NUCLEAR SPACES OF ALMOST PERIODIC FUNCTIONS WITH APPLICATION TO SINGULAR INTEGRAL OPERATORS

by

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INTRODUCTION

Let R^n be n-dimensional Euclidean space with the usual norm $|x|$ and inner product $<x,y>$, and let $\Sigma = \{x \in R^n : |x| = 1\}$ be its unit sphere. Σ is a compact, (n-1)-dimensional, C^∞ manifold; $C^\infty(\Sigma)$ is the space of C^∞ functions from Σ into the complex numbers, C . (The structure and properties of $C^\infty(\Sigma)$ are discussed in the Appendix.) If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers, we may define for $x \in R^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ and for $f \in C^\infty(R^n)$, $D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$. The Schwartz space $\mathcal{S}$ is the space consisting of all $u \in C^\infty(R^n)$ such that $\sup_{x \in R^n} |x^\beta D^\alpha u(x)| < \infty$ for all α and β as above. On $\mathcal{S}$ we have the Fourier transform $\mathfrak{F}$ defined by

$$[\mathfrak{F}(u)](\xi) = \hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{-i<x,\xi>} u(x)\, dx \quad .$$

L^q is the space of measurable functions on R^n with integrable q^{th}-powers. $\mathcal{S}$ is dense in L^q for all $q < \infty$.

Suppose $k : R^n \times (R^n \sim \{0\}) \to C$ is measurable, and for any (fixed) $x \in R^n$, $k(x,\cdot)$ is a C^∞ function, homogeneous of degree 0 (so, $k(x,\cdot)$ can be considered as an element of $C^\infty(\Sigma)$) . If the map $x \longmapsto k(x,\cdot)$ is essentially bounded on R^n with values in $C^\infty(\Sigma)$, then the Calderon-Zygmund Theorem [1] states that the operator from $\mathcal{S}$ to $\mathcal{S}$ defined by

$$(1) \qquad Ku(x) = (2\pi)^{-\frac{n}{2}} \int e^{i<x,\xi>} k(x,\xi)\, \hat{u}(\xi)\, d\xi$$

extends in a unique way to a bounded operator from L^q to L^q for any q, $1 < q < \infty$. We want to generalize this result, so that the map $x \longmapsto k(x,\cdot)$ may take values in spaces other than $C^\infty(\Sigma)$. In order to accomplish this, we employ the theory of topological tensor products and nuclear spaces developed by Grothendieck [4].

By a countably normed space we mean a Fréchet space whose topology is defined by a countable collection of norms $\{\| \|_p\}_{p=1}^\infty$ with $\| \|_p \le \| \|_{p+1}$ for all $p \ge 1$. If E and F are countably normed, we define on $E \otimes F$, the projective tensor topology which is given by norms $p \otimes q$ defined by

$$p \otimes q(u) = \inf\left\{ \sum_{j=1}^{n} \|x_j\|_{E_p} \|y_j\|_{F_q} : u = \sum_{j=1}^{n} x_j \otimes y_j \right\} .$$

The completion of $E \otimes F$ with respect to this topology is denoted by $E \hat{\otimes} F$. By a theorem of Grothendieck [4], page 51, any $\theta \in E \hat{\otimes} F$ is the sum of an absolutely convergent series, $\theta = \sum_{j=1}^{\infty} \lambda_j x_j \otimes y_j$, where $\{\lambda_j\}$ is a sequence of complex numbers such that $\sum_j |\lambda_j| < \infty$, $x_n \to 0$ in E, and $y_n \to 0$ in F. Using this theorem, one is led naturally (see [7], Chapter 47) to the following:

<u>0.1 Definition</u>: Suppose E and F are countably normed. Then a continuous, linear map $u : E \to F$ is <u>nuclear</u> if there are two bounded sequences $\{y_j : j = 1,2,\ldots\} \subset F$, $\{x_j' : j = 1,2,\ldots\} \subset E'$, the dual of E, and a sequence $\{\lambda_j\}$ of complex numbers with $\sum_j |\lambda_j| < \infty$ such

that u is given by

$$u(x) = \sum_{j=1}^{\infty} \lambda_j \langle x'_j, x\rangle y_j , \quad \text{for all} \quad x \in E .$$

The composition of a nuclear operator and a bounded operator (in either order) is nuclear. (See [7], page 479).

If we denote the completion of E with respect to $\| \|_{E_p}$ by E_p, we have for $m \geq p$, the natural inclusion $T_p^m : E_m \to E_p$. T_p^m is norm decreasing and hence continuous.

0.2 Definition: A countably normed space E is said to be nuclear if given any p , there is an $m > p$ such that T_p^m is a nuclear operator. If E and F are nuclear, then so is $E \hat{\otimes} F$. Any nuclear space is perfect; i.e., has the property that every closed and bounded subset is compact. (For proofs, see again [7], Chapter 47).

In Chapter I, we generalize the Calderon-Zygmund Theorem as follows: Let C_B^∞ denote the space of C^∞ functions on R^1, all of whose derivatives are bounded. C_B^∞ is countably normed by $\|f\|_p = \max_{0 \leq j \leq p} \sup_{t \in R} |f^{(j)}(t)|$. Suppose that E is a subspace of C_B^∞, nuclear in the induced topology. Then, defining $E^\# = \{f^\# \in C^\infty(R^+) : f^\#(e^t) \in E\}$, where $R^+ = \{t \in R : t > 0\}$, we consider the space $\mathcal{E}^\# = E^\# \hat{\otimes} C^\infty(\Sigma)$. We show that $\mathcal{E}^\#$ is topologically isomorphic to a subspace of $C^\infty(R^+;C^\infty(\Sigma))$, the space of C^∞ functions on R^+ with values in $C^\infty(\Sigma)$. Thus, $\mathcal{E}^\#$ can be viewed as a subspace of $C^\infty(R^n \sim \{0\})$. Then if $k : R^n \times \mathcal{E}^\# \to C$ is measurable and if the map $x \longmapsto k(x,\cdot)$ is essentially bounded from R^n into $\mathcal{E}^\#$, then K as in (1) again defines a bounded operator from L^q to L^q for all q, $1 < q < \infty$. Note that if we take E to be the constants, then $\mathcal{E}^{\#}$ is $C^\infty(\Sigma)$, and we have the usual Calderon-Zygmund Theorem.

We begin the proof by introducing the space M_q^q (multipliers on L^q) $= \{T \in \mathcal{S}' : \exists\, c > 0$ with $\|\hat{T} * \varphi\|_{L^q} \le c\|\varphi\|_{L^q}$ for all $\varphi \in \mathcal{S}\}$. Here $\mathcal{S}'$ is the dual of $\mathcal{S}$, and $\hat{T} * \varphi$ is the convolution of $\hat{T}$ with φ (see [7], Chapter 27). Thus if $T \in M_q^q$, then $\hat{T}$ extends to a bounded operator (under convolution) from L^q to L^q. We denote its norm, as a map between Banach spaces, by $\|\hat{T}\|_{L_q^q}$. The crux of the proof lies in showing that $\mathcal{E}^{\#}$ is continuously included in M_q^q, for all q, $1 < q < \infty$. This is done using the following theorem of Mikhlin and Hormander [5].

<u>0.3 Theorem</u>: If $f \in L^\infty$, and for some $p > n/2$, there is a $B > 0$ with

$$\sup_{\lambda>0} \sum_{|\alpha|\le p} \int_{1<|x|<2} |D^\alpha M_\lambda f(x)|^2 \, dx < B^2 \,, \tag{2}$$

then $f \in M_q^q$, $1 < q < \infty$, and $\|f\|_{M_q^q} \le C_q B$, where $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $[M_\lambda f](x) = f(\lambda x)$, and $\|f\|_{M_q^q} = \|\hat{f}\|_{L_q^q}$. Clearly (f,p,B) satisfies (2) if and only if $(M_\mu f,p,B)$ does for $\mu > 0$. Thus we are led to consider spaces which are closed under dilation. In order to see what this assumption means for E, we must examine the isomorphism $\sim$ mapping $C^\infty(R^n \sim \{0\})$ onto $C^\infty(R;C^\infty(\Sigma))$, defined by $[\tilde{f}(t)](\omega) = f(e^t\omega)$

or $f(x) = \tilde{f}(\log|x|)(\frac{x}{|x|})$, since $\mathcal{E}^{\#}$ is isomorphic to $E \hat{\otimes} C^{\infty}(\Sigma)$ under $\sim$. Now, for $\mu > 0$ and $f \in \mathcal{E}^{\#}$

$$\begin{aligned} M_{\mu} f(x) &= f(\mu x) = \tilde{f}(\log(\mu|x|)) \left(\frac{\mu x}{|\mu x|}\right) \\ &= \tilde{f}(\log \mu + \log|x|) \left(\frac{x}{|x|}\right) \\ &= \tilde{f}_{\log \mu}(\log|x|) \left(\frac{x}{|x|}\right) \quad , \end{aligned}$$

where $\tilde{f}_r(t) = \tilde{f}(t+r)$ as elements of $C^{\infty}(\Sigma)$. Thus, we are led to assume that E is translation invariant.

In Chapter II we show that a translation invariant, perfect subspace E of C_B^{∞} is a space of almost periodic functions. If f is almost periodic, its spectrum Λ_f is the set of its Fourier exponents. We define the spectrum Λ_E of E to be the union $\bigcup_{f \in E} \Lambda_f$ and show that it must be discrete. We characterize E as the set $\{f \in C_B^{\infty} : f$ is almost periodic and $\Lambda_f \subseteq \Lambda_E\}$. Thus E is determined by its spectrum. Furthermore, if Λ_E is either linearly independent over the rationals or satisfies a certain growth condition, E is topologically isomorphic to a space of functions on its spectrum. Our conjecture is that, without restriction on Λ_E, we always have such an isomorphism. The above partial results, however, are the best we have been able to prove. We conclude the chapter by characterizing nuclearity of such a space of functions on a discrete set, and thus determine a class of spaces $\mathcal{E}^{\#}$ which generalize the space of C^{∞} functions, homogeneous of degree 0 in the Calderon-Zygmund Theorem.

In order to maintain cohesion, we withhold until the Appendix all proofs and discussions which would otherwise lead to unnecessary digression.

CHAPTER I

Let E be a subspace of C_B^∞, nuclear in the induced topology. Define $\# : C_B^\infty \to C^\infty(R^+)$ by $f^\#(t) = f(\log t)$. $\#$ is one-one since $f(t) = f^\#(e^t)$, and we topologize $E^\# = \{f^\# : f \in E\}$ so that $\#$ is an isometry (and therefore $E^\#$ is nuclear), with the following norms:

$$(1) \qquad \|f^\#\|_p = \max_{0 \leq j \leq p} \sup_{t \in R} |f^{(j)}(t)| , \qquad p = 0,1,2,\ldots .$$

These are precisely the defining norms for C_B^∞ and hence for E. It will be convenient to have another collection of norms which defines the same topology on $E^\#$. So, we set

$$(2) \qquad \|f^\#\|_p' = \max_{0 \leq j \leq p} \sup_{t \in R^+} |t^j f^{\#(j)}(t)| .$$

In order to show the equivalence of the topologies defined by (1) and (2) we need a lemma, the proof of which appears in the Appendix.

