Conformal Geometry of Plane Domains
and
Holomorphic Iterated Function Systems

by

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Abstract

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We select a sequence of holomorphic functions from a hyperbolic domain Ω into a subdomain X. Consider the backward iterated function system corresponding to this sequence. By Montel's theorem, this system is a normal family. Therefore, it does have a set of limit functions, which we call the accumulation points of this system. The accumulation points are either open maps from Ω into X or constants. The constants can be inside X or on its boundary.

Suppose Ω is the unit disk Δ. Lorentzen and Gill showed that if X is relatively compact in Δ then every iterated function system has a unique accumulation point which is a constant inside X. In other words, they showed that relative non-compactness of X is necessary in order to have a boundary point as the accumulation point of an iterated function system. Beardon, Carne, Minda and Ng (see [2])

defined the notion of hyperbolic Bloch domain. These domains can be non-compact but satisfy a certain condition (see section 2.1). Keen and Lakic showed that if X does not have this property and c is a boundary point of X, we can find an iterated function system with the constant c as a limit function.

Our main result is that if c is a boundary point of a non-relatively compact subdomain of Δ, there always exists an iterated function system with the constant c as a limit function. In other words, we show that relative non-compactness of X in Δ is a sufficient condition to have c as a limit function.

In [10], Keen and Lakic defined new densities that generalize the hyperbolic density for a domain. One is a generalization of the Kobayashi density and the other is a generalization of Caratheodory density. We show that for a large class of domains Ω, with certain property that we define in chapter 5, the hyperbolic density on a hyperbolic domain X is equal to the generalized Kobayashi density. As a result, if X is a Kobayashi-Lipschitz subdomain of Ω it is a Caratheodory-Lipschitz subdomain as well.

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I would like to warmly thank my mother, to whom this thesis is dedicated, for all the support and encouragement she has given me. She has always been my endless source of inspiration. It also gives me great pleasure to thank my sisters and my brother for their encouragement.

Last but not least, I would like to express my deepest appreciation and gratitude to my deceased father, who was my first and greatest teacher and showed me the joy of learning and the difference between knowing and not knowing.

To My Mother

Contents

Chapter 1

Basic Facts

In this chapter we state basic facts and theorems in complex analysis which are used throughout this thesis. All of them are standard materials that can be found in a complex analysis or Riemann surface textbook.

1.1 The Schwarz Lemma

Theorem 1.1.1 *(The Schwarz Lemma) If* $f : \Delta \to \Delta$ *is holomorphic and* $f(0) = 0$*, then* $|f(z)| \leq |z|$ *and* $|f'(0)| \leq 1$*. Equality holds if and only if* f *is a rotation about the origin.*

Theorem 1.1.2 *(Schwarz-Pick Lemma) If* $f : \Delta \to \Delta$ *is holomorphic and* ρ *is the hyperbolic metric density on the unit disk then*

$$\rho(f(t)).|f'(t)| \leq \rho(t) \tag{1.1.1}$$

and

$$\rho(f(t), f(s)) \leq \rho(t, s). \tag{1.1.2}$$

These inequalities mean that f is both an infinitesimal and a global contraction with respect to hyperbolic metric.

1.2 The Riemann Mapping Theorem

Theorem 1.2.1 *(The Riemann mapping Theorem) If X is a simply connected proper subdomain of the plane then it is conformally isomorphic to the unit disk.*

1.3 Riemann Surfaces and the Uniformization Theorem

Definition 1.3.1 *A Riemann Surface S is a connected complex manifold of dimension 1. In other words, S is a connected Hausdorff space, locally homeomorphic to $\mathbb{R}^2$, with a maximal set of charts $\{U_\alpha, \varphi_\alpha\}$ (i.e. $\{U_\alpha\}$ is an open covering of S and each $\varphi_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism from U_α onto its image) such that all the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are holomorphic whenever $U_\alpha \cap U_\beta$ is nonempty.*

Theorem 1.3.1 *(The Uniformization Theorem)*

Let S be a non-compact Riemann surface. The universal covering space of S, denoted by $\widetilde{S}$, is conformally isomorphic to either the complex plane $\mathbb{C}$ or the unit disk Δ.

Definition 1.3.2 *If a Riemann surface admits the unit disk Δ as the universal covering, it is called a hyperbolic surface.*

Theorem 1.3.2 *Suppose f is a holomorphic map from a hyperbolic surface S_1 into a hyperbolic surface S_2 with $f(a) = b$. Let $\pi_1 : \Delta \to S_1$ and $\pi_2 : \Delta \to S_2$ be holomorphic covering maps with $\pi_1(0) = a$ and $\pi_2(0) = b$. Then f lifts to a holomorphic self map $\tilde{f}$ of the unit such that $\tilde{f}(0) = 0$ and for every t in Δ we have*

$$(f \circ \pi_1)(t) = (\pi_2 \circ \tilde{f})(t)$$

Theorem 1.3.3 *Suppose π is a universal holomorphic covering map from the unit disk onto Ω. Let w_1 and w_2 be in Ω and let t_1 be a preimage of w_1. Then there exists a preimage t_2 of w_2 such that*

$$\rho(t_1, t_2) = \rho_\Omega(w_1, w_2)$$

(For the definition of ρ_Ω, see chapter 2)

Definition 1.3.3 *The automorphism group of a Riemann surface S is the group of all conformal isomorphisms from S onto S. This group is denoted by Aut(S).*

Theorem 1.3.4 *Let Δ be the unit disk and $\mathbb{H}$ be the upper half plane then*

$$Aut(\Delta) = \{f(z) = \lambda\frac{z-\alpha}{1-\overline{\alpha}z} \mid \alpha \in \Delta, \lambda \in \mathbb{C} \; with \; |\lambda| = 1\}$$

and

$$Aut(\mathbb{H}) = \{f(z) = \frac{ax+b}{cz+d} \mid a, b, c, d \in \mathbb{R} \; with \; ad - bc = 1\}.$$

1.4 Picard's Theorems

Theorem 1.4.1 *(Picard's Little Theorem)*

An entire function which omits two values must be a constant function.

Theorem 1.4.2 *(Picard's Great Theorem)*

Let f be an analytic function with an essential singularity at $z = c$. Then in any neighborhood of c, f realizes any complex number with one possible exception.

1.5 Normal Families and Montel's Theorem

Definition 1.5.1 *A family $\mathfrak{F}$ of holomorphic functions defined on a domain is called a Normal Family if every sequence $\{f_n\}$ in $\mathfrak{F}$*

contains a locally uniformly convergent subsequence $\{f_{n_j}\}$ *or a subsequence which tends locally uniformly to infinity.*

Theorem 1.5.1 *(Montel's Theorem) Let* $\mathfrak{F}$ *be a family of holomorphic maps defined on a domain and suppose there are two points in the plane which are missed by every map in* $\mathfrak{F}$*. Then* $\mathfrak{F}$ *is a normal family.*

Theorem 1.5.2 *(Generalization of Montel's Theorem) If* S_1 *is a Riemann surface and* S_2 *is a hyperbolic Riemann Surface then* $\mathcal{H}ol(S_1, S_2)$ *is a normal family.*

Chapter 2

Conformal Geometry in the Plane

2.1 Hyperbolic Density

Remark. In this section we deal with hyperbolic domains. These are domains with at least two boundary points. Therefore, they admit the unit disk as a universal covering space such that the projection map is holomorphic.

Definition 2.1.1 *The hyperbolic density on the unit disk* Δ *is defined as*

$$\rho(z) = \frac{1}{1-|z|^2} \qquad for \quad z \ \in \ \Delta.$$

Theorem 2.1.1 *The hyperbolic density is invariant under every map of* $Aut(\Delta)$. *That is* f *is in* $Aut(\Delta)$, *then*

$$\rho(f(t))|f'(t)| = \rho(t).$$

Definition 2.1.2 *The ρ-distance between two points z and w is defined as*

$$\rho(z, w) = \inf \int_{\gamma} \rho(t)|dt| \tag{2.1.1}$$

where the infimum is over all paths γ in Δ joining z to w.

Remark. The ρ-distance is a complete metric on the unit disk Δ. It is called the *hyperbolic* or *Poincaré* metric on the unit disk. Moreover, up to multiplication by a positive number, the hyperbolic metric is the unique Riemannian metric on the unit disk with a constant negative curvature.

Definition 2.1.3 *The hyperbolic density on a hyperbolic domain Ω is defined as*

$$\rho_{\Omega}(w) = \frac{\rho(t)}{|\pi'(t)|} ,$$

where ρ is the hyperbolic density on the unit disk and π is a holomorphic covering map from the unit disk onto Ω with $\pi(t) = w$.

Example 1. Consider the upper half plane $\mathbb{H}$ and the following covering map from the unit disk onto the upper half plane

$$\pi(z) = i\frac{1+z}{1-z}.$$

By computation, it follows that

$$\rho_{\mathbb{H}}(w) = \frac{1}{2Im(w)} \ .$$

Example 2. Let $\Delta^* = \Delta \setminus \{0\}$ be the unit disk punctured at the origin and let

$$\pi(z) = \exp(-\frac{1+z}{1-z})$$

be a covering map from the unit disk onto the punctured disk. By computation, we have

$$\rho_{\Delta^*}(w) = \frac{1}{2|w|\ln(|\frac{1}{w}|)}.$$

Example 3. Consider the round annulus $A = \{z : r < |z| < 1\}$, where r is a non-negative real number smaller than 1. By computation, we have

$$\rho_A(w) = \frac{\pi}{2|w|l\sin(\frac{\pi}{l}\ln\frac{1}{|w|})} \ ,$$

where $l = \ln\frac{1}{r}$.

In the same way as for the unit disk, we can define the distance between two points in a hyperbolic domain.

