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One-relator groups with torsion, virtually free-by-cyclic groups, and free-by-free groups

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ONE–RELATOR GROUPS WITH TORSION, VIRTUALLY FREE–BY–CYCLIC GROUPS, AND FREE–BY–FREE GROUPS

by

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PREFACE

We know, by the example of Baumslag and Solitar,

$$\langle\, a, b\,;\, a^{-1}b^2a = b^3 \,\rangle$$

that it is possible for a one-relator group to be non-Hopfian. We also know, from the paper of Pride [P], that every two generator one-relator group with torsion is Hopfian. Gilbert Baumslag has conjectured [B4] that any one-relator group with torsion is virtually free-by-cyclic. This work has its origin in an attempt to prove that conjecture, and so show, by appealing to two general theorems, one by Malcev and one by Baumslag, that any one-relator group with torsion is necessarily Hopfian.

We will consider some generalizations of the class of free groups which arise in the analysis of one-relator groups, and examine some properties these groups share with free groups. The properties we will examine include residual finiteness and Hopficity.

Chapter 1 contains preliminary definitions and some essential theorems. In Chapter 2 we consider the class of virtually free-by-cyclic groups. We show in Section 2.1, by explicit computations, that certain one relator groups with torsion are virtually free-by-cyclic. Specifically, we prove the theorems

Theorem 2.1. *The groups* $G_{p,q,k} = \langle\, x, y\,;\, (x^{-1}y^p x y^q)^k \,\rangle$, *where* $0 < p, q < k$, *and* $(p+q, k) = 1$, *are virtually free-by-cyclic.*

Theorem 2.5. *The groups* $H_m = \langle\, a, b\,;\, [a,b]^m \,\rangle$ *are virtually free-by-cyclic.*

Theorem 2.15. *The groups* $S_k = \langle\, a, b, g_i, i \in I\,;\, ([a,b]w)^k, w \text{ a word in } g_i \,\rangle$ *are virtually free-by-cyclic.*

In Section 2.2, we examine closure properties in the class of virtually free-by-cyclic groups, leading up to two results of this type:

Theorem. *Suppose* A *and* B *are virtually free-by-cyclic groups:* A *is a finite extension of* $M = \langle\, x, F_1 \,\rangle$ *and* B *is a finite extension of* $N = \langle\, y, F_2 \,\rangle$. *Let* $G = A \underset{x=y}{*} B$ *be their free product amalgamating* $\langle\, x \,\rangle$ *and* $\langle\, y \,\rangle$. *Then, under some extra conditions on the conjugates of* x *in* A *and the conjugates of* y *in* B, G *is virtually free-by-cyclic.*

Chapter 3 deals with the more general class of free-by-free groups. Section 3.1 is concerned with residual finiteness in free-by-free groups. It gives some conditions on an action of a free group on a free group of countably infinite rank which will produce a free-by-free extension which is residually finite. In Section 3.2, we look at the kernel of a homomorphism from a one-relator group onto a free group, and prove several theorems which describe conditions on the relator under which this kernel will be free.

In Chapter 4, we examine the class of virtually residually free groups. Some basic results involving closure properties are shown first. Using similar techniques to those of Chapter 2, we prove the main result:

Theorem 4.9. *Let A and B be finitely generated abelian groups and $G = A \underset{C}{*} B$. Then G is virtually residually free.*

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CONTENTS

CHAPTER 1
SOME DEFINITIONS AND BASIC THEOREMS

Section 1.1

Presentations and Products

Any group G which is generated by a subset of its elements is a quotient of a free group F of rank equal to the cardinality of that subset: $G \simeq F/N$, for some normal subgroup $N \lhd F$. We say that G *has the presentation*

$$\langle\, x_1, x_2, \dots ; \, r_1, r_2, \dots \,\rangle$$

when $G \simeq F/N$, where F is the free group on $X = \{x_1, x_2, \dots\}$ and N is the normal subgroup of F generated by the set $\{r_1, r_2, \dots\}$ of words in F. Since $F \xrightarrow{f} G$ is onto, G is generated by the set of images of X, $\{f(x_1), f(x_2), \dots\}$; we usually say that G is generated by X. The set $\{r_1, r_2, \dots\}$ is called a set of *relators* for G.

If H is a subgroup of G with the presentation above, we can obtain a presentation for H by the *Reidemeister-Schreier method* (see [MKS] §2.3):

First, take a *Schreier transversal* for H in G: a set of representatives of the cosets of H, such that there is one representative for each coset, and any initial segment of a representative is itself a coset representative. Such a set of representatives always exists. For any element w of G, let $\overline{w}$ denote the representative of the coset wH.

Then H is generated by the set of elements

$$s_{K,x} = Kx(\overline{Kx})^{-1},$$

where K is a Schreier representative and x is a generator of G. There is a procedure, call it τ, for rewriting any element of H given in the generators of G in these generators of H. τ is a symbol by symbol replacement of the G-generators by the appropriate generators $s_{K,x}$.

It can then be shown that H has the presentation

$$H = \langle\, s_{K,x}, \ldots ;\, \tau(Mr_iMK^{-1}), \ldots \rangle$$

where K ranges through the Schreier representatives and x through the generators of G which give $Kx \neq \overline{Kx}$, and M runs through the set of Schreier representatives.

If $G = \langle X\,;\,R\rangle$ and $H = \langle Y\,;\,S\rangle$, where the sets of generators X and Y are disjoint, then the *free product of G and H* , written $G * H$, is the group with presentation

$$\langle X \cup Y;\, R \cup S\rangle.$$

G and H are both embedded in $G * H$ in the obvious fashion. All the subgroups of such a free product can be completely described (see [LS] §III.3):

Kurosh Subgroup Theorem. *Let $P = G * H$ and let M be a subgroup of P. Then M is the free product of a free group and certain conjugates of subgroups of the free factors G and H.*

The groups called *Higman-Neumann-Neumann extensions* or *HNN extensions* are also given by a special type of presentation. Suppose $G = \langle X ; R \rangle$, and let $\{t_i, i \in I\}$ be a set disjoint from X. Furthermore, let A_i and B_i be isomorphic subgroups of G with a collection of isomorphisms $\phi_i : A_i \longrightarrow B_i, i \in I$. Then the group

$$G^* = \langle X, t_i, i \in I ; R, t_i^{-1} a t_i = \phi_i(a), a \in A_i \rangle$$

is called an *HNN extension* of G with base group G and stable letters $\{t_i\}$. G is actually embedded as a subgroup of G^* in the obvious way.

A more general product construction is the free product with amalgamation. Suppose that the groups G and H contain isomorphic subgroups $C_1 < G$ and $C_2 < H$, with the isomorphism $\phi : C_1 \longrightarrow C_2$. Then the *free product of G and H amalgamating $C_1 = C_2$* is the group with presentation

$$\langle X, Y ; R, S, c^{-1}\phi(c), c \in C_1 \rangle.$$

The free product with amalgamation just described will be written $G \underset{C_1=C_2}{*} H$. G and H are both subgroups of $G \underset{C_1=C_2}{*} H$.

There is also a theorem which describes the subgroups of a group of this type ([KS1]):

Theorem. *Suppose $P = G \underset{C_1=C_2}{*} H$ is a free product with amalgamation. Any subgroup M of P is an HNN extension of a tree product S, where each vertex group of S is the intersection with M of a conjugate of G or of H, and each edge group is the intersection with M of a conjugate of C_1. In addition, each associated subgroup is the intersection with M of a conjugate of C_1.*

We will actually require the complete details of this theorem, so it will be examined more fully in §2.2.

The last product construction we need to consider is called a central product. Again, suppose that we have subgoups G and H with $C < Z(G)$ and $D < Z(H)$, and suppose further that C and D are isomorphic via the isomorphism ϕ. Then the *central product of G and H amalgamating C and D* is the quotient of the direct product of G and H, $G \times H$, by the subgroup generated by $\{(c, (\phi(c))^{-1}), c \in C\}$. As with the other constructions, G and H are subgroups of this central product.

Section 1.2

Extensions and Properties of Groups

The group G is an *extension* of the group K by the group Q if K is a normal subgroup of G and $G/K \simeq Q$. Let π be the homomorphism from G onto Q with kernel K. If in addition there is a homomorphism $\beta : Q \longrightarrow G$ with $\pi\beta = 1_Q$, then G is called a *split extension* or *semidirect product* of K by Q. G is generated by its subgroups K and Q, and so there is a homomorphism $\theta : Q \longrightarrow Aut(K)$ given by conjugation in the group $G : \theta_y(k) = yky^{-1}$, for all $k \in K$ and all $y \in Q$.

We have occasion to consider the class of all groups sharing a given property, say property $\mathcal{P}$. A group G is said to be *residually* $\mathcal{P}$ if for each nontrivial $g \in G$, there is a normal subgroup N, with $g \notin N$, such that G/N has property $\mathcal{P}$. Residually finite groups are of particular interest. G is *residually finite* if for any $g \in G, g \neq 1$, there is a normal subgroup $N \lhd G$ with $g \notin N$ and G/N finite.

We say a group G is *virtually* $\mathcal{P}$ if it has a subgroup N of finite index with property $\mathcal{P}$. A group G is $\mathcal{P}$*-by-*$\mathcal{Q}$ if it is an extension of a group F with property $\mathcal{P}$ by a group H with property $\mathcal{Q}$, that is, F is a normal subgroup of G and G/F has property $\mathcal{Q}$. In particular, we will be interested in groups which are free-by-cyclic or free-by-free.

Also, a group G is *Hopfian* if there is no nontrivial normal subgroup N of G with $G/N \simeq G$. The finitely generated free groups are Hopfian and residually finite. There are one-relator groups which are not Hopfian, in particular, the example of Baumslag and Solitar (see [MKS] §4.4) of the one-relator group $G = \langle\, a, b\,;\, a^{-1}b^2a = b^3 \,\rangle$.

An important connection between the properties of residual finiteness and Hopficity is the result of Malcev (see [LS] §IV.4): A finitely generated residually finite group is Hopfian. This result and the theorem of Baumslag [B1] that a finitely generated free-by-cyclic group is residually finite allow us to conclude that a virtually free-by-cyclic group is Hopfian.

A result of Miller ([M] III.A Theorem 7) shows that any split extension of a finitely generated residually finite group by a residualy finite group is again residually finite. This shows that a free-by-free group G as above is residually finite if the free normal subgroup F is of finite rank.

Section 1.3

Notation

None of the notation used is unusual, and most symbols will be introduced as they are needed. Here is the notation which is not explained in the text.

$\alpha\beta$	The composition of the functions $\beta : F \longrightarrow G$ and $\alpha : G \longrightarrow H$
$H < G$	H is a subgroup of G
$N \lhd G$	N is a normal subgroup of G
$[a, b]$	$aba^{-1}b^{-1}$, the commutator of the elements a and b
b^a	aba^{-1}, the conjugate of b by a
$Z(G)$	the center of the group G
$ker\alpha$	the kernel of the homomorphism α
$K \rtimes Q$	a semidirect product of K by Q
1_H	the identity homomorphism on the group H
(m, n)	the greatest common factor of the natural numbers m and n
$gp(a, b, \dots, c)$	the subgroup of a group G generated by the elements $a, b, \dots, c$
$gp_G(a, b, \dots, c)$	the normal subgroup of a group G generated by the elements $a, b, \dots, c$

CHAPTER 2
VIRTUALLY FREE–BY–CYCLIC GROUPS

Section 2.1

Computations:

Some Virtually Free-by-cyclic Groups

In this section we prove that the groups of two different classes of one-relator groups where the relator is a proper power are virtually free-by-cyclic. A one-relator group has elements of finite order precisely when the relator is a proper power of some word, and so we refer to these one-relator groups as one-relator groups with torsion.

Denote by $G_{p,q,k}$ the group with presentation

$$\langle\, x, y\, ;\, (x^{-1}y^{p}xy^{q})^{k}\, \rangle.$$

This presentation differs from the nonhopfian one-relator group mentioned in Chapter 1 by the introduction of torsion. We will show that, with primality conditions on p, q, and k, these groups are virtually free-by-cyclic. $G_{p,q,k}$ will be decomposed as a series of extensions:

$$M \lhd H \lhd G_{p,q,k},$$

where G/H is finite, H/M is infinite cyclic, and M is free of countably infinite rank.

We observe that this will show that $G_{p,q,k}$ is residually finite. Since a finite extension of a residually finite group is residually finite, $G_{p,q,k}$ will be residually finite if H is, and any finitely generated free-by-cyclic group H is residually finite by [B1].

Theorem 1. *Suppose $G_{p,q,k}$ has a presentation as above. If $0 < p, q < k$, and $(p+q,k) = 1$, then $G_{p,q,k}$ is a finite extension of a free-by-cyclic group.*

Proof. We will prove the theorem in two steps.

Lemma 2. *If we map $G_{p,q,k} \xrightarrow{\alpha} C_k = \langle\, y\,;\, y^k \,\rangle$ in the obvious way, $K = ker\alpha = \langle\, w = y^k, x_i = y^i x y^{-i}, i = 0, 1, \ldots, k-1\,;\, r \,\rangle$ is a one-relator group.*

Proof. From the Reidemeister-Schreier process, we get the generators listed for K, and k relations:

$$r_i = y^i r y^{-i}, i = 0, 1, \ldots, k-1,$$

rewritten in the generators of K.

Let us examine the relation r_0:

$$r_0 = x_{s_1}^{-1} x_{t_1} \cdots x_{s_k}^{-1} x_{t_k} w$$

where $s_i = (i-1)(p+q) \mod k$ and $t_i = (i-1)(p+q)+p \mod k$. In each case, the value of the subscript s_i or t_i on a generator x is the sum $\mod k$ of the exponents on y in the subword of r preceding the instance of x being replaced. Each sequence $\{s_i\}$ and $\{t_i\}$ is a permutation of the set $\{0, 1, \ldots, k-1\}$, obtained by counting cyclically by length $p+q$, starting at $0 \mod k$ for $\{s_i\}$ and at $p \mod k$ for $\{t_i\}$. The primality condition guarantees that this is a permutation. Also, r_0 contains $p+q$ occurrences of w which are not explicitly written. w appears in the relation between $x_{s_i}^{-1}$ and x_{t_i} if $s_i \geq t_i$ and between x_{t_i} and $x_{s_{i+1}}^{-1}$ if $t_i \geq s_{i+1}$. This happens $p+q$ times.