<u>1.1 Lemma</u>: Given any integer $p \leq 0$, there exist constants c_j^p and d_j^p, $j = 0,1,\ldots,p$ such that for every $f^\# \in E^\#$, we have

$$(3) \qquad f^{\#(p)}(t) = \frac{1}{t^p} \sum_{j=0}^{p} c_j^p f^{(j)}(\log t) \quad \text{for all} \quad t \in R^+ , \quad \text{and}$$

$$(4) \qquad f^{(p)}(s) = \sum_{j=0}^{p} d_j^p e^{js} f^{\#(j)}(e^s) \quad \text{for all} \quad s \in R .$$

By (3),

$$|t^p f^{\#(p)}(t)| \le \sum_{j=0}^{p} |c_j^p| \; |f^{(j)}(\log t)| \le \sum_{j=0}^{p} |c_j^p| \; \|f^{\#}\|_p$$

for all $t \in R^+$, so $\|f^{\#}\|'_p \le c\|f^{\#}\|_p$, and the norms in (2) define a weaker topology than those in (1). On the other hand, by (4)

$$|f^{(p)}(\log s)| \le \sum_{j=0}^{p} |d_j^p| \; |s^j f^{\#(j)}(s)| \le \sum_{j=0}^{p} |d_j^p| \; \|f^{\#}\|'_p$$

for all $s \in R^+$, so the topologies agree.

Now, let Σ again denote the unit sphere in R^n and $C^\infty(\Sigma)$ the space of C^∞ functions on the manifold Σ. Then $C^\infty(\Sigma)$ is a nuclear, countably normed space, with norms $\| \; \|_{H^p(\Sigma)}$ (see Appendix, page 31). We define $\mathcal{E}^{\#} = E^{\#} \hat{\otimes} C^\infty(\Sigma)$ and remark that, since $E^{\#}$ and $C^\infty(\Sigma)$ are both nuclear, so is $\mathcal{E}^{\#}$. In the Appendix (page 33), we show that $\mathcal{E}^{\#}$ is topologically isomorphic to a subspace $E^{\#}(C^\infty(\Sigma))$ of $C^\infty(R^+;C^\infty(\Sigma))$, topologized by the norms

$$(5) \qquad \|F\|_p = \max_{0 \le j \le p} \; \sup_{t \in R^+} \; \|t^j F^{(j)}(t)\|_{H^p(\Sigma)} .$$

Throughout this chapter we will identify $\mathcal{E}^{\#}$ and $E^{\#}(C^\infty(\Sigma))$, so that we may evaluate elements of $\mathcal{E}^{\#}$ at points of R^+. Furthermore we will take the norms in (5) as the defining norms for $\mathcal{E}^{\#}$. As $C^\infty(R^+;C^\infty(\Sigma))$ is isomorphic to $C^\infty(R^n \sim \{0\})$, $\mathcal{E}^{\#}$ can be considered as a space of C^∞ functions on $R^n \sim \{0\}$.

We are now in a position to state the main theorem of this chapter.

<u>1.2 Theorem</u>: Suppose $k : R^n \times \mathcal{E}^{\#} \to C$ is measurable, and the map $x \longmapsto k(x,\cdot)$ is essentially bounded from R^n into $\mathcal{E}^{\#}$ (considered as a subspace of $C^{\infty}(R^n \sim \{0\})$) . Then, if we define, for $u \in \mathcal{S}$,

$$(6) \qquad Ku(x) = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{i<x,\xi>} k(x,\xi)\, \hat{u}(\xi)\, d\xi \ ,$$

K extends in a unique way to a bounded operator from L^q to L^q for all q , $1 < q < \infty$.

<u>Remark:</u> The proof will follow generally the same line as the proof of the Calderon-Zygmund Theorem. However, where one usually uses an expansion in spherical harmonics to represent $k(x,\cdot)$ in an absolutely convergent series (see equation (10), below), we need only the nuclearity of $\mathcal{E}^{\#}$, thus simplifying the argument somewhat.

<u>Proof of 1.2</u>: We begin by showing that $\mathcal{E}^{\#}$ is continuously included in M_q^q , $1 < q < \infty$. In order to do this we need to translate Theorem 0.3 of Mikhlin and Hormander, [5], stated for functions mapping $R^n \sim \{0\}$ into C , into a theorem regarding functions from R^+ into $C^{\infty}(\Sigma)$.

Let $\hat{\Sigma}$ denote the annulus $\{x \in R^n : \frac{1}{2} < |x| < 2\}$, and let $\sim : C^{\infty}(R^n \sim \{0\}) \to C^{\infty}(R^+;C^{\infty}(\Sigma))$ be the isomorphism defined by $[\tilde{f}(t)](\omega) = f(t\omega)$ where $t \in R^+$ and $\omega \in \Sigma$. By Theorem 3.6 of the Appendix, given any p , there exist constants $c_1, c_2 > 0$, such that

$$(7)\qquad c_1 \|f\|^2_{H^p(\hat{\Sigma})} \le \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|\tilde{f}^{(j)}(t)\|^2_{H^{p-j}(\Sigma)}\, dt \le c_2 \|f\|^2_{H^p(\hat{\Sigma})} .$$

Now, defining as before, $M_\lambda f(x) = f(\lambda x)$, we see that $(M_\lambda f)^{\sim(j)}(t) = \lambda^j \tilde{f}^{(j)}(\lambda t)$ as elements of $C^\infty(\Sigma)$, so (7) yields

$$(8)\qquad c_1 \sup_{\lambda>0} \|M_\lambda f\|^2_{H^p(\hat{\Sigma})} \le \sup_{\lambda>0} \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|\lambda^j \tilde{f}^{(j)}(\lambda t)\|^2_{H^{p-j}(\Sigma)}\, dt$$

$$\le c_2 \sup_{\lambda>0} \|M_\lambda f\|^2_{H^p(\Sigma)} .$$

Since $\sup_{\lambda>0} \|M_\lambda f\|^2_{H^p(\hat{\Sigma})} = \sup_{\lambda>0} \sum_{|\alpha|\le p} \int_{\frac{1}{2}<|x|<2} |D^\alpha (M_\lambda f)(x)|^2\, dx$ which is the expression appearing in Theorem 0.3, (8) enables us to restate the Mikhlin-Hormander Theorem [5] in the following form.

<u>1.3 Theorem</u>: Suppose $F : R^+ \to C^\infty(\Sigma)$ is measurable and $\sup_{\substack{t\in R^+\\ \omega\in\Sigma}} |[F(t)](\omega)| < \infty$.

If there exist $p > n/2$ and $B > 0$ such that

$$\sup_{\lambda>0} \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|\lambda^j F^{(j)}(\lambda t)\|^2_{H^{p-j}(\Sigma)}\, dt < B^2 ,$$

then $F \in M_q^q$, $1 < q < \infty$, and $\|F\|_{M_q^q} \le C_q B$. In our present context, if $F \in \mathcal{E}^\#$, using the norms $\|F\|_p$ defined in (5), we have

$\|F^{(j)}(t)\|^2_{H^{p-j}(\Sigma)} \le \dfrac{1}{t^{2j}} \|F\|^2_p$, so

$$\sup_{\lambda>0} \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|\lambda^j F^{(j)}(\lambda t)\|^2_{H^{p-j}(\Sigma)} \, dt$$

$$\leq \sup_{\lambda>0} \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \frac{\lambda^{2j}}{(\lambda t)^{2j}} \|F\|_p^2 \, dt = C_p \|F\|_p^2 .$$

Thus by Theorem 1.2, $F \in M_q^q$, $1 < q < \infty$, and $\|F\|_{M_q^q} \leq C_{qp} \|F\|_p$. Hence $\mathcal{E}_p^\#$ is continuosly included in M_q^q , $1 < q < \infty$.

We now choose an integer $p > n/2$ (this is a technical assumption which will not be used until the end of the argument). Since $\mathcal{E}^\#$ is nuclear, there is an $m > p$ such that the natural map $T : \mathcal{E}_m^\# \to \mathcal{E}_p^\#$ is nuclear; i.e., for any $F \in \mathcal{E}_m^\#$,

$$(9) \qquad F = T(F) = \sum_{j=1}^{\infty} \lambda_j \langle \Psi_j, F \rangle \varphi_j \qquad \text{in } \mathcal{E}_p^\# ,$$

where $\Psi_j \in (\mathcal{E}_m^\#)'$, $\|\Psi_j\| \leq 1$, $\varphi_j \in \mathcal{E}_p^\#$, $\|\varphi_j\| \leq 1$, and $\lambda_j \geq 0$, $\sum_j \lambda_j < \infty$. Since $k(x,\cdot) \in \mathcal{E}^\# \subset \mathcal{E}_m^\#$, from (9) we get

$$(10) \qquad k(x,\cdot) = \sum_{j=1}^{\infty} \lambda_j \langle \Psi_j, k(x,\cdot) \rangle \varphi_j \qquad \text{in } \mathcal{E}_p^\# .$$

Now, set $r_j(x,\xi) = \lambda_j \langle \Psi_j, k(x,\cdot) \rangle \varphi_j(\xi)$ and let R_j be the integral operator with kernel r_j , defined as before:

$$R_j u(x) = (2\pi)^{-\frac{n}{2}} \int e^{i<x,\xi>} \lambda_j <\Psi_j, k(x,\cdot)> \varphi_j(\xi)\, \hat{u}(\xi)\, d\xi$$

$$= \lambda_j <\Psi_j, k(x,\cdot)> \Phi_j\, u(x) ,$$

where Φ_j is the integral operator with kernel φ_j. Since $\varphi_j \in \mathcal{E}^{\#}_p$, and the inclusion of $\mathcal{E}^{\#}_p$ into M^q_q is continuous, Φ_j defines a bounded operator from L^q into L^q, and $\|\Phi_j\|_{L^q_q} \le C_{pq} \|\varphi_j\|_p$.

Furthermore, $\operatorname{ess\,sup}_{x \in R^n} \|k(x,\cdot)\|_{\mathcal{E}^{\#}_m} < \infty$, so R_j is also bounded from L^q to L^q. Computing its norm, we obtain

$$\|R_j\|_{L^q_q} \le \lambda_j \ \operatorname{ess\,sup} |<\Psi_j, k(x,\cdot)>|\ \|\Phi_j\|_{L^q_q}$$

$$\le \lambda_j \ \operatorname{ess\,sup} |<\Psi_j, k(x,\cdot)>|\ C_{qp} \|\varphi_j\|_p \le C\, \lambda_j .$$

Thus $\sum_{j=1}^{\infty} \|R_j\|_{L^q_q} \le C \sum_j \lambda_j < \infty$, and $\tilde{K} = \sum_j R_j$ defines a bounded operator from L^q into L^q, $1 < q < \infty$.