Definition 2.1.4 *Let Ω be a hyperbolic domain. The ρ_Ω-distance* *between two points* z *and* w *in* Ω *is defined as*

$$\rho_\Omega(z,w) = \inf \int_\gamma \rho_\Omega(t)|dt| \tag{2.1.2}$$

where the infimum is over all paths γ *in* Ω *joining* z *to* w.

REMARK. The ρ_Ω-distance defined above is a complete metric on Ω. It is called the *hyperbolic* or *Poincaré* metric on Ω.

Theorem 2.1.2 *(Generalized Schwarz-Pick lemma) Let* Ω *and* X *be two arbitrary hyperbolic domains and let* $f : \Omega \to X$ *be a holomorphic map. Then*

(i) $\rho_X(f(z))|f'(z)| \leq \rho_\Omega(z)$ *for every* $z \in \Omega$

and

(ii) $\rho_X(f(z), f(w)) \leq \rho_\Omega(z,w)$ *for every pair* $z, w \in \Omega$.

Corollary 2.1.1 *If* π *is a holomorphic covering from a hyperbolic domain* Ω *onto a hyperbolic domain* X, *then it is an infinitesimal isometry. That is*

$$\rho_X(\pi(z))|\pi'(z)| = \rho_\Omega(z)$$

Corollary 2.1.2 *Let* f *be a holomorphic map from a hyperbolic domain* Ω *into a subdomain* X *of* Ω. *Then*

$$\rho_\Omega(f(z))|f'(z)| \leq \rho_\Omega(z),$$

for every $z \in \Omega$.

$$\rho_X(f(z))|f'(z)| \leq \rho_X(z),$$

for every $z{\in}X$.

$$\rho_\Omega(f(z), f(w)) \leq \rho_\Omega(z, w),$$

for every $z, w{\in}\Omega$.

$$\rho_X(f(z), f(w)) \leq \rho_X(z, w),$$

for every $z, w{\in}X$.

Definition 2.1.5 *Let* X *be a subdomain of the hyperbolic domain* Ω. *Then*

(i) The infinitesimal Ω*-contraction constant of a holomorphic map* $f : \Omega \to X$ *is defined as*

$$m_\Omega(f) = \sup_{z\in\Omega} \frac{\rho_\Omega(f(z))|f'(z)|}{\rho_\Omega(z)}.$$

(ii) The uniform Ω*-contraction constant is defined as*

$$m_\Omega = \sup_{f\in\mathcal{H}ol(\Omega,X)} m_\Omega(f).$$

(iii)The infinitesimal X-contraction constant of a holomorphic map $f : \Omega \to X$ is defined as

$$m_X(f) = \sup_{z\in X} \frac{\rho_X(f(z))|f'(z)|}{\rho_X(z)}.$$

(iv) The uniform X-contraction constant is defined as

$$m_X = \sup_{f\in \mathcal{H}ol(\Omega,X)} m_X(f).$$

Remark. By using the Schwarz-Pick lemma, it is easy to deduce that all of these constants are less than or equal to 1.

Theorem 2.1.3 *If X is a proper subdomain of a hyperbolic domain Ω then we have*

(i) $\rho_\Omega(z) < \rho_X(z)$ for every $z\in X$

(ii) $\rho_\Omega(z, w) < \rho_X(z, w)$ for every pair $z, w\in X$.

(In both parts, the inequalities are strict.)

Proof. We prove part(i). Part(ii) is similar.

Let π_Ω be a covering map from Δ onto Ω with $\pi_\Omega(0) = z$ and let π_X be a covering map from Δ onto X with $\pi_X(0) = z$. By Theorem 1.3.2, the inclusion map $i : X \to \Omega$ lifts to a map $f : \Delta \to \Delta$ such that $f(0) = 0$ and $\pi_\Omega \circ f = i \circ \pi_X$. After differentiating both sides at 0 we get

$$\pi'_\Omega(0) \circ f'(0) = \pi'_X(0).$$

Now, if $\rho_\Omega(z) = \rho_X(z)$ then $|\pi'_\Omega(0)| = |\pi'_X(0)|$. Consequently, $|f'(0)| = 1$. By the Schwarz lemma, f must be a rigid rotation about the origin. In particular, f is onto. But, the inclusion map is not onto, which is a contradiction. $\square$

Definition 2.1.6 *Let X be a subdomain of a domain Ω. The infinitesimal contraction constant is defined as*

$$m(X, \Omega) = \sup_{z \in X} \frac{\rho_\Omega(z)}{\rho_X(z)}.$$

Since the inclusion map is a contraction , $m(X, \Omega) \leq 1$.

Definition 2.1.7 *A subdomain X of a domain Ω is a Lipschitz or ρ-Lip subdomain of Ω if*

$$m(X, \Omega) < 1.$$

Definition 2.1.8 *A subdomain X of a domain Ω is called relatively compact in Ω if there exits a compact set K such that $X \subset K \subset \Omega$. Equivalently, the ρ_Ω-diameter of X, which is defined as*

$$\sup_{z,w \in X} \rho_\Omega(z, w),$$

is finite.

Theorem 2.1.4 *If X is a relatively compact subdomain of Ω then it is a Lipschitz subdomain of Ω.*

Proof. Let K be a compact subdomain of Ω. Define a function $f : K \longrightarrow \mathbb{R}$ as follows:

$$f(z) = \frac{\rho_\Omega(z)}{\rho_X(z)}$$

for $z \in X$ and

$$f(z) = 0$$

for $z \in K \setminus X$. Since f is continuous and K is compact, f realizes its maximum. As we saw before,

$$\rho_\Omega(z) < \rho_X(z)$$

Consequently,

$$\sup_{z \in X} \frac{\rho_\Omega(z)}{\rho_X(z)} < 1.$$

□

Example. $\Delta^* = \Delta - \{0\}$ is a non-Lipschitz subdomain of Δ, because

$$\frac{\rho_\Delta}{\rho_{\Delta^*}} = \frac{2|z| \ln(|\frac{1}{z}|)}{1 - z^2} \longrightarrow 1,$$

as $|z| \longrightarrow 1$.

Example. The round annulus $A = \{z : r < |z| < 1\}$ is a non-Lipschitz subdomain of Δ, because

$$\frac{\rho_\Delta}{\rho_A} = \frac{2|z|l\sin(\frac{\pi}{l}\ln\frac{1}{|z|})}{\pi(1-z^2)} \to 1,$$

as $|z| \to 1$.

Theorem 2.1.5 *Suppose f is a conformal homeomorphism from the hyperbolic domain Γ onto $f(\Gamma)$. Then U is a Lipschitz subdomain of Γ if and only if $f(U)$ is a Lipschitz subdomain of $f(\Gamma)$.*

Proof. Since f is a conformal homeomorphism from Γ onto $f(\Gamma)$, we have

$$\rho_\Gamma(z) = |f'(z)|\rho_{f(\Gamma)}(f(z))$$

for every z in Γ. Similarly, As $f|_U$ is a conformal homeomorphism from U onto $f(U)$, we have

$$\rho_U(z) = |f'(z)|\rho_{f(U)}(f(z))$$

for every z in U. Therefore,

$$\frac{\rho_\Gamma(z)}{\rho_U(z)} = \frac{\rho_{f(\Gamma)}(t)}{\rho_{f(U)}(t)}$$

where z is in U and $t = f(z)$. Taking the supremum of the both sides we have

$$m(U, \Gamma) = m(f(U), f(\Gamma)).$$

This proves the theorem. □

Definition 2.1.9 *A subdomain X is called a ρ-Bloch subdomain of Ω if the supremum of all radii, measured with respect to ρ_Ω, of subdisks of Ω which are contained in X is finite. Equivalently, if there is a constant K such that for every point z in X there exists $w \in \Omega \setminus X$ with $\rho_\Omega(z, w) \leq K$.*

Beardon, Carne, Minda, and Ng showed that the Lipschitz condition and the Bloch condition are equivalent. Here we prove the equivalence of these two conditions in two steps. First, we prove it for the special case that X is a subdomain of the unit disk. Then, by using covering map, we prove it for the general case.

Lemma 2.1.1 *X is a Lipschitz subdomain of the unit disk Δ if and only if it a Bloch subdomain of Δ.*

Proof. Suppose X is a Bloch subdomain of Δ. Therefoe, there is a constant K such that for every z in X, there is a point w in $\Delta \setminus X$ such that $\rho(z, w) \leq K$. Let T be a conformal automorphism of the unit disk which sends w to 0 and let $T(z) = t$. We have,

$$\rho(0, t) = \rho(z, w) \leq K$$

This shows that t is bounded away from the boundary. On the other hand, we have

$$\frac{\rho(z)}{\rho_{\Delta\setminus\{w\}}(z)} = \frac{\rho(t)}{\rho_{\Delta\setminus\{0\}}(t)}$$

As X is a subdomain of $\Delta \setminus \{w\}$, we have $\rho_{\Delta\setminus\{w\}}(z) \leq \rho_X(z)$. Therefore,

$$\frac{\rho(z)}{\rho_X(z)} \leq \frac{\rho(t)}{\rho_{\Delta\setminus\{0\}}(t)}$$

As t tends to 0, the right side of inequality approaches 0. By using Theorem 2.1.3, $\frac{\rho(z)}{\rho_X(z)}$ is bounded away from 1. In other words, X is a Lipschitz subdomain of Δ.

Now, suppose X is a Lipschitz subdomain of Δ. Let z be an arbitrary point in X and D be a hyperbolic disk contained in X and centered at z with hyperbolic radius r. By a simple computation we know that if D' is a hyperbolic disk centered at the origin with hyperbolic radius r and Euclidean radius R we have $\rho_{D'}(0) = 1/R$ and R tends to 1 as r tends to infinity. Therefore, we have

$$R = \frac{\rho(0)}{\rho_{D'}(0)} = \frac{\rho(z)}{\rho_D(z)} \leq \frac{\rho(z)}{\rho_X(z)}$$

Consequently, $R \leq m(X, \Delta) < 1$. This shows that X is a Bloch subdomain of Δ.

