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NON-SPECTRAL REPRESENTATIONS IN OPEN WAVEGUIDES

by

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Abstract

NON-SPECTRAL REPRESENTATIONS IN OPEN WAVEGUIDES

by

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Both thin dielectric films and fibers are used in the construction of optical communication systems. These structures act as open waveguides, guiding a portion of the electromagnetic field along their axes in the form of surface waves, and allowing the remainder to escape via radiation. An understanding of the modal properties of the fields within such guides is therefore of fundamental interest to the understanding and design of optical communication systems.

The surface waves are longitudinally propagating solutions of the wave equation, and are associated with spectral pole singularities of the transverse Green's function. Besides these poles, which are finite in number, the transverse Green's function has an infinity of non-spectral poles, a few of which are known to give rise to "leaky

waves." These waves have limited spatial existence and are recognized as highly convergent portions of the radiation field.

In this dissertation the TE and TM polarized fields are analyzed in detail for a variety of dielectric waveguides with arbitrary analytic transverse permittivity variation. For each polarization it is shown that the field can be represented as an infinite sum of functions, including the finite number of spectral (surface wave) and an infinity of non-spectral (complex) solutions. Additional functions are also included. Both the field polarization and the behavior of the permittivity at the waveguide interface(s) are found to affect the form (and in the case of the dielectric fiber, the validity) of the representation.

The non-spectral field theory is used to analyze a grating coupler and a dielectric waveguide with a random wall variation. Expressions are obtained for the efficiency, scattered field, radiated power, and attenuation coefficient associated with the grating coupler, and for the ensemble average of the power lost by an incident surface wave due to the random wall variation. These results demonstrate that the non-spectral terms can be used as design parameters for optical communication devices.

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CHAPTER I

INTRODUCTION AND SUMMARY

The advent of the laser, a practical source of coherent radiation, has stimulated interest in the development of optical communication systems. The advantages cited for communication at optical frequencies are large bandwidth, high directivity of laser beams (with virtually no sidelobes), and the possibility of small antennas and devices [1]. The use of integrated circuit technology in the fabrication of such systems offers certain additional advantages. This "integrated optics" [2] approach would not only yield economies in size and cost, but would also eliminate problems associated with alignment and sensitivity to environmental changes, as well as allow high power density levels to be achieved using low overall drive power [3]. Thus, as presently envisioned, IC technology will be used to deposit thin (in the micron range) films of material on suitable substrates for the construction of such devices as lasers, modulators, couplers, resonators, and filters, to name a few [2]. To date, an abundant literature has accumulated describing both theoretical and experimental investigations [1]-[23].

The waveguide, in addition to being used as a

transmission medium, is a basic element in the construction of signal processing devices. At optical frequencies, however, the use of conventional metallic waveguides, with reflection losses on the order of 100 db/cm [4], is quickly ruled out in favor of dielectric waveguides, where losses of less than 0.0002 db/cm have been reported [5]. Hence an understanding of the characteristics of dielectric waveguides is necessary for the development of optical systems.

The dielectric waveguide is an open waveguide, capable of supporting surface waves. These modes occur as a result of the trapping of a portion of the electromagnetic field within the guide by total internal reflection. The portion of the field which is not trapped in this manner escapes from the guide in the form of radiation. As a result, the complete transverse spectrum of the guide consists of a discrete spectrum, associated with the surface modes, plus a continuous spectrum, associated with the radiation field [24].

For guides with large aspect ratios, a two dimensional analysis is appropriate. Either a longitudinal or transverse spectral representation of the field may be obtained [25], the former being an integral over a continuous spectrum, whereas the latter consists of both an integral over a continuous spectrum plus a finite sum over a discrete spectrum. Both are equivalent and may be obtained from one another by a deformation of the contour of integration.

In going from the longitudinal representation to the

transverse representation via such a contour deformation, the discrete spectrum arises from pole crossings, and exhibits itself by contributing a finite sum of residues, the surface mode solutions. The continuous spectrum arises because of the presence of a branch cut, a characteristic of all open systems, and exhibits itself by contributing a branch cut integral, the radiation field.

Regardless of the representation used, the integrals which arise cannot, in general, be evaluated in closed form. Consequently, the so-called steepest descent representation [24]-[27], in which the field integral is evaluated approximately for points in the far-field of the source, is often employed. This technique yields an asymptotic approximation to the radiation field by a contour deformation which carries the integration onto part of the non-spectral Riemann sheet for some regions of space. While only a finite number of poles are located on the spectral Riemann sheet, an infinite number are located on the non-spectral sheet [24]. In its excursion onto the non-spectral sheet, the steepest descent contour may cross several of these non-spectral poles, the residues of which are discrete, so-called "leaky wave", contributions to the field. These contributions are thus seen to represent highly convergent portions of the radiation field [28]. The steepest descent representation therefore demonstrates a close connection between certain non-spectral poles and the continuous spectrum.

A question which immediately suggests itself is

whether the remaining infinity of non-spectral poles have any theoretical significance. This question, which has its roots in problems in physics [29]-[32] and mathematics [33], has recently received some attention [34],[35]. In this thesis we present a method for generating non-spectral field representations in a variety of open waveguides. The method, based upon the Green's function technique for obtaining eigenfunction expansions in closed waveguides [36]-[38], demonstrates that each singularity, regardless of its location on the Riemann surface, may be associated with a partial field contribution to the total solution.

A review of spectral representations in planar waveguides is presented in the next chapter. The TE field in simple closed and open waveguides is used for illustrative purposes.

In Chapter III, the method for obtaining non-spectral representations is presented and applied to a variety of planar dielectric waveguides. It is found that the form of the representation depends upon both the field polarization and the behavior of the permittivity at the dielectric-free space interface(s).

The use of the theory is illustrated in Chapter IV where it is applied to some problems in integrated optics. Several quantities of interest are obtained and related to the terms of the non-spectral representation. The results indicate that the non-spectral contributions may be of use as design parameters for optical communication devices.

The theory is extended in Chapter V to include circularly cylindrical dielectric waveguides. It is found that expansions analogous to those obtained for planar guides can be obtained for circular guides, provided that the smoothness of the dielectric-free space transition is restricted according to the field polarization.

The concluding chapter summarizes the results in this thesis and offers suggestions for possible future study.

CHAPTER II

SPECTRAL FIELD REPRESENTATIONS IN PLANAR WAVEGUIDES

Spectral field representations in planar waveguides are reviewed in this chapter. In the first section a parallel plate waveguide is used as a simple example of closed waveguides. The transverse spectral representation of the TE field is first obtained, and is seen to involve a purely discrete spectrum. This is followed by the longitudinal spectral representation, which involves a purely continuous spectrum. The equivalence of the two representations is then demonstrated and discussed.

In the next section the simple open waveguide which results from removing one of the conducting plates is considered. First the longitudinal spectral representation is obtained; just as in the closed waveguide, it involves a purely continuous spectrum. This representation is then used to obtain the transverse spectral representation in the manner discussed in the previous section on closed waveguides. However, unlike in the closed waveguide, the transverse spectral representation in the open waveguide involves a continuous, as well as a discrete, spectrum.

1 Spectral Representations in Closed Waveguides

In this section we demonstrate how to obtain the transverse and longitudinal spectral representations for the field in closed waveguides, using as a simple prototype the TE field in a parallel plate waveguide. It is shown that the transverse spectral representation involves a purely discrete spectrum, while the longitudinal spectral representation involves a purely continuous spectrum. It is then shown that the discrete terms of the transverse spectral representation arise from pole singularities of the integrand of the longitudinal spectral representation.

1.1 Formulation of Problem

We consider the parallel plate waveguide, illustrated in Fig. 2.1, consisting of a dielectric slab of thickness a , constant permittivity ϵ and permeability μ_0 , bounded by perfectly conducting plates at $x=0$ and $x=a$. With a time variation $e^{j\omega t}$ assumed and suppressed, the two curl equations are

$$\nabla \times \vec{E} = -j\omega\mu_0 \vec{H} \quad (1.1.1a)$$

and

$$\nabla \times \vec{H} = j\omega\epsilon \vec{E} + \vec{J} \quad (1.1.1b)$$

where $\vec{E}$, $\vec{H}$, and $\vec{J}$ are the electric, magnetic, and current density field vectors, respectively. Taking the curl of (1.1.1a) and substituting (1.1.1b), we obtain the wave equation for the electric field:

$$\nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) + k^2 \vec{E} = j\omega\mu_0 \vec{J} \quad (1.1.2)$$

where

$$k^2 = \omega^2 \mu_0 \epsilon. \quad (1.1.3)$$

Since the problem is two dimensional, $\frac{\partial}{\partial y} \equiv 0$, and the curl equations (1.1.1) yield the independent TE and TM solutions. The TE solution consists of the three components E_y , H_x and H_z . From (1.1.1a) we find that H_x and H_z are determined from E_y according to

$$H_x = - \frac{j}{\omega \mu_0} \frac{\partial E_y}{\partial z} \quad (1.1.4a)$$

and

$$H_z = \frac{j}{\omega \mu_0} \frac{\partial E_y}{\partial x} \quad (1.1.4b)$$

Hence once E_y is known, the TE solution is determined. From (1.1.2) we find that E_y satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) E_y = j \omega \mu_0 J_y \quad (1.1.5)$$

in the region $0 < x < a$, $-\infty < z < \infty$. At the conducting plates the tangential electric field must vanish, so that E_y must also satisfy the boundary conditions

$$E_y(0, z) = 0 \quad (1.1.6a)$$

and

$$E_y(a, z) = 0. \quad (1.1.6b)$$

In addition, E_y must satisfy the radiation condition at infinity.

When $J_y \equiv 0$, we may use separation of variables and set

$$E_y(x, z) = X(x)Z(z). \quad (1.1.7)$$

Substituting this into (1.1.5), we obtain the one-dimensional problems

$$\left(\frac{d^2}{dx^2} + \lambda^2 \right) X = 0 \quad (1.1.8)$$

and

$$\left(\frac{d}{dz}\right)^2 + \beta^2)z = 0 \quad (1.1.9)$$

where the separation constants are related by

$$\lambda^2 + \beta^2 = k^2. \quad (1.1.10)$$

For certain values of λ , (1.1.8) has non-trivial solutions which satisfy the boundary conditions $X(0) = 0$ and $X(a) = 0$. These solutions are the transverse eigenfunctions. By summing these eigenfunctions, appropriately weighted, we may obtain the solution E_y for arbitrary excitation J_y . The resulting expansion is the transverse spectral representation of the solution. The longitudinal spectral representation is obtained analogously by integrating appropriately weighted solutions of (1.1.9) over the separation constant β .

If we introduce the TE Green's function $G(x, z; \xi, \eta)$ as the solution to (1.1.5) and (1.1.6) with

$$j\omega\mu_0 J_y(x, z) = -2\pi\delta(x-\xi)\delta(z-\eta) \quad (1.1.11)$$

where $0 < \xi < a$ and $-\infty < \eta < \infty$, then the solution for arbitrary J_y is expressed as

$$E_y(x, z) = -j \frac{\omega\mu_0}{2\pi} \int_{\eta=-\infty}^{\infty} \int_{\xi=0}^a J_y(\xi, \eta) G(x, z; \xi, \eta) d\xi d\eta. \quad (1.1.12)$$

Hence it is sufficiently general to obtain the spectral representations of the TE Green's function.

1.2 The Transverse Spectral Representation

The transverse eigenfunctions, normalized to satisfy

$$\int_0^a \phi_n(x) \phi_m(x) dx = \begin{cases} 1 & n = m \\ 0 & n \neq m, \end{cases} \quad (1.2.1)$$

are given by

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin \lambda_n x, \quad n = 1, 2, \dots \quad (1.2.2)$$

with eigenvalues

$$\lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, \dots \quad (1.2.3)$$

In order that G be expandable in terms of these eigenfunctions, the eigenfunctions must form a complete set. Since (1.1.8), along with the boundary conditions on X , defines a selfadjoint differential system, the eigenfunctions do form a complete set [1]. Alternatively, we see that (1.2.2) are the terms of a Fourier sine series, so that completeness is assured. We may therefore express G as

$$G(x, z; \xi, \eta) = \sum_{n=1}^{\infty} g_n(z) \phi_n(x) \quad (1.2.4)$$

where the g_n are functions of z to be determined. Multiplying both sides of (1.1.5) [with J_y given by (1.1.11)] by $\phi_n(x)$ and integrating from $x=0$ to $x=a$, we obtain

$$(k^2 - \lambda_n^2 + \frac{\partial^2}{\partial z^2}) \int_0^a G(x, z; \xi, \eta) \phi_n(x) dx = -2\pi \phi_n(\xi) \delta(z-\eta). \quad (1.2.5)$$

Substituting (1.2.4) and utilizing (1.2.1), we find that the $g_n(z)$ satisfy

$$(\frac{d^2}{dz^2} + \beta_n^2) g_n(z) = -2\pi \phi_n(\xi) \delta(z-\eta) \quad (1.2.6)$$

where

$$\beta_n^2 = k^2 - \lambda_n^2. \quad (1.2.7)$$

The solution to (1.2.6) satisfying the radiation condition at $|z| = \infty$ is found to be [1]

$$g_n(z) = -j \frac{\pi}{\beta_n} \phi_n(\xi) e^{-j\beta_n |z-\eta|}. \quad (1.2.8)$$

Hence, the transverse spectral representation of the TE

Green's function is

$$G(x, z; \xi, \eta) = -j\pi \sum_{n=1}^{\infty} \frac{1}{\beta_n} \phi_n(\xi) \phi_n(x) e^{-j\beta_n |z-\eta|}. \quad (1.2.9)$$

1.3 The Longitudinal Spectral Representation

The longitudinal eigenvalues form a continuum over $[0, \infty)$, so that instead of a discrete sum as in (1.2.9), the longitudinal spectral representation takes the form of an integral over a continuous spectrum:

$$G(x, z; \xi, \eta) = \int_{-\infty}^{\infty} g(x, \xi; \lambda) e^{-j\beta(z-\eta)} d\beta, \quad (1.3.1)$$

where the transverse Green's function g is to be determined. That such a representation is possible follows immediately upon recognizing that G and g form a Fourier transform pair:

$$g(x, \xi; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, z; \xi, \eta) e^{j\beta(z-\eta)} dz. \quad (1.3.2)$$

To obtain g explicitly, multiply both sides of (1.1.5) [with J_y given by (1.1.11)] by $\frac{1}{2\pi} e^{j\beta(z-\eta)}$ and integrate from $z = -\infty$ to $z = \infty$; this yields

$$\left(\frac{d^2}{dx^2} + \lambda^2\right) g(x, \xi; \lambda) = -\delta(x-\xi). \quad (1.3.3)$$

The boundary conditions (1.1.6) require that

$$g(0, \xi; \lambda) = 0 \quad (1.3.4a)$$

and

$$g(a, \xi; \lambda) = 0. \quad (1.3.4b)$$

The solution is found to be [1]

$$g(x, \xi; \lambda) = \frac{\sin \lambda x \sin \lambda (a - \xi)}{\lambda \sin \lambda a} H(\xi - x) + \frac{\sin \lambda \xi \sin \lambda (a - x)}{\lambda \sin \lambda a} H(x - \xi), \quad (1.3.5)$$

where H is the unit step function. Hence the longitudinal spectral representation is

$$G(x, z; \xi, \eta) = \int_{-\infty}^{\infty} \frac{\sin \lambda x_{<} \sin \lambda (a - x_{>})}{\lambda \sin \lambda a} e^{-j\beta(z-\eta)} d\beta, \quad (1.3.6)$$

where $x_{<}$ stands for the lesser of x and ξ , and $x_{>}$ stands for the greater of x and ξ .

1.4 Equivalence of the Representations

The transverse spectral representation can be obtained from the longitudinal spectral representation by evaluating the integral in (1.3.6). The evaluation is accomplished by deforming the contour of integration in (1.3.6) and applying the residue theorem. Consequently, the singularities of g as a function of β must be determined.

Although λ is a multivalued function of β [viz., eq. (1.1.10)], g is single valued. Hence the only singularities of g are the poles at

$$\pm \beta_n = \pm (k^2 - (n\pi/a)^2)^{1/2}, \quad n = 1, 2, \dots \quad (1.4.1)$$

Assuming that the dielectric is slightly lossy, the poles are shifted off of the real and imaginary axes as illustrated in Fig. 2.2. Defining the integrals

$$I_{N+} \triangleq \int_{-R_N}^{+R_N} g(x, \xi; \lambda) e^{-j\beta(z-\eta)} d\beta + \int_{C_{N+}} g(x, \xi; \lambda) e^{-j\beta(z-\eta)} d\beta \quad (1.4.2a)$$

when $z - \eta > 0$, and

$$I_{N-} \triangleq \int_{-R_N}^{+R_N} g(x, \xi; \lambda) e^{-j\beta(z-\eta)} d\beta + \int_{C_{N-}} g(x, \xi; \lambda) e^{-j\beta(z-\eta)} d\beta \quad (1.4.2b)$$

when $z-\eta < 0$, where C_{N+} and C_{N-} are semicircles of radii R_N in the lower and upper half β -plane, respectively, centered at the origin, and passing between the N th and $N+1$ st poles of g , we may apply the residue theorem and obtain

$$I_{N\pm} = \mp 2\pi j \sum_{n=1}^N \text{Res}[g(x, \xi; \lambda) e^{-j\beta(z-\eta)}]_{\beta=\pm\beta_n} \quad (1.4.3)$$

Now as R_N approaches infinity, the contributions to $I_{N\pm}$ from the semicircles $C_{N\pm}$ vanish, so that in the limit (1.4.2) becomes (1.3.6). Furthermore, as R_N approaches infinity, (1.4.3) becomes an infinite sum. Since the terms of the sum are

$$\begin{aligned} \text{Res}[g(x, \xi; \lambda) e^{-j\beta(z-\eta)}]_{\beta=\pm\beta_n} &= \frac{\sin \lambda_n \xi \sin \lambda_n x}{\lambda_n \left[\frac{d}{d\beta} (\sin \lambda a) \right]_{\beta=\pm\beta_n}} e^{\mp j\beta_n(z-\eta)} \\ &= \pm \frac{\sin \lambda_n \xi \sin \lambda_n x}{\beta_n a} e^{\mp j\beta_n(z-\eta)} \end{aligned} \quad (1.4.4)$$

we conclude that

$$G(x, z; \xi, \eta) = -j \frac{2\pi}{a} \sum_{n=1}^{\infty} \frac{1}{\beta_n} \sin \lambda_n \xi \sin \lambda_n x e^{-j\beta_n |z-\eta|} \quad (1.4.5)$$

which is identical to (1.2.9).

We have thus demonstrated the equivalence of the transverse and longitudinal spectral representations. Moreover, we have also shown that the terms of the transverse spectral representation are the residues of the integrand of the longitudinal spectral representation at the poles of g . Thus the discrete spectrum of the transverse differential system (1.1.8) is associated with the pole singularities of the transverse Green's function.

2 Spectral Representations in Open Waveguides

Spectral representations in open waveguides are discussed in this section, using as a simple prototype the TE field in a grounded dielectric slab waveguide. First the longitudinal spectral representation, an integral over a continuous spectrum, is determined. The transverse spectral representation is then obtained by a deformation of the contour of integration, and is seen to involve a continuous, as well as a discrete, spectrum.

2.1 Formulation of Problem

The grounded dielectric slab waveguide, illustrated in Fig. 2.3, is obtained from the parallel plate waveguide of Fig. 2.1 by removing the conducting plate at $x = a$. The region $x > a$ is free space. The analysis of the TE field is carried out as in the case of the parallel plate waveguide, except that the field now extends into the region $x > a$. Instead of (1.1.5) and (1.1.6), we now find that E_y must satisfy

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x) \right] E_y = j\omega\mu_0 J_y \quad (2.1.1)$$

in the region $0 < x < \infty$, $-\infty < z < \infty$, where

$$k^2(x) = \begin{cases} k^2 & , 0 < x < a \\ k_0^2 & , a < x \end{cases} \quad (2.1.2)$$

subject to

$$E_y(0, z) = 0 \quad (2.1.3)$$

and the radiation condition at infinity. In addition, continuity of E_y and H_z requires that E_y and $\frac{\partial}{\partial x} E_y$ be continuous

across $x = a$.

If we apply separation of variables when $J_y \equiv 0$, setting

$$E_y(x, z) = X(x)Z(z), \quad (2.1.4)$$

we find that

$$\left[\frac{d^2}{dx^2} + \tau^2 + k^2(x) - k_0^2 \right] X = 0 \quad (2.1.5)$$

and

$$\left(\frac{d^2}{dz^2} + \beta^2 \right) Z = 0 \quad (2.1.6)$$

where the separation constants are related by

$$\tau^2 + \beta^2 = k_0^2. \quad (2.1.7)$$

Comparing (2.1.5) and (2.1.6) with (1.1.8) and (1.1.9), respectively, we see that the longitudinal differential systems are the same for both open and closed waveguides, while the transverse differential systems differ. We shall therefore find it more convenient to first obtain the longitudinal spectral representation, and then obtain the transverse spectral representation by deforming the contour of integration in the manner previously demonstrated with the closed waveguide.

2.2 The Longitudinal Spectral Representation

As in the closed waveguide, we shall consider the TE Green's function which, for the open waveguide, is the solution to (2.1.1) with J_y given by (1.1.11). Since the longitudinal differential systems (2.1.6) and (1.1.9) are the same, we may express G as an integral over a continuous

spectrum:

$$G(x, z; \xi, \eta) = \int_{-\infty}^{\infty} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta \quad (2.2.1)$$

where the transverse Green's function is

$$g(x, \xi; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, z; \xi, \eta) e^{j\beta(z-\eta)} dz. \quad (2.2.2)$$

Multiplying both sides of (2.1.1) [with J_y given by (1.1.11)] by $\frac{1}{2\pi} e^{j\beta(z-\eta)}$ and integrating from $z = -\infty$ to $z = \infty$, we find

that g must satisfy

$$\left(\frac{d^2}{dx^2} + \tau^2 + k^2(x) - k_0^2\right)g(x, \xi; \tau) = -\delta(x-\xi). \quad (2.2.3)$$

The boundary condition (2.1.3) requires that

$$g(0, \xi; \tau) = 0 \quad (2.2.4a)$$

while the continuity and radiation conditions require that

$$g(x, \xi; \tau) = g(a, \xi; \tau) e^{-j\tau(x-a)}, \quad \text{Im}\tau \leq 0, \quad x \geq a. \quad (2.2.4b)$$

The solution of (2.2.3) and (2.2.4) is found to be [1]

$$\begin{aligned} g(x, \xi; \tau) = & \frac{\sin\lambda x [\lambda \cos\lambda(a-\xi) + j\tau \sin\lambda(a-\xi)]}{\lambda(\lambda \cos\lambda a + j\tau \sin\lambda a)} H(\xi-x) \quad (2.2.5) \\ & + \frac{\sin\lambda \xi [\lambda \cos\lambda(a-x) + j\tau \sin\lambda(a-x)]}{\lambda(\lambda \cos\lambda a + j\tau \sin\lambda a)} [H(x-\xi) - H(x-a)] \\ & + \frac{\sin\lambda \xi}{\lambda \cos\lambda a + j\tau \sin\lambda a} e^{-j\tau(x-a)} H(x-a), \quad \text{Im}\tau \leq 0 \end{aligned}$$

where λ is defined by (1.1.10). Hence the longitudinal spectral representation is

$$G(x, z; \xi, \eta) = \int_{-\infty}^{\infty} \frac{\sin\lambda x [\lambda \cos\lambda(a-x) + j\tau \sin\lambda(a-x)]}{\lambda(\lambda \cos\lambda a + j\tau \sin\lambda a)} e^{-j\beta(z-\eta)} d\beta \quad (2.2.6a)$$

when $0 \leq x \leq a$, and

$$G(x, z; \xi, \eta) = \int_{-\infty}^{\infty} \frac{\sin \lambda \xi}{\lambda \cos \lambda a + j \tau \sin \lambda a} e^{-j[\tau(x-a) + \beta(z-\eta)]} d\beta \quad (2.2.6b)$$

when $a \leq x$. Note that $\text{Im} \tau \leq 0$ in each of the integrands.

2.3 The Transverse Spectral Representation

The transverse spectral representation may be obtained directly from the longitudinal spectral representation by closing the contour of integration and applying the residue theorem. However, the situation is complicated by the fact that the transverse Green's function depends upon τ , a multivalued function of β [viz., eq.(2.1.7)]. Consequently, in order to remain on the spectral sheet of the β -plane (i.e., the sheet where $\text{Im} \tau \leq 0$) in (2.2.6), branch cuts must be introduced, beginning at the branch points $\beta = \pm k_0$, extending along the real axis to the origin, and then along the imaginary axis to $\mp \infty$. Assuming slight losses, the branch points and branch cuts are shifted away from the real and imaginary axes as illustrated in Fig. 2.4 [2], [3]. Also indicated are the finite number of surface wave poles of g located, in the limit of vanishingly small losses, between $k_0 < |\text{Re} \beta| < k$; the remaining infinity of complex poles of g are located on the non-spectral ($\text{Im} \tau > 0$) sheet of the β -plane [4].

Now let us define the integrals

$$I_{R+} \triangleq \int_{-R}^{+R} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta + \int_{C_+} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta$$

$$+ \int_{C_{b+}} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta \quad (2.3.1a)$$

when $z-\eta > 0$, and

$$I_{R-} \triangleq \int_{-R}^{+R} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta + \int_{C_-} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta \quad (2.3.1b)$$

$$+ \int_{C_{b-}} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta$$

when $z-\eta < 0$, where C_+ and C_- are semicircles (save for small gaps at the branch cuts) of radii $R > |k|$ in the lower and upper half β -plane, respectively, centered at the origin; and C_{b+} and C_{b-} are contours extending around the branch cuts in the lower and upper half β -plane, respectively. Applying the residue theorem, we find that

$$I_{R\pm} = \mp 2\pi j \sum_{n=s} \text{Res}[g(x, \xi; \tau) e^{-j\beta(z-\eta)}]_{\beta = \pm \beta_n} \quad (2.3.2)$$

$$= \begin{cases} 2\pi \sum_{n=s} \frac{\tau_n \sin \lambda_n \xi \sin \lambda_n x}{\beta_n (1 + j\tau_n a)} e^{\mp j\beta_n(z-\eta)}, & 0 \leq x \leq a \\ 2\pi \sum_{n=s} \frac{\tau_n \sin \lambda_n \xi \sin \lambda_n a}{\beta_n (1 + j\tau_n a)} e^{-j[\tau_n(x-a) \pm \beta_n(z-\eta)]}, & a \leq x < \infty \end{cases}$$

where the sum is taken over all surface wave poles. Since, as R approaches infinity, the first integral on the RHS of (2.3.1) approaches (2.2.6), while the integrals about $C_{\pm}$ approach zero, we may write

$$G(x, z; \xi, \eta) = \lim_{R \rightarrow \infty} \{ I_{R\pm} - \int_{C_{b\pm}} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta \} \quad (2.3.3)$$

where the (+) sign is taken for $z-\eta > 0$, and the (-) sign for

$z-\eta < 0$. In view of (2.3.2) and the fact that a change of the variable of integration from β to τ yields

$$\lim_{R \rightarrow \infty} \int_{C_{b \pm}} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\beta = - \int_{-\infty}^{\infty} \frac{1}{\beta} g(x, \xi; \tau) e^{-j\beta(z-\eta)} d\tau \quad (2.3.4)$$

where the integration is carried out on the sheet of the τ -plane where $\text{Im}\beta \leq 0$, we conclude that the transverse spectral representation is

$$G(x, z; \xi, \eta) = 2\pi \sum_{n=s} \frac{\tau_n \sin \lambda_n \xi \sin \lambda_n x}{\beta_n (1 + j\tau_n a)} e^{-j\beta_n |z-\eta|} \quad (2.3.5a)$$

$$+ \int_{-\infty}^{\infty} \frac{\tau \sin \lambda x < [\lambda \cos \lambda (a-x) + j\tau \sin \lambda (a-x)]}{\beta \lambda (\lambda \cos \lambda a + j\tau \sin \lambda a)} e^{-j\beta |z-\eta|} d\tau$$

when $0 \leq x \leq a$, and

$$G(x, z; \xi, \eta) = 2\pi \sum_{n=s} \frac{\tau_n \sin \lambda_n \xi \sin \lambda_n a}{\beta_n (1 + j\tau_n a)} e^{-j[\tau_n (x-a) + \beta_n |z-\eta|]} \quad (2.3.5b)$$

$$+ \int_{-\infty}^{\infty} \frac{\tau \sin \lambda \xi}{\beta \lambda (\lambda \cos \lambda a + j\tau \sin \lambda a)} e^{-j[\tau (x-a) + \beta |z-\eta|]} d\tau$$

when $a \leq x$, where $\text{Im}\beta \leq 0$ in each of the integrands.

2.4 Discussion

The transverse spectral representation, (2.3.5), consists of a finite sum plus an integral. The terms of the sum are the residues of the integrand of the longitudinal spectral representation (2.2.6) at the spectral (i.e., surface wave) poles of g , while the integral is a branch cut integral due to the branch point singularities of g . However, the x -dependent factors of the terms of the sum are the transverse eigenfunctions, and the variable of integra-

tion in the branch cut integral extends over the continuous spectrum of the transverse differential system (2.1.5) [1]. Hence we see that the discrete and continuous spectra of the transverse differential system are associated with the spectral pole and branch point singularities, respectively, of the transverse Green's function.

It is noted that the infinity of non-spectral poles of g (i.e., those located on the non-spectral sheet of the β - plane) do not contribute to the spectral representation of the open waveguide field. However, in approximating integrals of the type in (2.3.5), some of the non-spectral poles may be associated with field contributions in certain limited spatial regions [3]. In the next chapter we present a method which associates all of the singularities of the transverse Green's function, including the non-spectral poles, with contributions to the field of the planar open waveguide.

CHAPTER III

NON-SPECTRAL FIELD REPRESENTATIONS IN PLANAR OPEN WAVEGUIDES

Waveguides are used both as transmission media and as basic elements in the construction of signal processing devices. Metal walled closed waveguides are usually used at microwave frequencies, while open waveguides, consisting of thin dielectric films, are used at optical frequencies. In analyzing both types of guides either a longitudinal or a transverse spectral representation of their electromagnetic fields is customarily employed.

In closed waveguides the longitudinal spectral representation involves a continuous spectrum only, while the transverse spectral representation involves a purely discrete spectrum. As a result, the field solution is expressed as an integral over the continuous spectrum with the former, and as a sum over the discrete spectrum with the latter. From a physical viewpoint the transverse representation is often more useful, insofar as the discrete spectral contributions correspond to distinct and independent longitudinally propagating solutions with finite power content. In contrast, each term in the continuum of the longitudinal representation has no clear physical meaning; only their sum

total, as expressed by the integral, carries finite power.

Open waveguides differ notably from closed waveguides in their transverse spectral representation. As an inherent consequence of the fact that the field is not totally confined, but extends in the transverse direction exterior to the guide, the transverse representation contains a continuous, as well as a discrete, spectrum [1]. The task of determining the open waveguide field therefore must always involve the evaluation of an integral over a continuous spectrum, regardless of whether a longitudinal or a transverse representation is employed [2].

The saddle point method (or method of steepest descent) is an approximate method for evaluating such integrals which yields accurate results in the far-field of the source [2]-[4]. With this method the path of integration is deformed into another path, the steepest descent path. The integrand decays rapidly along this path, except in the vicinity of the saddle point of the argument of the exponential factor in the integrand. The integral is evaluated approximately by expanding the integrand in a Taylor series about this critical point and integrating termwise. The resulting series is an asymptotic expansion of the actual solution.

In addition to the asymptotic results, residue contributions from the poles of the integrand which are crossed in deforming the original path into the steepest descent path must be accounted for. Since the steepest descent path

is a function of the angular position of the observation point, different poles may be crossed for different regions of space. Furthermore, since portions of the steepest descent path generally cross branch cuts onto non-spectral Riemann sheets, poles crossed in deforming the contour are often non-spectral poles. The corresponding residues give rise to waves which decay exponentially in the longitudinal direction, but increase exponentially in the transverse direction exterior to the guide. If allowed to continue indefinitely in the transverse direction, these non-spectral eigenfunctions, or "leaky modes," would violate the radiation condition. However, since the pole crossings occur in certain angular regions of space only, these leaky waves have limited spatial existence; outside their angular regions they are not defined, and hence do not violate the radiation condition; within their regions of existence they are seen to be highly convergent portions of the radiation field [5].

The observation has been made that all practical closed waveguides are in fact open waveguides, since perfectly conducting walls are idealizations [6]. A rigorous approach to practical closed waveguides would thus necessarily be in terms of a transverse representation which contained a continuous spectrum. If the metallic walls have high enough conductivity, however, the assumption of infinite conductivity yields sufficiently accurate results. For waveguides whose boundaries are less highly conducting, the

accuracy of using the purely discrete spectrum of a waveguide with perfectly conducting walls suffers. In such instances an impedance type boundary condition can often be used [6]-[9]. This approach avoids the introduction of the continuous spectrum by replacing the actual boundaries with fictitious boundaries upon which the ratio of the field to its normal derivative is prescribed. It is recognized, however, that in every case a purely discrete spectrum is actually an approximation to the true spectrum that a more rigorous analysis would yield. Moreover, if these discrete modes were to be continued outside of the waveguide, they would generally not decay properly in the transverse direction, violating the radiation condition at infinity. Such modes are thus seen to be similar in behavior to the leaky modes of the ideal open waveguide, inasmuch as they are highly convergent approximations to the radiation field within a specific spatial region. For this reason, such wave types have been referred to as "quasimodes" in the literature [6].

It is of interest to note that while the field of a practical closed waveguide is accurately expanded as an infinite sum of discrete quasimodes, only a finite number of the non-spectral eigenvalues of the ideal open waveguide give rise to leaky modes. The remaining infinity of non-spectral eigenvalues and eigenfunctions have, until recently [10], [11], been assumed unrelated to the open waveguide field. In this chapter we present a general method which

allows the open waveguide field to be expressed as a discrete sum, including an infinity of terms generated by the non-spectral eigenvalues. The method depends upon the intimate relationship which exists between the Green's function and the eigenfunctions of the transverse differential system associated with the interior of the waveguide [12]-[15]. It is developed here for various classes of planar open waveguides.

1 The Dielectric Slab - Impedance Plane Waveguide: TE Polarization

In this section the method for obtaining non-spectral field representations in open waveguides is developed, using as a prototype the TE field excited in a dielectric slab mounted on an impedance plane. The method is then applied to other planar problems in subsequent sections.

1.1 Formulation of Problem

Consider the planar ($\frac{\partial}{\partial y} \equiv 0$) open waveguide consisting of a dielectric slab mounted on an impedance plane at $x = 0$ and bounded by free space for $x > a$. The permittivity of the slab is assumed to be an analytic function of x . A time variation $e^{j\omega t}$ is assumed and suppressed. The situation is illustrated in Fig. 3.1.

From the wave equation for the electric field

$$\nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) + k^2 \vec{E} = j\omega\mu_0 \vec{J} \quad (1.1.1)$$

we find, upon setting

$$j\omega\mu_0 \vec{J} = -2\pi\delta(x-\xi)\delta(z-\eta)\hat{a}_y, \quad (1.1.2)$$

where $0 < \xi < a$ and $-\infty < \eta < \infty$, that the TE Green's function satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x)\right]G(x,z;\xi,\eta) = -2\pi\delta(x-\xi)\delta(z-\eta) \quad (1.1.3)$$

in the region $0 < x < \infty$, $-\infty < z < \infty$, subject to

$$z_{11}\left[\frac{\partial}{\partial x}G(x,z;\xi,\eta)\right]_{x=0} + z_{12}G(0,z;\xi,\eta) = 0 \quad (1.1.4)$$

and the radiation condition at infinity. Furthermore, since the remaining field components H_z and H_x are related to E_y through the curl equations according to

$$H_z = -\frac{1}{j\omega\mu_0} \frac{\partial}{\partial x} E_y \quad (1.1.5a)$$

and

$$H_x = \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} E_y, \quad (1.1.5b)$$

it follows that continuity of the tangential field components across the dielectric-free space boundary at $x = a$ requires that

$$G(a^-, z; \xi, \eta) = G(a^+, z; \xi, \eta) \quad (1.1.6a)$$

and

$$\left[\frac{\partial}{\partial x}G(x,z;\xi,\eta)\right]_{x=a^-} = \left[\frac{\partial}{\partial x}G(x,z;\xi,\eta)\right]_{x=a^+}. \quad (1.1.6b)$$

The TE impedance at $x = 0$ is defined as the ratio

$$Z_{TE} \triangleq \frac{E_y}{H_z}\bigg|_{x=0} = j\omega\mu_0 \frac{z_{11}}{z_{12}} \quad (1.1.7)$$

and is assumed constant (viz., z_{11} and z_{12} are constant).

We express G via its longitudinal (Fourier) representation:

$$G(x,z;\xi,\eta) = \int_{-\infty}^{\infty} g(x,\xi;\tau) e^{-j\beta(z-\eta)} d\beta \quad (1.1.8a)$$

where the transverse Green's function is given by

$$g(x, \xi; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, z; \xi, \eta) e^{j\beta(z-\eta)} dz \quad (1.1.8b)$$

and where the external transverse wavenumber is

$$\tau = (k_0^2 - \beta^2)^{1/2}. \quad (1.1.9)$$

Multiplying both sides of (1.1.3) by $\frac{1}{2\pi} e^{j\beta(z-\eta)}$ and integrating from $z = -\infty$ to $z = \infty$, we obtain the differential equation for the transverse Green's function:

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) + \tau^2 \right] g(x, \xi; \tau) = -\delta(x - \xi) \quad (1.1.10)$$

where

$$\Delta k^2(x) \triangleq k^2(x) - k_0^2. \quad (1.1.11)$$

The boundary condition (1.1.4) requires that

$$z_{11}g'(0, \xi; \tau) + z_{12}g(0, \xi; \tau) = 0 \quad (1.1.12a)$$

where the prime indicates differentiation with respect to x . In order that G satisfy (1.1.6a) and the radiation condition, we require that

$$g(x, \xi; \tau) = g(a, \xi; \tau) e^{-j\tau(x-a)}, \quad x \geq a, \quad \text{Im } \tau \leq 0 \quad (1.1.13a)$$

while (1.1.6 b) requires

$$g'(a^-, \xi; \tau) = g'(a^+, \xi; \tau). \quad (1.1.13b)$$

Equations (1.1.10), (1.1.12a) and (1.1.13) determine the transverse Green's function. Although the differential system is defined over the interval $0 \leq x < \infty$, we may focus attention on the interval $0 \leq x \leq a$ by noting that (1.1.13) implies

$$g'(a, \xi; \tau) + j\tau g(a, \xi; \tau) = 0, \quad \text{Im } \tau \leq 0 \quad (1.1.12b)$$

so that the equivalent interior (to the guide) problem is

given by (1.1.10) and (1.1.12). Hence, for the purpose of performing calculations within the guide, the entire region $x > a$ may be replaced by an impedance type boundary condition at $x = a$. Unlike the impedance at $x = 0$, however, the impedance at $x = a$ is a function of the spectral parameter. Moreover, the sign of $\text{Im}\tau$ is restricted.

For a reason which will be explained shortly, we temporarily drop the restriction on the sign of $\text{Im}\tau$. We therefore generate the new differential system

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) + \tau^2\right]g(x, \xi; \tau) = -\delta(x - \xi) \quad (1.1.14)$$

subject to

$$z_{11}g'(0, \xi; \tau) + z_{12}g(0, \xi; \tau) = 0 \quad (1.1.15a)$$

and

$$g'(a, \xi; \tau) + j\tau g(a, \xi; \tau) = 0 \quad (1.1.15b)$$

This differential system, whose solution is clearly the analytic continuation of the transverse Green's function into the upper half τ -plane, is an example of a non-selfadjoint second order differential system [14] whose eigensystem is

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) + \tau_n^2\right]\phi_n(x) = 0 \quad (1.1.16)$$

subject to the boundary conditions

$$z_{11}\phi_n'(0) + z_{12}\phi_n(0) = 0 \quad (1.1.17a)$$

and

$$\phi_n'(a) + j\tau_n\phi_n(a) = 0. \quad (1.1.17b)$$

We now see that the requirement $\text{Im}\tau \leq 0$ would single out

only the spectral eigenfunctions (e.g. surface mode solutions) from (1.1.16) and (1.1.17). Since there are only a finite number of these [3], they do not form a complete set of functions with which an arbitrary field solution could be represented. By temporarily dropping the restriction upon the sign of $\text{Im}\tau$, we include the infinity of non-spectral eigenfunctions (i.e., those corresponding to $\text{Im}\tau_n > 0$) in the completeness discussion.

Although completeness theorems exist for the eigenfunctions generated by certain types of non-selfadjoint boundary value problems [14], the dependence of (1.1.17b) upon the spectral parameter precludes their use in the present case. Non-selfadjoint problems similar to the above have, however, received some attention in the literature. Specifically, for certain types of non-selfadjoint problems involving the parameter in the boundary conditions, Tamarkin [16] showed that it was possible to generate null eigenfunction series which discontinuously changed to a non-zero value at the boundaries. In his study of a problem in diffusion, Peek [17] obtained two eigenfunction expansions, differing by a null series which discontinuously changed at one of the boundaries. Hille [18] noted that Peek's problem belongs to the class of boundary value problems considered by Tamarkin.

Setting $z_{11} = 0$ in (1.1.17a) yields the eigensystem considered by Siegert [19] in his study of nuclear scattering. This problem received additional attention by Humblet

[20] and by Regge [21], [22], who deduced the completeness of the eigenfunctions over the interval $(0, 2a)$ even though the differential system is restricted to the interval $(0, a)$. This implies that eigenexpansions on the smaller interval $(0, a)$ must be non-unique. Kravitskii [23] investigated the eigensystem in further detail, and showed that it was possible to generate an infinite number of uniformly convergent eigenexpansions, further emphasizing the non-uniqueness of the set of eigenfunctions.

The possibility of applying these results to obtain a non-spectral field representation in an open waveguide was first discussed by Shatrov [10]. By expanding the Fourier transform of the externally excited field into partial fractions, Shatrov was able to generate a non-spectral representation for the TE field in a type of grounded dielectric slab waveguide. He subsequently obtained two additional representations [11] by using Kravitskii's theorem.

It is therefore clear that the dependence of the boundary conditions upon the spectral parameter can lead to non-unique eigenexpansions. Intuitively, however, it might be expected that a criterion exists which would eliminate all but one of these expansions. Langer [24] noted that Peek's non-unique expansions failed to converge to the correct value at the upper boundary. By insisting upon correct convergence at the boundaries, Langer obtained a single eigenexpansion, distinct from Peek's. If we use boundary convergence as a criterion in Shatrov's problem, we find

that the first of his representations satisfies it, while the remaining two do not.

We obtain an expansion of the transverse Green's function in terms of the eigenfunctions defined by (1.1.16) and (1.1.17) which satisfies the boundary convergence criterion. We achieve our results by expressing g in its partial fraction representation via contour integration in the complex τ -plane. To this end, we first discuss the pertinent analytic properties of g .

1.2 The Transverse Green's Function

Writing (1.1.14) as

$$Lg = \delta(x-\xi) \quad (1.2.1a)$$

where L is the formally selfadjoint operator

$$L = -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \quad (1.2.1b)$$

with

$$p(x) = 1 \quad (1.2.1c)$$

and

$$q(x) = -[\Delta k^2(x) + \tau^2] \quad (1.2.1d)$$

we may immediately obtain g from the general formula [13]

$$g(x, \xi; \tau) = \frac{y_0(x, \tau)y_1(\xi, \tau)}{J(\tau)} H(\xi-x) + \frac{y_0(\xi, \tau)y_1(x, \tau)}{J(\tau)} H(x-\xi) \quad (1.2.2)$$

where H is the unit step function, y_0 and y_1 are the two independent solutions of the homogeneous equation

$$Ly_{0,1} = 0 \quad (1.2.3)$$

which satisfy the respective boundary conditions

$$z_{11}y_0'(0, \tau) + z_{12}y_0(0, \tau) = 0 \quad (1.2.4a)$$

and

$$y_1'(a, \tau) + j\tau y_1(a, \tau) = 0, \quad (1.2.4b)$$

and J is the conjunct of y_0 and y_1 :

$$J(\tau) = p(x) [y_0'(x, \tau)y_1(x, \tau) - y_0(x, \tau)y_1'(x, \tau)]. \quad (1.2.5)$$

Obviously, from (1.2.2) and (1.2.5), the analytic properties of g depend upon the properties of y_0 and y_1 .