Suppose ν is in $\mathcal{S}$. We want to show that $\tilde{K}\nu = K\nu$. If we set $K_\ell \nu = \sum_{j=1}^{\ell} R_j \nu$, then $K_\ell \nu$ converges to $\tilde{K}\nu$ in L^q, so it suffices to show that $K_\ell \nu(x)$ goes to $K\nu(x)$ almost everywhere. Now,

$$|(K-K_\ell)\nu(x)| \le (2\pi)^{-\frac{n}{2}} \int |e^{i<x,\xi>}| \; |k(x,\cdot) - \sum_{j=1}^{\ell} \lambda_j <\Psi_j,k(x,\cdot)>\varphi_j(\xi)| \; |\hat{\nu}(\xi)|d\xi$$

$$\le C_\nu \sup_\xi |k(x,\xi) - \sum_{j=1}^{\ell} \lambda_j <\Psi_j,k(x,\cdot)> \varphi_j(\xi)| \quad .$$

We know that $k(x,\cdot) = \sum_j \lambda_j <\Psi_j,k(x,\cdot)> \varphi_j$ in $\mathcal{E}_p^{\#}$. We claim that this convergence is also uniform in ξ. Suppose $F_n \to F$ in $\mathcal{E}_p^{\#}$. Then certainly $\sup_{t\in R^+} \|F(t) - F_n(t)\|_{H^p(\Sigma)} \to 0$ as $n \to \infty$. But we have chosen $p > n/2$, so by Sobolev's Lemma [7], page 331, $F(t)$ and $F_n(t)$ are continuous functions, and $\|F(t) - F_n(t)\|_{C^0(\Sigma)} \to 0$. Hence

$\sup_{t\in R^+} \sup_{\omega\in\Sigma} |[F(t)](\omega) - [F_n(t)](\omega)| \to 0$ as $n \to \infty$. Therefore we have shown that

$$\sup_{\xi\in R^n\sim\{0\}} |k(x,\xi) - \sum_{j=1}^{\ell} \lambda_j <\Psi_j,k(x,\cdot)> \varphi_j(\xi)| \longrightarrow 0 \text{ as } \ell \longrightarrow \infty$$

and so, $K = \sum_j R_j$ in L_q^q, $1 < q < \infty$, which completes the proof.

CHAPTER II

As we showed in the Introduction, an examination of the Mikhlin-Hormander Theorem leads us to try to determine translation invariant, nuclear subspaces of C_B^∞. Before proceeding with this problem, we give the definition and some properties of almost periodic functions on the line. As a general reference we use [2], Chapter I.

2.1 Definition: a) A function $T : R \to C$ of the form $T(x) = \sum_{k=1}^{n} c_k e^{i \lambda_k x}$, where the c_k are complex and the λ_k are real, is called a trigonometric polynomial.

b) A function $f : R \to C$ is almost periodic if for any $\epsilon > 0$, there is a trigonometric polynomial T_ϵ such that $|f(x) - T_\epsilon(x)| < \epsilon$ for all $x \in R$.

2.2 Proposition: f is almost periodic if and only if, given real numbers t_j, $j = 1,2,\dots$, the sequence of functions $\{f_{t_j}\}_{j=1}^{\infty}$ has a uniformly convergent subsequence. (Here f_{t_j} denotes the function $x \longmapsto f(x - t_j)$.)

The set of almost periodic functions forms a complex vector space, closed under multiplication, translation and complex conjugation. Furthermore, if f is almost periodic then the map $x \longmapsto f(tx)$ is also, for any $t \in R$. Let $\mathcal{Q}.\mathcal{P}.$ denote the space of almost periodic functions. On $\mathcal{Q}.\mathcal{P}.$ we have the mean

$$(1) \qquad M(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{a-T}^{a+T} f(x)\, dx$$

defined independent of a, and if we set $\hat{f}(\lambda) = M(f\,e^{-i\lambda\,\cdot})$, we have Parseval's formula

$$\sum_{\lambda\in R} |\hat{f}(\lambda)|^2 = M(|f|^2) \quad . \tag{2}$$

Thus $\hat{f}(\lambda)$ is zero except for countably many λ, and if we write the set $\{\lambda \in R : \hat{f}(\lambda) \neq 0\}$ as $\{\lambda_1, \lambda_2, \ldots\}$, we can represent f in a unique, mean-convergent Fourier series $f \sim \sum_{k=1}^{\infty} \hat{f}(\lambda_k)\, e^{i\lambda_k x}$. The set $\{\lambda_1, \lambda_2, \ldots\}$ is called the spectrum of f and will be denoted by Λ_f . If E is a subspace of G. P., Λ_E, the spectrum of E, is the union $\bigcup_{f\in E} \Lambda_f$.

We can now state the following basic structure theorem.

<u>2.3 Theorem</u>: Suppose E is a closed subspace of C_B^∞ and that it is translation invariant and perfect, in the induced topology. Then

i) $E \subseteq$ G. P.,

ii) $f \in E,\ g \in$ G.P. $\Rightarrow f \underset{M}{*} g(x) = \lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x-t)g(t)dt \in E$,

and

iii) $\lambda \in \Lambda_E \Rightarrow e^{i\lambda x} \in E$.

<u>Proof</u>: i) Since E is translation invariant, given $t_i \in R$, $i = 1,2,\ldots$, and $f \in E$, $f_{t_i} \in E$, $i = 1,2,\ldots$. Clearly $\{f_{t_i}\}$ is bounded in E, and as E is perfect, it must have an E-convergent subsequence; i.e., a subsequence which converges, along with its derivatives of all orders, uniformly on the line. Hence by Proposition 2.2, we conclude not only

that $f \in G.P.$, but also that $f^{(j)} \in G.P.$ $j = 1,2,\ldots$.

ii) Let $f \in E$ and $g \in G.P.$ Consider the function $F : R \to E$ defined by $F(t) = g(t)f_t$. Since the map $t \longmapsto f_t$ is continuous, so is F . We claim that, for each $T > 0$, the Riemann integral

$$R_T = \frac{1}{2T} \int_{-T}^{T} F(t)\,dt \tag{4}$$

defines an element of E . To this end, suppose $\mu : C_B^\infty \to C$ is linear and continuous, and such that $\langle\mu,h\rangle = 0$ for all $h \in E$. Then

$\langle\mu,R_T\rangle = \frac{1}{2T} \langle\mu,\lim \sum_j (t_{j+1} - t_j)F(t_j)\rangle$ where the limit is taken in C_B^∞ as the norm of the partition of $[-T,T]$ goes to zero. Now, since μ is continuous, linear, and annihilates E ,

$$\langle\mu,R_T\rangle = \frac{1}{2T} \lim \sum_j (t_{j+1} - t_j)\langle\mu,F(t_j)\rangle = 0 .$$

Therefore, as μ was arbitrary, we can conclude from the Hahn-Banach Theorem that $R_T \in E$. Furthermore, since $f \longmapsto f(x)$ is a continuous map from E into C for each $x \in R$,

$$R_T(x) = \frac{1}{2T} \lim \sum_j (t_{j+1} - t_j)\; g(t_j)\; f(x - t_j) = \frac{1}{2T} \int_{-T}^{T} f(x - t)g(t)dt.$$

A simple application of the Mean Value Theorem (see Appendix,pg.45) shows that

$$\frac{\partial^j}{\partial x^j} R_T = f^{(j)} * g_T , \text{ where} \tag{5}$$

$$g_T(t) = \begin{cases} \frac{1}{2T}\, g(t) , & |t| \le T \\ 0 , & |t| > T . \end{cases}$$

From (5) we can conclude that R_T is bounded in E for all T :

$$\|R_T\|_{E_p} = \max_{0\leq j\leq p} \sup_{x\in R} |f^{(j)}*g_T(x)| \leq \max_{0\leq j\leq p} \sup_{x} \frac{1}{2T}\int_{-T}^{T} |f^{(j)}(x-t)g(t)|dt$$

$$\leq \max_{0\leq j\leq p} \|f^{(j)}\|_{L^\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|dt \longrightarrow \max \|f^{(j)}\|_{L^\infty} M(|g|) .$$

Now, as E is perfect, $\{R_T\}_{T\to\infty}$ has a limit point H in E . But for each $x \in R$, the function $t \mapsto f(x-t)g(t)$ is almost periodic, so $f \underset{M}{*} g(x)$, which is the mean (with respect to t) of this function, exists pointwise. Hence $H(x) = f \underset{M}{*} g(x)$ for each x , and $f \underset{M}{*} g$ is in E .

iii) Given $\lambda \in \Lambda_E$, choose $f \in E$ with $\hat{f}(\lambda) \neq 0$. Define a function $g \in \mathcal{Q}.\mathcal{P}.$ by $g(x) = \frac{1}{\hat{f}(\lambda)} e^{i\lambda x}$. Then, by ii) $f \underset{M}{*} g \in E$.

Now,

$$(6) \qquad f \underset{M}{*} g(x) = \lim_{T\to\infty} \frac{1}{2T}\frac{1}{\hat{f}(\lambda)}\int_{-T}^{T} f(x-t)\, e^{i\lambda t}\, dt$$

$$= \frac{1}{\hat{f}(\lambda)} \lim \frac{1}{2T}\int_{x-T}^{x+T} f(t')\, e^{i\lambda(x-t')}\, dt'$$

$$= \frac{1}{\hat{f}(\lambda)} \hat{f}(\lambda)\, e^{i\lambda x} = e^{i\lambda x} .$$

Therefore, $e^{i\lambda x} \in E$, and the proof is complete.

As a direct result of Theorem 2.3, we determine that the spectrum of E can have no accumulation point.

<u>2.4 Corollary</u>: Let E be as in Theorem 2.3. Then Λ_E is discrete.

<u>Proof</u>: Suppose there exist $\lambda_k \in \Lambda_E$, $k = 1,2,\ldots$ such that $\{\lambda_k\}$ is a Cauchy sequence. Then the functions $g_k(x) = e^{i\lambda_k x}$ are in E and, since $\sup\limits_k |\lambda_k| < \infty$, form a bounded set. Thus $\{g_k\}$ must have a uniformly convergent subsequence. But

$$\sup_x |g_k(x)-g_\ell(x)| = \sup |e^{i\lambda_k x} - e^{i\lambda_\ell x}| = \sup |e^{i(\lambda_k-\lambda_\ell)x} - 1|$$

$$\geq |e^{i(\lambda_k-\lambda_\ell)\left(\frac{\pi}{\lambda_k-\lambda_\ell}\right)} - 1| = 2 ,$$

so the g_k cannot have such a subsequence, and Λ_E must be discrete.

Before we can go further in determining the structure of E , we need the following lemma which is proved in [6], page 165.

<u>2.5 Lemma</u>: Given $\lambda_k \in R$, $k = 1,\ldots,M$, and $\epsilon > 0$, there exists a trigonometric polynomial B such that

1) $B(x) \geq 0$,
2) $M(B) = 1$, and
3) $1 > \hat{B}(\lambda_k) > 1 - \epsilon$, $k = 1,\ldots,M$.

Using Lemma 2.5 and the techniques of [6], we get the following approximation theorem for E .

