in [5]. He considered the set $P_n = \{\{i,j\} | i \neq j, i,j \in I_n\}$, the set of unordered pairs. Two pairs are 0th associates if their intersection has cardinality 0 and first associates if their intersection has cardinality 1. After calculating n_0, n_1, and $\{p_{ij}^k\}$, he raised the question of whether a two-class association scheme on $\binom{n}{2}$ objects with these parameters had to arise in the prescribed way. The given scheme was known in the theory of experimental designs as the "triangular association scheme", leading to the name tetrahedral association scheme for the scheme we are discussing. The results of Conner [5], Shrikhande [9], Chang [4], and Hoffman [7] established that the triangular association scheme is characterized by its parameters unless $n = 8$, when there are three exceptions.

The relation of first association in the triangular association scheme can be conceived as adjacency in a graph (and 0th association as adjacency in the complementary graph). Emphasizing the graph concept, Bose and Laskar [3] introduced the concept of a tetrahedral graph, which in our language is the relation of second association in the tetrahedral association scheme. Bose and Laskar showed that the tetrahedral graph is characterized by certain postulates if $n > 16$. Aigner [1] showed the same thing, using different techniques, for $n = 6$, 7, and 8. In the appendix we shall show that his results settle our problem for the same range of n.

We should also mention that results analogous to Bose and Laskar's for T_m graphs (the points of the graph are the $\binom{n}{m}$ m-sets of I_n, two points being adjacent if the intersection of the corresponding m-sets has cardinality $m - 1$) have been given by Dowling [6], and the corresponding questions about a T_m scheme for $m > 3$ remain to be investigated.

CHAPTER 3

A MATRIX FORMULATION OF THE PROBLEM

Definition 3.1. Let $\mathcal{A}_n$ be an association scheme with the parameters given in Table 1. Let incidence matrices A_0, A_1, and A_2 be defined by:

(3.1.1) A_i is a symmetric $\binom{n}{3} \times \binom{n}{3}$ matrix:

(3.1.2) if $a, b \in \mathcal{A}_n$, $a \neq b$, and $\alpha(a,b) = i$, then $(A_i)_{ab} = 1$;

(3.1.3) if $a, b \in \mathcal{A}_n$, $a \neq b$, and $\alpha(a,b) \neq i$, then $(A_i)_{ab} = 0$;

(3.1.4) if $a \in \mathcal{A}_n$, then $(A_i)_{aa} = 0$.

There is one row/column representing each $a \in \mathcal{A}_n$. We will call the A_i, as well as the matrix $A_1 + 2A_2$, association matrices. The matrix $A_1 + 2A_2$ has a special significance to us, in that it completely describes how any two elements of $\mathcal{A}_n$ are related. More specifically, if $a, b \in \mathcal{A}_n$ and $a \neq b$, then $(A_1 + 2A_2)_{ab} = \alpha(a,b)$. The diagonal entries of $A_1 + 2A_2$ are all zero.

The proof presented here hinges on the fact that $\lambda^1(2A_2 + A_1) = -3$. ($\lambda^1(M)$ denotes the smallest eigenvalue of M where M is any real symmetric matrix.) This follows from the fact that A_1 and A_2

can be simultaneously diagonalized, and when this is done the eigenvalue -3 of A_2 is paired with the eigenvalue 3 of A_1. We will calculate all the eigenvalues of A_1 and A_2, and this entails calculating the roots of the minimal polynomials of A_1 and A_2. As it turns out, these polynomials are of the fourth degree and can easily be factored. The work would be very laborious, except for the fact that $(A_i)^2$ and $(A_i)^3$ and A_iA_j can be expressed as a linear combination of A_0, A_1, and A_2. This is shown in the following theorem. I denotes the $\binom{n}{3} \times \binom{n}{3}$ identity matrix; J is the $\binom{n}{3} \times \binom{n}{3}$ matrix whose entries are all 1.

Theorem 3.2. With A_0, A_1, and A_2 defined as above, the following holds:

(3.2.1) $A_0 + A_1 + A_2 + I = J$

(3.2.2) $A_k^2 = n_k I + \sum_{i=0}^{2} p_{kk}^i A_i \qquad (k \in \{0,1,2\})$

(3.2.3) $A_k A_j = \sum_{i=0}^{2} p_{kj}^i A_i = A_j A_k \qquad (k \neq j, k,j \in \{0,1,2\})$

(3.2.4) A_i commutes with $J \qquad (i = \{0,1,2\})$.

Proof. (3.2.1) merely expresses the fact that any two elements of A_n are either 0^{th}, first or second associates.

$(A_k^2)_{ab}$ is the dot product of the a-row and the b-column. If $a \neq b$, this is the count of the number of elements of A_n which are

k^{th} associates of both a and b, and this, by definition, is p^i_{kk} where $\alpha(a,b) = i$. Now $(A_i)_{ab} = 1$ (by definition (3.1.2)) so $p^i_{kk}A_i$ has p^i_{kk} in the ab position. Also, $(A_j)_{ab} = 0$, where $j \in \{0,1,2\}$ but $j \neq i$ (by definition (3.1.3)). Therefore $p^j_{kk}A_j$ has a 0 in the ab position. So if we let $M = \sum_{i=0}^{2} p^i_{kk}A_i$, then $(M)_{ab} = p^i_{kk} = (A_k)^2_{ab}$ and $(M)_{aa} = \sum_{i=0}^{2} p^i_{kk}(A_i)_{aa} = 0$. On the diagonal of A^2_k we have the dot product of any row vector with itself (A_k is symmetric) which is just the constant row sum n_k. Therefore $n_kI + M$ agrees with A^2_k on and off the diagonal. This establishes (3.2.2).

To show (3.2.3) we first remark that if a, $b \in A_n$ and $a \neq b$, then $(A_kA_j)_{ab} = p^i_{kj}$ where $i = \alpha(a,b)$. Also $(A_kA_j)_{aa} = 0$, since the a-row of A_k and the a-column of A_j cannot have a 1 in the same position. (For this would imply that some $x \in A_n$ is both a j^{th} and a k^{th} associate of a.) Therefore $A_kA_j = \sum_{i=0}^{2} p^i_{kj}A_i$. Finally, A_j and A_k commute, because $(A_kA_j)_{ab} = p^i_{kj} = p^i_{jk} = (A_jA_k)_{ab}$.

(3.2.4) is a consequence of the fact that the row and column sums of A_i are all the constant n_i; i.e., $(A_iJ)_{ab} = n_i = (JA_i)_{ab}$.

QED

We're ready now to tackle the problem of calculating the eigenvalues of A_1 and A_2.

Since A_1 and A_2 are symmetric 0 - 1 matrices with zeros along the diagonal and constant row and column sums, they can be regarded as adjacency matrices of regular graphs Γ_1 and Γ_2. Our next theorem is due to Hoffman [7]. We start with a graph Γ with adjacency matrix A.

<u>Theorem 3.3</u>. There is a polynomial p such that $p(A) = J$ iff Γ is a regular connected graph.

<u>Proof</u>. See [2, p. 15].

<u>Theorem 3.4</u>. Γ_1 and Γ_2 are regular connected graphs.

<u>Proof</u>. As stated above, Γ_i is regular because A_i has fixed row/column sums. To show that Γ_i is connected we must show that given $a, b \in A_n$, $a \neq b$, there is a chain of elements of A_n, each one an i^{th} associate of its neighbors in the chain, which starts at a and ends at b. Let $\alpha(a,b) = j$. We will first construct a chain of first associates from a to b. If $j = 1$, then (a,b) is such a chain. If $j \neq 1$, then $p^j_{11} > 0$ (Table 1c). Thus there is a $c \in A_n$ such that $\alpha(c,a) = \alpha(c,b) = 1$; (a,c,b) is such a chain. Next we will construct a chain of second associates from a to b. If $j = 2$, then (a,b) is such a chain. If $j = 1$, then $p^j_{22} > 0$ (Table 1c), so there exists a $d \in A_n$ such that $\alpha(d,a) = \alpha(d,b) = 2$; (a,d,b) is such a chain. If $j = 0$, then $p^j_{22} = 0$, so there is no such chain of length 3. However, $p^0_{21} = 9$ so there exists an $e \in A_n$ which is a second associate of a and a first associate of b. Also $p^1_{22} = 4$, so there exists an $f \in A_n$ such that $\alpha(e,f) = \alpha(f,b) = 2$. Then (a,e,f,b) is such a chain. Therefore Γ_1 and Γ_2 are connected.

QED

Corollary 3.5. There are polynomials p_1 and p_2 such that $p_1(A_1) = p_2(A_2) = J$.

Proof. This an immediate consequence of Theorems 3.3 and 3.4.

The next theorem, which is Corollary 3.3 in [2], derives a formula for the eigenvalues and the minimal polynomial of A, the adjacency matrix of Γ.

Theorem 3.6. Let Γ be a regular connected graph with n vertices, and let the distinct eigenvalues of Γ be $k > \lambda_1 > \ldots > \lambda_{s-1}$. Then if $q(\lambda) = \Pi(\lambda - \lambda_i)$, where the product is over the range $1 \leq i \leq s - 1$, we have

(3.6.1) $$J = \left(\frac{n}{q(k)}\right)q(A).$$

Proof. See [2, p. 15].

In the course of the proof of Corollary 3.3 it is shown that the minimal polynomial of A is a multiple of $(\lambda - k)q(\lambda)$. Also, the largest eigenvalue of A is the valence k of Γ, corresponding to the eigenvector $\bar{u}$ whose entries are all 1. This implies the following: Let k_i denote the largest eigenvalue of A_i $(i = 1,2)$. Then

(3.6.2) $$k_1 = n_1 = \frac{3}{2}(n - 3)(n - 4)$$

(3.6.3) $$k_2 = n_2 = 3(n - 3).$$

3.7. The Plan of Attack. The results of 3.3–3.6 suggest the following method for calculating the eigenvalues of A_1 and A_2. We look for a polynomial q_i such that $q_i(A_i)$ is a multiple of J. Then the minimal polynomial of A_i is a multiple of $(\lambda - k_i)q(\lambda)$. As we shall see in the following section, the powers of A_i can be expressed as a linear combination of A_0, A_1, and A_2, so the problem reduces to one of finding a solution to a set of linear equations where the unknown variables are the coefficients of the polynomial q_i.

CHAPTER 4

THE EIGENVALUES OF A_2

4.1. The Calculation of A_2^2 and A_2^3. Referring to (3.2.2) we find that

$$A_2^2 = n_2 I + \sum_{i=0}^{2} p_{22}^{i} A_i .$$

Substituting the values of n_2 and p_{22}^{i} $(i = 0, 1, 2)$ given in Table 1b and 1c into the equation above, we obtain

$$A_2^2 = 3(n - 3)I + 4A_1 + (n - 2)A_2 . \tag{4.1.1}$$

Now $A_2^3 = A_2(A_2^2)$. Using (4.1.1), this becomes

$$A_2^3 = 3(n - 3)A_2 + 4A_1A_2 + (n - 2)A_2^2 . \tag{4.1.2}$$

According to (3.2.3), $A_1A_2 = p_{12}^{0}A_0 + p_{12}^{1}A_1 + p_{12}^{2}A_2$. Substituting the values of p_{12}^{0}, p_{12}^{1}, and p_{12}^{2} given in Table 1c into this expression we get

$$A_1A_2 = 9A_0 + 2(n - 4)A_1 + 2(n - 4)A_2 . \tag{4.1.3}$$

Substituting (4.1.3) and (4.1.1) into (4.1.2), collecting like terms, and simplifying, we obtain

(4.1.4) $$A_2^3 = 3(n - 2)(n - 3)I + 36A_0 + (12n - 40)A_1 + (n^2 + 7n - 37)A_2.$$

We will next calculate values for the coefficients of q_2.

<u>Lemma 4.2</u>. Suppose $c_0A_0 + c_1A_1 + c_2A_2 + c_3I = 0$. Then $c_1 = c_2 = c_3 = c_0 = 0$.

<u>Proof</u>. A_0, A_1, A_2, and I are (0,1) matrices covering different positions in the matrix (indeed, their sum is J), so they are clearly linearly independent.

We are seeking $q_2(x) = r_0 + r_1x + r_2x^2 + r_3x^3$ such that

(4.2.1) $$q_2(A) = r_0I + r_1A_2 + r_2A_2^2 + r_3A_2^3 = J,$$

and we hope to find that the roots of q_2 are integral. Substitute (4.1.1) and (4.1.4) into (4.2.1); then set the resulting expression equal to $A_0 + A_1 + A_2 + I$. (They are both equal to J), collect like terms, and simplify. We obtain an expression of the form $A_0 + A_1 + A_2 + I = \rho_0A_0 + \rho_1A_1 + \rho_2A_2 + \rho I$. Rearranging, $(1 - \rho_0)A_0 + (1 - \rho_1)A_1 + (1 - \rho_2)A_2 + (1 - \rho)I = 0$. Lemma 4.2 implies that $1 = \rho_0$, $1 = \rho_1$,

$1 = \rho_2$, and $1 = \rho$. Thus when all the above mentioned operations are performed, we are left with

(4.2.2)
$$\begin{aligned}
\rho &= 1 = r_0 + 3(n-3)r_2 + 3(n-2)(n-3)r_3 \\
\rho_0 &= 1 = 36r_3 \\
\rho_1 &= 1 = 4r_2 + (12n-40)r_3 \\
\rho_2 &= 1 = r_1 + (n-2)r_2 + (n^2 + 7n - 37)r_3.
\end{aligned}$$

Solving this linear system for r_0, r_1, r_2, and r_3, we find that

(4.2.3)
$$\begin{aligned}
r_0 &= \frac{3}{36}(2n-9)(n-7) \\
r_1 &= \frac{1}{36}(2n^2 - 32n - 111) \\
r_2 &= \frac{1}{36}(19 - 3n) \\
r_3 &= \frac{1}{36} \quad .
\end{aligned}$$

Therefore,

(4.2.4)
$$\begin{aligned}
36q_2(x) = {} & 3(2n-9)(n-7) + (2n^2 - 32n - 111)x \\
& + (19 - 3n)x^2 + x^3.
\end{aligned}$$

4.3. Calculation of the Roots of q_2. We could find the roots of q_2 (as given in (4.2.4)) by means of a formula for the roots of a cubic equation, but the following approach is easier. We claim that if T_2 denotes the corresponding association matrix of second associates for $\mathcal{T}_n$, then $\lambda^1(T_2) = -3$. This is proved as follows.

Let ${}_{\binom{n}{2}}L^{\binom{n}{3}}$ be a (0,1) matrix whose rows correspond to the triples $\{i,j,k\}$ of $\mathcal{T}_n$ and whose columns correspond to the pairs $\{\ell,m\}$ where ℓ, $m \in I_n$. Let $(L)_{pq} = 1$ iff pair p is contained in triple q. Then ${}_{\binom{n}{3}}LL^{T\,\binom{n}{3}}$ is positive semi-definite (this is true of any real matrix) and singular (because L is not square). Therefore $\lambda^1(LL^T) = 0$. Now, an elementary consequence of the way L and T_2 are defined is that $LL^T = 3I + T_2$. Therefore, $\lambda^1(T_2) = \lambda^1(LL^T - 3I)$ $= \lambda^1(LL^T) - 3 = -3$, since LL^T and I can be simultaneously diagonalized.

In light of the fact that we are trying to establish an isomorphism between $\mathcal{A}_n$ and $\mathcal{T}_n$, -3 is an excellent candidate for the minimal eigenvalue of A_2. In fact, if we substitute $x = -3$ into (4.2.4) we find that $q_2(-3) = 0$.

Now, q_2 is a monic polynomial with integral coefficients and an integral root (-3). This establishes that $(x - k_2)q_2(x)$ is the minimal polynomial of A_2. Using the fact that -3 is a root of q_2, we can factor (4.2.4) as follows:

$$(4.3.1) \qquad q_2(x) = (x + 3)(x^2 + bx + c)$$

$$= x^3 + (3 + b)x^2 + (3b + c)x + 3c.$$

Equating coefficients of like terms in (4.3.1) and (4.2.4), we obtain

$$b = 16 - 3n$$

$$c = (2n - 9)(n - 7)$$

so

$$(4.3.2) \qquad q_2(x) = (x + 3)(x - (2n-9))(x - (n-7)).$$

Therefore, the distinct eigenvalues of A_2 are -3, $n - 7$, $2n - 9$, and $k_2 = 3(n - 3)$. Since $n > 8$, $\lambda^1(A_2) = -3$. Incidentally, these are also the eigenvalues of T_2. (The preceding calculations depend only on Theorem 3.2, which also applies to T_0, T_1, and T_2, since the parameters for T_n and A_n are the same.)