$\square$

Theorem 2.1.6 *([2]) X is a Lipschitz subdomain of a hyperbolic domain Ω if and only if it a Bloch subdomain of Ω.*

PROOF. Let π be a universal holomorphic covering map from the unit disk onto Ω. Let Y be a connected component of $\pi^{-1}(X)$. Therefore, the restriction of π to Y is a holomorphic covering map onto X and we have

$$\frac{\rho_X(z)}{\rho_\Omega(z)} = \frac{\rho_Y(w)}{\rho(w)},$$

where $\pi(w) = z$ and w is in Y. This shows that $m(X, \Omega) = m(Y, \Delta)$. Therefore, X is a Lipschitz subdomain of Ω if and only if Y is a Lipschitz subdomain of Δ.

Suppose X is a Bloch subdomain of Ω. Therefore, there exists a constant K such that for every z in X there exists z' in $\Omega \setminus X$ with $\rho_\Omega(z, z') \leq K$. Let w be a preimage of z in Y. By Theorem 1.3.3, there is w' in Δ such that $\pi(w') = z'$ and $\rho_\Omega(z, z') = \rho(w, w')$. It is clear that w' is not in Y. This shows that Y is a Bloch subdomain of Δ. By Lemma 2.1.1, Y is a Lipschitz subdomain of Δ. Consequently, X is a Lipschitz subdomain of Ω. Now, suppose X is a Lipschitz subdomain of Ω. Therefore Y is a Lipschitz subdomain of Δ. By Lemma 2.1.1, Y is a Bloch subdomain of Δ. Therefore, there exists a constant K such that for every w in Y there exits w' in $\Delta \setminus Y$

with $\rho(w, w') \leq K$. Let z be in X and let w be a preimage of z in Y. Therefore, there exits w' on the boundary of Y such that $\rho(w, w') \leq K$. Let $z' = \pi(w')$. It is clear that z' is in $\Omega \setminus X$. In addition, $\rho_\Omega(z, z') \leq \rho(w, w') \leq K$. This shows that X is a Bloch subdomain of Ω. $\square$

2.2 Kobayashi Density

In the previous section we defined hyperbolic density for a hyperbolic domain as

$$\rho_X(z) = \frac{\rho(t)}{|\pi'(t)|}$$

where π is a holomorphic covering from the unit disk Δ onto X and $\pi(t) = z$. On the other hand, due to the contraction property of holomorphic maps we have

$$\rho_X(z) \leq \frac{\rho(t)}{|f'(t)|},$$

where f is a holomorphic map from the unit disk Δ into X and $f(t) = z$. Therefore, the hyperbolic density can be equivalently defined by using the following formula

$$\rho_X(z) = \inf \frac{\rho(t)}{|f'(t)|},$$

where the infimum is over all holomorphic maps from the unit disk Δ into X and all t in the unit disk such that $f(t) = z$.

This equivalent definition, introduced first by Kobayashi, has some advantages. One of the advantages is that it can be defined for any domain, even non-hyperbolic ones. By pre-composing by appropriate automorphism of the unit disk, we can assume that $t = 0$ in the above definition.

Definition 2.2.1 *The standard Kobayashi density for a plane domain X is defined by*

$$\kappa_X(z) = \inf_{f \in \mathcal{H}ol(\Delta, X), f(0)=z} 1/|f'(0)|,$$

where Δ is the unit disk.

Example. Let $X = \mathbb{C} - \{0\}$ and let $z \in X$. Fix z and then consider the following sequence of functions in $\mathcal{H}ol(\Delta, X)$.

$$f_n(t) = ze^{nLog(t+1)},$$

where $n \in \mathbb{N}$ and Log is the principal branch. As we see, $f_n(0) = z$ and $f_n'(0) = nz$. By the definition of the Kobayashi density we have

$$\kappa_X(z) \leq \frac{\rho_\Delta(0)}{|f_n'(0)|} = \frac{1}{n|z|},$$

for every $n \in \mathbb{N}$. Consequently,

$$\kappa_X(z) = 0$$

The other advantage of Kobayashi's definition is that it can be generalized by using an arbitrary hyperbolic domain instead of using the unit disk as the source. This generalization was made by Keen and Lakic as follows. Let's take a closer look at the standard Kobayashi density defined on a domain X. Every holomorphic map from the unit disk into X pushes the hyperbolic density of the unit disk forward onto X. The Kobayashi density is the infimum push-forward. So, instead of the unit disk we can put any domain admitting hyperbolic metric as the source. Therefore, we have

Definition 2.2.2 *Let Ω be a hyperbolic plane domain and let X be a plane domain. The generalized Kobayashi density for z in X is defined by*

$$\kappa_X^{\Omega}(z) = \inf \frac{\rho_\Omega(w)}{|f'(w)|},$$

where ρ_Ω is the hyperbolic density on Ω and the infimum is over all holomorphic functions f from Ω to X and all points w in Ω such that $f(w) = z$.

By integrating the generalized Kobayashi density on X, we can define the distance between two points in X.

Definition 2.2.3 *The generalized κ-distance between two points z*

and w is defined as

$$\kappa_X^\Omega(z,w) = \inf \int_\gamma \kappa_X^\Omega(t)|dt| \tag{2.2.1}$$

where the infimum is over all paths γ in X joining z to w.

REMARK. If X is a hyperbolic domain, the generalized κ-distance defined above is a complete metric on X (see [10] for the proof). Note that unlike the usual metrics, the generalized κ-distance between two points is allowed to be infinite.

The generalized Kobayashi density is greater than or equal to the hyperbolic density.

Theorem 2.2.1 *If X is a hyperbolic domain then*

$$\rho_X(z) \leq \kappa_X^\Omega(z),$$

for every $z \in X$.

PROOF. As we saw in the previous section, if f is a holomorphic map from a hyperbolic domain Ω to hyperbolic domain X then it is a contraction. Therefore if $f(w) = z$, we have

$$\rho_X(z) \leq \frac{\rho_\Omega(w)}{|f'(w)|}$$

Taking the infimum we have

$$\rho_X(z) \leq \kappa_X^{\Omega}(z).$$

□

As we mentioned before, if the source is the unit disk the hyperbolic density and the Kobayashi density coincide on a hyperbolic domain X. In fact, this identity remains valid if the source is a covering space for X.

Theorem 2.2.2 *If Ω is a covering space for a hyperbolic domain X, then for every z in X we have*

$$\rho_X(z) = \kappa_X^{\Omega}(z).$$

PROOF. By proposition 2.2.1

$$\rho_X(z) \leq \kappa_X^{\Omega}(z).$$

So, in order to show the equality it is sufficient to show the inequality in the opposite direction. Because Ω is a covering space for X, there is a holomorphic covering π from Ω onto X. Since π is surjective and is an infinitesimal isometry there is w in Ω such that

$$\rho_X(z) = \frac{\rho_{\Omega}(w)}{|\pi'(w)|},$$

where $\pi(w) = z$. On the other hand, by the definition of the Kobayashi density we have

$$\kappa_X^{\Omega}(z) \leq \frac{\rho_{\Omega}(w)}{|\pi'(w)|}.$$

So we have

$$\kappa_X^{\Omega}(z) \leq \rho_X(z)$$

And consequently, for every z in X,

$$\rho_X(z) = \kappa_X^{\Omega}(z).$$

$\square$

Corollary 2.2.1 *If X is a hyperbolic domain, then for every $z \in X$ we have*

$$\kappa_X^{X}(z) = \rho_X(z).$$

If we replace the source domain by a conformally isomorphic domain, the generalized Kobayashi density doesn't change.

Theorem 2.2.3 *Let Ω_1, Ω_2 be two hyperbolic domains. If Ω_1 and Ω_2 are conformally isomorphic, then*

$$\kappa_X^{\Omega_1}(z) = \kappa_X^{\Omega_2}(z),$$

for every $z \in X$.

Proof.

Let $\varphi : \Omega_1 \to \Omega_2$ be a conformal homeomorphism.

By definition,

$$\kappa_X^{\Omega_2}(z) = \inf \frac{\rho_{\Omega_2}(t)}{|f'(t)|},$$

where ρ_{Ω_2} is the hyperbolic density on Ω_2 and the infimum is over all holomorphic functions f from Ω_2 to X and all points t in Ω_2 such that $f(t) = z$. For every t there exists ω in Ω_1 with $\varphi(\omega) = t$. As φ is a conformal homeomorphism, we have

$$\rho_{\Omega_1}(\omega) = \rho_{\Omega_2}(t)|\varphi'(\omega)|.$$

By the chain rule we have

$$\varphi'(\omega)f'(t) = (f \circ \varphi)'(\omega).$$

Therefore,

$$\frac{\rho_{\Omega_2}(t)}{|f'(t)|} = \frac{\rho_{\Omega_1}(\omega)}{|(f \circ \varphi)'(\omega)|}.$$

By taking the infimum of both sides, we conclude

$$\kappa_X^{\Omega_1}(z) \leq \kappa_X^{\Omega_2}(z).$$

By a similar argument we have

$$\kappa_X^{\Omega_2}(z) \leq \kappa_X^{\Omega_1}(z).$$

Consequently,

$$\kappa_X^{\Omega_1}(z) = \kappa_X^{\Omega_2}(z).$$

□

Corollary 2.2.2 *If X is hyperbolic and Ω is simply connected then*

$$\kappa_X^{\Omega} \equiv \rho_X.$$

Proof. Let z be in X. We have

$$\rho_X(z) = \kappa_X^{\Delta}(z)$$

Since Ω is is a simply connected hyperbolic domain it is conformally isomorphic to the unit disk. Therefore, by theorem 2.2.3 we have

$$\kappa_X^{\Delta}(z) = \kappa_X^{\Omega}(z).$$

And consequently,

$$\kappa_X^{\Omega}(z) = \rho_X(z).$$

□

As we saw in the previous section, every holomorphic map is a contraction with respect to the hyperbolic density. We have a similar property with respect to the generalized Kobayashi density.