<u>2.6 Theorem</u>: Let E again be as in Theorem 2.3. Suppose that $f \in C_B^\infty \cap \mathcal{A}.\mathcal{P}.$ and that $\Lambda_f \subseteq \Lambda_E$. Then there exists a sequence of trigonometric polynomials $\{P_n\}_{n=1}^\infty$ such that $\Lambda_{P_n} \subseteq \Lambda_f$ and the limit

in C_B^∞ of the P_n is f .

<u>Proof</u>: Write $\Lambda_f = \{\lambda_1, \lambda_2, \ldots\}$ and let B_n be the trigonometric polynomial of the lemma, for $\{\lambda_k : 1 \le k \le n\}$ and $\epsilon = 1/n$. We claim that $P_n = f \underset{M}{*} B_n$ has the desired properties. First of all, we can write $P_n(x)$ as $\sum_{\lambda \in R} \hat{B}_n(\lambda)\, f \underset{M}{*} e^{i\lambda x}$ which by a computation like the one in (6) is $\sum_{\lambda \in R} \hat{B}_n(\lambda)\, \hat{f}(\lambda)\, e^{i\lambda x}$ so that $\hat{P}_n(\lambda) = \hat{B}_n(\lambda)\, \hat{f}(\lambda)$. Therefore we have the inclusions $\Lambda_{P_n} \subseteq \Lambda_f \subseteq \Lambda_E$. Furthermore, since the set $\{\lambda : \hat{B}(\lambda) \neq 0\}$ is finite, P_n is a trigonometric polynomial, and by iii) of Theorem 2.3, P_n is in E . Thus we are left only to prove that $P_n \to f$ in C_B^∞ . Using the same argument which gave (5) (see Appendix, pg. 45) we prove that $P_n^{(j)} = f^{(j)} \underset{M}{*} B_n$, so

$$\|P_n\|_{E_p} \le \max_{0 \le j \le p} \|f^{(j)}\|_{L^\infty} M(B_n) = \max_{0 \le j \le p} \|f^{(j)}\|_{L^\infty} .$$

Thus the sequence $\{P_n\}$ is bounded in E and, as E is perfect, has an accumulation point P . We know that $1 > \hat{B}_n(\lambda_k) > 1 - 1/n$, $1 \le k \le n$, so for any k , $\lim \hat{P}_n(\lambda_k) = \lim \hat{B}_n(\lambda_k)\, \hat{f}(\lambda_k) = \hat{f}(\lambda_k)$. We have already shown that if $\lambda \notin \Lambda_f$, then $\lambda \notin \Lambda_{P_n}$, so $\hat{P}(\lambda) = \hat{f}(\lambda)$ for all $\lambda \in R$, and by the uniqueness of the Fourier series, $P = f$. Thus f is the only limit point of the P_n , and $P_n \to f$ in C_B^∞ .

Note that, since E is closed, we have shown that f is in E , which yields the following fact which will prove extremely useful.

<u>2.7 Corollary</u>: If E is as above, then $E = \{f \in C_B^\infty \cap \text{G.P.} : \Lambda_f \subseteq \Lambda_E\}$.

Hence E is determined by its spectrum. Of more importance will be the fact that, under appropriate restrictions on Λ_E, E is topologically isomorphic to a space F of functions on Λ_E. Furthermore, we will give necessary and sufficient conditions for F to be nuclear. In order to get these results we need to relate the seminorms in E to ℓ^1-norms of Fourier coefficients of functions in E.

<u>2.8 Proposition</u>: Suppose $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is a countable collection of real numbers which are linearly independent over the rationals. Then, if $f \in C_B^\infty \cap$ G.P. and $\Lambda_f \subseteq \Lambda$, $\|f\|_{L^\infty} = \sum_{k=1}^{\infty} |\hat{f}(\lambda_k)|$.

<u>Proof</u>: Let P_n be as in Theorem 2.6. P_n converges to f in C_B^∞, and therefore uniformly, so $\|f\|_{L^\infty} = \lim_n \|P_n\|_{L^\infty}$. Now, by Kroneker's Theorem (see, for instance, [6], page 181), since the λ_k are linearly independent, $\|P_n\|_{L^\infty} = \sum_k |\hat{P}_n(\lambda_k)|$ (this sum is finite). We already know that $\lim_n \hat{P}_n(\lambda_k) = \hat{f}(\lambda_k)$ for all k, so it suffices to show

$$\lim_n \sum_k |\hat{P}_n(\lambda_k)| = \sum_k \lim_n |\hat{P}_n(\lambda_k)| .$$

Using Fatou's Lemma, we obtain

$$\sum_k \lim_n |\hat{P}_n(\lambda_k)| \leq \lim_n \sum_k |\hat{P}_n(\lambda_k)| .$$

On the other hand, since $\hat{P}_n(\lambda) = \hat{f}(\lambda)\, \hat{B}_n(\lambda)$ and $|\hat{B}_n(\lambda)| \leq 1$, for all n and all $\lambda \in R$, $|\hat{P}_n(\lambda_k)| \leq |\hat{f}(\lambda_k)|$. Therefore,

$\lim_n \sum_k |\hat{P}_n(\lambda_k)| \leq \sum_k |\hat{f}(\lambda_k)|$ and the proof is complete.

Proposition 2.8 becomes much more useful when we recall that if f is in E, then $f^{(j)}$ is almost periodic for all j. Furthermore, the Fourier series of $f^{(j+1)}$ is obtained by differentiating the Fourier series of $f^{(j)}$, term by term (see [2], page 27); i.e., by induction,

$$(7) \qquad f^{(j)} \sim \sum_k \hat{f}(\lambda_k)(i\lambda_k)^j e^{i\lambda_k x} .$$

Thus, if the λ_k are independent Proposition 2.8 implies

$$(8) \qquad \sum_k |\hat{f}(\lambda_k)|\ |\lambda_k|^j = \|f^{(j)}\|_{L^\infty} < \infty$$

for all $f \in E$, and all $j = 0,1,2,\ldots$.

Again suppose $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is a collection of real numbers which are linearly independent over the rationals. With equation (8) in mind, we define the space of sequences

$$(9) \qquad F = F(\Lambda) = \{c = \{c_k\}_{k=1}^\infty : c_k \in C \quad \text{and}$$

$$\rho_j(c) = \sum_k |c_k|\ |\lambda_k|^j < \infty \qquad j = 0,1,2,\ldots\} .$$

Set $E = E(\Lambda) = \{f \in C_B^\infty \cap \text{G.P.} : \Lambda_f \subseteq \Lambda\}$ and define $\alpha : E \to F$ by $\alpha(f) = \{\hat{f}(\lambda_k)\}_{k=1}^\infty$.

<u>2.9 Proposition</u>: With the above notation, α is a topological isomorphism of E onto F .

<u>Proof</u>: By equation (8), $\rho_j(\alpha(f)) = \|f^{(j)}\|_{L^\infty} < \infty$, so α is well-

defined. The continuity of α follows from the same equality and the fact that $\|f^{(j)}\|_{L^\infty} \leq \|f\|_{E_j}$. Using the uniqueness of Fourier coefficients, we conclude that it is one-one. We claim that α is onto: For $c \in F$, define $f(x) = \sum_{k=1}^{\infty} c_k e^{i\lambda_k x}$. Since $\|c\|_{\ell^1} = \rho_0(c) < \infty$, the partial sums $\sum_{k=1}^{N} c_k e^{i\lambda_k x}$ converge uniformly to f , so by Definition 2.1 b), f is almost periodic. In the same way $\{c_k(i\lambda_k)\}_{k=1}^{\infty} \in \ell^1$, so $\sum_{k=1}^{N} c_k(i\lambda_k) e^{i\lambda_k x}$ converges uniformly as $N \to \infty$ and so must converge to the derivative of f . Thus, by induction on j , $\sum_{k=1}^{N} c_k(i\lambda_k)^j e^{i\lambda_k x} \longrightarrow f^{(j)}$ uniformly, and by Proposition 2.8, $\|f^{(j)}\|_{L^\infty} = \rho_j(c) < \infty$ and $f \in C_B^\infty$. Clearly $\Lambda_f \subseteq \Lambda$, and we have f in E with $\alpha(f) = c$. Thus, α is continuous, one-one, and onto; so by the Open Mapping Theorem it suffices to show that E and F are Fréchet spaces. Since they are countably normed, we are left only to prove them both to be complete.

<u>2.10 Lemma</u>: E is complete.

<u>Proof</u>: Suppose f_n is Cauchy in E . Then, since C^0 is complete, f_n has a uniform limit, f . Since the $f_n^{(j)}$ are Cauchy in C^0 for any j , f is j-times continuously differentiable and $f_n^{(j)}$ converges uniformly to $f^{(j)}$. Thus $f_n \to f$ in C_B^∞ . f is clearly almost periodic, and as $\Lambda_{f_n} \subseteq \Lambda$, we have $\Lambda_f \subseteq \Lambda$, so f is in E .

2.11 Lemma: F is complete.

Proof: Suppose $c^n = \{c_k^n\}_{k=1}^{\infty}$ is Cauchy in F. Then c^n is Cauchy in ℓ^1 and has an ℓ^1-limit $c = \{c_k\}$. Similarly for any $j \geq 1$, $d^n = \{c_k^n |\lambda_k|^j\}_{k=1}^{\infty}$ is ℓ^1-Cauchy and has an ℓ^1-limit $d = \{d_k\}$. Certainly for each k we have $c_k^n \to c_k$ and $d_k^n \to d_k$. But, since $d_k^n = c_k^n |\lambda_k|^j$, $d_k = c_k |\lambda_k|^j$, and therefore $c \in F$, $c_n \to c$ in F, and F is complete.

It is important to note that Proposition 2.9 remains valid if the linear independence of Λ is replaced by a growth condition on the λ_k.

2.12 Proposition: Suppose there exists an integer $p > 0$ such that $\sum_{k=1}^{\infty} |\lambda_k|^{-p} < \infty$. Then $\alpha : E \to F$, defined as above, is a topological isomorphism.

Remark: The following argument is patterned after one in [3], page 82.

Proof: If f is in E, we have as before,

$$f^{(j)}(x) = \sum_{k=1}^{\infty} \hat{f}(\lambda_k)(i\,\lambda_k)^j e^{i\lambda_k x}, \tag{10}$$

so that, by (2), $\sum_k |\hat{f}(\lambda_k)|^2 \, |\lambda_k|^{2j} = M(|f|^2) < \infty$ for all $j = 0,1,\ldots$.