<u>4.4. The Eigenvalues of</u> A_1. According to (4.1.1), $A_2^2 = 3(n - 3)I + 4A_1 + (n - 2)A_2$. Let U be an orthogonal matrix which diagonalizes A_2: $U^{-1}A_2U = F$. We can assume that U is chosen so that

(4.4.1)
$$F = \begin{bmatrix} f_1 & & & & 0 \\ & \ddots f_2 & & & \\ & & \ddots f_3 & & \\ & 0 & & \ddots & f_4 \end{bmatrix}$$

where f_i occurs m_i times in succession (m_i denotes the implicity of the eigenvalue f_i), and

(4.4.2)
$$f_1 = -3$$
$$f_2 = n - 7$$
$$f_3 = 2n - 9$$
$$f_4 = 3(n - 3).$$

Then $U^{-1}A_2^2U = (U^{-1}A_2U)(U^{-1}A_2U) = F^2$. Therefore, using (4.4.1), we have

(4.4.3)
$$F^2 = U^{-1}(3(n - 3)I + 4A_1 + (n - 2)A_2)U$$
$$= 3(n - 3)U^{-1}IU + 4U^{-1}A_1U + (n - 2)U^{-1}A_2U$$
$$= 3(n - 3)I + 4U^{-1}A_1U + (n - 2)F.$$

Rearranging,

(4.4.4) $$U^{-1}A_1U = \frac{1}{4}[F^2 - (n - 2)F - 3(n - 3)I].$$

Now F^2, $(n - 2)F$, and $3(n - 3)I$ are all diagonal matrices; therefore, $U^{-1}A_1U$ is diagonal, and of the form

(4.4.5) $$U^{-1}A_1U = \begin{pmatrix} e_1 & & & 0 \\ & \ddots\, e_2 & & \\ 0 & & \ddots\, e_3 & \\ & & & \ddots\, e_4 \end{pmatrix}$$

where

(4.4.6) $$e_i = \frac{1}{4}[f_i^2 - (n - 2)f_i - 3(n - 3)]$$

and the e_i occur m_i times in succession. e_i is therefore an eigenvalue of A_1 of multiplicity m_i, and we say that e_i is paired with f_i, since e_i and f_i occupy corresponding positions on the diagonal when A_1 and A_2 are simultaneously diagonalized by U.

Now let's substitute the values given in (4.4.2) for the f_i into (4.4.6); we obtain

(4.4.7)
$$e_1 = 3$$
$$e_2 = 11 - 2n$$
$$e_3 = \frac{1}{2}(n - 9)(n - 4)$$
$$e_4 = \frac{3}{2}(n - 3)(n - 4) = 3\binom{n-3}{2}.$$

This value for e_4 checks with the value for the largest eigenvalue of A_1 given in (3.6.2) .

4.5. Calculation of $\lambda^1(A_1 + 2A_2)$. Earlier we mentioned that it is of great importance to the work in this paper that $\lambda^1(A_1 + 2A_2) = -3$. We are now ready to prove this equality.

Theorem 4.5. The eigenvalues of $A_1 + 2A_2$ are $e_i + 2f_i$, $i \in \{1,2,3,4\}$.

Proof.
$$U^{-1}(A_1 + 2A_2)U = (U^{-1}A_1 + 2U^{-1}A_2)U$$
$$= U^{-1}A_1U + 2U^{-1}A_2U = E + 2F.$$
QED

Corollary 4.5. $\lambda^1(A_1 + 2A_2) = -3.$

Proof.
$$e_1 + 2f_1 = (11 - 2n) + 2(n - 7) = -3$$
$$e_2 + 2f_2 = 3 + 2(-3) = -3$$
$$e_3 + 2f_3 = \frac{1}{2}(n - 4)(n - 9) + 2(2n - 9) = \frac{1}{2}n(n - 5)$$

$$e_4 + 2f_4 = 3\binom{n-3}{2} + 3n - 9 = \frac{3}{2}(n - 2)(n - 3).$$

So the eigenvalues of $A_1 + 2A_2$ are -3, $\frac{n}{2}(n - 5)$, and $\frac{3}{2}(n - 2)(n - 3)$. Since $n > 8$, the latter two eigenvalues both exceed -3. Therefore $\lambda^1(A_1 + 2A_2) = -3$. QED

CHAPTER 5

TWO CONSEQUENCES OF THE CAUCHY INTERLACING THEOREM

Theorem 5.1. (Cauchy). Let A be an n-square hermitian matrix with characteristic roots $\lambda_1 \geq \cdots \geq \lambda_n$. Let B be a k-square principal submatrix of A with characteristic roots $\mu_1 \geq \cdots \geq \mu_k$. Then

$$\lambda_s \geq \mu_s \geq \lambda_{n-k+s}, \qquad s = 1, \ldots, k.$$

This statement of the so-called interlacing theorem, as well as its proof, can be found in [8, p. 203].

Corollary 5.2. Let A be as in Theorem 5.1, and let M be a k-square matrix $(k < n)$. If $\lambda^1(M) < \lambda^1(A)$, then M is not a principal submatrix of A.

Proof. $\lambda^1(M)$ and $\lambda^1(A)$ denote the minumum eigenvalue of M and A respectively. Assuming that M is a principal submatrix of A, then in the language of Theorem 5.1 we have $\lambda^1(M) = \mu_k$ and $\lambda^1(A) = \lambda_n$. But then

$$\mu_k < \lambda_n$$

which violates Theorem 5.1 with $s = k$.

Corollary 5.3. Let M be a symmetric $(0,1,2)$ matrix with zeros on the diagonal. If $\det(M + 3I) < 0$, then M is not a principal submatrix of $A_1 + 2A_2$.

Proof. Let k be the order of M_1 and let $g_1, \ldots, g_k$ denote the k (not necessarily distinct) eigenvalues of M. Then $g_1 + 3, \ldots, g_k + 3$ are the eigenvalues of $M + 3I$ (M and $3I$ commute, so they can be simultaneously diagonalized). A well known result of matrix theory is that the determinant of a matrix is the product of the eigenvalues, so

$$\det(M + 3I) = \prod_{j=1}^{k} (g_j + 3). \tag{5.3.1}$$

If $\det(M + 3I) < 0$, we can conclude from (5.3.1) that at least one of the factors on the right is negative; say $g_j + 3 < 0$. Then $\lambda^1(M) \leq g_j < -3 = \lambda^1(A_1 + 2A_2)$. Corollary 5.2 implies that M is not a principal submatrix of $A_1 + 2A_2$. QED

Corollary 5.3 and the similar Corollary 5.9 are the primary tools that we shall use in analyzing the structure of A_n.

Definition 5.4. Let $a \in A_n$. We define G_a, the graph associated with a, as follows:

(1) The elements of G_a are the second associates of a. (Therefore $|G_a| = n_2 = 3(n - 3)$.)

(2) b, $c \in G_a$, $b \neq c$ are adjacent iff $\alpha(b,c) = 2$. We'll write $e(b,c) = 1$ if b and c are adjacent.

(3) b, $c \in G_a$, $b \neq c$, are not adjacent iff $\alpha(b,c) = 1$. We'll write $e(b,c) = 0$ if b and c are not adjacent.

Remark. If b, $c \in G_a$, $b \neq c$, then $\alpha(b,c) \neq 0$, since both b and c are second associates of a, and $p_{22}^{0} = 0$.

Terminology. Elements of G_a adjacent to a given $b \in G_a$ are called neighbors of b. The set of neighbors of b in G_a is denoted by $N_a(b)$.

Definition 5.5. Let $a \in A_n$, let $G_a = \{b_1, b_2, \ldots, b_{3(n-3)}\}$. Then $A(G_a)$, the adjacency matrix of G_a, is defined as follows:

(1) $A(G_a)$ is a symmetric $3(n-3)$-square matrix. The rows/columns of $A(G_a)$ correspond to the elements of b_j of G_a.

(2) If j, $k \in I_{3(n-3)}$ and $j \neq k$, then $(A(G_a))_{jk} = e(b_j, b_k)$.

(3) If $j \in I_{3(n-3)}$, then $(A(G_a))_{jj} = 0$.

Theorem 5.6. For all $a \in A_n$, $A(G_a) + J - I$ is a principal submatrix of $A_1 + 2A_2$.

Proof. A direct consequence of Definitions 5.4 and 5.5 is that if j, $k \in I_{3(n-3)}$, then

$$(A(G_a) + J - I)_{jk} = \alpha(b_j, b_k) = (A_1 + 2A_2)_{b_j b_k}$$

Corollary 5.7. G_a is regular of degree $n - 2$.

Proof. The number of elements of G_a which are adjacent to a given element b of G_a is just the number of elements of A_n which are second associates of both a and b (Definition 5.4), and this number is $p_{22}^2 = n - 2$.

Corollary 5.8. $\lambda^1(A(G_a)) \geq -2$.

Proof. Corollary 5.7 says that row and column sums of $A(G_a)$ are all $n - 2$. Let $\bar{u} = (1, \ldots, 1)$ be the $1 \times 3(n - 3)$ unit vector. Then $\bar{u}$ is an eigenvector of $A(G_a)$, since $A(G_a)\bar{u} = (n - 2)\bar{u}$. Furthermore, since $A(G_a)$ is symmetric and regular, $n - 2$ is the largest eigenvalue. Now, if $\lambda^1(A(G_a)) = n - 2$, then since $n > 8$, $\lambda^1(A(G_a)) > -2$ so we are done. If not, then let $\bar{x}$ be an eigenvector corresponding to $\gamma = \lambda^1(A(G_a))$. γ and $n - 2$ are different eigenvalues, so $\bar{x}$ and $\bar{u}$ lie in orthogonal subspaces, i.e., $\bar{x} \cdot \bar{u} = 0$. Therefore $J\bar{x} = 0$.

Now let $B = A(G_a) + J - I$. Theorem 5.6 asserts that B is a principal submatrix of $A_1 + 2A_2$, so from Corollary 5.2 and Corollary 4.5 we conclude that $\lambda^1(B) \geq -3$. Now,

$$(5.8.1) \qquad B\bar{x} = A(G_a)\bar{x} + J\bar{x} - I\bar{x} = \gamma\bar{x} + 0 - \bar{x} = (\gamma - 1)\bar{x}.$$

Therefore, $\bar{x}$ is also an eigenvector of B, belonging to the eigenvalue $\gamma - 1$. Therefore, (5.8.2) $$\gamma - 1 \geq \lambda^1(B) \geq -3.$$

Finally (5.8.2) implies that $\gamma \geq -2$. QED

<u>Corollary 5.9</u>. Let M be a symmetric integer matrix of zeros and ones, with zeros along the diagonal. If $\det(M + 2I) < 0$, then M is <u>not</u> a principal submatrix of $A(G_a)$, for all $a \in \mathcal{A}_n$.

<u>Proof</u>. The proof of Corollary 5.9 is identical to the proof of Corollary 5.3, with "3" replaced everywhere by "2" and "$A_1 + 2A_2$" replaced everywhere by "$A(G_a)$".

<u>Theorem 5.10</u>. Let M be a symmetric matrix with constant row sums such that $\det(M + 3I) = 0$. Suppose that $\bar{w}$ is an eigenvector corresponding to the eigenvalue -3; suppose, further, that the sum of the coordinates of $\bar{w}$ is not zero. Then M is not a principal submatrix of $A_1 + 2A_2$.

<u>Theorem 5.11</u>. Let M be a symmetric matrix with constant row sums such that $\det(M + 2I) = 0$. Suppose that $\bar{w}$ is an eigenvector corresponding to the eigenvalue -2; suppose, further, that the sum of the coordinates of $\bar{w}$ is not zero. Then M is not a principal submatrix of $A(G_a)$, for all $a \in \mathcal{A}_n$.

<u>Proof</u>. Both Theorems 5.10 and 5.11 are consequences of the following well known property of eigenvalues and eigenvectors of a symmetric (real) matrix. Let A be a (real) symmetric matrix whose

least eigenvalue is β, and whose maximum eigenvalue is $\alpha > \beta$, with an eigenvector $\bar{r}$ corresponding to α. Let M be a principal submatrix of A, δ the least eigenvalue of M, and $\bar{w}$ an eigenvector of M corresponding to δ.

If $\delta = \beta$, then $\bar{w}$ is orthogonal to the projection of $\bar{r}$ on the subspace corresponding to M. If it so happens that the maximum eigenvalue α has the unit vector $(1, \ldots, 1)$ as an eigenvector, then we take $\bar{r} = (1, \ldots, 1)$ and conclude that $0 = \bar{w} \cdot \bar{r}|_M$ = the sum of the coordinates of $\bar{w}$. We have already observed, in the proof of Theorem 3.6, that for a symmetric matrix with constant row sum s, the largest eigenvalue is s, and the unit vector is a corresponding eigenvector. Observing that $A_1 + 2A_2$ has constant row sum $n_1 + 2n_2 = \frac{3}{2}(n-3)(n-4) + 6(n-3)$, and that for each $a \in A_n$, $A(G_a)$ has constant row sum $n - 2$, we obtain Theorems 5.10 and 5.11.

CHAPTER 6

AN IMPORTANT THEOREM ABOUT G_a

We first define some of the terms used in the following work.

Definition 6.1. Let x, $y \in G_a$, $x \neq y$. Then $\Delta(x,y)$ denotes the number of points of G_a adjacent to both x and y.

Definition 6.2. Let S be a collection of points of G_a. Then S is an independent set (we speak of s independent points, where $s = |S|$) if no two points of S are adjacent.

Definition 6.3. Let b, $x_1, \ldots, x_k \in G_a$. Then $\{b;x_1, \ldots, x_k\}$ is a k-claw if $\{x_1, \ldots, x_k\}$ is an independent set and $e(b,x_i) = 1$, $i = 1, \ldots, k$.

Definition 6.4. Let p, q, r, s be four distinct points of G_a. Then $(pqrs)$ is a four cycle if $e(p,q) = e(q,r) = e(r,s) = e(s,p) = 1$ and $e(p,r) = e(q,s) = 0$.

In this chapter we show that, given any two independent points x, $y \in G_a$, $\Delta(x,y) = 2$; furthermore, that the two points which are neighbors of both x and y are themselves not adjacent. We also prove two other revealing facts about G_a: G_a does not contain four independent points, and G_a does not contain a 3-claw. The proofs rely on the matrix techniques elaborated in Corollaries 5.3 and 5.9 and Theorems 5.10 and 5.11.

The points of G_a must relate to a 4-cycle in one of three ways. This is explained below.

<u>Theorem 6.5</u>. Let (p,q,r,s) be a 4-cycle in G_a, and let x be a fifth point of G_a. Then

(1) x cannot be adjacent to exactly one of p, q, r, and s,

(2) x cannot be adjacent to exactly three of p, q, r, and s,

(3) x cannot be adjacent to exactly two points of the 4-cycle unless they are themselves adjacent.

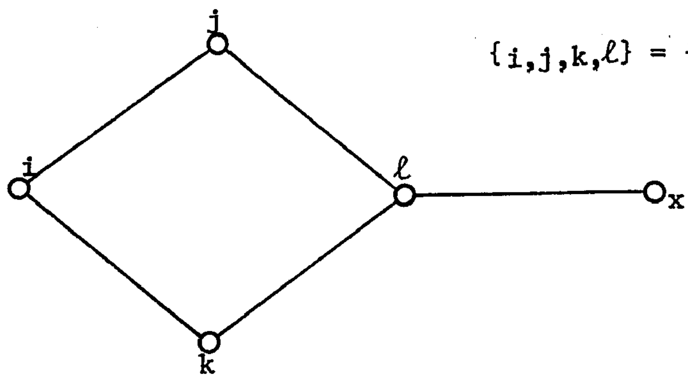
<u>Proof</u>. Refer to Figure 1 below. If x is adjacent to exactly one of p, q, r, and s, then the incidence matrix of $\{p,q,r,s,x\}$ is row-equivalent to

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

But M cannot be a principal submatrix of $A(G_a)$ according to Corollary 5.9, since $\det(M + 2I) = -4 < 0$. (1) follows.

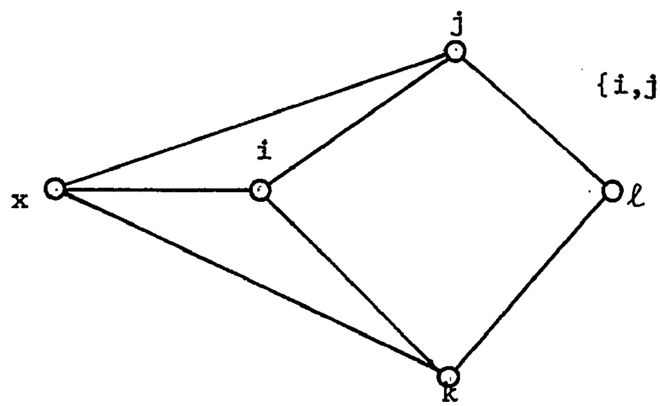
If x is adjacent to exactly three of p, q, r, and s, then the incidence matrix of $\{p,q,r,s,x\}$ is row equivalent to

(1)



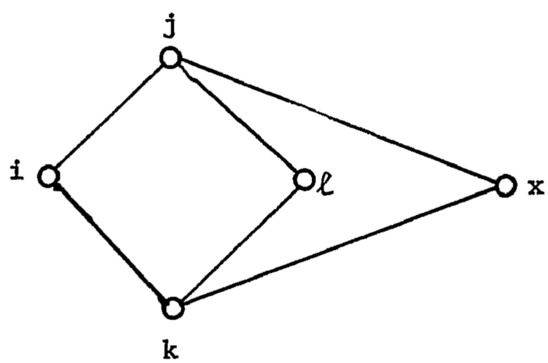
$\{i,j,k,\ell\} = \{p,q,r,s\}$

(2)



$\{i,j,k,\ell\} = \{p,q,r,s\}$

(3)



$\{i,j,k,\ell\} = \{p,q,r,s\}$

Figure 1. Three impossible subgraphs of G_a.