Now we see easily that $r_m = y^m r y^{-m}$ is a cyclic permutation of r_0, for we get by rewriting

$$r_m = x_{\sigma_1}^{-1} x_{\tau_1} \cdots x_{\sigma_k}^{-1} x_{\tau_k}$$

where $\sigma_i = s_i + m \mod k$ and $\tau_i = t_i + m \mod k$. So viewed cyclically the sequence σ_i is the same as the sequence s_i and the sequence τ_i is the same as the sequence t_i. The $p+q$ occurrences of w are placed as in r_0: between $x_{\sigma_i}^{-1}$ and x_{τ_i} if $\sigma_i \geq \tau_i$ and between x_{τ_i} and $x_{\sigma_{i+1}}^{-1}$ if $\tau_i \geq \sigma_{i+1}$. Thus K has only the single relation which we will call r. □

Lemma 3. *K is free-by-cyclic.*

Proof. Notice that each x_i, in particular x_0, appears with exponent sum 0 in r. In fact, x_0 appears once with exponent -1, since $s_0 = 0$, and once with exponent +1, since $t_m = 0$ for one m. Suppose we map $K \xrightarrow{\beta} \langle x_0 \rangle$. We get

$$M = ker\beta = \big\langle w_j, j \in \mathbf{Z}, x_{i,l}, i = 1, 2, \ldots, k-1, l \in \mathbf{Z}\, ;\, r_n, n \in \mathbf{Z} \big\rangle$$

with $w_j = x_0^j w x_0^{-j}$, $x_{i,l} = x_0^l x_i x_0^{-l}$ and

$$r_n = x_{t_1,n-1} x_{s_2,n-1}^{-1} \cdots x_{s_m,n-1}^{-1} x_{s_{m+1},n}^{-1} \cdots x_{s_k,n}^{-1} x_{t_k,n} w_n$$

where m is such that $t_m = 0$. Most of the occurrences of generators of type w_j have not been explicitly written.

Since $0 < p, q < k$, we know that $m \neq 1, k$.

$$t_k = (k-1)(p+q) + p \mod k = q \mod k \neq 0 \text{ since } q < k.$$

$$t_1 = p \mod k \neq 0 \text{ since } p < k.$$

Because of the special nature of the permutations $\{s_i\}$ and $\{t_i\}$, $t_{m+n} \mod k = (m+n-1)(p+q)+p \mod k = t_m + n(p+q) \mod k = s_{n+1}$. Thus $s_2 = t_{m+1}$. If we call this number s, we see that in each relation r_j, there is a unique occurrence of the maximally and minimally subscripted generators $x_{s,j}$ and $x_{s,j-1}$. We conclude that M is free because of this well known result about one-relator groups (see [B3]):

Lemma. *Let $G = \langle b, x, \ldots, c; r = 1 \rangle$ be a group with one defining relation, and suppose that b occurs with exponent sum 0 in r. Suppose also that m and M are respectively the minimum and maximum subscripts occurring on x in r_0. If $m < M$ and both x_m and x_M occur only once in r_0, then $gp_G(x, \ldots, c)$ is free on the generators $x_m, x_{m+1}, \ldots, x_{M-1}, \ldots, c_i, i \in \mathbf{Z}$.*

□

We should notice that the free-by-cyclic subgroup K is not free, because in $\overline{K}$, the abelianization of K, the element $\overline{w}$ has finite order.

There is a theorem (Theorem B of [C]), proved by Stallings and Swan using cohomological methods, which says

Theorem. *A torsion-free group containing a free subgroup of finite index is free.*

With this theorem and the observation that K is not free, we can show

Theorem 4. *$G_{p,q,k}$ is not virtually free.*

Proof. Suppose $G_{p,q,k}$ had a free subgroup F of finite index. Then $F \cap K$ would be a free subgroup, also of finite index in G since K is. Thus K, which is a torsion free one-relator group, has a free subgroup $F \cap K$ of finite index. The theorem above says that K must be free, a contradiction. □

Another class of one-relator groups with torsion, based on the fundamental groups of orientable surfaces, can be shown to be virtually free-by-cyclic by similar computational methods.

Theorem 5. *The groups $H_m = \langle\, a, b\,;\, [a,b]^m \,\rangle$ are virtually free-by-cyclic.*

We need to find a finite quotient of H_m which gives a kernel which is free-by-cyclic. Since the relation $[a,b]^m$ is in the commutator subgroup, we need a nonabelian quotient. We will treat the cases m odd and m even separately.

Lemma 6. *When m is odd, there is a homomorphism f from H_m onto the m-th dihedral group $D_m = \langle\, r, s\,;\, r^m, s^2, srs^{-1} = r^{-1} \,\rangle$.*

Proof. Send $a \mapsto r$, $b \mapsto s$, and then $[a,b] \mapsto r^{-1}s^{-1}rs = r^{-2}$. Since m is odd, r^{-2} is of order exactly m in D_m. Thus the map extends to a homomorphism f. □

Lemma 7. *L, the kernel of f, is free-by-cyclic.*

First we find a presentation for L. The set of words $\{a^i b^j, 0 \leq i \leq m-1, j = 0, 1\}$ is a Schreier transversal for H_m mod L.

Lemma 8. *L has a presentation*

$$L = \langle\ x_i, \beta_i, 0 \leq i \leq m-1;$$

$$r_0 = x_0 \beta_{m-1} \beta_{m-2}^{-1} x_{m-2} \beta_{m-3} \beta_{m-4}^{-1} \cdots x_1 \beta_0$$

$$x_1^{-1} x_3^{-1} \cdots x_{m-2}^{-1} x_0^{-1} x_2^{-1} \cdots x_{m-1}^{-1} \beta_{m-1}^{-1} \cdots x_2 \beta_1 \beta_0^{-1}\ \rangle.$$

Proof. Using the Reidemeister-Schreier method we get as generators for L:

$$x_i = a^i b a b^{-1} a^{-\overline{(i-1)}}$$

$$\beta_i = a^i b^2 a^{-i}$$

$$\alpha = a^m$$

where $0 \leq i \leq m-1$ and $\overline{(i-1)}$ denotes $(i-1) \mod m$.

We rewrite the words $a^i b^j [a, b]^m (a^i b^j)^{-1}$, $2m$ of them, in these generators, and get only two distinct relations for L.

$[a, b]^m$ rewrites as

$$(r_1) \qquad x_2^{-1} x_4^{-1} \cdots x_{m-1}^{-1} \alpha x_1^{-1} x_3^{-1} \cdots x_{m-2}^{-1} x_0^{-1}.$$

When we conjugate $[a, b]^m$ by a we get the relation

$$x_3^{-1} \cdot x_5^{-1} \cdots x_0^{-1} \alpha^{-1} \cdot \alpha \cdot x_2^{-1} \cdot x_4^{-1} \cdots x_{m-1}^{-1} \cdot \alpha x_1^{-1}$$

which is a cyclic permutation of the relation (r_1).

$b[a,b]^m b^{-1}$ rewrites as

$$(r_2) \qquad x_0\beta_{m-1}\beta_{m-2}^{-1}x_{m-2}\beta_{m-3}\beta_{m-4}^{-1}\cdots x_1\beta_0\alpha^{-1}\beta_{m-1}^{-1}\cdots x_2\beta_1\beta_0^{-1}.$$

Now if we conjugate $b[a,b]^m b^{-1}$ by a and rewrite, we get

$$x_1\alpha^{-1}\cdot\alpha\beta_0\alpha^{-1}\cdot\beta_{m-1}^{-1}\cdot x_{m-1}\cdot\beta_{m-2}\cdot\beta_{m-3}^{-1}\cdots x_2\cdot\beta_1\alpha^{-1}\cdot$$
$$\alpha\beta_0^{-1}\alpha^{-1}\cdot x_0\cdots x_3\cdot\beta_2\beta_1^{-1}$$

which is a cyclic permutation of (r_2).

Thus L has the presentation

$$\langle\, \alpha, x_i, \beta_i\,;\, r_1, r_2 \,\rangle.$$

We can use Tietze transformations to rewrite L. (r_1) says that

$$\alpha^{-1} = x_1^{-1}x_3^{-1}\cdots x_{m-2}^{-1}x_0^{-1}x_2^{-1}\cdots x_{m-1}^{-1},$$

and so L is generated by $\{x_i, \beta_i, 0 \le i \le m-1\}$ and has one relator

$$(r_0) \qquad x_0\beta_{m-1}\beta_{m-2}^{-1}x_{m-2}\beta_{m-3}\beta_{m-4}^{-1}\cdots x_1\beta_0\cdot$$
$$x_1^{-1}x_3^{-1}\cdots x_{m-2}^{-1}x_0^{-1}x_2^{-1}\cdots x_{m-1}^{-1}\beta_{m-1}^{-1}\cdots x_2\beta_1\beta_0^{-1}. \quad \square$$

Lemma 9. *L has an infinite cyclic quotient $\langle x_0 \rangle$. The kernel of the homomorphism $L \xrightarrow{h} \langle x_0 \rangle$ is free.*

Proof. x_0 appears in r_0 with exponent sum 0, so we can map $L \xrightarrow{h} \langle x_0 \rangle$ by

$$x_1 \longmapsto x_0$$

$$g \longmapsto 1, \text{ all other generators } g.$$

Call the kernel of h, M. The generators of M are

$$x_{i,k} = x_0^k x_i x_0^{-k}, i = 1, \dots m-1, k \in \mathbf{Z}$$

$$\beta_{i,k} = x_0^k \beta_i x_0^{-k}, i = 0, \dots, m-1, k \in \mathbf{Z}.$$

Notice that in r_0, each β_i appears once with exponent +1 and once with exponent -1. To find the relations of M, we first rewrite r_0 to get

$$(r_{0,0}) \qquad \beta_{m-1,1}\beta_{m-2,1}^{-1}x_{m-2,1}\beta_{m-3,1}\beta_{m-4,1}^{-1}\cdots x_{1,1}\beta_{0,1}$$

$$x_{1,1}^{-1}x_{3,1}^{-1}\cdots x_{m-2,1}^{-1}x_{2,0}^{-1}\cdots x_{m-1,0}^{-1}\beta_{m-1,0}^{-1}\cdots x_{2,0}\beta_{1,0}\beta_{0,0}^{-1}.$$

The rest of the relations are

$$r_{0,k} = r_{0,0}(x_{i,k}, x_{i,k+1}, \beta_{i,k}, \beta_{i,k+1}).$$

We see that in the relation $r_{0,0}$, the generators $\beta_{0,0}$ and $\beta_{0,1}$ each appear uniquely, so M is actually free on the generators

$$x_{i,k}, i = 1, \dots, m-1, k \in \mathbf{Z} \text{ and } \beta_{0,0}. \quad \square$$

We can also show

Theorem 10. *H_m is not virtually free.*

Proof. All we must show is that L, the torsion free subgroup of finite index, is not free, and the rest of the proof proceeds as in Theorem 4.

A result of Whitehead (see [LS] II.5) says that a one-relator group G on n generators with non-trivial relation is free if and only if the relation is a member of some free basis for the free group on the generators of G. If this is so, then G is free of rank $n-1$.

L is a one-relator group and the relation r_0 has exponent sum 0 on each of its generators. So the abelianization $\overline{L}$ is free abelian of rank $2m$, which is the number of generators in the one-relator presentation of L. Thus L cannot be free of rank $2m-1$, and so is not free. $\square$

The case where m is even proceeds in a similar fashion, except that we must use a slightly different finite quotient group.

Lemma 11. *When m is even, there is a homomorphism f from H_m onto the $2m$-th dihedral group $D_{2m} = \langle\, r, s\,;\, r^{2m}, s^2, srs^{-1} = r^{-1}\,\rangle$.*

Proof. Again, send $a \mapsto r$, $b \mapsto s$, and so $[a, b] \mapsto r^{-2}$. In D_{2m}, r^{-2} has order exactly m, so the map extends to a homomorphism g. □

Lemma 12. *L, the kernel of f, is free-by-cyclic.*

Again, the proof is in two pieces.

Lemma 13. *L has a presentation*

$$L = \langle\; x_i, \beta_i, 1 \le i \le 2m-1;$$

$$s_0 = x_2^1 x_4^{-1} \cdots x_{2m-2}^{-1} \beta_{2m-1} \beta_{2m-2}^{-1} x_{2m-2} \beta_{2m-3} \beta_{2m-4}^{-1} \cdots x_2 \beta_1 \cdot$$

$$x_1^{-1} x_3^{-1} \cdots x_{2m-1}^{-1} \beta_{2m-1}^{-1} x_{2m-1} \beta_{2m-2} \beta_{2m-3}^{-1} \cdots x_3 \beta_2 \beta_1^{-1} x_1 \rangle.$$

Proof. $\{a^i b^j, 0 \le i \le 2m-1, j = 0, 1\}$ is a Schreier transversal for H_m mod L, and so L is generated by

$$x_i = a^i b a b^{-1} a^{-\overline{(i-1)}}$$

$$\beta_i = a^i b^2 a^{-i}$$

$$\alpha = a^{2m}.$$

where $0 \le i \le 2m-1$ and $\overline{(i-1)}$ denotes $(i-1) \mod 2m$.

We get $4m$ relations for L, but only 4 distinct ones.

$[a, b]^m$ is rewritten as

$$(s_1) \qquad x_2^{-1} x_4^{-1} \cdots x_{2m-2}^{-1} x_0^{-1}.$$

We conjugate $[a,b]^m$ by a, and find that $a[a,b]^m a^{-1}$ rewrites as

$$(s_2) \qquad x_3^{-1} \cdots x_{2m-3}^{-1} x_{2m-1}^{-1} \alpha x_1^{-1}.$$

$b[a,b]^m b^{-1}$ rewrites as

$$(s_3) \qquad x_0 \beta_{2m-1} \beta_{2m-2}^{-1} x_{2m-2} \beta_{2m-3} \beta_{2m-4}^{-1} \cdots x_2 \beta_1 \beta_0^{-1}.$$

Then we conjugate $b[a,b]^m b^{-1}$ by a and get $ab[a,b]^m (ab)^{-1}$, which is rewritten as

$$(s_4) \qquad x_1 \beta_0 \alpha^{-1} \beta_{2m-1}^{-1} x_{2m-1} \beta_{2m-2} \beta_{2m-3}^{-1} \cdots x_3 \beta_2 \beta_1^{-1}.$$

Any other relation $a^i b^j [a,b]^m (a^i b^j)^{-1}$ is a cyclic permutation of one of these, for $aa[a,b]^m a^{-1} a^{-1}$ rewrites as

$$x_4^{-1} \cdot x_6^{-1} \cdots x_0^{-1} \alpha^{-1} \cdot \alpha x_2^{-1},$$

which reduces to a cyclic permutation of (s_1), and $aab[a,b]^m (ab)^{-1} a^{-1}$ rewrites as

$$x_2 \cdot \beta_1 \alpha^{-1} \cdot \alpha \beta_0^{-1} \alpha^{-1} \cdot \alpha x_0 \cdot \beta_{2m-1} \cdot \beta_{2m-2}^{-1} \cdots x_4 \beta_3 \beta_2^{-1},$$

which reduces to a cyclic permutation of (s_3).