Theorem 2.2.4 *Every holomorphic map is a contraction with respect to the corresponding generalized Kobayashi density. In other words, if f is a holomorphic map from a plane domain X into a plane domain Y, we have*

$$\kappa_Y^\Omega(f(z))|f'(z)| \leq \kappa_X^\Omega(z),$$

for every $z \in X$.

Proof. Let r be an arbitrary positive number. By the definition of $\kappa_X^\Omega(z)$, there exists a holomorphic function $g \in \mathcal{H}ol(E, F)$ and a point $\omega \in \Omega$ with $f(\omega) = z$ such that

$$\frac{\rho_\Omega(\omega)}{|g'(\omega)|} \leq \kappa_X^\Omega(z) + r.$$

By the definition we also have

$$\kappa_Y^\Omega((f \circ g)(\omega)) \leq \frac{\rho_\Omega(\omega)}{|(f \circ g)'(\omega)|}.$$

Since by chain rule we have $(fog)'(\omega) = f'(z)g'(\omega)$, we can equivalently write

$$\kappa_Y^\Omega(f(z))|f'(z)| \leq \frac{\rho_\Omega(\omega)}{|g'(\omega)|}.$$

So by the first and the third inequalities, we can conclude that

$$\kappa_Y^\Omega(f(z))|f'(z)| \leq \kappa_X^\Omega(z) + r.$$

As r was an arbitrary positive number we have

$$\kappa_Y^\Omega(f(z))|f'(z)| \leq \kappa_X^\Omega(z).$$

□

Definition 2.2.4 *Let X be a subdomain of Ω. The Kobayashi contraction constant is defined as*

$$m\kappa(X,\Omega) = \sup_{z\in X} \frac{\kappa_\Omega^\Omega(z)}{\kappa_X^\Omega(z)}.$$

Definition 2.2.5 *A subdomain X is a Kobayashi-Lipschitz or κ-Lip subdomain of Ω if $m\kappa(X,\Omega) < 1$.*

Theorem 2.2.5 *If X is a ρ-Lip subdomain of Ω then it is also a κ-Lip subdomain of Ω.*

PROOF. Since $\rho_X(z) \leq \kappa_X^\Omega(z)$ and $\kappa_\Omega^\Omega \equiv \rho_\Omega$, we have $\frac{\kappa_\Omega^\Omega(z)}{\kappa_X^\Omega(z)} \leq \frac{\rho_\Omega(z)}{\rho_X(z)}$, for every $z \in X$. Therefore, the supremum of the left side of the inequality is less than or equal the supremum of the right side of inequality. This proves the theorem. □

Remark. The converse is not necessarily true. Look at the following example. EXAMPLE. Suppose $\Omega = \mathbb{C} - \{-1, 1\}$ and $X = \mathbb{C} - \{-1, 0, 1\}$. Let f be a holomorphic map from Ω into X. By Picard's

Great Theorem, f can not have essential singularities at either -1 or 1. Therefore, f can be extended to a rational map. The complement of Ω has two points. So the image of f can not miss more than two points, unless f is a constant function. Consequently, the only holomorphic maps from Ω into X are constants. This means that $\kappa_X^{\Omega}(z) = \infty$, for every $z \in X$. So, $m\kappa(X, \Omega) = 0 < 1$. This shows X is a κ-Lip subdomain of Ω. On the other hand, the punctured disks about -1 and 1 contain hyperbolic disks of arbitrary large radius. Therefore, X is not a Bloch subdomain of Ω. By Theorem 2.1.6, X is not a ρ-Lipschitz subdomain of Ω.

Theorem 2.2.6 *Let X be a subdomain of the hyperbolic domain Ω. The Kobayashi contraction constant $m\kappa(X, \Omega)$ and the uniform Ω-contraction constant m_Ω are equal. Equivalently, X is a κ-Lip subdomain of Ω if and only if there exists $k < 1$, such that for every $f \in \mathcal{H}ol(E, F)$ and every pair of points $z, w \in \Omega$ we have*

$$\frac{\rho_\Omega(f(z), f(w))}{\rho_\Omega(z, w)} \leq k.$$

Proof. For every $f \in \mathcal{H}ol(\Omega, X)$, we have

$$\rho_\Omega(f(z)) \leq m\kappa(X, \Omega)\kappa_X^{\Omega}(f(z))$$

and

$$\kappa_X^\Omega(f(z))|f'(z)| \leq \kappa_\Omega^\Omega(z).$$

The first inequality is implied by the definition of $m\kappa(X,\Omega)$ and the second is true because any holomorphic map is a contraction with respect to the generalized Kobayashi density (Theorem 2.2.4). By these inequalities and considering the fact that $\kappa_\Omega^\Omega = \rho_\Omega$, we have

$$\frac{\rho_\Omega(f(z))|f'(z)|}{\rho_\Omega(z)} \leq m\kappa(X,\Omega).$$

This is true for every $z \in \Omega$ and every $f \in \mathcal{H}ol(\Omega, X)$. Therefore,

$$m_\Omega \leq m\kappa(X,\Omega).$$

Now we show the reverse inequality.

We have $m_\Omega(f) \leq m_\Omega$, for every $f \in \mathcal{H}ol(\Omega, X)$. Therefore,

$$\frac{\rho_\Omega(f(z))|f'(z)|}{\rho_\Omega(z)} \leq m_\Omega.$$

Equivalently,

$$\rho_\Omega(f(z)) \leq m_\Omega \frac{\rho_\Omega(z)}{|f'(z)|}.$$

Let t be a point in X. Take the supremum of the right side of the last inequality, over all $f \in \mathcal{H}ol(\Omega, X)$ and all $z \in \Omega$ with $f(z) = t$. We conclude that

$$\rho_\Omega(t) \leq m_\Omega \kappa_X^\Omega(t).$$

Equivalently,

$$\frac{\rho_\Omega(t)}{\kappa_X^\Omega(t)} \leq m_\Omega.$$

This is true for every $t \in X$, therefore we have

$$m\kappa(X, \Omega) \leq m_\Omega.$$

□

In the same way as we defined a Bloch subdomain with respect to the hyperbolic metric, we define a Bloch subdomain with respect to the generalized Kobayashi metric. Then we investigate the relation between the Kobayashi-Bloch and the Kobayashi-Lipschitz conditions.

Definition 2.2.6 *A subdomain X of Ω is called a κ-Bloch subdomain of Ω if there exists a constant K such that for every z in X there exists w in $\Omega \setminus X$ with $\kappa_\Omega^\Omega(z, w) \leq K$.*

As we saw before, $\kappa_\Omega^\Omega = \rho_\Omega$. So we have

Theorem 2.2.7 *A subdomain X of Ω is a κ-Bloch subdomain of Ω if and only if it is a ρ-Bloch subdomain of Ω.*

Corollary 2.2.3 *If X is a κ-Bloch subdomain of Ω, then X is a κ-Lipschitz subdomain of Ω.*

Proof. It is easily deduced by using Theorem 2.2.7, Theorem 2.1.6, and Theorem 2.2.5. □

The converse is not necessarily true. Earlier in this section we showed that $X = \mathbb{C} - \{-1, 0, 1\}$ is not a ρ-Lip subdomain of $\Omega = \mathbb{C} - \{-1, 1\}$. Therefore, it is not a κ-Bloch subdomain of Ω. But, X is a κ-Lip subdomain of Ω.

2.3 Caratheodory Density

Definition 2.3.1 *The standard Caratheodory density for a plane domain* Ω *is defined by*

$$c^{\Omega}(\omega) = \sup \rho(f(\omega))|f'(\omega)|,$$

where ρ *is the hyperbolic density on the unit disk* Δ *and the supremum is over all* $f \in \mathcal{H}ol(\Omega, \Delta)$.

Every holomorphic map pulls the hyperbolic density of the unit disk back onto Ω. The Caratheodory density is the supremum of the pullbacks. Therefore, in order to generalize the standard Caratheodory density, instead of using the unit disk we can use another hyperbolic domain as the target.

Definition 2.3.2 *Let X be a hyperbolic plane domain. The generalized Carathodory density for ω in Ω is defined by*

$$c_X^{\Omega}(w) = \sup \rho_X(f(w)|f'(w)|,$$

where ρ_X is the hyperbolic density on X and the supremum is over all $f \in \mathcal{H}ol(\Omega, X)$.

By integrating the generalized Caratheodory density on Ω, we can define the pseudo-distance between two points in Ω.

Definition 2.3.3 *The generalized Caratheodory pseudo-distance between two points z and w in Ω is defined as*

$$c_X^{\Omega}(z, w) = \inf \int_{\gamma} c_X^{\Omega}(t)|dt| \tag{2.3.1}$$

where the infimum is over all paths γ in Ω joining z to w.

Remark. The Caratheodoy pseudo-distance defined above is a pseudo-metric on the domain Ω. It has all the conditions of a metric with the exception that the distance between two distinct points is allowed to be zero.

The generalized Caratheodory density is smaller than or equal to the hyperbolic density.

Theorem 2.3.1 *If* Ω *is a hyperbolic domain then*

$$c_X^{\Omega}(w) \leq \rho_{\Omega}(w),$$

for every $w \in \Omega$.