For convenience we normalize y_0 and y_1 according to

$$y_0(0, \tau) = -z_{11}, \quad y_0'(0, \tau) = z_{12} \quad (1.2.6a)$$

$$y_1(a, \tau) = 1, \quad y_1'(a, \tau) = -j\tau \quad (1.2.6b)$$

and express them as the linear combinations

$$y_0(x, \tau) = A_{11} U_1(x, \tau) + A_{12} U_2(x, \tau) \quad (1.2.7a)$$

and

$$y_1(x, \tau) = A_{21} U_1(x, \tau) + A_{22} U_2(x, \tau), \quad (1.2.7b)$$

where U_1 and U_2 are the two independent solutions of

$$LU_{1,2} = 0 \quad (1.2.8)$$

which satisfy the respective boundary conditions

$$U_1(0, \tau) = 1, \quad U_1'(0, \tau) = 0 \quad (1.2.9a)$$

and

$$U_2(0, \tau) = 0, \quad U_2'(0, \tau) = 1. \quad (1.2.9b)$$

Now from the general theory of differential equations [12], it follows that U_1 and U_2 are integral functions of the parameter τ^2 . Since y_0 and y_1 are given by (1.2.7), they too will be integral functions of τ^2 if, and only if, the

constants (with respect to x) A_{ij} are also integral functions of τ^2 . To determine these constants, we substitute (1.2.7) into (1.2.6) and utilize (1.2.9) and the fact that the conjunct of U_1 and U_2 is independent of x [13]; we obtain

$$A_{11} = -z_{11}, \quad A_{12} = z_{12} \quad (1.2.10a)$$

and

$$A_{21} = U_2'(a, \tau) + j\tau U_2(a, \tau), \quad A_{22} = -U_1'(a, \tau) - j\tau U_1(a, \tau). \quad (1.2.10b)$$

From (1.2.7) and (1.2.10) it follows that y_0 is an integral function of τ^2 , while y_1 is not. The reason for this difference is the nature of the boundary conditions (1.2.4): the coefficients in (1.2.4a) are (independent of, and therefore) integral functions of the spectral parameter, whereas the coefficients in (1.2.4b) are not.

Substituting these results into the expression for g indicates that the transverse Green's function is regular throughout the τ^2 -plane, except at the zeros of J and at the branch point $\tau^2 = 0$. In obtaining the spectral representation for g , the presence of the branch point necessitates the introduction of an appropriate branch cut which, in turn, gives rise to the continuous spectrum. However, in the τ -plane the situation is different. Since U_1 and U_2 are integral functions of τ^2 , they are also integral functions of τ . It therefore follows from (1.2.7) that y_0 and y_1 are integral functions of τ too. This indicates that the transverse Green's function is regular throughout the τ -plane

except for pole singularities at the zeros of J . Consequently, g is a meromorphic function of τ .

According to the Mittag-Leffler theorem [25], every meromorphic function can be decomposed into partial fractions. We shall obtain the decomposition of g via contour integration in the τ -plane [26]. Since the method requires knowledge of the behavior of g for large $|\tau|$, we first obtain asymptotic (in τ) expansions of y_0 and y_1 , from which the asymptotic behavior of g will be deduced.

1.3 Asymptotic Expansion of $y_0(x, \tau)$ and $y_1(x, \tau)$

We construct y_0 and y_1 from two independent solutions, E_+ and E_- , of (1.2.3) having the asymptotic forms [27]

$$E_+(x, \tau) \sim e^{j\tau x \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x)} \quad (1.3.1a)$$

and

$$E_-(x, \tau) \sim e^{-j\tau x \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x)}. \quad (1.3.1b)$$

In order to determine the $a_k(x)$ and $b_k(x)$, (1.3.1) is substituted into (1.2.3) to obtain

$$e^{j\tau x \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k} [\Delta k^2(x) a_k(x) + 2j\tau a_k'(x) + a_k''(x)] = 0 \quad (1.3.2a)$$

and

$$e^{-j\tau x \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} [\Delta k^2(x) b_k(x) - 2j\tau b_k'(x) + b_k''(x)] = 0 \quad (1.3.2b)$$

which, upon equating like powers of τ to zero, leads to the recursion relationships

$$a_k'(x) = -\frac{1}{2} [\Delta k^2(x) a_{k-1}(x) + a_{k-1}''(x)], \quad k=0, 1, 2, \dots \quad (1.3.3a)$$

and

$$b_k'(x) = -\frac{1}{2} [\Delta k^2(x) b_{k-1}(x) + b_{k-1}'(x)], \quad k = 0, 1, 2, \dots \quad (1.3.3b)$$

where negative subscripted terms are interpreted as zero.

From (1.3.3) we see that the $a_k(x)$ and $b_k(x)$ are uniquely determined once their values at a single point in $[0, a]$ are specified. These values are obtained by constructing y_0 and y_1 as linear combinations of E_+ and E_- and enforcing the boundary conditions (1.2.6). The details of the calculation are in Appendix A. The results are, for y_0

$$y_0(x, \tau) \sim e^{j\tau x \sum} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + e^{-j\tau x \sum} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k a_k(x) \quad (1.3.4)$$

where the $a_k(x)$ are determined recursively by (1.3.3a) subject to

$$\begin{aligned} a_0 &\equiv -\frac{z_{11}}{2} \\ a_1(0) &= \frac{z_{12}}{2} \\ a_k(0) &= 0, \quad k = 2, 4, 6, \dots \\ a_k(0) &= -a_{k-1}'(0), \quad k = 3, 5, 7, \dots \end{aligned} \quad (1.3.5)$$

and for y_1

$$y_1(x, \tau) \sim e^{-j\tau(a-x) \sum} \sum_{k=\nu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + e^{j\tau(a-x) \sum} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x) \quad (1.3.6)$$

where the $a_k(x)$ and $b_k(x)$ are determined recursively by (1.3.3) subject to

$$\begin{aligned}
a_{\mu+2} &\equiv -(1/2)^{\mu+2} p_{\mu} \\
a_k(a) &= (-1)^{k-1} b_k(a), \quad k=\mu+3, \mu+4, \dots \\
b_0 &\equiv 1 \\
b_k(a) &= 0, \quad k = 1, 2, \dots, \mu+1 \\
b_k(a) &= -\frac{1}{2} b'_{k-1}(a), \quad k = \mu+2, \mu+3, \dots, 2\mu+3 \\
b_k(a) &= -\frac{1}{2} [b'_{k-1}(a) + (-1)^{k-1} a'_{k-1}(a)], \quad k=2\mu+4, 2\mu+5, \dots
\end{aligned} \tag{1.3.7}$$

where p_{μ} is the first non-vanishing derivative of $\Delta k^2(x)$ at $x = a$ (taken from the left):

$$p_{\mu} = \frac{d^{\mu}}{dx^{\mu}} \{\Delta k^2(x)\}_{x=a} \quad (1.3.8)$$

The order of this derivative, μ , is physically a measure of the smoothness of the change in the dielectric constant across $x = a$.

1.4 Asymptotic Behavior of $g(x, \xi; \tau)$

It is clear from (1.2.2) and (1.2.5) that the asymptotic behavior of g can be obtained by substitution of the asymptotic forms for y_0 and y_1 . From (1.3.4)-(1.3.8) we find that for sufficiently large $|\tau|$

$$y_0(x, \tau) = -\frac{z_{11}}{2} \{e^{j\tau x} [1+O(\frac{1}{\tau})] + e^{-j\tau x} [1+O(\frac{1}{\tau})]\} \quad (1.4.1)$$

and

$$y_1(x, \tau) = e^{j\tau(a-x)} [1+O(\frac{1}{\tau})] - \frac{p_{\mu}}{(2j\tau)^{\mu+2}} e^{-j\tau(a-x)} [1+O(\frac{1}{\tau})] \quad (1.4.2)$$

where O is the order symbol [27]. Upon substitution into (1.2.2), we obtain

$$\begin{aligned}
g(x, \xi; \tau) &= F_0(\tau, \xi - x - a) [1 + O(\frac{1}{\tau})] \\
&+ F_0(\tau, \xi + x - a) [1 + O(\frac{1}{\tau})] - \frac{p_\mu}{2} F_{\mu+2}(\tau, -\xi - x + a) [1 + O(\frac{1}{\tau})] \\
&- \frac{p_\mu}{2} F_{\mu+2}(\tau, -\xi + x + a) [1 + O(\frac{1}{\tau})], \quad 0 \leq x \leq \xi \leq a
\end{aligned} \tag{1.4.3}$$

where we have defined

$$F_k(\tau, w) \stackrel{\Delta}{=} - \frac{z_{11} e^{-j\tau w}}{2(j\tau)^k J(\tau)} \quad (1.4.4)$$

A similar expression is obtained for $0 \leq \xi \leq x \leq a$

The asymptotic behavior of g clearly depends upon J . Since J is independent of x , we may set $x=0$ in (1.2.5) and use (1.2.6a) to obtain

$$J(\tau) = z_{12} y_1(0, \tau) + z_{11} y_1'(0, \tau) \quad (1.4.5)$$

From (1.3.6) we have

$$y_1(0, \tau) \sim e^{-j\tau a_\Sigma} \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(0) + e^{j\tau a_\Sigma} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(0) \quad (1.4.6)$$

and

$$\begin{aligned}
y_1'(0, \tau) &\sim e^{-j\tau a_\Sigma} \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k [j\tau a_k(0) + a_k'(0)] \\
&+ e^{j\tau a_\Sigma} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k [-j\tau b_k(0) + b_k'(0)] \quad (1.4.7)
\end{aligned}$$

Therefore, we obtain the asymptotic expansion

$$\begin{aligned}
J(\tau) &\sim (z_{12} + j\tau z_{11}) e^{-j\tau a_\Sigma} \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(0) \\
&+ z_{11} e^{-j\tau a_\Sigma} \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k'(0) + (z_{12} - j\tau z_{11}) e^{j\tau a_\Sigma} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(0) \\
&+ z_{11} e^{j\tau a_\Sigma} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k'(0)
\end{aligned} \tag{1.4.8}$$

which for sufficiently large $|\tau|$, assuming that τ is not a zero of J and $z_{11} \neq 0$, behaves as

$$J(\tau) = -j\tau z_{11} \left\{ e^{j\tau a} \left[1 + O\left(\frac{1}{\tau}\right) \right] + \frac{p_\mu}{(2j\tau)^{\mu+2}} e^{-j\tau a} \left[1 + O\left(\frac{1}{\tau}\right) \right] \right\}. \quad (1.4.9)$$

Hence the asymptotic behavior of g depends upon that of

$$F_k(\tau, w) = \frac{e^{-j\tau w}}{2(j\tau)^{k+1} \left\{ e^{j\tau a} \left[1 + O\left(\frac{1}{\tau}\right) \right] + \frac{p_\mu}{(2j\tau)^{\mu+2}} e^{-j\tau a} \left[1 + O\left(\frac{1}{\tau}\right) \right] \right\}} \quad (1.4.10)$$

as w varies between $-a$ and a . Assuming that τ is not a zero of J , we obtain

$$F_k(\tau, w) = O(\tau^{-(k+1)}), \quad -a \leq w \leq 0, \quad \text{Im}\tau \leq 0 \quad (1.4.11a)$$

$$F_k(\tau, w) = O(e^{-|\text{Im}\tau|a} \tau^{-(k+1)}), \quad 0 \leq w \leq a, \quad \text{Im}\tau \leq 0 \quad (1.4.11b)$$

$$F_k(\tau, w) = O(e^{-|\text{Im}\tau|a} \tau^{\mu+1-k}), \quad -a \leq w \leq 0, \quad \text{Im}\tau > 0 \quad (1.4.11c)$$

and

$$F_k(\tau, w) = O(\tau^{\mu+1-k}), \quad 0 \leq w \leq a, \quad \text{Im}\tau > 0 \quad (1.4.11d)$$

for sufficiently large $|\tau|$. Noting that $0 \leq x \leq \xi \leq a$, we have

$$\begin{aligned} -a &\leq \xi - x - a \leq 0 \\ -a &\leq \xi + x - a \leq a \\ -a &\leq -\xi - x + a \leq a \\ 0 &\leq -\xi + x + a \leq a \end{aligned} \quad (1.4.12)$$

so that (1.4.3) and (1.4.11) yield

$$\begin{aligned} g(x, \xi; \tau) &= O(\tau^{-1}) + O(\tau^{-1}) \\ &+ O(\tau^{-(\mu+3)}) + O(e^{-|\text{Im}\tau|a} \tau^{-(\mu+3)}), \quad \text{Im}\tau \leq 0 \end{aligned} \quad (1.4.13a)$$

and

$$g(x, \xi; \tau) = O(e^{-|\text{Im}\tau|a} \tau^{\mu+1}) + O(\tau^{\mu+1}) \quad (1.4.13b)$$

$$+ O(\tau^{-1}) + O(\tau^{-1}), \quad \text{Im}\tau > 0$$

for the transverse Green's function in the region $0 \leq x \leq \xi \leq a$. By interchanging x and ξ , the transverse Green's function is obtained in the region $0 \leq \xi \leq x \leq a$. Because of this symmetry, the asymptotic behavior of g is the same in both regions. Hence, (1.4.13) gives the asymptotic behavior of g for all x and ξ in the closed interval $[0, a]$. It should be noted that this result is independent of the constant impedance at $x = 0$.

1.5 Partial Fraction Expansion

Knowing the asymptotic behavior of g , we may now proceed to determine its partial fraction expansion via contour integration [26]. Initially we consider the integral

$$I_N = \int_{C_N} \frac{g(x, \xi; \tau)}{\tau^P} d\tau \quad (1.5.1)$$

where C_N is a circle of radius R_N , centered at the origin and passing between the N th and $N+1$ st poles of g . The number P is to be chosen the least positive integer that insures I_N converges to zero as $R_N \rightarrow \infty$. Since $\tau = R_N e^{j\theta}$ on C_N , we find

$$|I_N| \leq R_N^{1-P} \int_0^{2\pi} |g(x, \xi; R_N e^{j\theta})| d\theta. \quad (1.5.2)$$

Substituting (1.4.13) into (1.5.2), we find that the second term on the RHS of (1.4.13b) contributes most to the integral (see Appendix B), yielding

$$I_N = O(R_N^{\mu+2-P}). \quad (1.5.3)$$

Obviously, then, we must choose

$$P = \mu + 3 \quad (1.5.4)$$

in order to obtain the required result

$$\lim_{R_N \rightarrow \infty} I_N = 0. \quad (1.5.5)$$

From (1.5.1), (1.5.4), and (1.5.5) we may therefore conclude that

$$\lim_{R_N \rightarrow \infty} \int_{C_N} \frac{g(x, \xi; s)}{s^{\mu+2} (s-\tau)} ds = 0. \quad (1.5.6)$$

The partial fraction expansion of g is obtained by applying the residue theorem to (1.5.6). Since the integrand has poles at $s = \tau_n$ (the poles of g), $s = \tau$, and a $\mu+2$ nd order pole at $s = 0$, we see that the sum of the corresponding residues must be zero. Hence,

$$\frac{g(x, \xi; \tau)}{\tau^{\mu+2}} + \frac{1}{(\mu+1)!} \frac{d^{\mu+1}}{ds^{\mu+1}} \left(\frac{g(x, \xi; s)}{s-\tau} \right)_{s=0} + \sum_{n=1}^{\infty} \frac{1}{\tau_n^{\mu+2}} \frac{\text{Res } g(x, \xi; \tau_n)}{\tau_n - \tau} = 0. \quad (1.5.7)$$

Rearranging, we arrive at the desired partial fraction expansion:

$$g(x, \xi; \tau) = \frac{\tau^{\mu+2}}{(\mu+1)!} \frac{d^{\mu+1}}{ds^{\mu+1}} \left(\frac{g(x, \xi; s)}{\tau-s} \right)_{s=0} + \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^{\mu+2} \frac{\text{Res } g(x, \xi; \tau_n)}{\tau - \tau_n}. \quad (1.5.8)$$

This expansion may be put in a more convenient form. Using Leibniz's theorem [28] [see eq. (A29)] the first term on the RHS of (1.5.8) may be expressed as

$$\frac{\tau^{\mu+2}}{(\mu+1)!} \frac{d^{\mu+1}}{ds^{\mu+1}} \left(\frac{g(x, \xi; s)}{\tau-s} \right)_{s=0} = \sum_{n=0}^{\mu+1} \frac{\tau^n}{n!} \left[\frac{d^n}{d\tau^n} g(x, \xi; \tau) \right]_{\tau=0} \quad (1.5.9)$$

which is recognized as the $\mu+2$ nd partial sum of a Maclaurin series expansion of g as a function of τ . In Appendix C it is shown, assuming simple poles, that

$$\text{Res } g(x, \xi; \tau_n) = \frac{-1}{2\tau_n} \phi_n(x) \phi_n(\xi) \quad (1.5.10)$$

where the normalized eigenfunctions are

$$\phi_n(x) = \frac{y_0(x, \tau_n)}{\left[\int_0^a y_0^2(x, \tau_n) dx - j \frac{y_0^2(a, \tau_n)}{2\tau_n} \right]^{1/2}} \quad (1.5.11)$$

Hence the final form of the expansion is

$$g(x, \xi; \tau) = \sum_{n=0}^{\mu+1} \tau^n g_n(x, \xi) - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^{\mu+2} \frac{\phi_n(x) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)} \quad (1.5.12)$$

where

$$g_n(x, \xi) \triangleq \frac{1}{n!} \left[\frac{d^n}{d\tau^n} g(x, \xi; \tau) \right]_{\tau=0} \quad (1.5.13)$$

Expansion (1.5.12) is similar to that which is obtained in closed waveguides, insofar as it involves an infinite sum of eigenfunctions. However, only a finite number of these eigenfunctions are spectral (i.e., give rise to surface modes). Moreover, an additional finite sum of terms appears which has no counterpart in closed waveguides. By substituting (1.5.12) into (1.1.14) and (1.1.15) and equating like powers of τ , these terms can be shown to be solutions to the auxiliary boundary value problems.

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) \right] g_n(x, \xi) = -g_{n-2}(x, \xi), \quad n=0, 1, \dots, \mu+1 \quad (1.5.14)$$

such that

$$z_{11}g_n'(0, \xi) + z_{12}g_n(0, \xi) = 0 \quad (1.5.15a)$$

and

$$g_n'(a, \xi) = -jg_{n-1}(a, \xi), \quad (1.5.15b)$$

where $g_{-2}(x, \xi) \triangleq \delta(x - \xi)$ and $g_{-1}(x, \xi) \triangleq 0$.

Furthermore, the total number of these terms, as well as the exponent of the weighting factor in the infinite sum, depends upon the smoothness parameter μ .

1.6 Non-Spectral Representation

The interior field is obtained from (1.5.12) and (1.1.8a). However since, from (1.1.9), τ is a multivalued function of β , g has branch points at $\beta = \pm k_0$, and appropriate branch cuts must be introduced in the β -plane before the integration is performed. These branch cuts are chosen to implement the previously discarded restriction on the sign of $\text{Im}\tau$; accordingly, they are chosen to separate the spectral sheet, where the integration is to be performed (viz., the sheet upon which $\text{Im}\tau < 0$), from the non-spectral sheet (upon which $\text{Im}\tau > 0$). The appropriate branch cuts therefore begin at $\beta = \pm k_0$, extend along the real axis to the origin, and then along the imaginary axis to $\mp \infty$ [2], [4]. Assuming slight losses, the branch points and branch cuts are shifted away from the axes as shown in Fig. 3.2. Consequently,

$$G(x, z; \xi, \eta) = \int_{-\infty}^{\infty} \left\{ \sum_{n=0}^{\mu+1} \tau^n g_n(x, \xi) - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^{\mu+2} \frac{\phi_n(x) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)} \right\} e^{-j\beta(z-\eta)} d\beta, \quad 0 \leq x \leq a \quad (1.6.1)$$

where the integration is performed along the real axis in Fig. 3.2.

The transverse representation is obtained by closing the contour of integration in (1.6.1) in the lower (upper) half plane for $z-\eta$ positive (negative), and applying the residue theorem. In order to remain on the spectral sheet, however, the contours must be warped around the branch cuts as shown in Fig. 3.2. Application of the residue theorem therefore yields (1.6.1) as an integral around a semi-circle, plus a branch cut integral, plus a sum of residues from the spectral poles. From (1.4.13a) and Appendix B it follows that the integrals along $C_{\pm}$ vanish. As a result, after changing the variable of integration from β to τ in the branch cut integrals and reversing the order of integration and summation, we find that the transverse representation of the interior field is

$$\begin{aligned}
 G(x, z; \xi, \eta) = & \sum_{n=0}^{\mu+1} g_n(x, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j\beta|z-\eta|} d\tau \\
 & - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(x) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+3} \frac{1}{\beta(\tau-\tau_n)} e^{-j\beta|z-\eta|} d\tau \\
 & - j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(x) \phi_n(\xi) e^{-j\beta_n|z-\eta|}, \quad 0 \leq x \leq a
 \end{aligned} \tag{1.6.2}$$

where $\text{Im } \beta, \beta_n \leq 0$ in each of the terms. This result demonstrates that each singularity of the transverse Green's function may be associated with discrete contributions to the total field $G(x, z; \xi, \eta)$: The first sum on the RHS represents contributions to the radiation field from the branch point

singularity of the transverse Green's function ($\beta = \pm k_0$ in the β -plane); those terms of the sum corresponding to even values of n vanish because their integrands are odd functions of τ . The infinite sum is generated by the spectral and non-spectral poles of g , and also represents contributions to the radiation field; the amount contributed by each term of the sum depends upon the smoothness parameter μ through the weighting factor in each integrand. The last sum consists of surface waves; they are generated by the spectral poles and are the familiar discrete contributions which appear in the spectral representation of the field.

Each term of the non-spectral representation has the form of a product of a function of x with a function of z , just as do the terms in the transverse spectral representation in a closed waveguide. However, (1.6.2) does not satisfy the wave equation term by term, nor are the functions of x mutually orthogonal. Furthermore, only the surface waves have a tabulated z -variation.

Knowing the field within the waveguide, it may readily be extended to the exterior region. From (1.1.8a), (1.1.13a) and (1.5.12), we obtain

$$\begin{aligned}
 G(x, z; \xi, \eta) &= \sum_{n=0}^{\mu+1} g_n(a, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j[\tau(x-a)+\beta|z-\eta|]} d\tau \\
 &- \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(a) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+3} \frac{1}{\beta(\tau-\tau_n)} e^{-j[\tau(x-a)+\beta|z-\eta|]} d\tau \\
 &- j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(a) \phi_n(\xi) e^{-j[\tau_n(x-a)+\beta_n|z-\eta|]}, \quad x \geq a. \quad (1.6.3)
 \end{aligned}$$

With the exception of the surface waves, the form of each of these terms is more complicated than the corresponding terms in (1.6.2). However, the far-field may readily be found using the method of steepest descent [2], [4]. We obtain

$$G(r, \theta) \sim \left(\frac{2\pi}{k_0 r}\right)^{1/2} e^{-j(k_0 r - \pi/4)} \left\{ \sum_{n=0}^{\mu+1} (k_0 \cos \theta)^{n+1} g_n(a, \xi) - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n}\right)^{\mu+3} \frac{\phi_n(a) \phi_n(\xi)}{k_0 \cos \theta - \tau_n} \right\} \\ - j \pi \sum_{n=\ell} \frac{\phi_n(a) \phi_n(\xi)}{\beta_n} e^{-j[\tau_n(x-a) + \beta_n |z-\eta|]} H(|\theta| - |\theta_n|), \\ |\theta| \neq \pi/2. \quad (1.6.4)$$

In the last sum, which extends over only the leaky waves, $\text{Im} \beta_n < 0$. The cylindrical coordinates r and θ are defined by

$$r \cos \theta = x - a, \quad r \sin \theta = z - \eta, \quad (1.6.5)$$

and θ_n is the capture angle of the corresponding leaky wave pole:

$$1 = \cosh \eta_n \cos(\sigma_n - \theta_n) \quad (1.6.6a)$$

where

$$\tau_n = k_0 \cos(\sigma_n + j \eta_n). \quad (1.6.6b)$$

The first two sums on the RHS of (1.6.4) contain cylindrical waves contributed by the branch point and pole singularities, respectively, of the transverse Green's function. The last sum contains inhomogeneous plane waves (i.e., leaky waves) contributed by those non-spectral poles satisfying (1.6.6). For sufficiently large $k_0 r$, the cylindrical waves dominate, while for smaller distances the leaky waves become important.

For τ_n satisfying $0 < \text{Re}\tau_n < k_0$ and $0 < \text{Im}\tau_n < \ll k_0$, the corresponding cylindrical waves peak in magnitude at an angle close to the capture angle of the corresponding leaky waves, a result consistent with the known effect of leaky waves on the radiation field [2], [4], [5].

It should be mentioned that since the response to the more general excitation

$$j\omega\mu_0 J_y(x, z) = -2\pi f(x) \delta(z-n) \quad (1.6.7)$$

where $f(x) \equiv 0$ for $x > a$, is given by

$$E_y(x, z) = \int_{-\infty}^{\infty} \langle g, f \rangle e^{-j\beta(z-n)} d\beta \quad (1.6.8a)$$

with

$$\langle g, f \rangle = \int_0^a g(x, \xi; \tau) f(\xi) d\xi, \quad (1.6.8b)$$

it follows that the non-spectral representation of the field due to an arbitrary internal line source parallel to the x -axis may be obtained directly from (1.6.2) and (1.6.3) by replacing $g_n(x, \xi)$ and $\phi_n(\xi)$ with $\langle g_n, f \rangle$ and $\langle \phi_n, f \rangle$, respectively.

Finally, it is noted that in the special case $z_{11}=0$ and $\mu=0$, the above reduces to Shatrov's [10] results. The present analysis indicates that a smoother dielectric transition at $x=a$ increases the number of terms contributed by the branch point, while changes in the impedance at $x=0$ have no effect upon the form of the representation.

2 Extension to Dielectric Slab: TE Polarization

The analysis of the previous section demonstrated that the non-spectral representation of the TE polarized field in the dielectric slab-impedance plane waveguide is affected by the smoothness of the transition across the dielectric-free space boundary. In this section the analysis is extended to the case of a dielectric slab that is not mounted on an impedance plane and, hence, has two dielectric-free space boundaries. It is found that the smoothness of the transitions across both dielectric-free space boundaries jointly affect the non-spectral representation.

2.1 The Transverse Green's Function

The waveguide, illustrated in Fig. 3.3, consists of a slab of dielectric material of thickness $2a$, bounded by free space for $|x| > a$. The permittivity is assumed to be an analytic function of x , and the TE excitation is provided by an internal current source $j\omega\mu_0 J_y = -2\pi\delta(x-\xi)\delta(z-\eta)$. The TE Green's function therefore satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x)\right]G(x, z; \xi, \eta) = -2\pi\delta(x-\xi)\delta(z-\eta) \quad (2.1.1)$$

with $-a < \xi < a$ and $-\infty < \eta < \infty$, subject to continuity of G and its normal derivative at $x = \pm a$, and the radiation condition at infinity. Transforming (2.1.1) according to (1.1.8), we find that the transverse Green's function satisfies

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) + \tau^2\right]g(x, \xi; \tau) = -\delta(x-\xi). \quad (2.1.2)$$

Since these equations are identical to (1.1.3) and (1.1.10), it follows that continuity of G and its normal derivative at $x=a$ and the radiation condition at infinity leads to the boundary condition (1.1.12b):

$$g'(a, \xi; \tau) + j\tau g(a, \xi; \tau) = 0, \quad \text{Im}\tau \leq 0. \quad (2.1.3a)$$

The boundary condition at $x=-a$ is obtained by noting that the continuity and radiation conditions require that

$$g'(-a^+, \xi; \tau) = g'(-a^-, \xi; \tau) \quad (2.1.4a)$$

and

$$g(x, \xi; \tau) = g(-a, \xi; \tau) e^{j\tau(x+a)}, \quad x \leq -a, \quad \text{Im}\tau \leq 0 \quad (2.1.4b)$$

which together imply

$$g'(-a, \xi; \tau) - j\tau g(-a, \xi; \tau) = 0, \quad \text{Im}\tau \leq 0. \quad (2.1.3b)$$

The equivalent interior problem is thus given by (2.1.2) and (2.1.3).

Temporarily dropping the restriction on the sign of $\text{Im}\tau$, we consider the analytic continuation of g into the upper half τ -plane:

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) + \tau^2 \right] g(x, \xi; \tau) = -\delta(x-\xi) \quad (2.1.5)$$

subject to

$$g'(a, \xi; \tau) + j\tau g(a, \xi; \tau) = 0 \quad (2.1.6a)$$

and

$$g'(-a, \xi; \tau) - j\tau g(-a, \xi; \tau) = 0. \quad (2.1.6b)$$

The solution may be obtained by noting that (2.1.5) is identical to (1.2.1); hence g may be expressed as (1.2.2):

$$g(x, \xi; \tau) = \frac{Y_0(x, \tau) Y_1(\xi, \tau)}{J(\tau)} H(\xi-x) + \frac{Y_0(\xi, \tau) Y_1(x, \tau)}{J(\tau)} H(x-\xi) \quad (2.1.7)$$

where now y_1 and y_0 are the two solutions to (1.2.3) which satisfy the respective boundary conditions

$$y_1'(a, \tau) + j\tau y_1(a, \tau) = 0 \quad (2.1.8a)$$

and

$$y_0'(-a, \tau) - j\tau y_0(-a, \tau) = 0 \quad (2.1.8b)$$

and $J(\tau)$ is their conjunct:

$$J(\tau) = y_0'(x, \tau)y_1(x, \tau) - y_0(x, \tau)y_1'(x, \tau). \quad (2.1.9)$$

If, for convenience, y_1 and y_0 are normalized according to

$$y_1(a, \tau) = 1, \quad y_1'(a, \tau) = -j\tau \quad (2.1.10a)$$

and

$$y_0(-a, \tau) = 1, \quad y_0'(-a, \tau) = j\tau \quad (2.1.10b)$$

and expressed as linear combinations of U_1 and U_2 , defined by (1.2.8) and (1.2.9), it is found that

$$\begin{aligned} y_1(x, \tau) = & [U_2'(a, \tau) + j\tau U_2(a, \tau)]U_1(x, \tau) \quad (2.1.11a) \\ & - [U_1'(a, \tau) + j\tau U_1(a, \tau)]U_2(x, \tau) \end{aligned}$$

and

$$\begin{aligned} y_0(x, \tau) = & [U_2'(-a, \tau) - j\tau U_2(-a, \tau)]U_1(x, \tau) \quad (2.1.11b) \\ & - [U_1'(-a, \tau) - j\tau U_1(-a, \tau)]U_2(x, \tau). \end{aligned}$$

Since U_1 and U_2 are integral functions of the parameter, these results indicate that in the τ^2 -plane g has a branch point singularity at the origin, as well as pole singularities at the zeros of J . As a function of τ , however, the only singularities of g are poles. Hence it follows that g is a meromorphic function of τ which, consequently, may be expanded into partial fractions. The asymptotic behavior of g will first be obtained so that the

partial fraction expansion may be found via contour integration.

2.2 Asymptotic Behavior of $g(x, \xi; \tau)$

The asymptotic behavior of g is obtained through (2.1.7), (2.1.9), and the asymptotic expansions of y_0 and y_1 . Comparing (2.1.11a) with (1.2.7b) and (1.2.10b), we immediately have from (1.3.6)

$$y_1(x, \tau) \sim e^{-j\tau(a-x)} \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + e^{j\tau(a-x)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x) \quad (2.2.1)$$

where the $a_k(x)$ and $b_k(x)$ are determined recursively from (1.3.3) subject to (1.3.7). For y_0 we find that (see Appendix A)

$$y_0(x, \tau) \sim e^{j\tau(a+x)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + e^{-j\tau(a+x)} \sum_{k=\nu+2}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x) \quad (2.2.2)$$

where the $a_k(x)$ and $b_k(x)$ are determined recursively from (1.3.3) subject to

$$b_{\nu+2} \equiv -(1/2)^{\nu+2} q_{\nu} \quad (2.2.3)$$

$$b_k(-a) = (-1)^{k-1} a_k(-a) \quad , \quad k=\nu+3, \nu+4, \dots$$

$$a_0 \equiv 1$$

$$a_k(-a) = 0 \quad , \quad k=1, 2, \dots, \nu+1$$

$$a_k(-a) = -\frac{1}{2} a'_{k-1}(-a) \quad , \quad k=\nu+2, \nu+3, \dots, 2\nu+3$$

$$a_k(-a) = -\frac{1}{2} [a'_{k-1}(-a) + (-1)^{k-1} b'_{k-1}(-a)] ,$$

$$k=2\nu+4, 2\nu+5, \dots$$

where q_{ν} is the first non-vanishing derivative of $\Delta k^2(x)$ at

$x = -a$ (taken from the right):

$$q_v = \frac{d^v}{dx^v} \{ \Delta k^2(x) \}_{x=-a}. \quad (2.2.4)$$

The order of this derivative, v , is physically a measure of the change in the dielectric constant across the lower boundary. These results indicate that for sufficiently large $|\tau|$,

$$y_1(x, \tau) = e^{j\tau(a-x)} [1 + O(\frac{1}{\tau})] - \frac{p_\mu}{(2j\tau)^{\mu+2}} e^{-j\tau(a-x)} [1 + O(\frac{1}{\tau})] \quad (2.2.5)$$

and

$$y_0(x, \tau) = e^{j\tau(a+x)} [1 + O(\frac{1}{\tau})] - \frac{q_v}{(-2j\tau)^{v+2}} e^{-j\tau(a+x)} [1 + O(\frac{1}{\tau})]. \quad (2.2.6)$$

Upon substitution into (2.1.7), we find that

$$\begin{aligned} g(x, \xi; \tau) = & F_0(\tau, \xi - x - 2a) [1 + O(\frac{1}{\tau})] - \frac{p_\mu}{2^{\mu+2}} F_{\mu+2}(\tau, -\xi - x) [1 + O(\frac{1}{\tau})] \\ & - \frac{q_v}{2^{v+2}} F_{v+2}(\tau, \xi + x) [1 + O(\frac{1}{\tau})] \\ & + (-1)^{v+2} \frac{q_v p_\mu}{2^{v+\mu+4}} F_{v+\mu+4}(\tau, -\xi + x + 2a) [1 + O(\frac{1}{\tau})] \end{aligned} \quad (2.2.7)$$

for $-a \leq x \leq \xi \leq a$, where we now have defined

$$F_k(\tau, w) \triangleq \frac{e^{-j\tau w}}{(j\tau)^k J(\tau)}. \quad (2.2.8)$$

In order to determine the asymptotic behavior of g , we need an asymptotic expression for $J(\tau)$. Since $J(\tau)$ is independent of x , set $x = -a$ in (2.1.9) and utilize (2.1.10b):

$$J(\tau) = j\tau y_1(-a, \tau) - y_1'(-a, \tau). \quad (2.2.9)$$

From (2.2.1) we have

$$y_1(-a, \tau) \sim e^{-j2\tau a \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k} a_k(-a) + e^{j2\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} b_k(-a) \quad (2.2.10)$$

and

$$\begin{aligned} y_1'(-a, \tau) \sim & e^{-j2\tau a \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k} [j\tau a_k(-a) + a_k'(-a)] \\ & + e^{j2\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} [-j\tau b_k(-a) + b_k'(-a)] \end{aligned} \quad (2.2.11)$$

so that

$$\begin{aligned} J(\tau) \sim & -e^{-j2\tau a \sum_{k=\mu+2}^{\infty} \left(\frac{1}{j\tau}\right)^k} a_k'(-a) \\ & + e^{j2\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} [j2\tau b_k(-a) - b_k'(-a)]. \end{aligned} \quad (2.2.12)$$

Utilizing equations (A20a) and (A29), we find that the $a_k(x)$ associated with $y_1(x, \tau)$ satisfy

$$\begin{aligned} a_k'(-a) &= 0, \quad k < \nu + \mu + 3 \\ a_{\nu+\mu+3}'(-a) &= -(-1)^{\nu+1} \frac{q_\nu p_\mu}{2^{\nu+\mu+3}} \end{aligned} \quad (2.2.13)$$

so that for sufficiently large $|\tau|$,

$$\begin{aligned} J(\tau) &= e^{-j2\tau a} (-1)^{\nu+1} \frac{q_\nu p_\mu}{(2j\tau)^{\nu+\mu+3}} [1 + O(\frac{1}{\tau})] \\ &+ j2\tau e^{j2\tau a} [1 + O(\frac{1}{\tau})]. \end{aligned} \quad (2.2.14)$$

The asymptotic behavior of g is thus determined by

$$F_k(\tau, w) = \frac{e^{-j\tau w}}{2(j\tau)^{k+1} \{ e^{j2\tau a} [1 + O(\frac{1}{\tau})] + (-1)^{\nu+1} e^{-j2\tau a} \frac{q_\nu p_\mu}{(2j\tau)^{\nu+\mu+4}} [1 + O(\frac{1}{\tau})] \}} \quad (2.2.15)$$

as w varies between $\pm 2a$. Assuming that τ is not a zero of $J(\tau)$, we find that

$$F_k(\tau, w) = O(\tau^{-(k+1)}), \quad -2a \leq w \leq 0, \quad \text{Im}\tau \leq 0 \quad (2.2.16a)$$

$$F_k(\tau, w) = O(e^{-|\text{Im}\tau| 2a} \tau^{-(k+1)}), \quad 0 \leq w \leq 2a, \quad \text{Im}\tau \leq 0 \quad (2.2.16b)$$

$$F_k(\tau, w) = O(e^{-|\text{Im}\tau| 2a} \tau^{\mu+\nu+3-k}), \quad -2a \leq w \leq 0, \quad \text{Im}\tau > 0 \quad (2.2.16c)$$

and

$$F_k(\tau, w) = O(\tau^{\mu+\nu+3-k}), \quad 0 \leq w \leq 2a, \quad \text{Im}\tau > 0 \quad (2.2.16d)$$

for sufficiently large $|\tau|$. Since $-a \leq x \leq \xi \leq a$ in (2.2.7),

$$\begin{aligned} -2a &\leq \xi - x \leq 0 \\ -2a &\leq -\xi - x \leq 2a \\ -2a &\leq \xi + x \leq 2a \\ 0 &\leq -\xi + x \leq 2a \end{aligned} \quad (2.2.17)$$

so that (2.2.7) and (2.2.16) yield

$$\begin{aligned} g(x, \xi; \tau) &= O(\tau^{-1}) + O(\tau^{-(\mu+3)}) + O(\tau^{-(\nu+3)}) \\ &\quad + O(e^{-|\text{Im}\tau| 2a} \tau^{-(\mu+\nu+5)}), \quad \text{Im}\tau \leq 0 \end{aligned} \quad (2.2.18a)$$

and

$$\begin{aligned} g(x, \xi; \tau) &= O(e^{-|\text{Im}\tau| 2a} \tau^{\mu+\nu+3}) + O(\tau^{\nu+1}) \\ &\quad + O(\tau^{\mu+1}) + O(\tau^{-1}), \quad \text{Im}\tau > 0 \end{aligned} \quad (2.2.18b)$$

for sufficiently large $|\tau|$ in the region $-a \leq x \leq \xi \leq a$. Inasmuch as the transverse Green's function is a symmetric function of x and ξ , (2.2.18) gives the asymptotic behavior of g for all x and ξ in the closed interval $[-a, a]$.

2.3 Non-Spectral Representation

We again consider the integral I_N defined by (1.5.1),

$$I_N = \int_{C_N} \frac{g(x, \xi; \tau)}{\tau^P} d\tau \quad (2.3.1)$$

in order to determine the least positive integer P which will insure that I_N converges to zero as $R_N \rightarrow \infty$. From (1.5.2),

$$|I_N| \leq R_N^{1-P} \int_0^{2\pi} |g(x, \xi; R_N e^{j\theta})| d\theta. \quad (2.3.2)$$

Substituting (2.2.18), we find that the first term on the RHS of (2.2.18b) contributes most to the integral (see Appendix B), yielding

$$I_N = O(R_N^{\mu+\nu+3-P}). \quad (2.3.3)$$

Therefore, to insure that $\lim_{R_N \rightarrow \infty} I_N = 0$, we must choose

$$P = \mu + \nu + 4. \quad (2.3.4)$$

This result indicates that

$$\lim_{R_N \rightarrow \infty} \int_{C_N} \frac{g(x, \xi; s)}{s^{\mu+\nu+3}(s-\tau)} ds = 0 \quad (2.3.5)$$

which, upon application of the residue theorem, yields the partial fraction expansion of g :

$$\begin{aligned} g(x, \xi; \tau) &= \frac{\tau^{\mu+\nu+3}}{(\mu+\nu+2)!} \frac{d^{\mu+\nu+2}}{ds^{\mu+\nu+2}} \left\{ \frac{g(x, \xi; s)}{\tau-s} \right\}_{s=0} \\ &+ \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^{\mu+\nu+3} \frac{\text{Res } g(x, \xi; \tau_n)}{\tau - \tau_n}. \end{aligned} \quad (2.3.6)$$

This expansion may be more conveniently expressed by noting that

$$\frac{\tau^{\mu+\nu+3}}{(\mu+\nu+2)!} \frac{d^{\mu+\nu+2}}{ds^{\mu+\nu+2}} \left\{ \frac{g(x, \xi; s)}{\tau-s} \right\}_{s=0} = \sum_{n=0}^{\mu+\nu+2} \frac{\tau^n}{n!} \left[\frac{d^n}{d\tau^n} g(x, \xi; \tau) \right]_{\tau=0} \quad (2.3.7)$$

which follows from Leibniz's theorem, and (see Appendix C)

$$\text{Res } g(x, \xi; \tau_n) = - \frac{1}{2\tau_n} \phi_n(x) \phi_n(\xi) \quad (2.3.8)$$

where the normalized eigenfunctions are

$$\phi_n(x) = \frac{y_0(x, \tau_n)}{\left[\int_{-a}^a y_0^2(x, \tau_n) dx - j \frac{y_0^2(a, \tau_n) + 1}{2\tau_n} \right]^{1/2}} \quad (2.3.9)$$

Hence the final form of the partial fraction expansion is

$$g(x, \xi; \tau) = \sum_{n=0}^{\mu+\nu+2} \tau^n g_n(x, \xi) - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^{\mu+\nu+3} \frac{\phi_n(x) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)} \quad (2.3.10)$$

where the functions

$$g_n(x, \xi) \triangleq \frac{1}{n!} \left[\frac{d^n}{d\tau^n} g(x, \xi; \tau) \right]_{\tau=0} \quad (2.3.11)$$

can be shown, by substituting (2.3.10) into (2.1.5) and (2.1.6), to be solutions of the auxiliary boundary value problems

$$\left[\frac{d^2}{dx^2} + \Delta k^2(x) \right] g_n(x, \xi) = -g_{n-2}(x, \xi), \quad n=0, 1, 2, \dots, \mu+\nu+2 \quad (2.3.12)$$

subject to

$$g_n'(a, \xi) = -j g_{n-1}(a, \xi) \quad (2.3.13a)$$

and

$$g_n'(-a, \xi) = j g_{n-1}(-a, \xi) \quad (2.3.13b)$$

where $g_{-2}(x, \xi) \triangleq \delta(x - \xi)$ and $g_{-1}(x, \xi) \triangleq 0$.

Comparison of (2.3.10) with (1.5.12) indicates that the effect of the second dielectric - free space boundary is to add $\nu+1$ additional terms to the finite sum and to increase

the exponent of the weighting factor in the infinite sum by $\nu + 1$. Hence the non-spectral field representation, which is obtained by inverse Fourier transformation of (2.3.10), may be formally obtained by changing μ to $\mu + \nu + 1$ in (1.6.2) and (1.6.3):

$$G(x, z; \xi, \eta) = \sum_{n=0}^{\mu+\nu+2} g_n(x, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j\beta|z-\eta|} d\tau - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(x) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+\nu+4} \frac{1}{\beta(\tau-\tau_n)} e^{-j\beta|z-\eta|} d\tau - j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(x) \phi_n(\xi) e^{-j\beta_n|z-\eta|}, \quad |x| \leq a \quad (2.3.14)$$

and

$$G(x, z; \xi, \eta) = \sum_{n=0}^{\mu+\nu+2} g_n(\pm a, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j[\tau(|x|-a)+\beta|z-\eta|]} d\tau - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(\pm a) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+\nu+4} \frac{1}{\beta(\tau-\tau_n)} e^{-j[\tau(|x|-a)+\beta|z-\eta|]} d\tau - j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(\pm a) \phi_n(\xi) e^{-j[\tau_n(|x|-a)+\beta_n|z-\eta|]}, \quad |x| \geq a \quad (2.3.15)$$

where the plus sign is taken for $x > a$, and the minus sign is taken for $x < -a$. Similarly, for the far-field

$$G(r, \theta) \sim \left(\frac{2\pi}{k_0 r}\right)^{1/2} e^{-j(k_0 r - \pi/4)} \left\{ \sum_{n=0}^{\mu+\nu+2} (k_0 \cos \theta)^{n+1} g_n(\pm a, \xi) - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n}\right)^{\mu+\nu+4} \frac{\phi_n(\pm a) \phi_n(\xi)}{k_0 \cos \theta - \tau_n} \right\} - j\pi \sum_{n=s} \frac{\phi_n(\pm a) \phi_n(\xi)}{\beta_n} e^{-j[\tau_n(|x|-a)+\beta_n|z-\eta|]} H(|\theta| - |\theta_n|), \quad |\theta| \neq \pi/2 \quad (2.3.16)$$

where $r \sin \theta = z - \eta$ and $r \cos \theta = |x| - a$.