We simplify the presentation via Tietze transformations, noting that

$$(s_2) \qquad \alpha = x_{2m-1} x_{2m-3} \cdots x_3^{-1} x_1$$

$$(s_3) \qquad x_0 = \beta_0 \beta_1^{-1} x_2^{-1} \cdots \beta_{2m-4} \beta_{2m-3}^{-1} x_{2m-2}^{-1} \beta_{2m-2} \beta_{2m-1}^{-1}$$

$$(s_4) \qquad \beta_0 = x_1^{-1} \beta_1 \beta_2^{-1} x_3^{-1} \cdots \beta_{2m-3} \beta_{2m-2}^{-1} x_{2m-1}^{-1} \beta_{2m-1} \alpha$$

allow us to remove the generators α, x_0, and β_0, and the relations s_2, s_3, s_4, and rewrite the relation s_1 to get the presentation described in the lemma.

Notice that L has $2(2m-1)$ generators and each generator appears in s_0 once with exponent +1 and once with exponent -1. $\square$

Lemma 14. *L has an infinite cyclic quotient $\langle x_1 \rangle$. The kernel of the homomorphism $L \xrightarrow{u} \langle x_1 \rangle$ is free.*

Proof. We map $L \xrightarrow{u} \langle x_1 \rangle$ by sending

$$x_1 \longmapsto x_1$$

$$g \longmapsto 1, \text{all other generators } g$$

Call the kernel of u, K. The generators of K are the conjugates by x_1 of all the other generators of L:

$$x_{i,k} = x_1^k x_i x_1^{-k}, i = 2, \ldots 2m-1, k \in \mathbf{Z}$$

$$\beta_{i,k} = x_1^k \beta_i x_1^{-k}, i = 1, \ldots, 2m-1, k \in \mathbf{Z}$$

When we rewrite the relation s_0, we get

$$s_{0,0} = x_{2,0}^{-1} x_{4,0}^{-1} \cdots x_{2m-2,0}^{-1} \beta_{2m-1,0} \beta_{2m-2,0}^{-1} x_{2m-2,0} \beta_{2m-3,0} \beta_{2m-4,0} \cdots x_{2,0} \beta_{1,0} \cdot$$

$$x_{3,-1}^{-1} \cdots x_{2m-1,-1}^{-1} \beta_{2m-1,-1}^{-1} x_{2m-1,-1} \beta_{2m-2,-1} \beta_{2m-3,-1}^{-1} \cdots x_{3,-1} \beta_{2,-1} \beta_{1,-1}^{-1} \cdot$$

The other relations are of the form

$$s_{0,k} = s_{0,0}(x_{i,k-1}, x_{i,k}, \beta_{i,k-1}, \beta_{i,k})$$

as k ranges through the integers.

K is free since in $s_{0,0}$, the generators $\beta_{1,-1}$ and $\beta_{1,0}$ appear uniquely. $\square$

Theorem 15. *H_m is not virtually free.*

Proof. Again, we need only show that L, the torsion free subgroup of finite index, is not free. As with Theorem 10, the previous result of this type, this follows from the observation that $\overline{L}$ is free abelian of rank $2(2m-1)$, which is the number of generators in the one-relator presentation for L. $\square$

Theorem 5 can be generalized to the groups $S_k = \langle a, b, g_l, l \in I\,;\, ([a,b]w)^k,$ w a word in $g_l \rangle$. The proofs involve essentially the same computations as in Theorem 5.

Theorem 16. *The groups S_k are virtually free-by-cyclic.*

The homomorphism onto the appropriate dihedral group $D_m = \langle r, s\,;\, r^m, s^2,$ $srs^{-1} = r^{-1} \rangle$, where $m = k$ if k is odd and $m = 2k$ if k is even, is defined by sending $a \longmapsto r, b \longmapsto s, g_l \longmapsto 1$.

In each case, we get

Lemma 17. *L, the kernel of the homomorphism $S_k \longrightarrow D_m$, is free-by-cyclic.*

Let us denote by $g_{l,i,j}$ the conjugate of g_l by a^ib^j, $a^ib^jg_l(a^ib^j)^{-1}$, and by $w_{i,j}$ the word $a^ib^jw(a^ib^j)^{-1} = w(g_{l,i,j})$. $g_{l,i,j}$ and thus $w_{i,j}$ are in L.

Lemma 18. *When k is odd, L has the presentation*

$$\langle \beta_i, 0 \le i \le k-1, x_i, 0 \le i \le k-1, g_{l,i,j}, l \in I, 0 \le i \le k-1, 0 \le j \le 1;$$

$$x_0\beta_{m-1}\beta_{m-2}^{-1}w_{m-2,1}x_{m-2}\beta_{m-3}\beta_{m-4}^{-1}w_{m-4,1}\cdots x_1\beta_0x_1^{-1}w_{1,0}x_3^{-1}w_{3,0}\cdots x_{m-2}^{-1}w_{m-2,0}\cdot$$

$$x_0^{-1}w_{0,0}x_2^{-1}w_{2,0}\cdots x_{m-1}^{-1}w_{m-1,0}\beta_{m-1}^{-1}w_{m-1,1}\cdots x_2\beta_1\beta_0^{-1}w_{0,1} \rangle.$$

Proof. We should notice that again we have the Schreier transversal $\{a^i b^j\}$ for $0 \leq i \leq k-1, 0 \leq j \leq 1$, and the listed generators are just the Schreier generators.

In order to rewrite the relations, notice that the representative of $([a,b]w)^{l-1}[a,b]$, $\overline{([a,b]w)^{l-1}[a,b]} = a^{2l}$. So when we rewrite $([a,b]w)^k$ we replace the generator g appearing in the l-th occurrence of w by $a^{2l}ga^{-2l} = g_{2l,0}$, and $([a,b]w)^k$ becomes

$$x_2^{-1}w_{2,0}x_4^{-1}w_{4,0}\cdots x_{m-1}^{-1}w_{m-1,0}\alpha x_1^{-1}w_{1,0}x_3^{-1}w_{3,0}\cdots x_{m-2}^{-1}w_{m-2,0}x_0^{-1}w_{0,0}.$$

When we rewite $a([a,b]w)^k a^{-1}$, we get

$$x_3^{-1}w_{3,0}x_5^{-1}w_{5,0}\cdots x_0^{-1}\alpha^{-1}\cdot \alpha w_{0,0}\alpha^{-1}\cdot \alpha \cdot x_2^{-1}w_{2,0}\cdot x_{m-1}^{-1}w_{m-2,0}\cdot \alpha x_1^{-1}w_{1,0},$$

which is a cyclic permutation of the relation above.

Similarly, we rewrite $b([a,b]w)^k b^{-1}$ and get

$$x_0\beta_{m-1}\beta_{m-2}^{-1}w_{m-2,1}x_{m-2}\beta_{m-3}\beta_{m-4}^{-1}w_{m-4,1}\cdots x_1\beta_0\alpha^{-1}\beta_{m-1}^{-1}w_{m-1,1}\cdots x_2\beta_1\beta_0^{-1}w_{0,1},$$

and conjugating by a only gives a cyclic permuation of this relation.

Finally, we notice that we can remove the generator α and get the single relation

$$\begin{aligned} t = {} & x_0\beta_{m-1}\beta_{m-2}^{-1}w_{m-2,1}x_{m-2}\beta_{m-3}\beta_{m-4}^{-1}w_{m-4,1}\cdots x_1\beta_0\cdot \\ & x_1^{-1}w_{1,0}x_3^{-1}w_{3,0}\cdots x_{m-2}^{-1}w_{m-2,0}x_0^{-1}w_{0,0}x_2^{-1}w_{2,0}\cdots x_{m-1}^{-1}w_{m-1,0}\cdot \\ & \beta_{m-1}^{-1}w_{m-1,1}\cdots x_2\beta_1\beta_0^{-1}w_{0,1}. \end{aligned}$$

□

Lemma 19. *L maps onto the infinite cyclic group* $\langle x_0 \rangle$ *with free kernel.*

Proof. The kernel of this map has a presentation that is very similar to that of M in Lemma 9. The differences are the addition of generators which are the conjugates by x_0 of the generators $g_{l,i,j}$ and the appearance in each relation of the conjugates by x_0 of the words $w_{i,j}$. Nonetheless, when t is rewritten as t_0, we find unique occurrences of $\beta_{0,0}$ and $\beta_{0,1}$, and thus the kernel is free. □

Lemma 20. *When k is even and* $m = 2k$, *L has the presentation*

$$\langle\, x_i, \beta_i, 0 \le i \le m-1, g_{l,i,j}, l \in I, 0 \le i \le 2k-1, 0 \le j \le 1;$$

$$x_2^{-1} w_{2,0} x_4^{-1} w_{4,0} \cdots x_{2m-2}^{-1} w_{2m-2,0} \beta_{2m-1} \beta_{2m-2}^{-1} w_{2m-2,1} \cdot$$

$$x_{2m-2} \beta_{2m-3} \beta_{2m-4}^{-1} w_{2m-4,1} \cdots x_2 \beta_1 x_1^{-1} w_{1,0} x_3^{-1} w_{3,0} \cdots x_{2m-1}^{-1} w_{2m-1,0} \cdot$$

$$\beta_{2m-1} w_{2m-1,1} x_{2m-1} \beta_{2m-2} \beta_{2m-3}^{-1} w_{2m-3,1} \cdots x_3 \beta_2 \beta_1^{-1} w_{1,0} x_1 w_{0,1} \,\rangle.$$

We will omit the details of the proof. It proceeds just like that of Lemma 13, with the extra considerations of $g_{l,i,j}$ and $w_{i,j}$ as in Lemma 18.

Lemma 21. *L maps onto the infinite cyclic group* $\langle x_1 \rangle$ *with free kernel.*

Again, the details of the proof are like those of Lemma 14 with the additional considerations of Lemma 19.

A specific instance of Theorem 16 is of interest.

Theorem 22. *The groups* $\langle\, a_i, b_i, 1 \le i \le n\,;\, ([a_1, b_1] \cdots [a_n, b_n])^k \,\rangle$ *are virtually free-by-cyclic.*

Section 2.2

Closure Properties in Virtually Free-by-Cyclic Groups

One approach to showing that one-relator groups with torsion are virtually free-by-cyclic would be to exploit the Magnus decomposition of a one-relator group via HNN extensions and free products with amalgamation. Any one-relator group is a subgroup of an HNN extension with one stable letter of a one-relator group, where the associated subgroups are free and the relator in the base group is shorter than the original relator. If the original relator is a proper power, then the relator of the base group is also. Thus we can apply the same process to the base group, and express it as a subgroup of an HNN extension of a one-relator group, with still shorter relator. Putting this together with the well known embedding of an HNN extension in a free product with amalgamation gives this result from [KS2]:

Theorem. *Any one-relator group with torsion can be produced by starting with a finite cyclic group and applying finitely many times the operations of taking a free product with amalgamation of two groups already produced and taking a subgroupof a group already produced.*

To apply this decomposition to the problem of showing a one-relator group with torsion is virtually free-by-cyclic, we would need to know that the class of virtually free-by-cyclic groups is closed under constructions of this type. Some of those questions are addressed in this section.

Some Basic Results.

To prove the first results, we will need some lemmas for groups that are free-by-cyclic.

Let G_1 and G_2 be free-by-cyclic groups, $G_1/F_1 \simeq \langle x_1 \rangle$ and $G_2/F_2 \simeq \langle x_2 \rangle$. We show, in Lemma 3.1, that an extension of this type is a split extension. Thus G_i is generated by its free subgoup F_i and the single additional element x_i of infinite order.

The class of free-by-cyclic groups is closed under the taking of subgroups because of a very general result.

Proposition 1. *If $\mathcal{P}$ and $\mathcal{Q}$ are properties of groups which are inherited by subgroups, and G is $\mathcal{P}$-by-$\mathcal{Q}$, then any subgroup M of G is also $\mathcal{P}$-by-$\mathcal{Q}$.*

Proof. Suppose $G/K \simeq H$, where K has property $\mathcal{P}$ and H has property $\mathcal{Q}$. Either M is a subgroup of K and so has property $\mathcal{P}$, or $M \neq M \cap K$ and then $M/M \cap K \simeq KM/K$, which is a subgroup of H. In either case, M is $\mathcal{P}$-by-$\mathcal{Q}$. □

Corollary 2. *Any subgroup of a free-by-cyclic group is free-by-cyclic.*

Proposition 3. *$H = G_1 * G_2$ is free-by-cyclic.*

Proof. Map $H \longrightarrow \langle x_1 \rangle * \langle x_2 \rangle \longmapsto \langle x_1 \rangle$. The first homorphism is just the map from $G_1 * G_2 \longrightarrow G_1/F_1 * G_2/F_2$; the second homomorphism sends $x_1 \mapsto x_1$, and $x_2 \mapsto x_1$.

Call the composition of these maps γ. $ker\gamma$ is the normal closure in H of the groups F_1, F_2 and $\langle x_1 x_2^{-1} \rangle$. By the Kurosh subgroup theorem, $ker\gamma$ is in fact equal to the free product of $\langle x_1 x_2^{-1} \rangle$ and conjugates of the groups F_1 and F_2. All of these free factors are free. □

Proposition 4. *If G is free-by-cyclic, then $Z(G)$ is either*

(1) *trivial,*

(2) *infinite cyclic, or*

(3) *all of G, and $G \simeq \mathbf{Z} \times \mathbf{Z}$.*

Proof. $G/F \simeq \langle x \rangle$, and $Z(G) \cap F < Z(F)$, which is trivial unless $F = \langle y \rangle$. If F is of rank 2 or more, then $Z(G) < \langle x \rangle$ and so either (1) or (2) is true. If $F = \langle y \rangle$, then $G \simeq \langle y \rangle \rtimes \langle x \rangle$. There are only two possibilities for G:

(1) $G \simeq \langle y \rangle \times \langle x \rangle$, so (3) above is true, or

(2) $G \simeq \langle y, x\, ;\, xyx^{-1} = y^{-1} \rangle$, in which case (2) above is true since $Z(G) = gp(x^2)$.