Proof. As we saw in the previous section, if f is a holomorphic map from a hyperbolic domain Ω to hyperbolic domain X then it is a contraction. Therefore, we have

$$\rho_X(f(w))|f'(w)| \leq \rho_{\Omega}(w).$$

Taking the supremum over all holomorphic maps from Ω into X, we have

$$c_X^{\Omega}(w) \leq \rho_{\Omega}(w).$$

□

Theorem 2.3.2 *If* Ω *is a covering space for* X, *then we have*

$$c_X^{\Omega}(w) = \rho_{\Omega}(w).$$

Proof. Suppose π is a holomorphic covering map from Ω onto X. Since π is a local isometry, for every w in Ω we have

$$\rho_X(\pi(w))|\pi'(w)| = \rho_{\Omega}(w)$$

So, by the definition of the Caratheodory density we have

$$\rho_\Omega(w) \le c_X^\Omega(w)$$

On the other hand by theorem 2.3.1

$$c_X^\Omega(w) \le \rho_\Omega(w)$$

Consequently, we have

$$c_X^\Omega(w) = \rho_\Omega(w).$$

□

Corollary 2.3.1 *If Ω is a simply connected hyperbolic domain, then for every $w \in \Omega$ we have*

$$c_X^\Omega(w) = \rho_\Omega(w).$$

Corollary 2.3.2 *If Ω is a hyperbolic domain, then for every $w \in \Omega$ we have*

$$c_\Omega^\Omega(w) = \rho_\Omega(w).$$

Definition 2.3.4 *Let X be a subdomain of Ω. The Caratheodory contraction constant is defined as*

$$mc(X,\Omega) = \sup_{z \in X} \frac{c_X^\Omega(z)}{c_X^X(z)}.$$

Definition 2.3.5 *A subdomain X is a Caratheodory-Lipschitz or c-Lip subdomain of Ω if*

$$mc(X,\Omega) < 1.$$

By using an argument similar to the one that we made for the Kobayashi density, we can show that every holomorphic map from one domain into another domain is a contraction with respect to the generalized Caratheodory density. More, precisely we have

Theorem 2.3.3 *If f is a holomorphic map from a domain Ω_1 into a domain Ω_2,we have*

$$c_X^{\Omega_2}(f(w))|f'(w)| \leq c_X^{\Omega_1}(w).$$

Theorem 2.3.4 *Let X be a subdomain of the hyperbolic domain Ω. The Caratheodory contraction constant $mc(X,\Omega)$ and the uniform X-contraction constant m_X are equal. Equivalently, X is a c-Lip subdomain of Ω if and only if there exists $k < 1$, such that for every $f \in \mathcal{H}ol(E,F)$ and every pair of points $z, w \in X$ we have*

$$\frac{\rho_X(f(z), f(w))}{\rho_X(z,w)} \leq k.$$

Proof. By using Corollary 2.3.2 and Theorem 2.3.3, for every holomorphic map f from Ω into X we have

$$\rho_X(f(w))|f'(w)| = c_X^X(f(w))|f'(w)| \leq c_X^{\Omega}(w) \leq mc(X,\Omega)\rho_X(w).$$

This shows $m_X \leq mc(X, \Omega)$. On the other hand, for every holomorphic map f from Ω into X we have

$$\rho_X(f(w))|f'(w)| \leq m_X \rho_X(w)$$

Now, by taking supremum over all f we conclude that

$$mc(X, \Omega) \leq m_X.$$

Consequently, $mc(X, \Omega) = m_X$. $\square$

Chapter 3

Iterated Function Systems

3.1 Introduction

Let Ω be a hyperbolic domain. Suppose that the maps $f_1, f_2, f_3, \ldots$ form a sequence of holomorphic maps from the domain Ω into a subdomain $X \subset \Omega$.

Consider the backward compositions

$$F_n = f_1 \circ f_2 \circ \ldots f_{n-1} \circ f_n. \tag{3.1.1}$$

The sequence $\{F_n\}$ is called a *backward iterated function system,* abbreviated to *IFS*. The sequence $\{F_n\}$ is a family of holomorphic maps from Ω into the subdomain $X \subset \Omega$. Since X is a hyperbolic domain, by Montel's theorem, the iterated function system $\{F_n\}$ is a normal family. Therefore, there is a subsequence of the sequence $\{F_n\}$ which converges locally uniformly to a holomorphic map F or to infinity. The limit functions of an *IFS* are called accumulation points; these are either open maps from Ω into X or constants. The constant accumulation points are in $\overline{X}$.

3.2 Definitions and Theorems

Definition 3.2.1 *An iterated function system* $\{F_n\}$ *is called degenerate if it has only constant accumulation points. If the constant accumulation point is unique, the iterated function system is called super degenerate.*

For example, if Ω and X are the unit disk and all the maps in an iterated function system are the same and this common map is not a conformal automorphism of the unit disk, then the iterated function system is super degenerate. This is, in fact, the Denjoy-Wolf Theorem.

Theorem 3.2.1 *(Denjoy-Wolf Theorem) Let* f *be a holomorphic self map of the unit disk* Δ *that is not a conformal automorphism. Then the iterates* $f^{\circ n}$ *of* f *converge locally uniformly in* Δ *to a constant value* c, *where* $|c| \leq 1$.

Definition 3.2.2 *A subdomain* X *of* Ω *is called degenerate if every iterated function system generated by a sequence of maps in*

$\mathcal{H}ol(\Omega, X)$ is degenerate. X is called super degenerate in Ω if every iterated function system generated by a sequence of maps in $\mathcal{H}ol(\Omega, X)$ is super degenerate.

In [2], Beardon, Carne, Minda, and Ng found a condition, based on the hyperbolic metrics on Ω and X, that implies the degeneracy of X in Ω.

Theorem 3.2.2 *([2]) If X is a Bloch subdomain of Ω, then X is degenerate.*

Therefore, by Theorem 2.1.6, we have

Theorem 3.2.3 *If X is a Lipschitz subdomain of Ω, then X is degenerate.*

REMARK. The converse of the theorem above is not necessarily true. For example, let

$$\Omega = \mathbb{C} \setminus \{-\frac{1}{2}, \frac{1}{2}\}$$

and

$$X = \Delta \setminus \{-\frac{1}{2}, \frac{1}{2}\}.$$

Let f be an arbitrary holomorphic map from Ω into X. Since f is bounded, both of the singularities are removable and f extends to a holomprphic map $\tilde{f}$ from the entire plane into Δ. By Liouville's Theorem, $\tilde{f}$ is a constant map. Therefore the accumulation points of any iterated function system are constants. In other words, X is a degenerate subdomain of Ω. But it is not a Lipschitz subdomain of Ω. The reason is that the punctured disks about $-1/2$ and $1/2$ contain hyperbolic disks of arbitrary large radius. Therefore, X is not a Bloch subdomain of Ω. By Theorem 2.1.6, X is not a Lipschitz subdomain of Ω.

Since every relatively compact subdomain is a Lipschitz subdomain, we can conclude that every relatively compact subdomain is degenerate. Lorentzen and Gill showed that in this case X is super degenerate in Ω. In addition, they showed that for every iterated function system the unique constant accumulation point is in X not on its boundary.

Theorem 3.2.4 *(Lorentzen-Gill Theorem) If an iterated function system $\{F_n\}$ is formed from functions in $\mathcal{H}ol(\Omega, X)$, where X is a relatively compact subdomain of Ω, then the system $\{F_n\}$ is super degenerate (consequently, X is super degenerate in Ω). Moreover, the unique constant accumulation point of $\{F_n\}$ is located in X, not* *on its boundary.*

Proof. As X is a relatively compact subdomain of Ω, the diameter of X with respect to ρ_Ω is finite. We denote it by D.

Since every relatively compact subdomain is a Lipschitz subdomain, we have

$$m(X, \Omega) < 1$$

By Theorem 2.2.5 and Theorem 2.2.6, we have

$$m_\Omega < 1$$

We claim that if f is an arbitrary holomorphic map from Ω into X and w_1 and w_2 are two arbitrary points in Ω we have $\rho_\Omega(f(w_1), f(w_2)) \leq m_\Omega \rho_\Omega(w_1, w_2)$. To prove this claim, suppose γ is a path whose hyperbolic length realizes the hyperbolic distance between w_1 and w_2. We have

$$\rho_\Omega(f(w_1), f(w_2)) \leq \rho_\Omega(f(\gamma)) \leq m_\Omega \rho_\Omega(\gamma) = m_\Omega \rho_\Omega(w_1, w_2)$$

This proves the claim.

Let z be a point in Ω. By using the above claim and the fact that a holomorphic map is a contraction, we have

$$\rho_\Omega(F_n(z), F_{n+m}(z)) \leq \rho_\Omega(f_2 \circ \ldots \circ f_n(z), f_2 \circ \ldots \circ f_{n+m}(z))$$

and

$$\rho_\Omega(f_2 \circ \ldots \circ f_n(z), f_2 \circ \ldots \circ f_{n+m}(z)) \le m_\Omega^{n-2} \rho_\Omega(f_n(z), f_n \circ \ldots \circ f_{n+m}(z)).$$

So, we have

$$\rho_\Omega(F_n(z), F_{n+m}(z)) \le m_\Omega^{n-2} D.$$

Therefore, $\{F_n(z)\}$ is a Cauchy sequence. Since Ω is a complete metric space with respect to the hyperbolic metric, $\{F_n(z)\}$ converges to a point in X.

Now, we show the limit of $\{F_n\}$ is a constant function. Let z and w be two distinct points of Ω. By using the above claim, we have

$$\rho_\Omega(F_n(z), F_n(w)) \le m_\Omega^{n-1} \rho_\Omega(f_n(z), f_n(w)).$$

Therefore, we have

$$\rho_\Omega(F_n(z), F_n(w)) \le m_\Omega^{n-1} D.$$

By using the triangle equality we can conclude that the sequences $\{F_n(z)\}$ and $\{F_n(w)\}$ have the same limit. $\square$

Chapter 4

New Results in Iterated Function Systems

4.1 Statement of the results

Consider the backward iterated function systems generated by the maps in $\mathcal{H}ol(\Delta, X)$, where Δ is the unit disk and X is a subdomain of Δ. Let c be a point on the boundary of X. We want to determine whether c can be an accumulation point of some IFS.