3 The Dielectric Slab - Impedance Plane Waveguide: TM Polarization

In this section the non-spectral representation of the TM polarized field in the dielectric slab-impedance plane waveguide is obtained. It is found that the representation is affected by the smoothness of the dielectric-free space transition in a manner which is quantitatively different from the TE case.

3.1 Formulation of Problem

Since the wave equation for the magnetic field is

$$\nabla^2 \vec{H} + k^2 \vec{H} + \frac{\nabla \epsilon}{\epsilon} \times \nabla \times \vec{H} = j\omega \epsilon \vec{V} \quad (3.1.1)$$

where the excitation $\vec{V}$ is defined as

$$j\omega \epsilon \vec{V} = \frac{\nabla \epsilon}{\epsilon} \times \vec{J} - \nabla \times \vec{J} \quad (3.1.2)$$

we find, upon setting

$$j\omega \epsilon \vec{V} = -2\pi \epsilon \delta(x-\xi) \delta(z-\eta) \hat{a}_y, \quad (3.1.3)$$

where $0 < \xi < a$ and $-\infty < \eta < \infty$, that the TM Green's function for the dielectric slab-impedance plane waveguide satisfies

$$\frac{1}{\epsilon(x)} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x) - \frac{\epsilon'(x)}{\epsilon(x)} \frac{\partial}{\partial x} \right] G(x, z; \xi, \eta) = -2\pi \delta(x-\xi) \delta(z-\eta) \quad (3.1.4)$$

in the region $0 < x < \infty$, $-\infty < z < \infty$, subject to

$$z_{21} \left[\frac{\partial}{\partial x} G(x, z; \xi, \eta) \right]_{x=0} + z_{22} G(0, z; \xi, \eta) = 0 \quad (3.1.5)$$

and the radiation condition at infinity. In addition, since the curl equations relate H_y to the remaining field components E_z and E_x according to

$$E_z = \frac{1}{j\omega\epsilon} \frac{\partial}{\partial x} H_y \quad (3.1.6a)$$

and

$$E_x = -\frac{1}{j\omega\epsilon} \frac{\partial}{\partial z} H_y, \quad (3.1.6b)$$

it follows that continuity of the tangential field components across the dielectric-free space boundary requires that

$$G(a^-, z; \xi, \eta) = G(a^+, z; \xi, \eta) \quad (3.1.7a)$$

and

$$\frac{1}{\epsilon(a^-)} \left[\frac{\partial}{\partial x} G(x, z; \xi, \eta) \right]_{x=a^-} = \frac{1}{\epsilon_0} \left[\frac{\partial}{\partial x} G(x, z; \xi, \eta) \right]_{x=a^+}. \quad (3.1.7b)$$

The TM impedance at $x = 0$ is defined as the ratio

$$Z_{TM} \triangleq \frac{E_z}{H_y} \Big|_{x=0} = -j\omega\epsilon(0) \frac{z_{22}}{z_{21}} \quad (3.1.8)$$

and is assumed constant. It is further assumed that $\epsilon(x)$ does not vanish.

Transforming (3.1.4) according to (1.1.8), we find that the transverse Green's function satisfies

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) + \tau^2 \right] g(x, \xi; \tau) = -\delta(x - \xi). \quad (3.1.9)$$

The boundary condition (3.1.5) requires that

$$z_{21} g'(0, \xi; \tau) + z_{22} g(0, \xi; \tau) = 0. \quad (3.1.10a)$$

The radiation condition and (3.1.7a) will be satisfied if

$$g(x, \xi; \tau) = g(a, \xi; \tau) e^{-j\tau(x-a)}, \quad x \geq a, \quad \text{Im}\tau \leq 0 \quad (3.1.11a)$$

while (3.1.7b) requires that

$$g'(a^-, \xi; \tau) = \epsilon_R(a^-) g'(a^+, \xi; \tau) \quad (3.1.11b)$$

where

$$\epsilon_R(x) \triangleq \frac{\epsilon(x)}{\epsilon_0} . \quad (3.1.12)$$

While (3.1.9), (3.1.10a) and (3.1.11) determine the transverse Green's function for $0 < x < \infty$, we may concentrate on the interior of the guide by noting that (3.1.11) implies

$$g'(a, \xi; \tau) + j\tau \epsilon_R(a) g(a, \xi; \tau) = 0, \quad \text{Im} \tau \leq 0 \quad (3.1.10b)$$

where the terms are evaluated from the left. Hence for the purpose of performing calculations within the guide, the entire region $x > a$ may be replaced by a spectral parameter dependent impedance at $x = a$. The equivalent interior problem is therefore given by (3.1.9) and (3.1.10).

We shall temporarily lift the restriction on the sign of $\text{Im} \tau$ in (3.1.10b) and consider instead the solution to

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) + \tau^2 \right] g(x, \xi; \tau) = -\delta(x - \xi) \quad (3.1.13)$$

subject to

$$z_{21} g'(0, \xi; \tau) + z_{22} g(0, \xi; \tau) = 0 \quad (3.1.14a)$$

and

$$g'(a, \xi; \tau) + j\tau \epsilon_R(a) g(a, \xi; \tau) = 0 \quad (3.1.14b)$$

which is clearly the analytic continuation of the transverse Green's function into the upper half τ -plane. By doing so, we shall find it possible to generate a field representation which includes all of the eigenfunctions of the corresponding eigensystem

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) + \tau_n^2 \right] \phi_n(x) = 0 \quad (3.1.15)$$

subject to

$$z_{21}\phi_n'(0) + z_{22}\phi_n(0) = 0 \quad (3.1.16a)$$

and

$$\phi_n'(a) + j\tau\epsilon_R(a)\phi_n(a) = 0 \quad (3.1.16b)$$

rather than just the spectral ($\text{Im}\tau_n \leq 0$) eigenfunctions. We accomplish this by expanding g into partial fractions via contour integration in the τ -plane. To this end we first discuss the pertinent analytic properties of g .

3.2 The Transverse Green's Function

Writing (3.1.13) as

$$Lg = \delta(x-\xi) \quad (3.2.1a)$$

where L is the formally selfadjoint operator

$$L = -\frac{d}{dx} \left[p(x) \frac{d}{dx} \right] + q(x) \quad (3.2.1b)$$

with

$$p(x) = \frac{1}{\epsilon(x)} \quad (3.2.1c)$$

and

$$q(x) = -\frac{1}{\epsilon(x)} \{ \Delta k^2(x) + \tau^2 \} \quad (3.2.1d)$$

we may immediately obtain g from the general formula [13]

$$g(x, \xi; \tau) = \frac{y_0(x, \tau)y_1(\xi, \tau)}{J(\tau)} H(\xi-x) + \frac{y_0(\xi, \tau)y_1(x, \tau)}{J(\tau)} H(x-\xi) \quad (3.2.2)$$

where y_0 and y_1 are the two independent solutions of the homogeneous equation

$$Ly_{0,1} = 0 \quad (3.2.3)$$

satisfying the respective boundary conditions

$$z_{21}y_0'(0, \tau) + z_{22}y_0(0, \tau) = 0 \quad (3.2.4a)$$

and

$$y_1'(a, \tau) + j\tau\epsilon_R(a) y_1(a, \tau) = 0, \quad (3.2.4b)$$

and J is their conjunct:

$$J(\tau) = \frac{1}{\epsilon(x)} [y_0'(x, \tau)y_1(x, \tau) - y_0(x, \tau)y_1'(x, \tau)]. \quad (3.2.5)$$

For convenience, we normalize y_0 and y_1 according to

$$y_0(0, \tau) = -z_{21}, \quad y_0'(0, \tau) = z_{22} \quad (3.2.6a)$$

$$y_1(a, \tau) = 1, \quad y_1'(a, \tau) = -j\tau\epsilon_R(a) \quad (3.2.6b)$$

and express them as the linear combinations

$$y_0(x, \tau) = B_{11}V_1(x, \tau) + B_{12}V_2(x, \tau) \quad (3.2.7a)$$

and

$$y_1(x, \tau) = B_{21}V_1(x, \tau) + B_{22}V_2(x, \tau) \quad (3.2.7b)$$

where V_1 and V_2 are the two independent solutions of

$$LV_{1,2} = 0 \quad (3.2.8)$$

which satisfy the respective boundary conditions

$$V_1(0, \tau) = 1, \quad V_1'(0, \tau) = 0 \quad (3.2.9a)$$

and

$$V_2(0, \tau) = 0, \quad V_2'(0, \tau) = 1. \quad (3.2.9b)$$

The B_{ij} are determined by substituting (3.2.7) into (3.2.6) and utilizing (3.2.9) and the fact that the conjunct of V_1 and V_2 is independent of x [13]; we obtain

$$B_{11} = -z_{21}, \quad B_{12} = z_{22} \quad (3.2.10a)$$

and

$$B_{21} = \frac{\varepsilon(0)}{\varepsilon(a)} [V_2'(a, \tau) + j\tau \varepsilon_R(a) V_2(a, \tau)],$$

$$B_{22} = -\frac{\varepsilon(0)}{\varepsilon(a)} [V_1'(a, \tau) + j\tau \varepsilon_R(a) V_1(a, \tau)]. \quad (3.2.10b)$$

From (3.2.7), (3.2.10) and the fact that V_1 and V_2 are integral functions of τ^2 [12], it follows that y_0 is an integral function of τ^2 , while y_1 is not. As a function of τ , however, both y_0 and y_1 are integral. From these results it follows that as a function of τ^2 , the transverse Green's function has poles at the zeros of J and a branch point at the origin; as a function of τ , the only singularities are poles. Hence we see that g is a meromorphic function of τ which may therefore be expanded into partial fractions. In order to obtain the partial fraction expansion via contour integration, we must first obtain the asymptotic behavior of g . This is accomplished by determining the asymptotic expansions of y_0 and y_1 and using (3.2.2).

3.3 Asymptotic Expansion of $y_0(x, \tau)$ and $y_1(x, \tau)$

We construct y_0 and y_1 from two independent solutions, E_+ and E_- , of (3.2.3) having the asymptotic forms [27]

$$E_+(x, \tau) \sim \varepsilon^{\frac{1}{2}}(x) e^{j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) \quad (3.3.1a)$$

and

$$E_-(x, \tau) \sim \varepsilon^{\frac{1}{2}}(x) e^{-j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x). \quad (3.3.1b)$$

In order to determine the $a_k(x)$ and $b_k(x)$, (3.3.1) is substituted into (3.2.3) to obtain

$$\epsilon^{-\frac{1}{2}}(x) e^{j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k [\Xi(x) a_k(x) + 2j\tau a_k'(x) + a_k''(x)] = 0 \quad (3.3.2a)$$

and

$$\epsilon^{-\frac{1}{2}}(x) e^{-j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k [\Xi(x) b_k(x) - 2j\tau b_k'(x) + b_k''(x)] = 0 \quad (3.3.2b)$$

where

$$\Xi(x) \triangleq \Delta k^2(x) - \frac{3}{4} \left[\frac{\epsilon'(x)}{\epsilon(x)} \right]^2 + \frac{1}{2} \frac{\epsilon''(x)}{\epsilon(x)}. \quad (3.3.3)$$

Upon equating like powers of τ to zero, we obtain the recursion relationships

$$a_k'(x) = -\frac{1}{2} [\Xi(x) a_{k-1}(x) + a_{k-1}''(x)], \quad k=0,1,2,\dots \quad (3.3.4a)$$

and

$$b_k'(x) = -\frac{1}{2} [\Xi(x) b_{k-1}(x) + b_{k-1}''(x)], \quad k=0,1,2,\dots \quad (3.3.4b)$$

where negative subscripted terms are interpreted as zero.

The determination of the $a_k(x)$ and $b_k(x)$ from (3.3.4) is made unique once their values are specified at a single point in $[0,a]$. This is accomplished by constructing y_0 and y_1 as linear combinations of E_+ and E_- and enforcing the boundary conditions (3.2.6). The details of the calculation are given in Appendix A. The results are, for y_0

$$y_0(x, \tau) \sim \epsilon^{\frac{1}{2}}(x) e^{j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + \epsilon^{\frac{1}{2}}(x) e^{-j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k a_k(x) \quad (3.3.5)$$

where the $a_k(x)$ are determined recursively from (3.3.4a) subject to

$$a_0 = - \frac{z_{21}}{2 \epsilon^{\frac{1}{2}}(0)}$$

$$a_1(0) = \frac{1}{2 \epsilon^{\frac{1}{2}}(0)} [z_{22} + \frac{\epsilon'(0)}{2 \epsilon(0)} z_{21}] \quad (3.3.6)$$

$$a_k(0) = 0, \quad k=2,4,6,\dots$$

$$a_k(0) = -a'_{k-1}(0), \quad k=3,5,7,\dots,$$

and for y_1 ,

$$y_1(x, \tau) \sim \epsilon^{\frac{1}{2}}(x) e^{-j\tau(a-x)} \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) \quad (3.3.7)$$

$$+ \epsilon^{\frac{1}{2}}(x) e^{j\tau(a-x)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x),$$

where the $a_k(x)$ and $b_k(x)$ are determined recursively from (3.3.4) subject to

$$\left. \begin{aligned} a_0 &\equiv - \frac{P_0}{2k_0^2 \epsilon^{\frac{1}{2}}(a)} \\ a_1(a) &= - \frac{P_1}{4k^2(a) \epsilon^{\frac{1}{2}}(a)} \\ a_k(a) &= - \frac{1}{2} [a'_{k-1}(a) + (-1)^{k-1} b'_{k-1}(a)], \quad k=2,3,\dots \\ b_0 &\equiv \frac{k^2(a) + k_0^2}{2k_0^2 \epsilon^{\frac{1}{2}}(a)} \\ b_k(a) &= -(-1)^k a_k(a), \quad k=1,2,\dots \end{aligned} \right\} \mu=0$$

$$\left. \begin{aligned} a_\mu &\equiv - \frac{P_\mu}{2^{\mu+1} k_0^2 \epsilon^{\frac{1}{2}}(a)} \\ a_k(a) &= - \frac{1}{2} [a'_{k-1}(a) + (-1)^{k-1} b'_{k-1}(a)], \quad k=\mu+1, \mu+2, \dots \\ b_0 &\equiv \epsilon_0^{-\frac{1}{2}} \\ b_k(a) &= 0, \quad k=1,2,\dots, \mu-1 \\ b_k(a) &= -(-1)^k a_k(a), \quad k=\mu, \mu+1, \dots \end{aligned} \right\} \mu \geq 1$$

(3.3.8)

where p_m is the m th derivative of $\Delta k^2(x)$ at $x = a$ (taken from the left):

$$p_m = \frac{d^m}{dx^m} \{ \Delta k^2(x) \}_{x=a}, \quad m=0,1,2,\dots \quad (3.3.9)$$

The integer μ , we recall, is the smallest m for which p_m is non-zero.

3.4 Asymptotic Behavior of $g(x, \xi; \tau)$

The asymptotic behavior of g is obtained by substituting the asymptotic forms for y_0 and y_1 into (3.2.2). From (3.3.5)-(3.3.9) we find that for sufficiently large

$$|\tau|, \\ y_0(x, \tau) = -\frac{z_{21}}{2} \left(\frac{\varepsilon(x)}{\varepsilon(0)} \right)^{\frac{1}{2}} \{ e^{j\tau x} [1+O(\frac{1}{\tau})] + e^{-j\tau x} [1+O(\frac{1}{\tau})] \} \quad (3.4.1)$$

and

$$y_1(x, \tau) = \frac{1}{2k_0^2} \left(\frac{\varepsilon(x)}{\varepsilon(a)} \right)^{\frac{1}{2}} \{ (k^2(a) + k_0^2) e^{j\tau(a-x)} [1+O(\frac{1}{\tau})] \\ - \frac{p_\mu}{(2j\tau)^\mu} e^{-j\tau(a-x)} [1+O(\frac{1}{\tau})] \} \quad (3.4.2)$$

so that substitution into (3.2.2) yields

$$g(x, \xi; \tau) = (k^2(a) + k_0^2) F_0(\tau, \xi - x - a) [1+O(\frac{1}{\tau})] \\ + (k^2(a) + k_0^2) F_0(\tau, \xi + x - a) [1+O(\frac{1}{\tau})] - \frac{p_\mu}{2^\mu} F_\mu(\tau, -\xi - x + a) [1+O(\frac{1}{\tau})] \\ - \frac{p_\mu}{2^\mu} F_\mu(\tau, -\xi + x + a) [1+O(\frac{1}{\tau})], \quad 0 \leq x \leq \xi \leq a \quad (3.4.3)$$

where we have defined

$$F_k(\tau, w) \triangleq -\frac{z_{21}}{4k_0^2} \left(\frac{\varepsilon(x)\varepsilon(\xi)}{\varepsilon(0)\varepsilon(a)} \right)^{\frac{1}{2}} \frac{e^{-j\tau w}}{(j\tau)^k k_J(\tau)}. \quad (3.4.4)$$

A similar expression is obtained for $0 \leq \xi \leq x \leq a$.

The asymptotic behavior of g follows from (3.4.3), (3.4.4), and the asymptotic behavior of J . Since J is independent of x , we may set $x = 0$ in (3.2.5) and use (3.2.6a) to obtain

$$J(\tau) = \epsilon^{-1}(0) [z_{22}y_1(0, \tau) + z_{21}y_1'(0, \tau)]. \quad (3.4.5)$$

From (3.3.7) we have

$$\begin{aligned} y_1(0, \tau) \sim \epsilon^{\frac{1}{2}}(0) e^{-j\tau a \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(0)} \\ + \epsilon^{\frac{1}{2}}(0) e^{j\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(0)} \end{aligned} \quad (3.4.6)$$

and

$$\begin{aligned} y_1'(0, \tau) \sim \epsilon^{\frac{1}{2}}(0) e^{-j\tau a \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k} [j\tau a_k(0) + \frac{\epsilon'(0)}{2\epsilon(0)} a_k(0) + a_k'(0)] \\ + \epsilon^{\frac{1}{2}}(0) e^{j\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} [-j\tau b_k(0) + \frac{\epsilon'(0)}{2\epsilon(0)} b_k(0) + b_k'(0)]. \end{aligned} \quad (3.4.7)$$

Hence

$$\begin{aligned} J(\tau) \sim \epsilon^{-\frac{1}{2}}(0) (z_{22} + z_{21} \frac{\epsilon'(0)}{2\epsilon(0)} + j\tau z_{21}) e^{-j\tau a \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k} a_k(0) \\ + \epsilon^{-\frac{1}{2}}(0) z_{21} e^{-j\tau a \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k} a_k'(0) \\ + \epsilon^{-\frac{1}{2}}(0) (z_{22} + z_{21} \frac{\epsilon'(0)}{2\epsilon(0)} - j\tau z_{21}) e^{j\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} b_k(0) \\ + \epsilon^{-\frac{1}{2}}(0) z_{21} e^{j\tau a \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k} b_k'(0) \end{aligned} \quad (3.4.8)$$

which for sufficiently large $|\tau|$, assuming that τ is not a zero of J and $z_{21} \neq 0$, behaves as

$$J(\tau) = \frac{-j\tau z_{21}}{2k_0^2 \epsilon^{\frac{1}{2}}(0) \epsilon^{\frac{1}{2}}(a)} \{ (k^2(a) + k_0^2) e^{j\tau a} [1 + O(\frac{1}{\tau})] + \frac{P_\mu}{(2j\tau)^\mu} e^{-j\tau a} [1 + O(\frac{1}{\tau})] \}. \quad (3.4.9)$$

Therefore, the asymptotic behavior of g is determined by

$$F_k(\tau, w) = \frac{\varepsilon^{\frac{1}{2}}(x) \varepsilon^{\frac{1}{2}}(\xi)}{2(j\tau)^{k+1}} \cdot \frac{e^{-j\tau w}}{[k^2(a) + k_0^2] e^{j\tau a} [1 + O(\frac{1}{\tau})] + \frac{p_\mu}{(2j\tau)^\mu} e^{-j\tau a} [1 + O(\frac{1}{\tau})]} \quad (3.4.10)$$

as w varies between $-a$ and a , for τ not a zero of J . We have

$$F_k(\tau, w) = O(\tau^{-(k+1)}) , \quad -a \leq w \leq 0 , \quad \text{Im}\tau \leq 0 \quad (3.4.11a)$$

$$F_k(\tau, w) = O(e^{-|\text{Im}\tau| a_\tau^{-(k+1)}}) , \quad 0 \leq w \leq a , \quad \text{Im}\tau \leq 0 \quad (3.4.11b)$$

$$F_k(\tau, w) = O(e^{-|\text{Im}\tau| a_\tau^{\mu-k-1}}) , \quad -a \leq w \leq 0 , \quad \text{Im}\tau > 0 \quad (3.4.11c)$$

and

$$F_k(\tau, w) = O(\tau^{\mu-k-1}) , \quad 0 \leq w \leq a , \quad \text{Im}\tau > 0 \quad (3.4.11d)$$

for sufficiently large $|\tau|$. Since $0 \leq x \leq \xi \leq a$, we see that

$$\begin{aligned} -a &\leq \xi - x - a \leq 0 \\ -a &\leq \xi + x - a \leq a \\ -a &\leq -\xi - x + a \leq a \\ 0 &\leq -\xi + x + a \leq a \end{aligned} \quad (3.4.12)$$

so that (3.4.3) and (3.4.11) yield

$$g(x, \xi; \tau) = O(\tau^{-1}) + O(\tau^{-1}) + O(\tau^{-(\mu+1)}) + O(e^{-|\text{Im}\tau| a_\tau^{-(\mu+1)}}) , \quad \text{Im}\tau \leq 0 \quad (3.4.13a)$$

and

$$g(x, \xi; \tau) = O(e^{-|\text{Im}\tau| a_\tau^{\mu-1}}) + O(\tau^{\mu-1}) + O(\tau^{-1}) + O(\tau^{-1}) , \quad \text{Im}\tau > 0 \quad (3.4.13b)$$

for the transverse Green's function in the region $0 \leq x \leq \xi \leq a$.

Because g is symmetric in x and ξ , it follows that (3.4.13)

gives the asymptotic behavior of g for all x and ξ in the

closed interval $[0, a]$. Note that these results are independent of the constant impedance at $x = 0$.

3.5 Partial Fraction Expansion. Non-Spectral Representation

Knowing the asymptotic behavior of g , we may obtain its partial fraction expansion via contour integration in the τ -plane. This expansion may then be transformed according to (1.1.8a) to obtain the non-spectral field representation.

We again consider the integral I_N defined by (1.5.1),

$$I_N = \int_{C_N} \frac{g(x, \xi; \tau)}{\tau^P} d\tau \quad (3.5.1)$$

in order to determine the least positive integer P which will insure that I_N converges to zero as $R_N \rightarrow \infty$. From (1.5.2),

$$|I_N| \leq R_N^{1-P} \int_0^{2\pi} |g(x, \xi; R_N e^{j\theta})| d\theta. \quad (3.5.2)$$

Substituting (3.4.13) we find that the second term on the RHS of (3.4.13b) contributes most to the integral (see Appendix B), yielding

$$I_N = O(R_N^{\mu-P}). \quad (3.5.3)$$

Therefore, the least positive integer P for which

$$\lim_{R_N \rightarrow \infty} I_N = 0 \quad \text{is}$$

$$P = \mu + 1. \quad (3.5.4)$$

This result indicates that

$$\lim_{R_N \rightarrow \infty} \int_{C_N} \frac{g(x, \xi; s)}{s^{\mu+1}(s-\tau)} ds = 0 \quad (3.5.5)$$

which, upon application of the residue theorem, yields the partial fraction expansion of g :

$$g(x, \xi; \tau) = \frac{\tau^\mu}{(\mu-1)!} \frac{d^{\mu-1}}{ds^{\mu-1}} \left\{ \frac{g(x, \xi; s)}{\tau-s} \right\}_{s=0} + \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^\mu \frac{\text{Res } g(x, \xi; \tau_n)}{\tau - \tau_n}. \quad (3.5.6)$$

The first term, due to the μ th order pole of the integrand in (3.5.5), does not appear if $\mu = 0$.

This expression may be simplified by applying Leibniz's theorem to the first term on the RHS,

$$\frac{\tau^\mu}{(\mu-1)!} \frac{d^{\mu-1}}{ds^{\mu-1}} \left\{ \frac{g(x, \xi; s)}{\tau-s} \right\}_{s=0} = \sum_{n=0}^{\mu-1} \frac{\tau^n}{n!} \left[\frac{d^n}{d\tau} g(x, \xi; \tau) \right]_{\tau=0} \quad (3.5.7)$$

and by noting from Appendix C that

$$\text{Res } g(x, \xi; \tau_n) = -\frac{1}{2\tau_n} \phi_n(x) \phi_n(\xi) \quad (3.5.8)$$

where the normalized eigenfunctions are

$$\phi_n(x) = \frac{Y_0(x, \tau_n)}{\left[\int_0^a \frac{1}{\epsilon(x)} Y_0^2(x, \tau_n) dx - j \frac{Y_0^2(a, \tau_n)}{2\epsilon_0 \tau_n} \right]^{1/2}}. \quad (3.5.9)$$

Substituting these into (3.5.6) and the result into (3.1.13) and (3.1.14) we find that g may be expressed as

$$g(x, \xi; \tau) = \sum_{n=0}^{\mu-1} \tau^n g_n(x, \xi) - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^\mu \frac{\phi_n(x) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)} \quad (3.5.10)$$

where the functions

$$g_n(x, \xi) \triangleq \frac{1}{n!} \left[\frac{d^n}{d\tau} g(x, \xi; \tau) \right]_{\tau=0} \quad (3.5.11)$$

are solutions to the auxiliary boundary value problems

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) \right] g_n(x, \xi) = -g_{n-2}(x, \xi), \quad n=0, 1, 2, \dots, \mu-1 \quad (3.5.12)$$

subject to

$$z_{21} g_n'(0, \xi) + z_{22} g_n(0, \xi) = 0 \quad (3.5.13a)$$

and

$$g_n'(a, \xi) = -j \epsilon_R(a) g_{n-1}(a, \xi) \quad (3.5.13b)$$

where $g_{-2}(x, \xi) \triangleq \delta(x - \xi)$ and $g_{-1}(x, \xi) \triangleq 0$.

Comparing (3.5.10) with (1.5.12), we see that while the form of the partial fraction expansion of the transverse Green's function is the same for both the TE and TM cases, the effect of the smoothness of the dielectric-free space transition across $x = a$ is quantitatively different. Formally, (3.5.10) may be obtained from (1.5.12) by changing μ to $\mu-2$. Hence the non-spectral field representation, which is obtained by inverse Fourier transformation of (3.5.10), may be obtained formally by replacing μ with $\mu-2$ in (1.6.2) and (1.6.3):

$$G(x, z; \xi, \eta) = \sum_{n=0}^{\mu-1} g_n(x, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j\beta|z-\eta|} d\tau \quad (3.5.14)$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(x) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+1} \frac{1}{\beta(\tau - \tau_n)} e^{-j\beta|z-\eta|} d\tau$$

$$- j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(x) \phi_n(\xi) e^{-j\beta_n|z-\eta|}, \quad 0 \leq x \leq a$$

and

$$G(x, z; \xi, \eta) = \sum_{n=0}^{\mu-1} g_n(a, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j[\tau(x-a) + \beta|z-\eta|]} d\tau \quad (3.5.15)$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(a) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+1} \frac{1}{\beta(\tau - \tau_n)} e^{-j[\tau(x-a) + \beta|z-\eta|]} d\tau$$

$$- j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(a) \phi_n(\xi) e^{-j[\tau_n(x-a) + \beta_n|z-\eta|]}, \quad x \geq a$$

where $\text{Im } \beta, \beta_n \leq 0$ in each of the terms. Similarly, for the far-field,

$$\begin{aligned}
G(r, \theta) \sim & \left(\frac{2\pi}{k_0 r}\right)^{\frac{1}{2}} e^{-j(k_0 r - \pi/4)} \left\{ \sum_{n=0}^{\mu-1} (k_0 \cos \theta)^{n+1} g_n(a, \xi) \right. \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n}\right)^{\mu+1} \frac{\phi_n(a) \phi_n(\xi)}{k_0 \cos \theta - \tau_n} \Big\} \\
& - j\pi \sum_{n=\ell}^{\infty} \frac{\phi_n(a) \phi_n(\xi)}{\beta_n} e^{-j[\tau_n(x-a) + \beta_n |z-\eta|]} H(|\theta| - |\theta_n|), \\
& |\theta| \neq \pi/2.
\end{aligned}
\tag{3.5.16}$$

We recall that the last sum is taken only over leaky waves.

These results demonstrate that, as in the TE case, each singularity of the transverse Green's function may be associated with discrete contributions to the total field. However, unlike the TE case, the branch point contribution to the TM field (the first term on the RHS of each of the above) vanishes when $\mu = 0$ (i.e., when the permittivity is discontinuous across $x = a$).

4 Extension to Dielectric Slab: TM Polarization

The preceding analysis is extended in this section to the case of a dielectric slab. It is found that the joint effect of the smoothness of the dielectric-free space transitions is quantitatively different from the TE case.

4.1 The Transverse Green's Function

We assume that the permittivity of the waveguide in Fig. 3.3 does not vanish and that the internal excitation is given by (3.1.3). The TM Green's function therefore satisfies

$$\frac{1}{\epsilon(x)} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x) - \frac{\epsilon'(x)}{\epsilon(x)} \frac{\partial}{\partial x} \right] G(x, z; \xi, \eta) = -2\pi \delta(x-\xi) \delta(z-\eta) \tag{4.1.1}$$

with $-a < \xi < a$ and $-\infty < \eta < \infty$, subject to continuity of G and $\frac{1}{\epsilon} \frac{\partial}{\partial x} G$ at $x = \pm a$, and the radiation condition at infinity. Transforming (4.1.1) according to (1.1.8), we find that the transverse Green's function satisfies

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) + \tau^2 \right] g(x, \xi; \tau) = -\delta(x - \xi). \quad (4.1.2)$$

Since these equations are identical to (3.1.4) and (3.1.9), it follows that the continuity requirements at $x = a$ and the radiation condition at infinity lead to the boundary condition (3.1.10b):

$$g'(a, \xi; \tau) + j\tau \epsilon_R(a) g(a, \xi; \tau) = 0, \quad \text{Im} \tau \leq 0 \quad (4.1.3a)$$

where the terms are evaluated from the left. We may deduce the boundary condition at $x = -a$ by noting that the continuity and radiation conditions require that

$$g'(-a^+, \xi; \tau) = \epsilon_R(-a^+) g'(-a^-, \xi; \tau) \quad (4.1.4a)$$

and

$$g(x, \xi; \tau) = g(-a, \xi; \tau) e^{j\tau(x+a)}, \quad x \leq -a, \quad \text{Im} \tau \leq 0 \quad (4.1.4b)$$

which together imply

$$g'(-a, \xi; \tau) - j\tau \epsilon_R(-a) g(-a, \xi; \tau) = 0, \quad \text{Im} \tau \leq 0 \quad (4.1.3b)$$

where the terms are evaluated from the right. The equivalent interior problem is thus given by (4.1.2) and (4.1.3).

Temporarily dropping the restriction on the sign of $\text{Im} \tau$, we consider the analytic continuation of g into the upper half τ -plane:

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) + \tau^2 \right] g(x, \xi; \tau) = -\delta(x - \xi) \quad (4.1.5)$$

subject to

$$g'(a, \xi; \tau) + j\tau \epsilon_R(a)g(a, \xi; \tau) = 0 \quad (4.1.6a)$$

and

$$g'(-a, \xi; \tau) - j\tau \epsilon_R(-a)g(-a, \xi; \tau) = 0. \quad (4.1.6b)$$

The solution may be obtained by noting that (4.1.5) is identical to (3.2.1); hence g may be expressed as (3.2.2):

$$g(x, \xi; \tau) = \frac{y_0(x, \tau)y_1(\xi, \tau)}{J(\tau)} H(\xi - x) + \frac{y_0(\xi, \tau)y_1(x, \tau)}{J(\tau)} H(x - \xi) \quad (4.1.7)$$

where now y_1 and y_0 are the two solutions of (3.2.3) satisfying, respectively,

$$y_1'(a, \tau) + j\tau \epsilon_R(a)y_1(a, \tau) = 0 \quad (4.1.8a)$$

and

$$y_0'(-a, \tau) - j\tau \epsilon_R(-a)y_0(-a, \tau) = 0 \quad (4.1.8b)$$

and J is their conjunct:

$$J(\tau) = \frac{1}{\epsilon(x)} [y_0'(x, \tau)y_1(x, \tau) - y_0(x, \tau)y_1'(x, \tau)]. \quad (4.1.9)$$

If, for convenience, y_0 and y_1 are normalized according to

$$y_1(a, \tau) = 1, \quad y_1'(a, \tau) = -j\tau \epsilon_R(a) \quad (4.1.10a)$$

and

$$y_0(-a, \tau) = 1, \quad y_0'(-a, \tau) = j\tau \epsilon_R(-a) \quad (4.1.10b)$$

and expressed as linear combinations of V_1 and V_2 , defined by (3.2.8) and (3.2.9), it is found that

$$\begin{aligned} y_1(x, \tau) &= \frac{\epsilon(0)}{\epsilon(a)} [V_2'(a, \tau) + j\tau \epsilon_R(a)V_2(a, \tau)] V_1(x, \tau) \\ &- \frac{\epsilon(0)}{\epsilon(a)} [V_1'(a, \tau) + j\tau \epsilon_R(a)V_1(a, \tau)] V_2(x, \tau) \end{aligned} \quad (4.1.11a)$$

and

$$\begin{aligned}
y_0(x, \tau) = & \frac{\varepsilon(0)}{\varepsilon(-a)} [V_2'(-a, \tau) - j\tau \varepsilon_R(-a) V_2(-a, \tau)] V_1(x, \tau) \quad (4.1.11b) \\
& - \frac{\varepsilon(0)}{\varepsilon(-a)} [V_1'(-a, \tau) - j\tau \varepsilon_R(-a) V_1(-a, \tau)] V_2(x, \tau).
\end{aligned}$$

Since V_1 and V_2 are integral functions of the parameter, these results indicate that in the τ^2 -plane g has a branch point singularity at the origin, as well as pole singularities at the zeros of J . As a function of τ , however, the only singularities of g are poles. Hence it follows that g is a meromorphic function of τ which, consequently, may be expanded into partial fractions. We first obtain the asymptotic behavior of g so that its partial fraction expansion may be obtained via contour integration.

4.2 Asymptotic Behavior of $g(x, \xi; \tau)$

The asymptotic behavior of g follows from (4.1.7), (4.1.9), and the asymptotic expansions of y_0 and y_1 . Comparing (4.1.11a) with (3.2.7b) and (3.2.10b), we immediately have from (3.3.7)

$$\begin{aligned}
y_1(x, \tau) \sim & \varepsilon^{\frac{1}{2}}(x) e^{-j\tau(a-x)} \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) \quad (4.2.1) \\
& + \varepsilon^{\frac{1}{2}}(x) e^{j\tau(a-x)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x)
\end{aligned}$$

where the $a_k(x)$ and $b_k(x)$ are determined recursively from (3.3.4) subject to (3.3.8). For y_0 we find that (see Appendix A)

$$\begin{aligned}
y_0(x, \tau) \sim & \varepsilon^{\frac{1}{2}}(x) e^{j\tau(a+x)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) \quad (4.2.2) \\
& + \varepsilon^{\frac{1}{2}}(x) e^{-j\tau(a+x)} \sum_{k=\nu}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x)
\end{aligned}$$

where the $a_k(x)$ and $b_k(x)$ are determined recursively from (3.3.4), subject to

$$\left. \begin{aligned}
 b_0 &\equiv -\frac{q_0}{2k_0^2 \varepsilon^{\frac{1}{2}}(-a)} \\
 b_1(-a) &\equiv -\frac{q_1}{4k^2(-a) \varepsilon^{\frac{1}{2}}(-a)} \\
 b_k(-a) &= -\frac{1}{2} [b'_{k-1}(-a) + (-1)^{k-1} a'_{k-1}(-a)], \quad k=2,3,\dots \\
 a_0 &\equiv \frac{k^2(-a) + k_0^2}{2k_0^2 \varepsilon^{\frac{1}{2}}(-a)} \\
 a_k(-a) &= -(-1)^k b_k(-a), \quad k=1,2,\dots \\
 b_v &\equiv -\frac{q_v}{2^{v+1} k_0^2 \varepsilon^{\frac{1}{2}}} \\
 b_k(-a) &= -\frac{1}{2} [b'_{k-1}(-a) + (-1)^{k-1} a'_{k-1}(-a)], \quad k=v+1, v+2, \dots \\
 a_0 &\equiv \varepsilon_0^{-\frac{1}{2}} \\
 a_k(-a) &= 0, \quad k=1,2,\dots, v-1 \\
 a_k(-a) &= -(-1)^k b_k(-a), \quad k=v, v+1, \dots
 \end{aligned} \right\} \begin{array}{l} v=0 \\ v \geq 1 \end{array}$$

(4.2.3)

where q_m is the m th derivative of $\Delta k^2(x)$ at $x = -a$ (taken from the right):

$$q_m = \frac{d^m}{dx^m} \{\Delta k^2(x)\}_{x=-a}. \quad (4.2.4)$$

These results indicate that for sufficiently large $|\tau|$,

$$\begin{aligned}
 y_1(x, \tau) &= \frac{1}{2k_0^2} \left(\frac{\varepsilon(x)}{\varepsilon(a)} \right)^{\frac{1}{2}} \{ [k^2(a) + k_0^2] e^{j\tau(a-x)} [1 + O(\frac{1}{\tau})] \\
 &\quad - \frac{P_\mu}{(2j\tau)^\mu} e^{-j\tau(a-x)} [1 + O(\frac{1}{\tau})] \}
 \end{aligned} \quad (4.2.5)$$

and

$$y_0(x, \tau) = \frac{1}{2k_0^2} \left(\frac{\varepsilon(x)}{\varepsilon(-a)} \right)^{1/2} \{ (k^2(-a) + k_0^2) e^{j\tau(a+x)} [1 + O(\frac{1}{\tau})] - \frac{q_v}{(-2j\tau)^v} e^{-j\tau(a+x)} [1 + O(\frac{1}{\tau})] \} \quad (4.2.6)$$

which, upon substitution into (4.1.7), yields

$$\begin{aligned} g(x, \xi; \tau) &= (k^2(-a) + k_0^2) (k^2(a) + k_0^2) F_0(\tau, \xi - x - 2a) [1 + O(\frac{1}{\tau})] \\ &\quad - (k^2(-a) + k_0^2) \frac{p_\mu}{2^\mu} F_\mu(\tau, -\xi - x) [1 + O(\frac{1}{\tau})] \\ &\quad - (k^2(a) + k_0^2) \frac{q_v}{2^v} F_v(\tau, \xi + x) [1 + O(\frac{1}{\tau})] \\ &\quad + \frac{p_\mu q_v}{2^{\mu+v}} F_{\mu+v}(\tau, -\xi + x + 2a) [1 + O(\frac{1}{\tau})], \end{aligned} \quad (4.2.7)$$

for $-a \leq x \leq \xi \leq a$, where

$$F_k(\tau, w) \triangleq \frac{1}{4k_0^4} \left(\frac{\varepsilon(x)\varepsilon(\xi)}{\varepsilon(-a)\varepsilon(a)} \right)^{1/2} \frac{e^{-j\tau w}}{(j\tau)^k J(\tau)}. \quad (4.2.8)$$

Hence, the asymptotic behavior of g can be obtained once the asymptotic behavior of J is found.

Setting $x = -a$ in (4.1.9), we find that

$$J(\tau) = \varepsilon^{-1}(-a) [j\tau \varepsilon_R(-a) y_1(-a, \tau) - y_1'(-a, \tau)], \quad (4.2.9)$$

where we have made use of (4.1.10b). Noting that

$$\begin{aligned} y_1(-a, \tau) &\sim \varepsilon^{1/2}(-a) e^{-j2\tau a} \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau} \right)^k a_k(-a) \\ &\quad + \varepsilon^{1/2}(-a) e^{j2\tau a} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau} \right)^k b_k(-a) \end{aligned} \quad (4.2.10)$$

and

$$y_1'(-a, \tau) \sim \varepsilon^{1/2}(-a) e^{-j2\tau a} \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau} \right)^k [j\tau a_k(-a) + \frac{\varepsilon'(-a)}{2\varepsilon(-a)} a_k(-a) + a_k'(-a)]$$

$$+ \epsilon^{\frac{1}{2}}(-a) e^{j2\tau a} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k [-j\tau b_k(-a) + \frac{\epsilon'(-a)}{2\epsilon(-a)} b_k(-a) + b_k'(-a)] \quad (4.2.11)$$

we obtain

$$J(\tau) \sim \epsilon^{-\frac{1}{2}}(-a) \{ e^{-j2\tau a} \sum_{k=\mu}^{\infty} \left(\frac{1}{j\tau}\right)^k [j\tau(\epsilon_R(-a)-1)a_k(-a) - \frac{\epsilon'(-a)}{2\epsilon(-a)} a_k(-a) \quad (4.2.12)$$

$$-a_k'(-a)] + e^{j2\tau a} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k [j\tau(\epsilon_R(-a)+1)b_k(-a) - \frac{\epsilon'(-a)}{2\epsilon(-a)} b_k(-a) - b_k'(-a)] \}.$$

Utilizing equations (A20a) and (A29), we find that the $a_k(x)$ associated with $y_1(x, \tau)$ satisfy

$$a_k'(-a) = 0, \quad k < \mu + \nu - 1$$

$$a_{\mu+\nu-1}'(-a) = -(-1)^{\nu+1} \frac{p_{\mu} q_{\nu}}{2^{\mu+\nu+1} \epsilon^{\frac{1}{2}}(a) k_0^4}, \quad \nu \geq 2 \quad (4.2.13)$$

so that for sufficiently large $|\tau|$ it follows that

$$J(\tau) = e^{-j2\tau a} \frac{p_{\mu} q_{\nu}}{4k_0^4 [\epsilon(-a)\epsilon(a)]^{\frac{1}{2}}} \cdot \frac{(-1)^{\nu+1}}{(2j\tau)^{\mu+\nu-1}} [1 + O(\frac{1}{\tau})] \\ + e^{j2\tau a} \frac{j\tau(k^2(-a) + k_0^2)(k^2(a) + k_0^2)}{2k_0^4 [\epsilon(-a)\epsilon(a)]^{\frac{1}{2}}} [1 + O(\frac{1}{\tau})]. \quad (4.2.14)$$

The asymptotic behavior of g is thus determined by

$$F_k(\tau, w) = \frac{(\epsilon(x)\epsilon(\xi))^{\frac{1}{2}}}{2(j\tau)^{k+1}} \\ \cdot \frac{e^{-j\tau w}}{(k^2(-a) + k_0^2)(k^2(a) + k_0^2) e^{j2\tau a} [1 + O(\frac{1}{\tau})] + (-1)^{\nu+1} \frac{p_{\mu} q_{\nu}}{(2j\tau)^{\mu+\nu}} e^{-j2\tau a} [1 + O(\frac{1}{\tau})]} \quad (4.2.15)$$

as w varies between $\pm 2a$. Away from the zeros of J we find that

$$F_k(\tau, w) = O(\tau^{-(k+1)}), \quad -2a \leq w \leq 0, \quad \text{Im}\tau \leq 0 \quad (4.2.16a)$$

$$F_k(\tau, w) = O(e^{-|\operatorname{Im}\tau| 2a_\tau^{-(k+1)}}), \quad 0 \leq w \leq 2a, \quad \operatorname{Im}\tau \leq 0 \quad (4.2.16b)$$

$$F_k(\tau, w) = O(e^{-|\operatorname{Im}\tau| 2a_\tau^{\mu+\nu-k-1}}), \quad -2a \leq w \leq 0, \quad \operatorname{Im}\tau > 0 \quad (4.2.16c)$$

and

$$F_k(\tau, w) = O(\tau^{\mu+\nu-k-1}), \quad 0 \leq w \leq 2a, \quad \operatorname{Im}\tau > 0 \quad (4.2.16d)$$

for sufficiently large $|\tau|$. Since $-a \leq x \leq \xi \leq a$ in (4.2.7),

$$\begin{aligned} -2a &\leq \xi - x - 2a \leq 0 \\ -2a &\leq -\xi - x \leq 2a \\ -2a &\leq \xi + x \leq 2a \\ 0 &\leq -\xi + x + 2a \leq 2a \end{aligned} \quad (4.2.17)$$

so that (4.2.7) and (4.2.16) yield

$$\begin{aligned} g(x, \xi; \tau) &= O(\tau^{-1}) + O(\tau^{-(\mu+1)}) \\ &+ O(\tau^{-(\nu+1)}) + O(e^{-|\operatorname{Im}\tau| 2a_\tau^{-(\mu+\nu+1)}}), \quad \operatorname{Im}\tau \leq 0 \end{aligned} \quad (4.2.18a)$$

and

$$\begin{aligned} g(x, \xi; \tau) &= O(e^{-|\operatorname{Im}\tau| 2a_\tau^{\mu+\nu-1}}) + O(\tau^{\nu-1}) \\ &+ O(\tau^{\mu-1}) + O(\tau^{-1}), \quad \operatorname{Im}\tau > 0 \end{aligned} \quad (4.2.18b)$$

for sufficiently large $|\tau|$ in the region $-a \leq x \leq \xi \leq a$. Because g is symmetric in x and ξ , (4.2.18) describes the asymptotic behavior of g for all x and ξ in the closed interval $[-a, a]$.