Therefore, using Cauchy-Schwartz, for any j

$$\sum_k |\hat{f}(\lambda_k)| \, |\lambda_k|^j = \sum_k |\hat{f}(\lambda_k)| \, \frac{|\lambda_k|^{j+p}}{|\lambda_k|^p}$$

$$\leq \Big(\sum_k |\lambda_k|^{-2p}\Big)^{\frac{1}{2}} \Big(\sum_k |\hat{f}(\lambda_k)|^2 \, |\lambda_k|^{2(j+p)}\Big)^{\frac{1}{2}} < \infty .$$

Thus α is well-defined. Furthermore, we have shown

$$\rho_j(\alpha(f)) \le c\ M(|f^{(j+p)}|^2) = c \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f^{(j+p)}(x)|^2\, dx$$

$$\le c \sup_{x\in R} |f^{(j+p)}(x)|^2 \le c\ \|f\|_{E_{j+p}}^2 \quad ,$$

so α is continuous. Again the uniqueness of the Fourier coefficients implies that α is one-one. To show that α is onto, choose $c \in F$, $c = \{c_k\}$, and define $f(x) = \sum_k c_k e^{i\lambda_k x}$. As before $f \in C^\infty \cap G.P.$ and $\Lambda_f \subseteq \Lambda$, so it suffices to show that $f^{(j)}$ is bounded for each j. Taking absolute values in (10), we see that

$$|f^{(j)}(x)| \le \sum_k |\hat{f}(\lambda_k)|\ |\lambda_k|^j < \infty \quad ,$$

so $\|f^{(j)}\|_{L^\infty} < \infty$. As the linear independence of Λ played no role in proving E and F to be Fréchet spaces, the Open Mapping Theorem again implies that α is a topological isomorphism.

Thus, if Λ satisfies either 2.9 or 2.12, $E(\Lambda)$ is isomorphic to $F(\Lambda)$. The strength of this result lies in the fact that we can characterize very nicely the desired properties of $F(\Lambda)$.

<u>2.13 Theorem</u>: Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ be an arbitrary countable collection of real numbers and define $F = F(\Lambda)$ as in equation (9). Then

a) F is perfect $\Leftrightarrow \Lambda$ is finite or $\lim_k |\lambda_k| = \infty$.

b) F is nuclear $\Leftrightarrow \exists\ p > 0$ such that $\sum_k |\lambda_k|^{-p} < \infty$.

Proof: If Λ is finite, $F(\Lambda)$ is finite dimensional and therefore nuclear (and perfect). Now assume Λ is infinite.

a) $\rightarrow$: Suppose $\lim_k |\lambda_k|$ is not ∞ ; i.e., the limit is either finite or does not exist. In either case there must exist a bounded, infinite subcollection λ_{k_n} of the λ_k , $|\lambda_{k_n}| \leq M$, $n = 1,2,\ldots$. Then defining $e^n \in F$ by $e^n_j = \delta^{k_n}_j$, $\{e^n\}^\infty_{n=1}$ is bounded in F , but $\rho_0(e^n - e^m) = 2$ for all $n \neq m$, so F is not perfect.

a) $\leftarrow$: Now suppose $\lim_k |\lambda_k| = \infty$ and that $|\lambda_k|$ is increasing with k . Let $\{c^n\}^\infty_{n=1}$ be a bounded sequence in F ; i.e., for any j there is a $B_j > 0$ such that

$$\rho_j(c^n) = \sum_k |c^n_k|\ |\lambda_k|^j \leq B_j\ , \quad \text{for all } n\ . \tag{11}$$

Then the sequence of k^{th} entries $\{c^n_k\}^\infty_{n=1}$ is bounded in C for each k , and thus, by the standard diagonalization proceedure and passage to a subsequence, we can assume that c^n is such that $\{c^n_k\}^\infty_{n=1}$ is Cauchy for each k . Let $c_k = \lim_{n\to\infty} c^n_k$ and set $c = \{c_k\}^\infty_{k=1}$.

Then for any N and any j ,

$$\sum_{k=1}^{N} |c_k|\ |\lambda_k|^j = \lim_n \sum_{k=1}^{N} |c^n_k|\ |\lambda_k|^j \leq B_j \quad ,$$

so taking limits as $N \to \infty$ we obtain $\sum_{k=1}^{\infty} |c_k| \, |\lambda_k|^j \le B_j$ for all j.

Thus $c \in F$. We now want to show that $c^n \to c$ in F. It clearly suffices to show

$$\overline{\lim_n} \sum_{k=K+1}^{\infty} |c_k - c_k^n| \, |\lambda_k|^j \longrightarrow 0 \quad \text{as} \quad K \longrightarrow \infty \ .$$

But, since $|\lambda_k|$ is increasing,

$$\overline{\lim_n} \sum_{k=K+1}^{\infty} |c_k - c_k^n| \, |\lambda_k|^j \le \overline{\lim_n} \frac{1}{|\lambda_K|} \sum_{k=K+1}^{\infty} |c_k - c_k^n| \, |\lambda_k|^{j+1}$$

$$\le \frac{1}{|\lambda_K|} \, 2B_{j+1} \longrightarrow 0 \quad \text{as} \quad K \longrightarrow \infty \ .$$

b) → : Suppose F is nuclear. Then there is a $p > 0$ such that the natural inclusion $I : F_p \to F_0$ is nuclear. The map $S : F_0 \to F_p$ defined by $S(\{c_k\}) = \{c_k |\lambda_k|^{-p}\}$ is linear and continuous, since

$\rho_p(S(c)) = \sum_k |c_k| \, |\lambda_k|^{-p} \, |\lambda_k|^p = \rho_0(c)$. Therefore the composition

$T = S \circ I : F_p \to F_p$ is nuclear; i.e.,

$$T(c) = \sum_{j=1}^{\infty} \mu_j \, \langle \Psi^j, c \rangle \, \varphi^j \quad \text{in} \quad F_p$$

where $\mu_j \ge 0$ with $\sum_{j=1}^{\infty} \mu_j < \infty$, $\varphi^j \in F_p$ with $\|\varphi^j\|_{F_p} \le 1$, and $\Psi^j \in F_p'$ with $\|\Psi^j\|_{F_p'} \le 1$. Now, F_p is $\ell^1(\nu)$, where ν is the measure on the integers defined by $\nu(k) = |\lambda_k|^p$, so $F_p' = \ell^\infty(\nu)$, which is just ℓ^∞ on the set $\{k : \lambda_k \neq 0\}$. The duality is defined by $\langle \Psi,\varphi\rangle = \sum_{k=1}^{\infty} \Psi_k \varphi_k |\lambda_k|^p$ for $\Psi = \{\Psi_k\} \in F_p'$ and $\varphi = \{\varphi_k\} \in F_p$.

Let $e^n \in F_p$ be defined by $e_k^n = \delta_k^n$. Then $Te^k = |\lambda_k|^{-p} e^k$, and $\langle \Psi^j, e^k\rangle = \Psi_k^j |\lambda_k|^p$. Therefore

$$|\lambda_k|^{-p} e^k = \sum_{j=1}^{\infty} \mu_j \Psi_k^j |\lambda_k|^p \varphi^j ,$$

so, letting $\pi_k : F_p \to C$ be the k^th^ projection, $\pi_k(\varphi) = \varphi_k$, we obtain

$$|\lambda_k|^{-p} = \pi_k \Big(\sum_{j=1}^{\infty} \mu_j \Psi_k^j |\lambda_k|^p \varphi^j \Big) = \sum_j \mu_j \Psi_k^j \varphi_k^j |\lambda_k|^p .$$

Then for any integer N,

$$\sum_{k=1}^{N} |\lambda_k|^{-p} = \sum_{j=1}^{\infty} \mu_j \sum_{k=1}^{N} \Psi_k^j \varphi_k^j |\lambda_k|^p \le \sum_{j=1}^{\infty} \mu_j \Big| \sum_{k=1}^{N} \Psi_k^j \varphi_k^j |\lambda_k|^p \Big|$$

$$\le \sum_{j=1}^{\infty} \mu_j |\langle \Psi^j, \varphi^j\rangle| \le \sum_j \mu_j \|\Psi^j\| \, \|\varphi^j\| \le \sum_j \mu_j < \infty .$$

Therefore $\sum_{k=1}^{\infty} |\lambda_k|^{-p}$ is finite.

b) $\leftarrow$: Finally, assume $\sum |\lambda_k|^{-p} < \infty$. Given n we must pick m such that the natural map $F_m \to F_n$ is nuclear. Consider $m = n + p$, and define:

$$e^k \in F_m \text{ by } e_j^k = \delta_j^k ,$$

$$\Psi_k \in F_m' \text{ by } \langle \Psi_k, c\rangle = |\lambda_k|^{n+p} c_k ,$$

$\varphi_k \in F_n$ by $\varphi_k = |\lambda_k|^{-n} e^k$, and

$\mu_k \geq 0$ by $\mu_k = |\lambda_k|^{-p}$.

Then $\|\Psi_k\| = 1$, $\|\varphi_k\| = 1$, and $\sum_k \mu_k < \infty$. Furthermore, for any $c \in F_{n+p}$ we have

$$\sum_{k=1}^{\infty} \mu_k \langle \Psi_k, c \rangle \varphi_k = \sum_k |\lambda_k|^{-p} |\lambda_k|^{n+p} c_k |\lambda_k|^{-p} e^k = \sum_k c_k e^k = c \ .$$

Thus if we define $T(c) = \sum_k \mu_k \langle \Psi_k, c \rangle \varphi_k$, T *is* the canonical map from F_{n+p} to F_n and it is nuclear, which completes the proof of 2.11.

Hence we have shown that, given $\{\lambda_1, \lambda_2, \ldots\}$ such that for some p , $\sum |\lambda_k|^{-p} < \infty$, $E(\Lambda) = \{f \in C_B^\infty \cap G.P. \mid \Lambda_f \subseteq \Lambda\}$ is nuclear, and so $\mathcal{E}^{\#} = E^{\#} \hat{\otimes} C^\infty(\Sigma)$ will generalize $C^\infty(\Sigma)$ in the Calderon-Zygmund Theorem.

APPENDIX

SECTION I. Lemma 1.1.

We want to show that for any $p \geq 0$ there exist constants c_j^p, d_j^p, $j = 0,1,\ldots,p$ such that for all $f \in E$ and $f^\# \in E^\#$ defined by $f^\#(t) = f(\log t)$, we have

$$(1) \qquad f^{\#(p)}(t) = \frac{1}{t^p} \sum_{j=0}^{p} c_j^p \, f^{(j)}(\log t) \qquad \forall\, t \in R^+ \,, \quad \text{and}$$

$$(2) \qquad f^{(p)}(s) = \sum_{j=0}^{p} d_j^p \, e^{js} \, f^{\#(j)}(e^s) \qquad \forall\, s \in R \,.$$

The proofs use an induction on p. Since $f^\#(t) = f(\log t)$, (1) is true with $c_0^0 = 1$ for $p = 0$. Suppose (1) is true for $p < m$. Then

$$f^{\#(m)}(t) = \frac{\partial}{\partial t}\left[\frac{1}{t^{m-1}} \sum_{j=0}^{m-1} c_j^{m-1} \, f^{(j)}(\log t)\right]$$

$$= \frac{1}{t^m} \sum_{j=0}^{m} c_j^m \, f^{(j)}(\log t) \,, \qquad \text{where}$$

$$c_j^m = \begin{cases} 0 & j = 0,\ m > 0 \\ -(m-1)c_j^{m-1} + c_{j-1}^{m-1} & 1 < j < m \\ c_{m-1}^{m-1} & j = m \end{cases} \,.$$

Again the case $p = 0$ is obvious with $d_0^0 = 1$. If (2) holds for $p < m$, then

$$f^{(m)}(s) = \frac{\partial}{\partial s}\left[\sum_{j=0}^{m-1} d_j^{m-1}\, e^{js} f^{\#(j)}(e^s)\right] = \sum_{j=0}^{m} d_j^m\, e^{js} f^{\#(j)}(e^s), \text{ where}$$

$$d_j^m = \begin{cases} 0 & j = 0,\ m > 0 \\ j d_j^{m-1} + d_{j-1}^{m-1} & 0 < j < m \\ d_{m-1}^{m-1} & j = m \end{cases} .$$

SECTION II. $C^\infty(\Sigma)$.