As we mentioned in the previous chapter, Lorentzen and Gill showed that if X is relatively compact in Δ, then all the accumulation points are constants and are inside X (not on the boundary of X). In other words, they showed that the non-relative compactness of X in Δ is necessary to have a boundary point c as an accumulation point.

In [9], Keen and Lakic showed that if X is a non-Bloch subdomain of Δ and c is a boundary point of X, we can find an IFS with c as an accumulation point. They also exhibited two special classes of non-relatively compact Bloch subdomains that do admit an IFS whose limit point does lie on the boundary.

In this chapter we show that if c is a boundary point of a non-relatively compact subdomain of Δ, there exists an iterated function system with c as an accumulation point. In other words, we show that the non-relative compactness of X in Δ is a sufficient condition for c to be an accumulation point of some IFS. To prove this result, first we prove the following theorem

Theorem 1 *(The Key Theorem) Let X be any non-relatively compact subdomain of Δ. Let $a_1, a_2, a_3, \ldots$ be a sequence of distinct points in $\Delta \setminus \{0\}$. Then there exists a holomorphic function $f : \Delta \to \Delta$ and a sequence of points $x_1, x_2, x_3, \ldots \in X$ such that $f(0) = 0$ and for all $i = 1, 2, 3, \ldots,$ $f(x_i) = a_i$.*

Then we use it in proving the following main theorem

Theorem 2 *Suppose X is a non-relatively compact subdomain of Δ. Then, for any boundary point $c \in \partial X$, there is an IFS with c as an accumulation point.*

This chapter is organized as follows. In section 2, we state a lemma, a remark on it, and the key theorem which will be used in proving the main theorem. In section 3, we state and prove the main theorem. Section 4 is devoted to an interesting example. In section 5, we show that if X is a Bloch subdomain of Ω and all the functions f_i are equal to some $f \in \mathcal{H}ol(\Omega, X)$, then the iterated function system can not realize any boundary point as an accumulation point.

4.2 Key Facts

When Keen and Lakic were studying the possibility of the existence of finitely many distinct accumulation points for an IFS with a non-relatively compact target domain, they came up with the following lemma.

Lemma 4.2.1 *(Keen-Lakic) Suppose X is a non-relatively compact subdomain of the unit disk Δ. Let $a_1, \ldots, a_n$ be n distinct points in $\Delta \setminus \{0\}$. Then there exists a holomorphic function $f : \Delta \to \Delta$ and points $x_1, \ldots, x_n \in X$ such that $f(x_i) = a_i/x_i$, for all $i = 1, \ldots, n$.*

Notation. In the proof we use the notation

$$A(a, z) = \frac{z - a}{1 - \bar{a}z},$$

where a and z are in the unit disk. A simple computation shows

$$A(a, A(-a, z)) = z.$$

PROOF. First we will choose points $x_1, \dots, x_n \in X$, one at a time, such that they satisfy a number of inequalities which are essential in constructing f. Then by using these points, we introduce the following recursive relations

$$A(x_1, z)g_1(A(x_1, z)) = A(\frac{a_1}{x_1}, f(z)) \tag{4.2.1}$$

For $k = 2, \dots, n$,

$$A(x_k, z)g_k(A(x_k, z)) = A(\frac{b_{(k-1)k}}{a_{(k-1)k}}, g_{(k-1)}(A(x_{(k-1)}, z))), \tag{4.2.2}$$

where for $1 \leq j \leq k \leq n$, a_{jk} is defined as

$$a_{jk} = A(x_j, x_k) \tag{4.2.3}$$

For $2 \leq k \leq n$, b_{1k} is defined as

$$b_{1k} = A(\frac{a_1}{x_1}, \frac{a_k}{x_k}) \tag{4.2.4}$$

and for $2 \leq j < k \leq n$, b_{jk} is defined as

$$b_{jk} = A(\frac{b_{(j-1)j}}{a_{(j-1)j}}, \frac{b_{(j-1)k}}{a_{(j-1)k}}) \tag{4.2.5}$$

In order that these recursive relations work, we need to have

$$|\frac{a_i}{x_i}| < 1 \tag{4.2.6}$$

for $1 \leq i \leq n$ and also

$$|\frac{b_{jk}}{a_{jk}}| < 1 \tag{4.2.7}$$

for $j < k$.

For fixed j and all $k > j$, when $|x_k|$ approaches 1, $|a_{jk}|$ approaches 1 as well.

Let $j = 1$. As $|x_k|$ tends to 1, we have

$$\limsup |b_{1k}| \leq |A(\frac{a_1}{x_1}, a_k e^{\theta_k})| = B_{1k} < 1,$$

where θ_k is chosen so that $\arg a_k e^{\theta_k} = \arg \frac{a_1}{x_1} + \pi$ and B_{1k} is maximum. Now we can choose x_1 such that, if the remaining x_i are close enough to the boundary of the unit disk, inequalities 4.2.6 and 4.2.7 are satisfied with $j = 1$.

Let $j = 2$. As $|x_k|$ tends to 1, we have

$$\limsup |b_{2k}| \leq |A(\frac{b_{12}}{a_{12}}, b_{1k} e^{\theta_k})| = B_{2k} < 1,$$

where again θ_k is chosen so that $\arg a_k e^{\theta_k} = \arg \frac{b_{12}}{a_{12}} + \pi$ and B_{2k} is maximum. Now we can choose x_2 such that, if the remaining x_i are close enough to the boundary of the unit disk, inequalities 4.2.6

and 4.2.7 are satisfied with $j = 2$. By repeating the same argument we can choose $x_3, \ldots, x_n$, in turn, to satisfy the inequalities 4.2.6 and 4.2.7.

Now, let $g_n(z) \equiv 0$ and work back through the recursive relations above to obtain f. It is easy to check that by the above construction we have $f(x_i) = \frac{a_i}{x_i}$ for $1 \leq i \leq n$.

□

Remark. In the Lemma above, n points $a_1, \ldots, a_n$ were given and we chose points $x_1, \ldots, x_n$ which satisfy inequalities 4.2.6 and 4.2.7. Now suppose we add one point a_{n+1} to the list of the given points $a_1, \ldots, a_n$ $(a_{n+1} \neq a_i, for 1 \leq i \leq n)$. The first n chosen points $x_1, \ldots, x_n$ don't need to change because we only need to choose x_{n+1} to satisfy the following inequalities

$$|\frac{a_{n+1}}{x_{n+1}}| < 1 \tag{4.2.8}$$

$$|\frac{b_{j(n+1)}}{a_{j(n+1)}}| < 1, \tag{4.2.9}$$

for $j \leq n$.

By an argument similar to the one that we made in the proof of the lemma above, by choosing x_{n+1} close enough to the boundary of the unit disk all the inequalities are satisfied. In the case that n points $a_1, \ldots, a_n$ were given and we choose $x_1, \ldots, x_n$, the function f was constructed in terms of $a_1, \ldots, a_n$ and $x_1, \ldots, x_n$. When a_{n+1} is added to the list of given points and we choose one more point x_{n+1}, the new constructed function, say g, is given in terms of $a_1, \ldots, a_{n+1}$ and $x_1, \ldots, x_{n+1}$. g is different from f. In fact, f is of degree $n-2$ and g is of degree $n-1$. But, at each x_i, $1 \leq i \leq n$, f and g have the same value. That is

$$f(x_i) = g(x_i) = a_i/x_i,$$

for $1 \leq i \leq n$.

In addition, $g(x_{n+1}) = a_{n+1}/x_{n+1}$.

Theorem 4.2.1 *(The Key Theorem) Let X be a non relatively compact subdomain of Δ. Let $a_1, a_2, a_3, \ldots$ be a sequence of infinitely many distinct points in $\Delta \setminus \{0\}$. Then there exists a holomorphic function $f : \Delta \to \Delta$ and a sequence of points $x_1, x_2, x_3, \ldots \in X$ such that for all $i = 1, 2, 3, \ldots$, $f(x_i) = a_i$ and $f(0) = 0$.*

PROOF. First, we construct a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in the following way: Consider the first point of this sequence, a_1. By lemma 4.2.1, there is a point $x_1 \in X$ and a function $f_1 : \Delta \to \Delta$

such that $f_1(x_1) = a_1/x_1$. Next, consider the first two points of the sequence, a_1 and a_2. By the remark above, x_1 does not need to change and there exists a point x_2 and a function $f_2 : \Delta \to \Delta$ such that $f_2(x_1) = a_1/x_1$ and $f_2(x_2) = a_2/x_2$. In general, in order to construct f_n, we consider the first n points of the sequence, $a_1, \ldots, a_n$. By the remark above, the points $x_1, \ldots, x_{n-1}$, that were selected for constructing f_{n-1}, do not need to change and there exists a point x_n and a function $f_n : \Delta \to \Delta$ such that $f_n(x_i) = a_i/x_i$, for $1 \leq i \leq n$. Therefore, we have a sequence of functions $f_1, f_2, f_3, \ldots$and a sequence of points $x_1, x_2, x_3, \ldots \in X$ such that for every fixed n,

$$f_n(x_i) = a_i/x_i,$$

for $i \leq n$. That is

$$f_1(x_1) = a_1/x_1$$

$$f_2(x_1) = a_1/x_1,\ f_2(x_2) = a_2/x_2$$

$$f_3(x_1) = a_1/x_1,\ f_3(x_2) = a_2/x_2,\ f_3(x_3) = a_3/x_3$$

$$\vdots$$

$f_n(x_1) = a_1/x_1,\ f_n(x_2) = a_2/x_2,\ f_n(x_3) = a_3/x_3,\ \ldots, f_n(x_n) = a_n/x_n$

Now, for every $n \geq 1$, define

$$h_n(z) = z f_n(z)$$

It is clear that $h_n(0) = 0$ and for every fixed n,

$$h_n(x_i) = a_i$$

for $i \leq n$.