4.3 Non-Spectral Representation

Substituting (4.2.18) into (1.5.1),

$$I_N = \int_{C_N} \frac{g(x, \xi; \tau)}{\tau^P} d\tau, \quad (4.3.1)$$

and noting (1.5.2),

$$|I_N| \leq R_N^{1-P} \int_0^{2\pi} |g(x, \xi; R_N e^{j\theta})| d\theta, \quad (4.3.2)$$

we find that the first term on the RHS of (4.2.18b) contributes most to I_N (see Appendix B), yielding

$$I_N = O(R_N^{\mu+\nu-1-P}). \quad (4.3.3)$$

Therefore

$$P = \mu + \nu \quad (4.3.4)$$

is the least positive integer which insures that $\lim_{R_N \rightarrow \infty} I_N = 0$.

This indicates that

$$\lim_{R_N \rightarrow \infty} \int_{C_N} \frac{g(x, \xi; s)}{s^{\mu+\nu-1} (s-\tau)} ds = 0 \quad (4.3.5)$$

which, upon application of the residue theorem, yields the partial fraction expansion

$$\begin{aligned} g(x, \xi; \tau) &= \frac{\tau^{\mu+\nu-1}}{(\mu+\nu-2)!} \frac{d^{\mu+\nu-2}}{ds^{\mu+\nu-2}} \left\{ \frac{g(x, \xi; s)}{\tau-s} \right\}_{s=0} \\ &+ \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n} \right)^{\mu+\nu-1} \frac{\text{Res } g(x, \xi; \tau_n)}{(\tau-\tau_n)}. \end{aligned} \quad (4.3.6)$$

This can be simplified by noting that, from Leibniz's theorem,

$$\frac{\tau^{\mu+\nu-1}}{(\mu+\nu-2)!} \frac{d^{\mu+\nu-2}}{ds^{\mu+\nu-2}} \left\{ \frac{g(x, \xi; s)}{\tau-s} \right\}_{s=0} = \sum_{n=0}^{\mu+\nu-2} \frac{\tau^n}{n!} \left[\frac{d^n}{d\tau^n} g(x, \xi; \tau) \right]_{\tau=0}, \quad (4.3.7)$$

and, from Appendix C,

$$\text{Res } g(x, \xi; \tau_n) = -\frac{1}{2\tau_n} \phi_n(x) \phi_n(\xi) \quad (4.3.8)$$

where the normalized eigenfunctions are

$$\phi_n(x) = \frac{Y_0(x, \tau_n)}{\left[\int_{-a}^a \frac{1}{\epsilon(x)} Y_0^2(x, \tau_n) dx - j \frac{Y_0^2(a, \tau_n)+1}{2\epsilon_0 \tau_n} \right]^{1/2}}. \quad (4.3.9)$$

Hence,

$$g(x, \xi; \tau) = \sum_{n=0}^{\mu+\nu-2} \tau^n g_n(x, \xi) - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+\nu-1} \frac{\phi_n(x) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)} \quad (4.3.10)$$

where the functions

$$g_n(x, \xi) \triangleq \frac{1}{n!} \left[\frac{d^n}{d\tau^n} g(x, \xi; \tau) \right]_{\tau=0} \quad (4.3.11)$$

can be shown, by substituting (4.3.10) into (4.1.5) and (4.1.6), to be solutions of the auxiliary boundary value problems

$$\frac{1}{\epsilon(x)} \left[\frac{d^2}{dx^2} - \frac{\epsilon'(x)}{\epsilon(x)} \frac{d}{dx} + \Delta k^2(x) \right] g_n(x, \xi) = -g_{n-2}(x, \xi), \quad n=0, 1, 2, \dots, \mu+\nu-2 \quad (4.3.12)$$

subject to

$$g_n'(a, \xi) = -j\epsilon_R(a) g_{n-1}(a, \xi) \quad (4.3.13a)$$

and

$$g_n'(-a, \xi) = j\epsilon_R(-a) g_{n-1}(-a, \xi) \quad (4.3.13b)$$

where $g_{-2}(x, \xi) \triangleq \delta(x - \xi)$ and $g_{-1}(x, \xi) \triangleq 0$.

Expansion (4.3.10) is valid as long as neither μ nor ν vanish. In such cases (4.3.10) is still valid with the understanding that both the first sum on the RHS and the exponent of the weighting factor in the infinite sum are taken as zero if $\mu + \nu \leq 1$.

Comparison of (4.3.10) with (2.3.10) indicates that the effect of the smoothness parameters is different for the TE and TM polarizations. We may formally obtain (4.3.10) from (2.3.10) by changing μ and ν to $\mu-2$ and $\nu-2$, respectively. Therefore it follows from (2.3.14) and (2.3.15)

that the non-spectral field representation is

$$\begin{aligned}
 G(x, z; \xi, \eta) = & \sum_{n=0}^{\mu+\nu-2} g_n(x, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j\beta|z-\eta|} d\tau \\
 & - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(x) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+\nu} \frac{1}{\beta(\tau-\tau_n)} e^{-j\beta|z-\eta|} d\tau \\
 & - j\pi \sum_{n=s}^{\infty} \frac{1}{\beta_n} \phi_n(x) \phi_n(\xi) e^{-j\beta_n|z-\eta|}, \quad |x| \leq a
 \end{aligned} \quad (4.3.14)$$

and

$$\begin{aligned}
 G(x, z; \xi, \eta) = & \sum_{n=0}^{\mu+\nu-2} g_n(\pm a, \xi) \int_{-\infty}^{\infty} \frac{\tau^{n+1}}{\beta} e^{-j[\tau(|x|-a)+\beta|z-\eta|]} d\tau \\
 & - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(\pm a) \phi_n(\xi) \int_{-\infty}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\mu+\nu} \frac{1}{\beta(\tau-\tau_n)} e^{-j[\tau(|x|-a)+\beta|z-\eta|]} d\tau \\
 & - j\pi \sum_{n=s}^{\infty} \frac{1}{\beta_n} \phi_n(\pm a) \phi_n(\xi) e^{-j[\tau_n(|x|-a)+\beta_n|z-\eta|]}, \quad |x| \geq a
 \end{aligned} \quad (4.3.15)$$

where the plus sign is taken for $x > a$, and the minus sign is taken for $x < -a$. Similarly, for the far-field

$$\begin{aligned}
 G(r, \theta) \sim & \left(\frac{2\pi}{k_0 r}\right)^{\frac{1}{2}} e^{-j(k_0 r - \pi/4)} \left\{ \sum_{n=0}^{\mu+\nu-2} (k_0 \cos \theta)^{n+1} g_n(\pm a, \xi) \right. \\
 & \left. - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n}\right)^{\mu+\nu} \frac{\phi_n(\pm a) \phi_n(\xi)}{k_0 \cos \theta - \tau_n} \right\} \\
 & - j\pi \sum_{n=s}^{\infty} \frac{\phi_n(\pm a) \phi_n(\xi)}{\beta_n} e^{-j[\tau_n(|x|-a)+\beta_n|z-\eta|]} H(|\theta| - |\theta_n|), \quad |\theta| \neq \pi/2
 \end{aligned} \quad (4.3.16)$$

where $r \sin \theta = z - \eta$ and $r \cos \theta = |x| - a$.

CHAPTER IV

SOME APPLICATIONS TO PROBLEMS IN INTEGRATED OPTICS

The coupling of energy into and out of thin dielectric films is a basic problem in optical communication systems. The grating [1]-[3] and prism [4]-[6] couplers are two examples of devices designed to perform this function. These devices are obtained by modifying an otherwise perfect dielectric waveguide. In the first case, one of the boundaries is altered from straightness in order to scatter the incident field either into or out of the guide. In the second case, a prism is introduced into the region exterior to the guide; energy exchange occurs as a result of evanescent field coupling.

Besides intentional modifications designed to manipulate the field, imperfections introduced during the manufacture of thin films are unavoidably present. These include boundary deviations from straightness and irregularities within the film itself, resulting in undesirable energy exchange between guided modes and the radiation field [7].

If the structure is such that the variation from perfection is slight, perturbation methods may be used in its analysis. In this chapter, a boundary perturbation technique [8], [9] is used to determine the effect of a

perturbed boundary upon the field of a dielectric waveguide. Both deterministic and statistical boundary perturbations are considered. In the first instance, a grating coupler, resulting from the introduction of a sinusoidal boundary variation of finite length, is analyzed. Expressions for the efficiency, scattered field, radiated power, and attenuation coefficient are obtained and related to the terms of the non-spectral field representation of the unperturbed guide. This is followed by an analysis of a dielectric waveguide with a random variation of one of its walls. An expression for the ensemble average of the power lost by an incident surface mode to other surface modes is obtained. The average loss due to radiation is also considered, and expressed in terms of the non-spectral representation of the unperturbed guide.

1 Boundary Perturbation Formulation

In this section the boundary perturbation technique [8], [9] is reviewed. It is shown that the first order scattered field is due to an effective surface current, caused by the interaction of the boundary perturbation with the incident field.

1.1 Waveguide with Perturbed Boundary

Consider the dielectric waveguide with a perturbed boundary, illustrated in Fig. 4.1. Defining the perturbed boundary by

$$S(x, z) = x - a[1 + \epsilon f(z)] = 0 \quad (1.1.1)$$

where $\hat{\epsilon}$ is a small number such that $|\hat{\epsilon}f(z)| < 1$, it follows that the TE electric field satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x) \right] E(x, z) = 0 \quad (1.1.2)$$

where

$$k^2(x) = \begin{cases} \omega^2 \mu_0 \epsilon_0 & , \quad x > a[1 + \hat{\epsilon}f(z)] \\ \omega^2 \mu_0 \epsilon(x) & , \quad -a < x < a[1 + \hat{\epsilon}f(z)] \\ \omega^2 \mu_0 \epsilon_0 & , \quad x < -a \end{cases} \quad (1.1.3)$$

subject to continuity of tangential electric and magnetic components across the boundaries. At the perturbed boundary this requires that

$$E(a^+[1 + \hat{\epsilon}f(z)], z) = E(a^-[1 + \hat{\epsilon}f(z)], z) \quad (1.1.4a)$$

and

$$\hat{n} \cdot \nabla_{xz} E(a^+[1 + \hat{\epsilon}f(z)], z) = \hat{n} \cdot \nabla_{xz} E(a^-[1 + \hat{\epsilon}f(z)], z) \quad (1.1.4b)$$

where ∇_{xz} is the gradient operator and $\hat{n}$ is the unit vector normal to the perturbed surface. Since the direction of the normal is the same as the gradient of S , we have

$$\hat{n} = \frac{\nabla_{xz} S}{|\nabla_{xz} S|} = \frac{\hat{a}_x - \hat{a}_z (a \hat{\epsilon} df/dz)}{[1 + (a \hat{\epsilon} df/dz)^2]^{1/2}} \quad (1.1.5a)$$

where $\hat{a}_x$ and $\hat{a}_z$ are unit vectors in the x and z directions, respectively, and

$$\hat{n} \cdot \nabla_{xz} = \frac{\partial/\partial x - (a \hat{\epsilon} df/dz) \partial/\partial z}{[1 + (a \hat{\epsilon} df/dz)^2]^{1/2}}. \quad (1.1.5b)$$

Now since the field should reduce to that of an unperturbed waveguide when $\hat{\epsilon}$ is zero, we assume that the field is an analytic function of $\hat{\epsilon}$ and write

$$E(x, z) = \sum_{n=0}^{\infty} \epsilon^n E^{(n)}(x, z). \quad (1.1.6)$$

Next we expand the $E^{(n)}$ in a Taylor series about the unperturbed boundary at $x = a$:

$$E^{(n)}(x, z) = \sum_{k=0}^{\infty} \frac{1}{k!} (x-a)^k \frac{\partial^k}{\partial x^k} E^{(n)}(a^{\pm}, z) \quad (1.1.7a)$$

where a^+ is used for $x > a$, and a^- for $x < a$. Similarly for the derivatives

$$\frac{\partial}{\partial x} E^{(n)}(x, z) = \sum_{k=0}^{\infty} \frac{1}{k!} (x-a)^k \frac{\partial^{k+1}}{\partial x^{k+1}} E^{(n)}(a^{\pm}, z) \quad (1.1.7b)$$

and

$$\frac{\partial}{\partial z} E^{(n)}(x, z) = \sum_{k=0}^{\infty} \frac{1}{k!} (x-a)^k \frac{\partial^{k+1}}{\partial z \partial x^k} E^{(n)}(a^{\pm}, z). \quad (1.1.7c)$$

Since (1.1.4) and (1.1.6) require that

$$\sum_{n=0}^{\infty} \epsilon^n E^{(n)}(a^+[1+\epsilon f(z)], z) = \sum_{n=0}^{\infty} \epsilon^n E^{(n)}(a^-[1+\epsilon f(z)], z) \quad (1.1.8a)$$

and

$$\sum_{n=0}^{\infty} \epsilon^n \hat{n} \cdot \nabla_{xz} E^{(n)}(a^+[1+\epsilon f(z)], z) = \sum_{n=0}^{\infty} \epsilon^n \hat{n} \cdot \nabla_{xz} E^{(n)}(a^-[1+\epsilon f(z)], z), \quad (1.1.8b)$$

substitute $x-a=a\epsilon f(z)$ into (1.1.7), and the result into

(1.1.8). Using (1.1.5b), (1.1.8) becomes

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} [a\epsilon f(z)]^k \epsilon^n \frac{\partial^k}{\partial x^k} E^{(n)}(a^+, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} [a\epsilon f(z)]^k \epsilon^n \frac{\partial^k}{\partial x^k} E^{(n)}(a^-, z) \quad (1.1.9a)$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} [a\epsilon f(z)]^k \epsilon^n \left[\frac{\partial^{k+1}}{\partial x^{k+1}} E^{(n)}(a^+, z) - a\epsilon \frac{df}{dz} \frac{\partial^{k+1}}{\partial z \partial x^k} E^{(n)}(a^+, z) \right] \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} [a\epsilon f(z)]^k \epsilon^n \left[\frac{\partial^{k+1}}{\partial x^{k+1}} E^{(n)}(a^-, z) - a\epsilon \frac{df}{dz} \frac{\partial^{k+1}}{\partial z \partial x^k} E^{(n)}(a^-, z) \right]. \end{aligned} \quad (1.1.9b)$$

Equating like powers of ϵ , we obtain the boundary conditions for the various $E^{(n)}$. For $n=0$ and $n=1$ we obtain, respectively,

$$E^{(0)}(a^+, z) = E^{(0)}(a^-, z) \quad (1.1.10a)$$

$$\frac{\partial}{\partial x} E^{(0)}(a^+, z) = \frac{\partial}{\partial x} E^{(0)}(a^-, z) \quad (1.1.10b)$$

and

$$E^{(1)}(a^+, z) + af(z) \frac{\partial}{\partial x} E^{(0)}(a^+, z) = E^{(1)}(a^-, z) + af(z) \frac{\partial}{\partial x} E^{(0)}(a^-, z) \quad (1.1.11a)$$

$$\begin{aligned} \frac{\partial}{\partial x} E^{(1)}(a^+, z) - a \frac{df}{dz} \frac{\partial}{\partial z} E^{(0)}(a^+, z) + af(z) \frac{\partial^2}{\partial x^2} E^{(0)}(a^+, z) \\ = \frac{\partial}{\partial x} E^{(1)}(a^-, z) - a \frac{df}{dz} \frac{\partial}{\partial z} E^{(0)}(a^-, z) + af(z) \frac{\partial^2}{\partial x^2} E^{(0)}(a^-, z). \end{aligned} \quad (1.1.11b)$$

Higher order terms may similarly be obtained. The boundary conditions for the zero order field are those for the unperturbed ($\epsilon=0$) field. The boundary conditions for the first order scattered field are in terms of the zero order field. Similarly, the boundary conditions on the higher order fields will be in terms of the lower order fields, so that an iteration procedure for the boundary conditions is obtained.

By substituting (1.1.10) into (1.1.11) we obtain the boundary conditions for the first order scattered field in terms of the zero order field:

$$E^{(1)}(a^+, z) = E^{(1)}(a^-, z) \quad (1.1.12a)$$

and

$$\frac{\partial}{\partial x} E^{(1)}(a^+, z) - \frac{\partial}{\partial x} E^{(1)}(a^-, z) = j\omega\mu_0 K(z) \quad (1.1.12b)$$

where the equivalent current sheet is defined by

$$j\omega\mu_0 K(z) \Delta a f(z) \frac{\partial^2}{\partial x^2} [E^{(0)}(a^-, z) - E^{(0)}(a^+, z)] - a \frac{df}{dz} \frac{\partial}{\partial z} [E^{(0)}(a^-, z) - E^{(0)}(a^+, z)]. \quad (1.1.13)$$

Since the second term on the RHS of (1.1.13) is proportional to the change in the normal component of the unperturbed magnetic field across $x=a$, it vanishes. Moreover from (1.1.2) and (1.1.10a) we determine that

$$\frac{\partial^2}{\partial x^2} [E^{(0)}(a^-, z) - E^{(0)}(a^+, z)] = -\Delta k^2(a) E^{(0)}(a, z) \quad (1.1.14)$$

where

$$\Delta k^2(a) \triangleq k^2(a^-) - k_0^2 \quad (1.1.15)$$

so that

$$j\omega\mu_0 K(z) = -a \Delta k^2(a) f(z) E^{(0)}(a, z) \quad (1.1.16)$$

is the equivalent current sheet, due to the interaction of the unperturbed field with the perturbed boundary, which generates the first order scattered field. Expressing the volume current density $J(x, z)$ in terms of the current sheet

$$J(x, z) = K(z) \delta(x-a) \quad (1.1.17)$$

we see that the first order scattered field satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x) \right] E^{(1)}(x, z) = j\omega\mu_0 K(z) \delta(x-a) \quad (1.1.18)$$

where $k^2(x)$ is given by (1.1.3) with $\hat{\epsilon} \approx 0$. The solution follows directly from the formula

$$E^{(1)}(x, z) = -j\frac{\omega\mu_0}{2\pi} \int_{\eta=-\infty}^{\infty} \int_{\xi=0}^a J(\xi, \eta) G(x, z; \xi, \eta) d\xi d\eta \quad (1.1.19)$$

where G is the TE Green's function defined by (2.1.1) of the third chapter. Substituting (1.1.17) into (1.1.19), we

obtain

$$E^{(1)}(x, z) = -j\omega\mu_0 \int_{-\infty}^{\infty} \tilde{K}(\beta) g(x, a; \tau) e^{-j\beta z} d\beta \quad (1.1.20)$$

where g is the transverse Green's function, defined by (1.1.8) of the third chapter, and

$$\tilde{K}(\beta) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} K(z) e^{j\beta z} dz. \quad (1.1.21)$$

2 Grating Coupler

A grating coupler may be constructed from a dielectric waveguide by introducing a periodic variation in one of its walls. When the perturbed boundary, or grating, is externally illuminated, it scatters a portion of the incident field into the surface modes of the waveguide. Conversely, a surface mode incident on the grating region is scattered out of the guide. We analyze both situations utilizing the boundary perturbation formulation and the non-spectral representation of the unperturbed waveguide.

2.1 Input Coupler

Let the coupler, illustrated in Fig. 4.2, be excited by a TE polarized laser beam whose waist is located at $x_1=0$. Assuming a Gaussian beam profile, we have

$$E_w(0, z_1) = A e^{-(z_1/w)^2} \quad (2.1.1)$$

where w is the half beam width, the point at which the field decreases to $1/e$ of its maximum value. In the absence of the grating, the total field in the region $x>a$ is the sum of an incident field due to (2.1.1) and a field reflected by

the waveguide. This total field may be approximated by [10]

$$E^{(0)}(x, z) = A[e^{-(z_i/w)^2} e^{jk_0 x_i} + \Gamma(\theta) e^{-(z_r/w)^2} e^{jk_0 x_r}] \quad (2.1.2)$$

where $\Gamma(\theta)$ is the TE reflection coefficient for a plane wave incident at the angle θ upon the waveguide. The coordinates are related via

$$x_{i,r} = \pm[x - (a \pm h)] \cos \theta - (z + h \tan \theta) \sin \theta \quad (2.1.3a)$$

and

$$z_{i,r} = \pm[x - (a \pm h)] \sin \theta + (z + h \tan \theta) \cos \theta. \quad (2.1.3b)$$

By defining

$$w_0 \triangleq \frac{w}{\cos \theta} \quad (2.1.4)$$

and

$$E_0 \triangleq A[1 + \Gamma(\theta)] e^{-jh k_0 \sec \theta} \quad (2.1.5)$$

we may express the total zero order field at $x=a$ as

$$E^{(0)}(a, z) = E_0 e^{-(z/w_0)^2} \quad (2.1.6)$$

which, according to (1.1.16), interacts with the grating to produce an effective current sheet

$$j\omega\mu_0 K(z) = -a\Delta k^2(a) f(z) E_0 e^{-jzk_0 \sin \theta} e^{-(z/w_0)^2} \quad (2.1.7)$$

which generates the first order scattered field via (1.1.20). Inasmuch as we are interested in the energy coupled into the guide, it is clear that we must determine the surface mode contributions to $E^{(1)}$. From (2.3.10) of the third chapter we have

$$g(x, a; \tau) = \sum_{n=0}^{\nu+2} \tau^n g_n(x, a) - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\nu+3} \frac{\phi_n(a) \phi_n(x)}{2\tau_n(\tau - \tau_n)} \quad (2.1.8)$$

where, having assumed $\Delta k^2(a) \neq 0$, we have set $\mu = 0$. Since the surface modes arise from the residues contributed by the terms in the infinite sum with $\text{Im}\tau_n < 0$,

$$\text{Res}\left[\left(\frac{\tau}{\tau_s}\right)^{\nu+3} \frac{\phi_s(a)\phi_s(x)}{2\tau_s(\tau-\tau_s)} e^{-j\beta|z|}\right]_{\beta=\beta_s} = j\frac{\pi}{\beta_s} \phi_s(a)\phi_s(x) e^{-j\beta_s|z|}, \quad (2.1.9)$$

we may determine their contribution to $E^{(1)}$ from (1.1.20), (2.1.9) and the convolution theorem; we obtain

$$E_S^{(1)}(x, z) = -ja\Delta k^2(a) \sum_{n=s} \frac{1}{2\beta_n} \phi_n(a)\phi_n(x) \int_{-\infty}^{\infty} f(t) E^{(0)}(a, t) e^{-j\beta_n|z-t|} dt. \quad (2.1.10)$$

Since we may write

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) E^{(0)}(a, t) e^{-j\beta_n|z-t|} dt &= e^{-j\beta_n z} \int_{-\infty}^z f(t) E^{(0)}(a, t) e^{j\beta_n t} dt \\ &+ e^{j\beta_n z} \int_z^{\infty} f(t) E^{(0)}(a, t) e^{-j\beta_n t} dt, \end{aligned} \quad (2.1.11)$$

we see that (2.1.10) is a sum of both forward and backward propagating modes. The forward traveling component of (2.1.10) is therefore

$$E_{FS}^{(1)}(x, z) = -ja\Delta k^2(a) \sum_{n=s} \frac{1}{2\beta_n} \phi_n(a)\phi_n(x) I_n(z) \quad (2.1.12)$$

where

$$I_n(z) \triangleq \int_{-\infty}^z e^{-j\beta_n z} \int_{-\infty}^z f(t) E^{(0)}(a, t) e^{j\beta_n t} dt. \quad (2.1.13)$$

Hence the amount of forward coupled field depends upon the grating shape and the input beam through I_n .

Let us consider the periodic grating characterized by

$$f(z) = p(z) \sin\left(\frac{2\pi}{d} z\right) \quad (2.1.14)$$

where $p(z)$ is initially assumed to be unity. From (2.1.6), (2.1.13) and (2.1.14) we have

$$I_n(z) = I_{n+}(z) + I_{n-}(z) \quad (2.1.15)$$

where

$$I_{n\pm}(z) = \pm \frac{1}{2j} E_0 e^{-j\beta_n z} \int_{-\infty}^z e^{j(\beta_n \pm 2\pi/d - k_0 \sin\theta)t} e^{-(t/w_0)^2} dt. \quad (2.1.16)$$

Since [11]

$$\int_{-\infty}^z e^{-(at^2 + 2bt)} dt = 1/2\sqrt{\pi/a} e^{b^2/a} \{1 + \operatorname{erf}(\sqrt{a}z + \frac{b}{\sqrt{a}})\} \quad (2.1.17)$$

where $\operatorname{erf}(y)$ is the error function, we obtain for (2.1.16)

$$I_{n\pm}(z) = \pm \frac{\sqrt{\pi}}{4j} E_0 w_0 e^{-j\beta_n z} e^{-[(\beta_n \pm 2\pi/d - k_0 \sin\theta)w_0/2]^2} \cdot \{1 + \operatorname{erf}[z/w_0 - j(\beta_n \pm 2\pi/d - k_0 \sin\theta)w_0/2]\}. \quad (2.1.18)$$

Inasmuch as the angle θ of the incident beam is variable, we may attempt to choose it so that one of the exponential damping factors in (2.1.18) is unity, thereby maximizing the contribution from I_{n+} or I_{n-} . Now since the unperturbed β_n for a surface mode satisfies $\beta_n > k_0$, while $k_0 \sin\theta \leq k_0$, it follows that only I_{n-} can be maximized, and then only for sufficiently small d (on the order of a wavelength) by setting $\theta = \theta_n$, where

$$k_0 \sin\theta_n = \beta_n - \frac{2\pi}{d}. \quad (2.1.19)$$

Thus the grating allows us to phase match the input beam to any particular surface mode via (2.1.19). This implies that only a single term in (2.1.12) will contribute significantly to the coupled field. Assuming this term to be the lowest

order surface mode ($n=1$), we have

$$I_1(z) = -\frac{\sqrt{\pi}}{4j} E_0 w_0 e^{-j\beta_1 z} [1 + \operatorname{erf}(z/w_0) - e^{-(2\pi w_0/d)^2} (1 + \operatorname{erf}(z/w_0 - j2\pi w_0/d))] . \quad (2.1.20)$$

If, as is usual, the beam width $2w$ is large compared to the wavelength, I_{1+} may be neglected compared to I_{1-} , so that

$$I_1(z) \approx -\frac{\sqrt{\pi}}{4j} E_0 w_0 e^{-j\beta_1 z} [1 + \operatorname{erf}(z/w_0)] . \quad (2.1.21)$$

Hence from (2.1.12) we obtain for the forward coupled field

$$E_{FS}^{(1)}(x, z) \approx \frac{\sqrt{\pi}}{8\beta_1} E_0 w_0 a \Delta k^2(a) \phi_1(a) \phi_1(x) e^{-j\beta_1 z} [1 + \operatorname{erf}(z/w_0)] . \quad (2.1.22)$$

This expression, however, should be amended to account for two facts. First, the grating is actually finite in extent, so that $p(z)$ must vanish outside a finite interval. However, it is clear that the integrand in (2.1.16) decays rapidly for $|t| > w_0$, so that by choosing

$$p(z) = \begin{cases} 1 & -Md < z < Nd \\ 0 & \text{elsewhere} \end{cases} \quad (2.1.23)$$

for sufficiently large M , we may ignore the effect of the finite extent of the grating for $z < Nd$. For $z > Nd$ we obtain

$$E_{FS}^{(1)}(x, z) \approx \frac{\sqrt{\pi}}{8\beta_1} E_0 w_0 a \Delta k^2(a) \phi_1(a) \phi_1(x) e^{-j\beta_1 z} [1 + \operatorname{erf}(Nd/w_0)] . \quad (2.1.24)$$

Second, although the β_g of the unperturbed waveguide are real, in the presence of the grating they must acquire an imaginary part to account for the leakage of the field back out of the guide through the grating. Instead of (2.1.22) and (2.1.24), (2.1.18) now yields

$$E_{FS}^{(1)}(x, z) \approx \frac{\sqrt{\pi}}{8\beta_1} E_0 w_0 a \Delta k^2 (a) \phi_1(a) \phi_1(x) e^{-j(\beta_1 - j\alpha_1)z} e^{(\alpha_1 w_0/2)^2} \cdot [1 + \operatorname{erf}(z/w_0 - \alpha_1 w_0/2)] \quad (2.1.25)$$

for $z \leq Nd$, and

$$E_{FS}^{(1)}(x, z) \approx \frac{\sqrt{\pi}}{8\beta_1} E_0 w_0 a \Delta k^2 (a) \phi_1(a) \phi_1(x) e^{-j\beta_1 z} e^{-\alpha_1 Nd} e^{(\alpha_1 w_0/2)^2} \cdot [1 + \operatorname{erf}(Nd/w_0 - \alpha_1 w_0/2)] \quad (2.1.26)$$

for $z \geq Nd$. The attenuation coefficient α_1 is given approximately by (2.2.20).

The efficiency of the coupler is defined by

$$\eta = \frac{P_{FS}}{P_{inc}} \quad (2.1.27)$$

where P_{inc} is the power incident on the grating coupler, and P_{FS} is the power remaining in the guide for $z > Nd$. Integrating the $-x_i$ component of the Poynting vector,

$$S_{-x_i} = \operatorname{Re}[-1/2 E_{inc}^{(0)} H_{z_i}^*], \quad (2.1.28)$$

where $E_{inc}^{(0)}$ is given by the first term on the RHS of (2.1.2) and $H_{z_i}^*$ is the complex conjugate of

$$H_{z_i} = -\frac{1}{j\omega\mu_0} \frac{\partial}{\partial x_i} E_{inc}^{(0)}, \quad (2.1.29)$$

from $z_i = -\infty$ to $z_i = \infty$, we obtain

$$\begin{aligned} P_{inc} &= 1/2 \sqrt{\frac{\epsilon_0}{\mu_0}} A^2 \int_{-\infty}^{\infty} e^{-2(z_i/w)^2} dz_i \\ &= A^2 \frac{w}{2} \sqrt{\frac{\pi \epsilon_0}{2\mu_0}}. \end{aligned} \quad (2.1.30)$$

Similarly, the power entering the region $z > Nd$ is found to be

$$P_{FS} = \frac{\pi}{128\omega\mu_0\beta_1} [\epsilon a \Delta k^2(a) E_0 w_0 \phi_1(a)]^2 e^{2[(\alpha_1 w_0/2)^2 - \alpha_1 Nd]} \cdot [1 + \text{erf}(Nd/w_0 - \alpha_1 w_0/2)]^2. \quad (2.1.31)$$

Examination of the efficiency as a function of Nd , the point at which the grating is terminated, indicates that η gradually increases for increasing Nd until it reaches a peak at $Nd = (Nd)_{\max}$, and then decreases to zero as Nd becomes infinite. For small $\alpha_1 w_0$, η increases very slowly, reaches a low peak, and then decays slowly. Increasing $\alpha_1 w_0$ causes the peak to shift to the left and become larger and narrower. Further increases in $\alpha_1 w_0$ result in an earlier, shallow peak [6],[12]. Hence the coupler can be made optimal by first determining the value of $\alpha_1 w_0 = (\alpha_1 w_0)_{\text{opt}}$ which results in the largest peak value for η , and then terminating the grating at $Nd = (Nd)_{\max}$ corresponding to this peak. For a given beamwidth these values, together with (2.2.20) for α_1 , determine $(M+N)_{\text{opt}}$, the optimum length of the coupler.

2.2 Output Coupler

When used as an output coupler, the incident field is a surface mode, say

$$E^{(0)}(x,z) = \phi_1(x) e^{-j\beta_1 z}, \quad (2.2.1)$$

which is scattered out of the guide by the grating. The effective current sheet due to the incident mode is, from (1.1.16),

$$j\omega\mu_0 K(z) = -a \Delta k^2(a) \phi_1(a) f(z) e^{-j\beta_1 z}. \quad (2.2.2)$$

Using (1.1.21), (2.1.14) and (2.1.23) we find

$$j\omega\mu_0\tilde{K}(\beta) = -j\frac{2a\Delta k^2(a)}{d}\phi_1(a)e^{-j\beta(M-N)d/2}\frac{\sin[(\beta-\beta_1)(M+N)d/2]}{(\beta-\beta_1)^2 - (2\pi/d)^2} \quad (2.2.3)$$

so that, from (1.1.20) and (2.1.8), the first order scattered field is

$$E^{(1)}(x, z) = j\frac{2a\Delta k^2(a)}{d}\phi_1(a)\int_{-\infty}^{\infty}\left\{\frac{\sin[(\beta-\beta_1)(M+N)d/2]}{(\beta-\beta_1)^2 - (2\pi/d)^2}\right\} \\ \cdot \left[\sum_{n=0}^{v+2}\tau_n g_n(x, a) - \sum_{n=1}^{\infty}\left(\frac{\tau}{\tau_n}\right)^{v+3}\frac{\phi_n(a)\phi_n(x)}{2\tau_n(\tau-\tau_n)}\right]e^{-j\beta[z+(M-N)d/2]}d\beta. \quad (2.2.4)$$

An expression for the field radiated from the coupler may be obtained by applying the method of steepest descent to (2.2.4). The result is, for sufficiently large r ,

$$E^{(1)}(r, \theta) \sim \frac{2a\Delta k^2(a)}{d}\phi_1(a)\sqrt{\frac{2\pi}{k_0 r}}e^{-j(k_0 r - 3\pi/4)}k_0 \cos\theta \\ \cdot \left[\frac{\sin[(k_0 \sin\theta - \beta_1)(M+N)d/2]}{(k_0 \sin\theta - \beta_1)^2 - (2\pi/d)^2}\right]\left[\sum_{n=0}^{v+2}(k_0 \cos\theta)^n g_n(\pm a, a) - \sum_{n=1}^{\infty}\left(\frac{k_0 \cos\theta}{\tau_n}\right)^{v+3}\frac{\phi_n(a)\phi_n(\pm a)}{2\tau_n(k_0 \cos\theta - \tau_n)}\right], \quad |\theta| \neq \pi/2 \quad (2.2.5)$$

where $z+(M-N)d/2 = r \sin\theta$; the plus sign is taken and $x-a = r \cos\theta$ for $x > a$; and the minus sign is taken and $x+a = -r \cos\theta$ for $x < -a$.

The above expression for the far-field indicates that each of the non-spectral components of the transverse Green's function contributes a cylindrical wave whose amplitude depends upon a factor due to the effective current

sheet, (2.2.3). Now if d is sufficiently small, (2.2.3) peaks in the visible region at $\theta=\theta_1$, where

$$k_0 \sin \theta_1 = \beta_1 - 2\pi/d, \quad (2.2.6)$$

the peak increasing with grating length, approaching a delta function as the length becomes infinite. Therefore by choosing a long grating with a short period, the incident surface mode will be scattered out of the coupler into narrow beams at the angles $\theta=\theta_1$. To obtain the radiated power, the r component of the Poynting vector,

$$S_r = \text{Re}[1/2 E_y H_\theta^*] \quad , \quad (2.2.7)$$

where H_θ^* is the complex conjugate of

$$H_\theta = \frac{j}{\omega \mu_0} \frac{\partial}{\partial r} E_y \quad , \quad (2.2.8)$$

must be obtained. From (1.1.6) and (2.2.7) we find

$$S_r(r, \theta) = \frac{\epsilon^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} |E^{(1)}(r, \theta)|^2. \quad (2.2.9)$$

The radiated power follows from

$$P_{\text{rad}} = \int_{-\pi/2}^{\pi/2} r S_r(r, \theta) d\theta. \quad (2.2.10)$$

However, since the dominant contributions to P_{rad} occur in narrow beams which peak at $\theta=\theta_1$, we may change the variable of integration from θ to

$$\beta = k_0 \sin \theta \quad (2.2.11)$$

and note that

$$\delta(\beta) = \lim_{L \rightarrow \infty} \frac{L}{\pi} \text{Sa}^2(\beta L) \quad (2.2.12)$$

to obtain the approximate result

$$P_{\text{rad}} \approx \frac{\gamma(M+N)d}{8\omega\mu_0} [\hat{\epsilon} a \Delta k^2(a) \phi_1(a)]^2 k_0 \cos \theta_1 \left\{ \sum_{n=0}^{v+2} (k_0 \cos \theta_1)^n g_n(a, a) \right\}$$

$$\begin{aligned}
& - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta_1}{\tau_n} \right)^{\nu+3} \frac{\phi_n^2(a)}{\tau_n (k_0 \cos \theta_1 - \tau_n)} \Big|^2 + \Big| \sum_{n=0}^{\nu+2} (k_0 \cos \theta_1)^n g_n(-a, a) \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta_1}{\tau_n} \right)^{\nu+3} \frac{\phi_n(a) \phi_n(-a)}{\tau_n (k_0 \cos \theta_1 - \tau_n)} \Big|^2 \}. \quad (2.2.13)
\end{aligned}$$

This result indicates that the radiated power may be increased by increasing either the amplitude or the length of the grating, or by increasing the dielectric difference across $x=a$. The contributions from the terms of the non-spectral representation depend upon the ϕ_n , τ_n and g_n of the particular dielectric slab.

Since (2.2.13) is actually the power extracted from the incident surface mode, it follows that β_1 must acquire an imaginary part in the presence of the grating. Hence (2.2.1) becomes

$$E^{(0)}(x, z) = \phi_1(x) e^{-j\beta_1 z} e^{-\alpha_1(z+Md)} \quad (2.2.14)$$

for $-Md < z < Nd$. We may determine α_1 approximately by equating the power lost by the incident mode to the power radiated from the grating. Calculating the z component of the Poynting vector,

$$S_z = \text{Re}[-1/2 E_y H_x^*] \quad (2.2.15)$$

where H_x^* is the complex conjugate of

$$H_x = \frac{-j}{\omega \mu_0} \frac{\partial}{\partial z} E_y, \quad (2.2.16)$$

we obtain

$$S_z(x, z) = \frac{\beta_1}{2\omega \mu_0} |E^{(0)}(x, z)|^2. \quad (2.2.17)$$

Upon integrating (2.2.17) from $x=-\infty$ to $x=\infty$, we find

$$P_{inc}(z) = \frac{1}{2\omega\mu_0} e^{-2\alpha_1(z+Md)}, \quad -Md < z < Nd. \quad (2.2.18)$$

Therefore, the power lost by the incident surface mode after passing through the grating is

$$\begin{aligned} \Delta P_{inc} &= P_{inc}(-Md) - P_{inc}(Nd) \\ &= \frac{\beta_1}{2\omega\mu_0} (1 - e^{-2\alpha_1(M+N)d}). \end{aligned} \quad (2.2.19)$$

Equating (2.2.19) with (2.2.13), we obtain an approximate expression for the attenuation coefficient:

$$\begin{aligned} \alpha_1^2 &= \frac{1}{2(M+N)d} \ln \left(1 - \frac{(M+N)d}{4\beta_1} [\epsilon a \Delta k^2(a) \phi_1(a)]^2 k_0 \cos \theta_1 \right. \\ &\quad \cdot \left\{ \left| \sum_{n=0}^{v+2} (k_0 \cos \theta_1)^n g_n(a, a) - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta_1}{\tau_n} \right)^{v+3} \frac{\phi_n^2(a)}{\tau_n (k_0 \cos \theta_1 - \tau_n)} \right|^2 \right. \\ &\quad \left. \left. + \left| \sum_{n=0}^{v+2} (k_0 \cos \theta_1)^n g_n(-a, a) - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta_1}{\tau_n} \right)^{v+3} \frac{\phi_n(a) \phi_n(-a)}{\tau_n (k_0 \cos \theta_1 - \tau_n)} \right|^2 \right\} \right). \end{aligned} \quad (2.2.20)$$

The above results demonstrate the dependence of both the radiated field and the attenuation coefficient upon the ϕ_n , τ_n , and g_n of the particular dielectric slab. In view of (2.1.31) and the discussion which follows it, it is clear that the optimum coupling length is similarly dependent. Hence the ϕ_n , τ_n , and g_n enter as parameters in the design of a sinusoidal grating coupler.

3 Random Wall Variation

The analysis of the effect of a random wall variation upon an incident surface mode is similar to the analysis of the output coupler, except that the boundary perturbation

is no longer deterministic. Consequently, the quantities of interest are the ensemble averages of the power scattered into other surface modes and the power scattered out of the guide.

3.1 Mode Coupling

Let the upper wall of an otherwise perfectly straight waveguide deviate randomly from straightness in the region $|z| < L/2$. It then follows from (1.1.16), (1.1.20) and (1.1.21) that when the surface mode

$$E^{(0)}(x, z) = \phi_1(x) e^{-j\beta_1 z} \quad (3.1.1)$$

is incident on this region, it will interact with the perturbation and give rise to the first order scattered field

$$E^{(1)}(x, z) = a \Delta k^2(a) \phi_1(a) \int_{-\infty}^{\infty} F(\beta - \beta_1) g(x, a; \tau) e^{-j\beta z} d\beta. \quad (3.1.2)$$

The contribution to (3.1.2) due to the surface modes is, from (2.1.10) and (2.1.11), given by

$$E_S^{(1)}(x, z) = E_{FS}^{(1)}(x, z) + E_{BS}^{(1)}(x, z) \quad (3.1.3)$$

where the forward and backward scattered fields are, respectively,

$$E_{FS}^{(1)}(x, z) = -\frac{j}{2} a \Delta k^2(a) \phi_1(a) \sum_{n=s} \left\{ \frac{1}{\beta_n} \phi_n(a) \phi_n(x) e^{-j\beta_n z} \int_{-\infty}^z f(t) e^{j(\beta_n - \beta_1)t} dt \right\} \quad (3.1.4)$$

and

$$E_{BS}^{(1)}(x, z) = -\frac{j}{2} a \Delta k^2(a) \phi_1(a) \sum_{n=s} \left\{ \frac{1}{\beta_n} \phi_n(a) \phi_n(x) e^{j\beta_n z} \int_z^{\infty} f(t) e^{j(-\beta_n - \beta_1)t} dt \right\}. \quad (3.1.5)$$

The amount of power converted from the incident surface mode

into other forward traveling surface modes and carried out of the perturbed region is obtained by integrating the Poynting vector, (2.2.15), from $x=-\infty$ to $x=\infty$. Noting that the surface modes are orthonormal, we obtain

$$P_{FS} = \frac{1}{8\omega\mu_0} [\hat{e} a \Delta k^2(a) \phi_1(a)]^2 \sum_{n=s \neq 1} \left\{ \frac{1}{\beta_n} \phi_n^2(a) \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} f(t) f(t') \cdot e^{j(\beta_n - \beta_1)(t-t')} dt dt' \right\}. \quad (3.1.6)$$

Similarly, for the power backscattered into other modes,

$$P_{BS} = \frac{1}{8\omega\mu_0} [\hat{e} a \Delta k^2(a) \phi_1(a)]^2 \sum_{n=s \neq 1} \left\{ \frac{1}{\beta_n} \phi_n^2(a) \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} f(t) f(t') \cdot e^{j(-\beta_n - \beta_1)(t-t')} dt dt' \right\}. \quad (3.1.7)$$

Let us assume that $f(z)$ is a member of a stationary random process; hence its autocorrelation function is a function of $z-z'$:

$$E\{f(z)f(z')\} = r(z-z'). \quad (3.1.8)$$

Therefore the expected value of the forward and backward scattered powers is

$$E\{P_{FS,BS}\} = \frac{1}{8\omega\mu_0} [\hat{e} a \Delta k^2(a) \phi_1(a)]^2 \sum_{n=s \neq 1} \left\{ \frac{1}{\beta_n} \phi_n^2(a) \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} r(t-t') e^{j(\pm\beta_n - \beta_1)(t-t')} dt dt' \right\}. \quad (3.1.9)$$

The double integral may be expressed more conveniently as [7], [13]

$$\int_{-L/2}^{L/2} \int_{-L/2}^{L/2} r(t-t') e^{j(\pm\beta_n - \beta_1)(t-t')} dt dt' = \int_{-L}^L (L-|u|) r(u) e^{j(\pm\beta_n - \beta_1)u} du. \quad (3.1.10)$$

Since $r(u)$ must decrease to zero as $|u|$ increases to infinity, we may assume that $L \gg |u|$ over the range where $r(u)$ contributes appreciably and extend the range of integration to infinity, thereby approximating (3.1.10) by

$$\int_{-L}^L (L - |u|) r(u) e^{j(\pm\beta_n - \beta_1)u} du \approx 2\pi L R(\pm\beta_n - \beta_1) \quad (3.1.11)$$

where R is the Fourier transform of r :

$$R(\beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(z) e^{j\beta z} dz. \quad (3.1.12)$$

Substituting (3.1.11) into (3.1.9), we obtain

$$E\{P_{FS,BS}\} = \frac{\pi L}{4\omega\mu_0} [\hat{e} a \Delta k^2(a) \phi_1(a)]^2 \sum_{n \neq 1} \frac{1}{\beta_n} \phi_n^2(a) R(\pm\beta_n - \beta_1). \quad (3.1.13)$$

For a given waveguide, this result indicates that the power loss due to mode conversion depends upon the power spectrum $R(\beta)$ of the random process $f(z)$. For a Gaussian process

$$r(z) = \sigma^2 e^{-\left(\frac{z-z'}{B}\right)^2} \quad (3.1.14)$$

where σ is the rms deviation of the wall from perfect straightness and B is the correlation length, we find, using (2.1.17), that

$$R(\beta) = \frac{\sigma^2}{2\sqrt{\pi}} B e^{-\left(\frac{1}{2}B\beta\right)^2}. \quad (3.1.15)$$

Hence the total average power loss is

$$E\{P_{FS}\} + E\{P_{BS}\} = \frac{\sqrt{\pi}}{8\omega\mu_0} LB [\hat{e} \sigma a \Delta k^2(a) \phi_1(a)]^2 \sum_{n \neq 1} \frac{1}{\beta_n} \phi_n^2(a) \{e^{-[B/2(\beta_n - \beta_1)]^2} + e^{-[B/2(\beta_n + \beta_1)]^2}\}. \quad (3.1.16)$$

This result indicates that the perturbation scatters more of

the incident power into the forward traveling modes than into the backward traveling modes.