□

Corollary 5. *Neither the class of free-by-cyclic groups nor the class of virtually free-by-cyclic groups is closed under direct products.*

Proof. $G = (\mathbf{Z} \times \mathbf{Z}) \times \mathbf{Z}$ is not free-by-cyclic. Any subgroup of G of finite index is free abelian of rank 3, so G is not virtually free-by-cyclic either. □

Now we can proceed to the results on virtually free-by-cyclic groups. We should notice that by the result of M. Hall (see [LS] §IV.4) that a finitely generated group has only finitely many subgroups of a given finite index and by Corollary 2, we may assume that a virtually free-by-cyclic group is actually a finite extension of a free-by-cyclic group. Suppose that H_i/G_i is finite, and G_i is free-by-cyclic as above.

Proposition 6. *If H is virtually free-by-cyclic and M is a subgroup of H, then M is virtually free-by-cyclic.*

Proof. By Corollary 2, $M \cap G$ is free-by-cyclic, and $M/M \cap G \simeq MG/G$ is finite. □

Proposition 7. *A free product of two virtually free-by-cyclic groups is virtually free-by-cyclic.*

Proof. Let $C = H_1 * H_2$, and consider the homomorphism $f : C \longrightarrow H_1/G_1 \times H_2/G_2$. Suppose we list the elements of $H_1/G_1 : \{a_i, i \in I_1\}$, and the elements of $H_2/G_2 : \{b_j, j \in I_2\}$. By the Kurosh Subgroup Theorem, N, the kernel of f, is the free product of the free group with generators $\{[a_i, b_j]\}$ and all the groups of the form $N \cap H_1^{b_j} = G_1^{b_j}$ and $N \cap H_2^{a_i} = G_2^{a_i}$. Thus N is virtually free-by-cyclic, by Proposition 3. □

Free Products with Amalgamation.

The main theorem of this section is of this form:

Theorem. *Suppose A and B are virtually free by cyclic groups: A is a finite extension of $M = \langle x, F_1 \rangle$ and B is a finite extension of $N = \langle y, F_2 \rangle$. Let*

$$G = A \underset{x=y}{*} B$$

be their free product amalgamating $\langle x \rangle$ and $\langle y \rangle$. Then, under some extra conditions on the conjugates of x in A and the conjugates of y in B, G is virtually free-by-cyclic.

An essential tool for this section is the subgroup theorem for free products with amalgamation of Karass and Solitar ([KS1]), so we will begin with a detailed explanation of the theorem.

Suppose

$$G = A \underset{U=V}{*} B$$

is the free product of the groups A and B, amalgamating the isomorphic subgroups U and V. The subgroup theorem says that any subgroup H of G is an HNN extension of a tree product S, where each vertex group of S is the intersection with H of a conjugate of A or of B, and each edge group is the intersection with H of a conjugate of U. In addition, each associated subgroup is the intersection with H of a conjugate of U.

The proof of this theorem, and the description of the specific conjugates involved in the subgroup, uses a Kurosh rewriting process, a generalization of the

Reidemeister-Schreier method mentioned in Chapter 1. G is generated by its subgroups A and B. Suppose A is generated by the elements $a_1, a_2, \ldots, u_1, u_2, \ldots$, called α-generators, and B is generated by the β-generators $b_1, b_2, \ldots, v_1, v_2, \ldots$. Also, U is generated by the u-generators $u_1, u_2, \ldots$, the v-generators $v_1, v_2, \ldots$ generate V, and u_i and v_i define the same element in G.

A *compatible regular extended Schreier system* or *cress* consists of a pair of coset representative functions, giving α-representatives and β-representatives, with the following properties:

(A) If a representative N ends in a symbol z of type α (β), $N = Mz$, then N and M are both representatives of type α (β).

(B) If the α-symbols are deleted from the end of each α-representative, the resulting collection of words forms a double coset representative system, called the α-double coset representative system, for G mod (H, A). The β-representatives have the same property.

(C) If the u-symbols are deleted from the end of each α-representative, the resulting collection of words is a double coset representative system, called the u-double coset representative system, for G mod (H, U). The v-symbols and β-representatives have the same property.

(D) An α-representative does not end in a v-generator; a β-representative does not end in a u-generator.

(E) If K is both a u- and a v-double coset representative, then $KP(u_i)$ is an

α-representative if and only if $KP(v_i)$ is a β-representative, where $P(u_i)$ is a word in the u-symbols and $P(v_i)$ is the same word in the v-symbols.

We will use the notation ${}^{\alpha}W$ for the α-representative of the element W, and ${}^{\beta}W$ for the β-representative of W.

We can obtain a set of generators for H by using the Kurosh method (see [MKS] §4.3):

H is generated by the elements $s_{K,a} = Ka\,{}^{\alpha}(Ka)^{-1}$, for all α-generators a and all α-representatives K, $s_{L,b} = Lb\,{}^{\beta}(Lb)^{-1}$, for all β-generators b and β-representatives L, and $t_L = L\,{}^{\alpha}(L)^{-1}$ for all β-representatives L.

There is an alternative view of a cress which is better for explicit computations. Each α-representative K can be expressed as a product of three words: $K = D_\alpha E_u P$, where P is the maximal terminal segment of K containing only u-generators, and $E_u P$ is the maximal terminal segment containing only α-generators. Any β-representative can be similarly expressed as a product of three words $D_\beta E_v P$.

The following conditions are equivalent to conditions (A) - (D) in the definition of a cress:

(1) The collection $\{D_\alpha\}$ is a double coset representative system for G mod (H, A), and each $\{D_\alpha\}$ is a β-representative ending in a b-symbol, unless $\{D_\alpha\} = 1$. An analogous statement holds for $\{D_\beta\}$.

(2) The collection $\{D_\alpha E_u\}$ is a double coset representative system for G mod (H, U). An analogous statement holds for the collection $\{D_\beta E_v\}$.

(3) The collection $\{E_uP\}$ of words which go with a fixed D_α form a Schreier system for A mod $A \cap D_\alpha^{-1}HD_\alpha$. An analogous statement holds for a fixed D_β.

(4) The collection $\{E_u\}$ of words which go with a fixed D_α form a Schreier system for A mod $(D_\alpha^{-1}HD_\alpha, U)$. An analogous statement holds for a fixed D_β.

(5) The collection $\{P\}$ of words which go with a fixed $\{D_\alpha E_u\}$ form a Schreier system for U mod $U \cap (D_\alpha E_u)^{-1}H(D_\alpha E_u)$. An analogous statement holds for a fixed $D_\beta E_v$.

A pair of coset representative functions with these properties can always be constructed using an inductive procedure.

Using this terminology available, we can state the subgroup theorem fully:

Subgroup Theorem for Free Products with Amalgamation. *Suppose*

$$G = A \underset{U=V}{*} B,$$

and consider H, a subgroup of G. Suppose there is a cress for G mod H with α-representatives $\{D_\alpha E_u\}$ and β-representatives $\{D_\beta E_v\}$. Then H is generated by those $t_{D_\beta E_v}$ such that $D_\beta E_v$ is neither an α- nor a β-double coset representative, together with all the subgroups $H \cap D_\alpha A D_\alpha^{-1} = H \cap A^{D_\alpha}$ and $H \cap D_\beta B D_\beta^{-1} = H \cap B^{D_\beta}$. Furthermore,

(1) *The $t_{D_\beta E_v}$ listed generate a free group.*

(2) *The subgroups* $H \cap A^{D_\alpha}$ *and* $H \cap B^{D_\beta}$ *generate a tree product* S, *with an edge connecting a vertex* $H \cap A^{D_\alpha}$ *to a vertex* $H \cap B^{D_\beta}$ *whenever either* D_α *or* $D_\beta = 1$, *or* D_α *is obtained by deleting the last syllable from* D_β *or vice versa. If* D *is the longer of the* D_α, D_β, *then the amalgamated subgroup is* $H \cap U^D$.

(3) H *is an HNN extension of* S:

$$H = \langle\, S, t_{D_\beta E_v} \;;$$

$$rel\ S, t_{D_\beta E_v}(H \cap (D_\alpha E_u)U(D_\alpha E_u)^{-1})t_{D_\beta E_v}^{-1} = H \cap (D_\beta E_v)U(D_\beta E_v)^{-1}\,\rangle$$

where $D_\alpha E_u$ *is the* u*-double coset representative of* $D_\beta E_v$.

With this terminology and structure available, we can state and prove the first version of our Theorem.

Theorem 8. *Suppose* A *and* B *are virtually free-by-cyclic groups. Specifically, let* A/M *be finite, with* $\{r_i, 1 \leq i \leq k\}$ *the coset representatives of* M *in* A, *and* B/N *be finite, with* $\{s_j, 1 \leq j \leq l\}$ *the coset representatives of* N *in* B. *Also suppose that* M *is a cyclic extension by* x *of the free group* F_1, *and* N *is a cyclic extension by* y *of the free group* F_2. *Form*

$$G = A \underset{x=y}{*} B,$$

the free product of A *and* B, *amalgamating* $\langle x \rangle$ *and* $\langle y \rangle$. *Then, if the conjugates of* x *in* A *and of* y *in* B *satisfy this condition:*

$$\text{for each } r_i,\ x^{r_i} = x \bmod M', \text{ and}$$

$$\text{for each } s_j,\ y^{s_j} = y \bmod N',$$

G is virtually free-by-cyclic.

G has an obvious normal subgroup of finite index, since $G \xrightarrow{\alpha} A/M \times B/N$. Let $K = ker\alpha$; we will show that K is free-by-cyclic.

Lemma 9. *The collections $\{s_j r_i\}$ and $\{r_i s_j\}$ are α- and β-representatives of a cress for G mod H.*

Proof. We will write out the properties for the α-representatives only; the proof for the β-representatives is the same.

(1) $\{s_j\}$ is a double coset representative system for G mod (K, A).

(2) $\{s_j r_i\}$ is a double coset representative system for G mod $(K, \langle x \rangle)$:

Since $\langle x \rangle < K$, each $(K, \langle x \rangle)$-double coset is simply a K-coset.

(3) The collection $\{r_i\}$ is a Schreier system for A mod $A \cap s_j^{-1} K s_j$:

K is normal in G, so $A \cap s_j^{-1} K s_j = A \cap K = M$, and $\{r_i\}$ is a Schreier system for A mod M.

(4) The collection $\{r_i\}$ is a Schreier system for A mod $(A \cap s_j^{-1} K s_j, \langle x \rangle)$:

This follows from the property above and the fact that $\langle x \rangle < K$.

(5) The empty collection is a Schreier system for $\langle x \rangle$ mod $\langle x \rangle \cap (s_j r_i)^{-1} K (s_j r_i)$:

$\langle x \rangle \cap (s_j r_i^{-1} K (s_j r_i) = \langle x \rangle \cap K = \langle x \rangle$.

(E) If K is both a u- and a v-double coset representative, then $KP(u_i)$ is an α-representative if and only if $KP(v_i)$ is a β-representative:

The only representatives K which are both u- and v-double coset representatives are $\{s_j\}$ and $\{r_i\}$ and the empty word. If $KP(u_i)$ is an α-

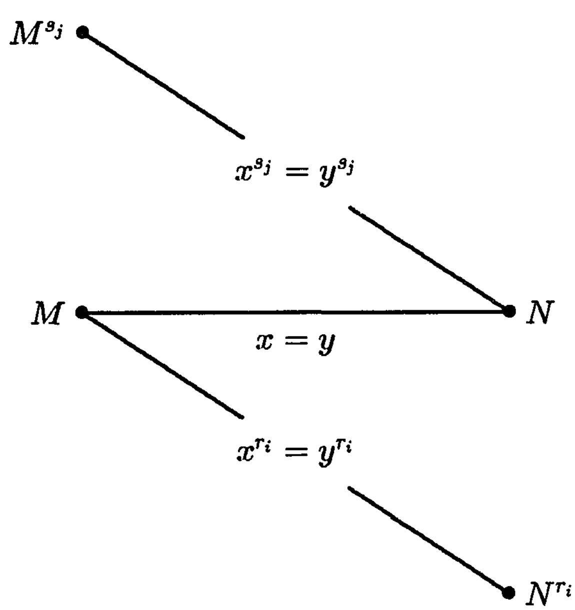
representative, then P is empty, in which case K is also a β-representative.

$\square$

Now that we have a cress for G mod K, we can describe K fully:

Lemma 10. *K is an HNN extension of a tree product S. The vertex subgroups of S are the groups M^{s_j} and N^{r_i}. The edge subgroups are $\langle x\rangle^{s_j}$ and $\langle y\rangle^{r_i}$.*

The tree for S is



The stable letters of the HNN extension are

$$\tau_{ij} = r_i s_j r_i^{-1} s_j^{-1}$$

and their action is given by

$$\tau_{ij}(x^{r_i})^{s_j}\tau_{ij}^{-1} = (y^{s_j})^{r_i}.$$

Proof. From the subgroup theorem, the vertex subgroups of S are the groups

$$K \cap s_j A s_j^{-1} = K \cap A^{s_j} = M^{s_j}$$

$$K \cap r_i B r_i^{-1} = K \cap B^{r_i} = N^{r_i}$$

since $K \cap A^{s_j} = ker\alpha|_{A^{s_j}}$.

The edge subgroups are

$$K \cap \langle x \rangle^{s_j} = \langle x \rangle^{s_j} \hookrightarrow M^{s_j}$$

$$\hookrightarrow N$$

$$K \cap \langle y \rangle^{r_i} = \langle y \rangle^{r_i} \hookrightarrow N^{r_i}$$

$$\hookrightarrow M.$$

which gives the tree as drawn.

The stable letters and associated subgroups follow in a similar fashion from the subgroup theorem. □

Notice that each M^{s_j} is simply an isomorphic copy of M, and similarly N^{r_i} is isomorphic to N. Thus S is a tree product of finitely many copies of M and of N, with amalgamations of the copy x^{s_j} of x in M^{s_j} with the image in N of y under conjugation by s_j and of the copy y^{r_i} of y in N^{r_i} with the image in M of x under conjugation by r_i.

The element $(x^{r_i})^{s_j}$ is the copy in the vertex group M^{s_j} of the conjugate of x by r_i. The stable letter τ_{ij} conjugates this element to $(y^{s_j})^{r_i}$, the copy in N^{r_i} of (y^{s_j}).