Suppose $V_1,\dots,V_m$ is a cover of Σ by coordinate neighborhoods. Let $\mu_i: \Sigma \to R$, $i = 1,\dots,m$, be a C^∞ partition of unity subordinate to V_i , and let $\psi_i: V_i \to R^{n-1}$, $i = 1,\dots,m$, be C^∞ coordinate maps. Then $C^\infty(\Sigma)$ is the space of all functions $f: \Sigma \to C$ such that $\mu_i f \circ \psi_i^{-1}$ is a C^∞ function from R^{n-1} into C , $i = 1,\dots,m$. $C^\infty(\Sigma)$ is topologized by the Hilbert norms

$$\|f\|_p^2 = \sum_{i=1}^{m} \sum_{|\alpha|\le p} \|D^\alpha(\mu_i f \circ \psi_i^{-1})\|^2_{L^2(R^{n-1})} = \sum_{i=1}^{m} \|\mu_i f \circ \psi_i^{-1}\|^2_{H^p(R^{n-1})} .$$

The completion of $C^\infty(\Sigma)$ with respect to $\|\ \|_p$ is denoted by $H^p(\Sigma)$ and is a Hilbert space. Thus $C^\infty(\Sigma)$ is countably normed with norms coming from inner products. Such a space is called countably Hilbert.

Remark: Had we chosen another open cover, partition of unity, or coordinate maps, the spaces $C^\infty(\Sigma)$ and $H^p(\Sigma)$ would be the same, and the new norms would be equivalent to the old.

3.1 Definition: A map $T: H_1 \to H_2$, where H_1 and H_2 are Hilbert spaces is *nuclear* if T can be written

$$(3) \qquad T(x) = \sum_{k=1}^{\infty} \lambda_k \langle x,\xi_k\rangle_2 \varphi_k \text{ , where } \{\lambda_k\} \in \ell^1 ,$$

$\{\xi_k\}$ is an orthonormal basis in H_1 , and $\{\varphi_k\}$ is an orthonormal basis in H_2 . ([7], page 494 shows that this is equivalent to 0.1 if E and F are Hilbert spaces.) A countably Hilbert space H is *nuclear* if given any p there is an $m > p$ such that the natural inclusion $T_p^m : H_m \to H_p$ is nuclear in the sense of (3).

<u>3.2 Proposition</u>: $C^\infty(\Sigma)$ is nuclear.

<u>Proof</u>: We will only sketch the proof. Given p, choose $m > p + n - 1$, where $n - 1$ is the dimension of Σ. If U is a bounded open subset of R^{n-1} we define $H_0^k(U)$ to be the closure with respect to $\| \ \|_{H^k(R^{n-1})}$ of the space of C^∞ functions on R^{n-1} with compact support contained in U. By [8], page 279, the natural inclusion of $H_0^m(U)$ into $H_0^p(U)$ is nuclear. Now consider the composition T_i,

$$\begin{tikzcd}
H^m(\Sigma) \arrow[r, "a"] & H^m(\Sigma) \arrow[r, "b"] & H_0^m(\psi_i V_i) \arrow[r, "c"] & H_0^p(\psi_i V_i) \arrow[r, "d"] & H^p(\Sigma) \\
f \arrow[r, mapsto] & \mu_i f \arrow[r, mapsto] & \mu_i f \circ \psi_i^{-1} \arrow[r, mapsto] & \mu_i f \circ \psi_i^{-1} \arrow[r, mapsto] & \mu_i f \ .
\end{tikzcd}$$

a is norm decreasing and therefore continuous, b and d are isometries and c is nuclear. Hence T_i is nuclear, and by [3], page 49, $\sum_{i=1}^{m} T_i$ is also. But $\left(\sum_{i=1}^{m} T_i\right) f = f$, so $\sum T_i$ is the natural inclusion, and $C^\infty(\Sigma)$ is nuclear.

SECTION III. $E^{\#}(C^{\infty}(\Sigma))$.

In order to improve the notation we will denote $E^{\#}$ by F and $C^{\infty}(\Sigma)$ by G , so G_p will be $H^p(\Sigma)$. $F(G)$ is the space of functions $u: R^+ \to F$ such that given any p and any orthonormal basis $\{\varphi_n\}$ of G_p , the map $t \longmapsto \langle u(t),\varphi_n\rangle_p$ is in F_p and

$$(4) \qquad \|u\|_p^2 = \sum_{n=1}^{\infty} \|\langle u(\cdot),\varphi_n\rangle_p\|_{F_p}^2$$

$$= \sum_{n=1}^{\infty} \max_{0\le j\le p} \sup_{t\in R^+} |\langle t^j u^{(j)}(t),\varphi_n\rangle_p|^2 < \infty \quad .$$

3.3 Proposition: The norms $\|u\|_p$ are equivalent to

$$\|u\|_p' = \max_{0\le j\le p} \sup_t \|t^j u^j(t)\|_{G_p} \quad .$$

Proof: For any orthonormal basis $\{\varphi_n\}$ of G_p , we have

$$\|u\|_p'^2 = \max_{0\le j\le p} \sup_t \sum_{n=1}^{\infty} |\langle t^j u^{(j)}(t),\varphi_n\rangle_p|^2$$

$$\le \sum_{n=1}^{\infty} \max \sup |\langle t^j u^{(j)}(t),\varphi_n\rangle_p|^2 = \|u\|_p^2 \quad .$$

Conversely, since G is nuclear $t^j u^{(j)}(t)$ can be represented in G_p as

$$t^j u^{(j)}(t) = \sum_{\ell=1}^{\infty} \lambda_\ell \langle t^j u^{(j)}(t),\psi_\ell\rangle_k \Phi_\ell \ , \quad \text{where}$$

$\{\lambda_\ell\} \in \ell^1, \{\psi_\ell\}$ is an orthonormal basis in G_k , and $\{\Phi_\ell\}$ is an orthonormal basis in G_p . Hence

$$\|u\|_p^2 = \sum_{n=1}^{\infty} \max_{0\le j\le p} \sup_t \Big| \sum_{\ell=1}^{\infty} \lambda_\ell <t^j u^{(j)}(t), \Psi_\ell>_k <\Phi_\ell, \varphi_n>_p \Big|^2$$

$$\le \sum_{n=1}^{\infty} \Big\{ \sum_{\ell=1}^{\infty} |\lambda_\ell| \, |<\Phi_\ell, \varphi_n>_p|^2 \Big\} \sum_{\ell=1}^{\infty} |\lambda_\ell| \max_{0\le \ell\le p} \sup_t |<t^j u^{(j)}(t), \Psi_\ell>_k|^2$$

$$\le \sum_{\ell=1}^{\infty} |\lambda_\ell| \sum_{n=1}^{\infty} |<\Phi_\ell, \varphi_n>_p|^2 \sum_{\ell=1}^{\infty} |\lambda_\ell| \max_{0\le j\le k} \sup_t \|t^j u^{(j)}(t)\|_{G_k}^2$$

$$= 2 \Big(\sum_{\ell} |\lambda_\ell| \Big) \|u\|_k'^2 .$$

Note that Proposition 3.3 implies that, had we defined $\|u\|_p$ in (4) with other orthonormal bases, we would have gotten the same topology for $F(G)$.

<u>3.4 Proposition</u>: $F(G)$ is complete.

<u>Proof</u>: Since the topology of $F(G)$ is given by a countable collection of norms, it suffices to show that $F(G) = \bigcap_p F(G)_p$, where $F(G)_p$ is the completion of $F(G)$ with respect to $\|u\|_p$. Clearly $F(G) \subset F(G)_p$ for all p so it is in the intersection. Suppose $u \in \bigcap_p F(G)_p$. Given $p \ \exists \ u_m \in E(F)$ so that $u_m \to u$ in p-norm; i.e.,

$$\max_{0\le j\le p} \sup_t \|t^j u^{(j)}(t) - t^j u_m^{(j)}(t)\|_{G_p} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty .$$

Then certainly $\max \sup |<t^j u^{(j)}(t), \varphi_n>_p - <t^j u_m^{(j)}(t), \varphi_n>_p| \to 0$ as $m \to \infty$, and since F_p is complete, the map $t \longmapsto <u(t), \varphi_n>_p$ is in F_p .

Furthermore

$$\sum_n \max_{0\leq j\leq p} \sup_t |\langle t^j u^{(j)}(t), \varphi_n\rangle_p|^2 \leq \overline{\lim_{m\to\infty}} \|u_m\|_p^2 < \infty ,$$

since u_m must be bounded in G_p , so u is in $F(G)$, and the proof is complete.

Now define $\alpha: F \times G \to F(G)$ by $\alpha(f,g)(t) = f(t)g$. α is well-defined and continuous, since $\langle f(t)g, \varphi_n\rangle_p = \langle g, \varphi_n\rangle_p\, f(t)$ is in F_p , and $\|\alpha(f,g)\|^2_{F(G)_p} = \|f\|^2_{F_p}\, \|g\|^2_{G_p}$. The projective tensor topology is universal with respect to continuous bilinear forms, so α determines $\tilde{\alpha}: F \otimes G \to F(G)$, and $\tilde{\alpha}$ is continuous. Furthermore, as $F(G)$ is complete, $\tilde{\alpha}$ extends to $\hat{\alpha}: F \hat{\otimes} G \to F(G)$, and $\hat{\alpha}$ is continuous.

<u>3.5 Proposition</u>: $\hat{\alpha}$ is one-one.