3.2 Radiated Power

An expression for the far-field scattered out of the guide may be obtained by evaluating (3.1.2) by the method of steepest descent. For sufficiently large r we find that

$$E^{(1)}(r, \theta) \sim a \Delta k^2(a) \phi_1(a) k_0 \cos \theta \sqrt{\frac{2\pi}{k_0 r}} e^{-j(k_0 r - \pi/4)} F(k_0 \sin \theta - \beta_1) \cdot g(\pm a, a; k_0 \cos \theta), \quad |\theta| \neq \pi/2 \quad (3.2.1)$$

where $z = r \sin \theta$; the plus sign is taken and $x - a = r \cos \theta$ when $x > a$; and the minus sign is taken and $x + a = -r \cos \theta$ when $x < -a$. The expected value of the radiated power follows from (2.2.9) and (2.2.10):

$$E\{P_{\text{rad}}\} = \frac{\epsilon^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} [a \Delta k^2(a) \phi_1(a)]^2 \frac{2\pi}{k_0} \int_{-\pi/2}^{\pi/2} (k_0 \cos \theta)^2 E\{|F(k_0 \sin \theta - \beta_1)|^2\} \cdot [|g(a, a; k_0 \cos \theta)|^2 + |g(-a, a; k_0 \cos \theta)|^2] d\theta. \quad (3.2.2)$$

Now since

$$F(\beta) = \frac{1}{2\pi} \int_{-L/2}^{L/2} f(z) e^{j\beta z} dz \quad (3.2.3)$$

it follows from (3.1.8) and (3.1.11) that

$$E\{|F(\beta - \beta_1)|^2\} = \frac{L}{2\pi} R(\beta - \beta_1). \quad (3.2.4)$$

Substituting this with (2.1.8) into (3.2.2), we obtain

$$E\{P_{\text{rad}}\} = \frac{L}{2\omega\mu_0} [\epsilon a \Delta k^2(a) \phi_1(a)]^2 \int_{-\pi/2}^{\pi/2} R(k_0 \sin \theta - \beta_1) \left[\left| \sum_{n=0}^{v+2} (k_0 \cos \theta)^{n+1} \right. \right. \\ \left. \cdot g_n(a, a) - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n} \right)^{v+4} \frac{\phi_n(a)}{k_0 \cos \theta - \tau_n} \right|^2 + \left| \sum_{n=0}^{v+2} (k_0 \cos \theta)^{n+1} g_n(-a, a) \right. \\ \left. \left. - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n} \right)^{v+4} \frac{\phi_n(-a)}{k_0 \cos \theta - \tau_n} \right|^2 \right] d\theta$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n} \right)^{\nu+4} \frac{\phi_n(a) \phi_n(-a)}{k_0 \cos \theta - \tau_n} \Big| d\theta. \quad (3.2.5)$$

Unfortunately, this result cannot be simplified further, and numerical methods must be resorted to. It does, however, illustrate the manner in which the non-spectral terms contribute to the expected value of the radiated power. For those waveguides whose non-spectral representations require only a few terms for a sufficiently accurate description of the field, the numerical effort required to evaluate (3.2.5) would be reduced accordingly.

CHAPTER V

NON-SPECTRAL FIELD REPRESENTATIONS IN CIRCULARLY CYLINDRICAL OPEN WAVEGUIDES

In Chapter III we obtained non-spectral representations of the fields in planar dielectric waveguides. The method we used utilized the fact that the transverse Green's functions for the TE and TM fields are meromorphic functions of the external transverse wavenumber; hence they could be decomposed into partial fractions displaying the principal parts at both the spectral and non-spectral poles. The resulting field expansions demonstrated that each of the singularities of the transverse Green's functions could be associated with partial contributions to the total field.

In the present chapter we extend the analysis to include the TE and TM fields in circularly cylindrical dielectric waveguides. As a consequence of the geometry, however, the transverse Green's functions have a logarithmic singularity which requires that the method used in planar waveguides be modified for use in circular waveguides. It is found that the class of waveguides that may be treated is restricted by the polarization and the smoothness of the dielectric-free space transition across the guide boundary.

1 Formulation of Problem

The axially symmetric, circularly cylindrical open waveguide, illustrated in Fig. 5.1, consists of a dielectric rod of radius a whose permittivity is a non-vanishing analytic function of the radial coordinate. With a suppressed time variation $e^{j\omega t}$, the electric and magnetic fields satisfy the respective wave equations.

$$\nabla^2 \vec{E} + \omega^2 \mu_0 \epsilon \vec{E} - \nabla(\nabla \cdot \vec{E}) = j\omega \mu_0 \vec{J} \quad (1.1)$$

and

$$\nabla^2 \vec{H} + \omega^2 \mu_0 \epsilon \vec{H} + \frac{\nabla \epsilon}{\epsilon} \times \nabla \times \vec{H} = j\omega \epsilon \vec{V} \quad , \quad (1.2)$$

where $\vec{V} \triangleq \frac{j}{\omega \epsilon} [\nabla \times \vec{J} - \frac{\nabla \epsilon}{\epsilon} \times \vec{J}]$. In addition, the fields must satisfy the radiation condition at infinity, and their tangential components must be continuous across $\rho=a$.

By substituting the cylindrical forms of the operators and taking $\frac{\partial}{\partial \phi} \equiv 0$, we obtain for the TE field

$$\frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} E_\phi \right) + \rho \frac{\partial^2}{\partial z^2} E_\phi + \rho \left(\omega^2 \mu_0 \epsilon - \frac{1}{\rho^2} \right) E_\phi = j\omega \mu_0 \rho J_\phi \quad (1.3)$$

and for the TM field

$$\frac{\partial}{\partial \rho} \left(\frac{\rho}{\epsilon} \frac{\partial}{\partial \rho} H_\phi \right) + \frac{\rho}{\epsilon} \frac{\partial^2}{\partial z^2} H_\phi + \frac{\rho}{\epsilon} \left(\omega^2 \mu_0 \epsilon - \frac{\epsilon'}{\rho \epsilon} - \frac{1}{\rho^2} \right) H_\phi = j\omega \rho V_\phi \quad (1.4)$$

Since the remaining field components are given by

$$H_\rho = \frac{1}{j\omega \mu_0} \frac{\partial}{\partial z} E_\phi \quad , \quad H_z = - \frac{1}{j\omega \mu_0 \rho} \frac{\partial}{\partial \rho} (\rho E_\phi) \quad (1.5)$$

in the TE case and

$$E_\rho = - \frac{1}{j\omega \epsilon} \frac{\partial}{\partial z} H_\phi \quad , \quad E_z = \frac{1}{j\omega \epsilon \rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \quad (1.6)$$

in the TM case, it follows that the tangential components will be continuous across $\rho=a$ if

$$E_{\phi}(a^-, z) = E_{\phi}(a^+, z) \quad , \quad \left\{ \frac{\partial}{\partial \rho} (\rho E_{\phi}) \right\}_{\rho=a^-} = \left\{ \frac{\partial}{\partial \rho} (\rho E_{\phi}) \right\}_{\rho=a^+} \quad (1.7)$$

and

$$H_{\phi}(a^-, z) = H_{\phi}(a^+, z) \quad , \quad \left\{ \frac{1}{\epsilon} \frac{\partial}{\partial \rho} (\rho H_{\phi}) \right\}_{\rho=a^-} = \left\{ \frac{1}{\epsilon} \frac{\partial}{\partial \rho} (\rho H_{\phi}) \right\}_{\rho=a^+} \quad (1.8)$$

respectively.

The Green's functions $G(\rho, z; \xi, \eta)$ are the solutions to the above when the excitations are impulsive. Setting

$$\rho J_{\phi} = j \frac{2\pi}{\omega \mu_0} \delta(\rho - \xi) \delta(z - \eta), \quad \rho V_{\phi} = j \frac{2\pi}{\omega} \delta(\rho - \xi) \delta(z - \eta) \quad (1.9)$$

with $0 < \xi < a$ and $-\infty < \eta < \infty$, and expressing each G via its longitudinal (Fourier) representation

$$G(\rho, z; \xi, \eta) = \int_{-\infty}^{\infty} g(\rho, \xi; \tau) e^{-j\beta(z-\eta)} d\beta, \quad (1.10a)$$

where the transverse Green's function is given by

$$g(\rho, \xi; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\rho, z; \xi, \eta) e^{j\beta(z-\eta)} dz, \quad (1.10b)$$

we find that

$$\frac{d}{d\rho} \left(\rho \frac{d}{d\rho} g \right) + \rho (\Delta k^2(\rho) + \tau^2 - \frac{1}{\rho^2}) g = -\delta(\rho - \xi) \quad (1.11)$$

in the TE case, and

$$\frac{d}{d\rho} \left(\frac{\rho}{\epsilon} \frac{d}{d\rho} g \right) + \frac{\rho}{\epsilon} (\Delta k^2(\rho) + \tau^2 - \frac{\epsilon'}{\rho\epsilon} - \frac{1}{\rho^2}) g = -\delta(\rho - \xi) \quad (1.12)$$

in the TM case. In the above, the external transverse wave-number is

$$\tau = (k_0^2 - \beta^2)^{1/2} \quad (1.13)$$

and

$$\Delta k^2(\rho) \triangleq k^2(\rho) - k_0^2. \quad (1.14)$$

Since (1.11) is Bessel's equation of order one when $\rho > a$, the radiation condition and continuity requirements (1.7) imply that

$$g(\rho, \xi; \tau) = \frac{g(a, \xi; \tau)}{H_1^{(2)}(\tau a)} H_1^{(2)}(\tau \rho), \quad \rho \geq a, \quad -\pi < \arg \tau < 0 \quad (1.15a)$$

and

$$g'(a^-, \xi; \tau) = g'(a^+, \xi; \tau) \quad (1.15b)$$

in the TE case, where $H_1^{(2)}(\tau \rho)$ is the Hankel function of the second kind and the prime indicates differentiation with respect to ρ . Similarly, from (1.12) and (1.8) we obtain

$$g(\rho, \xi; \tau) = \frac{g(a, \xi; \tau)}{H_1^{(2)}(\tau a)} H_1^{(2)}(\tau \rho), \quad \rho \geq a, \quad -\pi < \arg \tau < 0 \quad (1.16a)$$

and

$$\frac{1}{\epsilon_R(a^-)} [a g'(a^-, \xi; \tau) + g(a^-, \xi; \tau)] = a g'(a^+, \xi; \tau) + g(a^+, \xi; \tau) \quad (1.16b)$$

in the TM case, where

$$\epsilon_R(\rho) \triangleq \frac{\epsilon(\rho)}{\epsilon_0}. \quad (1.17)$$

In order to obtain unique solutions, an additional constraint is necessary. The homogenous forms of (1.11) and (1.12) each possess a regular singularity at the origin [1]. Furthermore, only one of any pair of linearly independent solutions of the homogeneous forms is bounded at the origin (see Appendix D). Since g is constructed from a pair of such solutions, and since the bounded solution vanishes at the origin, it follows that we must choose

$$g(0, \xi; \tau) = 0 \quad (1.18)$$

for both polarizations.

Equations (1.11), (1.15), (1.18) and (1.12), (1.16), (1.18) determine, respectively, the TE and TM transverse Green's functions. Although the differential systems are defined over all ρ , we may focus our attention on the region $\rho \leq a$ by noting that (1.15) and (1.16) imply, in turn,

$$H_1^{(2)}(\tau a) g'(a, \xi; \tau) - H_1^{(2)'}(\tau a) g(a, \xi; \tau) = 0, \quad -\pi < \arg \tau < 0 \quad (1.19)$$

and

$$H_1^{(2)}(\tau a) g'(a, \xi; \tau) - [\epsilon_R(a) H_1^{(2)'}(\tau a) + \frac{\epsilon_R(a) - 1}{a} H_1^{(2)}(\tau a)] \cdot g(a, \xi; \tau) = 0, \quad -\pi < \arg \tau < 0 \quad (1.20)$$

where the terms are evaluated from the left, so that the equivalent interior problem is given by (1.11), (1.18) and (1.19) for TE polarization, and by (1.12), (1.18) and (1.20) for TM polarization. Hence for the purpose of performing calculations within the guide, the entire region $\rho \geq a$ may be replaced by a boundary condition at $\rho = a$ which involves the spectral parameter through the Hankel function and its derivative.

We note from (1.19) and (1.20) that $\arg \tau$ is restricted. However, this restriction may be temporarily lifted, and by doing so we generate the new interior problems

$$\frac{d}{d\rho} \left(\rho \frac{d}{d\rho} g \right) + \rho [\Delta k^2(\rho) + \tau^2 - \frac{1}{\rho^2}] g = -\delta(\rho - \xi) \quad (1.21)$$

subject to

$$H_1^{(2)}(\tau a) g'(a, \xi; \tau) - H_1^{(2)'}(\tau a) g(a, \xi; \tau) = 0 \quad (1.22a)$$

$$g(0, \xi; \tau) = 0 \quad (1.22b)$$

in the TE case, and

$$\frac{d}{d\rho} \left(\frac{\rho}{\epsilon} \frac{d}{d\rho} g \right) + \frac{\rho}{\epsilon} (\Delta k^2(\rho) + \tau^2 - \frac{\epsilon'}{\rho \epsilon} - \frac{1}{\rho^2}) g = -\delta(\rho - \xi) \quad (1.23)$$

subject to

$$H_1^{(2)}(\tau a) g'(a, \xi; \tau) - [\epsilon_R(a) H_1^{(2)'}(\tau a) + \frac{\epsilon_R(a) - 1}{a} H_1^{(2)}(\tau a)] g(a, \xi; \tau) = 0 \quad (1.24a)$$

$$g(0, \xi; \tau) = 0 \quad (1.24b)$$

in the TM case. These differential systems, whose solutions are clearly the analytic continuation of the respective transverse Green's functions out of the region $-\pi < \arg \tau < 0$, are examples of non-selfadjoint second order systems [2], whose eigensystems are, respectively,

$$\frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \phi_n \right) + \rho (\Delta k^2(\rho) + \tau_n^2 - \frac{1}{\rho^2}) \phi_n = 0 \quad (1.25)$$

$$H_1^{(2)}(\tau_n a) \phi_n'(a) - H_1^{(2)'}(\tau_n a) \phi_n(a) = 0 \quad (1.26a)$$

$$\phi_n(0) = 0 \quad (1.26b)$$

and

$$\frac{d}{d\rho} \left(\frac{\rho}{\epsilon} \frac{d}{d\rho} \phi_n \right) + \frac{\rho}{\epsilon} (\Delta k^2(\rho) + \tau_n^2 - \frac{\epsilon'}{\rho \epsilon} - \frac{1}{\rho^2}) \phi_n = 0 \quad (1.27)$$

$$H_1^{(2)}(\tau_n a) \phi_n'(a) - [\epsilon_R(a) H_1^{(2)'}(\tau_n a) + \frac{\epsilon_R(a) - 1}{a} H_1^{(2)}(\tau_n a)] \phi_n(a) = 0 \quad (1.28a)$$

$$\phi_n(0) = 0. \quad (1.28b)$$

We now see that the restriction $-\pi < \arg \tau < 0$ would single out only the spectral eigenfunctions (e.g., surface mode solutions) from (1.25)-(1.28). Since there are only a finite number of these [3], they do not form a complete set of functions with which an arbitrary field solution could be represented. By temporarily removing the restriction on $\arg \tau$, we include an infinity of non-spectral eigenfunctions in the completeness discussion.

The completeness of the eigenfunction sets generated by certain non-selfadjoint boundary value problems which either exclude [2], or include [4]-[7], the spectral parameter from the boundary conditions has been established. In Chapter III we discussed the non-selfadjoint differential systems with parameter dependent boundary conditions generated by planar open waveguides, and obtained representations which included the non-spectral eigenfunctions. We shall see, however, that the present geometry introduces an added complexity which requires that the method used to generate non-spectral representations in planar guides be modified for use in circular guides.

2 The Transverse Green's Function

Writing (1.21) and (1.23) as

$$Lg = \delta(\rho - \xi), \quad (2.1a)$$

where L is the formally selfadjoint differential operator

$$L = -\frac{d}{d\rho} \left[p(\rho) \frac{d}{d\rho} \right] + q(\rho) \quad (2.1b)$$

with

$$p(\rho) = \begin{cases} \rho, & \text{for TE polarization} \\ \frac{\rho}{\epsilon}, & \text{for TM polarization} \end{cases} \quad (2.1c)$$

and

$$q(\rho) = \begin{cases} -\rho [\Delta k^2(\rho) + \tau^2 - \frac{1}{\rho^2}], & \text{for TE polarization} \\ -\frac{\rho}{\epsilon} [\Delta k^2(\rho) + \tau^2 - \frac{\epsilon'}{\rho \epsilon} - \frac{1}{\rho^2}], & \text{for TM polarization,} \end{cases} \quad (2.1d)$$

we may immediately obtain g from the general formula [8]

$$g(\rho, \xi; \tau) = \frac{y_0(\rho, \tau) y_1'(\xi, \tau)}{J(\tau)} H(\xi - \rho) + \frac{y_0'(\xi, \tau) y_1(\rho, \tau)}{J(\tau)} H(\rho - \xi) \quad (2.2)$$

where H is the unit step function, y_0 and y_1 are the two independent solutions of the homogeneous equation

$$Ly_{0,1} = 0 \quad (2.3)$$

which satisfy the respective boundary conditions

$$y_0(0, \tau) = 0 \quad (2.4a)$$

and

$$H_1^{(2)}(\tau a) y_1'(a, \tau) - H_1^{(2)'}(\tau a) y_1(a, \tau) = 0, \quad \text{for TE polarization} \quad (2.4b)$$

$$H_1^{(2)}(\tau a) y_1'(a, \tau) - [\epsilon_R(a) H_1^{(2)'}(\tau a) + \frac{\epsilon_R(a) - 1}{a} H_1^{(2)}(\tau a)] y_1(a, \tau) = 0 \quad \text{for TM polarization}$$

and J is the conjunct of y_0 and y_1 :

$$J(\tau) = p(\rho) [y_0'(\rho, \tau) y_1(\rho, \tau) - y_0(\rho, \tau) y_1'(\rho, \tau)] \quad (2.5)$$

Obviously, from (2.2) and (2.5), the analytic properties of g depend upon the properties of y_0 and y_1 .

For convenience, we normalize y_0 and y_1 according to

$$y_0(0, \tau) = 0, \quad y_0'(0, \tau) = 1 \quad (2.6a)$$

$$y_1(a, \tau) = H_1^{(2)}(\tau a), \quad y_1'(a, \tau) = \begin{cases} H_1^{(2)'}(\tau a), & \text{for TE polarization} \\ \epsilon_R(a) H_1^{(2)'}(\tau a) + \frac{\epsilon_R(a) - 1}{a} H_1^{(2)}(\tau a), & \text{for TM polarization} \end{cases} \quad (2.6b)$$

and express them as the linear combinations

$$y_0(\rho, \tau) = A_{11}U_1(\rho, \tau) + A_{12}U_2(\rho, \tau) \quad (2.7a)$$

and

$$y_1(\rho, \tau) = A_{21}U_1(\rho, \tau) + A_{22}U_2(\rho, \tau) \quad (2.7b)$$

where U_1 and U_2 are the regular and irregular solutions, respectively, discussed in Appendix D. For convenience, we normalize U_1 so that

$$U_1'(0, \tau) = 1. \quad (2.8)$$

The constants (with respect to ρ) A_{ij} are determined by substituting (2.7) into (2.6). We obtain

$$A_{11} = 1, \quad A_{12} = 0 \quad (2.9a)$$

and, upon noting that the conjunct of U_1 and U_2 is independent of ρ [8],

$$A_{21} = -\frac{\rho p(a)}{2p(\rho)} \Big|_{\rho=0} [y_1(a, \tau) U_2'(a, \tau) - y_1'(a, \tau) U_2(a, \tau)] \quad (2.9b)$$

$$A_{22} = \frac{\rho p(a)}{2p(\rho)} \Big|_{\rho=0} [y_1(a, \tau) U_1'(a, \tau) - y_1'(a, \tau) U_1(a, \tau)].$$

Since U_1 and U_2 are integral functions of τ^2 , they are also

integral functions of τ . From (2.7), (2.9) and (2.6b) it then follows that y_0 is an integral function of τ , while y_1 is not. The reason for this difference is the nature of the boundary conditions (2.4): (2.4a) is independent of τ , while (2.4b) depends upon the Hankel function, a non-integral function of τ .

Substituting these results into (2.2), we find that the transverse Green's function may be expressed as

$$g(\rho, \xi; \tau) = \frac{\rho p(a)}{2p(\rho)} \bigg|_{\rho=0} \left[U_1(\rho, \tau) U_1(\xi, \tau) \frac{A_{21}}{A_{22}} + U_1(\rho, \tau) U_2(\xi, \tau) H(\xi - \rho) \right. \\ \left. + U_1(\xi, \tau) U_2(\rho, \tau) H(\rho - \xi) \right] \quad (2.10)$$

which, from (2.9b), (2.6b) and the known properties of the Hankel function [9], indicates that g is a regular function of τ , except for pole singularities at the zeros of A_{22} and a branch point singularity at $\tau=0$. This is to be contrasted with the planar geometry, where it was shown that g was a meromorphic function of τ . This difference is attributed to the fact that, as a function of τ^2 , the transverse Green's function has a branch point singularity which is algebraic in the planar geometry, and logarithmic in the cylindrical geometry. As a result, we shall find that the representation we obtain for the transverse Green's function includes, in addition to pole contributions, a branch cut integral. Since the representation will be obtained via contour integration, we will need to know the behavior of g for large $|\tau|$. We proceed by first determining the asymptotic expansions of y_0 and y_1 .

3 Asymptotic Expansions of y_0 and y_1

We may construct y_0 and y_1 from two independent solutions, U_+ and U_- , of (2.3) having the asymptotic forms [10]

$$U_+(\rho, \tau) \sim e^{j\tau\rho} \rho^{-\frac{1}{2}}(\rho) \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(\rho) \quad (3.1a)$$

and

$$U_-(\rho, \tau) \sim e^{-j\tau\rho} \rho^{-\frac{1}{2}}(\rho) \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(\rho). \quad (3.1b)$$

In order to determine the $a_k(\rho)$ and $b_k(\rho)$, (3.1) is substituted into (2.3) to obtain

$$e^{j\tau\rho} \rho^{-\frac{1}{2}}(\rho) \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k [\gamma(\rho)a_k(\rho) + 2j\tau a_k'(\rho) + a_k''(\rho)] = 0 \quad (3.2a)$$

and

$$e^{-j\tau\rho} \rho^{-\frac{1}{2}}(\rho) \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k [\gamma(\rho)b_k(\rho) - 2j\tau b_k'(\rho) + b_k''(\rho)] = 0 \quad (3.2b)$$

where

$$\gamma(\rho) \triangleq \begin{cases} \Delta k^2(\rho) - \frac{3}{4\rho^2}, & \text{for TE polarization} \\ \Delta k^2(\rho) - \frac{3}{4} \left(\frac{\epsilon'}{\epsilon}\right)^2 + \frac{1}{2} \frac{\epsilon''}{\epsilon} - \frac{1}{2\rho} \frac{\epsilon'}{\epsilon} - \frac{3}{4\rho^2}, & \text{for TM polarization.} \end{cases} \quad (3.3)$$

Upon equating like powers of τ to zero, we obtain the recursion relationships

$$a_k'(\rho) = -\frac{1}{2} [\gamma(\rho)a_{k-1}(\rho) + a_{k-1}''(\rho)], \quad k=0,1,2,\dots \quad (3.4a)$$

and

$$b_k'(\rho) = -\frac{1}{2} [\gamma(\rho)b_{k-1}(\rho) + b_{k-1}''(\rho)], \quad k=0,1,2,\dots \quad (3.4b)$$

where negative subscripted terms are interpreted as zero.

From (3.4) we see that the terms of the formal expansions (3.1) are determined to within additive constants. In the case of y_1 , these constants may be found by assuming a linear combination of the forms (3.1) and imposing the boundary conditions (2.6b). However, the resulting expansion will diverge at $\rho=0$. In the case of y_0 , the divergence of any linear combination of the forms (3.1) at $\rho=0$ does not permit the determination of the additive constants by imposing the boundary conditions (2.6a). These difficulties arise because $\rho=0$ is a singular point of the differential equation (2.3). As a result, the forms (3.1) are valid only away from $\rho=0$; in a region including the singular point, a different approach is necessary.

The approach we use is due to Langer [11]. By means of successive approximations, we obtain solutions of the differential equation which converge to the exact solutions for sufficiently large $|\tau|$. The results indicate the connection between the asymptotic expansions (3.1) and the asymptotic expansions valid in the neighborhood of $\rho=0$. The details are given in Appendix E. We have, for y_0 ,

$$y_0(\rho, \tau) \sim \left[\frac{r(\rho)}{r(0)} \right]^{\frac{1}{2}} \left(\frac{2}{\pi \tau^3 \rho} \right)^{\frac{1}{2}} \left[e^{j(\tau \rho - 3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau} \right)^k a_k(\rho) + e^{-j(\tau \rho - 3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau} \right)^k b_k(\rho) \right], \quad \rho=O(1) \quad (3.5a)$$

and, for sufficiently large $|\tau|$,

$$y_0(\rho, \tau) = \left[\frac{r(\rho)}{r(0)} \right]^{\frac{1}{2}} \frac{2}{\tau} [J_1(\tau \rho) + \tau \rho \sum_{n=1}^{\infty} \rho^n O(1)], \quad \rho=O\left(\frac{1}{\tau}\right) \quad (3.5b)$$

where $a_0(\rho) = b_0(\rho) \equiv 1$ and

$$r(\rho) = \begin{cases} 1 & , \text{ for TE polarization} \\ \epsilon(\rho) & , \text{ for TM polarization.} \end{cases} \quad (3.6)$$

For y_1 we have

$$y_1(\rho, \tau) \sim \left(\frac{2}{\pi \tau p(\rho)}\right)^{1/2} \left[e^{j(\tau \rho - 2\tau a + 3\pi/4)} \sum_{k=\lambda}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(\rho) + e^{-j(\tau \rho - 3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(\rho) \right], \quad \rho = O(1) \quad (3.7a)$$

and, for sufficiently large $|\tau|$,

$$y_1(\rho, \tau) = R(\rho) \left[A(H_1^{(2)}(\tau \rho) + \frac{1}{\tau \rho} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1) \right) - \left(\frac{1}{j2\tau}\right)^{\lambda} p_{\mu} e^{-j2(\tau a - 3\pi/4)} \{ H_1^{(1)}(\tau \rho) + \frac{1}{\tau \rho} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1) \} \right], \quad \rho = O\left(\frac{1}{\tau}\right) \quad (3.7b)$$

where μ is the order of the first non-vanishing derivative of $\Delta k^2(\rho)$ at $\rho=a$ (taken from the left), and is physically a measure of the smoothness of the dielectric transition across $\rho=a$; p_{μ} is the value of this derivative,

$$p_{\mu} = \frac{d^{\mu}}{d\rho^{\mu}} (\Delta k^2(\rho))_{\rho=a} \quad (3.8)$$

and, in the TE case,

$$\lambda = \mu + 2, \quad a_{\mu+2}(\rho) \equiv -(1/2)^{\mu+2} p_{\mu}, \quad (3.9)$$

$$b_0(\rho) \equiv 1, \quad R(\rho) \equiv 1, \quad A = 1$$

while in the TM case,

$$\lambda = \mu, \quad a_{\mu}(\rho) \equiv -(1/2)^{\mu+1} \frac{1}{k_0^2 \epsilon^{1/2}(a)} p_{\mu},$$

$$b_0(\rho) \equiv \frac{k^2(a) + k_0^2}{2k_0^2 \epsilon^{1/2}(a)}, \quad R(\rho) = \frac{1}{2k_0^2} \left[\frac{\epsilon(\rho)}{\epsilon(a)} \right]^{1/2}, \quad (3.10)$$

$$A = k^2(a) + k_0^2$$

4 Asymptotic Behavior of $g(\rho, \xi; \tau)$

The asymptotic behavior of g follows from (2.2), (2.5) and the asymptotic behavior of y_0 and y_1 . From (3.5)-(3.10) we find that for sufficiently large $|\tau|$

$$y_0(\rho, \tau) = \left[\frac{x(\rho)}{x(0)} \right]^{\frac{1}{2}} \left(-\frac{2}{\pi \tau^3 \rho} \right)^{\frac{1}{2}} \{ e^{j(\tau \rho - 3\pi/4)} [1 + O(\frac{1}{\tau})] + e^{-j(\tau \rho - 3\pi/4)} [1 + O(\frac{1}{\tau})] \}, \quad \rho = O(1) \quad (4.1a)$$

$$y_0(\rho, \tau) = O(1), \quad \rho = O(\frac{1}{\tau}) \quad (4.1b)$$

$$y_1(\rho, \tau) = \left(\frac{2x(\rho)}{\pi \tau \rho} \right)^{\frac{1}{2}} \{ e^{-j(\tau \rho - 3\pi/4)} [1 + O(\frac{1}{\tau})] + (\frac{1}{j\tau})^{\lambda} a_{\lambda} e^{j(\tau \rho - 2\tau a + 3\pi/4)} [1 + O(\frac{1}{\tau})] \}, \quad \rho = O(1) \quad (4.2a)$$

and

$$y_1(\rho, \tau) = \frac{1}{\tau \rho} [O(1) + (\frac{1}{\tau})^{\lambda} e^{-j2\tau a} O(1)], \quad \rho = O(\frac{1}{\tau}). \quad (4.2b)$$

where O is the order symbol [10], and the functions so represented in (4.1b) and (4.2b) do not vanish. By substituting these into (2.2) when $0 \leq \rho \leq \xi \leq a$ we find that

$$g(\rho, \xi; \tau) = \frac{2}{\pi} \left[\frac{x(\rho)x(\xi)}{x(0)\rho\xi} \right]^{\frac{1}{2}} \{ \hat{F}_2(\tau, \xi - \rho) [1 + O(\frac{1}{\tau})] - j\hat{F}_2(\tau, \xi + \rho) [1 + O(\frac{1}{\tau})] + (-j)^{\lambda} a_{\lambda} \hat{F}_{\lambda+2}(\tau, 2a - \xi - \rho) [1 + O(\frac{1}{\tau})] + (-j)^{\lambda+1} a_{\lambda} \hat{F}_{\lambda+2}(\tau, 2a - \xi + \rho) [1 + O(\frac{1}{\tau})] \}, \quad \rho = O(1), \quad \xi = O(1) \quad (4.3a)$$

$$g(\rho, \xi; \tau) = \left[\frac{2x(\xi)}{\pi \xi} \right]^{\frac{1}{2}} O(1) \{ \hat{F}_{3/2}(\tau, \xi) [1 + O(\frac{1}{\tau})] + (-j)^{\lambda} a_{\lambda} e^{j3\pi/4} \hat{F}_{\lambda+3/2}(\tau, 2a - \xi) [1 + O(\frac{1}{\tau})] \}, \quad \rho = O(\frac{1}{\tau}), \quad \xi = O(1) \quad (4.3b)$$

and

$$g(\rho, \xi; \tau) = \frac{\rho}{\xi} [O(1) \hat{F}_1(\tau, 0) + O(1) \hat{F}_{\lambda+1}(\tau, 2a)], \quad \rho = O(\frac{1}{\tau}), \quad \xi = O(\frac{1}{\tau}) \quad (4.3c)$$

where we have defined

$$\hat{F}_k(\tau, w) \triangleq \frac{e^{-j\tau w}}{\tau^k J(\tau)}. \quad (4.4)$$

Similar expressions may be obtained for $0 \leq \xi \leq \rho \leq a$.

Since $J(\tau)$ is independent of ρ , we may use (4.1b) and (4.2b) in (2.5) to obtain

$$J(\tau) = \frac{1}{\tau} [O(1) + (\frac{1}{\tau})^\lambda e^{-j2\tau a} O(1)]. \quad (4.5)$$

Noting that $0 \leq \rho \leq \xi \leq a$ for each of the terms in (4.3), it therefore follows that the asymptotic behavior of g is determined by

$$\hat{F}_k(\tau, w) = \frac{e^{-j\tau w}}{\tau^{k-1} [O(1) + (\frac{1}{\tau})^\lambda e^{-j2\tau a} O(1)]} \quad (4.6)$$

as w varies between 0 and $2a$. Assuming that τ is not a zero of $J(\tau)$, we obtain

$$\hat{F}_k(\tau, w) = \begin{cases} O(\tau^{1-k}), & \text{Im}\tau < 0 \\ O(\tau^{\lambda+1-k}), & \text{Im}\tau \geq 0 \end{cases} \quad (4.7)$$

for sufficiently large $|\tau|$. Using this in (4.3) we conclude that

$$g(\rho, \xi; \tau) = O(\tau^{\lambda-1}) \quad (4.8)$$

for sufficiently large $|\tau|$ in the region $0 \leq \rho \leq \xi \leq a$. By interchanging ρ and ξ , the transverse Green's function is obtained in the region $0 \leq \xi \leq \rho \leq a$. Because of this symmetry, the asymptotic behavior of g is given by (4.8) for all ρ and ξ in the interval $[0, a]$.

5 Decomposition of $g(\rho, \xi; \tau)$. Non-Spectral Representation

The decomposition of the transverse Green's function into a form which exhibits contributions from both spectral and non-spectral poles can now be obtained via contour integration in the τ -plane. We consider the integral

$$I_N = \int_{C_{Nb}} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds \quad (5.1)$$

where C_{Nb} is a closed contour consisting of a circle (save for small gaps at the branch cut) of radius R_N , centered at the origin and passing between the N th and $N+1$ st poles of g , and a path C_b extending around the branch cut. The branch cut, while necessary to assure single-valuedness of the integrand, is otherwise arbitrary. However, in order to insure that the representation we obtain includes both the spectral poles and the leaky wave poles (viz., those non-spectral poles which are captured in a steepest descent field approximation), we have limited the asymptotic analysis in Appendix E (and hence the results of the previous two sections) to the region $|\arg \tau| < \pi$. We therefore cut the plane along the negative real axis and choose the branch corresponding to the principal value of the argument. See Fig. 5.2.

Along C_N , $s = R_N e^{j\theta}$, $|\theta| < \pi$, so that

$$\left| \int_{C_N} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds \right| \leq \int_{-\pi}^{\pi} \frac{|g(\rho, \xi; R_N e^{j\theta})|}{R_N^\lambda (1 - \frac{|\tau|}{R_N})} d\theta \quad (5.2)$$

which, on account of (4.8), approaches zero as R_N approaches infinity. Therefore, assuming that the branch cut integral

converges, we may apply the residue theorem to (5.1) and obtain

$$\lim_{R_N \rightarrow \infty} I_N = 2\pi j \left[\frac{g(\rho, \xi; \tau)}{\tau^\lambda} + \sum_{n=1}^{\infty} \frac{\text{Res } g(\rho, \xi; \tau_n)}{\tau_n^\lambda (\tau_n - \tau)} \right] = \int_{C_b} \frac{g(\rho, \xi; s)}{s^\lambda (s - \tau)} ds, \quad (5.3)$$

for $|\arg \tau| < \pi$.

In Appendix F we show that the branch cut integral is defined if

$$\lambda < 3 \quad (5.4)$$

and may be expressed as

$$\int_{C_b} \frac{g(\rho, \xi; s)}{s^\lambda (s - \tau)} ds = \frac{8K(a)}{j\pi a} \int_0^\infty \frac{e^{-j\pi} y_0(\rho, s) y_0(\xi, s)}{F_2(s) [2F_1(s) + 3F_2(s)] s^\lambda (s - \tau)} ds + \begin{cases} 0 & , \text{ for } \lambda = 0, 2 \\ \frac{2\pi j}{\tau} g(\rho, \xi; 0) & , \text{ for } \lambda = 1 \end{cases} \quad (5.5)$$

where

$$K(a) = \begin{cases} 1 & , \text{ for TE polarization} \\ \epsilon_R(a) & , \text{ for TM polarization} \end{cases} \quad (5.6)$$

and

$$F_{1,2}(s) = H_1^{(1,2)}(sa) y_0'(a, s) - \{K(a) H_1^{(1,2)'}(sa) + \frac{K(a) - 1}{a} H_1^{(1,2)}(sa)\} \cdot y_0(a, s). \quad (5.7)$$

In Appendix G we show, assuming simple poles, that

$$\text{Res } g(\rho, \xi; \tau_n) = - \frac{1}{2\tau_n} \phi_n(\rho) \phi_n(\xi) \quad (5.8)$$

where the normalized eigenfunctions are

$$\phi_n(\rho) = \frac{y_0(\rho, \tau_n)}{\left[\int_0^a p(\rho) y_0^2(\rho, \tau_n) d\rho - \frac{1}{2\tau_n} p(a) K(a) W(\tau_n) \right]^{1/2}} \quad (5.9a)$$

with

$$W(\tau) = \left(\frac{y_0(a, \tau)}{H_1^{(2)}(\tau a)} \right)^2 \left[H_1^{(2)'}(\tau a) \frac{d}{d\tau} H_1^{(2)}(\tau a) - H_1^{(2)}(\tau a) \frac{d}{d\tau} H_1^{(2)'}(\tau a) \right]. \quad (5.9b)$$

Substituting (5.5)-(5.9) into (5.3), we arrive at the representation

$$g(\rho, \xi; \tau) = g_0(\rho, \xi) - \frac{4}{\pi^2 a} \int_0^\infty e^{-j\pi} \left(\frac{\tau}{s} \right)^\lambda \frac{y_0(\rho, s) y_0(\xi, s)}{(s-\tau) F_2(s) [2F_1(s) + 3F_2(s)]} ds \\ - \sum_{n=1}^\infty \left(\frac{\tau}{\tau_n} \right)^\lambda \frac{\phi_n(\rho) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)}, \quad |\arg \tau| < \pi \quad (5.10a)$$

where

$$g_0(\rho, \xi) \triangleq \begin{cases} 0 & , \text{ for } \lambda = 0, 2 \\ g(\rho, \xi; 0), & \text{ for } \lambda = 1 \end{cases} \quad (5.10b)$$

Representation (5.10) is similar to that which is obtained in the planar geometry, insofar as it includes an infinite sum of spectral and non-spectral eigenfunctions, plus additional terms generated by $\tau=0$. However, in the planar geometry these additional terms form a truncated Maclaurin series, whereas here they include a branch cut integral due to the presence of the branch point at $\tau = 0$.

The non-spectral representation of the field is obtained from (5.10) and (5.10a). However, since τ is a multivalued function of β , g has branch points at $\beta = \pm k_0$ and appropriate branch cuts must be introduced before the

integration is carried out. These branch cuts are chosen to implement the previously discarded restriction on $\arg \tau$. Accordingly, we introduce branch cuts beginning at $\beta = \pm k_0$, extending along the real axis to the origin, and then along the imaginary axis to $\mp \infty$; and evaluate (1.10a) on the sheet corresponding to $-\pi < \arg \tau < 0$ [12], [13]. Assuming slight losses, the branch points and branch cuts are shifted away from the axes as shown in Fig. 5.3. Consequently,

$$G(\rho, z; \xi, \eta) = \int_{-\infty}^{\infty} \left\{ (g_0(\rho, \xi) - \frac{4}{\pi^2 a} \int_0^{\infty} e^{-j\pi} \left(\frac{\tau}{s}\right)^{\lambda} \frac{Y_0(\rho, s) Y_0(\xi, s)}{(s-\tau) F_2(s) [2F_1(s) + 3F_2(s)]} ds \right. \\ \left. - \sum_{n=1}^{\infty} \left(\frac{\tau}{\tau_n}\right)^{\lambda} \frac{\phi_n(\rho) \phi_n(\xi)}{2\tau_n(\tau - \tau_n)} \right\} e^{-j\beta(z-\eta)} d\beta, \quad 0 \leq \rho \leq a \quad (5.11)$$

where the integration is carried out along the real axis in Fig. 5.3.

The transverse representation is obtained by closing the contour of integration in the lower (upper) half plane for $z-\eta$ positive (negative) and applying the residue theorem. In order to remain on the spectral sheet, however, the contours must be warped around the branch cuts as shown in Fig. 5.3. Application of the residue theorem therefore yields G as an integral around a semi-circle, plus a branch cut integral, plus a sum of residues from the spectral (i.e., $-\pi < \arg \tau_n < 0$) poles. The integral about the semi-circle vanishes, so that upon changing the variable of integration from β to τ in the branch cut integrals and reversing the order of integration and summation, we obtain for the interior field the transverse representation

$$\begin{aligned}
G(\rho, z; \xi, \eta) = & g_0(\rho, \xi) \int_{-\infty}^{\infty} e^{-j\pi \frac{\tau}{\beta}} e^{-j\beta|z-\eta|} d\tau \\
& - \frac{4}{\pi^2 a} \int_{-\infty}^{\infty} e^{-j\pi \frac{\tau}{\beta}} \int_0^{\infty} e^{-j\pi \frac{s}{\beta}} \left(\frac{\tau}{s}\right)^{\lambda+1} \frac{s y_0(\rho, s) y_0(\xi, s)}{\beta(s-\tau) F_2(s) [2F_1(s) + 3F_2(s)]} e^{-j\beta|z-\eta|} ds d\tau \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(\rho) \phi_n(\xi) \int_{-\infty}^{\infty} e^{-j\pi \frac{\tau}{\tau_n}} \left(\frac{\tau}{\tau_n}\right)^{\lambda+1} \frac{1}{\beta(\tau-\tau_n)} e^{-j\beta|z-\eta|} d\tau \\
& - j\pi \sum_{n=s} \frac{1}{\beta_n} \phi_n(\rho) \phi_n(\xi) e^{-j\beta_n|z-\eta|}, \quad 0 \leq \rho \leq a
\end{aligned} \tag{5.12}$$

where $\text{Im}\beta, \beta_n \leq 0$ in each of the terms. This result demonstrates that the branch point and both spectral and non-spectral pole singularities of the transverse Green's function may be associated with contributions to the total field $G(\rho, z; \xi, \eta)$: The first two terms on the RHS represent contributions to the radiation field from the branch point singularity ($\beta = \pm k_0$ in the β -plane); the first term vanishes because its integrand is an odd function of τ , and the second term reflects the logarithmic nature of the branch point. The infinite sum is generated by the spectral and non-spectral poles in the region $|\arg \tau| < \pi$, and also represents contributions to the radiation field; the amount contributed by each term of the sum depends upon the polarization and the smoothness parameter μ through the exponent of the weighting factor in each integrand. The last sum consists of surface waves; they are generated by the spectral poles, and are the familiar discrete contributions which appear in the spectral representation of the field.