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

But M cannot be a principal submatrix of $A(G_a)$ according to Corollary 5.9, since $\det(M + 2I) = -4 < 0$, so we have (2).

Finally, if x is adjacent to two non-adjacent points in the 4-cycle, then the incidence matrix of $\{p,q,r,s,x\}$ is row-equivalent to

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

But M cannot be a principal submatrix of $A(G_a)$ since $\det(M + 2I) = -16 < 0$, so we have (3). QED

We work toward proving the nonexistence of a 3-claw in G_a.

<u>Lemma 6.6</u>. Let $a \in A_n$, x, $y \in G_a$, $e(x,y) = 0$. Then there exists at least one $z \in G_a$ such that $e(z,x) = e(z,y) = 0$.

<u>Proof</u>. Suppose that such a z does not exist. Then for all $z \in G_a$, $z \neq x$, $z \neq y$, one of three situations must hold:

$$(6.6.1) \qquad e(z,x) = 1 \quad \text{and} \quad e(z,y) = 0$$

$$(6.6.2) \qquad e(z,x) = 0 \quad \text{and} \quad e(z,y) = 1$$

$$(6.6.3) \qquad e(z,x) = e(z,y) = 1$$

Let the number of elements of G_a for which (6.6.1) holds be ρ. Let the number of elements of G_a for which (6.6.2) holds be τ. The number of elements for which (6.6.3) holds is $\Delta(x,y)$. Then

$$(6.6.4) \qquad |G_a| = 3(n - 3) = 2 + \rho + \tau + \Delta(x,y).$$

Also, the number of elements of G_a adjacent to x is

$$(6.6.5) \qquad n - 2 = \rho + \Delta(x,y).$$

Similarly,

$$(6.6.6) \qquad n - 2 = \tau + \Delta(x,y).$$

Solving (6.6.5) and (6.6.6) for ρ and τ and substituting in (6.6.4), we obtain

$$3(n - 3) = 2 + [(n - 2) - \Delta(x,y)] + [(n - 2) - \Delta(x,y)] + \Delta(x,y).$$

This reduces to

$$\Delta(x,y) = 7 - n.$$

This is impossible, since $n > 8$. QED

Lemma 6.7. Let a, x, y, and z be as in Lemma 6.6. Then $\Delta(x,y) \in \{2,3\}$, $\Delta(x,z) \in \{2,3\}$, and $\Delta(y,z) \in \{2,3\}$ in G_a.

Proof. Since x and y are first associates and $p^1_{22} = 4$, there are four elements of A_n, including a, which are second associates of both x and y. Suppose none, or only one of the remaining three lie in G_a. Then at least two of the remaining three lie outside of G_a, call them s and t. Let $\gamma = \alpha(s,t)$. Consider the association matrix M determined by $\{x,y,a,s,t\}$:

$$M = \begin{array}{c} \begin{array}{ccccc} x & y & a & s & t \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & \gamma \\ 2 & 2 & 1 & \gamma & 0 \end{pmatrix} \end{array}.$$

Now γ can't be 0, since s and t are both second associates of x, and $p^0_{22} = 0$. Therefore $\gamma = 1$ or $\gamma = 2$. Suppose $\gamma = 1$. Then $\det(M + 3I) = -32 < 0$. Corollary 5.3 implies that M is not a principal submatrix of $A_1 + 2A_2$. Therefore $\gamma \neq 1$. Suppose $\gamma = 2$. Then $\det(M + 3I) = -8 < 0$. Again Corollary 5.3 implies that M is not a principal submatrix of $A_1 + 2A_2$. Therefore $\gamma \neq 2$. Therefore at most one of the elements of A_n which are second associates of x and y lies outside of G_a, consequently $\Delta(x,y) \in \{2,3\}$ in G_a. Identical arguments show that $\Delta(y,z) \in \{2,3\}$ in G_a and $\Delta(x,z) \in \{2,3\}$ in G_a.

Lemma 6.8. There is no 4-claw in G_a.

Proof. Suppose $\{b;x,y,z,w\}$ is a 4-claw in G_a. Let M be the association matrix of the set $\{a,b,x,y,z,w\}$:

$$M = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then $\det(M + 3I) = -8 < 0$, so Corollary 5.3 implies that M is <u>not</u> a principal submatrix of $A_1 + 2A_2$. Therefore, there is no 4-claw in G_a.

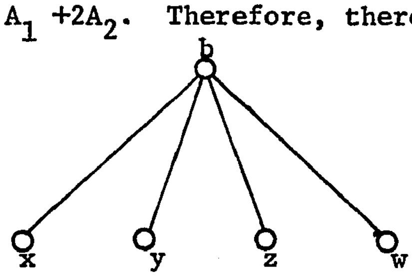


Figure 2. The 4-claw.

Definition 6.9. Let S be an independent set of points in G_a, and let $T \subseteq S$. Then $J(T;S)$ denotes the set of points in $G_a - S$ which are adjacent to each point of T and to no point of $S - T$. Alternatively, we may write $J(x_1, \ldots, x_k;S)$ where $T = \{x_1, \ldots, x_k\}$. If T is empty we write $J(\phi;S)$. Also, $g(T;S)$ or $g(x_1, \ldots, x_k;S)$ denote the cardinality of $J(T;S)$. If there is no ambiguity, we will write $J(T)$, $J(x_1, \ldots, x_k)$, $g(T)$, $g(x_1, \ldots, x_k)$, $J(\phi)$, and $g(\phi)$.

Theorem 6.10. Let $S = \{x,y,z\}$ be an independent set of points in G_a. Then $g(x,y) + g(x,z) + g(y,z) + 2g(x,y,z) - g(\phi) = 6$.

Proof. Counting up the different kinds of points in G_a, we get

(6.10.1) $$|G_a| = 3(n - 3) = 3 + g(\phi) + [g(x) + g(y) + g(z)] + [g(x,y) + g(x,z) + g(y,z)] + g(x,y,z).$$

Next, counting neighbors of x, y, and z, we obtain

(6.10.2) $$n - 2 = g(x) + g(x,y) + g(x,z) + g(x,y,z)$$

(6.10.3) $$n - 2 = g(y) + g(x,y) + g(y,z) + g(x,y,z)$$

(6.10.4) $$n - 2 = g(z) + g(x,z) + g(y,z) + g(x,y,z).$$

Adding (6.10.2)-(6.10.4) together, we get

$$(6.10.5)\qquad 3(n-2) = [g(x) + g(y) + g(z)]$$

$$+ 2[g(x,y) + g(x,z) + g(y,z)] + 3g(x,y,z).$$

Subtracting (6.10.1) from (6.10.5) and rearranging terms, we get

$$(6.10.6)\qquad g(x,y) + g(x,z) + g(y,z) + 2g(x,y,z) + g(\phi) = 6. \qquad \text{QED}$$

<u>Theorem 6.11</u>. If $\{x,y,z\}$ is an independent set, then $g(x,y,z) \leq 1$.

<u>Proof</u>. Suppose $g(x,y,z) \geq 2$; then there exist b, $c \in G_a$, $b \neq c$, such that $\{b;x,y,z\}$ and $\{c;x,y,z\}$ are 3-claws. Now, if $e(b,c) = 0$, look at the adjacency matrix M of $\{b,c,x,y,z\}$:

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\det(M + 2I) = -16 < 0$, so Corollary 5.9 implies that M is <u>not</u> a principal submatrix of $A(G_a)$.

On the other hand, if $e(b,c) = 1$, then the adjacency matrix of $\{b,c,x,y,z\}$ is

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now $\det(M + 2I) = 0$, so we can't appeal to Corollary 5.9. However, the vector $\overline{v} = (1,1,-1,-1,-1)$ is an eigenvector of M corresponding to the eigenvalue -2, and the sum of coordinates of $\overline{v}$ is not 0. It follows from Theorem 5.11 that M is not a principal submatrix of $A(G_a)$. Therefore $g(x,y,z) \leq 1$. QED

Theorem 6.12. Suppose $\{g;x,y,z\}$ is a 3-claw in G_a. Then there does not exist a point $c \in G_a$ such that c is adjacent to b and exactly one of x, y, and z.

Proof. Without loss of generality, we may assume that $e(c,x) = e(c,y) = 0$ and $e(c,z) = 1$. Then the association matrix of $\{x,y,z,c,b,a\}$ is

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 \end{pmatrix}.$$

Now $\det(M + 3I) = 0$, and $\bar{v} = (-3,-3,-2,-2,4,4)$ is an eigenvector corresponding to the eigenvalue -3. The sum of the coordinates of $\bar{v}$ is not zero, so by Theorem 5.10, M is not a principal submatrix of $A_1 + 2A_2$. Therefore such a point c does not exist.

Theorem 6.13. If $S = \{x,y,z\}$ is an independent set in G_a, and $g(x,y,z;S) = 1$, then $g(\phi;S) = n - 9$.

Proof. Let $\{b;x,y,z\}$ be the 3-claw assumed above, and let c be a neighbor of b. Then c can't be independent of $\{x,y,z\}$, for then $\{b;x,y,z,c\}$ would be a 4-claw, which by Lemma 6.8 does not exist. Also, c can't be adjacent to exactly one of x, y, and z (Theorem 6.12). Furthermore, c can't be adjacent to each of x, y, and z (Theorem 6.11). Therefore c is adjacent to exactly two of x, y, and z.

The converse is also true; if w is adjacent to exactly two of x, y, and z, then $e(w,b) = 1$. This is proved as follows. Suppose $e(w,i) = e(w,j) = 1$ where $i, j \in S$ and $i \neq j$. Suppose further that $e(w,b) = 0$. Then (b,i,w,j) is a 4-cycle. Then k (where $\{i,j,k\} = S$) is adjacent to exactly one point of the 4-cycle (namely b); however, we know this to be impossible (Theorem 6.5). Therefore $e(w,b) = 1$.

Let us now count the neighbors of b: besides x, y, and z we have the points of G_a adjacent to any two of x, y, and z, i.e.,

(6.13.1) $$n - 2 = 3 + g(x,y) + g(x,z) + g(y,z).$$

Substituting $g(x,y,z) = 1$ into (6.10.6) and rearranging we obtain

(6.13.2) $$g(x,y) + g(x,z) + g(y,z) = 4 + g(\phi).$$

Substituting (6.13.2) into (6.13.1) and rearranging we obtain

(6.13.3) $$g(\phi) = n - 9.$$ QED

Theorem 6.14. If $S = \{x,y,z\}$ is an independent set in G_a, and $g(x,y,z) = 1$, then $g(\phi,S) \in \{0,1,2\}$.

Proof. We rearrange (6.13.2) to obtain

(6.14.1) $$g(\phi) = g(x,y) + g(x,z) + g(y,z) - 4.$$

It follows directly from the definition of the symbols $\Delta(i,j)$, $g(i,j)$, and $g(i,j,k)$ that

(6.14.2) $g(x,y) = \Delta(x,y) - g(x,y,z) = \Delta(x,y) - 1$ (Since $g(x,y,z) = 1$)

$g(x,z) = \Delta(x,z) - g(x,y,z) = \Delta(x,z) - 1$ " "

$g(y,z) = \Delta(y,z) - g(x,y,z) = \Delta(y,z) - 1$ " "

Now Lemma 6.7 states that $\Delta(x,y)$, $\Delta(x,z)$, and $\Delta(y,z)$ are each either 2 or 3. Therefore $g(x,y)$, $g(x,z)$, and $g(y,z) \in \{1,2\}$ and $3 \leq g(x,y) + g(x,z) + g(y,z) \leq 6$. It follows that $-1 \leq g(x,y) + g(x,z) + g(y,z) - 4 \leq 2$. Substituting this in (6.14.1) we obtain

$$-1 \leq g(\phi) \leq 2.$$

$g(\phi)$ is, by definition, nonnegative, so

$$0 \leq g(\phi) \leq 2.$$ QED

Lemma 6.15. Let $S = \{x,y,z\}$ be an independent set in G_a. If $g(x,y,z) = 1$, $g(\phi;S) > 0$, and $w \in J(\phi;S)$, then w is adjacent to at most one point in each of $J(x;S)$, $J(y;S)$, and $J(z;S)$.

Proof. Let b be the single element in $J(x,y,z)$. Let $v \in S$; if $J(v;S)$ contains at most one point, then the conclusion is automatically fulfilled for $J(v;S)$. Suppose $J(v;S)$ contains at least two points, p and q.

Case 1. $e(p,q) = 0$. Suppose the conclusion is false, i.e., suppose $e(w,p) = e(w,q) = 1$. Theorem 6.12 implies that $e(b,p) = e(b,q) = 0$. Thus (v,p,w,q) is a 4-cycle, call it C. Now, $e(b,w) = 0$ (otherwise $\{b;x,y,z,w\}$ is a 4-claw, which violates Lemma 6.8), so b is adjacent to exactly one point of C (namely v). But this violates Theorem 6.5. Therefore w is not adjacent to both p and q.

Case 2. $e(p,q) = 1$. Suppose again that $e(w,p) = e(w,q) = 1$. Let $\{r,s,t\} = \{x,y,z\}$ and consider the incidence matrix M of $\{b,r,s,t,p,q,w\}$ (see also Figure 3).

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

$\det(M + 2I) = 0$, and M has the following eigenvector $\bar{e}$ corresponding to -2: $(2,-2,-1,-1,1,1,-1)$. But the sum of the coordinates of $\bar{e}$ is not zero, so by Theorem 5.11 M cannot be a principal submatrix of $A(G_a)$. Therefore w is not adjacent to both p and q. The conclusion of Lemma 6.15 follows immediately.

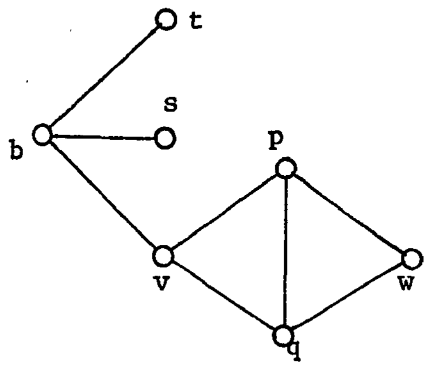


Figure 3. The graph of Lemma 6.15, Case 2.

Theorem 6.16. If $S = \{x,y,z\}$ is an independent set in G_a and $\{b;x,y,z\}$ is a 3-claw in G_a, then $g(\phi;S) = 0$.

Proof. Suppose $g(\phi) > 0$. Let w be independent of x, y, and z, i.e., $w \in J(\phi;S)$. Then $e(b,w) = 0$, for if $e(b,w) = 1$, then $\{b;x,y,z,w\}$ is a 4-claw, which, according to Lemma 6.8, does not exist in G_a. Now $\Delta(b,w) = 2$ or 3 (Lemma 6.7) and these mutual neighbors of b and w lie in $N_a(b) = S \cup J(x,y;S) \cup J(x,z;S) \cup J(y,z;S)$. This follows from the last part of the first paragraph of the proof of Theorem 6.13. Also, Lemma 6.15 implies that w is adjacent to at most three elements in $J(x;S) \cup J(y;S) \cup J(z;S)$. Since

$$G_a = \{b\} \cup S \cup J(x;S) \cup J(y;S) \cup J(z;S) \cup J(x,y;S)$$

$$\cup J(x,z;S) \cup J(y,z;S) \cup J(\phi;S),$$

We can conclude that w is adjacent to at most $3 + 3 = 6$ points in $G_a - J(\phi;S)$. Since w has $n - 2$ neighbors, at least $(n - 2) - 6 = n - 8$ of them lie in $J(\phi;S)$. According to Theorem 6.13, $g(\phi;S) = n - 9$, so we are led to conclude that

$$n - 8 \leq n - 9.$$

This contradiction proves that $g(\phi;S) = 0$. QED

Theorem 6.17. If $S = \{x,y,z\}$ is an independent set in G_a, then $g(x,y,z) = 0$.

Proof. Theorem 6.17 states that G_a does not contain a 3-claw.

Suppose the assertion is false; i.e., suppose there exists a $b \in G_a$ such that $\{b;x,y,z\}$ is a 3-claw (Figure 4). Then $g(x,y,z) \geq 1$. Theorem 6.11 established that $g(x,y,z) \leq 1$, so $g(x,y,z) = 1$. Theorem 6.16 showed that $g(\phi;S) = 0$, so (6.14.1) reduces to

$$g(x,y) + g(x,z) + g(y,z) = 4. \qquad (6.17.1)$$

Each of the terms in (6.17.1) is, by definition, nonnegative. We claim further that each term is positive. Let $\{i,j,k\} = \{x,y,z\}$. Suppose one of the terms, call it $g(i,j)$, is zero. Then $\Delta(i,j) = g(i,j) + g(i,j,k) = 0 + 1 = 1$. This violates Lemma 6.7, which states that $\Delta(i,j)$ must be 2 or 3. Therefore each term in (6.17.1) is positive, and this, in turn, implies that one of the terms equals 2 and the remaining two terms equal 1. We may, w.l.o.g., assume that $g(x,y) = 2$ and $g(x,z) = g(y,z) = 1$. Let s be the (unique) element in $J(y,z)$; thus $e(s,y) = e(s,z) = 1$ and $e(s,x) = 0$. The points a, b, and s are second associates of y and z. Since $\alpha(y,z) = 1$ and $p_{22}^{1} = 4$, there exists a fourth element of A_n, call it w, such that $\alpha(w,y) = \alpha(w,z) = 2$. We claim that $w \notin G_a$. For if $w \in G_a$, then $w \in J(y,z)$ or $w \in J(x,y,z)$, which implies that either $w = s$ or $w = b$, i.e., w is not a fourth mutual second associate of y and z at all. Therefore $w \notin G_a$ (and of course, $w \neq a$).