The sequence $h_1, h_2, h_3, \ldots$ is a normal family, so by Montel's Theorem there is a subsequence which converges locally uniformly to a holomorphic map f from the unit disk to the unit disk with $f(0) = 0$ and $f(x_i) = a_i$ for $i = 1, 2, 3, ...$

$\square$

Remark. Here, we should emphasize that f is not a constant function because it takes distinct values $a_1, a_2, a_3, \ldots$.

4.3 The Main Theorem

Theorem 4.3.1 *Suppose X is a non-relatively compact subdomain of Δ. Then, for any boundary point $c \in \partial X$, there is an IFS with c*

as a limit function.

PROOF. Let $c_0, c_1, c_2, c_3, \ldots$ be a sequence of distinct points in X which converges to c. We claim that there exists a function $f_1 : \Delta \to X$ and a sequence of points $c_{11}, c_{12}, c_{13}, \ldots \in X$ such that for all $i = 1, 2, 3, \ldots$, $f(c_{1i}) = c_i$ and $f_1(0) = c_0$.

Now, we prove this claim. Since X is a hyperbolic domain, it has a holomorphic covering $g : \Delta \to X$ such that $g(0) = c_0$. Let $a_1, a_2, a_3, \ldots$ be a sequence in Δ that is mapped onto the sequence $c_1, c_2, c_3, \ldots$ by g. By Theorem 4.2.1, there is a function $h : \Delta \to \Delta$ and a sequence of points $c_{11}, c_{12}, c_{13}, \ldots \in X$ such that for all $i = 1, 2, 3, \ldots$, $h(c_{1i}) = a_i$ and $h(0) = 0$. Define $f_1(z) = g \circ h(z)$. It is clear that f_1 is a function from the unit disk into X with $f(c_{1i}) = c_i$ and $f_1(0) = c_0$. This proves our claim. By the same argument there is a function $f_2 : \Delta \to X$ and a sequence of points $c_{212}, c_{213}, c_{214}, \ldots \in X$ such that for all $i = 2, 3, 4, \ldots$, $f_2(c_{21i}) = c_{1i}$ and $f_2(0) = c_{11}$.

By using the same argument repeatedly, we can obtain a sequence of functions $f_1, f_2, f_3, \ldots$ from Δ to X such that for every fixed n,

$$f_n(c_{n(n-1)\ldots 21i}) = c_{(n-1)\ldots 21i},$$

for $i = n, n+1, n+2, \ldots$ and $f_n(0) = c_{(n-1)(n-2)\ldots 1(n-1)}$.

Now, let's look at the backward compositions and and their values at 0.

$F_1(0) = f_1(0) = c_0,$

$F_2(0) = f_1 o f_2(0) = c_1,$

$F_3(0) = f_1 o f_2 o f_3(0) = c_2$

and in general, $F_n(0) = f_1 o f_2 o ... o f_n(0) = c_{n-1}$.

The following diagram explains it more clearly.

$$f_1 : \Delta \to X$$

$$0 \mapsto c_0$$

$$c_{11} \mapsto c_1$$

$$c_{12} \mapsto c_2$$

$$c_{13} \mapsto c_3$$

$$c_{14} \mapsto c_4$$

$$c_{15} \mapsto c_5$$

.

.

.

$$f_2 : \Delta \to X$$

$$0 \mapsto c_{11}$$

$$c_{212} \mapsto c_{12}$$

$$c_{213} \mapsto c_{13}$$

$$c_{214} \mapsto c_{14}$$

$$c_{215} \mapsto c_{15}$$

$$c_{216} \mapsto c_{16}$$

.

.

.

$$f_3 : \Delta \to X$$

$$0 \mapsto c_{212}$$

$$c_{3213} \mapsto c_{213}$$

$$c_{3214} \mapsto c_{214}$$

$$c_{3215} \mapsto c_{215}$$

$$c_{3216} \mapsto c_{216}$$

$$c_{3217} \mapsto c_{217}$$

.

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Since the sequence c_n converges to c, any convergent subsequence of F_n will also converge to c. $\square$

Corollary 4.3.1 *Suppose X is a non-relatively compact subdomain of Δ. Then, there is an IFS, $\{F_n\}$, with the following property: If p is a point in $\overline{X}$, it can be realized* [1] *by one of the accumulation points of $\{F_n\}$. In other words, there is an IFS, for which any point in $\overline{X}$, can be realized by one of its accumulation points.*

Proof. Let $\{c_n\}$ be a sequence consisting of all rational points (i.e. points with rational coordinates) in X. Using the same argument that we made in the proof of previous theorem, an IFS $\{F_n\}$ can be constructed with the property that $F_n(0) = c_{n-1}$. Let p be a point in $\overline{X}$. Since $\{c_n\}$ is dense in $\overline{X}$, there is a subsequence $\{c_{n_j}\}$ which converges to p. Now, consider the corresponding subsequence $\{F_{n_j+1}\}$. We have $F_{n_j+1}(0) = c_{n_j}$. Consequently, $\{F_{n_j+1}(0)\}$ converges to p. Therefore, the limit of the subsequence $\{F_{n_j+1}\}$, which is a limit function of the IFS, realizes p. $\square$

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[1] A point p is realized by a function f if $f(a) = p$ for some point a in the domain of f.

4.4 An interesting example

Suppose X is the region which is bounded by an ideal triangle, whose vertices are on the unit circle. Suppose a_n is a sequence consisting of all rational points (i.e. points with rational coordinates). By Theorem 4.2.1, there exits a holomorphic function $f : \Delta \to \Delta$ and a sequence of points $x_1, x_2, x_3, \ldots \in X$ such that for all $i = 1, 2, 3, \ldots$, $f(x_i) = a_i$ and $f(0) = 0$. Consider the restriction of f to the subdomain X and call it g. Now the sequence a_n which is a dense subset of the unit disk is in the range of g.

One might think that the range of g should be equal to the unit disk but it can not be. This is because the domain of g is X which has finite hyperbolic area π and a holomorphic map contracts hyperbolic area. The fact is that the range of g is an open dense subset of the unit disk which is a proper subset of the unit disk (and as we saw, it has finite hyperbolic area).

A simple example of a dense subdomain of the unit disk with finite area is the following.

Example. Let a_n be the sequence in the unit disk consisting of all points with rational coordinates. Around each point a_n, consider a hyperbolic disk, D_n, centered at a_n with hyperbolic area less than or equal to $\frac{1}{2^n}$. The union of these disks is an open subset of the unit disk with finite hyperbolic area. Let B_n be a region containing a_n and a_{n+1}, with hyperbolic area less than or equal to $\frac{1}{2^n}$.

Let $W = \bigcup_{n=1}^{\infty}(D_n \cup B_n)$. W is a subdomain of the unit disk with hyperbolic area less than or equal to 2 and is also dense.

4.5 Self Iteration

Theorem 4.5.1 *Let X be a ρ-Bloch subdomain of the unit disk and let $f : \Delta \to X$ be an arbitrary holomorphic map. Then no boundary point $p \in \partial X$ can be the limit of $\{f^{\circ n}\}$.*

Proof. Let $f : \Delta \to X$ be holomorphic. By Theorem 2.3.4, f is a strong contraction. Therefore, there exists $k < 1$ such that $\rho_X(f(z), f(w)) \leq k\rho_X(z, w)$ for all z and w in X. Now, let's look at the orbit of a point z_0 in X. We claim that $\{f^{\circ n}(z_0)\}$ can not approach to a boundary point. If it does then $\rho_X(z_0, f^{\circ n}(z_0))$ approaches infinity. On the other hand

$$\rho_X(f^{\circ(n-1)}(z_0), f^{\circ n}(z_0)) \leq k\rho_X(f^{\circ(n-2)}(z_0), f^{\circ(n-1)}(z_0))$$

for every $n \geq 2$. Therefore, if $z_1 = f(z_0)$, by using the triangle inequality we have

$$\rho_X(f^{\circ n}(z_0), z_0) \leq \rho_X(f^{\circ n}(z_0), f^{\circ(n-1)}(z_0)) + \rho_X(f^{\circ(n-1)}(z_0), f^{\circ(n-2)}(z_0)) + \ldots + \rho_X(z_1, z_0) \tag{4.5.1}$$

In addition, for every $i = 2, \ldots, n$ we have

$$\rho_X(f^{\circ i}(z_0), f^{\circ(i-1)}(z_0)) \leq k^{i-1} \rho_X(z_1, z_0) \tag{4.5.2}$$

Therefore, by inequalities 4.5.1 and 4.5.2 we have

$$\rho_X(f^{\circ n}(z_0), z_0) \leq \sum_{i=1}^{i=n} k^{i-1} \rho_X(z_1, z_0)$$

As n tends to infinity, the right side is finite (because $k < 1$), but the left side goes to infinity. $\square$