Lemma 11. *K maps onto the infinite cyclic group if the conjugates of x in A*

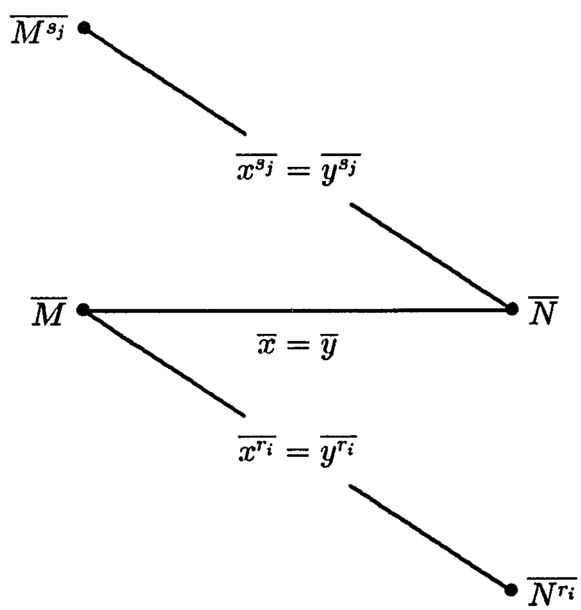
and of y in B satisfy this condition:

$$\text{for each } r_i,\ x^{r_i} = x \bmod M',\ \text{and}$$

$$\text{for each } s_j,\ y^{s_j} = y \bmod N'.$$

Proof. We define a homomorphism $\phi : K \longrightarrow \langle x \rangle$. We first abelianize K and then map $\overline{K} = K/K' \longrightarrow \langle x \rangle$. Now $\overline{K} = \langle \overline{\tau_{ij}} \rangle \times \langle \overline{S}\,;\, \overline{(x^{r_i})^{s_j}} = \overline{(y^{s_j})^{r_i}} \rangle$. Here we mean by $\langle \overline{\tau_{ij}} \rangle$ an abelian group presentation of the free abelian group generated by $\{\tau_{ij}\}$. $\langle \overline{S}\,;\, \overline{(x^{r_i})^{s_j}} = \overline{(y^{s_j})^{r_i}} \rangle$ is the abelian group presentation of the quotient of $\overline{S}$, the abelianization of S, by the subgroup generated by $\{(x^{r_i})^{s_j} \cdot ((y^{s_j})^{r_i})^{-1}\}$. The additional relations come from the abelianization of the associated subgroup relations in K.

$\overline{S}$ is a central product given by the tree for S:



Let's look more closely at the vertex groups of $\overline{S}$. For instance,

$$\overline{M} = \langle\, \overline{x}\, \rangle \times M_1,$$

and we know that for each r_i, $\overline{x^{r_i}} = \overline{x}$. Also, M_1 is F_1/M'.

We can send $\overline{M} \longrightarrow \langle\, x\, \rangle$ by mapping

$$\overline{x} \longmapsto x$$

$$t \longmapsto 1,\ t \in M_1.$$

The other vertex goups of $\overline{S}$ have the same type of structure, so we can define our map on all of $\overline{S}$ in a similar fashion. For instance, look at

$$\overline{N^{r_i}} = \langle\, \overline{y^{r_i}}\, \rangle \times \overline{N_1^{r_i}}.$$

Again, $\overline{N_1^{r_i}} = F_2^{r_i}/(N^{r_i})'$, and $\overline{(y^{s_j})^{r_i}} = \overline{y^{r_i}}$, for each s_j. So we map $\overline{N^{r_i}} \longrightarrow \langle\, x\, \rangle$ by sending

$$\overline{y^{r_i}} \longmapsto x$$

$$t \longmapsto 1,\ t \in \overline{N_1^{r_i}}$$

Notice that we have been careful to ensure that these maps agree on the amalgamated subgroups of $\overline{S}$:

$$\overline{x} \longmapsto x \quad \text{and} \quad \overline{y} \longmapsto x$$

$$\overline{x^{r_i}} = \overline{x} \longmapsto x \quad \text{and} \quad \overline{y^{r_i}} \longmapsto x$$

$$\overline{y^{s_j}} = \overline{y} \longmapsto x \quad \text{and} \quad \overline{x^{s_j}} \longmapsto x$$

There are some additional relations introduced by the abelianization of the asociated subgroup relations of the HNN extension K. The map we have constructed respects these relations. Look at a typical relation $\overline{(x^{r_i})^{s_j}} = \overline{(y^{s_j})^{r_i}}$:

$$\text{In } \overline{M^{s_j}},\ \overline{(x^{r_i})^{s_j}} = \overline{x^{s_j}} \longmapsto x.$$

$$\text{In } \overline{N^{r_i}},\ \overline{(y^{s_j})^{r_i}} = \overline{y^{r_i}} \longmapsto x.$$

Finally, we can extend this map to $\overline{K}$ by sending

$$\overline{\tau_{ij}} \longmapsto 1.$$

The homomorphism $\phi : K \longrightarrow \langle\, x \,\rangle$ is then the composition

$$K \longrightarrow \overline{K} \longrightarrow \langle\, x \,\rangle. \quad \square$$

Call the kernel of ϕ, L. We need to show that L is free. Since L is a subgroup of G, it also has the structure of an HNN extension of a tree product. To analyze L in detail, we need a cress for G mod L.

Lemma 12. *The sets of coset representatives* $\{s_j r_i x^m\}$ *and* $\{r_i s_j y^m\}$, $m \in \mathbf{Z}$, *form a cress for* G *mod* L.

Proof. We will write out the proof for the α-representatives only; the proof for the β-representatives is similar.

First, $\{s_j r_i x^m\}$ is a set of coset representatives for G mod L, since $\{s_j r_i\}$ is a set of representatives for G mod K and $\{x^m\}$ is a set of representatives for K mod L.

(1) $\{s_j\}$ is a double coset representative system for G mod (L, A):

Since $\{s_j r_i x^m\}$ is a set of coset representatives for G mod L, $g \in G$ is equal to a unique element $l \cdot s_j r_i x^m$, so $g \in L s_j A$.

(2) $\{s_j r_i\}$ is a double coset representative system for G mod $(L, \langle x \rangle)$:

In a manner similar to the property above, $g \in L s_j r_i \langle x \rangle$.

(3) The collection $\{r_i x^m\}$ is a Schreier system for A mod $A \cap (s_j)^{-1} L (s_j)$:

Since $L < K$ which is normal in G, $(s_j)^{-1} L (s_j) < K$. Thus, $A \cap (s_j)^{-1} L (s_j) = A \cap K \cap (s_j)^{-1} L (s_j) = M \cap (s_j)^{-1} L (s_j)$.

We need to show that $M \cap (s_j)^{-1} L (s_j) = M \cap L$. Suppose that $m \in M \cap (s_j)^{-1} L (s_j)$. Then $m^{s_j} \in M^{s_j} \cap L$. But $\phi(m) = \phi(m^{s_j})$, since the homomorphism ϕ respects the isomorphism $M \simeq M^{s_j}$. Thus if $m^{s_j} \in L = ker\phi$, then $m \in L$ and so $m \in M \cap L$. The other inclusion follows from the same argument.

$\{r_i x^m\}$ is clearly a Schreier system for A mod $M \cap L$, for $\{r_i\}$ is a Schreier system for A mod M and $\{x^m\}$ is a Schreier system for M mod $M \cap L$.

(4) The collection $\{r_i\}$ is a Schreier system for A mod $(A \cap s_j^{-1} L s_j, \langle x \rangle)$:

From the item above, we know that each $a \in A$ is in a unique coset $(A \cap {s_j}^{-1} L s_j) r_i x^m$. Then $(A \cap {s_j}^{-1} L s_j) r_i \langle x \rangle = \bigcup_m (A \cap {s_j}^{-1} L s_j) r_i x^m$, so the $\{r_i\}$ form the desired Schreier system.

(5) The set $\{x^m\}$ is a Schreier system for $\langle x \rangle$ mod $\langle x \rangle \cap (s_j r_i)^{-1} L (s_j r_i)$:

$\{x^m\}$ is a system of representatives for $\langle x \rangle$ mod $\langle x \rangle \cap T$ exactly when $\langle x \rangle \cap T = 1$. If $t \in \langle x \rangle \cap (s_j r_i)^{-1} L (s_j r_i)$, then $(s_j r_i) t (s_j r_i)^{-1} \in \langle (x^{r_i})^{s_j} \rangle \cap L$.

But $\phi(x) = \phi((x^{r_i})^{s_j})$, so ϕ is one-to-one on $\langle\, (x^{r_i})^{s_j} \,\rangle$, and t must be the identity.

(E) If K is both a u- and a v-double coset representative, then $KP(u_i)$ is an α-representative if and only if $KP(v_i)$ is a β-representative:

The only representatives which are both u- and a v-double coset representatives are the element 1 and the elements $\{r_i\}$ and $\{s_j\}$. For any of these representatives K, $KP(u_i) = Kx^m$ is an α-representative if and only if $KP(v_i) = Ky^m$ is a β-representative.

□

The final piece to be shown is:

Proposition 13. *L is a free group.*

Proof. This will be shown with two lemmas.

Lemma 14. *L is a free product of a free group and certain subgroups of conjugates of A and of B.*

Proof. We know L is an HNN extension of a tree product whose vertex groups are

$$L \cap A^{s_j} = L \cap K \cap A^{s_j} = L \cap M^{s_j} \text{ and}$$

$$L \cap B^{r_i} = L \cap N^{r_i}.$$

The stable letters are

$$\tau_{ij} = r_i s_j r_i^{-1} s_j^{-1}.$$

We must show that the associated and amalgamated subgroups are trivial. An amalgamated subgroup is equal to

$$L \cap \langle x \rangle = L \cap \langle y \rangle, \text{ or}$$

$$L \cap \langle x \rangle^{s_j} = L \cap \langle y \rangle^{s_j}, \text{ or}$$

$$L \cap \langle y \rangle^{r_i} = L \cap \langle x \rangle^{r_i}.$$

First, $L \cap \langle x \rangle = 1$: this group is $ker\phi \cap \langle x \rangle$, which is trivial since ϕ was constructed to be one-to-one on $\langle x \rangle$.

Similarly, ϕ is one-to-one on $\langle y \rangle^{s_j}$ and on $\langle x \rangle^{r_i}$, so the other amalgamated subgroups are trivial.

An associated subgroup is of the form

$$L \cap \langle (x^{r_i})^{s_j} \rangle.$$

Again, ϕ is one-to-one on the group $\langle (x^{r_i})^{s_j} \rangle$, which is the copy of $\langle x^{r_i} \rangle$ residing in M^{s_j}, so the group $L \cap \langle (x^{r_i})^{s_j} \rangle$ is trivial too. □

Lemma 15. *All the factors of L are free.*

Proof. We know that one factor of L is the free group generated by

$$\tau_{ij} = r_i s_j r_i^{-1} s_j^{-1} \text{ with both } r_i, s_j \neq 1.$$

Each other factor is of the form

$$A^{s_j} \cap L \quad \text{or} \quad B^{r_i} \cap L.$$

But $L < K$, so

$$A^{s_j} \cap L = A^{s_j} \cap K \cap L = M^{s_j} \cap L \text{ and}$$

$$B^{r_i} \cap L = B^{r_i} \cap K \cap L = N^{r_i} \cap L.$$

Recall that $M = \langle x, F_1 \rangle$ is free-by-cyclic. $M \cap L$, which is the kernel of $\phi|_M$, is equal to $gp_M(M', F_1)$. But $M' < F_1$, since for any $f \in F_1$, $f^x \in F_1$. Thus $gp_M(M', F_1) = F_1$ is free. Similarly, each of the other groups of the form $M^{s_j} \cap L$ and $N^{r_i} \cap L$ are free. So L is free. □

Notice that the condition on the conjugates x^{r_i} in A and y^{s_j} in B of Theorem 8 was not used in the proofs of Lemmas 9 and 10. The second version of this Theorem uses some alternate conditions to get the other results needed to show that the free product with amalgamation is virtually free-by-cyclic.

Theorem 8′. *Let*

$$G = A \underset{x=y}{*} B$$

with A and B virtually free-by-cyclic as before. Then G is virtually free by cyclic if A and B satisfy these two conditions:

(1) *x has infinite order in A/A', and y has infinite order in B/B'.*

(2) *Every subgroup of M or of N of infinite index is free.*

The kernel K of the homomorphism $G \xrightarrow{\alpha} A/M \times B/N$ has the same structure as before from Lemma 10. Our goal is again to map $K \longrightarrow \langle x \rangle$ by working through $K/K' = \overline{K}$, and to show that the kernel of this composition is free.

We should first notice this consequence of the condition on the order of x in A/A'.

Lemma 16. *If x has infinite order in A/A', then for each pair r_i and r_m of different M-coset representatives, where r_i may be the identity, either $x^{r_i} = x^{r_m}$ mod M', or x^{r_i} and x^{r_m} generate independent infinite cycles in M/M'.*

Proof. Suppose for some r_i, $x^{r_i} = x^k \bmod M'$ where $k \neq 1$. Then $x^{-k}x^{r_i} = m$, for $m \in M'$, or $x^{-k+1}x^{-1}r_i x r_i^{-1} = m$, and so $x^{-k+1} \in A'$.

The proof for any other pair of representatives is similar. □

Lemma 17. *K maps onto the infinite cyclic group if the conjugates of x in A and of y in B satisfy this condition:*

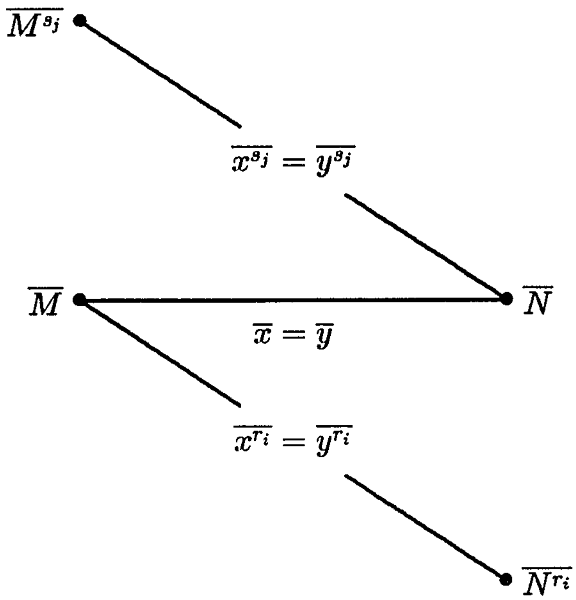
(a) *For each pair r_i and r_m of different M-coset representatives, where r_i may be the identity, either $x^{r_i} = x^{r_m}$ mod M', or x^{r_i} and x^{r_m} generate independent infinite cycles in M/M', and similarly for pairs of representatives s_j and conjugates y^{s_j} in N/N'.*

Proof. In this case, the abelianization of K,

$$\overline{K} = \langle \overline{\tau}_{ij} \rangle \times \langle \overline{S}\, ; \overline{(x^{r_i})^{s_j}} = \overline{(y^{s_j})^{r_i}} \rangle$$

is more complicated.