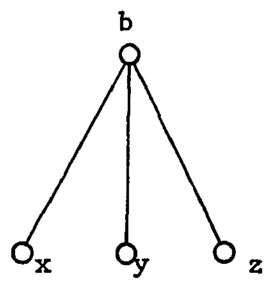


Figure 4. The 3-claw.

We will prove the nonexistence of the 3-claw by showing that $Q = \{a,b,x,y,z,s,w\}$ is not a principal submatrix of $A_1 + 2A_2$. To show this, we need to know how w associates to the remaining elements of Q. First of all, $w \notin G_a$ implies that either $\alpha(w,a) = 0$ or $\alpha(w,a) = 1$. But $\alpha(w,a) = 0$ is impossible, since both w and a are second associates of y and $p_{22}^{0} = 0$. So

$$\alpha(w,a) = 1. \tag{6.17.2}$$

Next we claim that $\alpha(w,x) \neq 2$. For if $\alpha(w,x) = 2$, then w, a, b and the two points in $J(x,y)$ are each second associates of x and y, violating $p^1_{22} = 4$. Therefore

(6.17.3) $$\alpha(w,x) = 0 \text{ or } 1.$$

Next, since w and b are both second associates of y,

(6.17.4) $$\alpha(w,b) \neq 0.$$

Also, we claim that $\alpha(w,b) \neq 2$. Suppose $\alpha(w,b) = 2$. Then $w \in G_b$ along with a, z, x, and y. Since $\alpha(w,b) = 2 = \alpha(x,b)$, we cannot have $\alpha(w,x) = 0$. Therefore, referring to (6.17.3) we see that $\alpha(w,x) = 1$, and $e(w,x) = 0$ in G_b. Thus (a,y,w,z) is a 4-cycle C in G_b, and $x \in G_b$ is adjacent to exactly one point of C (namely a), which by Theorem 6.5 is impossible. Therefore $\alpha(w,b) \neq 2$. Referring to (6.17.4) we conclude that

(6.17.5) $$\alpha(w,b) = 1.$$

Next, $\alpha(w,s) \neq 0$, since $\alpha(s,y) = \alpha(w,y) = 2$. Therefore

(6.17.6) $$\alpha(w,s) = 1 \text{ or } 2.$$

Finally, what about $\alpha(s,b)$? Since s, $b \in G_b$, $\alpha(s,b) = 1$ or 2.

Suppose $\alpha(s,b) = 1$. Then (b,y,x,z) is a 4-cycle C in G_a, and x is adjacent to exactly one point of C (namely b), which by Theorem 6.5 is impossible. Therefore,

(6.17.7) $$\alpha(s,b) = 2.$$

Now, using (6.17.2)-(6.17.7) we can construct the association matrix P of $\{a,b,y,z,x,s,w\}$:

$$P = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 0 & 1 & p \\ 2 & 2 & 2 & 2 & 1 & 0 & r \\ 1 & 1 & 2 & 2 & p & r & 0 \end{pmatrix}$$

where $p = \alpha(w,x)$ and $r = \alpha(w,s)$. Expanding $\det(P + 3I)$ we obtain

(6.17.8) $$\det(P + 3I) = -4[1 + (r - 2)^2 + 2(p + 1)^2].$$

Clearly, this is a negative quantity for all values of r and p. Therefore P is not a principal submatrix of $A_1 + 2A_2$. Hence G_a contains no 3-claw. QED

Corollary 6.18. Let $S = \{x,y,z\}$ be an independent set in G_a. Then

$$g(\phi;S) = g(x,y;S) + g(x,z;S) + g(y,z;S) - 6. \tag{6.18.1}$$

Proof. This is an immediate consequence of Theorems 6.10 and 6.17.

Corollary 6.19. Let $x,\ y \in S$, where S is an independent set in G_a. Then $g(x,y;S) = \Delta(x,y) = 2$ or 3.

Proof. Since G_a does not contain a 3-claw (Theorem 6.17), no mutual neighbor of x and y in G_a is adjacent to a point of $S - \{x,y\}$. Therefore, the 2 or 3 mutual neighbors of x and y (Lemma 6.7) are all counted in $g(x,y;S)$. QED

Since $g(x,y;S)$ does not depend on S, from now on we simply write $g(x,y)$.

We now come to the theorems most difficult to prove: that G_a does not contain four independent points, for $n = 9, \ldots, 16$. Unfortunately we found it necessary to consider each value of n in the range $9 \leq n \leq 13$ as a separate case. There are various commonalities in these cases, however, and we elucidate them in the following lemmas.

Definition 6.20. Suppose that $r,\ s \in G_a$, $r \neq s$, and r is not adjacent to s. We define $G_a(r,s) = G_a - \{r,s\} - N_a(r) - N_a(s)$.

Lemma 6.21. Let $\{p,q,r,s\}$ be an independent set in G_a. Then there is no point in $G_a - G_a(r,s)$ which is adjacent to both p and q.

Proof. Suppose α is such a point. Certainly $\alpha \notin \{r,s\}$, since by definition $p, q \notin N_a(r) \cup N_a(s)$. Therefore α is adjacent to $\gamma \in \{r,s\}$, and $\{\alpha;p,q,\gamma\}$ is a 3-claw, which violates Theorem 6.17. (Figure 5).

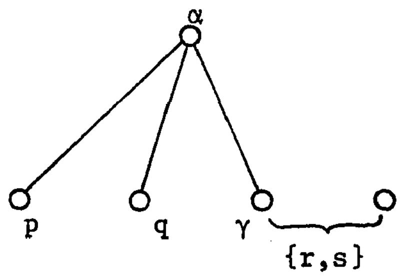


Figure 5. The graph of Lemma 6.21.

Lemma 6.22. Suppose $r, s \in G_a$ and $e(r,s) = 0$. Then $e(G_a(r,s),\ J(r,s) \cup \{r,s\}) = 0$.

Proof. Let $\alpha \in G_a(r,s)$, $\beta \in J(r,s) \cup \{r,s\}$. We must show that $e(\alpha,\beta) = 0$. If $\beta \in \{r,s\}$, then we see from Definition 6.20 that $e(\alpha,\beta) = 0$. If $\beta \in J(r,s)$ and $e(\alpha,\beta) = 1$, then $\{\beta;r,s,\alpha\}$ is a 3-claw, violating Theorem 6.17. Hence $e(\alpha,\beta) = 0$. QED

Lemma 6.23. If $p, q \in G_a\{r,s\}$ and $e(p,q) = 0$, then there are at least two and at most three points in $G_a\{r,s\}$ adjacent to both p and q.

Proof. Lemma 6.7 states that $\Delta(p,q) = 2$ or 3 in G_a, but none of these mutual neighbors of p and q can lie in $G_a - G_a(r,s)$ (Lemma 6.21). Therefore, they must lie in $G_a(r,s)$.

Lemma 6.24. Suppose r, $s \in G_a$, $r \neq s$ and $\Delta(r,s) = 3$. Then there is no point in $G_a(r,s)$ adjacent to every other point of $G_a(r,s)$.

Proof. Suppose that such a point $\alpha \in G_a(r,s)$ exists. Then, letting $S = \{\alpha,r,s\}$,

$$(6.24.1) \qquad G_a = N_a(\alpha) \cup N_a(r) \cup N_a(s) \cup S.$$

Referring to Corollary 6.18, we can write

$$(6.24.2) \qquad 0 = g(\phi;S) = g(\alpha,r) + g(\alpha,s) + g(r,s) - 6.$$

Corollary 6.19 implies that $g(\alpha,r)$ and $g(\alpha,s)$ are each 2 or 3; by hypothesis, $g(r,s) = \Delta(r,s) = 3$. Therefore

$$g(\alpha,r) + g(\alpha,s) + g(r,s) - 6 \geq 2 + 2 + 3 - 6 = 1,$$

and we are led to the absurdity $0 \geq 1$. Lemma 6.24 follows.

Lemma 6.25. G_a does not contain six mutually independent points.

Proof. If $\{x_1, \ldots, x_6\}$ is an independent set in G_a, then the association matrix of $\{a,x_1, \ldots, x_6\}$ is

$$M = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

But $\det(M + 3I) = 0$, and $\bar{e} = (-4,1,1,1,1,1,1)$ is an eigenvector of M corresponding to the eigenvalue -3. The coordinates of $\bar{e}$ add up to 2, so Theorem 5.11 implies that M is not a principal submatrix of $A_1 + 2A_2$.

Lemma 6.26. G_a does not contain five mutually independent points.

Proof. Suppose $T = \{i,j,k,\ell,m\}$ is an independent set in G_a. Let $S = \{k,\ell,m\}$. From Corollary 6.18 we conclude that

$$(6.26.1) \qquad g(\phi;S) = g(k,\ell) + g(k,m) + g(\ell,m) - 6.$$

Now G_a does not contain six independent points (Lemma 6.25), so every point in $J(\phi;S) - \{i,j\}$ is adjacent to i, j, or both i and j. Therefore, (6.26.2) $$g(\phi;S) = g(i;T) + g(j;T) + g(i,j) + 2.$$

Combining (6.26.1) and (6.26.2) we obtain

(6.26.3) $$g(i;T) + g(j;T) + g(i,j) + 2 = g(k,\ell) + g(k,m) + g(\ell,m) - 6.$$

Now, $g(i,j) = \Delta(i,j)$. (Corollary 6.19). Therefore, $g(i,j) \geq 2$, and the left side of (6.26.3) is at least 4. Similarly, $g(k,\ell) = \Delta(k,\ell) \leq 3$; $g(k,m) = \Delta(k,j) \leq 3$; and $g(\ell,m) = \Delta(\ell,m) \leq 3$. Therefore the right hand side of (6.26.3) is at most $3 + 3 + 3 - 6 = 3$, and we have obtained a contradiction.

<u>Lemma 6.27</u>. Suppose that $S = \{i,j,k,\ell\}$ is an independent set in G_a. Then

(6.27.1) $$\sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) = n + 5.$$

<u>Proof</u>. Corollary 6.18 implies that

(6.27.2) $$g(\phi;S - \{i\}) = g(j,k) + g(j,\ell) + g(k,\ell) - 6.$$

Now, since by Lemma 6.26 G_a does not contain five independent points, every point of G_a which is adjacent to none of j, k, and ℓ must either be i or adjacent to i. Therefore, the number of these points is

$$1 + g(i;S) = g(\phi;S - \{i\}). \tag{6.27.3}$$

Combining (6.27.2) and (6.27.3) we get

$$1 + g(i) = g(j,k) + g(j,\ell) + g(k,\ell) - 6. \tag{6.27.4}$$

Also, since each vertex in G_a has valence $n - 2$,

$$n - 2 = g(i) + g(i,j) + g(i,k) + g(i,\ell). \tag{6.27.5}$$

Adding (6.27.5) and (6.27.4) and simplifying, we obtain (6.27.1).

<u>Lemma 6.28</u>. If $S = \{i,j,k,\ell\}$ is an independent set in G_a, then

$$\sum_{\alpha \in S} g(\alpha;S) = 2(n - 9). \tag{6.28.1}$$

<u>Proof</u>. Solving (6.27.4) for $g(i)$ we obtain

$$g(i) = g(j,k) + g(j,\ell) + g(k,\ell) - 7. \tag{6.28.2}$$

Analogous equations can be written for $g(j)$, $g(k)$, and $g(\ell)$. Adding all of these together, we get

$$\sum_{\alpha \in S} g(\alpha) = 2 \sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) - 28. \tag{6.28.3}$$

Substituting (6.27.1) into (6.28.3) and simplifying, we obtain (6.28.1).

Theorem 6.29. If $n > 13$, then G_a does not contain four independent points.

Proof. Let $S = \{i,j,k,\ell\}$ be an independent set in G_a, and suppose $n > 13$. Referring to (6.27.1), we deduce that

$$\sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) > 18. \tag{6.29.1}$$

However, each $g(\alpha,\beta) \leq 3$, so the sum on the left cannot exceed 18. This contradiction establishes Theorem 6.29.

Theorem 6.30. If $n = 13$, then G_a does not contain four independent points.

Proof. Suppose that $S = \{i,j,k,\ell\}$ is an independent set in G_a. Since $n = 13$, (6.27.1) becomes

$$\sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) = 18. \tag{6.30.1}$$

The sum on the left has six terms, each less than or equal to three; therefore each $g(\alpha,\beta)$ must equal three. Applying this to (6.27.5) we conclude that for $\gamma \in S$, $g(\gamma) = 2$.

Now let $G_a(\ell)$ be the subgraph of G_a induced by $G_a - \{\ell\} - N_a(\ell)$. Since G_a has $3(13-3) = 30$ points, and $n - 2 = 11$, $G_a(\ell)$ has $30 - 11 - 1 = 18$ points. We claim that $G_a(\ell)$ is regular of valence 8. This is true of i, j, and k, since $g(i,\ell) = g(j,\ell) = g(k,\ell) = 3$ (see above), so i, j, and k are each adjacent to $11-3 = 8$ points of $G_a(\ell)$. Consider next a vertex x in $G_a(\ell)$ adjacent to i, but not to j or k. Since $\{x,j,k,\ell\}$ is an independent set, it follows from (6.30.1) that $g(x,\ell) = 3$. Therefore x is adjacent to $11-3 = 8$ points of $G_a(\ell)$. Similarly, the points in $J(j;S)$ and $J(k;S)$ are adjacent to 8 points of $G_a(\ell)$.

Finally, suppose $y \in G_a(\ell)$ is adjacent to two of i, j, and k, and not adjacent to the third, call it α, ($\alpha \in \{i,j,k\}$). Since $\{y,\alpha,\ell\} = R$ is an independent set, we obtain from Corollary 6.18 that $g(\phi;R) = g(y,\alpha) + g(y,\ell) + g(\alpha,\ell) - 6$. We know that $g(\alpha,\ell) = 3$, $g(y,\alpha) \geq 2$, and $g(y,\ell) \geq 2$; thus $g(\phi;R) \geq 1$. Thus there is a $z \in G_a(\ell)$ which is not adjacent to y, α, or ℓ, i.e., $\{z,y,\alpha,\ell\}$ is an independent set. It follows from (6.29.1) that $g(y,\ell) = 3$, and therefore y is adjacent to $11-3 = 8$ points of $G_a(\ell)$. All points of $G_a(\ell)$ have been accounted for, so our claim has been substantiated.

Next, let x and y be independent points in $G_a(\ell)$. We established above that $g(x,\ell) = g(y,\ell) = 3$. Since $\{x,y,\ell\}$ is an independent set, Corollary 6.18 implies that there is a $z \in G_a(\ell)$ not adjacent to any of $\{x,y,\ell\}$, i.e., $\{x,y,\ell,z\}$ is an independent set. Thus (6.30.1) implies that $g(x,y) = 3$. These three points adjacent to x and y must all lie in $G_a(\ell)$, for if one of them, call it β, lies in $G_a - G_a(\ell)$, then $\{\beta;x,y,\ell\}$ is a 3-claw.

In summary $G_a(\ell)$ has 18 vertices, each of valence 8, and two vertices of $G_a(\ell)$ which are not adjacent are mutually adjacent to three vertices in $G_a(\ell)$. Let $x \in G_a(\ell)$, and let $Y = \{y|y \in G_a(\ell), y \neq x, \text{ and } e(y,x) = 0\}$. The subgraph H of $G_a(\ell)$ induced by Y has 9 points, since x has valence 8 in $G_a(\ell)$. For each $y \in H$, $g(x,y) = 3$ (see above) and these three neighbors of y are not in H (since they are adjacent to x). Thus the remaining five neighbors of y lie in H. Thus H is regular, of valence 5, and $|H| = 9$.

Now for any graph, the sum of the valences of the points in the graph is twice the number of edges, which must be an even number; but for H the sum of the valences is $9 \times 5 = 45$. Therefore, for $n = 13$, G_a does not contain four independent points. QED

For $n = 9$, 10, 11, and 12 we take a different approach. We assume the existence of four independent points in G_a: x, y, z, and w. We then calculate all the essentially distinct possibilities for $\{g(\alpha)\}$ and $\{g(\alpha,\beta)\}$ where α and β range over $\{x,y,z,w\}$ and $\alpha \neq \beta$. We pick an appropriate pair (α,β) from $\{x,y,z,w\}$ and look at $G_a(\alpha,\beta)$. Using the tools at our disposal (Lemmas 6.21-6.28) we try to derive a contradiction.

Henceforth we call $S = \{x,y,z,w\}$.

Theorem 6.31. If $n = 12$, then G_a does not contain four independent points.