<u>Proof</u>: Suppose $v \in F \hat{\otimes} G$, and $\hat{\alpha}(v) = 0$. By Grothendieck's Theorem (see Introduction), v can be represented as an absolutely convergent series $v = \sum_{j=1}^{\infty} \mu_j f_j \otimes g_j$, where $\{\mu_j\} \in \ell^1$, $f_j \to 0$ in F , and $g_j \to 0$ in G . As before, given any p , g_j can be represented in G_p as $g_j = \sum_{n=1}^{\infty} \lambda_n \langle g_j, \psi_n\rangle_k \Phi_n$. Define

$$v_{NM} = \sum_{n=1}^{N} \lambda_n \Big(\sum_{j=1}^{M} \mu_j \langle g_j, \psi_n\rangle_k f_j \Big) \otimes \Phi_n .$$

Then v_{NM} is in $F \otimes G_p$, and $v_{NM} \to v$ in $(F \otimes G)_p$, since

$$\|v - v_{NM}\|_{(F\otimes G)_p} = \Big\| \sum_{j=M+1}^{\infty} \mu_j f_j \otimes g_j + \sum_{j=1}^{M} \mu_j f_j \otimes \Big(\sum_{n=N+1}^{\infty} \lambda_n \langle g_j, \psi_n \rangle_k \Phi_n \Big) \Big\|_p$$

$$\le c \sum_{j=M+1}^{\infty} |\mu_j| + c \sum_{j=1}^{\infty} |\mu_j| \Big[\sum_{n=N+1}^{\infty} |\lambda_n|^2 \Big]^{\frac{1}{2}} \longrightarrow 0 \quad \text{as} \quad N,M \longrightarrow \infty \quad .$$

Thus, if p is sufficiently large, since $\hat{\alpha}$ is continuous, there is an m such that $\hat{\alpha}(v_{NM}) \to 0$ in $F(G)_m$. But $\hat{\alpha}(v_{NM})(t) = \sum_{n=1}^{N} \lambda_n \sum_{j=1}^{M} \mu_j \langle g_j, \psi_n \rangle_k f_j(t) \Phi_n$ which goes in $F(G)_p$ to the same expression with N and M replaced by ∞, since using $\{\Phi_n\}$ in (4)

$$\Big\| \sum_{n=1}^{\infty} \lambda_n \Big[\sum_{j=1}^{\infty} \mu_j \langle g_j, \psi_n \rangle_k f_j(\cdot) \Big] \Phi_n - \hat{\alpha}(v_{NM})(\cdot) \Big\|^2_{F(G)_p}$$

$$\le \sum_{n=1}^{N} |\lambda_n|^2 \max_{0 \le \ell \le p} \sup_t \Big| \sum_{j=M+1}^{\infty} \mu_j \langle g_j, \psi_n \rangle_k t^{\ell} f_j^{(\ell)}(t) \Big|^2$$

$$+ \sum_{n=N+1}^{\infty} |\lambda_n|^2 \max \sup \Big| \sum_{j=1}^{\infty} \mu_j \langle g_j, \psi_n \rangle_k t^{\ell} f_j^{(\ell)}(t) \Big|^2$$

$$\le \sum_{j=1}^{\infty} |\mu_j| \sum_{n=1}^{\infty} |\lambda_n|^2 \; |\langle g_j, \psi_n \rangle_k|^2 \; \|f_j\|^2_{F_p} \sum_{j=M+1}^{\infty} |\mu_j|$$

$$+ \sum_{n=N+1}^{\infty} |\lambda_n|^2 \sum_{j=1}^{\infty} |\mu_j| \; |\langle g_j, \psi_n \rangle_k|^2 \sum_{j=1}^{\infty} |\mu_j| \; \|f_j\|^2_{F_p}$$

$$\le c \sum_{j=M+1}^{\infty} |\mu_j| + c \sum_{n=N+1}^{\infty} |\lambda_n|^2 \longrightarrow 0 \quad \text{as} \quad N,M \longrightarrow \infty \quad .$$

Therefore,

$$\sum_{n=1}^{\infty} \lambda_n \Big(\sum_{j=1}^{\infty} \mu_j \langle g_j, \psi_n \rangle_k \, f_j(\cdot) \Big) \Phi_n = 0 \quad \text{in} \quad F(G)_p \, ,$$

and since Φ_n is a basis

$$h_n = \lambda_n \sum_{j=1}^{\infty} \mu_j \langle g_j, \psi_n \rangle_k \, f_j = 0 \quad \text{in} \quad F_p \, .$$

But then $v = \sum_n h_n \otimes \Phi_n$ is 0 in $(F \otimes G)_p$, so $v = 0$ in $F \hat{\otimes} G$, and $\hat{\alpha}$ is one-one.

Finally we show that $\hat{\alpha}$ is onto and is therefore a topological isomorphism. Let u be in $F(G)$. Given p and t, $u(t) = \sum_{n=1}^{\infty} \lambda_n \langle u(t), \psi_n \rangle_k \Phi_n$ in G_p. Then $\langle u(\cdot), \psi_n \rangle_k$ is in $F_k \subset F_p$ and is bounded in F_p. Φ_n is bounded in G_p and since $\{\lambda_n\} \in \ell^1$, $v = \sum_{n=1}^{\infty} \lambda_n \langle u(\cdot), \psi_n \rangle_k \otimes \Phi_n$ defines an element of $(F \otimes G)_p$. Extending $\hat{\alpha}$ to $(F \otimes G)_p$ we see that $\hat{\alpha}(v)(t) = \lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n \langle u(t), \psi_n \rangle_k \Phi_n = u(t)$ in G_p. Hence $\hat{\alpha}(v)(t) = u(t)$ for all $t \in R^+$, so $\hat{\alpha}(v) = u$. Then, $\hat{\alpha}$ is onto, since for any $u \in F(G)$

$$\hat{\alpha}^{-1}(u) \in \bigcap_p (F \otimes G)_p = F \hat{\otimes} G \, ,$$

and by the Open Mapping Theorem it is a topological isomorphism.

We have therefore shown that $\mathcal{E}^{\#} = E^{\#} \hat{\otimes} C^{\infty}(\Sigma) \cong E^{\#}(C^{\infty}(\Sigma))$ and that we may use the norms

$$\|u\|_p' = \max_{0\leq j\leq p} \sup_{t\in R^+} \|t^j u^{(j)}(t)\|_{H^p(\Sigma)}$$

as the defining norms for $\mathcal{E}^{\#}$.

SECTION IV. Estimates for $\| \; \|_{H^p(\hat{\Sigma})}$.

3.6 Theorem: Given any p , there exist constants $c_1, c_2 > 0$ such that, for all $f \in C^\infty(R^n \sim \{0\})$

$$c_1 \|f\|^2_{H^p(\hat{\Sigma})} \le \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|\tilde{f}^{(j)}(t)\|^2_{H^{p-j}(\Sigma)} dt \le c_2 \|f\|^2_{H^p(\hat{\Sigma})}$$

where $\hat{\Sigma} = \{x \in \mathbf{R}^n : \frac{1}{2} < |x| < 2\}$ and $\tilde{f} \in C^\infty(R^+; C^\infty(\Sigma))$ is defined by $[\tilde{f}(t)](\omega) = f(t\omega)$.

Proof: The argument will be in two steps, the first of which requires the following preliminaries. Let U denote the set $\{x \in R^n : x_n = 1\}$, $\hat{U}$ the set $\{x \in R^n : \frac{1}{2} < x_n < 2\}$. Let $V_1, \ldots, V_m$ be coordinate neighborhoods for Σ , with C^∞ coordinate maps $\psi_i : V_i \to U (\cong R^{n-1})$. Let μ_i , $i = 1, \ldots, m$ be a C^∞ partition of unity subordinate to V_i and define $W_i = V_i \times (\frac{1}{2}, 2)$. Now, extend μ_i to $\hat{\mu}_i : W_i \to R$, homogeneous of degree 0 , and extend ψ_i to $\hat{\psi}_i : W_i \to \hat{U}$ by $\hat{\psi}_i(t\omega) = (\psi_{i,1}(\omega), \ldots, \psi_{i,n-1}(\omega), t)$ where $\omega \in \Sigma$, $\frac{1}{2} < t < 2$, and $\psi_{i,j}$ is the jth component of function of ψ_i , $1 \le i \le m$, $1 \le j \le n-1$. Note that if we write $x = (x', t)$, $x' \in R^{n-1}$, then $\hat{\psi}_i^{-1}(x) = t\, \psi_i^{-1}(x', 1)$.

3.7 Lemma: Given p , there exist positive constants c_1 and c_2 such that, for any $f \in C^\infty(R^n \sim \{0\})$

$$c_1 \|f\|^2_{H^p(\hat{\Sigma})} \le \sum_{i=1}^{m} \|\hat{\mu}_i\, f \circ \hat{\psi}_i^{-1}\|^2_{H^p(\hat{U})} \le c_2 \|f\|^2_{H^p(\hat{\Sigma})} .$$

Proof: $f = \sum_{i=1}^{m} \hat{\mu}_i f$ on $\hat{\Sigma}$, so $\|f\|^2_{H^p(\hat{\Sigma})} \leq m \sum_{i=1}^{m} \|\hat{\mu}_i f\|_{H^p(W_i)}$. Furthermore, since $\hat{\mu}_i$ can be extended to a C^∞ function on all of $R^n \sim \{0\}$, and W_i has compact closure in $R^n \sim \{0\}$, $\hat{\mu}_i$ and all its partial derivatives are bounded on W_i. Thus, by Leibnitz's formula, $\exists\ C' > 0$, independent of f, such that $\|\hat{\mu}_i f\|^2_{H^p(W_i)} \leq C'\|f\|^2_{H^p(\hat{\Sigma})}$. Hence it suffices to show the equivalence of $\|\hat{\mu}_i f\|^2_{H^p(W_i)}$ and $\|\hat{\mu}_i f \circ \hat{\psi}_i^{-1}\|^2_{H^p(\hat{U})}$.

Note that, as with $\hat{\mu}_i$, $\hat{\psi}_i^{-1}$ can be extended to a C^∞ function on $\{x \in R^n\colon x_n > 0\}$, and so the functions $\hat{\psi}^{-1}_{i,j}$ and their partial derivatives are bounded on the set $\pi_j(\psi_i(\hat{\Sigma}))$ since it also has compact closure. The proof of the lemma now follows by induction on p. For convenience we fix i and denote $\hat{\mu}_i f$ by g, $\hat{\psi}_i^{-1}$ by α, and W_i by W.

$$p = 0\colon \|g \circ \alpha\|^2_{H^0(\hat{U})} = \int_{\hat{U}} |g \circ \alpha(y)|^2\, dy = \int_W |g(x)|^2 \frac{dx}{|J_\alpha(\alpha^{-1}(x))|}$$

$$\leq c \int_W |g(x)|^2\, dx = c\, \|g\|^2_{H^0(W)} .$$

The last inequality follows from the fact that the Jacobian J_α does not change sign on $\{x \in R^n\colon x_n > 0\}$ so it is bounded away from zero in $\hat{U}$. By the same argument $\|g\|^2_{H^0(W)} = \|g \circ \alpha \circ \alpha^{-1}\|^2_{H^0(W)} \leq c\, \|g \circ \alpha\|^2_{H^0(\hat{U})}$.