Again, $\overline{S}$ is a central product:



However,

$$\overline{M} = \langle \overline{x} \rangle \times \langle \overline{x^{r_{k_2}}} \rangle \times \ldots \langle \overline{x^{r_{k_m}}} \rangle \times M_2.$$

Here we have used condition (a) of Lemma 17 and have listed a representative of each of the infinite cycles in $\overline{M}$ generated by $\overline{x}, \overline{x^{r_2}}, \ldots, \overline{x^{r_l}}$. In fact, we know that for each r_i, x^{r_i} is equal to one of $x^{r_{k_l}}$, modulo M'. Now send $\overline{M} \longrightarrow \langle x \rangle$ by

$$\overline{x} \longmapsto x$$

$$\overline{x^{r_{k_i}}} \longmapsto x$$

$$t \longmapsto 1, \; t \in M_2.$$

Each vertex group of $\overline{S}$ has a similar structure, so we can define our map on all of $\overline{S}$ in the same fashion. For instance,

$$\overline{N^{r_i}} = \langle \overline{y^{r_i}} \rangle \times \langle \overline{(y^{s_{l_2}})^{r_i}} \rangle \times \ldots \langle \overline{(y^{s_{l_n}})^{r_i}} \rangle \times N_2,$$

and we send $\overline{N^{r_i}}$ to $\langle\, x\,\rangle$ by

$$\overline{y^{r_i}} \longmapsto x$$

$$\overline{(y^{s_{l_j}})^{r_i}} \longmapsto x$$

$$t \longmapsto 1,\ t \in M_2.$$

We have been careful to ensure that the map agrees on the amalgamated subgroups of this central product. For example,

$$\overline{x^{r_i}} = \overline{x^{r_{k_i}}} \longmapsto x \quad \text{and} \quad \overline{y^{r_{k_i}}} \longmapsto x.$$

This map also respects the relations of the form

$$\overline{(x^{r_i})^{s_j}} = \overline{(y^{s_j})^{r_i}},$$

since the definition on $\overline{M^{s_j}}$ and on $\overline{N^{r_i}}$ is done in the same fashion as the definition on M.

Extend this map to $\overline{K}$ by sending

$$\overline{\tau_{ij}} \longmapsto 1.$$

and again the homomorphism $\phi : K \longrightarrow \langle\, x\,\rangle$ is the composition,

$$K \longrightarrow \overline{K} \longrightarrow \langle\, x\,\rangle.$$

□

Lemma 18. *Under condition (a), the sets of coset representatives* $\{s_j r_i x^m\}$ *and* $\{r_i s_j y^m\}$ *form a cress for* G *mod* L.

Even though condition (a) on conjugation is more general, the proof of Lemma 18 is the same as the proof of Lemma 12.

Let $L = ker\phi$. We must finally show:

Proposition 19. *L is free.*

Lemma 20. *L is a free product of a free group and subgroups of conjugates of A and of B.*

This follows from the Subgroup Theorem just as Lemma 14 does.

Lemma 21. *Each of the free factors of L is free.*

Proof. L is a free product of a free group and the groups

$$M^{s_j} \cap L \quad \text{and} \quad N^{r_i} \cap L,$$

just as before. In this case, each of the groups $M^{s_j} \cap L$ and $N^{r_i} \cap L$ are free by condition (2), since, for example, $M^{s_j} \cap L = ker\phi|_M$ is of infinite index in M^{s_j}. So L is free. □

CHAPTER 3
FREE–BY–FREE GROUPS

In this chapter only, we will use the other familiar notation for conjugation. Until further notice, by a^b, we will mean $b^{-1}ab$, and thus $a^{bc} = (bc)^{-1}a(bc) = (a^b)^c$.

Section 3.1

Some Residually Finite Free-by-Free Groups

In this section we will examine the residual finiteness of free-by-free groups. A finitely generated free-by-cyclic group is always residually finite; the one-relator groups with torsion of Chapter 2 were shown to be residually finite by finding a free-by-cyclic subgroup of finite index.

A general result due to Miller ([M] III.A Theorem 7) on the residual finiteness of certain extensions says, in particular, that an extension of a finitely generated free group by a free group is residually finite. The question we need to consider then involves extensions of a free group of countable rank by a free group of finite rank. We want to examine the actions of a free group on a countably generated free group which give residual finiteness in the semidirect product. This situation arises when we look at free quotients of one-relator groups; the normal subgroup involved is usually infinitely generated.

We should first make note of some basic facts about free-by-free groups. It is easy to show, using the universal mapping property of free groups, that any extension of a group by a free group splits, so every free-by-free group is in fact a split extension.

Lemma 1. *Suppose G is an extension of a group H by a free group F, $G/H \simeq F$. Then the extension is a split extension.*

Proof. Call the isomorphism $\alpha : G/H \simeq F$. Suppose F is free on the set $\{f_1, f_2, \dots, f_k, \dots\}$. Then we can map this set into G by sending each f_i to g_i, a representative of the H-coset $\alpha^{-1}(f_i)$. By the freeness of F, there is a homomorphism $t : F \longrightarrow G$ with $t(f_i) = g_i$, and thus $\alpha t = 1_F$. □

Using this, we can write a free-by-free group G as an HNN extension. If $G/F_1 \simeq F_2$, where $F_1 = \langle\, x_j, j \in J \,\rangle$ and $F_2 = \langle\, t_i, i \in I \,\rangle$ are free on the indicated generators, then $G = \langle\, x_j, j \in J, t_i, i \in I\, ;\, t_i^{-1} x_j t_i = \phi_i(x_j) \,\rangle$, where ϕ_i is an automorphism of F_2. In this case, the associated subgroups of the HNN extension are both the entire base group F_1.

Proposition 2. *Consider $F_k = \langle\, x_1, \dots, x_k \,\rangle$ as before and $F = \langle\, y_1, y_2, \dots, y_n, h_1, h_2, \dots, h_m, \dots \rangle$. Call $H = \langle\, h_1, h_2, \dots, h_m, \dots \rangle$, and $F_n = \langle\, y_1, y_2, \dots, y_n \,\rangle$. Suppose F_k acts on F by*

$$y_i^{x_j} = w_{i,j}(\widetilde{y_s})$$

$$h_i^{x_j} = h_i.$$

That is, F_k acts trivially on H, and nontrivially only on the finitely generated subgroup F_n of F. If $G = F \rtimes F_k$ via this action, then G is residually finite.

Proof. Take $g \in G, g \neq 1$. Then $g = vw, v \in F_k, w \in F$, using the normal form theorem for HNN extensions ([LS] §IV.2). Now w involves only finitely many elements of F, in particular only finitely many of the h_i's. Call the maximum subscript of the h_i's in g, r. Let $M = gp_F(h_{r+1}, h_{r+2}, \dots) = gp(h_i^z, i > r, z \in F)$ and $K = gp_G(h_{r+1}, h_{r+2}, \dots) = gp(h_i^u, i > r, u \in G)$.

$K = M$. Since $F < G$, $M < K$. Notice that if $v \in F_k, h \in H$, then $h^v = h$. Now look at a generator of K, h_i^u. If $u = v_1 u_1, v_1 \in F_k, u_1 \in F$, then

$$h_i^{v_1 u_1} = (h_i^{v_1})^{u_1} = h_i^{u_1} \in M.$$

Call $H_r = \langle h_1, h_2, \dots, h_r \rangle$. $H_r < H$, and $H/M \simeq H_r$. $G \simeq (F_n * H) \rtimes F_k$, and we can map $G \longrightarrow G/K \simeq G/M \simeq (F_n * G_r) \rtimes F_k$. The generators listed for M are Nielsen-reduced, so any product of them will have an occurrence of h_i, for some $i > r$. Thus $g \notin M$, and is nontrivial in $(F_n * H_r) \rtimes F_k$, which is residually finite by [M], so G is residually finite. □

Recall that if F is the free group of countable rank, we have $Aut_f(F) < Aut(F)$, where $Aut_f(F)$ is the subgroup generated by all the automorphisms α of F such that α fixes all but a finite number of the generators of F.

Proposition 3. *If we have $F_n \xrightarrow{g} Aut_f(F) < Aut(F)$, then $G = F \rtimes_g F_n$ is residually finite.*

Proof. Let $k = max_i \{m : f_m$ is moved by some $x_i\}$. Then $F_n \xrightarrow{g} Aut(F_k) < Aut_f(F)$, that is, the action of F_n on F is as in the previous proposition. $\square$

Section 3.2

One-relator Groups which are Free-by-free

In Chapter 2, we considered some one-relator groups which are free-by-cyclic. We used several times a result (see [B1]), quoted in the proof of Lemma 2.1.3, which gives conditions under which the kernel of a map from a one-relator group onto an infinite cyclic group will be free. The consideration of free-by-free one-relator groups led to the development of a similar test to determine whether the kernel of a homomorphism from a one-relator group onto a free group is free. The method involves explicit analysis of the presentation of the kernel.

Lemma 1. *Suppose* $G = \langle\, \alpha, \beta, \dots, \gamma, a, b, \dots, c\,;\, r\,\rangle$ *, a one-relator group, has a free group* F *as a homomorphic image. Suppose the generators* $a, b, \dots, c$ *are in the kernel* L*, and the images of the generators* $\alpha, \beta, \dots, \gamma$ *freely generate* F*. Then* L *has the presentation*

$$\langle\, a_w, b_w, \dots, c_w, w \in F\,;\, r_w, w \in F\,\rangle$$

where the set of relations has the following multiplication property:

$$\text{If } r_1 = r_1(a_{v_1}, \dots, a_{v_l}, b_{y_1}, \dots, b_{y_m}, \dots, c_{u_1}, \dots, c_{u_n}),$$

$$\text{then } r_w = (a_{v_1 w}, \dots, a_{v_l w}, b_{y_1 w}, \dots, b_{y_m w}, \dots, c_{u_1 w}, \dots, c_{u_n w}).$$

Proof. We get the generators and relations from the Reidemeister-Schreier process. Here a_w, $w \in F$, represents $w^{-1}aw$. We will refer to these generator as the a_w

family. The relations are all of the form $w^{-1}rw$, $w \in F$, rewritten in the generators listed for L. Then

$$r_1 = r_1(a_{v_1}, \ldots, a_{v_l}, b_{y_1}, \ldots, b_{y_m}, \ldots, c_{u_1}, \ldots, c_{u_n}),$$

which means that r_1 is a word in the listed conjugates of $a, b, \ldots, c$: $a_{v_1}, \ldots, a_{v_l}$, and so on. And therefore

$$\begin{aligned} r_w &= w^{-1}rw \\ &= r_1(w^{-1}a_{v_1}w, \ldots, w^{-1}a_{v_l}w, w^{-1}b_{y_1}w, \ldots, w^{-1}b_{y_m}w, \ldots, w^{-1}c_{u_1}w, \ldots, w^{-1}c_{u_n}w) \\ &= r_1(a_{v_1w}, \ldots, a_{v_lw}, b_{y_1w}, \ldots, b_{y_mw}, \ldots, c_{u_1w}, \ldots, c_{u_nw}). \end{aligned}$$

since $a_{v_1w} = w^{-1}a_{v_1}w$. □

Thus $G \simeq L \rtimes F$, and we can see that F acts on L by translating generators. More specifically, for each generator g_w of L and each generator x of F,

$$g_w^x = g_{wx}.$$

The next result gives conditions under which a group L with a presentation as in Lemma 1 will be free.

Theorem 2. *Suppose the group L has a presentation as in the lemma above, with the family of relations having the multiplication property. Suppose that in the relation r_1 there is, for some family of generators a_w, a unique occurrence of the generator a_{v_1} with $v_1 = 1 \in F$, and a unique occurrence of the generator a_{v_2}, where $l(v_2)$ is maximal among $l(v_1), \ldots, l(v_l)$, the lengths of the subscripts on* *the generators in the family a_w. Suppose also that for all of these subscripts v_i, $i = 2, \ldots, l$, $v_2 v_i$ is reduced as written, that is, if $v_2 = ug^e$, then no v_i, including v_2, begins with g^{-e}. Under these conditions, L is free on a subset of the generators for L listed in Lemma 1, containing all of $b_w, \ldots, c_w$, and only some of a_w.*

Proof. For each $w \in F$, we will use Tietze transformations to remove a generator and a relation from the presentation of L, ending with a presentation of L with no relations. We will consider separately two different types of words w.

If $l(w) > 0$, then either

(1) $w = g^{-e}s$, or

(2) $w = k^f s$, and $k^f g^e \neq 1$.

Consider first case (2), and let $w = 1$. In r_1, there is a unique occurrence of the generator a_{v_2} with $v_2 = ug^e$ the subscript of maximal length. Thus no other generator a_{v_i} has v_i beginning with ug^e. So we rewrite the relation r_1 to say a_{v_2} is a product of generators from other families g_w and generators a_{v_i} with v_i not beginning with ug^e. Then we can use another Tietze transformation to remove the generator a_{v_2} and the relation and rewrite all occurrences of a_{v_2} without using any generator a_{v_i} where v_i begins with ug^e.

Consider any word $w = k^f s$ as in case (2). Assume as inductive hypothesis that for every v of the form of case (2) with $l(v) < l(w)$, we have removed the relation r_v and the generator $a_{ug^e v}$ and have rewritten each occurrence of $a_{ug^e v}$ in terms of other generators using no generators of type a_w where w is a word beginning with

ug^e.

In r_w, the family of generators a_v becomes :

$$a_{k^f s}, a_{ug^e k^f s}, a_{v_3 k^f s}, \dots, a_{v_m k^f s}.$$

We know that $ug^e k^f s$ is freely reduced as written. Since $l(ug^e) \geq l(v_i)$, for all $i = 3, \dots, n$, $l(ug^e k^f s) \geq l(v_i k^f s)$. If $l(ug^e k^f s) = l(v_i k^f s)$, then $l(ug^e) = l(v_i)$, so v_i does not begin with ug^e. If $l(ug^e k^f s) > l(v_i k^f s)$, either $v_i k^f s$ does not begin with ug^e, or by the inductive hypothesis $a_{v_i k^f s}$ has been rewritten using only generators from other families and generators a_v, where v does not begin with ug^e. Thus we can remove the relation r_w and the generator $a_{ug^e k^f s}$ and rewrite this generator wherever it appears without using any generator a_v, v beginning with ug^e.