Proof. Substituting $n = 12$ into (6.27.1) and (6.28.1) we obtain

$$\sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) = 17, \tag{6.31.1}$$

and

$$\sum_{\alpha \in S} g(\alpha) = 6. \tag{6.31.2}$$

Now, there are $\binom{4}{2} = 6$ terms in the sum in (6.31.1), and each $g(\alpha,\beta)$ is 2 or 3. Therefore, in order for equality to hold in (6.31.1), precisely one term must be 2 and the remaining five terms must be 3. Without loss of generality, we set $g(x,y) = 2$; the remaining $g(\alpha,\beta)$ are 3. Applying this to (6.28.2) we find that $g(x) = g(y) = 2$ and $g(z) = g(w) = 1$.

We analyze $G_a(x,y)$. $|G_a(x,y)| = |\{w,z\}| + |J(w)| + |J(z)| + |J(w,z)| = 2 + 1 + 1 + 3 = 7$. We label the five points in $G_a(x,y) - \{w,z\}$ p_1, p_2, p_3, p_4, and p_5 and stipulate that $\{p_1,p_2,p_3\} = J(w,z)$, $\{p_4\} = J(w)$, and $\{p_5\} = J(z)$. We will show that $G_a(x,y)$ must be as diagramed in Figure 6.

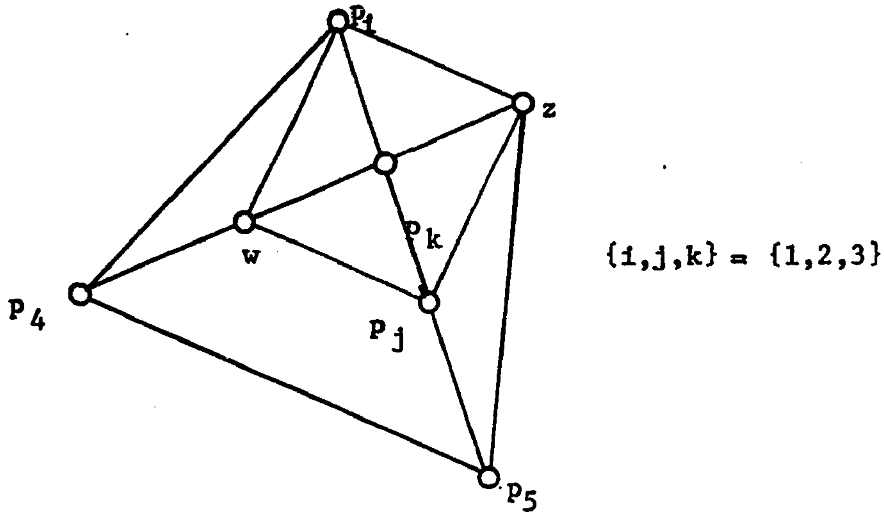


Figure 6. The graph of Theorem 6.31.

<u>Lemma 6.32</u>. $\{p_1,p_2,p_3\}$ is not a 3-clique.

<u>Proof</u>. Suppose that $\{p_1,p_2,p_3\}$ is a 3-clique.

Suppose further that $e(p_4,p_5) = 0$. Then there is a point in $G_a(x,y)$ which is adjacent to both p_4 and p_5 , call it α . $\alpha \notin \{w,z\}$ (by definition) so $\alpha \in \{p_1,p_2,p_3\}$. Thus α is adjacent to all 6 remaining points of $G_a(x,y)$. Let $R = \{\alpha,x,y\}$. Then we have the following sum:

$$\begin{aligned}|G_a| = 27 &= |R| + g(\alpha;R) + g(x;R) + g(y;R) \\ &\quad + g(x,y;R) + g(\alpha,x;R) + g(\alpha,y;R) \\ &= 3 + 6 + 2 + 2 + 2 + g(\alpha,x;R) + g(\alpha,y;R) .\end{aligned}$$

Rearranging, $12 = g(\alpha,x;R) + g(\alpha,y;R)$.

This is impossible, since the right hand terms are each bounded above by 3 . Therefore $e(p_4,p_5) = 1$. Now, $e(p_4,z) = 0$ so Lemma 6.23 implies that p_4 is adjacent to at least one of p_1,p_2,p_3, call it p_i . $e(p_5,p_i) = 0$, otherwise p_i is analogous to α above and the same contradiction may be derived. Thus (p_4,p_i,z,p_5) is a 4-cycle. Lemma 6.23 further implies that p_5 is adjacent to at least one of p_1,p_2,p_3, call it p_j , $j \neq i$ (see above). $e(p_j,p_4) = 0$, otherwise p_j is, again, analogous to α above. But now, p_j is adjacent to exactly three points of the 4-cycle, namely p_i, z, and p_5, and this violates Theorem 6.5. Since p_i, $p_j \in \{p_1,p_2,p_3\}$, $\{p_1,p_2,p_3\}$ is not a 3-clique. QED

Lemma 6.33. Let p_i and p_j be as in Lemma 6.32, and let $\{i,j,k\} = \{1,2,3\}$. Then p_k is adjacent to both p_i and p_j.

Proof. We first observe that since $e(p_i,p_j) = 0$ and p_i and p_j are each adjacent to both w and z, (w,p_i,z,p_j) is a 4-cycle. Since p_k is also adjacent to w and z, we deduce from Lemma 6.5 that p_k is adjacent to p_i and p_j.

Lemma 6.34. $G_a(x,y)$ is as diagramed in Figure 6.

Proof. Regarding the situation with respect to the 4-cycle (w,p_i,z,p_j), p_4 is adjacent to w and not z (by definition) so p_4 is adjacent to precisely one of p_i, p_j. Similarly, p_5 is adjacent to exactly one of p_i, p_j.

We claim that p_4 and p_5 are not adjacent to the same point of $\{p_i,p_j\}$. Suppose this is false; suppose, w.l.o.g., that p_4 and p_5 are both adjacent to p_i. Then neither is adjacent to p_j. Lemma 6.23 implies that p_4 and p_j have at least two mutual neighbors, and these cannot be among p_5, p_i, and z, since $e(z,p_4) = e(p_i,p_j) = e(p_5,p_j) = 0$. Therefore $e(p_4,p_k) = 1$. Similarly, p_5 and p_j have at least two neighbors, these cannot be among p_4, p_i, and w, so $e(p_5,p_k) = 1$. But now we have shown that p_k is adjacent to every other point of $G_a(x,y)$. This violates Lemma 6.24. Thus our claim is valid.

Therefore one of $\{p_4,p_5\}$, say p_4, is adjacent to p_i; then p_5 is adjacent to p_j. We claim that $e(p_4,p_5) = 1$. Suppose $e(p_4,p_5) = 0$. None of p_i, w, p_j, and z is adjacent to both p_4 and p_5, so at most one point of $G_a(x,y)$ is a mutual neighbor of p_4 and p_5, namely p_k. But Lemma 6.23 implies that if $e(p_4,p_5) = 0$, then there must be two or three mutual neighbors. Therefore $e(p_4,p_5) = 1$.

We also have that $e(p_4,w) = e(w,p_j) = e(p_j,p_5) = 1$ and $e(p_4,p_j) = e(p_5,w) = 0$. This means that (p_4,w,p_j,p_5) is a 4-cycle. We know that $e(p_k,p_w) = e(p_k,p_j) = 1$, so by Lemma 6.5, p_k is either adjacent to both p_4 and p_5 or to neither. If p_k is adjacent to both p_4 and p_5, then p_k is adjacent to every other point of $G_a(x,y)$, and this cannot be (Lemma 6.24). Thus p_k isn't adjacent to p_4 or p_5. Now we have determined all adjacencies in $G_a(x,y)$ and they are as diagrammed in Figure 6. QED

We proceed with the proof of Theorem 6.31.

Let $P = G_a - G_a(x,y) - \{x,y\} - J(x,y)$. Then $|P| = 27 - 7 - 2 - 2 = 16$. Let α be a point in $G_a(x,y)$, β a point in $(x,y) \cup J(x,y)$. Then $e(\alpha,\beta) = 0$. For if $\beta \in \{x,y\}$, then by definition $e(\alpha,\beta) = 0$. And if $\beta \in J(x,y)$, $e(\alpha,\beta) = 1$, then $\{\beta;\alpha,x,y\}$ is a 3-claw, which cannot be. Therefore all the neighbors of β lie in $G_a(x,y) - \{x,y\} - J(x,y)$.

Let us count edges incident to points in $G_a(x,y)$. G_a is regular of valence $12 - 2 = 10$, so the sum of the valences of points in $G_a(x,y)$ is $7 \times 10 = 70$. Thus

$$70 = 2e(G_a(x,y),G_a(x,y)) + e(G_a(x,y),P) = 26 + e(G_a(x,y),P).$$

Therefore $e(G_a(x,y),P) = 44$.

Now $|J_x| = |J_y| = 2$. Let β be a point in $J_x \cup J_y$ and let $\alpha \in \{w,z,p_1,p_2,p_3\}$. Then $e(\alpha,\beta) = 0$. For if $\alpha \in \{w,z\}$, then by definition $e(\alpha,\beta) = 0$. And if $\alpha \in \{p_1,p_2,p_3\}$ and $e(\alpha,\beta) = 1$, then $\{\alpha;w,z,\beta\}$ is a 3-claw, which cannot be. Therefore β can be adjacent to at most two points of $G_a(x,y)$, namely p_4 and p_5. Consequently, $e(J_x,G_a(x,y)) \leq 4$ and $e(J_y,G_a(x,y)) \leq 4$, i.e., $e(J_x \cup J_y,G_a(x,y)) \leq 8$. Let $Q = P - J_x - J_y$. Then $e(G_a(x,y),Q) \geq 44 - 8 = 36$. Thus the average number of edges from points in Q to points in $G_a(x,y)$ is at least $\frac{36}{|Q|} = \frac{36}{12} = 3$.

Let $\beta \in Q$ and let R be the set of points in $G_a(x,y)$ adjacent to β. Then Lemma 6.21 implies that R is a clique. Examining $G_a(x,y)$ (see Figure 6) we see that $G_a(x,y)$ does not contain a 4-clique. Therefore β is adjacent to at most three points of $G_a(x,y)$, and since this coincides with the average calculated above, we may conclude that every point of Q is adjacent to precisely three points of $G_a(x,y)$.

Consequently the inequalities above may be replaced by equalities: in particular, we are interested in the fact that $e(J_x \cup J_y,G_a(x,y)) = 8$. We infer from this that p_4 is adjacent to each point of $J_x \cup J_y$. Thus p_4 has three neighbors in $G_a(x,y)$ and four neighbors in $J_x \cup J_y$, leaving three neighbors in Q for a total of ten.

Now Q is the union of $J(x,w)$, $J(x,z)$, $J(y,w)$, and $J(y,z)$. Since $p_4 \in J(w)$, p_4 has no neighbors in $J(x,z) \cup J(y,z)$; otherwise we obtain a 3-claw $\{\beta; z, \gamma, p_4\}$, where β is the neighbor of p_4 in question and $\gamma \in \{x,y\}$. Therefore, the three neighbors of p_4 that lie in Q must lie in $J(x,w) \cup J(y,w)$. Now $e(p_4,x) = e(p_4,y) = 0$, so $\Delta(p_4,x) \leq 3$ and $\Delta(p_4,y) \leq 3$. (Lemma 6.7). Since $e(p_4,J_x)$ $= e(p_4,J_y) = 2$ (see above), p_4 can have at most one neighbor in each of $J(x,w)$, $J(y,w)$. Therefore P_4 has at most two neighbors in Q. But we have just shown that p_4 has three such neighbors. This contradiction establishes Theorem 6.31.

Theorem 6.35. If $n = 11$ then G_a does not contain four independent points.

Proof. Substituting $n = 11$ into (6.27.1) and (6.28.1) we obtain

(6.35.1) $$\sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) = 16$$

and

(6.35.2) $$\sum_{\alpha \in S} g(\alpha) = 4.$$

Each summand in (6.35.1) is 2 or 3, so in order for equality to hold, two terms must be 2 and four terms must be 3. We can label the points of S so that one of two cases holds.

Case (a). $g(x,w) = g(y,z) = 2$ and $g(x,y) = g(x,z) = g(y,w) = g(z,w) = 3$. Applying this data to (6.28.2) we find that $g(x) = g(y) = g(z) = g(w) = 1$.

Case (b). $g(y,w) = g(y,z) = 2$ and $g(x,y) = g(z,w) = g(x,z) = g(x,w) = 3$. Applying this to (6.28.2) we find that $g(y) = 2$, $g(x) = 0$, and $g(z) = g(w) = 1$.

We analyze each case separately.

Case (a). We look at $G_a(x,y)$. As in the proof of Theorem 6.31, $G_a(x,y)$ consists of seven points, which we label (as before) p_1, p_2, p_3, (the elements of $J(w,z)$), p_4 (the sole element of $J(w)$), p_5 (the sole element of $J(z)$), and, of course, w and z. Now, if the reader will examine the proofs of Lemmas 6.32, 6.33, and 6.34, he will find that they did not depend on the value of n. Therefore $G_a(x,y)$ is as diagrammed in Figure 6.

Define P as before:

$$P = G_a - G_a(x,y) - \{x,y\} - J(x,y). \tag{6.35.3}$$

Let $\alpha \in G_a(x,y)$. Then α is not adjacent to x or y (by definition) and α is not adjacent to any point β in $J(x,y)$ (or we have a 3-claw $\{\beta;x,y,\alpha\}$). Therefore the neighbors of α which do not lie in $G_a(x,y)$ must be in P; i.e.,

$$e(\alpha,P) = 9 - e(\alpha,G_a(x,y)). \tag{6.35.4}$$

Notice that G_a is regular of valence $11-2 = 9$. Referring to Figure 6 and repeatedly using (6.35.4) we obtain the following:

$$e(p_4,P) = e(p_5,P) = 6$$

$$e(w,P) = e(z,P) = e(p_1,P) = e(p_2,P) = e(p_3,P) = 5.$$

Thus $e(G_a(x,y),P) = 2 \times 6 + 5 \times 5 = 37$. Referring to (6.35.3) we see that

$$(6.35.5) \qquad P = J(x,z) \cup J(x,w) \cup J(y,z) \cup J(y,w) \cup J(x) \cup J(y),$$

and so

$$(6.35.6) \qquad |P| = g(x,z) + g(x,w) + g(y,z) + g(y,w) + g(x)$$

$$+ g(y) = 3 + 2 + 2 + 3 + 1 + 1 = 12.$$

Therefore the average number of neighbors in $G_a(x,y)$ of points in P is

$$\frac{e(G_a(x,y),P)}{|P|} = \frac{37}{12} = 3\frac{1}{12}.$$

Therefore at least one $\beta \in P$ has four neighbors in $G_a(x,y)$. Lemma

6.21 implies that these four points must be a 4-clique. Referring to Figure 6 we see that $G_a(x,y)$ does not contain a 4-clique. This contradiction shows that Case (a) is vacuous.

Case (b). Again, we look at $G_a(x,y)$. From this perspective all the significant parameters are the same as in Case (a); the proof carries through word for word, except that (6.35.6) would be rewritten as

$$|P| = 3 + 3 + 2 + 2 + 0 + 2 = 12.$$

Theorem 6.36. If $n = 10$, then G_a does not contain four independent points.

Proof. Substituting $n = 10$ into (6.27.1) and (6.28.1) we obtain

(6.36.1) $$\sum_{\substack{\alpha \neq \beta \\ \alpha,\beta \in S}} g(\alpha,\beta) = 15$$

and

(6.36.2) $$\sum_{\alpha \in S} g(\alpha) = 2.$$

Each summand in (6.36.1) is 2 or 3, so in order for equality to hold, three terms must be 3 and three terms must be 2. Up to relabeling, there are three distinct ways for (6.36.1) to be satisfied.

Case (a). $g(x,y) = g(x,w) = g(y,z) = 3$ and $g(x,z) = g(y,w) = g(z,w) = 2$. Applying this data to (6.28.2) we find that

$$g(z) = g(w) = 1 \quad \text{and} \quad g(x) = g(y) = 0.$$

Case (b). $g(x,y) = g(y,z) = g(x,z) = 3$ and $g(x,w) = g(y,w) = g(z,w) = 2$. Applying this data to (6.28.2) we find that

$$g(w) = 2 \quad \text{and} \quad g(x) = g(y) = g(z) = 0.$$

Case (c). $g(x,y) = g(x,z) = g(x,w) = 3$ and $g(y,z) = g(y,w) = g(z,w) = 2$. Applying this data to (6.28.2) we find that

$$g(x) = g(y,z) + g(y,w) + g(z,w) - 7 = 2 + 2 + 2 - 7 = -1.$$

Since $g(x)$ is, by definition, nonnegative, Case (c) is vacuous.

Case (a). $G_a(x,y)$ consists of six points: z, w, the two points of $J(z,w)$, the single point of $J(z)$, and the single point of $J(w)$. Label the latter four points p_1, p_2, p_3, and p_4 respectively.

<u>Subcase (1).</u> $e(p_1,p_2) = 0$.