Suppose the lemma has been proved for $p \leq m$. An easy argument shows that the norm $\|f\|^2_{H^{m+1}(\hat{U})}$ is equivalent to the sum

$$\|f\|^2_{H^m(\hat{U})} + \sum_{j=1}^{n} \left\|\frac{\partial f}{\partial y_j}\right\|^2_{H^m(\hat{U})} \text{ , so}$$

$$\|g \circ \alpha\|^2_{H^{m+1}(\hat{U})} \le c \left[\|g \circ \alpha\|^2_{H^m(\hat{U})} + \sum_{j=1}^{n} \left\|\frac{\partial g \circ \alpha}{\partial y_j}\right\|^2_{H^m(\hat{U})}\right]$$

$$\le c' \|g\|^2_{H^m(W)} + c \sum_{j=1}^{n} \left\|\frac{\partial g \circ \alpha}{\partial y_j}\right\|^2_{H^m(\hat{U})} \ .$$

Now, since α_i is bounded along with all of its derivatives, an application of the chain rule yields

$$\left\|\frac{\partial g \circ \alpha}{\partial y_j}\right\|^2_{H^m(\hat{U})} \le c \sum_{i=1}^{n} \left\|\frac{\partial g}{\partial x_i} \circ \alpha\right\|^2_{H^m(\hat{U})} \ .$$ By the induction step,

$$\left\|\frac{\partial g}{\partial x_i} \circ \alpha\right\|_{H^m(\hat{U})} \le c \left\|\frac{\partial g}{\partial x_i}\right\|_{H^m(W)} \text{ , so we have}$$

$$\|g \circ \alpha\|^2_{H^{m+1}(\hat{U})} \le c'' \left(\|g\|^2_{H^m(W)} + \sum_{j=1}^{n} \left\|\frac{\partial g}{\partial x_j}\right\|^2_{H^m(W)}\right) \le c''' \|g\|^2_{H^{m+1}(W)} \ .$$

As before, the reverse inequality follows from this one, and the lemma is proved. Now we are left to show the equivalence of

$$\sum_{i=1}^{m} \|\hat{\mu}_i f \circ \hat{\psi}_i^{-1}\|^2_{H^p(\hat{U})} \quad \text{and} \quad \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|\tilde{f}^{(j)}(t)\|^2_{H^{p-j}(\Sigma)}\, dt \ .$$

Suppose $|\alpha| \le p$ and $\alpha_n = j$, $0 \le j \le p$. Then if $\epsilon^n = (0,0,\ldots,1)$,

(5) $$\sum_{i=1}^{m} \int_{\hat{U}} |D^{\alpha}(\hat{\mu}_i f \circ \hat{\psi}_i^{-1})(x',t)|^2 \, dx' \, dt$$

$$= \sum_{i=1}^{m} \int_{\frac{1}{2}}^{2} \int_{U} |D^{\alpha - j\epsilon^n}\left(\frac{\partial}{\partial x_n}\right)^j \hat{\mu}_i f \circ \hat{\psi}_i^{-1}(x',t)|^2 \, dx' \, dt \quad .$$

If we define $[(\mu_i f \circ \psi_i^{-1})^{\wedge}(t)](x',1) = \hat{\mu}_i f \circ \hat{\psi}_i^{-1}(x',t)$, the expressions in (5) are equal to

$$\sum_{i=1}^{m} \int_{\frac{1}{2}}^{2} \int_{U} |D^{\alpha - j\epsilon^n}\left[(\mu_i f \circ \psi_i^{-1})^{\wedge(j)}(t)\right](x',1)|^2 \, dx' \, dt \; , \quad \text{so}$$

$$\sum_{i=1}^{m} \|\hat{\mu}_i f \circ \hat{\psi}_i^{-1}\|_{H^p(\hat{U})} = \sum_{i=1}^{m} \sum_{j=0}^{p} \sum_{\substack{|\alpha| \leq p \\ \alpha_n = j}} \int_{\frac{1}{2}}^{2} \int_{U} |D^{\alpha - j\epsilon^n}\left[(\mu_i f \circ \psi_i^{-1})^{\wedge(j)}(t)\right](x',1)|^2 \, dx' \, dt$$

$$= \sum_{i=1}^{m} \sum_{j=0}^{p} \int_{\frac{1}{2}}^{2} \|(\mu_i f \circ \psi_i^{-1})^{\wedge(j)}(t)\|^2_{H^{p-j}(U)} \, dt \quad .$$

By definition, $\|\tilde{f}^{(j)}(t)\|^2_{H^{p-j}(\Sigma)} = \sum_{i=1}^{m} \|\mu_i[\tilde{f}^{(j)}(t)] \circ \psi_i^{-1}\|^2_{H^{p-j}(U)}$, so to complete the proof it suffices to show that $(\mu_i f \circ \psi_i^{-1})^{\wedge(j)}(t)(x',1) = \mu_i[\tilde{f}^{(j)}(t)] \circ \psi_i^{-1}(x',1)$. We employ induction on j .

$j = 0$: Since $\hat{\mu}_i$ is homogeneous of degree 0 , and $\hat{\psi}_i^{-1}(x',t) = t\,\psi_i^{-1}(x',1)$, we have

$$\begin{aligned}(\mu_i f \circ \psi_i^{-1})^{\wedge}(t)(x',1) &= \hat{\mu}_i f \circ \hat{\psi}_i^{-1}(x',t)\\ &= \hat{\mu}_i(t\psi_i^{-1}(x',1))\ f(t\psi_i^{-1}(x',1))\\ &= \mu_i(\psi_i^{-1}(x',1))[\tilde{f}(t)](\psi_i^{-1}(x',1))\\ &= \mu_i[\tilde{f}(t)] \circ \psi_i^{-1}(x',1) \quad .\end{aligned}$$

Now assume equality has been shown for $j < m + 1$. Since $[\tilde{f}^{(j)}(t)](\omega) = \left(\frac{\partial}{\partial t}\right)^j f(t\omega)$,

$$\begin{aligned}\mu_i[\tilde{f}^{(j)}(t)] \circ \psi_i^{-1}(x',1) &= \hat{\mu}_i(t\psi_i^{-1}(x',1))\left(\frac{\partial}{\partial t}\right)^j f(t\psi_i^{-1}(x',1))\\ &= \hat{\mu}_i\left[\left(\frac{\partial}{\partial t}\right)^j f\right] \circ \hat{\psi}_i^{-1}(x',t) \quad .\end{aligned}$$

On the other hand, $(\mu_i f \circ \psi_i^{-1})^{\wedge(j)}(t)(x',1) = \left(\frac{\partial}{\partial x_n}\right)^j \hat{\mu}_i f \circ \hat{\psi}_i^{-1}(x',t)$,

so we are left only to show

$$\frac{\partial}{\partial x_n}\left[\hat{\mu}_i\left[\left(\frac{\partial}{\partial t}\right)^m f\right] \circ \hat{\psi}_i^{-1}\right](x',t) = \hat{\mu}_i\left[\left(\frac{\partial}{\partial t}\right)^{m+1} f\right] \circ \hat{\psi}_i^{-1}(x',t) \quad .$$

$$(6) \qquad \frac{\partial}{\partial x_n}\left[\hat{\mu}_i\left[\left(\frac{\partial}{\partial t}\right)^m f\right] \circ \hat{\psi}_i^{-1}\right](x',t)$$

$$= \left(\frac{\partial}{\partial t}\right)^m f(\hat{\psi}_i^{-1}(x',t)) \sum_{q=1}^{n} \frac{\partial\hat{\mu}_i}{\partial x_q}(\hat{\psi}_i^{-1}(x',t))\psi_{i,q}^{-1}(x',1)$$

$$+ \hat{\mu}_i(\hat{\psi}_i^{-1}(x',t)) \sum_{q=1}^{n} \frac{\partial}{\partial x_q}\left[\left(\frac{\partial}{\partial t}\right)^m f\right](\hat{\psi}_i^{-1}(x',t))\psi_{i,q}^{-1}(x',1) \quad .$$

If we define $\omega_q(x) = \frac{x_q}{\|x\|}$, then

$$\frac{\partial}{\partial t}\, g(x) = \sum_{q=1}^{n} \frac{\partial g}{\partial x_q}(x)\, \omega_q(x) \quad .$$

Finally, since $\omega_q(\hat{\psi}_i^{-1}(x',t)) = \psi_{i,q}^{-1}(x',1)$, (6) can be recognized as

$$\left(\frac{\partial}{\partial t}\right)^m f(\hat{\psi}_i^{-1}(x',t)) \;\; \frac{\partial\hat{\mu}_i}{\partial t}\left(t\hat{\psi}_i^{-1}(x',1)\right) + \hat{\mu}_i(\hat{\psi}_i^{-1}(x',t))\left(\frac{\partial}{\partial t}\right)^{m+1} f(\hat{\psi}_i^{-1}(x',t))$$

$$= \hat{\mu}_i \left[\left(\frac{\partial}{\partial t}\right)^{m+1} f \right] \circ \hat{\psi}_i^{-1}(x',t)$$

since $\hat{\mu}_i$ is homogeneous of degree 0 . Thus the proof is complete.

SECTION V: Derivatives of the mean convolution.

For $f \in C_B^\infty$ and $g \in \mathcal{A}.\mathcal{P}.$ we have defined $R_T(x) = \frac{1}{2T} \int_{-T}^{T} f(x-t)g(t)dt =$ $f * g_T$, where

$$g_T(t) = \begin{cases} \frac{1}{2T}\, g(t) \ , & |t| \leq T \\ 0 \ , & |t| > T \end{cases}$$

In order to show that $R_T^{(j)}(x) = f^{(j)} * g_T(x)$, we need first to prove that

$$\lim_{h \to 0} \frac{1}{2T} \int_{-T}^{T} \Big[\frac{f(x+h-t)-f(x-t)}{h} - f'(x-t)\Big] g(t)dt = 0 \ .$$

Now

$$\frac{f(x+h-t)-f(x-t)}{h} - f'(x-t) = \int_0^1 f'(x-t+\theta h) - f'(x-t)d\theta$$

and, since f' is uniformly continuous and g is bounded, given $\epsilon > 0$ $\exists$ $c > 0$ such that, if $h < c$

$$\sup_{\substack{t \in R \\ 0 \leq \theta \leq 1}} |f'(x-t+\theta h) - f'(x-t)| < \frac{\epsilon}{\sup_t |g(t)|} \ .$$

Then, for all $h < c$

$$\frac{1}{2T} \int_{-T}^{T} \Big[\frac{f(x+h-t)-f(x-t)}{h} - f'(x-t)\Big] g(t)dt$$

$$\leq \sup_{t \in R} \Big|\frac{f(x-h+t)-f(x-t)}{h} - f'(x-t)\Big| g(t)$$

$$\leq \sup_t \int_0^1 |f'(x-t+\theta h) - f'(x-t)|d\theta \ |g(t)| < \epsilon \ .$$

Thus $R_T'(x) = f' * g_T(x)$. The induction step is identical since $f^{(j)}$ is uniformly continuous for all j . Hence $R_T^{(j)}(x) = f^{(j)} * g_T(x)$, and since the estimates above were all independent of T ,

$$(f \underset{M}{*} g)^{(j)}(x) = f^{(j)} \underset{M}{*} g(x) \ .$$

BIBLIOGRAPHY

[1] A.P. Calderon and A. Zygmund, Singular integral operators and differential equations, Amer. J. Math., 79(1957), 289-309.

[2] C. Corduneanu, Almost Periodic Functions, Interscience Publishers, New York,(1968).

[3] I.M. Gelfand and N.Ya. Vilenkin,Generalized Functions: Applications of Harmonic Analysis, Vol. 4, Academic Press, (1964).

[4] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Memoirs Amer. Math. Soc., Providence, R.I., (1955).

[5] L. Hormander, Estimates for translation invariant operators in L^p spaces, Acta Math., 104(1960), 93-139.

[6] Y. Katznelson, An Introduction to Harmonic Analysis, John Wiley & Sons, Inc., New York, (1968).

[7] F. Treves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, (1967).

[8] K. Yosida, Functional Analysis, Springer-Verlag, (1966).

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