Now let's look at (1): $w = g^{-e}s$. Here the initial case is $w = g^{-e}$. In the relation r_w, the generators of type a_v are

$$a_{g^{-e}}, a_{v_2 g^{-e}} = a_{ug^{-e}g^e} = a_u, a_{v_3 g^{-e}}, \dots, a_{v_m g^{-e}},$$

and none of v_i, $i = 2, \dots, n$ begins with g^{-e}. We know that the generator $a_{g^{-e}}$ appears uniquely, so we can use Tietze transformations to remove this generator and this relation, and rewrite every appearance of $a_{g^{-e}}$ using only generators from other families and generators of type a_v where v does not begin with g^{-e}.

Now take any word w of type (1). Assume the inductive hypothesis: for every v of the form of case (1) with $l(v) < l(w)$, we have removed the relation r_v and the generator $a_{g^{-e}v}$ and have rewritten each occurrence of $a_{g^{-e}v}$ without any generators of type a_s where s is a word beginning with g^{-e}.

Since $w = g^{-e}s$, the relation r_w contains the generators from the a_v family

$$a_{g^{-e}s}, a_{v_2g^{-e}s} = a_{ug^{-e}g^{e}s} = a_{us}, a_{v_3g^{-e}s}, \dots , a_{v_ng^{-e}s}.$$

Recall that none of v_i, $i = 1, \dots , n$, begins with g^{-e}, and for $i \geq 2, v_i \neq 1$. So if any $v_ig^{-e}s$ other than $i = 1$ begins with g^{-e}, it is because all of v_i has cancelled in the product. In that case,

$$l(v_ig^{-e}s) < l(s) + 1 - l(v_i) < l(s) + 1 = l(g^{-e}s).$$

Thus by the inductive hypothesis, all the occurrences in r_w of generators $a_{v_ig^{-e}s}$ where $v_ig^{-e}s$ begins with g^{-e}, save the generator $a_{g^{-e}s}$, have been rewritten so as not to use any generator of the type a_v, with v beginning g^{-e}. We can therefore remove this relation r_w and the generator $a_{g^{-e}s}$, and rewrite each appearance of $a_{g^{-e}s}$ without using any a_v, v beginning with g^{-e}.

L now has a presentation with no relations, and so is free on the remaining families of generators $b_w, \dots , c_w$, $w \in F$, and generators a_t, $t \in F$, t begins with neither g^{-e} nor ug^{e}. $\square$

We should notice here that we have a refined view of the extension $G \simeq L \rtimes F$. Assume for convenience that e, the exponent of the distinguished generator g of F mentioned in Theorem 2, is actually 1. (An automorphism of F can arrange this if necessary.) Then the generators x of F act on L by

$y_w^x = y_{wx}$, for all generators x and y a generator of L other than a,

$a_t^x = a_{tx}$, for $x \neq g$,

$a_t^g = a_{tg}$, for $t \neq u$,

$a_u^g = V(a_t, b_w, \dots, c_w)$,

where $V(a_t, b_w, \dots, c_w)$ is the word in the new generators for L which expresses the deleted generator a_{ug}.

The following result gives another condition on the family of relations which will allow us to conclude that the kernel of a map onto a free group is free.

Theorem 3. *Suppose the group L has a presentation as in Lemma 1. Suppose that in the relation r_1 there is, for some family of generators a_w, a unique occurrence of the generator a_{v_1} and a unique occurrence of the generator a_{v_2}, where $l(v_1)$ is maximal among $l(v_1), \dots, l(v_m)$, the lengths of the subscripts on the generators in the family a_w. Suppose also that for all of these subscripts v_i, $i \neq 2$, $v_i v_2^{-1}$ and $v_2^{-1} v_i$ are reduced as written. Under these conditions, L is free on a subset of the generators listed, containing all of $b_w, \dots, c_w$, and only some of a_w.*

Proof. We will change the labelling on the family of relations, and show that with this relabelling the conditions of Theorem 2 apply.

Let s_w represent the relation $r_{wv_2^{-1}}$, so that with the new labels $s_1 = r_{v_2^{-1}}$. Now in s_1, there is a unique occurrence of a_1 and of $a_{v_1 v_2^{-1}}$ coming from, respectively, a_{v_2} and a_{v_1} in r_1. Since the length of v_1 is maximal among the subscripts in the family a_w, and $v_1 v_2{}^{-1}$ is reduced as written, then the length of $v_1 v_2{}^{-1}$ is maximal among the subscripts on the family a_w in the relation s_1. Also, the product of new subscripts, $(v_1 v_2{}^{-1})(v_i v_2{}^{-1})$, is reduced as written, since $v_i{}^{-1} v_2$ and $v_2 v_i{}^{-1}$ are both reduced as written by hypothesis.

Therefore the group L with the relations s_w satisfies the hypotheses of Theorem 2, and so L is free on a subset of the generators as stated. □

The next result gives another condition of the same general type under which the group L will be free.

Theorem 4. *Suppose that we have a group L, with the presentation*

$$L = \langle\, a_w, b_w, \ldots , c_w \,;\, r_w \,\rangle$$

indexed over $w \in F = \langle\, \alpha, \beta, \ldots , \gamma \,\rangle$, and the family of relations has the multiplication property. Suppose also that in r_1 for the family of generators a_w, there is a unique occurrence of a_{v_1}, where $l(v_1)$ is maximal among $l(v_1), \ldots , l(v_m)$, and for some other family of generators b_y, there is a unique occurrence of b_{y_1} where $l(y_1)$ is maximal among $l(y_1), \ldots , l(y_m)$, and also $v_1 y_1^{-1}$ is freely reduced as written in F. Then L is free on a subset of the generators listed, containing all of g_w with $g \neq a, b$ and only some of a_w and b_w.

Proof. For each $w \in F$, we will use Tietze transformations to remove a generator and a relation from the presentation of L and thereby produce a presentation of L with no relations.

Note that $v_1 = ug^e$, u a word in F, and g some generator, $e = \pm 1$, and $y_1 = th^d$, t a word in F, h a generator and $d = \pm 1$. Since $v_1 y_1^{-1}$ is freely reduced as written, either $g \neq h$, or if $g = h$, then $e - d \neq 0$.

The proof proceeds by induction on $l(w)$, with separate consideration of two different types of words w:

Consider any w in F with $l(w) > 0$. Either

(1) $w = g^{-e}s$, or

(2) $w = k^f s$, and $k^f g^e \neq 1$.

We will consider case (2) first.

$w = 1$: In r_1, a_{v_1} occurs uniquely and $l(v_1)$ is maximal, so no other $v_2, \ldots, v_m$ begins with ug^e. We can remove this relation r_1 and rewrite all the appearances of this generator a_{v_1} in terms of generators from other families and a_w's, where w does not begin with ug^e.

Now consider w with $l(w) > 0$, and $w = k^f s$, $k^f g^e \neq 1$. Assume the inductive hypothesis: for every v of this form with $l(v) < l(w)$, we have removed the relation r_v and the generator $a_{ug^e v}$ and have rewritten each occurrence of $a_{ug^e v}$ in terms of other generators using no generators of type $a_{ug^e s}$.

In r_w, a_{v_1} becomes $a_{v_1 w} = a_{ug^e k^f s}$, where there is no cancellation in $ug^e k^f s$. All other a_{v_i} become $a_{v_i k^f s}$, with $l(v_i k^f s) \leq l(ug^e k^f s)$, since $l(v_1)$ is maximal. If $l(v_i) < l(v_1)$, then $l(v_i k^f s) < l(ug^e k^f s)$. If $l(v_i) = l(v_1)$, then either $l(v_i k^f s) = l(ug^e k^f s)$, or $l(v_i k^f s) < l(ug^e k^f s)$, because some reduction occurs in $v_i k^f s$.

Let's look at the generators $a_{v_ik^fs}$ in r_w.

If $l(v_i) < l(v_1)$, then either v_ik^fs doesn't begin with ug^e , or the generator $a_{v_ik^fs}$ has already been rewritten, via the inductive hypothesis, using other generators with no occurrence of generators of type a_{ug^es}.

If $l(v_ik^fs) = l(ug^ek^fs)$, then $l(v_i) = l(ug^e)$, thus v_ik^fs doesn't begin with $ug^e = v_1$.

So by induction, we can remove each relation r_w, where w does not begin with g^e, and also remove each generator of the form a_{ug^ew}, for all such $w \in F_k$.

Now for case (1):

Consider first $w = g^{-e}$. Then examine b_{y_1}. We know $y_1 = th^d$, where either $h \neq g$, or if $h = g$, then $e - d \neq 0$. Thus $y_1w = y_1g^{-e}$ is freely reduced as written, and so b_{y_1} appears in r_w as $b_{y_1g^{-e}}$ which is still of maximal length among the subscripts of the family b_y in r_w, and is the only subscript in this family beginning with y_1g^{-e}.

So we can remove the relation $r_{g^{-e}}$ and the generator $b_{y_1g^{-e}}$ and rewrite this generator in terms of other families of generators and b_w's in which w doesn't begin with $y_1g^{-e} = th^dg^{-e}$.

The induction step proceeds exactly as in case (2). We remove all the generators of the family b_w where w begins with $y_1g^{-e} = th^dg^{-e}$.

Thus L is free on the generators g_w with $g \neq a, b$, $w \in F$, $a_v, v \in F$, v does not begin with ug^e, and $b_y, y \in F$, y does not begin with th^dg^{-e}. □

We can use these results to show that certain one relator groups with torsion are virtually free-by-free. In these cases, the torsion-free subgroup of finite index can be shown to have a free quotient of large rank.

Proposition 5. *The groups* $G_k = \langle\, x, y\,;\, (x^{-1}yxy^{k-2})^k \,\rangle, k \geq 3$, *are virtually free-by-free. If* $k = 2m+1$ *or* $k = 2m$, *the subgroup of finite index has a free quotient of rank* $m+1$.

Proof. For this Proposition, we return to the notation of Chapter 2: $x^y = yxy^{-1}$, and $x_i = y^i x y^{-i}$ We recall from §2.1 that G_k has a subgroup K of index k which has the presentation

$$\langle\, w, x_i, i = 0, \ldots, k-1\,;\, x_{s_1}^{-1}x_{t_1}x_{s_2}^{-1}x_{t_2}\cdots x_{s_k}^{-1}x_{t_k}w \,\rangle$$

where $x_i = y^i x y^{-i}$ and $w = y^k$. The subscripts s_i and t_i are given by

$$s_i = (i-1)(k-1) \mod k = -(i-1) \mod k, \text{ and}$$

$$t_i = ((i-1)(k-1)+1) \mod k = -(i-2) \mod k.$$

We will consider the cases $k = 2m$ and $k = 2m+1$ separately. When $k = 2m$, the relation for G_k is actually

$$x_0^{-1}x_1x_{k-1}^{-1}wx_0x_{k-2}^{-1}x_{k-1}wx_{k-3}^{-1}x_{k-2}w\ldots x_2^{-1}x_3wx_1^{-1}x_2w.$$

To get the free quotient of rank $m+1$ for k even, we first replace the $m-1$ generators $x_3, x_5, \ldots, x_{k-1}$ as follows. Let $t_1 = x_3wx_1^{-1}$, so $x_3 = t_1x_1w^{-1}$. Generally $t_i = x_{2i+1}wx_{2i-1}^{-1}$, so $x_{2i+1} = t_ix_{2i-1}w^{-1} = t_it_{i-1}\ldots t_1x_1w^{-i}$ The last replacement is $t_{m-1} = x_{k-1}wx_{k-3}^{-1}$, which gives $x_{k-1} = t_{m-1}t_{m-2}\cdots t_1x_1w^{-(m-1)}$.

When we rewrite the relation using the generators t_i, we get

$$x_0^{-1}x_1w^{m-1}x_1^{-1}t_1^{-1}\cdots t_{m-1}^{-1}wx_0x_{k-2}^{-1}t_{m-2}x_{k-2}w\dots x_2^{-1}t_1x_2w.$$

Now map $K \longrightarrow G = \langle x_1, x_{2i}, i = 0, \dots, m-1 \rangle$, a free group of rank $m+1$, by sending

$$t_i \longrightarrow 1, \text{ for all i}$$

$$w \longrightarrow 1$$

$$x_{2i} \longrightarrow x_{2i}, \; i = 0, \dots, m-1$$

$$x_1 \longrightarrow x_1.$$

The kernel of this map is generated by

$$w_v, t_{1,v}, \dots, t_{m-1,v}$$

as v ranges through the elements of G. The relations of the kernel are the family u_v, indexed over $v \in G$,

$$u_v = w^{m-1}_{x_1^{-1}x_0v}t^{-1}_{1,x_0v}\cdots t^{-1}_{m-1,x_0v}w_{x_0v}t_{m-2,x_{k-2}v}w_v\dots t_{1,x_2v}w_v.$$

By Theorem 3, the kernel is free; the generators t_{1,x_0} and t_{1,x_2} in u_1 satisfy the hypothesis.

For $k = 2m+1$, the single relation for G_k is similar:

$$x_0^{-1}x_1x_{k-1}^{-1}wx_0x_{k-2}^{-1}x_{k-1}wx_{k-3}^{-1}x_{k-2}w\dots$$

$$x_3^{-1}x_4wx_2^{-1}x_3wx_1^{-1}x_2w.$$

We again rewrite generators, this time replacing the m generators $x_2, x_4, \ldots, x_{k-1}$. Let $t_1 = x_1^{-1}x_2$, so $x_2 = x_1t_1$. Then we contine changing generators: $t_i = x_{2i}wx_{2(i-1)}^{-1}$, and so $x_{2i} = t_it_{i-1}\cdots t_2x_1t_1w^{-(i-1)}$. The final substitution is $t_m = x_{k-1}wx_{k-3}^{-1}$, which gives $x_{k-1} = t_mt_{m-1}\cdots t_2x_1t_1w^{-(m-1)}$.