If $e(p_1,p_2) = 0$, then (w,p_1,z,p_2) is a 4-cycle. Since $e(p_4,w) = 1$ and $e(p_4,z) = 0$ (by definition), Lemma 6.5 implies that p_4 is adjacent to exactly one of p_1 and p_2. Without loss of generality we may assume that $e(p_4,p_1) = 1$ and $e(p_4,p_2) = 0$. Similarly p_3 is adjacent to exactly one of p_1 and p_2. Suppose $e(p_3,p_1) = 1$ and $e(p_3,p_2) = 0$. Then z is the only point of $G_a(x,y)$ adjacent to both p_3 and p_2, and $e(p_2,p_3) = 0$; this violates Lemma 6.23. Therefore $e(p_3,p_1) = 0$ and $e(p_3,p_2) = 1$.

Now, Lemma 6.23 implies that at least two points of $G_a(x,y)$ are adjacent to both z and p_4. The neighbors of z in $G_a(x,y)$ are p_1, p_2, and p_3; we know p_1 is adjacent to p_4 and p_2 is not (see above). Therefore $e(p_3,p_4) = 1$. $G_a(x,y)$ is depicted in Figure 7.

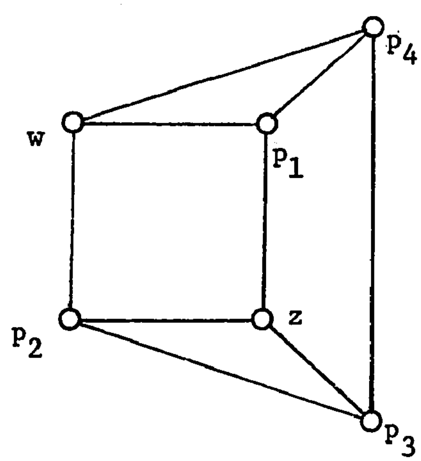


Figure 7. The graph of Theorem 6.36, Subcase (1).

We see that $G_a(x,y)$ is regular with valence 3. Since G_a has valence $10 - 2 = 8$, each point of $G_a(x,y)$ is adjacent to $8 - 3 = 5$ points of $P = J(x,z) \cup J(x,w) \cup J(y,z) \cup J(y,w) \cup J(x) \cup J(y)$. P consists of $2 + 3 + 3 + 2 + 0 + 0 = 10$ points. Therefore the average number of neighbors in $G_a(x,y)$ of points in P is $\frac{6\times5}{10} = 3$. $G_a(x,y)$ does not contain a 4-clique (see Figure 7), so Lemma 6.21

implies that no point of P is adjacent to four points of $G_a(x,y)$. Therefore every point of P is adjacent to exactly three points of $G_a(x,y)$.

Let $\beta \in J(x,w)$. The three neighbors of β in $G_a(x,y)$ are a 3-clique that contains w. Referring to Figure 7 we see that they are w, p_1, and p_4. Now $|J(x,w)| = g(x,w) = 3$, so let $\gamma \in J(x,w)$, $\gamma \neq \beta$. Both γ and β are adjacent to p_1, p_4, w, and x. Thus γ must be adjacent to β, for if not, then we have $\Delta(\gamma,\beta) \geq 4$ which violates $\Delta(p,q) = 2$ or 3 which holds for all non-adjacent pairs p and q in G_a. But now, we see that every point of G_a is adjacent to at least one point of $\{\beta,z,y\}$: w, x, p_1, p_4 and the two other points of $J(x,w)$ are adjacent to β; p_2, p_3, and the points in $J(x,z) \cup J(y,z)$ are adjacent to z; the points in $J(y,w) \cup J(y,z) \cup J(y,x)$ are adjacent to y. This accounts for every point of $G_a - \{\beta,z,y\}$ Therefore,

$$g(\phi;\{\beta,z,y\}) = 0.$$

But from Corollary 6.18,

$$g(\phi;\{\beta,z,y\}) = g(\beta,z) + g(\beta,y) + g(y,z) - 6.$$

We know that $g(y,z) = 3$ and $g(\beta,z) \geq 2$ and $g(\beta,y) \geq 2$. Therefore $g(\phi;\{\beta,z,y\})$ is at least 1. This contradiction proves that Subcase (1) is vacuous.

<u>Subcase (2)</u>. $e(p_1,p_2) = 1$.

From the facts that $e(p_4,z) = e(p_3,w) = 0$, we are assured by Lemma 6.23 that p_4 is adjacent to at least two points of $\{p_1,p_2,p_3\}$ and that p_3 is adjacent to at least two points of $\{p_1,p_2,p_4\}$. Lemma 6.24 implies that neither p_1 nor p_2 is adjacent to both p_3 and p_4. We can conclude that $e(p_3,p_4) = 1$, and p_4 is adjacent to one of $\{p_1,p_2\}$, all it p_i, and p_3 is adjacent to the other, call it p_j, $\{i,j\} = \{1,2\}$; of course, $e(p_4,p_j) = e(p_3,p_i) = 0$. This gives us all adjacencies in $G_a(x,y)$ (see Figure 8).

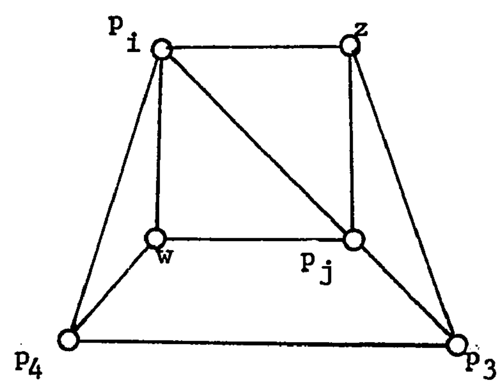


Figure 8. The graph of Theorem 6.36, Subcase (2).

Thus (p_4,w,p_j,p_3) is a 4-cycle and p_i is adjacent to exactly three points of the 4-cycle, namely p_4, w, and p_j. This violates Theorem 6.5, so Subcase (2) is vacuous; i.e., Case (a) is vacuous.

We now turn our attention to Case (b). Once again, Case (b) is defined by:

$$g(x,y) = g(y,z) = g(x,z) = 3 \quad \text{and} \quad g(x,w) = g(y,w) = g(z,w) = 2$$

$$g(w) = 2 \quad \text{and} \quad g(x) = g(y) = g(z) = 0.$$

We label the two points in $J(w,z)$ p_1 and p_2, and we label the two points in $J(w)$ p_3 and p_4.

First, we claim that $e(p_1,p_2) \neq 0$. Suppose $e(p_1,p_2) = 0$. By definition, p_4 and z are not adjacent, so they have at least two mutual neighbors, and they must be p_1 and p_2 since these are the only neighbors of z in $G_a(x,y)$. But then p_4 is adjacent to exactly three points (namely, w, p_1, and p_2) of the 4-cycle (w,p_1,z,p_2) and this violates Theorem 6.5. Therefore $e(p_1,p_2) \neq 0$.

Suppose $e(p_1,p_2) = 1$. As before, p_4 must be adjacent to p_1 and p_2. Similarly p_3 is adjacent to p_1 and p_2. But then p_1 is adjacent to every other point of $G_a(x,y)$, and this violates Lemma 6.24. Therefore $e(p_1,p_2) \neq 1$. Therefore Case (b) is vacuous. We have already established that Case (c) is vacuous, so we have proven Theorem 6.36.

<u>Theorem 6.37</u>. If $n = 9$, then G_a does not contain four independent points.

<u>Proof</u>. Substituting $n = 9$ into (6.27.1) and (6.28.1) we obtain

(6.37.1) $$\sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\in S}} g(\alpha,\beta) = 14$$

and

(6.37.2) $$\sum_{\alpha\in S} g(\alpha) = 0.$$

Each summand in (6.37.1) is 2 or 3, so in order for equality to hold, two terms must be 3 and the remaining four terms must be 2. Up to relabeling, there are two distinct ways for (6.37.1) to be satisfied.

Case (a). $g(x,y) = g(x,z) = 3$ and $g(x,w) = g(y,z) = g(y,w) = g(z,w) = 2$. Applying this data to (6.28.2) we obtain

$$g(x) = g(y,z) + g(y,w) + g(z,w) - 7 = 2 + 2 + 2 - 7 = -1.$$

Therefore Case (a) is vacuous.

Case (b). $g(x,y) = g(z,w) = 3$ and $g(x,z) = g(x,w) = g(y,z) = g(y,w) = 2$. Applying this data to (6.28.2) we obtain $g(x) = g(y) = g(z) = g(w) = 0$, which is consistent with (6.37.2).

We look at $G_a(x,y)$, which consists of five points: w, z, and the three points of $J(w,z)$. Let $\alpha \in G_a(x,y)$, and let R denote the independent set $\{\alpha,x,y\}$. Corollary 6.18 implies that

(6.37.3) $$g(\phi;R) = g(\alpha,x) + g(\alpha,y) + g(x,y) - 6,$$

Let $\beta \in J(\phi;R)$. Then $\beta \in G_a(x,y)$. Since $\{\beta,\alpha,x,y\} = Q$ is an independent set, (6.37.2) tells us that every neighbor of β must also be adjacent to exactly one of α, x, and y. Since G_a is regular of valence $9 - 2 = 7$, this implies that

$$(6.37.4) \qquad g(\beta,\alpha) + g(\beta,x) + g(\beta,y) = 7.$$

Combining (6.37.1), (6.37.3) and (6.37.4) we obtain

$$\sum_{\substack{p,q\in Q \\ p\neq q}} g(p,q) = 14 = 7 + (g(\phi;R) + 6).$$

We conclude that $g(\phi;R) = 1$; i.e., each point $\alpha \in G_a(x,y)$ is adjacent to all but one of the remaining points of $G_a(x,y)$. However, there is no graph on five points which is regular of valence 3; therefore G_a cannot contain four independent points. QED

In summary, what we have established so far is that for $n \geq 9$, G_a contains neither a 3-claw nor four independent points. We now prove that for any two independent points x and y in G_a, $\Delta(x,y) = 2$.

<u>Theorem 6.38</u>. Let x and y be non-adjacent points of G_a. Then x and y have precisely two mutual neighbors.

<u>Proof</u>. Lemma 6.6 states that there exist a point $z \in G_a$ such that $\{x,y,z\}$ is an independent set. Theorem 6.10 states that

(6.38.1) $$g(x,y) + g(x,z) + g(y,z) + 2g(x,y,z) - g(\phi) = 6.$$

G_a does not contain a 3-claw (Theorem 6.17), so $g(x,y,z) = 0$. G_a does not contain four independent points (Theorems 6.29, 6.30, 6.31, 6.35, 6.36, and 6.37) so $g(\phi) = 0$. Thus (6.38.1) simplifies to

(6.38.2) $$g(x,y) + g(x,z) + g(y,z) = 6.$$

Each term on the left hand side of (6.38.2) is 2 or 3 (Lemma 6.7), so in order for equality to hold, each term must be 2. QED

In the following last result of this chapter we determine how the four mutual second associates of two given first associates relate to each other.

<u>Theorem 6.39</u>. Let x, $y \in A_n$, $\alpha(x,y) = 1$. Let a, b, c, and d be the four elements of A_n which are second associates of both x and y. Then the association matrix M of $\{x,y,a,b,c,d\}$ is row-equivalent to

(6.39.1) $$\begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 0 \end{pmatrix}$$

Proof. x and y are non-adjacent points in G_a; so $\Delta(x,y) = 2$ in G_a. Without loss of generality we may assume that the two mutual neighbors of x and y in G_a are c and d. Consider G_b; x, $y \in G_b$ and $e(x,y) = 0$ in G_b, so $\Delta(x,y) = 2$ in G_b. We already know that $b \notin G_a$, so $\alpha(a,b) \neq 2$, i.e., $\alpha(a,b) = 1$, and therefore $a \notin G_b$. Therefore c, $d \in G_b$, and $\alpha(b,c) = \alpha(b,d) = 2$. Now consider G_c; x, $y \in G_c$ and $e(x,y) = 0$ in G_c so $\Delta(x,y) = 2$ in G_c. From our work above, a and b are second associates of c; therefore $d \notin G_c$, for otherwise $\Delta(x,y) = 3$ in G_c. Therefore $\alpha(c,d) = 1$. Summarizing the association relations among x, y, a, b, c, and d in matrix form we obtain (6.39.1). QED

CHAPTER 7

THE STRUCTURE OF G_a

7.1. Introduction. In this section we shall use the results of Chapter 6 to prove Theorems 7.9 and 7.10, which completely describe G_a. The reader is advised to compare this description of G_a with the graph of second associates of a given triple $\{i,j,k\}$ in T_n, which he can easily construct for himself. We begin by developing two graph and matrix tools.

Theorem 7.2. Suppose p, q, $r \in G_a$, $e(p,q) = e(p,r) = 1$, and $e(q,r) = 0$. Then q, p, r belong to exactly one 4-cycle (p,q,s,r) in G_a; i.e., there exists a unique $s \in G_a$ such that $e(s,q) = e(s,r) = 1$ and $e(s,p) = 0$.

Proof. $e(q,r) = 0$ so $\Delta(q,r) = 2$ in G_a. p is one such mutual neighbor of q and r, let s denote the other. Theorem 6.39 assures us that $\alpha(s,p) = 1$, i.e., $e(s,p) = 0$.

Definition 7.3. The 4-cycle discussed above will be denoted by $C(p;q,r)$. The fourth point of the 4-cycle $C(p;q,r)$ will be called $c(p;q,r)$.

Theorem 7.4. $\lambda^1(A(G_a)) = -2$.

Proof. G_a contains a 4-cycle (see above), and the adjacency matrix of the 4-cycle is

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

M is a principal submatrix of $A(G_a)$ and $\lambda^1(M) = -2$. (M has the eigenvector $(1,-1,1,-1)$ corresponding to -2.)

Theorem 7.5. Let $p \in G_a$. Then $N_a(p)$ is the disjoint union of two independent cliques.

Proof. First, we claim that $N_a(p)$ is not a clique. Suppose $N_a(p)$ is a clique; let $Q = G_a - \{p\} - N_a(p)$. Let $q \in Q$. Then $q \notin N_a(p)$ so $e(p,q) = 0$. Therefore $\Delta(p,q) = 2$ in G_a. Suppose x is adjacent to both p and q. Since, by assumption, $N_a(p)$ is a clique and $x \in N_a(p)$, x is adjacent to p, the $n - 3$ remaining neighbors of p, and q; i.e., to a total of $n - 1$ points. This contradicts the fact that G_a is regular, of valence $n - 2$.

Consequently, there exist q and $r \in N_a(p)$ such that $e(q,r) = 0$. Now suppose that $s \in N_a(p)$ and $s \notin \{q,r\}$. Then s must be adjacent to at least one of $\{q,r\}$, for otherwise $\{p,q,r,s\}$ would be a 3-claw. Also, S can't be adjacent to both q and r, for then the association matrix of $\{q,r,a,w,p,s\}$, where w is the second associate of both q and r that lies outside of G_a is

$$\begin{array}{c} \begin{matrix} q & r & a & w & p & s \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & \textcircled{2} \\ 2 & 2 & 2 & 2 & \textcircled{2} & 0 \end{bmatrix} \end{array}$$

which violates Theorem 6.39. (The circled entries should be 1.) Now suppose t, $s \in N_a(p) \cap N_a(q)$ and $e(t,s) = 0$. The paragraph above implies that $e(t,r) = e(s,r) = 0$. Therefore, $\{p;r,s,t\}$ is a 3-claw. This cannot be, so if t, $s \in N_a(p) \cap N_a(q)$, then $e(t,s) = 1$. Therefore $N_a(p) \cap N_a(q)$ is a clique. Obviously, q is adjacent to every point in $N(p) \cap N(q)$, so $D_1 = (N_a(p) \cap N_a(q)) \cup \{q\}$ is a clique. By the same reasoning, $D_2 = (N(p) \cap N(r)) \cap \{r\}$ is a clique. D_1 and D_2 are the disjoint independent cliques whose existence was asserted in the statement of Theorem 7.5. QED

<u>Theorem 7.6</u>. Let $p \in G_a$, q, $r \in N_a(p)$, $e(q,r) = 0$. Let D_1, D_2 be the disjoint cliques described in Theorem 7.5, and let u, $u' \in D_1$ and v, $v' \in D_2$. Suppose also that $(u,v) \neq (u',v')$; then $c(p;u,v) \neq c(p;u'v')$.

Proof. Let $s = c(p;u,v)$. Then (p,u,s,v) is a 4-cycle (by definition), so $e(p,s) = 0$. Theorem 6.38 assures us that $\Delta(p,s) = 2$, so u and r are the only elements of G_a adjacent to both p and s. Since by assumption at least one of $\{u',v'\}$ doesn't belong to $\{u,v\}$, at least one of $\{u',v'\}$ isn't adjacent to s. Therefore $c(p;u',v') \neq s = c(p;u,v)$, for otherwise, $\Delta(p,s) > 2$. QED

Theorem 7.7. $\{|D_1|,|D_2|\} = \{2,n-4\}$.