Knowing the field within the waveguide, it may readily be extended to the exterior region. From (1.10a), (1.15a), (1.16a) and (5.10), we obtain

$$\begin{aligned}
 G(\rho, z; \xi, \eta) = & g_0(a, \xi) \int_{-\infty}^{\infty} e^{-j\pi} \frac{\tau H_1^{(2)}(\tau \rho)}{\beta H_1^{(2)}(\tau a)} e^{-j\beta|z-\eta|} d\tau \\
 & - \frac{4}{\pi^2 a} \int_{-\infty}^{\infty} e^{-j\pi} \int_0^{\infty} \left(\frac{\tau}{s}\right)^{\lambda+1} \frac{s y_0(a, s) y_0(\xi, s)}{\beta(s-\tau) F_2(s) [2F_1(s) + 3F_2(s)]} \\
 & \quad \cdot \frac{H_1^{(2)}(\tau \rho)}{H_1^{(2)}(\tau a)} e^{-j\beta|z-\eta|} ds d\tau \\
 & - \frac{1}{2} \sum_{n=1}^{\infty} \phi_n(a) \phi_n(\xi) \int_{-\infty}^{\infty} e^{-j\pi} \left(\frac{\tau}{\tau_n}\right)^{\lambda+1} \frac{1}{\beta(\tau-\tau_n)} \frac{H_1^{(2)}(\tau \rho)}{H_1^{(2)}(\tau a)} e^{-j\beta|z-\eta|} d\tau \\
 & - j\pi \sum_{n=1}^{\infty} \frac{1}{\beta_n} \phi_n(a) \phi_n(\xi) \frac{H_1^{(2)}(\tau_n \rho)}{H_1^{(2)}(\tau_n a)} e^{-j\beta_n|z-\eta|}, \quad \rho \geq a. \quad (5.13)
 \end{aligned}$$

With the exception of the surface waves, the form of each of these terms is more complicated than the corresponding terms in (5.12). However, by using the method of steepest descent [13], the far-field is found to have the simpler form

$$\begin{aligned}
 G(r, \theta) \sim & \frac{2}{H_1^{(2)}(ak_0 \cos \theta)} \frac{e^{-jk_0 r}}{r} \{ g_0(a, \xi) \\
 & - \frac{4}{\pi^2 a} \int_0^{\infty} e^{-j\pi} \left(\frac{k_0 \cos \theta}{s}\right)^{\lambda} \frac{y_0(a, s) y_0(\xi, s)}{(s - k_0 \cos \theta) F_2(s) [2F_1(s) + 3F_2(s)]} ds \\
 & - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{k_0 \cos \theta}{\tau_n}\right)^{\lambda} \frac{\phi_n(a) \phi_n(\xi)}{\tau_n (k_0 \cos \theta - \tau_n)} \} \\
 & + \sum_{n=1}^{\infty} \left(\frac{j2\pi}{\tau_n}\right)^{\frac{1}{2}} \frac{\phi_n(a) \phi_n(\xi)}{\beta_n H_1^{(2)}(\tau_n a)} e^{-j(\tau_n \rho + \beta_n |z-\eta|)} H(|\theta| - |\theta_n|),
 \end{aligned}$$

$$|\theta| \neq \pi/2. \quad (5.14)$$

In the last sum, which extends over only the leaky waves, $\text{Im}\beta_n < 0$. The coordinates r and θ are defined by

$$r \cos\theta = \rho, \quad r \sin\theta = z - \eta \quad (5.15)$$

and θ_n is the capture angle of the corresponding leaky wave pole:

$$1 = \cosh \eta_n \cos(\sigma_n - \theta_n) \quad (5.16a)$$

where

$$\tau_n = k_0 \cos(\sigma_n + j\eta_n). \quad (5.16b)$$

The first two terms on the RHS of (5.14) are spherical waves contributed by the branch point singularity of the transverse Green's function. The third term consists of an infinite sum of spherical waves contributed by the pole singularities of the transverse Green's function in the region $|\arg\tau| < \pi$. The last sum consists of inhomogeneous conical waves (i.e., leaky waves) contributed by those non-spectral poles in $0 < \arg\tau < \pi$ satisfying (5.16). For sufficiently large $k_0 r$, the spherical waves dominate, while for smaller distances the leaky waves become important. For τ_n satisfying $0 < \text{Re}\tau_n < k_0$ and $0 < \text{Im}\tau_n < k_0$, the corresponding spherical waves peak in magnitude at an angle close to the capture angle of the corresponding leaky waves, a result consistent with the known effect of leaky waves on the radiation field [12], [13].

It noted that since the response to the more general excitations

$$\left. \begin{array}{l} j\omega\mu_0\rho J_\phi(\rho, z) \\ j\omega\rho V_\phi(\rho, z) \end{array} \right\} = -2\pi f(\rho)\delta(z-\eta) \quad (5.17)$$

where $f(\rho) \equiv 0$ for $\rho > a$, is given by

$$\left. \begin{array}{l} E_\phi(\rho, z) \\ H_\phi(\rho, z) \end{array} \right\} = \int_{-\infty}^{\infty} \langle g, f \rangle e^{-j\beta(z-\eta)} d\beta \quad (5.18a)$$

with

$$\langle g, f \rangle \triangleq \int_0^a g(\rho, \xi; \tau) f(\xi) d\xi, \quad (5.18b)$$

it follows that the non-spectral representation of the field due to an arbitrary internal ϕ -directed ring source may be obtained directly from (5.12) and (5.13) by replacing $g_0(\rho, \xi)$, $y_0(\xi, s)$ and $\phi_n(\xi)$ with $\langle g_0, f \rangle$, $\langle y_0, f \rangle$ and $\langle \phi_n, f \rangle$, respectively.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

Non-spectral field representations have been discussed and developed for a variety of dielectric waveguides. It has been shown that planar waveguides, both with and without impedance plane backings, having arbitrary analytic transverse permittivity variations, admit expansions which associate a partial field contribution to each of the singularities of the transverse Green's function: spectral poles, non-spectral poles, and branch point. The precise form of these contributions was shown to be dependent upon the field polarization and the smoothness of the transition across the dielectric-free space interface(s).

In changing from the longitudinal to the transverse representation, it was demonstrated that all contributions give rise to cylindrical waves in the far-field. In addition, the contributions from the spectral poles give rise to surface waves, and the contributions from certain non-spectral poles (viz., the leaky wave poles) give rise to leaky waves when evaluated by the method of steepest descent. The cylindrical waves associated with the leaky waves were seen to exhibit a resonant behavior, consistent with the known

effect of leaky waves on the far-field.

The use of such expansions was illustrated in the analysis of a grating coupler and of a dielectric guide with a random surface irregularity by obtaining non-spectral expressions for various quantities of interest. The results indicated that the terms of the non-spectral representation may be of use as design parameters for optical communication devices.

The method used to obtain non-spectral representations in planar waveguides utilized the fact that the transverse Green's functions for the guides considered were meromorphic in the external transverse wavenumber. Such was not the case in the circularly cylindrical geometry, so that the method had to be suitably modified. It was shown that expansions analogous to those obtained for planar guides could be obtained for circular guides, provided that the smoothness of the dielectric-free space transition was restricted according to the polarization of the field.

A numerical study of the convergence properties of the non-spectral representations would be desirable. Such a study might yield useful information relating the parameters of a particular guide to the number of terms of the expansion necessary to obtain sufficiently accurate results. Initially, a particularly simple waveguide might be considered, one whose eigenfunctions are well known, so that computer time could be devoted to obtaining the eigenvalues and determining the number of eigenfunctions necessary for

sufficiently accurate results in any particular application. As an example, we mention that we have studied the TE eigenvalues of the waveguide consisting of a dielectric slab of constant permittivity mounted on a perfect conductor. The unnormalized eigenfunctions are

$$\phi_n(x) = \sin \lambda_n x \quad , \quad (6.1a)$$

where

$$\lambda_n = (\Delta k^2 + \tau_n^2)^{1/2} \quad . \quad (6.1b)$$

We found that the eigenvalue equation could be expressed as

$$U_n = \pm A \sin U_n \quad , \quad (6.2a)$$

where

$$U_n = \lambda_n a \quad (6.2b)$$

and

$$A = \Delta k a \quad , \quad (6.2c)$$

and calculated the U_n for various values of A by using an interval halving technique on a digital computer. By comparing the results with the asymptotic formulas

$$\text{Re}U_n = (2n-1)\pi/2 - \frac{2 \ln[(2n-1)\pi/A]}{(2n-1)\pi} \quad (6.3a)$$

$$\text{Im}U_n = \pm \ln[(2n-1)\pi/A] \quad (6.3b)$$

which are obtained from (6.2), we found that as A was increased, the accuracy of (6.3) also increased. For $A > 1$, (6.3) yielded results accurate to within 0.6% of the computer generated results. Since $A=1$ corresponds to the onset of the appearance of the lowest order surface mode, it

appears that the asymptotic estimates of the complex eigenvalues may be sufficiently accurate for problems where one or more surface modes can be excited; in which case, computer time need only be spent in determining the surface mode eigenvalues. It would be of interest to determine whether this is also true for the more general class of waveguides considered in this thesis.

Further theoretical work might be directed toward extending the non-spectral method to include multilayered and anisotropic waveguides. It is also suggested that the possibility of developing a "coupled mode" theory be investigated, although admittedly such a theory would be complicated by the non-orthogonality of the eigenfunctions.

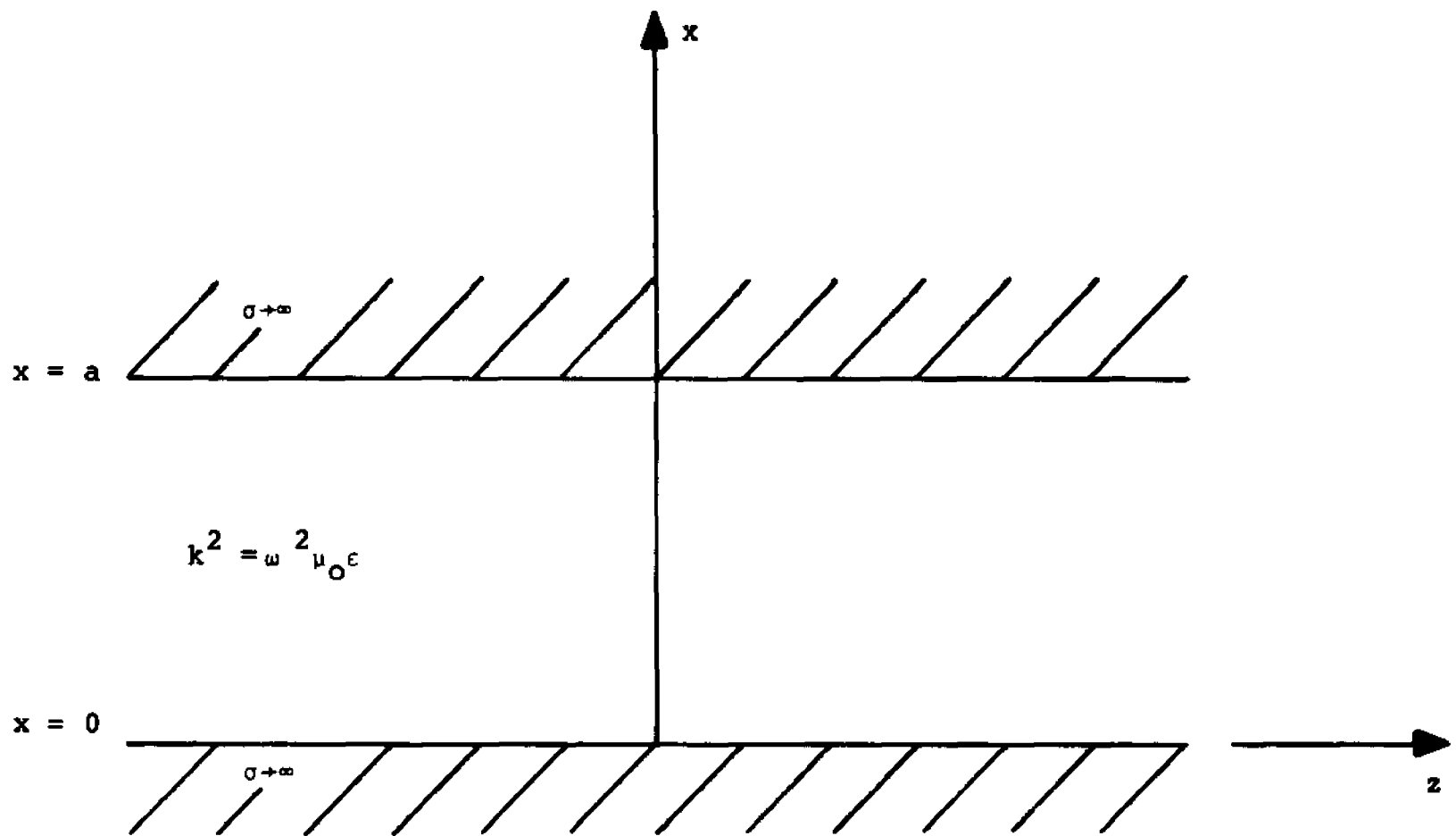
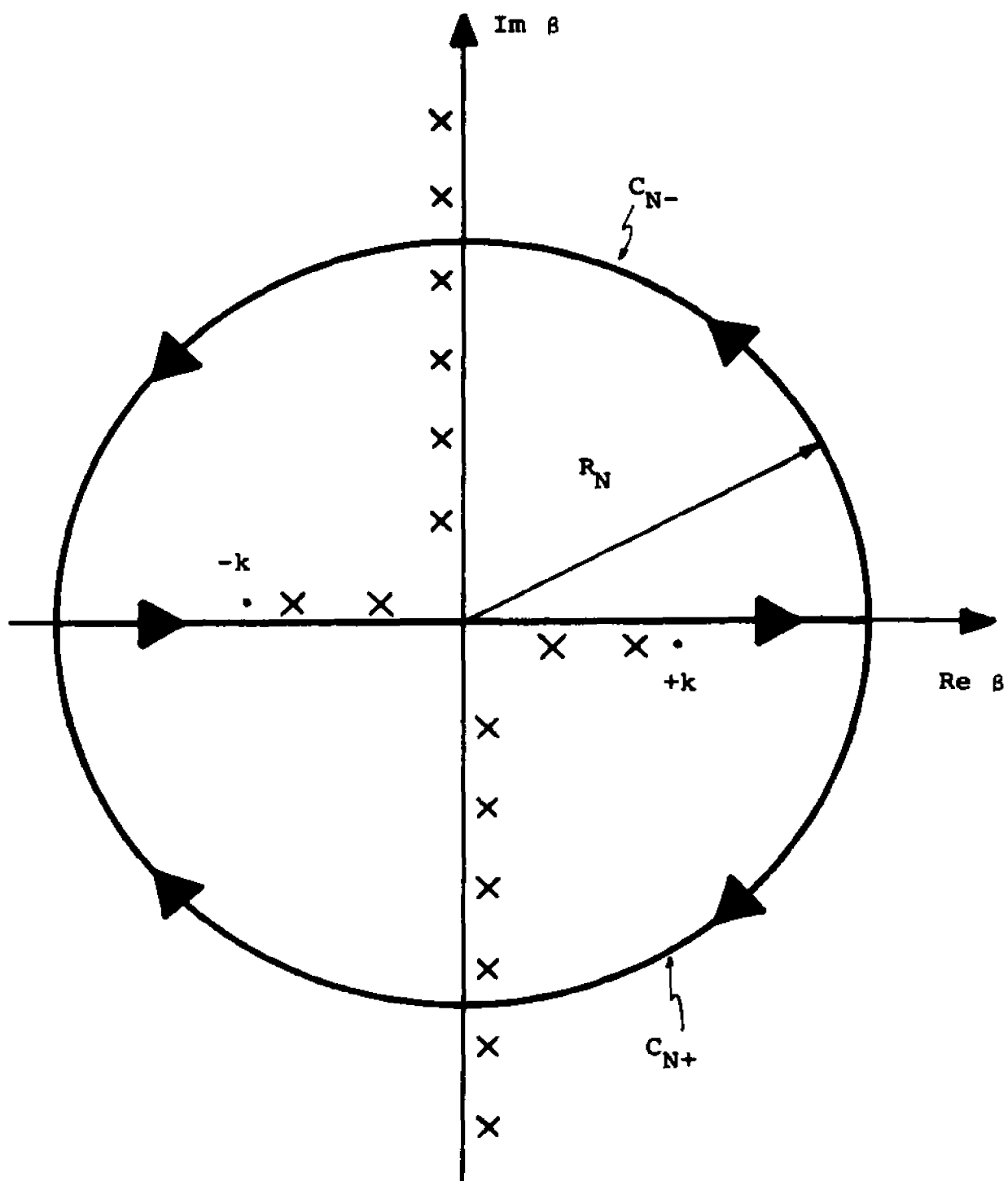