Chapter 5

New Results In Conformal Geometry

5.1 Special case

As we saw in Theorem 2.2.1, if X is a hyperbolic domain, for every z in X, we have

$$\rho_X(z) \leq \kappa_X^\Omega(z)$$

Let Ω be a subdomain of Δ. In this section we state a condition on Ω, under which, for every subdomain X of Ω and every z in X we have

$$\rho_X(z) = \kappa_X^\Omega(z)$$

Before talking about that condition, we give an example, in which the inequality is strict. That is

$$\rho_X(z) < \kappa_X^\Omega(z)$$

Example. Let Ω be a domain obtained by removing countably infinitely many points from the unit disk Δ to make a Bloch subdomain Ω. To do this we remove the points so that no hyperbolic disk with ρ-radius bigger than 1 can fit in Ω and any pair of removed points can not have a ρ-distance less than $1/2$. For this purpose, consider all hyperbolic circles centered at the origin with ρ-radius $n/2$, where n=1,2,3,.... On each circle we remove finitely many points such that the distance between any two points is neither less that 1/2 nor bigger that 1. Now we construct a subdomain X of Ω in the following way: Let a_n be the sequence of the points that we removed for making Ω. At each a_n, remove a closed hyperbolic disk centered at a_n with ρ-radius 1/5. It is clear that all of these closed disks are mutually disjoint. Let z be point in X and let r be any positive number. By the definition of $\kappa_X^\Omega(z)$, there exist $f : \Omega \to X$ and w in Ω such that $f(w) = z$ and

$$\frac{\rho_\Omega(w)}{|f'(w)|} \leq \kappa_X^\Omega(z) + r$$

f is not constant, because $\kappa_X^\Omega(z)$ is not infinity. Now we prove this claim. Let D be the Euclidean disk contained in X with center z and Euclidean radius s. Define $g : \Omega \to X$ as

$$g(t) = st + z$$

Therefore, $\kappa_X^\Omega(z) \leq \frac{\rho(0)}{g'(0)} = \frac{1}{s}$. This shows f can not be constant. Since f is a bounded holomorphic map and Ω has only punctures, all the punctures are removable singularities. Moreover, X has no punctures. Therefore, f can be extended to holomorphic map $\tilde{f}$ from Δ into X. So we can write the above inequality as

$$\frac{\rho_\Omega(w)}{|\tilde{f}'(w)|} \leq \kappa_X^\Omega(z) + r \tag{5.1.1}$$

By Theorem 2.1.6, Ω is a Lipschitz subdomain of the unit disk. In the other words, $m(\Omega, \Delta) < 1$. Therefore,

$$\frac{\rho_\Delta(w)}{m(\Omega, \Delta)} \leq \rho_\Omega(w) \tag{5.1.2}$$

Since, $\tilde{f}$ is a contraction, we have

$$\rho_X(z) \leq \frac{\rho_\Delta(w)}{|\tilde{f}'(w)|} \tag{5.1.3}$$

By equations 5.1.1, 5.1.2, and 5.1.3, we have

$$\frac{\rho_X(z)}{m(\Omega, \Delta)} \leq \kappa_X^\Omega(z) + r \tag{5.1.4}$$

This inequality is true for every $r > 0$, so we have

$$\frac{\rho_X(z)}{m(\Omega, \Delta)} \leq \kappa_X^\Omega(z) \tag{5.1.5}$$

Since $m(\Omega, \Delta) < 1$, we can conclude that

$$\rho_X(z) < \kappa_X^\Omega(z) \tag{5.1.6}$$

In the example above, Ω is a Lipschitz subdomain of the unit disk. In fact, we will show that if Ω is a non-Lipschitz subdomain of the unit disk and X is any hyperbolic domain, then the generalized Kobayashi density κ_X^Ω and the hyperbolic density ρ_X are equivalent on X.

Theorem 5.1.1 *Suppose Ω is a non-Lipschitz subdomain of the unit disk and X is any hyperbolic domain. Then*

$$\rho_X(z) = \kappa_X^\Omega(z)$$

for every z in X.

Proof. We know that

$$\rho_X(z) \leq \kappa_X^\Omega(z).$$

Therefore, in order to show that $\rho_X(z) = \kappa_X^\Omega(z)$, we need only to show that $\kappa_X^\Omega(z) \leq \rho_X(z)$.

As X is hyperbolic, there is a holomorphic covering

$$\pi : \Delta \to X.$$

Since π is a covering, it is locally an isometry and we have

$$\rho_\Delta(\omega) = \rho_X(z).|\pi'(\omega)|, \tag{5.1.7}$$

where $\pi(\omega) = z$. By precomposing by a Möbius map we can let ω be anywhere in Ω.

Now, let f be the restriction of π to Ω. Therefore,

$$\kappa_X^{\Omega}(z) \leq \frac{\rho_\Omega(\omega)}{|f'(\omega)|} = \frac{\rho_\Omega(\omega)}{|\pi'(\omega)|}.$$

As we saw

$$|\pi'(\omega)| = \frac{\rho_\Delta(\omega)}{\rho_X(z)}.$$

Consequently,

$$\kappa_X^{\Omega}(z) \leq \frac{\rho_\Omega(\omega)}{\rho_\Delta(\omega)}.\rho_X(z).$$

Since Ω is a non-Lipschitz subdomain of Δ, $\frac{\rho_\Omega(\omega)}{\rho_\Delta(\omega)}$ can be made as close as we wish to 1 by choosing ω properly. Therefore,

$$\kappa_X^{\Omega}(z) \leq \rho_X(z).$$

□

Corollary 5.1.1 *Suppose Ω is a non-Lipschitz subdomain of the unit disk. A subdomain X is a κ-Lip subdomain of Ω if and only if it is a ρ-Lip subdomain of Ω.*

Corollary 5.1.2 *Suppose Ω is a non-Lipschitz subdomain of the unit disk. If X is a κ-Lip subdomain of Ω then it is a c-Lip subdomain of Ω.*

5.2 Generalization

In this section we want to state similar results for a large class of domains in the plane.

Definition 5.2.1 *A domain* Ω *in the complex plane is called quasi-bounded if the smallest simply connected plane domain containing* Ω *is a proper subset of the complex plane* $\mathbb{C}$*. The smallest simply connected domain containing* Ω *is denoted by* $\hat{\Omega}$*.*

Example. If Ω is the round annulus $\{z : r < |z| < 1\}$, then it is quasi-bounded and $\hat{\Omega} = \Delta$.

Example. If $\Omega = \mathbb{C} \setminus \{-1, 1\}$, then it is not quasi-bounded because the smallest simply connected domain containing Ω is $\mathbb{C}$.

Before stating and proving the main theorem of this section, we need to state one remark.

Remark. As we saw in Theorem 2.2.4, every holomorphic map is an infinitesimal contraction with respect to the generalized Kobayashi density. Therefore, if a map $f : X \to Y$ is a conformal homoemorhism then it is an infinitesimal isometry with respect to the

Kobayash density. That is,

$$\kappa_Y^{\Omega}(f(z))|f'(z)| = \kappa_X^{\Omega}(z),$$

for every z in X.

Theorem 5.2.1 *Suppose Ω is quasi-bounded and is a non-Lipschitz subdomaiin of $\hat{\Omega}$. Then for any hyperbolic domain X we have*

$$\rho_X(z) = \kappa_X^{\Omega}(z)$$

for every z in X.

PROOF. Since Ω is quasi-bounded, by the Riemann mapping Theorem there is a conformal homeomorphism f from $\hat{\Omega}$ onto Δ. By Theorem 2.1.5 $f(\Omega)$ is a non-Lipschitz subdomain of Δ. Set $\Lambda = f(\Omega)$. By Theorem 5.1.1

$$\rho_X(t) = \kappa_X^{\Lambda}(t), \tag{5.2.1}$$

for every t in X.

As Ω and Λ are conformally isomorphic, by Theorem 2.2.3 we have

$$\kappa_X^{\Omega}(z) = \kappa_X^{\Lambda}(z), \tag{5.2.2}$$

for every z in X.

By equations 5.2.1 and 5.2.2, we conclude that

$$\rho_X(z) = \kappa_X^{\Omega}(z).$$

□

Corollary 5.2.1 *Suppose Ω is a non-Lipschitz subdomain of $\hat{\Omega}$. A subdomain X is a κ-Lip subdomain of Ω if and only if it is a ρ-Lip subdomain of Ω.*

PROOF. By Corollary 2.2.1 we have

$$\kappa_{\Omega}^{\Omega} = \rho_{\Omega}. \tag{5.2.3}$$

In addition, by Theorem 5.2.1 we have

$$\rho_X(z) = \kappa_X^{\Omega}(z). \tag{5.2.4}$$

Therefore, we conclude that

$$m\kappa(X, \Omega) = \sup_{z \in X} \frac{\kappa_{\Omega}^{\Omega}(z)}{\kappa_X^{\Omega}(z)} = \sup_{z \in X} \frac{\rho_{\Omega}(z)}{\rho_X(z)} = m(X, \Omega) \tag{5.2.5}$$

□

Corollary 5.2.2 *Suppose Ω is a non-Lipschitz subdomain of $\hat{\Omega}$. If X is a κ-Lip subdomain of Ω then it is a c-Lip subdomain of Ω.*

Proof. By Corollary 2.3.2 we have

$$c_X^X(z) = \rho_X(z) \tag{5.2.6}$$

In addition, by Theorem 2.3.1

$$c_X^\Omega(z) \leq \rho_\Omega(z). \tag{5.2.7}$$

Therefore, we have

$$mc(X,\Omega) = \sup_{z\in X} \frac{c_X^\Omega(z)}{c_X^X(z)} \leq \sup_{z\in X} \frac{\rho_\Omega(z)}{\rho_X(z)} = m(X,\Omega). \tag{5.2.8}$$

On the other hand, by Corollary 5.2.1, X is a κ-Lip subdomain of Ω if and only if it is a ρ-Lip subdomain of Ω. This completes the proof.

□

Corollary 5.2.3 *Suppose Ω is a non-Lipschitz subdomain of $\hat{\Omega}$. If all the functions $f \in \mathcal{H}ol(\Omega, X)$ are uniformly strong contractions with respect to ρ_Ω then all of them are uniformly contractions with respect to ρ_X.*

Proof. By Theorem 2.2.6, X is a κ-Lip subdomain of Ω if and only if all the functions $f \in \mathcal{H}ol(\Omega, X)$ are uniformly strong contractions with respect to ρ_Ω and by Theorem 2.3.4, X is a c-Lip subdomain of Ω if and only if all the functions $f \in \mathcal{H}ol(\Omega, X)$ are uniformly strong contractions with respect ρ_X. On the other hand, by Corollary 5.2.2, if X is a κ-Lip subdomain of Ω then it is a c-Lip subdomain of Ω. This completes the proof. $\square$

Appendix A

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