Proof. First we will show that

$$G_a = \{p\} \cup D_1 \cup D_2 \cup \{c(p;u,v) | u \in D_1, v \in D_2\}. \tag{7.7.1}$$

Let $x \in G_a - \{p\} - D_1 - D_2$. Then $e(x,p) = 0$, and therefore by Theorem 6.38 p and x have two mutual neighbors, u and v, which both lie in $N_a(p) = D_1 \cup D_2$. Theorem 6.39 implies that $e(u,v) = 0$, so w.l.o.g. we may assume that $u \in D_1$ and $v \in D_2$, since by Theorem 7.5 D_1 and D_2 are disjoint independent cliques. By definition, this means that $x = c(p;u,v)$. Thus, we have (7.7.1). Theorem 7.6 implies that

$$|\{c(p;u,v) | u \in D_1, v \in D_2\}| = |D_1| \cdot |D_2|. \tag{7.7.2}$$

Therefore, counting points in (7.7.1) we obtain

(7.7.3) $$3(n - 3) = 1 + |D_1| + |D_2| + |D_1| \cdot |D_2|$$

$$= (1 + |D_1|)(1 + |D_2|).$$

Since $D_1 \cup D_2 = N_a(p)$ we must have $|D_1| + |D_2| = n - 2$, or

(7.7.4) $$|D_2| = n - 2 - |D_1|.$$

Substituting (7.7.4) into (7.7.3) we obtain

(7.7.5) $$3(n - 3) = (|D_1| + 1)(n - 2 - |D_1| + 1)$$

$$= -|D_1|^2 + (n - 2)D_1 + n - 1.$$

Rearranging (7.7.5) yields

(7.7.6) $$|D_1|^2 - (n - 2)D_1 + (2n - 8) = 0$$

$$(|D_1| - (n - 4))(D_1 - 2) = 0.$$

Therefore, $|D_1| = n - 4$ or 2. Combining this with (7.7.4) we get

$$\{|D_1|, |D_2|\} = \{2, n - 4\}.$$ QED

Lemma 7.8. Let $\{i,j\} = \{1,2\}$, let $u \in D_i$, and let v, $w \in D_j$, $v \neq w$. Then $c(p;u,v)$ is adjacent to $c(p;u,w)$.

Proof. If the statement above is false, then since by Theorem 7.6 $c(p;u,v) \neq c(p;u,w)$, $\{u;p,c(p;u,v),c(p;u,w)\}$ is a 3-claw, which is impossible.

Theorem 7.9. G_a is the disjoint union of three cliques, each with $n - 3$ elements.

Proof. We assume, w.l.o.g., that $|D_1| = n - 4$ and $|D_2| = 2$. We label the two points of D_2 q and r. We define

(7.9.1)
$$C_1 = \{p\} \cup D_1$$
$$C_2 = \{q\} \cup \{c(p;q,u)\,|\,u \in D_1\}$$
$$C_3 = \{r\} \cup \{c(p;r,u)\,|\,u \in D_1\}.$$

We claim that C_1, C_2 and C_3 are disjoint cliques, each with $n - 3$ elements, whose union is G'_a. First, from the way the C_i are defined, and from Theorem 7.6, we know that the C_i are disjoint. D_1 is a clique of order $n - 4$ lying in $N_a(p)$, so C_1 is a clique of order $n - 3$. Lemma 7.8 implies that $\{c(p;q,u)\,|\,u \in D_1\}$ and $\{c(p;r,u)\,|\,u \in D_1\}$ are each $n - 4$ cliques; also q is adjacent to each $c(p;q,u)$, and r is adjacent to each $c(p;r,u)$ (by definition), so C_2 and C_3 are each cliques of order $n - 3$. Since

$$C_1 \cup C_2 \cup C_3 \subseteq G_a \quad \text{and} \quad C_1 \cup C_2 \cup C_3 = |C_1| + |C_2| + |C_3| = 3(n-3)$$

$$= |G_a|,$$

(the C_i are disjoint) we have established Theorem 7.9.

Theorem 7.10. Let $p \in G_a$. Then there exist unique q, $r \in G_a$ such that p, q, and r belong to distinct $n - 3$ cliques in G_a and $e(p,q) = e(q,r) = e(p,r) = 1$.

Proof. Let q and r be defined as in Theorem 7.9. Then p, q, and r lie in the distinct $n - 3$ cliques C_1, C_2, and C_3, respectively. Also, $e(p,q) = e(p,r) = 1$ since q, $r \in D_2 \subseteq N_a(p)$; furthermore, $e(q,r) = 1$, since D_2 is a clique (Theorem 7.5). By definition, no $c(p;q,u)$ or $c(p;r,u)$, where $u \in D_1$, is adjacent to p, so q and r are the unique points of C_2 and C_3 adjacent to p. This establishes Theorem 7.10.

We summarize Theorems 7.9 and 7.10 in Figure 9 below.

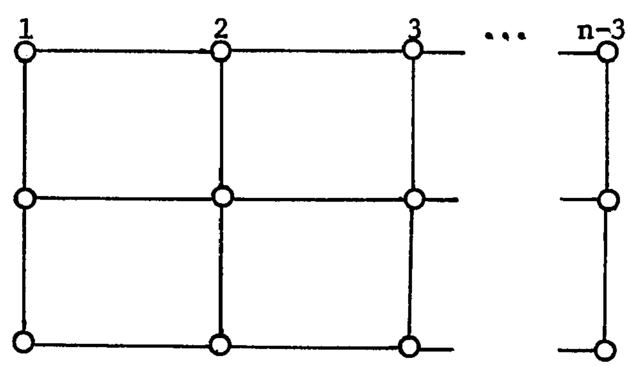


Figure 9. What G_a looks like.

This diagram is interpreted as follows: two points are adjacent iff they lie on a common horizontal or vertical line.

CHAPTER 8

THE RECONSTRUCTION OF THE TRIPLES

In this final chapter we prove that A_n is isomorphic to T_n and derive a specific isomorphic mapping from A_n onto T_n.

Definition 8.1. Let $a \in A_n$, $i \in \{1,2,3\}$. In Theorem 7.9 we introduced (n-3)-cliques C_1, C_2 and C_3 whose disjoint union is G_a. Hereinafter we will label these same cliques $C_i(a)$ to indicate the dependence on a. We define

$$S_i(a) = C_i(a) \cup \{a\},$$

and refer to $S_i(a)$ as an "associated clique" or "AC" for short.

Theorem 8.2. Let $a \in A_n$, Then a lies in exactly three ACs.

Proof. According to Definition 8.1, a certainly lies in $S_1(a)$, $S_2(a)$, and $S_3(a)$. Suppose a lies in an AC, call it S. Then every element of $S - \{a\}$ is a second associate of a. Therefore, $S - \{a\}$ is an (n-3)-clique in G_a and must coincide with $C_1(a)$, $C_2(a)$, or $C_3(a)$ (Theorem 7.9). Consequently, S must be one of $S_1(a)$, $S_2(a)$, $S_3(a)$.

Theorem 8.3. There are $\binom{n}{2}$ distinct ACs.

Proof. Each AC has $n - 2$ elements, each element of A_n lies in exactly 3 ACs (Theorem 8.3), and $|A_n| = \binom{n}{3}$. Therefore $(n - 2) \times (\#\text{ distinct ACs}) = 3\binom{n}{3}$. Rearranging this,

$$\#\text{ distinct ACs} = \frac{3}{n-2}\binom{n}{3} = \frac{3}{n-2} \cdot \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)}{2} = \binom{n}{2}.$$

Theorem 8.4. If S and R are distinct ACs then $|S \cap R| = 0$ or 1.

Proof. Suppose $x, y \in S \cap R$, $x \neq y$. Then $S - \{x\}$ and $R - \{x\}$ are both $n - 3$ cliques in G_x. Since G_x is the disjoint union of three $(n-3)$-cliques, and since y belongs to both $S - \{x\}$ and $R - \{x\}$, we must have $S - \{x\} = R - \{x\}$. Therefore $S = R$.

Theorem 8.5. If $x, y \in A_n$ are second associates, then there is exactly one AC containing both x and y.

Proof. If there are two distinct ACs containing x and y, then their intersection has cardinality at least two, in violation of Theorem 8.4. QED

Theorems 8.4 and 8.5 suggest a geometric interpretation of L_n. In the diagrams accompanying the following text we represent the ACs as lines, and the elements of ACs as points on lines.

Definition 8.6. We define a graph L_n as follows: the points of L_n are the $\binom{n}{2}$ distinct ACs. Two distinct ACs are adjacent iff they have a non-empty intersection.

Lemma 8.7. Let $a \in A_n$. Then there does not exist an AC $R \in L_n$ such that $a \notin R$ and R is adjacent to $S_1(a)$, $S_2(a)$, and $S_3(a)$.

Proof. The situation described in the lemma is diagrammed in Figure 10 below.

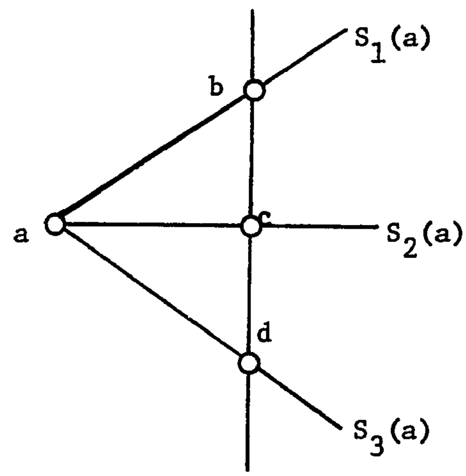


Figure 10. The graph of Lemma 8.7.

Suppose we have such an AC, call it R. We define $\{b\} = R \cap S_1(a)$, $\{c\} = R \cap S_2(a)$, $\{d\} = R \cap S_3(a)$. There must exist $x \in R$ such that $x \notin G_a$, for otherwise R is an $(n-2)$-clique in G_a, which by Theorem 7.9 is impossible. Thus x must be a 0^{th} or first associate of a. Now $\alpha(x,a) = 0$ is impossible, since b is a second associate of both x and a, and $p^0_{22} = 0$ (Table 1). $\alpha(x,a) = 1$ is also impossible, since the association matrix of $\{x,a,b,c,d\}$ would be

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix}$$

which is ruled out by Theorem 6.39. QED

It is known [7] that if $n \neq 8$ and if the points of a graph of order $\binom{n}{2}$ satisfy the following conditions, then the graph is isomorphic to $L(K_n)$, the line graph of the complete graph on n vertices:

(1) The number of points adjacent to any given point is $2(n-2)$;

(2) If two points are adjacent, then there are $n-2$ points adjacent to both;

(3) If two points are <u>not</u> adjacent, then there are four points adjacent to both.

We will show that L_n satisfies conditions (1)-(3).

<u>Theorem 8.8</u>. Condition (1) holds for L_n.

<u>Proof</u>. Let $R \in L_n$ be given and let the elements of R be $a_1, \ldots, a_{n-2}$. Without loss of generality, we may identify R as $S_3(a_k)$ for each $k \in I_{n-2}$. The $2(n-2)$ ACs given by $S_i(a_k)$, $k \in I_{n-2}$, $i \in I_2$ are all adjacent to R, since $R \cap S_i(a_k) = S_3(a_k) \cap S_i(a_k) = \{a_k\}$.

We claim that these neighbors of R are all distinct. Suppose not; then for some k, $m \in I_{n-2}$, $k \neq m$, and some i, $j \in I_2$, $S_i(a_k) = S_j(a_m)$. Now $a_m \in S_j(a_m)$, so $a_m \in S_i(a_k)$. Also, $a_k \in S_i(a_k)$. Therefore, $S_i(a_k)$ is the unique AC determined by a_k and a_m (Theorem 8.5), and by hypothesis, this is R. But this is impossible, for then G_{a_k} would consist of only the two $(n-3)$-cliques $R - \{a_k\}$ and $S_\ell(a_k) - \{a_k\}$, where $\{i,\ell\} = I_2$. Therefore, $S_i(a_k) \neq S_j(a_m)$.

By similar reasoning we conclude that $S_1(a_k) \neq S_2(a_k)$ for each $k \in I_{n-2}$. Hence the $S_i(a_k)$ are $2(n - 2)$ distinct neighbors of R.

Finally, R has no other neighbors, since if P is adjacent to R, then $P \cap R = \{a_k\}$ for some $k \in I_{n-2}$, and therefore $P \in \{S_1(a_k), S_2(a_k)\}$. QED

<u>Theorem 8.9</u>. Condition (2) holds for L_n: if two ACs are adjacent, then there are $n - 2$ ACs adjacent to both.

<u>Proof</u>. Let $P = \{a_1, a_2, \ldots, a_{n-2}\}$ and $Q = \{a_1, b_2, \ldots, b_{n-2}\}$ be two adjacent ACs (having a_1 in common). Define $S_1(a_1) = P$ and $S_2(a_1) = Q$; then $S_3(a_1)$ consists of a_1 and $n - 3$ remaining elements which we denote by $c_2, \ldots, c_{n-2}$. $S_3(a_1)$ is adjacent to both P and Q. Without loss of generality, we assume that the indices are chosen so that a_k, b_k and c_k are mutual second associates, for each $k \in \{2, \ldots, n - 2\}$ (that such an arrangement is possible is guaranteed by Theorem 7.10).

According to Theorem 8.5 there is, for each $k \in \{2, \ldots, n-2\}$, a unique AC containing a_k and b_k; call it R_k. There is no other AC containing both a_k and an element of $Q - a_1$ because b_k is the element of $Q - \{a_1\}$ which is a second associate of a_k. Therefore, $S_3(a_1)$, R_2, ..., and R_{n-2} are the only ACs adjacent to both P and Q. QED

The next three lemmas are preliminaries to Theorem 8.14, where we prove that condition (3) holds for L_n.

<u>Lemma 8.10</u>. Let $S = \{a_1, \ldots, a_{n-2}\}$ be an AC. Define $S_3(a_k) = S$ for all $k \in I_{n-2}$. Then for all $j \neq k$, $j, k \in I_{n-2}$, and $i \in I_2$, $S_i(a_k)$ is adjacent to exactly one of $S_1(a_j)$, $S_2(a_j)$; i.e., $|S_i(a_k) \cap (S_1(a_j) \cup S_2(a_j)| = 1$.

<u>Proof</u>. $S_i(a_k)$ can't be adjacent to $S_1(a_j)$ <u>and</u> $S_2(a_j)$, for then S, $S_1(a_j)$, $S_2(a_j)$, and $S_i(a_k)$ relate to each other in a way forbidden by Lemma 8.7. Now a_j, and a_k are second associates, so the number of elements of A_n which are second associates of both is $p_{22}^2 = n - 2$. $n - 4$ of these belong to S (everything in S except for a_j and a_k). The remaining two elements must lie in

$$(S_1(a_k) \cup S_2(a_k)) \cap (S_1(a_j) \cup S_2(a_j))$$

$$= [S_1(a_k) \cap (S_1(a_j) \cup S_2(a_j))] \cup [S_2(a_k) \cap (S_1(a_j) \cup S_2(a_j))].$$

Now $S_i(a_k)$ can't be adjacent to $S_1(a_j)$ <u>and</u> $S_2(a_j)$ (see above), so in order for the union of the bracketed terms to contain two elements, each bracketed term must contain exactly one element. QED

<u>Lemma 8.11</u>. Let S and R be any two distinct, nonadjacent points of L_n. Suppose that there exists a $Q \in L_n$ adjacent to S and R. Then there are precisely four points in L_n adjacent to S and R.

<u>Remark</u>. This is not quite condition (3), since we assume the existence of at least one point adjacent to S and R. Later we'll show that such a Q must exist.

<u>Proof</u>. Let $\{p\} = S \cap Q$ and $\{q\} = R \cap Q$. Regarding R as $S_1(q)$ and Q as $S_3(q)$ we see that $S_2(q)$ must be adjacent to S, since $S_1(q) = R$ is not (Lemma 8.10). Similarly, regarding Q as $S_3(p)$ and S as $S_1(p)$, we conclude that $S_2(p)$ must be adjacent to R.

Let $S_2(p) \cap R = \{r\}$ and $S_2(q) \cap S = \{s\}$. Theorems 7.9 and 7.10 applied to G_q, together with the facts that (1) p is a second associate of s and r; and (2) p, r, and s all lie in different $(n-3)$-cliques of G_q, imply that s and r are second associates. Theorem 8.5 implies that there exists a unique AC containing both s and r; call it W. The situation is diagrammed below in Figure 11.

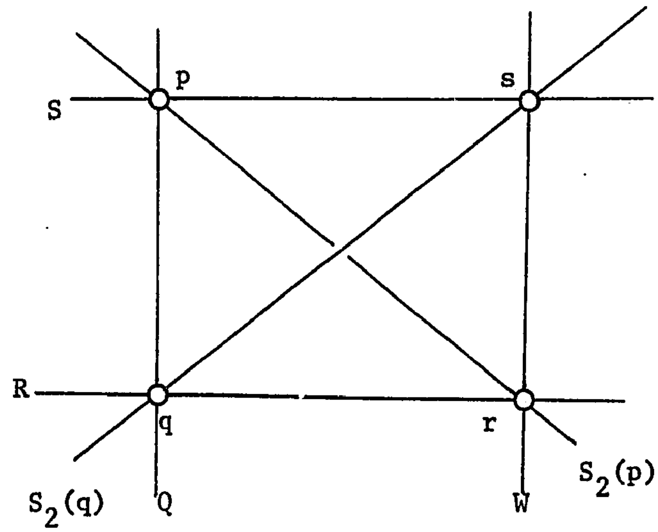


Figure 11. The graph of Lemma 8.11.

So far we have proved that there are at least four ACs adjacent to S and R: Q, $S_2(p)$, $S_2(q)$, and W. To prove that there are no more, we need the following lemma.