Fig. 2.1 Parallel plate waveguide

Fig. 2.2 Complex β -plane

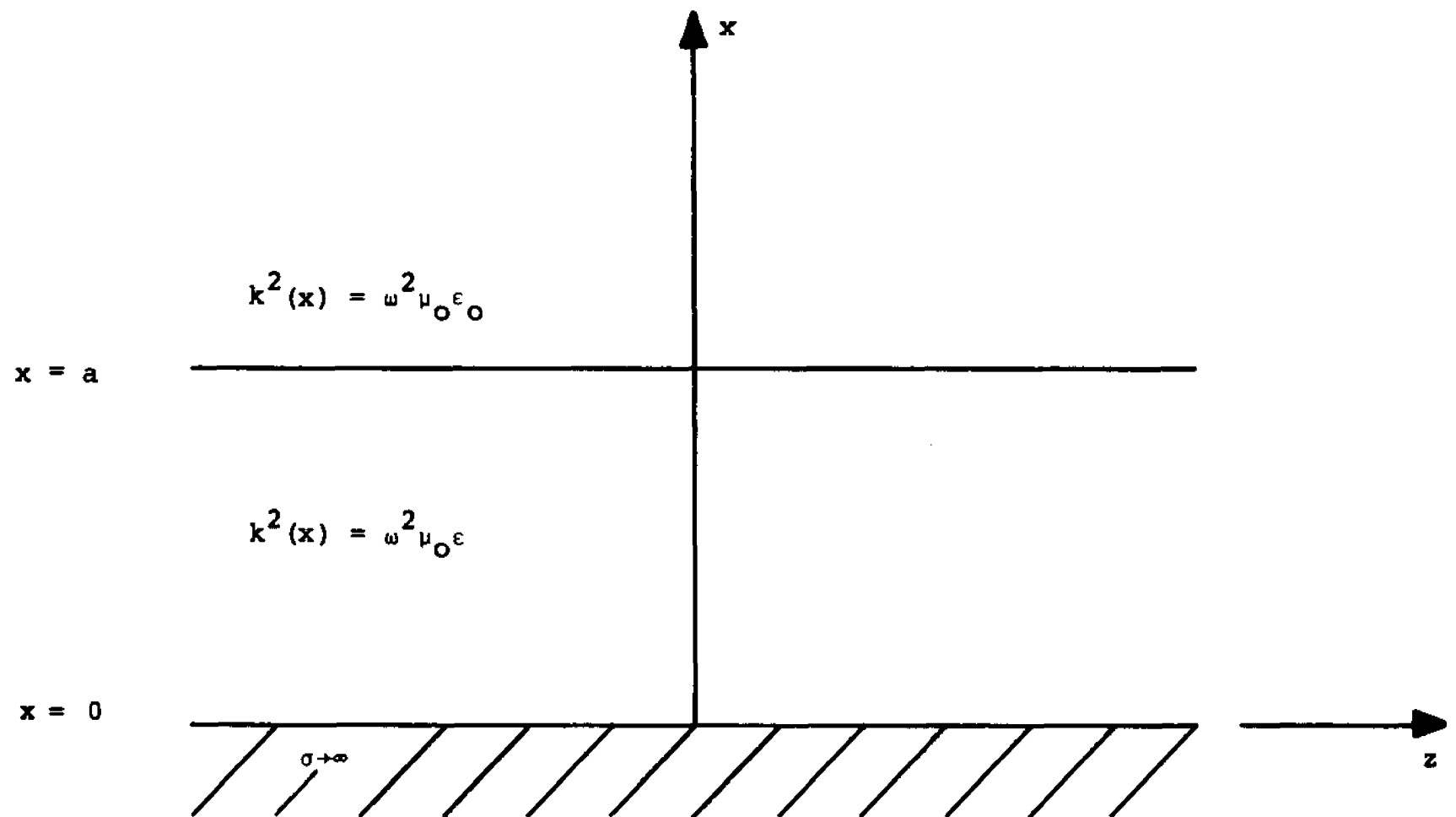


Fig. 2.3 Grounded dielectric slab waveguide

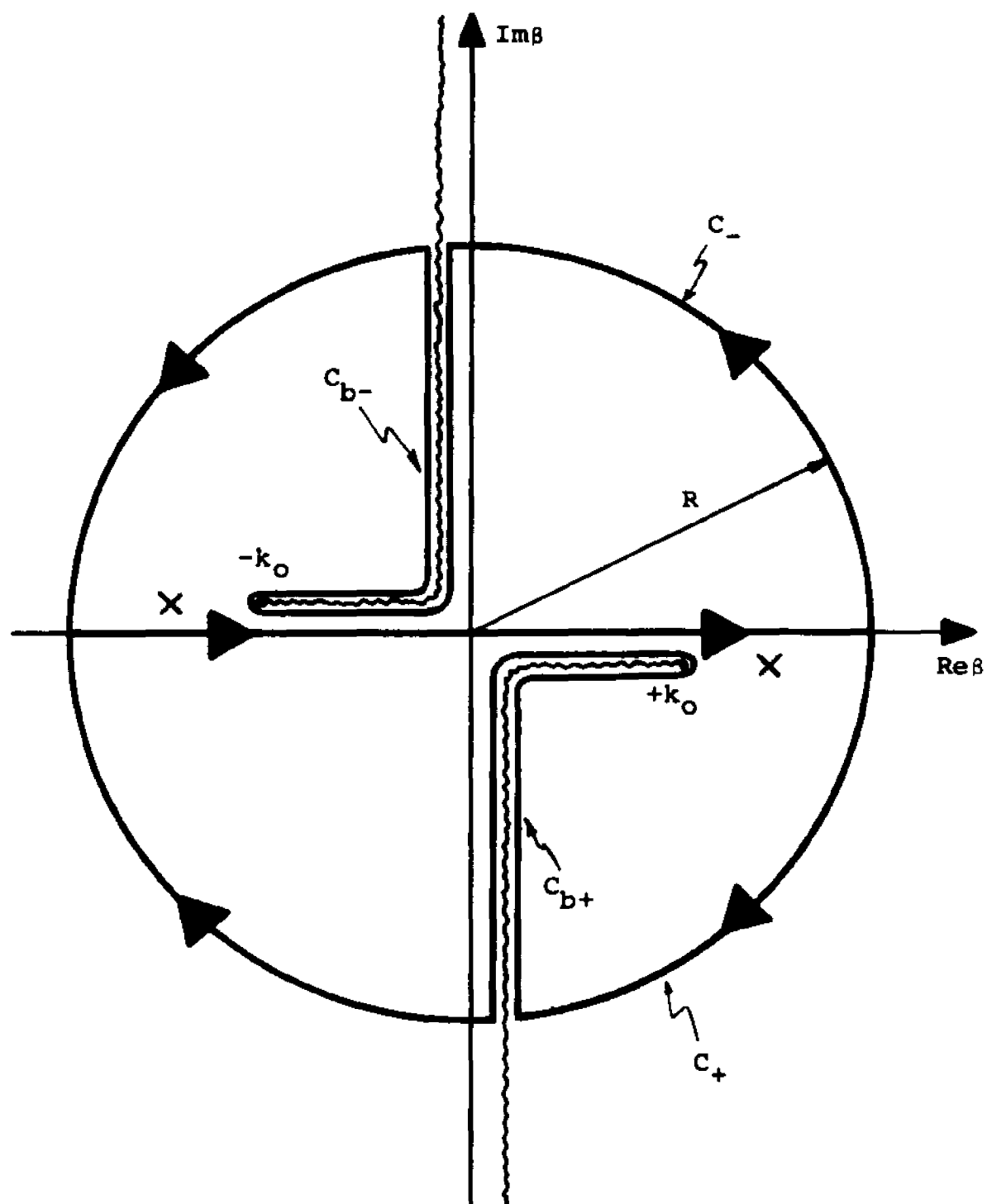


Fig. 2.4 Spectral sheet of complex β -plane

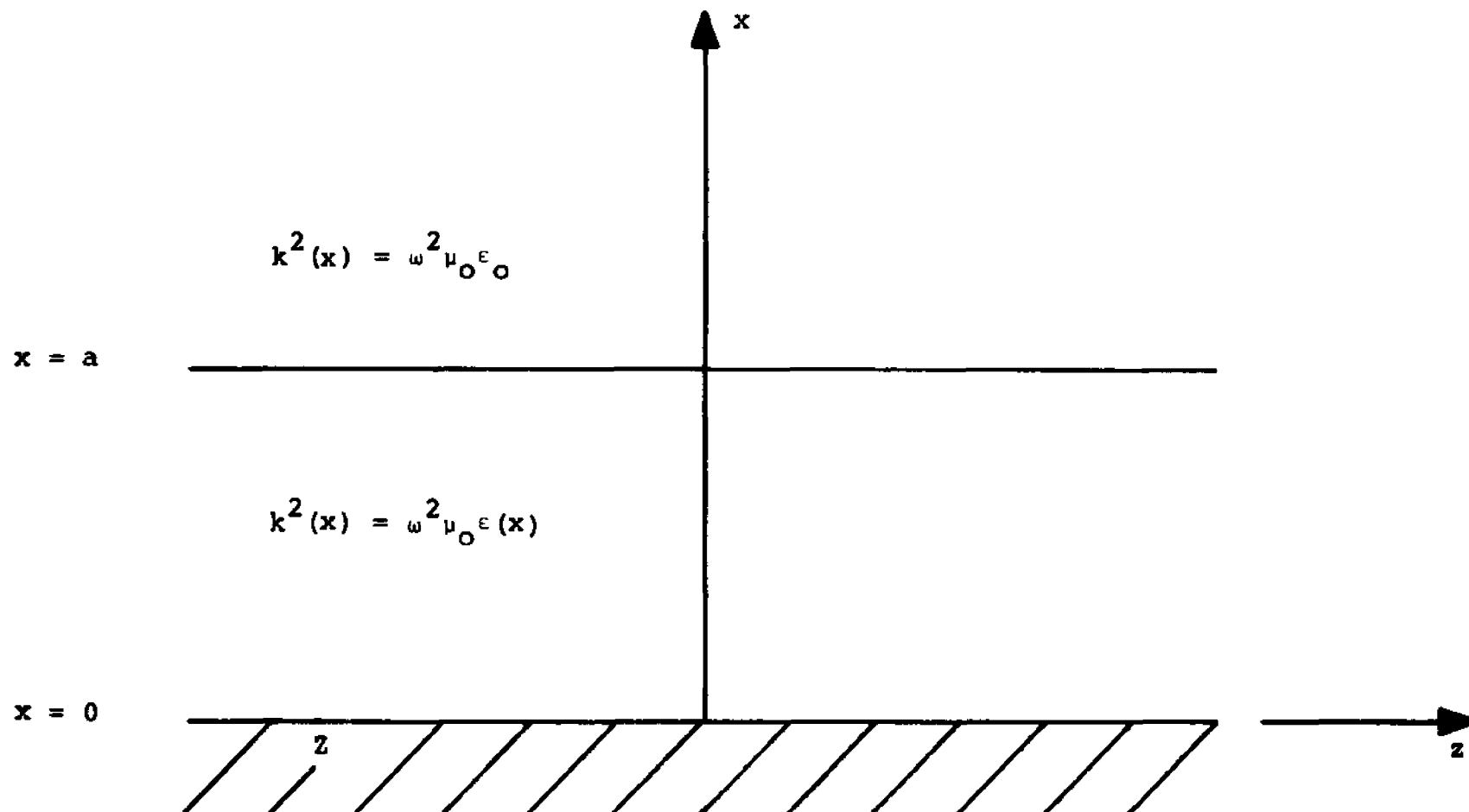


Fig. 3.1 Dielectric slab-impedance plane waveguide

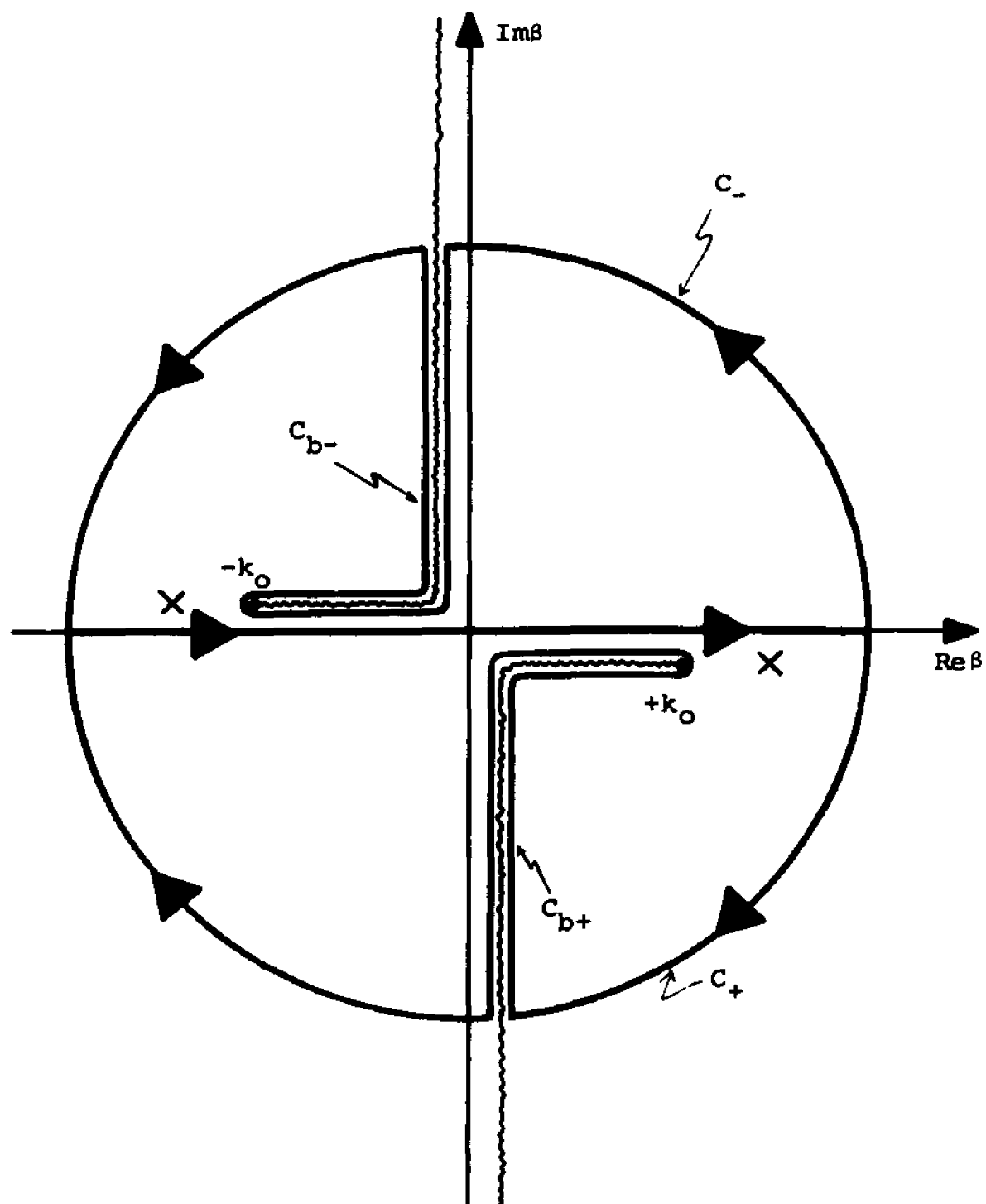


Fig. 3.2 Spectral sheet of complex β -plane

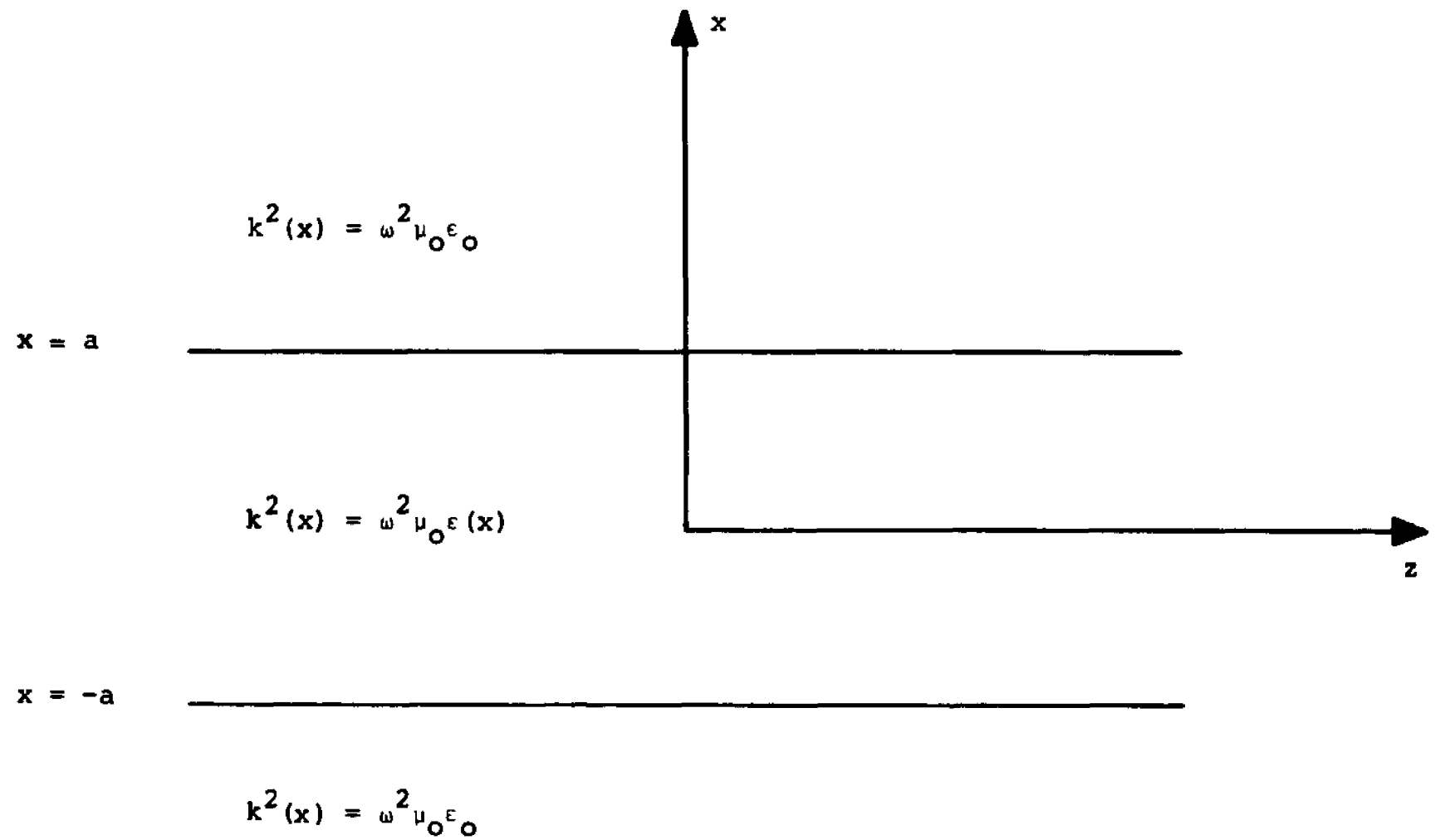


Fig. 3.3 Dielectric slab waveguide

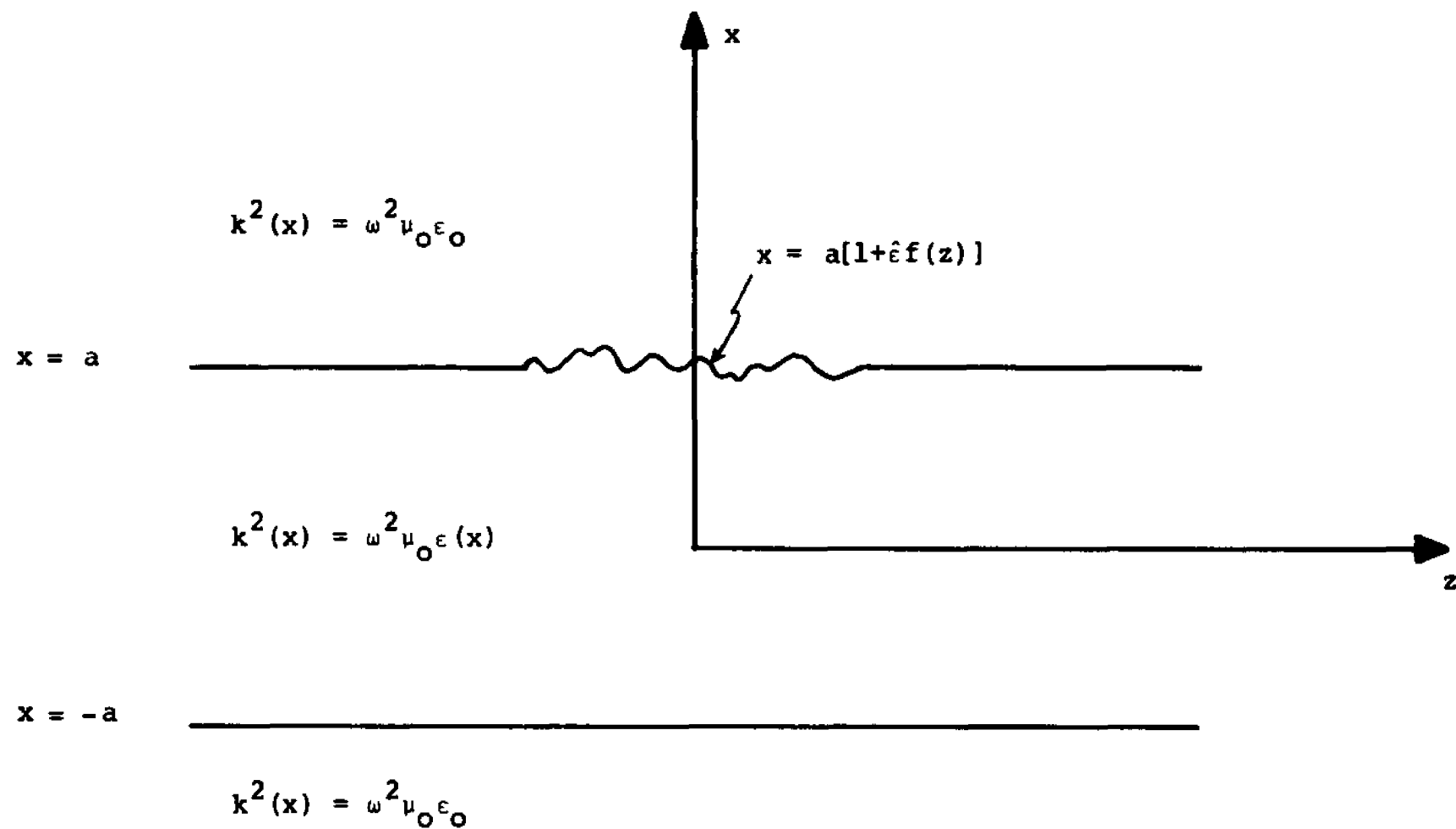


Fig. 4.1 Dielectric slab waveguide with perturbed boundary

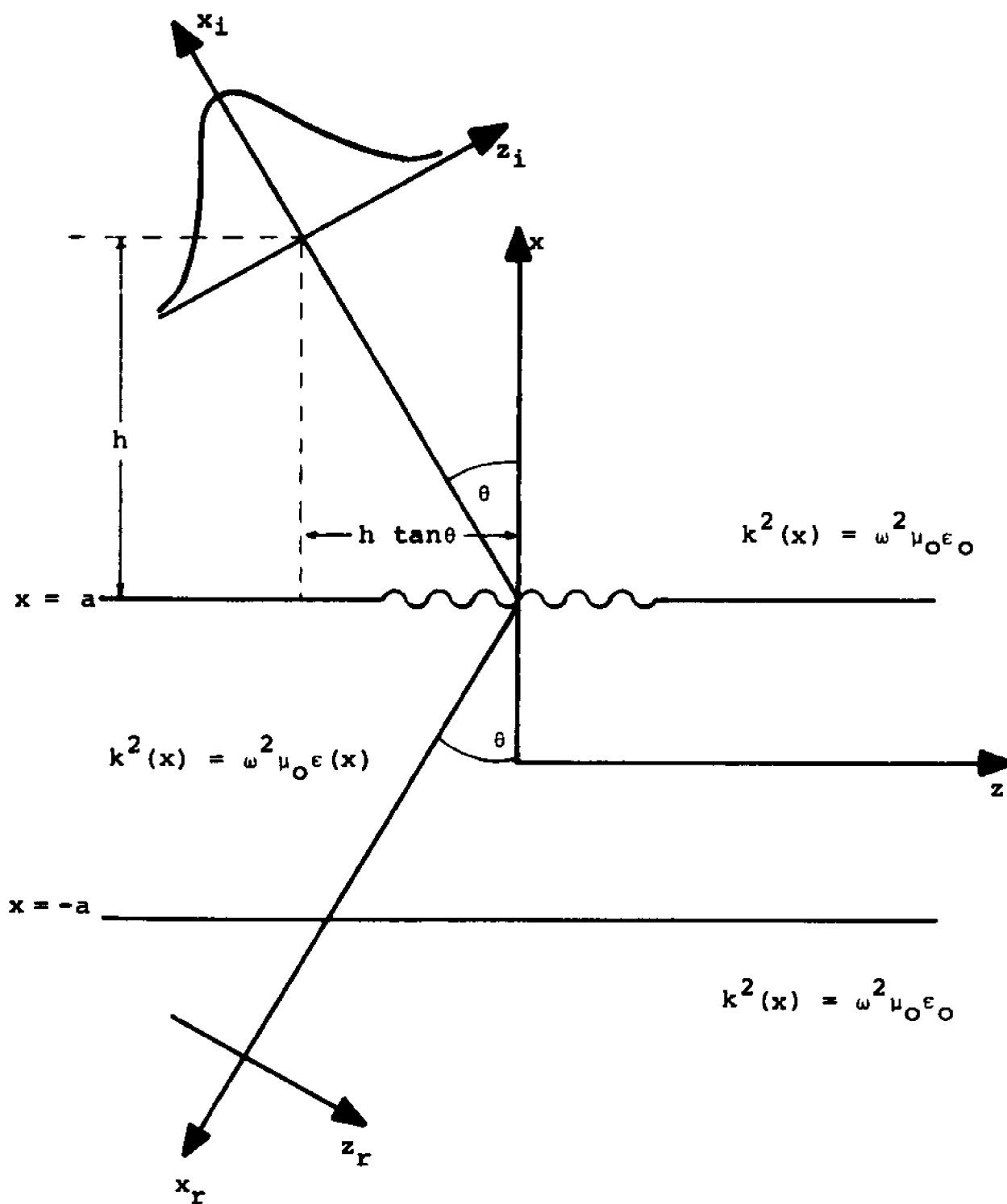


Fig. 4.2 Geometry of grating coupler

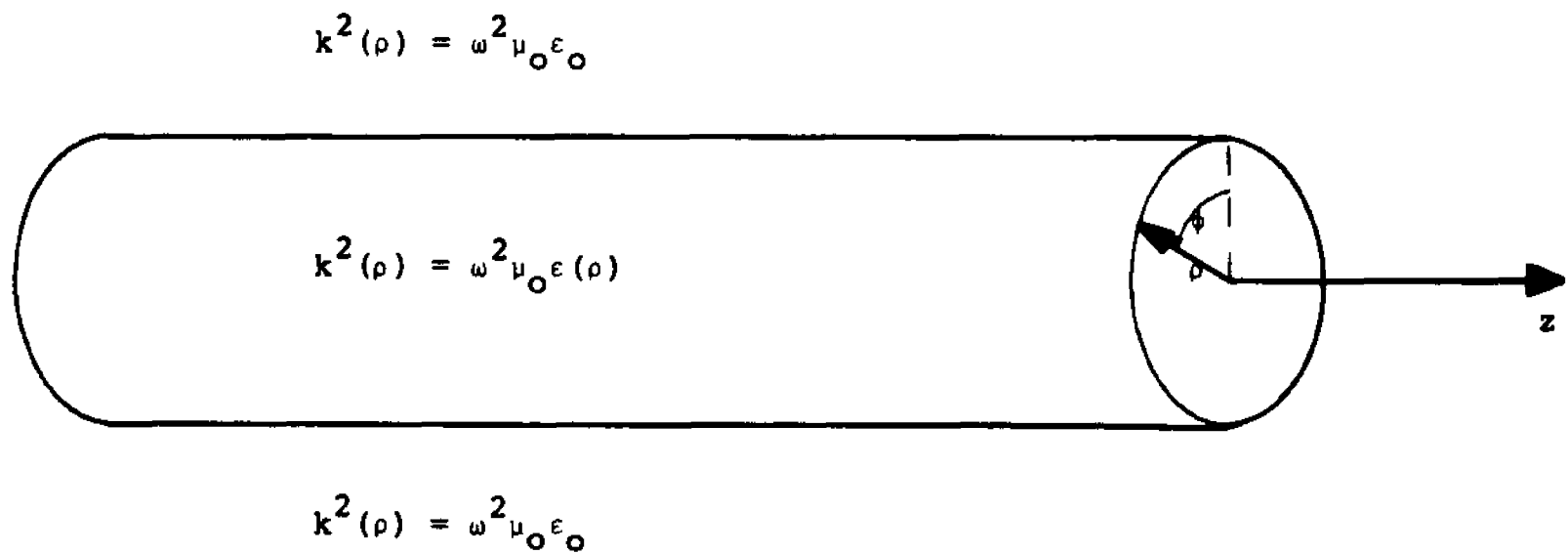


Fig. 5.1 Circularly cylindrical dielectric waveguide

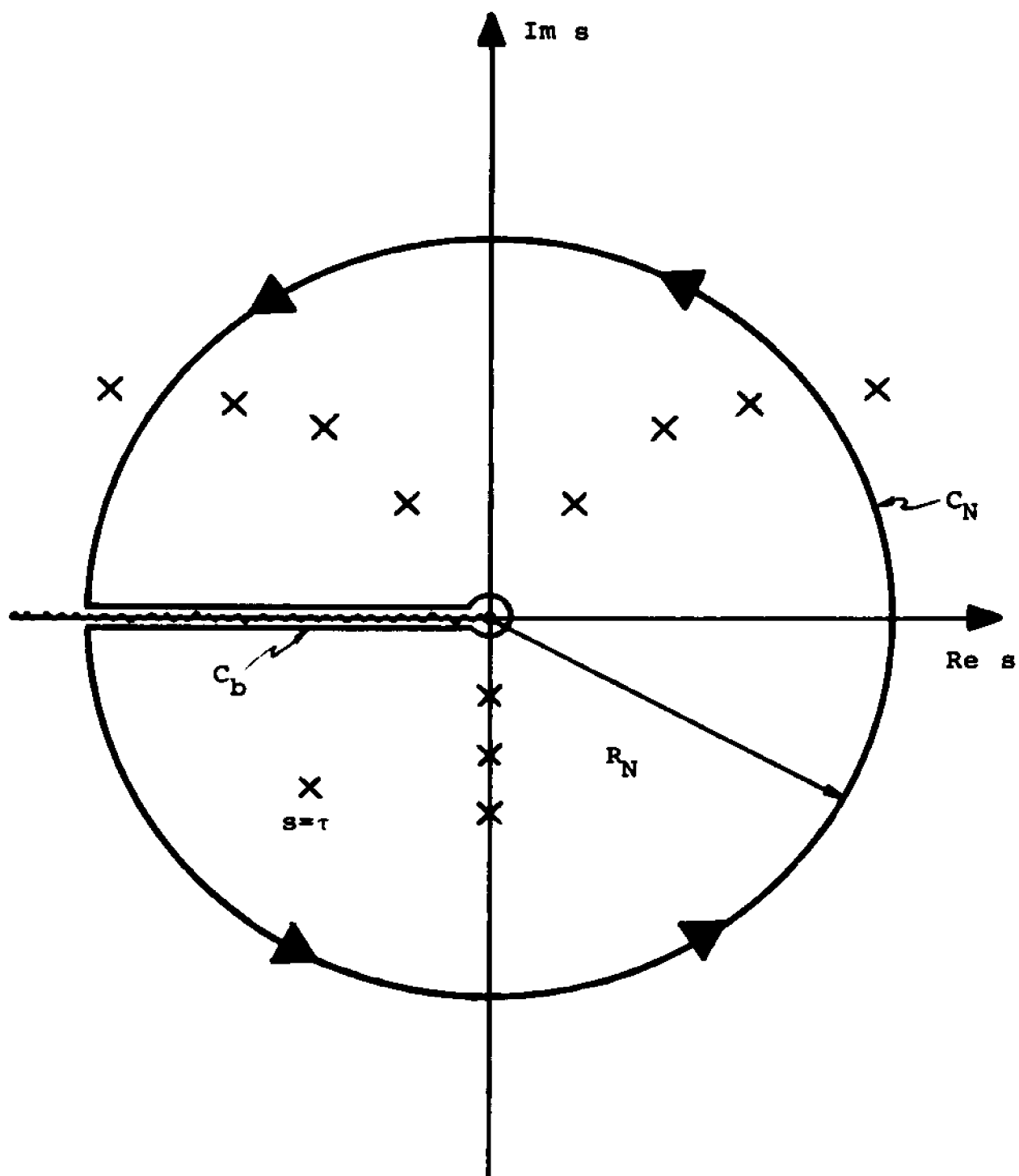


Fig. 5.2 Principal sheet of complex s -plane

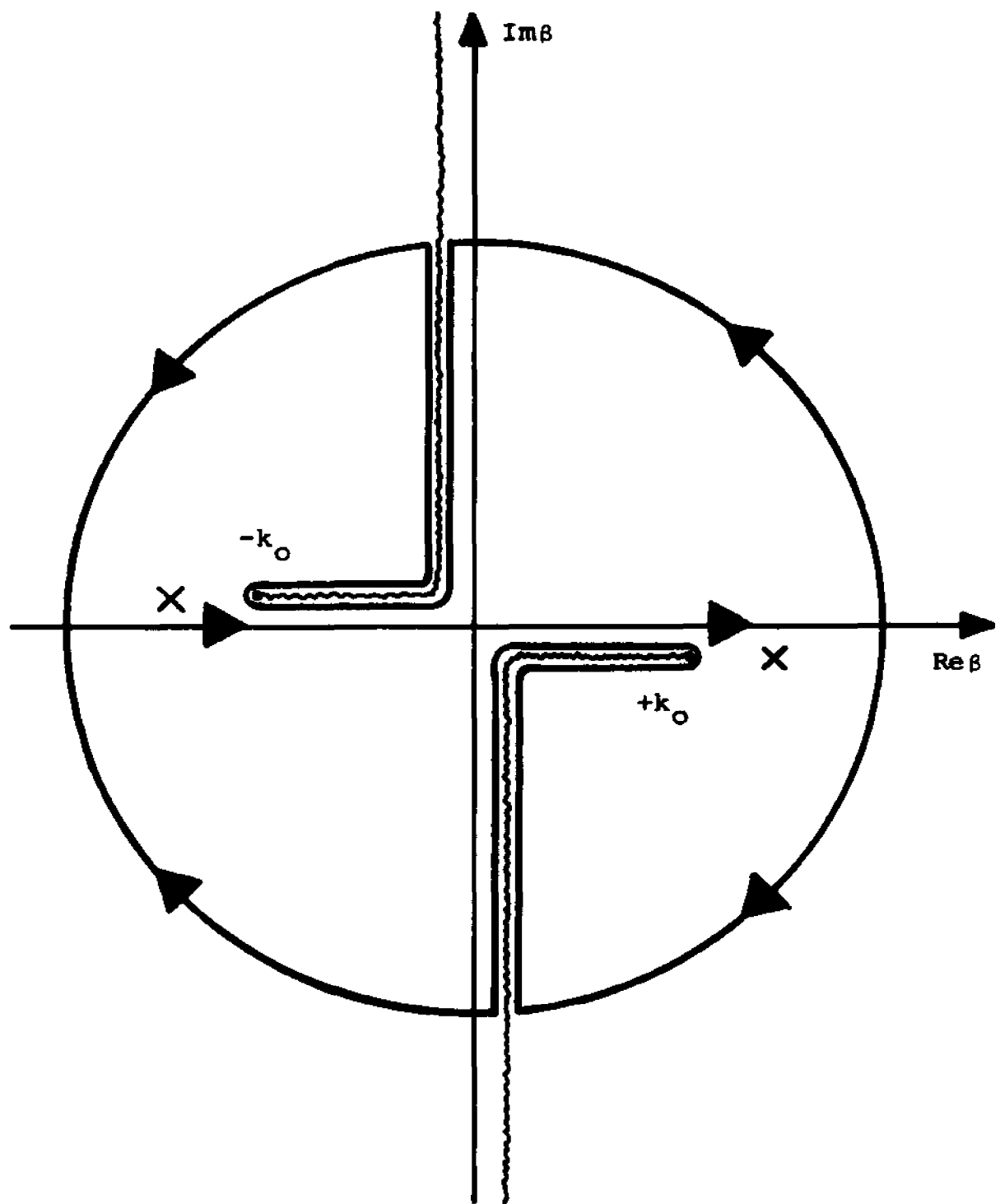


Fig. 5.3 Spectral sheet of complex β -plane

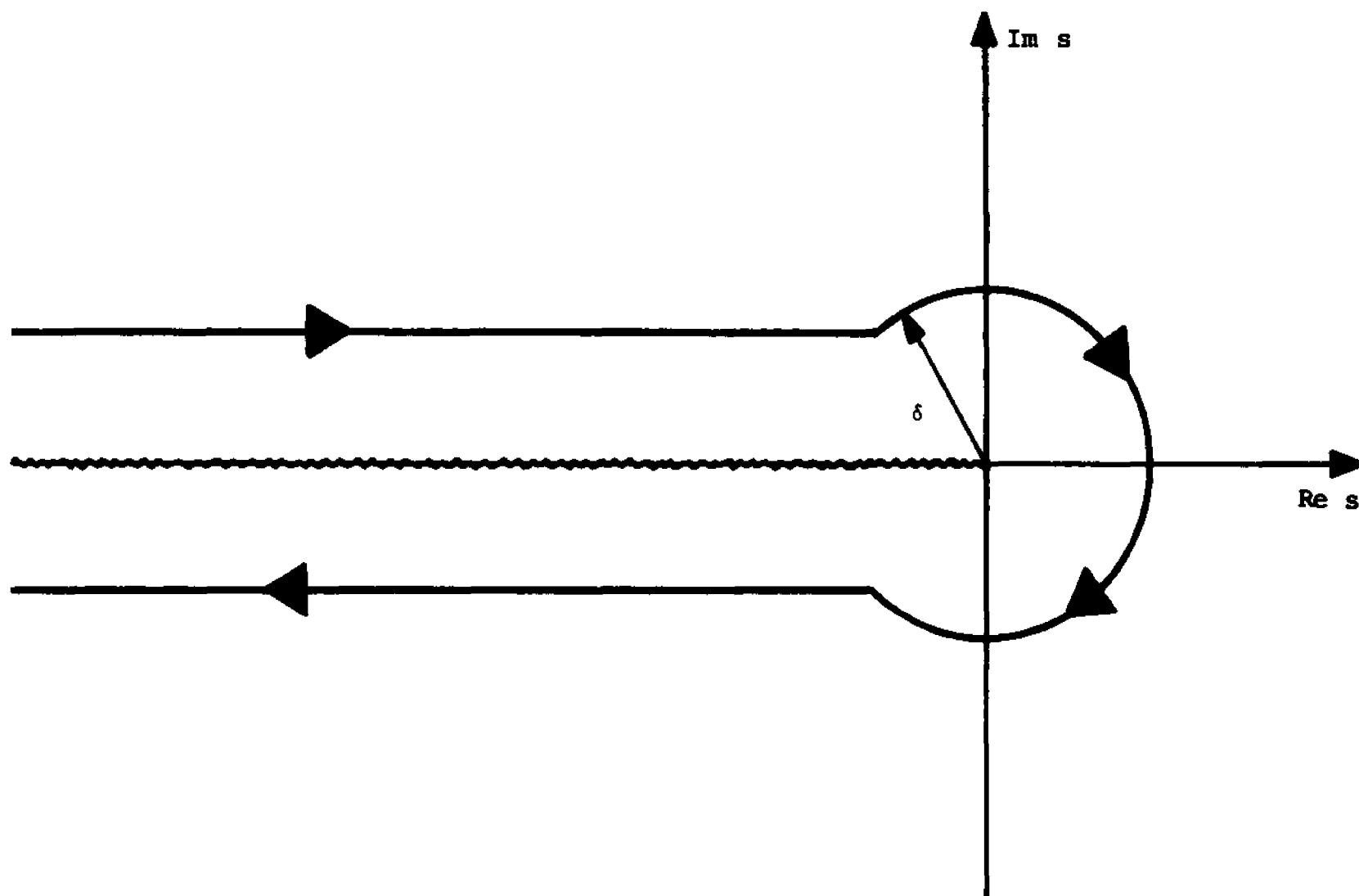


Fig. F-1 Path of integration for equation (F1)

APPENDIX A

ASYMPTOTIC SOLUTIONS IN PLANAR WAVEGUIDES

The asymptotic expansion of the function $y_0(x, \tau)$ associated with the dielectric slab-impedance plane waveguide is given by

$$y_0(x, \tau) \sim p^{-1/2}(x) e^{j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + p^{-1/2}(x) e^{-j\tau x} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x) \quad (\text{A1})$$

where the $a_k(x)$ and $b_k(x)$ are determined to within additive constants by the recursion relationships.

$$a_k'(x) = -\frac{1}{2} [\Omega(x) a_{k-1}(x) + a_{k-1}''(x)] \quad (\text{A2a})$$

and

$$b_k'(x) = -\frac{1}{2} [\Omega(x) b_{k-1}(x) + b_{k-1}''(x)]. \quad (\text{A2b})$$

The determination is made unique by imposing the boundary conditions

$$y_0(0, \tau) = -z_{11} \quad (\text{A3a})$$

and

$$y_0'(0, \tau) = z_{12} \quad (\text{A3b})$$

from which we obtain

$$\sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^{k-1} a_k(0) - \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^{k-1} b_k(0) = -j\tau p^{1/2}(0) z_{11} \quad (\text{A4})$$

and

$$\sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^{k-1} [a_k(0) + r(0)a_{k-1}(0) + a'_{k-1}(0)] \quad (\text{A5})$$

$$+ \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^{k-1} [b_k(0) + r(0)b_{k-1}(0) + b'_{k-1}(0)] = p^h(0) z_{i2}.$$

In the above we have defined

$$\begin{aligned} \Omega(x) &\triangleq \Delta k^2(x) \\ i &\triangleq 1 \\ r(x) &\triangleq 0 \end{aligned} \quad (\text{A6a})$$

for TE polarization, and

$$\begin{aligned} \Omega(x) &\triangleq \Xi(x) \\ i &\triangleq 2 \\ r(x) &\triangleq \frac{1}{2} \frac{\epsilon'(x)}{\epsilon(x)} \end{aligned} \quad (\text{A6b})$$

for TM polarization.

Equating like powers of τ in (A4), we obtain

$$a_0(0) + b_0(0) = -p^h(0) z_{i1} \quad (\text{A7a})$$

and

$$a_k(0) + (-1)^k b_k(0) = 0, \quad k \geq 1. \quad (\text{A7b})$$

Adding (A4) to (A5) and equating like powers of τ , we also obtain

$$a_0(0) = -\frac{1}{2} p^h(0) z_{i1} \quad (\text{A8a})$$

$$a_1(0) = -\frac{1}{2} \{r(0)[a_0(0) + b_0(0)] - p^h(0) z_{i2}\} \quad (\text{A8b})$$

and

$$a_k(0) = -\frac{1}{2} \{r(0)[a_{k-1}(0) + (-1)^{k-1} b_{k-1}(0)] + a'_{k-1}(0) + (-1)^{k-1} b'_{k-1}(0)\} \quad (\text{A8c})$$

$$k \geq 2.$$

From (A7) and (A8), we see that

$$a_0(0) = b_0(0) = -\frac{1}{2} p^{\frac{1}{2}}(0) z_{i1} \quad (\text{A9a})$$

$$a_1(0) = b_1(0) = \frac{1}{2} p^{\frac{1}{2}}(0) [r(0) z_{i1} + z_{i2}] \quad (\text{A9b})$$

and

$$a_k(0) = -(-1)^k b_k(0) = -\frac{1}{2} [a'_{k-1}(0) + (-1)^{k-1} b'_{k-1}(0)], \quad k \geq 2. \quad (\text{A9c})$$

It therefore follows from (A2) and (A9) that

$$a_k(x) = b_k(x), \quad \forall k \quad (\text{A10})$$

where

$$\begin{aligned} a_0(x) &\equiv -\frac{1}{2} p^{\frac{1}{2}}(0) z_{i1} \\ a_1(0) &= -\frac{1}{2} p^{\frac{1}{2}}(0) [r(0) z_{i1} + z_{i2}] \end{aligned} \quad (\text{A11})$$

$$a_k(0) = 0, \quad k = 2, 4, 6, \dots$$

$$a_k(0) = -a'_{k-1}(0), \quad k = 3, 5, 7, \dots$$

Similarly, the $a_k(x)$ and $b_k(x)$ in the asymptotic expansion of the function $y_1(x, \tau)$,

$$\begin{aligned} y_1(x, \tau) \sim p^{-\frac{1}{2}}(x) e^{-j\tau(a-x)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + p^{-\frac{1}{2}}(x) e^{j\tau(a-x)} \\ \cdot \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x), \end{aligned} \quad (\text{A12})$$

may be determined uniquely by imposing the boundary conditions

$$y_1(a, \tau) = 1 \quad (\text{A13a})$$

and

$$y'_1(a, \tau) = -j\tau K(a) \quad (\text{A13b})$$

where

$$K(x) \triangleq \begin{cases} 1 & \text{for TE polarization} \\ \epsilon_R(x) & \text{for TM polarization.} \end{cases} \quad (\text{A14})$$

We obtain

$$\sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(a) + \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(a) = p^{\frac{1}{2}}(a) \quad (\text{A15})$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k [a_k(a) + r(a) a_{k-1}(a) + a'_{k-1}(a)] \\ & - \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k [b_k(a) + r(a) b_{k-1}(a) + b'_{k-1}(a)] = -p^{\frac{1}{2}}(a) K(a). \end{aligned} \quad (\text{A16})$$

Equating like powers of τ in (A15), we obtain

$$a_0(a) + b_0(a) = p^{\frac{1}{2}}(a) \quad (\text{A17a})$$

and

$$a_k(a) + (-1)^k b_k(a) = 0, \quad k \geq 1. \quad (\text{A17b})$$

Adding (A15) to (A16) and equating like powers of τ , we also obtain

$$a_0(a) = -\frac{1}{2} p^{\frac{1}{2}}(a) [1 - K(a)] \quad (\text{A18a})$$

and

$$\begin{aligned} a_k(a) = & -\frac{1}{2} (r(a) [a_{k-1}(a) + (-1)^{k-1} b_{k-1}(a)] + a'_{k-1}(a) + (-1)^{k-1} \\ & \cdot b'_{k-1}(a)), \quad k \geq 1. \end{aligned} \quad (\text{A18b})$$

From (A17) and (A18) we see that

$$b_0(a) = \frac{1}{2} p^{\frac{1}{2}}(a) [1 + K(a)] \quad (\text{A19a})$$

$$a_1(a) = -\frac{1}{2} r(a) p^{\frac{1}{2}}(a) \quad (\text{A19b})$$

and

$$a_k(a) = -\frac{1}{2} [a_{k-1}'(a) + (-1)^{k-1} b_{k-1}'(a)], \quad k \geq 2. \quad (A19c)$$

We now show, by induction, that for $k \geq 1$ (A2) is equivalent to

$$a_k'(x) = \sum_{m=0}^{k-1} (-1/2)^{m+1} \frac{d^m}{dx^m} \{ \Omega(x) a_{k-1-m}(x) \} \quad (A20a)$$

and

$$b_k'(x) = \sum_{m=0}^{k-1} (-1/2)^{m+1} \frac{d^m}{dx^m} \{ \Omega(x) b_{k-1-m}(x) \}. \quad (A20b)$$

To do so, assume (A20) is true for some $k \geq 1$. Now from (A2a),

$$a_{k+1}'(x) = -\frac{1}{2} [\Omega(x) a_k(x) + a_k''(x)]. \quad (A21)$$

Differentiating (A20a) and substituting the result into (A21), we obtain

$$\begin{aligned} a_{k+1}'(x) &= -\frac{1}{2} \Omega(x) a_k(x) + \sum_{m=0}^{k-1} (-1/2)^{m+2} \frac{d^{m+1}}{dx^{m+1}} \{ \Omega(x) a_{k-1-m}(x) \} \\ &= \sum_{m=0}^k (-1/2)^{m+1} \frac{d^m}{dx^m} \{ \Omega(x) a_{k-m}(x) \} \end{aligned} \quad (A22)$$

which is (A20a) with k replaced by $k+1$. Since (A20a) is true for $k=1$, it therefore follows that it is true for $k \geq 1$. Similarly for (A20b).

Now let us consider the Taylor series expansion of $\Omega(x)$ about $x = a$, valid in the region $x < a$. We have

$$\Omega(x) = \sum_{m=0}^{\infty} \frac{1}{m!} r_m (x-a)^m \quad (A23)$$

where

$$r_m = \frac{d^m}{dx^m} \Omega(x) \Big|_{x=a}, \quad (A24)$$

the derivatives being taken from the left. Since $\Omega(x)$ is a function of $\Delta k^2(x)$, we also consider

$$\Delta k^2(x) = \sum_{m=0}^{\infty} \frac{1}{m!} p_m (x-a)^m \quad (A25)$$

and assume that

$$p_m = 0, \quad m \leq \mu-1 \quad (A26)$$

so that p_μ is the first non-zero derivative of $\Delta k^2(x)$ at $x = a$. From the definition of $\Omega(x)$, it is clear that

$$r_m = p_m \quad (A27)$$

in the TE case, while in the TM case

$$r_m = p_m + \frac{d^m}{dx^m} \left\{ \frac{1}{2} \frac{\epsilon''(x)}{\epsilon(x)} - \frac{3}{4} \left[\frac{\epsilon'(x)}{\epsilon(x)} \right]^2 \right\} \Big|_{x=a}. \quad (A28)$$

Noting Leibniz's theorem,

$$\frac{d^m}{dx^m} [f(x)g(x)] = \sum_{r=0}^m \frac{m!}{(m-r)!r!} f^{(r)}(x) g^{(m-r)}(x), \quad (A29)$$

we see that

$$\frac{d^m}{dx^m} \left[\frac{\epsilon''(x)}{\epsilon(x)} \right] = \sum_{r=0}^m \frac{m!}{(m-r)!r!} \epsilon^{(r+2)}(x) \left[\frac{1}{\epsilon(x)} \right]^{(m-r)}. \quad (A30)$$

Since

$$\epsilon^{(m)}(a) = \frac{p_m}{\omega^2 \mu_0}, \quad m \geq 1 \quad (A31)$$

it follows that

$$\frac{d^m}{dx^m} \left[\frac{\epsilon''(x)}{\epsilon(x)} \right] \Big|_{x=a} = 0, \quad m \leq \mu-3. \quad (A32)$$

Similarly, repeated application of (A29) yields

$$\frac{d^m}{dx^m} \left[\left(\frac{\epsilon'(x)}{\epsilon(x)} \right)^2 \right] = \sum_{r=0}^m \sum_{s=0}^r \sum_{t=0}^{m-r} \frac{m!}{(r-s)!(m-r-t)!s!t!} \epsilon^{(s+1)}(x) \cdot \epsilon^{(t+1)}(x) \left[\frac{1}{\epsilon(x)} \right]^{(r-s)} \left[\frac{1}{\epsilon(x)} \right]^{(m-r-t)} \quad (A33)$$

so that

$$\frac{d^m}{dx^m} \left[\left(\frac{\epsilon'(x)}{\epsilon(x)} \right)^2 \right] \Big|_{x=a} = 0, \quad m \leq \mu-2. \quad (A34)$$

Hence it follows that

$$r_m = 0, \quad m \leq \lambda-1 \quad (A35)$$

where

$$\lambda = \begin{cases} \mu & \text{for TE polarization} \\ \mu-2 & \text{for TM polarization.} \end{cases} \quad (A36)$$

Substituting (A35) into (A20) and using (A29), we obtain

$$a'_k(a) = 0, \quad 0 \leq k \leq \lambda \quad (A37a)$$

$$a'_k(a) = \sum_{m=\lambda}^{k-1} \sum_{n=\lambda}^m \left(-\frac{1}{2}\right)^{m+1} \frac{m!}{(m-n)!n!} r_n a_{k-1-m}^{(m-n)}(a), \quad k \geq \lambda+1 \quad (A37b)$$

$$b'_k(a) = 0, \quad 0 \leq k \leq \lambda \quad (A37c)$$

and

$$b'_k(a) = \sum_{m=\lambda}^{k-1} \sum_{n=\lambda}^m \left(-\frac{1}{2}\right)^{m+1} \frac{m!}{(m-n)!n!} r_n b_{k-1-m}^{(m-n)}(a), \quad k \geq \lambda+1. \quad (A37d)$$

Substituting (A37) into (A19c) and collecting our results, we have

$$a_0(x) \equiv \frac{1}{2} p^{\frac{1}{2}}(a) [1-K(a)] \quad (A38a)$$

$$a_1(a) = -\frac{1}{2} r(a) p^{\frac{1}{2}}(a) \quad (A38b)$$

$$a_k(a) = 0, \quad 2 \leq k \leq \lambda+1 \quad (A38c)$$

$$a_k(a) = -\frac{1}{2} [a'_{k-1}(a) + (-1)^{k-1} b'_{k-1}(a)], \quad k \geq \max(\lambda+2, 2) \quad (\text{A38d})$$

$$b_0(x) \equiv \frac{1}{2} p^h(a) [1+K(a)] \quad (\text{A38e})$$

$$b_k(a) = -(-1)^k a_k(a), \quad k \geq 1. \quad (\text{A38f})$$

These values are critically affected by the polarization and the behavior of the dielectric constant at the dielectric-free space interface.

In the TE case, $K(x) \equiv 1$ and $q(x) \equiv 0$. Substituting this into (A38) and using (A20a) and (A37), we find that

$$a_k(x) \equiv 0, \quad k \leq \mu+1$$

$$a_{\mu+2}(x) \equiv -(1/2)^{\mu+2} p_\mu$$

$$a_k(a) = -\frac{1}{2} (-1)^{k-1} b'_{k-1}(a), \quad \mu+3 \leq k \leq 2\mu+3$$

$$a_k(a) = -\frac{1}{2} [a'_{k-1}(a) + (-1)^{k-1} b'_{k-1}(a)], \quad k \geq 2\mu+4 \quad (\text{A39})$$

$$b_0(x) \equiv 1$$

$$b_k(a) = 0, \quad 1 \leq k \leq \mu+1$$

$$b_k(a) = -(-1)^k a_k(a), \quad k \geq \mu+2$$

In the TM case, $a_0(x)$ vanishes if $\mu \geq 1$ and $a_1(a)$ vanishes if $\mu \geq 2$. Substituting this into (A38) and using (A20a) and (A37), we find that

$$\begin{aligned}
a_k(x) &\equiv 0 & , k \leq \mu-1 \\
a_\mu(x) &\equiv -\left(\frac{1}{2}\right)^{\mu+1} \frac{1}{k_0^2 \epsilon^{1/2}(a)} p_\mu \\
a_1(a) &= -\frac{1}{4} \frac{\epsilon_0}{k_0^2 \epsilon^{3/2}(a)} p_1 & , \mu=0 \quad (A40) \\
a_k(a) &= -\frac{1}{2} [a_{k-1}'(a) + (-1)^{k-1} b_{k-1}'(a)] , k \geq \max(\mu+1, 2) \\
b_0(x) &\equiv \frac{p_0 + 2k_0^2}{2k_0^2 \epsilon^{1/2}(a)} \\
b_k(a) &= 0 & , 1 \leq k \leq \mu-1 \\
b_k(a) &= -(-1)^k a_k(a) & , k \geq \max(\mu, 1)
\end{aligned}$$

The asymptotic expansion of the function $y_0(x, \tau)$ associated with the dielectric slab waveguide is obtained in an analogous fashion. We have

$$y_0(x, \tau) \sim p^{-1/2}(x) e^{j\tau(a+x)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(x) + p^{-1/2}(x) e^{-j\tau(a+x)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(x) \quad (A41)$$

subject to the boundary conditions

$$y_0(-a, \tau) = 1 \quad (A42a)$$

and

$$y_0'(-a, \tau) = j\tau K(-a) . \quad (A42b)$$

Paralleling the analysis of $y_1(x, \tau)$, we obtain

$$b_k(x) \equiv 0, \quad k \leq v+1$$

$$b_{v+2}(x) \equiv -(1/2)^{v+2} q_v$$

$$b_k(-a) = -\frac{1}{2} (-1)^{k-1} a'_{k-1}(-a), \quad v+3 \leq k \leq 2v+3 \quad (A43)$$

$$b_k(-a) = -\frac{1}{2} [b'_{k-1}(-a) + (-1)^{k-1} a'_{k-1}(-a)], \quad k \geq 2v+4$$

$$a_0(x) \equiv 1$$

$$a_k(-a) = 0, \quad 1 \leq k \leq v+1$$

$$a_k(-a) = -(-1)^k b_k(-a), \quad k \geq v+2$$

for TE polarization, and

$$b_k(x) \equiv 0, \quad k \leq v-1$$

$$b_v(x) \equiv -(1/2)^{v+1} \frac{1}{k_0^2 \epsilon^{1/2}(-a)} q_v$$

$$b_1(-a) = -\frac{1}{4} \frac{\epsilon_0}{k_0^2 \epsilon^{3/2}(-a)} q_1, \quad v=0 \quad (A44)$$

$$b_k(-a) = -\frac{1}{2} [b'_{k-1}(-a) + (-1)^{k-1} a'_{k-1}(-a)], \quad k \geq \max(v+1, 2)$$

$$a_0(x) \equiv \frac{q_0 + 2k_0^2}{2k_0^2 \epsilon^{1/2}(-a)}$$

$$a_k(-a) = 0, \quad 1 \leq k \leq v-1$$

$$a_k(-a) = -(-1)^k b_k(-a), \quad k \geq \max(v, 1)$$

for TM polarization. In the above, q_m is the m th derivative of $\Delta k^2(x)$ at $x = -a$, taken from the right, with $q_m = 0$ for $0 \leq m \leq v-1$.

APPENDIX B

CONTOUR INTEGRALS

Consider the integral

$$I = \int_C e^{-A \operatorname{Im} \tau} \tau^q d\tau \quad (B1)$$

where C is a semicircle in the upper half τ -plane, A is a positive constant or zero, and q is a positive integer or zero. Since $\tau = R e^{j\theta}$ on the contour, where R is the radius of the semicircle, we have

$$I = j R^{q+1} \int_{\theta=0}^{\pi} e^{j\theta - AR \sin \theta} d\theta. \quad (B2)$$

Now since

$$\sin \theta \geq \begin{cases} \frac{2}{\pi} \theta & , \quad 0 \leq \theta \leq \pi/2 \\ 2 - \frac{2}{\pi} \theta & , \quad \pi/2 \leq \theta \leq \pi \end{cases} \quad (B3)$$

we have

$$|I| \leq R^{q+1} \left\{ \left| \int_0^{\pi/2} e^{j\theta - AR \frac{2}{\pi} \theta} d\theta \right| + \left| \int_{\pi/2}^{\pi} e^{j\theta - AR (2 - \frac{2}{\pi} \theta)} d\theta \right| \right\}. \quad (B4)$$

Evaluating the integrals

$$\int_0^{\pi/2} e^{j\theta - AR \frac{2}{\pi} \theta} d\theta = \frac{1 - j e^{-AR}}{AR \frac{2}{\pi} - j} \quad (B5)$$

and

$$\int_{\pi/2}^{\pi} e^{j\theta - AR(2 - \frac{2}{\pi}\theta)} d\theta = -\left(\frac{1 + je^{-AR}}{AR\frac{2}{\pi} + j}\right), \quad (B6)$$

we conclude that for sufficiently large R

$$I = O(R^q), \quad A \neq 0 \quad (B7a)$$

and

$$I = O(R^{q+1}), \quad A = 0. \quad (B7b)$$

APPENDIX C

RESIDUE CALCULATIONS FOR PLANAR WAVEGUIDES

The transverse Green's function

$$g(x, \xi; \tau) = \frac{y_0(x, \tau) y_1(\xi, \tau)}{J(\tau)} H(\xi - x) + \frac{y_0(\xi, \tau) y_1(x, \tau)}{J(\tau)} H(x - \xi) \quad (C1)$$

has a residue at $\tau = \tau_n$ given by

$$\text{Res } g(x, \xi; \tau_n) = \frac{y_0(x, \tau_n) y_1(\xi, \tau_n)}{\left. \frac{d}{d\tau} J(\tau) \right|_{\tau=\tau_n}} H(\xi - x) + \frac{y_0(\xi, \tau_n) y_1(x, \tau_n)}{\left. \frac{d}{d\tau} J(\tau) \right|_{\tau=\tau_n}} H(x - \xi). \quad (C2)$$

Since the conjunct

$$J(\tau) = p(x) [y_0'(x, \tau) y_1(x, \tau) - y_0(x, \tau) y_1'(x, \tau)] \quad (C3)$$

is independent of x , we may set $x = a$ and note that

$$y_1(a, \tau) = 1, \quad y_1'(a, \tau) = -j\tau K(a) \quad (C4)$$

where $K(x)$ is defined by (A14), to obtain

$$J(\tau) = p(a) [y_0'(a, \tau) + j\tau K(a) y_0(a, \tau)]. \quad (C5)$$

Therefore,

$$\frac{d}{d\tau} J(\tau) = p(a) \left[\frac{d}{d\tau} y_0'(a, \tau) + j\tau K(a) \frac{d}{d\tau} y_0(a, \tau) + jK(a) y_0(a, \tau) \right]. \quad (C6)$$

Now y_0 and y_1 are solutions to

$$[p(x) y'(x, \tau)]' + p(x) [\Delta k^2(x) + \tau^2] y(x, \tau) = 0. \quad (C7)$$

Multiplying this by $\frac{\partial}{\partial \tau} y(x, \tau)$, we obtain

$$\frac{\partial}{\partial \tau} y(x, \tau) [p(x) y'(x, \tau)]' + p(x) [\Delta k^2(x) + \tau^2] y(x, \tau) \frac{\partial}{\partial \tau} y(x, \tau) = 0. \quad (C8)$$

Differentiate (C7) with respect to τ and multiply the result by $y(x, \tau)$:

$$y(x, \tau) [p(x) \frac{\partial}{\partial \tau} y'(x, \tau)]' + p(x) [\Delta k^2(x) + \tau^2] y(x, \tau) \frac{\partial}{\partial \tau} y(x, \tau) + 2\tau p(x) y^2(x, \tau) = 0. \quad (C9)$$

Subtracting (C9) from (C8) and integrating by parts, we obtain the relationship

$$[p(x) \{y'(x, \tau) \frac{\partial}{\partial \tau} y(x, \tau) - y(x, \tau) \frac{\partial}{\partial \tau} y'(x, \tau)\}]_{x_1}^{x_2} = 2\tau \int_{x_1}^{x_2} p(x) y^2(x, \tau) dx. \quad (C10)$$

Setting $y=y_0$, $x_2=a$, and $x_1=0$ in the impedance plane problems, and noting that $\frac{\partial}{\partial \tau} z_{ij} = 0$, we find that

$$y_0'(a, \tau) \frac{\partial}{\partial \tau} y_0(a, \tau) - y_0(a, \tau) \frac{\partial}{\partial \tau} y_0'(a, \tau) = \frac{2\tau}{p(a)} \int_0^a p(x) y_0^2(x, \tau) dx. \quad (C11)$$

In the slab problems we set $y=y_0$, $x_2=a$, $x_1=-a$, and note that

$$\begin{aligned} \frac{\partial}{\partial \tau} y_0(-a, \tau) &= \frac{\partial}{\partial \tau} [1] = 0 \\ \frac{\partial}{\partial \tau} y_0'(-a, \tau) &= \frac{\partial}{\partial \tau} [j\tau K(-a)] = jK(-a) \end{aligned} \quad (C12)$$

to find

$$\begin{aligned} y_0'(a, \tau) \frac{\partial}{\partial \tau} y_0(a, \tau) - y_0(a, \tau) \frac{\partial}{\partial \tau} y_0'(a, \tau) + j \frac{p(-a)}{p(a)} K(-a) \\ = \frac{2\tau}{p(a)} \int_{-a}^a p(x) y_0^2(x, \tau) dx. \end{aligned} \quad (C13)$$

Solving for $\frac{\partial}{\partial \tau} y_0'(a, \tau)$ and substituting into (C6), we obtain

$$\begin{aligned} \frac{d}{d\tau} J(\tau) &= p(a) \left[\frac{y_0'(a, \tau)}{y_0(a, \tau)} + j\tau K(a) \right] \frac{\partial}{\partial \tau} y_0(a, \tau) + jp(a) K(a) y_0(a, \tau) \\ &\quad - \frac{2\tau}{y_0(a, \tau)} \int_0^a p(x) y_0^2(x, \tau) dx \end{aligned} \quad (C14)$$

for the impedance plane problems, and

$$\begin{aligned} \frac{\partial}{\partial \tau} J(\tau) = & p(a) \left[\frac{y_0'(a, \tau)}{y_0(a, \tau)} + j\tau K(a) \right] \frac{\partial}{\partial \tau} y_0(a, \tau) + jp(a)K(a)y_0(a, \tau) \\ & + jp(-a)K(-a)\frac{1}{y_0(a, \tau)} - \frac{2\tau}{y_0(a, \tau)} \int_{-a}^a p(x)y_0^2(x, \tau) dx \quad (C15) \end{aligned}$$

for the slab problems.

At $\tau = \tau_n$ the conjunct vanishes, so that from (C5)

$$y_0'(a, \tau_n) + j\tau_n K(a)y_0(a, \tau_n) = 0. \quad (C16)$$

Therefore, from (C14) and (C15),

$$\left. \frac{d}{d\tau} J(\tau) \right|_{\tau=\tau_n} = jp(a)K(a)y_0(a, \tau_n) - \frac{2\tau_n}{y_0(a, \tau_n)} \int_0^a p(x)y_0^2(x, \tau_n) dx \quad (C17)$$

and

$$\begin{aligned} \left. \frac{d}{d\tau} J(\tau) \right|_{\tau=\tau_n} = & jp(a)K(a)y_0(a, \tau_n) + jp(-a)K(-a)\frac{1}{y_0(a, \tau_n)} \\ & - \frac{2\tau_n}{y_0(a, \tau_n)} \int_a^{\infty} p(x)y_0^2(x, \tau_n) dx \quad (C18) \end{aligned}$$

in the impedance plane and slab problems, respectively. The vanishing of $J(\tau_n)$ also implies that y_0 and y_1 are proportional at $\tau = \tau_n$; since $y_1(a, \tau_n) = 1$, it follows that

$$y_1(x, \tau_n) = \frac{y_0(x, \tau_n)}{y_0(a, \tau_n)}. \quad (C19)$$

Substituting (C17)-(C19) into (C2) and choosing as eigenfunctions

$$\phi_n(x) = \frac{y_0(x, \tau_n)}{\left[-\frac{y_0'(a, \tau_n)}{2\tau_n} \left. \frac{d}{d\tau} J(\tau) \right|_{\tau=\tau_n} \right]^{1/2}} \quad (C20)$$

we obtain

$$\text{Res } g(x, \xi; \tau_n) = -\frac{1}{2\tau_n} \phi_n(x) \phi_n(\xi). \quad (C21)$$

APPENDIX D

REGULAR AND IRREGULAR SOLUTIONS

Solutions of the second order homogeneous differential equation

$$LU \equiv \frac{d^2 U}{d\rho^2} + \hat{p}(\rho) \frac{dU}{d\rho} + \hat{q}(\rho) U = 0 \quad (D1)$$

where

$$\hat{p}(\rho) = \begin{cases} \frac{1}{\rho} & , \text{ for TE polarization} \\ \frac{1}{\rho} [1 - \rho \frac{\epsilon'}{\epsilon}] & , \text{ for TM polarization} \end{cases} \quad (D2)$$

and

$$\hat{q}(\rho) = \begin{cases} \Delta k^2 + \tau^2 - \frac{1}{\rho^2} & , \text{ for TE polarization} \\ \Delta k^2 + \tau^2 - \frac{1}{\rho^2} [1 + \rho \frac{\epsilon'}{\epsilon}] & , \text{ for TM polarization} \end{cases} \quad (D3)$$

may be obtained by initially assuming the formal series

$$U(\alpha, \rho) = \rho^\alpha \sum_{n=0}^{\infty} C_n(\alpha) \rho^n. \quad (D4)$$

Since $\rho \hat{p}(\rho)$ and $\rho^2 \hat{q}(\rho)$ are regular at the origin, we can expand these functions into their Taylor series

$$\rho \hat{p}(\rho) = \sum_{m=0}^{\infty} \hat{p}_m \rho^m, \quad \rho^2 \hat{q}(\rho) = \sum_{m=0}^{\infty} \hat{q}_m \rho^m. \quad (D5)$$

Formal substitution into (D1) yields

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)C_n(\alpha)\rho^{n+\alpha-2} + \sum_{m=0}^{\infty} \hat{p}_m \rho^{m-1} \sum_{n=0}^{\infty} (n+\alpha)C_n(\alpha)\rho^{n+\alpha-1} \\ + \sum_{m=0}^{\infty} \hat{q}_m \rho^{m-2} \sum_{n=0}^{\infty} C_n(\alpha)\rho^{n+\alpha} = 0. \quad (D6)$$

Equating like powers of ρ to zero, we obtain the indicial equation

$$C_0(\alpha)F(\alpha) \triangleq C_0(\alpha)[\alpha(\alpha-1) + \hat{p}_0\alpha + \hat{q}_0] = 0 \quad (D7)$$

and the recurrence relation

$$C_n(\alpha)F(\alpha+n) + \sum_{\ell=0}^{n-1} C_{\ell}(\alpha)\{(\ell+\alpha)\hat{p}_{n-\ell} + \hat{q}_{n-\ell}\} = 0, \quad n=1,2,\dots \quad (D8)$$

The indicial equation (D7) yields α_1 and α_2 , the exponents at the regular singularity:

$$\alpha_1 = 1, \quad \alpha_2 = -1. \quad (D9)$$

The formal solution (D4) with $\alpha = \alpha_1$ can be shown to converge for $0 \leq \rho < a$. Thus the regular solution of (D1) is

$$U_1(\rho) = U(\alpha_1, \rho) \quad (D10)$$

for arbitrary C_0 .

For $\alpha = \alpha_2$ we find that $C_2(\alpha_2)$ is unbounded, so that a second solution of the form (D4) is not obtained. To obviate this difficulty, we choose $C_0(\alpha) = \alpha - \alpha_2$. It follows that $C_0(\alpha_2) = C_1(\alpha_2) = 0$, but $C_2(\alpha_2)$ is now indeterminate. Choosing an arbitrary value for $C_2(\alpha_2)$ and applying (D8) yields $C_{n+2}(\alpha_2)$ as linear multiples of $C_2(\alpha_2)$. This produces the solution

$$U(\alpha_2, \rho) = \rho^{\alpha_2} \sum_{n=0}^{\infty} C_n(\alpha_2)\rho^n = cU_1(\rho) \quad (D11)$$

where c is some constant. Hence the choice $C_0 = \alpha - \alpha_2$ in

(D4) yields a second solution which is merely a constant multiple of the first.

To obtain a second independent solution, we evaluate LU with $U = U(\alpha, \rho)$, $C_0(\alpha) = \alpha - \alpha_1$ and the remaining $C_n(\alpha)$ given by the recurrence relation (D8), but where α is not a solution of the indicial equation (D7). We obtain

$$L\{U(\alpha, \rho)\} = (\alpha - \alpha_2)F(\alpha)\rho^\alpha = (\alpha - \alpha_2)^2(\alpha - \alpha_1)\rho^\alpha. \quad (D12)$$

Because of the repeated factor we have

$$L\left\{\left(\frac{\partial U}{\partial \alpha}\right)_{\alpha=\alpha_2}\right\} = 0, \quad (D13)$$

so that $\left(\frac{\partial U}{\partial \alpha}\right)_{\alpha=\alpha_2}$ is a second independent solution of (D1).

From (D4) we calculate

$$\frac{\partial U}{\partial \alpha} = \ln \rho U(\alpha, \rho) + \rho^\alpha \sum_{n=0}^{\infty} C'_n(\alpha) \rho^n. \quad (D14)$$

Setting $\alpha = \alpha_2$ and noting (D11), we obtain the irregular solution of (D1)

$$U_2(\rho) = c \ln \rho U_1(\rho) + \rho^{\alpha_2} \sum_{n=0}^{\infty} C'_n(\alpha_2) \rho^n \quad (D15)$$

which converges for $0 < \rho < a$.

It is noted that since $\hat{p}$ and $\hat{q}$ are both integral functions of τ^2 , (D8), (D10) and (D15) demonstrate that U_1 and U_2 are also integral functions of τ^2 .