Then the relation, when rewritten using the generators t_i, becomes

$$x_0^{-1}x_1w^{m-1}t_1^{-1}x_1^{-1}t_2^{-1}\ldots t_m^{-1}wx_0x_{k-2}^{-1}t_{m-1}x_{k-2}w\ldots x_3^{-1}t_2x_3wt_1w.$$

Map $K \longrightarrow H = \langle\, x_1, x_{2i}, 0 \leq i \leq m-1 \,\rangle$, a free group of rank $m+1$, in the obvious fashion. The kernel of this map is generated by

$$w_v, t_{1,v}, \ldots, t_{m,v}$$

as v ranges through H, and the relations of the kernel, again indexed over $v \in H$, are

$$u_v = w^{m-1}_{x_1^{-1}x_0v}t^{-1}_{1,x_1^{-1}v}t^{-1}_{2,x_0v}\cdots t^{-1}_{m,x_0v}w_{x_0v}t_{m-1,x_{k-2}v}w_v\cdots t_{2,x_3v}w_vt_{1,v}w_v.$$

The generators $t_{1,x_1^{-1}}$ and $t_{1,1}$ in u_1 satisfy the hypothesis of Theorem 2, so again the kernel is free. $\square$

The one-relator kernels in the examples above have already been shown to be free-by-cyclic in §2.1. By generalizing from these groups, one might be led to conjecture that any one-relator group which is free-by-free is actually free-by-cyclic. The following group provides a counter-example to this conjecture.

Proposition 6. *The group* $H = \langle\, x, y, b\,;\, x^{-1}y^{-1}byxy^{-1}x^{-1}bxyb \,\rangle$ *is free-by-free but not free-by-cyclic.*

Proof. (We return to the notation $b_w = w^{-1}bw$).

To see H is free-by-free, first map $H \xrightarrow{\phi} \langle x, y \rangle$ by sending

$$b \longrightarrow 1$$

$$x \longrightarrow x$$

$$y \longrightarrow y.$$

Using the Reidemeister-Schreier method, we find a presentation for the kernel K of ϕ. The generators of K are the elements

$$b_w = w^{-1}bw, \text{ for all } w \in \langle x, y \rangle,$$

and the relations are

$$r_w = b_{yxw}b_{xyw}b_w, \text{ for all } w \in \langle x, y \rangle.$$

The generators b_1 and b_{yx} satisfy the hypothesis of Theorem 2, and so K is free on the generators b_t, where t does not begin with either x^{-1} or with yx.

Now suppose we map H onto an infinite cyclic group, $H \xrightarrow{w} \langle t \rangle$. Notice that $H/H' = \langle x \rangle \times \langle y \rangle \times \langle b\, ;\, b^3 \rangle$. Thus we know that if $z \in H$ has $w(z) = t$, then $z = x^m y^n h$ for some integers m and n and $h \in H'$. Then $w(x^m y^n) = t$, and we can take as a Schreier transversal for *kerw* the set of elements $(x^m y^n)^k$, for all integers k. Observe then that the representative of x, $\overline{x} = (x^m y^n)^p$ and the representative of y, $\overline{y} = (x^m y^n)^q$, and one of p or q is non-zero.

We then get as generators for $ker\alpha = M$

$$(x^m y^n)^{-k} x\overline{(x^m y^n)^k x} = (x^m y^n)^{-k} x (x^m y^n)^{k+p} = x_k$$

$$(x^m y^n)^{-k} y\overline{(x^m y^n)^k y} = (x^m y^n)^{-k} y (x^m y^n)^{k+q} = y_k$$

$$(x^m y^n)^{-k} b\overline{(x^m y^n)^k b} = (x^m y^n)^{-k} b (x^m y^n)^{k} = b_k,$$

and as relations the collection

$$x_{k-p}^{-1} y_{k-p-q}^{-1} b_{k-p-q} y_{k-p-q} x_{k-p} y_{k-q}^{-1} x_{k-p-q}^{-1} b_{k-p-q} x_{k-p-q} y_{k-q} b_k$$

for all integers k.

We can easily see that the group M is not free by examining M/M'. Modulo the commutator subgroup, the collection of relations becomes

$$b_{k-m}^2 b_k$$

where $m = p + q$ and k runs through all integers.

If $m \neq 0$, then M/M' has as a subgroup the group generated by b_m with presentation as abelian group

$$\langle\, b_{km}\, ;\, b_{(k-1)m}^2 b_{km}, k \in \mathbf{Z} \,\rangle$$

which is isomorphic to the dyadic rationals. Thus M/M' is not free abelian, and so M is not free.

If $m = 0$, then M/M' has the presentation as abelian group

$$\langle\, b_k\, ;\, b_k^3, k \in \mathbf{Z} \,\rangle$$

which is not free abelian, and so M again is not free. $\square$

CHAPTER 4
VIRTUALLY RESIDUALLY FREE GROUPS

The principal result of this section, on free products with amalgamation of finitely generated abelian groups, uses the subgroup theorem for free products with amalgamation described in detail in Chapter 2. We will also need some basic closure results for residually free groups. These lemmas will be shown for residually $\mathcal{P}$ groups in general.

Lemma 1. *If $\mathcal{P}$ is a property of groups which is closed under free products, then residually $\mathcal{P}$ is closed under free products.*

Proof. Let $G = A * B$ be such a free product of two residually $\mathcal{P}$ groups and $u = a_1 b_1 \cdots a_n b_n$ be a nontrivial element of G. Assume that $a_1 \neq 1$.

We know that for each $a \neq 1$, there is an onto homomorphism $f : A \longrightarrow F$, with F having property $\mathcal{P}$ and $f(a) \neq 1$, and for each $b \neq 1$, an onto homomorphism $g : B \longrightarrow G$, with G having property $\mathcal{P}$ and $g(b) \neq 1$. Now take a homomorphism $f : A \longrightarrow F$ for which $f(a_1) \neq 1$. If $b_1 \neq 1$, take a homomorphism $g : B \longrightarrow G$ for which $g(b_1) \neq 1$. If $b_1 = 1$, that is, $u \in A$, take for the homomorphism $g : B \longrightarrow G$ any of these homorphisms.

Map

$$G \xrightarrow{h=f*g} F * G$$

and observe that $h(u) \neq 1$ in the group $F * G$ which has property $\mathcal{P}$. □

Lemma 2. *For any property $\mathcal{P}$, residually $\mathcal{P}$ is closed under direct products.*

Proof. Consider $A \times B$, the direct product of two residually $\mathcal{P}$ groups, and $w = a \times b$ a nontrivial element. Say, for instance, that $a \neq 1$.. Then we can map $f : A \longrightarrow F$, where F has property $\mathcal{P}$ and $f(a) \neq 1$. Now map

$$A \times B \overset{h=f\times t}{\longrightarrow} F,$$

where $t : B \longrightarrow F$ is the trivial map, and $h(w) \neq 1$. □

Lemma 3. *If $\mathcal{P}$ is a property inherited by subgroups, then so is residually $\mathcal{P}$.*

Proof. Let H be a residually $\mathcal{P}$ group and $M < H$. Then if $m \neq 1$ in M, we have a map $\phi : H \longrightarrow X$, with X having property $\mathcal{P}$ and $\phi(m) \neq 1$. Thus we have the homomorphism $\phi|_M : M \longrightarrow \phi(M)$, where $\phi(M)$ inherits property $\mathcal{P}$ from X. □

Lemma 4. *If G is finitely generated and virtually virtually $\mathcal{P}$, where $\mathcal{P}$ is inherited by subgroups, then G is virtually $\mathcal{P}$.*

Proof. Suppose G/H is finite, H/N is finite, and N has property $\mathcal{P}$. By a well-known theorem of M. Hall (see [LS] IV.4), N contains a subgroup L, normal in G and of finite index in G, and L has property $\mathcal{P}$. □

Proposition 5. *Suppose $\mathcal{P}$ is a property of groups such that when G has property $\mathcal{P}$, then $G/Z(G)$ has property $\mathcal{P}$. Then if H is residually $\mathcal{P}$, so is $H/Z(H)$.*

Proof. Let H be a group which is residually $\mathcal{P}$, and let $gZ(H) \neq Z(H)$ be a nontrivial element of $H/Z(H)$. Then there is some $h \in H$ with $h^{-1}h^g \neq 1$. Because H is residually $\mathcal{P}$, there is $N \lhd H$, with $h^{-1}h^g \notin N$, and H/N has property $\mathcal{P}$.

Now $H/Z(H)$ acts on H/N by conjugation:

$$(gN)^{xZ(H)} = g^x N,$$

so we have the homomorphism

$$H/Z(H) \longrightarrow \frac{H/N}{Z(H/N)}$$

which maps $H/Z(H)$ onto a group which has property $\mathcal{P}$. Let L be the kernel of this map. $L = gp(xZ(H))$ such that for every $t \in G, t^{-1}t^x \in N$. We know $gZ(H) \notin L$ since we have h with $h^{-1}h^g \notin N$.

Thus $H/Z(H)$ is residually $\mathcal{P}$. □

We will also require two easy lemmas on finitely generated abelian groups.

Lemma 6. *If C is a subgroup of A, a finitely generated abelian group, then C is a direct factor of a subgroup of finite index.*

Proof. $A/C = T/C \times F/C$, where T/C is finite and F/C is free abelian.

Then $F = A_2 \times C$, with A_2 free abelian. A/F is finite, since it is a quotient of T/C. □

Lemma 7. *Let A be a free abelian group of rank m and $g_1, g_2, \dots, g_n$ be non-trivial elements of A. Then there is a homomorphism $\phi : A \to A^-$, A^- of rank $m-1$, with $\phi(g_i) \neq 0$, $1 \leq i \leq n$.*

Proof.

$$A = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_m \rangle, \text{ and}$$

$$g_i = x_1^{\alpha_{1,i}} x_2^{\alpha_{2,i}} \dots x_m^{\alpha_{m,i}}.$$

We define $\phi : A \to A^-$ by

$$x_i \to x_i, \qquad 1 \leq i \leq m-1$$

$$x_m \to x_1^e$$

where e is chosen so that $\phi(g_i) \neq 0$.

$\phi(g_i) = x_1^{\alpha_{1,i}+e\alpha_{m,i}} x_2^{\alpha_{2,i}} \dots x_{m-1}^{\alpha_{m-1,i}}$. If both $\alpha_{1,i} = 0$ and $\alpha_{m,i} = 0$, then some other $\alpha_{k,i} \neq 0$, so $\phi(g_i) \neq 0$, no matter what choice of e.

If $\alpha_{1,i} \neq 0$ or $\alpha_{m,i} \neq 0$, then the equation $\alpha_{1,i} + e\alpha_{m,i} = 0$ has at most one solution s_i, so if we choose e any integer $\notin \bigcup s_i$, we will get $\phi(g_i) \neq 0$, for all i. □

Corollary 8. *With A, $g_1, \dots, g_n$ as above, there is a homomorphism $\phi : A \to \langle x \rangle$, with $\phi(g_i) \neq 0$, $1 \leq i \leq n$. A free abelian group is residually infinite cyclic.*

Theorem 9. *Let A and B be finitely generated abelian groups and $G = A \underset{C}{*} B$. Then G is virtually residually free.*

Proof. By Lemma 6, C is virtually a direct factor of A and of B: there are $A_1 < A$, $B_1 < B$, with A/A_1 finite, and $A_1 = A_2 \times C$, A_2 free abelian, B/B_1 finite, and $B_1 = B_2 \times C$, B_2 free abelian. Suppose $A = \sum_{i=1}^{n} r_i A_1$ and $B = \sum_{i=1}^{m} s_i B_1$.

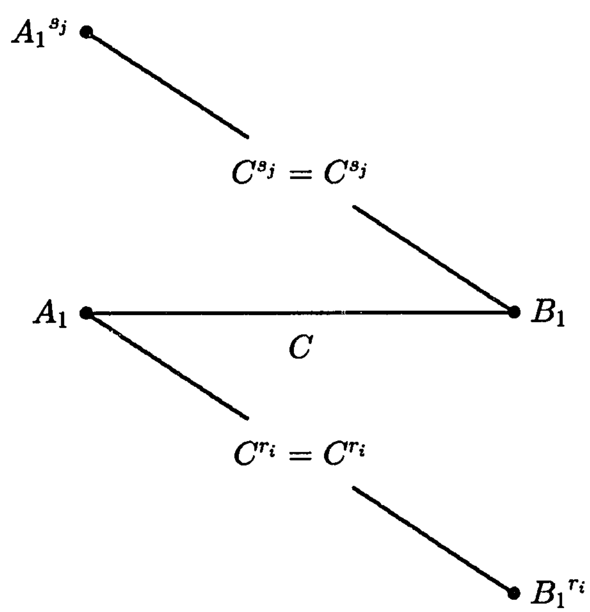
Map $G \xrightarrow{\phi} A/A_1 \times B/B_1$ by

$$a \longmapsto a\,A_1$$

$$b \longmapsto b\,B_1$$

which respects the amalgamated subgroup C.

We can describe K = kernel ϕ using the subgroup theorem for amalgamated free products. $K = gp_G(A_1, B_1, [r_i, s_j])$ is an HNN extension with stable letters $g_{i,j} = [r_i, s_j]$ of a tree product S, where S is generated by the subgroups $A_1, A_1^{s_2}, \ldots, A_1^{s_m}, B_1, \ldots, B_1^{r_n}$, with amalgamations as indicated by the tree:



Since A and B are abelian and $C < Z(G)$, actually $B_1^{r_i} = B_2^{r_i} \times C$ and $A_1^{s_i} = A_2^{s_j} \times C$, and the amalgamation indicated by each edge is of the direct factor C.

So the tree product can be rewritten as

$$S = (A_2 \times C) \underset{C}{*} (B_2^{r_2} \times C) \underset{C}{*} \dots \underset{C}{*} (B_2^{r_n} \times C) \underset{C}{*} (B_2 \times C) \underset{C}{*} (A_2^{s_2} \times C) \underset{C}{*} \dots \underset{C}{*} (A_2^{s_m} \times C)$$
$$= (A_2 * A_2^{s_2} * \dots A_2^{s_m} * B_2 * B_2^{r_2} * \dots * B_2^{r_n}) \times C$$

The last equality occurs because the indicated direct factors generate S, are normal in S, and have trivial intersection, since $C \cap A_2^{s_j} = 1 = C \cap B_2^{r_i}$.

Furthermore, the action of the stable letters $g_{i,j}$ is given by

$$g_{i,j}^{-1} C^{s_j r_i} g_{i,j} = C^{r_i s_j}.$$

Again, since $C < Z(G)$, this amounts to

$$g_{i,j}^{-1} C g_{i,j} = C.$$

So the entire group K looks like

$$K = (\langle g_{i,j} \rangle * A_2 * A_2^{s_2} * \dots * A_2^{s_m} * B_2 * B_2^{r_2} * \dots * B_2^{r_n}) \times C$$

Now K is virtually residually free. By Corollary 8, any free abelian group is residually free. The direct factor of K which is a free product is then a free product of residually free groups and so is residually free by Lemma 1. Since C is a direct product of a finite group and a free abelian group, K is a direct product of a finite group and two residually free groups, and so is virtually residually free by Lemma 2.

Finally, by Lemma 4, G itself is virtually residually free. □

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