Lemma 8.12. Let S, R, and Q be as in Lemma 8.11. Let $x \in R$, $x \neq q$, r; define $S_3(x) = R$. Then

$$|(Q \cup S_2(p)) \cap (S_1(x) \cup S_2(x))| = 2.$$

Proof. This is fairly direct consequence of Lemma 8.10, which implies that Q is adjacent to exactly one $S_1(x)$, $S_2(x)$; so set $Q \cap (S_1(x) \cup S_2(x)) = \{y\}$. Lemma 8.10 also implies that $S_2(p)$ is adjacent to exactly one of $S_1(x)$, $S_2(x)$, so set $S_2(p) \cap (S_1(x) \cup S_2(x)) = z$. Now, $y \neq z$, because $y \in Q$, $z \in S_2(p)$, and $Q \cap S_2(p) = \{p\}$. The situation is depicted in Figure 12 below.

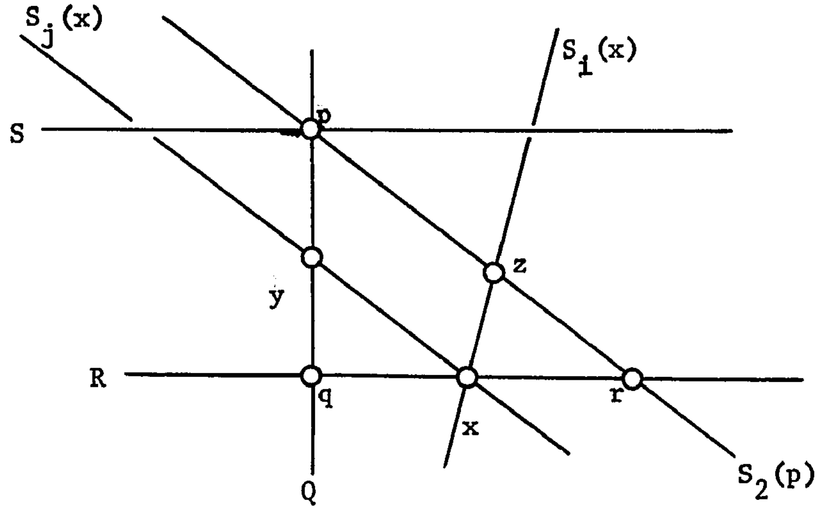


Figure 12. The graph of Lemma 8.12.

To continue with the proof of Lemma 8.11; y and z described in Lemma 8.12 can't lie in S, because

(1) $y, \ z \in Q \cup S_2(p)$;

(2) $(Q \cup S_2(p)) \cap S = \{p\}$; and

(3) by hypothesis, $x \neq q, \ r$, i.e., x is not a second associate of p, and therefore $S_1(x) \cup S_2(x)$ does not contain p.

Also, x and p can't be 0^{th} associates, because q is a second associate of both x and p, and $p_{22}^{0} = 0$. Therefore, x and p are first associates. Since $p_{22}^{1} = 4$, q, r, y, and z are the only elements of A_n which are second associates of both x and p.

Therefore $S_1(x)$ and $S_2(x)$ can't be adjacent to S, for if one of them were, say $S_1(x)$, i.e., $S_1(x) \cap S = \{w\} \neq \phi$, then w would be a fifth second associate of both x and p. Since this is true of all $x \in R$ $(x \neq q,r)$, the only points of L_n adjacent to S and R are Q, $S_2(p)$, $S_2(q)$, and W. QED

<u>Definition 8.13</u>. Let X be a non-empty subset of L_n. We define $\mathcal{N}(X)$ to be the set of ACs in $L_n - X$ which are adjacent to at least one AC in X.

<u>Theorem 8.14</u>. Condition (3) holds for L_n.

<u>Proof</u>. Let $S \in L_n$. We will show that

$$L_n = \mathcal{N}(S) \cup \mathcal{N}(\mathcal{N}(S)). \tag{8.14.1}$$

First, we remark that $\mathcal{N}(\mathcal{N}(S))$ consists of S and ACs which are not adjacent to S. This follows directly from Definition 8.13. Therefore, if (8.14.1) is valid and R is an AC not adjacent to S, then $R \in \mathcal{N}(\mathcal{N}(S))$ and so R is adjacent to at least one $Q \in \mathcal{N}(S)$. Condition (3) follows (see the remark following Lemma 8.11).

Let Y be the set of elements of A_n contained in some AC adjacent to S, but not contained in S. Each element of Y lies in three ACs (Theorem 8.2), exactly two of which are adjacent to S (Lemma 8.10). Let's count the elements of Y. There are $2(n - 2)$ ACs adjacent to S (Theorem 8.8), and each such AC contains $n - 3$ elements of A_n which do not belong to S. Furthermore, each element of Y is contained in two such ACs (see above), and so is counted twice in the above fashion. Thus

$$|Y| = \frac{1}{2} \times 2(n - 2) \times (n - 3) = (n - 2)(n - 3) \tag{8.14.2}$$

Next, consider an AC other than S in $N(N(S))$, call it R. In Lemma 8.11 we showed that there are four ACs adjacent to both S and R, and that these ACs interesect R in precisely two elements. In other words, each $R \in N(N(S)) - \{S\}$ contains exactly two elements of Y. Furthermore, each element of Y lies in exactly one AC in $N(N(S)) - \{S\}$ (Lemma 8.10). Therefore

(8.14.3) $$|N(N(S)) - \{S\}| = \frac{1}{2}|Y| = \frac{1}{2}(n-2)(n-3).$$

Therefore,

(8.14.4) $$|N(N(S))| + |N(S)| = |N(N(S)) - \{S\}| + 1 + |N(S)|$$

$$= \frac{1}{2}(n-2)(n-3) + 1 + 2(n-2)$$

$$= \binom{n}{2} = |L_n|.$$

Since $N(N(S)) \cup N(S)$ is a disjoint union, (8.14.1) follows immediately from (8.14.4). QED

<u>Corollary 8.15</u>. L_n is isomorphic to $L(K_n)$.

<u>Proof</u>. This follows from Theorems 8.8, 8.9, and 8.14 and the remarks preceding Theorem 8.8.

Our next task is to use this fact to show that there is an isomorphic mapping between A_n and T_n.

Let ψ be an isomorphic mapping from L_n onto $L(K_n)$.

Let $\psi_1 : L_n \times L_n \times L_n \to L(K_n) \times L(K_n) \times L(K_n)$ be defined by:

$$\psi_1(\{S,Q,R\}) = \{\psi(S),\psi(R),\psi(Q))\}$$

where S, Q, and R are three distinct ACs. ψ_1 is 1 - 1, onto.

Let $G \subset L(K_n) \times L(K_n) \times L(K_n)$ be defined as follows. Let x, y, $z \in L(K_n)$. Then $\{x,y,z\} \in G$ iff x, y, and z are the edges of a triangle in K_n, i.e., $\{x,y,z\}$ can be represented as $\{\{i,j\},\{j,k\},\{i,k\}\}$.

Let $\Delta : G \to T_n$ be defined by:

$$\Delta\{\{i,j\},\{j,k\},\{i,k\}\} = \{i,j,k\}.$$

Δ is 1 - 1, onto.

Let $B \subset L_n \times L_n \times L_n$ be defined by: $\{P,Q,R\} \in B$ iff P, Q, R are distinct associated cliques with a non-empty intersection. According to Definition 8.1 and Theorem 8.2, B consists precisely of triples of the form $\{S_1(a),S_2(a),S_3(a)\}$ where $a \in A_n$.

Let ρ be the 1 - 1, onto mapping from A_n onto B defined by

$$\rho(a) = \{S_1(a),S_2(a),S_3(a)\} \qquad (a \in A_n).$$

Let $\phi = \Delta \circ \psi_1 \circ \rho$.

Theorem 8.16. ϕ is an isomorphic mapping from $\mathcal{A}_n$ onto $\mathcal{T}_n$.

The proof of Theorem 8.16, which acomplishes the primary objective of this paper, requires several steps. The mapping ϕ is diagrammed as follows:

$$a \underset{\rho^{-1}}{\overset{\rho}{\rightleftarrows}} \{S_1(a), S_2(a), S_3(a)\} \underset{\psi_1^{-1}}{\overset{\psi_1}{\rightleftarrows}} \{\psi(S_1(a)), \psi(S_2(a)), \psi(S_3(a))\} \underset{\Delta^{-1}}{\overset{\Delta}{\rightleftarrows}} \{i,j,k\}.$$

Lemma 8.17. Three distinct points in $L(K_n)$, call them p_1, p_2, and p_3, are the edges of a triangle in K_n iff

(1) $\{p_1, p_2, p_3\}$ is a triangle in $L(K_n)$, and

(2) the number of points of $L(K_n)$ which are adjacent to p_3 and p_2 but not p_1 is $n - 3$.

Proof. There are only two kinds of triangles in $L(K_n)$:

(1) $\{\{i,j\},\{j,k\},\{i,k\}\}$ and

(2) $\{\{i,j\},\{i,k\},\{i,\ell\}\}$

where i, j, k, $\ell \in I_n$.

Represented as arcs in K_n, we have the two cases

(1) and (2)

In Case (1), the points of $L(K_n)$ which are adjacent to p_3 and p_2 but not p_1 are $\{j,m\}$, where $m \in I_n - \{i,j,k\}$; there are $n - 3$ of these.

In Case (2) the only point of $L(K_n)$ which is adjacent to p_3 and p_2 but not p_1 is $\{\ell,k\}$. QED

Now, if we can show that $\psi(S_1(a))$, $\psi(S_2(a))$, and $\psi(S_3(a))$ meet conditions (1) and (2) of Lemma 8.17, then we can conclude that they are the arcs of a triangle in K_n, and therefore $\{\psi(S_1(a)),\psi(S_2(a)),\psi(S_3(a))\} \in \mathcal{G}$.

First, $S_1(a)$, $S_2(a)$, and $S_3(a)$ are all adjacent (they all contain a), and therefore the $\psi(S_i(a))$ are all adjacent; thus, condition (1) is satisfied.

Next, we label the elements of $S_3(a)$ as follows:

$$S_3(a) = \{a,b_2, \ldots, b_{n-2}\},$$

According to Lemma 8.10, for each $K \in \{2, \ldots, n - 2\}$, $S_2(a)$ is adjacent to exactly one of $S_1(b_k)$, $S_2(b_k)$. We can relabel all of the $S_i(b_k)$ $(i = 1, 2,\ k = 2, \ldots, n - 2)$ so that $S_2(b_k)$ is adjacent to $S_2(a)$, and $S_1(b_k)$ is not. Lemma 8.10 also implies that $S_2(b_k)$ is <u>not</u> adjacent to $S_1(a)$. Of course, $S_2(b_k)$ is adjacent to $S_3(a)$, since both contain b_k. Therefore, $S_2(b_k)$ is as required by condition (2), and the number of $S_2(b_k)$ is $n - 3$, since k ranges from 2 to $n - 2$. Therefore, condition (2) is satisfied by the $S_i(a)$, i.e., they are the arcs of a triangle in K_n. Therefore, we have proved

Lemma 8.18. Let $a \in A_n$. Then $\{\psi(S_1(a)),\psi(S_2(a)),\psi(S_3(a))\} \in G$.

Proof. To prove Theorem 8.16, we need to prove the following:

(1) ϕ is well defined;

(2) ϕ is $1-1$ and onto;

(3) if $a, b \in A_n$ are 0^{th} associates, then $\phi(a)$ and $\phi(b)$ are 0^{th} associates in T_n;

(4) if $a, b \in A_n$ are first associates, then $\phi(a)$ and $\phi(b)$ are first associates;

(5) if $a, b \in A_n$ are second associates, then $\phi(a)$ and $\phi(b)$ are second associates.

(1) Lemma 8.18 implies that ϕ is well defined.

(2) ρ, ψ_1, and Δ are $1-1$, therefore ϕ is $1-1$. Also $|A_n| = |T_n| = \binom{n}{3}$ and ϕ is $1-1$; therefore ϕ is onto.

(3) Let $a, b \in A_n$ be 0^{th} associates. Then the $S_i(a)$ are not adjacent to the $S_i(b)$, $i = 1, 2, 3$ (for if $S_i(a) \cap S_j(b) = \{p\} \neq \phi$, then p would be a second associate to two 0^{th} associates; but $p_{22}^0 = 0$ so this is impossible). And, of course, the $S_i(a)$ are distinct from the $S_i(b)$ (for if $S_i(a) = S_j(b)$, then a and b must be second associates). Therefore no $\psi(S_i(a))$ is adjacent or equal to a $\psi(S_j(b))$. Therefore $\phi(a)$ and $\phi(b)$ have no edges in common, and

no edge of $\phi(a)$ is adjacent to an edge of $\phi(b)$. Therefore $\phi(a)$ and $\phi(b)$ have no vertices in common. Therefore $\phi(a)$ and $\phi(b)$ are 0^{th} associates.

(4) Let a, $b \in A_n$ be first associates. Since $p^1_{22} = 4$, there exists a $c \in A_n$ which is a second associate of both a and b. Therefore $c = S_i(a) \cap S_j(b)$ for some i, $j \in I_3$. Therefore $S_i(a)$ is adjacent to $S_j(b)$. Therefore $\psi(S_i(a))$ is adjacent to $\psi(S_j(b))$. So if $\psi(S_i(a)) = \{k,\ell\}$, then $\psi(S_j(b)) = \{k,m\}$ (k, ℓ, and m are distinct elements of I_n).

Now, none of the ACs containing a coincide with an AC containing b. (If they did, then a and b would either be the same or second associates.) Therefore the $\psi(S_k(a))$s and the $\psi(S_k(b))$s are all distinct $(k \in I_3)$. Therefore, $\phi(a)$ and $\phi(b)$ don't have a common edge. Therefore $\phi(a)$ and $\phi(b)$ have just one common vertex, i.e., $\phi(a)$ and $\phi(b)$ are first associates.

(5) Let a, $b \in A_n$ be second associates, and let $S_3(a) = S_3(b)$ be the AC containing both a and b. Let $\psi(S_3(a)) = \{i,j\}$. Then $\phi(a) = \{i,j,k\}$ and $\phi(b) = \{i,j,\ell\}$, $k \neq \ell$, are second associates. This completes the proof of Theorem 8.16. QED

APPENDIX

In [1] Aigner showed that any graph satisfying properties (P1)-(P4) below, with $n \in \{6,7,8\}$, is isomorphic to the tetrahedral graph, whose points can be identified with $\binom{n}{3}$ unordered triplets on n symbols, with two points adjacent iff their corresponding triplets have two symbols in common. Letting $d(x,y)$ denote the distance between two points x and y, and $\Delta(x,y)$ the number of points adjacent to both x and y, we see that the tetrahedral graph G has the following properties:

(P1) The number of points is $\binom{n}{3}$

(P2) G is connected and regular of degree $3(n - 3)$.

(P3) If $d(x,y) = 1$, then $\Delta(x,y) = n - 2$.

(P4) If $d(x,y) = 2$, then $\Delta(x,y) = 4$.

Now let $n \in \{6,7,8\}$, and let A be an association scheme satisfying properties (1), (2), and (3) given on page 1 of this paper. Referring to Table 1(a), (b) $(n_2 = 3(n - 3))$, (c) $(p^2_{22} = n - 2$ and $p^1_{22} = 4)$, and Theorem 3.4, we see that the graph H induced on A by the relation of second association satisfies (P1)-(P4). Therefore it is isomorphically equivalent to the tetrahedral graph.

We claim that A is isomorphically equivalent to the tetrahedral association scheme. Since H is tetrahedral, the points of H (i.e., the points of A) can be identified as unordered triplets of an n-set, two points being adjacent iff they are second associates. To establish the claim above we must show:

(1) If two elements of A are first associates, then the corresponding triples have one symbol in common.

(2) If two elements of A are 0th associates, then the corresponding triples have no symbols in common.

Let $t(w)$ denote the triple corresponding to w, for each $w \in A$.

Suppose $x, y \in A$ and $\alpha(x,y) = 1$. Then $|t(x) \cap t(y)| \neq 2$, since $\alpha(x,y) \neq 2$. There is a $z \in A$ such that $\alpha(x,z) = \alpha(y,z) = 2$, since $p^1_{22} = 4 > 0$. Now $|t(z) \cap t(x)| = |t(z) \cap t(y)| = 2$, and therefore $|t(x) \cap t(y)| \neq 0$. Therefore $|t(x) \cap t(y)| = 1$. This proves (1).

Suppose $x, y \in A$ and $\alpha(x,y) = 0$. then $|t(x) \cap t(y)| \neq 2$, since $\alpha(x,y) \neq 2$. Suppose that $|t(x) \cap t(y)| = 1$. Let $\{i,j,k\}$ be a triplet having two symbols in common with each of $t(x)$ and $t(y)$, and let $z \in A$ satisfy $t(z) = \{i,j,k\}$. Then $\alpha(x,z) = \alpha(y,z) = 2$. Since $\alpha(x,y) = 0$, we have violated the condition that $p^0_{22} = 0$. Therefore, $|t(x) \cap t(y)| = 0$ and we have established (2).

Therefore the claim above is valid.

Restating the conclusion of this paper, we have proven that for $n \geq 6$, the parameters of the tetrahedral association scheme characterize the scheme itself.

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