APPENDIX E

ASYMPTOTIC SOLUTIONS IN CIRCULARLY CYLINDRICAL WAVEGUIDES

With the substitution

$$U(\rho) = w(\rho) \exp\left[-\frac{1}{2} \int \hat{p}(\rho) d\rho\right] \quad (\text{E1})$$

equation (D1) is transformed into

$$\left(\frac{d^2}{d\rho^2} + \hat{q}(\rho) - \frac{1}{2} \hat{p}'(\rho) - \frac{1}{4} \hat{p}^2(\rho)\right)w(\rho) = 0. \quad (\text{E2})$$

The additional substitutions

$$\rho = \frac{x^2}{4}, \quad w(\rho) = x^{1/2} \psi(x) \quad (\text{E3})$$

transforms (E2) into

$$\left(\frac{d^2}{dx^2} + \frac{1}{4} x^2 \Delta k^2 - \frac{15}{4x^2}\right)\psi(x) = -\Delta(x)\psi(x) \quad (\text{E4})$$

where

$$\Delta(x) = \begin{cases} \frac{1}{4} x^2 \Delta k^2(\rho) & , \text{ for TE polarization} \\ \frac{1}{4} x^2 \left[\Delta k^2(\rho) - \frac{3}{4} \left(\frac{\epsilon'}{\epsilon}\right)^2 + \frac{1}{2} \frac{\epsilon''}{\epsilon} \right] - \frac{1}{2} \frac{\epsilon'}{\epsilon} & , \text{ for TM polarization.} \end{cases} \quad (\text{E5})$$

By variation of parameters, it follows that the general solution of (E4) in the interval $[0, 2a^{1/2}]$ (corresponding to the interval $0 \leq \rho \leq a$) may be written as

$$\psi(x) = C_1 \beta_1(x) + C_2 \beta_2(x) + \int_{x_0}^x \frac{\beta_p(x) \beta_q(y) - \beta_p(y) \beta_q(x)}{W(\beta_p, \beta_q)} \Delta(y) \psi(y) dy \quad (\text{E6})$$

where $\beta_{1,2}$ and $\beta_{p,q}$ are independent pairs of solutions to (E4) when $\Delta \equiv 0$, $C_{1,2}$ are constants, x_0 is a fixed point in $[0, 2a^{1/2}]$ and

$$W(\beta_p, \beta_q) = \beta_p(y) \frac{d}{dy} \beta_q(y) - \beta_q(y) \frac{d}{dy} \beta_p(y). \quad (E7)$$

In order to study the behavior of the solutions as a function of τ as well as x , we introduce the change in variable

$$z = \frac{1}{4} \tau x^2, \quad z_1 = \frac{1}{4} \tau y^2 \quad (E8)$$

so that (E6) becomes

$$\psi(z) = C_1 \beta_1(z) + C_2 \beta_2(z) + \frac{1}{\tau^{1/2}} \int_{z_0}^z \frac{\beta_p(z) \beta_q(z_1) - \beta_p(z_1) \beta_q(z)}{z_1^{1/2} W(\beta_p, \beta_q)} \Delta(z_1) \psi(z_1) dz_1. \quad (E9)$$

A particular solution ψ_1 , corresponding to the solution U_1 which vanishes at $\rho = 0$, may be obtained by choosing

$$\begin{aligned} C_1 &= 1, \quad C_2 = 0, \quad z_0 = 0 \\ \beta_1(z) &= \beta_p(z) = z^{1/4} J_1(z) \\ \beta_q(z) &= z^{1/4} N_1(z) \end{aligned} \quad (E10)$$

so that

$$\psi_1(z) = z^{1/4} J_1(z) + \frac{\pi}{2\tau} \int_0^z \left(\frac{z}{z_1}\right)^{1/4} [J_1(z) N_1(z_1) - J_1(z_1) N_1(z)] \Delta(z_1) \psi_1(z_1) dz_1. \quad (E11)$$

Alternatively, (E11) may be written as

$$\psi_1(z) = z^{1/4} J_1(z) + \frac{1}{4} \pi \int_0^z \left(\frac{z}{z_1}\right)^{1/4} [H_1^{(1)}(z) H_1^{(2)}(z_1) - H_1^{(1)}(z_1) H_1^{(2)}(z)] \cdot \Delta(z_1) \psi_1(z_1) dz_1, \quad (\text{E12})$$

which is obtained by choosing

$$C_1 = 1, \quad C_2 = 0, \quad z_0 = 0$$

$$\beta_1(z) = z^{1/4} J_1(z) \quad (\text{E13})$$

$$\beta_p(z) = z^{1/4} H_1^{(1)}(z)$$

$$\beta_q(z) = z^{1/4} H_1^{(2)}(z)$$

We shall obtain the behavior of ψ_1 by applying successive approximations to (E11) when the magnitude of z is sufficiently small (denoted symbolically by $|z| \leq N$), and to (E12) when the magnitude of z is sufficiently large ($|z| > N$).

Solution when $|z| \leq N$

In this region we have

$$J_1(z) = zO(1), \quad N_1(z) = \frac{1}{z}O(1), \quad (\text{E14})$$

with $O(1)$ signifying a bounded function, so that

$$\psi_1(z) = z^{1/4} J_1(z) + I\{\psi_1\} \quad (\text{E15})$$

where

$$I\{f\} = \frac{1}{2} \pi \int_0^z \left(\frac{z}{z_1}\right)^{1/4} \left[\frac{z}{z_1} O(1) - \frac{z_1}{z} O(1)\right] \Delta(z_1) f(z_1) dz_1. \quad (\text{E16})$$

Upon setting $t = z_1/z$, this may be written as

$$I\{f\} = \frac{\pi z}{2} \int_0^1 [t^{-5/4} O(1) - t^{3/4} O(1)] \Delta(tz) f(tz) dt. \quad (\text{E17})$$

Let $z^{1/4}J_1(z)$ be an initial approximation to $\psi_1(z)$. Upon substitution into the RHS of (E15), we are led to a new estimate of $\psi_1(z)$. This current estimate may be similarly used to generate a newer estimate, and so on. This procedure leads to the formal solution

$$\psi_1(z) = \sum_{n=0}^{\infty} \psi_1^{(n)}(z) \quad (\text{E18})$$

where $\psi_1^{(0)}(z) \triangleq z^{1/4}J_1(z)$ and

$$\psi_1^{(n+1)}(z) \triangleq I\{\psi_1^{(n)}\}, \quad n = 0, 1, 2, \dots \quad (\text{E19})$$

Since Δ is bounded, (E14), (E17) and (E19) yield

$$\begin{aligned} \psi_1^{(1)}(z) &= \frac{\pi z^{9/4}}{2\tau} \int_0^1 [O(1) - t^2 O(1)] O(1) dt \\ &= \frac{z^{9/4}}{\tau} O(1), \quad \text{when } |z| \leq N. \end{aligned} \quad (\text{E20})$$

Continuing the iteration procedure, we obtain the general relation

$$\psi_1^{(n)}(z) = \frac{z^{n+5/4}}{\tau^n} O(1), \quad \text{when } |z| \leq N \quad (\text{E21})$$

from which it is evident that (E18) is uniformly convergent over the region $|z| \leq N$ when $|\tau|$ is sufficiently large. Therefore, from (E1), (E3), (E18) and (E21), we have

$$U_1(\rho) = [2r(\rho)\tau^{1/2}]^{1/2} [J_1(\tau\rho) + \tau\rho \sum_{n=1}^{\infty} \rho^n O(1)] \quad (\text{E22})$$

when $|\tau|$ is sufficiently large and $\rho \leq \frac{N}{|\tau|}$, where

$$r(\rho) \triangleq \begin{cases} 1 & , \text{ for TE polarization} \\ \epsilon(\rho) & , \text{ for TM polarization.} \end{cases} \quad (\text{E23})$$

Solution when $|z| > N$

For $|z| < N$ we have

$$H_1^{(1,2)}(z) = \frac{1}{z} O(1) \quad (E24)$$

while for $|z| > N$,

$$H_1^{(1,2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm j(z-3\pi/4)} [1 + O(\frac{1}{z})], \quad |\arg z| < \pi. \quad (E25)$$

Since the variable of integration in (E12) extends from a region of small magnitudes to a region of large magnitudes, we may use these to express (E12) as

$$\begin{aligned} \psi_1(z) = & z^{1/4} J_1(z) + \frac{j\pi}{4\tau} \left\{ \left(\frac{2}{\pi}\right)^{1/2} \int_0^{z_N} z^{-1/4} z_1^{-5/4} [e^{j(z-3\pi/4)} O(1) \right. \\ & - e^{-j(z-3\pi/4)} O(1)] \Delta(z_1) \psi_1(z_1) dz_1 + \frac{2}{\pi} \int_{z_N}^z z^{-1/4} z_1^{-3/4} [e^{j(z-z_1)} O(1) \\ & \left. - e^{-j(z-z_1)} O(1)] \Delta(z_1) \psi_1(z_1) dz_1 \right\}, \end{aligned} \quad (E26)$$

where $|z_N| = N$.

Proceeding in the manner following (E17), we obtain the formal solution (E18) with (E19), where now we define

$$\begin{aligned} I\{f\} = & \frac{j\pi}{4\tau} \left\{ \left(\frac{2}{\pi}\right)^{1/2} \int_0^{z_N} z^{-1/4} z_1^{-5/4} [e^{j(z-3\pi/4)} O(1) - e^{-j(z-3\pi/4)}] \right. \\ & \cdot \Delta(z_1) f(z_1) dz_1 + \frac{2}{\pi} \int_{z_N}^z z^{-1/4} z_1^{-3/4} [e^{j(z-z_1)} O(1) - e^{-j(z-z_1)} O(1)] \\ & \left. \cdot \Delta(z_1) f(z_1) dz_1 \right\}. \end{aligned} \quad (E27)$$

Noting that for $|z| > N$

$$\begin{aligned} J_1(z) = & \left(\frac{1}{2\pi z}\right)^{1/2} \{ e^{j(z-3\pi/4)} [1 + O(\frac{1}{z})] + e^{-j(z-3\pi/4)} [1 + O(\frac{1}{z})] \}, \\ & |\arg z| < \pi, \end{aligned} \quad (E28)$$

we may use this with (E19), (E20) and (E27) to obtain

$$\begin{aligned} \psi_1^{(1)}(z) = & \frac{j\pi}{4\tau} z^{-1/4} \left\{ \left(\frac{2}{\pi}\right)^{1/2} \int_0^{z_N} [e^{j(z-3\pi/4)} O(1) - e^{-j(z-3\pi/4)} O(1)] \right. \\ & \cdot O(1) dz_1 + \left(\frac{2}{\pi}\right)^{1/2} \int_{z_N}^z z_1^{-1} e^{\pm j(z_1-3\pi/4)} [e^{j(z-z_1)} O(1) \\ & \left. - e^{-j(z-z_1)} O(1)] O(1) dz_1 \right\}, \end{aligned} \quad (E29)$$

where the plus sign is used when $-\pi < \arg z < 0$ and the minus sign is used when $0 < \arg z < \pi$. For $-\pi < \arg z < 0$ (E29) yields

$$\begin{aligned} \psi_1^{(1)}(z) = & \frac{j\pi}{4\tau} z^{-1/4} e^{j(z-3\pi/4)} \left\{ \left(\frac{2}{\pi}\right)^{1/2} \int_0^{z_N} [O(1) - e^{-j2(z-3\pi/4)} O(1)] \right. \\ & \cdot O(1) dz_1 + \left(\frac{2}{\pi}\right)^{1/2} \int_{z_N}^z z_1^{-1} [O(1) - e^{-j2(z-z_1)} O(1)] O(1) dz_1 \left. \right\}. \end{aligned} \quad (E30)$$

Since the first integral on the RHS is bounded, while the second is of the order $\ln \tau$, we find that

$$\psi_1^{(1)}(z) = z^{-1/4} \frac{\ln \tau}{\tau} e^{j(z-3\pi/4)} O(1), \text{ when } |z| > N \text{ and } -\pi < \arg z < 0. \quad (E31)$$

Continuing the iteration procedure, we obtain

$$\psi_1^{(n)}(z) = z^{-1/4} \left(\frac{\ln \tau}{\tau}\right)^n e^{j(z-3\pi/4)} O(1), \text{ when } |z| > N \text{ and } -\pi < \arg z < 0. \quad (E32)$$

In the region $0 < \arg z < \pi$, the form obtained differs from (E32) only in the sign of the argument of the exponential:

$$\psi_1^{(n)}(z) = z^{-1/4} \left(\frac{\ln \tau}{\tau}\right)^n e^{-j(z-3\pi/4)} O(1), \text{ when } |z| > N \text{ and } 0 < \arg z < \pi. \quad (E33)$$

The convergence of (E18) is therefore assured when $|\tau|$ is

sufficiently large. Hence, from (E1), (E3), (E18), (E32) and (E33), we have

$$U_1(\rho) = [2r(\rho)\tau^{1/2}]^{1/2} [J_1(\tau\rho) + \frac{1}{(\tau\rho)^{1/2}} (e^{j(\tau\rho-3\pi/4)} \sum_{n=1}^{\infty} (\frac{\ln\tau}{\tau})^n O(1) + e^{-j(\tau\rho-3\pi/4)} \sum_{n=1}^{\infty} (\frac{\ln\tau}{\tau})^n O(1))] \quad (E34)$$

when $|\tau|$ is sufficiently large, $-\pi < \arg\tau < \pi$ and $\rho > \frac{N}{|\tau|}$.

Also of interest are the particular solutions ψ_+ and ψ_- which decay with increasing $|\tau|$ in the regions $0 < \arg\tau < \pi$ and $-\pi < \arg\tau < 0$, respectively. Initially, we assume that R_τ , the domain of τ , includes sufficiently large $|\tau|$, but is bounded. We designate by τ_{M+} and τ_{M-} those points in R_τ for which $\text{Im}\tau_{M-} \leq \text{Im}\tau \leq \text{Im}\tau_{M+}$ for all τ in R_τ . Then for ψ_+ we may choose

$$\begin{aligned} C_1 &= 1, C_2 = 0, z_0 \triangleq z_{M+} = \tau_{M+} a \\ \beta_1(z) &= \beta_p(z) = z^{1/4} H_1^{(1)}(z) \\ \beta_q(z) &= z^{1/4} H_1^{(2)}(z) \end{aligned} \quad (E35)$$

or, alternatively,

$$\begin{aligned} C_1 &= 1, C_2 = 0, z_0 = z_{M+} \\ \beta_1(z) &= z^{1/4} H_1^{(1)}(z) \\ \beta_p(z) &= z^{1/4} J_1(z) \\ \beta_q(z) &= z^{1/4} N_1(z) \end{aligned} \quad (E36)$$

to obtain, respectively,

$$\begin{aligned} \psi_+(z) = & z^{1/4} H_1^{(1)}(z) + \frac{j\pi}{4\tau} \int_{z_{M+}}^z \left(\frac{z}{z_1}\right)^{1/4} [H_1^{(1)}(z) H_1^{(2)}(z_1) - H_1^{(1)}(z_1) \\ & \cdot H_1^{(2)}(z)] \Delta(z_1) \psi_+(z_1) dz_1 \end{aligned} \quad (\text{E37})$$

and

$$\begin{aligned} \psi_+(z) = & z^{1/4} H_1^{(1)}(z) + \frac{\pi}{2\tau} \int_{z_{M+}}^z \left(\frac{z}{z_1}\right)^{1/4} [J_1(z) N_1(z_1) - J_1(z_1) N_1(z)] \\ & \cdot \Delta(z_1) \psi_+(z_1) dz_1. \end{aligned} \quad (\text{E38})$$

For ψ_- we may choose

$$\begin{aligned} C_1 &= 0, \quad C_2 = 1, \quad z_0 = z_{M-} \triangleq \tau_{M-} a \\ \beta_P(z) &= z^{1/4} H_1^{(1)}(z) \\ \beta_2(z) &= \beta_Q(z) = z^{1/4} H_1^{(2)}(z) \end{aligned} \quad (\text{E39})$$

or

$$\begin{aligned} C_1 &= 0, \quad C_2 = 1, \quad z_0 = z_{M-} \\ \beta_2(z) &= z^{1/4} H_1^{(2)}(z) \\ \beta_P(z) &= z^{1/4} J_1(z) \\ \beta_Q(z) &= z^{1/4} N_1(z) \end{aligned} \quad (\text{E40})$$

which respectively yield

$$\begin{aligned} \psi_-(z) = & z^{1/4} H_1^{(2)}(z) + \frac{j\pi}{4\tau} \int_{z_{M-}}^z \left(\frac{z}{z_1}\right)^{1/4} [H_1^{(1)}(z) H_1^{(2)}(z_1) - H_1^{(1)}(z_1) \\ & \cdot H_1^{(2)}(z)] \Delta(z_1) \psi_-(z_1) dz_1 \end{aligned} \quad (\text{E41})$$

and

$$\begin{aligned} \psi_-(z) = & z^{1/4} H_1^{(2)}(z) + \frac{\pi}{2\tau} \int_{z_{M-}}^z \left(\frac{z}{z_1}\right)^{1/4} [J_1(z) N_1(z_1) - J_1(z_1) N_1(z)] \\ & \cdot \Delta(z_1) \psi_-(z_1) dz_1. \end{aligned} \quad (\text{E42})$$

For large $|z|$, (E37) and (E41) will be used; (E38) and (E42) will be used for small $|z|$.

Solution when $|z| > N$

We assume that the contours of integration in (E37) and (E41) have been chosen such that $|z_1| > N$ along them. Using (E25) in (E37) we obtain

$$\psi_+(z) = z^{1/4} H_1^{(1)}(z) + \frac{j}{2\tau} z^{-1/4} \int_{z_{M+}}^z z_1^{-3/4} [e^{j(z-z_1)} O(1) - e^{-j(z-z_1)} O(1)] \Delta(z_1) \psi_+(z_1) dz_1. \quad (E43)$$

Defining

$$I\{f\} \triangleq \frac{j}{2\tau} z^{1/4} \int_{z_{M+}}^z z_1^{-3/4} [e^{j(z-z_1)} O(1) - e^{-j(z-z_1)} O(1)] \Delta(z_1) f(z_1) dz_1, \quad (E44)$$

we obtain, by successive approximations, the formal solution

$$\psi_+(z) = \sum_{n=0}^{\infty} \psi_+^{(n)}(z) \quad (E45)$$

where $\psi_+^{(0)} \triangleq z^{1/4} H_1^{(1)}(z)$ and

$$\psi_+^{(n+1)}(z) \triangleq I\{\psi_+^{(n)}\}. \quad (E46)$$

For the first approximation, we obtain

$$\psi_+^{(1)}(z) = \frac{j}{4\tau} \left(\frac{2}{\pi}\right)^{1/2} z^{-1/4} e^{j(z-3\pi/4)} \int_{z_{M+}}^z z_1^{-1} [O(1) - e^{-j2(z-z_1)} O(1)] O(1) dz_1 \quad (E47)$$

Since $\text{Im} z < \text{Im} z_1$ along the contour, the integral is of the order $\ln \tau$; therefore

$$\psi_+^{(1)}(z) = z^{-1/4} \frac{\ln \tau}{\tau} e^{j(z-3\pi/4)} O(1), \text{ when } |z| > N \text{ and } 0 < \arg z < \pi. \quad (E48)$$

Continuing the iteration procedure, we find that

$$\psi_+^{(n)}(z) = z^{-1/4} \left(\frac{\ln \tau}{\tau}\right)^n e^{j(z-3\pi/4)} O(1), \text{ when } |z| > N \text{ and } 0 < \arg z < \pi, \quad (E49)$$

so that (E45) converges for sufficiently large $|\tau|$.

Consequently, we have

$$U_+(\rho) = [2r(\rho)\tau^{1/2}]^{1/2} [H_1^{(1)}(\tau\rho) + \frac{1}{(\tau\rho)^{1/2}} e^{j(\tau\rho-3\pi/4)} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1)] \quad (E50)$$

when $|\tau|$ is sufficiently large, $0 < \arg \tau < \pi$ and $\rho > \frac{N}{|\tau|}$.

Following the same procedure for ψ_- , we find that

$$U_-(\rho) = [2r(\rho)\tau^{1/2}]^{1/2} [H_1^{(2)}(\tau\rho) + \frac{1}{(\tau\rho)^{1/2}} e^{-j(\tau\rho-3\pi/4)} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1)] \quad (E51)$$

when $|\tau|$ is sufficiently large, $-\pi < \arg \tau < 0$ and $\rho > \frac{N}{|\tau|}$.

Solution when $|z| \leq N$

The variable of integration in (E38) and (E42) extends from a region of large magnitudes to a region of small magnitudes. Using (E14), (E28) and the fact that for $|z| > N$

$$N_1(z) = \left(\frac{1}{2\pi z}\right)^{1/2} \{e^{j(z-5\pi/4)} [1+O(\frac{1}{z})] + e^{-j(z-5\pi/4)} [1+O(\frac{1}{z})]\}, \quad |\arg z| < \pi \quad (E52)$$

we may express (E38) as

$$\psi_+(z) = z^{1/4} H_1^{(1)}(z) + \frac{\pi}{2\tau} \left\{ \left(\frac{1}{2\pi}\right)^{1/2} \int_{z_M+}^{z_N} z_1^{-3/4} [z_1^{5/4} e^{-j(z_1-5\pi/4)} \{O(1) + e^{j2(z_1-5\pi/4)} O(1)\} - z_1^{-3/4} e^{-j(z_1-3\pi/4)} \{O(1) + e^{j2(z_1-3\pi/4)} O(1)\}] \right\}$$

$$\cdot \Delta(z_1) \psi_+(z_1) dz_1 + \int_{z_N}^z [z^{5/4} z_1^{-5/4} O(1) - z^{-3/4} z_1^{3/4} O(1)] \Delta(z_1) \psi_+(z_1) dz_1 \quad (E53)$$

where $|z_N| = N$. Applying successive approximations we obtain the formal solution (E45) with (E46), where now

$$\begin{aligned} I(f) &= \frac{\pi}{2\tau} \left(\left(\frac{1}{2\pi} \right)^{1/2} \int_{z_{M+}}^{z_N} z_1^{-3/4} [z^{5/4} e^{-j(z_1-5\pi/4)} \{O(1) + e^{j2(z_1-5\pi/4)} \right. \\ &\quad \cdot O(1)\} - z^{-3/4} e^{-j(z_1-3\pi/4)} \{O(1) + e^{j2(z_1-3\pi/4)} O(1)\}] \Delta(z_1) f(z_1) dz_1 \\ &\quad \left. + \int_{z_N}^z [z^{5/4} z_1^{-5/4} O(1) - z^{-3/4} z_1^{3/4} O(1)] \Delta(z_1) f(z_1) dz_1 \right) \quad (E54) \end{aligned}$$

Using (E24) and (E25), we obtain for the first approximation

$$\begin{aligned} \psi_+^{(1)}(z) &= \frac{\pi}{2\tau} z^{-3/4} \left\{ \frac{1}{\pi} \int_{z_{M+}}^{z_N} z_1^{-1} [jz^2 \{O(1) + e^{j2(z_1-5\pi/4)} O(1)\} \right. \\ &\quad \left. - \{O(1) + e^{j2(z_1-3\pi/4)} O(1)\}] O(1) dz_1 + \int_{z_N}^z \left[\left(\frac{z}{z_1} \right)^2 O(1) - O(1) \right] O(1) dz_1 \right\}. \quad (E55) \end{aligned}$$

The first of the integrals on the RHS is of the order $\ln \tau$, while the second is bounded. Hence

$$\psi_+^{(1)}(z) = z^{-3/4} \frac{\ln \tau}{\tau} O(1), \text{ when } |z| \leq N. \quad (E56)$$

Continuing the iteration procedure, we obtain

$$\psi_+^{(n)}(z) = z^{-3/4} \left(\frac{\ln \tau}{\tau} \right)^n O(1), \text{ when } |z| \leq N \quad (E57)$$

so that (E45) converges for sufficiently large $|\tau|$. Therefore we have

$$U_+(\rho) = [2r(\rho)\tau^{1/2}]^{1/2} [H_1^{(1)}(\tau\rho) + \frac{1}{\tau} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1)] \quad (\text{E58})$$

when $|\tau|$ is sufficiently large and $\rho \leq \frac{N}{|\tau|}$.

Following the same procedure for ψ_- , we obtain

$$U_-(\rho) = [2r(\rho)\tau^{1/2}]^{1/2} [H_1^{(2)}(\tau\rho) + \frac{1}{\tau\rho} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1)] \quad (\text{E59})$$

when $|\tau|$ is sufficiently large and $\rho \leq \frac{N}{|\tau|}$.

The Solutions y_0 and y_1

Since the solution y_0 must vanish at $\rho=0$, it follows that y_0 is proportional to U_1 . Considering the normalization required by (2.8), we therefore conclude that

$$y_0(\rho, \tau) = \left[\frac{r(\rho)}{r(0)}\right]^{1/2} \frac{2}{\tau} [J_1(\tau\rho) + \tau\rho \sum_{n=1}^{\infty} \rho^n O(1)], \quad (\text{E60})$$

when $|\tau|$ is sufficiently large and $\rho \leq \frac{N}{|\tau|}$, and

$$y_0(\rho, \tau) = \left[\frac{r(\rho)}{r(0)}\right]^{1/2} \frac{2}{\tau} [J_1(\tau\rho) + \left(\frac{1}{\tau\rho}\right)^{1/2} \{e^{j(\tau\rho-3\pi/4)} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1) + e^{-j(\tau\rho-3\pi/4)} \sum_{n=1}^{\infty} \left(\frac{\ln \tau}{\tau}\right)^n O(1)\}], \quad (\text{E61})$$

when $|\tau|$ is sufficiently large and $\rho > \frac{N}{|\tau|}$.

The asymptotic expansion of y_0 , valid away from $\rho = 0$, is given by a linear combination of the forms (3.1). By substituting the asymptotic expansion of J_1 into (E61), the correct linear combination becomes apparent; we obtain

$$y_0(\rho, \tau) \sim \left[\frac{r(\rho)}{r(0)}\right]^{1/2} \left(\frac{2}{\pi\tau\rho}\right)^{1/2} [e^{j(\tau\rho-3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(\rho) + e^{-j(\tau\rho-3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(\rho)], \quad \rho \neq 0(1) \quad (\text{E62})$$

where the $a_k(\rho)$ and $b_k(\rho)$ are determined to within additive constants by (3.4), with $a_0(\rho)=b_0(\rho)\equiv 1$.

We may determine the asymptotic expansion of y_1 away from $\rho=0$ by assuming the linear combination

$$y_1(\rho, \tau) \sim \left(\frac{2}{\pi\tau}\right)^{1/2} e^{j(\tau\rho - 2\tau a + 3\pi/4)} p^{-1/2}(\rho) \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k a_k(\rho) \\ + \left(\frac{2}{\pi\tau}\right)^{1/2} e^{-j(\tau\rho - 3\pi/4)} p^{-1/2}(\rho) \sum_{k=0}^{\infty} \left(\frac{1}{-j\tau}\right)^k b_k(\rho) \quad (E63)$$

and enforcing the boundary conditions (2.6 b). Noting the asymptotic expansions

$$\left. \begin{aligned} H_1^{(2)}(\tau\rho) &\sim \left(\frac{2}{\pi\tau\rho}\right)^{1/2} e^{-j(\tau\rho - 3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k Q_k(\rho) \\ H_1^{(2)'}(\tau\rho) &\sim \left(\frac{2}{\pi\tau\rho}\right)^{1/2} e^{-j(\tau\rho - 3\pi/4)} \sum_{k=0}^{\infty} \left(\frac{1}{j\tau}\right)^k [-j\tau Q_k(\rho) \\ &\quad - \frac{2k+1}{2} Q_k(\rho)] \end{aligned} \right\}, \quad (E64)$$

where

$$Q_k(\rho) \triangleq \frac{\Gamma(3/2+k)}{k! \Gamma(3/2-k) (2\rho)^k}, \quad (E65)$$

we find, with the help of the recurrence formula

$$Q_k(\rho) = \frac{1}{2k\rho} (1/2+k)(3/2-k) Q_{k-1}(\rho), \quad (E66)$$

which follows from the recurrence formula for the Gamma function, that

$$a_k(a) + (-1)^k b_k(a) = Q_k(a), \quad k=0,1,2,\dots \quad (E67)$$

$$a_0(a) - b_0(a) = -Q_0(a)$$

$$\left. \begin{aligned} a_k(a) &= -\left(\frac{k-1}{2a}\right) Q_{k-1}(a) - \frac{1}{2} [a_{k-1}'(a) + (-1)^{k-1} \\ &\quad \cdot b_{k-1}'(a)], \quad k=1,2,3,\dots \end{aligned} \right\} \quad (E68)$$

for TE polarization, and

$$a_k(a) + (-1)^k b_k(a) = \epsilon^{-1/2}(a) Q_k(a) \quad (\text{E69})$$

$$a_0(a) - b_0(a) = -\epsilon^{-1/2}(a) \epsilon_R(a) Q_0(a)$$

$$a_k(a) = -\frac{1}{2a} \epsilon^{-1/2}(a) Q_{k-1}(a) \left\{ k-1 + [\epsilon_R(a)-1] \frac{(2k-3)(2k-1)}{8k} + \frac{a \epsilon'(a)}{2 \epsilon(a)} \right\}, \quad (\text{E70})$$

$$k = 1, 2, 3, \dots$$

for TM polarization. These equations may be used to find the additive constants necessary to uniquely determine the $a_k(\rho)$ and $b_k(\rho)$ from (3.4).

With the help of (3.4) and (E66), it can be shown from (E67)-(E70) that

$$a_k(\rho) \equiv 0, \quad k \leq \mu+1$$

$$a_{\mu+2}(\rho) \equiv -(1/2)^{\mu+2} p_\mu Q_0(a) \quad (\text{E71})$$

$$b_0(\rho) \equiv Q_0(a)$$

in the TE case, and

$$a_k(\rho) \equiv 0, \quad k \leq \mu-1$$

$$a_\mu(\rho) \equiv -(1/2)^{\mu+1} \frac{1}{k_0^2 \epsilon^{1/2}(a)} p_\mu Q_0(a) \quad (\text{E72})$$

$$b_0(\rho) \equiv \frac{p_0 + 2k_0^2}{2k_0^2 \epsilon^{1/2}(a)} Q_0(a)$$

in the TM case, where μ is the order of the first non-vanishing derivative of $\Delta k^2(\rho)$ at $\rho=a$ (taken from the left):

$$\Delta k^2(\rho) = \sum_{m=0}^{\infty} \frac{1}{m!} p_m (\rho-a)^m; \quad p_m \equiv 0, \quad m < \mu. \quad (\text{E73})$$

Actually, we have determined (E71) and (E72) by direct calculation only for $\mu \leq 3$. However, we speculate that these results are valid for all μ .

Substituting (E71) and (E72) into (E63) and comparing the results with the leading terms in (E50) and (E51), it becomes evident that

$$y_1(\rho, \tau) = 2^{-1/2} \tau^{-1/4} [U_-(\rho) - (\frac{1}{j2\tau})^{\mu+2} p_\mu e^{-j2(\tau a - 3\pi/4)} U_+(\rho)] \quad (E74)$$

in the TE case, and

$$y_1(\rho, \tau) = 2^{-1/2} \tau^{-1/4} \frac{1}{2k_0^2 \epsilon^{1/2}(a)} [(p_0 + 2k_0^2) U_-(\rho) - (\frac{1}{j2\tau})^\mu p_\mu \cdot e^{-j2(\tau a - 3\pi/4)} U_+(\rho)] \quad (E75)$$

in the TM case. Hence it follows from (E58) and (E59) that when $|\tau|$ is sufficiently large and $\rho \leq \frac{N}{|\tau|}$,

$$y_1(\rho, \tau) = [H_1^{(2)}(\tau\rho) + \frac{1}{\tau\rho} \sum_{n=1}^{\infty} (\frac{1n\tau}{\tau})^n O(1) - (\frac{1}{j2\tau})^{\mu+2} p_\mu e^{-j2(\tau a - 3\pi/4)} \{H_1^{(1)}(\tau\rho) + \frac{1}{\tau\rho} \sum_{n=1}^{\infty} (\frac{1n\tau}{\tau})^n O(1)\}] \quad (E76)$$

for TE polarization, and

$$y_1(\rho, \tau) = \frac{1}{2k_0^2} [\frac{\epsilon(\rho)}{\epsilon(a)}]^{1/2} [(p_0 + 2k_0^2) (H_1^{(2)}(\tau\rho) + \frac{1}{\tau\rho} \sum_{n=1}^{\infty} (\frac{1n\tau}{\tau})^n O(1)) - (\frac{1}{j2\tau})^\mu p_\mu e^{-j2(\tau a - 3\pi/4)} \{H_1^{(1)}(\tau\rho) + \frac{1}{\tau\rho} \sum_{n=1}^{\infty} (\frac{1n\tau}{\tau})^n O(1)\}] \quad (E77)$$

for TM polarization.

APPENDIX F

BRANCH CUT INTEGRAL

We consider the integral

$$I = \int_{C_b} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds \quad (F1)$$

on the sheet corresponding to the principal value of the arguments of s and τ . The path C_b , illustrated in Fig. F-1, consists of the straight lines above and below the branch cut, plus the arc of a circle of radius δ about the branch point, as $\delta \rightarrow 0$.

Changing the variable of integration from s to $se^{j2\pi}$ along the straight line above the branch cut, we may write

$$\int_{s=\delta e^{-j\pi}}^{\delta e^{j\pi}} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds + \int_{s=\delta e^{-j\pi}}^{\infty e^{-j\pi}} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds = \int_{s=\delta e^{-j\pi}}^{\infty e^{-j\pi}} \frac{g(\rho, \xi; s) - g(\rho, \xi; se^{j2\pi})}{s^\lambda (s-\tau)} ds \quad (F2)$$

Substituting (2.10) for g and noting that $U_{1,2}(\rho, s)$ are integral functions of s , we find that

$$\int_{s=\delta e^{-j\pi}}^{\infty e^{-j\pi}} \frac{g(\rho, \xi; s) - g(\rho, \xi; se^{j2\pi})}{s^\lambda (s-\tau)} ds = \frac{p(a)r(0)}{2} \int_{s=\delta e^{-j\pi}}^{\infty e^{-j\pi}} \frac{U_1(\rho, s)U_1(\xi, s)}{s^\lambda (s-\tau)} ds$$

$$\cdot \left[\frac{A_{21}(s)}{A_{22}(s)} - \frac{A_{21}(se^{j2\pi})}{A_{22}(se^{j2\pi})} \right] ds. \quad (F3)$$

By using equations (2.7)-(2.9) and the relation

$$H_1^{(2)}(ze^{j2\pi}) = 3H_1^{(2)}(z) + 2H_1^{(1)}(z) \quad (F4)$$

we may express (F3) as

$$\begin{aligned} & \int_{s=\delta e^{-j\pi}}^{\infty e^{-j\pi}} \frac{U_1(\rho, s) U_1(\xi, s)}{s^\lambda (s-\tau)} \left[\frac{A_{21}(s)}{A_{22}(s)} - \frac{A_{21}(se^{j2\pi})}{A_{22}(se^{j2\pi})} \right] ds \\ &= \frac{8K(a)}{j\pi a} \int_{s=\delta e^{-j\pi}}^{\infty e^{-j\pi}} \frac{Y_0(\rho, s) Y_0(\xi, s)}{F_2(s) [2F_1(s) + 3F_2(s)] s^\lambda (s-\tau)} ds \quad (F5) \end{aligned}$$

where

$$K(\rho) = \begin{cases} 1 & , \text{ for TE polarization} \\ \epsilon_R(\rho) & , \text{ for TM polarization} \end{cases} \quad (F6)$$

and

$$F_{1,2}(s) = H_1^{(1,2)}(sa) Y_0'(a, s) - \{ K(a) H_1^{(1,2)'}(sa) + \frac{K(a)-1}{a} H_1^{(1,2)}(sa) \} Y_0(a, s) \quad (F7)$$

Since in the neighborhood of $s=0$ $U_1(\rho, s)$ is bounded (see Appendix D) and

$$H_1^{(1,2)}(s\rho) = \mp j \frac{2}{\pi s \rho} + \frac{\rho s}{2} \left\{ 1 \mp j \frac{2}{\pi} [\gamma - 1/2 + \ln(\frac{\rho s}{2})] \right\} + O(s^3 \ln s), \quad (F8)$$

where γ is Euler's constant, it follows that the integrand in (F5) is $O(s^{2-\lambda})$ as $s \rightarrow 0$. Hence, to insure that (F5) converges as $\delta \rightarrow 0$, we require that

$$\lambda < 3. \quad (F9)$$

For the integral about the circular arc enclosing the branch point, we first estimate $g(\rho, \xi; s)$ as $s \rightarrow 0$. From Appendix D we see that $U_{1,2}(\rho, s)$ are integral functions of

s^2 which, near the origin, have the form

$$U_{1,2}(\rho, s) = B_{1,2}(\rho) [1 + C_{1,2}(\rho) s^2 + O(s^4)]. \quad (F10)$$

Furthermore, noting (F7), we find that $A_{21,22}(s)$ have the form

$$A_{21,22}(s) = \frac{1}{s} [d_{1,2} + O(s^2 \ln s)]. \quad (F11)$$

Substituting these into (2.10) indicates that

$$g(\rho, \xi; s) = D + O(s^2 \ln s). \quad (F12)$$

Hence

$$\int_{C_\delta} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds = D \int_{C_\delta} \frac{1}{s^\lambda (s-\tau)} ds + \int_{C_\delta} \frac{O(s^2 \ln s)}{s^\lambda (s-\tau)} ds. \quad (F13)$$

By setting $s = \delta e^{j\theta}$ and integrating from $\theta=\pi$ to $\theta=-\pi$, we find that as $\delta \rightarrow 0$ the second integral on the RHS of (F13) vanishes for the allowed λ , given by (F9). The first integral also vanishes, except when $\lambda=1$. Noting that $D=g(\rho, \xi; 0)$, we therefore have

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} \frac{g(\rho, \xi; s)}{s^\lambda (s-\tau)} ds = \begin{cases} 0 & , \text{ for } \lambda = 0, 2 \\ \frac{2\pi j}{\tau} g(\rho, \xi; 0), & \text{ for } \lambda = 1 \end{cases} \quad (F14)$$

APPENDIX G

RESIDUE CALCULATIONS FOR CIRCULARLY CYLINDRICAL WAVEGUIDES

The transverse Green's function

$$g(\rho, \xi; \tau) = \frac{Y_0(\rho, \tau) Y_1(\xi, \tau)}{J(\tau)} H(\xi - \rho) + \frac{Y_0(\xi, \tau) Y_1(\rho, \tau)}{J(\tau)} H(\rho - \xi) \quad (G1)$$

has a residue at $\tau = \tau_n$ given by

$$\text{Res } g(\rho, \xi; \tau_n) = \frac{Y_0(\rho, \tau_n) Y_1(\xi, \tau_n)}{\frac{d}{d\tau} J(\tau_n)} H(\xi - \rho) + \frac{Y_0(\xi, \tau_n) Y_1(\rho, \tau_n)}{\frac{d}{d\tau} J(\tau_n)} H(\rho - \xi). \quad (G2)$$

Since the conjunct

$$J(\tau) = p(\rho) [y_0'(\rho, \tau) y_1(\rho, \tau) - y_0(\rho, \tau) y_1'(\rho, \tau)] \quad (G3)$$

is independent of ρ , we may set $\rho = a$ and note that

$$y_1(a, \tau) = H_1^{(2)}(\tau a), \quad y_1'(a, \tau) = K(a) H_1^{(2)'}(\tau a) + \frac{K(a) - 1}{a} H_1^{(2)}(\tau a) \quad (G4)$$

where $K(\rho)$ is defined by (F6), to obtain

$$J(\tau) = p(a) [H_1^{(2)}(\tau a) y_0'(a, \tau) - \{K(a) H_1^{(2)'}(\tau a) + \frac{K(a) - 1}{a} H_1^{(2)}(\tau a)\} y_0(a, \tau)] \quad (G5)$$

Therefore

$$\begin{aligned} \frac{d}{d\tau} J(\tau) = & p(a) [H_1^{(2)}(\tau a) \frac{d}{d\tau} y_0'(a, \tau) + y_0'(a, \tau) \frac{d}{d\tau} H_1^{(2)}(\tau a) \\ & - \{K(a) H_1^{(2)'}(\tau a) + \frac{K(a) - 1}{a} H_1^{(2)}(\tau a)\} \frac{d}{d\tau} y_0(a, \tau)] \end{aligned}$$

$$- \{K(a) \frac{d}{d\tau} H_1^{(2)'}(\tau a) + \frac{K(a)-1}{a} \frac{d}{d\tau} H_1^{(2)}(\tau a)\} y_0(a, \tau). \quad (G6)$$

Now y_0 and y_1 are solutions to

$$[p(\rho) y'(\rho, \tau)]' + p(\rho) [\Delta k^2(\rho) + \tau^2 - \frac{K'(\rho)}{\rho K(\rho)} - \frac{1}{\rho^2}] y(\rho, \tau) = 0. \quad (G7)$$

Multiplying this by $\frac{\partial}{\partial \tau} y(\rho, \tau)$, we obtain

$$\frac{\partial}{\partial \tau} y(\rho, \tau) [p(\rho) y'(\rho, \tau)]' + p(\rho) [\Delta k^2(\rho) + \tau^2 - \frac{K'(\rho)}{\rho K(\rho)} - \frac{1}{\rho^2}] y(\rho, \tau) \frac{\partial}{\partial \tau} y(\rho, \tau) = 0. \quad (G8)$$

Differentiate (G7) with respect to τ and multiply the result by $y(\rho, \tau)$:

$$y(\rho, \tau) [p(\rho) \frac{\partial}{\partial \tau} y'(\rho, \tau)]' + p(\rho) [\Delta k^2(\rho) + \tau^2 - \frac{K'(\rho)}{\rho K(\rho)} - \frac{1}{\rho^2}] y(\rho, \tau) \frac{\partial}{\partial \tau} y(\rho, \tau) + 2\tau p(\rho) y^2(\rho, \tau) = 0. \quad (G9)$$

Subtracting (G9) from (G8) and integrating by parts, we obtain the relationship

$$[p(\rho) \{y'(\rho, \tau) \frac{\partial}{\partial \tau} y(\rho, \tau) - y(\rho, \tau) \frac{\partial}{\partial \tau} y'(\rho, \tau)\}]_{\rho_1}^{\rho_2} = 2\tau \int_{\rho_1}^{\rho_2} p(\rho) y^2(\rho, \tau) d\rho. \quad (G10)$$

Setting $y=y_0$, $\rho_2=a$, $\rho_1=0$, and noting that $y_0(0, \tau)=0$, we find

$$y_0'(a, \tau) \frac{\partial}{\partial \tau} y_0(a, \tau) - y_0(a, \tau) \frac{\partial}{\partial \tau} y_0'(a, \tau) = \frac{2\tau}{p(a)} \int_0^a p(\rho) y_0^2(\rho, \tau) d\rho. \quad (G11)$$

Solving for $\frac{\partial}{\partial \tau} y_0'(a, \tau)$ and substituting into (G6), we obtain

$$\frac{d}{d\tau} J(\tau) = p(a) [H_1^{(2)}(\tau a) \frac{y_0'(a, \tau)}{y_0(a, \tau)} - \{K(a) H_1^{(2)'}(\tau a) + \frac{K(a)-1}{a} H_1^{(2)}(\tau a)\}]$$

$$+ \frac{\partial}{\partial \tau} y_0(a, \tau) + p(a) [y_0'(a, \tau) \frac{d}{d\tau} H_1^{(2)}(\tau a) - \{K(a) \frac{d}{d\tau} H_1^{(2)'}(\tau a) + \frac{K(a)-1}{a}$$

$$\frac{d}{d\tau} H_1^{(2)}(\tau a) \} y_0(a, \tau)] - 2\tau \frac{H_1^{(2)}(\tau a)}{y_0(a, \tau)} \int_0^a p(\rho) y_0^2(\rho, \tau) d\rho. \quad (G12)$$

At $\tau = \tau_n$ the conjunct vanishes, so that from (G5)

$$H_1^{(2)}(\tau_n a) y_0'(a, \tau_n) - \{ K(a) H_1^{(2)'}(\tau_n a) + \frac{K(a)-1}{a} H_1^{(2)}(\tau_n a) \} y_0(a, \tau_n) = 0. \quad (G13)$$

Therefore, from (G12) and (G13),

$$\begin{aligned} \frac{d}{d\tau} J(\tau_n) = & \frac{p(a)K(a)}{H_1^{(2)}(\tau_n a)} [H_1^{(2)'}(\tau_n a) \frac{d}{d\tau} H_1^{(2)}(\tau_n a) - H_1^{(2)}(\tau_n a) \frac{d}{d\tau} H_1^{(2)'}(\tau_n a)] \\ & \cdot y_0(a, \tau_n) - 2\tau_n \frac{H_1^{(2)}(\tau_n a)}{y_0(a, \tau_n)} \int_0^a p(\rho) y_0^2(\rho, \tau_n) d\rho. \end{aligned} \quad (G14)$$

The vanishing of $J(\tau_n)$ also implies that y_0 and y_1 are proportional at $\tau = \tau_n$; since $y_1(a, \tau_n) = H_1^{(2)}(\tau_n a)$, it follows that

$$y_1(\rho, \tau_n) = \frac{H_1^{(2)}(\tau_n a)}{y_0(a, \tau_n)} y_0(\rho, \tau_n). \quad (G15)$$

Substituting (G14) and (G15) into (G2) and choosing as eigenfunctions

$$\phi_n(\rho) = \left[-\frac{1}{2\tau_n} \frac{y_0(a, \tau_n)}{H_1^{(2)}(\tau_n a)} \frac{d}{d\tau} J(\tau_n) \right]^{-1/2} y_0(\rho, \tau_n) \quad (G16)$$

we obtain

$$\text{Res } g(\rho, \xi; \tau_n) = -\frac{1}{2\tau_n} \phi_n(\rho) \phi_n(\xi). \quad (G17)$$

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